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Determination of the optimal periodic maintenance policy under imperfect repair assumption

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Abstract

An appropriate maintenance policy is essential to reduce expenses and risks related to repairable systems failures. The usual assumptions of minimal or perfect repair at failures are not suitable for many real systems, requiring the application of Imperfect Repair models. In this work, the classes Arithmetic Reduction of Age and Arithmetic Reduction of Intensity, proposed by Doyen and Gaudoin (2004) are explored. Likelihood functions for such models are derived, and the parameters are estimated, allowing to compute reliability indicators to forecast the future behavior of the failure process. Under the classic Imperfect Repair virtual age model presented by Kijima et al. (1988) (particular case of Aithmetic Reduction of Age class), a periodic Preventive Maintenance policy is proposed, which estimates optimal time intervals for Preventive Maintenance, in order to minimize (preventive and corrective) maintenance costs. Under a dynamic perspective, it is showed how this policy can be improved, using each failure observation in order to recalculate the optimal time to Preventive Maintenance for a particular system, considering the effect of the repair action. These policies are applicable to any Imperfect Repair model. Monte Carlo simulation studies are implemented in order to evaluate the performance of the proposed methods. Those methods are applied to a real situation regarding the maintenance of engines of off-road trucks used in a mining company. These results bring valuable information to support decision making regarding Preventive Maintenance policy.

Contents

List of Figures

List of Tables

1 Acronyms

ABAO As Bad as Old

AGAN As Good as New

ARA Arithmetic Reduction of Age

ARI Arithmetic Reduction of Intensity

IR Imperfect Repair

MCF Mean Cumulative Function

MR Minimal Repair

MTTF Mean Time to Failure

NHPP Nonhomogeneous Poisson Process

PLP Power Law Process

PM Preventive Maintenance

PR Perfect Repair

ROCOF Rate of Occurence of Failures

2 Introduction

Since the 80's, operation and maintenance activities of industrial plants have been recognized as being as important to successful corporative strategies as the product development and manufacturing activities. Operation and maintenance actions play a critical role in a wide range of issues such as security and environmental factors and company profitability levels. It is not difficult to find real-life situations where the environment–safety–profitability triad is present, transforming maintenance into a critical activity. One example is the extracting of oil and gas sector. In 2006, the Petrobras (a brazilian company with business in this sector) announced the discovery of oil reservoir down the sea, below the salt layer. The total depth, i.e., the distance from the sea surface to oil reservoirs below the salt layer can reach 8,000 meters. Consequently, the adoption of appropriate policies for maintenance of the involved equipment is critical, since failures will have a strong impact on the environment, workers safety, and company profitability.

In industries whose production line is composed of complex machines arranged in a serial layout, a single failure may stop the production as a whole, resulting in huge losses and disorders. Similarly, faults in the vehicle fleet carriers can generate costs associated to the displacement of winches and maintenance staff, the rental car to cover the route and even costs related to the loss of cargo. In any of the mentioned cases, the sum of costs arising from the occurrence of failure translates into losses that lead companies to exceed its budget target, resulting in financial distress and damage to its image, and the consequences of the latter are often immeasurable.

For these reasons, the equipment maintenance, once seen as a "necessary evil", is considered now a strategical activity, indispensable to production, besides being one of the foundations of every industrial activities. The maintenance focus, which was only corrective (ie, after the equipment failure), has became preventive (planned preventative maintenance). Periodicities of interventions have been defined and the maintenance management has evolved further with the use of predictive and inspection techniques. All the tools have been developed in the search of an increased operational reliability.

Maintenance is defined by the Brazilian Association of Technical Standards (ABNT

- Associação Brasileira de Normas Técnicas, 1994) as the combination of all technical and administrative actions, including supervision, to maintain or replace an item into a state where it can perform a required function. While the Corrective Maintenance is performed after the equipment failure, the Preventive Maintenance (PM) consists of interventions performed in predetermined intervals or according to predefined criteria, aiming to reduce the probability of failure or degradation in the normal operation of a system.

According to Ascher and Feingold (1984), a repairable system (machine, industrial equipment, software, etc.) is the one that, after failing to perform satisfactorily one or more of its functions, can return to its operating condition by some repair (replacement or repair of a component) without needing to replace the system as a whole. Probabilistic and statistical models to analyze and optimize the performance of repairable systems have been widely discussed in the literature. Such models must describe the occurrence of events (failures) over time and the effect of corrective maintenance (repairs). Thus, any study to determine an appropriate PM policy for repairable systems must be based on such models.

In the literature, the most explored assumption about the effect of corrective maintenance is the Minimal Repair (MR), where it is assumed that each repair action focuses on correcting only the component that originated the failure, leaving the system in the same condition as it was before the failure (As Bad as Old (ABAO)). Under this assumption, the associated failure process can be described by a Nonhomogeneous Poisson Process (NHPP). According to Muralidharan (2008), in a NHPP, the probability of failure in a small time interval depends only on the age of the system. Thus, the failure intensity function, which provides the instantaneous probability that an event (failure) occurs at time t, conditional on the process history, depends only on t, in the absence of covariates.

The determination of PM policies under the MR assumption has been explored in the literature since the work of Barlow and Hunter (1960). They presented two types of policies, one more useful for simple systems (PM by age), and another one for complex systems (PM by blocks). The latter proposes to conduct MRs until a predetermined time, when the system must be replaced or undergo perfect PM. It is noteworthy that the perfect PM aims to restore the system to like new condition (As

Good as New (AGAN)), and in the present work, we assume that every PM have this effect, so perfect PM will be referred to simply as PM.

The work of Barlow and Hunter (1960) was the driver of a large number of studies that aimed to determine PM policies under MR. Among these, we mention Morimura (1970), Park (1979), Phelps (1981), Barlow and Proschan (1987), Park *et al.* (2000), and Wang (2002). In particular, Gilardoni and Colosimo (2007) used a Power Law Process (PLP) to model the occurrence of failures, and determined the optimal frequency of PM by minimizing a cost function.

The MR assumption seems plausible for systems consisting of many components, each one having its own failure mode, as the repair of the failed component does not change the failure rate of the system. However, in practice, this assumption may not be reasonable for many systems. According to Kijima *et al.* (1988), for systems composed of only a few vulnerable components, it is more appropriate to consider that the repair brings the state of a failed system to an intermediate level between the completely new and pre-failure. In addition, the mobilization of a maintenance crew to fix a fault often leads to more actions related to system maintenance than those specifically related to the repair. Several authors have studied the behavior of systems subject to this kind of action, called Imperfect Repair (IR).

Brown and Proschan (1983) investigated the failure process assuming that the maintenance action performed after a failure is a perfect repair (AGAN) with probability equal to p, and IR with probability equal to $1-p$. The resulting model is known as *B–P Model*, and originated several works such as Block *et al.* (1985), Whitaker and Samaniego (1989), Sheu and Griffith (1992) and Cui *et al.* (2004).

Kijima *et al.* (1989) introduced the idea of *virtual age* of a system, which is a positive function of its real age and its failures history. The virtual age model proposed introduces one parameter, denoted by θ ($0 \leq \theta \leq 1$), which represents the effect of repairs, and includes ABAO and AGAN as special cases (θ =0 and $\theta = 1$, respectively). Doyen and Gaudoin (2004) proposed two classes of IR models. In the first class, the effect of the repair is expressed by a reduction in the failure intensity (ARI model), while in the second, the effect of the repair is expressed by a reduction in the virtual age of the system (ARA model).

Although several studies have focused on the estimation of the parameters involved in IR models (Shin *et al.*, 1996; Yanez *et al.*, 2002; Pan and Rigdon, 2009; Doyen and Gaudoin, 2006 and 2011, and Corset *et al.*, 2012), few investigations have been done in the use of such models for determining optimal PM policies, and therefore is the central theme of this work.

2.1 Problem definition

The main motivation for this work was a practical problem in a Brazilian mining company. In the mining sector, the production process is highly dependent on large equipment. To maintain a constant supply of ore to the treatment plant, it is necessary to replace an off-road truck that operates in the transport between the mining front and homogenization cell, as soon as a failure occurs. As it is necessary to keep extra off-road trucks, to act as backups, failures in these systems should be avoided in order to minimize the number of necessary spare trucks, and increase the fleet availability.

According to Abranches (2013), in the company unit under study, there are several managers who assist the mining activity, and among them, there is a manager related to the maintenance of large components such as electric motors and diesel engines, and other types of industrial equipment. Diesel engines are mainly used in off-road trucks, which are capable of carrying hundreds of tons of material daily. Figure 2.1 exhibits such a truck. These trucks have a high degree of embedded technology, which enables the use of modern georeferenced systems. These systems make real-time routing of the trucks between various points in the mine, such as mining fronts, the cell homogenization and barren areas disposal.

The reliability of these trucks depends on the reliability of many of its components such as the engine, the weighbridge, the tracking system and the cockpit. The engine is responsible for propelling the truck, while the correct operation of the weighbridge is related to its productivity in cargo transportation. The tracking system is used to determine the routes to be followed between the points of loading and unloading, as well as allowing control of other systems, and the cockpit is related to safety and occupational health of the operator.

Thus, each one of these systems is considered separately, so that anyone may be individually replaced in case of failure. The problem under study refers only to diesel

Figure 2.1: Example of an off-road truck

engines. These can be taken from a truck, for maintenance actions, and replaced by another so that the truck can operate again in the shortest possible time. Because of this, when a component is stopped to PM, there is the cost associated to the idle truck that the engine was serving, due to the time for replacement by another in perfect condition.

Each one of these engines goes through a PM program. According to recommendations from the manufacture's manual, systems must undergo PMs at a predetermined frequency (15,000 hours). However, due to a larger sporadically demand, in practice it is not always possible. Failures can also occur in these engines, even in the ones where PMs are held periodically. In such cases, the failed engine must undergo a corrective maintenance to restore its use conditions.

For a group of engines, data related to their functioning were collected. The accumulated working hours were stored, as well as the number of hours when each PM or failure occurred. As detailed below, statistical analyzes in these data showed that the repair actions taken after failures are neither MR nor perfect repairs. Therefore, the use of an IR model that considers the degree of repair actions is required.

It is of interest to avoid a break in these engines, and consequently to reduce the necessity for corrective maintenance, which has higher cost (on average approximately 23% larger) than the PM (even disregarding the indirect costs arising from problems previously cited). The problem which has motivated this work is the establishment of an optimal PM policy for the off-road truck engines. By optimal here, it is meant a policy which minimizes the total maintenance cost, i.e., the costs associated to PM and corrective maintenance actions. Thus, it is possible to improve the reliability of these engines, which are essential to the operation of the production process from the mining company.

2.2 Objectives

This work aims at the development of statistical models for the analysis of failure and equipment repairs data, in order to subsidize the elaboration of an optimal maintenance policy. The focus is on situations where the system is subjected to a program of periodic preventive maintenance (assumed here to be perfect) and corrective maintenance in the occurrence of failures. However, we work here with the more general assumption of imperfect repair at failures. The goal is to determine the optimal frequency of PMs, where the "optimal" goes in the sense of minimizing the total maintenance cost.

The following specific objectives can be enumerated:

- The preparation of a detailed study about the classes of IR models proposed by Doyen and Gaudoin, 2004 (ARA and ARI models). For such models, we intend to explore:
  - The use of maximum likelihood estimation method to obtain estimates for the involved parameters;
  - The selection of the best fitted model, considering different memories in each class, using criteria for model selection;
  - The prediction of the future failure process of repairable systems using reliability indicators based on such models.
- The determination of a PM policy under IR which specifies the optimal PM frequency;
- The determination of a PM policy under IR with a dynamic perspective, which allows to incorporate information from each new failure in the system to recalculate the optimal time for the next PM action;

- The performance comparison between the two proposed policies in terms of maintenance costs, under different scenarios.

These results are then applied to the practical motivating situation described in the previous section, aiming to provide the maintenance factory more analytical information.

3 Layout of the text

The core of the text is a collection of three articles dealing with statistical models for IR. The three articles were developed with the following co-authors: Marta Afonso Freitas (Departamento de Engenharia de Produção, UFMG), Enrico Colosimo (Departamento de Estatística, UFMG) and Gustavo Gilardoni (Departamento de Estatística, UnB).

The first article considers ARI and ARA classes of models proposed by Doyen and Gaudoin (2004). The estimation in such models is explored and applied to a real dataset. Note that these data do not refer to those described in Section 2.1, but were used at this stage due to the extensive record of failures, allowing us to explore the differences between the studied models. This work is summarized in the first article, entitled "ARA and ARI Imperfect Repair Models: A Case Study in a Brazilian Mining Company", submitted to the journal *Reliability Engineering and System Safety*. This article is presented in Section 4.

The second article of this thesis derives a statistical procedure to estimate a periodicity PM policy under the following assumptions: (1) perfect repair in PM, and (2) IR after each failure. This policy presents optimal time intervals for PM, in order to minimize the total expected cost with maintenance actions. Its usage is illustrated as a solution to the problem that motivated this work, described in Section 2.1. This article, entitled "Optimal Periodic Maintenance Policy under Imperfect Repair: A Case Study of Off-Road Engines", submitted to the journal *European Journal of Operational Research* and currently under revision, is presented in Section 5.

The third and final article in this thesis, presented in Section 6, discusses both the determination and practical implementation of an optimal PM policy under IR. This policy considers the information provided by new failures observed in a repairable system, allowing to recalculate the optimal time for the next PM, based on the effect of the repair performed. The proposed method is also applied as a solution to the problem of the off-road engines. This study comprises the article entitled "Dynamics of the Optimal Maintenance Policy under Imperfect Repair Models", which is being revised for submission.

Finally, Section 7 closes the work with some final remarks. The dataset associated to the practical problem is presented in Appendix A, while the codes refered to the proposed procedures, implemented in R language, are available in Appendix B.

4 ARA and ARI Imperfect Repair Models: A Case Study in a Brazilian Mining Company

4.1 Abstract

An appropriate maintenance policy is essential to reduce expenses and risks related to equipment failures. A fundamental aspect to be considered when specifying such policies is to be able to predict the reliability of the systems under study. The usual assumptions of minimal or perfect repair at failures are not appropriate for many real systems, requiring the application of imperfect repair models. In this paper, the classes Arithmetic Reduction of Age and Arithmetic Reduction of Intensity models proposed by Doyen and Gaudoin (2004) are explored. Likelihood functions for such models are derived, assuming Power Law Process and a memory of general order. Based on this, point and interval estimates were obtained for a real dataset involving failures in trucks used by a Brazilian mining company considering models with different memories. Specific statistical measures were used for model selection. Model parameters, namely, shape and scale for Power Law Process, and the efficiency of repair were estimated for the best fitted model. They provided evidences that the trucks tend to fail more frequently over time, justifying the necessity for preventive maintenance, and also, that the repairs after failures tend to leave the equipment in a state between as good as new and as bad as old. The Estimation of model parameters allowed to derive reliability indicators to forecast the future behavior of the failure process. These results are a valuable information for the mining company. They can be used to support decision making regarding preventive maintenance policy.

4.2 Introduction

In the industrial scenario, appropriate Preventive Maintenance (PM) policies are essential to reduce risks of equipment failures, which lead to potential expenses and unsafe conditions. There is an extensive literature toward specifying such policies, as the papers from Barlow and Hunter (1960), Morimura (1970), Nakagawa (1986), Jayabalan and Chaudhuri (1992), Wu and Clements-Croome (2005), Gilardoni and Colosimo (2007), Bartholomew-Biggs *et al.* (2009), Wu and Zuo (2010), and Remy *et al.* (2013). In general, these works are concerned with the study and optimization of PM policies, through the minimization of maintenance cost functions.

These cost functions, and consequently, the resulting PM times, depend on the model and the parameters. In general, these parameters are not known in practice, and, therefore, must be estimated from data. These estimates provide valuable information about the systems under study, and allow to (1) assess the aging speed and the efficiency of repair actions taken after failures; (2) estimate predictive reliability indicators such as failure intensity and Mean Time to Failure (MTTF) and (3) use these estimates in a PM optimization procedure (Doyen and Gaudoin, 2011). This paper is concerned with the two first issues.

When considering models for repairable systems, a critical point is how to account for the effect of repair actions taken after failures. In this sense, the most explored assumptions are Minimal Repair (MR), which returns the system to the condition just before the failure (ABAO), and Perfect Repair (PR), which leaves the system as if it were new (AGAN). These assumptions were discussed in many works such as Barlow and Hunter (1960), Phelps (1981), Barlow and Proschan (1987), Zhao and Xie (1996), Park *et al.* (2000), and Wang (2002), among others.

Nevertheless, a more realistic assumption for many systems is the Imperfect Repair (IR) condition. It means that the system returns to an intermediate state between MR and PR. Nowadays some studies have explored this assumption, among them are Kijima *et al.* (1988), Brown and Proschan (1983), Malik (1979), Shin *et al.* (1996), Yanez *et al.* (2002), Pan and Rigdon (2009), and Corset *et al.* (2012). The former one have proposed the idea of a virtual age model. It is important to emphasize that under MR or PR assumptions, the model parameters are basically those related to the wear-out speed of the systems, while under IR approach, an additional parameter describes the effect of repair actions. Therefore it is necessary to develop an estimation method to take this new parameter into account.

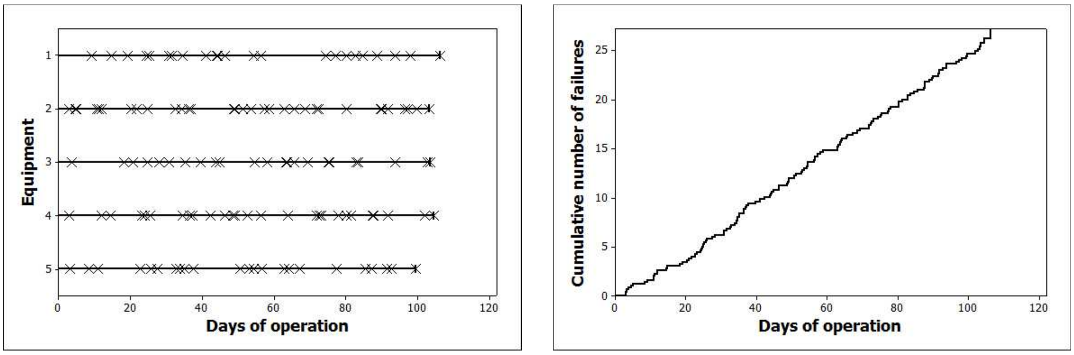
Doyen and Gaudoin (2004) proposed two new classes of IR models. In the first class, ARA, the repair efficiency is expressed by a reduction in the systems virtual age. In the second class, ARI, the repair efficiency is characterized by the reduction in the intensity function of the failure process. These models are defined by its memory of order m, where m refers to the maximum number of previous failure times involved in the calculation of the intensity function. The virtual age model proposed by Kijima *et al.* (1988) corresponds to a particular case of these classes, ARA model with memory of order 1, denoted by ARA_1.

The present study was conducted in order to investigate the adequacy of ARA and ARI models to a real data set. It was motivated by a situation involving maintenance problems in dump trucks owned by a Brazilian mining company. An unexpected failure in these equipments is extremely costly and harmful due to safety aspects, and also due to operational issues, such as delays in cargo delivery, overtime for employees, unavailability of equipment, and realignment of maintenance resources causing delays in scheduled maintenance on other systems. Each truck is a complex system, so each repair action may involve the replacement or repair of many or only a small fraction of its constituent parts. Thus, assuming MR or PR at failures may result in a simplification that does not correspond to the real condition. This fact motivated the use of IR models in this data set.

Data set consists of failure records in a sample of five trucks from the mining company fleet. Data were collected from July to October 2012, when 129 failures were observed, each one followed by a repair. Figure 4.1 (a) shows events (failures) vs. operation time (in days), where each line corresponds to a sample unit, and each "x" symbol represents a failure time. The data for the five trucks were failure truncated, meaning that the last observation for each one corresponds to a failure time. Visually, no trend in failures over time can be observed from this graph. Figure 4.1 (b) exhibits the mean observed cumulative number of failures. Globally, this curve is neither concave nor convex, so neither improvement nor degradation of the observed equipment can be detected through this visual inspection.

The goal here is to (1) identify which model better fits the data, considering ARA and ARI with different memory values; (2) obtain point and interval parameter estimates for the best fitted model, specially, for the effect of repair parameter; (3) use these estimates to compute reliability indicators for the trucks, and then, provide information to base the decision-making related to PM policies in the mining company.

The rest of the paper is organized as follows. In Section 4.3 the intensity functions for ARA and ARI classes of models from Doyen and Gaudoin (2004) are presented,



(a) (b)

Figure 4.1: (a) Failure times in days of operation for each truck (horizontal lines are trucks and "x" are failures); (b) Cumulative number of failures *versus* days of operation.

along with an example for a better understanding of the differences between them. Likelihood functions for these models are presented in Section 4.4. Measures to compare the models are also discussed, and predictive reliability indicators are derived. This methodology is then applied to the dump trucks data set, and the results are presented in Section 4.5. Finally, Section 4.6 ends the paper with some concluding remarks.

4.3 ARA and ARI classes of models

Assuming that failures in a repairable system are equivalently defined by the processes $\{N(t)\}_{t\geq 0}$, or $\{T_i\}_{i\geq 1}$, where $N(t)$ denotes the number of observed failures up to time t, T_i corresponds to the time elapsed up to the i^{th} failure, and that a repair action (with negligible duration) is taken after each failure, the distribution of such processes is completely determined by the failure intensity (or simply intensity) function defined by

$$\rho(t) = \lim_{\delta t \to 0} \frac{P(N(t+\delta t) - N(t) = 1 | \Im_{t^-})}{\delta t}, \quad \forall t \geq 0 \tag{4.1}$$

where $\Im_{t^-}$ is the minimal filtration defined by the history set of all failure times occurred before t. It can be shown (Aalen, 1978) that the Mean Cumulative Function (MCF) of the process is $\Lambda(t) = E[N(t)] = \int_0^t E[\rho(u)]du$.

Before the first repair action, the system intensity function is the rate of occurence of failures (ROCOF) function, given by

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(N(t+\delta t) - N(t) = 1)}{\delta t}. \tag{4.2}$$

Under MR assumption, the failure process is a NHPP, and $\rho(t) = \lambda(t)$. A versatile and extensively explored parametric form under this assumption is the PLP, with ROCOF function $\lambda(t) = (\beta/\eta)(t/\eta)^{\beta-1}$ and its MCF is given by $\Lambda(t) = \int_0^t \lambda(u)du = (t/\eta)^\beta$ (Crow, 1974). Here, η is a scale parameter, and β is a shape parameter. When $\beta > 1$, $\lambda(t)$ increases in t and the system is deteriorating.

Under the IR approach, the model proposed by Kijima *et al.* (1988) has the virtual age of a system in time t expressed by

$$V_t = V(t; N(t); T_1, T_2, \ldots, T_{N(t)}), \tag{4.3}$$

where $T_{N(t)}$ denotes the elapsed time since the startup of the system and the $N(t)^{th}$ failure. Under this model, each repair reduces the virtual age of the system, and the effect of repair is represented by a parameter denoted by θ ($0 \leq \theta \leq 1$), including ABAO and AGAN as special cases ($\theta = 1$ and $\theta = 0$, respectively).

In ARA class of models proposed by Doyen and Gaudoin (2004), it is assumed that the system failure intensity at time t (real age) is equal to its ROCOF at time V_t (virtual age), where $V_t \leq t$. Also, between two consecutive failures, its failure intensity is horizontally parallel to its ROCOF. ARA models are defined by its memory parameter m, so in ARA$_m$ model, it is assumed that the repair reduces the increment in system age since the last m failures, and its failure intensity function is given by

$$\rho_{ARA_m}(t) = \lambda(t - (1-\theta) \sum_{j=0}^{min(m-1, N(t)-1)} \theta^j T_{N(t)-j}). \tag{4.4}$$

At each repair, the θ^j component in this equation makes older repairs have less effect on reducing the systems virtual age.

An extreme special case of ARA model, namely ARA$_1$, assumes that the repair effect is to reduce the increment in system age only by the last failure, while in the other extreme, ARA$_\infty$ assumes that each repair reduces the virtual age of the system in a quantity proportional to its age immediately before repair.

ARI is another class of models proposed by Doyen and Gaudoin (2004), where each repair action reduces not the virtual age, but the failure intensity function of the system. In this class, between two consecutive failures, its failure intensity is vertically parallel to its ROCOF. In ARI_m model, it is assumed that the repair reduces the increment in failure intensity since the last m failures, and its failure intensity function is given by:

$$\rho_{ARI_m}(t) = \lambda(t) - (1-\theta) \sum_{j=0}^{min(m-1,N(t)-1)} \theta^j \lambda(T_{N(t)-j}). \tag{4.5}$$

Similarly to ARA, ARI_1 and ARI_∞ are the extreme special cases of ARI class of models.

In order to illustrate ARA and ARI models, suppose a repairable system whose failure process has a PLP ROCOF with parameters $\beta = 3$, $\eta = 1$, and effect of repair parameter $\theta = 0.5$. Suppose that failures are observed at times $T_1 = 1.2$ and $T_2 = 1.9$. According to Equations 4.4 and 4.5, the failure intensity functions for ARA_1 and ARI_1 models are expressed, respectively, by:

$$\rho_{ARA_1}(t) = \begin{cases} \lambda(t) = 3t^2, \quad 0 \leq t < 1.2 \\ \lambda(t - 0.5 \times 1.2) = 3(t-0.6)^2, \quad 1.2 \leq t < 1.9 \\ \lambda(t - 0.5 \times 1.9) = 3(t-0.95)^2, \quad 1.9 \leq t < \ldots \\ \ldots \quad \ldots \end{cases}$$

$$\rho_{ARI_1}(t) = \begin{cases} \lambda(t) = 3t^2, \quad 0 \leq t < 1.2 \\ \lambda(t) - 0.5 \times \lambda(1.2) = 3t^2 - 0.5 \times 3(1{,}2)^2, \quad 1.2 \leq t < 1.9 \\ \lambda(t) - 0.5 \times \lambda(1.9) = 3t^2 - 0.5 \times 3(1{,}9)^2, \quad 1.9 \leq t < \ldots \\ \ldots \quad \ldots \end{cases}$$

Figure 4.2 exhibits ARA_1 (left) and ARI_1 (right) failure intensity functions for this system. It can be observed how the intensity function decreases after failures under each model. After $T_1 = 1.2$, for example, while under ARA_1 the virtual age decreases to $0.5 \times 1.2 = 0.6$, under ARI_1, the intensity function decreases to $0.5 \times \lambda(1.2)$. It means that, after each failure, while under ARA model the virtual age decreases, under ARI model, the intensity function decreases, what explains the models nomenclature.

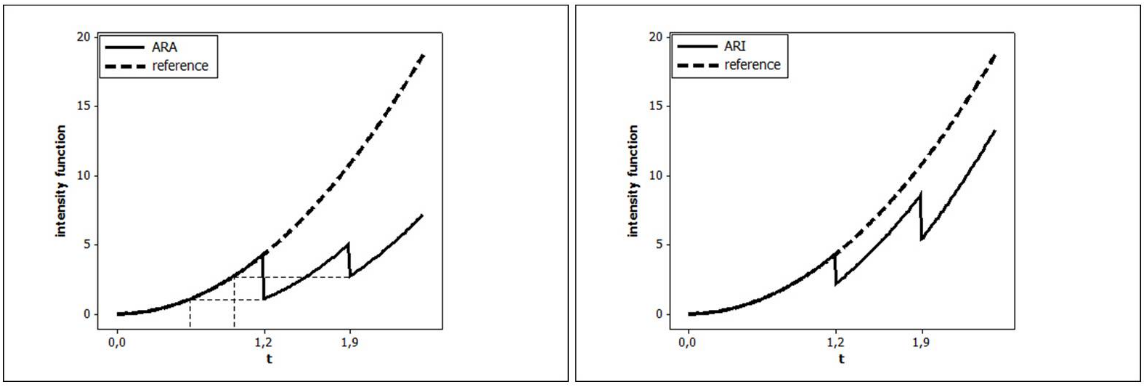


Figure 4.2: ARA_1 and ARI_1 failure intensity functions for PLP intensity function with $\beta = 3$, $\eta = 1$, and $\theta = 0.5$, and observed failure times $T_1 = 1.2$ and $T_2 = 1.9$.

4.4 Estimation in ARA and ARI classes of models

In this section, likelihood functions associated to the intensity functions 4.4 and 4.5 are derived. They are used to get Maximum Likelihood (ML) estimators for the parameters. Measures for choosing the best model are also discussed, and finally, reliability indicators, such as Reliability function and MTTF for each model are derived.

4.4.1 Parameters estimation: The likelihood functions

Consider k identical repairable systems, $k = 1,2,\ldots,$ where the failures occur independently, and assume the following conditions:

- At each failure, a repair action of degree θ is performed.
- n_i failures are observed in the $i-th$ system, $i = 1,2,\ldots,k$.
- $N = \sum_{i=1}^{k} n_i$ is the total number of observed failures in the systems.
- Let $T_{i,j}$ $(i = 1,2,\ldots,k,\ j = 1,2,\ldots,n_i)$ be random variables representing the failure times for the $i-th$ system, recorded as the time since the initial start-up of the system $(T_{i,1} < T_{i,2} < \ldots < T_{i,n_i})$. For time truncated systems, n_i is a random variable, and for failure truncated systems, n_i is fixed. In addition, let

$t_{i,j}$ denote their observed values (data), and $T_i = (T_{i,1}; T_{i,2}; \ldots ; T_{i,n_i})^t$ be the $(n_i \times 1)$ random vector of failure times for the i^{th} system.

- If the $i-th$ system is time truncated, it is observed until the predetermined time t_i^* occurs, and if it is failure truncated, it is observed until the predetermined number of failures n_i occurs. So, the last observation time refers to a censor in t_i^* for time truncated, or a failure in t_{i,n_i} for failure truncated systems.

- Let μ denote the vector of model parameters, which includes the parameters indexing the ROCOF and the repair efficiency parameter θ. For example, assuming PLP for the ROCOF, we have $\mu = (\beta; \eta; \theta)^t$.

A likelihood function for this process must combine the joint probability density of the k systems failure times. Using the failure intensity functions in Equations 4.4 and 4.5, the likelihood functions for ARA$_m$ and ARI$_m$ models are given, respectively, by:

$$
\begin{aligned}
L_{ARA_m}(\mu) =& \\
&= \prod_{i=1}^{k} \prod_{j=1}^{n_i} \{ \lambda(t_{i,j} - (1-\theta) \sum_{p=0}^{min(m-1,j-2)} \theta^p t_{i,j-1-p}) \times \\
&\times e^{-\Lambda(t_{i,j}-(1-\theta)\sum_{p=0}^{min(m-1,j-2)} \theta^p t_{i,j-1-p}) + \Lambda(t_{i,j-1}-(1-\theta)\sum_{p=0}^{min(m-1,j-2)} \theta^p t_{i,j-1-p})} \} \times \\
&\times e^{-\Lambda(t_i^*-(1-\theta)\sum_{p=0}^{min(m-1,n_i-1)} \theta^p t_{i,n_i-p}) + \Lambda(t_{i,n_i}-(1-\theta)\sum_{p=0}^{min(m-1,n_i-1)} \theta^p t_{i,n_i-p})}
\end{aligned}
$$

and

$$
\begin{aligned}
L_{ARI_m}(\mu) =& \\
&= \prod_{i=1}^{k} \prod_{j=1}^{n_i} \{ [\lambda(t_{i,j}) - (1-\theta) \sum_{p=0}^{min(m-1,j-2)} \theta^p \lambda(t_{i,j-1-p})] \times \\
&\times e^{-\Lambda(t_{i,j})+\Lambda(t_{i,j-1})+(1-\theta)[t_{i,j}-t_{i,j-1}]\sum_{p=0}^{min(m-1,j-2)} \theta^p \lambda(t_{i,j-1-p})} \} \times \\
&\times e^{-\Lambda(t_i^*)+\Lambda(t_{i,n_i})+(1-\theta)[t_i^*-t_{i,n_i}]\sum_{p=0}^{min(m-1,n_i-1)} \theta^p \lambda(t_{i,n_i-p})},
\end{aligned}
$$

where, if the system is failure truncated, $t_i^* = t_{i,n_i}$. These likelihood functions can then be rewritten assuming a PLP for the ROCOF. So, in order to find the ML estimates $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$ of β, η, θ respectively, the following log-likelihood functions must be numerically maximized for the ARA$_m$ and ARI$_m$ classes:

$$
\begin{aligned}
l_{ARA_m}(\mu) = logL_{ARA_m}(\mu) = \\
&= \left(\sum_{i=1}^{k} n_i\right) log(\beta) - \beta \left(\sum_{i=1}^{k} n_i\right) log(\eta) + \\
&+(\beta-1)\left(\sum_{i=1}^{k}\sum_{j=1}^{n_i} log(t_{i,j} - (1-\theta)\sum_{p=0}^{min(m-1,j-2)} \theta^p t_{i,j-1-p})\right) + \\
&+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left[-\left(\frac{t_{i,j} - (1-\theta)\sum_{p=0}^{min(m-1,j-2)}\theta^p t_{i,j-1-p}}{\eta}\right)^{\beta} + \left(\frac{t_{i,j-1} - (1-\theta)\sum_{p=0}^{min(m-1,j-2)}\theta^p t_{i}}{\eta}\right.\right. \\
&+\sum_{i=1}^{k}\left[-\left(\frac{t_i^* - (1-\theta)\sum_{p=0}^{min(m-1,n_i-1)}\theta^p t_{i,n_i-p}}{\eta}\right)^{\beta} + \left(\frac{t_{i,n_i} - (1-\theta)\sum_{p=0}^{min(m-1,n_i-1)}\theta^p t_{i,n_i-p}}{\eta}\right)\right.
\end{aligned}
$$

and

$$
\begin{aligned}
l_{ARI_m}(\mu) = logL_{ARI_m}(\mu) = \\
&= \left(\sum_{i=1}^{k} n_i\right) log(\beta) - \beta \left(\sum_{i=1}^{k} n_i\right) log(\eta) + \\
&+\sum_{i=1}^{k}\sum_{j=1}^{n_i} log\left(t_{i,j}^{\beta-1} - (1-\theta)\sum_{p=0}^{min(m-1,j-2)} \theta^p t_{i,j-1-p}^{\beta-1}\right) + \\
&+\eta^{-\beta}\left[\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(-t_{i,j}^{\beta} + t_{i,j-1}^{\beta} + (1-\theta)\beta[t_{i,j} - t_{i,j-1}]\sum_{p=0}^{min(m-1,j-2)}\theta^p t_{i,j-1-p}^{\beta-1}\right)\right] + \\
&+\eta^{-\beta}\left[\sum_{i=1}^{k}\left(-t_i^{*\beta} + t_{i,n_i}^{\beta} + (1-\theta)\beta[t_i^* - t_{i,n_i}]\sum_{p=0}^{min(m-1,n_i-1)}\theta^p t_{i,n_i-p}^{\beta-1}\right)\right]
\end{aligned}
$$

Asymptotic theory based on the Normal distribution can be used to construct confidence intervals for the parameters.

Depending on the number of observed failures for the systems under study, ARA and ARI models can be fitted with different memory values (m). For example, if the maximum number of failure times observed for the systems is 5, m can assume the values 1 to 5, with $m = 5$ corresponding to $m = \infty$. However, fitting many models to a data set requires model selection, and a basic way to achieve this is through the maximum values of the estimated log-likelihoods, which will be denoted here by $\hat{L}$. Since all models explored have the same number of parameters, the comparison using $\hat{L}$ corresponds to the one obtained from Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) measures (Akaike, 1974 and Schwarz, 1978).

Another useful measure to compare models was proposed by Burnham and Anderson (2004). It is based on the scaling criteria values,

$$\Delta_r = \hat{L}_{max} - \hat{L}_r, \quad (r = 1, \ldots, R) \tag{4.6}$$

where $\hat{L}_{max}$ is the maximum of the R different $\hat{L}$ values, considering that R different models were fitted. This transformation forces the best model to have $\Delta = 0$, while the rest of the models have positive values. These values can then be used to calculate weights, based on the normalization of the model likelihoods such that they sum to 1 and can be treated as probabilities,

$$w_r = \frac{exp(-\Delta_r/2)}{\sum_{r=1}^{R} exp(-\Delta_r/2)}. \tag{4.7}$$

So, the w_r are useful as the "weight of evidence" in favor of model r as being the best model in the set.

After fitting and choosing the best model for the data set under study, reliability indicators can be derived from the estimated parameters. The next section presents some of the indicators that can be derived from models 4.4 and 4.5.

4.4.2 Predictive reliability indicators

In order to forecast for future systems behavior, ARA and ARI intensity functions can be used to compute predictive reliability indicators. Considering that the last observed failure time is $T_n = t_n$, it can be of interest the time to next failure $T_{n+1} - T_n$, given the history up to time $T_n = t_n$ ($\Im_{T_n}$).

The following functions can characterize the future behavior of the process and they might be useful for the system engineering manager:

- The reliability function at time $T_n = t_n$ is

$$\begin{aligned} R_{T_n}(t) &= P(T_{n+1} - t_n > t|\Im_{T_n}) = P(N(t_n,t_n + t] = 0|\Im_{T_n}) \\ &= exp\left\{-\int_{t_n}^{t_n+t} \rho(u)du\right\}. \end{aligned} \quad (4.8)$$

  where $t_n \le u \le t_n + t < T_{n+1}$. This function express the probability that the system will work without failure for a given time t after T_n, given all the past history of the failure process.

- The MTTF at time $T_n = t_n$ is the mean time to the next failure occurring after T_n, given by

$$MTTF_{T_n} = E[T_{n+1} - t_n|\Im_{T_n}] = \int_0^{\infty} R_{T_n}(u)du, \quad (4.9)$$

  where R_{T_n} is the Equation 4.8.

Replacing the intensity functions 4.4 and 4.5 in Equation 4.8, the reliability functions for models ARA$_m$ and ARI$_m$ are given, respectively, by

$$\begin{aligned} R_{T_n,ARA_m}(t) &= exp\left\{-\int_{t_n}^{t_n+t} \lambda(u - (1-\theta)\sum_{j=0}^{min(m-1,n-1)} \theta^j t_{n-j})du\right\} \\ &= exp\left\{-\Lambda(t - (1-\theta)\sum_{j=1}^{min(m-1,n-1)} \theta^j t_{n-j})\right\} \times \\ &\times \; exp\left\{\Lambda(-(1-\theta)\sum_{j=1}^{min(m-1,n-1)} \theta^j t_{n-j})\right\} \end{aligned}$$

and

$$
\begin{aligned}
R_{T_n,ARI_m}(t) &= exp\left\{-\int_{t_n}^{t_n+t} \lambda(u) - (1-\theta) \sum_{j=0}^{min(m-1,n-1)} \theta^j \lambda(t_{n-j}) du\right\} \\
&= exp\left\{\Lambda(t_n) - \Lambda(t_n+t) + t(1-\theta) \sum_{j=0}^{min(m-1,n-1)} \theta^j \lambda(t_{n-j})\right\}
\end{aligned}
$$

which, in the particular case of a ROCOF modeled by a PLP, become

$$
\begin{aligned}
R_{T_n,ARA_m}(t) &= exp\left\{-\left(\frac{t-(1-\theta)\sum_{j=1}^{min(m-1,n-1)} \theta^j t_{n-j}}{\eta}\right)^{\beta}\right\} \times \\
&\times \; exp\left\{\left(\frac{-(1-\theta)\sum_{j=1}^{min(m-1,n-1)} \theta^j t_{n-j}}{\eta}\right)^{\beta}\right\} \qquad (4.10)
\end{aligned}
$$

and

$$
\begin{aligned}
R_{T_n,ARI_m}(t) &= exp\left\{\left(\frac{t_n}{\eta}\right)^{\beta} - \left(\frac{t_n+t}{\eta}\right)^{\beta}\right\} \times \\
&\times \; exp\left\{t(1-\theta) \sum_{j=0}^{min(m-1,n)-1)} \theta^j \frac{\beta}{\eta}\left(\frac{t_{n-j}}{\eta}\right)^{\beta-1}\right\}. \qquad (4.11)
\end{aligned}
$$

After using the likelihood functions from Section 4.4.1, the ML estimates $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$ can be replaced in Equations 4.10 and 4.11 and then in MTTF formula 4.9, so that the future behavior of a given system can be predicted. The next section presents the results of this methodology when applied to the dump trucks data set.

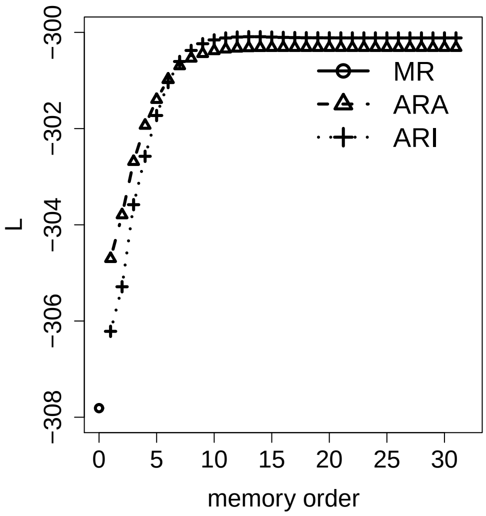
4.5 Dump trucks data set revisited

In the following, we apply ARA_m and ARI_m models with several memories to the trucks data set described in Section 4.2. ML estimates for PLP (β and η) and for the effect of repair (θ) parameters are obtained through the numerical maximization of the log-likelihood functions derived in Section 4.4, using a script written in R (www.R-project.org, v.2.15). Estimates are also obtained considering the MR PLP model. Initially, the purpose is to identify which model provides the best fit to the data. Table 4.1 exhibits the results for MR ($\theta = 1$), ARA_m and ARI_m with $m = 1,4,8,10,13,20,\infty$. For the data set under study, $m = 31$ corresponds to $m = \infty$, since the maximum number of failures observed was 31. Other memory values in the interval were also considered, but the results were ommited since they do not add valuable information to our conclusions.

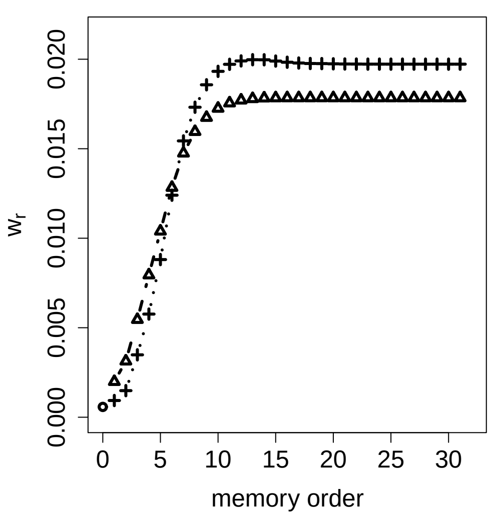
Table 4.1: Point and interval (95% confidence level) estimates for PLP (β, η) and effect of repair (θ) parameters, and the values of the maximum of the log-likelihood function ($\hat{L}$) under each fitted model.

Model:	MR	ARA_1	ARA_4	ARA_8
$\hat{\beta}$	1.14(0.96,1.35)	1.33(1.05,1.69)	1.60(1.26,2.03)	1.77(1.37,2.28)
$\hat{\eta}$	5.92(3.54,9.93)	4.94(3.41,7.15)	6.23(4.53,8.58)	7.27(5.22,10.12)
$\hat{\theta}$	-	0.02(0.0001,0.53)	0.39(0.24,0.63)	0.56(0.40,0.78)
$\hat{L}$	−307.1811	−304.7039	−301.9342	−300.5397
	ARA_{10}	ARA_{13}	ARA_{20}	ARA_∞
$\hat{\beta}$	1.79(1.38,2.32)	1.80(1.39,2.35)	1.81(1.39,2.35)	1.81(1.39,2.35)
$\hat{\eta}$	7.48(5.32,10.52)	7.58(5.35,10.76)	7.59(5.35,10.79)	7.59(5.35,10.79)
$\hat{\theta}$	0.58(0.42,0.81)	0.60(0.43,0.84)	0.60(0.42,0.84)	0.60(0.42,0.84)
$\hat{L}$	−300.3833	−300.3218	−300.3164	−300.3165
		ARI_1	ARI_4	ARI_8
$\hat{\beta}$		1.42(1.06,1.91)	1.81(1.58,2.07)	1.87(1.66,2.11)
$\hat{\eta}$		4.18(2.57,6.79)	6.26(4.52,8.66)	7.26(5.44,9.70)
$\hat{\theta}$		0.23(0.05,1.00)	0.50(0.38,0.66)	0.64(0.51,0.81)
$\hat{L}$		−306.2146	−302.5764	−300.3757
	ARI_{10}	ARI_{13}	ARI_{20}	ARI_∞
$\hat{\beta}$	1.88(1.67,2.11)	1.89(1.69,2.11)	1.90(1.71,2.11)	1.90(1.71,2.11)
$\hat{\eta}$	7.48(5.63,9.93)	7.63(5.74,10.11)	7.65(5.76,10.17)	7.65(5.76,10.17)
$\hat{\theta}$	0.66(0.52,0.85)	0.67(0.52,0.87)	0.67(0.52,0.87)	0.67(0.52,0.87)
$\hat{L}$	−300.1566	−300.0904	−300.1135	−300.1155

Using $\hat{L}$ as the criterion for model selection, it can be observed from Table 4.1 that, for ARA and ARI models, in general, increasing the memory (m value) improves the model adequacy (greater $\hat{L}$). Using a memory value equal or above 8 does not make a great difference in the model fit, which can be seen especially in the behavior of $\hat{L}$ values in Figure 4.3 (a). Weight values (Equation 4.7) plotted in Figure 4.3 (b) show the particular superiority of ARI_{13} fit, which presents the higher "weight of evidence"



(a)



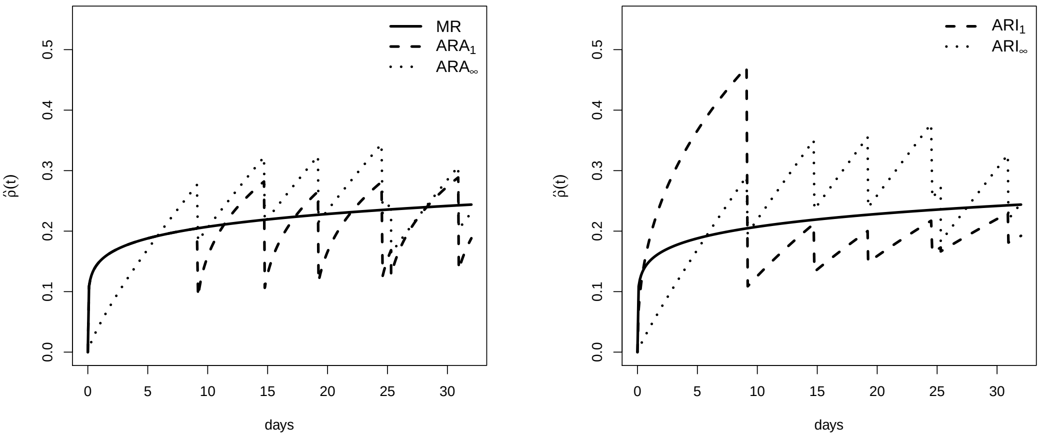
(b)

Figure 4.3: Criteria values for model selection versus memory order (m), under each fitted model for the trucks data set. Dot, triangles and crosses represent values for MR, ARA and ARI models, respectively. The y-axis in each graph are: (a) Maximum of log-likelihood, and (b) Weight - Eq. 4.7.

in its favor int the data set. However, in order to simplify the conclusions in this paper, ARI∞ model was used as the best one, since its criteria measures are very similar to the ones obtained for ARI_{13}.

For ARI_∞, the estimated β parameter was 1.90, with 95% confidence interval given by 1.71 to 2.11, indicating that the equipment failure intensity function increases with time. It means that the systems are under intrinsic ageing. The trucks tend to fail more frequently over time, justifying the necessity for PM. Point and interval estimates for the effect of repair parameter θ are 0.67 (0.52,0.87). It can be observed that it is significantly different than the two extreme cases ABAO ($\theta = 1$) and AGAN ($\theta = 0$). It means that the repairs after failures tend to leave the equipment in a state between AGAN and ABAO. Any attempt to establish a PM policy for these trucks must take these parameter values into account, as assuming the basic assumptions of MR or PR can lead to serious errors.

Based on the estimated parameter values from Table 4.1, Figure 4.4 presents the plots of the estimated intensity functions under the extreme models MR, ARA_1, ARI_1, ARA_∞, and ARI_∞, considering the first failure times for one of the sample trucks.

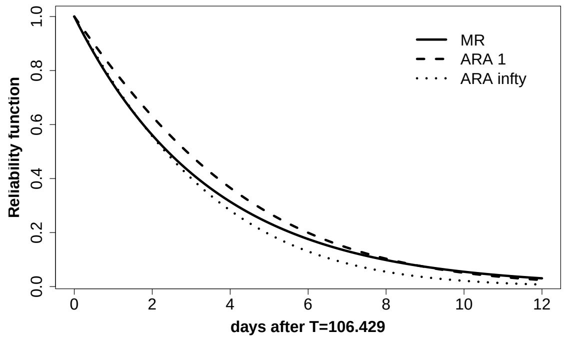


(a) (b)

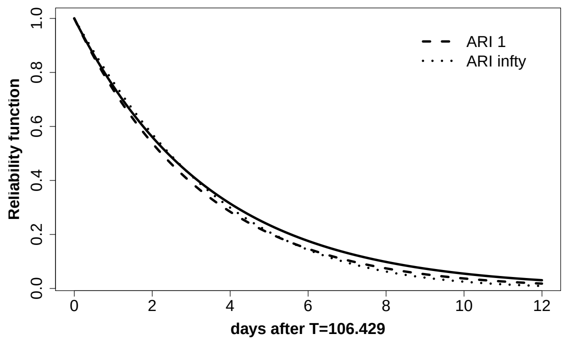
Figure 4.4: Estimated intensity functions under the fitted models, considering the first five failure times for one of the sample trucks. MR, ARA_1 and ARA_∞ models are showed on the left, while ARI_1 and ARI_∞ are on the right.

Is is remarkable the difference in the behavior of the intensity functions between the worst fitted models (MR, ARA_1, and ARI_1) and the best one, ARI_∞.

The reliability indicators derived in Section 4.4.2 are an useful tool for providing information about the future behavior of these repairable systems. In order to illustrate, it is considered one of the five sample trucks, whose last observed failure time is $T_{23} = 106.429$ days. We can predict the behavior of the time to next failure (T_{24}), given the history up to time T_{23} ($\Im_{T_{23}^-}$). Figure 4.5 shows the reliability functions $\hat{R}_{T_{23}}(t)$ calculated for MR, ARA (Equation 4.10) and ARI (Equation 4.11) models, using the ML parameters estimates. For the five models considered in the graphs, the reliability function goes from one to zero in 12 days, that is, the estimated probability of this system work without failure for 12 days after $T_{23} = 106.429$ is zero, given all the past of the failure process. While the fitted curve under model ARA_∞ is very close to the best fitted model one (ARI_∞), the reliability functions under ARA_1 and MR models tend to overestimate the survival probabilities. This behavior can also be observed in the corresponding MTTF values from Table 4.2. The mean time to the next failure occuring after T_{23} is 3.1 days according to the best fitted model, ARI_∞, while under ARA_1 model is 3.7 days.



(a)



(b)

Figure 4.5: Estimated reliability functions at $T_{23} = 106.429$ days for the trucks data set under the fitted models.

Table 4.2: Estimated MTTF values at $T_{23} = 106.429$ days for the trucks data set under the fitted models.

Model:	MR	ARA_1	ARA_∞	ARI_1	ARI_∞
	3.450	3.719	3.023	3.128	3.144

4.6 Conclusions and final remarks

In this paper, the problem related to the specification of a PM policy for repairable systems was studied under the primary focus of predicting the reliability of the systems using a well fitted model, under the general assumption of IR at failures. From ARA and ARI classes of models, likelihood functions were derived, and estimation methods were presented in order to obtain point and interval estimates for the parameters of the models. Assuming a PLP ROCOF, ARA and ARI models with different values of memory were fitted and compared to the classical MR model, using a dataset of failures records in dump trucks used by a mining company. Criteria methods were applied in order to choose the best fitted model. Estimation of model parameters allowed to forecast future behavior of the failure process, through reliability indicators.

According to criteria measures for model selection, the best fitted model was ARI_∞. Based on this model, the estimated shape of aging speed (β) parameter was 1.90, with 95% confidence interval given by 1.71 to 2.11, and the effect of repair parameter θ was estimated as 0.67 (0.52 to 0.87). These values give evidences that the trucks tend to fail more frequently over time, and also, that the repairs after failures tend to leave the equipment in a state between AGAN and ABAO. Illustrative predictive reliability indicators for a specific truck in the sample showed that the estimated probability of this system to work without failure for 12 days after the last observed failure time is zero, given all the past of the failure process. Additionally, the estimated MTTF after the last failure indicated that the mean time to the next failure for this system is 3.128 days.

It is important to point out that the trucks studied in this paper were originally designed to operate in road highways, but instead of this, are used by the mining company under much more severe conditions. Therefore, the PM policy suggested by the trucks manufacturer is not applicable, making it necessary for the mining company to define a PM policy based on the real working conditions of these equipment. Parameter estimates derived in this paper, such as the shape of aging speed and the effect of repair parameters, and also, the predictive reliability indicators, were obtained considering a history of failure times for the systems under the operating conditions inside the mine. They represent the real behavior of this failure process. Therefore, they provide important information to the decision-making process related to PM policies in the mining company.

5 Optimal Periodic Maintenance Policy Under Imperfect Repair: A Case Study of Off-Road Engines

5.1 Abstract

In the repairable systems literature one can find a great number of papers that propose maintenance policies under the assumption of minimal repair after each failure (such repair leaves the system in the same condition as it was just before the failure - *as bad as old*). This paper derives a statistical procedure to estimate the optimal Preventive Maintenance (PM) periodic policy, under the following two assumptions: (1) perfect repair at each PM action (i.e., the system returns to the *as good as new* state) and (2) imperfect system repair after each failure (the system returns to an intermediate state between *as bad as old* and *as good as new*). Models for imperfect repair have already been presented in the literature. However inference procedures for the quantities of interest have not yet been fully studied. In the present paper, statistical methods, including the likelihood function, Monte Carlo simulation, and bootstrap resample methods, are used in order to: (1) estimate the degree of efficiency of repair and (2) obtain the optimal preventive maintenance check points that minimize the expected total cost. This study was motivated by a real situation involving off-road engines maintenance.

5.2 Introduction

5.2.1 Motivating situation: Off-road engine maintenance data

Off-road trucks are designed to operate in harsh conditions and, consequently, they are used in every conceivable industry where rough terrain goes with the territory (mining, drilling, etc.). In mining companies particularly, off-road trucks are used to transport high production and, for that matter, the good performance of this equipment is essential to the financial health of this kind of business. Due to the high cost of these systems, one great concern is the implementation of good maintenance policies in order to prolong their life and reduce any expenses generated by the occurence of unexpected failures. Engine failures, for example, cost millions of dollars to the global mining industry directly (replacement and corrective repair actions) and indirectly through the inconveniences caused by those failures, such as loss of production, security risks, and reallocation of maintenance resources.

This paper was motivated by a real situation concerning engine failures in off-road trucks used by a Brazilian mining company. This company keeps a database with detailed descriptions of all maintenance actions performed on their off-road engines. The data used in this paper are a subset of this database, and include preventive (scheduled) and corrective (nonscheduled) maintenance records for a sample of 143 diesel engines. There were 50 Preventive Maintenance (PM) actions during the follow-up period, each assumed to be a perfect repair returning the engine to AGAN condition. Therefore it is fair to say that a *new* system has been put up into observation just after the PM action.

Consequently, as far as the data analysis is concerned, these 50 PM actions leaded to 50 *new* systems. The *final* database consisted of 143+50=193 diesel engines and 208 failure time were recorded. In addition, among the 193 engines, 52 were right censored since their last inspection time corresponded to a system removal for a PM.

The perfect repair assumption was verified through a statistical test comparing the 143 original systems to the 50 *new* ones generated by the PM actions. No evidence of difference between the failures behavior of the two groups was found.

Figure 5.1 plots the mean cumulative number of failures versus time (in hours of operation) for the sample of 193 engines. The convex shape of this function indicates that failures tend to occurr more frequently as the system age increases, in other words, the times between failures tend to get shorter with advancing age. This deteriorating behavior of the engines indicates that PM actions are essential to ensure the reliability of these equipment.

According to the mining company, the cost of a corrective maintenance performed after an unexpected failure is 23% higher than the cost of preventive maintenance. Hence, the company needs to adopt a maintenance policy that favors PM actions, as opposed to repair actions taken after failures.

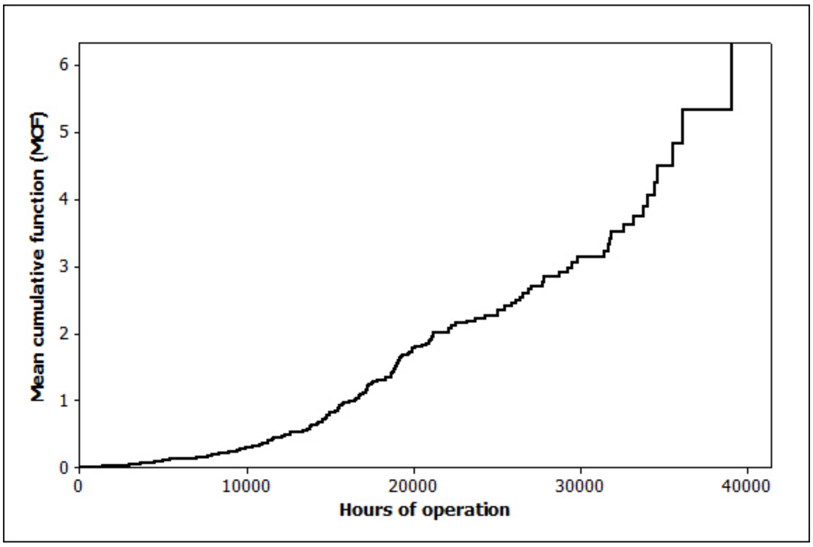


Figure 5.1: Mean cumulative number of failures versus time for the 193 off-road engines.

5.2.2 Background and literature

In the maintenance literature one can find a great number of papers that propose PM policies under the assumption of Minimal Repair (MR) on failures. In other words, it is assumed that the repair after each failure does not materially change the condition of the system: the repair restores the system to its status immediately before failure (ABAO). Barlow and Hunter (1960) used elementary renewal theory to obtain two types of PM policies, one which is more useful for simple systems (age replacement policy) and another for complex systems (block replacement policy). Related work can be found in Morimura (1970), Phelps (1981), Barlow and Proschan (1987), Park *et al.* (2000), Wang (2002) and Jaturonnatee *et al.* (2006).

Gilardoni and Colosimo (2007) considered statistical inference for the optimal PM periodicity under MR using maximum likelihood estimation. More precisely, they applied Barlow and Hunter's block replacement policy to a real data set concerning failures histories of power transformers. Assuming perfect PM actions (which restore the system to AGAN condition) and MR for failures, the authors came up with a closed form expression for the optimal PM policy, given by check points at every τ units of time. Gilardoni and Colosimo (2011, 2013) also considered nonparametric estimation and bootstrap confidence intervals for the optimal periodicity under MR.

However, in many practical situations, more realistic notions of repair, intermediate between the two extremes AGAN and ABAO, might be needed. In other words, many repairs actions are more likely to be Imperfect Repairs (IR), and any attempt to elaborate an optimal maintenance policy must take the actual degree of efficiency of these repairs into account. Many models have already been proposed for IR effects (for a review see, for example, Pham and Wang ,1996). Among them are the virtual age models proposed by Kijima *et al.* (1988) and Kijima (1989). In particular, Kijima *et al.* (1988) adapted the block replacement policy by Barlow and Hunter (1960) to the assumption of IR, where the degree of efficiency of the repair is represented by the parameter θ ($0 \leq \theta \leq 1$) and includes ABAO and AGAN as special cases ($\theta = 1$ and $\theta = 0$, respectively). The authors developed a virtual age model to describe the operation over time of a system which is repaired by IRs. But opposed to the MR case developed by Gilardoni and Colosimo (2007), under IR assumption there is no closed form expression to find the optimal PM periodicity. To overcome this difficulty, an approximation procedure was proposed by the authors, but its usage depends on the knowledge of the repair efficiency (θ) and the distribution of the lifetimes of the systems being studied. Numerical examples were provided assuming the particular case of a Gamma distribution, but the model was not statistically studied. More recently, Wu and Zuo (2010) presented a collection of PM models under IR assumption, but statistical estimation for the models parameters was also not discussed.

Only a few IR models have been statistically studied, and the emphasis has been on the estimation of the repair efficiency parameter. For virtual age models, some empirical studies on maximum likelihood estimators have been published: Shin *et al.* (1996), Yanez *et al.* (2002) and Doyen and Gaudoin (2004). All these articles are based on simulation results. In particular, Doyen and Gaudoin (2004) proposed two new classes of IR models. In the first class of models, the repair effect is expressed by a reduction in the failure intensity (the so called Arithmetic Reduction of Intensity or ARI models). In the second class, the repair effect is expressed by a reduction in the system's virtual age (the so called Arithmetic Reduction of Age or ARA models). A numerical statistical study on the quality of the model parameters estimators was presented. It is noteworthy that the virtual age model proposed by Kijima *et al.*

(1988) corresponds to the particular case of ARA model with memory 1, namely, ARA_1. More recently, Doyen and Gaudoin (2006, 2011) proposed the joint estimation of aging and maintenance efficiency and also, Pan and Rigdon (2009) and Corset *et al.* (2012) used Bayesian analysis for the ARA and ARI classes of models, but the focus was not on optimal maintenance policies.

More recently, Remy *et al.* (2013) presented a case study of technical and economic optimization of the periodicity of predetermined PM actions carried out on a repairable industrial system from an EDF electric power plant. Using several model selection criteria, the authors came up with the best model, namely [CM ARA_∞; PM AGAN], in other words, corrective maintenance (CM) actions were modeled via an ARA model of order infinity and PM actions modeled as renewals. The parameters associated to the intensity function and efficiency of repair were estimated. Then, using an economic indicator as the optimization criterion, the authors came up with an equation for C_{TOT}, the predictive total maintenace cost. Since C_{TOT} is a random variable, the optimal periodicity w is the value that minimizes the expected value of C_{TOT}. The search for this optimal value was made through a Monte Carlo procedure where $N = 5 \times 10^4$ trajectories were drawn, given a set of fictitious but realistic input data.

In the present work we also deal with the estimation of the effect of the repair under a situation where PM actions are modeled as perfect (AGAN) and the corrective actions considered IRs. The main difference of this work with the one of Remy *et al.* (2013) is that the search for the optimal periodicity was done via an approximation of the mean (cumulative number of failures) function and not by a search for a given set of input data.

5.2.3 The problem

From a modelling point of view, $\{N(t)\}_{t\geq 0}$ (where $N(t)$ denotes the number of observed failures up to time t) is a stochastic point process, with mean function $\Lambda(t) = E[N(t)]$ and failure intensity function

$$\rho(t) = \lim_{\delta t \to 0} \frac{P(N(t+\delta t) - N(t) = 1 | \Im_t^-)}{\delta t}, \quad \forall t \geq 0 \tag{5.1}$$

where $\Im_t^-$ represents the history up to time t (informally, one could think of $\Im_t^-$ as the information provided by the failure times $0 < t_1 < \cdots < t_{N(t)} < t$).

It can be shown (see, e.g. Aalen, 1978) that $\Lambda(t) = \int_0^t E[\rho(s)]\, ds$.

Before the first maintenance action, the system failure intensity is the rate of occurrence of failures (ROCOF), given by

$$\lambda(t) = \lim_{\delta t \to 0} \frac{E[N(t + \delta t) - N(t)]}{\delta t}. \tag{5.2}$$

While $\rho(t)$ depends on the history $\Im_t^-$, the ROCOF $\lambda(t)$ is not conditional and hence depends only on t. The ROCOF characterizes the reliability the system would have if it was not maintained.

Under the MP assumption, it is assumed that the effect of maintenance is to leave the system in the same state as it was just before failure. The underlying failure process in this case is a Nonhomogeneous Poisson Process (NHPP), and the failure intensity function $\rho(t)$ equals the ROCOF. In other words, $\rho(t) = \lambda(t)$, $t \geq 0$.

Under the IR assumption, some functional forms for $\rho(t)$ have been proposed in the literature. In particular, for ARA_1 model, it can be shown that the failure intensity is given by (Doyen and Gaudoin, 2004)

$$\rho(t) = \lambda(t - (1 - \theta)T_{N(t)}), \tag{5.3}$$

where T_n is a random variable representing the real age of the system at the n^{th} failure (the elapsed time since the initial start-up of the system) and $\lambda(.)$ is the ROCOF corresponding to the condition of MR.

If $\theta = 1$, it is assumed that an MR is performed (NHPP). Futhermore, $\theta = 0$ indicates that the system is renewed after each repair and the resulting process is a Renewal Process. Under this model, after each repair the virtual age of the system is reduced by the multiplicative constant $1 - \theta$.

If the intensity function is the one given in Equation 5.3, it can be shown (Kijima *et al.*, 1988) that the expected number of failures (or mean function) at time τ is given by

$$\Lambda(\tau) = \int_0^\tau E[\lambda(t - (1-\theta)T_{N(t)})]dt. \tag{5.4}$$

This equation is usually referred to in the literature as the *general renewal function*, or *g-renewal function*. Unfortunately, there is no closed form solution for this equation, except for special cases, such as $\theta = 1$ (MR), or $\theta = 0$ (perfect repair) with the underlying failure times exponentially distributed.

For the general case ($0 \leq \theta \leq 1$), a deterministic approximation method for Equation 5.4 was proposed by Kijima *et al.* (1988). However, applying such an approximation to real systems (which is of great practical interest) is not possible, because this method does not enable estimating the parameters from the failure history data. Later on, Yevkin and Krivtsov (2012) also proposed an approximation for the *g-renewal function*, but no inference procedure with desirable statistical properties was proposed.

Although no closed form solution may be obtained for the *g-renewal function*, this function still can be estimated from the data.

In view of the limitations of the approximations cited above, this paper proposes a procedure to obtain estimators for Equation 5.4 using the observed failure history. The proposed method aims at dealing with the following three issues at the same time, namely: **(1)** the estimation of the parameters of the intensity function $\rho(t)$ (Equation 5.3), **(2)** the calculation of an estimator for the mean function $\Lambda(t)$ (Equation 5.4), and **(3)** the combination of (1) and (2) to find the optimal PM policy, *i.e.*, to obtain the optimal PM check points (or periodicity τ) that minimize the expected total cost (preventive and corrective maintenance actions) under an IR environment. The method is applied to the failure history of off-road engines.

The outline of this paper is as follows. In Section 5.3, the cost function to be minimized is presented. Section 5.4 deals with statistical methods. In particular, the expression of the likelihood function needed to estimate the model parameters, namely, the intensity function and the efficiency of repair parameters, is derived. In Section 5.5, a method to use these parameter estimates to find the optimal PM policy (given by check points at every τ units of time) is proposed. In fact, this is the main contribution of this paper. First of all, a procedure to estimate $\Lambda(t)$ *from the data* is presented. Next, it is described how to use these estimates to determine the optimal PM periodicity for a predetermined ratio of costs (C_{PM}/C_{IR}), where C_{PM} and C_{IR} denote the PM and IR actions costs, respectively. Also, a Monte Carlo experiment was run to study the performance of the proposed method. Section 5.6 presents the results of this study, and also illustrates the proposed methodology with simulations. The method is applied to the off-road engines maintenance data and the results are presented in Section 5.7 (point and interval estimates for τ are provided). Conclusions and final comments end the paper in Section 5.8.

5.3 Cost Function and Optimal PM Under an ARA$_1$ Model

Consider a system which is subject to failure, and that is put in operation at time $t = 0$. Assume the following conditions:

- PM check points are scheduled after every τ units of time;
- at each PM check point, a maintenance of fixed cost C_{PM} is executed, which instantly returns the system to AGAN condition;
- between successive PM check points, an IR of degree θ ($0 \leq \theta \leq 1$) is done after each failure, where $\theta = 1$ represents an MR (ABAO condition) and $\theta = 0$ a perfect repair (AGAN);
- the expected cost for each IR action is C_{IR}, that is, for each period defined by successive PM check points, the expected total cost is equal to the expected cost per failure times the expected number of failures;
- repair costs and failure times are independent;
- repair times are neglected.

Assume that PM is performed every t units of time. The long run expected maintenance cost $C(t)$ per unit time for the system is given by (Gilardoni and Colosimo, 2007)

$$C(\tau) = \frac{C_{PM} + C_{IR}E[N(\tau)]}{\tau}, \quad \tau > 0 \tag{5.5}$$

Under ARA_1 model, $E[N(t)]$ is given by Equation 5.4. The objective here is to find an optimal PM interval τ which minimizes Equation 5.5. Then, the PM policy that minimizes $C(t)$ is the value τ that satisfies

$$D(\tau) = \tau\lambda(\tau) - \Lambda(\tau) = \frac{C_{PM}}{C_{IR}}. \tag{5.6}$$

where $\lambda(\tau) = \frac{d}{d\tau}\Lambda(\tau)$ is the ROCOF function for the system.

However, under IR assumption no closed form solution can be obtained for the *g-renewal function* $\Lambda(\tau)$ and, consequently, for Equation 5.6.

In this paper, a procedure to deal jointly with the three following issues is proposed: (1) the estimation of the parameters involved in Equation 5.3, (2) the calculation of an estimator for the mean function $\Lambda(t)$ (Equation 5.4) from which the ROCOF $\lambda(t)$ may be derived, and (3) the combination of (1) and (2) to solve Equation 5.6 for τ, i.e., to find the optimal PM policy.

The individual values of the costs C_{PM} and C_{IR} are not necessary. Instead, only the ratio between them needs to be considered, which simplifies the application in practice.

The first step for finding the optimal PM policy is to estimate the model parameters. Section 5.4 introduces some additional notation and presents the likelihood function for ARA_1 model. In particular, inference procedures for the parameters involved in Equation 5.3 using a ROCOF modeled by a Power Law Process are presented.

5.4 Parameter Estimation: The Likelihood Function

Consider k identical repairable systems, $k = 1,2,\ldots$, where the failures occur independently. There are basically two ways to observe data from a repairable system. When the data collection stops after a predermined number of failures, the data are said to be failure truncated. On the other hand, when the data collection stops at a predetermined time t, the data are said to be time truncated. The likelihood function is constructed here assuming that among the k observed repairable systems, k_1 are time truncated and k_2 are failure truncated, $k_1,k_2 = 1,2,\ldots,k$ and $k_1 + k_2 = k$.

Assume the following conditions:

- At each failure, a repair action of degree θ is executed.

- n_i failures are observed in the i^{th} time truncated system, $i = 1,2,\ldots,k_1$, and n_j^* failures are observed in the j^{th} failure truncated system, $j = 1,2,\ldots,k_2$.

- $N = \sum_{i=1}^{k_1} n_i + \sum_{j=1}^{k_2} n_j^*$ is the total number of observed failures in the systems.

- The i^{th} time truncated system is observed until the predetermined time t_i^*, and the j^{th} failure truncated system is observed until the predetermined number of failures n_j^* occurs.

- Let $T_{i,l}$ $(i = 1,2,\ldots,k_1,\ l = 1,2,\ldots,n_i)$ be random variables representing the failure times for the i^{th} time truncated system, recorded as the time since the initial start-up of the system $(T_{i,1} < T_{i,2} < \ldots < T_{i,n_i})$. For time truncated systems, it is a random number of variables. In addition, let $t_{i,l}$ denote their observed values (data), and $T_i = (T_{i,1}; T_{i,2}; \ldots; T_{i,n_i})^t$ be the $(n_i \times 1)$ random vector of failure times for the i^{th} time truncated system.

- Let $T_{j,m}(j = 1,2,\ldots,k_2,\ m = 1,2,\ldots,n_j^*)$ be random variables representing the failure times for the j^{th} failure truncated system, thus being a fixed number of variables $(T_{j,1} < T_{j,2} < \ldots < T_{j,n_j^*})$. Let $t_{j,m}$ denote their observed values. In addition, let $T_j = (T_{j,1}; T_{j,2}; \ldots; T_{j,n_j^*})^t$ be the $(n_j^* \times 1)$ random vector of failure times for the j^{th} failure truncated system.

- Let $N(t)$ be a random variable representing the number of failures in the interval $(0,t]$.

- Let μ denote the vector of model parameters. It includes the parameters indexing the process intensity function and the repair efficiency parameter θ.

A likelihood function appropriate to model this process must combine the joint probability density function of the k systems global times, and is given by

$$L(\mu) = \prod_{i=1}^{k_1} [f_{T_i|N(t_i^*)}(t_{i,1},\ldots,t_{i,n_i}|n_i) P(N(t_i^*) = n_i)] \times \prod_{j=1}^{k_2} f_{T_j}(t_{j,1},\ldots,t_{j,n_j^*}) \qquad (5.7)$$

if $k \geq 1$, $k = k_1 + k_2$.

The contributions of the k_1 time truncated systems and of the k_2 failure truncated systems to the likelihood function are represented by the first and second terms of Equation 5.7, respectively.

As discussed above, the induced intensity function under the ARA_1 model is shown in Equation 5.3. Thus, it is assumed that the failure process has an intensity function which is conditional on the effective maintenance times. For this model, the likelihood function in Equation 5.7 becomes

$$\begin{aligned} L(\mu) &= \prod_{i=1}^{k_1}\left[\left(\prod_{l=1}^{n_i}\lambda(t_{i,l}-(1-\theta)t_{i,l-1})e^{\Lambda(\theta t_{i,l})-\Lambda(t_{i,l}-(1-\theta)t_{i,l-1})}\right)e^{-\Lambda(t_i^*-(1-\theta)t_{i,n_i})}\right]\times \\ &\times \prod_{i=1}^{k_2}\left[\prod_{m=1}^{n_j^*}\lambda(t_{j,m}-(1-\theta)t_{j,m-1})e^{\Lambda(\theta t_{j,m-1})-\Lambda(t_{j,m}-(1-\theta)t_{j,m-1})}\right]. \end{aligned} \tag{5.8}$$

If the Power Law Process (PLP - Crow, 1974) is used, then the ROCOF function in Equation 5.3 and its corresponding mean function are given, respectively, by

$$\lambda(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}, \quad \eta,\beta,t > 0, \tag{5.9}$$

and

$$\Lambda(t) = \int_0^t \rho_R(u)du = \left(\frac{t}{\eta}\right)^{\beta}, \tag{5.10}$$

where β (the shape parameter) represents how the system deteriorates or improves over time ($\beta > 1$ indicates an increasing intensity function), and η is a scale parameter.

Therefore, the likelihood function (Equation 5.8) can then be rewritten as a function of the PLP intensity function parameters (β e η), and also of θ, the efficiency of the repair parameter, yielding

$$
\begin{aligned}
L(\mu) &= \beta^{\sum_{i=1}^{k_1} n_i + \sum_{j=1}^{k_2} n_j^*} \eta^{-\beta(\sum_{i=1}^{k_1} n_i + \sum_{j=1}^{k_2} n_j^*)} \left[\prod_{i=1}^{k_1} \prod_{l=1}^{n_i} (t_{i,l-1} - (1-\theta) t_{i,l-1})^{\beta-1} \right] \times \\
&\times \left[\prod_{j=1}^{k_2} \prod_{m=1}^{n_j^*} (t_{j,m} - (1-\theta) t_{j,m-1})^{\beta-1} \right] \times \\
&\times \; exp \left\{ \sum_{i=1}^{k_1} \left[\sum_{l=1}^{n_i} \left(\frac{\theta t_{i,l}}{\eta} \right)^{\beta} - \left(\frac{t_{i,l} - (1-\theta) t_{i,l-1}}{\eta} \right)^{\beta} \right] - \left(\frac{t_i^* - (1-\theta) t_{i,n_i}}{\eta} \right)^{\beta} \right\} \times \\
&\times \; exp \left\{ \sum_{j=1}^{k_2} \left[\sum_{m=1}^{n_j^*} \left(\frac{\theta t_{j,m-1}}{\eta} \right)^{\beta} - \left(\frac{t_{j,m} - (1-\theta) t_{j,m-1}}{\eta} \right)^{\beta} \right] \right\}. \qquad (5.11)
\end{aligned}
$$

Statistical inference is needed in order to estimate the model parameters and to compute the reliability indicators from the failure data. In order to find the Maximum Likelihood Estimates (MLEs) $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$ of the parameters, the following log-likelihood function must be numerically maximized:

$$
\begin{aligned}
l(\mu) &= Nlog(\beta) - \beta N log(\eta) + \\
&+ (\beta - 1) \left(\sum_{i=1}^{k_1} \sum_{l=1}^{n_i} log(t_{i,l} - (1-\theta) t_{i,l-1}) + \sum_{j=1}^{k_2} \sum_{m=1}^{n_j^*} log(t_{j,m} - (1-\theta) t_{j,m-1}) \right) + \\
&+ \left\{ \sum_{i=1}^{k_1} \left[\sum_{l=1}^{n_i} \left(\frac{\theta t_{i,l}}{\eta} \right)^{\beta} - \left(\frac{t_{i,l} - (1-\theta) t_{i,l-1}}{\eta} \right)^{\beta} \right] - \left(\frac{t_i^* - (1-\theta) t_{i,n_i}}{\eta} \right)^{\beta} \right\} + \\
&+ \left\{ \sum_{j=1}^{k_2} \left[\sum_{m=1}^{n_j^*} \left(\frac{\theta t_{j,m}}{\eta} \right)^{\beta} - \left(\frac{t_{j,m} - (1-\theta) t_{j,m-1}}{\eta} \right)^{\beta} \right] - \left(\frac{\theta t_{j,n_j^*}}{\eta} \right)^{\beta} \right\}. \qquad (5.12)
\end{aligned}
$$

Asymptotic theory for the Normal distribution can then be used to construct approximate confidence intervals for the parameters.

5.5 Proposed Method to Obtain the Optimal PM under IR

In a practical situation, it is necessary to estimate the optimal maintenance periodicity τ using the failure history of the systems under study. It is fair to say that under the MR assumption, the procedure to obtain the optimal maintenance periodicity is straightforward. For instance, if one assumes the PLP, $\lambda(t)$ and $\Lambda(t)$ have closed form expressions (Equations 5.9 and 5.10) which can be plugged into Equation 5.6 and solved for τ, leading to the analytic solution (Gilardoni and Colosimo, 2007):

$$\tau = \eta \left[\frac{C_{PM}}{(\beta - 1)C_{MR}} \right]^{(1/\beta)}, \tag{5.13}$$

Then, using the invariance property of the maximum likelihood method, the MLE of τ is obtained by plugging $\hat{\eta}$ and $\hat{\beta}$ (the MLEs of η and β, respectively) into Equation 5.13. Asymptotic theory can be used to construct confidence intervals.

However, the same approach can not be followed under the IR assumption, since, as mentioned in Section 5.3, there is no closed form solution for the *g-renewal function* given by Equation 5.4. In this section, a method to estimate the mean and intensity functions ($\Lambda(t)$ and $\lambda(t)$, respectively), *from the data* is proposed. These estimates are then used in the cost function (Equation 5.6) to calculate a point estimate of the PM periodicity parameter τ. Subsequently, confidence intervals for τ are obtained using bootstrap resampling.

The mean function $\Lambda(t)$ is estimated using a combination of MLE for the parameters involved (θ,β,η), Monte Carlo simulation, and the Nelson–Aalen nonparametric procedure (Aalen, 1978), also known as the Mean Cumulative Function (MCF). In the sequel, we will use the term MCF whenever we refer to the Nelson–Aalen estimate. The steps of the proposed method are illustrated using the PLP but it can be applied to any other parametric form chosen for the ROCOF. The steps are described below:

- **Step 1: Maximum Likelihood estimation of the model parameters.** Use the failure history and log-likelihood function (Equation 5.12) to obtain the MLEs $\hat{\beta}$, $\hat{\eta}$ (PLP parameters) and $\hat{\theta}$ (repair efficiency).

- **Step 2: Estimation of the mean function** $\Lambda(t)$: Monte Carlo simulation of failure histories and calculation of the MCF.

    - **Step 2.1: Monte Carlo simulation.** Use the estimated values $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$ to generate failure histories for K systems (K large; here K=10,000) truncated in time T, using, for each system, the procedure described in the Appendix.

– **Step 2.2: Calculation of the MCF $\hat{\Lambda}(t)$ using the failure time histories of the K simulated systems.** As the failure processes generated in Step 2.1 are time truncated, the MCF is simply the mean number of observed failures between 0 and T on the K trajectories.

- **Step 3: Estimation of the optimal periodicity τ.** In order to use the cost function given by Equation 5.6, it is necessary to find estimates for the functions $\lambda(t)$ and $\Lambda(t)$. In Step 2, the MCF was used as an estimate for $\Lambda(t)$. However, the MCF is a step function, so its derivative is almost everywhere zero, and an estimate for $\lambda(t)$ cannot be directly obtained from this. So, we use here the nonparametric estimate given by the right derivative of the Greatest Convex Minorant (GCM) (Boswell, 1966):

  1. The GCM of $\hat{\Lambda}(t)$, namely $\hat{\Lambda}_{S_G}(t)$, is given by:

$$\hat{\Lambda}_{S_G}(t) = sup\{g(t) : g \text{ is convex and } g(u) \leq \hat{\Lambda}(u) \text{ for all } u\}; \qquad (5.14)$$

  2. Then, $\hat{\lambda}_{S_G}(t) = \hat{\Lambda}'_{S_G}(t+0)$ (the right derivative of $\hat{\Lambda}_{S_G}(t)$).

In the MR case, the failure process is a NHPP and Boswell (1966) showed that the constrained nonparametric MLE of $\lambda(t)$, namely, $\hat{\lambda}_{C-NPMLE}$, is equal to $\hat{\lambda}_{S_G}(t)$ (constrained here, in the sense that the MLE is obtained subject to monotonicity constraints). In addition, Gilardoni and Colosimo (2011) showed that this is also valid for several systems (under MR assumption) if one overlaps them via a TTT-transformation.

In the method proposed here, it is not necessary to use a TTT-transformation, since all the systems are truncated at the same time in Step 2.1. There is no garantee that this property is valid under the IR assumption but we have used the GCM procedure since it has provided a good fit.

In the Appendix we give the algorithm to compute the step function $\hat{\lambda}_{C-NPMLE}$ as the right derivative of the GCM of the MCF described by Boswell (1966) and based on the failure history of one system only (under the MR assumption).

Appropriate confidence intervals for τ are obtained using nonparametric bootstrap resampling (Efron and Tibshirani, 1986). The method consists of resampling with replacement B (B large) samples from the original database. Each resample has the same size as the original data set. For the off-road engines, for example, the database consists of 193 engines. Thus, each one of the B resamples has 193 engines resampled with replacement from the original database.

The method described in Steps 1 to 3 is then applied to each of the B resamples, thereby obtaining B bootstrap estimates for τ, denoted here by τ_i^b ($i = 1,2,\ldots,B$). The $100(1-\alpha)\%$ percentile CI for τ is then given by the limits $(\hat{\tau}_{[l]}^b,\hat{\tau}_{[u]}^b)$, where $\hat{\tau}_{[i]}^b$, ($i = 1,2,\ldots,B$) are the bootstrap estimates sorted in increasing order, $l = B \times (\alpha/2)$ and $u = B \times (1-\alpha/2)$, l and u rounded to the smallest and largest nearest integers, respectively.

5.6 Simulation Study

Before we apply the proposed method to the real data set, it is instructive to investigate, through a simulation study, the performance of the point and interval estimates for τ generated by this procedure.

Unfortunately, under the IR assumption, it is not possible to calculate the real value of the optimum PM periodicity parameter, which in turn, makes also impossible the calculation of the usual performance measures (such as the median relative absolute error). One workable alternative is to study the performance of the proposed method under the MR assumption since in this case, the value of τ can be analytically calculated through Equation 5.13, for given β and η and cost ratio values. The results of a Monte Carlo simulation study under the MR assumption are presented in Section 5.6.1.

Although the performance measures cannot be computed in the IR case it is possible to illustrate the method on simulated data from different scenarios. The results of such application to simulated data are presented on Section 5.6.2.

All the simulations were done using a script written in R, a language and environment for statistical computing (www.R-project.org, v.2.15).

5.6.1 Monte Carlo Simulation Study

The failure times were generated using three sets of PLP parameter values:

1. $\beta = 1.5$, $\eta = 16{,}716$
2. $\beta = 2.1$, $\eta = 16{,}716$
3. $\beta = 3.0$, $\eta = 16{,}716$

The second set of parameter values are the MLEs obtained with the off-road engines data, assuming PLP and MR. This assumption is not valid for this case, as will be shown in Section 5.7, but the numerical values are used here just as a reference for the study. The other two cases illustrate data generated under different conditions but still with increasing intensity functions.

Initially, for each set of PLP parameter values considered, $M = 1{,}000$ samples of $n = 193$ failure processes were generated employing the method described in Step 2.1, Section 5.5, using $K = 10{,}000$; truncation time $T = 40{,}000$ and $\theta = 1$. The value for n was chosen to coincide with the engines data set sample size. Assuming different cost ratios, point estimates for τ were then obtained for the samples generated, following Steps 1–3 in Section 5.5.

Moreover, for each one of the $M = 1{,}000$ samples generated for each set of PLP parameters, 95% confidence intervals (CI) for τ were calculated, assuming different cost ratios. The nonparametric bootstrap resampling method was used to build CIs, using $B = 1{,}000$.

The following perfomance measures were calculated for each scenario (by *scenario* we mean a combination of PLP parameter values and cost ratios), and the results are shown in Table 5.1:

- **for the point estimates of** τ:
    1. median (absolute) relative error (MRE): where the relative error is the error divided the real value of τ;
    2. interquartile range of the (absolute) relative error (IQR.RE).
- **for the 95% confidence intervals (95% CI) for** τ:
    1. observed coverage: $1 - \hat{\alpha}$;
    2. median relative CI length (MRL): median CI length divided by the real τ.

It can be observed that the coverages of the Bootstrap CI were quite close to the nominal value 0.95 (minimum=0.941; maximum=1.00). The only exception is the coverage value for the scenario $\beta = 1.5$ and cost ratio 1.23. The MRL values seem to increase with the cost ratio values, but they do not seem to be associated with the observed coverage. For the point estimates, both MRE and IQR.RE values increase with the cost ratio, for the three scenarios.

It is important to note that all the performance measures get smaller with the increase of the shape PLP parameter β, i.e. for mean functions with higher degrees of convexity.

Of course, our conclusions are limited to the scenarios considered above. In addition, they are restricted since they were drawn based on simulation results originating under the minimal repair assumption. But it is fair to say that, in general, these simulation results attest to the credibility of the proposed method.

5.6.2 Illustrating the Method with Simulated Data under the IR assumption

Using the algorithm described in Appendix, $n = 193$ failure processes were generated under different combinations of β (1.5, 2.1 and 3.0) and θ (0.1, 0.5 and 0.9) parameters, assuming an ARA_1 virtual age model with a PLP ROCOF, with η scale parameter set to 16,716. The parameter values, as well as the sample size n, were based on the real data described in Section 5.2.1, for which more details are given in next section. The parameters combinations represent a wide variety of scenarios, in terms of the efficiency of repairs and the shape of the intensity function (aging speed). Assuming that, for each sample of generated processes, maintenance actions costs have the ratios $C_{PM}/C_{IR} = 1.23$, 1/5 and 1/15, Steps 1–3 from Section 5.5 were followed for each sample, resulting in point and interval (95% bootstrap CIs with $B = 10{,}000$) estimates for the optimal PM periodicity, τ.

The results are displayed in Table 5.2 and Figure 5.2, from which we can observe that:

- **The effect of C_{PM}/C_{IR}:** The optimal periodicity τ decreases along with the decrease of the cost ratio C_{PM}/C_{IR}. This is an intuitive result, since expensive

Table 5.1: Simulation Study Results-Performance measures (for point and interval estimates) for each simulated scenario (assuming MR and PLP with η (scale) = 16716.53)

	$\beta = 1.5$				
C_{MR}/C_{PM}	1.23	3	5	10	15
Point Estimates (performance measures)					
MRE[(1)]	0.048	0.047	0.052	0.063	0.069
IQR.RE[(2)]	0.056	0.063	0.062	0.072	0.086
95% CI (Performance measures)					
MRL[(3)]	0.168	0.227	0.268	0.338	0.386
$(1-\hat{\alpha})$[(4)]	0.704	0.956	0.990	0.999	1.000
	$\beta = 2.1$				
C_{MR}/C_{PM}	1.23	3	5	10	15
Point Estimates (performance measures)					
MRE	0.023	0.033	0.037	0.046	0.055
IQR.RE	0.027	0.038	0.048	0.062	0.063
95% CI (performance measures)					
MRL	0.116	0.155	0.183	0.229	0.260
$(1-\hat{\alpha})$	0.995	0.983	0.980	0.969	0.970
	$\beta = 3.0$				
C_{MR}/C_{PM}	1.23	3	5	10	15
Point Estimates (performance measures)					
MRE	0.018	0.025	0.030	0.035	0.038
IQR.RE	0.022	0.029	0.036	0.043	0.048
95% CI (performance measures)					
MRL	0.058	0.114	0.134	0.168	0.192
$(1-\hat{\alpha})$	0.944	0.941	0.945	0.951	0.963

(1) median relative (absolute) error; (2) interquartile range of the (absolute) relative error; (3) median relative CI length; (4) observed coverage. Results based on $M = 1000$ samples of size $n = 193$ systems. In all cases the bootstrap (nonparametric) sample size is $B = 1000$.

repair actions must be avoided with more frequently PM actions. On the other hand, very frequently PM inspections tend to generate unnecessary expenses, so, the proposed policy balances both principles through Equation 5.6.

- **The effect of the parameter θ:** Greater values of θ, representing conditions closer to MR are associated to higher frequency of PM (smaller τ values). However, it can be observed that this effect of θ parameter becomes smaller when the cost of a IR action gets higher (when compared to the PM cost). For $C_{PM}/C_{IR} = 1/15$ (Figure 5.2 (c)), for example, no specific pattern is observed.

- **The effect of the parameter** β: Larger β values are asssociated to a faster aging process. In general, under scenarios with larger β values, the procedure tends to suggest a higher frequency of PM (i.e. smaller τ values).

  However, in the more critical scenario, $C_{PM}/C_{IR} = 1/15$, it is not possible to establish a specific relationship. Under this condition, the range in τ values is small, so the high cost of IR seems to be the most important factor to define when to perform PM actions - neither the effect of repair (θ) nor the aging speed (β) parameters contribute to this decision.

Table 5.2: Simulated Data under IR assumption- Point and interval estimates (95% bootstrap CIs with $B = 10{,}000$) for the optimal PM periodicity by cost ratio (C_{IR}/C_{PM}) for each sample of $n = 193$ failure processes generated under different combinations of β and θ parameters ($\eta = 16.716$).

β	θ	$C_{PM}/C_{IR} = 1/1.23$	$C_{PM}/C_{IR} = 1/5$	$C_{PM}/C_{IR} = 1/15$
1.5	0.1	39,966(36,654; 39,999)	11,167(9,303; 14,928)	4,392(4,066; 5,933)
1.5	0.5	35,144(26,937; 39,888)	9,240(8,797; 12,994)	5,229(3,911; 5,840)
1.5	0.9	25,825(22,671; 28,827)	9,184(8,228; 10,694)	4,162(3,573; 5,273)
2.1	0.1	39,927(33,536; 39,999)	8,981(7,676; 9,488)	4,456(4,069; 5,354)
2.1	0.5	19,030(16,932; 20,250)	8,288(7,135; 8,694)	4,408(3,904; 5,252)
2.1	0.9	15,627(14,387; 16,270)	7,536(6,960; 8,518)	4,717(3,946; 5,239)
3.0	0.1	33,419(30,387; 39,802)	8,348(7,763; 9,218)	5,585(5,177; 6,593)
3.0	0.5	13,792(13,697; 15,425)	8,057(7,423; 8,694)	5,314(4,860; 6,138)
3.0	0.9	12,162(11,843; 13,061)	7,826(7,100; 8,224)	5,416(4,774; 5,829)

5.7 Off-Road Engines Maintenance Data Revisited

In this section, we return to the situation described in Section 5.2.1, i.e, the off-road engines maintenance data presented in Figure 5.1. In that figure, the mean observed cumulative number of failures showed a convex shape, indicating that the ROCOF function is increasing for these systems, therefore justifying PM. The goal here is to

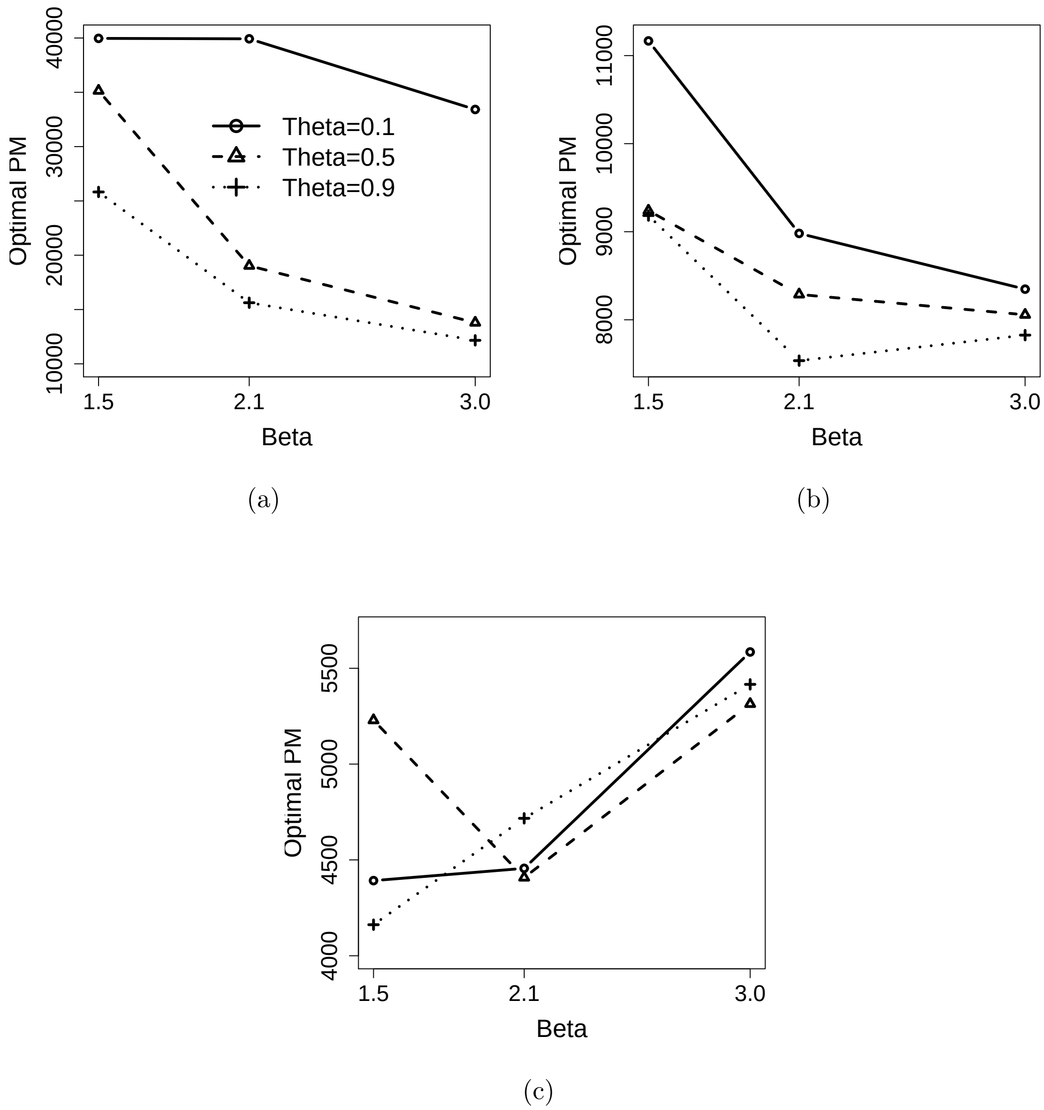


Figure 5.2: Point estimates for the optimal PM periodicity for each generated sample, assuming C_{IR}/C_{PM} = (a) 1/1.23, (b) 1/5, and (c) 1/15.

obtain the optimal PM check points that minimize the expected total cost, through the method described in Section 5.5. The results are summarized in Figure 5.3 and Table 5.3. The main points are listed below:

- MR assumption: The following point and interval (95% confidence intervals based on Normal approximation) MLEs were obtained for the PLP parameters: $\hat{\beta} = 2.126$ $(1.916; 2.358)$ and $\hat{\eta} = 16{,}715$ $(15{,}604; 17{,}905)$. Figures 5.3 (a) and 5.3 (b) (dotted lines) show the MLEs of the mean and intensity functions respectively. These estimates were obtained using the invariance property of the

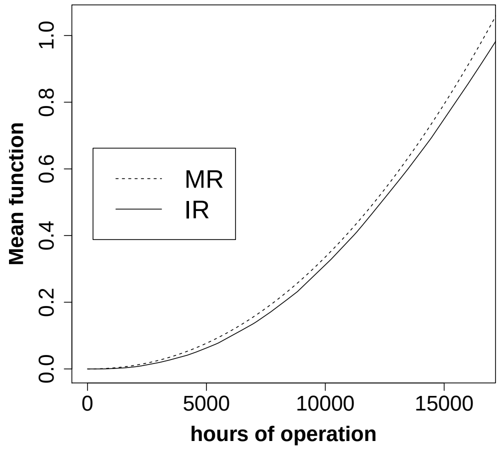
MLEs and Equations 5.9 and 5.10.

Next, following Gilardoni and Colosimo (2007) approach, $\hat{\Lambda}$ and $\hat{\lambda}$ were then plugged into Equation 5.6, resulting in the closed form for τ given by expression 5.13. Figure 5.3 (c) (dotted line) exhibits the function $\hat{D}(t)$.

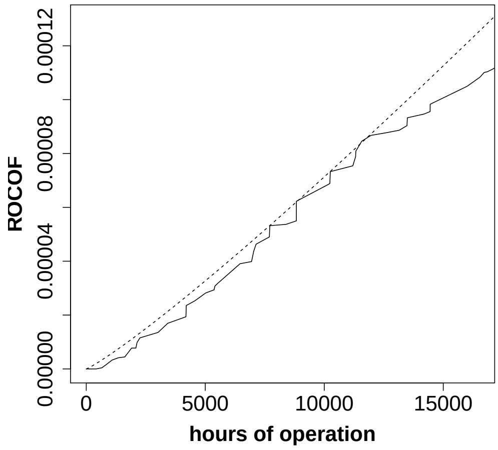
- IR assumption: Following the methodology proposed in Section 5.5, the likelihood function described in Section 5.4 was used to obtain MLEs and 95% CIs for the PLP parameters ($\hat{\beta} = 2.458$ $(2.185; 2.765)$ and $\hat{\eta} = 15{,}586$ $(14{,}605; 16{,}633)$) and the effect of repair parameter ($\hat{\theta} = 0.471$ $(0.330; 0.673)$) (Step 1). Next we follow Steps 2 and 3 to estimate the mean function $\Lambda(t)$, defined in Equation 5.4, which does not have a closed solution. Firstly, the MCF was obtained from $K = 10{,}000$ failure processes simulated with the truncation time $T = 40{,}000$ h (that corresponds to the time range in the observed data). Secondly, the GCM of the MCF provided a estimation for $\Lambda(t)$ (Figure 5.3 (a), solid line), whose derivation generated an estimation for $\lambda(t)$, the ROCOF of this process (Figure 5.3 (b), solid line). After replacing these estimated functions in Equation 5.6, the function $\hat{D}(t)$ was obtained (Figure 5.3 (c), solid line).

It is noteworthy that the estimated value for θ and the corresponding confidence interval suggest that the repair actions after failures are neither minimal ($\theta = 1$) nor perfect repairs ($\theta = 0$). Therefore, the traditional modeling MR assumption is inappropriate for the off-road engines, reassuring one as to the relevance of the methodology proposed in this paper. Therefore, any conclusions taken under the MR assumption are subject to bias.

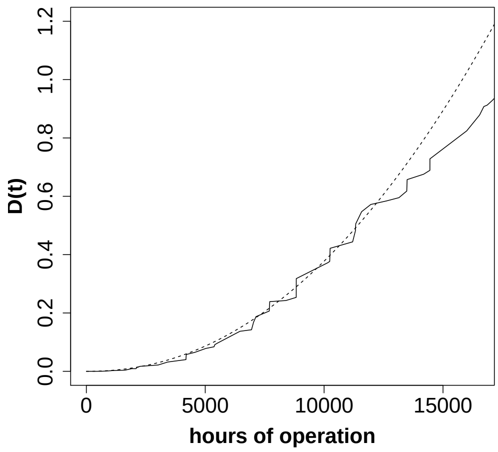
The functions plotted in Figure 5.3 allow a comparison between the estimates obtained under MR and IR assumptions. It can be observed that the mean function estimated under MR overestimates the mean acumulated number of failures at each time, since the dotted line (Figure 5.3 (a)) is above the solid line over all of the time interval. While PLP ROCOF (or intensity) is caracterized as being a continuous curve, the same pattern is not observed when the combination PLP + ARA_1 model is used, as it can be seen in Figure 5.3 (b) (solid line). Under the IR assumption each observed failure causes a decrease of degree $1 - \theta$ in the ROCOF. The estimated value



(a)



(b)



(c)

Figure 5.3: Estimated functions for the engines data, under MR (dotted lines) and IR (solid lines): (a) $\hat{\Lambda}(t)$, (b) $\hat{\lambda}(t)$ and (c) $\hat{D}(t)$ (Equation 5.6 6), versus time.

of θ turned out to be significantly different from 1 but the ROCOF estimated under MR assumption is not able to detect this behavior.

Finally, the function $D(t)$ provides the optimal PM check points. According to Equation 5.6, the policy that minimizes the expected total cost is the value τ that satisfies $D(\tau) = C_{PM}/C_{IR}$. Figure 5.3 (c) shows the graphs of this function under MR (dotted line) and IR (solid line) assumptions. Interpolations may be done fixing values for C_{PM}/C_{IR} in y-scale, what provides estimations for τ in x-scale. Recall that according to the mining company, the cost of a corrective maintenance performed after an unexpected failure is 23% higher than the cost of preventive maintenance.

Therefore, using the ratio $C_{PM}/C_{IR} = 1/1.23$, the optimal maintenance periodicity obtained under IR was 15,815 hours (or approximately 659 days).

Using the same approach, point estimates for the optimal PM periodicity τ were calculated for some alternative scenarios of the cost ratio (C_{PM}/C_{IR}), as showed in Table 5.3. Also, bootstrap confidence intervals were obtained for τ, using $B = 10{,}000$. Although asymptotic theory can be used to construct confidence intervals under the MR assumption, for the sake of comparison, bootstrap confidence intervals were also obtained for that case.

Table 5.3: Off road engines data- Optimal Periodic Maintenance Policy by Cost Ratio (C_{IR}/C_{PM}), under MR ($\hat{\tau}_{MR}$) and IR ($\hat{\tau}_{IR}$) assumptions, and bootstrap (B=10,000) confidence intervals

C_{PM}/C_{IR}	$\hat{\tau}_{MR}$	95% CI	$\hat{\tau}_{IR}$	95% CI
1/1.23	14,345	(13,304; 15,511)	15,815	(13,632; 18,082)
1/3	9,429	(8,898; 9,995)	9,207	(8,608; 10,173)
1/5	7,414	(6,974; 7,901)	7,500	(6,720; 8,125)
1/10	5,350	(4,949; 5,820)	5,593	(4,847; 6,227)
1/15	4,421	(4,028; 4,888)	4,621	(4,017; 5,386)

5.8 Final Remarks

In this paper, an optimal preventive maintenance (PM) check point was obtained for the case of repairable systems subject to perfect preventive maintenance actions (which returns them to AGAN condition) and imperfect repairs (IR) after a failure. The IRs were assumed to be of degree θ ($0 \leq \theta \leq 1$, unknown), following an ARA_1 model. The motivating practical situation concerned failure histories of off-road truck engines used by a mining company.

The estimates of the model parameters were obtained jointly by the Maximum Likelihood method, namely, the PLP parameters and the degree of repair θ.

Next, a method for estimating the mean function $\Lambda(t)$ under an ARA_1 model was presented. The method combined Monte Carlo simulation and the calculation of the (nonparametric) Mean Cumulative Function. This procedure made it possible to estimate the optimal preventive maintencance check point (τ) for the practical situation under study. Confidence intervals were also obtained for this quantity, using bootstrap resampling. The results were compared with the ones obtained under a minimal repair assumption.

Recall that, by using an ARA-1 type model, the main challenge here was not only to come up with a good approximation to the g-renewal function involved in the expression of the Mean Function ($\Lambda(t)$), but one that was differentiable. With this in mind, one solution was to use the Greatest Convex Minorant (GCM) of the MCF (Step 3 of the procedure).

In a previous analysis, a simpler and more intuitive alternative was implemented, namely, the fitting of a polynomial curve to the MCF points. Unfortunately, simulation results based on this approach turned out to be unsatisfying.

On the other hand, the simulation results presented in Section 5.6 showed that GCM method seems to be a good candidate for approximating MCFs calculated from failure processes with convex shaped intensity functions. In fact, these are the cases where preventive maintenance is justified. In addition, since the MCF is a step function, by using a large value of K in the Monte Carlo simulation, one garantees the smoothness of the MCF curve, which in turn leads to a good GCM fitting.

As far as the practical results are concerned, two important pieces of information were provided to the mining company:

1. the degree of repair: the estimated value of θ and the corresponding confidence interval indicated that the repair actions after failures are neither minimal nor perfect repairs

2. optimal preventive maintenance check points: in order to minimize the total expected maintenance cost, the mining company needs to implement PM after every 15,709 h, i.e., around 22 months.

Appendix

Algorithm used to generate K systems, under $\mathbf{ARA}_1$ and PLP (Step 2.1-Section 4).

Use the estimated values $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$ to generate failure histories for K systems (K large; here K=10,000) using the following procedure for each system:

1. **Suppose the m^{th} system failure ocurred at time t_m.**

2. Let $x = t_{m+1} - t_m$ be the elapsed time from the m^{th} to the $(m+1)^{th}$ failure.

   Hence,

$$\begin{aligned} F_{X|t_m}(x) &= P(X \leq x|t_m) = 1 - P(X > x|t_m) \\ &= 1 - P(N(x + t_m) - N(t_m) = 0|t_m) \\ &= 1 - e^{-\Lambda(x+t_m-(1-\theta)t_m)+\Lambda(\theta t_m)}. \end{aligned} \tag{5.15}$$

3. Next, the $(m+1)^{th}$ failure time is obtained through the steps [i] to [iii] below:

   i Generate u, a value of a random variable following a continuous Uniform(0,1) distribution.

   ii Solve $F_{X|t_m}(x) = u$.

   iii Calculate $t_{m+1} = t_m + x$.

   Note that [ii] corresponds to

$$t_{m+1} = t_m + x = \Lambda^{-1}[\Lambda(\theta t_m) - log(1-u)] + (1-\theta)t_m. \tag{5.16}$$

   In addition, since the PLP is being used (Equation 5.9) and the point estimates (MLEs) $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$ have already been calculated in Step 2.1, then

$$\hat{\Lambda}(t) = \left(\frac{t}{\hat{\eta}}\right)^{\hat{\beta}} \quad and \quad \hat{\Lambda}^{-1}(t) = \hat{\eta}t^{1/\hat{\beta}}. \tag{5.17}$$

   These expressions (along with the MLE $\hat{\theta}$) are then placed into Equation 5.16. Note that by making $m = 0$ $(t_0 = 0)$ in Equation 5.15, $x = t_1$, the time to the first failure. In addition, since the PLP is used, Equation 5.15 takes the form of a Weibull cumulative distribution function, as expected.

4. Finally, generate a failure history for a time truncated system (truncated at time T), by using steps [i]–[iii] recursively. Failure times t_1,t_2,...,t_n are generated until $t_{n+1} > T$.

Algorithm used to calculate $\hat{\lambda}_{C-NPMLE}(t)$.

Suppose that only one system is observed and let $N(t) = \sum_{j=1}^{n} I(t \ \geq \ t_j)$ be the number of failures up to time t, where $0 < t_1 < \ldots < t_n < T$ are the observed failure times. For a time truncated system, the *at risk* process is $Y(t) = I(0 \ \leq t \ \leq T)$. In practice, $\hat{\lambda}_{C-NPMLE}(t)$ can be computed using the following algorithm:

- set $i_0 = 0$;
- repeat until $i_{E+1} = m + 1$. Set i_{h+1} to be the index which minimizes the slopes between $(i_{i_h}, i_h - 1)$ and $(t_i, i - 1)$ $(i = i_h + 1, \ldots, n + 1)$;
- the constrained NPMLE is then given by $\hat{\lambda}_{C-NPMLE}(t) = (i_{j+1} - i_j)/(t_{i_{j+1}} - t_j)$ whenever $t_{i_j} < t \leq t_{i_{j+1}}$.

6 Dynamics of the Optimal Maintenance Policy under Imperfect Repair Models

6.1 Abstract

It is discussed both determination and practical implementation of an optimal preventive maintenance policy under imperfect repair which takes into account the information provided by observing the failure history of a repairable system. The proposed policy is applied to a real situation involving maintenance in off-road engines owned by a Brazilian mining company. A simulation study compares the performance between the optimal maintenance policy presented and the periodicity policy from Toledo *et al.* (2013), and concludes that the cost of maintaining a repairable system is significantly lower when the proposed policy is considered.

6.2 Introduction

In this paper, in order to propose an optimal dynamic maintenance policy for repairable systems, we will assume that at any time the operator of the repairable system can decide to perform a (perfect) *preventive maintenance* (PM) action, which leaves the system in AGAN condition. A PM policy specifies the moments at which the system is maintained. On the other hand, every time the system fails between two successive PM actions, it is necessary to perform a repair action to bring it back to operating condition. Usually it is assumed that such action is a *minimal repair* (MR), in the sense that it leaves the system at the same condition it was immediately before failing (i.e., ABAO). Under MR, the optimal PM periodicity can be determined after noting that, in this case, the failure history of the system follows a nonhomogeneous Poisson process (NHPP) (Gilardoni and Colosimo, 2007). The argument goes essentially as follows.

Let $N(t)$ be the number of failures in $(0,t]$ and define the mean function $\Lambda(t) = E[N(t)]$, the *rate of occurrence of failures* (ROCOF) function

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(N(t+\delta t) - N(t) = 1)}{\delta t}, \quad \forall t \geq 0, \tag{6.1}$$

and assume that $\lambda(.)$ is increasing. Also, the intensity of failure function at time t is defined by

$$\lambda(t|\Im(t^-)) = \lim_{\delta t \to 0} \frac{P(N(t+\delta t) - N(t) = 1|\Im_t^-)}{\delta t}, \quad \forall t \geq 0. \tag{6.2}$$

where $\Im_{t^-}$ is the minimal filtration defined by the history set of all failure times occurred before t. It can be shown (Aalen, 1978) that $\Lambda(t) = \int_0^t E[\lambda(s|\Im(s^-))]ds = \int_0^t \lambda(s)ds$. Under the MR assumption, the past of failure times is not taken into account, so equations 6.1 and 6.2 are equal.

If the PM and MR actions have fixed costs C_{PM} and C_{MR} and one decides to perform PMs at every τ units of time, the expected cost per unit of time would be

$$C(\tau) = [C_{PM} + C_{MR}\Lambda(\tau)]/\tau. \tag{6.3}$$

Differentiating, one finds the optimal periodicity τ_{OPT} as the solution of

$$D(\tau) = \tau\lambda(\tau) - \Lambda(\tau) = C_{PM}/C_{MR}. \tag{6.4}$$

In practice, λ and Λ are unknown and hence have to be estimated and plugged in 6.4 to obtain an estimate of τ_{OPT} (Gilardoni and Colosimo, 2007, 2011 or Gilardoni *et al.*, 2013).

Assuming MR actions at each failure may be to restrictive in practical applications. More realistic models assume that, after each failure, an *imperfect repair* (IR) action leaves the system at some point between the AGAN and ABAO conditions. For instance, Kijima *et al.* (1988) introduced a class of *virtual age models* to describe the operation of a system under IR. In its simplest form, the model assumes a ROCOF $\lambda(t)$ and that, after a failure, the repair reduces the actual age of the model by a factor θ. Hence we have actual age t and virtual age $V(t) = t - (1-\theta)t_{N(t)}$, so that the intensity of failure at time t is actually $\lambda(V(t))$, where $0 < t_1 < \ldots < t_n < \ldots$ are the failure times. The parameter θ measures the degree of the IR; $\theta = 0$ and $\theta = 1$ correspond respectively to PR and MR.

The problem of specifying an *optimal* PM periodicity under IR has already been explored by Toledo *et al.* (2013) and Remy *et al.* (2013). However, this approach may be somewhat shortsighted, in the sense that it does not take into account the history of the process and hence makes sense only in the scenario where $N(t)$ has independent increments. If, on the other hand, the IR model results in a process with dependent increments, then the history has to be considered when determining an optimal policy. The use of the term "dynamic" in the definition of the optimal PM policy proposed in this paper refers to this consideration.

The rest of this paper is organized as follows. Section 6.3 deals with the optimal PM policy under *any* IR model. The main finding – see equation 6.8 below, states there exists a constant, independent of the history of the process, such that a PM action should be performed whenever the conditional intensity of the IR failure process given the available information attains that constant. Section 6.4 deals briefly with the associated inference problem (i.e., estimation of the optimal PM policy) in the virtual age model. Section 6.5 presents the results of the proposed methodology when applied to a real situation, involving maintenance in off-road engines. In Section 6.6 are discussed the results of a simulation study in order to compare the performance of the proposed methodology, based on a dynamic PM policy, with a PM periodicity policy. Finally, Section 6.7 ends the paper with some final comments.

6.3 Optimal PM policy under IR

As before, suppose a failure process $N(t)$ subject exclusively to IR actions and define $\Lambda(t|s_0) = \Lambda(t|\Im(s_0^-)) = E[N(s_0+t) - N(s_0)|\Im(s_0^-)]$, where $\Im(s_0^-)$ is the failure history up to the moment immediately before s_0. In other words, $\Lambda(t|s_0)$ is the expected number of failures during the next t units of time given the history of the process up to s_0. We will consider the following assumptions: First, both PM and IR actions are performed instantaneously. Second, the costs of the PM and IR actions are independent of the history of the system and have expectation C_{PM} and C_{IR} respectively. Third, for each s_0, $\Lambda(t|s_0)$ is differentiable for every t and its derivative $\lambda(t|s_0) = \lambda(t|\Im(s_0^-))$ is continuous and strictly increasing.

A *maintenance policy* for the time interval (s_0,S) specifies the number of PMs n and its moments $s_0 + \tau_1 < s_0 + \tau_1 + \tau_2 < \ldots < s_0 + \tau_1 + \ldots + \tau_n$, where $\tau_i > 0$ and $\sum_{i=1}^{n} \tau_i < S - s_0$. We will write $M(s_0,S) = (n; \tau_1, \ldots, \tau_n)$. Given a maintenance policy $M(s_0,S)$, its expected cost given $\Im(s_0^-)$ is

$$C[M(s_0,S)] = nC_{PM} + C_{IR}\{\Lambda(\tau_1|s_0) + \sum_{j=2}^{n} \Lambda(\tau_i|0) + \Lambda(S - s_0 - \tau_1 - \ldots - \tau_n|0)\}. \quad (6.5)$$

A maintenance policy $M_{OPT}(s_0,S)$ is optimal if $C[M_{OPT}(s_0,S)] \leq C[M(s_0,S)]$ for every other $M(s_0,S)$. Since the first PM action renews the system, it should be clear that, if $M_{OPT}(s_0,S) = (n,\tau_1,\ldots,\tau_n)$, then $M_{OPT}(0,S - s_0 - \tau_1) = (n-1,\tau_2,\ldots,\tau_n)$. This shows that, to solve the general case, it is important to understand the problem with $s_0 = 0$.

- **The problem without information** $(s_0 = 0)$

  In order to obtain the optimal policy we will proceed in two stages. First, we will assume the number n of PMs fixed and will obtain the optimal PM moments. Then, we will discuss how to obtain the optimal n for an infinite horizon (i.e., for $S \to \infty$).

  Considering n fixed in 6.5 and differentiating with respect to τ_i we get that $\lambda(\tau_i|0) = \lambda(S - \tau_1 - \ldots - \tau_n)$ for $i = 1,\ldots,n$. Since $\lambda(.|0)$ is strictly increasing, this implies that $\tau_i = S/(n+1)$. In other words, in this case the optimal policy specifies in fact an optimal period.

  Now, to obtain the optimal n, we substitute the previous times again in 6.5 to obtain $c(n) = nC_{PM} + (n+1)C_{IR}\Lambda(\frac{S}{n+1}|0) = S\frac{n+1}{S}[C_{PM} + C_{IR}\Lambda(\frac{S}{n+1}|0)] - C_{PM}$. Hence, to obtain the optimal n one should minimize $c^*(n) = \frac{n+1}{S}[C_{PM} + C_{IR}\Lambda(\frac{S}{n+1}|0)]$– compare with 6.3. Since the function $f(\tau) = [C_{PM}+C_{IR}\Lambda(\tau|0)]/\tau$ is convex, it follows that the optimal n is either $n_{OPT} = [S/\tau^* - 1]$ or $n_{OPT} = [S/\tau^* - 1] + 1$, where τ^* is the minimizer of $f(\tau)$ and $[a]$ is the integer part of a. Putting these considerations together and letting $S \to \infty$, it follows that the optimal PM policy calls for PM actions at every

$$\tau_{OPT} = \lim_{S\to\infty} \frac{S}{n_{OPT}+1} = B^{-1}(C_{PM}/C_{IR}), \quad (6.6)$$

  where we have defined $B(t) = t\lambda(t|0) - \Lambda(t|0) = \int_0^t u\lambda'(u|0)du$. This is essentially the same solution given in 6.4.

- **The general case** ($s_0 > 0$).

Consider now an optimal policy $M(s_0,S) = (n,\tau_1,\ldots,\tau_n)$. For large S it follows from 6.6 that $\tau_2 = \ldots = \tau_n = B^{-1}(C_{PM}/C_{IR})$ and $n = (S - s_0 - \tau_1)/B^{-1}(C_{PM}/C_{IR})$. Hence, to obtain the optimal policy we have now to optimize with respect to the remaining variable τ_1. Substituting in 6.5 and differentiating with respect to τ_1 we get that the optimal policy $M(s_0,S)$ must satisfy

$$\lambda(\tau_{1,OPT}|s_0) = \lambda[B^{-1}(C_{PM}/C_{IR})|0], \tag{6.7}$$

$n_{OPT} = \frac{S-s_0-\tau_{1,OPT}}{B^{-1}(C_{PM}/C_{IR})}$ and $\tau_{2,OPT} = \ldots = \tau_{n,OPT} = B^{-1}(C_{PM}/C_{IR})$.

Note that in purely dynamical implementation of this solution, the only relevant equation is 6.7. This is because one monitors the history of the system (s_0) up to a time which solves 6.7, at which moment a PM action is performed and a renewal occurs. Then, one monitors again the history of the renewed system (again s_0) and so on. In other words, we never get to apply the last two equations. For this reason, we call equation 6.7 the *fundamental law of preventive maintenance.*

Moreover, although 6.7 may suggest that in order to implement the optimal policy one has to evaluate $\lambda(t|s_0)$ for every possible s_0, if PM actions can be scheduled without delay, one only need actually to evaluate $\lambda(t|t)$. More precisely, denote by $\tau_{OPT}(s_0)$ the solution of 6.7. In implementing the optimal policy, the operator of the system monitors the failure history and at each s_0 computes $\tau_{OPT}(s_0)$. In general he or she would have that $\tau_{OPT}(s_0) > s_0$ and will keep going without performing a PM. In other words, the only way that a PM action would be eventually performed is if for some s_0 one has that $\tau_{OPT}(s_0) = s_0$. In other words, a PM action would be performed if and only if $\lim_{s_0 \to \tau_1} \lambda(\tau_1|s_0) = \lambda(B^{-1}(C_{PM}/C_{IR})|0)$. This is quite nice, because usually $\lambda(t|t)$ is much easier to compute than $\lambda(t|s_0)$. For instance, for the simple virtual age model, it is easy to show that $\lambda(t|t) == \lambda[t - (1-\theta)t_{N(t)}] = \lambda[V(t)]$

(see Kijima *et al.*, 1988). Hence, 6.7 becomes now

$$\lambda[\tau_{1,OPT} - (1-\theta)t_{N(\tau_{1,OPT})}] = \lambda[B^{-1}(C_{PM}/C_{IR})|0], \tag{6.8}$$

or equivalently,

$$\tau_{1,OPT} - t_{N(\tau_1,OPT)} = \lambda^{-1}\{\lambda[B^{-1}(C_{PM}/C_{IR})|0]\} - \theta t_{N(\tau_1,OPT)}. \tag{6.9}$$

In other words, a PM action will occur whenever the virtual age attains the value $\lambda^{-1}\{\lambda[B^{-1}(C_{PM}/C_{IR})|0]\}$.

6.4 Statistical inference for the virtual age model

Consider the virtual age model with a ROCOF modeled by a Power Law Process (PLP), given by $\lambda(t) = (\beta/\eta)(t/\eta)^{\beta-1}$ with $\beta > 1$. Suppose that the system is observed up to time T with observed failure times $0 < t_1 < \ldots < t_n < T$. Let $V(t_i) = \theta t_i$ and $V(t_i^-) = t_i - (1-\theta)t_{i-1}$. The likelihood is

$$L(\beta,\eta,\theta) = \left(\prod_{i=1}^{n} \lambda[V(t_i^-)]e^{\Lambda[V(t_i)]-\Lambda[V(t_i^-)]}\right) e^{\Lambda[T-(1-\theta)t_n]}, \tag{6.10}$$

where $\Lambda(t) = \int_0^t \lambda(u)du = (t/\eta)^\beta$ (see Toledo *et al.*, 2013, for details).

In order to estimate the right hand side of 6.8 we proceed as follows. First, we maximize numerically the likelihood 6.10 to obtain the maximum likelihood estimates (MLEs) $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$. Then, to estimate $\Lambda(t|0)$, we simulate many systems with the estimated parameters and use a Nelson-Aalen estimator. Following Gilardoni and Colosimo (2011), an estimate of $\lambda(t|0)$ which takes into account the monotonicity constraint can now be obtained as the derivative of the *greatest convex minorant* (GCM) of the Nelson-Aalen estimate $\hat{\Lambda}(t|0)$. The estimates $\hat{\Lambda}(t|0)$ and $\hat{\lambda}(t|0)$ can now be used to obtain $\hat{B}(t) = t\hat{\lambda}(t|0) - \hat{\Lambda}(t|0)$. Inverting $\hat{B}(t)$ we get an estimate of $B^{-1}(C_{PM}/C_{IR})$. It is noteworthy that this procedure corresponds to the method proposed in Toledo *et al.* (2013), where $B^{-1}(C_{PM}/C_{IR})$ corresponds to the optimal PM periodicity policy for the systems under study.

Since we have already computed $\hat{\lambda}(t|0)$, this means that we obtain an estimate of the right hand side of 6.8. Likewise, the right hand side of 6.9 can be estimated now after noting that $\lambda^{-1}(x) = \eta[\eta x/\beta]^{1/(\beta-1)}$. This estimation provides the values associated to the optimal dynamic PM policy, which is the object of this paper.

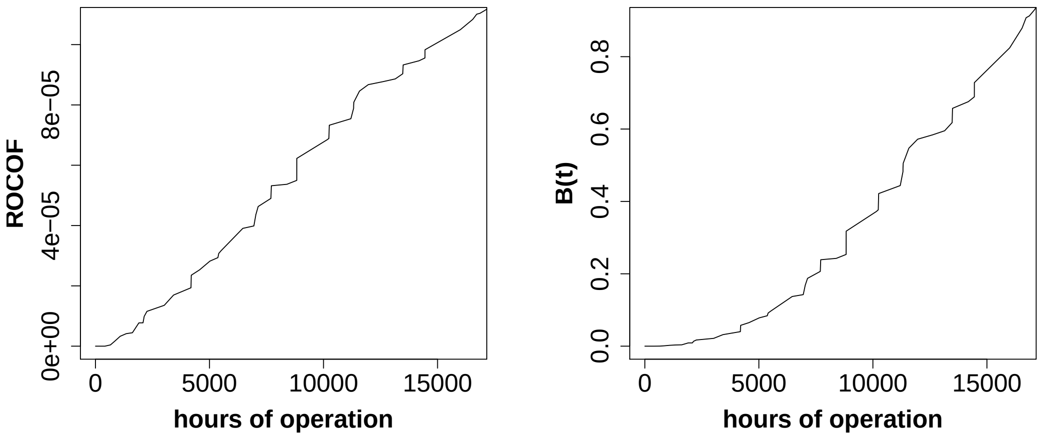
Two comments are in order here. First, the Monte Carlo simulation has to be done only once during the entire process, because the right hand sides of 6.8 and 6.9 involve only $\Lambda(t|0)$ and $\lambda(t|0)$. Second, the size of the simulation can be taken large enough to make its precision at least an order of magnitude larger than the precision of the MLEs of the parameters, so that in practice the relevant uncertainty in the final estimates would depend only on the precision of the MLEs. In next section, point and intervalar estimates for the optimal dynamic PM policy are going to be obtained through equations 6.8 and 6.9 for a real situation involving repairable systems.

6.5 Application for a real data set

In this section we are going to apply the *fundamental law of preventive maintenance*, defined in Section 6.3, together with the likelihood function derived in Section 6.4, in order to estimate the optimal dynamic PM policy for a data set involving failures in off-road engines from a Brazilian mining company. Later, the dynamics associated to this optimal PM is going to be explored through an example.

The data set consists of 208 failure times, measured in hours of operation (each failure followed by an immediate repair), observed in a sample of 193 diesel engines. In addition, among the 193 engines, 52 were right censored (their last inspection time corresponded to a removal for a PM), so 141 were failure truncated. Using Kijima *et al.* (1988) simple virtual age model, along with the likelihood function 6.10 for a ROCOF modeled by a PLP, MLEs for the parameters are obtained as $\hat{\beta} = 2.458$, $\hat{\eta} = 15.586$ and $\hat{\theta} = 0.471$. More details about this estimation procedure can be found in Toledo *et al.* (2013), where this data set was firstly explored.

The next step consists of using the estimated parameters applying Toledo *et al.* (2013) method, in order to obtain $\hat{B}(t) = t\hat{\lambda}(t|0) - \hat{\Lambda}(t|0)$. The estimated functions $\hat{\lambda}(t|0)$ and $\hat{B}(t)$ for this data set, obtained using 10.000 Monte Carlo interations, are shown in Figures 6.1 (a) and (b), respectively. For a given ratio C_{PM}/C_{IR}, an estimate of $B^{-1}(C_{PM}/C_{IR})$ for the systems under study can be directly obtained from



(a) (b)

Figure 6.1: Estimated functions for the engines data set using Toledo *et al.* (2013) approach: (a)$\hat{\lambda}(t|0)$ versus t and (b) $\hat{B}(t)$ versus t, t measured in hours of operation.

the function showed in Figure 6.1 (b). Assuming different values of C_{PM}/C_{IR} (specially the ratio 1.23 declared by the mining company), the estimated point values for $B^{-1}(C_{PM}/C_{IR})$, along with 95% intervalar estimates (based on parametric Bootstrap procedure with 10,000 iterations) were calculated and are shown in Table 6.1. It is noteworthy that these values correspond to the optimal PM periodicity estimation based on Toledo *et al.* (2013) method.

Since we already have computed $\hat{\lambda}(t|0)$ (Figure 6.1 (a)), the values $\hat{\lambda}[\hat{B}^{-1}(C_{PM}/C_{IR})|0]$ can be obtained considering each ratio of costs. For a new system, which is being accompanied from the time $t = 0$, equation 6.9 becomes:

$$\tau_{VA} = \lambda^{-1}\{\lambda[B^{-1}(C_{PM}/C_{IR})|0]\}, \tag{6.11}$$

where τ_{VA} is the optimal PM periodicity based on the virtual age, in other words, it corresponds to the optimal dynamic PM policy. For the engines data set, the right hand side of this equation can then be estimated from $\hat{\lambda}^{-1}(x) = \hat{\eta}[\hat{\eta}x/\hat{\beta}]^{1/(\hat{\beta}-1)}$. The column $\hat{\tau}_{VA}$ from Table 6.1 presents the estimated values for the off-road engines, based on Equation 6.11, for different values of costs ratio. Confidence intervals based on parametric Bootstrap were also obtained for τ_{VA}, using the following procedure:

1. From the estimated parameters $\hat{\beta}$, $\hat{\eta}$ and $\hat{\theta}$, generate $n = 193$ failure processes (truncated in time T = 40,000) under the virtual age model with a ROCOF modeled by a PLP;

2. Estimate $\hat{\beta}^{(b)}$, $\hat{\eta}^{(b)}$ and $\hat{\theta}^{(b)}$ numerically maximizing the likelihood in 6.10;

3. Use Toledo *et al.* (2013) procedure to obtain the functions $\hat{\lambda}(t|0)$ and $\hat{B}^{-1}(t)$;

4. For a given ratio C_{PM}/C_{IR}, estimate $\tau_{VA}^{(b)}$ using Equation 6.11;

5. Repeat steps 1 to 4 (10,000 times), thereby obtaining 10,000 Bootstrap estimates for τ_{VA} ($\hat{\tau}_{VA}^{(1)}, \ldots, \hat{\tau}_{VA}^{(10,000)}$). $100(1 - \alpha\%)$ percentile confidence interval for τ_{VA} can then be obtained using the 10,000 sorted estimated values.

The 95% confidence intervals for τ_{VA} are also exhibited in Table 6.1.

For instance, assuming that the expected cost of an IR is 23% higher than the expected cost of a PM action ($C_{PM}/C_{IR} = 1/1.23$), the values in the table can be interpreted the following way:

- The optimal PM periodicity is 15,815 hours, with 95% confidence interval from 13,632 to 18,082 hours. For a new system, this is the time that must be expected for the first PM action, independent on how many failures occur until there.

- The optimal PM periodicity based on the virtual age is 11,373 hours, with 95% confidence interval from 10,978 to 12,023 hours. For a new system, this is the time that must be expected for the first PM action, *if no failure occurs until there*. If any failure is observed before 11,373 hours, the optimal time no the next PM must be recalculated, using equation 6.9. This is what we call dynamics of the optimal preventive PM.

Based on this interpretation, let's suppose we are observing an engine from the same population of off-road engines that composed our sample. We estimated $\hat{\tau}_{VA}$ = 11,373 hours, but the first failure occurred in t_1 = 7,000 hours. Equation 6.9 becomes then

Table 6.1: Estimations for the off-road engines data set, under different values of costs ratios (C_{PM}/C_{IR}): Optimal PM periodicity ($\hat{B}^{-1}(C_{PM}/C_{IR})$) based on Toledo *et al.* (2013) approach; Optimal dyamic PM ($\hat{\tau}_{VA}$), and bootstrap (B = 10,000) 95% confidence intervals (values are in hours)

C_{PM}/C_{IR}	$\hat{B}^{-1}(C_{PM}/C_{IR})$	95% CI	$\hat{\tau}_{VA}$	95% CI
1/1.23	15,815	(13,632; 18,082)	11,373	(10,978; 12,023)
1/3	9,207	(8,608; 10,173)	8,141	(7,572; 8,647)
1/5	7,500	(6,720; 8,125)	6,537	(6,043; 7,097)
1/10	5,593	(4,847; 6,227)	4,694	(4,403; 5,404)
1/15	4,621	(4,017; 5,386)	4,031	(3,656; 4,596)

$$\begin{aligned}
\tau_{1,OPT} - t_1 &= \hat{\lambda}^{-1}\{\hat{\lambda}[\hat{B}^{-1}(1/1.23)|0]\} - \hat{\theta}t_1 \\
&= 11{,}373 - 0.471 \times 7{,}000 \\
&= 8{,}076. \qquad (6.12)
\end{aligned}$$

So, the first PM action, that would occur in $\hat{\tau}_{VA}$ = 11,373, must be done 8,076 hours after t_1 = 7,000. In other words, the first PM action was postponed to 15,076 hours. This phenomenon is due to the dynamic aspect of the PM policy proposed in this paper: Since the first IR action, with effect of repair $\hat{\theta}$ = 0.471, renewed the system in some degree, reducing its virtual age at t_1, from 7,000 hours to $V(t_1)$ = 0.471×7,000 = 3,297 hours, the time to the first PM should be recalculated to express this reduction.

It is important to emphasize that this dynamics is continuous: For the system in the example, if another failure occurs before the recalculated value $\hat{\tau}_{VA}$ = 15,076, Equation 6.9 must be recalculated again. This process will be repeated every time a failure occurs. In fact, a PM action only will be performed if the time $\hat{\tau}$, recalculated from the last observed failure, is reached without any new failure.

In the next section we discuss the results of a simulation study conducted in order to compare the performance of the optimal PM periodicity policy, with the one proposed here, the optimal dynamic PM.

6.6 Simulation study

In this section we describe a simulation study in order to compare the performance of the two maintenance policies - the one proposed in Toledo *et al.* (2013), and the one proposed in this paper, which for simplification will be called here, respectively, periodic and dynamic. The simulations were done using a script written in R, a language and environment for statistical computing (www.R-project.org, v.2.15). The performance was defined as the mean cost per unit time with maintenance activities (preventive - PM and corrective - IR) for systems subjected to such policies, denoted by C_P and C_D, for the periodic and dynamic policy, respectively. Failure times were generated using Monte Carlo method associated to PLP, according to the simple virtual age model. Failure processes were generated under the combination of the following parameters:

- $\beta = 1.5$, 2.0, 2.5 and 3.0
- $\eta = 15{,}000$
- $\theta = 0.1$, 0.3, 0.5, 0.7, 0,9

where β and η are the shape and scale PLP parameters, respectively, and θ is the effect of repair parameter. The η value remained fixed since it does not have an impact on the comparative results.

On the other hand, the maintenance costs ratio (PM actions cost divided by IR actions cost), denoted by C_{PM}/C_{IR}, does have an impact on the policies performance results. The values 1/3, 1/5 and 1/15 were used on the simulations. So, successive failure times were generated under $120 = 4 \times 5 \times 3 \times 2$ different scenarios. These were defined considering (i) four β values, (ii) five θ values, (iii) three costs ratio values and (iv) two policies (periodic and dynamic). The following procedure was then used to run simulations for each maintenance policy:

1. **Periodic Maintenance Policy**

   The method proposed in Toledo *et. al.* (2013), provides, for each scenario, the optimal PM periodicity (τ_P). Therefore, in the simulation study, failure processes were generated for N systems under each scenario, up to τ_P. The number of failures observed in the time interval $(0,\tau_P]$ for each system was then computed.

2. **Dynamic Maintenance Policy**

   For a new system, this policy provides an initial time for PM (τ_{VA}), and recalculates this time if a failure occurs before the initial τ_{VA}, through the *fundamental law of preventive maintenance.* In the simulation study, failure processes were generated for N systems under each scenario. Every time a failure occurred before τ_{VA}, this failure time was used to recalculate Equation 6.9, which in turn provided a new perspective for τ_{VA}. For each system, the procedure ended when the first PM was effectively conducted.

The mean maintenance costs per unit time for systems subject to periodic and dynamic maintenance policies (C_P and C_D respectively) along with Monte Carlo standard errors (SE), were obtained as follows.

- For the i^{th} system ($i = 1,2,\ldots,N$) subject to policy j, $j = P$ (Periodic), D (Dynamic):
  - the total number of failures (repairs) to the first PM, denoted by $N_{j,i}$, was computed;
  - the maintenance cost per unit time, given by:

$$C_{j,i} = \frac{C_{PM} + N_{j,i} \times C_{IR}}{t_{j,i}} \tag{6.13}$$

    was calculated, where $t_{j,i}$ corresponds to the time up to the first PM for the i^{th} system subjected to policy j.

- Finally, the mean cost per unit time for policy j (C_j) and the respective SE (SE_j) were calculated as the average and standard error for $C_{j,i}$ ($i = 1,2,\ldots,N$), respectively.

In this article, failure times for $N = 10{,}000$ systems were generated for each policy. Table 6.2 shows C_P and C_D point estimates for each policy, along with the Monte Carlo standard errors. In addition, Figure 6.2 exhibit the curves of the relationships between the point estimates C_j ($j = P$,D) and the values of β and θ used in the simulation, by scenario. The results obtained are listed below:

- **the effect of the degree of repair parameter (θ)**: the degree of repair does have an effect on the mean costs per unit time. It can be observed from Figures 6.2(a) to 6.2(c) that, for both policies, the mean costs decrease along with the decrease on the value of θ. In other words, the better the repair (*i.e.*, θ closer to 0; a perfect repair) the lower the mean cost. However, this effect is more evident for the dynamic policy. In addition, for the same degree of repair, the mean costs for the dynamic policy are lower than the respective ones for the Periodic policy. Note that we have assumed that repair actions with different degrees have the same cost. In other words, the costs C_{IR} were assumed to be the same no matter which value of θ was used. Also, a very important point to be observed from Table 6.2 values is that the differences in costs between the two policies are inversely proportional to θ values. In general, higher values of θ (closer to 1 - MR) are related to lower differences between the costs under the two policies. This is an evidence that, under the MR assumption (independent increments), the periodic policy is a reasonable approach (although its associated costs are still higher than those related to the dynamic policy). However, when each repair has a considerable effect of renewing the system ($\theta < 1$, dependent increments), the dynamic policy has a substantial better performance.

- **the effect of the cost ratios (C_{PM}/C_{IR})**: if we analyze Figures 6.2(a) to 6.2(c) for each policy separately, we can see that for lower values of C_{PM}/C_{IR} (*i.e.*, higher values of C_{IR}), the mean costs per unit time for a given policy tend to be practically the same, no matter the effect of the repair. This pattern can be observed for both policies. However, this effect is more evident for the dynamic policy. In practice this result indicates that if one is dealing with much higher C_{IR} costs (see for example the case $C_{PM}/C_{IR} = 1/15$) there is not much difference if one assumes IR (of any degree) or MR.

- **the effect of the shape parameter PLP (β)**: in both policies, an increase in β leads to a decrease in the mean cost. This can be explained by the fact that under such policies a stronger convexity for the intensity function (in PLP case, that means a higher value of the shape parameter) determines shorter intervals between PM actions and, consequently, less failures are observed in the systems.

- Finally last but not least important, the point estimates of the mean costs are in general lower for the dynamic policy than for the periodic.

We have based our conclusions on the sample averages C_j (j = P,D) since the sample size used for the simulations was rather large (N = 10,000 systems) and should be pretty close to the populational mean cost per unit time value.

Table 6.2: Mean cost per unit time point estimates[(1)] and standard errors[(1,2)] for the periodic (C_P, SE_P) and dynamic (C_D, SE_D) policies

		$C_{PM}/C_{IR} = 1/3$		$C_{PM}/C_{IR} = 1/5$		$C_{PM}/C_{IR} = 1/15$	
β	θ	Periodic	Dynamic	Periodic	Dynamic	Periodic	Dynamic
1.5	0.1	2.37 (0.01)	1.99 (0.01)	3.49 (0.03)	2.85 (0.02)	7.16 (0.13)	6.26 (0.10)
2.0	0.1	2.19 (0.02)	1.82 (0.01)	2.91 (0.03)	2.39 (0.02)	5.22 (0.10)	4.67 (0.06)
2.5	0.1	1.94 (0.01)	1.63 (0.01)	2.39 (0.02)	2.10 (0.01)	3.87 (0.07)	3.39 (0.04)
3.0	0.1	1.77 (0.01)	1.52 (0.01)	2.17 (0.02)	1.89 (0.01)	3.11 (0.06)	2.72 (0.03)
1.5	0.3	2.45 (0.02)	2.15 (0.02)	3.53 (0.03)	2.99 (0.03)	7.54 (0.14)	6.06 (0.10)
2.0	0.3	2.19 (0.02)	1.91 (0.01)	2.87 (0.03)	2.47 (0.02)	5.23 (0.10)	4.50 (0.07)
2.5	0.3	1.99 (0.02)	1.71 (0.01)	2.46 (0.03)	2.15 (0.02)	3.85 (0.07)	3.36 (0.05)
3.0	0.3	1.78 (0.01)	1.61 (0.01)	2.14 (0.02)	1.90 (0.01)	2.95 (0.05)	2.79 (0.04)
1.5	0.5	2.54 (0.02)	2.24 (0.02)	3.64 (0.04)	3.11 (0.03)	7.64 (0.14)	6.75 (0.11)
2.0	0.5	2.26 (0.02)	2.02 (0.01)	2.89 (0.03)	2.59 (0.02)	5.18 (0.10)	4.46 (0.07)
2.5	0.5	1.98 (0.02)	1.82 (0.01)	2.48 (0.03)	2.24 (0.02)	3.82 (0.07)	3.54 (0.05)
3.0	0.5	1.80 (0.01)	1.65 (0.01)	2.13 (0.02)	1.98 (0.02)	3.11 (0.05)	2.95 (0.04)
1.5	0.7	2.57 (0.02)	2.36 (0.02)	3.67 (0.04)	3.34 (0.03)	7.60 (0.14)	7.01 (0.12)
2.0	0.7	2.28 (0.02)	2.11 (0.02)	3.00 (0.03)	2.70 (0.03)	5.22 (0.10)	4.86 (0.08)
2.5	0.7	2.04 (0.02)	1.87 (0.01)	2.45 (0.03)	2.29 (0.02)	3.92 (0.08)	3.60 (0.06)
3.0	0.7	1.80 (0.01)	1.69 (0.01)	2.16 (0.02)	2.04 (0.02)	3.07 (0.06)	3.03 (0.05)
1.5	0.9	2.58 (0.02)	2.55 (0.02)	3.66 (0.04)	3.67 (0.04)	7.98 (0.15)	7.45 (0.14)
2.0	0.9	2.33 (0.02)	2.29 (0.02)	3.02 (0.03)	2.92 (0.03)	5.20 (0.10)	4.98 (0.09)
2.5	0.9	2.03 (0.02)	1.98 (0.02)	2.48 (0.03)	2.49 (0.03)	3.93 (0.07)	3.82 (0.07)
3.0	0.9	1.79 (0.01)	1.81 (0.01)	2.16 (0.02)	2.12 (0.02)	3.17 (0.06)	3.02 (0.05)

(1) reported values should be multiplied by 10^{-4}; (2) Monte Carlo standard erros

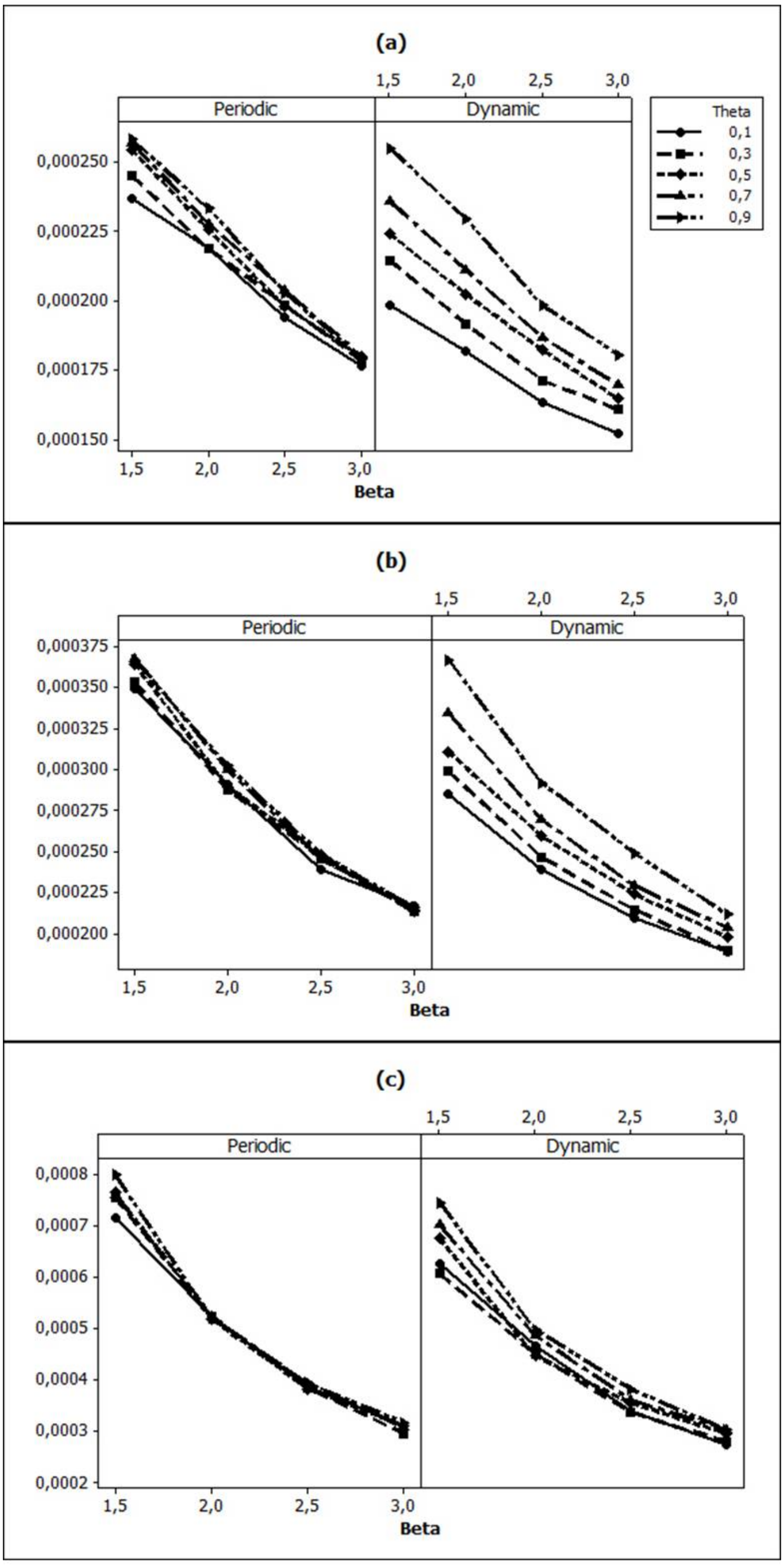


Figure 6.2: Mean cost per unit time, C_{PM}/C_{IR} = (a)1/3; (b)1/5; (c)1/15.

6.7 Conclusions and final remarks

In this paper, a dynamic method to estimate the optimal PM policy for repairable systems was proposed. The method is based on the virtual age model presented by Kijima *et al.* (1988), which considers the effect of repair actions taken after failures. In constrast to optimal PM periodicity policies found in literature, the proposed approach allows, when defining an optimal PM policy for a system, to incorporate the information provided by any failure that may arise, updating the decision process of when to perform PM.

The proposed approach was presented, including the derivation of the *fundamental law of preventive maintenance*, which states there exists a constant, independent of the history of the process, such that a PM action should be performed whenever the conditional intensity of the IR failure process given the available information attains that constant. Also, the associated inference problem to estimate the optimal PM policy in the virtual age model was discussed.

The main result of the paper, Equation 6.9, states, for a new system, an optimal time for the PM (τ_{VA}), based on its virtual age. Also, if any failure occur before τ_{VA}, this equation can be used to update the optimal time, considering the wear-out speed of the system, and also, the effect of repair actions taken after failures. When applied to a real situation involving maintenance in off-road engines from a Brazilian mining company, assuming a PLP process for the ROCOF, the involved parameters were estimated as $\hat{\beta} = 2.458$, $\hat{\eta} = 15.586$ and $\hat{\theta} = 0.471$. Assuming a costs ratio of $C_{PM}/C_{IR} = 1/1.23$ for these systems, the estimated optimal PM periodicity based on the virtual age was 11,373 hours. In other words, for a new system, this is the time that must be expected for the first PM action, *if no failures occur until there.* If any failure is observed before 11,373 hours, the optimal time no the next PM must be recalculated, using equation 6.9. Confidence intervals were also obtained, and the resulted values were compared to those related to the optimal PM periodicity policy (Toledo *et al.* (2013) unpublished manuscript).

The simulation study conducted in order to compare the performance of the proposed methodology (based on a dynamic PM policy) with a PM periodicity policy, showed that the cost of maintenance (including PM and IR) actions is significantly lower under the dynamic policy. This result is a strong evidence that the method proposed in this paper has a better performance (in termos of costs) than the ones already presented in literature. While the approach based on the optimal PM periodicity does not take into account the history of the particular process under consideration, the dynamic method presented here allows to use the virtual age of the system as the reference for deciding when to run the next PM action, basing the decision-making process with much more information.

7 Final remarks

The statistical models for repairable systems studied in this thesis make it possible to deal with the general assumption of IR in the occurrence of failures. Optimal PM policies are specified, either under the periodic perspective (Section 5) or dynamic perspective (Section 6) and those are the main contributions of this work.

In the off-road trucks engines problem, while the manufacturer suggested PM actions at every 15,000 hours of operation, the obtained results in this work show that:

- The optimal PM periodicity is 15,815 hours, with 95% confidence interval from 13,632 to 18,082 hours. For a new system, this is the time that must be expected for the first PM action, independent on how many failures occur until there.
- This estimation can be improved based on the optimal PM periodicity based on the virtual age: 11,373 hours, with 95% confidence interval from 10,978 to 12,023 hours. For a new system, this is the time that must be expected for the first PM action, if no failure occur until there. If any failure is observed, the optimal time must be recalculated, using the method presented in Section 6.

Among the several possibilities for future work related to the topic, we highlight the flexibilization of the studied models regarding the effect of PMs. In the practical situation of diesel engines, a statistical analysis lead to conclude that the systems that undergo PMs return to the AGAN condition, presenting the same behavior of failure intensity of a new system. However, for many situations this assumption may not be valid.

The development of IR models that also incorporate the PM effect is, therefore, the main continuation point of this work. This effect can be modeled as a reduction in either the age of the system, or the failure intensity. Therefore, it is expected that the results presented in this thesis motivate future work in this direction.

A Off-road engines data set

Table A1: Off-road engines dataset

Unit	Failure and PM times (hours)				Unit	Failure and PM times (hours)			
1	18315*	32133	50934		26	16546			
2	16137*	34722*	53990		27	14291	24375	39107	
3	12666	22143			28	12510*	26323		
4	15012*	18065			29	13662*	24300	40731	
5	10629*	21717*	22663		30	10993			
6	12303*	28665*	42483	50844	31	8613*	24229		
7	10215*	23355*	38169	44595	32	19789			
8	9927*	23103*	32323*	47880	33	16348*	27891	32218	
9	11583*	26173			34	7722*	25024		
10	22347*	23783			35	22349			
11	12647				36	20430*	28731		
12	9288*	13280	18846		37	15616			
13	12825*	22358	35392		38	13824*	33824		
14	16686*	31230*	48023		39	7707			
15	7931*	16173			40	10746*	14808	15924*	20998
16	16056*	25776*	42972		41	15025	31479	33268	
17	15804*	19040			42	17262	34668		
18	10427				43	5283			
19	12154	16174			44	18363	34042		
20	14328*	20808*	28093		45	19115			
21	13460				46	16335*	35496		
22	8532*	25838	43022		47	17271	31691	31909	32318
23	15768*	22718			48	14634*			
24	10548*	13699			49	11681	15389	17066	
25	12096*	25830*	36641		50	15016	29260		

Censoring times due to preventive maintenance are indicated by *.

Unit	Failure and PM times (hours)				Unit	Failure and PM times (hours)	
51	12261	25054			76	11350	
52	18872	25054			77	17074	
53	15799	28777			78	16234	
54	19010	21098			79	14595	15653
55	17201	25535			80	7790	
56	15524	25939	26414		81	17375	
57	8064*	11729	20419	23684	82	9369	
58	11631	20110	23759		83	9725	
59	17217				84	17602	
60	19195				85	12545	
61	10552	19702			86	14572	
62	14665	16731			87	8601	11370
63	13936	14739	18731		88	10023	
64	13812				89	8430	
65	15233	21004			90	14934	
66	6157	17380			91	14337	15587
67	18975				92	14876	
68	10882	17496			93	7723	7971
69	20811				94	16598	
70	17876				95	15547	
71	13638				96	5447	
72	18274	20533			97	12037	
73	14467				98	11594	
74	19018				99	11114	
75	8984				100	10962*	

Unit	Failure and PM times (hours)				Unit	Failure and PM times (hours)	
101	13839				126	10839	29533
102	13658				127	7957	26200
103	13444				128	14111	18723
104	12781				129	18762	
105	12633				130	8463	
106	11321				131	16879	18966
107	9993				132	2750	
108	15246*	28691	30129		133	5076*	9581
109	20514*	25947	37332		134	14818	
110	17532*	21814			135	5031	7134
111	19196	36154			136	7064	
112	12157	19376	27831	33781	137	9465	
113	18337	35612			138	11305	
114	11380				139	9972	
115	10228	23275	31770		140	5218	
116	8398	18705	21115		141	4497	
117	9108*	18772	30304		142	3330	
118	12465	26559			143	2205	
119	11324	29857					
120	12646*	32589					
121	4581*	9615					
122	14356						
123	11646*	26703	30526				
124	14400*	23719					
125	15872	27746					

B R codes

B.1 Point and intervalar estimation of the parameters in ARA$_m$+PLP model, using MLE

```
################################################################################
## Lendo o arquivo de dados, que deve ser composto de 3 colunas: a primeira,
## contendo a identificacao das unidades, a segunda, contendo os tempos
## globais, e a terceira, indicando se a medicao eh uma falha (1) ou censura (0)
################################################################################
file_tij<-read.table(paste(apppath,"tij_motores.txt",sep=""),header=TRUE)
################################################################################
## Declaracao da funcao de log-verossimilhanca
################################################################################
m<-1 #memoria do modelo ARA
mloglik<-function(pars,dados){
  beta<-exp(pars[1])
  eta<-exp(pars[2])
  theta<-exp(pars[3])
  soma1<-0
  soma2<-0
  soma3<-0
  N<-sum(dados[,3])
  n<-length(dados[,1])
  j<-1
  for (i in 2:(n+1)){
    vector<-NULL
    if(i>n){
      vector<-dados[j:(i-1),c(1,2,3)]
      ## os passos seguintes são executados apenas para o último sistema
      ni<-length(vector[,1])
if(vector[ni,3]==1){#se o sistema é truncado por falha
```

```
  soma1<-soma1+log(vector[1,2])
  soma2<-soma2-(vector[1,2]/eta)^beta
  if(ni>1){
    for (k in 2:ni){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]
}
      }
      soma1<-soma1+log(vector[k,2]-(1-theta)*soma12)
      soma2<-soma2-((vector[k,2]-(1-theta)*soma12)/eta)^beta+
      +((vector[k-1,2]-(1-theta)*soma12)/eta)^beta
    }
  }
}
if(vector[ni,3]==0){#se o sistema é truncado por tempo
  if(ni==1){
    soma3<-soma3-(vector[ni,2]/eta)^beta
  }
  if(ni==2){
    soma1<-soma1+log(vector[1,2])
    soma2<-soma2-(vector[1,2]/eta)^beta
    soma3<-soma3-((vector[ni,2]-(1-theta)*vector[1,2])/eta)^beta+
    +((vector[1,2]-(1-theta)*vector[1,2])/eta)^beta
  }
  if(ni>2){
    soma1<-soma1+log(vector[1,2])
    soma2<-soma2-(vector[1,2]/eta)^beta
```

```
    for (k in 2:sum(vector[,3])){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]
}
      }
      soma1<-soma1+log(vector[k,2]-(1-theta)*soma12)
      soma2<-soma2-((vector[k,2]-(1-theta)*soma12)/eta)^beta+
      +((vector[k-1,2]-(1-theta)*soma12)/eta)^beta
    }
    soma32<-0
      for (p in 0:(min(m-1,ni-2))){
soma32<-soma32+(theta^p)*vector[ni-1-p,2]
}
    soma3<-soma3-((vector[ni,2]-(1-theta)*soma32)/eta)^beta+
    +((vector[ni-1,2]-(1-theta)*soma32)/eta)^beta
  }
}
      ####################################################################
      j<-i
    }
    else if(dados[i,1]>dados[i-1,1]){#entra apenas quando identifica que
    completou os dados de um sistema
      vector<-dados[j:(i-1),c(1,2,3)]#constroi uma matriz contendo apenas
      os dados de um sistema
      ## os passos seguintes são executados para cada sistema
      ni<-length(vector[,1])
if(vector[ni,3]==1){#se o sistema é truncado por falha
```

```
  soma1<-soma1+log(vector[1,2])
  soma2<-soma2-(vector[1,2]/eta)^beta
  if(ni>1){
    for (k in 2:ni){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]
}
      }
      soma1<-soma1+log(vector[k,2]-(1-theta)*soma12)
      soma2<-soma2-((vector[k,2]-(1-theta)*soma12)/eta)^beta+
      +((vector[k-1,2]-(1-theta)*soma12)/eta)^beta
    }
  }
}
if(vector[ni,3]==0){#se o sistema é truncado por tempo
  if(ni==1){
    soma3<-soma3-(vector[ni,2]/eta)^beta
  }
  if(ni==2){
    soma1<-soma1+log(vector[1,2])
    soma2<-soma2-(vector[1,2]/eta)^beta
    soma3<-soma3-((vector[ni,2]-(1-theta)*vector[1,2])/eta)^beta+
    +((vector[1,2]-(1-theta)*vector[1,2])/eta)^beta
  }
  if(ni>2){
    soma1<-soma1+log(vector[1,2])
    soma2<-soma2-(vector[1,2]/eta)^beta
```

```
    for (k in 2:sum(vector[,3])){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]
}
      }
      soma1<-soma1+log(vector[k,2]-(1-theta)*soma12)
      soma2<-soma2-((vector[k,2]-(1-theta)*soma12)/eta)^beta+
      +((vector[k-1,2]-(1-theta)*soma12)/eta)^beta
    }
    soma32<-0
      for (p in 0:(min(m-1,ni-2))){
soma32<-soma32+(theta^p)*vector[ni-1-p,2]
}
    soma3<-soma3-((vector[ni,2]-(1-theta)*soma32)/eta)^beta+
    +((vector[ni-1,2]-(1-theta)*soma32)/eta)^beta
  }
}
      ####################################################################
      j<-i
    }
  }
###############
  logl<-N*log(beta)-beta*N*log(eta)+(beta-1)*soma1+soma2+soma3
  return(-1*logl)
#  return(logl)
}
###########################################################################
```

```
## Otimizacao da funcao de de log-verossimilhanca:
###########################################################################
ml.fit<-optim(c(log(1.8),log(7),log(0.5)),mloglik,dados=file_tij,hessian=T)
###########################################################################
## Extraimos as EMV de log(beta), log(eta) e log(theta);
## Calculamos os DP's estimados usando a informacao observada;
## Calculamos as estimativas pontuais dos parametros e seus respectivos
## IC de 95% (i.e. exponenciamos os ICs obtidos para o log dos parametros)
###########################################################################
hat.logbeta<-ml.fit$par[1]
hat.logeta<-ml.fit$par[2]
hat.logtheta<-ml.fit$par[3]
# Os desvios-padrao das estimativas dos parametros sao obtidos por:
# 1) Obtem-se a matriz hessiana. O negativo da hessiana eh a
# "informacao observada".
# 2) Usando a funcao 'solve', calcula-se o inverso da hessiana, que eh a
# matriz de covariancia da aproximacao normal padrao.
# 3) Calcula-se a raiz quadrada usando-se 'sqrt' das variancias.
sd.hat.logbeta<-sqrt(solve(ml.fit$hessian)[1,1])
sd.hat.logeta<-sqrt(solve(ml.fit$hessian)[2,2])
sd.hat.logtheta<-sqrt(solve(ml.fit$hessian)[3,3])
hat_beta<-exp(hat.logbeta)
hat_eta<-exp(hat.logeta)
hat_theta<-exp(hat.logtheta)
IC_beta<-exp(c(hat.logbeta-1.96*sd.hat.logbeta,hat.logbeta+
 +1.96*sd.hat.logbeta))
IC_eta<-exp(c(hat.logeta-1.96*sd.hat.logeta,hat.logeta+1.96*sd.hat.logeta))
IC_theta<-exp(c(hat.logtheta-1.96*sd.hat.logtheta,hat.logtheta+
 +1.96*sd.hat.logtheta))
hat_beta
IC_beta
hat_eta
```

IC_eta

hat_theta

IC_theta

ml.fit

B.2 Point and intervalar estimation of the parameters in ARI_m+PLP model, using MLE

```
file_tij<-read.table(paste(apppath,"tij_caminhoes.txt",sep=""),header=TRUE)
###########################################################################
## Declaracao da funcao de log-verossimilhanca
###########################################################################
m<-1 #memoria do modelo ARI
mloglik<-function(pars,dados){
  beta<-exp(pars[1])
  eta<-exp(pars[2])
  theta<-exp(pars[3])
  soma1<-0
  soma2<-0
  soma3<-0
  N<-sum(dados[,3])
  n<-length(dados[,1])
  j<-1
  for (i in 2:(n+1)){
    vector<-NULL
    if(i>n){
      vector<-dados[j:(i-1),c(1,2,3)]
       ni<-length(vector[,1])
if(vector[ni,3]==1){#se o sistema é truncado por falha
  soma1<-soma1+log(vector[1,2]^(beta-1))
  soma2<-soma2-(vector[1,2]^beta)
  if(ni>1){
    for (k in 2:ni){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]^(beta-1)
      }
```

```
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]^(beta-1)
}
      }
      soma1<-soma1+log(vector[k,2]^(beta-1)-(1-theta)*soma12)
      soma2<-soma2+(-vector[k,2]^beta+vector[k-1,2]^beta+
      +(1-theta)*beta*(vector[k,2]-vector[k-1,2])*soma12)
    }
  }
}
if(vector[ni,3]==0){#se o sistema é truncado por tempo
  if(ni==1){
    soma3<-soma3-(vector[ni,2]^beta)
  }
  if(ni==2){
    soma1<-soma1+log(vector[1,2]^(beta-1))
    soma2<-soma2-(vector[1,2]^beta)
    soma3<-soma3+(-vector[ni,2]^beta+vector[1,2]^beta+
    +(1-theta)*beta*(vector[ni,2]-vector[1,2])*vector[1,2]^(beta-1))
  }
  if(ni>2){
    soma1<-soma1+log(vector[1,2]^(beta-1))
    soma2<-soma2-(vector[1,2]^beta)
    for (k in 2:sum(vector[,3])){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]^(beta-1)
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]^(beta-1)
```

```
}
      }
      soma1<-soma1+log(vector[k,2]^(beta-1)-(1-theta)*soma12)
      soma2<-soma2+(-vector[k,2]^beta+vector[k-1,2]^beta+
      +(1-theta)*beta*(vector[k,2]-vector[k-1,2])*soma12)
    }
    soma32<-0
      for (p in 0:(min(m-1,ni-2))){
soma32<-soma32+(theta^p)*vector[ni-1-p,2]
}
    soma3<-soma3+(-vector[ni,2]^beta+vector[ni-1,2]^beta+
    +(1-theta)*beta*(vector[ni,2]-vector[ni-1,2])*soma32)
  }
}
      j<-i
    }
    else if(dados[i,1]>dados[i-1,1]){
      vector<-dados[j:(i-1),c(1,2,3)]
      ni<-length(vector[,1])
if(vector[ni,3]==1){#se o sistema é truncado por falha
  soma1<-soma1+log(vector[1,2]^(beta-1))
  soma2<-soma2-(vector[1,2]^beta)
  if(ni>1){
    for (k in 2:ni){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]^(beta-1)
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]^(beta-1)
}
```

```
      }
      soma1<-soma1+log(vector[k,2]^(beta-1)-(1-theta)*soma12)
      soma2<-soma2+(-vector[k,2]^beta+vector[k-1,2]^beta+
      +(1-theta)*beta*(vector[k,2]-vector[k-1,2])*soma12)
    }
  }
}
if(vector[ni,3]==0){#se o sistema é truncado por tempo
  if(ni==1){
    soma3<-soma3-(vector[ni,2]^beta)
  }
  if(ni==2){
    soma1<-soma1+log(vector[1,2]^(beta-1))
    soma2<-soma2-(vector[1,2]^beta)
    soma3<-soma3+(-vector[ni,2]^beta+vector[1,2]^beta+
    +(1-theta)*beta*(vector[ni,2]-vector[1,2])*vector[1,2]^(beta-1))
  }
  if(ni>2){
    soma1<-soma1+log(vector[1,2]^(beta-1))
    soma2<-soma2-(vector[1,2]^beta)
    for (k in 2:sum(vector[,3])){
      soma12<-0
      if (k==2){
      soma12<-vector[1,2]^(beta-1)
      }
      else{
for (p in 0:(min(m-1,k-2))){
  soma12<-soma12+(theta^p)*vector[k-1-p,2]^(beta-1)
}
      }
      soma1<-soma1+log(vector[k,2]^(beta-1)-(1-theta)*soma12)
      soma2<-soma2+(-vector[k,2]^beta+vector[k-1,2]^beta+
```

```
      +(1-theta)*beta*(vector[k,2]-vector[k-1,2])*soma12)
    }
    soma32<-0
      for (p in 0:(min(m-1,ni-2))){
soma32<-soma32+(theta^p)*vector[ni-1-p,2]
}
    soma3<-soma3+(-vector[ni,2]^beta+vector[ni-1,2]^beta+
    +(1-theta)*beta*(vector[ni,2]-vector[ni-1,2])*soma32)
  }
}
      j<-i
    }
  logl<-N*log(beta)-beta*N*log(eta)+soma1+(eta^(-beta))*soma2+(eta^(-beta))*soma
  return(-1*logl)
}
########################################################################
## Otimizacao da funcao de de log-verossimilhanca:
########################################################################
ml.fit<-optim(c(log(1.5),log(5),log(0.5)),mloglik,dados=file_tij,hessian=T)
########################################################################
hat.logbeta<-ml.fit$par[1]
hat.logeta<-ml.fit$par[2]
hat.logtheta<-ml.fit$par[3]
sd.hat.logbeta<-sqrt(solve(ml.fit$hessian)[1,1])
sd.hat.logeta<-sqrt(solve(ml.fit$hessian)[2,2])
sd.hat.logtheta<-sqrt(solve(ml.fit$hessian)[3,3])
hat_beta<-exp(hat.logbeta)
hat_eta<-exp(hat.logeta)
hat_theta<-exp(hat.logtheta)
IC_beta<-exp(c(hat.logbeta-1.96*sd.hat.logbeta,hat.logbeta+
 +1.96*sd.hat.logbeta))
IC_eta<-exp(c(hat.logeta-1.96*sd.hat.logeta,hat.logeta+
```

```
 +1.96*sd.hat.logeta))
IC_theta<-exp(c(hat.logtheta-1.96*sd.hat.logtheta,
 ,hat.logtheta+1.96*sd.hat.logtheta))
hat_beta
IC_beta
hat_eta
IC_eta
hat_theta
IC_theta
ml.fit
```

B.3 Point estimation of the optimal PM periodicity under ARA_1+PLP model

```
cost.ratio<-1/1.23
K<-10000 #tamanho do Monte Carlo
T<-40000 #tempo de truncamento
beta.hat<-2.458
eta.hat<-15586
theta.hat<-0.471
# GERAPLPRI toma T, beta, eta e theta e gera as falhas de um
 PL+ARA1 truncado em T
GERAPLPRI<-function(beta,teta,eta,trunc){
tempos<-NULL
t<-0
while(t<trunc) {
  tempos<-c(tempos,t)
  u<-runif(1)
  t<-(1-teta)*t+eta*((teta*t/eta)^beta-log(1-u))^(1/beta)
  }
  return(tempos[-1])
}
falhas<-NULL
for(k in 1:K){
  falhas<-c(falhas,GERAPLPRI(beta=beta.hat,eta=eta.hat,teta=theta.hat,trunc=T))
}
falhas<-sort(falhas)
mcf.abs<-c(0,falhas,T)
mcf.ords<-c(0,0:(length(falhas)-1),length(falhas))/K
GCM1<-function(t,ords){
  ext<-c(rep(0,length(t)-1),1)
  indice<-1
  while(indice<length(t)){
```

```
    ext[indice]<-1
    slopes<-(ords[-(1:indice)]-ords[indice])/(t[-(1:indice)]-t[indice])
    indice<-indice+which.min(slopes)
  }
  return(list(tempos=t[ext==1],ords=ords[ext==1]))
}
gcm<-GCM1(mcf.abs,mcf.ords)
int.ords<-c(0,diff(gcm$ords)/diff(gcm$tempos))
B_t<-gcm$tempos*int.ords-gcm$ords
tau<-approx(x=B_t,y=gcm$tempos,xout=cost.ratio)$y
tau
```

B.4 Interval estimation (Bootstrap) of the optimal PM periodicity under ARA_1+PLP model

```
file_tij<-read.table(paste(apppath, "tij_motores.txt",sep=""),header=TRUE)
###########################################################################
## Reamostrando do arquivo de dados original para construir o Bootstrap
###########################################################################
reamostragem<-function(tij){
  num_elem <-tij[length(tij[, 1]),1]
  x<-sample(1:num_elem, num_elem, replace=T)
  #tij_B<-matrix(ncol=3,nrow=0)
  acc<-0
  for(i in 1:num_elem) { # para cada elmento do resample
    for(j in 1:length(tij[,1])){  # para cada elemento da matrix de dados
      if(tij[j,1]==x[i]){
        acc<-1+acc                      #acumular isso no vetor
      }
    }
  }
  patati<-1
  tij_B<-matrix(ncol=3,nrow=acc)
  for(i in 1:num_elem) { # para cada elmento do resample
    for(j in 1:length(tij[,1])){ # para cada elemento da matrix de dados
      if(tij[j,1]==x[i]){
        tij_B[patati,1]<-i#tij[j,1] #bind value to the last array position
        tij_B[patati,2]<-tij[j,2] #bind value to the last array position
        tij_B[patati,3]<-tij[j,3] #bind value to the last array position
        patati<-patati+1
      }
    }
  }
  return(tij_B)
```

```
}
## EMV para cada reamostra
mloglik<-function(pars,dados){
  N<-0
  beta<-exp(pars[1])
  eta<-exp(pars[2])
  theta<-exp(pars[3])
  Total<-length(dados[, 1])

  soma1<-0
  soma2<-0
  soma3<-0
  soma4<-0

  if (dados[1,3]==0){
    soma3<-soma3+(dados[1,2]/eta)^beta
  }
  if (dados[1,3]==1 & dados[1,1]!=dados[2,1]){
    soma3<-soma3+((theta*dados[1,2])/eta)^beta
  }

  if (dados[1,3]==1){
    soma2<-soma2+(dados[1,2]/eta)^beta
    soma4<-soma4+log(dados[1,2])
    soma1<-soma1+((theta*dados[1,2])/eta)^beta
  }
  if(dados[1,3] == 1) {
    N<-N+1
  }

  for(i in 2:Total) {
    if (dados[i,3]==1 & dados[i,1]!=dados[(i-1),1]){
```

```
    soma2<-soma2+(dados[i,2]/eta)^beta
  }else if (dados[i,3]==1 & dados[i,1]==dados[(i-1),1]){
    soma2<-soma2+((dados[i,2]-(1-theta)*dados[(i-1),2])/eta)^beta
  }
  #----
  if (dados[i,3]==0 & dados[i,1]==dados[(i-1),1]){
    soma3<-soma3+((dados[i,2]-(1-theta)*dados[(i-1),2])/eta)^beta
  }else if (dados[i,3]==0 & dados[i,1]!=dados[(i-1),1]){
    soma3<-soma3+(dados[i,2]/eta)^beta
  }
  #------
  if (dados[i,3]==1 & dados[i,1]!=dados[(i-1),1]){
    soma4<-soma4+log(dados[i,2])
  }else if (dados[i,3]==1 & dados[i,1]==dados[(i-1),1]){
    soma4<-soma4+log((dados[i,2]-(1-theta)*dados[(i-1),2]))
  }
  #------
  if (dados[i,3]==1){
    soma1<-soma1+((theta*dados[i,2])/eta)^beta
    N<-N+1
  }
}
for(i in 2:(Total-1)) {
  if (dados[i,3]==1 & dados[i,1]!=dados[(i+1),1]){
    soma3<-soma3+((theta*dados[i,2])/eta)^beta
  }
}
if (dados[Total,3]==1){
  soma3<-soma3+((theta*dados[Total,2])/eta)^beta
}

logl<-soma1-soma2-soma3+N*log(beta)-beta*N*log(eta)+(beta-1)*soma4
```

```
  return(-1*logl)
}
bootstrap<-function(B){
  ptm <- proc.time() #inicia contador de tempo de processamento
  matparam<-matrix(ncol=3,nrow=B)
  for(i in 1:B){
    tryCatch({
      tij_B<-reamostragem(file_tij)
      ml.fit<-optim(c(log(2.45),log(15500),log(0.47)),
       ,mloglik,dados=tij_B,hessian=F)
      hat_beta<-exp(ml.fit$par[1])
      hat_eta<-exp(ml.fit$par[2])
      hat_theta<-exp(ml.fit$par[3])
      matparam[i,1]<-hat_beta
      matparam[i,2]<-hat_eta
      matparam[i,3]<-hat_theta
    },error=function(err)
    {
      matparam[i,1]<-NA
      matparam[i,2]<-NA
      matparam[i,3]<-NA
    })
  }
  cat("\n->Bootstrap: Tempo de processamento uso do CPU ",proc.time() - ptm)
  return(matparam)
}
## Com base nos parametros estimados para cada reamostra de tij, ajusta a MCF e
## estima Lambda(t), lambda(t) e B(t) pelo Maximo Minorante Convexo (GCM)
GERAPLPRI<-function(beta,teta,eta,trunc){
  tempos<-NULL
  t<-0
    while(t<trunc) {
```

```
        tempos<-c(tempos,t)
        u<-runif(1)
        t<-(1-teta)*t+eta*((teta*t/eta)^beta-log(1-u))^(1/beta)
      }
  return(tempos[-1])
}
GCM1<-function(t,ords){
  ext<-c(rep(0,length(t)-1),1)
  indice<-1
  while(indice<length(t)){
    ext[indice]<-1
    slopes<-(ords[-(1:indice)]-ords[indice])/(t[-(1:indice)]-t[indice])
    indice<-indice+which.min(slopes)
  }
  return(list(tempos=t[ext==1],ords=ords[ext==1]))
}
mlgt<-function(B,K,T,cost.ratio,mat.parametros){
tau_gcm<-vector("numeric",B)
  beta.hat<-vector("numeric",B)
  eta.hat<-vector("numeric",B)
  theta.hat<-vector("numeric",B)
  saida<-matrix(ncol=4,nrow=B)
  idx<-1
  for(i in 1:B){
    beta.hat[i]<-mat.parametros[i,1]
    eta.hat[i]<-mat.parametros[i,2]
    theta.hat[i]<-mat.parametros[i,3]
    if(!( is.na(beta.hat[i]) || is.na(eta.hat[i]) || is.na(theta.hat[i]))){
      ###
      falhas<-NULL
      for(k in 1:K){
        falhas<-c(falhas,GERAPLPRI(beta=beta.hat[i],eta=eta.hat[i],
```

```
      ,teta=theta.hat[i],trunc=T))
    }
    falhas<-sort(falhas)
    mcf.abs<-NULL
    mcf.abs<-c(0,falhas,T)
    mcf.ords<-NULL
    mcf.ords<-c(0,0:(length(falhas)-1),length(falhas))/K
    gcm<-NULL
    gcm<-GCM1(mcf.abs,mcf.ords)
    int.ords<-NULL
    int.ords<-c(0,diff(gcm$ords)/diff(gcm$tempos))
    B_t<-NULL
    B_t<-gcm$tempos*int.ords-gcm$ords
    gcm$tempos
    tau_gcm[i]<-approx(x=B_t,y=gcm$tempos,xout=cost.ratio)$y
    idx<-idx+1
    saida[i,1]<-beta.hat[i]
    saida[i,2]<-eta.hat[i]
    saida[i,3]<-theta.hat[i]
    saida[i,4]<-tau_gcm[i]
    ###
  }else{
    #cat("\nNA's encontrados, ignorando")
    tau_gcm[i]<-NA
  }
 }
 IC_gcm<-quantile(tau_gcm,probs=c(0.025,0.975),na.rm=TRUE)
 print(saida)
 return(IC_GCM=IC_gcm)
}
cost.ratio<-1/1.23
K<-10000
```

```
T<-40000

B<-10000

mat.parametros<-bootstrap(B)

mlgt(B,K,T,cost.ratio,mat.parametros)
```

B.5 Estimating the periodicity in virtual age (Dinamic Policy) under the model ARA_1+PLP and Bootstratp CIs

```
#################################################################
## Objetivo: Estimar a periodicidade de manutenção preventiva na
## idade virtual baseada na Politica Dinamica, para um conjunto
## de sistemas sob estudo que seguem o modelo ARA 1 + PLP com
## parâmetros estimados beta, eta e theta, dada uma razao de custos
## Cmp/Cmc. Fornece a estimativa pontual e IC Bootstrap parametrico:
## 1) Utilizando a Equacao Fundamental da Manutencao Preventiva,
## estima a periodicidade (tau) utilizando Monte Carlo com K sistemas.
## 2) O Bootstrap parametrico consiste em gerar n sistemas com os
## parametros beta, eta e theta, e estimar a periodicidade, B vezes.
## 3) O IC percentilico eh obtido com base nas B reamostras Bootstrap.
## 4) O resultado final a ser armazenado consiste na estimativa
## pontual da periodicidade, seu IC e os B valores de beta, eta,
theta e tau estimados para cada uma das B reamostras.
#################################################################
### Parametros de entrada ########################################

#Razão Cmp/Cmc
cost.ratio<-1/1.23

#Parametros ARA1+PLP
beta<-2.458
theta<-0.471
eta<-15586

#tempo de truncamento para gerar os sistemas
T<-60000

K<-10000 #Monte Carlo para estimar a MCF
```

```
B<-10000 #Bootstrap para construir ICs
n<-193 #Numero de sistemas simulados no Bootstrap parametrico

### Estimaçao tau pontual#########################################

GERAPLPRI<-function(beta,theta,eta,trunc){
  tempos<-NULL
  t<-0
  while(t<trunc) {
    tempos<-c(tempos,t)
    u<-runif(1)
    t<-(1-theta)*t+eta*((theta*t/eta)^beta-log(1-u))^(1/beta)
  }
  return(tempos[-1])
}

GCM<-function(t,ords){
  ext<-c(rep(0,length(t)-1),1)
  indice<-1
  while(indice<length(t)){
    ext[indice]<-1
    slopes<-(ords[-(1:indice)]-ords[indice])/(t[-(1:indice)]-t[indice])
    indice<-indice+which.min(slopes)
  }
  return(list(tempos=t[ext==1],ords=ords[ext==1]))
}

falhas<-NULL
for(i in 1:K){
  falhas<-c(falhas,GERAPLPRI(beta,theta,eta,T))
}
falhas<-sort(falhas)
```

```
mcf.abs<-c(0,falhas,T)
mcf.ords<-c(0,0:(length(falhas)-1),length(falhas))/K
#plot(mcf.abs,mcf.ords)
gcm<-GCM(mcf.abs,mcf.ords)
#plot(gcm$tempos,gcm$ords)
int.ords<-c(0,diff(gcm$ords)/diff(gcm$tempos))
#plot(gcm$tempos,int.ords)
B_t<-gcm$tempos*int.ords-gcm$ords
#plot(gcm$tempos,B_t)
hat.tau_Periodica<-approx(x=B_t,y=gcm$tempos,xout=cost.ratio)$y
lambda<-approx(y=int.ords,x=gcm$tempos,xout=hat.tau_Periodica,
 ,method="constant")$y
Tau_pontual_VA<-eta*((lambda*eta)/beta)^(1/(beta-1)) #periodicidade
#na idade virtual (sob a politica dinamica)

### Estimaçao IC Bootstrap #####################################

mloglikRI<-function(parsRI,dadosRI){
  N_RI<-0
  betaRI<-exp(parsRI[1])
  etaRI<-exp(parsRI[2])
  thetaRI<-exp(parsRI[3])
  TotalRI<-length(dadosRI[, 1])
  soma1RI<-0
  soma2RI<-0
  soma3RI<-0
  soma4RI<-0
  if (dadosRI[1,3]==0){
    soma3RI<-soma3RI+(dadosRI[1,2]/etaRI)^betaRI
  }
  if (dadosRI[1,3]==1 & dadosRI[1,1]!=dadosRI[2,1]){
    soma3RI<-soma3RI+((thetaRI*dadosRI[1,2])/etaRI)^betaRI
```

```
}
if (dadosRI[1,3]==1){
  soma2RI<-soma2RI+(dadosRI[1,2]/etaRI)^betaRI
  soma4RI<-soma4RI+log(dadosRI[1,2])
  soma1RI<-soma1RI+((thetaRI*dadosRI[1,2])/etaRI)^betaRI
}
if(dadosRI[1,3] == 1) {
  N_RI<-N_RI+1
}
for(i in 2:TotalRI) {
  if (dadosRI[i,3]==1 & dadosRI[i,1]!=dadosRI[(i-1),1]){
    soma2RI<-soma2RI+(dadosRI[i,2]/etaRI)^betaRI
  }else if (dadosRI[i,3]==1 & dadosRI[i,1]==dadosRI[(i-1),1]){
    soma2RI<-soma2RI+((dadosRI[i,2]-(1-thetaRI)*
    *dadosRI[(i-1),2])/etaRI)^betaRI
  }
  #----
  if (dadosRI[i,3]==0 & dadosRI[i,1]==dadosRI[(i-1),1]){
    soma3RI<-soma3RI+((dadosRI[i,2]-(1-thetaRI)*
    *dadosRI[(i-1),2])/etaRI)^betaRI
  }else if (dadosRI[i,3]==0 & dadosRI[i,1]!=dadosRI[(i-1),1]){
    soma3RI<-soma3RI+(dadosRI[i,2]/etaRI)^betaRI
  }
  #------
  if (dadosRI[i,3]==1 & dadosRI[i,1]!=dadosRI[(i-1),1]){
    soma4RI<-soma4RI+log(dadosRI[i,2])
  }else if (dadosRI[i,3]==1 & dadosRI[i,1]==dadosRI[(i-1),1]){
    soma4RI<-soma4RI+log((dadosRI[i,2]-(1-thetaRI)*dadosRI[(i-1),2]))
  }
  #------
  if (dadosRI[i,3]==1){
    soma1RI<-soma1RI+((thetaRI*dadosRI[i,2])/etaRI)^betaRI
```

```
      N_RI<-N_RI+1
    }
  }
  for(i in 2:(TotalRI-1)) {
    if (dadosRI[i,3]==1 & dadosRI[i,1]!=dadosRI[(i+1),1]){
      soma3RI<-soma3RI+((thetaRI*dadosRI[i,2])/etaRI)^betaRI
    }
  }
  if (dadosRI[TotalRI,3]==1){
    soma3RI<-soma3RI+((thetaRI*dadosRI[TotalRI,2])/etaRI)^betaRI
  }
  loglRI<-soma1RI-soma2RI-soma3RI+N_RI*log(betaRI)-
   -betaRI*N_RI*log(etaRI)+(betaRI-1)*soma4RI
  return(-1*loglRI)
}

Bootstrap<-function(n){
#Gera n processos ARA 1 + PLP
  falhas<-NULL
  falhas<-c(GERAPLPRI(beta,theta,eta,T))
  matriz_dados1<-matrix(ncol=3,nrow=(length(falhas)+1))
  for (i in 1:(length(falhas))){
    matriz_dados1[i,1]<-1
    matriz_dados1[i,2]<-falhas[i]
    matriz_dados1[i,3]<-1
  }
  matriz_dados1[length(falhas)+1,1]<-1
  matriz_dados1[length(falhas)+1,2]<-T
  matriz_dados1[length(falhas)+1,3]<-0
  matriz_dados<-NULL
  for (j in 2:n){
    falhas<-c(GERAPLPRI(beta,theta,eta,T))
```

```
    matriz_dados_prov<-NULL
    matriz_dados_prov<-matrix(ncol=3,nrow=(length(falhas)+1))
    for (i in 1:(length(falhas))){
      matriz_dados_prov[i,1]<-j
      matriz_dados_prov[i,2]<-falhas[i]
      matriz_dados_prov[i,3]<-1
    }
    matriz_dados_prov[length(falhas)+1,1]<-j
    matriz_dados_prov[length(falhas)+1,2]<-T
    matriz_dados_prov[length(falhas)+1,3]<-0
    matriz_dados<-rbind(matriz_dados,matriz_dados_prov)
 }
 file_tij<-rbind(matriz_dados1,matriz_dados)
#Estima os parametros sob RI e estima tau sob a politica dinamica
  ml.fitRI<-optim(c(log(beta),log(eta),log(theta)),mloglikRI,
   ,dadosRI=file_tij,hessian=F)
  beta_RI<-exp(ml.fitRI$par[1])
  eta_RI<-exp(ml.fitRI$par[2])
  theta_RI<-exp(ml.fitRI$par[3])
  falhas<-NULL
  for(i in 1:K){
    falhas<-c(falhas,GERAPLPRI(beta_RI,theta_RI,eta_RI,T))
  }
  falhas<-sort(falhas)
  mcf.abs<-c(0,falhas,T)
  mcf.ords<-c(0,0:(length(falhas)-1),length(falhas))/K
  gcm<-GCM(mcf.abs,mcf.ords)
  int.ords<-c(0,diff(gcm$ords)/diff(gcm$tempos))
  B_t<-gcm$tempos*int.ords-gcm$ords
  hat.tau_Periodica<-approx(x=B_t,y=gcm$tempos,xout=cost.ratio)$y
  lambda<-approx(y=int.ords,x=gcm$tempos,xout=hat.tau_Periodica,
   ,method="constant")$y
```

```
  Tau_VA_B<-eta*((lambda*eta)/beta)^(1/(beta-1)) #periodicidade na
   #idade virtual (sob a politica dinamica)
  return(list(hat.beta=beta_RI,hat.eta=eta_RI,hat.theta=theta_RI,
   ,hat.tau_Periodica=hat.tau_Periodica,Tau_VA=Tau_VA_B))
}

beta_B<-NULL
eta_B<-NULL
theta_B<-NULL
Tau_periodica<-NULL
Tau_VA<-NULL

for (i in 1:B){
 result<-Bootstrap(n)
 C<-data.frame(result)
 beta_B[i]<-C[1,1]
 eta_B[i]<-C[1,2]
 theta_B[i]<-C[1,3]
 Tau_periodica[i]<-C[1,4]
 Tau_VA[i]<-C[1,5]
}

IC_Tau_VA<-quantile(Tau_VA,probs=c(0.025,0.975),na.rm=TRUE)
Tau_pontual_VA
IC_Tau_VA
setwd("Desktop")
write.table(beta_B,"beta_Bootstrap.txt", sep = "\t",eol="\n\r",
 ,row.names=FALSE, col.names=FALSE)
write.table(eta_B,"eta_Bootstrap.txt", sep = "\t",eol="\n\r",
 ,row.names=FALSE, col.names=FALSE)
write.table(theta_B,"theta_Bootstrap.txt", sep = "\t",eol="\n\r",
 ,row.names=FALSE, col.names=FALSE)
```

```
write.table(Tau_periodica,"Tau_periodica_Bootstrap.txt", sep = "\t",
 ,eol="\n\r",row.names=FALSE, col.names=FALSE)
write.table(Tau_VA,"Tau_dinamica_Bootstrap.txt", sep = "\t",eol="\n\r",
 ,row.names=FALSE, col.names=FALSE)
```

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