

Universidade Federal de Minas Gerais
Instituto de Ciências Exatas
Departamento de Matemática

On varieties of superalgebras with graded
involution and small $\ast$ -graded colength

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*Aos meus queridos pais,
Ari e Maria Aparecida.*

*”E, Jesus, respondendo disse-lhes: Tendes fé em Deus;
porque em verdade vos digo que qualquer que disser a este monte:
Ergue-te e lança-te no mar; e não duvidar em seu coração,
mas crer que se fará aquilo que se diz, tudo o que disser lhe será feito.
Por isso vos digo que tudo o que pedirdes orando,
crede que o recebereis, e tê-lo-eis.”
(Marcos 11:22-24)*

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Abstract

In this thesis, we present the extension of some classical results of PI-theory to the class of $*$ -superalgebras, that is, superalgebras endowed with a graded involution. Let $\mathcal{V} = \text{var}^{gr}(A)$ be a variety generated by a finite dimensional $*$ -superalgebra A over a field F of characteristic zero. Giambruno, dos Santos and Vieira proved in [9] that the $*$ -graded codimension sequence of $\mathcal{V}$ is polynomially bounded if, and only if, $\mathcal{V}$ does not contain five $*$ -superalgebras: D_* the commutative algebra $D = F \oplus F$ endowed with the exchange involution and trivial grading; D^{gr} the commutative algebra with grading $F(1, 1) \oplus F(1, -1)$ and endowed with trivial involution; D^{gr} the commutative algebra with non-trivial grading and endowed with the exchange involution; M_* and M^{gr} , where M is a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices, endowed with the reflection involution and with trivial and non-trivial grading, respectively. As a consequence the algebras D_* , D^{gr} , D^{gr} , M_* and M^{gr} generate the only varieties generated by finite dimensional $*$ -superalgebras of almost polynomial growth. We expound here the classification of all subvarieties of these five varieties of almost polynomial growth, that was given in [21] and [12] in different contexts. We also exhibit the decompositions of the $*$ -graded cocharacters of all minimal subvarieties of $\text{var}^{gr}(D_*)$, $\text{var}^{gr}(D^{gr})$, $\text{var}^{gr}(D^{gr})$, $\text{var}^{gr}(M_*)$ and $\text{var}^{gr}(M^{gr})$ and compute their $*$ -graded colengths. Finally, we classify the varieties generated by finite dimensional $*$ -superalgebras such that their sequence of $*$ -graded colengths is bounded by three.

Keywords: Polynomial identity, superalgebra, algebra with involution, codimension, cocharacter, bounded colength.

Resumo expandido

Seja F um corpo de característica zero, $F\langle X \rangle$ a álgebra associativa livre gerada por um conjunto enumerável X sobre F e seja A uma álgebra associativa sobre F . Denotamos por $Id(A) \subseteq F\langle X \rangle$ o T -ideal das identidades polinomiais de A . Escrevemos $\mathcal{V} = var(A)$ para denotar a variedade gerada pela álgebra A e $Id(\mathcal{V}) = Id(A)$. Uma vez que todo T -ideal é um ideal das identidades polinomiais satisfeitas por uma dada variedade de álgebras, muitas vezes é conveniente traduzir um problema sobre T -ideais numa linguagem de variedade de álgebras.

Uma maneira efetiva de estudar T -ideais é determinando alguns invariantes numéricos que podem ser atribuídos ao T -ideal para dar uma descrição quantitativa. Um invariante numérico muito útil é a sequência de codimensões. Tal sequência foi introduzida por Regev em [28] e mede a taxa de crescimento dos polinômios pertencentes a um T -ideal dado. Regev provou que se A é uma PI-álgebra, isto é, A satisfaz uma identidade polinomial não nula, então a sequência de codimensões $c_n(A)$, $n = 1, 2, \dots$, é limitada exponencialmente. Depois, Kemer mostrou que dada qualquer PI-álgebra A , a sequência de codimensões não pode ter crescimento intermediário, isto é, ou cresce exponencialmente ou é polinomialmente limitada. Kemer também provou que a sequência de codimensões é limitada polinomialmente se, e somente se, a variedade de álgebras gerada por A não contém a álgebra de Grassmann $\mathcal{G}$ de um espaço vetorial de dimensão infinita e não contém a álgebra $UT_2(F)$ das matrizes triangulares superiores 2×2 sobre F . Portanto, $var(\mathcal{G})$ e $var(UT_2(F))$ são as únicas variedades de crescimento quase polinomial.

O estudo das variedades de crescimento polinomial foi feito extensivamente nos anos seguintes ([3], [5], [19]). Estes resultados foram estendidos para álgebras com estruturas adicionais tais como superálgebras, álgebras graduadas por um grupo, álgebras com involução, involução graduada e superinvolução, permitindo o estudo das identidades correspondentes ([4], [7], [8], [9], [12], [17], [20], [21]).

Na literatura, nós temos a classificação completa de todas as subvariedades das variedades de crescimento quase polinomial em diferentes linguagens ([12], [19], [20], [21]). Os autores também classificam todas as suas subvariedades minimais de crescimento polinomial. Recordamos que $\mathcal{V}$ é uma variedade minimal de crescimento polinomial n^k se assintoticamente $c_n(\mathcal{V}) \approx an^k$, para algum $a \neq 0$, e $c_n(\mathcal{U}) \approx bn^t$, com $t < k$, para qualquer subvariedade própria $\mathcal{U}$ de $\mathcal{V}$. A relevância das variedades minimais de crescimento polinomial está no fato de que estas variedades são os blocos construtores que permitiram aos autores dar uma classificação completa das

subvariedades das variedades de crescimento quase polinomial (veja também [6]).

Outro invariante numérico associado a uma álgebra A é a sequência de cocomprimentos $l_n(A)$ que conta, para cada representação, a multiplicidade dada pelo número de somandos irredutíveis na decomposição do cocaracter $\chi_n(A)$, para $n \geq 1$. Um problema interessante envolvendo tal invariante é classificar todas as variedades de álgebras tais que a sequência de cocomprimentos é limitada por uma constante. Sabemos que existe uma equivalência entre crescimento polinomial e variedade com cocomprimento limitado por uma constante no caso de PI-álgebras, superálgebras e álgebras com involução (ver [24], [27], [30]). Nós ainda não temos um resultado similar no caso de superálgebras com involução graduada ou no caso de álgebras com superinvolução.

Também é conhecida a classificação das variedades de álgebras tais que a sequência de cocomprimentos é limitada por uma constante fixa, em casos pequenos. Por exemplo, em [3] Giambruno e La Mattina classificaram PI-álgebras com sequência de cocomprimentos limitada por $k = 2$ e depois La Mattina deu em [18] tal classificação para $k = 4$. Em linguagem de superálgebras, Vieira classificou em [31] todas as supervarieties tais que a sequência de cocomprimentos graduados é limitada por 2. Recentemente, em um trabalho conjunto com La Mattina e Vieira (ver [23]), nós classificamos todas as variedades de álgebras com involução tais que a sequência de cocomprimentos é limitada por 3.

Nesta tese, trabalhamos com superálgebras sobre um corpo de característica zero munidas de uma involução tal que as componentes homogêneas são invariantes sob a involução. Mais precisamente, dizemos que uma superálgebra $A = A^{(0)} \oplus A^{(1)}$ munida de uma involução $*$ é uma **-superálgebra* se $(A^{(0)})^* = A^{(0)}$ e $(A^{(1)})^* = A^{(1)}$. Neste caso, dizemos que $*$ é uma *involução graduada*.

Consideramos a álgebra $D = F \oplus F$ e denotamos por D_* a álgebra comutativa D com graduação trivial e munida da involução $(a, b)^* = (b, a)$, chamada involução troca; D^{gr} será a álgebra comutativa D com graduação dada por $F(1, 1) \oplus F(1, -1)$ e munida da involução trivial, enquanto D^{gr_i} será a álgebra comutativa D com graduação $F(1, 1) \oplus F(1, -1)$ e munida da involução troca.

Em seguida, definimos M como sendo a seguinte subálgebra de $UT_4(F)$

$$M = \left\{ \begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in F \right\}.$$

Denotamos por M_* a álgebra M com graduação trivial e munida da involução reflexão, i.e. a involução obtida através da reflexão da matriz ao longo de sua diagonal secundária.

Escrevemos M^{gr_i} para denotar a álgebra M munida com a involução reflexão e

com graduação dada por

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

Em [9], Giambruno, dos Santos e Vieira provaram que uma $\ast$ -superálgebra de dimensão finita A tem crescimento polinomial se, e somente se, a variedade gerada por A não contém as $\ast$ -superálgebras $D_\ast$, D^{gr} , D^{gri} , $M_\ast$ e M^{gri} . Consequentemente, estas cinco $\ast$ -superálgebras geram as únicas variedades de crescimento quase polinomial geradas por $\ast$ -superálgebras de dimensão finita.

O propósito inicial desta tese era classificar as subvariedades das cinco variedades de crescimento quase polinomial geradas por $\ast$ -superálgebras de dimensão finita e explicitar a decomposição dos cocaracteres $\ast$ -graduados das subvariedades minimais encontradas. Ao mesmo tempo, Ioppolo e La Mattina classificaram em [12] as subvariedades das variedades geradas por álgebras de dimensão finita munidas de uma superinvolução e como crescimento quase polinomial. Como a classificação deles é uma extensão da nossa, tivemos que avançar um pouco mais. Coletando os resultados sobre os cocomprimentos $\ast$ -graduados das $\ast$ -superálgebras que geram subvariedades minimais nas variedades de crescimento quase polinomial, obtemos uma lista de $\ast$ -superálgebras com cocomprimento $\ast$ -graduado pequeno. O objetivo principal desta tese é classificar todas as variedades geradas por $\ast$ -superálgebras de dimensão finita tais que a sequência de cocomprimentos é limitada por 3, apresentando uma lista completa de $\ast$ -superálgebras geradoras de dimensão finita.

Organizamos esta tese em três capítulos dispostos da seguinte maneira.

No Capítulo 1 recordamos brevemente alguns resultados sobre PI-álgebras, superálgebras e álgebras com involução, e apresentamos as principais propriedades sobre $\ast$ -superálgebras e resultados sobre crescimento polinomial das codimensões $\ast$ -graduadas. Também definimos o principal objeto de estudo desta tese, que é o cocomprimento $\ast$ -graduado $l_n^{gri}(A)$ de uma $\ast$ -superálgebra A , e explicamos como calculá-lo usando vetores de altura máxima.

No Capítulo 2 apresentamos a classificação das subvariedades das variedades de crescimento quase polinomial não-comutativas, $var^{gri}(M_\ast)$ e $var^{gri}(M^{gri})$. Também exibimos a decomposição do cocaracter $\ast$ -graduado das subvariedades minimais pertencentes às variedades $var^{gri}(M_\ast)$ e $var^{gri}(M^{gri})$; e calculamos o cocomprimento $\ast$ -graduado delas.

No Capítulo 3 classificamos as subvariedades das variedades de crescimento quase polinomial comutativas, $var^{gri}(D_\ast)$, $var^{gri}(D^{gr})$ e $var^{gri}(D^{gri})$, explicitamos a decomposição do cocaracter $\ast$ -graduado e calculamos o cocomprimento $\ast$ -graduado das subvariedades minimais pertencente a elas. Finalmente, estudamos outras $\ast$ -superálgebras com cocomprimento $\ast$ -graduado pequeno para enfim caracterizar todas as $\ast$ -superálgebras de dimensão finita que geram variedades com cocomprimento $\ast$ -graduado limitado por 3.

As principais técnicas utilizadas neste trabalho são métodos da teoria de representações do grupo simétrico S_n e o estudo do comportamento assintótico dos graus de S_n -representações irredutíveis. Sugerimos ao leitor os livros [13] e [14] para o estudo de S_n -representações e os livros [1] e [11] para mais informações sobre a teoria de PI-álgebras.

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Introduction

Let F be a field of characteristic zero, $F\langle X \rangle$ be the free associative algebra on a countable set X over F and A be an associative algebra over F . Let $Id(A) \subseteq F\langle X \rangle$ denote the T -ideal of polynomial identities of A . We write $\mathcal{V} = var(A)$ to denote the variety generated by the algebra A and $Id(\mathcal{V}) = Id(A)$. Since every T -ideal is an ideal of polynomial identities satisfied by a given variety of algebras, it is often convenient to translate a given problem on T -ideals into the language of varieties of algebras.

An effective way to study T -ideals is that of determining some numerical invariants that can be attached to the T -ideal to give a quantitative description. A very useful numerical invariant is the sequence of codimensions. Such sequence was introduced by Regev in [28] and measures the rate of growth of the polynomials lying in a given T -ideal. Regev proved that if A is a PI-algebra, that is, satisfies a non-trivial polynomial identity, then the sequence of codimensions $c_n(A)$, $n = 1, 2, \dots$, is exponentially bounded. Later, Kemer showed that given any PI-algebra A , the sequence of codimension cannot have intermediate growth, that is, either grows exponentially or is polynomially bounded. Kemer also proved that the codimension is polynomially bounded if, and only if, the variety of algebras generated by A does not contain the Grassmann algebra $\mathcal{G}$ of an infinite dimensional vector space and also does not contain the algebra $UT_2(F)$ of 2×2 upper triangular matrices over F . Hence, $var(\mathcal{G})$ and $var(UT_2(F))$ are the only varieties of almost polynomial growth.

The study of varieties of polynomial growth was extensively made in later years (e.g., [3], [5], [19]). These results have been extended to algebras with an additional structure such as superalgebras, group graded algebras, algebras with involution, graded involution and superinvolution, allowing to study the corresponding identities (e.g., [4], [7], [8], [9], [12], [17], [20], [21]).

In literature, we have the complete classification of all subvarieties of the varieties of almost polynomial growth in different languages (e.g. [12], [19], [20], [21]). The authors also classify all their minimal subvarieties of polynomial growth. We recall that $\mathcal{V}$ is a minimal variety of polynomial growth n^k if asymptotically $c_n(\mathcal{V}) \approx an^k$, for some $a \neq 0$, and $c_n(\mathcal{U}) \approx bn^t$, with $t < k$, for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$. The relevance of the minimal varieties of polynomial growth relies in the fact that these were the building blocks that allowed the authors to give a complete classification of the subvarieties of the varieties of almost polynomial growth. (see also [6]).

Another numerical invariant associated to the algebra A is the sequence of

colengths $l_n(A)$ counting, for each representation, the multiplicity given by the number of irreducible summands in the decomposition of the cocharacter $\chi_n(A)$, for $n \geq 1$. One interesting problem involving such invariant is to classify all the varieties of algebras such that the sequence of colengths is bounded by a constant. We know that there is an equivalence between polynomial growth of codimensions and varieties with sequence of colength bounded by a constant in case of PI-algebras, superalgebras and algebras with involution (see [24], [27], [30]). We still don't have a similar result in case of superalgebras neither with graded involution nor with superinvolution.

It is also known the classification of varieties of algebras such that the sequence of colengths is bounded by a fixed constant in small cases. For example, in [3] Giambruno and La Mattina classified PI-algebras with sequence of colengths bounded by $k = 2$ and later La Mattina, in [18], gave such classification for $k = 4$. In superalgebra language, Vieira classified in [31] all supervarieties such that the sequence of graded colengths is bounded by 2. Recently, in a joint work with La Mattina and Vieira (see [23]), we classified all the varieties of algebras with involution such that the sequence of colengths is bounded by 3.

In this thesis, we work with superalgebras over a field of characteristic zero endowed with an involution such that the homogeneous components are invariant under the involution. More precisely, we say that a superalgebra $A = A^{(0)} \oplus A^{(1)}$ endowed with an involution $*$ is a $*$ -superalgebra if $(A^{(0)})^* = A^{(0)}$ and $(A^{(1)})^* = A^{(1)}$. In this case, we say that $*$ is a graded involution.

We consider the algebra $D = F \oplus F$ and we denote by D_* the commutative algebra D with trivial grading and endowed with the involution $(a, b)^* = (b, a)$, called exchange involution; D^{gr} will be the commutative algebra D with the grading $F(1, 1) \oplus F(1, -1)$ and endowed with trivial involution and $D^{gr i}$ will be the commutative algebra D with the grading $F(1, 1) \oplus F(1, -1)$ and endowed with the exchange involution.

Next, we define M to be the following subalgebra of $UT_4(F)$

$$M = \left\{ \begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in F \right\}.$$

We denote by M_* the algebra M with trivial grading and endowed with reflection involution, i.e. the involution obtained by flipping the matrix along its secondary diagonal.

We write $M^{gr i}$ to denote the algebra M endowed with reflection involution and with grading given by

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

In [9], Giambruno, dos Santos and Vieira proved that a finite dimensional $\ast$ -superalgebra A has polynomial growth if, and only if, the variety generated by A does not contain the $\ast$ -superalgebras $D_\ast$, D^{gr} , D^{gri} , $M_\ast$ and M^{gri} . As consequence these five $\ast$ -superalgebras generate the only varieties of almost polynomial growth generated by finite dimensional $\ast$ -superalgebras.

The initial purpose of this thesis was to classify the subvarieties of the five varieties of finite dimensional $\ast$ -superalgebras with almost polynomial growth and to explicit the decomposition of the $\ast$ -graded cocharacter of the minimal subvarieties found. At the same time, Ioppolo and La Mattina classified in [12] the subvarieties of the varieties of finite dimensional superalgebras endowed with a superinvolution and with almost polynomial growth. Since their classification is an extension of ours, we have to advance a little more. Collecting the results on the $\ast$ -graded colengths of the $\ast$ -superalgebras that generate minimal subvarieties lying in the varieties of almost polynomial growth, we obtain a list of $\ast$ -superalgebras with $\ast$ -graded small colength. The main goal of this thesis is to classify all the varieties generated by finite dimensional $\ast$ -superalgebras such that the sequence of colengths is bounded by 3, by giving a complete list of finite dimensional generating $\ast$ -superalgebras.

We organized this thesis in three chapters disposed in the following way.

In Chapter 1 we briefly recall some results about PI-algebras, superalgebras and algebras with involution and present the principal properties of $\ast$ -superalgebras and results about the polynomial growth of the $\ast$ -graded codimensions. We also define the main object of study of this thesis, that is, the $\ast$ -graded colength $l_n^{gri}(A)$ of a $\ast$ -superalgebra A , and explain how to calculate it by using highest weight vectors.

In Chapter 2 we present the classification of the subvarieties of the noncommutative varieties of almost polynomial growth, $var^{gri}(M_\ast)$ and $var^{gri}(M^{gri})$. We also exhibit the decomposition of the $\ast$ -graded cocharacter of the minimal subvarieties lying in $var^{gri}(M_\ast)$ and $var^{gri}(M^{gri})$; and compute the $\ast$ -graded colength of them.

In Chapter 3 we classify the subvarieties of the commutative varieties of almost polynomial growth, $var^{gri}(D_\ast)$, $var^{gri}(D^{gr})$ and $var^{gri}(D^{gri})$, explicit the decomposition of the $\ast$ -graded cocharacter and calculate the $\ast$ -graded colength of the minimal subvarieties lying in them. Finally, we study other $\ast$ -superalgebras with small $\ast$ -graded colength in order to characterize all finite dimensional $\ast$ -superalgebras that generate the varieties of $\ast$ -graded colength bounded by 3.

The main techniques employed in this work are methods of representation theory of the symmetric group S_n and computations of the asymptotic behavior for the degrees of the irreducible S_n -representations. We refer to the reader the books [13] and [14] for the study of S_n -representations, and the books [1] and [11] for more about the theory of PI-algebras.

Chapter 1

*-Superalgebras

In this chapter we briefly present some results about PI-algebras, superalgebras and algebras with involution. We are more interested in superalgebras over a field of characteristic zero endowed with involution such that the homogeneous components are invariant under the involution, called *-superalgebras.

Here, we define the free associative *-superalgebra and introduce the *-graded polynomial identities on *-superalgebras and the *-graded codimension sequence. We define the main objects of study of this thesis which are the *-graded cocharacter and *-graded colength sequences of a *-superalgebra A .

We also present the classification of the *-supervarieties with almost polynomial growth, in finite dimensional case, given in [9]. The authors proved that there exists only five *-superalgebras under this condition. Such *-superalgebras will be useful to obtain our main goal, which is to classify the *-superalgebras with *-graded colength bounded by three.

1.1 PI-algebras, superalgebras and *-algebras

Let A be an associative algebra over F , a field of characteristic zero. We consider $F\langle X \rangle$ to be the free associative algebra on X over F , where $X = \{x_1, x_2, \dots\}$ is a countable set of noncommutative variables. We say that a polynomial $f(x_1, \dots, x_n) \in F\langle X \rangle$ is an *identity* of A if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$ and, in this case, we write $f \equiv 0$ in A . If A satisfies a non-zero identity, we say that A is a *PI-algebra*.

We denote by $Id(A) = \{f \in F\langle X \rangle \mid f \equiv 0 \text{ on } A\}$ the ideal of all identities satisfied by A . We have that $Id(A)$ is a *T-ideal* of $F\langle X \rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X \rangle$. In [15], Kemer proved that all *T-ideal* I is finitely generated by the set of multilinear polynomials, that is, there exist $f_1, \dots, f_m$ multilinear polynomials such that $I = \langle f_1, \dots, f_m \rangle_T$, in characteristic zero.

We consider the space P_n of all multilinear polynomials of degree n in $x_1, \dots, x_n$ in the free algebra $F\langle X \rangle$. Let S_n be the symmetric group of degree n . If $\sigma \in S_n$, we

define a natural action on the space P_n as follows: $\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. We have that

$$P_n(A) := \frac{P_n}{P_n \cap Id(A)}$$

has a structure of left S_n -module. The dimension of the space $P_n(A)$ is called *n-th codimension of A* and is denoted by $c_n(A) = \dim_F \frac{P_n}{P_n \cap Id(A)}$.

Moreover, let $\chi_n(A)$ denote the character of the S_n -module $P_n(A)$. We have that its decomposition into irreducible S_n -characters is given by $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ and $\chi_n(A)$ is called the *n-th cocharacter of A*. We can also define the *n-th colength of A* as $l_n(A) = \sum_{\lambda \vdash n} m_\lambda$.

For instance, a commutative algebra is a PI-algebra since $[x_1, x_2] := x_1x_2 - x_2x_1$, is an identity of A . If A is a nilpotent algebra, such that $A^n = 0$, then $x_1 \cdots x_n$ is an identity of A .

Let UT_2 be the 2×2 upper triangular matrix algebra over F . We have that $Id(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle_T$, and $c_n(UT_2)$ grows exponentially.

It is well known that any finite dimensional algebra is also a PI-algebra. An important example of infinite dimensional PI-algebra is the unitary Grassmann algebra $\mathcal{G}$. We can write

$$\mathcal{G} = \langle 1, e_1, e_2, \dots \mid e_i e_j = -e_j e_i \rangle.$$

Moreover, we have that $\mathcal{G}$ can be written as a direct sum of the vector subspaces

$$\mathcal{G}^{(0)} = \text{span}_F \{e_{i_1} \cdots e_{i_{2k}} \mid 1 \leq i_1 < \dots < i_{2k}, k \geq 0\} \text{ and}$$

$$\mathcal{G}^{(1)} = \text{span}_F \{e_{j_1} \cdots e_{j_{2p+1}} \mid 1 \leq j_1 < \dots < j_{2p+1}, p \geq 0\}.$$

We know that $Id(\mathcal{G}) = \langle [x_1, x_2, x_3] \rangle_T$, and $c_n(\mathcal{G})$ also grows exponentially.

For an algebra A , we denote by $\text{var}(A)$ the *variety* of algebras generated by A , i.e., $\text{var}(A) = \{B \mid Id(A) \subseteq Id(B)\}$. We say that the algebras A and B are *T-equivalent* if and only if $Id(A) = Id(B)$. In this case, we write $A \sim_T B$.

In [28], Regev proved that if A is a PI-algebra, then the sequence of the codimensions of A is exponentially bounded. In [16], Kemer proved that the sequence $c_n(A)$ is polynomially bounded if, and only if, neither the infinite dimensional Grassmann algebra $\mathcal{G}$ nor the algebra $UT_2(F)$ of the 2×2 upper triangular matrices lie in $\text{var}(A)$.

We recall that an algebra A has *almost polynomial growth*, if the sequence of the codimensions of A grows exponentially but the sequence of the codimensions of any proper subvariety of $\text{var}(A)$ is polynomially bounded. Hence $\mathcal{G}$ and $UT_2(F)$ are the only algebras of almost polynomial growth.

If the algebra A has a decomposition $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}$ and $A^{(1)}$ are subspaces such that $A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)}$ and $A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}$, then we say that A is a $\mathbb{Z}_2$ -graded algebra or a *superalgebra* over F and we write $A = (A^{(0)}, A^{(1)})$.

Let $F\langle Y \cup Z \rangle$ denote the free associative superalgebra, where y_i and z_i denotes variables of degree 0 and variables of degree 1, respectively. Let $A = (A^{(0)}, A^{(1)})$ be a superalgebra over F , a field of characteristic zero. We say that a polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m)$ in the free associative superalgebra $F\langle Y \cup Z \rangle$ is a *graded identity* of A , if $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_n \in A^{(0)}$ and $b_1, \dots, b_m \in A^{(1)}$.

If characteristic of F is equal to zero, the ideal $Id^{gr}(A)$ of the graded identities satisfied by A is an ideal invariant under all endomorphisms of $F\langle Y \cup Z \rangle$ preserving the grading and is completely determined by its multilinear polynomials.

We denote by P_n^{gr} the space of multilinear polynomials of degree n in $y_1, z_1, \dots, y_n, z_n$ and if $k = (a_1, \dots, a_n; \sigma)$ is an element of the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$, we define a natural action on the space P_n^{gr} as follows: $ky_i = y_{\sigma(i)}$ and $kz_i = z_{\sigma(i)}$ or $-z_{\sigma(i)}$ according to whether $a_{\sigma(i)} = 1$ or -1 , respectively. We consider now the space

$$P_n^{gr}(A) := \frac{P_n^{gr}}{(P_n^{gr} \cap Id^{gr}(A))}.$$

We denote by $c_n^{gr}(A) = \dim_F P_n^{gr}(A)$ the dimension of $P_n(A)$, and we call this number of the *n -th graded codimensions of A* .

Moreover, $P_n^{gr}(A)$ has a $\mathbb{Z}_2 \wr S_n$ -modulo structure and its $\mathbb{Z}_2 \wr S_n$ -character, denoted by $\chi_n^{gr}(A)$ is called the *n -th graded cocharacter of A* . By considering the decomposition into irreducible $\mathbb{Z}_2 \wr S_n$ -characters $\chi_n^{gr}(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \chi_{\lambda,\mu}$, we can

define the *n -th graded colength of A* as $l_n^{gr}(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}$.

For a superalgebra A , we denote by $var^{gr}(A)$ the variety of superalgebras (or supervariety) generated by A , and we say that the superalgebras A and B are T_2 -equivalent (and we write $A \sim_{T_2} B$) if, and only if $Id^{gr}(A) = Id^{gr}(B)$.

Any algebra A can be viewed as a superalgebra with trivial grading, that is, $A = A \oplus \{0\}$. We let UT_2 and $\mathcal{G}$ denote such algebras with trivial grading. So we easily see that $Id^{gr}(UT_2) = \langle [y_1, y_2][y_3, y_4], z \rangle_{T_2}$ and $Id^{gr}\mathcal{G} = \langle [y_1, y_2, y_3], z \rangle_{T_2}$.

Now, we consider $\mathcal{G}^{gr} = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)}$, called the canonical grading of the Grassmann algebra, and UT_2^{gr} the algebra UT_2 with the grading

$$\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \oplus \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

We also know that $Id^{gr}(\mathcal{G}^{gr}) = \langle [y_1, y_2], [y, z], z_1 z_2 + z_2 z_1 \rangle_{T_2}$ and $Id^{gr}(UT_2^{gr}) = \langle [y_1, y_2], z_1 z_2 \rangle_{T_2}$.

Finally, we consider the commutative algebra $F \oplus cF$, where $c^2 = 1$, with grading (F, cF) . We have $Id^{gr}(F \oplus cF) = \langle [y_1, y_2], [y, z], [z_1, z_2] \rangle_{T_2}$, $c_n^{gr}(F \oplus cF) = 2^n$ and $\chi_n^{gr}(F \oplus cF) = \sum_{j=0}^n \chi_{(n-j), (j)}$.

These five algebras are very useful to characterize superalgebras with sequence of graded codimension polynomially bounded.

Theorem 1.1.1. [8, Theorem 2] *Let $\mathcal{V}$ be a variety of superalgebras. Then $\mathcal{V}$ has polynomial growth if and only if $\mathcal{G}, \mathcal{G}^{gr}, UT_2(F), UT_2(F)^{gr}, F \oplus cF \notin \mathcal{V}$, where $c^2 = 1$.*

As a consequence of this theorem, we have that $\mathcal{G}, \mathcal{G}^{gr}, UT_2(F), UT_2(F)^{gr}$ and $F \oplus cF$ are the only supervarieties with almost polynomial growth. Their subvarieties were completely classified by La Mattina, in [19] and [20].

An anti-automorphism $*$ of order at most 2 of an algebra A over F is called an *involution* that is, $(a^*)^* = a$, $(a + b)^* = a^* + b^*$, $(ab)^* = a^*b^*$, $(\alpha a)^* = \alpha a^*$, $\forall a, b \in A, \forall \alpha \in F$. An algebra A endowed with a involution $*$ is called a **-algebra*.

We have $A = A^+ \oplus A^-$ where A^+ is the subspace formed by all symmetric elements, i.e. such that $a^* = a$, and A^- is the subspace of all skew elements, i.e. such that $a^* = -a$, with $a \in A$.

Let $F\langle X, * \rangle$ be the free associative algebra with involution on X over F . It is useful to consider $F\langle X, * \rangle = F\langle Y \cup Z \rangle$ as generated by symmetric and skew variables. We say that a polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y \cup Z \rangle$ is a **-identity* of A if $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_n \in A^+$ and $b_1, \dots, b_m \in A^-$.

The involution case is analogous the $\mathbb{Z}_2$ -graded case. The ideal $Id^*(A)$ of all *-identities of an F -algebra with involution A is a T^* -ideal of $F\langle Y \cup Z \rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle Y \cup Z \rangle$ commuting with the involution $*$, and is completely determined by its multilinear polynomials.

We consider the space P_n^* of all multilinear polynomials of degree n in $y_1, z_1, \dots, y_n, z_n$ in the free algebra with involution $F\langle Y \cup Z \rangle$. Let H_n be the hyperoctahedral group of degree n . If $k = (a_1, \dots, a_n; \sigma)$ is an element of the hyperoctahedral group H_n , we define a natural action on the space P_n^* as follows: $ky_i = y_{\sigma(i)}$ and $kz_i = z_{\sigma(i)}$ or $-z_{\sigma(i)}$ according to whether $a_{\sigma(i)} = 1$ or -1 , respectively. We have that

$$P_n^*(A) := \frac{P_n^*}{P_n^* \cap Id^*(A)}$$

has a structure of left H_n -module. The dimension of the space $P_n^*(A)$ is called the n -th **-codimension* of A and is denoted by $c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap Id^*(A)}$.

By considering $\chi_n^*(A)$ the character of the H_n -module $P_n^*(A)$, we have that its decomposition into irreducible H_n -characters is given by $\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda, \mu} \chi_{\lambda, \mu}$

and $\chi_n^*(A)$ is called the n -th **-cocharacter* of A . We can also define the n -th **-co-length* of A as $l_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}$.

For a $*$ -algebra A we denote by $var^*(A)$ the variety of $*$ -algebras (or $*$ -variety) generated by A , i.e. $var^*(A) = \{B | Id^*(A) \subseteq Id^*(B)\}$. We say that the $*$ -algebras A and B are T^* -equivalents if and only if $Id^*(A) = Id^*(B)$. In this case, we write $A \sim_{T^*} B$.

Notice that if A is a commutative algebra, then the identity map is an involution of A , and is called the *trivial involution*.

We consider now the commutative algebra $D = F \oplus F$ with trivial grading and exchange involution $(a, b)^* = (b, a)$. This algebra was presented by Giambruno and Mishchenko [7]. We know that $Id^*(D) = \langle [y_1, y_2], [y, z], [z_1, z_2] \rangle_{T^*}$, $c_n^*(D) = 2^n$ and $\chi_n^*(D) = \sum_{j=0}^n \chi_{(n-j), (j)}$.

Next, we define M to be the following subalgebra of the algebra $UT_4(F)$ of 4×4 upper triangular matrices:

$$M = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

We consider M endowed with reflection involution, i.e., the involution obtained by flipping the matrix along its secondary diagonal

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

This $*$ -algebra was presented and studied by Mishchenko and Valenti, in [25]. We know that $Id^*(M) = \langle z_1 z_2 \rangle_{T^*}$ and if $\chi_n^*(M) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \chi_{\lambda,\mu}$ then $m_{(n), \emptyset} = 1$,

$$m_{\lambda,\mu} = q + 1 \quad \text{if} \quad \begin{cases} \lambda = (p + q, p) \text{ and } \mu = \emptyset, \text{ for all } p, q \geq 0 \\ \lambda = (p + q, p, 1) \text{ and } \mu = \emptyset, \text{ for all } p \geq 1, q \geq 0 \\ \lambda = (p + q, p) \text{ and } \mu = (1), \text{ for all } p \geq 1, q \geq 0 \end{cases}$$

and $m_{\lambda,\mu} = 0$, otherwise. Moreover, the sequence of the $*$ -codimensions of M grows exponentially.

In [7], Giambruno and Mishchenko characterized $*$ -algebras whose sequence of $*$ -codimensions is polynomially bounded.

Theorem 1.1.2. [7, Theorem 4.7] *Let $\mathcal{V}$ be a variety of algebras with involution. Then $\mathcal{V}$ has polynomial growth if and only if $D, M \notin \mathcal{V}$.*

This proves that D and M are the only $*$ -varieties with almost polynomial growth. The classification of the subvarieties inside them is completely given by La Mattina and Martino in [21].

Next, we introduce the structure of $*$ -superalgebras, and present some results that generalize what we see here in the context of PI-algebras, superalgebras and algebras with involution.

1.2 *-Superalgebras and the *-graded codimension

Let F be field of characteristic zero and consider A an associative algebra over F . Remind that an involution on the algebra A is just an antiautomorphism of order at most 2 on A , which we shall denote by $*$. In this case, we write $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ for the sets of symmetric and skew elements of A , respectively.

An involution $*$ on a superalgebra $A = A^{(0)} \oplus A^{(1)}$ that preserves the homogeneous components $A^{(0)}$ and $A^{(1)}$, that is, $(A^{(0)})^* = A^{(0)}$ and $(A^{(1)})^* = A^{(1)}$, is called *graded involution*. A superalgebra A endowed with a graded involution $*$ is called **-superalgebra*.

We remind that if $A = (A^{(0)} \oplus A^{(1)})$ is a superalgebra, then $\varphi \in \text{Aut}(A)$ defined by $\varphi(a_0 + a_1) = a_0 - a_1$, where $a_0 \in A^{(0)}, a_1 \in A^{(1)}$, is an automorphism of order at most 2. Moreover, any automorphism $\varphi \in \text{Aut}(A)$ of order at most 2 determines a $\mathbb{Z}_2$ -grading on A by setting $A^{(0)} = \{a + \varphi(a) \mid a \in A\}$ and $A^{(1)} = \{a - \varphi(a) \mid a \in A\}$.

The connection between the superstructure and the involution on A is given in the next lemma. The demonstration was made by R. B. dos Santos in his doctoral thesis and we will omit here.

Lemma 1.2.1. *Let A be a superalgebra over a field F of characteristic different from 2 endowed with an involution $*$ and φ the automorphism of order at most 2 determined by the superstructure. Then A is a $*$ -superalgebra if and only if $* \circ \varphi = \varphi \circ *$.*

As a consequence of this lemma, if A is a superalgebra over a field F of characteristic different from 2 endowed with an involution $*$, then A is a $*$ -superalgebra if, and only if, the subspaces A^+ and A^- are graded subspaces. As a consequence, any $*$ -superalgebra can be written as a sum of 4 subspaces

$$A = (A^{(0)})^+ \oplus (A^{(1)})^+ \oplus (A^{(0)})^- \oplus (A^{(1)})^-.$$

Let X be a countable set of noncommutative variables. We write the set X as the disjoint union of four countable sets $X = Y_0 \cup Y_1 \cup Z_0 \cup Z_1$, where $Y_0 = \{y_{1,0}, y_{2,0}, \dots\}$, $Y_1 = \{y_{1,1}, y_{2,1}, \dots\}$, $Z_0 = \{z_{1,0}, z_{2,0}, \dots\}$ and $Z_1 = \{z_{1,1}, z_{2,1}, \dots\}$.

We can define the free $*$ -superalgebra $\mathcal{F} = F\langle X | \mathbb{Z}_2, * \rangle$ of countable rank on X by giving a superstructure on $\mathcal{F}$ where we require that the variables of $Y_0 \cup Z_0$ are homogeneous of degree 0 and those of $Y_1 \cup Z_1$ are homogeneous of degree 1. We also define an involution on $\mathcal{F}$ by requiring that the variables of $Y_0 \cup Y_1$ are symmetric and those of $Z_0 \cup Z_1$ are skew.

Consider $\mathcal{F}^{(0)}$ to be the span of all monomials in the variables of X which have an even number of variables of degree 1 and $\mathcal{F}^{(1)}$ to be the span of all monomials in the variables of X which have an odd number of variables of degree 1. Then $(\mathcal{F}^{(0)})^* = \mathcal{F}^{(0)}$ and $(\mathcal{F}^{(1)})^* = \mathcal{F}^{(1)}$ and so $\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ has a structure of $*$ -superalgebra. The elements of $\mathcal{F}$ are called $(\mathbb{Z}_2, *)$ -polynomials.

Let

$$f = f(y_{1,0}, \dots, y_{m,0}, y_{1,1}, \dots, y_{n,1}, z_{1,0}, \dots, z_{p,0}, z_{1,1}, \dots, z_{q,1}) \in \mathcal{F}.$$

We say that f is a $(\mathbb{Z}_2, *)$ -identity for the $*$ -superalgebra A , and we write $f \equiv 0$ on A , if

$$f(a_{1,0}^+, \dots, a_{m,0}^+, a_{1,1}^+, \dots, a_{n,1}^+, a_{1,0}^-, \dots, a_{p,0}^-, a_{1,1}^-, \dots, a_{q,1}^-) = 0,$$

for all $a_{1,0}^+, \dots, a_{m,0}^+ \in (A^{(0)})^+$, $a_{1,1}^+, \dots, a_{n,1}^+ \in (A^{(1)})^+$, $a_{1,0}^-, \dots, a_{p,0}^- \in (A^{(0)})^-$ and $a_{1,1}^-, \dots, a_{q,1}^- \in (A^{(1)})^-$.

It is clear that any algebra with involution $*$ endowed with trivial grading is a $*$ -superalgebra. Also, notice that for a commutative superalgebra A , the identity map is a graded involution on A .

The *ideal of $(\mathbb{Z}_2, *)$ -identities of A* is the set

$$Id^{gr_i}(A) = \{f \in \mathcal{F} | f \equiv 0 \text{ on } A\}$$

and we can notice that $Id^{gr_i}(A)$ is a T_2^* -ideal of $\mathcal{F}$, i.e. an ideal invariant under all endomorphisms of $\mathcal{F}$ that preserves the superstructure and commutes with the involution.

Since $\text{char}(F) = 0$, $Id^{gr_i}(A)$ is determined by its multilinear polynomials and so we define

$$P_n^{gr_i} = \text{span}_F \{w_{\sigma(1)} \cdots w_{\sigma(n)} | \sigma \in S_n, w_i = y_{i,g_i} \text{ or } w_i = z_{i,g_i}, g_i = 0, 1\},$$

the space of multilinear polynomials in the first n variables. As in the ordinary case, $Id^{gr_i}(A)$ is finitely generated as a T_2^* -ideal and we use the notation $\langle f_1, \dots, f_m \rangle_{T_2^*}$ to indicate that $Id^{gr_i}(A)$ is generated, as a T_2^* -ideal, by $f_1, \dots, f_m \in \mathcal{F}$.

The dimension of the quotient space

$$P_n^{gr_i}(A) := \frac{P_n^{gr_i}}{Id^{gr_i}(A) \cap P_n^{gr_i}}$$

is called *the n -th $*$ -graded codimension of A* and it is denoted by $c_n^{gr_i}(A)$.

Given a $*$ -superalgebra A , we shall denote by $\text{var}^{gr_i}(A)$ the variety of $*$ -superalgebras generated by A , that is, $\text{var}^{gr_i}(A)$ is the class of all $*$ -superalgebras B such that $Id^{gr_i}(A) \subseteq Id^{gr_i}(B)$. Consequently, $\text{var}^{gr_i}(A) = \text{var}^{gr_i}(B)$ if, and only if $Id^{gr_i}(A) = Id^{gr_i}(B)$. In this case, we say that A is T_2^* -equivalent to B and we denote by $A \sim_{T_2^*} B$.

We say that a $*$ -superalgebra A has *polynomial growth*, if there exist constants α, t such that $c_n^{gr_i}(A) \leq \alpha n^t$, for all $n \geq 1$. If there exists a constant β such that $c_n^{gr_i}(A) \approx \beta^n$, for all $n \geq 1$, then we say that $c_n^{gr_i}(A)$ grows exponentially. Moreover, if $\mathcal{V}$ is a variety generated by the $*$ -superalgebra A , then we write $c_n^{gr_i}(\mathcal{V}) = c_n^{gr_i}(A)$ and the growth of $\mathcal{V}$ is the growth of $c_n^{gr_i}(\mathcal{V})$.

If A is a $*$ -superalgebra, we can also consider its identities, $*$ -identities and graded identities. Since we can identify in a natural way P_n, P_n^* and P_n^{gr} with suitable subspaces of $P_n^{gr_i}$, in what follows we shall consider $Id(A) \subseteq Id^*(A) \subseteq Id^{gr_i}(A)$ and $Id(A) \subseteq Id^{gr}(A) \subseteq Id^{gr_i}(A)$. The relation among the corresponding codimensions is given in the following.

Lemma 1.2.2. [9, Lemma 3.1] *Let A be a $*$ -superalgebra. Then for any $n \geq 1$, we have*

1. $c_n(A) \leq c_n^*(A) \leq c_n^{gr_i}(A)$;
2. $c_n(A) \leq c_n^{gr}(A) \leq c_n^{gr_i}(A)$;
3. $c_n^{gr_i}(A) \leq 4^n c_n(A)$.

By [28], remind that an algebra A is a PI-algebra if, and only if $c_n(A)$ is exponentially bounded. Thus, as an immediate consequence of the previous lemma, we have the following.

Corollary 1.2.3. [9, Corollary 3.2] *Let A be a $*$ -superalgebra. Then A is a PI-algebra if, and only if its sequence of $*$ -graded codimensions $\{c_n^{gr_i}(A)\}_{n \geq 1}$ is exponentially bounded.*

Moreover, since any finite dimensional algebra A is a PI-algebra we have that if A is a finite dimensional $*$ -superalgebra, then the sequence of $*$ -graded codimensions $\{c_n^{gr_i}(A)\}_{n \geq 1}$ is exponentially bounded.

From now on, we denote by D_* the algebra $D = F \oplus F$ with trivial grading and exchange involution $(a, b)^* = (b, a)$. We also consider D^{gr} to be the algebra $D = F \oplus F$ with grading $D = F(1, 1) \oplus F(1, -1)$ and trivial involution. Then, D_* and D^{gr} are $*$ -superalgebras and we have:

1. $Id^{gr_i}(D_*) = \langle Id^*(D_*), y_{1,1}, z_{1,1} \rangle_{T_2^*}$ and $c_n^{gr_i}(D_*) = c_n^*(D_*) = 2^n$;
2. $Id^{gr_i}(D^{gr}) = \langle Id^{gr}(D^{gr}), z_{1,0}, z_{1,1} \rangle_{T_2^*}$ and $c_n^{gr_i}(D^{gr}) = c_n^{gr}(D^{gr}) = 2^n$.

Now, we consider the $*$ -superalgebra D^{gri} to be the commutative algebra D with the grading $F(1, 1) \oplus F(1, -1)$ and endowed by the exchange involution.

We say that a polynomial $f \in P_n^{gri}$ is a *proper $*$ -polynomial*, if it is a linear combination of elements of the type

$$y_{i_1,1} \cdots y_{i_r,1} z_{j_1,0} \cdots z_{j_s,0} z_{l_1,1} \cdots z_{l_t,1} w_1 \cdots w_m$$

where $w_1, \dots, w_m$ are left normed Lie commutators in the variables from $Y_0 \cup Z_0 \cup Y_1 \cup Z_1$. Notice that the symmetric even variables appear only inside the commutators.

Lemma 1.2.4. *We have $Id^{gri}(D^{gri}) = \langle z_{1,0}, y_{1,1} \rangle_{T_2^*}$ and $c_n^{gri}(D^{gri}) = 2^n$, for every $n \geq 1$.*

Proof. Since D^{gri} is a commutative algebra, $((D^{gri})^{(0)})^- = 0$ and $((D^{gri})^{(1)})^+ = 0$, we get $z_{1,0}, y_{1,1} \in Id^{gri}(D^{gri})$ and D^{gri} satisfies the commutators $[y_{1,0}, y_{2,0}]$, $[z_{1,1}, y_{1,0}]$ and $[z_{1,1}, z_{2,1}]$. Let us consider $I = \langle z_{1,0}, y_{1,1} \rangle_{T_2^*}$, then we have $I \subseteq Id^{gri}(D^{gri})$. Let us check the opposite inclusion.

Let f be a $(\mathbb{Z}_2, *)$ -identity of D^{gri} . By the standard multilinearization process and since D^{gri} is an algebra with 1, we can assume f is a multilinear proper polynomial of degree $t > 0$. After reducing the polynomial f modulo I we obtain $f = \alpha z_{1,1} \cdots z_{t,1}$. By making the evaluation $z_{i,1} = (1, -1)$, for all $1 \leq i \leq t$, we get $f = \alpha(1, (-1)^t) \neq 0$. But since $f \in Id^{gri}(D^{gri})$, we must have $\alpha = 0$ and so $Id^{gri}(D^{gri}) = I$.

It also proves that for all $t \geq 1$ the polynomial $\{z_{1,1} \cdots z_{t,1}\}$ is a basis for the proper polynomial of degree t modulo $Id^{gri}(D^{gri})$ and so $\gamma_t^{gri}(D^{gri}) = 1$, for all $t \geq 0$. Hence,

$$c_n^{gri}(D^{gri}) = \sum_{j=0}^n \binom{n}{j} = 2^n.$$

□

Recall that M is the algebra

$$M = \left\{ \left(\begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in F \right) \right\}.$$

From now on, we let M_* be the algebra M with trivial grading and reflection involution. Then, M_* is a $*$ -superalgebra and we have $Id^{gri}(M_*) = \langle Id^*(M_*), y_{1,1}, z_{1,1} \rangle_{T_2^*}$ and $c_n^{gri}(M_*)$ grows exponentially.

We denote by M^{gri} the algebra M with the grading

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

and endowed with the reflection involution. Also, M^{gri} is a $*$ -superalgebra and we have $(M^{(0)})^+ = M^{(0)}$, $(M^{(0)})^- = \{0\}$, $(M^{(1)})^+ = F(e_{12} + e_{34})$ and $(M^{(1)})^- = F(e_{12} - e_{34})$. Notice that $z_{1,0}$ and $x_{1,1}x_{2,1}$ are identities of M^{gri} , where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$. Let us denote by I the T_2^* -ideal generated by the polynomials $z_{1,0}$ and $x_{1,1}x_{2,1}$.

Remark 1.2.5. For any polynomial $f \in F\langle X | \mathbb{Z}_2, * \rangle$ we have $x_{1,1}fx_{2,1} \in I$.

Proof. We may clearly assume that f is a monomial of homogeneous degree 0. Since $[x_{1,1}, f] \in F\langle X | \mathbb{Z}_2, * \rangle^{(1)}$, we get

$$x_{1,1}fx_{2,1} = [x_{1,1}, f]x_{2,1} + fx_{1,1}x_{2,1} \equiv 0 \pmod{I}.$$

□

Theorem 1.2.6. [9, Theorem 6.3] $Id^{gri}(M^{gri}) = \langle z_{1,0}, x_{1,1}x_{2,1} \rangle_{T_2^*}$. Moreover, $c_n^{gri}(M^{gri})$ grows exponentially.

Proof. Since, by Lemma 1.2.2, $c_n^*(M_*) \leq c_n^{gri}(M^{gri})$ and $c_n^*(M_*)$ grows exponentially, we get that $c_n^{gri}(M^{gri})$ grows exponentially. Let $I = \langle z_{1,0}, x_{1,1}x_{2,1} \rangle_{T_2^*}$. By the discussion above, $I \subseteq Id^{gri}(M^{gri})$.

We shall prove that if $f \in Id^{gri}(M^{gri})$, then $f \equiv 0 \pmod{I}$. To this end, we may clearly assume that f is a multilinear polynomial of degree, say, n . Since $[y_{i,0}, y_{j,0}] \in I$, we have $y_{\sigma(1),0} \cdots y_{\sigma(n),0} \equiv y_{1,0} \cdots y_{n,0} \pmod{I}$ for any $\sigma \in S_n$. Moreover, by Remark 1.2.5 we have $x_{1,1}fx_{2,1} \in I$, for any polynomial $f \in F\langle X | \mathbb{Z}_2, * \rangle$. Then we get that either $f \equiv \alpha y_{1,0} \cdots y_{n,0} \pmod{I}$, for some $\alpha \in F$, or f can be written $\pmod{I}$ as a linear combination of monomials of the type

$$y_{i_1,0} \cdots y_{i_t,0} x_{1,1} y_{i_{t+1},0} \cdots y_{i_{n-1},0},$$

where $0 \leq t \leq n-1$, $i_1 < \cdots < i_t$ and $i_{t+1} < \cdots < i_{n-1}$.

In the first case, by making the evaluation $y_{i,0} = 1$, for $i = 1, \dots, n$, we get $\alpha = 0$ and so $f \in I$, as wished.

In the second case, write

$$f \equiv \sum_{t=0}^{n-1} \sum_{1 \leq i_1 < \cdots < i_t \leq n-1} \alpha_{i_1, \dots, i_t} y_{i_1,0} \cdots y_{i_t,0} x_{1,1} y_{i_{t+1},0} \cdots y_{i_{n-1},0} \pmod{I},$$

with $\alpha_{i_1, \dots, i_t} \in F$. If for some $i_1 < \cdots < i_t$, $\alpha_{i_1, \dots, i_t} \neq 0$, we make the evaluation $y_{i_1,0} = \cdots = y_{i_t,0} = e_{11} + e_{44}$, $y_{i_{t+1},0} = \cdots = y_{i_{n-1},0} = e_{22} + e_{33}$ and $x_{1,1} = e_{12} + e_{34}$, in case $x_{1,1}$ is symmetric, or $x_{1,1} = e_{12} - e_{34}$, in case $x_{1,1}$ is skew. It is easily seen that f evaluates to $\alpha_{i_1, \dots, i_t} (e_{11} + e_{44})(e_{12} \pm e_{34})(e_{22} + e_{33}) = \alpha_{i_1, \dots, i_t} e_{12} \pmod{I}$ and $\alpha_{i_1, \dots, i_t} = 0$. Thus, $f \in I$ and the proof is complete. □

The following result allows us to assume F an algebraically closed field, whenever we are studying the T_2^* -ideals and the $*$ -graded codimensions.

Lemma 1.2.7. [9, Lemma 8.1] *Let F be a field of characteristic zero, $\bar{F}$ its algebraic closure and A a $*$ -superalgebra over F . Then the algebra $\bar{A} = A \otimes_F \bar{F}$ has an induced structure of $*$ -superalgebra, $(c_n^F(A))^{gri} = (c_n^{\bar{F}}(\bar{A}))^{gri}$. Furthermore, $Id^{gri}(A) = Id^{gri}(\bar{A})$, viewed as $*$ -superalgebras over F .*

1.3 The $*$ -graded cocharacter and the $\langle n \rangle$ -cocharacter

The *wreath product* between $\mathbb{Z}_2 \times \mathbb{Z}_2$ and S_n is the group defined by

$$\mathbb{H}_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr S_n = \{((g_1, h_1), \dots, (g_n, h_n); \sigma) \mid (g_i, h_i) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \sigma \in S_n\}$$

with multiplication given by

$$((g_1, h_1), \dots, (g_n, h_n); \sigma)((a_1, b_1), \dots, (a_n, b_n); \tau) = ((\bar{g}_1, \bar{h}_1), \dots, (\bar{g}_n, \bar{h}_n); \sigma\tau),$$

where $\bar{g}_i = g_i a_{\sigma^{-1}(i)}$ and $\bar{h}_i = h_i b_{\sigma^{-1}(i)}$, for all $1 \leq i \leq n$.

We have that $\mathbb{H}_n$ acts on P_n^{gri} by the following

$$\begin{aligned} ((g_1, h_1), \dots, (g_n, h_n); \sigma) \cdot y_{i, t_i} &= y_{\sigma(i), g_i + g_{\sigma(i)}} \\ ((g_1, h_1), \dots, (g_n, h_n); \sigma) \cdot z_{i, t_i} &= \begin{cases} z_{\sigma(i), g_i + g_{\sigma(i)}}, & \text{if } h_{\sigma(i)} = 1 \\ -z_{\sigma(i), g_i + g_{\sigma(i)}}, & \text{if } h_{\sigma(i)} = * \end{cases} \end{aligned}$$

The cocharacter of the $\mathbb{H}_n$ -module $P_n^{gri}(A)$ is called the n -th $*$ -graded cocharacter of the $*$ -superalgebra A , and it is denoted by $\chi_n^{gri}(A)$.

For an integer number $n \geq 1$, we write $n = n_1 + n_2 + n_3 + n_4$ as a sum of four non-negative integers and write $\langle n \rangle = (n_1, n_2, n_3, n_4)$. A multipartition $\langle \lambda \rangle = (\lambda(1), \dots, \lambda(4)) \vdash \langle n \rangle$ is such that $\lambda(i) = (\lambda(i)_1, \lambda(i)_2, \dots) \vdash n_i$, for $1 \leq i \leq 4$. Since $\text{char}(F) = 0$, there exists a one-to-one correspondence between the irreducible $\mathbb{H}_n$ -characters and the multipartitions $\langle \lambda \rangle \vdash \langle n \rangle$.

Hence, we can write the $\mathbb{H}_n$ -character of P_n^{gri} as

$$\chi_n^{gri}(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle},$$

where $\chi_{\langle \lambda \rangle}$ is the irreducible $\mathbb{H}_n$ -character associated to the multipartition $\langle \lambda \rangle$ and $m_{\langle \lambda \rangle} \geq 0$ is the corresponding multiplicity. We denote by $l_n^{gri}(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle}$ the n -th $*$ -graded colength of A .

We define $P_{\langle n \rangle}$ to be the space of multilinear $(\mathbb{Z}_2, *)$ -polynomials in which the first n_1 variables are symmetric of homogeneous degree 0, the next n_2 variables are symmetric of homogeneous degree 1, the next n_3 variables are skew of homogeneous degree 0 and the next n_4 variables are skew of homogeneous degree 1.

We can notice that for any choice of $\langle n \rangle = (n_1, n_2, n_3, n_4)$ there are $\binom{n}{\langle n \rangle}$ subspaces isomorphic to $P_{\langle n \rangle}$ where $\binom{n}{\langle n \rangle} = \binom{n}{n_1, n_2, n_3, n_4}$ denotes the multinomial coefficient and it is clear that $P_{\langle n \rangle}$ is embedded into P_n^{gri} . Also we have that

$$P_n^{gri} \cong \bigoplus_{\langle n \rangle} \binom{n}{\langle n \rangle} P_{\langle n \rangle}.$$

Let us consider $P_{\langle n \rangle}(A) := \frac{P_{\langle n \rangle}}{P_{\langle n \rangle} \cap Id^{gri}((A))}$ and $c_{\langle n \rangle}(A) = \dim_F P_{\langle n \rangle}(A)$. By the above, it is also clear that

$$c_n^{gri}(A) = \sum_{\langle n \rangle} \binom{n}{\langle n \rangle} c_{\langle n \rangle}(A). \quad (1.3.1)$$

Remark 1.3.1. If A and B are *-superalgebras, then $A \oplus B$ is a *-superalgebra and $Id^{gri}(A \oplus B) = Id^{gri}(A) \cap Id^{gri}(B)$. Furthermore, $c_n^{gri}(A \oplus B) \leq c_n^{gri}(A) + c_n^{gri}(B)$ and the equality holds if and only if

$$\dim \frac{P_n^{gri}}{P_n^{gri} \cap Id^{gri}(A) \cap Id^{gri}(B)} = \dim \frac{P_n^{gri}}{P_n^{gri} \cap Id^{gri}(A)} + \dim \frac{P_n^{gri}}{P_n^{gri} \cap Id^{gri}(B)}.$$

This is equivalent to say that $\dim P_n^{gri} = \dim(P_n^{gri} \cap Id^{gri}(A) + P_n^{gri} \cap Id^{gri}(B))$, and, so, any polynomial in P_n^{gri} can be written as a sum of multilinear polynomials in $Id^{gri}(A)$ and in $Id^{gri}(B)$.

Similarly $c_{\langle n \rangle}(A \oplus B) = c_{\langle n \rangle}(A) + c_{\langle n \rangle}(B)$ if, and only if any polynomial in $P_{\langle n \rangle}$ can be written as a sum of multilinear polynomials in $P_{\langle n \rangle} \cap Id^{gri}(A)$ and in $P_{\langle n \rangle} \cap Id^{gri}(B)$.

According to the construction of the spaces $P_{\langle n \rangle}$, with $\langle n \rangle = (n_1, n_2, n_3, n_4)$, $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ acts on $P_{\langle n \rangle}$ by permuting the respective variables, that is, for $f \in P_{\langle n \rangle}$ and $(\sigma_1, \dots, \sigma_4) \in S_{\langle n \rangle} = S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ we have

$$(\sigma_1, \dots, \sigma_4)f(y_{1,0}, \dots, y_{n_1,0}, y_{1,1}, \dots, y_{n_2,1}, z_{1,0}, \dots, z_{n_3,0}, z_{1,1}, \dots, z_{n_4,1}) =$$

$$f(y_{\sigma_1(1),0}, \dots, y_{\sigma_1(n_1),0}, y_{\sigma_2(1),1}, \dots, y_{\sigma_2(n_2),1}, z_{\sigma_3(1),0}, \dots, z_{\sigma_3(n_3),0}, z_{\sigma_4(1),1}, \dots, z_{\sigma_4(n_4),1})$$

and so $P_{\langle n \rangle}$ is a $S_{\langle n \rangle}$ -module. Since T_2^* -ideals are invariant under the given action, we have that $P_{\langle n \rangle}(A)$ also inherits a structure of $S_{\langle n \rangle}$ -module.

It is well known that there exists a one-to-one correspondence between the irreducible $S_{\langle n \rangle}$ -characters and the multipartitions $\langle \lambda \rangle \vdash \langle n \rangle$. We also know that the irreducible $S_{\langle n \rangle}$ -characters are the outer tensor product of irreducible characters of $S_{n_1}, \dots, S_{n_4}$, respectively. Then, we denote by $\chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)}$ the irreducible $S_{\langle n \rangle}$ -character corresponding to $\langle \lambda \rangle$ and by $d_{\lambda(1)} \dots d_{\lambda(4)}$ its degree, where $d_{\lambda(i)}$ is given by the hook formula $d_{\langle \lambda \rangle} = \binom{n}{\langle \lambda \rangle} d_{\lambda_1} d_{\lambda_2} d_{\lambda_3} d_{\lambda_4}$.

By complete reducibility, we can write the character $\chi_{\langle n \rangle}(A)$ of $P_{\langle n \rangle}(A)$ as

$$\chi_{\langle n \rangle}(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m'_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(4)}, \quad (1.3.2)$$

where $m'_{\langle\lambda\rangle}$ are the corresponding multiplicities. We call $\chi_{\langle n\rangle}(A)$ the n th $\langle n\rangle$ -cocharacter of A .

The following result establishes a relation between the $*$ -graded cocharacter and the $\langle n\rangle$ -cocharacter of a $*$ -superalgebra.

Theorem 1.3.2. *If $P_n^{gri}(A)$ has $\mathbb{H}_n$ -character $\chi_n^{gri}(A) = \sum_{\langle\lambda\rangle \vdash \langle n\rangle} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}$ and $P_{\langle n\rangle}(A)$ has $S_{\langle n\rangle}$ -character $\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle \vdash \langle n\rangle} m'_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$, then $m_{\langle\lambda\rangle} = m'_{\langle\lambda\rangle}$, for all multipartition $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$ such that $\lambda(i) \vdash n_i, i = 1, 2, 3, 4$ and $n = n_1 + n_2 + n_3 + n_4$.*

Now, we consider $F_m := F_m \langle y_{1,0}, \dots, y_{m,0}, y_{1,1}, \dots, y_{m,1}, z_{1,0}, \dots, z_{m,0}, z_{1,1}, \dots, z_{m,1} \rangle$ and let $F_m^n := F_m^n \langle y_{1,0}, \dots, y_{m,0}, y_{1,1}, \dots, y_{m,1}, z_{1,0}, \dots, z_{m,0}, z_{1,1}, \dots, z_{m,1} \rangle$ be the subspace of the homogeneous polynomials with degree $n \geq m$. Then $GL_m \times GL_m \times GL_m \times GL_m$ acts diagonally in F_m^n , and so F_m^n has a $GL_m \times GL_m \times GL_m \times GL_m$ -modulo structure. We also have that $F_m^n \cap Id^{gri}(A)$ is invariant under this action.

Hence, the space

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap Id^{gri}(A)}$$

is a $GL_m \times GL_m \times GL_m \times GL_m$ -modulo. We denote by $\Psi_n^{gri}(A)$, the $GL_m \times GL_m \times GL_m \times GL_m$ -modulo $F_m^n(A)$.

The $GL_m \times GL_m \times GL_m \times GL_m$ representation theory shows that there exists an one-to-one correspondence between irreducible $GL_m \times GL_m \times GL_m \times GL_m$ -modulos and multipartitions $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$ of n such that $h(\lambda(i)) \leq m$, where $h(\lambda(i))$ denotes the number of boxes of the first column of $\lambda(i), i = 1, 2, 3, 4$. We denote by $\Psi_{\langle\lambda\rangle}$ the irreducible $GL_m \times GL_m \times GL_m \times GL_m$ -character corresponding to the multipartition $\langle\lambda\rangle$.

Since $\text{char}(F) = 0$, we may write

$$\Psi_n^{gri}(A) = \sum_{\substack{\langle\lambda\rangle \vdash \langle n\rangle \\ h(\langle\lambda\rangle) \leq m}} \bar{m}_{\langle\lambda\rangle} \Psi_{\langle\lambda\rangle},$$

where $\bar{m}_{\langle\lambda\rangle} \geq 0$ is the respective multiplicity and $h(\langle\lambda\rangle) = \max\{h(\lambda(i)), i = 1, 2, 3, 4\}$.

We also have that all irreducible $GL_m \times GL_m \times GL_m \times GL_m$ -modulo from F_m^n is cyclic, and is generated by a non-zero polynomial of the type

$$f_{\langle\lambda\rangle} = \prod_{\substack{i=1 \\ \lambda(3)_1}}^{\lambda(1)_1} St_{h_i(\lambda(1))}(y_{1,0}, \dots, y_{h_i(\lambda(1)),0}) \prod_{\substack{i=1 \\ \lambda(4)_1}}^{\lambda(2)_1} St_{h_i(\lambda(2))}(y_{1,1}, \dots, y_{h_i(\lambda(2)),1}) \\ \prod_{i=1} St_{h_i(\lambda(3))}(z_{1,0}, \dots, z_{h_i(\lambda(3)),0}) \prod_{i=1} St_{h_i(\lambda(4))}(z_{1,1}, \dots, z_{h_i(\lambda(4)),1}) \sum_{\sigma \in S_n} \alpha_{\sigma} \sigma,$$

where $\alpha_\sigma \in F$ and the direct action of S_n under F_m^n is defined as the place permutation. This polynomial $f_{\langle\lambda\rangle}$ is called *highest weight vector corresponding to the multipartition $\langle\lambda\rangle$* .

We consider the multitableaux $T_{\langle\lambda\rangle}$ which is filled by placing the numbers in ascending order from top to bottom column by column. Its corresponding highest weight vector is called *the standard highest weight vector* and we write

$$f_{T_{\langle\lambda\rangle}} = \prod_{i=1}^{\lambda(1)_1} St_{h_i(\lambda(1))}(y_{1,0}, \dots, y_{h_i(\lambda(1)),0}) \prod_{i=1}^{\lambda(2)_1} St_{h_i(\lambda(2))}(y_{1,1}, \dots, y_{h_i(\lambda(2)),1}) \\ \prod_{i=1}^{\lambda(3)_1} St_{h_i(\lambda(3))}(z_{1,0}, \dots, z_{h_i(\lambda(3)),0}) \prod_{i=1}^{\lambda(4)_1} St_{h_i(\lambda(4))}(z_{1,1}, \dots, z_{h_i(\lambda(4)),1})$$

We consider $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(4)})$ a multitableaux. We know that every polynomial $f_{\langle\lambda\rangle}$ can be written as a unique linear combination of polynomials of the type

$$f_{T_{\langle\lambda\rangle}} = \prod_{i=1}^{\lambda(1)_1} St_{h_i(\lambda(1))}(y_{1,0}, \dots, y_{h_i(\lambda(1)),0}) \prod_{i=1}^{\lambda(2)_1} St_{h_i(\lambda(2))}(y_{1,1}, \dots, y_{h_i(\lambda(2)),1}) \\ \prod_{i=1}^{\lambda(3)_1} St_{h_i(\lambda(3))}(z_{1,0}, \dots, z_{h_i(\lambda(3)),0}) \prod_{i=1}^{\lambda(4)_1} St_{h_i(\lambda(4))}(z_{1,1}, \dots, z_{h_i(\lambda(4)),1}) \sigma^{-1},$$

where σ is the only permutation of S_n that changes the standard multitableaux to the multitableaux $T_{\langle\lambda\rangle}$. The polynomial $f_{T_{\langle\lambda\rangle}}$ is called *highest weight vector corresponding to the multitableaux $T_{\langle\lambda\rangle}$* .

Theorem 1.3.3. *If $P_n^{gri}(A)$ has $\mathbb{H}_n$ -character $\chi_n^{gri}(A) = \sum_{\langle\lambda\rangle \vdash \langle n \rangle} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}$ and $F_m^n(A)$ has $GL_m \times GL_m \times GL_m \times GL_m$ -character $\Psi_n^{gri}(A) = \sum_{\substack{\langle\lambda\rangle \vdash \langle n \rangle \\ h(\langle\lambda\rangle) \leq m}} \bar{m}_{\langle\lambda\rangle} \Psi_{\langle\lambda\rangle}$, then we have*

$m_{\langle\lambda\rangle} = \bar{m}_{\langle\lambda\rangle}$, for all multipartition $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$ such that $\lambda(i) \vdash n_i$, $i = 1, 2, 3, 4$, $n = n_1 + n_2 + n_3 + n_4$ and $h(\langle\lambda\rangle) \leq m$.

Remark 1.3.4. The multiplicity $\bar{m}_{\langle\lambda\rangle} \neq 0$ if, and only if, there exists a multitableaux $T_{\langle\lambda\rangle}$ such that $f_{T_{\langle\lambda\rangle}} \notin Id^{gri}(A)$. Furthermore, $\bar{m}_{\langle\lambda\rangle}$ is equal to the maximum number of vectors $f_{T_{\langle\lambda\rangle}}$ which are linearly independent in $F_m^n(A)$.

Previously, we have presented the cocharacter of D^{gr} , under a view of superalgebras, and of D_* and M_* , under a view of $*$ -algebras. Then, it is easy to find the $*$ -graded cocharacter of these $*$ -superalgebras. We have the following:

$$1. \chi_n^{gri}(D_*) = \sum_{j=0}^n \chi_{(n-j), \emptyset, (j), \emptyset} \text{ and } \chi_n^{gri}(D^{gr}) = \sum_{j=0}^n \chi_{(n-j), (j), \emptyset, \emptyset};$$

$$2. \text{ If } \chi_n^{gri}(M_*) = \sum_{\langle\lambda\rangle \vdash \langle n \rangle} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle} \text{ then } m_{((n), \emptyset, \emptyset, \emptyset)} = 1,$$

$$m_{\langle\lambda\rangle} = q + 1 \text{ if } \begin{cases} \langle\lambda\rangle = ((p+q, p), \emptyset, \emptyset, \emptyset), \text{ for all } p, q \geq 0 \\ \langle\lambda\rangle = ((p+q, p, 1), \emptyset, \emptyset, \emptyset), \text{ for all } p \geq 1, q \geq 0 \\ \langle\lambda\rangle = ((p+q, p), \emptyset, (1), \emptyset), \text{ for all } p \geq 1, q \geq 0 \end{cases}$$

and $m_{\langle \lambda \rangle} = 0$ otherwise.

From now on, we will use the representation theory of the general linear group to compute the decomposition of the *-graded cocharacter of a *-superalgebra.

Lemma 1.3.5. *For every $n \geq 1$ we have $\chi_n^{gri}(D^{gri}) = \sum_{j=0}^n \chi_{(n-j), \emptyset, \emptyset, (j)}$.*

Proof. Fixed $j \geq 0$, we consider $f_j = y_{1,0}^{n-j} z_{1,1}^j$ the standard highest weight vector corresponding to the multipartition $((n-j), \emptyset, \emptyset, (j))$. By evaluating $y_{1,0} = (1, 1)$ and $z_{1,1} = (1, -1)$ we have $f_j(y_{1,0}, z_{1,1}) = (1, (-1)^j) \neq 0$. Then for all $j \geq 0$ we have $m_{(n-j), \emptyset, \emptyset, (j)} \geq 1$ and so

$$c_n^{gri}(D^{gri}) \geq \sum_{j=0}^n d_{(n-j), \emptyset, \emptyset, (j)} = \sum_{j=0}^n \binom{n}{j} = c_n^{gri}(D^{gri}).$$

Hence, $\chi_n^{gri}(D^{gri}) = \sum_{j=0}^n \chi_{(n-j), \emptyset, \emptyset, (j)}$. □

Theorem 1.3.6. [9, Theorem 6.4] *If $\chi_n^{gri}(M^{gri}) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}$ then*

$$m_{\langle \lambda \rangle} = \begin{cases} 1, & \text{if } \langle \lambda \rangle = ((n), \emptyset, \emptyset, \emptyset) \\ q+1, & \text{if } \langle \lambda \rangle = ((p+q, p), (1), \emptyset, \emptyset) \\ q+1, & \text{if } \langle \lambda \rangle = ((p+q, p), \emptyset, \emptyset, (1)) \\ 0, & \text{otherwise,} \end{cases}$$

where $p, q \geq 0$ and $2p + q + 1 = n$.

Proof. By Theorem 1.2.6, $c_n^{gri}(M^{gri})$ grows exponentially. Hence, M^{gri} generates a *-supervariety of exponential growth.

We start by computing the decomposition of the *-graded cocharacter of M^{gri} into irreducible characters. Let

$$\chi_n^{gri}(M^{gri}) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle} \tag{1.3.3}$$

be the decomposition of the *-graded cocharacter of M^{gri} .

Now, since $z_{1,0}$ is an identity of M^{gri} , if $\chi_{\langle \lambda \rangle}$ appears with non-zero multiplicity in (1.3.3), we must have $\lambda(3) = 0$. Moreover, by Remark 1.2.5, two variables of homogeneous degree 1 cannot appear in any non-zero monomial (mod $Id^{gri}(M^{gri})$). Thus $m_{\langle \lambda \rangle} \neq 0$ in (1.3.3) implies that either $\langle \lambda \rangle = (\lambda(1), (1), \emptyset, \emptyset)$ or $\langle \lambda \rangle = (\lambda(1), \emptyset, \emptyset, (1))$ or $\langle \lambda \rangle = (\lambda(1), \emptyset, \emptyset, \emptyset)$. Since $\dim_F((M^{(0)})^+) = 2$, any polynomial alternating on three symmetric variables of homogeneous degree 0 vanishes in M^{gri} . By standard arguments this says that $m_{\langle \lambda \rangle} \neq 0$ implies that $\lambda(1) = (p+q, p)$, where $p \geq 0, q \geq 0$, is a partition with at most two parts.

As in the proof of the previous theorem, we have that symmetric variables of homogeneous degree 0 commute (mod $Id^{gri}(M^{gri})$). Hence, we have that $m_{\langle \lambda \rangle} \neq 0$ implies that either $\langle \lambda \rangle = ((n), \emptyset, \emptyset, \emptyset)$ or $\langle \lambda \rangle = ((p+q, p), \emptyset, \emptyset, (1))$ or $\langle \lambda \rangle = ((p+q, p), (1), \emptyset, \emptyset)$, where $p \geq 0$, $q \geq 0$ and $n = 2p + q + 1$.

We claim that $m_{((p+q, p), \emptyset, \emptyset, (1))} = m_{((p+q, p), (1), \emptyset, \emptyset)} = q + 1$. To this end, we follow closely the proof of [25, Lemma 2] (or [29, Theorem 3]), taking into account the due changes.

Define, for $0 \leq i \leq q$, the polynomials

$$a_{p,q}^{(i)}(y_{1,0}, y_{2,0}, x_{1,1}) = y_{1,0}^i \underbrace{\bar{y}_{1,0} \cdots \bar{y}_{1,0}}_p x_{1,1} \underbrace{\bar{y}_{2,0} \cdots \bar{y}_{2,0}}_p y_{1,0}^{q-i},$$

where $-$ and $\sim$ mean alternation on the corresponding variables and $x_{1,1} = y_{1,1}$ or $x_{1,1} = z_{1,1}$.

Then we can show that the polynomials $a_{p,q}^{(i)}$ are highest weight vectors corresponding to Young multitableaux and they are linearly independent (mod $Id^{gri}(M^{gri})$). Hence, $m_{((p+q, p), \emptyset, \emptyset, (1))} = m_{((p+q, p), (1), \emptyset, \emptyset)} = q + 1$ as claimed. Also, through an obvious evaluation, it is clear that $m_{((n), \emptyset, \emptyset, \emptyset)} = 1$, for all $n \geq 1$. $\square$

As a consequence of Poincaré-Birkhoff-Witt Theorem, we have that if A is a *-superalgebras A with 1, then its $(\mathbb{Z}_2, *)$ -identities follow from its proper $(\mathbb{Z}_2, *)$ -identities. Hence, in order to study $(\mathbb{Z}_2, *)$ -identities of unitary *-superalgebras, we study the proper ones.

We denote by Γ_n^{gri} the subspace of P_n^{gri} of proper *-polynomials and establish $\Gamma_0^{gri} = \text{span} \{1\}$. The sequence of proper *-graded codimensions is defined as

$$\gamma_i^{gri}(A) = \dim \frac{\Gamma_n^{gri}}{\Gamma_n^{gri} \cap Id^{gri}(A)}, \quad n = 0, 1, 2, \dots$$

For a unitary *-superalgebra the relation between *-graded codimension and proper *-graded codimension, is given by

$$c_n^{gri}(A) = \sum_{i=0}^n \binom{n}{i} \gamma_i^{gri}(A), \quad n = 0, 1, 2, \dots$$

For every $i \geq 1$, we have that Γ_{k+i}^{gri} is a consequence of Γ_k^{gri} , it means that $\Gamma_{k+i}^{gri} \subseteq \langle \Gamma_k^{gri} \rangle_{T_2^*}$. As a consequence, we have the following.

Lemma 1.3.7. *Let A be a *-superalgebra with 1. If for some $k \geq 2$, $\gamma_k^{gri}(A) = 0$ then $\gamma_m^{gri}(A) = 0$ for all $m \geq k$.*

Since $\Gamma_n^{gri}(A) = \frac{\Gamma_n^{gri}}{\Gamma_n^{gri} \cap Id^{gri}(A)}$ is a $\mathbb{H}_n$ -submodulo of $P_n^{gri}(A)$, we consider its $\mathbb{H}_n$ -character $\psi_n^{gri}(A)$, called *proper n -th *-graded cocharacter* of A .

1.4 Almost polynomial growth

In this section, we present some results about the classification of *-superalgebras with polynomial growth. We start with some results about the structure of *-superalgebras which were given in [9].

Let A be a *-superalgebra, φ the automorphism of order 2 determined by the $\mathbb{Z}_2$ -grading and I an ideal of A . We say that I is a *-graded ideal, if $I^\varphi = I$ and $I^* = I$. A *-superalgebra A is a *simple *-superalgebra* if $A^2 \neq \{0\}$ and A has no non-zero *-graded ideals.

The next theorem is a generalization of Wedderburn-Malcev Theorem.

Theorem 1.4.1. [9, Theorem 7.3] *Let A be a finite dimensional *-superalgebra over a field F of characteristic zero. Then:*

1. $J(A)$ is a *-graded ideal;
2. If F is algebraically closed, then $A = A_1 \oplus \cdots \oplus A_m + J(A)$, where each algebra $A_i, i = 1, \dots, m$, is a simple *-superalgebra.

In [9] the authors characterized the finite dimensional simple *-superalgebras over an algebraically closed field F of characteristic zero. They also characterized the finite dimensional *-superalgebras of *-graded codimensions polynomially bounded.

Theorem 1.4.2. [9, Theorem 7.6] *Let A be a finite dimensional simple *-superalgebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following *-superalgebras:*

1. $M_{k,l}(F)$, with $k \geq 1, k \geq l \geq 0$, with transpose or symplectic involution (the symplectic involution can occur only when $k = l$);
2. $M_{k,l}(F) \oplus M_{k,l}(F)^{op}$, with $k \geq 1, k \geq l \geq 0$, with induced grading and exchange involution;
3. $M_n(F) + cM_n(F)$, with involution given by $(a+cb)^\dagger = a^* - cb^*$, where $*$ denotes the transpose or symplectic involution;
4. $M_n(F) + cM_n(F)$, with involution given by $(a+cb)^\dagger = a^* + cb^*$, where $*$ denotes the transpose or symplectic involution;
5. $(M_n(F) + cM_n(F)) \oplus (M_n(F) + cM_n(F))^{op}$, with grading

$$(M_n(F) \oplus M_n(F)^{op}, c(M_n(F) \oplus M_n(F)^{op}))$$

and exchange involution.

Theorem 1.4.3. [9, Theorem 8.3] *Let A be a finite dimensional *-superalgebra over an algebraically closed field F of characteristic zero. Then $c_n^{gr}(A)$ is polynomially bounded if and only if*

1. $c_n(A)$ is polynomially bounded;
2. $A = B + J(A)$, where B is a maximal semisimple subalgebra of A with trivial induced $\mathbb{Z}_2$ -grading and trivial induced involution.

In [9] the authors proved that a finite dimensional *-superalgebra A has polynomial growth if, and only if, $Id^{gr}(A) = Id^{gr}(B)$ for some finite dimensional *-superalgebra B having an explicit decomposition into suitable subalgebras with induced graded involution $*$.

Theorem 1.4.4. [2, Theorem 3.5] *Let A be a finite dimensional *-superalgebra over a field F of characteristic zero. Then $c_n^{gr}(A)$ is polynomially bounded if and only if $var^{gr}(A) = var^{gr}(B_1 \oplus \cdots \oplus B_m)$, where each B_i is a finite dimensional *-superalgebra over F such that $\dim_F B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$.*

This result will be very useful in the next chapter. Whenever we want to prove some property about a *-superalgebra A such that $c_n^{gr}(A)$ is polynomially bounded, we can study the properties of *-superalgebras of the type $F + J$, and then recover the property about A .

Recall that if $A = F + J$ is a finite dimensional algebra over F where $J = J(A)$ is its Jacobson radical, then J can be decomposed into the direct sum of B -bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11} \quad (1.4.1)$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. In a similar way, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, r, s \in \{0, 1\}$, $J_{ir}J_{rs} \subseteq J_{is}$, $J_{ik}J_{rs} = 0$ for $k \neq r$ and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B .

We also have that the given modules are graded and if the algebra A has an involution $*$, then J_{00} and J_{11} are stable under the involution whereas $J_{01}^* = J_{10}$.

We say that a *-superalgebra A has *almost polynomial growth*, or, A is an *APG *-superalgebra*, if the sequence of the *-graded codimensions of A grows exponentially but any proper subvariety of A has polynomial growth.

We have seen in Theorems 1.1.1 and 1.1.2 that $var^{gr}(D^{gr})$, $var^*(D_*)$ and $var^*(M_*)$ are varieties of almost polynomial growth, according to the point of view. Then we have the following.

Lemma 1.4.5. [9, Theorem 5.1] *$var^{gr}(D_*)$, $var^{gr}(M_*)$ and $var^{gr}(D^{gr})$ are APG *-supervarieties.*

Proof. Since the grading on D_* is trivial, we have that

$$Id^{gr}(D_*) = \langle Id^*(D_*), y_{1,1}, z_{1,1} \rangle_{T_2^*},$$

the T_2^* -ideal generated by $Id^*(D_*)$, $y_{1,1}$, $z_{1,1}$. Also $c_n^{gri}(D_*) = c_n^*(D_*)$, hence $c_n^{gri}(D_*)$ grows exponentially. Let $\mathcal{U}$ be a proper subvariety of $var^{gri}(D_*)$. Since $\mathcal{U} \subset var^{gri}(D_*)$, we have that $y_{1,1}, z_{1,1} \in Id^{gri}(\mathcal{U})$. Hence $Id^{gri}(\mathcal{U}) = \langle Id^*(\mathcal{U}), y_{1,1}, z_{1,1} \rangle_{T_2^*}$, $c_n^{gri}(\mathcal{U}) = c_n^*(\mathcal{U})$ and $c_n^{gri}(\mathcal{U})$ is polynomially bounded. Hence, $var^{gri}(D_*)$ is an APG *-supervariety. Analogously, $var^{gri}(M_*)$ is an APG *-supervariety.

Similarly, since D^{gr} has trivial involution, we have that

$$Id^{gri}(D^{gr}) = \langle Id^{gr}(D^{gr}), z_{1,0}, z_{1,1} \rangle_{T_2^*},$$

the T_2^* -ideal generated by $Id^{gr}(D^{gr})$, $z_{1,0}$, $z_{1,1}$. Also $c_n^{gri}(D^{gr}) = c_n^{gr}(D^{gr})$, then $c_n^{gri}(D^{gr})$ grows exponentially. Let $\mathcal{U}$ be a proper subvariety of $var^{gri}(D^{gr})$. Since $\mathcal{U} \subset var^{gri}(D^{gr})$, we get $z_{1,0}, z_{1,1} \in Id^{gri}(\mathcal{U})$. Hence $Id^{gri}(\mathcal{U}) = \langle Id^{gr}(\mathcal{U}), z_{1,0}, z_{1,1} \rangle_{T_2^*}$, $c_n^{gri}(\mathcal{U}) = c_n^{gr}(\mathcal{U})$ and $c_n^{gri}(\mathcal{U})$ is polynomially bounded. Hence, $var^{gri}(D^{gr})$ is also an APG *-supervariety. □

Lemma 1.4.6. *D^{gri} generates an APG *-supervariety.*

Proof. First notice that $Id^{gri}(A) \not\subseteq Id^{gri}(D^{gri})$ if, and only if, $z_{1,1}^r \in Id^{gri}(A)$, for some $r \geq 1$.

In fact, if $z_{1,1}^r \in Id^{gri}(A)$, for some $r \geq 1$, then $Id^{gri}(A) \not\subseteq Id^{gri}(D^{gri})$, by Lemma 1.2.4. Suppose now that $Id^{gri}(A) \subseteq Id^{gri}(D^{gri})$. Then there exists $f \in Id^{gri}(A)$ such that $f \notin Id^{gri}(D^{gri})$, then we must have $f = f(y_{1,0}, \dots, y_{r,0}, z_{1,1}, \dots, z_{n-r,1})$, since $z_{1,0}, y_{1,1} \in Id^{gri}(D^{gri})$.

We may assume f multilinear and so f does not vanish in a basis of D^{gri} . Consider $a = (1, 1)$ and $b = (1, -1)$ and notice that $\{a\}$ and $\{b\}$ form a basis for $((D^{gri})^{(0)})^+$ and $((D^{gri})^{(1)})^-$, respectively. Since $b^2 = a$ is a even symmetric element and $f \notin Id^{gri}(D^{gri})$, we have:

$$0 \neq f(a, \dots, a, b, \dots, b) = f(b^2, \dots, b^2, b, \dots, b) = \alpha b^{n+r},$$

where $\alpha \neq 0$ is equal to the sum of the coefficients of f . Since $z_{1,1}^2$ is an even symmetric monomial, it follows that $f(z_{1,1}^2, \dots, z_{1,1}^2, z_{1,1}, \dots, z_{1,1}) = \alpha z_{1,1}^{n+r} \in Id^{gri}(A)$ and since $\alpha \neq 0$ it implies that $z_{1,1}^{n+r} \in Id^{gri}(A)$.

Let $A \in var^{gri}(D^{gri})$ such that $var^{gri}(A) \subsetneq var^{gri}(D^{gri})$. Then for any $(\lambda, \emptyset, \emptyset, \mu)$, if $m_{\lambda, \emptyset, \emptyset, \mu}$ and $m'_{\lambda, \emptyset, \emptyset, \mu}$ are the multiplicity of $\chi_{\lambda, \emptyset, \emptyset, \mu}$ in $\chi_n^{gri}(A)$ and $\chi_n^{gri}(D^{gri})$, respectively, then $m_{\lambda, \emptyset, \emptyset, \mu} \leq m'_{\lambda, \emptyset, \emptyset, \mu}$. Thus we have, by Lemma 1.3.5,

$$\chi_n^{gri}(A) = \sum_{j=0}^n m_j \chi_{(n-j), \emptyset, \emptyset, (j)},$$

where $m_j \in \{0, 1\}$. We have that even symmetric variables and odd skew variables commute modulo $Id^{gri}(D^{gri})$ and, by the previous remark, we have that $z_{1,1}^r \in Id^{gri}(A)$ for some $r \geq 1$. This implies that $m_j = 0$ for all $j \geq r$. Thus

$$c_n^{gri}(A) \leq \sum_{j=0}^{r-1} \chi_{(n-j), \emptyset, \emptyset, (j)} = \sum_{j=0}^{r-1} \binom{n}{j} \approx \frac{1}{(r-1)!} n^{r-1}.$$

Hence, $c_n^{gri}(A)$ is polynomially bounded. $\square$

Next, we characterize varieties of polynomial growth which are generated by finite dimensional *-superalgebras, by excluding from them the *-superalgebras $D_*, M_*, D^{gr}, D^{gri}$ and M^{gri} . We start with the following lemmas.

Lemma 1.4.7. [9, Lemma 8.4] *Let A and B be *-superalgebras. If B has trivial grading and $B \notin var^{gri}(A)$, then $B \notin var^*(A^{(0)})$.*

Proof. Clearly, $Id^{gri}(A^{(0)}) = \langle Id^*(A^{(0)}), y_{1,1}, z_{1,1} \rangle_{T_2^*}$ and also $Id^{gri}(B) = \langle Id^*(B), y_{1,1}, z_{1,1} \rangle_{T_2^*}$. Hence, if $B \in var^*(A^{(0)})$, then $B \in var^{gri}(A^{(0)})$. Since $A^{(0)}$ is a subalgebra of A , $var^{gri}(A^{(0)}) \subseteq var^{gri}(A)$ which says that $B \in var^{gri}(A)$. $\square$

Lemma 1.4.8. [9, Lemma 8.5] *Let A be a finite dimensional *-superalgebra over an algebraically closed field of characteristic zero. Let $A = A_1 \oplus \dots \oplus A_k + J$ be a Wedderburn-Malcev decomposition of A , where $A_1, \dots, A_k$ are simple *-superalgebras. If for some $i, l \in \{1, \dots, k\}, i \neq l$, we have that $A_i^{(0)} J^{(1)} A_l^{(0)} \neq \{0\}$, then $M^{gri} \in var^{gri}(A)$.*

Proof. Suppose that there exist $i, l \in \{1, \dots, k\}, i \neq l$, such that $A_i^{(0)} J^{(1)} A_l^{(0)} \neq \{0\}$ and let $a \in A_i^{(0)}, b \in A_l^{(0)}, j' \in J^{(1)}$ such that $aj'b \neq 0$. If e_1 and e_2 are the unit elements of $A_i^{(0)}$ and $A_l^{(0)}$, respectively, then $e_1 aj' b e_2 \neq 0$ and if we set $aj'b = j$, we have $e_1 j e_2 \neq 0$ with $j \in J^{(1)}$.

Let $k \geq 1$ be the largest integer such that $e_1 J e_2 \subseteq J^k$ and let $A' = A/J^{k+1}$. Since J is a *-graded ideal, A' is a *-superalgebra and $A' \in var^{gri}(A)$.

Let $\bar{e}_1, \bar{e}_2, \bar{j}$ be the images of e_1, e_2, j in A' , respectively. Since $\bar{J} = J(A') = J/J^{k+1}$, we have that $\bar{e}_1 \bar{J} \bar{e}_2 \neq \{0\}$. Let $C = \text{span}\{\bar{e}_1, \bar{e}_2, \bar{e}_1 \bar{j} \bar{e}_2, \bar{e}_2 \bar{j}^* \bar{e}_1\}$. Since e_1 and e_2 are orthogonal idempotents and $e_1 J e_2 J = e_2 J e_1 J \subseteq J^{k+1}$ we get that C is a subalgebra of A' . Moreover, C is a *-superalgebra and $(C^{(0)})^+ = \text{span}\{\bar{e}_1, \bar{e}_2\}$, $(C^{(0)})^- = \{0\}$, $(C^{(1)})^+ = \text{span}\{\bar{e}_1 \bar{j} \bar{e}_2 + \bar{e}_2 \bar{j}^* \bar{e}_1\}$ and $(C^{(1)})^- = \text{span}\{\bar{e}_1 \bar{j} \bar{e}_2 - \bar{e}_2 \bar{j}^* \bar{e}_1\}$. Recalling the multiplication table of M^{gri} we obtain that the map $\psi : C \rightarrow M^{gri}$ defined by $\bar{e}_1 \mapsto e_{11} + e_{44}, \bar{e}_2 \mapsto e_{22} + e_{33}, \bar{e}_1 \bar{j} \bar{e}_2 \mapsto e_{12}, \bar{e}_2 \bar{j}^* \bar{e}_1 \mapsto e_{34}$ is an isomorphism of *-superalgebras. Hence $M^{gri} \in var^{gri}(C) \subseteq var^{gri}(A') \subseteq var^{gri}(A)$ and we are done. $\square$

In order to prove the following theorem, we will need to introduce one more concept about the exponent of an algebra.

Let A be a PI-algebra over a field F of characteristic zero. It is well known that $c_n(A)$ is exponentially bounded and, in [10], Giambruno and Zaicev proved that $exp(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer called the PI-exponent of the algebra A . Moreover, $c_n(A)$ is polynomially bounded if, and only if, $exp(A) \leq 1$.

The authors also determinate a way to compute the exponent. Let A be a finite dimensional algebra over an algebraically closed field F of characteristic zero and let B be a maximal semisimple subalgebra of A . Then

$$\exp(A) = \max_i \dim_F(C_1^{(i)} + \cdots + C_k^{(i)}),$$

where $C_1^{(i)}, \dots, C_k^{(i)}$ are distinct simple subalgebras of B and

$$C_1^{(i)} J C_2^{(i)} J \cdots J C_{k-1}^{(i)} J C_k^{(i)} \neq \{0\}.$$

In the following theorem, we characterize the APG *-varieties generated by finite dimensional *-superalgebras.

Theorem 1.4.9. [9, Theorem 8.6] *Let A be a finite dimensional *-superalgebra over a field of characteristic zero. Then $c_n^{gr}(A)$ is polynomially bounded if and only if $M_*, D_*, D^{gr}, D^{gr}, M^{gr} \notin \text{var}^{gr}(A)$.*

Proof. By Lemma 1.2.7, we may assume that the field F is algebraically closed. Suppose that $c_n^{gr}(A)$ is polynomially bounded. Since, by Theorem 1.4.5 and by Theorem 1.2.6, the *-graded codimensions of M_*, D_*, D^{gr}, D^{gr} and M^{gr} grow exponentially, we get that $M_*, D_*, D^{gr}, D^{gr}, M^{gr} \notin \text{var}^{gr}(A)$.

Conversely, suppose that $M_*, D_*, D^{gr}, D^{gr}, M^{gr} \notin \text{var}^{gr}(A)$. Let $A = B + J$ be a Wedderburn-Malcev decomposition of A , where B is a maximal semisimple *-superalgebra. Write $B = A_1 \oplus \cdots \oplus A_k$, where the A_i 's are simple *-superalgebras. Then

$$A^{(0)} = B^{(0)} + J^{(0)} = A_1^{(0)} \oplus \cdots \oplus A_k^{(0)} + J^{(0)}$$

is an algebra with involution and with trivial grading. Since, by Lemma 1.4.7, $M_*, D_* \notin \text{var}^{gr}(A^{(0)})$, we have, by Theorem 1.1.2, that $c_n^*(A^{(0)}) = c_n^{gr}(A^{(0)})$ is polynomially bounded. Also $A_i^{(0)} \cong F$, for all $i = 1, \dots, k$, and $*$ is the identity map on $B^{(0)}$. Since $c_n(A^{(0)}) \leq c_n^*(A^{(0)})$ is polynomially bounded, $\exp(A^{(0)}) \leq 1$ and so $A_i^{(0)} J^{(0)} A_l^{(0)} = \{0\}$, for all $i, l \in \{1, \dots, k\}, i \neq l$.

Next, we consider B and we have $A_i = A_i^{(0)} \oplus A_i^{(1)}$, for all $i = 1, \dots, k$. Since A_i 's are simple superalgebras, by Theorem 1.4.2 and by the above, either $A_i \cong F$ or $A_i \cong F + cF$ with trivial involution or $A_i \cong F + cF$ with the involution given by $(a + cb)^* = a - cb$, for $i = 1, \dots, k$. If, for some i , $A_i \cong F + cF$ with trivial involution, then $D^{gr} \in \text{var}^{gr}(A)$, a contradiction. If, for some i , $A_i \cong F + cF$ with the involution given by $(a + cb)^* = a - cb$, then $D^{gr} \in \text{var}^{gr}(A)$, another contradiction. Thus B has trivial grading and trivial involution.

Now, suppose that there exist $i, l \in \{1, \dots, k\}, i \neq l$, such that $A_i J A_l = A_i^{(0)} J^{(1)} A_l^{(0)} \neq \{0\}$. Then, by Lemma 1.4.8, $M^{gr} \in \text{var}^{gr}(A)$, a contradiction. Therefore, we have that, for all $i, l \in \{1, \dots, k\}, i \neq l$, $A_i J A_l = \{0\}$. By the properties of $\exp(A)$, we have that $\exp(A) \leq 1$ and $c_n(A)$ is polynomially bounded. Hence, by Theorem 1.4.3, $c_n^{gr}(A)$ is polynomially bounded and this completes the proof of the theorem. $\square$

As an immediate consequence of the above theorem, we have that if A is a finite dimensional $*$ -superalgebra over a field of characteristic zero, then the sequence $c_n^{gr}(A)$, $n \geq 1$, is either polynomially bounded or grows exponentially. Moreover, we classify all the APG $*$ -supervarieties generated by finite dimensional $*$ -superalgebras.

Corollary 1.4.10. *[9, Corollary 8.8] $var^{gr}(M_*)$, $var^{gr}(D_*)$, $var^{gr}(D^{gr})$, $var^{gr}(D^{gr})$ and $var^{gr}(M^{gr})$ are the only $*$ -supervarieties of almost polynomial growth generated by finite dimensional $*$ -superalgebras.*

We say that $\mathcal{V}$ is a *minimal variety of polynomial growth n^k* if asymptotically $c_n^{gr}(\mathcal{V}) \approx an^k$, for some $a \neq 0$, and $c_n^{gr}(\mathcal{U}) \approx bn^t$, with $t < k$, for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$.

In the next chapters, we classify all the subvarieties of the APG $*$ -varieties generated by finite dimensional $*$ -superalgebras, and exhibit the decompositions of the $*$ -graded cocharacters of all minimal subvarieties of $var^{gr}(M_*)$, $var^{gr}(M^{gr})$, $var^{gr}(D_*)$, $var^{gr}(D^{gr})$ and $var^{gr}(D^{gr})$ and compute their $*$ -graded colengths. We will collect such results to classify the varieties such that the sequence of $*$ -graded colengths of them is bounded by three.

Chapter 2

The APG noncommutative *-superalgebras

In [21] and [12], the authors considered the algebra M as an algebra with involution and as an algebra with superinvolution, and classified all subvarieties of the $*$ -variety $var^*(M_*)$ and of the variety with superinvolution $var^{sup}(M)$, respectively.

In this chapter, we clarify the concept of superinvolution, explain why the classification given by those authors implies the classification of all subvarieties of the $*$ -supervarieties $var^{gr_i}(M_*)$ and $var^{gr_i}(M^{gr_i})$, and establish the results given in [21] and [12] in the language of $*$ -superalgebras.

We also compute the $*$ -graded colength of the minimal subvarieties obtained. Such results about the minimal subvarieties lying in $var^*(M_*)$ have been recently submitted for publication in our joint work with La Mattina and Vieira [23], in $*$ -algebras language. The results of this chapter will be collected in order to classify the $*$ -superalgebras with $*$ -graded colength bounded by three in the last chapter of this thesis.

2.1 Subvarieties of the $var^{gr_i}(M_*)$

Recall that

$$M_* = \left\{ \begin{pmatrix} u & r & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & v \\ 0 & 0 & 0 & u \end{pmatrix} \mid u, r, s, v \in F \right\}$$

with trivial grading and endowed with the reflection involution. Moreover, we have $Id^{gr_i}(M_*) = \langle z_{1,0}z_{2,0}, y_{1,1}, z_{1,1} \rangle_{T_2^*}$.

The purpose of this section is to construct $*$ -superalgebras belonging to the variety generated by the algebra M_* .

Since M_* is a $*$ -superalgebra with trivial grading, we can see the algebra M_* only with the involution algebra structure. Then, we have that the classification of the $*$ -superalgebras, up to T_2^* -equivalence, inside $\text{var}^{gr i}(M_*)$ and the classification of the $*$ -algebras inside the $\text{var}^*(M_*)$ are equivalent.

The classification of all subvarieties inside the $*$ -variety generated by the $*$ -algebra M_* was given in [21, Theorem 7] by La Mattina and Martino in 2015. Here, we restate such results in $*$ -superalgebra language.

In order to describe the subvarieties of $\text{var}^{gr i}(M_*)$, we start by considering, for any fixed $k \geq 2$, the algebra UT_{2k} of $2k \times 2k$ upper triangular matrices over F and $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1} \in UT_{2k}$, where e'_{ij} 's are the usual matrix units. Also we consider the subalgebras $N_{k,*}$, $U_{k,*}$ and $A_{k,*}$ of UT_{2k} introduced in [21].

For $k \geq 2$, we denote by $N_{k,*}$ the subalgebra of UT_{2k} :

$$N_k = \text{span}_F\{I_{2k}, E, \dots, E^{k-2}; e_{12} - e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}\}$$

with trivial grading and endowed with the reflection involution, where I_{2k} denotes the $(2k) \times (2k)$ identity matrix. Notice that

$$(N_{k,*}^{(0)})^+ = \text{span}_F\{I_{2k}, E, \dots, E^{k-2}, e_{13} + e_{2k-2,2k}, \dots, e_{1k} + e_{k+1,2k}\} \text{ and}$$

$$(N_{k,*}^{(0)})^- = \text{span}_F\{e_{12} - e_{2k-1,2k}, e_{13} - e_{2k-2,2k}, \dots, e_{1k} - e_{k+1,2k}\}.$$

Then we have $N_{k,*} \in \text{var}^{gr i}(M_*)$, since $z_{1,0}z_{2,0}$ is a $(\mathbb{Z}_2, *)$ -identity of $N_{k,*}$.

Similarly, for any $k \geq 2$, we denote by $U_{k,*}$ the algebra:

$$U_k = \text{span}_F\{I_{2k}, E, \dots, E^{k-2}; e_{12} + e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}\}$$

with trivial grading and endowed with the reflection involution. Notice that we also have $U_{k,*} \in \text{var}^{gr i}(M_*)$, since

$$(U_{k,*}^{(0)})^+ = \text{span}_F\{I_{2k}, E, \dots, E^{k-2}, e_{12} + e_{2k-1,2k}, e_{13} + e_{2k-2,2k}, \dots, e_{1k} + e_{k+1,2k}\},$$

$$(U_{k,*}^{(0)})^- = \text{span}_F\{e_{13} - e_{2k-2,2k}, \dots, e_{1k} - e_{k+1,2k}\}.$$

For example, for $k = 2, 4$, we have $N_{2,*} = \begin{pmatrix} a & -b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$, $U_{2,*} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}$,

$$N_{4,*} = \begin{pmatrix} a & b & c & d & 0 & 0 & 0 & 0 \\ 0 & a & f & g & 0 & 0 & 0 & 0 \\ 0 & 0 & a & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & f & g & h \\ 0 & 0 & 0 & 0 & 0 & a & f & i \\ 0 & 0 & 0 & 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \text{ and } U_{4,*} = \begin{pmatrix} a & b & c & d & 0 & 0 & 0 & 0 \\ 0 & a & f & g & 0 & 0 & 0 & 0 \\ 0 & 0 & a & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & f & g & h \\ 0 & 0 & 0 & 0 & 0 & a & f & i \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Remark 2.1.1. [22, Remark 8] Let A be a $\ast$ -superalgebra. If $z_{1,0} \cdots z_{m,0}$ is a $(\mathbb{Z}_2, \ast)$ -identity of A , for some $m \geq 1$, then

$$z_{1,0}w_1z_{2,0}w_2 \cdots w_{m-1}z_{m,0}$$

is a $(\mathbb{Z}_2, \ast)$ -identity of A , where $w_1, \dots, w_m$ are monomials of $F\langle X | \mathbb{Z}_2, \ast \rangle$ in variables of homogeneous degree 0.

Proof. Notice that for $s \in (A^{(0)})^+$, $k \in (A^{(0)})^-$ we have $sk + ks \in (A^{(0)})^-$, then $ks = -sk + k'$, for some $k' \in (A^{(0)})^-$. If we evaluate the polynomial

$$z_{1,0}w_1z_{2,0}w_2 \cdots w_{m-1}z_{m,0}$$

in $A^{(0)}$, after a repeated application of the relation $ks = -sk + k'$, we can write the evaluation as a linear combination of monomials each one containing at least m consecutive skew even elements. Since the product of m skew even elements of A is zero, we obtain that all evaluation in the polynomial $z_{1,0}w_1z_{2,0}w_2 \cdots w_{m-1}z_{m,0}$ is also zero. Hence the proof is completed. $\square$

We notice that $U_{2,\ast} \sim_{T_2^\ast} F$ is the commutative algebra with trivial grading and trivial involution. The result about the $(\mathbb{Z}_2, \ast)$ -identities and the $\ast$ -graded codimensions of $N_{k,\ast}$ and $U_{k,\ast}$ follows bellow.

Lemma 2.1.2. *For the $\ast$ -superalgebras $N_{2,\ast}$ and $U_{2,\ast}$ we have*

1. [21, Lemma 3] $Id^{gri}(U_{2,\ast}) = \langle z_{1,0}, y_{1,1}, z_{1,1} \rangle_{T_2^\ast}$,
2. [22, Lemma 10] $Id^{gri}(N_{2,\ast}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}] \rangle_{T_2^\ast}$,
3. $c_n^{gri}(U_{2,\ast}) = 1$ and $c_n^{gri}(N_{2,\ast}) = n + 1$.

Proof. In order to prove the item (1), just notice that $(U_{2,\ast}^{(0)})^+ = \text{span}_F\{I_4, e_{12} + e_{3,4}\}$ and $(U_{2,\ast}^{(0)})^- = 0$, then we get that $U_{2,\ast}$ is $T_2^\ast$ -equivalent to a commutative algebra with trivial grading and trivial involution. Hence $Id^{gri}(U_{2,\ast}) = \langle z_{1,0}, y_{1,1}, z_{1,1} \rangle_{T_2^\ast}$ and $c_n^{gri}(U_{2,\ast}) = 1$.

Now we study the algebra $N_{2,\ast}$. Let $I = \langle [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], z_{1,0}z_{2,0}, y_{1,1}, z_{1,1} \rangle_{T_2^\ast}$. Since $(N_{2,\ast}^{(0)})^+ = \text{span}_F\{I_4\}$ and $(N_{2,\ast}^{(0)})^- = \text{span}_F\{e_{12} - e_{3,4}\}$ we can easily check that $I \subset Id^{gri}(N_{2,\ast})$.

Let f be a $(\mathbb{Z}_2, \ast)$ -identity of $N_{2,\ast}$, we may assume f multilinear of degree n . Since $y_{1,1}, z_{1,1}, z_{1,0}z_{2,0} \in I$, by the previous remark, we can write f modulo I as a linear combination of the polynomials

$$y_{1,0} \cdots y_{n,0}, y_{i_1,0} \cdots y_{i_{n-1},0} z_{1,0}, i_1 < \dots < i_{n-1}.$$

Now let f be a linear combination of these polynomials. By the multihomogeneity of $T_2^\ast$ -ideals we may assume $f = \alpha y_{1,0} \cdots y_{n,0}$ or $f = \beta y_{1,0} \cdots y_{n-1,0} z_{n,0}$.

By evaluating in $y_{1,0} = \dots = y_{n,0} = I_4$, we get $\alpha = 0$. Also, the evaluation $y_{1,0} = \dots = y_{n-1,0} = I_4$ and $z_{n,0} = e_{12} - e_{34}$ gives $\beta = 0$.

It shows that these polynomials are linearly independent modulo $P_n^{gri} \cap Id^{gri}(N_{2,*})$. Since $P_n^{gri} \cap I \subseteq P_n^{gri} \cap Id^{gri}(N_{2,*})$, it follows that $I = Id^{gri}(N_{2,*})$ and the polynomials above form a basis of $P_n^{gri}(\text{mod } P_n^{gri} \cap Id^{gri}(N_{2,*}))$. Hence $c_n^{gri}(N_{2,*}) = 1 + n$. $\square$

Lemma 2.1.3. [21, Lemma 2] Let $k \geq 3$. Then

1. $Id^{gri}(N_{k,*}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, [y_{1,0}, \dots, y_{k-1,0}] \rangle_{T_2^*}$.
2. $c_n^{gri}(N_{k,*}) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} (2j-1) + \binom{n}{k-1} (k-1) \approx qn^{k-1}$, for some $q > 0$.

Proof. Let $I = \langle [y_{1,0}, \dots, y_{k-1,0}], z_{1,0}z_{2,0}, y_{1,1}, z_{1,1} \rangle_{T_2^*}$. We can see that $I \subset Id^{gri}(N_{k,*})$. We shall prove the opposite inclusion. Let $f \in Id^{gri}(N_{k,*}^{gri})$ be a multilinear polynomial. Since $N_{k,*}$ is a unitary algebra, we can assume f is a proper polynomial. After reducing f modulo I , we obtain the following:

(i) If $\deg f \geq k$, we have $f \equiv 0$.

(ii) If $\deg f = k-1$, so f is a linear combination of polynomials

$$[z_{i,0}, y_{i_1,0}, \dots, y_{i_{k-2},0}], \text{ for } i = 1, \dots, k-1, \quad i_1 < \dots < i_{k-2}.$$

(iii) If $\deg f = s < k-1$, so f is a linear combination of polynomials

$$[z_{i,0}, y_{i_1,0}, \dots, y_{i_{s-1},0}], [y_{j,0}, y_{j_1,0}, \dots, y_{j_{s-1},0}],$$

where $i = 1, \dots, s$, $i_1 < \dots < i_{s-1}$ and $j > j_1 < \dots < j_{s-1}$.

Hence, modulo I , we can assume that for some $1 \leq s \leq k-1$

$$f = \sum_{i=1}^s \alpha_i [z_{i,0}, y_{i_1,0}, \dots, y_{i_{s-1},0}] + \sum_{j=2}^s \beta_j [y_{j,0}, y_{j_1,0}, \dots, y_{j_{s-1},0}].$$

Suppose that there exists j such that $\beta_j \neq 0$. By making the evaluation $z_{i,0} = 0$, for all $i = 1, \dots, s$, $y_{j,0} = e_{13} + e_{2k-2,2k}$, $y_{j_m} = E$, for all $m = 1, \dots, s-1$, we get $\beta_j = 0$, a contradiction. Then, $\beta_j = 0$, for all $2 \leq j \leq s$.

Now, suppose that there exists i such that $\alpha_i \neq 0$. By evaluating in $z_{i,0} = e_{12} - e_{2k-1,2k}$, $z_{t,0} = 0$, for all $t \neq i$, $y_{i_m} = E$, for all $m = 1, \dots, s-1$ we get the result $\alpha_i = 0$, a contradiction. Then, $\alpha_i = 0$, for all $1 \leq i \leq s$.

The arguments above say that $f \in I$, and so $I = Id^{gri}(N_{k,*})$. Moreover, we also have proved that the proper *-graded codimensions are:

$$\gamma_s^{gri}(N_{k,*}) = \begin{cases} 0, & \text{if } s \geq k \\ s, & \text{if } s = k-1 \\ 2s-1, & \text{if } 0 \leq s < k-1 \end{cases}$$

Then we conclude that $c_n^{gri}(N_{k,*}) = 1 + \sum_{j=1}^{k-2} \binom{n}{j}(2j-1) + \binom{n}{k-1}(k-1)$.

□

By using similar arguments as in the proof of the Lemma 2.1.3, we can prove the following results about the $(\mathbb{Z}_2, *)$ -identities and $*$ -graded codimensions of $U_{k,*}$ and $N_{k,*} \oplus U_{k,*}$, for $k \geq 3$.

Lemma 2.1.4. [21, Lemma 3] *Let $k \geq 3$. Then*

1. $Id^{gri}(U_{k,*}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, [z_{1,0}, y_{1,0}, \dots, y_{k-2,0}] \rangle_{T_2^*}$.
2. $c_n^{gri}(U_{k,*}) = 1 + \sum_{j=1}^{k-2} \binom{n}{j}(2j-1) + \binom{n}{k-1}(k-2) \approx qn^{k-1}$, for some $q > 0$.

Notice that if $t > k$ then $N_{t,*} \oplus U_{k,*} \sim_{T_2^*} N_{t,*}$, on other hand, if $t < k$ we have $N_{t,*} \oplus U_{k,*} \sim_{T_2^*} U_{k,*}$. Moreover, if $k = t = 2$ then $N_{2,*} \oplus U_{2,*} \sim_{T_2^*} N_{2,*}$.

Lemma 2.1.5. [21, Lemma 4] *If $k \geq 3$, then*

1. $Id^{gri}(N_{k,*} \oplus U_{k,*}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, [y_{1,0}, y_{2,0}, \dots, y_{k,0}], [z_{1,0}, y_{1,0}, \dots, y_{k-1,0}] \rangle_{T_2^*}$,
2. $c_n^{gri}(N_{k,*} \oplus U_{k,*}) = 1 + \sum_{j=1}^{k-1} \binom{n}{j}(2j-1) \approx qn^{k-1}$, for some $q > 0$.

Now for $k \geq 2$, we denote by $A_{k,*}$ the subalgebra of UT_{2k} :

$$A_k = \text{span}_F\{e_{11} + e_{2k,2k}, E, \dots, E^{k-2}; e_{12}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-1,2k}\},$$

with trivial grading and endowed with the reflection involution. Notice that

$$(A_{k,*}^{(0)})^+ = \text{span}_F\{e_{11} + e_{2k,2k}, E, \dots, E^{k-2}, e_{12} + e_{2k-1,2k}, e_{13} + e_{2k-2,2k}, \dots, e_{1k} + e_{k+1,2k}\} \text{ and}$$

$$(A_{k,*}^{(0)})^- = \text{span}_F\{e_{12} - e_{2k-1,2k}, e_{13} - e_{2k-2,2k}, \dots, e_{1k} - e_{k+1,2k}\}.$$

Then we have $A_{k,*} \in \text{var}^{gri}(M_*)$, since $z_{1,0}z_{2,0}$ is a $(\mathbb{Z}_2, *)$ -identity of $A_{k,*}$.

For example, for $k = 2, 4$, we have

$$A_{2,*} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad A_{4,*} = \begin{pmatrix} a & b & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & f & g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 & f & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}.$$

Let $St_3(y_1, y_2, y_3) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)}$ denote the standard polynomial of degree 3. About the $*$ -identities and the $*$ -graded codimensions of the algebras $A_{k,*}$ we have the following.

Lemma 2.1.6. *For the *-algebra $A_{2,*}$, we have*

1. $[22, \text{Lemma 11}] Id^{gri}(A_{2,*}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, St_3(y_{1,0}, y_{2,0}, y_{3,0}), y_{1,0}z_{1,0}y_{2,0} \rangle_{T_2^*}$
2. $c_n^{gri}(A_{2,*}) = 4n - 1$, for $n \geq 3$.

Proof. Let $I = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, St_3(y_{1,0}, y_{2,0}, y_{3,0}), y_{1,0}z_{1,0}y_{2,0} \rangle_{T_2^*}$. We can easily check that $I \subseteq Id^{gri}(A_{2,*})$. Let us show the opposite inclusion.

First, notice that since $z_{1,0}z_{2,0} \in Id^{gri}(A_{2,*})$, by Remark 2.1.1 we have $z_{1,0}wz_{2,0} \in Id^{gri}(A_{2,*})$ for any monomial w of $F\langle X | \mathbb{Z}_2, * \rangle$ in variables of homogeneous degree 0. Then, since $y_{1,1}, z_{1,1} \equiv 0$ on $A_{2,*}$ we must have $P_{n_1, n_2, n_3, n_4}^{gri}(A_{2,*}) = \{0\}$, if $n_2 > 0$ or $n_4 > 0$ or $n_3 \geq 2$. Thus by (1.3.1),

$$c_n^{gri}(A_{2,*}) = c_{n,0,0,0}^{gri}(A_{2,*}) + nc_{n-1,0,1,0}^{gri}(A_{2,*}). \quad (2.1.1)$$

We start by considering $P_{n,0,0,0}^{gri}(A_{2,*})$. By Poincaré-Birkhoff-Witt theorem, every monomial in $y_{1,0}, \dots, y_{n,0}$ can be written as a linear combination of products of the type

$$y_{i_1,0} \cdots y_{i_s,0} w_1 \cdots w_m \quad (2.1.2)$$

where $w_1, \dots, w_m$ are left normed Lie commutators in the $y'_{i,0}$ s and $i_1 < \dots < i_s$. Since $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}], y_{1,0}[y_{2,0}, y_{3,0}]y_{4,0} \in I$, then modulo $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}]$, at most one commutator can appear in (2.1.2), i.e. elements in (2.1.2) are polynomials of type

$$y_{1,0} \cdots y_{n,0} \text{ or } y_{i_1,0} \cdots y_{i_s,0} [y_{r,0}, y_{j_1,0}, \dots, y_{j_t,0}] \text{ with } r > j_i < \dots < j_t.$$

Moreover, modulo $y_{1,0}[y_{2,0}, y_{3,0}]y_{4,0}$ we have

$$[y_{r,0}, y_{j_1,0}, \dots, y_{j_t,0}] = [y_{r,0}, y_{j_1,0}]y_{j_2,0} \cdots y_{j_t,0} \pm y_{j_2,0} \cdots y_{j_t,0}[y_{r,0}, y_{j_1,0}].$$

Then, modulo I , every polynomial in $P_{n,0,0,0}^{gri}$ can be written as a linear combination of elements of the type

$$[y_{r,0}, y_{1,0}]y_{2,0} \cdots \widehat{y}_{r,0} \cdots y_{n,0}, \quad y_{i_1,0} \cdots y_{i_{n-2},0} [y_{i,0}, y_{j,0}] \quad \text{and} \quad y_{1,0} \cdots y_{n,0}. \quad (2.1.3)$$

Notice that elements of the first type only appear in case $s = 0$ in (2.1.2). Now since $[y_{1,0}, y_{2,0}]w[y_{3,0}, y_{4,0}] \in I$, where w is a monomial in $y'_{i,0}$ s, then the variables out of the commutator in the polynomials of the second type in (2.1.3) can be ordered. Moreover, since $St_3(y_{1,0}, y_{2,0}, y_{3,0}) \in I$, then $y_{1,0}[y_{2,0}, y_{3,0}] \equiv y_{2,0}[y_{1,0}, y_{3,0}] + y_{3,0}[y_{2,0}, y_{1,0}]$ can be applied and we obtain that the polynomials

$$[y_{r,0}, y_{1,0}]y_{2,0} \cdots \widehat{y}_{r,0} \cdots y_{n,0}, \quad y_{2,0} \cdots \widehat{y}_{r,0} \cdots y_{n,0} [y_{r,0}, y_{1,0}] \quad \text{and} \quad y_{1,0} \cdots y_{n,0}, \quad (2.1.4)$$

generate $P_{n,0,0,0}^{gri}$ modulo $P_{n,0,0,0}^{gri} \cap I$.

Let $f \in P_{n,0,0,0}^{gri} \cap Id^{gri}(A_{2,*})$ be a linear combination of the polynomials in (2.1.4) and write

$$f = \alpha y_{1,0} \cdots y_{n,0} + \sum_{j=1}^n \alpha_j [y_{j,0}, y_{1,0}] y_{2,0} \cdots \widehat{y}_{r,0} \cdots y_{n,0} + \beta_j y_{2,0} \cdots \widehat{y}_{r,0} \cdots y_{n,0} [y_{j,0}, y_{1,0}].$$

First, by making $y_{i,0} = e_{11} + e_{44}$ for all $i = 1, \dots, n$, we get $\alpha(e_{11} + e_{44}) = 0$, then $\alpha = 0$. Now, for a fixed j , we make the evaluation $y_{j,0} = e_{12} + e_{34}$ and $y_{i,0} = e_{11} + e_{44}$, for all $i \neq j$, and we get $\alpha_j e_{34} - \beta_j e_{12} = 0$. Then $\alpha_j = \beta_j = 0$.

These arguments prove that $P_{n,0,0,0}^{gri} \cap I = P_{n,0,0,0}^{gri} \cap Id^{gri}(A_{2,*})$ and the polynomials in (2.1.4) form a basis for $P_{n,0,0,0}^{gri}(A_{2,*})$. Thus $c_{n,0,0,0}^{gri}(A_{2,*}) = 1 + 2(n-1) = 2n-1$.

We now consider $P_{n-1,0,1,0}^{gri}(A_{2,*})$. Since $y_{1,0}z_{1,0}y_{2,0} \in I$, then $P_{n-1,0,1,0}^{gri}$ can be generated modulo $P_{n-1,1,0,0}^{gri} \cap I$ by the monomials

$$z_{n,0}y_{1,0} \cdots y_{n-1,0} \quad \text{and} \quad y_{1,0} \cdots y_{n-1,0}z_{n,0}. \quad (2.1.5)$$

We claim that these polynomials form a basis of $P_{n-1,1,0,0}^{gri}(A_{2,*})$. In fact, let $f \in P_{n-1,0,1,0}^{gri} \cap Id^{gri}(A_{2,*})$ be a linear combination of the polynomials in (2.1.5),

$$f = \alpha z_{n,0}y_{1,0} \cdots y_{n-1,0} + \beta y_{1,0} \cdots y_{n-1,0}z_{n,0}.$$

By making the evaluation $z_{n,0} = e_{12} - e_{34}$ and $y_{i,0} = e_{11} + e_{44}$, for all $i \neq n$, we get $-\alpha e_{34} + \beta e_{12} = 0$, and so $\alpha = \beta = 0$. It follows that $P_{n-1,0,1,0}^{gri} \cap Id^{gri}(A_{2,*}) = P_{n-1,0,1,0}^{gri} \cap I$ and the affirmation is proved. Thus $c_{n-1,1,0,0}^{gri}(A_{2,*}) = 2$.

Hence, by the multihomogeneity of T_2^* -ideals, $Id^{gri}(A_{2,*}) = I$, and according to (2.1.1) we have $c_n^{gri}(A_{2,*}) = 2n-1 + 2n = 4n-1$. $\square$

Remark 2.1.7. Consider $k \geq 3$, $I_1 = \langle [y_{1,0}, y_{2,0}] [y_{3,0}, y_{4,0}], [y_{1,0}, y_{2,0}] y_{3,0} \cdots y_{k+1,0} \rangle_{T_2^*}$ and $I_2 = \langle [y_{1,0}, y_{2,0}] [y_{3,0}, y_{4,0}], y_{3,0} \cdots y_{k+1,0} [y_{1,0}, y_{2,0}] \rangle_{T_2^*}$. In a similar way as the [19, Lemma 3.1] we can prove that

$$c_{n,0,0,0}^{gri}(I_1) = c_{n,0,0,0}^{gri}(I_2) = 1 + \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1).$$

Moreover, if I is the T_2^* -ideal $I_1 \cap I_2$ then

$$I = \langle [y_{1,0}, y_{2,0}] [y_{3,0}, y_{4,0}], y_{1,0} \cdots y_{k-1,0} [y_{k,0}, y_{k+1,0}] y_{k+2,0} \cdots y_{2k,0} \rangle_{T_2^*}.$$

From Remark 1.3.1, we have the strict inequality

$$c_{n,0,0,0}^{gri}(I) < c_{n,0,0,0}^{gri}(I_1) + c_{n,0,0,0}^{gri}(I_2)$$

since $y_{1,0} \cdots y_{n,0}$ is a polynomial in $P_{n,0,0,0}^{gri}$ which is not in $(P_{n,0,0,0}^{gri} \cap I_1) + (P_{n,0,0,0}^{gri} \cap I_2)$. Furthermore $I \cap P_{n,0,0,0}^{gri} \subset Id^{gri}(A_{k,*}) \cap P_{n,0,0,0}^{gri}$, then we have

$$c_{n,0,0,0}^{gri}(A_{k,*}) \leq c_{n,0,0,0}^{gri}(I) < c_{n,0,0,0}^{gri}(I_1) + c_{n,0,0,0}^{gri}(I_2) = 2 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1). \quad (2.1.6)$$

Lemma 2.1.8. *Let $k \geq 2$. Then*

1. [21, Lemma 1] $Id^{gri}(A_{k,*}) = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, y_{1,0} \cdots y_{k-1,0}z_{1,0}y_{k,0} \cdots y_{2k-2,0}, y_{1,0} \cdots y_{k-2,0}St_3(y_{k-1,0}, y_{k,0}, y_{k+1,0})y_{k+2,0} \cdots y_{2k-1,0} \rangle_{T_2^*},$
2. [23, Lemma 3.10] $c_n^{gri}(A_{k,*}) = 1 + 2 \sum_{j=0}^{k-2} \binom{n}{j}(n-j) + 2 \sum_{j=0}^{k-2} \binom{n}{j}(n-j-1) \approx qn^{k-1},$ for some $q > 0$.

Proof. The result has already been proved for $k = 2$ in Lemma 2.1.6 so we consider $k \geq 3$. Let $I = \langle y_{1,1}, z_{1,1}, z_{1,0}z_{2,0}, y_{1,0} \cdots y_{k-2,0}St_3(y_{k-1,0}, y_{k,0}, y_{k+1,0})y_{k+2,0} \cdots y_{2k-1,0}, y_{1,0} \cdots y_{k-1,0}z_{1,0}y_{k,0} \cdots y_{2k-2,0} \rangle_{T_2^*}$. We can check that $I \subset Id^{gri}(A_{k,*})$. Let us prove the opposite inclusion.

Since $z_{1,0}z_{2,0} \in Id^{gri}(A_{k,*})$, similarly to the proof of Lemma 2.1.6, by Remark 2.1.1, we have $z_{1,0}wz_{2,0} \in Id^{gri}(A_{k,*})$ for any monomial w of $F\langle X|_{\mathbb{Z}_2}, * \rangle$ in variables of homogeneous degree 0. Then, since $y_{1,1}, z_{1,1} \equiv 0$ on $A_{k,*}$, we must have $P_{n_1, n_2, n_3, n_4}^{gri}(A_{k,*}) = \{0\}$, if $n_2 > 0$ or $n_4 > 0$ or $n_3 \geq 2$. Thus by (1.3.1),

$$c_n^{gri}(A_{k,*}) = c_{n,0,0,0}^{gri}(A_{k,*}) + nc_{n-1,0,1,0}^{gri}(A_{k,*}). \quad (2.1.7)$$

Let us study the dimensions of $P_{n,0,0,0}^{gri}(A_{k,*})$ and of $P_{n-1,0,1,0}^{gri}(A_{k,*})$.

We start by considering $P_{n,0,0,0}^{gri}(A_{k,*})$. We claim that the following polynomials in $P_{n,0,0,0}^{gri}$

$$y_1 \cdots y_n, \quad y_{i_1} \cdots y_{i_t}[y_r, y_m]y_{j_1} \cdots y_{j_s}, \quad y_{p_1} \cdots y_{p_u}[y_a, y_b]y_{q_1} \cdots y_{q_v} \quad (2.1.8)$$

where $t < k-1$, $i_1 < \cdots < i_t$, $r > m < j_1 < \cdots < j_s$ and $v < k-1$, $a > b < p_1 < \cdots < p_u$, $q_1 < \cdots < q_v$ form a basis of $P_{n,0,0,0}^{gri}(A_{k,*})$.

In fact, let $f \in P_{n,0,0,0}^{gri} \cap Id^{gri}(A_{k,*})$. Since $y_{1,0} \cdots y_{k-1,0}[y_{k,0}, y_{k+1,0}]y_{k+2,0} \cdots y_{2k,0} \in I$, then we can write f modulo I as a linear combination of

$$f = \alpha y_{1,0} \cdots y_{n,0} + \sum_{\substack{t < k-1 \\ \text{or} \\ s < k-1}} \sum_{r, I, J} \alpha_{r, I, J} y_{i_1,0} \cdots y_{i_t,0}[y_{r,0}, y_{m,0}]y_{j_1,0} \cdots y_{j_s,0}$$

where $t + s = n - 2$ and for any fixed t and s , $I = \{i_1, \dots, i_t\}$ and $J = \{j_1, \dots, j_s\}$. If $t < k-1$ then $i_1 < \cdots < i_t$ and $r > m < j_1 < \cdots < j_s$ and if $s < k-1$ then $r > m < i_1 < \cdots < i_t$ and $j_1 < \cdots < j_s$.

First, suppose that $\alpha \neq 0$. Then by making the evaluation $y_{1,0} = \cdots = y_{n,0} = e_{11} + e_{2k,2k}$ we get $\alpha(e_{11} + e_{2k,2k}) = 0$ and so $\alpha = 0$, a contradiction. So $\alpha = 0$.

Now suppose that $\alpha_{r, I, J} \neq 0$, for some $t < k-1, r, I$ and J . Then by making the evaluation $y_{i_1,0} = \cdots = y_{i_t,0} = E$, $y_{r,0} = e_{12} + e_{2k-1,2k}$ and $y_{m,0} = y_{j_1,0} = \cdots = y_{j_s,0} = e_{11} + e_{2k,2k}$ we get $\alpha_{r, I, J} e_{2k-t'-1,2k} - \alpha_{r, J, I} e_{1,2+t'} = 0$, thus $\alpha_{r, I, J} = \alpha_{r, J, I} = 0$, a contradiction. Similarly, if $\alpha_{r, J, I} \neq 0$, for some $s < k-1, r, I$ and J , by making the evaluation $y_m = y_{i_1,0} = \cdots = y_{i_t,0} = e_{11} + e_{2k,2k}$, $y_{r,0} = e_{12} + e_{2k-1,2k}$ and $y_{j_1,0} = \cdots = y_{j_s,0} = E$ we get $\alpha_{r, I, J} = \alpha_{r, J, I} = 0$, a contradiction as before. It

follows that $f \in P_{n,0,0,0}^{gri} \cap I$ and these polynomials are linearly independent modulo $P_{n,0,0,0}^{gri} \cap Id^{gri}(A_{k,*})$.

Therefore, by counting, we have $1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1)$ polynomials in (2.1.8) and since they are linearly independent modulo $P_{n,0,0,0}^{gri} \cap Id^*(A_{k,*})$ we have

$$1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1) \leq c_{n,0,0,0}^{gri}(A_{k,*}).$$

On the other hand, by (2.1.6) we get $c_{n,0,0,0}^*(A_{k,*}) < 2 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1)$. Thus we conclude that $c_{n,0,0,0}^*(A_{k,*}) = 1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1)$.

Now we consider $P_{n-1,0,1,0}^{gri}(A_{k,*})$. Since $y_{1,0} \dots y_{k-1,0} z_{1,0} y_{k,0} \dots y_{2k-2,0} \in Id^{gri}(A_{k,*})$, then $P_{n-1,0,1,0}^{gri}$ can be generated, modulo $Id^{gri}(A_{k,*})$, by the monomials

$$y_{i_1,0} \dots y_{i_t,0} z_{n,0} y_{j_1,0} \dots y_{j_s,0} \quad (2.1.9)$$

where $i_1 < \dots < i_t$, $j_1 < \dots < j_s$ and we have $t < k-1$ or $s < k-1$.

Next, we show that these polynomials are linearly independent modulo $Id^{gri}(A_{k,*})$. In fact, let $f \in P_{n-1,0,1,0}^{gri} \cap Id^{gri}(A_{k,*})$ be a linear combination of the polynomials above and write

$$f = \sum_{\substack{t \leq k-1 \\ \text{or} \\ s \leq k-1}} \sum_{I,J} \alpha_{I,J} y_{i_1,0} \dots y_{i_t,0} z_{n,0} y_{j_1,0} \dots y_{j_s,0}$$

where $t + s = n-1$ and for any fixed t and s , $i_1 < \dots < i_t$, $j_1 < \dots < j_s$, $I = \{i_1, \dots, i_t\}$ and $J = \{j_1, \dots, j_s\}$.

Suppose $\alpha_{I,J} \neq 0$, for some $t < k-1$, I and J . By making the evaluation, just like in the proof of [21, Lemma 1], $z_{n,0} = e_{12} - e_{2k-1,2k}$, $y_{i_1,0} = \dots = y_{i_t,0} = E$ and $y_{j_1,0} = \dots = y_{j_s,0} = e_{11} + e_{2k,2k}$, we get $-\alpha_{I,J} e_{2k-t-1,2k} + \alpha_{J,I} e_{1,2+t} = 0$, thus $\alpha_{I,J} = \alpha_{J,I} = 0$, a contradiction.

Suppose now $\alpha_{J,I} \neq 0$, for some $s < k-1$, I and J . Then the evaluations $z_{n,0} = e_{12} - e_{2k-1,2k}$, $y_{i_1,0} = \dots = y_{i_t,0} = e_{11} + e_{2k,2k}$ and $y_{j_1,0} = \dots = y_{j_s,0} = E$ give $\alpha_{J,I} = 0$, a contradiction. Thus we have $f \in I$ and the polynomials in (2.1.9) form a basis of $P_{n-1,0,1,0}^{gri}(A_{k,*})$. By counting, we get $c_{n-1,0,1,0}^{gri}(A_{k,*}) = 2 \sum_{j=0}^{k-2} \binom{n-1}{j}$. So

$$nc_{n-1,0,1,0}^{gri}(A_{k,*}) = 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j).$$

Finally, by the multihomogeneity of T_2^* -ideals and by (2.1.7), we have $Id^{gri}(A_{k,*}) = I$ and

$$c_n^{gri}(A_{k,*}) = 1 + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j-1) + 2 \sum_{j=0}^{k-2} \binom{n}{j} (n-j).$$

□

Remark 2.1.9. Since M_* has trivial grading, then any $A \in \text{var}^{gri}(M^{gri})$ also has trivial grading. By using [21, Theorem 3] we have that if $A \in \text{var}^{gri}(M_*)$ has polynomial growth, then

$$A \sim_{T_2^*} (B_1 \oplus \dots \oplus B_m),$$

for some finite dimensional *-superalgebras $B_i, 1 \leq i \leq m$ such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. This means that either $B_i \cong J(B_i)$ is nilpotent or $B_i \cong F + J(B_i)$.

Next we present the classification of the minimal subvarieties of $\text{var}^{gri}(M_*)$. We will omitted the proofs because of the similarity to the proof of the equivalent results for subvarieties of $\text{var}^{gri}(M^{gri})$ that will be studied in the next section.

Theorem 2.1.10. [21, Theorem 6] *Let A be a *-superalgebra such that $\text{var}^{gri}(A) \subsetneq \text{var}^{gri}(M_*)$. Then A is T_2^* -equivalent to one of the following *-superalgebras: N , $N_{t,*} \oplus N$, $U_{t,*} \oplus N$, $A_{k,*} \oplus N$, $N_{t,*} \oplus U_{t,*} \oplus N$, $U_{t,*} \oplus A_{k,*} \oplus N$, $N_{t,*} \oplus A_{k,*} \oplus N$, $N_{t,*} \oplus U_{t,*} \oplus A_{k,*} \oplus N$ for some $k, t \geq 2$, where N is a nilpotent *-superalgebra and C is a commutative algebra with trivial grading and trivial involution.*

Corollary 2.1.11. [21, Corollary 1] *A *-superalgebra $A \in \text{var}^{gri}(M_*)$ generates a minimal variety of polynomial growth if and only if either $A \sim_{T_2^*} N_{k,*}$ or $A \sim_{T_2^*} U_{t,*}$ or $A \sim_{T_2^*} A_{r,*}$, for some $k, r \geq 2$ and $t > 2$.*

2.2 Subvarieties of $\text{var}^{gri}(M^{gri})$

In [12], Ioppolo and La Mattina considered the algebra M^{sup} to be the algebra M with superinvolution and classify all subvarieties of the variety $\text{var}^{sup}(M^{sup})$, from a point of view of algebras with superinvolution.

A *superinvolution* on a superalgebra $A = A^{(0)} \oplus A^{(1)}$ is a map $*$: $A \rightarrow A$ such that $(a^*)^* = a$ for all $a \in A$ and $(ab)^* = (-1)^{(\deg a)(\deg b)} b^* a^*$, for any homogeneous elements $a, b \in A$. Here $\deg c$ denotes the homogeneous degree of $c \in A^{(0)} \cup A^{(1)}$.

Notice that if $A = A^{(0)} \oplus A^{(1)}$ is a superalgebra such that $(A^{(1)})^2 = 0$ then the superinvolutions on A coincide with the graded involutions on A . In fact, suppose that $*$ is a superinvolution on A . Given $a, b \in A$, we write $a = a_0 + a_1$, $b = b_0 + b_1$, where $a_0, b_0 \in A^{(0)}$ and $a_1, b_1 \in A^{(1)}$. So

$$\begin{aligned} (ab)^* &= ((a_0 + a_1)(b_0 + b_1))^* = (a_0 b_0 + a_0 b_1 + a_1 b_0 + \underbrace{a_1 b_1}_0)^* \\ &= b_0^* a_0^* + b_1^* a_0^* + b_0^* a_1^* - \underbrace{b_1^* a_1^*}_0 = (b_0^* + b_1^*)(a_0^* + a_1^*) \\ &= b^* a^*. \end{aligned}$$

Then we have that $*$ is a graded involution on A .

Conversely, suppose that $*$ is a graded involution on $A = A^{(0)} \oplus A^{(1)}$ such that $(A^{(1)})^2 = 0$. Given $a_0, b_0 \in A^{(0)}$ and $a_1, b_1 \in A^{(1)}$ we always have

$$(a_0 b_0)^* = b_0^* a_0^*, \quad (a_0 b_1)^* = b_1^* a_0^*, \quad (a_1 b_0)^* = b_0^* a_1^*, \quad (a_1 b_1)^* = b_1^* a_1^* = 0 = -b_1^* a_1^*.$$

Thus we have that $*$ is a superinvolution on A .

Since $M^{gr i} = M^{(0)} \oplus M^{(1)}$ is a superalgebra such that $(M^{(1)})^2 = 0$, we conclude that the classification of subvarieties of the superalgebra M with superinvolution coincides with the classification of subvarieties of $var^{gr i}(M^{gr i})$. So the results we present here are in agreement with the results obtained by Ioppolo and La Mattina in [12].

The purpose of this section is to construct $*$ -superalgebras belonging to the variety generated by the algebra $M^{gr i}$. Notice that we can see $M^{gr i}$ as the algebra M with the reflection involution and the elementary grading induced by $\mathbf{g} = (0, 1, 0, 1) \in \mathbb{Z}_2^4$. By Lemma 1.2.6, recall that $Id^{gr i}(M^{gr i}) = \langle z_{1,0}, x_{1,1}x_{2,1} \rangle_{T_2^*}$, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$.

For all $k \geq 2$ we define $N_k^{gr i}, U_k^{gr i}$ and $A_k^{gr i}$ to be the algebras N_k, U_k and A_k , respectively, with the elementary grading induced by $\mathbf{g} = (0, \underbrace{1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{k-1}, 1) \in \mathbb{Z}_2^{2k}$ and endowed with the reflection involution.

We start by considering the algebra $N_k^{gr i}$. Since

$$(N_k^{gr i})^{(0)} = \text{span}_F\{I, E, \dots, E^{k-2}\} \quad \text{and}$$

$$(N_k^{gr i})^{(1)} = \text{span}_F\{e_{12} - e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}\},$$

we can notice that $(N_k^{gr i})^{(0)}$ is a commutative subalgebra of N_k and moreover $z_{1,0} \equiv 0$ in $N_k^{gr i}$. We also observe $x_{1,1}x_{2,1} \equiv 0$ for $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$ are $(\mathbb{Z}_2, *)$ -identities of $N_k^{gr i}$. Hence, we have $N_k^{gr i} \in var^{gr i}(M^{gr i})$, for any $k \geq 2$.

Similarly, we consider the algebra $U_k^{gr i}$, for $k \geq 2$. We notice that, since

$$(U_k^{gr i})^{(0)} = \text{span}_F\{I, E, \dots, E^{k-2}\} \quad \text{and}$$

$$(U_k^{gr i})^{(1)} = \text{span}_F\{e_{12} + e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}\},$$

we have that $z_{1,0} \equiv 0$ and $x_{1,1}x_{2,1} \equiv 0$ for $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$ are $(\mathbb{Z}_2, *)$ -identities of $U_k^{gr i}$. Then, we also have $U_k^{gr i} \in var^{gr i}(M^{gr i})$, for any $k \geq 2$.

Let us start with the particular case $k = 2$.

Lemma 2.2.1. *For the $*$ -superalgebras $N_2^{gr i}$ and $U_2^{gr i}$ we have*

1. $Id^{gr i}(N_2^{gr i}) = \langle z_{1,0}, y_{1,1}, z_{1,1}z_{2,1} \rangle_{T_2^*}$,
2. $Id^{gr i}(U_2^{gr i}) = \langle z_{1,0}, z_{1,1}, y_{1,1}y_{2,1} \rangle_{T_2^*}$,
3. $c_n^{gr i}(N_2^{gr i}) = c_n^{gr i}(U_2^{gr i}) = 1 + n$.

Proof. Let us consider the algebra N_2^{gri} , the arguments are the same for U_2^{gri} . Let $I = \langle z_{1,0}, y_{1,1}, z_{1,1}z_{2,1} \rangle_{T_2^*}$. We can easily see that $I \subset Id^{gri}(N_2^{gri})$. We shall verify the opposite inclusion. Let $f \in Id^{gri}(N_2^{gri})$ be a multilinear polynomial. Since N_2^{gri} is a unitary algebra, we can assume f a proper polynomial.

If $\deg f \geq 2$ then $f \equiv 0$, modulo I . Now if $\deg f = 1$, so modulo I , we get $f = \alpha z_{1,1}$. By evaluating in $z_{1,1} = e_{12} - e_{34}$ we get $\alpha = 0$, then $f \in I$. Hence $I = Id^{gri}(N_2^{gri})$. Moreover, we have $\gamma_0^{gri}(N_2^{gri}) = \gamma_1^{gri}(N_2^{gri}) = 1$ then $c_n^{gri}(N_2^{gri}) = 1 + n$. $\square$

We can see, by the previous lemma, that U_2^{gri} is a commutative algebra with trivial involution and elementary grading induced by $\mathbf{g} = (0, 1, 1, 0) \in \mathbb{Z}_2^4$. On the other hand, N_2^{gri} is a commutative algebra with non-trivial involution and grading.

Next we describe the $(\mathbb{Z}_2, *)$ -identities and $*$ -graded codimensions of N_k^{gri} and U_k^{gri} , for any $k \geq 3$.

Lemma 2.2.2. [12, Theorem 4.4] *If $k \geq 3$, then*

1) $Id^{gri}(N_k^{gri}) = \langle z_{1,0}, x_{1,1}x_{2,1}, [y_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \rangle_{T_2^*}$, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$.

2) $c_n^{gri}(N_k^{gri}) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} 2j + \binom{n}{k-1} (k-1) \approx qn^{k-1}$, for some $q > 0$.

Proof. Let $I = \langle z_{1,0}, x_{1,1}x_{2,1}, [y_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \rangle_{T_2^*}$, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$. It is clear that $I \subset Id^{gri}(N_k^{gri})$. We shall prove the opposite inclusion. Let $f \in Id^{gri}(N_k^{gri})$ be a multilinear polynomial. Since N_k^{gri} is a unitary algebra, we can assume f a proper multilinear proper. By reducing f modulo I we get:

(i) By Remark 1.2.5, for any polynomial $f \in F\langle X | \mathbb{Z}_2, * \rangle$, we have $x_{1,1}fx_{2,1} \in I$. Since $[z_{1,1}, y_{1,0}] \in F\langle Y_1 \rangle$, it means that for any evaluation in $[z_{1,1}, y_{1,0}]$ we get an odd symmetric element, then if $\deg f \geq k$, we have $f \equiv 0$.

(ii) If $\deg f = k - 1$, so f is a linear combination of polynomials

$$[z_{i,1}, y_{i_1,0}, \dots, y_{i_{k-2},0}], \text{ for } i = 1, \dots, k-1 \text{ } i_1, \dots, i_{k-2}.$$

(iii) If $\deg f = s < k - 1$, so f is a linear combination of polynomials

$$[z_{i,1}, y_{i_1,0}, \dots, y_{i_{s-1},0}], [y_{j,1}, y_{j_1,0}, \dots, y_{j_{s-1},0}],$$

where $i = 1, \dots, s$, $i_1 < \dots < i_{s-1}$ and $j_1 < \dots < j_{s-1}$.

Hence module I , we may assume that for some $1 \leq s \leq k$

$$f = \sum_{i=1}^s \alpha_i [z_{i,1}, y_{i_1,0}, \dots, y_{i_{s-1},0}] + \sum_{i=1}^s \beta_i [y_{i,1}, y_{i_1,0}, \dots, y_{i_{s-1},0}].$$

Suppose that there exists i such that $\alpha_i \neq 0$. By making the evaluation in $y_{j,1} = 0$, for all $j = 1, \dots, s$, $z_{i,1} = e_{12} - e_{2k-1,2k}$, $z_{j,i} = 0$, for all $j \neq i$, $y_{i_m} = E$, for

all $m = 1, \dots, s-1$, we get $\alpha_i(e_{1,s+1} + (-1)^s e_{2k-s,2k}) = 0$, this implies that $\alpha_i = 0$, a contradiction. Then, $\alpha_i = 0$, for all $1 \leq i \leq s$.

Now, suppose that there exists j such that $\beta_j \neq 0$. By evaluating in $z_{t,i} = 0$, for all $t \neq j$, $y_{j,1} = e_{13} + e_{2k-2,2k}$, $y_{j_m} = E$, for all $m = 1, \dots, s-1$ we get the result $\beta_j(e_{1,s+2} + (-1)^{s+1} e_{2k-s-1,2k}) = 0$, this implies that $\beta_j = 0$, and this is a contradiction. Then, $\beta_j = 0$, for all $1 \leq j \leq s$.

Hence, we get $I = Id^{gri}(N_k^{gri})$ and we have the proper *-graded codimensions

$$\gamma_s^{gri}(N_k^{gri}) = \begin{cases} 0, & \text{if } s \geq k \\ s, & \text{if } s = k-1 \\ 2s, & \text{if } 1 \leq s < k-1 \\ 1, & \text{if } s = 0 \end{cases}$$

Then we conclude that $c_n^{gri}(N_k^{gri}) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} 2j + \binom{n}{k-1} (k-1)$.

□

Similarly to the previous lemma we can prove the following results about the $(\mathbb{Z}_2, *)$ -identities and *-graded codimensions of U_k^{gri} and $N_k^{gri} \oplus U_k^{gri}$, for $k \geq 2$.

Lemma 2.2.3. [12, Theorem 4.5] *If $k \geq 3$, then*

1) $Id^{gri}(U_k^{gri}) = \langle z_{1,0}, x_{1,1}x_{2,1}, [z_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \rangle_{T_2^*}$, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$.

2) $c_n^{gri}(U_k^{gri}) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} 2j + \binom{n}{k-1} (k-1) \approx qn^{k-1}$, for some $q > 0$.

Notice that if $t > k$ then $N_t^{gri} \oplus U_k^{gri} \sim_{T_2^*} N_t^{gri}$, on the other hand if $t < k$ so $N_t^{gri} \oplus U_k^{gri} \sim_{T_2^*} U_k^{gri}$.

Lemma 2.2.4. [12, Theorem 4.6] *If $k \geq 2$, then*

1) $Id^{gri}(N_k^{gri} \oplus U_k^{gri}) = \langle z_{1,0}, x_{1,1}x_{2,1}, [x_{1,1}, y_{1,0}, \dots, y_{k-1,0}] \rangle_{T_2^*}$, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$.

2) $c_n^{gri}(N_k^{gri} \oplus U_k^{gri}) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} 2j \approx qn^{k-1}$, for some $q > 0$.

Finally, for $k \geq 2$, we consider the algebra A_k^{gri} . We notice that

$$(A_k^{gri})^{(0)} = \text{span}_F\{e_{11} + e_{2k,2k}, E, \dots, E^{k-2}\} \quad \text{and}$$

$$(A_k^{gri})^{(1)} = \text{span}_F\{e_{12}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}, e_{2k-1,2k}\},$$

then we also have that $z_{1,0} \equiv 0$ and $x_{1,1}x_{2,1} \equiv 0$ for $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$ are $(\mathbb{Z}_2, *)$ -identities of U_k^{gri} . Hence, for any $k \geq 2$, $A_k^{gri} \in \text{var}^{gri}(M^{gri})$.

The result about the $(\mathbb{Z}_2, *)$ -identities and *-graded codimensions of A_k^{gri} is the following.

Lemma 2.2.5. [12, Theorem 5.1] Let $k \geq 2$. Then

1) $Id^{gri}(A_k^{gri}) = \langle z_{1,0}, x_{1,1}x_{2,1}, y_{1,0} \cdots y_{k-1,0}x_{1,1}y_{k,0} \cdots y_{2k-2,0} \rangle_{T_2^*}$ where $x_i = y_i$ or $x_i = z_i$, for $i = 1, 2$.

2) $c_n^{gri}(A_k^{gri}) = 1 + 4 \sum_{j=0}^{k-2} \binom{n}{j} (n-j) \approx qn^{k-1}$, for some $q > 0$.

Proof. Let $R = \langle z_{1,0}, x_{1,1}x_{2,1}, y_{1,0} \cdots y_{k-1,0}x_{1,1}y_{k,0} \cdots y_{2k-2,0} \rangle_{T_2^*}$ where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$. We have $R \subset Id^{gri}(A_k^{gri})$ and we shall verify the opposite inclusion. Let $f \in Id^{gri}(A_k^{gri})$, we may assume f a multilinear polynomial of degree n . By Remark 1.2.5, we have $x_{1,1}fx_{2,1} \in R$. Then, modulo R , we have that f is a linear combination of the polynomials

$$y_{1,0} \cdots y_{n,0}, \quad y_{i_1,0} \cdots y_{i_r,0} y_{l,1} y_{j_1,0} \cdots y_{j_s,0}, y_{p_1,0} \cdots y_{p_u,0} z_{t,1} y_{q_1,0} \cdots y_{q_v,0}, \quad (2.2.1)$$

with $r + s = u + v = n - 1$, $1 \leq l, t \leq n$, $i_1 < \cdots < i_r$, $j_1 < \cdots < j_s$, $p_1 < \cdots < p_u$ and $q_1 < \cdots < q_v$.

So, we write f as a linear combination of the polynomials in (2.2.1)

$$\begin{aligned} f = & \delta y_{1,0} \cdots y_{n,0} + \sum_{\substack{r < k-1 \\ \text{or} \\ s < k-1}} \sum_{I, J, l} \alpha_{I, J, l} y_{i_1,0} \cdots y_{i_r,0} y_{l,1} y_{j_1,0} \cdots y_{j_s,0} \\ & + \sum_{\substack{u < k-1 \\ \text{or} \\ v < k-1}} \sum_{P, Q, t} \beta_{P, Q, t} y_{p_1,0} \cdots y_{p_u,0} z_{t,1} y_{q_1,0} \cdots y_{q_v,0}, \end{aligned}$$

where $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_s\}$, $P = \{p_1, \dots, p_u\}$ and $Q = \{q_1, \dots, q_v\}$.

First, suppose $\delta \neq 0$. By making the evaluation $y_{i,0} = e_{11} + e_{2k,2k}$ and $y_{l,1} = z_{t,1} = 0$, for all $1 \leq i, l, t \leq n$, we get $\delta(e_{11} + e_{2k,2k}) = 0$, a contradiction. So we must have $\delta = 0$.

Suppose $\alpha_{I, J, l} \neq 0$, for some fixed $r < k - 1$, I, J, l . By making the evaluation $z_{t,1} = 0$, for all $1 \leq t \leq n$, $y_{j,1} = 0$, for all $j \neq l$, $y_{l,1} = e_{12} + e_{2k-1,2k}$, $y_{i_1,0} = \cdots = y_{i_r,0} = E$ and $y_{j_1,0} = \cdots = y_{j_s,0} = e_{11} + e_{2k,2k}$ we get that $\alpha_{I, J, l} e_{2k-r-1,2k} + \alpha_{J, I, l} e_{1,r+2} = 0$ implies $\alpha_{I, J, l} = \alpha_{J, I, l} = 0$, a contradiction. Similarly, if $\alpha_{I, J, l} \neq 0$, for some fixed $s < k - 1$, I, J, l . By making the evaluation $z_{t,1} = 0$, for all $1 \leq t \leq n$, $y_{j,1} = 0$, for all $j \neq l$, $y_{l,1} = e_{12} + e_{2k-1,2k}$, $y_{i_1,0} = \cdots = y_{i_r,0} = e_{11} + e_{2k,2k}$ and $y_{j_1,0} = \cdots = y_{j_s,0} = E$ we get $\alpha_{I, J, l} = 0$, a contradiction. Then we must have $\alpha_{I, J, l} = 0$, for all I, J, l .

In a similar way, we may prove that the coefficients $\beta_{P, Q, t} = 0$, for all P, Q, t .

Hence, we conclude that $I = Id^{gri}(A_k^{gri})$ and the polynomials in (2.2.1) are linearly independent modulo $Id^{gri}(A_k^{gri})$. By counting those polynomials, we have

$$c_n^{gri}(A_k^{gri}) = 1 + 4 \sum_{j=0}^{k-2} \binom{n}{j} (n-j).$$

□

Next we shall prove that N_k^{gri} , U_k^{gri} and A_k^{gri} generate minimal varieties of polynomial growth.

Remark 2.2.6. In [12, Corollary 4.3], Ioppolo and La Mattina proved that if $A \in \text{var}^{gri}(M^{gri})$ is a *-superalgebra over an algebraically closed field F , then $\text{var}^{gri}(A) = \text{var}^{gri}(B)$, for some finite dimensional *-superalgebra B .

As a consequence of this result and of Theorem 1.4.4 we have that if $A \in \text{var}^{gri}(M^{gri})$ has polynomial growth then $A \sim_{T_2^*} (B_1 \oplus \dots \oplus B_m)$, for some finite dimensional *-superalgebras B_i , $1 \leq i \leq m$ such that $\dim_{J(B_i)} \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. This means that either $B_i \cong J(B_i)$ is nilpotent or $B_i \cong F + J(B_i)$.

Remark 2.2.7. Let $A = F + J_{00} + J_{10} + J_{01} + J_{11}$ be a *-superalgebra. If A satisfies the ordinary identity $[x_1, \dots, x_t]$ for some $t \geq 2$, then $J_{10} = J_{01} = 0$.

Proof. The proof is trivial, just notice that $[J_{10}, \underbrace{F, \dots, F}_{t-1}] = J_{10}$ and $[J_{01}, \underbrace{F, \dots, F}_{t-1}] = J_{01}$. Hence $J_{10} = J_{01} = 0$ and $A = (F + J_{11}) \oplus J_{00}$. $\square$

Theorem 2.2.8. [12, Theorem 4.7 and Theorem 4.8] For all $k \geq 2$, N_k^{gri} and U_k^{gri} generate minimal varieties of polynomial growth.

Proof. We shall prove for N_k^{gri} and the proof of the result is similar for U_k^{gri} .

We start by considering $k = 2$. Let $A \in \text{var}^{gri}(N_2^{gri})$ such that $c_n^{gri}(A) \approx qn$, for some $q > 0$. By Remark 2.2.6, we may assume $A = B_1 \oplus \dots \oplus B_m$ such that $\dim_F B_i < \infty$ and $\dim_F \frac{B_i}{J(B_i)} \leq 1$. Since

$$c_n^{gri}(A) \leq c_n^{gri}(B_1) + \dots + c_n^{gri}(B_m),$$

then there exists B_i such that $c_n^{gri}(B_i) \approx bn$, for some $b > 0$. We have that N_2^{gri} satisfies the ordinary identity $[x_1, x_2]$, then by Remark 2.2.7, we get $J_{10}(B_i) = 0$ and $J_{01}(B_i) = 0$. Hence $F + J(B_i) = (F + J_{11}(B_i)) \oplus J_{00}(B_i)$ and, for n large enough, we have $c_n^{gri}(F + J(B_i)) = c_n^{gri}(F + J_{11}(B_i))$. In order to show that $A \sim_{T_2^*} N_2^{gri}$, it is enough to verify that $F + J_{11}(B_i) \sim_{T_2^*} N_2^{gri}$, so we assume that A is a unitary algebra.

Since $c_n^{gri}(A) \approx bn$, we get $c_n^{gri}(A) = 1 + n\gamma_1^{gri}(A)$, with $\gamma_1^{gri}(A) \neq 0$. Since $Id^{gri}(N_2^{gri}) \subseteq Id^{gri}(A)$, we have $\gamma_1^{gri}(A) \leq \gamma_1^{gri}(N_2^{gri})$. By Lemma 2.2.1, we conclude that $\gamma_1^{gri}(A) = \gamma_1^{gri}(N_2^{gri}) = 1$. Hence $c_n^{gri}(A) = c_n^{gri}(N_2^{gri})$, for all $n = 0, 1, 2, \dots$. Then we have $A \sim_{T_2^*} N_2^{gri}$.

Now we consider $k \geq 3$. Let $A \in \text{var}^{gri}(N_k^{gri})$ such that $c_n^{gri}(A) \approx qn^{k-1}$, for some $q > 0$, we shall prove that $A \sim_{T_2^*} N_k^{gri}$. By using the same arguments of the first part, there exists B_i such that $c_n^{gri}(B_i) \approx bn^{k-1}$, for some $b > 0$. Since N_k^{gri} satisfies the ordinary identity $[x_1, \dots, x_k]$, by Remark 2.2.7 we get $F + J(B_i) = (F + J_{11}(B_i)) \oplus J_{00}(B_i)$ and, for n large enough, we have $c_n^{gri}(F + J(B_i)) = c_n^{gri}(F + J_{11}(B_i))$. Thus we may assume that A is a unitary algebra, without loss of generality.

Since $c_n^{gri}(A) \approx bn^{k-1}$, we get $c_n^{gri}(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^{gri}(A)$ and $\gamma_i^{gri}(A) \neq 0$ for all $0 \leq i \leq k-1$, by Lemma 1.3.7.

Now, since $Id^{gri}(N_k^{gri}) \subset Id^{gri}(A)$, we have that $\frac{\Gamma_n^{gri}}{\Gamma_n^{gri} \cap Id^{gri}(A)}$ is isomorphic to a quotient module of $\frac{\Gamma_n^{gri}}{\Gamma_n^{gri} \cap Id^{gri}(N_k^{gri})}$. Then, if $\psi_i^{gri}(A) = \sum_{\langle \lambda \rangle \vdash i} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}$ and $\psi_i^{gri}(N_k^{gri}) = \sum_{\langle \lambda \rangle \vdash i} m'_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}$ are the i -th proper $*$ -graded cocharacters of A and N_k^{gri} , respectively, we must have $m_{\langle \lambda \rangle} \leq m'_{\langle \lambda \rangle}$, for all $\langle \lambda \rangle \vdash i$ and $0 \leq i \leq k-1$.

For $\langle \lambda \rangle = (\emptyset, \emptyset, \emptyset, \emptyset) \vdash 0$, $\langle \lambda \rangle = (\emptyset, (1), \emptyset, \emptyset) \vdash 1$ and $\langle \lambda \rangle = (\emptyset, \emptyset, \emptyset, (1)) \vdash 1$, we have $m_{\langle \lambda \rangle} = m'_{\langle \lambda \rangle} = 1$. Now for each $i = 2, \dots, k-2$, let $f_1 = [z_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{i-1}]$ and $f_2 = [y_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{i-1}]$ be the highest weight vectors corresponding to the partitions $\langle \lambda \rangle = ((i-1), \emptyset, \emptyset, (1))$ and $\langle \lambda \rangle = ((i-1), (1), \emptyset, \emptyset)$, respectively. We have $f_1, f_2 \notin Id^{gri}(N_k^{gri})$, for all $i = 1, \dots, k-2$. Moreover, we have that $[z_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{k-2}]$ is a highest weight vector corresponding to $\langle \lambda \rangle = ((k-2), \emptyset, \emptyset, (1))$ which is not a $*$ -graded identity of N_k^{gri} .

Then, for all $i = 1, \dots, k-2$ we have that $\chi_{((i-1), \emptyset, \emptyset, (1))}$, $\chi_{((i-1), (1), \emptyset, \emptyset)}$ and $\chi_{((k-2), \emptyset, \emptyset, (1))}$ appear in the decomposition of the i -th proper $*$ -graded cocharacters of N_k^{gri} with non-zero multiplicities. Since

$$\gamma_{k-1}^{gri}(N_k^{gri}) = k-1 = \deg \chi_{((k-2), \emptyset, \emptyset, (1))} \quad \text{and}$$

$$\gamma_i^{gri}(N_k^{gri}) = 2i = \deg \chi_{((i-1), \emptyset, \emptyset, (1))} + \deg \chi_{((i-1), (1), \emptyset, \emptyset)},$$

for all $1 \leq i \leq k-2$, we obtain $\psi_i^{gri}(N_k^{gri}) = \chi_{((i-1), \emptyset, \emptyset, (1))} + \chi_{((i-1), (1), \emptyset, \emptyset)}$, for all $1 \leq i \leq k-2$, and $\psi_{k-1}^{gri}(N_k^{gri}) = \chi_{((k-2), \emptyset, \emptyset, (1))}$.

Hence, since $\gamma_{k-1}^{gri}(A) \neq 0$, we also get $\psi_{k-1}^{gri}(A) = \chi_{((k-2), \emptyset, \emptyset, (1))}$. Moreover, for all $1 \leq i \leq k-2$, we must have $\psi_i^{gri}(A) = \chi_{((i-1), \emptyset, \emptyset, (1))} + \chi_{((i-1), (1), \emptyset, \emptyset)}$. In fact, suppose $\psi_i^{gri}(A) = \chi_{((i-1), (1), \emptyset, \emptyset)}$, for some $1 \leq i \leq k-2$, this implies that $[z_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{i-1}] \in Id^{gri}(A)$ and so $[z_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{k-1}] \in Id^{gri}(A)$, thus $\gamma_{k-1}^{gri}(A) = 0$, a contradiction. Similarly, if $\psi_i^{gri}(A) = \chi_{((i-1), \emptyset, \emptyset, (1))}$ for some $1 \leq i \leq k-2$, then $[y_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{i-1}] \in Id^{gri}(A)$. Now, notice that $[z_{1,1}, y_{1,0}] \in F\langle Y_1 \rangle$, so we also have $[\underbrace{z_{1,1}, y_{1,0}, \dots, y_{1,0}}_{i-1}] \in Id^{gri}(A)$ and $\gamma_{k-1}^{gri}(A) = 0$, a contradiction.

Thus we must have $\psi_i^{gri}(A) = \chi_{((i-1), \emptyset, \emptyset, (1))} + \chi_{((i-1), (1), \emptyset, \emptyset)}$, for all $1 \leq i \leq k-2$, and $\psi_{k-1}^{gri}(A) = \chi_{((k-2), \emptyset, \emptyset, (1))}$. Hence, for all $n \geq 1$ we get

$$c_n^{gri}(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^{gri}(A) = 1 + \sum_{j=1}^{k-2} \binom{n}{j} 2j + \binom{n}{k-1} (k-1) = c_n^{gri}(N_k^{gri}),$$

and, since $Id^{gri}(N_k^{gri}) \subseteq Id^{gri}(A)$, we conclude that $Id^{gri}(N_k^{gri}) = Id^{gri}(A)$.

□

Next we prove that A_k^{gri} generates a minimal variety of polynomial growth.

Remark 2.2.9. Let $A = F + J \in var^{gri}(A_k^{gri})$ with $J = J_{00} \oplus J_{10} \oplus J_{01} \oplus J_{11}$. Then $J_{11}^{(1)} = 0$.

Proof. In fact, since $y_{1,0} \cdots y_{k-1,0} x_{1,1} y_{k,0} \cdots y_{2k-2,0} \in Id^{gri}(A_k^{gri})$, where $x_{1,1} = y_{1,1}$ or $x_{1,1} = z_{1,1}$, we notice that

$$\underbrace{F \cdots F}_{k-1} (J_{11}^{(1)})^+ \underbrace{F \cdots F}_{k-1} = (J_{11}^{(1)})^+ = 0 \quad \text{and} \quad \underbrace{F \cdots F}_{k-1} (J_{11}^{(1)})^- \underbrace{F \cdots F}_{k-1} = (J_{11}^{(1)})^- = 0.$$

Hence $J_{11}^{(1)} = 0$.

□

Theorem 2.2.10. [12, Theorem 5.2] For all $k \geq 2$, A_k^{gri} generates a minimal variety of polynomial growth.

Proof. Let $A = F + J$ be a *-superalgebra with $J_{10} \neq 0$ (hence $J_{01} \neq 0$) such that $A \in var^{gri}(A_k^{gri})$ with $c_n^{gri}(A) \approx qn^{k-1}$, for some $q > 0$. We claim that $A \sim_{T_2^*} A_k^{gri}$.

By the previous remark, we have $A = F + J_{00} \oplus J_{10} \oplus J_{01} \oplus J_{11}$ with $J_{11}^{(1)} = 0$. Suppose that $J_{10}((J_{00}^{(0)})^+)^{k-2} = 0$, it also says that $((J_{00}^{(0)})^+)^{k-2} J_{01} = 0$. We claim that if $J^m = 0$ then for all $n \geq m$, the polynomials

$$f_1 = y_{i_1,0} \cdots y_{i_t,0} y_{1,0} \cdots y_{k-2,0} y_{1,1} y_{k,0} \cdots y_{2k-4,0} y_{j_1,0} \cdots y_{j_l,0},$$

$$f_2 = y_{i_1,0} \cdots y_{i_t,0} y_{1,0} \cdots y_{k-2,0} z_{1,1} y_{k,0} \cdots y_{2k-4,0} y_{j_1,0} \cdots y_{j_l,0},$$

with $t + l + 2k - 3 = n$ are $(\mathbb{Z}_2, *)$ -identities of A .

In fact, since f_1 and f_2 are multilinear polynomials, it is enough to evaluate the variables on a basis of A which is the union of a basis of $J_{00}, J_{10}, J_{01}, J_{11}$ and 1_F . Since $J^m = 0$, if we evaluate all variables in J , we get $f_i \equiv 0, i = 1, 2$. So, at least one variable must be evaluated in 1_F . Now, since $(J_{11}^{(1)})^+ = (J_{11}^{(1)})^- = 0$, we need to evaluate the variables $y_{1,1}$ and $z_{1,1}$ in $J_{10} + J_{01}$. We can see that, since $J_{10}((J_{00}^{(0)})^+)^{k-2} = 0$ and $((J_{00}^{(0)})^+)^{k-2} J_{01} = 0$, we get $f_i \equiv 0, i = 1, 2$, for all evaluation in A . Thus, $f_1, f_2 \in Id^{gri}(A)$.

Let $I \subseteq Id^{gri}(A_k^{gri})$ be the T_2^* -ideal generated by f_1, f_2 plus the generators of the T_2^* -ideal $Id^{gri}(A_k^{gri})$. For any $n \geq m$, the following set of polynomials

$$\{y_{1,0} \cdots y_{n,0}\} \cup \{y_{i_1,0} \cdots y_{i_r,0} y_{l,1} y_{j_1,0} \cdots y_{j_s,0}, y_{i_1,0} \cdots y_{i_r,0} z_{l,1} y_{j_1,0} \cdots y_{j_s,0}\}$$

where $r < k - 2$ or $s < k - 2$, $i_1 < \dots < i_r$, $j_1 < \dots < j_s$ and $1 \leq l \leq n$, generate $P_n^{gri}(\text{mod } P_n^{gri} \cap Id^{gri}(I))$. Thus

$$c_n^{gri}(A) \leq 1 + 4 \sum_{j=0}^{k-3} \binom{n}{j} (n-j) \approx bn^{k-2},$$

for some $b > 0$, a contradiction.

Hence, we must have $J_{10}((J_{00}^{(0)})^+)^{k-2} \neq 0$ and also $((J_{00}^{(0)})^+)^{k-2} J_{01} \neq 0$. Let $a \in J_{10}$, $b_1, \dots, b_{k-2} \in (J_{00}^0)^+$ be such that $ab_1 \cdots b_{k-2} \neq 0$, then we also have $b_{k-2}^* \cdots b_1^* a^* \neq 0$ with $a^* \in J_{01}$, $b_1^*, \dots, b_{k-2}^* \in (J_{00}^0)^+$.

Let $f \in Id^{gri}(A)$ be a multilinear polynomial of degree n . By Lemma 2.2.5, we can write f , modulo $Id^{gri}(A_k^{gri})$, like:

$$\begin{aligned} f = & \delta y_{1,0} \cdots y_{n,0} + \sum_{\substack{r < k-1 \\ \text{or} \\ s < k-1}} \sum_{I, J, l} \alpha_{I, J, l} y_{i_1,0} \cdots y_{i_r,0} y_{l,1} y_{j_1,0} \cdots y_{j_s,0} \\ & + \sum_{\substack{u < k-1 \\ \text{or} \\ v < k-1}} \sum_{P, Q, t} \beta_{P, Q, t} y_{p_1,0} \cdots y_{p_u,0} z_{t,1} y_{q_1,0} \cdots y_{q_v,0}, \end{aligned}$$

where $I = \{i_1, \dots, i_r\}$, $J = \{j_1, \dots, j_s\}$, $P = \{p_1, \dots, p_u\}$ and $Q = \{q_1, \dots, q_v\}$.

By evaluating $y_{i,0} = e_{11} + e_{2k,2k}$ and $y_{l,1} = z_{t,1} = 0$, for all $1 \leq i, l, t \leq n$, we get $\delta(e_{11} + e_{2k,2k}) = 0$. So, we must have $\delta = 0$.

Fixed $s < k - 1$, I, J, l . By making the evaluation $z_{t,1} = 0$, for $1 \leq t \leq n$, $y_{j,1} = 0$, for all $j \neq l$, $y_{l,1} = a + a^*$, $y_{j_p,0} = b_p$, for $1 \leq p \leq s$ and $y_{i_m,0} = 1_F$, for $1 \leq m \leq r$, we get $\alpha_{I, J, l} ab_1 \cdots b_s + \alpha_{J, I, l} b_1 \cdots b_s a^* = 0$. Since $ab_1 \cdots b_s \in J_{10}$ and $b_1 \cdots b_s a^* \in J_{01}$ are non-zero and linearly independent, this implies $\alpha_{I, J, l} = \alpha_{J, I, l} = 0$. Similarly, fixed $r < k - 1$, I, J, l . By making the evaluation $z_{t,1} = 0$, for all $1 \leq t \leq n$, $y_{j,1} = 0$, for all $j \neq l$, $y_{l,1} = a + a^*$, $y_{j_p,0} = 1_F$, for $1 \leq p \leq s$ and $y_{i_{r-m},0} = b_{m+1}^*$, for $0 \leq m \leq r - 1$, we get $\alpha_{I, J, l} b_r^* \cdots b_1^* a^* + \alpha_{J, I, l} ab_r^* \cdots b_1^* = 0$. Again, since $ab_r^* \cdots b_1^* \in J_{10}$ and $b_r^* \cdots b_1^* a^* \in J_{01}$ are non-zero and linearly independent, this also implies $\alpha_{I, J, l} = 0$. Then we must have $\alpha_{I, J, l} = 0$, for all I, J, l .

Now fixed $v < k - 1$, P, Q, t . By making the evaluation $z_{j,1} = 0$, for all $j \neq t$, $z_{t,1} = a - a^*$, $y_{q_i,0} = b_i$, for $1 \leq i \leq v$ and $y_{p_m,0} = 1_F$, for $1 \leq m \leq u$, we get $\beta_{P, Q, t} ab_1 \cdots b_v + \beta_{P, Q, t} b_1 \cdots b_v a^* = 0$, then it implies $\beta_{P, Q, t} = \beta_{P, Q, t} = 0$. Similarly, fixed $u < k - 1$, I, J, l . By making the evaluation $z_{t,1} = 0$, for all $j \neq t$, $z_{t,1} = a - a^*$, $y_{q_i,0} = 1_F$, for $1 \leq i \leq v$ and $y_{p_{u-m},0} = b_{m+1}^*$, for $0 \leq m \leq u - 1$, we get $\beta_{P, Q, t} b_u^* \cdots b_1^* a^* + \beta_{P, Q, t} ab_u^* \cdots b_1^* = 0$, then it implies $\beta_{P, Q, t} = \beta_{P, Q, t} = 0$. Then we must have $\beta_{P, Q, t} = 0$, for all P, Q, t .

Thus, we conclude that $I = Id^{gri}(A_k^{gri})$ and, hence, $Id^{gri}(A) = Id^{gri}(A_k^{gri})$.

Now, we consider $A \in var^{gri}(A_k^{gri})$ such that $c_n^{gri}(A) \approx qn^{k-1}$, for some $q > 0$, in general case. By Remark 2.2.6 we may assume $A = B_1 \oplus \cdots \oplus B_m$, where $B_1, \dots, B_m$ are finite dimensional *-superalgebras such that $\dim_F \frac{B_i}{J(B_i)} \leq 1$. since

$$c_n^{gri}(A) \leq c_n^{gri}(B_1) + \cdots + c_n^{gri}(B_m),$$

then there exists B_i such that $c_n^{gri}(B_i) \approx bn^{k-1}$, for some $b > 0$, with $B_i \cong F + J(B_i)$. Hence, since $B_i \in var^{gri}(A_k^{gri})$ we get $B_i \sim_{T_2^*} A_k^{gri}$, according to the result of the first part of the proof.

Thus, we have

$$var^{gri}(A_k^{gri}) = var^{gri}(B_i) \subseteq var^{gri}(A) \subseteq var^{gri}(A_k^{gri}).$$

Hence, we conclude that $A \sim_{T_2^*} A_k^{gri}$. $\square$

The next result presents the classification up to T_2^* -equivalence of all unitary *-superalgebras that generate a proper subvariety of $var^{gri}(M^{gri})$.

Lemma 2.2.11. *Let $A \in var M^{gri}$ be a unitary *-superalgebra that generates a proper subvariety of $var^{gri}(M^{gri})$. Then either $A \sim_{T_2^*} C$ or $A \sim_{T_2^*} N_k^{gri}$ or $A \sim_{T_2^*} U_k^{gri}$ or $A \sim_{T_2^*} N_k^{gri} \oplus U_k^{gri}$, for some $k \leq 2$, where C is a commutative algebra with trivial grading and trivial involution.*

Proof. By Corollary 1.4.10, M^{gri} has almost polynomial growth, then if A generates a proper subvariety of $var^{gri}(M^{gri})$ so we have, for some $k \geq 1$,

$$c_n^{gri}(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^{gri}(A) \approx an^{k-1}.$$

If $k = 1$, then $\Gamma_1 \subseteq Id^{gri}(A)$. Thus we have $z_{1,0} \equiv y_{1,1} \equiv z_{1,1} \equiv 0$ in A . Hence $A \sim_{T_2^*} C$, where C is a commutative algebra with trivial grading and trivial involution.

If we assume $k = 2$, then we have $\gamma_2^{gri}(A) = 0$ and so $\Gamma_2^{gri} \subseteq Id^{gri}(A)$. Since $A \in var^{gri}(M^{gri})$, we have $z_{1,0} \equiv 0$ in A . So we have three cases to consider:

(i) $y_{1,1} \in Id^{gri}(A)$ and $z_{1,1} \notin Id^{gri}(A)$. Then $Id^{gri}(N_2^{gri}) \subseteq Id^{gri}(A)$, by Lemma 2.2.1. Since N_2^{gri} generates a minimal variety and $c_n^{gri}(A) \approx an$, by Theorem 2.2.8, thus we have $A \sim_{T_2^*} N_2^{gri}$.

(ii) $y_{1,1} \notin Id^{gri}(A)$ and $z_{1,1} \in Id^{gri}(A)$. We get $Id^{gri}(U_2^{gri}) \subseteq Id^{gri}(A)$, by Lemma 2.2.1. Since U_2^{gri} generates a minimal variety and $c_n^{gri}(A) \approx an$, by Theorem 2.2.8, thus we have $A \sim_{T_2^*} U_2^{gri}$.

(iii) $y_{1,1}, z_{1,1} \notin Id^{gri}(A)$. Since $c_n^{gri}(A) \approx an$ and $\Gamma_2^{gri} \subset Id^{gri}(A)$, in particular $[y_{1,1}, y_{1,0}]$ and $[z_{1,1}, y_{1,0}]$ are $(\mathbb{Z}_2, *)$ -identities of A . Then, by Lemma 2.2.4, we get $Id^{gri}(N_2^{gri} \oplus U_2^{gri}) \subseteq Id^{gri}(A)$. It is clear that $\gamma_1^{gri}(A) = 2$ and so $c_n^{gri}(A) = 1 + 2n = c_n^{gri}(N_2^{gri} \oplus U_2^{gri})$, for all $n \geq 0$. Hence $A \sim_{T_2^*} N_2^{gri} \oplus U_2^{gri}$.

Suppose now $k \geq 3$. Since $\gamma_{k-1}^{gri}(A) \neq 0$, at least one of the polynomials $[y_{1,1}, y_{1,0}, \dots, y_{k-2,0}]$ and $[z_{1,1}, y_{1,0}, \dots, y_{k-2,0}]$ is not a $(\mathbb{Z}_2, *)$ -identity of A .

First, if $[y_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \in Id^{gri}(A)$ and $[z_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \notin Id^{gri}(A)$, then $Id^{gri}(N_k^{gri}) \subseteq Id^{gri}(A)$, by Lemma 2.2.2. Since N_k^{gri} generates a minimal variety, by Lemma 2.2.8 and $c_n^{gri}(A) \approx an^{k-1}$, thus we have $A \sim_{T_2^*} N_k^{gri}$.

Now, if $[y_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \notin Id^{gri}(A)$ and $[z_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \in Id^{gri}(A)$, then it implies that $Id^{gri}(U_k^{gri}) \subseteq Id^{gri}(A)$, by Lemma 2.2.3. Since U_k^{gri} generates a minimal variety, by Lemma 2.2.8 and $c_n^{gri}(A) \approx an^{k-1}$, then we conclude $A \sim_{T_2^*} U_k^{gri}$.

Finally, suppose $[y_{1,1}, y_{1,0}, \dots, y_{k-2,0}], [z_{1,1}, y_{1,0}, \dots, y_{k-2,0}] \notin Id^{gri}(A)$. Since $\gamma_k^{gri}(A) = 0$, then all proper polynomial of degree k lies in $Id^{gri}(A)$, in particular $[y_{1,1}, y_{1,0}, \dots, y_{k-1,0}], [z_{1,1}, y_{1,0}, \dots, y_{k-1,0}] \in Id^{gri}(A)$. By Lemma 2.2.4, we get $Id^{gri}(N_k^{gri} \oplus U_k^{gri}) \subseteq Id^{gri}(A)$. Let us prove the opposite inclusion.

For each $i = 1, \dots, k-1$, let $f_1 = [z_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{i-1}]$ and $f_2 = [y_{1,1}, \underbrace{y_{1,0}, \dots, y_{1,0}}_{i-1}]$

be the highest weight vectors corresponding to the partitions $\langle \lambda \rangle_1 = ((i-1), \emptyset, \emptyset, (1))$ and $\langle \lambda \rangle_2 = ((i-1), \emptyset, (1), \emptyset)$, respectively. Since $f_1, f_2 \notin Id^{gri}(N_k^{gri} \oplus U_k^{gri})$, we have that $\chi_{\langle \lambda \rangle_1}$ and $\chi_{\langle \lambda \rangle_2}$ effectively appear in the decomposition of the i -th proper $\ast$ -graded cocharacters of $N_k^{gri} \oplus U_k^{gri}$ with non-zero multiplicities. Now, since $\gamma_i^{gri}(N_k^{gri} \oplus U_k^{gri}) = 2i = \deg \chi_{\langle \lambda \rangle_1} + \deg \chi_{\langle \lambda \rangle_2}$, we have for all $i = 1, \dots, k-1$

$$\psi_i^{gri}(N_k^{gri} \oplus U_k^{gri}) = \chi_{((i-1), \emptyset, \emptyset, (1))} + \chi_{((i-1), \emptyset, (1), \emptyset)}.$$

If $\psi_i^{gri}(A) = \sum_{\langle \lambda \rangle \vdash i} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}$ and $\psi_i^{gri}(N_k^{gri} \oplus U_k^{gri}) = \sum_{\langle \lambda \rangle \vdash i} m'_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}$ are respectively the i -th proper $\ast$ -graded cocharacters of A and $N_k^{gri} \oplus U_k^{gri}$, so we must have for all $\langle \lambda \rangle \vdash i$ and $0 \leq i \leq k-1$, $m_{\langle \lambda \rangle} \leq m'_{\langle \lambda \rangle}$. Moreover, we must have

$$\psi_i^{gri}(A) = \chi_{((i-1), \emptyset, \emptyset, (1))} + \chi_{((i-1), \emptyset, (1), \emptyset)},$$

for all $i = 1, \dots, k-1$, since $[y_{1,1}, y_{1,0}, \dots, y_{k-2,0}]$ and $[z_{1,1}, y_{1,0}, \dots, y_{k-2,0}]$ are not $(\mathbb{Z}_2, \ast)$ -identities of A . Then, for all $n \geq 0$, we get

$$c_n^{gri}(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^{gri}(A) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} 2j = c_n^{gri}(N_k^{gri} \oplus U_k^{gri}).$$

Hence $Id^{gri}(N_k^{gri} \oplus U_k^{gri}) = Id^{gri}(A)$ and so $A \sim_{T_2^*} N_k^{gri} \oplus U_k^{gri}$. $\square$

The classification of all proper subvarieties of the variety generated by M^{gri} is given by the following.

Theorem 2.2.12. [12, Theorem 5.3] *Let A be a $\ast$ -superalgebra such that $var^{gri}(A) \subsetneq var^{gri}(M^{gri})$. Then A is T_2^* -equivalent to one of the following $\ast$ -superalgebras: N , $C \oplus N$, $N_t^{gri} \oplus N$, $U_t^{gri} \oplus N$, $A_k^{gri} \oplus N$, $U_t^{gri} \oplus N_t^{gri} \oplus N$, $U_t^{gri} \oplus A_k^{gri} \oplus N$, $N_t^{gri} \oplus A_k^{gri} \oplus N$, $U_t^{gri} \oplus N_t^{gri} \oplus A_k^{gri} \oplus N$ for some $k, t \geq 2$, where N is a nilpotent $\ast$ -superalgebra and C is a commutative algebra with trivial grading and trivial involution.*

Corollary 2.2.13. *A $\ast$ -superalgebra $A \in var^{gri}(M^{gri})$ generates a minimal variety of polynomial growth if and only if either $A \sim_{T_2^*} N_k^{gri}$ or $A \sim_{T_2^*} U_k^{gri}$ or $A \sim_{T_2^*} A_k^{gri}$, for some $k \geq 2$.*

2.3 The *-graded cocharacter of the minimal subvarieties

In this section, we explicit the sequences of *-graded cocharacters and of *-graded colengths of the minimal varieties $var^{gri}(A) \subseteq var^{gri}(M_*)$ and $var^{gri}(A) \subseteq var^{gri}(M^{gri})$.

The results about the minimal subvarieties lying in $var^{gri}(M_*)$ are in a joint work with La Mattina and Vieira [23] which was recently submitted for publication in the language of *-varieties. Here we restate such results in *-superalgebra language.

We prove all theorems by using induction on k , so for each class of algebras $N_{k,*}$, $U_{k,*}$ and $A_{k,*}$ we start with a lemma about the sequence of the *-graded cocharacters in a particular case.

We start by the study of *-cocharacters and of *-colengths of the minimal varieties $var^{gri}(A_{k,*})$, for $k \geq 2$.

Lemma 2.3.1. *For the *-superalgebra $A_{2,*}$, we have*

1. $\chi_n^{gri}(A_{2,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + 2\chi_{(n-1,1),\emptyset,\emptyset,\emptyset} + 2\chi_{(n-1),\emptyset,(1),\emptyset}$,
2. $l_n^{gri}(A_{2,*}) = 5$.

Proof. By Lemma 2.1.6, it is known that $c_n^{gri}(A_{2,*}) = 4n - 1$ and notice that

$$d_{(n),\emptyset,\emptyset,\emptyset} + 2d_{(n-1),\emptyset,(1),\emptyset} + 2d_{(n-1,1),\emptyset,\emptyset,\emptyset} = 1 + 2n + 2(n-1) = c_n^{gri}(A_{2,*}).$$

Then, since $m_{(n),\emptyset,\emptyset,\emptyset} = 1$, if we find two linearly independent highest weight vectors for each pair of partitions $((n-1), \emptyset, (1), \emptyset)$ and $((n-1, 1), \emptyset, \emptyset, \emptyset)$ which are not identities of $A_{2,*}$, we may conclude that $\chi_n^{gri}(A_{2,*})$ has the wished decomposition.

In fact, let us consider the following highest weight vectors associated to the multipartition $((n-1), \emptyset, (1), \emptyset)$ and their corresponding multitableaux:

$$(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline \end{array}, \emptyset, \begin{array}{|c|} \hline n \\ \hline \end{array}, \emptyset) \text{ and } f_1 = y_{1,0}^{n-1} z_{1,0} \quad (2.3.1)$$

$$(\begin{array}{|c|c|c|c|c|} \hline 2 & 3 & \cdots & n-1 & n \\ \hline \end{array}, \emptyset, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \emptyset) \text{ and } f_2 = z_{1,0} y_{1,0}^{n-1}. \quad (2.3.2)$$

It is clear that, by making the evaluation $y_{1,0} = e_{11} + e_{44}$ and $z_{1,0} = e_{12} - e_{34}$, we get $f_1 = e_{12} \neq 0$ and $f_2 = -e_{34} \neq 0$. This implies that f_1 and f_2 are not $(\mathbb{Z}_2, *)$ -identities of $A_{2,*}$. Moreover by making the same evaluation we have that $\alpha f_1 + \beta f_2 = 0$ implies $\alpha = \beta = 0$, so these polynomials are linearly independent modulo $Id^{gri}(A_{2,*})$.

On the other hand, consider the following highest weight vectors associated to the multipartition $((n-1), \emptyset, (1), \emptyset)$ and their corresponding multitableaux:

$$(\begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & n \\ \hline 2 & & & \end{array}, \emptyset, \emptyset, \emptyset) \text{ and } g_1 = [y_{1,0}, y_{2,0}] y_{1,0}^{n-2} \quad (2.3.3)$$

$$\left(\begin{array}{|c|c|c|c|} \hline n-1 & 1 & \cdots & n-2 \\ \hline n & & & \\ \hline \end{array} , \varnothing , \varnothing , \varnothing \right) \text{ and } g_2 = y_{1,0}^{n-2}[y_{1,0}, y_{2,0}]. \quad (2.3.4)$$

By making the evaluation $y_{1,0} = e_{11} + e_{44}$ and $y_{2,0} = e_{12} + e_{34}$, we get $g_1 = -e_{34} \neq 0$ and $g_2 = e_{12} \neq 0$. This shows that g_1 and g_2 are not $(\mathbb{Z}_2, *)$ -identities of $A_{2,*}$ and by making the same evaluation we have that $\alpha g_1 + \beta g_2 = 0$ implies $\alpha = \beta = 0$, so these polynomials are linearly independent modulo $Id^{gri}(A_{2,*})$.

Thus, we finally have $\chi_n^{gri}(A_{2,*}) = \chi_{(n),\varnothing,\varnothing,\varnothing} + 2\chi_{(n-1),\varnothing,(1),\varnothing} + 2\chi_{(n-1,1),\varnothing,\varnothing,\varnothing}$ and $l_n^{gri}(A_{2,*}) = 5$.

□

Before giving the decomposition of $\chi_n^{gri}(A_{k,*})$, for any $k \geq 2$, we prove the following.

Remark 2.3.2. Let $k \geq 2$. Then

$$\begin{aligned} c_n^{gri}(A_{k,*}) &= d_{(n),\varnothing,\varnothing,\varnothing} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j,j),\varnothing,\varnothing,\varnothing} + \sum_{j=1}^{k-2} 2(k-j-1)d_{(n-j-1,j,1),\varnothing,\varnothing,\varnothing} \\ &\quad + \sum_{j=0}^{k-2} 2(k-j-1)d_{(n-j-1,j),\varnothing,(1),\varnothing}. \end{aligned}$$

Proof. We will use induction on k . By Lemma 2.3.1, we have

$$\chi_n^{gri}(A_{2,*}) = \chi_{(n),\varnothing,\varnothing,\varnothing} + 2\chi_{(n-1,1),\varnothing,\varnothing,\varnothing} + 2\chi_{(n-1),\varnothing,(1),\varnothing},$$

this implies that the result is true for $k = 2$.

Now we suppose the result is true for some $k \geq 2$. By Lemma 2.1.8, we have the following

$$\begin{aligned} c_n^{gri}(A_{k+1,*}) &= c_n^{gri}(A_{k,*}) + 2\binom{n}{k-1}(n-k) + 2\binom{n}{k-1}(n-k+1) \\ &= c_n^{gri}(A_{k,*}) + 2\sum_{j=1}^k d_{(n-j,j),\varnothing,\varnothing,\varnothing} + 2\sum_{j=1}^{k-1} d_{(n-j-1,j,1),\varnothing,\varnothing,\varnothing} + 2\sum_{j=0}^{k-1} d_{(n-j-1,j),\varnothing,(1),\varnothing} \\ &= d_{(n),\varnothing,\varnothing,\varnothing} + \sum_{j=1}^k 2(k+1-j)d_{(n-j,j),\varnothing,\varnothing,\varnothing} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j-1,j,1),\varnothing,\varnothing,\varnothing} \\ &\quad + \sum_{j=0}^{k-1} 2(k-j)d_{(n-j-1,j),\varnothing,(1),\varnothing} \end{aligned}$$

by using $\sum_{j=1}^k d_{(n-j,j),\varnothing,\varnothing,\varnothing} + \sum_{j=1}^{k-1} d_{(n-j-1,j,1),\varnothing,\varnothing,\varnothing} = \binom{n}{k-1}(n-k)$ and $\sum_{j=0}^{k-1} d_{(n-j-1,j),\varnothing,(1),\varnothing} = \binom{n}{k-1}(n-k+1)$. Thus, the result is true for any $k \geq 2$. □

We will adopt the convention where the symbols $-$, $=$ and $\sim$ indicate alternation on a given set of variables in the next lemmas. Thus, for instance, the notation $\bar{y}_1\bar{y}_1\tilde{y}_1y_4\bar{y}_2\bar{y}_2\tilde{y}_2\bar{y}_3$ indicates the polynomial

$$\sum_{\substack{\sigma \in S_3 \\ \rho, \tau \in S_2}} (\text{sign}\rho)(\text{sign}\sigma)(\text{sign}\tau)y_{\rho(1)}y_{\sigma(1)}y_{\tau(1)}y_4y_{\sigma(2)}y_{\rho(2)}y_{\tau(2)}y_{\sigma(3)}.$$

Theorem 2.3.3. *For $k \geq 2$, we have*

$$\begin{aligned}
 1. \quad \chi_n^{gri}(A_{k,*}) &= \chi_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-1} 2(k-j)\chi_{(n-j,j),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-2} 2(k-j-1)\chi_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset} \\
 &\quad + \sum_{j=0}^{k-2} 2(k-j-1)\chi_{(n-j-1,j),\emptyset,(1),\emptyset}, \\
 2. \quad l_n^{gri}(A_{k,*}) &= 3k^2 - 5k + 3.
 \end{aligned}$$

Proof. By the previous remark, we have, for any $k \geq 2$,

$$\begin{aligned}
 c_n^{gri}(A_{k,*}) &= d_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j,j),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-2} 2(k-j-1)d_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset} \\
 &\quad + \sum_{j=0}^{k-2} 2(k-j-1)d_{(n-j-1,j),\emptyset,(1),\emptyset}.
 \end{aligned}$$

It is clear that $m_{(n),\emptyset,\emptyset,\emptyset} = 1$. In order to prove the wished decomposition for $\chi_n^{gri}(A_{k,*})$, we shall prove that the irreducible characters $\chi_{(n-j,j),\emptyset,\emptyset,\emptyset}$, $\chi_{(n-l-1,l,1),\emptyset,\emptyset,\emptyset}$ and $\chi_{(n-t-1,t),\emptyset,(1),\emptyset}$, for $1 \leq j \leq k-1$, $1 \leq l \leq k-2$ and $0 \leq t \leq k-2$, appear in the decomposition of the cocharacter $\chi_n^{gri}(A_{k,*})$ with multiplicity $m_{(n-j,j),\emptyset,\emptyset,\emptyset} = 2(k-j)$, $m_{(n-l-1,l,1),\emptyset,\emptyset,\emptyset} = 2(k-l-1)$ and $m_{(n-t-1,t),\emptyset,(1),\emptyset} = 2(k-t-1)$, respectively.

(i) For the multipartition $((n-1, 1), \emptyset, \emptyset, \emptyset)$, for any $0 \leq p \leq k-2$ we consider the following pairs of multitableaux:

$$\left(\begin{array}{|c|c|c|c|c|c|} \hline p+1 & 1 & \cdots & p & p+3 & \cdots & n \\ \hline p+2 & & & & & & \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right)$$

$$\left(\begin{array}{|c|c|c|c|c|c|} \hline n-p-1 & 1 & \cdots & n-p-2 & n-p+1 & \cdots & n \\ \hline n-p & & & & & & \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right)$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_{1,0}^p [y_{1,0}, y_{2,0}] y_{1,0}^{n-p-2} \quad \text{and} \quad g_p = y_{1,0}^{n-p-2} [y_{1,0}, y_{2,0}] y_{1,0}^p.$$

By making the evaluation $y_{1,0} = e_{11} + e_{2k,2k} + E$ and $y_{2,0} = e_{12} + e_{2k-1,2k}$, we get

$$f_p(y_{1,0}, y_{2,0}) = e_{2k-p-2,2k} - e_{2k-p-1,2k} \quad \text{and} \quad g_p(y_{1,0}, y_{2,0}) = e_{1,p+2} - e_{1,p+3}.$$

Then, f_p and g_p are not $(\mathbb{Z}_2, *)$ -identities of $A_{k,*}$, for any $0 \leq p \leq k-2$, and these $2(k-1)$ polynomials are linearly independent modulo $Id^{gri}(A_{k,*})$. Hence $m_{(n-1,1),\emptyset,\emptyset,\emptyset} \geq 2(k-1)$.

(ii) Fixed $2 \leq j \leq k-1$, for the multipartition $((n-j, j), \emptyset, \emptyset, \emptyset)$ and for $0 \leq p \leq k-j-1$ and $w = n-p$, we consider the following pairs of multitableaux:

$$\begin{aligned}
& \left(\begin{array}{|c|c|c|c|} \hline p+1 & p+2 & \cdots & p+j \\ \hline p+j+1 & p+j+2 & \cdots & p+2j \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & p & p+2j+1 & \cdots & n \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right) \\
& \left(\begin{array}{|c|c|c|c|} \hline w-2j+1 & w-2j+2 & \cdots & w-j \\ \hline w-j+1 & w-j+2 & \cdots & w \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & w-2j & w+1 & \cdots & n \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right)
\end{aligned} \tag{2.3.5}$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_{1,0}^p \underbrace{y_{1,0} \cdots y_{1,0}}_j \underbrace{y_{2,0} \cdots y_{2,0}}_j y_{1,0}^{n-2j-p} \quad \text{and} \quad g_p = y_{1,0}^{n-2j-p} \underbrace{y_{1,0} \cdots y_{1,0}}_j \underbrace{y_{2,0} \cdots y_{2,0}}_j y_{1,0}^p.$$

By making the evaluation $y_{1,0} = e_{11} + e_{2k,2k} + E$ and $y_{2,0} = e_{11} + e_{2k,2k} + e_{12} + e_{2k-1,2k}$, we have $f_p(y_{1,0}, y_{2,0}) = \alpha e_{2k-p-j,2k}$ and $g_p(y_{1,0}, y_{2,0}) = \beta e_{1,j+p+1}$, with $\alpha \neq 0$ and $\beta \neq 0$. Then, for any $0 \leq p \leq k-j-1$, f_p and g_p are not $(\mathbb{Z}_2, *)$ -identities of $A_{k,*}$. Moreover, the same evaluation shows that these $2(k-j)$ polynomials are linearly independent modulo $Id^{gri}(A_{k,*})$. Thus $m_{(n-j,j),\emptyset,\emptyset,\emptyset} \geq 2(k-j)$, for any $2 \leq j \leq k-1$.

(iii) Now, fixed $1 \leq l \leq k-2$, for the multipartition $((n-l-1, l, 1), \emptyset, \emptyset, \emptyset)$ and for $0 \leq p \leq k-j-2$ and $w = n-p$, we consider the following pairs of multitableaux:

$$\begin{aligned}
& \left(\begin{array}{|c|c|c|c|} \hline p+l & p+1 & \cdots & p+l-1 \\ \hline p+l+1 & p+l+3 & \cdots & p+2l+1 \\ \hline p+l+2 & & & \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & p & p+2l+2 & \cdots & n \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right) \\
& \left(\begin{array}{|c|c|c|c|} \hline w-l-1 & w-2l & \cdots & w-l-2 \\ \hline w-l & w-l+2 & \cdots & w \\ \hline w-l+1 & & & \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & \cdots & w-2l-1 & w+1 & \cdots & n \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right)
\end{aligned} \tag{2.3.6}$$

and their corresponding highest weight vectors, respectively,

$$\begin{aligned}
f_p &= y_{1,0}^p \underbrace{y_{1,0} \cdots y_{1,0}}_{l-1} y_{1,0} y_{2,0} y_{3,0} \underbrace{y_{2,0} \cdots y_{2,0}}_{l-1} y_{1,0}^{n-p-2l-1} \quad \text{and} \\
g_p &= y_{1,0}^{n-p-2l-1} \underbrace{y_{1,0} \cdots y_{1,0}}_{l-1} y_{1,0} y_{2,0} y_{3,0} \underbrace{y_{2,0} \cdots y_{2,0}}_{l-1} y_{1,0}^p.
\end{aligned}$$

Evaluating $y_{1,0} = e_{11} + e_{2k,2k} + E$, $y_{2,0} = E$ and $y_{3,0} = e_{12} + e_{2k-1,2k}$, we get $f_p(y_{1,0}, y_{2,0}, y_{3,0}) = \alpha e_{2k-l-p-1,2k}$ and $g_p(y_{1,0}, y_{2,0}, y_{3,0}) = \beta e_{1,l+p+2}$, with $\alpha \neq 0$ and $\beta \neq 0$. Thus f_p and g_p , for any $0 \leq p \leq k-j-2$, are not $(\mathbb{Z}_2, *)$ -identities of $A_{k,*}$ and these $2(k-l-1)$ polynomials are linearly independent modulo $Id^{gri}(A_{k,*})$. Hence we have $m_{(n-l-1,l,1),\emptyset,\emptyset,\emptyset} \geq 2(k-l-1)$, for any $1 \leq l \leq k-2$.

(iv) Finally, fixed $0 \leq t \leq k-2$, for the multipartition $((n-t-1, t), \emptyset, (1), \emptyset)$ and for $0 \leq p \leq k-j-2$ and $w = n-p$, we consider the following pairs of multitableaux:

$$\left(\begin{array}{|c|c|c|} \hline p+1 & \cdots & p+t \\ \hline p+t+2 & \cdots & p+2t+1 \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|} \hline 1 & \cdots & p & p+2t+2 & \cdots & n \\ \hline \end{array}, \emptyset, \boxed{p+t+1}, \emptyset \right)$$

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline w-2t & \cdots & w-t-1 & 1 & \cdots & w-2t-1 & w+1 & \cdots & n \\ \hline w-t+1 & \cdots & w & & & & & & \\ \hline \end{array} \right), \emptyset, \boxed{w-t}, \emptyset \quad (2.3.7)$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_{1,0}^p \underbrace{y_{1,0}^- \cdots y_{1,0}^-}_t z_{1,0} \underbrace{y_{2,0}^- \cdots y_{2,0}^-}_t y_{1,0}^{n-p-2t-1} \quad \text{and}$$

$$g_p = y_{1,0}^{n-p-2t-1} \underbrace{y_{1,0}^- \cdots y_{1,0}^-}_t z_{1,0} \underbrace{y_{2,0}^- \cdots y_{2,0}^-}_t y_{1,0}^p.$$

By making the evaluation $y_{1,0} = e_{11} + e_{2k,2k} + E$ and $z_{1,0} = e_{12} - e_{2k-1,2k}$, in case $t = 0$, and $y_{1,0} = e_{11} + e_{2k,2k} + E$, $y_{2,0} = E$ and $z_{1,0} = e_{12} - e_{2k-1,2k}$ otherwise, we get $f_p(y_{1,0}, y_{2,0}, z_{1,0}) = \alpha e_{2k-t-p-1,2k}$ and $g_p(y_{1,0}, y_{2,0}, z_{1,0}) = \beta e_{1,t+p+1}$, with $\alpha \neq 0$ and $\beta \neq 0$. Thus $m_{(n-t-1,t),\emptyset,(1),\emptyset} \geq 2(k-t-1)$, for any $0 \leq t \leq k-2$, since f_p and g_p are not $(\mathbb{Z}_2, *)$ -identities of $A_{k,*}$, for all $0 \leq p \leq k-t-2$, and these $2(k-t-1)$ polynomials are linearly independent modulo $Id^{gri}(A_{k,*})$.

Hence, $\chi_n^{gri}(A_{k,*})$ has the wished decomposition. It is easy to show that $l_n^{gri}(A_{k,*}) = 3k^2 - 5k + 3, \forall k \geq 2$, and the result is proved. $\square$

Now, we study the $*$ -graded cocharacters and the $*$ -graded colengths of the minimal varieties $var^{gri}(N_{k,*})$ and $var^{gri}(U_{t,*})$, for all $k \geq 2$ and $t \geq 3$.

Lemma 2.3.4. *For the $*$ -superalgebra $N_{2,*}$, we have*

1. $\chi_n^{gri}(N_{2,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset}$
2. $l_n^{gri}(N_{2,*}) = 2$.

Proof. By Lemma 2.1.2 it is known that $c_n^{gri}(N_{2,*}) = 1 + n$ and notice that we have

$$d_{(n),\emptyset,\emptyset,\emptyset} + d_{(n-1),\emptyset,(1),\emptyset} = 1 + n = c_n^{gri}(N_{2,*}).$$

Then, since $m_{(n),\emptyset,\emptyset,\emptyset} = 1$, if we find a highest weight vector for the multipartition $((n-1), \emptyset, (1), \emptyset)$ which is not a $(\mathbb{Z}_2, *)$ -identity of $N_{2,*}$, we may conclude that $\chi_n^{gri}(N_{2,*})$ has the wished decomposition.

In fact, let $f_1 = y_{1,0}^{n-1} z_{1,0}$ be the highest weight vector associated to the multipartition $((n-1), \emptyset, (1), \emptyset)$ corresponding to the multitableaux:

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline \end{array} \right), \emptyset, \begin{array}{|c|} \hline n \\ \hline \end{array}, \emptyset \quad (2.3.8)$$

It is clear that, by making the evaluation $y_{1,0} = I$ and $z_{1,0} = e_{12} - e_{34}$, we get $f(y_{1,0}, z_{1,0}) = e_{12} - e_{34} \neq 0$. This implies that f is not a $(\mathbb{Z}_2, *)$ -identity of $N_{2,*}$. Hence, we have $\chi_n^{gri}(N_{2,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset}$ and $l_n^{gri}(N_{2,*}) = 2$. $\square$

Lemma 2.3.5. *For the $*$ -superalgebra $U_{3,*}$, we have*

1. $\chi_n^{gri}(U_{3,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1,1),\emptyset,\emptyset,\emptyset} + \chi_{(n-2,1,1),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset},$
2. $l_n^{gri}(U_{3,*}) = 4.$

Proof. By Lemma 2.1.4 it is known that $c_n^{gri}(U_{3,*}) = 1 + n + \frac{n(n-1)}{2}$ and notice that we have

$$\begin{aligned} c_n^{gri}(U_{3,*}) &= 1 + n + (n-1) + \frac{(n-1)(n-2)}{2} \\ &= d_{(n),\emptyset,\emptyset,\emptyset} + d_{(n-1),\emptyset,(1),\emptyset} + d_{(n-2,1),\emptyset,\emptyset,\emptyset} + d_{(n-1,1^2),\emptyset,\emptyset,\emptyset}. \end{aligned}$$

Then, since $m_{(n),\emptyset,\emptyset,\emptyset} = 1$, if we find a highest weight vector for each multipartition $((n-1), \emptyset, (1), \emptyset)$, $((n-1, 1), \emptyset, \emptyset, \emptyset)$ and $((n-2, 1^2), \emptyset, \emptyset, \emptyset)$ which is not $(\mathbb{Z}_2, *)$ -identity of $U_{3,*}$, we may conclude that $\chi_n^{gri}(U_{3,*})$ has the wished decomposition.

In fact, let $f = y_{1,0}^{n-1} z_{1,0}$ be the highest weight vector associated to the multipartition $((n-1), \emptyset, (1), \emptyset)$ and corresponding to the multitableaux:

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n-1 \\ \hline \end{array}, \emptyset, \begin{array}{|c|} \hline n \\ \hline \end{array}, \emptyset \right). \quad (2.3.9)$$

It is clear that, by making the evaluation $y_{1,0} = I$ and $z_{1,0} = e_{13} - e_{46}$, we get $f(y_{1,0}, z_{1,0}) = e_{13} - e_{46} \neq 0$, then f is not a $(\mathbb{Z}_2, *)$ -identity of $U_{3,*}$.

Now, we consider $g = [y_{1,0}, y_{2,0}] y_{1,0}^{n-2}$ the highest weight vector associated to the multipartition $((n-1, 1), \emptyset, \emptyset, \emptyset)$ and corresponding to the multitableaux:

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & n \\ \hline 2 & & & \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right). \quad (2.3.10)$$

By making the evaluation $y_{1,0} = I + e_{12} + e_{56}$ and $y_{2,0} = e_{23} + e_{45}$, we get $g(y_{1,0}, y_{2,0}) = e_{13} - e_{46} \neq 0$. Then g is not a $(\mathbb{Z}_2, *)$ -identity of $U_{3,*}$.

Finally, we consider $h = St_3(y_{1,0}, y_{2,0}, y_{3,0}) y_{1,0}^{n-3}$ the highest weight vector associated to the multipartition $((n-2, 1^2), \emptyset, \emptyset, \emptyset)$ and corresponding to the multitableaux:

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 4 & \cdots & n \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array}, \emptyset, \emptyset, \emptyset \right). \quad (2.3.11)$$

By making the evaluation, $y_{1,0} = I$, $y_{2,0} = e_{23} + e_{45}$ and $y_{3,0} = e_{12} + e_{56}$, we get $h(y_{1,0}, y_{2,0}, y_{3,0}) = -e_{13} + e_{46} \neq 0$ and it shows that h is not a $(\mathbb{Z}_2, *)$ -identity of $U_{3,*}$.

Then, we finally have $\chi_n^{gri}(U_{3,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-1,1),\emptyset,\emptyset,\emptyset} + \chi_{(n-2,1^2),\emptyset,\emptyset,\emptyset}$ and $l_n^{gri}(U_3) = 4$. $\square$

Next we make the following observation:

Remark 2.3.6. Let $k \geq 2$. Then

$$\begin{aligned} c_n^{gri}(N_{k,*}) &= d_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-3} (k-j-2) [d_{(n-j,j),\emptyset,\emptyset,\emptyset} + d_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] \\ &\quad + \sum_{j=0}^{k-2} (k-j-1) d_{(n-j-1,j),\emptyset,(1),\emptyset}. \end{aligned}$$

Proof. We will use induction on k . By Lemma 2.3.4, we have

$$\chi_n^{gri}(N_{2,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset},$$

it implies that the result is true for $k = 2$.

Now we suppose the result is true for some $k \geq 2$. By Lemma 2.1.3, we have

$$c_n^{gri}(N_{k+1,*}) = c_n^{gri}(N_{k,*}) + \binom{n}{k-1}(k-2) + \binom{n}{k}(k).$$

Hence, by using this, for all $r \geq 1$, $\sum_{j=0}^r d_{(n-j-1,j),\emptyset,(1),\emptyset} = \binom{n}{r}(n-r) = \binom{n}{r+1}(r+1)$ and $\sum_{j=1}^r [d_{(n-j,j),\emptyset,\emptyset,\emptyset} + d_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] = \binom{n}{r+1}r$, we get the following:

$$\begin{aligned} c_n^{gri}(N_{k+1,*}) &= c_n^{gri}(N_{k,*}) + \binom{n}{k-1}(k-2) + \binom{n}{k}k \\ &= c_n^{gri}(N_{k,*}) + \sum_{j=1}^{k-2} [d_{(n-j,j),\emptyset,\emptyset,\emptyset} + d_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] + \sum_{j=0}^{k-1} d_{(n-j-1,j),\emptyset,(1),\emptyset} \\ &= d_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-2} (k-j-1)[d_{(n-j,j),\emptyset,\emptyset,\emptyset} + d_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] \\ &\quad + \sum_{j=0}^{k-1} (k-j)d_{(n-j-1,j),\emptyset,(1),\emptyset} \end{aligned}$$

Thus the result is true for any $k \geq 2$. □

Theorem 2.3.7. *For $k \geq 2$, we have*

$$\begin{aligned} 1. \quad \chi_n^{gri}(N_{k,*}) &= \chi_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-3} (k-j-2) [\chi_{(n-j,j),\emptyset,\emptyset,\emptyset} + \chi_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] \\ &\quad + \sum_{j=0}^{k-2} (k-j-1) \chi_{(n-j-1,j),\emptyset,(1),\emptyset}, \\ 2. \quad l_n^{gri}(N_{k,*}) &= \frac{3k^2 - 11k + 14}{2}. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 2.3.3. By the previous remark, we have, for any $k \geq 2$,

$$\begin{aligned} c_n^{gri}(N_{k,*}) &= d_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-3} (k-j-2)[d_{(n-j,j),\emptyset,\emptyset,\emptyset} + d_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] \\ &\quad + \sum_{j=0}^{k-2} (k-j-1)d_{(n-j-1,j),\emptyset,(1),\emptyset}. \end{aligned}$$

It is clear that $m_{(n),\emptyset,\emptyset,\emptyset} = 1$. In order to prove the wished decomposition for $\chi_n^{gri}(N_{k,*})$, we shall prove that the irreducible characters $\chi_{(n-j,j),\emptyset,\emptyset,\emptyset}$, $\chi_{(n-l-1,l,1),\emptyset,\emptyset,\emptyset}$

and $\chi_{(n-t-1,t),\emptyset,(1),\emptyset}$, for $1 \leq j, l \leq k-3$ and $0 \leq t \leq k-2$, appear in the decomposition of the $*$ -graded cocharacter $\chi_n^{gr}(N_{k,*})$ with multiplicity $m_{(n-j,j),\emptyset,\emptyset,\emptyset} = k-j-2$, $m_{(n-l-1,l,1),\emptyset,\emptyset,\emptyset} = k-l-2$ and $m_{(n-t-1,1),\emptyset,(1),\emptyset} = k-t-1$, respectively.

(i) Fixed $1 \leq j \leq k-3$, for the multipartition $((n-j,j),\emptyset,\emptyset,\emptyset)$ and for $0 \leq p \leq k-j-3$, we consider the multitableaux (2.3.5) given in Lemma 2.3.3 whose corresponding highest weight vector is

$$f_p = y_{1,0}^{n-2j-p} \underbrace{y_{1,0}^{-1} \cdots y_{1,0}^{-1}}_j \underbrace{y_{2,0}^{-1} \cdots y_{2,0}^{-1}}_j y_{1,0}^p.$$

By making the evaluation $y_{1,0} = I + E$ and $y_{2,0} = I + e_{13} + e_{2k-2,2k}$ we get

$$f_p(y_{1,0}, y_{2,0}) = \alpha \sum_{i=0}^{k-2} \binom{n-2j-p}{i} e_{2k-j-i-2,2k} + \beta \sum_{i=0}^p \binom{p}{i} e_{1,3+j+i},$$

with α and β non-zero values. Then, for any $0 \leq p \leq k-j-3$, f_p is not a $(\mathbb{Z}_2, *)$ -identity of $N_{k,*}$. Moreover, the same evaluation shows that these $(k-j-2)$ polynomials are linearly independent modulo $Id^{gr}(N_{k,*})$. Thus $m_{(n-j,j),\emptyset,\emptyset,\emptyset} \geq k-j-2$, for any $1 \leq j \leq k-3$.

(ii) Now, fixed $1 \leq l \leq k-3$, for the multipartition $((n-l-1,l,1),\emptyset,\emptyset,\emptyset)$ and $0 \leq p \leq k-j-3$, we consider the multitableaux (2.3.6) with the following corresponding highest weight vector:

$$g_p = y_{1,0}^{n-p-2l-1} \underbrace{y_{1,0}^{-1} \cdots y_{1,0}^{-1}}_{l-1} y_{1,0} y_{2,0} y_{3,0} \underbrace{y_{2,0}^{-1} \cdots y_{2,0}^{-1}}_{l-1} y_{1,0}^p.$$

Evaluating $y_{1,0} = I + E$, $y_{2,0} = E$ and $y_{3,0} = e_{13} + e_{2k-2,2k}$, we also get

$$g_p(y_{1,0}, y_{2,0}, y_{3,0}) = \alpha \sum_{i=0}^{k-2} \binom{n-2j-p}{i} e_{2k-j-i-2,2k} + \beta \sum_{i=0}^p \binom{p}{i} e_{1,3+j+i},$$

with α and β non-zero values. Thus g_p , for any $0 \leq p \leq k-j-3$, is not a $(\mathbb{Z}_2, *)$ -identity of $N_{k,*}$ and these $(k-l-2)$ polynomials are linearly independent modulo $Id^{gr}(N_{k,*})$. Hence, we have $m_{(n-l-1,l,1),\emptyset,\emptyset,\emptyset} \geq (k-l-2)$, for any $1 \leq l \leq k-3$.

(iii) Finally, fixed $0 \leq t \leq k-2$, for the multipartition $((n-t-1,t),\emptyset,(1),\emptyset)$ and for $0 \leq p \leq k-j-2$, we consider the multitableaux (2.3.7) and its corresponding highest weight vector

$$h_p = y_{1,0}^{n-p-2t-1} \underbrace{y_{1,0}^{-1} \cdots y_{1,0}^{-1}}_t z_{1,0} \underbrace{y_{2,0}^{-1} \cdots y_{2,0}^{-1}}_t y_{1,0}^p.$$

By making the evaluation $y_{1,0} = I + E$ and $z_{1,0} = e_{12} - e_{2k-1,2k}$, in case $t = 0$, and $y_{1,0} = I + E$, $y_{2,0} = E$ and $z_{1,0} = e_{12} - e_{2k-1,2k}$ otherwise, we get

$$h_p(y_{1,0}, y_{2,0}, z_{1,0}) = \alpha \sum_{i=0}^{k-2} \binom{n-2j-p}{i} e_{2k-j-i-1,2k} + \beta \sum_{i=0}^p \binom{p}{i} e_{1,2+j+i},$$

with α and β non-zero values. Thus $m_{(n-t-1,t),\emptyset,(1),\emptyset} \geq (k-t-1)$, for any $0 \leq t \leq k-2$, since h_p is not a $(\mathbb{Z}_2, *)$ -identity of $N_{k,*}$, for all $0 \leq p \leq k-t-2$, and these $(k-t-1)$ polynomials are linearly independent modulo $Id^{gri}(N_{k,*})$.

Hence, by the previous remark, $\chi_n^{gri}(N_{k,*})$ has the wished decomposition. It is easy to show that $l_n^{gri}(N_{k,*}) = \frac{3k^2 - 11k + 14}{2}$. $\square$

We will omit the proof of the following theorem, since we can prove it similarly to the proof of Theorem 2.3.7.

Theorem 2.3.8. *For $k \geq 3$, we have*

$$\begin{aligned} 1. \quad & \chi_n^{gri}(U_{k,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=1}^{k-2} (k-j-1) [\chi_{(n-j,j),\emptyset,\emptyset,\emptyset} + \chi_{(n-j-1,j,1),\emptyset,\emptyset,\emptyset}] \\ & + \sum_{j=0}^{k-3} (k-j-2) \chi_{(n-j-1,j),\emptyset,(1),\emptyset}, \\ 2. \quad & l_n^{gri}(U_{k,*}) = \frac{3k^2 - 9k + 8}{2}. \end{aligned}$$

Now, we explicit the sequences of $*$ -graded cocharacters and of $*$ -graded colengths of the minimal varieties $var^{gri}(A) \subseteq var^{gri}(M^{gri})$.

We start by computing the $*$ -cocharacters and the $*$ -colengths of the minimal varieties $var^{gri}(A_k^{gri})$, for $k \geq 2$. In order to demonstrate the decomposition of the $\chi_n^{gri}(A_k^{gri})$, for any $k \geq 2$, we need to prove the following results.

Lemma 2.3.9. *For the $*$ -superalgebra A_2^{gri} , we have*

$$\begin{aligned} 1. \quad & \chi_n^{gri}(A_2^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + 2\chi_{(n-1),(1),\emptyset,\emptyset} + 2\chi_{(n-1),\emptyset,\emptyset,(1)}, \\ 2. \quad & l_n^{gri}(A_2^{gri}) = 5. \end{aligned}$$

Proof. By Lemma 2.2.5 we have $c_n^{gri}(A_2^{gri}) = 1 + 4n$. We notice that

$$d_{(n),\emptyset,\emptyset,\emptyset} + 2d_{(n-1),(1),\emptyset,\emptyset} + 2d_{(n-1),\emptyset,\emptyset,(1)} = 1 + 4n = c_n^{gri}(A_2^{gri}).$$

Let us consider the following highest weight vectors associated to the multipartition $((n-1), (1), \emptyset, \emptyset)$ and their corresponding multitableaux:

$$(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline \end{array} , \begin{array}{|c|} \hline n \\ \hline \end{array} , \emptyset , \emptyset) \text{ and } f_1 = y_{1,0}^{n-1} y_{1,1} \quad (2.3.12)$$

$$(\begin{array}{|c|c|c|c|c|} \hline 2 & 3 & \cdots & n-1 & n \\ \hline \end{array} , \begin{array}{|c|} \hline 1 \\ \hline \end{array} , \emptyset , \emptyset) \text{ and } f_2 = y_{1,1} y_{1,0}^{n-1}. \quad (2.3.13)$$

It is clear that, by making the evaluation $y_{1,0} = e_{11} + e_{44}$ and $y_{1,1} = e_{12} + e_{34}$, we get $f_1(y_{1,0}, y_{1,1}) = e_{12} \neq 0$ and $f_2(y_{1,0}, y_{1,1}) = e_{34} \neq 0$. This implies that f_1 and f_2 are not $(\mathbb{Z}_2, *)$ -identities of A_2^{gri} and these polynomials are linearly independent modulo $Id^{gri}(A_2^{gri})$. So $m_{(n-1),(1),\emptyset,\emptyset} \geq 2$.

On the other hand, consider the following highest weight vectors associated to the multipartition $((n-1), \emptyset, \emptyset, (1))$ and their corresponding multitableaux:

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline \end{array}, \emptyset, \emptyset, \begin{array}{|c|} \hline n \\ \hline \end{array} \right) \text{ and } g_1 = y_{1,0}^{n-1} z_{1,1} \quad (2.3.14)$$

$$\left(\begin{array}{|c|c|c|c|c|} \hline 2 & 3 & \cdots & n-1 & n \\ \hline \end{array}, \emptyset, \emptyset, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right) \text{ and } g_2 = z_{1,1} y_{1,0}^{n-1}. \quad (2.3.15)$$

By making the evaluation $y_{1,0} = e_{11} + e_{44}$ and $z_{1,1} = e_{12} - e_{34}$, we get $g_1(y_{1,0}, z_{1,1}) = e_{12} \neq 0$ and $g_2(y_{1,0}, z_{1,1}) = -e_{34} \neq 0$. Then it implies that g_1 and g_2 are not $(\mathbb{Z}_2, *)$ -identities of A_2^{gri} and these polynomials are linearly independent modulo $Id^{gri}(A_2^{gri})$. So $m_{(n-1), \emptyset, \emptyset, (1)} \geq 2$.

Thus, we finally have $\chi_n^{gri}(A_2^{gri}) = \chi_{(n), \emptyset, \emptyset, \emptyset} + 2\chi_{(n-1), (1), \emptyset, \emptyset} + 2\chi_{(n-1), \emptyset, \emptyset, (1)}$ and $l_n^{gri}(A_2^{gri}) = 5$.

□

Remark 2.3.10. Let $k \geq 2$. Then

$$c_n^{gri}(A_k^{gri}) = d_{(n), \emptyset, \emptyset, \emptyset} + \sum_{j=0}^{k-2} 2(k-j-1)[d_{(n-j-1, j), (1), \emptyset, \emptyset} + d_{(n-j-1, j), \emptyset, \emptyset, (1)}]$$

Proof. We will use induction on k . By Lemma 2.3.9, we have

$$\chi_n^{gri}(A_2^{gri}) = \chi_{(n), \emptyset, \emptyset, \emptyset} + 2\chi_{(n-1), (1), \emptyset, \emptyset} + 2\chi_{(n-1), \emptyset, \emptyset, (1)}$$

it implies that $c_n^{gri}(A_2^{gri}) = d_{(n), \emptyset, \emptyset, \emptyset} + 2d_{(n-1), (1), \emptyset, \emptyset} + 2d_{(n-1), \emptyset, \emptyset, (1)}$ so the result is true for $k = 2$.

Now, we suppose the result is true for some $k \geq 2$. By Lemma 2.2.5, we have

$$\begin{aligned} c_n^{gri}(A_{k+1}^{gri}) &= c_n^{gri}(A_k^{gri}) + 4\binom{n}{k-1}(n-k+1) \\ &= c_n^{gri}(A_k^{gri}) + 2 \sum_{j=0}^{k-1} d_{(n-j, j-1), \emptyset, \emptyset, (1)} + 2 \sum_{j=0}^{k-1} d_{(n-j, j-1), (1), \emptyset, \emptyset} \\ &= d_{(n), \emptyset, \emptyset, \emptyset} + \sum_{j=0}^{k-1} 2(k-j)[d_{(n-j-1, j), (1), \emptyset, \emptyset} + d_{(n-j-1, j), \emptyset, \emptyset, (1)}] \end{aligned}$$

by using $\sum_{j=0}^{k-1} d_{(n-j-1, j), \emptyset, \emptyset, (1)} = \sum_{j=0}^{k-1} d_{(n-j-1, j), (1), \emptyset, \emptyset} = \binom{n}{k-1}(n-k+1)$. Thus, the result is true for any $k \geq 2$. □

Now we are in position to compute the *-graded cocharacter and the *-graded colength of A_k^{gri} , for any $k \geq 2$.

Theorem 2.3.11. *For $k \geq 2$, we have*

$$1. \chi_n^{gri}(A_k^{gri}) = \chi_{(n), \emptyset, \emptyset, \emptyset} + \sum_{j=0}^{k-2} 2(k-j-1)[\chi_{(n-j-1, j), (1), \emptyset, \emptyset} + \chi_{(n-j-1, j), \emptyset, \emptyset, (1)}],$$

$$2. \ l_n^{gri}(A_k^{gri}) = 2k^2 - 2k + 1.$$

Proof. By the previous remark, we have, for any $k \geq 2$,

$$c_n^{gri}(A_k^{gri}) = d_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=0}^{k-2} 2(k-j-1)[d_{(n-j-1,j),(1),\emptyset,\emptyset} + d_{(n-j-1,j),\emptyset,\emptyset,(1)}]$$

It is clear that $m_{(n),\emptyset,\emptyset,\emptyset} = 1$. In order to prove the wished decomposition for $\chi_n^{gri}(A_k^{gri})$, we shall prove that the irreducible characters $\chi_{(n-t-1,t),(1),\emptyset,\emptyset}$ and $\chi_{(n-t-1,t),\emptyset,\emptyset,(1)}$, for $0 \leq t \leq k-2$, both appear in the decomposition of the cocharacter $\chi_n^{gri}(A_k^{gri})$ with multiplicity $m_{(n-t-1,t),(1),\emptyset,\emptyset} = m_{(n-t-1,t),\emptyset,\emptyset,(1)} = 2(k-t-1)$.

Fixed $0 \leq t \leq k-2$ and for $0 \leq p \leq k-j-2$ and $w = n-p$, we consider the following pairs of multitableaux:

$$\begin{aligned} (T_{\lambda_1}, T_{\mu_1}) &= \left(\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline p+1 & \cdots & p+t & 1 & \cdots & p & p+2t+2 & \cdots & n \\ \hline p+t+2 & \cdots & p+2t+1 & & & & & & \\ \hline \end{array}, \begin{array}{|c|} \hline p+t+1 \\ \hline \end{array} \right) \\ (T_{\lambda_2}, T_{\mu_2}) &= \left(\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline w-2t & \cdots & w-t-1 & 1 & \cdots & w-2t-1 & w+1 & \cdots & n \\ \hline w-t+1 & \cdots & w & & & & & & \\ \hline \end{array}, \begin{array}{|c|} \hline w-t \\ \hline \end{array} \right) \end{aligned} \quad (2.3.16)$$

Now, for the multipartitions $((n-t-1, t), (1), \emptyset, \emptyset)$ and $((n-t-1, t), \emptyset, \emptyset, (1))$ consider the highest weight vectors

$$f_p = y_{1,0}^p \underbrace{y_{1,0}^{-1} \cdots y_{1,0}^{-1}}_t x_{1,1} \underbrace{y_{2,0}^{-1} \cdots y_{2,0}^{-1}}_t y_{1,0}^{n-p-2t-1}$$

corresponding to the multitableaux $(T_{\lambda_1}, T_{\mu_1}, \emptyset, \emptyset)$ and $(T_{\lambda_1}, \emptyset, \emptyset, T_{\mu_1})$ according to $x_{1,1} = y_{1,1}$ or $x_{1,1} = z_{1,1}$, respectively. And consider the highest weight vector

$$g_p = y_{1,0}^{n-p-2t-1} \underbrace{y_{1,0}^{-1} \cdots y_{1,0}^{-1}}_t x_{1,1} \underbrace{y_{2,0}^{-1} \cdots y_{2,0}^{-1}}_t y_{1,0}^p$$

corresponding to the multitableaux $(T_{\lambda_2}, T_{\mu_2}, \emptyset, \emptyset)$ and $(T_{\lambda_2}, \emptyset, \emptyset, T_{\mu_2})$ according to $x_{1,1} = y_{1,1}$ or $x_{1,1} = z_{1,1}$, respectively.

By making the evaluation $y_{1,0} = e_{11} + e_{2k,2k} + E$, $y_{2,0} = E$ (when $t > 0$), $y_{1,1} = e_{12} + e_{2k-1,2k}$ and $z_{1,0} = e_{12} - e_{2k-1,2k}$ we get $f_p(y_{1,0}, y_{2,0}, x_{1,1}) = \alpha e_{2k-t-p-1,2k}$ and $g_p(y_{1,0}, y_{2,0}, x_{1,1}) = \beta e_{1,t+p+1}$, with $\alpha \neq 0$ and $\beta \neq 0$, for $x_{1,1} = y_{1,1}$ or $x_{1,1} = z_{1,1}$. Thus, $m_{(n-t-1,t),(1),\emptyset,\emptyset} \geq 2(k-t-1)$ and $m_{(n-t-1,t),\emptyset,\emptyset,(1)} \geq 2(k-t-1)$, for any $0 \leq t \leq k-2$, since f_p and g_p are not $(\mathbb{Z}_2, *)$ -identities of A_k^{gri} , for all $0 \leq p \leq k-t-2$, and these $2(k-t-1)$ polynomials are linearly independent modulo $Id^{gri}(A_k^{gri})$.

$$\text{Since } c_n^{gri}(A_k^{gri}) = d_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=0}^{k-2} 2(k-j-1)[d_{(n-j-1,j),(1),\emptyset,\emptyset} + d_{(n-j-1,j),\emptyset,\emptyset,(1)}],$$

we conclude that $\chi_n^{gri}(A_k^{gri})$ has the wished decomposition. It is easy to show that $l_n^{gri}(A_k^{gri}) = 2k^2 - 2k + 1, \forall k \geq 2$, and the result is proved. $\square$

Finally, we study the $*$ -graded cocharacter of the minimal $*$ -superalgebras N_k^{gri} and U_k^{gri} for any $k \geq 2$.

Lemma 2.3.12. *For the *-superalgebras N_2^{gri} and U_2^{gri} , we have*

1. $\chi_n^{gri}(N_2^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)},$
2. $\chi_n^{gri}(U_2^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),(1),\emptyset,\emptyset},$
3. $l_n^{gri}(N_2^{gri}) = l_n^{gri}(U_2^{gri}) = 2.$

Proof. Let us consider the algebra N_2^{gri} . The arguments are similar for U_2^{gri} . By Lemma 2.2.1 it is known that $c_n^{gri}(N_2^{gri}) = n + 1$ and notice that we have

$$d_{(n),\emptyset,\emptyset,\emptyset} + d_{(n-1),\emptyset,\emptyset,(1)} = 1 + n = c_n^{gri}(N_2^{gri}).$$

We also have $m_{(n),\emptyset,\emptyset,\emptyset} = 1$. Consider $f = y_{1,0}^{n-1}z_{1,1}$ the standard highest weight vector corresponding to the multipartition $((n-1), \emptyset, \emptyset, (1))$. By making the evaluation $y_{1,0} = I$ and $z_{1,1} = e_{12} - e_{34}$, we get $f(y_{1,0}, z_{1,1}) = e_{12} - e_{34}$ and so $f \notin Id^{gri}(N_2^{gri})$ and this implies that $m_{(n-1),\emptyset,\emptyset,(1)} \geq 1$. Hence, by comparing the codimension, we must have $\chi_n^{gri}(N_2^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)}$ and $l_n^{gri}(N_2^{gri}) = 2$.

□

Now we may explicit the decomposition of the *-graded cocharacter of N_k^{gri} and U_k^{gri} for any $k \geq 2$. The computations are similar to the ones in Theorem 2.3.7.

Theorem 2.3.13. *For $k \geq 2$, we have*

1. $\chi_n^{gri}(N_k^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=0}^{k-3} (k-j-2)\chi_{(n-j-1,j),(1),\emptyset,\emptyset} + \sum_{j=0}^{k-2} (k-j-1)\chi_{(n-j-1,j),\emptyset,\emptyset,(1)},$
2. $\chi_n^{gri}(U_k^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \sum_{j=0}^{k-2} (k-j-1)\chi_{(n-j-1,j),(1),\emptyset,\emptyset} + \sum_{j=0}^{k-3} (k-j-2)\chi_{(n-j-1,j),\emptyset,\emptyset,(1)},$
3. $l_n^{gri}(N_k^{gri}) = l_n^{gri}(U_k^{gri}) = k^2 - 2k + 2.$

We end this section by collecting the *-superalgebras with small *-graded colength that appear in this chapter. We observe that $l_n^{gri}(N_{2,*}) = l_n^{gri}(U_2^{gri}) = l_n^{gri}(N_2^{gri}) = 2$ and $l_n^{gri}(A_{2,*}) = l_n^{gri}(A_2^{gri}) = 5$. Also, for all $k > 2$, the *-superalgebras $N_{k,*}, U_{k,*}, A_{k,*}, N_k^{gri}, U_k^{gri}, A_k^{gri}$ and any direct sum of two distinct *-superalgebras among them have *-graded colength greater than 3.

Chapter 3

*-Superalgebras with small colength

The study of the subvarieties of the commutative APG $\ast$ -superalgebras $\text{var}^{gr_i}(D_\ast)$ and $\text{var}^{gr_i}(D^{gr})$ has already been done in previous situations and in different contexts.

In this chapter, we recall the classification of the subvarieties of $\text{var}^{gr_i}(D_\ast)$ and $\text{var}^{gr_i}(D^{gr})$ given in [21, Theorem 7] and [20, Theorem 8.3] in the specific cases of varieties of algebras with involution and varieties of superalgebras, respectively.

Here we establish the results about those subvarieties in the $\ast$ -superalgebra language and, as a new contribution, we classify all subvarieties of $\text{var}^{gr_i}(D^{gr_i})$. We also compute the $\ast$ -graded colengths of all minimal subvarieties of the commutative APG $\ast$ -supervarieties considered above, based on the decomposition of the $\ast$ -graded cocharacter of each one of them.

We will use the results proved here and the results contained in Chapter 2 to demonstrate the main result of our thesis, that is to classify the $\ast$ -superalgebras with $\ast$ -graded colength bounded by three in the last section of this chapter.

3.1 Subvarieties of the APG commutative $\ast$ -super-varieties

The $\ast$ -superalgebra $D_\ast$ is the algebra $D = F \oplus F$ with trivial grading and endowed with the exchange involution $(a, b)^\ast = (b, a)$. So it is not difficult to see that the classification of $\ast$ -superalgebras, up to $T_2^\ast$ -equivalence, inside $\text{var}^{gr_i}(D_\ast)$ and the classification of the $\ast$ -algebras inside $\text{var}^\ast(D)$ are equivalent. This last classification was done by La Mattina and Martino in [21, Theorem 7].

Next, we present the $\ast$ -superalgebra $C_{k,\ast}$, which generates the only minimal subvariety of $\text{var}^{gr_i}(D_\ast)$ and restate the results in [21] in the language of $\ast$ -superalgebras.

In the end of this section, we exhibit the decomposition of $\chi_n(C_{k,*})$ and compute its $*$ -graded codimension.

For $k \geq 2$, we denote by I_k the $k \times k$ identity matrix and consider the matrix $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in UT_k$, where e'_{ij} s are the usual matrix units.

We denote by $C_{k,*}$ the commutative subalgebra of UT_k

$$C_k = \{ \alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F \}$$

with trivial grading and endowed with the involution given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$

For instance

$$((C_{4,*})^{(0)})^+ = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad ((C_{4,*})^{(0)})^- = \begin{pmatrix} 0 & b & 0 & d \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Recall that $Id^{gri}(D_*) = \langle y_{1,1}, z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], [z_{1,0}, z_{2,0}] \rangle_{T_2^*}$ and notice that $C_{k,*}$ satisfies all $(\mathbb{Z}_2, *)$ -identities of D_* . Hence $C_{k,*} \in var^{gri}(D_*)$. Below we present the $(\mathbb{Z}_2, *)$ -identities and the $*$ -graded codimension sequence of the $*$ -superalgebra $C_{k,*}$, for all $k \geq 2$.

Theorem 3.1.1. [21, Lemma 9] *Let $k \geq 2$. Then*

$$1. Id^{gri}(C_{k,*}) = \langle y_{1,1}, z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], [z_{1,0}, z_{2,0}], z_{1,0} \cdots z_{k,0} \rangle_{T_2^*}.$$

$$2. c_n^{gri}(C_{k,*}) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}.$$

Proof. Let $Q = \langle y_{1,1}, z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], [z_{1,0}, z_{2,0}], z_{1,0} \cdots z_{k,0} \rangle_{T_2^*}$. It is easily checked that $Q \subseteq Id^{gri}(C_{k,*})$, since $C_{k,*}$ is commutative with trivial grading and $((C_{k,*}^{(1)})^-)^k = 0$.

Let f be a $(\mathbb{Z}_2, *)$ -identity of $C_{k,*}$ of degree t . Since the $(\mathbb{Z}_2, *)$ -identities of a unitary $*$ -superalgebra follow from the proper ones, we may assume f is proper. Now, if we reduce the polynomial f modulo Q , we obtain f is the zero polynomial if $t \geq k$ and $f = \alpha z_{1,0} \cdots z_{t,0}$ if $t \leq k-1$. In the second case, if $\alpha = 0$, by evaluating $z_{i,0} = E_1$, for all $1 \leq i \leq t$, we get $f = \alpha E_1^t \neq 0$, a contradiction, because $f \in Id^{gri}(C_{k,*})$. Hence, we must have $\alpha = 0$, and so, $Id^{gri}(C_{k,*}) = Q$

This also proves that in case $t \leq k - 1$, the polynomial $z_{1,0} \cdots z_{t,0}$ forms a basis of the multilinear proper polynomials of degree t modulo $Id^{gri}(C_{k,*})$. Hence $\gamma_t^{gri}(C_{k,*}) = 1$ for $0 \leq t \leq k - 1$ and $\gamma_t^{gri}(C_{k,*}) = 0$ for $t \geq k$. Then we get

$$c_n^{gri}(C_{k,*}) = \sum_{j=0}^{k-1} \binom{n}{j}.$$

□

Notice that, by Lemma 2.1.2, we have $C_{2,*} \sim_{T_2^*} N_{2,*}$.

Remark 3.1.2. Since D_* is commutative with trivial grading, we may see the algebra D_* only with the involution algebra structure. Then if $A \in var^{gri}(D_*)$ we use the [21, Theorem 3] to show that if $c_n^{gri}(A)$ is polynomially bounded then

$$A \sim_{T_2^*} (B_1 \oplus \dots \oplus B_m),$$

for some finite dimensional *-superalgebras $B_i, 1 \leq i \leq m$ such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. It means that either $B_i \cong J(B_i)$ is nilpotent or $B_i \cong F + J(B_i)$.

So, in order to classify all subvarieties in $var^{gri}(D_*)$, we just need to know what happens with *-superalgebras of type $F + J$ that satisfy the $(\mathbb{Z}_2, *)$ -identities of D_* .

Before proving that $C_{k,*}$ generates a minimal *-supervariety of polynomial growth we need some results about *-superalgebras of the type $A = F + J$.

Lemma 3.1.3. *Let $A = F + J$ be a *-superalgebra with $J = J_{11} + J_{10} + J_{01} + J_{00}$. If A satisfies the $(\mathbb{Z}_2, *)$ -identities $[y_{1,0}, y_{2,0}] \equiv [z_{1,0}, y_{1,0}] \equiv [y_{1,1}, y_{1,0}] \equiv [z_{1,1}, y_{1,0}] \equiv 0$, then $J_{10} = J_{01} = 0$.*

Proof. In fact, suppose that there exists $a \in J_{10}^{(0)}$. Then we have $a + a^*$ is a symmetric element and $a^* - a$ is a skew element, both with degree 0. Since $[y_{1,0}, y_{2,0}] \equiv [z_{1,0}, y_{1,0}] \equiv 0$ in A , we have $[a + a^*, 1_F] = a^* - a = 0$ and $[a^* - a, 1_F] = a^* + a = 0$, thus $a = 0$. Hence $J_{10}^{(0)} = 0$ and $J_{01}^{(0)} = (J_{10}^{(0)})^* = 0$.

Similarly, we have $J_{10}^{(1)} = J_{01}^{(1)} = 0$. Hence $A = (F + J_{11}) \oplus J_{00}$. □

Corollary 3.1.4. *Let $A = F + J$ be a *-superalgebra with $J = J_{11} + J_{10} + J_{01} + J_{00}$. If $A \in var^{gri}(D_*)$ then $J_{10} = J_{01} = 0$.*

Lemma 3.1.5. *For any $k \geq 2$, $C_{k,*}$ generates a minimal *-supervariety of polynomial growth.*

Proof. Suppose that the algebra $A \in var^{gri}(C_{k,*})$ generates a subvariety of $var^{gri}(C_{k,*})$ and $c^{gri}(A) \approx qn^{k-1}$, for some $q > 0$. We shall prove that in this case $A \sim_{T_2^*} C_{k,*}$ and this will complete the proof.

By Remark 3.1.2 we may assume that

$$A = (B_1 \oplus \dots \oplus B_m),$$

where $B_1, \dots, B_m$ are finite dimensional *-superalgebras such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$.

Since $c_n^{gri}(A) \leq c_n^{gri}(B_1 \oplus \dots \oplus B_m)$, then there exists B_i such that $c_n^{gri}(B_i) \approx bn^{k-1}$, for some $b > 0$. Hence

$$\text{var}^{gri}(C_{k,*}) \supseteq \text{var}^{gri}(A) \supseteq \text{var}^{gri}(F + J(B_i)) \supseteq \text{var}^{gri}(F + J_{11}(B_i))$$

and $c_n^{gri}(B_i) = c_n^{gri}(F + J(B_i)) \approx bn^{k-1}$. By Corollary 3.1.4, since $F + J(B_i) \in \text{var}^{gri}(D_*)$, we get $F + J(B_i) = F + J_{11}(B_i) \oplus J_{00}(B_i)$ and $c_n^{gri}(F + J(B_i)) = c_n^{gri}(F + J_{11}(B_i))$, for n large enough. Then, we may assume that A is a unitary algebra.

Now, since $c_n^{gri}(A) \approx qn^{k-1}$ then $c_n^{gri}(A) = \sum_{j=0}^{k-1} \binom{n}{j} \gamma_j^{gri}(A)$, and, by Proposition 1.3.7, we must have $\gamma_j^{gri}(A) \neq 0$ for all $1 \leq j \leq k-1$. Since $A \in \text{var}^{gri}(C_{k,*})$, we have $\gamma_j^{gri}(A) \leq \gamma_j^{gri}(C_{k,*}) = 1$. Then $c_n^{gri}(A) = c_n^{gri}(C_{k,*})$ for all n and so, $A \sim_{T_2^*} C_{k,*}$. $\square$

At this point, we are in a position to classify the subvarieties of $\text{var}^{gri}(D_*)$.

Theorem 3.1.6. [21, Theorem 7] *Let A be a *-superalgebra such that $\text{var}^{gri}(A) \subsetneq \text{var}^{gri}(D_*)$. Then either $A \sim_{T_2^*} N$ or $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} C_{k,*} \oplus N$, for some $k \geq 2$, where N is a nilpotent *-superalgebra and C is a commutative *-superalgebra with trivial grading and trivial involution.*

Proof. If $\text{var}^{gri}(A) \subsetneq \text{var}^{gri}(D_*)$, then $c_n^{gri}(A) \approx qn^{k-1}$ for some $k \geq 0$, since $\text{var}^{gri}(D_*)$ has almost polynomial growth, by Theorem 1.4.5.

By Remark 3.1.2 we may assume that

$$A = B_1 \oplus \dots \oplus B_m,$$

where $B_1, \dots, B_m$ are finite dimensional *-superalgebras such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. If B_i is nilpotent for all $1 \leq i \leq m$, then we have $A \sim_{T_2^*} N$. Otherwise, by Corollary 3.1.4 we may assume $B_i \cong (F + J_{11}) \oplus J_{00}$ or B_i is a nilpotent *-superalgebra. Hence $A = B_1 \oplus \dots \oplus B_m = B \oplus N$ and

$$c_n^{gri}(A) = c_n^{gri}(B) = \sum_{j=0}^{k-1} \binom{n}{j} \gamma_j^{gri}(B),$$

for n large enough, where B is a unitary *-superalgebra.

If $k = 1$, then $\Gamma_1^{gri} \subseteq \text{Id}^{gri}(B)$, hence B is commutative with trivial grading and trivial involution, and so $A \sim_{T_2^*} C \oplus N$. If $k \geq 2$, we have $\Gamma_k^{gri} \subseteq \text{Id}^{gri}(B)$ then $B \in \text{var}^{gri}(C_{k,*})$. By Lemma 3.1.5, $C_{k,*}$ generates a minimal *-supervariety of polynomial growth. Since $c_n^{gri}(B) \approx qn^{k-1}$ and $c_n^{gri}(C_{k,*}) \approx q'n^{k-1}$ we obtain $B \sim_{T_2^*} C_{k,*}$, so $A \sim_{T_2^*} C_{k,*} \oplus N$. $\square$

A consequence of the previous theorem is the classification of all the $*$ -superalgebras generating minimal varieties lying in the variety generated by D_* .

Corollary 3.1.7. *[21, Corollary 3] A $*$ -superalgebra $A \in \text{var}^{gr}(D_*)$ generates a minimal $*$ -supervariety of polynomial growth if and only if $A \sim_{T_2^*} C_{k,*}$, for some $k \geq 2$.*

Next we describe the sequences of the $*$ -graded cocharacter and of the $*$ -graded colength of the only minimal variety lying in $\text{var}^{gr}(D_*)$.

Since D is commutative, any antiautomorphism of D is an automorphism, so D_* can be viewed as a superalgebra with grading $(D^{(0)}, D^{(1)})$ where $D^{(0)} = D_*^+ = F(1, 1)$ and $D^{(1)} = D_*^- = F(1, -1)$. Thus the descriptions of the sequence of the $*$ -graded cocharacter and the $*$ -graded colength of the minimal variety generated by $C_{k,*}$ correspond to the descriptions given for the minimal variety generated by C_k of $\text{var}^{gr}(F \oplus cF)$, with $c^2 = 1$, proved by Nascimento, dos Santos and Vieira in [26, Theorem 8.3]. Here we restate such results in $*$ -superalgebra language.

Theorem 3.1.8. *[26, Theorem 8.3] For $k \geq 2$, $\chi_n^{gr}(C_{k,*}) = \sum_{j=0}^{k-1} \chi_{(n-j), \emptyset, (j), \emptyset}$ and $l_n^{gr}(C_{k,*}) = k$.*

Proof. For any $0 \leq j \leq k-1$ we consider the highest weight vector $f_{\langle \lambda \rangle} = y_{1,0}^{n-j} z_{1,0}^j$ corresponding to the multipartition $\langle \lambda \rangle = ((n-j), \emptyset, (j), \emptyset)$. Since $j \leq k-1$, evaluating $y_{1,0} = I_k$ and $z_{1,0} = E_1$, we get $f_{\langle \lambda \rangle} = E_1^j \neq 0$ and so $m_{((n-j), \emptyset, (j), \emptyset)} \neq 0$, for all $j = 0, \dots, k-1$.

Thus, by using Theorem 3.1.1 we have

$$c_n^{gr}(C_{k,*}) \geq \sum_{j=0}^{k-1} d_{((n-j), \emptyset, (j), \emptyset)} = \sum_{j=0}^{k-1} \binom{n}{j} = c_n^{gr}(C_{k,*}).$$

We conclude that we must have $m_{((n-j), \emptyset, (j), \emptyset)} = 1$, for all $j = 1, \dots, k$ and zero in other cases. Hence

$$\chi_n^{gr}(C_{k,*}) = \sum_{j=0}^{k-1} \chi_{((n-j), \emptyset, (j), \emptyset)}.$$

As a consequence $l_n^{gr}(C_{k,*}) = k$ and we finish the proof. $\square$

Now we study the subvarieties of the $*$ -supervariety generated by D^{gr} , the algebra D with the grading $D^{gr} = F(1, 1) \oplus F(1, -1)$ and trivial involution.

Since D^{gr} is commutative with trivial involution, we can see D^{gr} only as a superalgebra and we have $\text{var}^{gr}(D^{gr}) = \text{var}^{gr}(F \oplus cF)$, with $c^2 = 1$. Hence, the classification of the $*$ -superalgebras, up to T_2^* -equivalence, inside $\text{var}^{gr}(D^{gr})$ and the classification of the superalgebras inside the $\text{var}^{gr}(F \oplus cF)$, with $c^2 = 1$, are

equivalent. This last classification was given by La Mattina in [20, Theorem 8.2]. Next we present such results in the language of *-superalgebras.

For $k \geq 2$, we have already considered the $k \times k$ identity matrix I_k and the matrix $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in UT_k$, where e'_{ij} s are the usual matrix units.

We denote by C_k^{gr} the commutative subalgebra of UT_k

$$C_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\},$$

with elementary $\mathbb{Z}_2$ -grading induced by $\mathbf{g} = (0, 1, 0, 1 \dots) \in \mathbb{Z}_2^k$ and trivial involution.

For example

$$((C_4^{gr})^{(0)})^+ = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad ((C_4^{gr})^{(1)})^+ = \begin{pmatrix} 0 & b & 0 & d \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remind that $Id^{gr}(D^{gr}) = \langle z_{1,0}, z_{1,1} \rangle_{T_2^*}$ and notice that $C_k^{gr} \in var^{gr}(D^{gr})$. The following result shows the $(\mathbb{Z}_2, *)$ -identities and the *-graded codimension sequence of the *-superalgebra C_k^{gr} , for all $k \geq 2$.

Theorem 3.1.9. [20, Theorem 8.1] *Let $k \geq 2$. Then*

1. $Id^{gr}(C_k^{gr}) = \langle z_{1,0}, z_{1,1}, y_{1,1} \dots y_{k,1} \rangle_{T_2^*}$.
2. $c_n^{gr}(C_k^{gr}) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}, n \rightarrow \infty$.

Proof. First of all notice that since $z_{1,0}, z_{1,1} \in Id^{gr}(C_k^{gr})$, then we have $[y_{1,0}, y_{2,0}], [y_{1,0}, y_{1,1}], [y_{1,1}, y_{2,1}] \in Id^{gr}(C_k^{gr})$. Now let $Q = \langle z_{1,0}, z_{1,1}, y_{1,1} \dots y_{k,1} \rangle_{T_2^*}$. It is easily checked that $Q \subseteq Id^{gr}(C_k^{gr})$, since C_k^{gr} is commutative with trivial involution and $((C_k^{gr})^{(1)})^+)^k = 0$.

Let f be a $(\mathbb{Z}_2, *)$ -identity of C_k^{gr} of degree t . Since the $(\mathbb{Z}_2, *)$ -identities of a unitary *-superalgebra follow from the proper ones, we may assume f is proper. Now, after reducing the polynomial f modulo Q , we obtain: f is the zero polynomial if $t \geq k$; and $f = \alpha y_{1,1} \dots y_{t,1}$ if $t \leq k-1$. In the second case, if $\alpha = 0$, by evaluating $y_{i,1} = E_1$, for all $1 \leq i \leq t$, we get $f = \alpha E_1^t \neq 0$, a contradiction, because $f \in Id^{gr}(C_k^{gr})$. Hence, we must have $\alpha = 0$, and so, $Id^{gr}(C_k^{gr}) = Q$.

This also proves that in case $t \leq k-1$, the polynomial $y_{1,1} \dots y_{t,1}$ forms a basis of the multilinear proper polynomials of degree t modulo $Id^{gr}(C_k^{gr})$. Hence $\gamma_t^{gr}(C_k^{gr}) = 1$ for $0 \leq t \leq k-1$, and $\gamma_t^{gr}(C_k^{gr}) = 0$ for $t \geq k$. Then we get

$$c_n^{gr}(C_k^{gr}) = \sum_{j=0}^{k-1} \binom{n}{j}.$$

□

Notice that, by Lemma 2.2.1, we have $C_2^{gr} \sim_{T_2^*} U_2^{gr}$.

In the next two results, we state that $var^{gr}(C_k^{gr})$ is a minimal *-supervariety of polynomial growth in $var^{gr}(D^{gr})$ and give the classification of the subvarieties of $var^{gr}(D^{gr})$. The proofs are very similar to the proofs of Lemma 3.1.5 and Theorem 3.1.6 and they will be omitted.

Lemma 3.1.10. *For any $k \geq 2$, C_k^{gr} generates a minimal *-supervariety of polynomial growth.*

Theorem 3.1.11. *[20, Theorem 8.2] Let A be a *-superalgebra such that $var^{gr}(A) \subsetneq var^{gr}(D^{gr})$. Then either $A \sim_{T_2^*} N$ or $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} C_k^{gr} \oplus N$, for some $k \geq 2$, where N is a nilpotent *-superalgebra and C is a commutative *-superalgebra with trivial grading and trivial involution.*

A consequence of the previous theorem is the classification of the *-superalgebras generating minimal varieties lying in the variety generated by D^{gr} .

Corollary 3.1.12. *[20, Corollary 8.2] A *-superalgebra $A \in var^{gr}(D^{gr})$ generates a minimal *-supervariety of polynomial growth if and only if $A \sim_{T_2^*} C_k^{gr}$, for some $k \geq 2$.*

We give below the descriptions of the sequences of *-graded cocharacters and of *-graded colengths of the only minimal variety lying in $var^{gr}(D^{gr})$. We have noticed previously that these descriptions correspond to the ones for the superalgebra C_k given by Nascimento, dos Santos and Vieira in [26, Theorem 8.3].

Theorem 3.1.13. *[26, Theorem 8.3] For $k \geq 2$, $\chi_n^{gr}(C_k^{gr}) = \sum_{j=0}^{k-1} \chi_{(n-j),(j),\emptyset,\emptyset}$ and $l_n^{gr}(C_k^{gr}) = k$.*

Proof. For any $1 \leq j \leq k-1$ we consider the highest weight vector $f_{\langle \lambda \rangle} = y_{1,0}^{n-j} y_{1,1}^j$ corresponding to the multipartition $\langle \lambda \rangle = ((n-j), (j), \emptyset, \emptyset)$. Evaluating $y_{1,0} = I_k$ and $y_{1,1} = E_1$, we get $f_{\langle \lambda \rangle} = E_1^j \neq 0$, since $j \leq k-1$ and so $m_{((n-j),(j),\emptyset,\emptyset)} \neq 0$, for all $j = 0, \dots, k-1$.

Thus by using Theorem 3.1.9 we have

$$c_n^{gr}(C_k^{gr}) \geq \sum_{j=0}^{k-1} d_{((n-j),(j),\emptyset,\emptyset)} = \sum_{j=0}^{k-1} \binom{n}{j} = c_n^{gr}(C_k^{gr}).$$

We conclude $m_{((n-j),(j),\emptyset,\emptyset)} = 1$, for all $j = 1, \dots, k$ and zero in other cases. Hence

$$\chi_n^{gr}(C_k^{gr}) = \sum_{j=0}^{k-1} \chi_{((n-j),(j),\emptyset,\emptyset)} \text{ and so } l_n^{gr}(C_k^{gr}) = k.$$

□

Finally we study the subvarieties of the *-supervariety generated by D^{gri} , the algebra D with the grading $F(1, 1) \oplus F(1, -1)$ and endowed with the exchange involution.

For $k \geq 2$, we have already considered the $k \times k$ identity matrix I_k and the matrix $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in UT_k$, where e'_{ij} s are the usual matrix units.

We denote by C_k^{gri} the commutative subalgebra of UT_k

$$C_k = \{ \alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F \}$$

with elementary $\mathbb{Z}_2$ -grading induced by $\mathbf{g} = (0, 1, 0, 1 \dots) \in \mathbb{Z}_2^k$ and endowed with the involution given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$

For instance

$$((C_4^{gri})^{(0)})^+ = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \text{ and } ((C_4^{gri})^{(1)})^- = \begin{pmatrix} 0 & b & 0 & d \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We know that $Id^{gri}(D^{gri}) = \langle z_{1,0}, y_{1,1} \rangle_{T_2^*}$. Notice that $C_k^{gri} \in var^{gri}(D^{gri})$.

Next we calculate the T_2^* -ideal and the *-graded codimension of C_k^{gri} .

Theorem 3.1.14. *Let $k \geq 2$. Then*

1. $Id^{gri}(C_k^{gri}) = \langle z_{1,0}, y_{1,1}, z_{1,1} \dots z_{k,1} \rangle_{T_2^*}$.
2. $c_n^{gri}(C_k^{gri}) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}, n \rightarrow \infty$.

Proof. First of all, notice that $z_{1,0}, y_{1,1} \in Id^{gri}(C_k^{gr})$, then we have $[y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,1}], [z_{1,1}, z_{2,1}] \in Id^{gri}(C_k^{gri})$. Now let $Q = \langle z_{1,0}, y_{1,1}, z_{1,1} \dots z_{k,1} \rangle_{T_2^*}$. Since

$$(((C_k^{gri})^{(0)})^-) = 0, (((C_k^{gri})^{(1)})^+) = 0 \text{ and } (((C_k^{gri})^{(1)})^-)^k = 0,$$

it is easily checked that $Q \subseteq Id^{gri}(C_k^{gri})$.

Let f be a $(\mathbb{Z}_2, *)$ -identity of C_k^{gri} of degree t . Since the $(\mathbb{Z}_2, *)$ -identities of a unitary *-superalgebra follow from the proper ones, we may assume f is proper. Now, if reduce the polynomial f modulo Q then we obtain: f is the zero polynomial if $t \geq k$; and $f = \alpha z_{1,1} \dots z_{t,1}$ if $t \leq k-1$. In this second case, if $\alpha = 0$, by

evaluating $z_{i,1} = E_1$, for all $1 \leq i \leq t$, we get $f = \alpha E_1^t \neq 0$, a contradiction, because $f \in Id^{gri}(C_k^{gri})$. Hence, we must have $\alpha = 0$, and so, $Id^{gri}(C_k^{gri}) = Q$.

This also proves that in case $t \leq k - 1$, the polynomial $z_{1,1} \cdots z_{t,1}$ forms a basis of the multilinear proper polynomials of degree t modulo $Id^{gri}(C_k^{gri})$. Hence $\gamma_t^{gri}(C_k^{gri}) = 1$ for $0 \leq t \leq k - 1$ and $\gamma_t^{gri}(C_k^{gri}) = 0$ for $t \geq k$. Then we get

$$c_n^{gri}(C_k^{gri}) = \sum_{j=0}^{k-1} \binom{n}{j}.$$

□

Notice that, by Lemma 2.2.1, we have $C_2^{gri} \sim_{T_2^*} N_2^{gri}$.

Next we prove that C_k^{gri} generates the only minimal *-supervariety of polynomial growth in $var^{gri}(D^{gri})$.

Remark 3.1.15. We may see D^{gri} only as a superalgebra by establishing $D^{gri} = D^{(0)} \oplus D^{(1)}$, where $D^{(0)} = ((D^{gri})^{(0)})^+$ and $D^{(1)} = ((D^{gri})^{(1)})^-$. This way, we have $var^{gr}(D^{gri}) = var^{gr}(F \oplus cF)$, with $c^2 = 1$. Hence, the classification of the *-superalgebras, up to T_*^2 -equivalence, inside $var^{gri}(D^{gri})$ and the classification of the superalgebras inside the $var^{gr}(F \oplus cF)$, with $c^2 = 1$, are equivalent.

By using [4, Proposition 4], this equivalence also implies that if $A \in var^{gri}(D^{gri})$ has polynomial growth then

$$A \sim_{T_2^*} (B_1 \oplus \dots \oplus B_m),$$

for some finite dimensional *-superalgebras $B_i, 1 \leq i \leq m$ such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. It means that either $B_i \cong J(B_i)$ is nilpotent or $B_i \cong F + J(B_i)$.

The next result is a consequence of Lemma 3.1.3.

Corollary 3.1.16. *Let $A = F + J$ be a *-superalgebra with $J = J_{11} + J_{10} + J_{01} + J_{00}$. If $A \in var^{gri}(D^{gri})$ then $J_{10} = J_{01} = 0$.*

Lemma 3.1.17. *For any $k \geq 2$, C_k^{gri} generates a minimal *-supervariety of polynomial growth.*

Proof. By Remark 3.1.15, if $A \in var^{gri}(C_k^{gri})$ generates a subvariety of $var^{gri}(C_k^{gri})$ and $c^{gri}(A) \approx qn^{k-1}$, for some $q > 0$, we may assume that

$$A = B_1 \oplus \dots \oplus B_m,$$

where $B_1, \dots, B_m$ are finite dimensional *-superalgebras such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$.

Also, by the proof of Lemma 3.1.5 we can assume that A is a unitary algebra.

Now, since $c_n^{gri}(A) \approx qn^{k-1}$ then $c_n^{gri}(A) = \sum_{j=0}^{k-1} \binom{n}{j} \gamma_j^{gri}(A)$, and, by Proposition 1.3.7, we must have $\gamma_j^{gri}(A) \neq 0$ for all $1 \leq j \leq k-1$. Since $A \in var^{gri}(C_k^{gri})$, we have $\gamma_j^{gri}(A) \leq \gamma_j^{gri}(C_k^{gri}) = 1$. It implies that $c_n^{gri}(A) = c_n^{gri}(C_k^{gri})$ for all n , thus $A \sim_{T_2^*} C_k^{gri}$. $\square$

Now we are in position to classify the subvarieties of $var^{gri}(D^{gri})$.

Theorem 3.1.18. *Let A be a *-superalgebra such that $var^{gri}(A) \subsetneq var^{gri}(D^{gri})$. Then either $A \sim_{T_2^*} N$ or $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} C_k^{gri} \oplus N$, for some $k \geq 2$, where N is a nilpotent *-superalgebra and C is a commutative *-superalgebra with trivial grading and trivial involution.*

Proof. By Theorem 1.4.6, if $var^{gri}(A) \subsetneq var^{gri}(D^{gri})$, then $c_n^{gri}(A) \approx qn^{k-1}$ for some $r \geq 0$. By the Remark 3.1.15 we may assume that

$$A = B_1 \oplus \dots \oplus B_m,$$

where $B_1, \dots, B_m$ are finite dimensional *-superalgebras such that $\dim \frac{B_i}{J(B_i)} \leq 1$, for all $1 \leq i \leq m$. By using the same arguments as in the proof of Theorem 3.1.6 we may write $A = B_1 \oplus \dots \oplus B_m = B \oplus N$ and

$$c_n^{gri}(A) = c_n^{gri}(B) = \sum_{j=0}^{k-1} \binom{n}{j} \gamma_j^{gri}(B),$$

for n large enough, where B is a unitary *-superalgebra.

If $k = 1$, then $\Gamma_1^{gri} \subseteq Id^{gri}(B)$, hence B is commutative with trivial grading and trivial involution, then $A \sim_{T_2^*} C \oplus N$. If $k \geq 2$, this implies that $\Gamma_k^{gri} \subseteq Id^{gri}(B)$, and so $B \in var^{gri}(C_k^{gri})$. By Lemma 3.1.17, we have C_k^{gri} generates a minimal *-supervariety of polynomial growth and since $c_n^{gri}(B) \approx qn^{k-1}$ and $c_n^{gri}(C_k^{gri}) \approx q'n^{k-1}$ we obtain that $B \sim_{T_2^*} C_k^{gri}$, then $A \sim_{T_2^*} C_k^{gri} \oplus N$. $\square$

A consequence of the previous theorem is the classification of all the *-superalgebras generating minimal varieties lying in $var^{gri}(D^{gri})$.

Corollary 3.1.19. *A *-superalgebra $A \in var^{gri}(D^{gri})$ generates a minimal *-supervariety of polynomial growth if and only if $A \sim_{T_2^*} C_k^{gri}$, for some $k \geq 2$.*

Next we describe the sequences of *-graded cocharacter and of *-graded colengths of the only minimal variety lying in $var^{gri}(D^{gri})$. As we have noticed before, these descriptions correspond to those ones for the superalgebra C_k given by Nascimento, dos Santos and Vieira in [26, Theorem 8.3].

Theorem 3.1.20. *For $k \geq 2$, $\chi_n^{gri}(C_k^{gri}) = \sum_{j=0}^{k-1} \chi_{(n-j), \emptyset, \emptyset, (j)}$ and $l_n^{gri}(C_k^{gri}) = k$.*

Proof. For any $0 \leq j \leq k-1$ we consider the highest weight vector $f_{\langle \lambda \rangle} = y_{1,0}^{n-j} z_{1,1}^j$ corresponding to the multipartition $\langle \lambda \rangle = ((n-j), \emptyset, \emptyset, (j))$. Since $j \leq k-1$, by evaluating $y_{1,0} = I_k$ and $z_{1,1} = E_1$, we get $f_{\langle \lambda \rangle} = E_1^j \neq 0$, and so $m_{((n-j), \emptyset, \emptyset, (j))} \neq 0$, for all $j = 0, \dots, k-1$.

Thus by using Theorem 3.1.14 we have

$$c_n^{gri}(C_k^{gri}) \geq \sum_{j=0}^{k-1} d_{((n-j), \emptyset, \emptyset, (j))} = \sum_{j=0}^{k-1} \binom{n}{j} = c_n^{gri}(C_k^{gri}).$$

We conclude that $m_{((n-j), \emptyset, \emptyset, (j))} = 1$, for all $j = 1, \dots, k$ and zero in other cases. Hence

$$\chi_n^{gri}(C_k^{gri}) = \sum_{j=0}^{k-1} \chi_{((n-j), \emptyset, \emptyset, (j))} \text{ and so } l_n^{gri}(C_k^{gri}) = k.$$

□

3.2 Some *-superalgebras with small *-graded co-length

For $k \geq 1$ we denote by G_k the Grassmann algebra with 1 on a k -dimensional vector space over F , i.e.,

$$G_k = \langle 1, e_1, \dots, e_k \mid e_i e_j = -e_j e_i \rangle.$$

We write G_k to mean G_k with trivial grading and write G_k^{gr} to mean G_k with canonical grading.

We also consider three involutions on G_k denoted by τ , ψ and ρ defined by

$$\tau : e_i \mapsto -e_i, \quad \psi : e_i \mapsto e_i \text{ and } \rho : e_i \mapsto (-1)^i e_i,$$

for all $1 \leq i \leq k$. We denote by $G_{2,*}$ and $G_{2,*}^{gr}$ the algebras G_2 and G_2^{gr} , respectively, endowed with the involution $*$ = τ , $*$ = ψ or $*$ = ρ . Observe that $G_{2,*}$ and $G_{2,*}^{gr}$ are *-superalgebras if $*$ = τ , $*$ = ψ or $*$ = ρ .

The algebra $G_{2,\tau}$ was initially studied by La Mattina and Misso in [22, Lemma 16] as an *-algebra, where the authors calculated its T^* -ideal and its *-codimension. After in a joint work with La Mattina and Vieira [23], we describe the *-cocharacter of $G_{2,\tau}$ and study the *-algebra $G_{3,\tau}$. Here we present such results in *-superalgebra language.

Lemma 3.2.1. *For the *-superalgebras $G_{2,\tau}$ and $G_{3,\tau}$ we have*

$$1. \text{Id}^{gri}(G_{2,\tau}) = \langle y_{1,1}, z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{2,0}], z_{1,0} z_{2,0} + z_{2,0} z_{1,0}, z_{1,0} z_{2,0} z_{3,0} \rangle_{T_2^*};$$

2. $c_n^{gri}(G_{2,\tau}) = 1 + n + \frac{n(n-1)}{2}$;
3. $\chi_n^{gri}(G_{2,\tau}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-2),\emptyset,(1^2),\emptyset}$ and $l_n^{gri}(G_{2,\tau}) = 3$.
4. $Id^{gri}(G_{3,\tau}) = \langle y_{1,1}, z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{2,0}], z_{1,0}z_{2,0} + z_{2,0}z_{1,0}, z_{1,0}z_{2,0}z_{3,0}z_{4,0} \rangle_{T_2^*}$;
5. $c_n^{gri}(G_{3,\tau}) = 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}$;
6. $\chi_n^{gri}(G_{3,\tau}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-2),\emptyset,(1^2),\emptyset} + \chi_{(n-3),\emptyset,(1^3),\emptyset}$ and $l_n^{gri}(G_{3,\tau}) = 4$.

Proof. We will prove the results about the *-superalgebra $G_{2,\tau}$ and, using similar argument, we can prove the results about $G_{3,\tau}$.

Let $I = \langle y_{1,1}, z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{2,0}], z_{1,0}z_{2,0} + z_{2,0}z_{1,0}, z_{1,0}z_{2,0}z_{3,0} \rangle_{T_2^*}$. By noticing that $(G_{2,\tau}^{(0)})^+ = \text{span}_F\{1\}$ and $(G_{2,\tau}^{(0)})^- = \text{span}_F\{e_1, e_2, e_1e_2\}$, we can check that $I \subseteq Id^{gri}(G_{2,\tau})$. Moreover, we can see that the polynomials

$$y_{1,0} \cdots y_{n,0}, y_{1,0} \cdots \widehat{y_{i,0}} \cdots y_{n,0}z_{i,0}, y_{1,0} \cdots \widehat{y_{i,0}} \cdots \widehat{y_{j,0}} \cdots y_{n,0}z_{i,0}z_{j,0}, \quad 1 \leq i < j \leq n,$$

generate $P_n^{gri}(\text{mod } P_n^{gri} \cap I)$. We claim that they are linearly independent modulo $Id^{gri}(G_{2,\tau})$.

If $f \in P_n^{gri} \cap Id^{gri}(G_{2,\tau})$ is a linear combination of the above polynomials, by multihomogeneity of T_2^* -ideals we may write that either $f = \alpha y_{1,0} \cdots y_{n,0}$, or $f = \beta y_{1,0} \cdots y_{n-1,0}z_{n,0}$, or $f = \delta y_{1,0} \cdots y_{n-2,0}z_{n-1,0}z_{n,0}$. If we evaluate $y_{1,0} = \dots = y_{n,0} = 1$ we get $\alpha = 0$. If we evaluate $y_{1,0} = \dots = y_{n-1,0} = 1, z_{n,0} = e_1$ we have $\beta = 0$. Finally if we evaluate $y_{1,0} = \dots = y_{n-2,0} = 1, z_{n-1,0} = e_1, z_{n,0} = e_2$ we obtain $\delta = 0$. Then this implies $f \in P_n^{gri} \cap I$ and so $Id^{gri}(G_{2,\tau}) = I$. This also proves that the above polynomials form a basis of $P_n^{gri}(\text{mod } P_n^{gri} \cap Id^{gri}(G_{2,\tau}))$ and so

$$c_n^{gri}(G_{2,\tau}) = 1 + n + \frac{n(n-1)}{2}.$$

In order to prove that $\chi_n^{gri}(G_{2,\tau}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-2),\emptyset,(1^2),\emptyset}$, we start by noticing that

$$d_{(n),\emptyset,\emptyset,\emptyset} + d_{(n-1),\emptyset,(1),\emptyset} + d_{(n-2),\emptyset,(1^2),\emptyset} = 1 + n + \frac{n(n-1)}{2} = c_n^{gri}(G_{2,\tau}).$$

Then, since $m_{(n),\emptyset,\emptyset,\emptyset} = 1$, we just need to find a highest weight vector for each multipartitions $((n-1), \emptyset, (1), \emptyset)$ and $((n-2), \emptyset, (1^2), \emptyset)$ which is not a $(\mathbb{Z}_2, *)$ -identity of $G_{2,\tau}$, to conclude that $\chi_n^{gri}(G_{2,\tau})$ has the wished decomposition.

In fact, let $f = y_{1,0}^{n-1} z_{1,0}$ and $g = y_{1,0}^{n-2} [z_{1,0}, z_{2,0}]$ be the highest weight vectors associated to the multipartitions $((n-1), \emptyset, (1), \emptyset)$ and $((n-2), \emptyset, (1^2), \emptyset)$, respectively, and corresponding to the multitableaux:

$$(\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n-1 \\ \hline \end{array}, \emptyset, \begin{array}{|c|} \hline n \\ \hline \end{array}, \emptyset) \text{ and } (\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 \\ \hline \end{array}, \emptyset, \begin{array}{|c|} \hline \frac{n-1}{n} \\ \hline \end{array}, \emptyset), \quad (3.2.1)$$

By evaluating $y_{1,0} = 1$, $z_{1,0} = e_1$ and $z_{2,0} = e_2$, we get $f = e_1 \neq 0$ and $g = 2e_1e_2 \neq 0$; then f and g are not $(\mathbb{Z}_2, *)$ -identities of $G_{2,*}$ and the proof is complete. $\square$

Next, we study the $*$ -superalgebra G_2^{gr} endowed with the involution τ , ψ and ρ . We have the following:

Lemma 3.2.2. *For the algebra $G_{2,\tau}^{gr}$ we have*

1. $Id^{gr_i}(G_{2,\tau}^{gr}) = \langle y_{1,1}, z_{1,0}z_{2,0}, z_{1,0}z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], z_{1,1}z_{2,1} + z_{2,1}z_{1,1}, z_{1,1}z_{2,1}z_{3,1} \rangle_{T_2^*}$
and $c_n^{gr_i}(G_{2,\tau}^{gr}) = 1 + 2n + \frac{n(n-1)}{2}$;
2. $\chi_n^{gr_i}(G_{2,\tau}^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)} + \chi_{(n-2),\emptyset,\emptyset,(1^2)}$ and $l_n^{gr_i}(G_{2,\tau}^{gr}) = 4$.
3. $Id^{gr_i}(G_{2,\psi}^{gr}) = \langle z_{1,1}, z_{1,0}z_{2,0}, z_{1,0}y_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], y_{1,1}y_{2,1} + y_{2,1}y_{1,1}, y_{1,1}y_{2,1}y_{3,1} \rangle_{T_2^*}$
and $c_n^{gr_i}(G_{2,\psi}^{gr}) = 1 + 2n + \frac{n(n-1)}{2}$;
4. $\chi_n^{gr_i}(G_{2,\psi}^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),(1),\emptyset,\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)} + \chi_{(n-2),(1),\emptyset,(1)}$ and $l_n^{gr_i}(G_{2,\psi}^{gr}) = 4$.
5. $Id^{gr_i}(G_{2,\rho}^{gr}) = \langle z_{1,0}, z_{1,1}z_{2,1}, y_{1,1}y_{2,1}, [y_{1,0}, y_{1,1}], [y_{1,0}, z_{1,1}], y_{1,1}z_{1,1} + z_{1,1}y_{1,1} \rangle_{T_2^*}$ and $c_n^{gr_i}(G_{2,\rho}^{gr}) = 1 + 2n + \frac{n(n-1)}{2}$;
6. $\chi_n^{gr_i}(G_{2,\psi}^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),(1),\emptyset,\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)} + \chi_{(n-2),(1),\emptyset,(1)}$ and $l_n^{gr_i}(G_{2,\rho}^{gr}) = 4$.

Proof. First we consider $G_{2,\tau}^{gr}$ and notice that we have $((G_{2,\tau}^{gr})^{(0)})^+ = \text{span}_F\{1\}$, $((G_{2,\tau}^{gr})^{(0)})^- = \text{span}_F\{e_1e_2\}$, and $((G_{2,\tau}^{gr})^{(1)})^- = \text{span}_F\{e_1, e_2\}$.

Let $J = \langle y_{1,1}, z_{1,0}z_{2,0}, z_{1,0}z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], z_{1,1}z_{2,1} + z_{2,1}z_{1,0}, z_{1,1}z_{2,1}z_{3,1} \rangle_{T_2^*}$. We can see $J \subseteq Id^{gr_i}(G_{2,\tau}^{gr})$ and that the polynomials

$$y_{1,0} \cdots y_{n,0}, y_{1,0} \cdots \widehat{y_{i,0}} \cdots y_{n,0}z_{i,0}, y_{1,0} \cdots \widehat{y_{i,0}} \cdots y_{n,0}z_{i,1}, y_{1,0} \cdots \widehat{y_{i,0}} \cdots \widehat{y_{j,0}} \cdots y_{n,0}z_{i,1}z_{j,1},$$

$1 \leq i < j \leq n$, generate $P_n^{gr_i}(\text{mod } P_n^{gr_i} \cap J)$. We claim that they are linearly independent modulo $Id^{gr_i}(G_{2,\tau}^{gr})$.

If $f \in P_n^{gr_i} \cap Id^{gr_i}(G_{2,\tau}^{gr})$ is a linear combination of the above polynomials, by multihomogeneity of T_2^* -ideals we may write that either $f = \delta y_{1,0} \cdots y_{n,0}$ or $f = \alpha y_{1,0} \cdots y_{n-1,0}z_{n,0}$, or $f = \beta y_{1,0} \cdots y_{n-1,0}z_{n,1}$, or $f = \gamma y_{1,0} \cdots y_{n-2,0}z_{n-1,1}z_{n,1}$. If we evaluate $y_{1,0} = \cdots = y_{n,0} = 1$ we get $\delta = 0$. If we evaluate $y_{1,0} = \cdots = y_{n-1,0} = 1$, $z_{n,0} = e_1e_2$ and $z_{n,1} = e_1$ we have $\alpha = \beta = 0$. Finally, if we evaluate $y_{1,0} = \cdots = y_{n-2,0} = 1$, $z_{n-1,1} = e_1$, $z_{n,1} = e_2$ we obtain $\gamma = 0$. Then this implies $f \in P_n^{gr_i} \cap J$ and so $Id^{gr_i}(G_{2,\tau}^{gr}) = J$. Moreover, this also proves that the above polynomials form a basis of $P_n^{gr_i}(\text{mod } P_n^{gr_i} \cap Id^{gr_i}(G_{2,\tau}^{gr}))$ and so $c_n^{gr_i}(G_{2,\tau}^{gr}) = 1 + 2n + \frac{n(n-1)}{2}$.

In order to prove that $\chi_n^{gr}(G_{2,\tau}^{gr})$ has the wished decomposition, we start by noticing that

$$d_{(n),\emptyset,\emptyset,\emptyset} + d_{(n-1),\emptyset,(1),\emptyset} + d_{(n-1),\emptyset,\emptyset,(1)} + d_{(n-2),\emptyset,\emptyset,(1^2)} = 1 + 2n + \frac{n(n-1)}{2} = c_n^{gr}(G_{2,\tau}^{gr}).$$

Then, since $m_{(n),\emptyset,\emptyset,\emptyset} = 1$, we just need to find a highest weight vector for each multipartitions $((n-1), \emptyset, (1), \emptyset)$, $((n-1), \emptyset, \emptyset, (1))$ and $((n-2), \emptyset, \emptyset, (1^2))$, which is not a $(\mathbb{Z}_2, *)$ -identity of $G_{2,\tau}^{gr}$ to conclude that $\chi_n^{gr}(G_{2,\tau}^{gr})$ has the wished decomposition.

We consider $f = y_{1,0}^{n-1} z_{1,0}$, $g = y_{1,0}^{n-1} z_{1,1}$ and $h = y_{1,0}^{n-2} [z_{1,1}, z_{2,1}]$, the standard highest weight vectors associated to the multipartitions $((n-1), \emptyset, (1), \emptyset)$, $((n-1), \emptyset, \emptyset, (1))$ and $((n-2), \emptyset, \emptyset, (1^2))$, respectively.

By making the evaluation $y_{1,0} = 1$, $z_{1,0} = e_1 e_2$, $z_{1,1} = e_1$ and $z_{2,1} = e_2$, we get $f = e_1 e_2 \neq 0$, $g = e_1 \neq 0$ and $h = 2e_1 e_2 \neq 0$; then f, g and h are not $(\mathbb{Z}_2, *)$ -identities of $G_{2,*}^{gr}$ and we have

$$\chi_n^{gr}(G_{2,\tau}^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)} + \chi_{(n-2),\emptyset,\emptyset,(1^2)}.$$

We can easily prove the results about $G_{2,\psi}^{gr}$ and $G_{2,\rho}^{gr}$ by noticing that

$$((G_{2,\psi}^{gr})^{(0)})^+ = \text{span}_F\{1\}, \quad ((G_{2,\psi}^{gr})^{(0)})^- = \text{span}_F\{e_1 e_2\}, \quad ((G_{2,\psi}^{gr})^{(1)})^+ = \text{span}_F\{e_1, e_2\},$$

$$((G_{2,\rho}^{gr})^{(0)})^+ = \text{span}_F\{1, e_1 e_2\}, \quad ((G_{2,\rho}^{gr})^{(1)})^- = \text{span}_F\{e_1\}, \quad ((G_{2,\rho}^{gr})^{(1)})^+ = \text{span}_F\{e_2\},$$

and following the same arguments of the first part of this lemma. □

Now we denote by $G_{2,\tau}^{gr}$ and $G_{2,\rho}^{gr}$ to be the $*$ -superalgebra G_2 with the grading $G_2 = (F1 + Fe_1) \oplus (Fe_2 + Fe_1 e_2)$ and endowed by the involution τ and ρ , respectively.

Lemma 3.2.3. *For the algebras $G_{2,\tau}^{gr}$ and $G_{2,\rho}^{gr}$ we have*

1. $Id^{gr}(G_{2,\tau}^{gr}) = \langle y_{1,1}, z_{1,0} z_{2,0}, z_{1,1} z_{2,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], z_{1,0} z_{1,1} + z_{1,1} z_{1,0} \rangle_{T_2^*}$;
2. $Id^{gr}(G_{2,\rho}^{gr}) = \langle z_{1,1}, z_{1,0} z_{2,0}, y_{1,1} y_{2,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], z_{1,0} y_{z,1} + y_{z,1} z_{1,0} \rangle_{T_2^*}$;
3. $c_n^{gr}(G_{2,\tau}^{gr}) = c_n^{gr}(G_{2,\rho}^{gr}) = 1 + 2n + \frac{n(n-1)}{2}$;
4. $\chi_n^{gr}(G_{2,\tau}^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)} + \chi_{(n-2),\emptyset,(1),(1)}$
5. $\chi_n^{gr}(G_{2,\rho}^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-1),(1),\emptyset,\emptyset} + \chi_{(n-2),(1),(1),\emptyset}$
6. $l_n^{gr}(G_{2,\tau}^{gr}) = l_n^{gr}(G_{2,\rho}^{gr}) = 4$.

Proof. Similarly to the previous lemma, we can easily check these results by noticing that

$$\begin{aligned} ((G_{2,\tau}^{gri})^{(0)})^+ &= \text{span}_F\{1\}, \quad ((G_{2,\tau}^{gri})^{(0)})^- = \text{span}_F\{e_1\}, \quad ((G_{2,\tau}^{gri})^{(1)})^- = \text{span}_F\{e_2, e_1e_2\}, \\ ((G_{2,\rho}^{gri})^{(0)})^+ &= \text{span}_F\{1\}, \quad ((G_{2,\rho}^{gri})^{(0)})^- = \text{span}_F\{e_1\}, \quad ((G_{2,\rho}^{gri})^{(1)})^+ = \text{span}_F\{e_2, e_1e_2\}. \end{aligned}$$

□

Moreover, we also can obtain the following results.

Lemma 3.2.4. *For the algebras $G_{2,\tau} \oplus C_{3,*}$, $G_{2,\tau} \oplus C_2^{gr}$ and $G_{2,\tau} \oplus C_2^{gri}$ we have*

1. $c_n^{gri}(G_{2,\tau} \oplus C_{3,*}) = n^2 + 1$ and $c_n^{gri}(G_{2,\tau} \oplus C_2^{gr}) = c_n^{gri}(G_{2,\tau} \oplus C_2^{gri}) = \frac{n^2 + 3n + 2}{2}$
2. $\chi_n^{gri}(G_{2,\tau} \oplus C_{3,*}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-2),\emptyset,(1^2),\emptyset} + \chi_{(n-2),\emptyset,(2),\emptyset}$;
3. $\chi_n^{gri}(G_{2,\tau} \oplus C_2^{gr}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-2),\emptyset,(1^2),\emptyset} + \chi_{(n-1),(1),\emptyset,\emptyset}$;
4. $\chi_n^{gri}(G_{2,\tau} \oplus C_2^{gri}) = \chi_{(n),\emptyset,\emptyset,\emptyset} + \chi_{(n-1),\emptyset,(1),\emptyset} + \chi_{(n-2),\emptyset,(1^2),\emptyset} + \chi_{(n-1),\emptyset,\emptyset,(1)}$;
5. $l_n^{gri}(G_{2,\tau} \oplus C_{3,*}) = l_n^{gri}(G_{2,\tau} \oplus C_2^{gr}) = l_n^{gri}(G_{2,\tau} \oplus C_2^{gri}) = 4$.

We finish this section by presenting the *-superalgebra $D_* \oplus D^{gr} \oplus D^{gri}$ whose proprieties will be important in order to classify the *-supervarieties with *-colength bounded by 3.

Lemma 3.2.5. *For the *-superalgebra $S = D_* \oplus D^{gr} \oplus D^{gri}$ we have:*

1. $Id^{gri}(S) = \langle z_{1,0}y_{1,1}, z_{1,0}z_{1,1}, y_{1,1}z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], [y_{1,0}, y_{1,1}], [y_{1,0}, z_{1,1}], [y_{1,1}, y_{2,1}], [z_{1,0}, z_{2,0}], [z_{1,1}, z_{2,1}] \rangle_{T_2^*}$;
2. $c_n^{gri}(S)$ grows exponentially.

Proof. Let $I = \langle z_{1,0}y_{1,1}, z_{1,0}z_{1,1}, y_{1,1}z_{1,1}, [y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], [y_{1,0}, y_{1,1}], [y_{1,0}, z_{1,1}], [y_{1,1}, y_{2,1}], [z_{1,0}, z_{2,0}], [z_{1,1}, z_{2,1}] \rangle_{T_2^*}$. Since $Id^{gri}(S) = Id^{gri}(D_*) \cap Id^{gri}(D^{gr}) \cap Id^{gri}(D^{gri})$ we have $I \subseteq Id^{gri}(S)$. Let us check the opposite inclusion.

Let f be a $(\mathbb{Z}_2, *)$ -identity of S . Since D^{gri} is an algebra with 1, we can assume f is a multilinear proper polynomial of degree $t > 0$. After reducing the polynomial f modulo I , we obtain that either $f = \alpha z_{1,0} \cdots z_{t,0}$, or $f = \alpha y_{1,1} \cdots y_{t,1}$, or $f = \alpha z_{1,1} \cdots z_{t,1}$. Denote by $a = (1, -1)$ and, for all $1 \leq i \leq t$, make the evaluation $z_{i,0} = (a, 0, 0)$, $y_{i,1} = (0, a, 0)$, $z_{i,1} = (0, 0, a)$ then we get $\alpha \neq 0$ in all cases. Since $f \in Id^{gri}(S)$, we must have $\alpha = 0$ and so $Id^{gri}(S) = I$.

It also proves that for all $t \geq 1$ the polynomials $\{z_{1,0} \cdots z_{t,0}\}$, $\{y_{1,1} \cdots y_{t,1}\}$, $\{z_{1,1} \cdots z_{t,1}\}$ form a basis for the proper polynomials of degree t modulo $Id^{gri}(S)$. Moreover, since D_*, D^{gr}, D^{gri} lies in $var^{gri}(S)$ and their *-graded codimensions grow exponentially, it follows that $c_n^{gri}(S)$ also grows exponentially. □

For all $k, r, t \geq 1$ the $*$ -superalgebra $C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}$ satisfies all $(\mathbb{Z}_2, *)$ -identities of $D_* \oplus D^{gr} \oplus D^{gri}$. In particular case, notice that $C_{1,*}, C_1^{gr}$ and C_1^{gri} are T_*^2 -equivalent to F . For example, we have $C_{1,*} \oplus C_t^{gr} \oplus C_r^{gri} \sim_{T_*^2} C_t^{gr} \oplus C_r^{gri}$, $C_{k,*} \oplus C_1^{gr} \oplus C_r^{gri} \sim_{T_*^2} C_{k,*} \oplus C_r^{gri}$ and $C_{k,*} \oplus C_t^{gr} \oplus C_1^{gri} \sim_{T_*^2} C_{k,*} \oplus C_t^{gr}$ for all $k, r, t \geq 1$.

We establish the T_*^2 -ideal, the $*$ -graded codimension and the $*$ -graded colength of this algebras in the following result.

Lemma 3.2.6. *For all $k, r, t \geq 1$ we have*

1. $Id^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = \langle Id^{gri}(S), z_{1,0} \cdots z_{k,0}, y_{1,1} \cdots y_{t,1}, z_{1,1} \cdots z_{r,1} \rangle_{T_*^2}$;
2. $c_n^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} + \sum_{j=1}^{t-1} \binom{n}{j} + \sum_{j=1}^{r-1} \binom{n}{j}$;
3. $\chi_n^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = \sum_{j=0}^{k-1} \chi_{(n-j), \emptyset, (j), \emptyset} + \sum_{j=1}^{t-1} \chi_{(n-j), (j), \emptyset, \emptyset} + \sum_{j=1}^{r-1} \chi_{(n-j), \emptyset, \emptyset, (j)}$
and $l_n^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = k + t + r - 2$.

Proof. Let $Q = \langle Id^{gri}(S), z_{1,0} \cdots z_{k,0}, y_{1,1} \cdots y_{t,1}, z_{1,1} \cdots z_{r,1} \rangle_{T_*^2}$. It is easily checked that $Q \subseteq Id^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri})$.

Let f be a $(\mathbb{Z}_2, *)$ -identity of $C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}$ of degree m . Since the $(\mathbb{Z}_2, *)$ -identities of a unitary $*$ -superalgebra follow from the proper ones, we may assume f is proper. Now, if we reduce the polynomial f modulo Q , we obtain that: either f is the zero polynomial if $m \geq \max\{k, t, r\}$; or $f = \alpha z_{1,0} \cdots z_{m,0}$ if $m < k$; or $f = \alpha y_{1,1} \cdots y_{m,1}$ if $m < t$; or $f = \alpha z_{1,1} \cdots z_{m,1}$ if $m < r$.

In the second case, if $m < k$ and $\alpha = 0$, by evaluating $z_{i,0} = (E_1, 0, 0)$, for all $1 \leq i \leq m$, we get $f = \alpha(E_1^m, 0, 0) \neq 0$, a contradiction, since $f \in Id^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri})$. Then we must have $\alpha = 0$ in the second case. The same result will be found in the third and fourth case. Hence $Id^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = Q$.

This also proves that in case $m < \max\{k, t, r\}$, those polynomials form a basis of the multilinear proper polynomials of degree m modulo $Id^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri})$.

$$c_n^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = 1 + \sum_{j=1}^{k-1} \binom{n}{j} + \sum_{j=1}^{t-1} \binom{n}{j} + \sum_{j=1}^{r-1} \binom{n}{j}.$$

For any $1 \leq j \leq k-1$ we consider the highest weight vector $f_{\langle \lambda \rangle} = y_{1,0}^{n-j} z_{1,0}^j$ corresponding to the multipartition $\langle \lambda \rangle = ((n-j), \emptyset, (j), \emptyset)$. Evaluating $y_{1,0} = (I_k, 0, 0)$ and $z_{1,0} = (E_1, 0, 0)$, we get $f_{\langle \lambda \rangle} = (E_1^j, 0, 0) \neq 0$, since $j \leq k-1$ and so $m_{((n-j), \emptyset, (j), \emptyset)} \neq 0$, for all $j = 1, \dots, k-1$.

For any $1 \leq j \leq t-1$ we consider the highest weight vector $f_{\langle \lambda \rangle} = y_{1,0}^{n-j} y_{1,1}^j$ corresponding to the multipartition $\langle \lambda \rangle = ((n-j), (j), \emptyset, \emptyset)$. Evaluating $y_{1,0} = (0, I_t, 0)$ and $y_{1,1} = (0, E_1, 0)$, we get $f_{\langle \lambda \rangle} = (0, E_1^j, 0) \neq 0$, since $j \leq t-1$ and so $m_{((n-j), (j), \emptyset, \emptyset)} \neq 0$, for all $j = 1, \dots, t-1$.

For any $1 \leq j \leq r-1$ we consider the highest weight vector $f_{\langle \lambda \rangle} = y_{1,0}^{n-j} z_{1,1}^j$ corresponding to the multipartition $\langle \lambda \rangle = ((n-j), \emptyset, \emptyset, (j))$. Evaluating $y_{1,0} = (0, 0, I_r)$ and $z_{1,1} = (0, 0, E_1)$, we get $f_{\langle \lambda \rangle} = (0, 0, E_1^j) \neq 0$, since $j \leq r-1$ and so $m_{((n-j), \emptyset, \emptyset, (j))} \neq 0$, for all $j = 1, \dots, r-1$.

Since $m_{((n), \emptyset, \emptyset, \emptyset)} = 1$ and using the codimension, we may conclude that

$$\chi_n^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = \sum_{j=0}^{k-1} \chi_{(n-j), \emptyset, (j), \emptyset} + \sum_{j=1}^{t-1} \chi_{(n-j), (j), \emptyset, \emptyset} + \sum_{j=1}^{r-1} \chi_{(n-j), \emptyset, \emptyset, (j)}$$

and so $l_n^{gri}(C_{k,*} \oplus C_t^{gr} \oplus C_r^{gri}) = k + t + r - 2$. $\square$

In particular case, we have the following *-graded colength

$$\begin{aligned} l_n^{gri}(C_{2,*} \oplus C_2^{gr}) &= l_n^{gri}(C_2^{gr} \oplus C_2^{gri}) = l_n^{gri}(C_{2,*} \oplus C_2^{gri}) = 3, \quad l_n^{gri}(C_{2,*} \oplus C_2^{gr} \oplus C_2^{gri}) = 4, \\ l_n^{gri}(C_{3,*} \oplus C_2^{gr}) &= l_n^{gri}(C_{3,*} \oplus C_2^{gri}) = 4, \quad l_n^{gri}(C_{2,*} \oplus C_3^{gr}) = l_n^{gri}(C_3^{gr} \oplus C_2^{gri}) = 4 \text{ and} \\ l_n^{gri}(C_{2,*} \oplus C_3^{gri}) &= l_n^{gri}(C_2^{gr} \oplus C_3^{gri}) = 4. \end{aligned}$$

3.3 The *-superalgebras with *-graded colength bounded by 3

In the previous sections and the previous chapter we saw some *-superalgebras with small *-graded colengths. For example: $C_{2,*}, C_2^{gr}, C_2^{gri}$ are *-superalgebras with *-graded colengths equal 2, the direct sum of two distinct *-superalgebras among them has *-graded colengths equal 3 and $G_{2,r}$ also have *-graded colengths equal 3.

In this section we shall classify the varieties generated by finite dimensional *-superalgebras with sequence of *-graded colengths bounded by three. The classification of the varieties of algebras with involution with sequence of *-colengths bounded by three was recently made in [23], now we want to prove a result in case of *-superalgebras that generalize the result obtained in [23].

In order to prove the main result of this thesis, we still need a few more lemmas about *-superalgebras of type $A = F + J$. Let us see what happens in this case.

Lemma 3.3.1. *If $A = F + J$ is a finite dimensional *-superalgebra where $J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$.*

1. [22, Lemma 14] *If $A_{2,*} \notin \text{var}^{gri}(A)$ then $J_{10}^{(0)} = J_{01}^{(0)} = 0$.*
2. *If $A_2^{gri} \notin \text{var}^{gri}(A)$ then $J_{10}^{(1)} = J_{01}^{(1)} = 0$.*

Hence if $A_{2,*}, A_2^{gri} \notin \text{var}^{gri}(A)$ then $J_{10} = J_{01} = 0$.

Proof. First, suppose that there exists $a \in J_{10}^{(0)}$ such that $a \neq 0$ and so $a^2 = 0$. Let B be the subalgebra of A generated by $1_F, a, a^*$ and let I be the $*$ -graded ideal generated by aa^*, a^*a . Then we can verify that $\overline{B} = B/I$ is linearly generated by $\overline{1_F}, \overline{a}, \overline{a^*}$. Notice that $\overline{B}$ has trivial grading and $\overline{a} + \overline{a^*}$ is a symmetric element, and $\overline{a} - \overline{a^*}$ is a skew element. It is easily seen that $\overline{B} \cong A_{2,*}$ through the isomorphism φ such that $\varphi(\overline{1_F}) = e_{11} + e_{44}$, $\varphi(\overline{a}) = e_{12}$, $\varphi(\overline{a^*}) = e_{34}$. Hence $A_{2,*} \in \text{var}^{gri}(A)$, a contradiction. So we must have $J_{10}^{(0)} = 0$ and $J_{01}^{(0)} = (J_{01}^{(0)})^* = 0$ and the first part of the lemma is proved.

Similarly, if there exists $a \in J_{10}^{(1)}$ such that $a \neq 0$ and so we also have $a^2 = 0$. Let B be the subalgebra of A generated by $1_F, a, a^*$ and let I be the $*$ -graded ideal generated by aa^*, a^*a . Then we can verify that $\overline{B} = B/I$ is linearly generated by $\overline{1_F}, \overline{a}, \overline{a^*}$. Notice that $\overline{a} + \overline{a^*}$ is a symmetric odd element and $\overline{a} - \overline{a^*}$ is a skew odd element. Then we can easily show that $\overline{B} \cong A_2^{gri}$ through the isomorphism seen in the first part of the lemma. Hence $A_2^{gri} \in \text{var}^{gri}(A)$, a contradiction. So we must have $J_{10}^{(1)} = 0$ and $J_{01}^{(1)} = (J_{01}^{(1)})^* = 0$. $\square$

In the classification of the $*$ -superalgebras with $*$ -graded colength at most 3, we must exclude the $*$ -superalgebras $A_{2,*}$ and A_2^{gri} , since by Lemmas 2.3.1 and 2.3.9 we have $\chi_n^{gri}(A_{2,*}) = \chi_n^{gri}(A_2^{gri}) = 5$. So from now on, we will study $*$ -superalgebras of the type $F + J_{11}$.

Lemma 3.3.2. *Let $B = F + J_{11}$ be a $*$ -superalgebra.*

1. *If $C_{i,*} \notin \text{var}^{gri}(B)$, for $i \geq 2$, then $z_{1,0}^{i-1} \equiv 0$ on B .*
2. *If $C_i^{gr} \notin \text{var}^{gri}(B)$, for $i \geq 2$, then $y_{1,1}^{i-1} \equiv 0$ on B .*
3. *If $C_i^{gri} \notin \text{var}^{gri}(B)$, for $i \geq 2$, then $z_{1,1}^{i-1} \equiv 0$ on B .*

Proof. We will proceed by the same way, in order to prove each item. First suppose that there exists $a \in J$ under the condition of each item such that $a^{i-1} \neq 0$ and consider the subalgebra R of B generated by 1 and a over F . Then if I is the $*$ -graded ideal generated by a^i , we have the algebra $\overline{R} = R/I$ has induced involution and $\overline{R} = \text{span}\{\overline{1}, \overline{a}, \overline{a^2}, \dots, \overline{a^{i-1}}\}$. Thus, the correspondence

$$\overline{1} \mapsto e_{11} + \dots + e_{ii}, \quad \overline{a} \mapsto e_{12} + \dots + e_{i-1,i}$$

defines an isomorphism between:

1. $\overline{R}$ and $C_{i,*}$, if $a \in (J_{11}^{(0)})^-$. Hence $C_{i,*} \in \text{var}^{gri}(B)$.
2. $\overline{R}$ and C_i^{gr} , if $a \in (J_{11}^{(1)})^+$. Hence $C_i^{gr} \in \text{var}^{gri}(B)$.
3. $\overline{R}$ and C_i^{gri} , if $a \in (J_{11}^{(1)})^-$. Hence $C_i^{gri} \in \text{var}^{gri}(B)$.

$\square$

Lemma 3.3.3. *Let $B = F + J_{11}$ be a *-superalgebra.*

1. *If $U_{3,*} \notin \text{var}^{gr}(B)$ then $[y_{1,0}, y_{2,0}] \equiv 0$ on B .*
2. *If $N_{3,*} \notin \text{var}^{gr}(B)$ then $[y_{1,0}, z_{1,0}] \equiv 0$ on B .*

Proof. Suppose, by contradiction, that $[y_{1,0}, y_{2,0}] \neq 0$. Let $a, b \in (J_{11}^{(0)})^+$ be such that $[a, b] \neq 0$ and consider the subalgebra R generated by $1, a, b$ over F , and let I be the *-graded ideal generated by $a^2, b^2, ab + ba$. So the *-superalgebra $\bar{R} = R/I$ is linearly generated by $\{\bar{1}, \bar{a}, \bar{b}, \bar{a}\bar{b}\}$ and we claim that $\text{Id}^{gr}(\bar{R}) = \text{Id}^{gr}(U_{3,*})$. Clearly $y_{1,1} \equiv z_{1,1} \equiv 0$, $z_{1,0}z_{2,0} \equiv 0$ and $[z_{1,0}, y_{1,0}] \equiv 0$ are $(\mathbb{Z}_2, *)$ -identities of $\bar{R}$, and so, $\text{Id}^{gr}(U_{3,*}) \subseteq \text{Id}^{gr}(\bar{R})$.

Let $f \in P_n^{gr} \cap \text{Id}^{gr}(\bar{R})$ be a multilinear polynomial of degree n . Notice that we can write $f \pmod{\text{Id}^{gr}(U_{3,*})}$ as:

$$f = \alpha y_{1,0} \cdots y_{n,0} + \sum_{1 \leq i < j \leq n} \alpha_{ij} y_{i,0} \cdots y_{i_{n-2},0} [y_{i,0}, y_{j,0}] + \sum_{i=1}^n \alpha_i y_{j_1,0} \cdots y_{j_{n-1},0} z_{i,0},$$

where $i_1 < i_2 < \cdots < i_{n-2}$ and $j_1 < j_2 < \cdots < j_{n-1}$. By making the evaluations $y_{1,0} = \cdots = y_{n,0} = \bar{1}$ and $z_{i,0} = 0$ for $i = 1, \dots, n$, we get $\alpha = 0$. Also, for a fixed $i < j$ the evaluation $y_{i,0} = \bar{a}$, $y_{j,0} = \bar{b}$, $y_{k,0} = \bar{1}$ for $k \notin \{i, j\}$ and $z_{l,0} = 0$ for $l = 1, \dots, n$, gives $\alpha_{ij} = 0$. Finally, the evaluation $z_{i,0} = [\bar{a}, \bar{b}]$, $y_{j,0} = \bar{1}$ for $j \neq i$ gives $\alpha_i = 0$. Hence $f \in \text{Id}^{gr}(U_{3,*})$, and so, $\text{Id}^{gr}(\bar{R}) \subseteq \text{Id}^{gr}(U_{3,*})$. Thus $U_{3,*} \in \text{var}^{gr}(B)$ and the proof of the first part is complete.

The second part of the lemma is proved similarly. We suppose that there exists $a \in (J_{11}^{(0)})^+$ and $b \in (J_{11}^{(0)})^-$ such that $[a, b] \neq 0$ and consider the subalgebra R generated by $1, a, b$ over F and let I be the *-graded ideal generated by $a^2, b^2, ab + ba$. So the *-superalgebra $\bar{R} = R/I$ is linearly generated by $\{\bar{1}, \bar{a}, \bar{b}, \bar{a}\bar{b}\}$ and satisfy the $(\mathbb{Z}_2, *)$ -identities $y_{1,1} \equiv z_{1,1} \equiv 0$, $z_{1,0}z_{2,0} \equiv 0$ and $[y_{1,0}, y_{2,0}] \equiv 0$. Thus $\text{Id}^{gr}(N_{3,*}) \subseteq \text{Id}^{gr}(\bar{R})$.

By using the same arguments of the first part of the lemma, we can prove that $\bar{R} \sim_{T_2^*} N_{3,*}$ and so $N_{3,*} \in \text{var}^{gr}(B)$. And the second part is also proved by contradiction. $\square$

Lemma 3.3.4. *Let $B = F + J_{11}$ be a *-superalgebra.*

1. *If $U_3^{gr} \notin \text{var}^{gr}(B)$ then $[y_{1,0}, y_{1,1}] \equiv 0$ on B .*
2. *If $N_3^{gr} \notin \text{var}^{gr}(B)$ then $[y_{1,0}, z_{1,1}] \equiv 0$ on B .*

Proof. Suppose, by contradiction, that $[y_{1,0}, y_{1,1}] \neq 0$. Let $a \in (J_{11}^{(0)})^+$ and $b \in (J_{11}^{(1)})^+$ be such that $[a, b] \neq 0$ and consider the subalgebra R generated by $1, a, b$ over F and let I be the *-graded ideal generated by $a^2, b^2, ab + ba$. So the *-superalgebra $\bar{R} = R/I$ is linearly generated by $\{\bar{1}, \bar{a}, \bar{b}, \bar{a}\bar{b}\}$ and we claim that $\text{Id}^{gr}(\bar{R}) = \text{Id}^{gr}(U_3^{gr})$.

Notice that $\bar{a} \in (J_{11}^{(0)})^+, \bar{b} \in (J_{11}^{(1)})^+$ and $\bar{a}\bar{b} \in (J_{11}^{(1)})^-$, so it is clear that $z_{1,0} \equiv 0$, $x_{1,1}x_{2,1} \equiv 0$, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$, for $i = 1, 2$, and $[z_{1,1}, y_{1,0}] \equiv 0$ are $(\mathbb{Z}_2, *)$ -identities of $\bar{R}$, and so, $Id^{gri}(U_3^{gri}) \subseteq Id^{gri}(\bar{R})$.

Let $f \in P_n^{gri} \cap Id^{gri}(\bar{R})$ be a multilinear polynomial of degree n . We can write $f \pmod{Id^{gri}(U_3^{gri})}$ as:

$$f = \alpha y_{1,0} \cdots y_{n,0} + \sum_{1 \leq i < j \leq n} \alpha_{ij} y_{i,1,0} \cdots y_{i_{n-2},0} [y_{i,0}, y_{j,1}] + \sum_{i=1}^n \alpha_i y_{j_1,0} \cdots y_{j_{n-1},0} z_{i,1},$$

where $i_1 < i_2 < \cdots < i_{n-2}$ and $j_1 < j_2 < \cdots < j_{n-1}$. By making the evaluations $y_{1,0} = \cdots = y_{n,0} = \bar{1}$ and $y_{i,1} = z_{i,1} = 0$ for $i = 1, \dots, n$, we get $\alpha = 0$. Also, for a fixed $i < j$ the evaluation $y_{i,0} = \bar{a}$, $y_{j,1} = \bar{b}$, $y_{k,0} = \bar{1}$ for $k \notin \{i, j\}$, and $z_{l,1} = 0$ for $l = 1, \dots, n$, gives $\alpha_{ij} = 0$. Finally, the evaluation $z_{i,1} = \bar{a}\bar{b}$, $y_{j,0} = \bar{1}$ for $j \neq i$ gives $\alpha_i = 0$. Hence $f \in Id^{gri}(U_3^{gri})$, and so, $Id^{gri}(\bar{R}) \subseteq Id^{gri}(U_3^{gri})$. Thus $U_3^{gri} \in var^{gri}(B)$ and the proof of the first part is complete.

The second part of the lemma is proved similarly, by contradiction. We suppose that there exists $a \in (J_{11}^{(0)})^+$ and $b \in (J_{11}^{(1)})^-$ such that $[a, b] \neq 0$ and consider the subalgebra R generated by $1, a, b$ over F and let I be the $*$ -graded ideal generated by $a^2, b^2, ab + ba$. By using the same arguments of the first part of the lemma, we can prove that $\bar{R} \sim_{T_2^*} N_3^{gri}$ and so $N_3^{gri} \in var^{gri}(B)$. $\square$

Lemma 3.3.5. *If $B = F + J_{11}$ is a $*$ -superalgebra such that*

1. $[z_{1,0}, z_{2,0}] \neq 0$ then $G_{2,\tau} \in var^{gri}(B)$.
2. $[z_{1,1}, z_{2,1}] \neq 0$ then $G_{2,\tau}^{gr} \in var^{gri}(B)$.
3. $[y_{1,1}, y_{2,1}] \neq 0$ then $G_{2,\psi}^{gr} \in var^{gri}(B)$.

Proof. Generally, we start by considering $a, b \in J$ under the conditions of each item such that $[a, b] \neq 0$. Let R be the subalgebra of B generated by $1, a, b$ and let I be the $*$ -graded ideal generated by $a^2, b^2, ab + ba$. So the $*$ -superalgebra $\bar{R} = R/I$ is linearly generated by $\{\bar{1}, \bar{a}, \bar{b}, \bar{a}\bar{b}\}$. Thus, the correspondence

$$\bar{1}_F \mapsto 1, \quad \bar{a} \mapsto e_1, \quad \bar{b} \mapsto e_2,$$

defines an isomorphism between:

1. $\bar{R}$ and $G_{2,\tau}$, if $a, b \in (J_{11}^{(0)})^-$. Hence $G_{2,\tau} \in var^{gri}(B)$.
2. $\bar{R}$ and $G_{2,\tau}^{gr}$, if $a, b \in (J_{11}^{(1)})^-$. Hence $G_{2,\tau}^{gr} \in var^{gri}(B)$.
3. $\bar{R}$ and $G_{2,\psi}^{gr}$, if $a, b \in (J_{11}^{(1)})^+$. Hence $G_{2,\psi}^{gr} \in var^{gri}(B)$.

□

Lemma 3.3.6. *If $B = F + J_{11}$ is a $\ast$ -superalgebra such that*

1. $y_{1,1}z_{1,1} \neq 0$ then $G_{2,\rho}^{gr} \in \text{var}^{gr}(B)$.
2. $z_{1,0}y_{1,1} \neq 0$ then $G_{2,\rho}^{gri} \in \text{var}^{gri}(B)$.
3. $z_{1,0}z_{1,1} \neq 0$ then $G_{2,\tau}^{gri} \in \text{var}^{gri}(B)$.

Proof. We start by assuming that the polynomial is not an $(\mathbb{Z}_2, \ast)$ -identity for B , so that there exist elements $a, b \in J$ under the condition of each item such that $ab \neq 0$. After we consider R to be the subalgebra of B generated by $1_F, a, b$ and I to be the $\ast$ -graded ideal generated by $a^2, b^2, ab + ba$. Then $\overline{R} = R/I$ is a $\ast$ -superalgebra linearly generated by $\overline{1}_F, \overline{a}, \overline{b}, \overline{ab}$. Thus, the correspondence $\overline{1}_F \mapsto 1, \overline{a} \mapsto e_1, \overline{b} \mapsto e_2$, defines an isomorphism between:

1. $\overline{R}$ and $G_{2,\rho}^{gr}$, if $a \in (J_{11}^{(1)})^-$ and $b \in (J_{11}^{(1)})^+$. Hence $G_{2,\rho}^{gr} \in \text{var}^{gr}(B)$.
2. $\overline{R}$ and $G_{2,\rho}^{gri}$, if $a \in (J_{11}^{(0)})^-$ and $b \in (J_{11}^{(1)})^+$. Hence $G_{2,\rho}^{gri} \in \text{var}^{gri}(B)$.
3. $\overline{R}$ and $G_{2,\tau}^{gri}$, if $a \in (J_{11}^{(0)})^-$ and $b \in (J_{11}^{(1)})^-$. Hence $G_{2,\tau}^{gri} \in \text{var}^{gri}(B)$.

□

Lemma 3.3.7. *Suppose that $B = F + J_{11}$ satisfies the identity $z_{1,0}z_{2,0} + z_{2,0}z_{1,0} \equiv 0$. If $z_{1,0}z_{2,0}z_{3,0} \neq 0$ then $G_{3,\tau} \in \text{var}^{gri}(B)$.*

Proof. Consider $a, b, c \in (J_{11}^{(0)})^-$ such that $abc \neq 0$. Let R be the subalgebra of B generated by $1, a, b, c$. Since $z_{1,0}z_{2,0} + z_{2,0}z_{1,0} \equiv 0$ in R we have $a^2 = b^2 = c^2 = 0$ and so $R = \text{span}\{1, a, b, c, ab, ac, bc, abc\}$. As a consequence, the correspondence

$$1 \mapsto 1, \quad a \mapsto e_1, \quad b \mapsto e_2, \quad c \mapsto e_3$$

defines an isomorphism between R and $G_{3,\tau}$. □

Lemma 3.3.8. *Let $B = F + J_{11}$ be a finite dimensional $\ast$ -superalgebra such that $B \in \text{var}^{gri}(D_\ast \oplus D^{gr} \oplus D^{gri})$. If $c_n^{gri}(B) \approx an^t$, for some constant a , then $B \sim_{T_2^\ast} B_1 \oplus B_2 \oplus B_3$ where $B_1 \in \text{var}^{gri}(D_\ast)$, $B_2 \in \text{var}^{gri}(D^{gr})$ and $B_3 \in \text{var}^{gri}(D^{gri})$.*

Proof. Notice that $F + J_{11}^{(0)}$ is a subalgebra of B . Moreover, by the $(\mathbb{Z}_2, \ast)$ -identities of $D_\ast \oplus D^{gr} \oplus D^{gri}$, we can show that $F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^+$ and $F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^-$ are subalgebras of B too. So obviously we have

$$\text{Id}^{gri}(B) \subseteq \text{Id}^{gri}((F + J_{11}^{(0)}) \oplus (F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^+) \oplus (F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^-)).$$

Conversely, let

$$f \in Id^{gri}((F + J_{11}^{(0)}) \oplus (F + (J_{11}^{(0)})^- + (J_{11}^{(1)})^+) \oplus (F + (J_{11}^{(0)})^- + (J_{11}^{(1)})^-))$$

be a multilinear polynomial of degree n .

Since $Id^{gri}(D_* \oplus D^{gr} \oplus D^{gri}) \subseteq Id^{gri}(B)$, we write f modulo $Id^{gri}(B)$ as either $f = \alpha y_{1,0} \cdots y_{n,0}$, or $f = \alpha_{p,q} y_{i_1,0} \cdots y_{i_p,0} z_{j_1,0} \cdots z_{j_q,0}$, or $f = \alpha_{p,q} y_{i_1,0} \cdots y_{i_p,0} y_{j_1,1} \cdots y_{j_q,1}$, or $f = \alpha_{p,q} y_{i_1,0} \cdots y_{i_p,0} z_{j_1,1} \cdots z_{j_q,1}$, where $p + q = n$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$.

If f is of the first type, by making the evaluation $y_{i,0} = 1_F$ for all $1 \leq i \leq n$, we get $\alpha = 0$ and so $f \in Id^{gri}(B)$. If f is of the other types, we also get $f \in Id^{gri}(B)$, since $f \in Id^{gri}(F + J_{11}^{(0)})$, $f \in Id^{gri}(F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^+)$ and $f \in Id^{gri}(F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^-)$. Hence, we have the equality

$$Id^{gri}(B) = Id^{gri}((F + J_{11}^{(0)}) \oplus (F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^+) \oplus (F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^-)).$$

Now since $F + J_{11}^{(0)} \in var^{gri}(D_*)$, $F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^+ \in var^{gri}(D^{gr})$ and $F + (J_{11}^{(0)})^+ + (J_{11}^{(1)})^- \in var^{gri}(D^{gri})$, we get the wished result. $\square$

At this point, we are in a position to prove the main result of this section which allows us to classify the varieties generated by a finite dimensional *-superalgebra with *-graded colengths bounded by 3, for n large enough.

Theorem 3.3.9. *Let A be an finite dimensional *-superalgebra over a field F of characteristic zero. The following conditions are equivalent.*

1. $l_n^{gri}(A) \leq 3$, for n large enough.
2. $A_{2,*}, A_2^{gri}, N_{3,*}, N_3^{gri}, U_{3,*}, U_3^{gri}, C_{4,*}, C_4^{gr}, C_4^{gri}, G_{3,\tau}, G_{2,\tau}^{gr}, G_{2,\psi}^{gr}, G_{2,\rho}^{gr}, G_{2,\tau}^{gri}, G_{2,\rho}^{gri}, G_{2,\tau} \oplus C_{3,*}, G_{2,\tau} \oplus C_2^{gr}, G_{2,\tau} \oplus C_2^{gri}, C_{3,*} \oplus C_2^{gr}, C_{3,*} \oplus C_2^{gri}, C_{2,*} \oplus C_3^{gr}, C_3^{gr} \oplus C_2^{gri}, C_{2,*} \oplus C_3^{gri}, C_2^{gr} \oplus C_3^{gri}, C_{2,*} \oplus C_2^{gr} \oplus C_2^{gri} \notin var^{gri}(A)$.
3. A is T_2^* -equivalent to N or $C \oplus N$ or $C_{2,*} \oplus N$ or $C_{3,*} \oplus N$ or $C_2^{gr} \oplus N$ or $C_3^{gr} \oplus N$ or $C_2^{gri} \oplus N$ or $C_3^{gri} \oplus N$ or $C_{2,*} \oplus C_2^{gr} \oplus N$ or $C_{2,*} \oplus C_2^{gri} \oplus N$ or $C_2^{gr} \oplus C_2^{gri} \oplus N$ or $G_{2,\tau} \oplus N$, where N is a nilpotent *-superalgebra and C is a commutative non-nilpotent algebra with trivial involution and trivial grading.

Proof. First, notice that the condition (1) implies the condition (2) since by Lemmas 2.3.1, 2.3.9 and by Theorem 2.3.13 we have $l_n^{gri}(N_3^{gri}) = l_n^{gri}(U_3^{gri}) = l_n^{gri}(A_{2,*}) = l_n^{gri}(A_2^{gri}) = 5$; by Lemma 2.3.5 and Theorems 2.3.7, 3.1.8, 3.1.13, 3.1.20 we have $l_n^{gri}(N_{3,*}) = l_n^{gri}(U_{3,*}) = l_n^{gri}(C_{4,*}) = l_n^{gri}(C_4^{gr}) = l_n^{gri}(C_4^{gri}) = 4$; by Lemmas 3.2.1, 3.2.2, 3.2.3 we have $l_n^{gri}(G_{3,\tau}) = l_n^{gri}(G_{2,\tau}^{gr}) = l_n^{gri}(G_{2,\psi}^{gr}) = l_n^{gri}(G_{2,\rho}^{gr}) = l_n^{gri}(G_{2,\tau}^{gri}) = l_n^{gri}(G_{2,\rho}^{gri}) = 4$ and by Lemmas 3.2.4 and 3.2.6 we also have $l_n^{gri}(G_{2,\tau} \oplus C_{3,*}) =$

$$l_n^{gr}(C_{2,*} \oplus C_3^{gr}) = l_n^{gr}(C_3^{gr} \oplus C_2^{gr}) = l_n^{gr}(C_{2,*} \oplus C_3^{gr}) = l_n^{gr}(C_2^{gr} \oplus C_3^{gr}) = l_n^{gr}(G_{2,\tau} \oplus C_2^{gr}) = l_n^{gr}(G_{2,\tau} \oplus C_2^{gr}) = l_n^{gr}(C_{2,*} \oplus C_2^{gr} \oplus C_2^{gr}) = 4.$$

Also, the condition (3) implies the condition (1), since for n large enough we have $l_n^{gr}(N) = 0$, $l_n^{gr}(C \oplus N) = 1$, by Theorems 3.1.8, 3.1.13, 3.1.20 we have $l_n^{gr}(C_{2,*} \oplus N) = l_n^{gr}(C_2^{gr} \oplus N) = l_n^{gr}(C_2^{gr} \oplus N) = 2$, $l_n^{gr}(C_{3,*} \oplus N) = l_n^{gr}(C_3^{gr} \oplus N) = l_n^{gr}(C_3^{gr} \oplus N) = 3$ and by Lemmas 3.2.1 and 3.2.6 we get $l_n^{gr}(G_{2,\tau} \oplus N) = l_n^{gr}(C_{2,*} \oplus C_2^{gr} \oplus N) = l_n^{gr}(C_{2,*} \oplus C_2^{gr} \oplus N) = l_n^{gr}(C_2^{gr} \oplus C_2^{gr} \oplus N) = 3$.

Suppose now that the condition (2) is satisfied, it means that we exclude all that twenty-five *-superalgebras from $var^{gr}(A)$. Since $C_{4,*} \in var^{gr}(D_*)$, $C_4^{gr} \in var^{gr}(D^{gr})$, $C_4^{gr} \in var^{gr}(D^{gr})$, $A_{2,*} \in var^{gr}(M_*)$ and $A_2^{gr} \in var^{gr}(M^{gr})$, it follows that $D_*, D^{gr}, D^{gr}, M_*, M^{gr} \notin var^{gr}(A)$. Hence, by Theorem 1.4.9, the *-graded codimensions of A are polynomially bounded. Since A is finite dimensional, by Theorem 1.4.4, we may assume that

$$A = B_1 \oplus \cdots \oplus B_m$$

is a direct sum of finite-dimensional *-superalgebras where either B_i is nilpotent or $B_i = F + J(B_i)$.

If B_i is nilpotent for all i , then A is a nilpotent *-superalgebra and we are done in this case.

Therefore we may assume that there exists $i = 1, \dots, m$ such that $B_i = F + J(B_i)$ and $J(B_i) = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$.

Since $A_{2,*}, A_2^{gr} \notin var^{gr}(B_i)$, by Lemma 3.3.1, we have $J_{01} = J_{10} = 0$, and so $B_i = (F + J_{11}) \oplus J_{00}$ is a direct sum of *-superalgebras. Then we study next $B = F + J_{11}$, since J_{00} is nilpotent.

Since $U_{3,*}, N_{3,*}, U_3^{gr}, N_3^{gr}, G_{2,\tau}, G_{2,\psi}, G_{2,\rho}, G_{2,\tau}, G_{2,\rho}^{gr} \notin var^{gr}(B)$, by Lemmas 3.3.3, 3.3.4, 3.3.5 (item 2 and 3) and 3.3.6, it follows that all these polynomials are $(\mathbb{Z}_2, *)$ -identities of B :

$$[y_{1,0}, y_{2,0}], [y_{1,0}, z_{1,0}], [y_{1,0}, y_{1,1}], [y_{1,0}, z_{1,1}], [y_{1,1}, y_{2,1}], [z_{1,1}, z_{2,1}], z_{1,1}y_{1,1}, z_{1,0}y_{1,1}, z_{1,0}z_{1,1}.$$

Now we have to consider two different cases $[z_{1,0}, z_{2,0}] \equiv 0$ and $[z_{1,0}, z_{2,0}] \not\equiv 0$ on B .

Suppose that $[z_{1,0}, z_{2,0}] \equiv 0$ then we have $B \in var^{gr}(D_* \oplus D^{gr} \oplus D^{gr})$, by Lemma 3.2.5. Since B is also polynomially bounded, by Lemma 3.3.8, we must have $B \sim_{T_2^*} B_1 \oplus B_2 \oplus B_3$ where $B_1 \in var^{gr}(D_*)$, $B_2 \in var^{gr}(D^{gr})$ and $B_3 \in var^{gr}(D^{gr})$. Now since $C_{4,*}, C_4^{gr}, C_4^{gr}, C_{3,*} \oplus C_2^{gr}, C_{3,*} \oplus C_2^{gr}, C_{2,*} \oplus C_3^{gr}, C_3^{gr} \oplus C_2^{gr}, C_{2,*} \oplus C_3^{gr}, C_2^{gr} \oplus C_3^{gr}, C_{2,*} \oplus C_2^{gr} \oplus C_2^{gr} \notin var^{gr}(B)$, we must have that B is T_2^* -equivalent to either C or $C_{2,*}$ or C_2^{gr} or C_2^{gr} or $C_{3,*}$ or C_3^{gr} or C_3^{gr} or $C_{2,*} \oplus C_2^{gr}$ or $C_{2,*} \oplus C_2^{gr}$ or $C_2^{gr} \oplus C_2^{gr}$.

Now assume that $[z_{1,0}, z_{2,0}] \not\equiv 0$ on B . So $G_{2,\tau} \in var^{gr}(B)$, by item (1) of Lemma 3.3.5. On the other hand, since $G_{2,\tau} \oplus C_{3,*}, G_{2,\tau} \oplus C_2^{gr}, G_{2,\tau} \oplus C_2^{gr} \notin var^{gr}(B)$ we must have $C_{3,*}, C_2^{gr}, C_2^{gr} \notin var^{gr}(B)$. Hence, by Lemma 3.3.2, $z_{1,0}^2 \equiv y_{1,1} \equiv z_{1,1} \equiv 0$

on B . After linearizing $z_{1,0}^2 \equiv 0$ we get $z_{1,0}z_{2,0} + z_{2,0}z_{1,0} \equiv 0$ on B . Finally, since $G_{3,\tau} \notin \text{var}^{gr}(B)$, by Lemma 3.3.7, we have that $z_{1,0}z_{2,0}z_{3,0} \equiv 0$. Hence, by Lemma 3.2.1, $\text{Id}^{gr}(G_{2,\tau}) \subseteq \text{Id}^{gr}(B)$ and it follows that B is T_2^* -equivalent to $G_{2,\tau}$.

Recalling that $A = B_1 \oplus \cdots \oplus B_m$ and putting together all pieces, we get the desired conclusion. $\square$

In particular case, we have the following classification of the *-supervarieties with *-graded colengths bounded by 2, for n large enough.

Corollary 3.3.10. *Let A be an finite dimensional *-superalgebra over a field F of characteristic zero. The following conditions are equivalent.*

1. $l_n^{gr}(A) \leq 2$, for n large enough.
2. $A_{2,*}, A_2^{gr}, N_{3,*}, N_3^{gr}, U_{3,*}, U_3^{gr}, C_{3,*}, C_3^{gr}, C_3^{gr}, G_{2,\tau}, G_{2,\tau}^{gr}, G_{2,\psi}^{gr}, G_{2,\rho}^{gr}, G_{2,\tau}^{gr}, G_{2,\rho}^{gr}, C_{2,*} \oplus C_2^{gr}, C_{2,*} \oplus C_2^{gr}, C_2^{gr} \oplus C_2^{gr} \notin \text{var}^{gr}(A)$.
3. A is T_2^* -equivalent to N or $C \oplus N$ or $C_{2,*} \oplus N$ or $C_2^{gr} \oplus N$ or $C_2^{gr} \oplus N$, where N is a nilpotent *-algebra and C is a commutative non-nilpotent algebra with trivial involution.

Proof. We easily see that the condition (1) implies (2) and the condition (3) implies (1). In order to prove that the condition (2) implies (3), notice that since $A_{2,*}, A_2^{gr}, C_{3,*}, C_3^{gr}, C_3^{gr} \notin \text{var}^{gr}(A)$, it follows that $M_*, M^{gr}, D_*, D^{gr}, D^{gr} \notin \text{var}^{gr}(A)$. Hence, by Theorem 1.4.9, the *-graded codimensions of A are polynomially bounded.

Moreover, notice that now we are excluding $G_{2,\tau}$ from $\text{var}^{gr}(A)$, by item (1) of Lemma 3.3.5, what implies that $[z_{1,0}, z_{2,0}] \equiv 0$ on A . Thus $N_{3,*}, N_3^{gr}, U_{3,*}, U_3^{gr}, C_{3,*}, C_3^{gr}, C_3^{gr}, G_{2,\tau}, G_{2,\tau}^{gr}, G_{2,\psi}^{gr}, G_{2,\rho}^{gr}, G_{2,\tau}^{gr}, G_{2,\rho}^{gr} \notin \text{var}^{gr}(A)$ imply that A satisfies all the $(\mathbb{Z}_2, *)$ -identities of $D_* \oplus D^{gr} \oplus D^{gr}$, i.e., $A \in \text{var}^{gr}(D_* \oplus D^{gr} \oplus D^{gr})$.

The rest of the proof is similar to the first part of the proof of previous theorem. Since $C_{2,*} \oplus C_2^{gr}, C_{2,*} \oplus C_2^{gr}, C_2^{gr} \oplus C_2^{gr} \notin \text{var}^{gr}(A)$, we will conclude that A is T_2^* -equivalent to N or $C \oplus N$ or $C_{2,*} \oplus N$ or $C_2^{gr} \oplus N$ or $C_2^{gr} \oplus N$, where N is a nilpotent *-superalgebra and C is a commutative non-nilpotent algebra with trivial involution and trivial grading. $\square$

In conclusion, we have the following classification: for any finite dimensional *-superalgebra A and n large enough,

1. $l_n^{gr}(A) = 0$ if, and only if, $A \sim_{T_2^*} N$.
2. $l_n^{gr}(A) = 1$ if, and only if, $A \sim_{T_2^*} C \oplus N$.
3. $l_n^{gr}(A) = 2$ if, and only if, either $A \sim_{T_2^*} C_{2,*} \oplus N$ or $A \sim_{T_2^*} C_2^{gr} \oplus N$ or $A \sim_{T_2^*} C_2^{gr} \oplus N$.

4. $l_n^{gr}(A) = 3$ if, and only if, either $A \sim_{T_2^*} C_{3,*} \oplus N$ or $A \sim_{T_2^*} C_3^{gr} \oplus N$ or $A \sim_{T_2^*} C_3^{gri} \oplus N$ or $A \sim_{T_2^*} G_{2,\tau} \oplus N$ or $A \sim_{T_2^*} C_{2,*} \oplus C_2^{gr} \oplus N$ or $A \sim_{T_2^*} C_{2,*} \oplus C_2^{gri} \oplus N$ or $A \sim_{T_2^*} C_2^{gr} \oplus C_2^{gri} \oplus N$,

where N is a nilpotent *-superalgebra and C is a commutative non-nilpotent algebra with trivial involution and trivial grading.

Final considerations

The theory of $*$ -graded identities of finite dimensional $*$ -superalgebras presented here generalizes the results for algebras with involution. In fact, if A is a $*$ -superalgebra with trivial involution, then $c_n^{gr}(A) = c_n^*(A)$ and $l_n^{gr}(A) = l_n^*(A)$.

In this thesis, we have classified the varieties generated by a finite dimensional $*$ -superalgebra A with $*$ -graded colength bounded by 3 by excluding twenty-five $*$ -superalgebras from the variety generated by A and giving a complete list of finite dimensional generating $*$ -superalgebras.

In [23], a recent joint work with La Mattina and Vieira, we proved that:

Theorem 3.3.11. *Let A be an algebra with involution over a field F of characteristic zero. The following conditions are equivalent.*

1. $l_n^*(A) \leq 3$, for n large enough.
2. $A_{2,*}, N_{3,*}, U_{3,*}, C_{4,*}, G_{3,\tau}, G_{2,\tau} \oplus C_{3,*} \notin \text{var}^*(A)$.
3. A is T^* -equivalent to N or $C \oplus N$ or $C_{2,*} \oplus N$ or $C_{3,*} \oplus N$, $G_{2,\tau} \oplus N$, where N is a nilpotent $*$ -algebra and C is a commutative non-nilpotent algebra with trivial involution.

Notice that Theorem 3.3.9 generalizes this result, in finite dimensional case, as expected. In fact, the algebras $A_{2,*}, N_{3,*}, U_{3,*}, C_{4,*}, G_{3,\tau}$ and $G_{2,\tau} \oplus C_{3,*}$ are the only $*$ -superalgebras with trivial grading that appear in the list of the excluded algebras in Theorem 3.3.9.

In [3], Giambruno and La Mattina proved the equivalence between algebras whose sequence of codimensions is bounded by a linear function and algebras with colength bounded by 2. In case of superalgebras and algebras with involution, we don't have this equivalence (see [31], [23]). Consequently, this also happens in $*$ -superalgebras case. In fact, the algebras $A_{2,*}$ and A_2^{gr} have $*$ -graded codimension bounded by a linear function but the $*$ -graded colength of them is 5.

The classification of the algebras A of at most linear codimension growth has already been made in [12] by Ioppolo and La Mattina, in language of algebras with superinvolution. In that classification, the authors gave a complete list of finite dimensional algebras with superinvolution generating the varieties of at most

linear codimension growth. We would also like to know a classification by excluding algebras with superinvolution from the variety generated by the algebra A .

For a finite dimensional *-superalgebra A , we consider its *-graded cocharacter

$$\chi_n^{gr}(A) = \sum_{\langle \lambda \rangle \vdash \langle n \rangle} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}$$

and we would like to classify finite dimensional *-superalgebras A such that the multiplicities $m_{\langle \lambda \rangle}$ are bounded by a constant K . Such classification has already been given in the setting of algebras [24], superalgebras [27] and algebras with involution [30].

We also would like to obtain a generalization for the Kemer's result for PI-algebra, that is, $c_n(A)$ is polynomially bounded if, and only if, the sequence of colengths is bounded by a constant. Such equivalence has already proved in case of finite generated superalgebras and finite generated algebras with involution by Vieira in [31] and [30], respectively. It is clear that if $l_n^{gr}(A)$ is bounded by a constant then $c_n^{gr}(A)$ is polynomially bounded. Now, we would like to know if the converse is true.

Finally, it seems to be interesting to study algebras with G -graded involution, that is, G -graded algebras endowed with a G -graded involution $*$, where G is a group. In this case, we would like to produce similar results as we have in other structures.

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