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CONSENSUS IN MULTI-AGENT SYSTEMS WITH INPUT SATURATION AND TIME-VARYING DELAYS

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Dissertation submitted to the Graduate Program in Electrical Engineering of the Federal University of Minas Gerais, in partial fulfillment of the requirements for the degree of Master in Electrical Engineering.

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“We who cut mere stones must always be envisioning cathedrals.”

Medieval Quarry Worker’s Creed

Abstract

This dissertation deals with the problem of consensus in multi-agent systems. Sufficient conditions for consensus are derived for networked systems constituted of identical agents with linear models of arbitrary order, input saturation, and subject to non-uniform input time-varying delays. To this end we use the framework of Lyapunov-Krasovskii theory–an extension of Lyapunov theory for delayed systems.

We use a linear combination of feedback matrices to represent the saturation non-linearity. Although this representation might introduce some conservatism, in the sense that the representation can cover a larger set than the actual saturated input, it allows to handle the problem via linear matrix inequalities. Additionally, this methodology is also convenient to extend the conditions of stability analysis and derive a systematic method to synthesize the feedback gains for the agents, in such a way that guarantees consensus. The presence of saturation might prevent the multi-agent system to attain consensus from any set of initial conditions, for this reason we propose an estimation for the region in which the consensus can always be reached.

The input time-varying delay is considered non-uniform over the network, possibly non-differentiable, and belonging to a closed set. The closed set considered may have a positive lower limit. Thus, the study is applicable to interval delays. Thereby, sufficient conditions for the analysis of consensus are expressed considering upper and lower bounds for the input delays. Moreover, we propose algorithms to analyse the consensus and to synthesize the feedback matrices with a set of initial conditions as large as possible that guarantees consensus. Both problems are formulated with linear matrix inequalities constraints.

Finally, throughout the text we present motivational examples that demonstrate the importance of developing conditions for networked systems considering time-delay and input saturation simultaneously. We also present examples to illustrate the effectiveness of the proposed method.

Resumo

Esta dissertação aborda o problema de consenso em sistemas multiagentes. São propostas condições suficientes que garantem o consenso para sistemas interconectados constituídos de agentes idênticos com modelo linear de ordem arbitrária, com saturação nos atuadores, e sujeitos a atrasos não-uniformes nas entradas. Para este fim o arcabouço da teoria de Lyapunov-Krasovskii é utilizado– uma extensão da teoria de Lyapunov apropriada para sistemas com atraso.

A representação da não-linearidade causada pela saturação é realizada através da combinação linear de matrizes de realimentação. Embora esta representação possa introduzir conservadorismo, no sentido de que a representação pode abranger um conjunto maior do que o gerado pela saturação na entrada, ela permite formular o problema por meio de desigualdades matriciais lineares. Além disso, esta metodologia é coveniente para estender as condições de análise de estabilidade e permite derivar um método sistemático para sintetizar os ganhos dos controladores dos agentes que levam ao consenso. A presença da saturação pode impedir que o sistema multiagentes atinja o consenso a partir de qualquer conjunto de condições iniciais, por essa razão é proposta uma estimativa para a região inicial na qual o consenso pode ser sempre alcançado.

Os atrasos nos atuadores dos agentes são considerados variantes no tempo, não-uniformes, e possivelmente não-diferenciáveis, pertencentes a um conjunto fechado. Este conjunto pode possuir um limite inferior diferente de zero, ou seja, a abordagem é aplicável a atrasos em intervalos. Deste modo, condições suficientes para a análise de consenso são expressas levando em consideração os limites inferior e superior dos atrasos de entrada. Ademais, são propostos algoritmos para analisar a capacidade do sistema entrar em consenso e para projetar as matrizes de ganho enquanto se busca o maior conjunto de condições iniciais que garantem o consenso. Ambos problemas são formulados com restrições na forma de desigualdades matriciais lineares.

Finalmente, exemplos são apresentados ao longo do texto de forma a ilustrar a importância do desenvolvimento de condições que garantam o consenso voltadas para sistemas multiagentes com agentes sujeitos a saturação e atrasos nos atuadores, simultaneamente. Também são apresentados exemplos que ilustram a eficácia dos métodos propostos.

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Muito obrigado a todos.

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List of Abbreviations

LMI	**L**inear **M**atrix **I**nequality
BMI	**B**i-linear **M**atrix **I**nequality

List of Symbols

$\mathbb{R}$	-	set of real numbers
$\vert\vert \cdot \vert\vert$	-	vector norm
M^T, M^{-1},	-	transpose and inverse of matrix M
$M > 0$ ($M \geq 0$)	-	matrix M is positive definite (semi-definite)
$M < 0$ ($M \leq 0$)	-	matrix M is negative definite (semi-definite)
$diag\{\cdot\}$	-	denotes a diagonal matrix
$Co\{\cdot\}$	-	denotes the convex hull
I_m	-	identity matrix of dimension m
$\mathbf{1}_n$	-	vector of ones in $\mathbb{R}^n$
$\mathbf{0}_n$	-	vector of zeros in $\mathbb{R}^n$
0	-	matrix of zeros with appropriate dimensions
$\mathcal{C}_\tau^n$	-	set of continuous functions on the interval $[-\tau, 0]$, in $\mathbb{R}^n$
$A \otimes B$	-	Kronecker product between matrices A and B
$\mathcal{G}$	-	graph
$\mathcal{V}$	-	vertex set of graph $\mathcal{G}$
$\mathcal{E}$	-	edges set of graph $\mathcal{G}$
$\mathcal{A}$	-	Adjacency Matrix associated with graph $\mathcal{G}$
$\mathcal{D}$	-	Degree Matrix associated with graph $\mathcal{G}$
$\mathcal{L}$	-	Laplacian Matrix associated with graph $\mathcal{G}$
$\boldsymbol{x}_i$	-	ith agent's state variables. Vector of dimension $\mathbb{R}^m$
$\boldsymbol{u}_i$	-	ith agent's input variables. Vector of dimension $\mathbb{R}^p$
$\boldsymbol{x}$	-	stacked state variables. Vector of dimension $\mathbb{R}^{nm}$
$\boldsymbol{u}$	-	stacked input variables. Vector of dimension $\mathbb{R}^{np}$
τ_i(t)	-	ith agent's input time-varying delay
τ	-	constant input delay
μ_m	-	bound for time-varying delay
$\boldsymbol{\phi}_i$	-	initial condition for ith agent
$u_{i(k)}$	-	kth entry of control input $\boldsymbol{u}_i$
$\boldsymbol{z}_i$	-	ith entry of disagreement of state variables. Vector of dimension $\mathbb{R}^m$
$\boldsymbol{z}$	-	stacked disagreement of state variables. Vector of dimension $\mathbb{R}^{m(n-1)}$
K	-	gain matrix of dimension $\mathbb{R}^{p\times m}$

u_{max}	-	limit of actuators
$\mathcal{S}_{DOC}$	-	domain of consensus
$\mathcal{L}(\cdot)$	-	linear region for multi-agent system with saturation
$\mathcal{I}_i$	-	vector of dimension $\mathbb{R}^n$ with 1 on ith position and 0 elsewhere

To my love, Carolina.

Chapter 1

Introduction

The wide range of applications for systems that work cooperatively have make the research of these arrangements remarkably attractive. This type of connection is desirable since it gives the networked system features like robustness to individual failure, network scalability, and the possibility to attain tasks otherwise very hard for single systems (for example, carry a heavy and sparse load). That is to say, there are situations where it is simpler to develop solutions for a group of systems than for a single one.

In robotics these networked systems are often called multi-agent systems, where each agent is an individual entity able to make decisions and interact with the environment. Agents in such networks are required to operate harmoniously with each other in order to achieve a global level objective, usually having limited access to information and using simple rules. It is possible to divide the control approach for multi-agent systems in two large groups, according to the kind of information that each agent has access and the group communication. Namely, there are the *centralized* and *decentralized* methods. On the former, the actions of each agent are planned by a central controller which have access to information of the whole group. Whereas the latter, the agents share information only with a subset of the group, called neighbors, and use local information to plan its own actions.

Although the centralized approach may be applicable in order to ensure the accomplishment of various tasks, it is not scalable in general and the complexity of these systems grows fast. On the other hand, decentralized methods impose that each agent have rules that use only local information, in this way, it may be possible to add agents without increasing the complexity of the task to be performed by the networked system.

In the field of multi-agent systems, consensus problem is a hot topic, partly due to the vast number of problems in groups of robots that can be studied as consensus (see Qin et al., 2017; Cao et al., 2013, and references therein for recent advances). The main aspect of the consensus problem is to make the network reach an agreement on pre-specified variables in a cooperative way using distributed protocols with the use of local information (that is, a decentralized approach).

There is a large and crescent core of literature on establishing conditions for consensus subject to different scenarios with the aim to analyse realistic conditions, such as intermittent communication (Li and Liu, 2018; Savino et al., 2016; Yang, Wang, and Ni, 2013), external disturbances (Yang, Wang, and Ni, 2013; Dal Col, Tarbouriech, and Zaccarian, 2016; Qin et al., 2015), faults (Semsar-Kazerooni and Khorasani, 2010; Chen et al., 2017), input saturation (Jesus et al., 2014; Deng and Yang, 2017; Li and Lin, 2017), or with challenges related to implementation like event-triggered (Dimarogonas, Frazzoli, and Johansson, 2012; Cui et al., 2018; Wang et al., 2017), signed networks (Altafini, 2013; Valcher and Misra, 2014) and heterogeneous systems (Li and Liu, 2018; Zheng and Wang, 2015; Wen et al., 2016). Within this large field, few relevant works can be found where the saturation and time-delay are addressed together, although this is not a rare situation in physical systems. To the best of the authors' knowledge, very few papers such as You et al., 2016, Yanumula, Kar, and Majhi, 2017, and more recently Ding, Zheng, and Guo, 2018 have considered the consensus with time-delay and saturating inputs.

Time-delay is an important feature to take into account on system stabilization, it is almost inevitable and its presence can lead to undesirable performance and even to instability. Its relevance does not decrease for multi-agent systems, as first shown by Olfati-Saber and Murray, 2004 it can prevent the network to reach consensus. Olfati-Saber and Murray have started the analysis of consensus in multi-agent systems with constant uniform time-delays and then their research was expanded for different types of delays, for example, considering multiple time-varying delays (Sun, Wang, and Xie, 2008) and nonuniform delays (Bliman and Ferrari-Trecate, 2008).

Regarding saturation, this constraint causes a nonlinear behavior in the closed-loop system that can lead to performance degradation, occurrence of limit cycles and, instability, which may also prevent the multi-agent system to reach consensus. There are basically two main strategies that are generally used in overall linear saturated control systems (Tarbouriech and Turner, 2009): the first one is to design the controller without considering the actuators limits and then to introduce modifications in order to minimize saturation effects (Oliveira et al., 2013;

Tarbouriech and Silva, 2005). In this approach the system behaves as linear whenever no saturation is encountered and the modifications become active only in the presence of saturation. The second strategy is to take into account the limits of actuators *a priori*, while trying to ensure performance requirements. This approach allows a clear trade-off between performance and the size of the region of stability (when global stabilization is not guaranteed) (Paim et al., 2002; Castelan et al., 2006). It is worth mentioning that within this strategy there are promising approaches to deal with saturation, for example Zhang, Wang, and Li, 2013 presents a distributed model predictive control to deal with actuator saturation and derive linear matrix inequalities conditions to solve it. Cheng et al., 2015 also suggests model predictive control protocols for multi-agent systems with dynamics of double-integrators reaches consensus. The problem can also be approached using fuzzy logic (Zhang et al., 2015; Wang and Tong, 2018), and others strategies. See the work of Wang et al., 2016 for a review on strategies for coordination of multi-agent systems subject to saturation.

This work fits into the second strategy in the context of consensus with input time-delay, that is, the controller synthesis is performed while trying to enlarge the size of the region that guarantee consensus taking into account the saturation.

1.1 History

The consensus problem has its roots on statistical analysis, Degroot, 1974 is one of the firsts to use ideas and terms that later composed the lexicon of multi-agent systems that lead the research line on consensus. There are lots of similarities, Degroot proposed a method for a group of individuals to reach an agreement on estimation of an unknown quantity given that each individual had its own probability distribution of this quantity.

Another field that preceded and influenced consensus in multi-agent systems was the graphical modeling on computers of flocks of birds and herds of land animals. The objective was to simulate the emergent behavior that occurs in large groups of animals, for example the alignment and movement at same speed that these groups tend to attain. A remarkable characteristic is that on these models each individual is designed to react to its neighborhood with simple rules. Reynolds, 1987 proposed an algorithm to simulate this behavior in a three dimensional space and established his results using simulations, without convergence proofs. Then, Vicsek et al., 1995 proposed a simpler model for particles moving on the plane, a particular case of Reynolds' results. In their paper Vicsek et al., 1995, provided a variety of simulations to show that the agents eventually move in the same direction in absence of centralized control and with varying neighborhood. The theoretical explanation of the results of Vicsek et al., 1995 was given later by Jadbabaie, Lin, and Morse, 2003.

In the context of control of multi-agent systems, the consensus problem arises later in the 90's with the problem of formation control in robotics and it inherits the theoretical development of statistics and graphical modeling (Suzuki and Yamashita, 1999; Singh et al., 2000; Liu, Passino, and Polycarpou, 2001; Mesbahi and Hadaegh, 2001). At this time the main objective was to find protocols to make a group of robots perform a given task in a cooperative way. Particularly, Suzuki and Yamashita, 1999 studied algorithms that distribute groups of robots to form geometrical patterns with rigorous proofs of convergence. Singh et al., 2000 suggested an adaptive control system for multiple unmanned aerial vehicles asymptotically track a desired shape, and Liu, Passino, and Polycarpou, 2001 proposed distributed models for members of a swarm which guarantee convergence for groups composed of these models. At the time an interesting interplay between Lyapunov theory and algebraic graph theory was shown in the works of Mesbahi and Hadaegh, 2001 and Fax and Murray, 2002. After that, the study of distributed networked systems using this representation became standard.

1.2 Dissertation contribution overview

This work can be viewed as a natural extension of the study developed by Savino, Souza, and Pimenta, 2014, in the sense that it is now considered models for the agents with saturating inputs and it is suggested a method to design the feedback gains for the consensus protocol.

This study belongs to a still little explored context. It is worth remarking that differently from the present work neither You et al., 2016 nor Yanumula, Kar, and Majhi, 2017 developed strategies to design the controller gains. They consider the particular case of classifying the network capacity to reach consensus and, additionally, they were concerned with consensus in a global or semi-global context. Thus, when the agents are open-loop unstable the above approaches are not applicable, since they consider fixed linear feedback controllers (Lin, 2019; Li and Lin, 2017). In this sense our approach expresses a more complete result.

In this dissertation, the consensus problem is translated to a stability analysis one and the saturation is written in terms of an auxiliary feedback matrix, as presented for linear systems by Hu, Lin, and Chen, 2002. The time-delay is considered nonuniform and, possibly, non-differentiable within a closed set (Savino, Souza, and Pimenta, 2014).

We propose sufficient conditions for stability analysis and synthesis of controllers for consensus in multi-agent systems. For this end we used the *Lyapunov-Krasovskii* theory (Fridman, 2014), and the conditions are derived in terms of linear matrix inequalities (LMIs), appropriated for linear agents with input saturation and input time-varying delays.

The contributions of this work come in twofold: firstly, it is proposed a method to study the consensus in terms of the stability of linear multi-agent systems subject to input saturation and input time-varying delays, based on the transformation carried by Sun and Wang, 2009. Secondly, we present a systematic method to synthesize the controller gains for agents with saturating inputs while maximizing the size of the estimation of the domain of attraction for the consensus.

The remaining of this text is divided in the following way:

- **Chapter 2:** A formal definition of consensus is given and it is shown examples to illustrate the importance of the study of multi-agent systems with input constraints and delays.
- **Chapter 3:** It is presented a transformation that allows consensus problems with input saturation and delays to be analyzed as stability problems.
- **Chapter 4:** In this chapter the proposed methods for consensus analysis and gains design are presented, as well as mathematical proofs of these results and numerical examples.
- **Chapter 5:** Finally, final considerations about the text and suggestions for future works are made.

Chapter 2

Preliminary theory and presentation

In this chapter we introduce the fundamental topic developed in this dissertation. We bring the definition of consensus and use motivational examples to illustrate the consensus under saturation and time-delays. Firstly, we show the use of graph theory to represent the communication topology of the multi-agent system, then the transformation of consensus problem into stability problem is presented and, finally, some background assumptions related with saturation and time-delay used throughout the text are presented using examples.

2.1 Graph theory

Interaction between agents in a multi-agent system is a *necessary* condition for cooperation and its representation has a vital role in the network analysis and synthesis, regardless the medium that the information exchange occurs. To take the dynamics of information exchange at the level of agents' model might not be an easy task, for this reason very often the agents' interactions are represented using graphs. With this abstraction the communication complexity becomes essentially combinatorial: it identifies which pairs of agents have interactions and to what degree. To illustrate this idea consider the group of five robots in Figure 2.1, where each dot on the left side represents a robot and the gray area represents its sensors' range. The interaction topology is abstracted on the right side as a graph, where the nodes represent the robots and an edge exists only if a robot's sensor is able to perceive a neighbor robot, the arrows represent the direction of the information flow.

The use of algebraic theory of graphs to model interactions between systems have been used by several authors on the context of networked systems. It was firstly used by Mesbahi and Hadaegh, 2001 and then this representation became popular (most of our references represent the multi-agent system communication in this way).

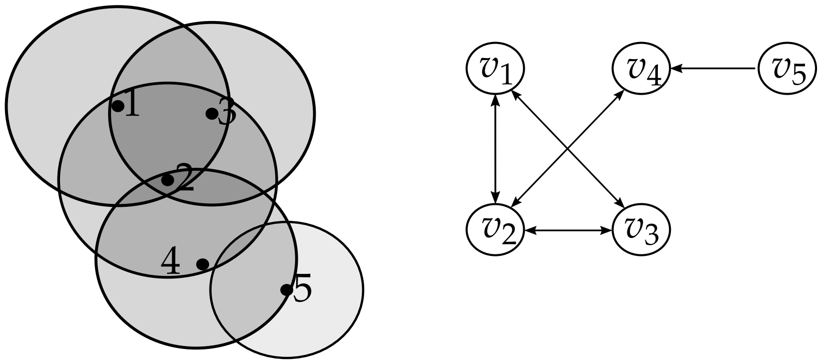


FIGURE 2.1: Graph representation of agents interaction.

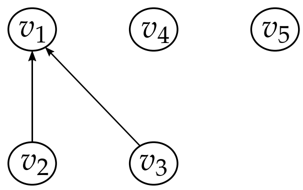
A *finite, directed, simple graph* is built upon a finite set called *vertex set* denoted by $\mathcal{V} = \{v_1, \ldots, v_n\}$ (Mesbahi and Egerstedt, 2010). Then, to formally define a graph $\mathcal{G}$ we use a particular subset of the Cartesian product $\mathcal{V} \times \mathcal{V}$, named *edges of* $\mathcal{G}$ and assign the symbol $\mathcal{E}$ to it. We want that each element of the edge set, e_{ij}, represents a directed information flow on the multi-agent system, to this end we construct $\mathcal{E}$ by choosing only ordered pairs that represent two agents that interact with each other. Thus, on each element $e_{ij} = (v_i, v_j)$ the first argument v_i is the parent node (the arrow's tail) and the second argument v_j is the child node (the arrow's head), so formally we have: $\mathcal{E} = \{(v_i, v_j) : v_j \text{ receive information from } v_i\}$. Sometimes we need to refer to the set of all vertices that interact with a specific vertex, v_i, specifically $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$ and we call it neighborhood of the vertex v_i. Additionally, we call adjacent node, or neighbor of v_i, a node v_j that belongs to the neighborhood $\mathcal{N}_i$. We define the Adjacency Matrix $\mathcal{A}$ by assigning its elements according the edge set:

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } \nexists e_{ji}, \\ 1, & \text{if } e_{ji} \in \mathcal{E}. \end{cases} \tag{2.1}$$

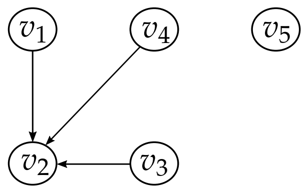
Then, the idea of moving from node to node by means of the edges comes naturally using the notion of adjacency. To speak about this we use the term *path* related to the graph $\mathcal{G}$, which is given by a sequence of nodes $\{v_1, \ldots, v_n\}$ such that for all $k = \{1, \ldots, n-1\}$, the nodes v_k and v_{k+1} are adjacent and v_{k+1} is a child node.

We define the Degree Matrix $\mathcal{D}$ associated with the graph $\mathcal{G}$ as a diagonal matrix with elements given by $d_{ii} = \sum_{j=1}^{n} a_{ij}$. Additionally, the Laplacian matrix associated to the graph is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$.

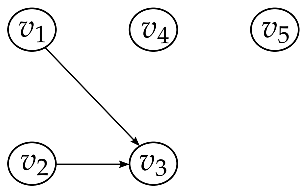
Throughout the text we use the idea of sub-graph of a graph $\mathcal{G}$, to facilitate notation. In general, a sub-graph of a graph $\mathcal{G}$ is a graph composed of a subset of nodes and edges of $\mathcal{G}$. In our context we will be interested in the sub-graphs composed of a node v_i along with all the other nodes and the so-called incoming edges of v_i. We assign the same symbol of the primary graph to the sub-graph, with subscript to specify that is a sub-graph, for example $\mathcal{G}_{ai}$ is the sub-graph of the ith node associated with the graph $\mathcal{G}$. Thus, the sub-graph of the ith node is a graph $\mathcal{G}_{ai} = (\mathcal{V}, \mathcal{E}_{ai})$, where $\mathcal{E}_{ai} = \{(v_i, v_j) \in \mathcal{E} : \forall v_j \in \mathcal{V}\}$. To illustrate this idea, the graph formed in Figure 2.1 is shown in Figure 2.2 with its sub-graphs.



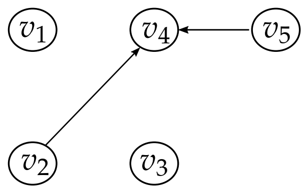
(A) $\mathcal{G}_{a1}$



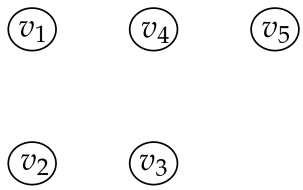
(B) $\mathcal{G}_{a2}$



(C) $\mathcal{G}_{a3}$



(D) $\mathcal{G}_{a4}$



(E) $\mathcal{G}_{a5}$

FIGURE 2.2: Sub-graphs formed by the nodes of Figure 2.1.

2.2 Consensus formulation

Consensus is one of the fundamental problems in multi-agent systems, as point out by Ren and Beard, 2008. Consensus algorithms have applications in several problems of multi-agent systems, for example the problem of rendezvous (Cortes, Martinez, and Bullo, 2006; Dong and Huang, 2018), formation control (Porfiri, Roberson, and Stilwell, 2007; Ren, Beard, and Atkins, 2007), and sensor networks (Spanos and Murray, 2005; Mohammadi and Asif, 2015). Consensus means to reach an agreement on a joint variable value. In other words, the agents share information and update its variable according to some rule until the network converges to a common value. Formally, the definition of consensus can be stated as:

Definition 2.1. *A multi-agent system with n agents and state variables* $\boldsymbol{x}_i(t) \in \mathbb{R}^m$*, where* $i \in \{1, \ldots, n\}$ *are the agents index, asymptotically reaches consensus on these variables if, for all* $i \neq j$*,*

$$\lim_{t \to \infty} (\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) = 0. \tag{2.2}$$

A consensus protocol is the interaction rule (control law) between agents that aims to drive the system toward consensus. The classical protocol discussed in this dissertation was introduced by Saber and Murray, 2003, and for the ith agent is given by

$$\boldsymbol{u}_i(t) = -\sum_{j=1}^{n} a_{ij} K \big(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\big), \tag{2.3}$$

where $\boldsymbol{u}_i(t) \in \mathbb{R}^p$ and $K \in \mathbb{R}^{p \times m}$ is a constant gain matrix. This is a simple distributed protocol that uses only local information with a linear relation between agents' states. The constant a_{ij} is the (i, j) entry of the Adjacency Matrix, and on the consensus protocol (2.3) it selects only the information provided by the neighbors of the ith agent. Note that this constant is equal to zero for non-neighbor agents.

2.2.1 Tree-type transformation

The consensus problem is said to be solved if Equation (2.2) is satisfied for all $i, j \in \mathcal{V}$. To facilitate the analysis we translate the consensus problem into a stability problem through a tree-type transformation (Sun and Wang, 2009). Define the disagreement of states variables as follows

$$\boldsymbol{z}_i(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_{i+1}(t), \tag{2.4}$$

for $i = \{1, \ldots, n-1\}$. All the disagreements of states are written in one equation for all agents using the transformation,

$$\boldsymbol{z}(t) = (U \otimes I_m)\boldsymbol{x}(t), \tag{2.5}$$

where $\boldsymbol{z}(t) = [\boldsymbol{z}_1^T(t) \ \ldots \ \boldsymbol{z}_{n-1}^T(t)]^T$, the symbol $\otimes$ represents the Kronecker product between two matrices, and I_m is an identity matrix of order m. The transformation from the disagreement back to original states is given by

$$\boldsymbol{x}(t) = \boldsymbol{1}_n \otimes \boldsymbol{x}_1(t) + (W \otimes I_m)\boldsymbol{z}(t), \tag{2.6}$$

where the matrices U and W are defined as,

$$U = \begin{bmatrix} \mathbf{1}_{n-1} & -I_{n-1} \end{bmatrix}, \tag{2.7}$$

$$W = \begin{bmatrix} \mathbf{0}_{n-1}^T \\ -I_{n-1} \end{bmatrix}, \tag{2.8}$$

with $\mathbf{1}_{n-1}$ and $\mathbf{0}_{n-1}$ being column vectors of size $n-1$, in which all elements are ones and zeros, respectively. Therefore, we can rephrase Definition 2.1 as follow,

Definition 2.2. *A multi-agent system with n agents and state variables $\boldsymbol{x}_i(t) \in \mathbb{R}^m$, where $i \in \{1, \ldots, n\}$ are the agents index, asymptotically reaches consensus if,*

$$\lim_{t \to \infty} \boldsymbol{z}(t) = 0. \tag{2.9}$$

Hence, we study the consensus problem by studying the stability of the transformed multi-agent system. This is simpler because it allows the use of several tools previously developed for stability analysis.

2.2.2 Consensus problem with saturation

We are interested in the problem of consensus subject to constrained control signal on the agents' inputs, according to the following assumption:

Assumption 2.1. *All the agents' input vectors, $\boldsymbol{u}_i(t)$, are constrained by symmetric limits, that is $-u_{max} \le u_{i(k)}(t) \le u_{max}$, for all $k \in \{1, \ldots, p\}$.*

In which $u_{max} \in \mathbb{R}$ is a scalar and the subscript in parenthesis selects the elements of the vector $\boldsymbol{u}_i(t)$. Moreover, the following assumption is made regarding the agents models:

Assumption 2.2. *All agents are identical and are modeled by the same equations.*

Thus, under Assumption 2.2 we restrict our study to heterogeneous networks with distinctions only on the time-delay, that is, networks with identical agents besides the delay. The study of heterogeneous multi-agent systems described by distinct models presents more challenges related with the overall network representation and has been less studied, some examples are the works of Zheng and Wang, 2015; Li and Liu, 2018; Mehrabian and Khorasani, 2016, to cite a few.

As pointed out, Assumption 2.1 is related to actuator physical limits, a practical constraint present in most real systems that can lead to performance degradation. This is a known problem and has been studied in different scenarios (see the work of Bernstein and Michel, 1995 for a chronological review on saturation until

1995, and the book of Tarbouriech et al., 2011 on the topic of stability and stabilization of linear systems with input saturation). Saturation is a common nonlinearity hard to model and on the study of new problems, such as consensus, it is important to understand its impact due to potential performance degradation.

Our goal is to address the scenario where the saturation can make the consensus unreachable for all agents' initial states. Even though open-loop linear models are considered for the agents, the presence of saturation makes the closed-loop models nonlinear. Thus, the nonlinear behavior that the Assumption 2.1 introduces makes the characterization of the admissible set of initial states crucial to study consensus under control constraints. This characterization is not necessary in all scenarios, there are cases where global or semi-global consensus can be guaranteed even with the presence of saturation, see for example the works of Dal Col, Tarbouriech, and Zaccarian, 2016; Meng, Zhao, and Lin, 2012; Yang et al., 2014; Yang et al., 2018, where only global or semi-global convergence was studied. When only global and semi-global convergence are investigated with linear controls, it is common to consider further assumptions regarding the agents behavior. The regional consensus, which is the case where the consensus can be achieved only when the agents' initial states belong to a region of the state space, is a more challenging problem because of the requirement of characterization of this region, and can incorporate the global and semi-global cases. To illustrate the necessity to characterize a region for consensus with saturation, lets take the system studied on Kim and Bien, 1994, considering the problem of consensus with consensus protocol (2.3):

Example 2.1. *Consider a multi-agent system with four agents, the graph representation of interactions of the agents is given in Figure 2.3, and with each agent described as,*

$$\begin{bmatrix} \dot{x}_i^1(t) \\ \dot{x}_i^2(t) \end{bmatrix} = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3.0 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} sat(\boldsymbol{u}_i(t)),$$

where the superscript on $\dot{x}_i^1(t)$ and $\dot{x}_i^2(t)$ refers to the state variables.

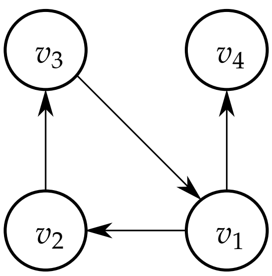


FIGURE 2.3: Example 2.1–Graph representation of agents interaction.

For $u_{max} = 0.5$ *and initial conditions,*

$$x_1 = \begin{bmatrix} -5.2 \\ 5.5 \end{bmatrix}; \; x_2 = \begin{bmatrix} -5.4 \\ 5.5 \end{bmatrix}; \; x_3 = \begin{bmatrix} 5.4 \\ -4.4 \end{bmatrix}; \; x_4 = \begin{bmatrix} 5.3 \\ -2.4 \end{bmatrix},$$

we have the following network evolution,

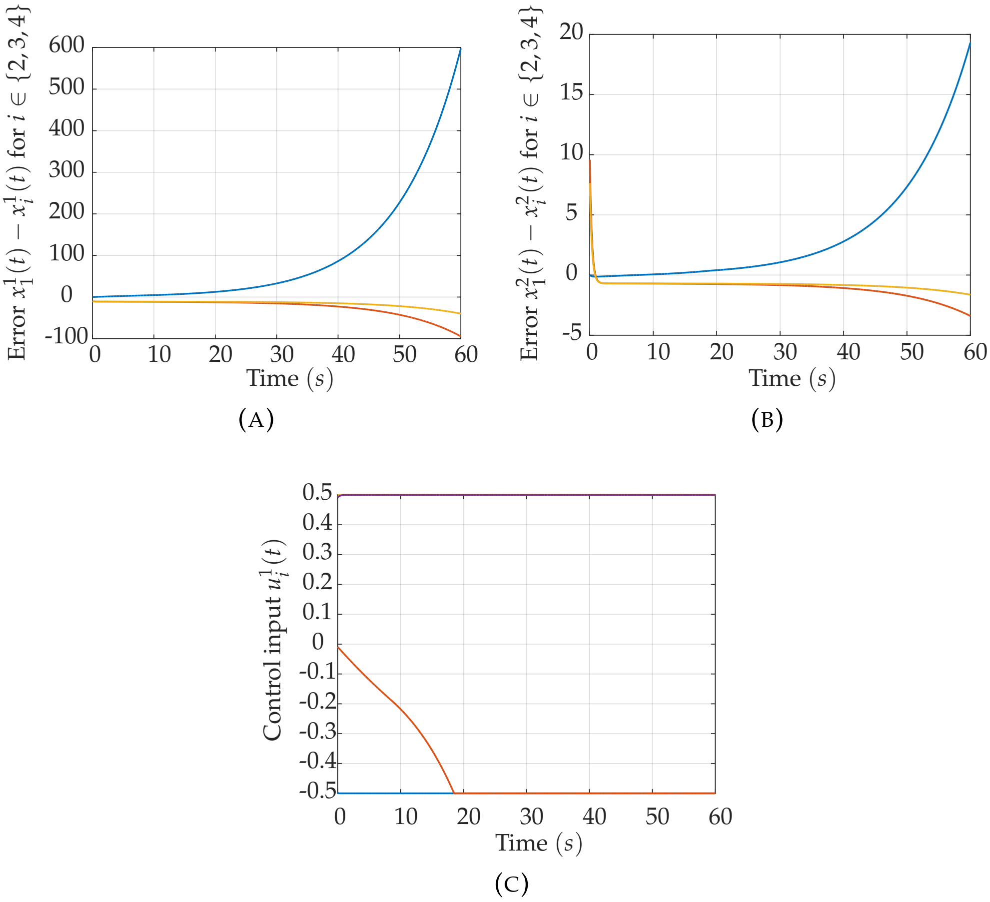


FIGURE 2.4: Example 2.1–Divergent trajectories of the multi-agent system with saturated input.

Figure 2.4 shows that the consensus is not reached and the agents' inputs are almost always saturated. If instead we consider the following initial conditions,

$$x_1 = \begin{bmatrix} -3.0 \\ 3.0 \end{bmatrix}; \; x_2 = \begin{bmatrix} -3.0 \\ 5.0 \end{bmatrix}; \; x_3 = \begin{bmatrix} 1.0 \\ -4.0 \end{bmatrix}; \; x_4 = \begin{bmatrix} 3.0 \\ -2.0 \end{bmatrix},$$

the network evolution on Figure 2.5 is obtained, which shows asymptotic convergence.

Example 2.1 allows to see that not all initial conditions lead to consensus when saturation is present, thus the characterization of a region where the consensus is guaranteed is compelling. With this in mind, we define a region on space in which the consensus can be attained,

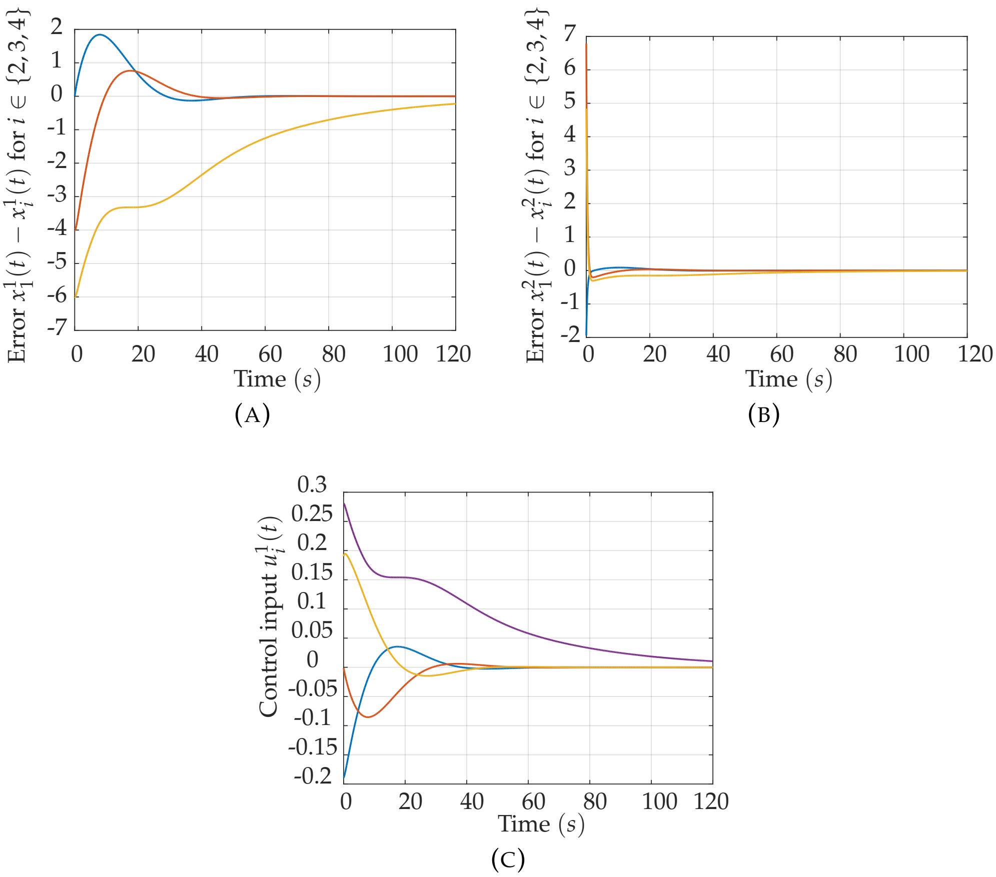


FIGURE 2.5: Example 2.1–Convergent trajectories of the multi-agent system with saturated input.

Definition 2.3. *The domain of consensus is a region $\mathcal{S}_{DOC} \in \mathbb{R}^{nm}$ in which for any $\boldsymbol{x}(t_0) \in \mathcal{S}_{DOC}$, the multi-agent system achieves consensus. The domain of consensus can be written in the following way,*

$$\mathcal{S}_{DOC}=\left\{\boldsymbol{x}(t_0)\in\mathbb{R}^{nm}\colon \lim_{t\to\infty} \boldsymbol{x}_i(t) = \boldsymbol{x}_j(t)\ \forall i,j \in \mathcal{V},\ i \neq j\right\}, \tag{2.10}$$

where

$$\boldsymbol{x}(t) = \left[\boldsymbol{x}_1^T(t), \cdots, \boldsymbol{x}_n^T(t)\right]^T.$$

Taking into account the behavior of each agent subject to input saturation on the space, it is possible to define two regions related to the saturation: i) a region where the input is saturated; and another ii) where the control signal is not saturated (that is, $-u_{max} \leq u_{i(k)}(t) \leq u_{max}$, for all $k \in \{1, \ldots, p\}$). Figure 2.6 depicts these two regions.

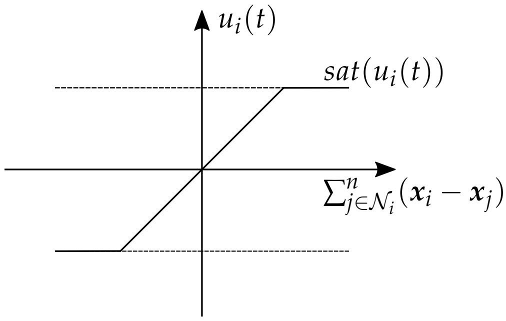


FIGURE 2.6: Representation of linear and saturated regions.

With the consensus protocol (2.3), the linear region for the multi-agent system is defined as,

Definition 2.4. *(Hu, Lin, and Chen, 2002) Let $H \in \mathbb{R}^{p\times m}$, and $\boldsymbol{h}_r$ represent its rth row, then:*

$$\mathcal{L}(H)= \left\{ \boldsymbol{x}(t) : \left| \boldsymbol{h}_r \sum_{j=1}^{n} a_{ij}(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) \right| \leq u_{max}, r = \{1,\ldots,p\},\ \forall i \in \mathcal{V} \right\}, \tag{2.11}$$

represents the linear region on $\mathbb{R}^{nm}$.

Hence, with Definition 2.4 the multi-agent system can be described with linear agents if $\boldsymbol{x}(t)$ belongs to $\mathcal{L}(H)$, and having nonlinear behavior otherwise. It is worth to remark that, without additional assumptions, the multi-agent system initializing within the linear region is not guaranteed to remain there.

Using the linear region it is possible to define a convex hull that always contains the saturated feedback control. The vertices of the convex hull are composed with the combination of the original linear feedback control, which does not take into account the saturation, and the linear region, which does not saturate. In such a way that the saturated feedback control is always within this set (see the work of Hu, Lin, and Chen, 2002 for details). Thus, we have the following proposition,

Lemma 2.1. *(Hu, Lin, and Chen, 2002) Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^p$, suppose that $|v_r| \leq u_{max}$ for all $r \in \{1,\ldots,p\}$, and $\mathbb{D} \in \mathbb{R}^{p\times p}$ be the set of $p \times p$ diagonal matrix whose diagonal elements are either 1 or 0. There are 2^p elements in $\mathbb{D}$, denote each element as D_k and let $D_k^- = I - D_k \in \mathbb{D}$. Then:*

$$sat(\boldsymbol{u}) \in Co\left\{ D_k\boldsymbol{u} + D_k^-\boldsymbol{v} : k \in \{1,\ldots,2^p\} \right\}, \tag{2.12}$$

where $Co\{\cdot\}$ denotes the convex hull.

With Lemma 2.1 we can represent the agents' saturation within a convex hull of linear controls.

2.2.3 Consensus problem with delay

Regarding the delay, we are concerned with the problem of the time that the agents need to process the received information and the actuators need to respond. This is known as input delay and in the context of multi-agent systems it is present locally on each agent i. To make it clear we state this in the form of the following assumption:

Assumption 2.3. *The consensus protocol has access to all neighbors agents' states instantaneously, but the controller action is delayed.*

In other words, under Assumption 2.3 we deal only with the delay caused by the time that takes for the execution of the control action. Additionally, we want to let different delays occur in each agent, as we consider only local delays it is reasonable that they are unrelated over the network. To make this explicit consider also the assumption:

Assumption 2.4. *Each agent of the multi-agent system is subject to independent time-varying delays.*

The problem of consensus with time-delay has been studied from different point of views with the aim to find maximum delays that allow consensus, for example the works of Olfati-Saber and Murray, 2004; Savino et al., 2013; Wen et al., 2016; You et al., 2016; Zhao et al., 2017.

On the consensus protocol (2.3) the input delay is represented as

$$\boldsymbol{u}_i(t - \tau_i(t)) = -\sum_{j=1}^{n} a_{ij} K\big(\boldsymbol{x}_i(t - \tau_i(t)) - \boldsymbol{x}_j(t - \tau_i(t))\big), \tag{2.13}$$

that is, the function $\tau_i(t)$ maps the delay, the subscript i identifies the agent, and the time dependency means that it is a time-varying delay.

To illustrate the degradation caused by time-delay consider the following example:

Example 2.2. *We use the same data of Example 2.1 without saturation and with constant input time-delays. Figure 2.7 shows the evolution of the multi-agent system for* $\tau = 3.0$ *seconds, and Figure 2.8 for* $\tau = 2.5$ *seconds.*

Clearly, depending on the time-delay there are trajectories that converge to consensus or diverge. This is not a new question, as said before, and there are

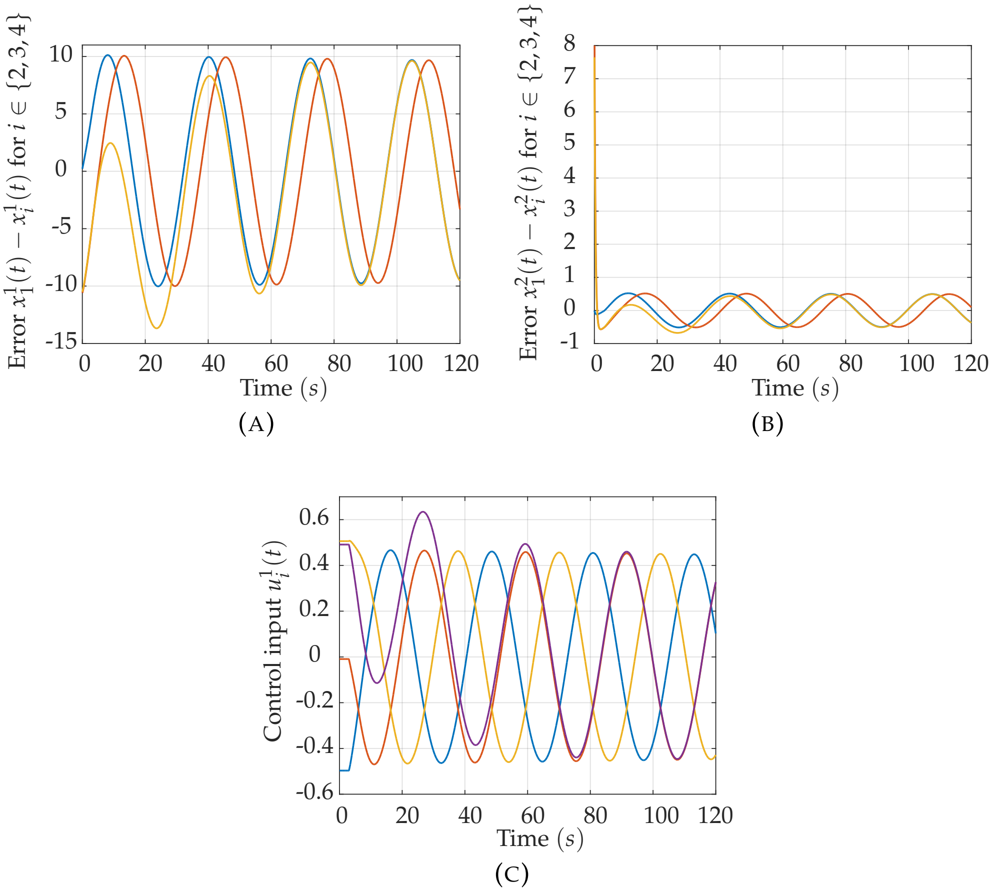


FIGURE 2.7: Example 2.2–Divergent trajectories of the multi-agent system with input time-delays.

a number of works that aims to derive necessary and sufficient conditions for consensus on networks under time-delays. In this work, we want to state that for the *same* system both, saturation and time-varying delays, may be present and conditions that ensure consensus for that scenario must be derived.

2.2.4 Consensus with saturation and time-delays

On this dissertation our main interest is the analysis and synthesis of controllers that lead multi-agent systems to consensus under Assumptions 2.1 to 2.4. These assumptions express more appropriate representation for networked systems than those generally analyzed in the literature. To the best of the authors' knowledge only You et al., 2016, Yanumula, Kar, and Majhi, 2017 and Ding, Zheng, and Guo, 2018 have investigated consensus under similar premises, and yet their analysis present important distinctions in relation to ours. Namely, the work of You et al., 2016:

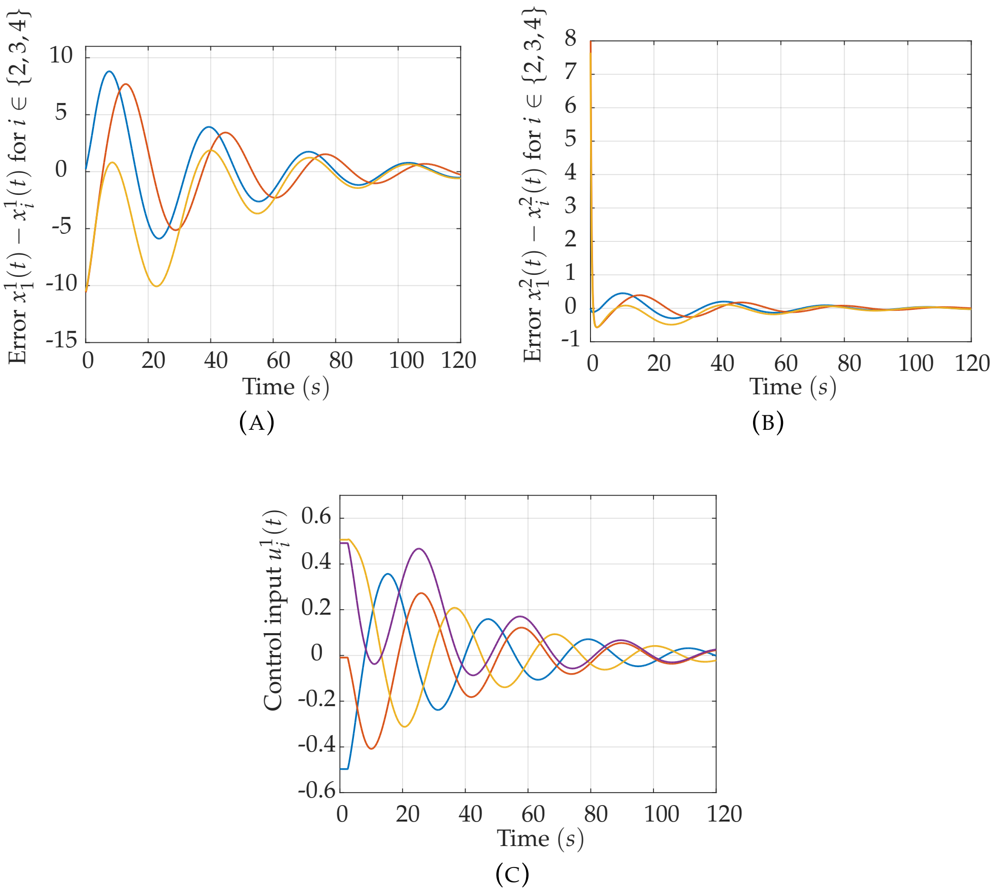


FIGURE 2.8: Example 2.2–Convergent trajectories of the multi-agent system with input time-delays.

i) deals only with the problem of leader-following consensus in the global sense, while our approach can be used on local consensus with and without leader;

ii) even though they said that the approach is appropriated for global consensus, no explicit assumption is made in order to guarantee that only systems with this characteristic will be analyzed. Thus, there might be cases where their conditions are satisfied and global consensus can not be reached (this is exemplified on Chapter 4);

iii) considers undirected switching communication topologies, while the network communication topology on this work is directed, but fixed;

iv) the agents on their work have a term to take into account uncertainties, and our agents are considered precisely known.

On the work of Yanumula, Kar, and Majhi, 2017:

i) are concerned only with second-order agents. Our agents have high-order dynamics, which can be used to model more complex agents;

ii) represents the saturation non-linearity by describing functions to estimate the existence of limit cycles (details on this representation can be found on the book of Slotine and Li, 1991), while we represent it as a regional uncertainty and derive an estimation for the region where the consensus is attainable;

iii) study saturation on the rate of a state, specifically consider agents with the following model

$$\dot{x}_i(t) = sat(v_i(t)),$$
$$\dot{v}_i(t) = u_i(t),$$

and we are interested in saturation in the control input;

iv) consider distinct, but constant input delays for the agents. We consider the more general case of distinct and time-varying delays.

And on the work of Ding, Zheng, and Guo, 2018:

i) also are concerned only with the problem of leader-following consensus, while we deal with consensus with and without leader;

ii) formulate the problem of consensus analysis with bilinear matrix inequalities. Although this formulation might have some advantages, it is a non-convex problem and in general there is no unified method to solve it;

iii) consider input saturation with local stabilization and estimate a set of admissible initial conditions that guarantee consensus, as we do. But they represent the saturation with a decentralized deadzone model (see the book of Tarbouriech et al., 2011 for details on this representation), this representation might give different results from ours on the estimation of the set of admissible initial conditions;

iv) consider *buffers* over the network used to synchronize the input delays, in such a way that the control action is updated only after *all* agents receive a confirmation signal. Moreover, their approach assumes that the ith agent receives the confirmation signal from all others agents, leading to a centralized type of communication. Our approach is more general, considering decentralized, distinct, and time-varying delays.

2.2.5 Additional lemmas

Integral inequalities are intensively used in the context of stability of time-delay systems to derive conditions in terms of linear matrix inequalities using Lyapunov-Krasovskii theory. The following two lemmas are used to derive the main results in Chapter 4.

Lemma 2.2. *(Sun, Liu, and Chen, 2009). For any constant matrix $M = M^T > 0$ and scalars $t > t - \tau \geq 0$ such that the following integrations are well defined, then*

$$\int_{-\tau}^{0}\int_{t+s}^{t} z^T(\epsilon)Mz(\epsilon)d\epsilon ds \geq \frac{2}{\tau^2}\int_{-\tau}^{0}\int_{t+s}^{t} z^T(\epsilon)d\epsilon ds M \int_{-\tau}^{0}\int_{t+s}^{t} z(\epsilon)d\epsilon ds.$$

Lemma 2.3. *(Seuret and Gouaisbaut, 2013). For any constant matrix $M = M^T > 0$ and scalars $t > t - \tau \geq 0$ such that the following integrations are well defined, then*

$$\int_{t-\tau}^{t} \dot{z}^T(\epsilon)M\dot{z}(\epsilon)d\epsilon \geq \frac{1}{\tau}\int_{t-\tau}^{t} \dot{z}^T(\epsilon)d\epsilon M \int_{t-\tau}^{t} \dot{z}(\epsilon)d\epsilon + \frac{3}{\tau}\Omega^T M\Omega,$$

with

$$\Omega = z(t-\tau) + z(t) - \frac{2}{\tau}\int_{t-\tau}^{t} z(\epsilon)d\epsilon.$$

2.3 Conclusions of the chapter

In this chapter the main assumptions that we are concerned with in this work were presented. Moreover, we presented a method typically used to describe the communication topology of networked systems based on graph theory.

The consensus problem is defined and the tree-type transformation is presented to translate the consensus problem into a stability problem. A systematic way to write the consensus in multi-agent systems with saturation and time-varying delay using this representation is presented in the next chapter.

Still in this chapter, motivational examples are given on consensus with saturated input and with time-delay in order to show the importance of the developed analysis. Finally, the characteristics of the proposed approach are compared with others similar works found in the literature.

Chapter 3

Problem formulation

In this chapter we manipulate the equations involved in the formulation of the consensus problem with saturation and input time-varying delay to obtain an equivalent stability problem. We use this representation to derive sufficient conditions for reaching consensus and to synthesize the agents' feedback matrix that leads to consensus.

The main points developed are: i) the use of convex combination of linear feedbacks to represent agents' saturation, ii) the simplification of the representation writing the multi-agent system in one equation using Kronecker product and algebraic graphs properties, and iii) the transformation of the consensus problem into a stability problem on the disagreement variables.

3.1 Dynamical network

Consider a multi-agent system consisting of n agents subject to saturation and time-delay input, in which the open-loop dynamics of the ith agent is given by

$$\dot{\boldsymbol{x}}_i(t) = A\boldsymbol{x}_i(t) + Bsat\big(\boldsymbol{u}_i(t - \tau_i(t))\big), \tag{3.1}$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^m$ is the agent's state variable, $\boldsymbol{u}_i(t - \tau_i(t)) \in \mathbb{R}^p$ the delayed control input, and A and B are real constant known matrices of appropriate dimensions. Initial conditions for the ith agent are given by

$$\boldsymbol{x}_i(t_0) = \boldsymbol{\phi}_i, \tag{3.2}$$

with $\boldsymbol{\phi}_i \in \mathcal{C}^m_{\max\{\tau_i(t)\}}$, in which $\mathcal{C}^m_{\max\{\tau_i(t)\}}$ represents the set of continuous functions on the interval $[t_0 - \tau_i(t), t_0]$. The function $sat(\cdot)$ maps the saturation as

$$\begin{aligned} sat(\boldsymbol{u}_i) &= [sat(u_{i(1)}) \cdots sat(u_{i(p)})]^T, \\ sat(u_{i(r)}) &= sgn(u_{i(r)})min(u_{max}, |u_{i(r)}|), \end{aligned} \tag{3.3}$$

for all $r = \{1, \dots, p\}$, and the scalar u_{max} represents the actuator's limit. Here, for simplicity, we slightly abuse notation using the same notation for vector saturation function and scalar function. It is assumed that all inputs are independent and have the same maximum value. Observe that due to the saturation it is impossible to steer the agents' states from any initial condition $\boldsymbol{x}_i(t_0) \in \mathbb{R}^m$ to the origin with static state-feedback control, if the matrix A in (3.1) have some eigenvalues with positive real part (Schmitendorf and Barmish, 1980). This implies that in the best scenario, with a stabilizing feedback, there is a region of the state space where for any initial condition exists a bounded control that stabilizes the agents. The existence of this set of initial conditions is guaranteed even if the saturation leads to instability because the system can start on the linear region and move asymptotically to a equilibrium point without saturate the inputs.

The time-varying delay that affects the ith agent's input is represented in (3.1) by $\tau_i(t)$ and is modeled by

$$\tau_i(t) = \tau + \mu_i(t), \; \forall i \in \{1, \dots, n\}, \tag{3.4}$$

where τ is a constant value and $\mu_i(t)$ a possibly non-differentiable time-varying perturbation that satisfies $|\mu_i(t)| \leq \mu_m < \tau$ with μ_m known, hence the delay belongs to an interval given by $\tau_i(t) \in [\tau - \mu_m, \tau + \mu_m]$. Figure 3.1 illustrates a delay with $\tau = 2.5s$ and $\mu_m = 1.0s$,

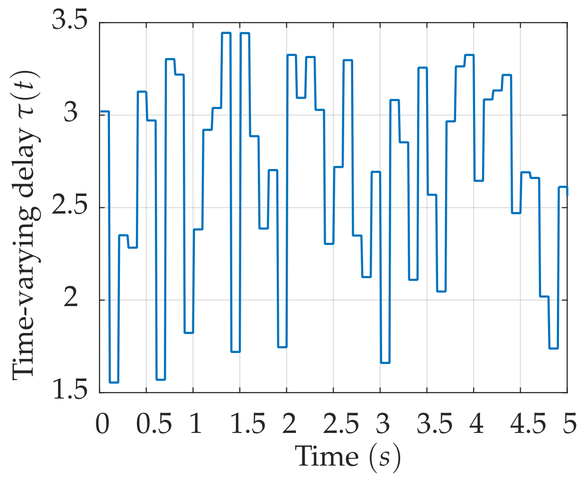


FIGURE 3.1: Representation of a time-varying delay.

Lemma 2.1 is used to represent the saturated input of the ith agent as a linear combination of linear feedbacks, with $\boldsymbol{u}$ given by equation (2.3) we chose $\boldsymbol{v}$

similarly, as

$$v_i(t) = -\sum_{j=1}^{n} a_{ij} H(x_i(t) - x_j(t)), \tag{3.5}$$

with $|v_{i(r)}| \leq u_{max}$ for all $r \in \{1, \ldots, p\}$. In this way it is possible to represent the saturation as combination of the original consensus protocol without saturation, and a consensus protocol within the linear region (note that $v_i(t)$ is chosen in such a way that it does not saturate). Therefore, we can write the agents' feedback as the exactly combination of the vertices of the convex hull (2.12) as follow,

$$sat(u_i(t)) = -\sum_{k=1}^{2^p} \sum_{j=1}^{n} \alpha_k \{ D_k K a_{ij}(x_i(t) - x_j(t)) + D_k^- H a_{ij}(x_i(t) - x_j(t)) \}, \tag{3.6}$$

with

$$\sum_{k=1}^{2^p} \alpha_k = 1, \qquad \alpha_k \geq 0.$$

The following example illustrate the representation of the saturation for a simple multi-agent system with two agents.

Example 3.1. *Consider that two agents that interacts with each other should synchronize. Assume that the states of interest $x_i(t)$, the feedback matrix K, and the auxiliary matrix H, are scalars. For the agent i=1, we can write the saturated input as,*

$$sat(u_1(t)) = -\sum_{k=1}^{2} \alpha_k \{ D_k K a_{12}(x_1(t) - x_2(t)) + D_k^- H a_{12}(x_1(t) - x_2(t)) \},$$

in this case the matrices D_k and D_k^- are given by,

$$D_k \in \{[0]; [1]\}, \;\; D_k^- \in \{[1], [0]\}.$$

Thus, making the sum on k give us,

$$sat(u_1(t)) = -\alpha_1 H(x_1(t) - x_2(t)) - \alpha_2 K(x_1(t) - x_2(t)),$$

which means that for each time instant we have two vertices, $K(x_1(t) - x_2(t))$ and $H(x_1(t) - x_2(t))$. Figure 3.2 illustrate this scenario.

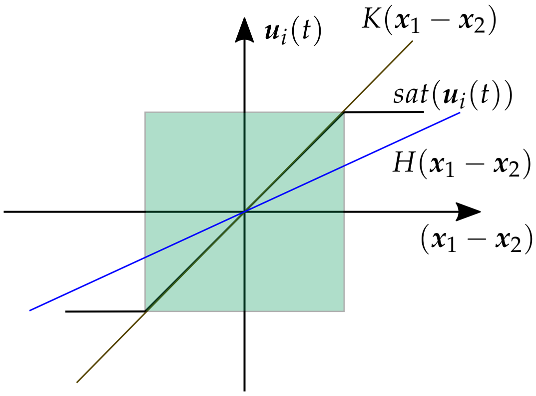


FIGURE 3.2: Example 3.1–Representation of the input saturation.

Note that within the linear region, the green area on Figure 3.2, the choice $\alpha_1 = 0$ *and* $\alpha_2 = 1$ *is straightforward. Outside this region we can assume a limit* u_{max} *for the saturation and rewrite the linear combination as,*

$$u_{max} = -\alpha_1 H(x_1(t) - x_2(t)) - (1 - \alpha_1)K(x_1(t) - x_2(t)),$$

which is satisfied by choosing appropriately the values of α_1.

Example 3.1 elucidate the representation of the saturation as a convex combination of the feedback control and an auxiliary feedback control. An important point shown is that the vertices of the convex region might vary with the agents' states, which implies that for an exact representation the values of α_k should be chosen for each instant. In this context, the following remark highlights this characteristic,

Remark 3.1. Representation (3.6) holds only for a particular convex combination that depends on the difference of agents states. This means that all trajectories of the saturated input on (3.1) can be generated by (3.6). However, the converse is not true, some values of (3.6) may not be generated by (3.1). Thus, this representation may introduce some conservatism (Tarbouriech et al., 2011).

3.2 On the multi-agent system saturation

The multi-agent system can be represented in one equation by using the Kronecker product, as

$$\dot{x}(t) = (I_n \otimes A)x(t) + (I_n \otimes B)sat\big(u(t - \tau_\bullet(t))\big), \tag{3.7}$$

where,

$$\boldsymbol{u}(t-\tau_\bullet(t)) = \begin{bmatrix} \boldsymbol{u}_1^T(t-\tau_1(t)) & \cdots & \boldsymbol{u}_n^T(t-\tau_n(t)) \end{bmatrix}^T. \tag{3.8}$$

Defining the vector $\mathcal{I}_i \in \mathbb{R}^n$ with 1 on the ith position and 0 elsewhere, for example for $i = 1$,

$$\mathcal{I}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T,$$

then (3.8) is rewritten as,

$$\boldsymbol{u}(t-\tau_\bullet(t)) = \sum_{i=1}^{n} \mathcal{I}_i \otimes \boldsymbol{u}_i(t-\tau_i(t)). \tag{3.9}$$

Thus, it is possible to write the stacked saturating input for the multi-agent system using equation (3.6) on equation (3.9),

$$\begin{aligned} & sat\big(\boldsymbol{u}(t-\tau_\bullet(t))\big) = \\ & \qquad -\sum_{i=1}^{n} \mathcal{I}_i \otimes \sum_{k=1}^{2^p} \sum_{j=1}^{n} \alpha_k (D_k K + D_k^- H) a_{ij} \big(\boldsymbol{x}_i(t-\tau_i(t)) - \boldsymbol{x}_j(t-\tau_i(t))\big). \end{aligned} \tag{3.10}$$

For simplicity, we define

$$K_k = \alpha_k (D_k K + D_k^- H).$$

On equation (3.10) we have represented the input for the networked system in a stacked form, considering saturation and time-varying delays. Next, some simplifications are performed in order to obtain a suitable model to apply the tree-type transformation.

The following equalities hold:

$$\sum_{j=1}^{n} a_{ij} \boldsymbol{x}_j(t-\tau_i(t)) = (\mathcal{I}_i^T \mathcal{A}_{ai} \otimes I_m) \boldsymbol{x}(t-\tau_i(t)), \tag{3.11}$$

$$\sum_{j=1}^{n} a_{ij} \boldsymbol{x}_i(t-\tau_i(t)) = (\mathcal{I}_i^T \mathcal{D}_{ai} \otimes I_m) \boldsymbol{x}(t-\tau_i(t)), \tag{3.12}$$

where the subscript on $\mathcal{A}_{ai}$ and $\mathcal{D}_{ai}$ refers to the sub-graph related to ith agent.

Noticing that $\mathcal{L}_{ai} = \mathcal{D}_{ai} - \mathcal{A}_{ai}$ and using (3.11) and (3.12), we can write (3.10) as,

$$sat\big(\boldsymbol{u}(t-\tau_{\bullet}(t))\big) = -\sum_{i=1}^{n}\sum_{k=1}^{2^p}(\mathcal{I}_i \otimes K_k)(\mathcal{I}_i^T\mathcal{L}_{ai} \otimes I_m)\boldsymbol{x}(t-\tau_i(t)). \tag{3.13}$$

Moreover, through the same idea used in equation (3.9) we write,

$$\boldsymbol{x}(t-\tau_i(t)) = \sum_{\ell=1}^{n}\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t)), \tag{3.14}$$

and,

$$\begin{aligned} sat\big(\boldsymbol{u}(t-\tau_{\bullet}(t))\big) =& -\sum_{i=1}^{n}\sum_{k=1}^{2^p}(\mathcal{I}_i \otimes K_k)(\mathcal{I}_i^T\mathcal{L}_{ai} \otimes I_m)\sum_{\ell=1}^{n}(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t))) \\ =& -\sum_{i=1}^{n}\sum_{k=1}^{2^p}\sum_{\ell=1}^{n}(\mathcal{I}_i \otimes K_k)(\mathcal{I}_i^T\mathcal{L}_{ai} \otimes I_m)(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t))). \end{aligned} \tag{3.15}$$

To simplify equation (3.15) we use the mixed-product property of Kronecker,

$$\begin{aligned} &(\mathcal{I}_i \otimes K_k)(\mathcal{I}_i^T\mathcal{L}_{ai} \otimes I_m)(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t))) \\ =&(\mathcal{I}_i \otimes K_k)(\mathcal{I}_i^T\mathcal{L}_{ai}\mathcal{I}_\ell) \otimes (I_m\boldsymbol{x}_\ell(t-\tau_i(t))) \\ =&\mathcal{I}_i \otimes (1 \otimes K_k)(\mathcal{I}_i^T\mathcal{L}_{ai}\mathcal{I}_\ell) \otimes (\boldsymbol{x}_\ell(t-\tau_i(t))) \\ =&\mathcal{I}_i \otimes (\mathcal{I}_i^T\mathcal{L}_{ai}\mathcal{I}_\ell) \otimes (K_k\boldsymbol{x}_\ell(t-\tau_i(t))) \\ =&(\mathcal{I}_i \otimes \mathcal{I}_i^T)(\mathcal{L}_{ai} \otimes K_k)(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t))). \end{aligned} \tag{3.16}$$

Making the sums on i, k, and ℓ in (3.16),

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{k=1}^{2^p}\sum_{\ell=1}^{n}(\mathcal{I}_i \otimes \mathcal{I}_i^T)(\mathcal{L}_{ai} \otimes K_k)(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t))) \\ =&\sum_{i=1}^{n}(\mathcal{I}_i \otimes \mathcal{I}_i^T)\sum_{k=1}^{2^p}(\mathcal{L}_{ai} \otimes K_k)\sum_{\ell=1}^{n}(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t-\tau_i(t))) \\ =&\sum_{i=1}^{n}\sum_{k=1}^{2^p}(\mathcal{L}_{ai} \otimes K_k)\boldsymbol{x}(t-\tau_i(t)), \end{aligned}$$

where we have used (3.9) and

$$\sum_{i=1}^{n}\mathcal{I}_i \otimes \mathcal{I}_i^T = I_n.$$

Hence, equation (3.10) becomes,

$$sat\big(\boldsymbol{u}(t-\tau_{\bullet}(t))\big)=-\sum_{i=1}^{n}\sum_{k=1}^{2^p}(\mathcal{L}_{ai}\otimes K_k)\boldsymbol{x}(t-\tau_i(t)).$$

Finally, the multi-agent system (3.7) can be written as,

$$\dot{\boldsymbol{x}}(t)=(I_n\otimes A)\boldsymbol{x}(t)-(I_n\otimes B)\sum_{i=1}^{n}\sum_{k=1}^{2^p}(\mathcal{L}_{ai}\otimes K_k)\boldsymbol{x}(t-\tau_i(t)). \tag{3.17}$$

Note that in (3.17) we have placed the multi-agent system saturation into a convex hull of a group of linear feedbacks.

3.3 Consensus as a stability problem

To translate the consensus problem of the stacked multi-agent system (3.17) into a stability problem we take the time derivative of equation (2.5) and substitute (3.17) on $\dot{\boldsymbol{x}}(t)$, this gives us

$$\dot{\boldsymbol{z}}(t)=(U\otimes I_m)(I_n\otimes A)\boldsymbol{x}(t)-(U\otimes I_m)(I_n\otimes B)\sum_{i=1}^{n}\sum_{k=1}^{2^p}\mathcal{L}_{ai}\otimes K_k\boldsymbol{x}(t-\tau_i(t)), \tag{3.18}$$

replacing (2.6) on (3.18),

$$\begin{aligned}\dot{\boldsymbol{z}}(t)=&(U\otimes I_m)(I_n\otimes A)\big((\mathbf{1}_n\otimes x_1(t))+(W\otimes I_m)\boldsymbol{z}(t)\big)\\&-(U\otimes I_m)(I_n\otimes B)\sum_{i=1}^{n}\sum_{k=1}^{2^p}\mathcal{L}_{ai}\otimes K_k\\&\times\big((\mathbf{1}_n\otimes x_1(t-\tau_i(t)))+(W\otimes I_m)\boldsymbol{z}(t-\tau_i(t))\big),\end{aligned}$$

then, through Kronecker properties,

$$\begin{aligned}\dot{\boldsymbol{z}}(t)=&(U\mathbf{1}_n)\otimes(Ax_1(t))+(UW\otimes A)\boldsymbol{z}(t)\\&-\sum_{i=1}^{n}\sum_{k=1}^{2^p}(U\otimes B)\big((\mathcal{L}_{ai}\mathbf{1}_n)\otimes(K_kx_1(t-\tau_i(t)))+(\mathcal{L}_{ai}W\otimes K_k)\boldsymbol{z}(t-\tau_i(t))\big),\end{aligned}$$

noticing that $U\mathbf{1}_n=0$, $\mathcal{L}_{ai}\mathbf{1}_n=0$, and $UW{=}I_{n-1}$, the following disagreement system is obtained

$$\dot{\boldsymbol{z}}(t)=(I_{n-1}\otimes A)\boldsymbol{z}(t)-\sum_{i=1}^{n}\sum_{k=1}^{2^p}(U\mathcal{L}_{ai}W\otimes BK_k)\boldsymbol{z}(t-\tau_i(t)). \tag{3.19}$$

Thus, as pointed out on Definition 2.2, we can study the stability of (3.19) as a means to study the consensus problem of the multi-agent system. This representation is appropriated to the use of tools designed to analyze the stability of linear systems. Particularly, in the next chapter we will derive conditions to stability and design stabilizing control through linear matrix inequalities aiming the consensus through the stability of (3.19).

3.4 Conclusions of the chapter

This chapter gives a procedure to write the consensus problem considering input saturation and time-varying delays as a stability problem.

Firstly, the agents' saturation is written as convex combination of linear state feedbacks and then the overall networked system is written in a compact form, an important remark associated with the conservativeness of the method is made. Secondly, a change of variables on the multi-agent system translates the consensus into a stability problem. This representation is used in Chapter 4 to derive conditions for stability and to synthesize the state feedback gains.

Chapter 4

Stability analysis and control synthesis

This chapter addresses the stability analysis and the control synthesis for multi-agent systems with time-delays and input saturation. First, we rely on the theoretical framework of Lyapunov-Krasovskii to derive sufficient conditions to allege if the multi-agent system is able to achieve consensus. Next, we estimate a domain of consensus using an upper bound of the Lyapunov-Krasovskii functional. In order to find a set of initial conditions as large as possible, an optimization problem with LMI constraints is suggested. Then, based on the results of the stability analysis, conditions to design a state feedback controller are proposed and an optimization problem is suggested to synthesize the agents' feedback matrix with an estimation for the domain of consensus.

4.1 Stability analysis

The following theorem presents sufficient conditions to certify stability of the system (3.19), and consequently, guarantee the consensus of the multi-agent system (3.17) with input saturation and time-varying delays.

Theorem 4.1. *Let the feedback matrix K and the scalars τ, μ_m, λ, be given. The multi-agent system (3.7) with consensus protocol (2.3) and time-delay $\tau_i(t) \in [\tau - \mu_m, \tau + \mu_m]$ for $\tau > 0$, $0 \leq \mu_m < \tau$, reaches consensus if there exist matrices: F, $P_1 = P_1^T$, P_2, $P_3 = P_3^T$, $Q = Q^T > 0$, $R = R^T > 0$, $S = S^T > 0$, and $Z = Z^T > 0$ with dimensions $m(n-1) \times m(n-1)$, and H with dimension $p \times m$ such that the following inequalities are satisfied:*

$$\begin{bmatrix} \Phi_k & \mu_m \Gamma_k \\ * & -\mu_m Z \end{bmatrix} < 0, \quad \forall k \in \{1, \ldots, 2^p\}, \tag{4.1}$$

$$\begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0, \tag{4.2}$$

and

$$\begin{bmatrix} P_1 & (W^T \mathcal{L}_{ai}^T \mathcal{I}_i) \otimes \boldsymbol{h}_r^T \\ (\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes \boldsymbol{h}_r & u_{max}^2 \end{bmatrix} \geq 0, \tag{4.3}$$
$$\forall i \in \{1, \dots, n\},\ \forall r \in \{1, \dots, p\}$$

where

$$\Phi_k = \Phi_P + \Phi_Z + \Phi_R + \Phi_{QS} + \Phi_{Fk}, \tag{4.4}$$

$$\Gamma_k = \begin{bmatrix} F\hat{B}_k \\ \lambda F\hat{B}_k \\ 0 \\ 0 \end{bmatrix}, \qquad \Phi_Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 2\mu_m Z & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

$$\Phi_P = \begin{bmatrix} P_2 + P_2^T & P_1 & -P_2 & P_3 \\ * & 0 & 0 & P_2 \\ * & * & 0 & -P_3 \\ * & * & * & 0 \end{bmatrix}$$

$$\Phi_R = \begin{bmatrix} -\frac{4}{\tau}R & 0 & -\frac{2}{\tau}R & \frac{6}{\tau^2}R \\ * & \tau R & 0 & 0 \\ * & * & -\frac{4}{\tau}R & \frac{6}{\tau^2}R \\ * & * & * & \frac{-12}{\tau^3}R \end{bmatrix}$$

$$\Phi_{QS} = \begin{bmatrix} Q - 2S & 0 & 0 & \frac{2}{\tau}S \\ * & \frac{\tau^2}{2}S & 0 & 0 \\ * & * & -Q & 0 \\ * & * & * & -\frac{2}{\tau^2}S \end{bmatrix}$$

$$\Phi_{Fk} = \begin{bmatrix} -F\bar{A} - \bar{A}^T F^T & F - \lambda \bar{A}^T F^T & -F\hat{B}_k & 0 \\ * & \lambda(F + F^T) & -\lambda F\hat{B}_k & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

and

$$\bar{A} = I_{n-1} \otimes A,$$
$$\hat{B}_k = -U\mathcal{L}W \otimes B(D_k K + D_k^- H).$$

Furthermore, an estimation for the domain of consensus for the multi-agent system is given by $\bar{\mathcal{X}}_{DOC} \leq 1$*, where*

$$\begin{aligned}\bar{\mathcal{X}}_{DOC}=&\delta_1^2\left[\lambda_{max}(P)(1+\tau^2)+\tau\lambda_{max}(Q)\right]\\&+\delta_2^2\left[\frac{\tau}{2}\lambda_{max}(R)+\frac{\tau^3}{6}\lambda_{max}(S)+2\mu_m\tau\lambda_{max}(Z)\right],\end{aligned} \tag{4.5}$$

the scalars δ_1 *and* δ_2 *are upper bounds for the initial states* $\boldsymbol{\phi}$ *and its time derivative* $\dot{\boldsymbol{\phi}}$*, and* $\lambda_{max}(\cdot)$ *represents the biggest eigenvalue of the matrix in its argument.*

Proof. The inequality (4.3) is equivalent to the set inclusion

$$\mathcal{E}(P_1,1) \subset \mathcal{L}(H),$$

where $\mathcal{E}(P_1,1)$ represents an ellipsoid defined as

$$\mathcal{E}(P_1,1) = \{\boldsymbol{z}(t) : \boldsymbol{z}^T(t)P_1\boldsymbol{z}(t) \leq 1\}.$$

This can be shown by writing (2.11) on the variable $\boldsymbol{z}(t)$ using equation (2.6). That is, with (3.11) and (3.12), the linear region (2.11) becomes a region in which,

$$\left|\boldsymbol{h}_r\sum_{j=1}^{n}a_{ij}(\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t))\right| = \left|\boldsymbol{h}_r(\mathcal{I}_i^T\mathcal{L}_{ai}\otimes I_m)\boldsymbol{x}(t)\right| \leq u_{max},$$

writing $\boldsymbol{x}(t)$ in a similar fashion as $\boldsymbol{x}(t-\tau_i(t))$ in (3.14) and applying the mixed-product property of Kronecker product,

$$\begin{aligned}\left|\boldsymbol{h}_r(\mathcal{I}_i^T\mathcal{L}_{ai}\otimes I_m)\boldsymbol{x}(t)\right|=&\left|\boldsymbol{h}_r(\mathcal{I}_i^T\mathcal{L}_{ai}\otimes I_m)\sum_{\ell=1}^{n}\mathcal{I}_\ell\otimes\boldsymbol{x}_\ell(t)\right|\\=&\left|\sum_{\ell=1}^{n}\boldsymbol{h}_r(\mathcal{I}_i^T\mathcal{L}_{ai}\otimes I_m)(\mathcal{I}_\ell\otimes\boldsymbol{x}_\ell(t))\right|\\=&\left|\sum_{\ell=1}^{n}\boldsymbol{h}_r(\mathcal{I}_i^T\mathcal{L}_{ai}\mathcal{I}_\ell)\otimes(I_m\boldsymbol{x}_\ell(t))\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{\ell=1}^{n} (1 \otimes \boldsymbol{h}_r)(\mathcal{I}_i^T \mathcal{L}_{ai} \mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t)) \right| \\
&= \left| \sum_{\ell=1}^{n} (\mathcal{I}_i^T \mathcal{L}_{ai} \mathcal{I}_\ell) \otimes \boldsymbol{h}_r \boldsymbol{x}_\ell(t) \right| \\
&= \left| \sum_{\ell=1}^{n} (\mathcal{I}_i^T \mathcal{L}_{ai} \otimes \boldsymbol{h}_r)(\mathcal{I}_\ell \otimes \boldsymbol{x}_\ell(t)) \right| \\
&= \left| (\mathcal{I}_i^T \mathcal{L}_{ai} \otimes \boldsymbol{h}_r) \boldsymbol{x}(t)) \right| \leq u_{max}.
\end{aligned}
$$

Finally, replacing $\boldsymbol{x}(t)$ with (2.6) gives,

$$
\begin{aligned}
&\left| (\mathcal{I}_i^T \mathcal{L}_{ai} \otimes \boldsymbol{h}_r)(\mathbf{1}_n \otimes \boldsymbol{x}_1(t) + (W \otimes I_m)\boldsymbol{z}(t)) \right| \\
=&\left| (\mathcal{I}_i^T \mathcal{L}_{ai} \otimes \boldsymbol{h}_r)(\mathbf{1}_n \otimes \boldsymbol{x}_1(t)) + (\mathcal{I}_i^T \mathcal{L}_{ai} \otimes \boldsymbol{h}_r)(W \otimes I_m)\boldsymbol{z}(t) \right| \\
=&\left| (\mathcal{I}_i^T \mathcal{L}_{ai} \mathbf{1}_n) \otimes (\boldsymbol{h}_r \boldsymbol{x}_1(t)) + (\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes (\boldsymbol{h}_r I_m)\boldsymbol{z}(t) \right| \\
=&\left| ((\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes (\boldsymbol{h}_r))\boldsymbol{z}(t) \right| \leq u_{max}.
\end{aligned}
$$

Therefore, the linear region can be written as,

$$
\mathcal{L}(H) = \left\{ \boldsymbol{x}(t) : \left| \big((\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes (\boldsymbol{h}_r)\big)\boldsymbol{z}(t) \right| \leq u_{max}, r = \{1, \ldots, p\}, \ \forall i \in \mathcal{V} \right\},
$$

and $\mathcal{E}(P_1, 1) \subset \mathcal{L}(H)$ implies that

$$
\left| \big((\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes (\boldsymbol{h}_r)\big)\boldsymbol{z}(t) \right| \leq u_{max}, \ \forall \boldsymbol{z}(t) \in \mathcal{E}(P_1, 1),
$$

for $r = \{1, \ldots, p\}$ and for all $i \in \mathcal{V}$. Then, the following inequality has to be satisfied:

$$
\begin{aligned}
&\Big(\boldsymbol{z}^T(t)\big((W^T \mathcal{L}_{ai}^T \mathcal{I}_i) \otimes \boldsymbol{h}_r^T\big)\Big)\Big(\big((\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes \boldsymbol{h}_r\big)\boldsymbol{z}(t)\Big) \leq u_{max}^2 \boldsymbol{z}^T(t) P_1 \boldsymbol{z}(t), \\
&\boldsymbol{z}^T(t)\Big(P_1 - \big((W^T \mathcal{L}_{ai}^T \mathcal{I}_i) \otimes \boldsymbol{h}_r^T\big) u_{max}^{-2} \big((\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes \boldsymbol{h}_r\big)\Big)\boldsymbol{z}(t) \geq 0,
\end{aligned}
$$

and by applying Schur complement we have

$$
\begin{bmatrix} P_1 & (W^T \mathcal{L}_{ai}^T \mathcal{I}_i) \otimes \boldsymbol{h}_r^T \\ (\mathcal{I}_i^T \mathcal{L}_{ai} W) \otimes \boldsymbol{h}_r & u_{max}^2 \end{bmatrix} \geq 0, \ \ r = \{1, \ldots, p\},
$$

for all $i \in \mathcal{V}$, which is equal to inequality (4.3).

Thus, if the inequality (4.3) is satisfied and $\mathcal{E}(P_1, 1)$ is an invariant set, implies that the saturation can be always expressed as (3.17). Thereafter we show the conditions for stability, to this end we choose the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned} V(z_t) =& \chi^T(t)P\chi(t) + \int_{t-\tau}^{t} \mathbf{z}^T(\epsilon)Q\mathbf{z}(\epsilon)d\epsilon \\ &+ \int_{-\tau}^{0}\int_{t+s}^{t} \dot{\mathbf{z}}^T(\epsilon)R\dot{\mathbf{z}}(\epsilon)d\epsilon ds \\ &+ \int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+s}^{t} \dot{\mathbf{z}}^T(\epsilon)S\dot{\mathbf{z}}(\epsilon)d\epsilon ds d\theta \\ &+ \int_{-\mu_m}^{\mu_m}\int_{t+s-\tau}^{t} \dot{\mathbf{z}}^T(\epsilon)Z\dot{\mathbf{z}}(\epsilon)d\epsilon ds, \end{aligned} \tag{4.6}$$

with $\chi^T(t) = \begin{bmatrix} \mathbf{z}^T(t) & \int_{t-\tau}^{t} \mathbf{z}^T(\epsilon)d\epsilon \end{bmatrix}$,

$$P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix},$$

and z_t corresponds to $\mathbf{z}(\sigma)$ with $\sigma \in [t-\tau-\mu_m, t]$.

To fulfill the condition for (4.6) be positive definite we impose $P > 0$, $Q > 0$, $R > 0$, $S > 0$ and $Z > 0$.

Sufficient conditions for $\dot{V}(z_t) < 0$ are developed next. Using the Newton-Leibniz formula related to states (3.19), we have the following null term

$$\begin{aligned} 0 =& 2\Lambda(t)\left[\dot{\mathbf{z}}(t) - \bar{A}\mathbf{z}(t) - \sum_{i=1}^{n}\hat{B}_i\mathbf{z}(t-\tau_i(t))\right] \\ =& 2\Lambda(t)\left[\dot{\mathbf{z}}(t) - \bar{A}\mathbf{z}(t) - \sum_{i=1}^{n}\hat{B}_i\mathbf{z}(t-\tau) + \sum_{i=1}^{n}\hat{B}_i\int_{-\tau_i(t)}^{-\tau}\dot{\mathbf{z}}(t+\epsilon)d\epsilon\right] \\ =& 2\Lambda(t)\left[\dot{\mathbf{z}}(t) - \bar{A}\mathbf{z}(t) - \hat{B}\mathbf{z}(t-\tau)\right] + v(t), \end{aligned} \tag{4.7}$$

where,

$$\Lambda(t) = \left[-\mathbf{z}^T(t)F - \lambda\dot{\mathbf{z}}^T(t)F\right], \tag{4.8}$$

$$\hat{B}_i = -\sum_{k=1}^{2^p}(U\mathcal{L}_{ai}W)\otimes BK_k, \tag{4.9}$$

$$\hat{B} = \sum_{i=1}^{n}\hat{B}_i, \tag{4.10}$$

$$v(t) = \sum_{i=1}^{n}\int_{-\tau_i(t)}^{-\tau} 2\Lambda(t)\hat{B}_i\dot{\mathbf{z}}(t+\epsilon)d\epsilon, \tag{4.11}$$

and λ is a free constant for adjustment. Applying the inequality $2\boldsymbol{a}^T\boldsymbol{b} \leq \boldsymbol{a}^T X \boldsymbol{a} + \boldsymbol{b}^T X^{-1}\boldsymbol{b}$ on (4.11), with

$$\begin{aligned}
\boldsymbol{a}^T &= \Lambda(t)\hat{B}_i,\\
\boldsymbol{b} &= \dot{\boldsymbol{z}}(t+\epsilon),\\
X^{-1} &= \frac{Z}{n},
\end{aligned}$$

since Z is assumed to be a definite positive matrix, we obtain

$$\begin{aligned}
v(t) \leq & \sum_{i=1}^{n}\int_{-\tau_i(t)}^{-\tau}\left(\Lambda(t)\hat{B}_i\right)nZ^{-1}\left(\Lambda(t)\hat{B}_i\right)^T d\epsilon\\
& + \sum_{i=1}^{n}\int_{-\tau_i(t)}^{-\tau}\dot{\boldsymbol{z}}^T(t+\epsilon)\frac{Z}{n}\dot{\boldsymbol{z}}(t+\epsilon)d\epsilon\\
\leq & \sum_{i=1}^{n}\mu_m\left(\Lambda(t)\hat{B}_i\right)nZ^{-1}\left(\Lambda(t)\hat{B}_i\right)^T\\
& + \int_{t-\tau-\mu_m}^{t-\tau+\mu_m}\dot{\boldsymbol{z}}^T(\epsilon)Z\dot{\boldsymbol{z}}(\epsilon)d\epsilon.
\end{aligned} \tag{4.12}$$

With (4.12) the null term (4.7) becomes,

$$\begin{aligned}
0 \leq & -2\lambda\dot{\boldsymbol{z}}^T(t)F\dot{\boldsymbol{z}}(t) + 2\boldsymbol{z}^T(t)F\bar{A}\boldsymbol{z}(t)\\
& + 2\lambda\dot{\boldsymbol{z}}^T(t)F\bar{A}\boldsymbol{z}(t) - 2\boldsymbol{z}^T(t)F\dot{\boldsymbol{z}}(t)\\
& + 2\boldsymbol{z}^T(t)F\hat{B}\boldsymbol{z}(t-\tau) + 2\lambda\dot{\boldsymbol{z}}^T(t)F\hat{B}\boldsymbol{z}(t-\tau)\\
& + \sum_{i=1}^{n}\mu_m\left(\Lambda(t)\hat{B}_i\right)nZ^{-1}\left(\Lambda(t)\hat{B}_i\right)^T\\
& + \int_{t-\tau-\mu_m}^{t-\tau+\mu_m}\dot{\boldsymbol{z}}^T(\epsilon)Z\dot{\boldsymbol{z}}(\epsilon)d\epsilon.
\end{aligned} \tag{4.13}$$

Then, we add equation (4.13) to the functional time-derivative,

$$
\begin{aligned}
\dot{V}(z_t) \leq & \chi^T(t)P\dot{\chi}(t) + \dot{\chi}^T(t)P\chi(t) + z^T(t)Qz(t) \\
& - z^T(t-\tau)Qz(t-\tau) + \tau\dot{z}^T(t)R\dot{z}(t) \\
& - \int_{t-\tau}^{t} \dot{z}^T(\epsilon)R\dot{z}(\epsilon)d\epsilon + \frac{\tau^2}{2}\dot{z}^T(t)S\dot{z}(t) \\
& - \int_{-\tau}^{0}\int_{t+s}^{t} \dot{z}^T(\epsilon)S\dot{z}(\epsilon)d\epsilon ds + 2\mu_m\dot{z}^T(t)Z\dot{z}(t) \\
& - \int_{t-\tau-\mu_m}^{t-\tau+\mu_m} \dot{z}^T(\epsilon)Z\dot{z}(\epsilon)d\epsilon - 2\lambda\dot{z}^T(t)F\dot{z}(t) \\
& + 2z^T(t)F\bar{A}z(t) + 2\lambda\dot{z}^T(t)F\bar{A}z(t) - 2z^T(t)F\dot{z}(t) \\
& + 2z^T(t)F\hat{B}z(t-\tau) + 2\lambda\dot{z}^T(t)F\hat{B}z(t-\tau) \\
& + \sum_{i=1}^{n} \mu_m\big(\Lambda(t)\hat{B}_i\big)nZ^{-1}\big(\Lambda(t)\hat{B}_i\big)^T \\
& + \int_{t-\tau-\mu_m}^{t-\tau+\mu_m} \dot{z}^T(t)Z\dot{z}(t)d\epsilon,
\end{aligned}
$$

and applying Lemmas 2.2 and 2.3 on the integrals with terms S and R, respectively, yields

$$
\begin{aligned}
\dot{V}(z_t) \leq & \chi^T(t)P\dot{\chi}(t) + \dot{\chi}^T(t)P\chi(t) + z^T(t)Qz(t) \\
& - z^T(t-\tau)Qz(t-\tau) + \tau\dot{z}^T(t)R\dot{z}(t) \\
& - \int_{t-\tau}^{t} \dot{z}^T(\epsilon)d\epsilon R \int_{t-\tau}^{t} \dot{z}(\epsilon)d\epsilon - \frac{3}{\tau}\Omega^T R\Omega \\
& + \frac{\tau^2}{2}\dot{z}^T(t)S\dot{z}(t) \\
& - \frac{2}{\tau^2}\int_{-\tau}^{0}\int_{t+s}^{t} \dot{z}^T(\epsilon)d\epsilon ds S \int_{-\tau}^{0}\int_{t+s}^{t} \dot{z}^T(\epsilon)d\epsilon ds \\
& + 2\mu_m\dot{z}^T(t)Z\dot{z}(t) \\
& - 2\lambda\dot{z}^T(t)F\dot{z}(t) + 2z^T(t)F\bar{A}z(t) \\
& + 2\lambda\dot{z}^T(t)F\bar{A}z(t) - 2z^T(t)F\dot{z}(t) \\
& + 2z^T(t)F\hat{B}z(t-\tau) + 2\lambda\dot{z}^T(t)F\hat{B}z(t-\tau) \\
& + \sum_{i=1}^{n} \mu_m\big(\Lambda(t)\hat{B}_i\big)nZ^{-1}\big(\Lambda(t)\hat{B}_i\big)^T
\end{aligned}
$$

$$
\begin{aligned}
\leq & \chi^T(t)P\dot{\chi}(t) + \dot{\chi}^T(t)P\chi(t) + z^T(t)Qz(t) \\
& -z^T(t-\tau)Qz(t-\tau) + \tau\dot{z}^T(t)R\dot{z}(t) \\
& -\big(z(t)-z(t-\tau)\big)^T R\big(z(t)-z(t-\tau)\big) - \frac{3}{\tau}\Omega^T R\Omega \\
& +\frac{\tau^2}{2}\dot{z}^T(t)S\dot{z}(t) - 2z^T(t)Sz(t) \\
& +\frac{4}{\tau}z^T(t)S\int_{t-\tau}^{t} z(\epsilon)d\epsilon \\
& -\frac{2}{\tau^2}\int_{t-\tau}^{t} z^T(\epsilon)d\epsilon S\int_{t-\tau}^{t} z^T(\epsilon)d\epsilon \\
& +2\mu_m\dot{z}^T(t)Z\dot{z}(t) \\
& -2\lambda\dot{z}^T(t)F\dot{z}(t) + 2z^T(t)F\bar{A}z(t) \\
& +2\lambda\dot{z}^T(t)F\bar{A}z(t) - 2z^T(t)F\dot{z}(t) \\
& +2z^T(t)F\hat{B}z(t-\tau) + 2\lambda\dot{z}^T(t)F\hat{B}z(t-\tau) \\
& +\sum_{i=1}^{n}\mu_m\big(\Lambda(t)\hat{B}_i\big)nZ^{-1}\big(\Lambda(t)\hat{B}_i\big)^T.
\end{aligned} \tag{4.14}
$$

Let $Y^T(t) = \begin{bmatrix} z^T(t) & \dot{z}^T(t) & z^T(t-\tau) & \int_{t-\tau}^{t} z^T(\epsilon)d\epsilon \end{bmatrix}$ and

$$
\Gamma_i = \begin{bmatrix} F\hat{B}_i \\ \lambda F\hat{B}_i \\ 0 \\ 0 \end{bmatrix},
$$

we can rewrite the product $\Lambda(t)\hat{B}_i$ as $Y^T(t)\Gamma_i$. Defining $\hat{\Phi}$ similarly to Φ_k in (4.4), but replacing $\hat{B}_k$ with $\hat{B}$ as (4.10), the matrix $\hat{\Phi}_F$ is written as,

$$
\hat{\Phi}_F = \begin{bmatrix} -F\bar{A} - \bar{A}^T F^T & F - \lambda\bar{A}^T F^T & -F\hat{B} & 0 \\ * & \lambda(F + F^T) & -\lambda F\hat{B} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}
$$

then, equation (4.14) becomes

$$
\begin{aligned}
\dot{V}(z_t) \leq & Y^T(t)\hat{\Phi}Y(t) + \sum_{i=1}^{n}\mu_m Y^T(t)\Gamma_i nZ^{-1}\Gamma_i^T Y(t) \\
= & Y^T(t)\left(\sum_{i=1}^{n}\left(\frac{1}{n}\hat{\Phi} + \mu_m\Gamma_i nZ^{-1}\Gamma_i^T\right)\right)Y(t).
\end{aligned} \tag{4.15}
$$

In order to have $\dot{V}(t) < 0$ for any $\Upsilon(t) \neq 0$, the parenthesis term in equation (4.15) must be negative. Applying Schur complement, we obtain

$$\sum_{i=1}^{n} \begin{bmatrix} \frac{1}{n}\hat{\Phi} & \mu_m \Gamma_i \\ * & \frac{-\mu_m}{n} Z \end{bmatrix} < 0. \tag{4.16}$$

Note from equations (4.9) and (4.10) that the ith term of the sum (4.16) is written as convex combinations on the variable α_k. Thus, to inequality (4.16) be satisfied we impose that each term of the sum must be negative in all vertices on variable α_k. Under Assumption 2.2 we have that all agents share the same set of vertices of state feedback matrices, in this manner we can write equation (4.10) as follows,

$$\begin{aligned} \hat{B} = \sum_{i=1}^{n} \hat{B}_i &= -\sum_{i=1}^{n}\sum_{k=1}^{2^p} (U\mathcal{L}_{ai}W) \otimes \alpha_k B(D_k K + D_k^- H) \\ &= -\sum_{i=1}^{n} (U\mathcal{L}_{ai}W) \otimes \sum_{k=1}^{2^p} \alpha_k B(D_k K + D_k^- H), \end{aligned}$$

then, noting that

$$\sum_{i=1}^{n} \mathcal{L}_{ai} = \mathcal{L},$$

on the vertices of $\hat{B}$ we have,

$$\begin{aligned} \hat{B}_k &= -\sum_{i=1}^{n} (U\mathcal{L}_{ai}W) \otimes B(D_k K + D_k^- H) \\ &= -(U\mathcal{L}W) \otimes B(D_k K + D_k^- H), \ \ \forall k \in \{1, \dots, 2^p\}. \end{aligned}$$

Therefore, the ith term of the sum (4.16) is written as,

$$\begin{bmatrix} \frac{1}{n}\hat{\Phi}_k & \mu_m \Gamma_{ik} \\ * & \frac{-\mu_m}{n} Z \end{bmatrix} < 0, \qquad \forall k \in \{1, \dots, 2^p\}, \tag{4.17}$$

that is, we are evaluating terms $\hat{B}_i$ and $\hat{B}$ on its vertices, as

$$\begin{aligned} \hat{B}_{ik} &= -(U\mathcal{L}_{ai}W) \otimes B(D_k K + D_k^- H), \\ \hat{B}_k &= -(U\mathcal{L}W) \otimes B(D_k K + D_k^- H), \ \ \forall k \in \{1, \dots, 2^p\}. \end{aligned}$$

Summing the n terms (4.17) gives,

$$\sum_{i=1}^{n} \begin{bmatrix} \frac{1}{n}\hat{\Phi}_k & \mu_m \Gamma_{ik} \\ * & \frac{-\mu_m}{n} Z \end{bmatrix} = \begin{bmatrix} \Phi_k & \mu_m \Gamma_k \\ * & -\mu_m Z \end{bmatrix} < 0, \quad \forall k \in \{1, \ldots, 2^p\},$$

which is equivalent to inequality (4.1). Therefore, if the conditions of Theorem (4.1) holds we have $V(z_t) > 0$ and $\dot{V}(z_t) < 0$, which implies that the multi-agent system reaches consensus asymptotically.

Furthermore, considering the upper bound on integral terms

$$\int_a^b f(x)dx \leq (a-b) \sup_{\eta \in [a,b]} f(\eta),$$

we have the following upper bound for the Lyapunov-Krasovskii functional

$$\begin{aligned} z^T(t)P_1 z(t) &\leq V(z_t) \\ &\leq \sup_{\eta \in [-\tau-\mu_m, 0]} ||\phi_z(\eta)||^2 \left[(1+\tau^2)\lambda_{max}(P) + \tau\lambda_{max}(Q)\right] \\ &\quad + \sup_{\eta \in [-\tau-\mu_m, 0]} ||\dot{\phi}_z(\eta)||^2 \left[\frac{\tau}{2}\lambda_{max}(R) + \frac{\tau^3}{6}\lambda_{max}(S) \right. \\ &\quad \left. + 2\mu_m \tau \lambda_{max}(Z)\right] = \bar{\mathcal{X}}_{DOC}, \end{aligned}$$

where $\phi_z(\eta)$ and $\dot{\phi}_z(\eta)$ are initial conditions in $z(t)$ and $\dot{z}(t)$, respectively. Hence, if $\bar{\mathcal{X}}_{DOC} \leq 1$ implies that $z^T(t)P_1 z(t) \leq 1$, and all the trajectories of $z(t)$ starting from $\bar{\mathcal{X}}_{DOC} \leq 1$ remains within $\mathcal{E}(P_1, 1)$. □

Theorem 4.1 presents sufficient conditions to guarantee consensus of multi-agent system with agents described by (3.1), particularly it characterize the multi-agent system stability described as equation (3.19), which implies in consensus. Additionally it gives an estimation for the region of consensus presented in equation (4.5).

It is important to remark that:

Remark 4.1. The development of the proof of Theorem 4.1 is based on the proof of Theorem 1 developed by Savino, Souza, and Pimenta, 2017. Particularly, the same Lyapunov-Krasovskii functional candidate and null term (4.7) is used to derive inequalities (4.1) and (4.2).

Remark 4.2. The adjustment parameter λ is free and no systematic procedure to tune is developed. In all examples its value is given, usually in the range $(0, 1]$.

4.1.1 Optimization problem

A procedure to use the conditions derived in Theorem 4.1 is proposed in this section. The difficulty on solve the matrix inequalities derived is highlighted in the follow.

Remark 4.3. It should be noticed that inequality (4.1) is a bilinear matrix inequality (BMI) due to Γ_k and Φ_{Fk}, if H and F are decision variables. A simple procedure to overcome this issue is given below as an iterative algorithm suggested by Tarbouriech et al., 2011, where at each step some variables are fixed and an LMI problem is solved.

Algorithm 4.1

1. Initialize H.
2. Solve the optimization problem (4.18).
3. Keep the value of F and solve the optimization problem (4.18) for H.
4. Go to step 2 until no significant change on the size of the estimation of domain of consensus is obtained.
5. Stop.

$$\begin{aligned} & min\ \ \gamma \\ s.t. & \begin{cases} a) & \text{LMI (4.1), (4.2), and (4.3)} \\ b) & \beta_1 I_{2m(n-1)} - P \geq 0, \\ c) & \beta_2 I_{m(n-1)} - Q \geq 0, \\ d) & \beta_3 I_{m(n-1)} - R \geq 0, \\ e) & \beta_4 I_{m(n-1)} - S \geq 0, \\ f) & \beta_5 I_{m(n-1)} - Z \geq 0, \end{cases} \end{aligned} \tag{4.18}$$

where

$$\gamma = \beta_1 + \tau^2\beta_1 + \tau\beta_2 + \frac{\tau}{2}\beta_3 + \frac{\tau^3}{6}\beta_4 + 2\mu_m\tau\beta_5,$$

and the scalars $\beta_i > 0$, for $i = \{1, \ldots, 5\}$, are introduced to bound terms on equation (4.5). Hence, an estimate for the domain of consensus is given by $\delta_{max} = 1/\sqrt{\gamma}$.

Remark 4.4. We aim to find an estimation for the domain of consensus as the follow,

$$\mathcal{X}_{DOC} = \{ \boldsymbol{x}(t_0) \in \mathbb{R}^{nm} : ||\boldsymbol{\phi}_1 - \boldsymbol{\phi}_i||_\infty \geq \delta_1, \\ ||\dot{\boldsymbol{\phi}}_1 - \dot{\boldsymbol{\phi}}_i||_\infty \geq \delta_2, \ \forall i \in \mathcal{V} \},$$

by maximizing the scalars $\delta_1, \delta_2 \in \mathbb{R}$. On the optimization problem (4.18) we set the parameters $\delta_1 = \delta_2$. Better solutions may be obtained by adjusting these parameters.

One question that may arise with the use of Algorithm 4.1 is: how to choose an initial value for the matrix H. If the multi-agent system is able to reach consensus without saturation with the feedback matrix K, then the convergence of the algorithm is ensured by making $H = K$. This happens because even with saturation there is a region on the space where the agents are close enough so the inputs does not saturate.

4.1.2 Numerical examples

This section shows examples to illustrate the use of Theorem 4.1. The method developed is compared with the work of You et al., 2016 to demonstrate that our approach is more appropriate for multi-agent system with input saturation and time-delays.

Example 4.1. *Consider a multi-agent system composed of five agents with directed and static communication topology, as shown in Figure 4.1, and Laplacian Matrix given by*

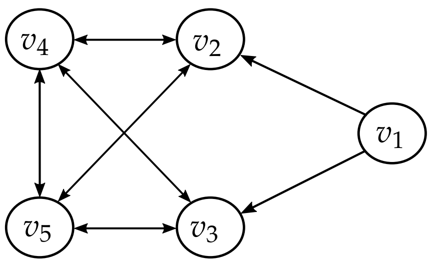


FIGURE 4.1: Example 4.1–Graph representation of agents interaction.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

Note that this is a leader-following problem–that is, all followers have to synchronize its states with the leader's states, v_1. It is assumed that the agents have the following model,

$$\begin{bmatrix} \dot{x}_i^1(t) \\ \dot{x}_i^2(t) \\ \dot{x}_i^3(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \\ x_i^3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} sat\big(\boldsymbol{u}_i(t - \tau_i(t))\big),$$

numbered $i = 2, \ldots, 5$, and the leader's model,

$$\begin{bmatrix} \dot{x}_1^1(t) \\ \dot{x}_1^2(t) \\ \dot{x}_1^3(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} x_1^1(t) \\ x_1^2(t) \\ x_1^3(t) \end{bmatrix}.$$

The gains in matrix K, for the consensus protocol on (2.3), are considered

$$K = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.7 \end{bmatrix},$$

and the limit of actuators $u_{max} = 2.5$.

For comparison purpose, due to the conditions of You et al., 2016, it is necessary to assume that the time-varying delays $\tau_i(t)$ are synchronous and satisfies $0 \leq \tau_i(t) \leq \tau$, and $\dot{\tau}_i(t) \leq \mu_m < 1$, where τ and μ_m are positive constants. Thus, to fulfill these conditions the time-varying delays are given by,

$$\tau_i(t) = \tau + \mu_m \sin(t), \quad \forall i \in \{2, \ldots 5\}.$$

Table 4.1 shows results for maximum allowed constant delays for Theorem 4.1 and Theorem 1 (You et al., 2016) with different values of the time-varying part. It was used $\lambda = 0.4$ on Theorem 4.1. To show that Theorem 4.1 is more appropriate than the results of You et al., 2016 to deal with consensus problem with saturation and time-varying delays, Figures 4.2 and 4.3 depicts simulations of the multi-agent system for different initial conditions and for time-varying delay $\tau_i(t) = 0.23 + 0.1 \sin(t)$. In Figure 4.2 it

TABLE 4.1: Example 4.1–Comparison of maximum input delay τ_{max} for various time-varying parts μ_m

	$\mu_m = 0.01$	$\mu_m = 0.05$	$\mu_m = 0.10$
Theorem 4.1	0.43	0.34	0.23
You et al., 2016	0.35	0.34	0.34

is shown the evolution of the network with initial conditions:

$$x_1 = \begin{bmatrix} 3.0 \\ 2.0 \\ 4.0 \end{bmatrix}; \; x_2 = \begin{bmatrix} 0.0 \\ 4.0 \\ 6.0 \end{bmatrix}; \; x_3 = \begin{bmatrix} 1.0 \\ 4.0 \\ 5.0 \end{bmatrix}; \; x_4 = \begin{bmatrix} 2.0 \\ 3.0 \\ 7.0 \end{bmatrix}; \; x_5 = \begin{bmatrix} 25.0 \\ 5.0 \\ 4.0 \end{bmatrix},$$

and in Figure 4.3 the evolution of the network with initial conditions:

$$x_1 = \begin{bmatrix} 1.2 \\ 2.0 \\ 4.3 \end{bmatrix}; \; x_2 = \begin{bmatrix} 0.8 \\ 1.58 \\ 3.88 \end{bmatrix}; \; x_3 = \begin{bmatrix} 1.62 \\ 2.42 \\ 4.72 \end{bmatrix}; \; x_4 = \begin{bmatrix} 1.5 \\ 2.3 \\ 4.0 \end{bmatrix}; \; x_5 = \begin{bmatrix} 0.9 \\ 1.9 \\ 4.7 \end{bmatrix},$$

The estimation for the domain of consensus was $\delta_1 = \delta_2 = 0.84$, *which is respected only on the second set of initial conditions. Observe that this value is related to variable* $z(t)$ *which is translated in* $x_i(t)$ *as each agent's distance from* $x_1(t)$ *(see equation (2.4)). Therefore, the Theorem 4.1 guarantees that for any initial condition within* $||x_1(t_0) - x_i(t_0)||_\infty \leq 0.84$, *the multi-agent system will reach consensus asymptotically.*

Table 4.1 of Example 4.1 shows that as the time-varying part of the delay (μ_m) increases the approach of You et al., 2016 guarantee larger values for the constant part (τ) than the conditions of Theorem 4.1. However, note that the conditions of Theorem 4.1 are guaranteeing consensus for delays within the set $[\tau - \mu_m, \tau + \mu_m]$, with nonuniform and non-differentiable delays, while You et al., 2016 consider synchronous and differentiable delays. Moreover, in Example 4.1 the networked system does not reach consensus for the first set of initial conditions (Figure 4.2), even though the conditions of Theorem 1 (You et al., 2016) has been satisfied. But it converges for the second set of initial conditions (Figure 4.3), when the estimation of the domain of consensus is respected. This fact shows that more assumptions should be made in order to obtain results more appropriate for global or semi-global convergence. With the method developed here the consensus is guaranteed on the estimation of the domain of consensus.

In the next example we study the influence of the communication topology on the size of the estimated domain of consensus. This is an interesting aspect

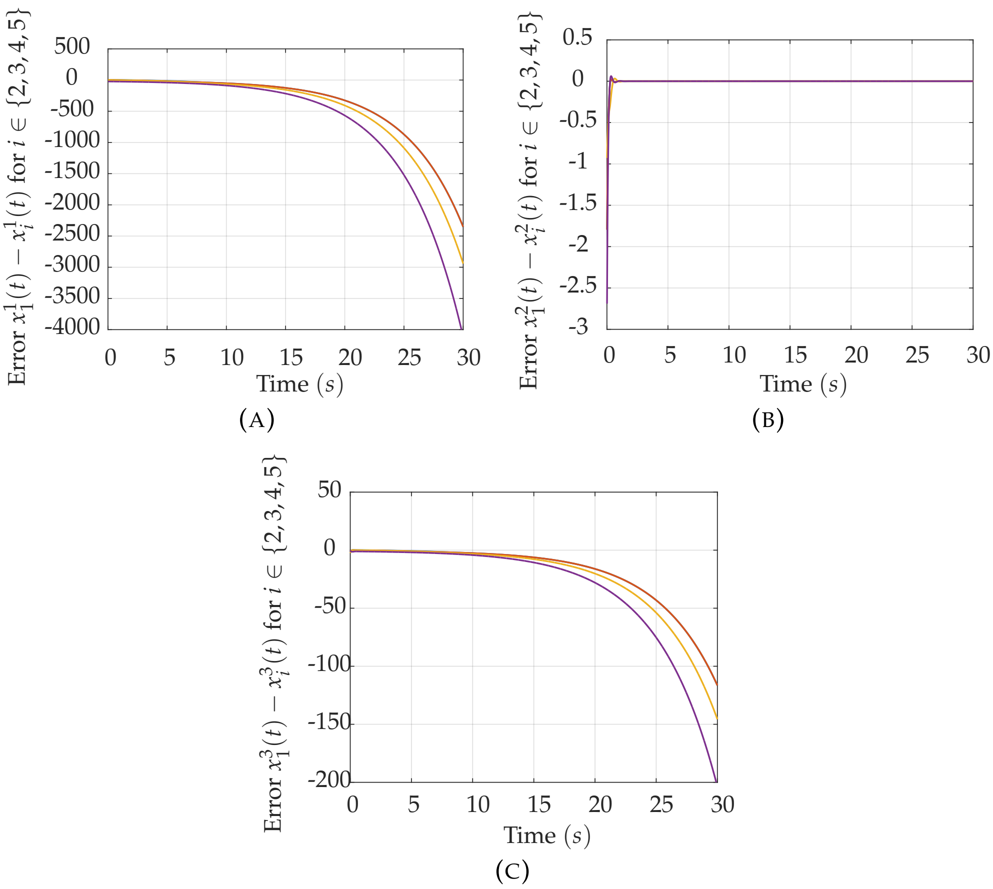


FIGURE 4.2: Example 4.1–Divergent trajectories of the multi-agent system with input time-delays.

considering the relation between the number of neighbors of an agent with the linear region and the construction of the domain of consensus.

Example 4.2. *In this example, we will explore how different connections affects the size of the estimation of domain of consensus. To this end, consider a network with six agents, each of them with the following dynamics,*

$$\begin{bmatrix} \dot{x}_i^1(t) \\ \dot{x}_i^2(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 1 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} sat\big(\boldsymbol{u}_i(t - \tau_i(t))\big), \;\; i = \{1, \dots, 6\}.$$

Lets consider five communication configurations for the agents, represented in Figure 4.4, with Laplacians represented as follows,

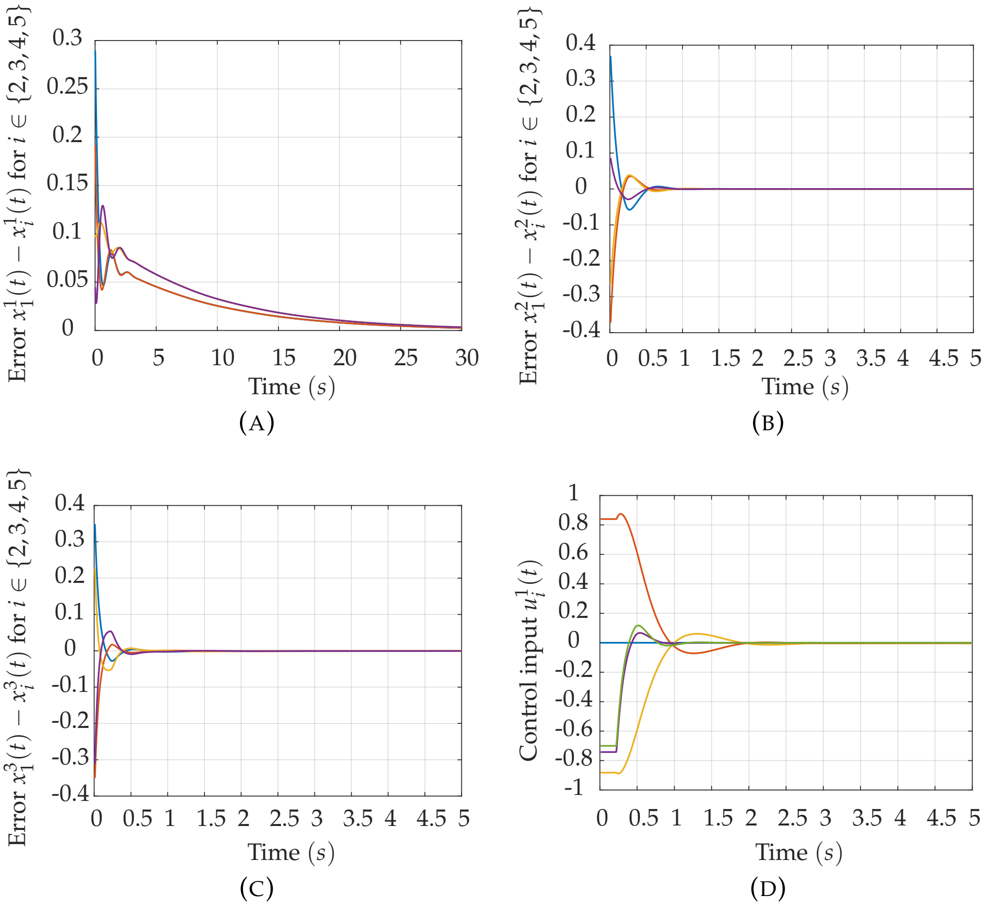


FIGURE 4.3: Example 4.1–Convergent trajectories of the multi-agent system with input time-delays.

$$\mathcal{L}_{(A)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathcal{L}_{(B)} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$\mathcal{L}_{(C)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{L}_{(D)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{L}_{(E)} = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

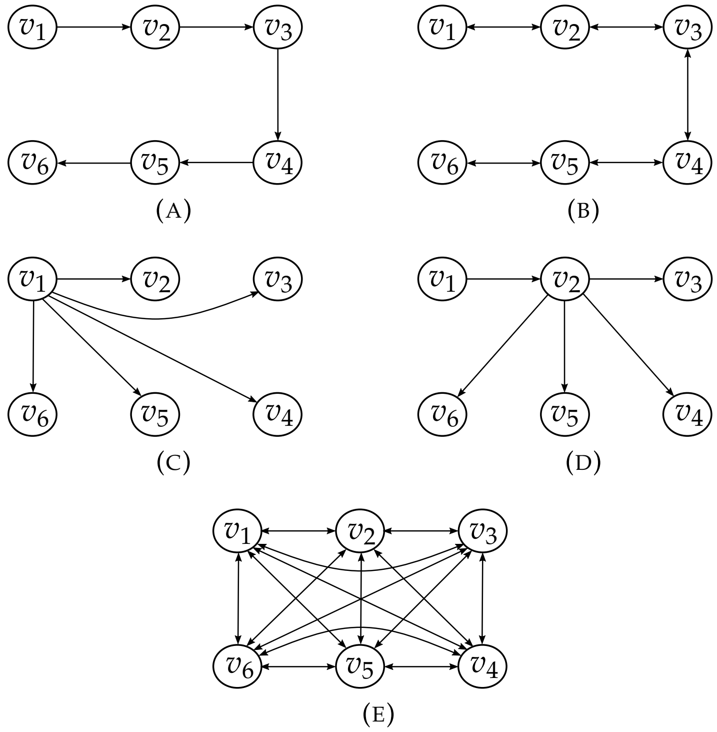


FIGURE 4.4: Example 4.2–Communication topology for five distinct configurations.

In Table 4.2 we present the size of the estimation of domain of consensus for each communication topology, calculated with Algorithm 4.1, with $\lambda = 0.7$, $\tau = 0.2$, $\mu_m = 0.1$ *and feedback gain matrix* $K = [0.1\ 0.5]$.

TABLE 4.2: Example 4.2–Comparison of the estimation of the domain of consensus for different communication arrangements

Configuration	(A)	(B)	(C)	(D)	(E)
Estimation of the domain of consensus δ_{max}	2.95	1.93	4.78	3.37	0.10

As shown in Table 4.2 configuration (C) leads to the largest estimation and configuration (E) to the smallest, which suggests that a more connected network leads to a small domain of consensus. This tendency might be caused by the protocol of consensus chosen, it increases the size of control action directly with the grow of the neighbor set, which causes saturation with less distance between agents.

4.2 Control Synthesis

In the following theorem, we present conditions to synthesize a stabilizing feedback control for each agent with an estimate of the region of consensus.

Theorem 4.2. *Let the scalars τ, μ_m, and λ be given. The multi-agent system (3.7) with consensus protocol (2.3) and time-delay $\tau_i(t) \in [\tau - \mu_m, \tau + \mu_m]$ for $\tau > 0$, $0 \leq \mu_m < \tau$ reaches consensus with state feedback matrix $K{=}\bar{K}F_m$, if there exist matrices: $\bar{F}{>}0$, $\bar{P}_1{=}\bar{P}_1^T$, $\bar{P}_2$, $\bar{P}_3{=}\bar{P}_3^T$, $\bar{Q}{=}\bar{Q}^T{>}0$, $\bar{R}{=}\bar{R}^T{>}0$, $\bar{S}{=}\bar{S}^T{>}0$, $\bar{Z}{=}\bar{Z}^T{>}0$ with dimensions $m(n{-}1) \times m(n{-}1)$, and $\bar{H}$, $\bar{K}$ with dimensions $p \times m$, such that the following conditions are satisfied:*

$$\begin{bmatrix} \bar{\Phi}_k & \mu_m \bar{\Gamma}_k \\ * & -\mu_m \bar{Z} \end{bmatrix} < 0, \ \ \forall k = \{1, \dots, 2^p\}, \tag{4.19}$$

$$\begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ * & \bar{P}_3 \end{bmatrix} > 0, \tag{4.20}$$

and

$$\begin{bmatrix} u_{max}^2 & \mathcal{I}_i^T \mathcal{L}_{ai} W \otimes \bar{\boldsymbol{h}}_r \\ W^T \mathcal{L}_{ai}^T \mathcal{I}_i \otimes \bar{\boldsymbol{h}}_r^T & \bar{P}_1 \end{bmatrix} \geq 0, \quad \forall i \in \{1, \dots, n\}, \ \forall r \in \{1, \dots, p\} \tag{4.21}$$

where

$$\bar{\Phi}_k = \bar{\Phi}_P + \bar{\Phi}_Z + \bar{\Phi}_R + \bar{\Phi}_{QS} + \bar{\Phi}_{Fk}, \tag{4.22}$$

$$\bar{\Gamma}_k = \begin{bmatrix} \bar{B}_k \\ \lambda \bar{B}_k \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{\Phi}_Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 2\mu_m \bar{Z} & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

$$\bar{\Phi}_P = \begin{bmatrix} \bar{P}_2 + \bar{P}_2^T & \bar{P}_1 & -\bar{P}_2 & \bar{P}_3 \\ * & 0 & 0 & \bar{P}_2 \\ * & * & 0 & -\bar{P}_3 \\ * & * & * & 0 \end{bmatrix}$$

$$\bar{\Phi}_R = \begin{bmatrix} -\frac{4}{\tau}\bar{R} & 0 & -\frac{2}{\tau}\bar{R} & \frac{6}{\tau^2}\bar{R} \\ * & \tau \bar{R} & 0 & 0 \\ * & * & -\frac{4}{\tau}\bar{R} & \frac{6}{\tau^2}\bar{R} \\ * & * & * & \frac{-12}{\tau^3}\bar{R} \end{bmatrix} \tag{4.23}$$

$$\bar{\Phi}_{QS} = \begin{bmatrix} \bar{Q} - 2\bar{S} & 0 & 0 & \frac{2}{\tau}\bar{S} \\ * & \frac{\tau^2}{2}\bar{S} & 0 & 0 \\ * & * & -\bar{Q} & 0 \\ * & * & * & -\frac{2}{\tau^2}\bar{S} \end{bmatrix}$$

$$\bar{\Phi}_{Fk} = \begin{bmatrix} \bar{A}\bar{F}^T + \bar{F}\bar{A}^T & \lambda\bar{F}\bar{A}^T - \bar{F}^T & \bar{B}_k & 0 \\ * & -\lambda(\bar{F}^T + \bar{F}) & \lambda\bar{B}_k & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

and,

$$\begin{aligned} \bar{A} &= I_{n-1} \otimes A, \\ \bar{B}_k &= -U\mathcal{L}W \otimes B(D_k\bar{K} + D_k^-\bar{H}). \end{aligned}$$

Furthermore, an estimation for the domain of consensus for the multi-agent system is given by $\bar{\mathcal{X}}_{DOC} \leq 1$*, where*

$$\begin{aligned} \bar{\mathcal{X}}_{DOC} =& \delta_1^2\big[\lambda_{max}\big((I_2 \otimes \bar{F}^{-1})\bar{P}(I_2 \otimes \bar{F}^{-1})\big)(1+\tau^2) \\ &+ \tau\lambda_{max}\big(\bar{F}^{-1}\bar{Q}\bar{F}^{-1}\big)\big] \\ &+ \delta_2^2\left[\frac{\tau}{2}\lambda_{max}\big(\bar{F}^{-1}\bar{R}\bar{F}^{-1}\big) + \frac{\tau^3}{6}\lambda_{max}\big(\bar{F}^{-1}\bar{S}\bar{F}^{-1}\big)\right. \\ &\left. + 2\mu_m\tau\lambda_{max}\big(\bar{F}^{-1}\bar{Z}\bar{F}^{-1}\big)\right]. \end{aligned} \tag{4.24}$$

Proof. The conditions (4.19) and (4.20) of Theorem 4.2 are derived from the inequalities (4.1) and (4.2). To derive them define $F = I_{n-1} \otimes F_m$, with $F_m > 0$, $F_m \in \mathbb{R}^{m\times m}$, and let $\bar{F} = F^{-1}$, $\bar{P}_i = \bar{F}P_i\bar{F}$ for $i = \{1,2,3\}$, $\bar{Z} = \bar{F}Z\bar{F}$, $\bar{R} = \bar{F}R\bar{F}$, $\bar{Q} = \bar{F}Q\bar{F}$, $\bar{S} = \bar{F}S\bar{F}$, and

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^T & \bar{P}_3 \end{bmatrix}.$$

Pre- and post-multiplying inequality (4.1) by $diag\{\bar{F}, \bar{F}, \bar{F}, \bar{F}, \bar{F}\}$ and $diag\{\bar{F}^T, \bar{F}^T, \bar{F}^T, \bar{F}^T, \bar{F}^T\}$, respectively, we obtain (4.19). The product $\hat{B}_k\bar{F}$ is given by

$$\begin{aligned} \hat{B}_k\bar{F} =& -\big(U\mathcal{L}W \otimes B(D_kK + D_k^-H)\big)\big(I_{n-1} \otimes F_m^{-1}\big) \\ =& -U\mathcal{L}WI_{n-1} \otimes B(D_kK + D_k^-H)F_m^{-1}. \end{aligned}$$

We define $\bar{K} = KF_m^{-1}$, $\bar{H} = HF_m^{-1}$, and

$$\bar{B}_k = -U\mathcal{L}W \otimes B(D_k\bar{K} + D_k^-\bar{H}),$$

and the state feedback gain for each agent can be computed by

$$K = \bar{K}F_m.$$

Moreover, inequality (4.21) is equivalent to (4.6), where $\bar{\boldsymbol{h}}_r$ is the rth row of $\bar{H}$. This can be shown by writing (2.11) on the variable $\boldsymbol{z}(t)$ using equation (2.6) and following the steps of the proof of Theorem 4.1. □

Remark 4.5. The conditions for synthesizing the feedback gains on Theorem 4.2 are expressed in terms of linear matrix inequalities, but the estimation for the domain of consensus is nonlinear. To design the feedback matrix while maximizing the domain of consensus we propose next a linear optimization problem, which can be solved with an appropriate method, such the ones provided in *Matlab LMI toolbox*.

4.2.1 Optimization problem

With feasible conditions on Theorem 4.2, we are interested in finding a set of initial conditions as large as possible for the domain of consensus. To this end, the following problem can be solved. For simplicity we select $\delta_1 = \delta_2$ (Remark 4.4 is applicable here),

$$\begin{aligned} & min \;\; \gamma \\ s.t. & \begin{cases} a) & \text{LMI (4.19), (4.20), and (4.21)} \\ b) & \begin{bmatrix} \beta_1 I_{2m(n-1)} & I_{2m(n-1)} \\ I_{2m(n-1)} & 2(I_2 \otimes \bar{F}) - \bar{P} \end{bmatrix} \geq 0, \\ c) & \begin{bmatrix} \beta_2 I_{m(n-1)} & I_{m(n-1)} \\ I_{m(n-1)} & 2\bar{F} - \bar{Q} \end{bmatrix} \geq 0, \\ d) & \begin{bmatrix} \beta_3 I_{m(n-1)} & I_{m(n-1)} \\ I_{m(n-1)} & 2\bar{F} - \bar{R} \end{bmatrix} \geq 0, \\ e) & \begin{bmatrix} \beta_4 I_{m(n-1)} & I_{m(n-1)} \\ I_{m(n-1)} & 2\bar{F} - \bar{S} \end{bmatrix} \geq 0, \\ f) & \begin{bmatrix} \beta_5 I_{m(n-1)} & I_{m(n-1)} \\ I_{m(n-1)} & 2\bar{F} - \bar{Z} \end{bmatrix} \geq 0, \end{cases} \end{aligned} \tag{4.25}$$

where

$$\gamma = \beta_1 + \tau^2\beta_1 + \tau\beta_2 + \frac{\tau}{2}\beta_3 + \frac{\tau^3}{6}\beta_4 + 2\mu_m\tau\beta_5,$$

and the scalars $\beta_i > 0$, for $i = \{1, \ldots, 5\}$, as in (4.18), are introduced to bound the terms on equation (4.24). Hence, an estimate for the domain of consensus is given by $\delta_{max} = 1/\sqrt{\gamma}$.

Remark 4.6. In order to handle the nonlinear matrix inequalities on the estimation for domain of consensus, the inequality $\bar{F}M^{-1}\bar{F} \geq 2\bar{F} - M$ was used multiple times. Although this can insert some conservatism, it leads to a linear optimization problem for synthesizing the agents' gains and obtain an estimation for the domain of consensus.

4.2.2 Numerical examples

Example 4.3. *Consider a multi-agent system consisting of four agents with directed communication topology represented in Figure 4.5,*

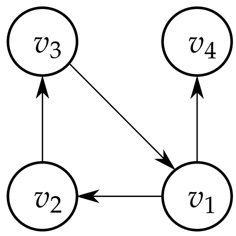


FIGURE 4.5: Example 4.3–Communication topology of multi-agent system.

and with Laplacian matrix given by

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

All agents are subject to input saturation and time-varying delays, the saturation limit considered is u_{max}=2.5*, and the bounds on the time-varying delay are given by* $\tau = 0.7$ *and* $\mu_m = 0.2$*. The ith agent dynamics is*

$$\begin{bmatrix} \dot{x}_i^1(t) \\ \dot{x}_i^2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} sat\big(\boldsymbol{u}_i(t - \tau_i(t))\big),$$

note that the open-loop dynamics is unstable.

The optimization problem (4.25) was used to design the feedback matrix for the agents, with $\lambda = 0.7$*, it yields*

$$K = \begin{bmatrix} 1.48 & -0.28 \end{bmatrix}. \tag{4.26}$$

The estimation for the domain of consensus was $\delta_1 = \delta_2 = 8.95$*, which, as commented in Example 4.1, ensures that for any initial condition with*

$$||\boldsymbol{x}_1(t_0) - \boldsymbol{x}_i(t_0)||_\infty \leq 8.95,$$

the multi-agent system will reach consensus asymptotically.

The consensus errors are shown in Figure 4.6 (A) and (B) for feedback matrix (4.26), the control inputs are shown in Figure 4.6 (C).

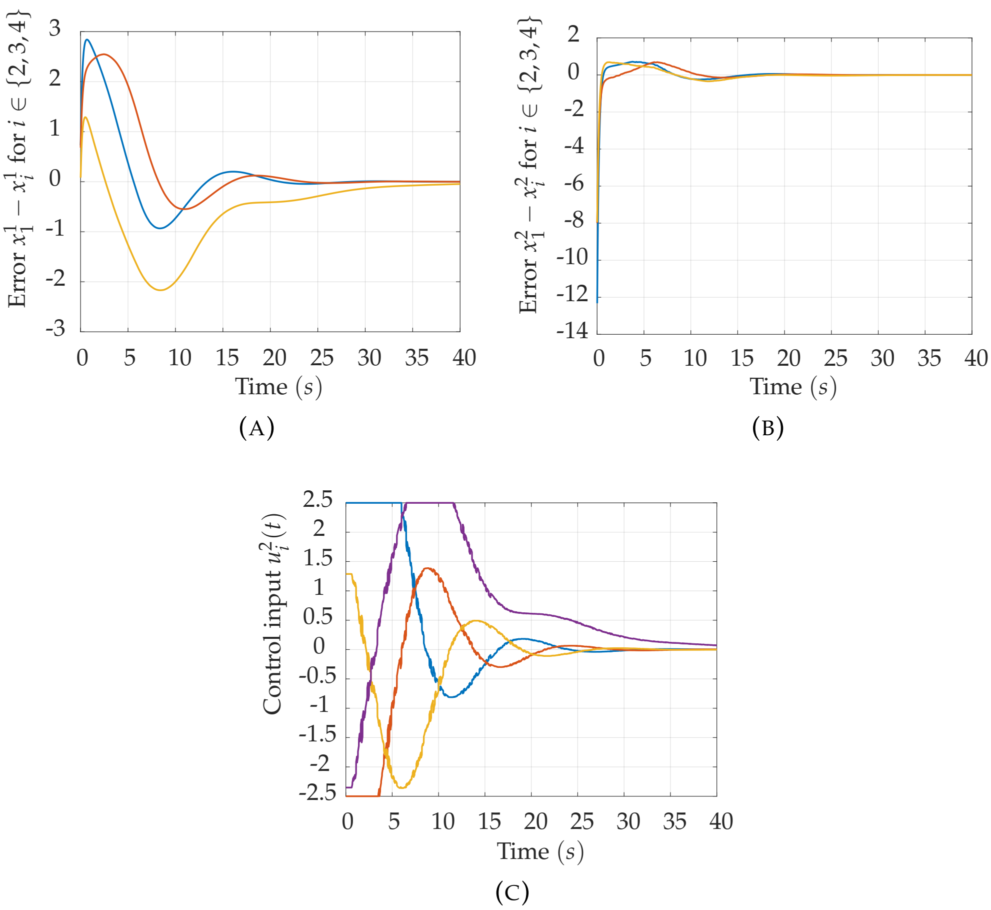


FIGURE 4.6: Example 4.3–Errors of states and input control of the multi-agent system.

4.3 Conclusions of the chapter

In this chapter we presented the main contributions of this dissertation. It was shown methods to analyze the consensus and to design gains of feedback matrices that guarantee consensus. For both methods an estimation for the region of consensus was given.

Throughout the development of the conditions of Theorem 4.1 some possibly conservative decisions were made–and propagated to Theorem 4.2. Specifically, the use of inequality $2\boldsymbol{a}^T\boldsymbol{b} \leq \boldsymbol{a}^T X \boldsymbol{a} + \boldsymbol{b}^T X^{-1} \boldsymbol{b}$, Lemma 2.2, and the imposition that all terms of the convex combination in (4.16) must be negative definite at all vertices with the same matrices. These decisions may have led to conservative results, in the sense that there may exist conditions that ensure consensus with larger region for consensus with tighter input saturation, and with larger time-delays.

Chapter 5

Conclusions

In this work we presented new sufficient conditions for the consensus problem on networked systems with fixed directed graph topology and agents subject to input saturation and time-varying delays. Furthermore, we proposed a method to estimate the region of guaranteed convergence.

The study was carried out considering multi-agent systems composed of agents with input time-varying delays that lie in a given interval. This interval might have a non-zero lower-bound and the stability is shown for a bounded set of delays. Moreover, we consider non-differentiable time-varying delays, a desirable feature since the variation of the delay is not always known and might even be random. This feature also fits well on the context of multi-agent systems because it allows to represent arbitrary delays for each agent within the same set and consequently to represent the whole networked system in a compact form. This is evident in the upper-bound choice of inequality (4.12).

Saturation was represented by a polytopic model that can reproduce the trajectories of the actual saturated closed-loop models for the agents. This representation eases the manipulation of the equations of the multi-agent system because the nonlinearity is removed, which allows the use of tools developed for the analysis of linear systems. Using this representation we have shown in Chapter 3 a way to translate the problem of consensus into a stability problem and it was proposed in Chapter 4 sufficient conditions to guarantee consensus. From these conditions a numerical algorithm was presented to estimate a region in which the consensus is attainable. Additionally, we presented a procedure to design the gains of distributed state feedback controllers for the agents with input saturation and time-varying delays. The feedback gains are designed in such a way the consensus is locally ensured with asymptotic convergence and an estimation of the domain of consensus is generated as large as possible.

During the development of the stability conditions in Chapter 4, some possible conservative inequalities were used to achieve a linear optimization problem and, in this way, to derive conditions that can be solved with efficient known methods. This perhaps led to estimation of small intervals for the time-varying delays, to the domain of consensus or/and to the saturation level tolerated by the multi-agent system.

Finally, the numerical examples in Chapter 4 showed that our approach is appropriate to address the consensus analysis with saturation and time-varying delays. Particularly, our results can deal with the problem of consensus with and without leader, with directed communication topology, non-differentiable input time-varying delays, and it estimates the region for consensus in all these cases. Furthermore, the effectiveness of the proposed conditions to find feedback matrices that guarantee the consensus in this scenario was illustrated.

5.1 Publications

Part of Theorem 4.1 is published by Silva, Souza, and Pimenta, 2018 in:

Silva, Thales C., Fernando O. Souza, and Luciano C. A. Pimenta (2018). "Consenso em sistemas multiagentes sujeitos a saturação e atrasos variantes no tempo". In: Anais do XXII Congresso Brasileiro de Automação. João Pessoa-PB, pp. 1–8.

Specifically, the saturation was represented with the particular choice of matrix H, that is $H = diag\{h_1, \ldots, h_p\}K$, with $h_k \in [\epsilon, 1]$ and $\epsilon > 0$, for all $k = 1, \ldots, p$ and without an estimation for the region of guaranteed convergence. Thus, the results exposed here are more general.

5.2 Future work

We approach the consensus problem under two main sources of possible degradation, the input time-varying delays and the input saturation. Some possible lines of investigation that give continuity to the results presented here are:

- To represent the saturation with a less conservative model. As indicated, our representation covers a larger set than the actual saturated signal, which implies that we are studying the stability on trajectories that the system may does not operates. The sector non-linearity model presented by Tarbouriech et al., 2011 may be a promising approach with less conservativeness.

- To consider time-varying communication topology. The communication between agents in multi-agent systems an intrinsic feature, thus it is valuable to derive conditions for systems with connections that change over time since it would represent more general scenarios.

- To study different Lyapunov-Krasovskii candidate functionals. There are new results on stability analysis of linear systems with time-varying delays that could be used to study the consensus through the disagreement system (3.19), this might give less conservative conditions.

- To consider agents with uncertain and nonlinear models. Consequently, investigate more realistic conditions for consensus. This might enlarge the real world applications for multi-agent systems with the vast theoretical framework of non-linear systems.

- To design a structure anti-windup to mitigate the performance degradation due to saturation. As pointed out, this structure could enlarge the linear region of operation of the agents. Thus, it would guarantee performance requirements for larger sets and improve the overall behavior of the multi-agent system.

- To analyze the problem with communication delay and input saturation. This problem must be investigated, because the communication is necessary in order to the consensus be attained and the possibility of occurrence of communication delay is high. In fact, there are involving studies in this context that could have its results extended for the scenario of input saturation. For example the works of Tian and Zhang, 2012, Wang et al., 2013, and Zhou and Lin, 2014.

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