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EXPANSIVE AND CONTINUUM-WISE HYPERBOLIC HOMEOMORPHISMS ON SURFACES

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Orientador: Prof. Dr. Bernardo Melo de Carvalho
Coorientador: Prof. Dr. Alberto Berly Sarmiento Vera

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ATA DA DEFESA DE DISSERTAÇÃO DE MESTRADO DO ALUNO RODRIGO ARRUDA RODRIGUES, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DO INSTITUTO DE CIÊNCIAS EXATAS DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 10 DE JULHO DE 2023.

Aos dez dias do mês de julho de 2023, às 11h00, em reunião pública virtual na Plataforma Zoom: https://salavirtual-udelar.zoom.us/j/81282639889 (conforme mensagem eletrônica da Pró-Reitoria de Pós-Graduação de 26/03/2020, com orientações para a atividade de defesa de dissertação durante a vigência da Portaria nº 1819), reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de dissertação do aluno **Rodrigo Arruda Rodrigues**, intitulada: *"Expansive and continuum-wise hyperbolic homeomorphisms on surfaces"*, requisito final para obtenção do Grau de mestre em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Alberto Berly Sarmiento Vera, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 10 de julho de 2023.

PROF. DR. ALBERTO BERLY SARMIENTO VERA
Presidente (UFMG)

Arbieto a.

PROF. DR. ALEXANDER EDUARDO ARBIETO MENDOZA
Examinador (UFRJ)

PROF. DR. ALFONSO ARTIGUE
Examinador (UDELAR, Uruguai)

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Resumo

Esta dissertação foi dividida em duas partes. Na primeira parte, estudamos a classificação de homeomorfismos expansivos em superfícies compactas dada por Hiraide, que afirma que todo homeomorfismo expansivo de uma superfície compacta é pseudo-Anosov. A segunda parte é um trabalho em conjunto com Carvalho e Sarmiento, onde trabalhamos para classificar os homeomorfismos cw-hiperbólicos em superfícies compactas. Mostramos que um homeomorfismo cw_F-hiperbólico que contém um número finito de espinhas é cw_2-hiperbólico. Em particular, mostramos que homeomorfismos cw_3-hiperbólicos contêm um número finito de espinhas, portanto a cw_3-hiperbolicidade implica em cw_2-hiperbolicidade.

Palavras-chave: Expansividade; hiperbolicidade; superficies.

Abstract

This dissertation has been divided into two parts. In the first part, we studied the classification of expansive homeomorphisms on compact surfaces given by Hiraide, which asserts that every expansive homeomorphism on a compact surface is pseudo-Anosov. The second part is the result of a collaborative work with Carvalho and Sarmiento, where we worked on classifying cw-hyperbolic homeomorphisms on compact surfaces. We showed that a cw_F-hyperbolic homeomorphism containing a finite number of spines is cw_2-hyperbolic. In particular, we demonstrated that cw_3-hyperbolic homeomorphisms contain a finite number of spines, thus implying that cw_3-hyperbolicity implies cw_2-hyperbolicity.

Keywords: Expansivity; hiperbolicity; surfaces.

Contents

1 Introduction

The concept of expansivity was introduced in the 1950s in [23] and is a property found in many classes of dynamical systems that exhibit chaotic behavior. In simplified terms, an expansive dynamical system is one in which any two trajectories can be distinguished by an observer with an instrument capable of distinguishing points at a distance greater than a certain constant $c > 0$. Examples of expansive systems include Anosov diffeomorphisms and Axiom A diffeomorphisms restricted to the non-wandering set. Generalizing these systems, the class of expansive homeomorphisms was created.

In 1979, Mañe made significant contributions to the field. In [19], various properties of stable and unstable sets for expansive homeomorphisms in metric spaces were demonstrated.

Afterward, independently, Hiraide [14] and Lewowicz [18] proved that expansive homeomorphisms on compact surfaces are conjugate to pseudo-Anosov diffeomorphisms.

In 1993, Kato [17] introduced the notion of continuum-wise expansive homeomorphisms. This property has been studied in a recent work by Artigue, Carvalho, Cordeiro, and Vieitez [6], where the concept of continuum-wise hyperbolicity was introduced.

In 2011, the concept of N-expansiveness was introduced in [20] by Morales. This property has been studied by Artigue, Pacífico and Vieitez [4] in the study of N-expansive homeomorphisms on surfaces. They exhibit an example of a 2-expansive homeomorphism on the bitorus that is not expansive, and prove that non-wandering 2-expansive homeomorphisms on surfaces are, indeed, expansive. The techniques of Lewowicz/Hiraide are applied to study the structure of bi-asymptotic sectors, which are discs bounded by a local stable and a local unstable arc.

Chapter 3 aims to present Hiraide's work on the classification of expansive homeomorphisms on compact surfaces.

Chapter 4 is dedicated to presenting the work developed in the master's project. Using the techniques of Hiraide and Lewowicz, we worked towards classifying continuum-wise hyperbolic homeomorphisms on compact surfaces.

2 Preliminaries

2.1 Basic notions in dynamical systems

The reference for this section is the book by M. Brin and G. Stuck [7].

A discrete-time dynamical system consists of a non-empty set X and a map $f : X \to X$. For $n \in \mathbb{N}$, the nth iterate of f is the n-fold composition $f^n = f \circ \cdots \circ f$, and we define f^0 to be the identity map. If f is invertible, then $f^{-n} = f^{-1} \circ \cdots \circ f^{-1}$ (n times). For an invertible map, since $f^{n+m} = f^n \circ f^m$, these iterates form a group.

Definition 2.1. *For a discrete-time dynamical system $f : X \to X$ and $x \in X$, we define the positive orbit of x by*

$$\mathcal{O}_f^+(x) = \{f^n(x)\}_{n\in\mathbb{N}}.$$

If f is invertible we define the negative orbit of x by

$$\mathcal{O}_f^-(x) = \{f^{-n}(x)\}_{n\in\mathbb{N}},$$

and we put the orbit of x as

$$\mathcal{O}_f(x) = \mathcal{O}_f^-(x) \cup \mathcal{O}_f^+(x).$$

The subscript f may be omitted if the context is clear.

Definition 2.2. *A point $x \in X$ is a periodic point of period $t > 0$ if $f^t(x) = x$. The smallest $t \in \mathbb{N}$ such that $f^t(x) = x$ is called de minimal period of x.*

If x is a periodic point of period 1 we say that x is a fixed point.

2.2 Topology

Definition 2.3. *Let X be a topological space. A path in X is a continuous map $\alpha : [0, 1] \to X$. A path α is said to be an arc if it is injective.*

Definition 2.4. *Let X be a topological space. We say that X is arcwise connected if for all $x \neq y$ there exists an arc $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.*

Peano space is a Hausdorff, compact, connected, locally connected, metrizable space.

Proposition 2.5. *Every Peano space is arcwise connected and locally arcwise connected.*

Lemma 2.6 (Zorn's Lemma). *Let S be a partially ordered set. If every totally ordered subset of S has an upper bound in S, then S contains a maximal element.*

Corollary 2.7. *Let S be a partially ordered set. There exists a totally ordered subset S_0 of S such that each element of $S \setminus S_0$ is not an upper bound of S_0.*

2.3 Covering maps

Definition 2.8. *Let* X *be a topological space. A covering map of* X *is a continuous map*

$$\pi : E \to X$$

such that there exists a discret space D *and for every* $x \in X$ *an open neighborhood* $U \subset X$*, such that*

$$\pi^{-1}(U) = \bigsqcup_{d \in D} V_d$$

and $\pi|_{V_d} : V_d \to U$ *is a homeomorphism for every* $d \in D$*. The degree of a covering is the cardinality of* D*.*

Definition 2.9. *A map is a p-fold branched covering if it is a covering map except for a nowhere dense set (branch set) and p is the covering degree.*

Example 2.10. *The projection map from a bouquet of* 4 *circles (see figure 1) to a circle is an example of* 4*-fold branched covering map.*

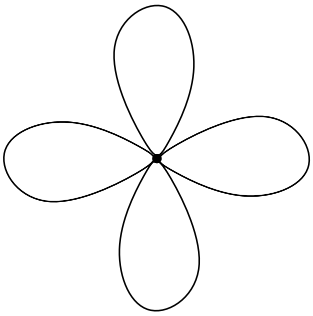


Figure 1 – Bouquet of 4 circles

2.4 Surfaces and foliations

2.4.1 Surfaces

Definition 2.11. *Let X be a topological space. A chart (U, ϕ) on X consists of an open subset U of X and a homeomorphism $\phi : U \to \mathbb{R}^n$ for some $n \in \mathbb{N}$.*

Definition 2.12. *Let X be a topological space. For $0 \leq r \leq \infty$, a C^r atlas on X is a collection of charts $\{(U_\lambda, \phi_\lambda)\}_{\lambda \in \lambda}$ satisfying,*

1. $X = \bigcup_{\lambda \in \Lambda} U_\lambda$,
2. *for all i and j in Λ, the transition map*

$$\phi_i \circ \phi_j^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a C^r map.

Definition 2.13. *Let M be a connected, Hausdorff and second-countable topological space. We say that M is a C^∞ manifold if there exists a C^∞ atlas $\{(U_\lambda, \phi_\lambda)\}_{\lambda \in \lambda}$ on M.*

Lemma 2.14. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open subsets such that U is homeomorphic to V. Then $n = m$.*

Corollary 2.15. *If M is a C^∞ manifold, then the codomain of all charts of $\{(U_\lambda, \phi_\lambda)\}_{\lambda \in \lambda}$ coincides. If $\mathbb{R}^n$ is the codomain, we write $\dim(M) = n$ and M is said to be a C^∞ n-dimensional manifold.*

Definition 2.16. *We say that S is a surface if it is a C^∞ 2-dimensional manifold. If S is a compact topological space, we say that S is a compact surface.*

2.4.2 C^0 foliations

Definition 2.17. *Let S be a compact surface.*

A C^0 foliation $\mathcal{F}$ on S consists of a C^0 atlas $\{(U_\lambda, \phi_\lambda)\}_{\lambda \in \lambda}$ of S such that the transition maps

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

are of the form

$$\phi_{ij}(x, y) = (\alpha_{ij}(x, y), \gamma_{ij}(y)).$$

The atlas used in the definition of $\mathcal{F}$ is called a foliated atlas.

For $y_0 \in \mathbb{R}$ the connected components of $\phi_i^{-1}(x, y_0)$ are called plaques of U_i. For a plaque α, ϕ^{-1} induces a topology called by plaque topology. Furthermore, note that for plaques $\alpha \subset U_i$ and $\beta \subset U_j$, the second property of foliation definition ensures that either $\alpha \cap \beta = \emptyset$ or $\alpha \cap \beta$ is open in α and β in the plaque topology.

2.4.3 C^0 singular foliations

For $p \in \mathbb{N}$, let $\pi_p : \mathbb{C} \to \mathbb{C}$ be the map which sends z to z^p. We define connected open subsets $\mathcal{D}_p$ $(p = 1, 2, \ldots)$ of $\mathbb{C}$ by

$$\mathcal{D}_2 = \{z \in \mathbb{C} : |Re(z)| < 1|, |\operatorname{Im}(z)| < 1\},$$

$$\mathcal{D}_1 = \pi_2(\mathcal{D}_2) \quad \text{and} \quad \mathcal{D}_p = \pi_p^{-1}(\mathcal{D}_1).$$

By definition of π_p and construction of $\mathcal{D}_p$, then $\pi_p : \mathcal{D}_p \to D_1$ is a p-fold branched covering map for every $p \in \mathbb{N}$.

Denote by $\mathcal{H}_2$ and $\mathcal{V}_2$ the horizontal and vertical foliations on $\mathcal{D}_2$ respectively. We define a decomposition $\mathcal{H}_1$ of $\mathcal{D}_1$ as the projection of $\mathcal{H}_2$ by $\pi_2 : \mathcal{D}_2 \to \mathcal{D}_1$. Similarly we define a decomposition $\mathcal{V}_1$. For $p \geq 3$, we define decompositions $\mathcal{H}_p$ and $\mathcal{V}_p$ of $\mathcal{D}_p$ as the lifting of $\mathcal{H}_1$ and $\mathcal{V}_1$ by $\pi_p : \mathcal{D}_p \to \mathcal{D}_1$ respectively (See figure 2). Since $\pi_p : \mathcal{D}_p \to D_1$ is a p-fold branched covering map, then $\mathcal{H}_p$ and $\mathcal{V}_p$ are well defined and for each element $\alpha \in \mathcal{H}_1$ we have $\#(\pi_p^{-1}(\alpha)) = p$. The same holds for $\mathcal{V}_1$.

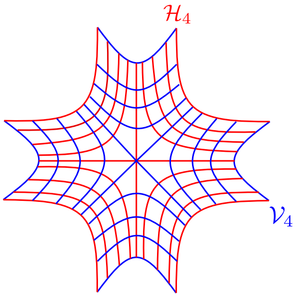


Figure 2 – $\mathcal{D}_4$ and decompositions $\mathcal{H}_4$ and $\mathcal{V}_4$

Definition 2.18. *A decomposition $\mathcal{F}$ of S is called a C^0 singular foliation if every $L \in \mathcal{F}$ is path connected and if for every $x \in S$ there are $p(x) \in \mathbb{N}$ and a C^0 chart $\varphi_x : U_x \to \mathbb{C}$ around x such that*

1. $\varphi_x(x) = 0$,
2. $\varphi_x(U_x) = \mathcal{D}_{p(x)}$,
3. *φ_x sends each connected component of $U_x \cap L$ onto some element of $\mathcal{H}_{p(x)}$ if $U_x \cap L \neq \emptyset$.*

Let $\mathcal{F}$ be a C^0 singular foliation on S. Each element of $\mathcal{F}$ is called a leaf and equipped with the leaf topology induced by plaque topology as in the previous section. The number $p(x)$ is called the number of separatrices at x. A regular point x is a point with $p(x) = 2$, and x is a singular point if $p(x) \neq 2$. We denote the set of singular points by $\operatorname{Sing}(\mathcal{F})$.

Denote by $\mathcal{RF}$ the C^0 foliation on $S \setminus \operatorname{Sing}(\mathcal{F})$.

Definition 2.19. *A C^0 singular foliation $\mathcal{F}$ on S is called minimal if every leaf of $\mathcal{RF}$ is dense in S.*

Definition 2.20. *Let $\mathcal{F}$ a C^0 singular foliation and P_i the projection from $\mathbb{C}$ onto the imaginary axis. An arc A of S is called a transversal of $\mathcal{F}$ if the interior of A is contained in $S \setminus \operatorname{Sing}(\mathcal{F})$ and if for every $x \in A \setminus \operatorname{Sing}(\mathcal{F})$ there is a C^0 chart $\varphi_x : U_x \to \mathbb{C}$ around x such that $P_i \circ \varphi_x|_{U_x \cap A}$ is injective.*

Definition 2.21. *Let A_0 and A_1 be transversals of $\mathcal{F}$. We say $A_0 \simeq A_1$ if there is a continuous map $H : [0,1] \times [0,1] \to S$ such that*

$$H_0 = H|_{[0,1]\times\{0\}} \text{ is a homeomorphism onto } A_0$$

and

$$H_1 = H|_{[0,1]\times\{1\}} \text{ is a homeomorphism onto } A_1,$$

and such that if $L \in \mathcal{F}$ then $H^{-1}(L) = B \times [0,1]$ for some $B \subset [0,1]$.

If $A_0 \simeq A_1$, the homeomorphism

$$H_1 \circ h \circ H_0^{-1} : A_0 \to A_1$$

is called a projection along the leaves, where $h : [0,1] \times \{0\} \to [0,1] \times \{1\}$ is the homeomorphism which send $(t,0)$ to $(t,1)$.

Definition 2.22. *A transverse invariant measure μ for $\mathcal{F}$ is a collection $\{\mu_A : A \text{ is a transversal}\}$ of finite Borel measures on all transversals of $\mathcal{F}$ such that*

1. *$\mu_A|_{A'} = \mu_{A'}$ if $A' \subset A$,*
2. *$\mu_{A_1} \circ h = \mu_{A_0}$ if $h : A_0 \to A_1$ is a projection along the leaves.*

A measured C^0 singular foliation $(\mathcal{F}, \mu)$ is a C^0 singular foliation equipped with a transversal invariant measure μ.

Definition 2.23. *Let $\mathcal{F}$ and $\mathcal{F}'$ be C^0 singular foliations on S. We say that $\mathcal{F}$ is transverse to $\mathcal{F}'$ if $\mathcal{F}$ and $\mathcal{F}'$ have the same number of separatrices $p(x)$ at all $x \in S$ and for every $x \in S$ has a C^0 chart $\varphi_x : U_x \to \mathbb{C}$ such that*

1. *$\varphi_x(x) = 0$,*
2. *$\varphi_x(U_x) = \mathcal{D}_{p(x)}$,*
3. *φ_x sends each connected component of $U_x \cap L$ onto some element of $\mathcal{H}_{p(x)}$ if $U_x \cap L \neq \emptyset$ for $L \in \mathcal{F}$,*
4. *φ_x sends each connected component of $U_x \cap L'$ onto some element of $\mathcal{V}_{p(x)}$ if $U_x \cap L' \neq \emptyset$ for $L' \in \mathcal{F}'$.*

Remark 2.24. *Let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be transverse* C^0 *singular foliations on* S*. Then* $\mathrm{Sing}(\mathcal{F}) = \mathrm{Sing}(\mathcal{F}')$*. If* A *is an arc in a leaf of* $\mathcal{F}$ *(resp.* $\mathcal{F}'$*), and the interior of* A *is contained in* $S \setminus \mathrm{Sing}(\mathcal{F})$*, then* A *is a transversal of* $\mathcal{F}'$ *(resp.* $\mathcal{F}$*).*

3 Classification of expansive homeomorphisms

3.1 Definitions and preliminaries

Let (X, d) be a compact metric space.

Definition 3.1. *$f : X \to X$ is said to be expansive if there exists $c > 0$ such that* $\mathrm{d}(f^n(x), f^n(y)) < c$ *for all $n \in \mathbb{Z}$, then $x = y$.*

Definition 3.2. *We say that a homeomorphism $f : S \to S$ is pseudo-Anosov if there are a constant $\lambda > 1$ and a pair of transverse measured C^0 foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$ with;*

1. *the number of separatrices at each singular point greater than* 2*,*
2. *every finite Borel measure of μ^s and of μ^u non-atomic and positive on all non-empty open sets such that*

$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \ \ f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

Let $f : X \to X$ a homeomorphism. For $x \in X$ we define the stable set $W^s(x)$ and the unstable set $W^u(x)$ by

$$W^s(x) = \{y \in X : \mathrm{d}(f^n(x), f^n(y)) \to 0 \ \ as \ \ n \to \infty\},$$

$$W^u(x) = \{y \in X : \mathrm{d}(f^{-n}(x), f^{-n}(y)) \to 0 \ \ as \ \ n \to \infty\}$$

and put

$$\mathcal{F}^\sigma_f = \{W^\sigma(x) : x \in X\} \qquad (\sigma = s, u).$$

If X is a compact surface and f is pseudo-Anosov, one can check that every leaf L of $\mathcal{F}^\sigma$ coincides with $W^\sigma(x)$ for all $x \in L$. Then we have $\mathcal{F}^\sigma = \mathcal{F}^\sigma_f$.

Proposition A. *Let $f : S \to S$ be an expansive homeomorphism. Then $\mathcal{F}^\sigma_f$ $(\sigma = s, u)$ have the following properties;*

1. *$\mathcal{F}^\sigma_f$ is a C^0 singular foliation,*
2. *every leaf $W^\sigma(x) \in \mathcal{F}^\sigma_f$ is homeomorphic to $L_p = \{z \in \mathbb{C} : \mathrm{Im}(z^{p/2}) = 0\}$ for some $p \geq 2$,*
3. *$\mathcal{F}^s_f$ is transverse to $\mathcal{F}^u_f$,*

4. $\mathcal{F}_f^\sigma$ is minimal.

Proposition B. *Let $f : S \to S$ be a homeomorphism and let $\mathcal{F}^s$ and $\mathcal{F}^u$ be transverse C^0 singular foliations on S. If $f(\mathcal{F}^\sigma) = \mathcal{F}^\sigma$ and $\mathcal{F}^\sigma$ is minimal for $\sigma = s, u$, then there are a constant $\lambda > 0$ and transverse invariant measures μ^σ for $\mathcal{F}^\sigma$ with every finite Borel measure of μ^σ non-atomic and positive on all non-empty open sets such that $f_*(\mu^s) = \lambda^{-1}\mu^s$ and $f_*(\mu^u) = \lambda\mu^u$.*

Let $f : X \to X$ homeomorphism. For $x \in X$ and $\varepsilon > 0$ we define the local stable set $W_\varepsilon^s(x)$ and local unstable set $W_\varepsilon^u(x)$ by

$$W_\varepsilon^s(x) = \{y \in X : \mathrm{d}(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n > 0\},$$

$$W_\varepsilon^u(x) = \{y \in X : \mathrm{d}(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \text{ for all } n > 0\}.$$

We denote by $C_\varepsilon^s(x)$ and $C_\varepsilon^u(x)$ the connected component of x in $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$, respectively.

These sets are closed subsets of X.

We can also define expansiveness in terms of stable and unstable local sets.

Lemma 3.3. *$f : X \to X$ is expansive with expansive constant $c > 0$ if, and only if, $W_c^s(x) \cap W_c^u(x) = \{x\}$ for all $x \in X$.*

Let $f : X \to X$ be expansive with expansive constant $c > 0$. Mañé proved in [19] that for every $\varepsilon > 0$ there is $N > 0$ such that

$$f^n(W_c^s(x)) \subset W_\varepsilon^s(f^n(x)), \quad f^{-n}(W_c^u(x)) \subset W_\varepsilon^u(f^{-n}(x)) \tag{3.1}$$

for all $n \geq N$ and all $x \in X$. Hence

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W_\varepsilon^s(f^n(x))), \quad W^u(x) = \bigcup_{n\geq 0} f^n(W_\varepsilon^s(f^{-n}(x))) \tag{3.2}$$

Theorem 1. *Every expansive homeomorphism of a compact surface is pseudo-Anosov.*

3.2 Uniform diameter of $C_\varepsilon^\sigma(x)$

In this section we will prove that for $\varepsilon > 0$, the diameter of $C_\varepsilon^s(x)$ is greater than δ for all $x \in X$ and some $\delta > 0$. Before proving this, some definitions and lemmas will be useful.

Let (X, d) be a compact metric space and denote by $\mathcal{C}(X)$ the set of all non-empty closed subsets of X.

Lemma 3.4. *If X is connected and $A \in \mathcal{C}(X)$ with $A \neq X$, then every connected component of A intersects the boundary of A in at least one point.*

Proof. See [11, p. 357]. ☐

We define the Hausdorff metric on $\mathcal{C}(X)$ by

$$H(A,B) = \inf\{\varepsilon > 0 : N_\varepsilon(A) \supset B, N_\varepsilon(B) \supset A\} \quad A, B \in \mathcal{C}(X).$$

where $N_\varepsilon(A)$ denotes the $\varepsilon-$neighborhood of A in X.

Lemma 3.5. *$\mathcal{C}(X)$ equipped with the Hausdorff metric is a compact space.*

Proof. See [22, p. 8-9]. ☐

Lemma 3.6. *Let $\varepsilon > 0$ and suppose that a pair of sequences $\{x_i\}_{i\in\mathbb{N}}$ of X and $\{B_i\}_{i\in\mathbb{N}}$ of $\mathcal{C}(X)$ converges to $x_\infty \in X$ and $B_\infty \in \mathcal{C}(X)$ respectively.*

If $B_i \subset W_\varepsilon^\sigma(x_i)$, for all $i \in \mathbb{N}$, then $B_\infty \subset W_\varepsilon^\sigma(x_\infty)$. ($\sigma = s, u$).

Proof. We give the proof for $\sigma = s$. Let $z \in B_\infty$. Since $B_i \to B_\infty$, there is a sequence $\{y_i\}_{i\in\mathbb{N}}$ with $y_i \in B_i$ for all $i \in N$ such that $y_i \to z$.

Since $B_i \subset W_\varepsilon^s(x_i)$, then $\mathrm{d}(f^n(x_i), f^n(y_i)) \leq \varepsilon$ for all $n \geq 0$. Since $x_i \to x_\infty$ and $y_i \to z$, then $\mathrm{d}(f^n(x_\infty), f^n(z)) < \varepsilon$ for all $n \geq 0$. Hence $z \in W_\varepsilon^s(x_\infty)$ and $B_\infty \subset W_\varepsilon^s(x_\infty)$.

With the same argument, we obtain for $\sigma = u$. ☐

Lemma 3.7. *Let $\{x_i\}_{i\in N}, x_\infty$, $\{B_i\}_{i\in N}$ and B_∞ be as in Lemma 3.6. Then*

1. *if $f^n(B_i) \subset B_\varepsilon(f^n(x_i))$ for all $0 \leq n \leq i$ and all $i \in N$, then $B_\infty \subset W_\varepsilon^u(x_\infty)$,*
2. *if $f^{-n}(B_i) \subset B_\varepsilon(f^{-n}(x_i))$ for all $0 \leq n \leq i$ and all $i \in N$, then $B_\infty \subset W_\varepsilon^u(x_\infty)$.*

From here we assume that $f : X \to X$ is expansive with expansive constant $c > 0$.

Lemma 3.8. *For all $0 < \varepsilon \leq \frac{c}{2}$, there exists $0 < \delta \leq \varepsilon$ such that*

1. *if $\mathrm{d}(x,y) \leq \delta$ and $\varepsilon \leq \max\{\mathrm{d}(f^i(x), f^i(y)) : \ 0 \leq i \leq n\} \leq 2\varepsilon$, then $\mathrm{d}(f^n(x), f^n(y)) \geq \delta$,*
2. *if $\mathrm{d}(x,y) \leq \delta$ and $\varepsilon \leq \max\{\mathrm{d}(f^i(x), f^i(y)) : \ -n \leq i \leq 0\} \leq 2\varepsilon$, then $\mathrm{d}(f^n(x), f^n(y)) \geq \delta$*

Proof. See Lemma 2.5 of [14]. ☐

Lemma 3.9. *For $0 < \varepsilon \leq \frac{c}{2}$, let $0 < \delta \leq \varepsilon$ be as in Lemma 3.7. If $A \subset X$ is connected and $x \in A$, then*

1. *if $A \subset B_\delta(x)$, $f^i(A) \cap \partial B_\varepsilon(f^i(x))$ for some $0 \leq i \leq n$ and $f^i(A) \subset B_{2\varepsilon}(f^i(x))$ for all $0 \leq i \leq n$, then $f^n(A) \cap \partial B_\delta(f^n(x) \neq \emptyset$,*
2. *if $A \subset B_\delta(x)$, $f^i(A) \cap \partial B_\varepsilon(f^i(x))$ for some $-n \leq i \leq 0$ and $f^i(A) \subset B_{2\varepsilon}(f^i(x))$ for all $-n \leq i \leq 0$, then $f^{-n}(A) \cap \partial B_\delta(f^{-n}(x) \neq \emptyset$.*

Proof. It follows directly from the previous Lemma. ☐

Lemma 3.10. *For $0 < \varepsilon \leq \frac{c}{2}$, let $0 < \delta \leq \varepsilon$ be as in Lemma 3.7. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a sequence of X and let $\Delta(x_i)$ denote the connected component of x_i in $B_\delta(x_i) \cap f^{-i}B_{\frac{\delta}{2}}(f^i(x_i))$ for all $i \in \mathbb{Z}$. Then*

1. *if for a sequence j of $\mathbb{Z}$ with $j \to \infty$*

$$\lim_{j\to\infty} x_j = x_\infty \quad \textit{and} \quad \lim_{j\to\infty} \Delta(x_j) = \Delta_\infty,$$

then $\Delta_\infty \subset W^s_\varepsilon(x_\infty)$,

2. *if for a sequence j of $\mathbb{Z}$ with $j \to -\infty$*

$$\lim_{j\to-\infty} x_j = x_{-\infty} \quad \textit{and} \quad \lim_{j\to-\infty} \Delta(x_j) = \Delta_{-\infty},$$

then $\Delta_\infty \subset W^s_\varepsilon(x_{-\infty})$.

Proof. See Lemma 2.7 of [14]. ☐

Lemma 3.11. *If X is non-trivial, connected, and locally connected, then for all $0 < \varepsilon \leq \frac{c}{2}$ and all $x \in X$*

$$\operatorname{int} W^\sigma_\varepsilon(x) = \emptyset \qquad \sigma = s, u$$

Proof. We give the proof for $\sigma = u$.

Fix $0 < \varepsilon \leq \frac{c}{2}$, $x \in X$ and let $0 < \delta \leq \varepsilon$ given by Lemma 3.7.

Suppose $y \in \operatorname{int} W^u_\varepsilon(x) \neq \emptyset$ and let $0 < \gamma \leq \delta$ such that $B_{2\gamma}(y) \subset \operatorname{int} W^u_\varepsilon(x)$.

Claim 1. *For all $0 < \eta \leq \delta$ there exists $n > 0$ such that $B_{\frac{\delta}{2}}(f^n(z)) \subset f^n(B_n(z))$ for all $z \in B_\gamma(y)$.*

Before we prove this claim, we show how it leads to a contradiction.

Since X is non-trivial and connected, then for each $k > 0$ there are $p_1, p_2, \dots, p_k \in B_\gamma(y)$ and we can choose $\eta_k \in (0, \gamma]$ such that

$$B_{\eta_k}(p_i) \cap B_{\eta_k}(p_j) = \emptyset \quad \text{for} \quad i \neq j.$$

By using Claim 3.2, for each $k > 0$, there exists $n_k > 0$ such that

$$B_{\frac{\delta}{2}}(f^{n_k}(p_i)) \subset f^{n_k}(B_{\eta_k}(p_i)) \quad \text{for} \quad i = 1, \dots, k.$$

Therefore $B_{\frac{\delta}{2}}(f^{n_k}(p_i)) \cap B_{\frac{\delta}{2}}(f^{n_k}(p_j)) \subset f^{n_k}(B_{\eta_k}(p_i)) \cap f^{n_k}(B_{\eta_k}(p_j)) = \emptyset$, since $B_{\eta_k}(p_i) \cap B_{\eta_k}(p_j) = \emptyset$ and f is a homeomorphism.

Hence $B_{\frac{\delta}{2}}(f^{n_k}(p_i)) \cap B_{\frac{\delta}{2}}(f^{n_k}(p_j)) = \emptyset$ for $i \neq j$ and for each $k > 0$ we get k mutually disjoint balls of radius $\frac{\delta}{2}$, contradicting the compactness of X and finishing the proof.

It remains to prove Claim 3.2. Suppose that the claim does not hold. Then there exists $0<\eta\leq\gamma$ and $\{z_n\}_{n\in\mathbb{N}}$ with $z_n\in B_\gamma(y)$ for all $n\in\mathbb{N}$ and

$$B_{\frac{\delta}{2}}(f^n(z_n))\not\subset f^n(B_\eta(z_n))$$

which implies that

$$f^{-n}(B_{\frac{\delta}{2}}(f^n(z_n)))\not\subset B_\eta(z_n) \tag{3.3}$$

Let $\Delta(z_n)$ be the connected component of z_n in $B_\delta(z_n)\cap f^{-n}(B_{\frac{\delta}{2}}(f^n(z_n)))$.

By using (3.3) and Lemma 3.4, we have $\Delta(z_n)\cap\partial B_\eta(z_n)\neq\emptyset$.

Since X and $\mathcal{C}(X)$ are compact, we can take a subsequence z_{n_j} such that $\Delta(z_{n_j})\to\Delta\in\mathcal{C}(X)$ and we have

$$\Delta\cap\partial B_\eta(z)\neq\emptyset \tag{3.4}$$

where $z_{n_j}\to z$.

On the other hand, Lemma 3.10 ensures that $\Delta\subset W_\varepsilon^s(z)$, and since $0<\eta\leq\gamma$, $z\in B_\gamma(y)$ and $B_{2\gamma}(y)\subset W_\varepsilon^u(x)$, then

$$B_\eta(z)\subset W_\varepsilon^u(x)$$

which implies that

$$B_\eta(z)\cap\Delta\subset W_\varepsilon^s(z)\cap W_\varepsilon^u(x)\subset W_{2\varepsilon}^s(z)\cap W_{2\varepsilon}^u(z)=\{z\}.$$

So $B_\eta(z)\cap\Delta=\{z\}$ and therefore $\Delta\cap\partial B_\eta(z)=\emptyset$, which is a contradiction with (3.4).

□

Proposition 3.12. *Let $f:X\to X$ be an expansive homeomorphism. If X is non-trivial, connected, and locally connected, then for every $\varepsilon>0$ there is $\delta>0$ such that for all $x\in X$*

$$C_\varepsilon^\sigma(x)\cap\partial B_\delta(x)\neq\emptyset \qquad \sigma=s,u,$$

where $C_\varepsilon^\sigma(x)$ denotes the connected component of $W_\varepsilon^\sigma(x)$ containing x.

Proof. Since if $0<\varepsilon<\varepsilon'$ then $C_\varepsilon^\sigma(x)\subset C_{\varepsilon'}^\sigma(x)$, is sufficient to prove for $0<\varepsilon<\frac{c}{4}$.

Fix $0<\varepsilon<\frac{c}{4}$ and let $0<\delta<\varepsilon$ given by Lemma 3.7. We give the proof for $\sigma=s$.

Let $x\in X$ and denote by $x(i)=f^i(x)$ for $i\geq 0$. Let j subsequence of i such that $x(j)\to x_\infty$. By Lemma 3.11 we have $\operatorname{int}W_{2\varepsilon}^u(x_\infty)=\emptyset$, then for $0<\eta\leq\delta$, take m_η such that

$$f^{-m_\eta}(B_{\frac{\eta}{2}})\not\subset B_{2\varepsilon}(f^{-m_\eta}(x_\infty)).$$

Then $m_\eta\to\infty$ as $\eta\to 0$. Let $j_\eta\geq m_\eta$ with $\mathrm{d}(x(j_\eta),x_\infty\leq\frac{\eta}{2}$, then $\operatorname{diam}f^{-m_\eta}B_\eta(x(j_\eta))>2\varepsilon$ and let $0<n_\eta\leq j_\eta$ such that

$$f^{-i}B_\eta(x(j_\eta))\subset B_\varepsilon(x(j_\eta-1)) \quad \text{ for } 0\leq i\leq n_\eta-1$$

and

$$f^{-n_\eta}B_\eta(x(j_\eta)) \not\subset B_\varepsilon(x(j_\eta - n_\eta)).$$

For $0 \le k \le i$, let $\Delta_k(x(i-k))$ the connected component of $x(i-k)$ in

$$B_\varepsilon(x(i-k)) \cap f^{-1}B_\varepsilon(x(i-k+1)) \cap \cdots \cap f^{-k+1}B_\varepsilon(x(i-1)) \cap f^{-k}B_\delta(x(i)) = A_{i,k}.$$

We see that $\Delta_{n_\eta}(x(j_\eta - n_\eta))$ contains the connected component $C(x(j_\eta - n_\eta))$ of $x(j_\eta - n_\eta)$ in $B_\varepsilon(x(j_\eta - n_\eta)) \cap f^{-n_\eta}B_\eta(x(j_\eta))$. Indeed, since $f^{-i}B_\eta(x(j_\eta)) \subset B_\varepsilon(x(j_\eta - i)$ for $0 \le i \le n_\eta - 1$, then

$$f^{-n_\eta}B_\eta(x(j_\eta)) = f^{-n_\eta+i}(f^{-i}B_\eta(x(j_\eta))) \subset f^{-n_\eta+i}B_\varepsilon(x(j_\eta - i)) \ \text{ for } 0 \le i \le n_\eta - 1.$$

Hence

$$f^{-n_\eta}B_\eta(x(j_\eta)) \subset f^{-i}B_\varepsilon(x(j_\eta - n_\eta + i)) \quad \text{for } 0 \le i \le n_\eta - 1.$$

Furthermore, $\eta < \delta$, then $f^{-n_\eta}B_\eta(x(j_\eta)) \subset f^{n_\eta}B_\delta(x(j_\eta))$. Hence

$$f^{-n_\eta}B_\eta(x(j_\eta)) \subset f^{-1}B_\varepsilon(x(j_\eta - n_\eta + 1)) \cap \cdots \cap f^{-n_\eta}B_\delta(x(j_\eta)),$$

which implies that $B_\varepsilon(x(j_\eta - n_\eta)) \cap f^{-n_\eta}B_\eta(x(j_\eta)) \subset A_{j_\eta,n_\eta}$, concluding this claim.

Since $B_\eta(x(j_\eta))$ is connected and $f^{-n_\eta}B_\eta(x(j_\eta)) \not\subset B_\varepsilon(x(j_\eta - n_\eta))$, Lemma 3.4 ensures that

$$C(x(j_\eta - n_\eta)) \cap \partial B_\varepsilon(x(j_\eta - n_\eta)) \neq \emptyset$$

and then

$$\Delta(0) \cap \partial B_\varepsilon(x(j_\eta - n_\eta)) \neq \emptyset, \tag{3.5}$$

where $\Delta(0) = \Delta_{n_\eta}(x(j_\eta - n_\eta))$.

For $k > 0$, let $\Delta(k)$ be the connected component of $x(j_\eta - n_\eta - k)$ in $f^{-1}(\Delta(k-1)) \cap B_\varepsilon(x(j_\eta - n_\eta - k))$. Then we have

$$f^i(\Delta(j_\eta - n_\eta)) \subset B_\varepsilon(x_i) \quad \text{for } 0 \le i \le j_\eta - 1 \tag{3.6}$$

$$f^{j_\eta}(\Delta(j_\eta - n_\eta)) \subset B_\delta(x(j_\eta)). \tag{3.7}$$

Claim 1. *$f^i(\Delta(j_\eta - n_\eta)) \cap \partial B_\varepsilon(x(i)) \neq \emptyset$ for some $0 \le i \le j_\eta - n_\eta$.*

Proof of Claim 1. Suppose by contradiction that $f^i(\Delta(j_\eta - n_\eta)) \cap \partial B_\varepsilon(x(i)) = \emptyset$ for all $0 \le i \le j_\eta - n_\eta$. Since $\Delta(j_\eta - n_\eta)$ is the connected component of $x(0)$ in $f^{-1}(\Delta(j_\eta - n_\eta - 1)) \cap B_\varepsilon(x(0))$ and $\Delta(j_\eta - n_\eta) \subset B_\varepsilon(x(0))$, then

$$f^{-1}(\Delta(j_\eta - n_\eta - 1)) \subset B_\varepsilon(x(0)),$$

and therefore

$$\Delta(j_\eta - n_\eta) = f^{-1}(\Delta(j_\eta - n_\eta - 1)).$$

Inductively we have

$$f^i\Delta(j_\eta - n_\eta) = \Delta(j_\eta - n_\eta - i) \quad \text{for all } 0 \leq i \leq j_\eta - n_\eta.$$

Hence $f^{j_\eta - n_\eta}(\Delta(j_\eta - n_\eta)) = \Delta(0)$, contradicting the fact that $\Delta(0) \cap \partial B_\varepsilon(x(j_\eta - n_\eta)) \neq \emptyset$. Therefore the claim holds.

Combining the claim, (3.6), (3.7) and Lemma 3.9, follows that $\Delta(j_\eta - n_\eta) \cap \partial B_\delta(x) \neq \emptyset$. By (3.6) and (3.7) we have $\Delta(j_\eta - n_\eta) \subset \Delta_{j_\eta}(x)$, hence $\Delta_{j_\eta}(x) \cap \partial B_\delta(x) \neq \emptyset$.

Since $j_\eta \to \infty$ as $\eta \to 0$, we can take $\{j'_\eta\}$ subsequence such that $\Delta_{j_\eta 1} \to \Delta_\infty \in \mathcal{C}(X)$. Then $\Delta_\infty \cap \partial B_\delta(x) \neq \emptyset$ and Δ_∞ is connected. Since $f^i(\Delta_{j'_\eta}(x)) \subset B_\varepsilon(f^i(x))$ for all $0 < i < j'_\eta$, Lemma 3.6 ensures that $\Delta_\infty \subset W^s_\varepsilon(x)$. Then $\Delta_\infty \subset C^s_\varepsilon(x)$ and therefore $C^s_\varepsilon(x) \cap \partial B_\delta(x) \neq \emptyset$, concluding the proof.

In the same way we obtain for $\sigma = u$.

□

3.3 Local connectedness of $C^\sigma_\varepsilon(x)$

Proposition 3.13. *Let $f : X \to X$ be an expansive homeomorphism with expansive constant $c > 0$. If X is a compact surface, then $C^\sigma_\varepsilon(x)$ ($\sigma = s, u$) are locally connected for all $x \in X$ and all $0 < \varepsilon \leq \frac{c}{2}$.*

Proof. We give the proof for $\sigma = s$. Fix $x \in X$ and $0 < \varepsilon \leq \frac{c}{2}$ and take $\delta > 0$ given by Proposition 3.12.

Suppose by contradiction that $C^s_\varepsilon(x)$ is not locally connected. Then we can take $y \in C^s_\varepsilon(x)$ and $0 < \gamma \leq \frac{\delta}{2}$ such that the connected component of y in $C^s_\varepsilon(x) \cap B_\gamma(y)$ does not contain $C^s_\varepsilon(x) \cap B_\lambda(y)$ for all $\lambda > 0$.

Denote by $\mathcal{K}$ the set of all connected components of $C^s_\varepsilon(x) \cap B_\gamma(y)$. Since $C^s_\varepsilon(x)$ is connected, Lemma 3.4 ensures that $K \cap \partial B_\gamma(y) \neq \emptyset$ for each $K \in \mathcal{K}$.

Fix $0 < t < \gamma$ and define $\mathcal{S} = \{K \in \mathcal{K} : K \cap B_t(y) \neq \emptyset\}$. Then $\mathcal{S}$ is an infinite set. Lemma 3.5 ensures the existence of a sequence $\{K_i\}_{i\in\mathbb{N}}$ of $\mathcal{S}$ with

$$K_i \cap K_j = \emptyset \qquad \text{for } i \neq j$$

and

$$K_i \to K_\infty \in \mathcal{C}(C^s_\varepsilon(x) \cap B_\gamma(y)).$$

Since K_i is connected for each $i \in \mathbb{N}$, then K_∞ is connected and is contained in a connected component of $C^s_\varepsilon(x) \cap B_\gamma(y)$. We may assume that $K_i \cap K_\infty = \emptyset$ for all i.

Let $T = B_\gamma(y) \setminus (\text{int}\, B_t(y))$ a ring delimited by $\partial B_\gamma(y)$ and $\partial B_t(y)$, $a_i \in K_i \cap \partial B_\gamma(y)$ and L_i the connected component of a_i in $T \cap K_i$. Since each K_i is connected and $K_i \cap B_t(y) \neq \emptyset$ there is $b_i \in L_i \cap \partial B_t(y)$. Since $K_i \cap K_j = \emptyset$, then $L_i \cap L_j = \emptyset$, $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Since $K_i \to K_\infty$, then (taking subsequences if necessary) $L_i \to L_\infty \in \mathcal{C}(X)$,

$a_i \to a_\infty \in \partial B_\gamma(y)$ and $b_i \to b_\infty \in \partial B_t(y)$ as $i \to \infty$. Since $K_i \cap K_\infty = \emptyset$ and $L_i \subset K_i$ then $L_\infty \subset K_\infty$ and we have $L_i \cap L_\infty = \emptyset$, $a_i \neq a_\infty$ and $b_i \neq b_\infty$ for each $i \in \mathbb{N}$.

We can choose $a_i a_\infty$ arcs in $\partial B_\gamma(y)$ jointing a_i and a_∞ (taking subsequences if necessary) such that

$$a_1 a_\infty \supsetneq a_2 a_\infty \supsetneq \ldots \supsetneq a_i a_\infty \supsetneq \ldots$$

and $b_i b_\infty$ in $\partial B_t(y)$ such that

$$b_1 b_\infty \supsetneq b_2 b_\infty \supsetneq \ldots \supsetneq b_i b_\infty \supsetneq \ldots.$$

Since $a_i \to a_\infty$ and $b_i \to b_\infty$, we have $\mathrm{d}(a_i, a_\infty) \to 0$ and $\mathrm{d}(b_i, b_\infty) \to 0$ as $i \to \infty$.

Since L_i is connected, for each $i \in \mathbb{N}$ we can take

$$z_i \in L_i \cap \partial B_{t+(\gamma-t)/2}(y),$$

then $z_i \in C^s_\varepsilon(x)$ and by expansiveness and the choice of ε we have $C^s_\varepsilon(x) \cap C^u_\varepsilon(z_i) = \{z_i\}$. Hence

$$(C^u_\varepsilon(z_i) \cup L_i) \cap L_{i-1} = \emptyset \quad \text{and} \quad (C^u_\varepsilon(z_i) \cup L_i) \cap L_{i+1} = \emptyset. \tag{3.8}$$

By (3.8) we can take N_{i-1} and N_{i+1} connected neighborhoods of L_{i-1} and L_{i+1} in T respectively with $N_{i-1} \cap N_{i+1} = \emptyset$, $N_{i-1} \cap (C^u_\varepsilon(z_i) \cup L_i) = \emptyset$ and $N_{i+1} \cap (C^u_\varepsilon(z_i) \cup L_i) = \emptyset$. Then there are arcs $A_{i-1} \subset N_{i-1}$ and $A_{i+1} \subset N_{i+1}$ with

$$A_{i-1} \cap (\partial B_t(y) \cup \partial B_\gamma(y)) = \{b_{i-1}, a_{i-1}\}$$

and

$$A_{i+1} \cap (\partial B_t(y) \cup \partial B_\gamma(y)) = \{b_{i+1}, a_{i+1}\}$$

such that A_{i-1}, A_{i+1} and $L_i \cup C^u_\varepsilon(z_i)$ are mutually disjoint.

Define

$$\Gamma = A_{i+1} \cup a_{i+1}a_{i-1} \cup A_{i-1} \cup b_{i+1}b_{i-1},$$

where $a_{i+1}a_{i-1}$ and $b_{i+1}b_{i-1}$ denotes the subarc in $a_1 a_\infty$ and $b_1 b_\infty$ jointing a_{i+1} and a_{i-1}, and b_{i+1} and b_{i-1} respectively. Then Γ is a simple closed curve and bounds a disk D in T.

Since $\gamma < \frac{\delta}{2}$ and $C^u_\varepsilon(z_i)$ is connected, Proposition 3.12 ensures that $C^u_\varepsilon(z_i) \cap \partial B_\gamma(y) \neq \emptyset$ and so $C^u_\varepsilon(z_i) \cap \Gamma \neq \emptyset$. Since $C^u_\varepsilon(z_i) \cap (A_{i-1} \cup A_{i+1}) = \emptyset$, then

$$C^u_\varepsilon(z_i) \cap a_{i-1}a_{i+1} \neq \emptyset \quad \text{or} \quad C^u_\varepsilon(z_i) \cap b_{i-1}b_{i+1} \neq \emptyset.$$

Without loss generality we assume that

$$w_i \in C^u_\varepsilon(z_i) \cap a_{i-1}a_{i+1} \neq \emptyset \quad i \geq 2.$$

Since $\mathrm{diam}(a_i a_\infty) \to 0$, we see that $w_i \to a_\infty$ as $i \to \infty$. Since $z_i \in L_i$ and $L_i \to L_\infty$ then z_i converges to some $z_\infty \in L_\infty$ and since $z_i \in \partial B_{t+(\gamma-t)/2}(y)$ for each $i \in \mathbb{N}$, then $z_\infty \in \partial B_{t+(\gamma-t)/2}(y)$. Since $w_i \in C^u_\varepsilon(z_i)$, Lemma 3.6 ensures that $a_\infty \in C^u_\varepsilon(z_\infty)$. Since $a_\infty, z_\infty \in L_\infty \subset C^s_\varepsilon(x)$, and f is expansive, then $a_\infty = z_\infty$, contradicting that $a_\infty \in \partial B_\gamma(y)$. Therefore $C^s_\varepsilon(x)$ is locally connected.

In the same way we obtain for $\sigma = u$. □

3.4 Topological structure of $C^\sigma_\varepsilon(x)$

Let (X,d) be a compact metric space and $f : X \to X$ be an expansive homeomorphism with expansive constant $c > 0$.

Lemma 3.14. *For every* $0 < \varepsilon \leq c$ *there exists* $\delta > 0$ *such that*

$$W^\sigma_\varepsilon(x) \cap B_\delta(x) = W^\sigma(x) \cap B_\delta(x) \qquad \sigma = s, u$$

for all $x \in X$*.*

Proof. See Lemma 5 in [19]. □

Lemma 3.15. *Let* $0 < \varepsilon \leq \frac{c}{2}$ *and let* A *and* B *be non-empty subsets of* X*. If* $W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \neq \emptyset$ *for all* $x \in A$ *and* $y \in B$*, then* $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ *consists of exactly one point* $\alpha(x,y)$ *and* $\alpha : A \times B \to X$ *is a continuous map.*

Proof. Let $0 < \varepsilon \leq \frac{c}{2}$. If $z_1, z_2 \in W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$, then $z_2 \in W^s_c(z_1) \cap W^u_c(z_2)$. Since c is an expansive constant for f, z_1 must be equal z_2. Hence $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ must consist of exactly one point.

To show that $\alpha : A \times B \to X$ is continuous, let (x_i, y_i) a sequence of $A \times B$ converging to some $(x,y) \in A \times B$ and put $z_i = \alpha(x_i, y_i)$. Since X is compact, there is a subsequence z_j of z_i such that z_j converges to some $z_\infty \in X$ as $j \to \infty$. Since $z_j \in W^s_\varepsilon(x_j)$, Lemma 3.6 ensures that

$$z_\infty \in W^s_\varepsilon(x).$$

In the same way, we have

$$z_\infty \in W^u_\varepsilon(x),$$

and therefore $z_\infty = \alpha(x,y)$. This shows that α is continuous. □

From here, let S **be a compact surface and** $f : S \to S$ **be an expansive homeomorphism** with expansive constant $c > 0$.

Fix $x \in S$ and $0 < \varepsilon \leq \frac{c}{2}$.

Lemma 3.16. $C^\sigma_\varepsilon(x)$ *is arcwise connected and locally arcwise connected.*

Proof. From Proposition 3.13 and Theorem 5.9 of [12], it follows that $C^\sigma_\varepsilon(x)$ is a Peano space. Hence, Theorem 6.29 of [12] ensures that $C^\sigma_\varepsilon(x)$ is arcwise connected and locally arcwise connected. □

Lemma 3.17. *For each pair* (y,z) *of distinct points of* $C^\sigma_\varepsilon(x)$ *there exists a unique arc jointing* y *amd* z *in* $C^\sigma_\varepsilon(x)$*.*

Proof. We prove for $\sigma = s$ and the proof for $\sigma = u$ follows the same way.

The previous lemma ensures the existence of arcs in $C^s_\varepsilon(x)$ jointing y and z. It remains to prove the uniqueness of such arc.

Since S is a compact surface, we can take $0 < \varepsilon' \leq \frac{c}{2}$ small enough such that $B_{\varepsilon'}(w)$ is a disk for all $w \in S$. Since f is uniformly continuous, we can take $0 < r \leq \varepsilon'$ such that $f(B_r(w)) \subset B_{\varepsilon'}(f(w))$ for all $w \in S$. By (3.1), there is $N > 0$ such that $f^n(W^s_c(x)) \subset W^s_r(f^n(x))$ for all $n \geq N$.

Arguing by contradiction, suppose that there are two distinct arcs in $C^s_\varepsilon(x)$ jointing y and z. Then we can find a simple closed curve $\Gamma \subset C^s_\varepsilon(x)$. Since $\Gamma \subset C^s_\varepsilon(x) \subset W^s_c(x)$ and $W^s_r(f^n(x)) \subset B_r(f^n(x))$, by the choice of $r > 0$ we have

$$f^n(\Gamma) \subset B_r(f^n(x)) \quad \text{for all } n \geq N.$$

Since $B_r(f^N(x))$ is a disk and $f^N(\Gamma)$ is a simple closed curve, then $f^N(\Gamma)$ bounds a disk D in $B_r(f^N(x))$. By the choice of $r > 0$, we have $f(B_r(f^N(x))) \subset B_{\varepsilon'}(f^{N+1}(x))$ and since $\partial D = f^N(\Gamma)$, then $f(D) \subset B_r(f^{N+1}(x))$. Inductively we obtain

$$f^i(D) \subset B_r(f^{N+i}(x)) \quad \text{for all } i \in \mathbb{N}.$$

This contradicts Lemma 3.11, because $\operatorname{int} D \neq \emptyset$. Hence there is a unique arc in $C^s_\varepsilon(x)$ jointing y and z. ☐

Let y and z distinct points of $C^\sigma_\varepsilon(x)$. We denote by $\sigma(y, z; x, \varepsilon)$ the unique arc in $C^\sigma_\varepsilon(x)$ jointing y and z. Since $C^\sigma_\varepsilon(x) \subset C^\sigma_{\frac{c}{2}}$, then $\sigma(y, z; x, \varepsilon) = \sigma(y, z; x, \frac{c}{2})$. Then we can simply omit ε from the notation and write

$$\sigma(y, z; x) = \sigma(y, z; x, \varepsilon).$$

We denote by $IC^\sigma_\varepsilon(x)$ the union of all open subarcs of $C^\sigma_\varepsilon(x)$ and define

$$BC^\sigma_\varepsilon(x) = C^\sigma_\varepsilon(x) \setminus (IC^\sigma_\varepsilon(x) \cup \{x\})$$

Lemma 3.18. *$BC^\sigma_\varepsilon(x) \neq \emptyset$ and*

$$C^\sigma_\varepsilon(x) = \bigcup_{b \in BC^\sigma_\varepsilon(x)} \sigma(x, b; x)$$

Proof. By Proposition 3.12, we have $C^\sigma_\varepsilon(x) \supsetneq \{x\}$, then fix another point $y \in C^\sigma_\varepsilon(x)$ and put

$$\mathcal{S} = \{\sigma(x, z; x) : \sigma(x, y; x) \subset \sigma(x, z; x)\}.$$

$\mathcal{S}$ is a partially ordered set with respect to inclusion, then Corollary 2.7 ensures that there is $\mathcal{S}_0$ a totally ordered subset such that each element of $\mathcal{S} \setminus \mathcal{S}_0$ is not an upper bound of $\mathcal{S}_0$.

Let L be the union of all elements of $\mathcal{S}_0$. Then $y \in L$. It is sufficient to show that $L = \sigma(x, b; x)$ for some $b \in C^\sigma_\varepsilon(x)$. Indeed, since $\mathcal{S}_0$ contains all its upper bounds, we have

$b \in BC^\sigma_\varepsilon(x)$ and so $BC^\sigma_\varepsilon(x) \neq \emptyset$. Since $y \in L$ and y is chosen arbitrarily, by Lemma 3.17 we have $C^\sigma_\varepsilon(x) = \bigcup_{b \in BC^\sigma_\varepsilon(x)} \sigma(x, b; x)$.

Let $\mathcal{U}$ be the collection of all injective maps from $[0,1)$ to $C^\sigma_\varepsilon(x)$ and define

$$\mathcal{U}_L = \{\alpha \in \mathcal{U} : \alpha(0) = x,\ \alpha([0,1)) \subset L\}.$$

Claim 1. *For all $\alpha \in \mathcal{U}_L$, there exists $\sigma(x, z; x) \in \mathcal{S}_0$ such that $\alpha([0,1)) \subset \sigma(x, z; x)$.*

Proof of Claim 1. Arguing by contradiction, suppose the existence of $\alpha_\infty \in \mathcal{U}_L$ such that for all $\sigma(x, z; x) \in \mathcal{S}_0$ there is $t \in [0,1)$ with $\alpha_\infty(t) \notin \sigma(x, z; x)$.

On the other hand, since $\alpha_\infty([0,1)) \subset L$, then $\alpha_\infty(t) \in L$ and hence there exists $\sigma(x, w; x) \in \mathcal{S}_0$ such that $\alpha_\infty(t) \in \sigma(x, w; x)$.

Since $\mathcal{S}_0$ is totally ordered, we have $\sigma(x, z; x) \subset \sigma(x, w; x)$. Furthermore, since $\alpha_\infty(0) = x$, then $\sigma(x, z; x) \subset \alpha_\infty([0, t))$ and hence $\sigma(x, z; x) \subset \alpha_\infty([0,1))$. By this fact and since $\sigma(x, z; x)$ is chosen arbitrarily we have $\alpha_\infty([0,1)) = L$. Hence there exists a sequence z_i such that

$$\sigma(x, z_i; x) \subset \sigma(x, z_{i+1}; x) \quad \text{for all } i \in \mathbb{N}$$

and

$$L = \bigcup_{i \in \mathbb{N}} \sigma(x, z_i; x).$$

By compactness there is z_j subsequence of z_i with $z_j \to z_\infty \in C^\sigma_\varepsilon(x)$.

Let $J = \sigma(x, z_\infty; x) \cap L$ if $z_\infty \neq x$ or $J = \{x\}$ if $z_\infty = x$. It is clear that $J \subsetneq L$, because if not, $z_\infty \neq x$ and $J = L$. Then $L \subset \sigma(x, z_\infty; x)$ and since $L = \alpha([0,1))$, we have $L \subsetneq \sigma(x, z_\infty; x)$, contradicting that L is the union of all elements of $\mathcal{S}_0$. Hence $J \subsetneq L$. By Lemma 3.17, J must be an arc or a singleton set and hence $J \subsetneq \sigma(x, z_l; x)$ for some $l \in \mathbb{N}$.

Since $\sigma(x, z_j; x) \supsetneq \sigma(x, z_l; x)$ for $j > l$, then $\sigma(z_\infty, z_j; x) \supsetneq \sigma(z_\infty, z_l; x)$ for $j > l$, and hence

$$\operatorname{diam}(\sigma(z_\infty, z_j; x)) \geq \operatorname{diam}(\sigma(z_\infty, z_l; x)).$$

Since $z_j \to z_\infty$, this contradicts that $C^s_\varepsilon(x)$ is arcwise locally connected and proves the claim.

Since $L \subset C^\sigma_\varepsilon(x)$, we can take a countable subset G of L such that the closure $\bar{G}$ of G in $C^\sigma_\varepsilon(x)$ contains L and construct a map $\alpha \in \mathcal{U}_L$ with $G \subset \alpha([0,1))$. Claim 1 ensures the existence of $\sigma(x, b; x) \in \mathcal{S}_0$ such that $\alpha([0,1)) \subset \sigma(x, b; x)$ and then $G \subset \sigma(x, b; x)$. Since $L \subset \bar{G}$ and $\sigma(x, b; x) \subset L$ and by the definition of L we have that $L = \sigma(x, b; x)$, finishing the proof. □

Lemma 3.19. *Let A be an arc in $C^\sigma_\varepsilon(x)$. If x is an end point of A, then there exists $b \in BC^\sigma_\varepsilon(x)$ such that $A \subset \sigma(x, b; x)$.*

Proof. Let $y \in A$ be another end point. Since $y \in C^s_\varepsilon(x)$, by Lemma 3.18 there exists $b \in BC^\sigma_\varepsilon(x)$ such that $y \in \sigma(x, b; x)$. By Lemma 3.17, there exists an unique arc in $C^s_\varepsilon(x)$ jointing x and y, then we have $A \subset \sigma(x, b; x)$. □

Let a, b and c points of $C^\sigma_\varepsilon(x)$ such that $a \neq b$ and $a \neq c$. We say that $\sigma(a,b;x) \sim \sigma(a,c;x)$ if $\sigma(a,b;x) \cap \sigma(a,c;x) \supsetneq \{x\}$. It is easy to see that $\sim$ is an equivalence relation.

Since $\sim$ is an equivalence relation we can define

$$P^\sigma_\varepsilon(x) = \#(\{\sigma(x,b;x) : \ b \in BC^\sigma_\varepsilon(x)\}/\sim).$$

Lemma 3.20. $P^\sigma_\varepsilon(x) = P^\sigma_{\frac{c}{2}}$ *for* $0 < \varepsilon \leq \frac{c}{2}$

Proof. By Lemma 3.19 we can find δ_1 and δ_2 such that $W^\sigma_\varepsilon(x) \cap B_{\delta_1}(x) = W^\sigma(x) \cap B_{\delta_1}(x)$ and $W^\sigma_{\frac{c}{2}}(x) \cap B_{\delta_2}(x) = W^\sigma(x) \cap B_{\delta_2}(x)$. Then we can find $\delta > 0$ such that

$$W^\sigma_\varepsilon(x) \cap B_\delta(x) = W^\sigma_{\frac{c}{2}} \cap B_\delta(x).$$

Let C be the connected component of x in $W^\sigma_\varepsilon(x) \cap B_\delta(x)$. Since C is connected, $C \subset C^\sigma_\varepsilon(x) \cap B_\delta(x)$ and hence C is the connnected component of x in $C^\sigma_\varepsilon(x) \cap B_\delta(x)$. Since $W^\sigma_\varepsilon(x) \cap B_\delta(x) = W^\sigma_{\frac{c}{2}} \cap B_\delta(x)$ then C is the connected component of x in $C^\sigma_{\frac{c}{2}} \cap B_\delta(x)$.

Lemma 3.19 ensures that if A is an arc in $C^s_\varepsilon(x)$ jointing x to some point $y \in C^s_\varepsilon(x) \subset C^s_{\frac{c}{2}}(x)$, then there exists $b \in BC^\sigma_{\frac{c}{2}}(x)$ such that $A \subset \sigma(x,b;x)$. Since the connected component of x coincides in $C^\sigma_\varepsilon \cap B_\delta(x)$ and $C^\sigma_{\frac{c}{2}} \cap B_\delta(x)$, we have $P^\sigma_\varepsilon(x) = P^\sigma_{\frac{c}{2}}(x)$. ☐

Since $P^\sigma_\varepsilon(x)$ is independent of $0 < \varepsilon \leq \frac{c}{2}$, we omit ε and write

$$P^\sigma(x) = P^\sigma_\varepsilon(x).$$

Definition 3.21. $\mathrm{Sing}^\sigma(f) = \{x \in S : \ P^\sigma(x) \geq 3\}$. *We call this set the set of singularities.*

Lemma 3.22. $\mathrm{Sing}^\sigma(f)$ *is a finite set* $(\sigma = s, u)$.

Proof. We give the proof for $\sigma = s$ and in the same way we obtain for $\sigma = u$.

Let $0 < \varepsilon \leq \frac{c}{6}$ and $0 < \delta \leq \varepsilon$ given by Proposition 3.12.

Let Δ be the set of the points $x \in S$ such that there exists a_1, a_2, a_3 distinct points of $C^s_\varepsilon(x)$ with

$$(x, a_i; x) \not\sim s(x, a_j; x) \quad \text{for } i \neq j$$

and

$$s(x, a_i; x) \cap \partial B_\delta(x) \neq \emptyset \quad i = 1, 2, 3.$$

It is clear by Lemma 3.19 that $\Delta \subset \mathrm{Sing}(f)$, since each a_i is associated with distincts $b_i \in BC^s_\varepsilon(x)$.

Claim 1. $\#\Delta \geq \#\,\mathrm{Sing}^s(f)$

Proof of Claim 1. Let $x \in \mathrm{Sing}^s(f)$, then $P^s(x) \geq 3$. First we show the existence of $m_0 > 0$ such that $f^{-i}(x) \in \Delta$ for all $i \geq m_0$. For that, for each $i > m_0$ we show the existence of three subarcs of $C^s_\varepsilon(x)$ containing x under hypothesis of Lemma 3.9 (2), then we conclude that these arcs intersects $\partial B_\delta(f^{-i}(x))$, which implies that $f^{-i}(x) \in \Delta$.

Since $P^s(x) \geq 3$ there are $a_k \in C^s_\varepsilon(x)$ $(k = 1, 2, 3)$ such that $s(x, a_k; x) \not\sim s(x, a_l; x)$ for $k \neq l$ and $s(x, a_k; x) \subset B_\delta(x)$. By expansiveness, for each $k = 1, 2, 3$ we can find $m_k > 0$ such that

$$f^{-i}((s(x, a_k; x)) \subset B_\varepsilon(x) \quad \text{for } 0 \leq i \leq m_k$$

and

$$f^{-m_k}(s(x, a_k; x)) \cap \partial B_\varepsilon(x) \neq \emptyset.$$

Let $A^k(m_k)$ be the connected component of $f^{-m_k}(x)$ in $f^{-m_k}(s(x, a_k; x)) \cap B_\varepsilon(f^{-m_k}(x))$. Then $A^k(m_k)$ is an arc of $C^s_\varepsilon(f^{-m_k}(x))$, $f^{-m_k}(x)$ is an end point and $A^k(m_k) \cap B_\varepsilon(f^{-m_k}(x)) \neq \emptyset$. By the choice of a_k we have $f^{m_k}(A^k(m_k)) \subset B_\delta(x)$.

For $i \geq m_k$, let $A^k(i)$ be the connected component of $f^{-i}(x)$ in $f^{-1}(A^k(i-1)) \cap B_\varepsilon(f^{-i}(x))$. Then $A^k(i)$ is an arc in $C^s_\varepsilon(f^{-i}(x))$ and $f^{-i}(x)$ is an end point. Furthermore, since $s(x, a_k; x) \not\sim s(x, a_l; x)$, we have $A^k(i) \not\sim A^l(i)$ for $k \neq l$.

Since $A^k(m_k) \cap B_\varepsilon(f^{-m_k}(x)) \neq \emptyset$, there exists $m_k \leq j \leq i$ such that $f^{i-j}(A^k(i)) \cap B_\varepsilon(f^{-j}(x)) \neq \emptyset$. Furthermore $f^i(A^k(i)) \subset f^{m_k}(A^k(m_k)) \subset B_\delta(x)$. Then $f^i(A^k(i))$ satisfies the hypothesis of Lemma 3.9 (2). Hence $A^k(i) \cap B_\delta(f^{-i}(x)) \neq \emptyset$ for $k = 1, 2, 3$ and $i \geq m_0$, where $m_0 = \max(m_1, m_2, m_3)$ and we conclude that $f^{-i}(x) \in \Delta$ for $i \geq m_0$.

To prove the claim we define an injection from $\mathrm{Sing}^s(f)$ to Δ as follows. Consider

$$\mathcal{S} = \bigcup_{x \in \mathrm{Sing}^s(f)} \mathcal{O}_f(x),$$

where $\mathcal{O}_f(x)$ denotes the orbit of x by f. Clearly $\mathrm{Sing}^s(f) \subset \mathcal{S}$. Then we define $\phi : \mathcal{S} \to \Delta$ by

$$\phi(f^i(x)) = f^i(x) \quad (i \in \mathbb{Z}) \quad \text{if } x \in \mathrm{Per}(f);$$

$$\phi(f^i(x)) = \begin{cases} f^{-m_0-2i}(x) & (i > 0) \\ f^{-m_0}(x) & (i = 0) \quad \text{if } x \notin \mathrm{Per}(f) \\ f^{-m_0+2i+1}(x) & (i < 0) \end{cases} .$$

By the choice of m_0 we have that ϕ is well defined and is an injection. Since $\mathrm{Sing}(f) \subset \mathcal{S}$ we induce an injection from $\mathrm{Sing}(f)$ to Δ. Hence $\#\Delta \geq \#\,\mathrm{Sing}(f)$ and Claim 1 is proven.

To prove the lemma we argue by contradiction. Suppose that $\mathrm{Sing}^s(f)$ is an infinite set. By the previous claim we obtain that Δ is an infinite set. Hence by compactness we can find $p \in S$ such that $\Delta \cap \operatorname{int} B_{\frac{\delta}{4}}(p)$ is infinite.

Denote by Δ_0 the subset of $\Delta \cap \operatorname{int} B_{\frac{\delta}{4}}(p)$ with the property that

$$C^s_\varepsilon(y) \cap C^s_\varepsilon(z) = \emptyset \quad \text{if } y, z \in \Delta_0$$

and

$$C^s_\varepsilon(y) \cap C^s_\varepsilon(z) \neq \emptyset \quad \text{if } y \in \Delta_0 \text{ and } z \in (\Delta \cap \operatorname{int} B_{\frac{\delta}{4}}(p)) \setminus \Delta_0.$$

We have two possible cases, Δ_0 is infinite or Δ_0 is finite. In both cases we arrive at a contradiction.

If Δ_0 is infinite. Then we can find a sequence x_i in Δ_0 with $x_i \neq x_j$ for $i \neq j$ converging to some $x_\infty \in \operatorname{int} B_{\frac{\delta}{4}}$.

By the definition of Δ and since $\Delta_0 \subset \Delta$, for each $x_i \in \Delta_0$ there exists distinct points $a^i_1, a^i_2, a^i_3 \in C^s_\varepsilon(x_i) \cap \partial B_{\frac{\delta}{2}}(p)$. Hence $\{a^i_k\}_{k=1}^3$ cuts $\partial B_{\frac{\delta}{2}}(p)$ in three open arcs.

Let x_i and x_j distinct points of Δ_0. Then $\{a^i_k\}_{k=1}^3$ cuts $\partial B_{\frac{\delta}{2}}(p)$ in three open arcs and by the definition of Δ_0 we have $\{a^j_k\}_{k=1}^3$ contained in the same arc of $(\partial B_{\frac{\delta}{2}}(p)) \setminus \{a^i_k\}_{k=1}^3$.

Let I_k be the open arcs for $\{a^1_k\}_{k=1}^3$ $(k = 1, 2, 3)$. By the above result we have

$$\{a^i_k\}_{k=1}^3 \subset I_{k(i)} \quad \text{for all } i > 1,$$

where $k(i) = 1, 2$ or 3.

For each $i > 1$, let $A_i \subset I_{k(i)}$ be the minimal arc jointing $\{a^i_k\}_{k=1}^3$. We may assume that a^i_1 and a^i_3 are end points of A_i. Then $\{a^i_k\}_{k=1}^3$ cuts A_i in two open arcs J^1_i and J^2_i.

For $i \neq j$ we have two possibilities, $A_i \cap A_j = \emptyset$ or $A_i \cap A_j \neq \emptyset$. If $A_i \cap A_j \neq \emptyset$, then $A_i, A_j \subset I_{k(i)}$. By the definition of Δ_0 we see that $A_j \subset A_i$ or $A_i \subset A_j$. If $A_j \subset A_i$ then $\{a^j_k\}_{k=1}^3 \subset J^1_i$ or $\{a^j_k\}_{k=1}^3 \subset J^2_i$.

Since Δ_0 is an infinite set, by the above result we have a sequence $\{A_{i_l}\}_{l\in\mathbb{N}}$ such that one of these cases holds.

1. $A_{i_l} \cap A_{i_m} = \emptyset \quad$ for $l \neq m$
2. $A_{i_{l+1}} \subset J^1_{i_l} \quad$ or $\quad A_{i_{l+1}} \subset J^2_{i_l} \quad$ for $l \in \mathbb{N}$.

Since a^i_1 and a^i_3 are end points of A_i, we can write $A_i = a^i_1 a^i_3$, where $a^i_l a^i_k$ denotes the minimal arc of A_i jointing a^i_l and a^i_k. In this way, the interior of $a^i_1 a^i_2$ and $a^i_2 a^i_3$ denotes J^i_1 and J^i_2, respectively.

Using these notations and taking subsequences if necessary, without loss of generality we may assume $A_{i_{l+1}} \subset J^1_{i_l}$ and rewrite the above cases as

1. $a^i_1 a^i_3 \cap a^j_1 a^j_3 = \emptyset \quad$ for $i \neq j$
2. $a^i_1 a^i_2 \supset a^{i+1}_1 a^{i+1}_3 \quad$ for $i \in \mathbb{N}$.

Case 1. Since $x_i \in B_{\frac{\delta}{4}}(p)$ and $a^i_2 \in B_{\frac{\delta}{2}}(p)$, there exists $z_i \in B_{\frac{3\delta}{8}}(p) \cap (s(x_i, a^i_2; x_i))$.

It is clear that $a^i_1 a^i_3 \cup s(x_i, a^i_1; x_1) \cup s(x_i, a^i_3; x_i)$ is the boundary of a disk D contained in $B_{\frac{\delta}{2}}(p)$. By Proposition 3.12 we have that $C^u_\varepsilon(z_i)$ intersects the boundary of D and by expansiveness this intersection must be contained in $a^i_1 a^i_3$. Then we can take $w_i \in C^u_\varepsilon(z_i) \cap a^i_1 a^i_3$.

By taking subsequences if necessary and by Lemma 3.5 we may assume that z_i, a^i_2 and $s(x_i, a^i_2; x_i)$ converges to $z_\infty \in \partial B_{\frac{3\delta}{8}}(p)$, $a_\infty \in \partial B_{\frac{\delta}{2}}(p)$, and $\Delta_\infty \in \mathcal{C}(B_{\frac{\delta}{4}})(p)$ respectively as $i \to \infty$. By Lemma 3.6 we have $a_\infty, z_\infty \in \Delta_\infty$ and $\Delta_\infty \subset C^s_\varepsilon(x_\infty)$. Hence $z_\infty \in C^s_\varepsilon(x_\infty)$.

Furthermore, since $\partial B_{\frac{\delta}{2}}(p)$ is compact, then $\mathrm{diam}(a_1^i a_3^i) \to 0$, which implies that w_i converges to a_∞. Since $w_i \in C^u_\varepsilon(z_i)$ for all $i \in \mathbb{N}$ and $z_i \to z_\infty$, by Lemma 3.6 we have that $a_\infty \in C^u_\varepsilon(z_\infty)$. Since $a_\infty, z_\infty \in C^s_\varepsilon(x_\infty)$, by expansiveness $a_\infty = z_\infty$, contradicting that $a_\infty \in \partial B_{\frac{\delta}{2}}(p)$ and $z_\infty \in \partial B_{\frac{3\delta}{8}}(p)$.

Case 2**.** Let $T = B_{\frac{\delta}{2}}(p) \setminus (\mathrm{int}\, B_{\frac{\delta}{2}}(p))$. Since δ is small, T is an annulus bounded by $\partial B_{\frac{\delta}{2}}(p)$ and $\partial B_{\frac{\delta}{2}}(p)$.

Since $x_i \in B_{\frac{\delta}{4}}(p)$ and $a_k^i \in \partial B_{\frac{\delta}{2}}(p)$, we can find

$$b_k^i \in \partial B_{\frac{\delta}{4}}(p) \cap s(x_i, a_k^i; x_i)$$

such that $s(b_k^i, a_k^i; x_i) \subset T$. By Lemma 3.5 we have $a_\infty \in \partial B_{\frac{\delta}{2}}(p)$, $b_\infty \in B_{\frac{\delta}{4}}(p)$ and $\Delta_\infty \in \mathcal{C}(T)$ such that a_2^i, b_2^i and $s(b_k^i, a_k^i; x_i)$ converges to, respectively. It is clear that $a_\infty, b_\infty \in \Delta_\infty$ and since $x_i \to x\infty$, by Lemma 3.6 we have $\Delta_\infty \subset W^s_\varepsilon(x_\infty)$.

By connectedness of $s(b_k^i, a_k^i; x_i)$, for each $i \in \mathbb{N}$ we find $z_i \in s(b_k^i, a_k^i; x_i) \cap \partial B_{\frac{3\delta}{8}}(p)$. Then z_i converges to $z_\infty \in \partial B_{\frac{3\delta}{8}}(p)$, since $z_i \in s(b_k^i, a_k^i; x_i)$, then $z_\infty \in \Delta_\infty \subset W^s_\varepsilon(x_\infty)$ and by Lemma 3.6 $C^u_\varepsilon(z_i)$ converges to a subset of $W^u_\varepsilon(z_\infty)$.

For $a, a' \in a_1^1 a_3^1$ let aa' denote the minimal arc in $a_1^1 a_3^1$ jointing a and a'. Since $a_1^i a_2^i \supset a_1^{i+1} a_3^{i+1}$, we have $a_\infty a_3^i \supset a$ and $a_\infty \neq a_1^i, a_2^i$. Furthermore, $a_2^i \in a_\infty a_3^i$ for all $i \in N$. Then we have

$$a_\infty a_3^1 \supset a_\infty a_2^1 \supset a_\infty a_3^2 \supset a_\infty a_2^2 \supset \cdots \tag{3.9}$$

Similarly, we obtain that

$$b_\infty b_3^1 \supset b_\infty b_2^1 \supset b_\infty b_3^2 \supset b_\infty b_2^2 \supset \cdots, \tag{3.10}$$

where bb' is defined in the same way as aa', for $b, b' \in \partial B_{\frac{\delta}{4}}(p)$.

Proposition 3.12 ensures that $C^u_\varepsilon(z_i) \cap \partial B_{\frac{\delta}{2}}(p) \neq \emptyset$ for $i \in N$. By expansiveness, (3.9) and (3.10) we obtain that

$$C^u_\varepsilon(z_i) \cap a_2^{i+1} a_3^i \neq \emptyset \qquad \text{or} \qquad C^u_\varepsilon(z_i) \cap b_2^{i+1} b_3^i \neq \emptyset.$$

We deal with the $w_i \in C^u_\varepsilon(z_i) \cap a_2^{i+1} a_3^i \neq \emptyset$ and the other case is obtained in the same way. Since $a_2^i \to a_\infty$ and $\partial B_{\frac{\delta}{2}}(p)$ is compact, then $\mathrm{diam}(a_2^{i+1} a_3^i)$ converges to 0. Hence w_i converges to a_∞.

Since $w_i \in C^u_\varepsilon(z_i)$ for $i \in \mathbb{N}$ and $w_i \to a_\infty$, then $a_\infty \in W^u_\varepsilon(z_\infty)$. Since $z_\infty, a_\infty \in W^u_\varepsilon(x_\infty)$, then $a_\infty = z_\infty$ by expansiveness. This contradicts that $a_\infty \in \partial B_{\frac{\delta}{2}}(p)$ and $z_\infty \in \partial B_{\frac{3\delta}{8}}(p)$, and concludes that Δ_0 is not infinite.

If Δ_0 is finite. Since Δ_0 is finite and Δ is infinite, for $y \in \Delta_0$ we can find a sequence $\{x_i\}$ in $\Delta \setminus \Delta_0$ with $x_i \neq x_j$ for $i \neq j$ such that $C^s_\varepsilon(y) \cap C^s_\varepsilon(x_i) \neq \emptyset$. Then $C^s_\varepsilon(x_i) \subset C^s_{3\varepsilon}(y)$ for all $i \in \mathbb{N}$.

By the choice of Δ there exists $a^i_k \in C^s_\varepsilon(x_i) \cap \partial B_{\frac{\delta}{2}}(p)$ ($i \in \mathbb{N}$ and $k = 1, 2, 3$) with $s(x_i, a^i_m; x_i) \not\sim s(x_i, a^i_n; x_i)$ for $m \neq n$.

Let

$$K = \{a^i_k : i \in \mathbb{N},\ k = 1, 2, 3\}.$$

We claim that K must be infinite. Indeed, if K is finite, then $\{a^i_k\}^3_{k=1} = \{a^j_k\}^3_k = 1$ for some $i \neq j$. Then $s(x_i, a^i_1; y) \cup s(x_j, a^i_1; y)$ and $s(x_i, a^i_2; y) \cup s(x_j, a^i_2; y)$ are distinct arcs in $C^s_{3\varepsilon}(y)$ jointing x_i and x_j, contradicting Lemma 3.17. Hence K must be infinite.

Since K is infinite, there are a subsequence x_l of x_i and a sequence a^l of K with $x_l \neq x_{l'}$ and $a^l \neq a^{l'}$ for $l \neq l'$ converging to $x_\infty \in B_{\frac{\delta}{4}}(p)$ and $a_\infty \in \partial B_{\frac{\delta}{2}}(p)$ respectively. Since $x_l, a^l \in C^s_{3\varepsilon}(y)$ for all l, then $x_\infty, a_\infty \in C^s_{3\varepsilon}(p)$. By Lemma 3.16 $C^s_{3\varepsilon}(p)$ is locally arcwise connected, then we find U and V disjoint arcwise connected neighborhoods of x_∞ and a_∞ in $C^s_{3\varepsilon}(p)$, respectively. Then $x_l, x_{l'} \in U$ and $a^l, a^{l'} \in V$ for l, l' sufficiently large. By arcwise connectedness of U we find an arc $\alpha \subset U \subset C^s_{3\varepsilon}(y)$ jointing x_l and $x_{l'}$. Hence $\alpha \cap V = \emptyset$. On the other hand there are two disjoint arcs in $C^s_{3\varepsilon}(y)$ jointing x_l and a^l, and $x_{l'}$ and $a^{l'}$, and since V is arcwise connected there is an arc $\beta \subset V \subset C^s_{3\varepsilon}(y)$ jointing a^l and $a^{l'}$. The union of these three arcs is an arc in $C^s_{3\varepsilon}(y)$ jointing x_l and $x_{l'}$ intersecting V. This contradicts Lemma 3.17 and proves that $\mathrm{Sing}^\sigma(f)$ is a finite set.

□

Lemma 3.23. *If* $P^\sigma(x) \geq 3$, *then* $x \in \mathrm{Per}(f)$, $\sigma = s, u$.

Proof. We give the proof for $\sigma = s$.

Suppose $P^\sigma(x) \geq 3$ for some $x \in X$. Since $f(W^\sigma_\varepsilon(x)) \subset W^\sigma_\varepsilon(f(x))$ we have $f(C^\sigma_\varepsilon(x)) \subset C^\sigma_\varepsilon(f(x))$ and then $P^\sigma(f(x)) \geq 3$. Inductively $P^\sigma(f^n(x)) \geq 3$ for all $n \in \mathbb{N}$. Since $\mathrm{Sing}(f)$ is finite and f is a homeomorphism, then $x \in \mathrm{Per}(f)$.

In the same way we obtain for $\sigma = u$. □

Lemma 3.24. *For every* $x \in S$, $P^\sigma(x)$ *is finite* ($\sigma = s, u$).

Proof. Fix $0 < \varepsilon \leq \frac{c}{2}$ and let $0 < \delta \leq \varepsilon$ given by 3.7. Suppose by contradiction that $P^\sigma(x)$ is infinite for some $x \in S$. Then $x \in \mathrm{Per}(f)$ by Lemma 3.23. Now we define

$$B = \{b \in BC^\sigma_\varepsilon(x) :\ \sigma(x, b; x) \cap \partial B_\delta(x) \neq \emptyset\}.$$

Since $P^\sigma(x)$ is infinite, Claim 1 of the proof of Lemma 3.22 ensures the existence of an infinite subset B' of B such that $\sigma(x, b_1; x) \nsim \sigma(x, b_2; x)$ for $b_1, b_2 \in B'$ with $b_1 \neq b_2$.

Since $C^\sigma_\varepsilon(x)$ is locally arcwise connected and B' is infinite, we can find an arcwise connected subset U of $C^\sigma_\varepsilon(x)$ with $\operatorname{diam} U < \delta$ containing distinct points b_1, b_2 of B' by taking an arcwise connected neighborhood of an accumulation point of B' in $C^\sigma_\varepsilon(x)$.

Since U is arcwise connected, by Lemma 3.17, we have $\sigma(x, b_1; x) \cup \sigma(x, b_2; x) \subset U$. Since $\sigma(x, b_1; x) \cap \partial B_\delta(x) \neq \emptyset$, we have $\operatorname{diam} U \geq \delta$, contradicting that $\operatorname{diam} U < \delta$ and finishing the proof. □

Let $x \in S$ and $0 < \varepsilon \leq \frac{c}{2}$ and let $y \in C^\sigma_\varepsilon(x) \setminus \{x\}$. We say that y is a branch point of $C^\sigma_\varepsilon(x)$ if there exists a_1 and a_2 distinct points of $BC^\sigma_\varepsilon(x)$ such that $\sigma(x, y; x) = \sigma(x, a_1; x) \cap \sigma(x, a_2; x)$.

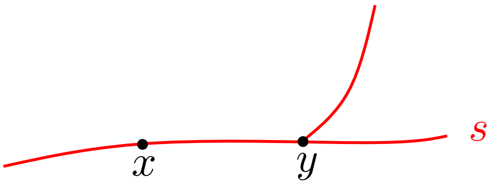


Figure 3 – Branch point y.

Lemma 3.25. *For $x \in S$ and $0 < \varepsilon \leq \frac{c}{4}$, $C^\sigma_\varepsilon(x)$ has at most one branch point ($\sigma = s, u$). If $P^\sigma(x) \geq 3$, then $C^\sigma_\varepsilon(x)$ has no branch points.*

Proof. We give the proof for $\sigma = s$ and for $\sigma = u$ follows the same way.

Let $y \in C^s_\varepsilon(x)$ and assume that y is a branch point. Since $C^s_{2\varepsilon}(y) \supset C^s_\varepsilon(x)$, Lemma 3.19 assures that $P^s(y) \geq 3$. By Lemma 3.23 we have that $y \in \mathrm{Per}(f)$. Therefore all branch points contained in $C^s_\varepsilon(x)$ are periodic.

If z is a branch point of $C^s_\varepsilon(x)$, then z is periodic and a branch point of $C^s_c(y)$. By (3.1), for $\varepsilon_n \to 0$ we have $i_n \to \infty$ such that

$$f^i(C^s_c(y)) \subset C^s_{\varepsilon_n}(y) \quad (i \geq i_n) \quad \text{and} \quad f^{i_n}(z) = z.$$

Since $z \in C^s_c(y)$, then $z \in C^s_{\varepsilon_n}(y)$ and by the above result we easily check that

$$z \in W^s_c(y) \cap W^u_c(y).$$

Since c is an expansive constant we have $y = z$. Therefore $C^s_\varepsilon(x)$ has at most one branch point. The second statement is obtained in the same way. □

Lemma 3.26. *For $x \in S$ and $0 < \varepsilon \leq \frac{c}{4}$, $BC^\sigma_\varepsilon(x)$ is a finite set ($\sigma = s, u$).*

Proof. It follows directly from Lemmas 3.24 and 3.25 □

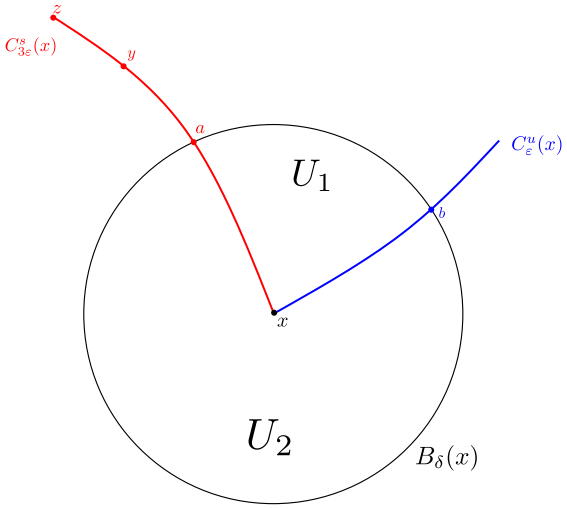
Lemma 3.27. *For every $x \in S$, $P^\sigma(x) \geq 2$.*

Proof. We give the proof for $\sigma = s$ and for $\sigma = u$ is obtained in the same way. Since $BC^s_{\frac{c}{2}}(x) \neq \emptyset$ we have $P^s(x) \geq 1$ for all $x \in S$, then it is sufficient to show that $P^s(x) \neq 1$.

Lemma 3.25 ensures the existence of $0 < 3\varepsilon \leq \frac{c}{4}$ such that $C^s_{3\varepsilon}(x)$ has no branch points, that is, it is an arc.

Arguing by contradiction, suppose $P^s(x) = 1$ for some $x \in S$. Then we find $z \in BC^s_{3\varepsilon}(x)$ such that $C^s_{3\varepsilon}(x) = s(x, z; x)$. Since $C^s_\varepsilon(x) \subset C^s_{3\varepsilon}(x)$ we have $C^s_\varepsilon(x) = s(x, y; x)$ for some $y \in s(x, z; x)$.

Being $0 < 2\delta \leq \varepsilon$ given by Proposition 3.12 we find $a \in C^s_\varepsilon(x) \cap \partial B_\delta(x)$ and $b \in C^u_\varepsilon(x) \cap \partial B_\delta(x)$ such that $s(x,a;x) \setminus \{a\} \subset \operatorname{int} B_\delta(x)$ and $u(x,b;x) \setminus \{b\} \subset \operatorname{int} B_\delta(x)$. By expansiveness, $L = s(x,a;x) \cup u(x,b;x)$ is an arc and cuts $B_\delta(x)$ in two connected components U_1, U_2.



Claim 1. *There are* $q \in s(x,a;x) \setminus \{x,a\}$, $q_1 \in U_1 \cap C^u_\varepsilon(q)$ *and* $q_2 \in U_2 \cap C^u_\varepsilon(q)$ *such that* $u(q,q_i;q) \subset U_i$ *for* $i = 1,2$.

Proof of Claim 1. If we prove the existence of such q, $q_1 \in U_1 \cap C^u_\varepsilon(q)$ and $q_2 \in U_2 \cap C^u_\varepsilon(q)$ such that $u(q_1,q_2;q) \subset B_\delta(x)$, then by expansiveness we have $u(q,q_i;q) \subset U_i$ for $i = 1,2$.

Let $p \in s(x,a;x)$ with $\mathrm{d}(x,q) = \frac{\delta}{2}$. Then $s(x,a;x) \cap C^u_{\frac{\delta}{4}}(p) = \{p\}$ by expansiveness. Furthermore, $u(x,b;x) \cap C^u_{\frac{\delta}{4}}(p) = \emptyset$, because if $w \in u(x,b;x) \cap C^u_{\frac{\delta}{4}}(p)$, since $w \in C^s_\varepsilon(x)$ then $p \in C^u_{2\varepsilon}(x)$, which implies that $p \in C^s_c(x) \cap C^u_c(x)$, contradicting the expansiveness. Then we conclude that $C^u_{\frac{\delta}{4}}(p) \cap L = \emptyset$.

By Proposition 3.12 we have that

$$C^u_{\frac{\delta}{4}}(p) \cap U_1 \neq \emptyset \quad \text{or} \quad C^u_{\frac{\delta}{4}}(p) \cap U_2 \neq \emptyset.$$

Combining this fact with the above result we have three possible cases;

1. $C^u_{\frac{\delta}{4}}(p) \cap U_i \neq \emptyset$ for $i = 1,2$.
2. $C^u_{\frac{\delta}{4}}(p) \cap U_1 \neq \emptyset$ and $C^u_{\frac{\delta}{4}}(p) \cap U_2 = \emptyset$
3. $C^u_{\frac{\delta}{4}}(p) \cap U_1 = \emptyset$ and $C^u_{\frac{\delta}{4}}(p) \cap U_2 \neq \emptyset$

In case 1 let $q = p$ and the result follows from expansiveness.

We give the proof of case 2 and case 3 follows the same way. Let $\{w_l\}_{l \in \mathbb{N}}$ a sequence in U_2 converging to p. Then $C^u_{\frac{\delta}{4}}(w_l)$ converges to some $\Delta_\infty \subset C^u_{\frac{\delta}{4}}(p)$ by Lemma 3.6. Since $C^u_{\frac{\delta}{4}}(p) \cap U_2 = \emptyset$, $w_l \to p$ as $l \to \infty$ and $\operatorname{diam} C^u_{\frac{\delta}{4}}(w_l)$ is bounded away from zero, then

$$\mathrm{d}(p, w_l) \leq \frac{\delta}{8} \quad \text{and} \quad C^u_{\frac{\delta}{4}}(w_l) \cap U_1 \neq \emptyset$$

for sufficiently large $l \in \mathbb{N}$. Then we find

$$q_1 \in U_1 \cap C^u_{\frac{\delta}{4}}(w_l) \quad \text{and} \quad q_2 \in U_2 \cap C^u_{\frac{\delta}{4}}(w_l).$$

Since $\mathrm{d}(p, w_l) \leq \frac{\delta}{8}$ and $\mathrm{d}(p, x) = \frac{\delta}{2}$, we have $C^u_{\frac{\delta}{4}}(w_l) \subset B_\delta(x)$, which implies the existence of $q \in s(x, a; x) \cap C^u_{\frac{\delta}{4}}(w_l)$. Hence $q_1, q_2 \in C^u_\varepsilon(q)$, because $C^u_{\frac{\delta}{4}}(w_l) \subset C^u_\varepsilon(q)$. Since there is an unique arc in $C^u_\varepsilon(q)$ jointing q_1, q_2, this arc must coincide with the arc in $C^u_{\frac{\delta}{4}}(w_l)$, which implies that $u(q_1, q_2; q) \subset B_\delta(x)$. Therefore the claim holds.

Take $q \in s(x, a; x) \setminus \{x, a\}$, $q_i \in C^u_\varepsilon(q)$ $(i = 1, 2)$ as in above claim. By Proposition 3.12 we can find

$$t_1 \in C^s_\varepsilon(q_1) \cap \partial B_\delta(x) \quad \text{and} \quad t_2 \in C^s_\varepsilon(q_2) \cap \partial B_\delta(x)$$

such that

$$s(q_1, t_1; q_1) \setminus \{t_1\} \subset \operatorname{int} B_\delta(x) \quad \text{and} \quad s(q_2, t_2; q_2) \setminus \{t_2\} \subset \operatorname{int} B_\delta(x).$$

By expansiveness we have

$$s(q_1, t_1; q_1) \cap s(q_2, t_2; q_2) = \emptyset,$$

$$s(q_1, t_1; q_1) \cap u(q_1, t_1; q_1) = \{q_1\},$$

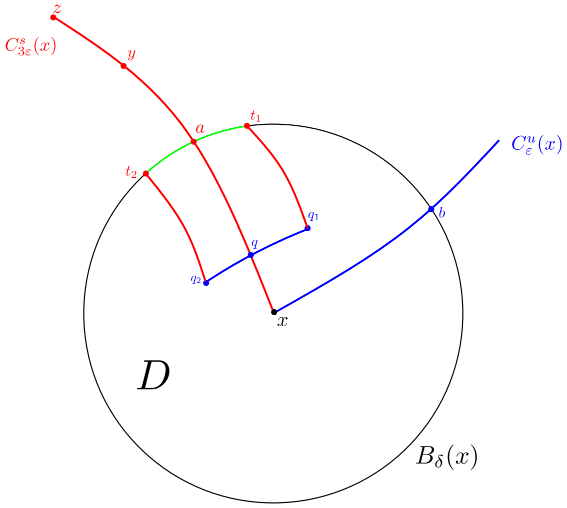
$$s(q_2, t_2; q_2) \cap u(q_2, t_2; q_2) = \{q_2\}.$$

If t_1t_2 denotes an arc in $\partial B_\delta(x)$ jointing t_1 and t_2, we define

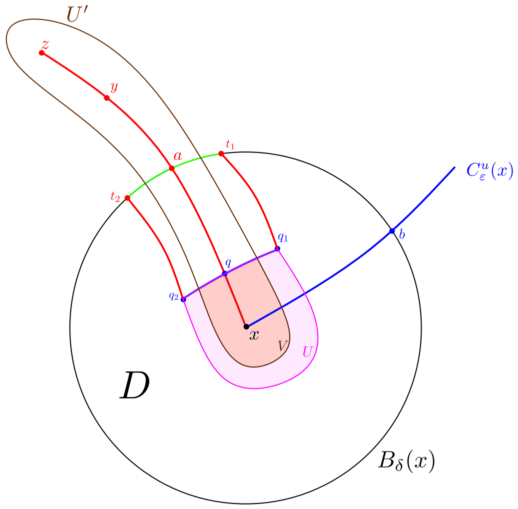
$$\Gamma = t_1t_2 \cup s(q_1, t_1; q_1) \cup u(q_1, q_2; q) \cup s(q_2, t_2; q_2).$$

From the properties of these arcs we have that Γ is a simple closed curve in $B_\delta(x)$.

Let D be the disk in $B_\delta(x)$ bounded by Γ. By expansiveness we have that $s(x, a; x) \cap u(q_1, q_2; x) = \{q\}$, then we may assume that $s(x, q; x) \subset D$ (if necessary we change the arc in $\partial B_\delta(x)$ jointing t_1, t_2.



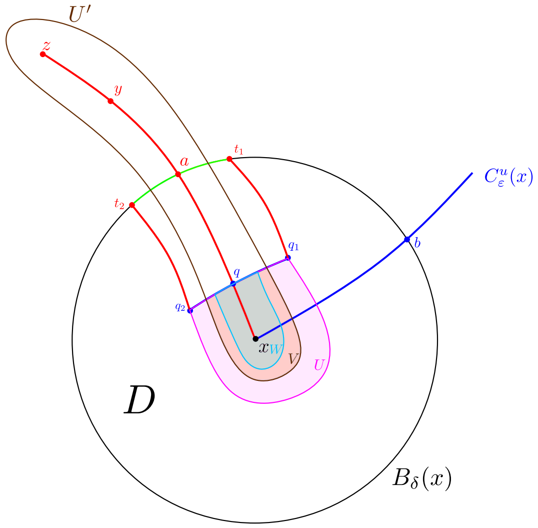
Since $s(x,q;x)\cap\Gamma=\{q\}$, there is a neighborhood U of $s(x,q;x)$ in D such that $U\cap\Gamma\subset u(q_1,q_2;q)$. Furthermore, since $s(x,q;x)\subset s(x,z;x)$, we have $s(x,z;x)\cap u(q_1,q_2;q)=\{q\}$ by expansiveness. Then we find a neighborhood U' of $s(x,z;x)$ in S such that $V\subset U$ where V denotes the connected component of $s(x,q;x)$ in $D\cap U'$. Then $s(x,q;x)=s(x,z;x)$.



For that U' we see that there exists a connected neighborhood W of $s(x,q;x)$ in D such that $W\subset V$ and $C^s_\varepsilon(w)\subset U'$. Indeed, if this is false we can find a sequence $\{w_l\}_{l\in\mathbb{N}}$ in D converging to some $w_\infty\in s(x,q;x)$ such that $C^s_\varepsilon(w_l)\not\subset U'$ for all $l\in\mathbb{N}$. By Lemma 3.5 we have $C^s_\varepsilon(w_l)$ converging to some $\Delta_\infty\in\mathcal{S}$ and $\Delta_\infty\subset C^s_\varepsilon(w_\infty)$ by Lemma 3.6. Since

$$w_\infty\in s(x,q;x)\subset s(x,a;x)\subset s(x,z;x)=C^s_{3\varepsilon}(x),$$

then $\Delta_\infty\subset C^s_\varepsilon(w_\infty)\subset C^s_{3\varepsilon}(x)\subset U'$. On the other hand, since $C^s_\varepsilon(w_l)\not\subset U'$ for all $l\in\mathbb{N}$ and $C^s_\varepsilon(w_l)$ converges to Δ_∞, then Δ_∞ must contain a point that not belongs to U', then $\Delta_\infty\not\subset U'$, leading us to a contradiction. This proves the existence of that W.



By Proposition 3.12 there is $e\in\partial B_\delta(x)\cap C^s_\varepsilon(w)$ for every $w\in W$. Since $W\subset D\subset B_\delta(x)$ and Γ is the boundary of D, then $s(w,e;w)$ intersects Γ. Hence we find $t\in s(w,e;w)\cap\Gamma$

with $s(w,t;w) \subset D$, and since $w \in W$, then $s(w,t;w) \subset U' \cap D$ and by connectedness of $s(w,t;w)$ we have $s(w,t;w) \subset V$. Since $V \subset U$ and $U \cap \Gamma = u(q_1,q_2;q)$, then $t \in u(q_1,q_2;q)$ and therefore $C^s_\varepsilon(w) \cap u(q_1,q_2;q) \neq \emptyset$.

Consider the sets

$$W_i = \{w \in W : C^s_\varepsilon(w) \cap u(q_i,q;q) \neq \emptyset\} \quad i = 1,2.$$

Note that by the above result, we have

$$W \setminus s(x,q;x) = W_1 \cup W_2.$$

It is checked that $W_1 \cap W_2 = \emptyset$. Indeed, if it does not hold, we find $w \in W_1 \cap W_2 \neq \emptyset$. Then $C^s_\varepsilon(w) \cap u(q_1,q_2;q) = \{q\}$ by expansiveness, and then $s(w,q;w) \subset V$. Since $q \in s(x,q;x) \subset C^s_\varepsilon(x)$ and $q \in C^s_\varepsilon(w)$, then $C^s_\varepsilon(w) \subset C^s_{3\varepsilon}(x) = s(x,z;x)$ and hence $s(w,q;w) \subset s(x,q;x)$, which contradicts that $w \in W \setminus s(x,q;x)$. Therefore $W_1 \cap W_2 = \emptyset$.

Now we see that W_i is closed in $W \setminus s(x,q;x)$ for $i = 1,2$. For that, take a sequence $\{w_l\}_{l\in\mathbb{N}}$ in W_i such that w_l converges to some $w_\infty \in W \setminus s(x,q;x)$. Then we find a sequence e_l of $C^s_\varepsilon(w_l) \cap u(q_i,q;q)$ and $e_\infty \in C^s_\varepsilon(w_\infty) \cap u(q_i,q;q)$. By Lemma 3.15 we have $e_l \to t_\infty$ as $l \to \infty$, hence $e_\infty \in u(q_i,q;q)$, which implies that $e_\infty \in W_i$. This shows that W_i is closed for $i = 1,2$.

Since W is connected, so is $W \setminus s(x,q;x)$, and since $W \setminus s(x,q;x) = W_1 \cup W_2$, and $W_1 \cap W_2 = \emptyset$, then $W_1 = \emptyset$ or $W_2 = \emptyset$ by the above results. Without loss of generality, we may assume $W \setminus s(x,q;x) = W_1$. For $w \in u(q_2,q;x) \setminus \{q\}$ we have a sequence w_l of W_1 such that w_l converges to w. Since $w_l \in W_1$, we find a sequence $e_l \in C^s_\varepsilon(w_l) \cap u(q_1,q;q)$. By Lemma 3.15 there is $e_\infty \in u(q_1,q;q)$ such that e_l converges to e_∞. Since $e_l \in C^s_\varepsilon(w_l)$ and $e_\infty(q_1,q;q)$, then $e_\infty \in C^s_\varepsilon(w) \cap u(q_1,q;q)$. By expansiveness, we must have $e_\infty = w$, contradicting the choice of $w \in u(q_2,q;q)\setminus\{q\}$, since $u(q_1,q;q) \cap u(q_2,q;q) = \{q\}$. Therefore $P^s(x) \neq 1$, this concludes the proof.

□

Lemma 3.28. *For every $0 < \varepsilon \leq \frac{c}{4}$ there exists $0 < \delta \leq \varepsilon$ such that*

$$\partial B_\delta(x) \cap \sigma(x,b;x) \neq \emptyset \qquad \sigma = s,u$$

for all $x \in S$ and all $b \in BC^\sigma_\varepsilon(x)$.

Proof. Let $0 < \delta \leq \varepsilon$ given by Lemma 3.14, then

$$W^\sigma_\varepsilon(x) \cap B_\delta(x) = W^\sigma_{2\varepsilon}(x) \cap B_\delta(x).$$

Arguing by contradiction, we assume that $\sigma(x,b;x) \subset \operatorname{int} B_\delta(x)$ for some $x \in S$ and $b \in BC^\sigma_\varepsilon(x)$. Then we can take $\gamma < \delta$ such that

$$\sigma(x,b;x) \subset \operatorname{int} B_\gamma(x).$$

Since $b \in C^\sigma_\varepsilon(x)$, then $C^\sigma_\varepsilon(b) \subset C^\sigma_{2\varepsilon}(x)$ and hence

$$C^\sigma_\varepsilon(b) \cap B_{\delta-\gamma}(b) \subset C^\sigma_{2\varepsilon}(x) \cap B_{\delta-\gamma}(b).$$

Then the connected component of b in $C^\sigma_\varepsilon(b) \cap B_{\delta-\gamma}(b)$ must be contained in the connected component of x in $C^\sigma_{2\varepsilon}(x) \cap B_\delta(x)$, since $B_{\delta-\gamma}(b) \subset B_\delta(x)$. Since $W^\sigma_\varepsilon(x) \cap B_\delta(x) = W^\sigma_{2\varepsilon}(x) \cap B_\delta(x)$, then the connected component of x in $C^\sigma_{2\varepsilon}(x) \cap B_\delta(x)$ coincides with that of x in $C^\sigma_\varepsilon(x) \cap B_\delta(x)$, hence the connected component of b in $C^\sigma_\varepsilon(b) \cap B_{\delta-\gamma}(b)$ is contained in that x in $C^\sigma_\varepsilon(x) \cap B_\delta(x)$. Therefore $P^\sigma(b) = 1$ by Lemma 3.17, which contradicts Lemma 3.25 and this completes the proof. □

For $0 < \varepsilon \leq \frac{c}{4}$, let $0 < \delta \leq \varepsilon$ given by Lemma 3.28. Lemma 3.25 ensures the existence of $0 < \varepsilon(x) \leq \frac{\delta}{2}$ such that $C^\sigma_\varepsilon(x) \cap B_{\varepsilon(x)}(x)$ has no branch points ($\sigma = s, u$), then we define

$$\partial B^\sigma_{\varepsilon(x)}(x) = \{a \in \partial B_{\varepsilon(x)} : \ \sigma(x, a; x) \setminus \{a\} \subset \operatorname{int} B_{\varepsilon(x)}(x)\}.$$

Lemma 3.29. *For every $x \in S$, $\#(\partial B^\sigma_{\varepsilon(x)}(x)) = P^\sigma(x) \quad (\sigma = s, u)$.*

Proof. It follows directly from Lemmas 3.19 and 3.28 □

Lemma 3.30. *For every $x \in S$, $\partial B^\sigma_{\delta(x)}(x)$ is a finite set with at least two points ($\sigma = s, u$). Let I^s_i ($1 \leq i \leq l$) be the open arcs in which $\partial B^s_{\varepsilon(x)}(x)$ cut $\partial B_{\varepsilon(x)}(x)$. Then every $y \in \partial B^u_{\varepsilon(x)}(x)$ is contained in some $I^s_i \in \{I^s_i : 1 \leq i \leq l\}$. Choose from $\partial B^u_{\varepsilon(x)}(x)$ another point different from y. Then the point is not contained in the same I^s_i. Exchanging s and u, one has the same result.*

Proof. The first statement follows directly from Lemmas 3.24, 3.27 and 3.29.

Arguing by contradiction, suppose $x \in S$ and distinct points $a, b \in \partial B^u_{\varepsilon(x)}(x) \cap I^s_i$ for some $1 \leq i \leq l$. Then

$$\Gamma = ab \cup u(x, a; x) \cup u(x, b; x)$$

is a simple closed curve and bounds a disk $D \subset B_{\varepsilon(x)}(x)$, where ab denotes the arc in I^s_i jointing a and b.

Let

$$\Sigma = \bigcup_{z \in \partial B^s_{\varepsilon(x)}(x)} s(x, z; x).$$

By expansiveness and the definition of Σ, we have $\Sigma \cap \Gamma = \{x\}$ and then $\Sigma \cap (D \setminus \Gamma) = \emptyset$.

For $\frac{\varepsilon}{2}$, let $\gamma > 0$ given by Lemma 3.28. Since $\Sigma \cap (D \setminus \Gamma) = \emptyset$, we can find U neighborhood of Σ in $B_{\varepsilon(x)}(x)$ such that $U \cap D \subset B_{\frac{\gamma}{4}}(x)$ and a sequence x_i with $x_i \in U \cap D$ for all $i \in \mathbb{N}$ and $x_i \to x$ as $i \to \infty$. By Lemma 3.5 we have that

$$C^s_{\frac{\varepsilon(x)}{2}}(x_i) \to \Delta_\infty \in \mathcal{C}(X),$$

where $\Delta_\infty \subset \Sigma$. Hence $C^s_{\frac{\varepsilon}{2}}(x_l) \subset U$ for sufficiently large l.

By Lemma 3.28 we have that $C^s_{\varepsilon(x_l)}(x_l)$ intersects $\partial B_{\frac{\gamma}{2}}(x)$ at at least two points a_1, a_2 with $s(x_l, a_1; x_l) \not\sim s(x_l, a_2; x_l)$. Since $U \cap D \subset B_\gamma(x)$, then

$$s(x_l, a_1; x_l) \cap (u(x, a; x) \cup u(x, a_2; x)) \neq \emptyset$$

and

$$s(x_l, a_2; x_l) \cap (u(x, a; x) \cup u(x, a_2; x)) \neq \emptyset.$$

This contradicts the expansiveness, because $0 < \varepsilon(x) \leq \frac{c}{4}$. □

Lemma 3.31. *$P^s(x) = P^u(x)$ for all $x \in S$.*

Proof. It follows directly from Lemmas 3.29 and 4.10. □

Lemma 3.32. *Let $0 < \varepsilon \leq \frac{c}{8}$. For every $x \in S$ there exists $0 < \eta < \varepsilon(x)$ such that if*

$$y \in B_\eta(x) \setminus \bigcup_{a \in \partial B^\sigma_{\varepsilon(x)}(x)} \sigma(x, a; x) \qquad \sigma = s, u$$

then $C^\sigma_\varepsilon(y)$ is an arc.

Proof. Since $\mathrm{Sing}(f)$ is a finite set and $P^\sigma(x) \geq 2$ for all $x \in X$, we can choose $\eta_0 > 0$ such that $P^\sigma(y) = 2$ for all

$$y \in B_{\eta_0}(x) \setminus \bigcup_{a \in \partial B^\sigma_{\varepsilon(x)}(x)} \sigma(x, a; x).$$

Suppose by contradiction that for all $n \in \mathbb{N}$, there exists $y_n \in B_{\frac{\eta_0}{n}}(x)$ such that $C^\sigma_\varepsilon(y_n)$ is not an arc. Since $P^\sigma(y_n) = 2$ for all $n \in \mathbb{N}$, then there is $z_n \in C^\sigma_\varepsilon(y_n)$ a branch point and then $z_n \in \mathrm{Sing}(f)$. Since $\mathrm{Sing}(f)$ is a finite set, we can assume that $z_n = z$ for all $n \in \mathbb{N}$. By Lemma 3.5 we can assume that $C^\sigma_\varepsilon(y_n) \to \Delta_\infty \in \mathcal{C}(X)$ and since $y_n \to x$ and $z \in C^\sigma_\varepsilon(y_n)$, then $\Delta_\infty \subset C^\sigma_\varepsilon(x)$ and $z \in C^\sigma_\varepsilon(x)$, so $x \in C^\sigma_{2\varepsilon}(z)$.

We have $x \in C^\sigma_{2\varepsilon}(z)$ and $y_n \in C^\sigma_{2\varepsilon}(z)$ for all n. Lemmas 3.18 and 3.26 ensures that $C^\sigma_\varepsilon(z)$ is a finite union of arcs and since $y_n \to x$ we have A_n arcs of $C^\sigma_{2\varepsilon}(z)$ with arbitrarily small diameter with $y_n, x \in A_n$. Since $z \in C^\sigma_\varepsilon(x)$, then $C^\sigma_{2\varepsilon}(z) \subset C^\sigma_{3\varepsilon}(x)$ and so $A_n \subset C^\sigma_{3\varepsilon}(x)$. Since the diameter of A_n is arbitrarily small, Lemma 3.14 ensures that $A_n \subset C^\sigma_\varepsilon(x)$ and then

$$A_n \subset \bigcup_{a \in \partial B^\sigma_{\varepsilon(x)}(x)} \sigma(x, a; x),$$

contradicting the choice of y_n and finishing the proof. □

3.5 Proof of Proposition A

In this section we give the proof for Proposition A.

Recall that for a homeomorphism $f : X \to X$ we put $\mathcal{F}_f^\sigma$ as

$$\mathcal{F}_f^\sigma = \{W^\sigma(x) : x \in X\} \quad (\sigma = s, u).$$

Hereafter let $f : S \to S$ be an expansive homeomorphism of a compact surface S with expansive constant $c > 0$.

Proposition A. *Let $f : S \to S$ be an expansive homeomorphism. Then $\mathcal{F}_f^\sigma$ ($\sigma = s, u$) have the following properties;*

1. *$\mathcal{F}_f^\sigma$ is a C^0 singular foliation,*
2. *every leaf $W^\sigma(x) \in \mathcal{F}_f^\sigma$ is homeomorphic to $L_p = \{z \in \mathbb{C} : \mathrm{Im}(z^{p/2}) = 0\}$ for some $p \geq 2$,*
3. *$\mathcal{F}_f^s$ is transverse to $\mathcal{F}_f^u$,*
4. *$\mathcal{F}_f^\sigma$ is minimal.*

By Lemma 3.31, for each $x \in S$ we have $P^s(x) = P^u(x)$, then we may omit s, u and write $p(x)$ to denote

$$p(x) = P^s(x) = P^u(x).$$

By Lemmas 3.24 and 3.27 we have

$$2 \leq p(x) < \infty \quad \text{for all } x \in S.$$

3.5.1 Construction of U_x

We recall that for $\varepsilon(x) > 0$, $C_\varepsilon^\sigma(x) \cap B_{\varepsilon(x)}(x)$ has no branch points and for $\sigma = s, u$, we put

$$\partial B_{\varepsilon(x)}^\sigma(x) = \{a \in \partial B_{\varepsilon(x)} : \ \sigma(x, a; x) \setminus \{a\} \subset \mathrm{int}\, B_{\varepsilon(x)}(x)\}.$$

Let $x \in S$, $\varepsilon(x) > 0$ and $p(x)$ as above. Then $\partial B_{\varepsilon(x)}^\sigma(x) \subset \partial B_{\varepsilon(x)}(x)$. Furthermore, since $\#(\partial B_{\varepsilon(x)}^\sigma(x)) = p(x)$ by Lemma 3.28 and $2 \leq p(x) < \infty$ by Lemmas 3.24, 3.27, then $\partial B_{\varepsilon(x)}^\sigma(x)$ cuts $\partial B_{\varepsilon(x)}(x)$ in $p(x)$ open arcs I_i^σ) $(1 \leq i \leq p(x))$. By Lemma 4.10, stable and unstable classes are alternating, then we have

$$\partial B_{\varepsilon(x)}^s(x) \subset \bigcup_{i=1}^{p(x)} I_{\varepsilon(x)}^u \quad \text{and} \quad \partial B_{\varepsilon(x)}^u(x) \subset \bigcup_{i=1}^{p(x)} I_{\varepsilon(x)}^s.$$

Moreover, Lemma 4.10 ensures that $\partial B_{\varepsilon(x)}^s(x) \cap I_i^u$ is exactly one point a_i^s and $\partial B_{\varepsilon(x)}^u(x) \cap I_i^s$ is exactly one point a_i^u for $1 \leq i \leq p(x)$.

Since $\#(\partial B^{\sigma}_{\varepsilon(x)}(x)) = p(x)$, we may assume that the boundary of I^s_i and I^u_i in $B_{\varepsilon(x)}(x)$ are $\{a^s_i, a^s_{i+1}\}$ and $\{a^u_{i-1}, a^u_i\}$ respectively, where $a^{\sigma}_{p(x)} = a^{\sigma}_0$ and $a^{\sigma}_{p(x)+1} = a^{\sigma}_1$. Then $\{a^s_i\} \cup I^s_i \cup \{a^s_{i+1}\}$ and $\{a^u_{i-1}\} \cup I^u_i \cup \{a^u_i\}$ are arcs in $\partial B_{\varepsilon(x)}(x)$ and we denote them by $a^s_i a^s_{i+1}$ and $a^u_{i-1} a^u_i$ respectively. It is clear that $a^u_i \in a^s_i a^s_{i+1}$ and $a^s_i \in a^u_{i-1} a^u_i$, then we denote by $a^s_i a^u_i$ the subarc in $a^s_i a^s_{i+1}$ jointing a^s_i and a^u_i. In the same way, we have $a^u_i a^s_{i+1}$.

By the above results we have that

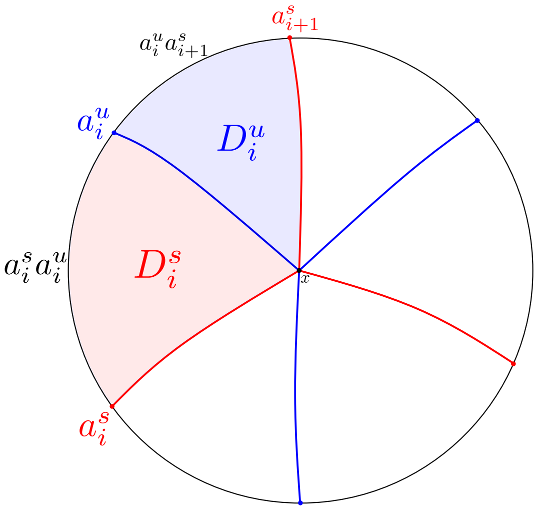
$$\Gamma^s_i = s(x, a^s_i; x) \cup a^s_i a^u_i \cup u(x, a^u_i; x)$$

and

$$\Gamma^u_i = u(x, a^u_i; x) \cup a^u_i a^s_{i+1} \cup s(x, a^s_{i+1}; x)$$

are simple closed curves and then Γ^s_i and Γ^u_i bounds disks D^s_i and D^u_i in $B_{\varepsilon(x)}(x)$, respectively. By the choice of $\varepsilon(x)$ it is clear that

$$D^s_i \cap D^u_i = u(x, a^u_i; x) \quad \text{ and } D^u_i \cap D^s_{i+1} = s(x, a^s_{i+1}; x).$$



By Lemma 3.32 we find $0 < \eta \leq \varepsilon(x)$ such that $C^\sigma_\eta(z)$ is an arc if $\mathrm{d}(x, z) < \eta$, which implies that $p(z) = 2$ for all $z \in B_\eta(x)$. For each $1 \leq i \leq p(x)$, let

$$y_i \in s(x, a^s_i; x) \quad \text{with} \quad \mathrm{d}(y_i, x) < \eta.$$

Then $p(y_i) = 2$ and by Lemma 3.28 we find $c_i(k) \in C^u_\varepsilon(y_i) \cap \partial B_{\varepsilon(x)}(x)$ such that $u(y_i, c_i(k); y_i) \setminus \{c_i(k)\} \subset \operatorname{int} B_{\varepsilon(x)}(x)$ for $k = 1, 2$. Since $y_i \in C^s_\varepsilon(x)$, by expansiveness we see that

$$u(y_i, c_i(k); y_i) \cap (u(x, a^u_{i-1}; x) \cup u(x, a^u_i; x)) = \emptyset \quad (k = 1, 2). \tag{3.11}$$

Combining (3.11) and the fact that $D^u_{i-1} \cup D^s_i$ is a disk bounded by

$$a^u_{i-1} a^s_i \cup a^s_i a^u_i \cup u(x, a^u_{i-1}; x) \cup u(x, a^u_i; x)$$

we have

$$u(y_i, c_i(k); y_i) \subset D^u_{i-1} \cup D^s_i \quad (k = 1, 2).$$

Furthermore, Lemma 4.10 and expansiveness ensures that $c_i(1)$ and $c_i(2)$ do not belong to the same D^u_{i-1} or D^s_i, then we may assume that $c_i(1) \in D^u_{i-1}$ and $c_i(2) \in D^s_i$, which implies that

$$c_i(1) \in a^u_{i-1} a^s_i \quad \text{and} \quad c_i(2) \in a^s_i a^u_i.$$

In the same way, let

$$z_i \in u(x, a^u_i; x) \quad \text{with} \quad \mathrm{d}(x, z_i) < \eta.$$

Hence we find $d_i(k) \in C^s_\varepsilon(z_i) \cap a^s_i a^s_{i+1}$ such that $s(z_i, d_i(k); z_i) \subset \operatorname{int} B_{\varepsilon(x)}(x)$ $(k = 1, 2)$ and

$$d_i(1) \in a^s_i a^u_i \quad \text{and} \quad d_i(2) \in a^u_i a^s_{i+1}.$$

Claim 1. *If* $\mathrm{d}(x, z_i)$ *is sufficietly small, then*

$$s(z_i, c_i(1); z_i) \cap u(y_i, c_1(2); y_i) \neq \emptyset, \tag{3.12}$$

$$s(z_i, c_i(2); z_i) \cap u(y_{i+1}, c_{i+1}(1); y_{i+1}) \neq \emptyset. \tag{3.13}$$

Proof of Claim 1. By the choice of $c_i(2)$ we obtain that $u(y_i; c_i(2); y_1)$ cuts D_i^s in exactly two components $D_i^s(-)$ and $D_i^s(+)$. Denote the subarc of $a_i^s a_i^u$ jointing $a_i^s c_i(2)$ by $a_i^s c_i(2)$, $c_i(2) a_i^u$ is also defined. It is clear that $a_i^s c_i(2)$ and $c_i(2) a_i^u$ are contained in $D_i^s(-)$ or $D_i^s(+)$. We may assume that $a_i^s c_i(2) \subset D_i^s(-)$, then $c_i(2) a_i^u \subset D_i^s(+)$.

By contradiction, suppose a sequence $\{z_i\}_{i \in \mathbb{N}}$ in $u(x, a_i^u; x)$ such that z_i converges to x and $s(z_i, d_i(1); z_1) \cap u(y_i, c_1(2); y_i) = \emptyset$ $(i \in \mathbb{N})$. Hence

$$d_i(1) \in c_i(2) a_i^u$$

and

$$s(z_i, d_i(1); z_i) \subset D_i^s(+) \quad \text{for all } i \in \mathbb{N}.$$

By Lemmas 3.5 and 3.6 we have that $d_i(1)$ converges to some $d_\infty \in c_i(2) a_i^u$ and $s(z_i, d_i(1); z_i)$ converges to some $\Delta_\infty \in \mathcal{C}(D_i^s(+))$ (take subsequences if necessary). Since $z_i \in C_\varepsilon^s(x)$ and $z_i \to x$, then

$$\Delta_\infty \subset C_\varepsilon^s(x),$$

because Δ_∞ is connected.

Since $x, d_\infty \in \Delta_\infty$, then $s(x, d_\infty; x) \subset \Delta_\infty$, hence

$$s(x, d_\infty; x) \subset C_\varepsilon^s(x).$$

Since $\Delta_\infty \subset D_i^s$, then

$$s(x, d_\infty; x) \subset D_i^s.$$

Moreover, $d_\infty \notin s(x, a_i^s; x)$, because $d_\infty \in c_i(2) a_i^u$. Combining these facts we have

$$s(x, a_i^s; x) \subset s(x, d_\infty; x),$$

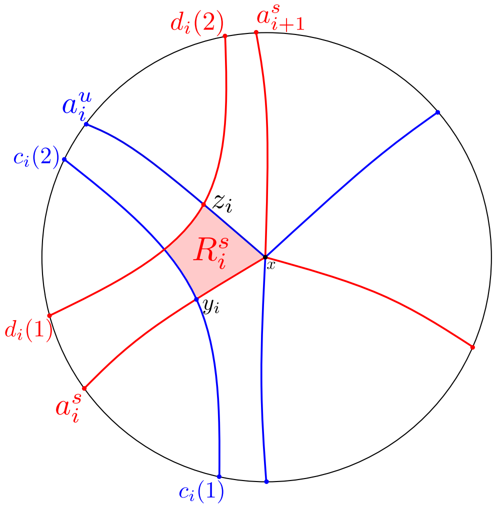
and since $s(x, a_i^s; x) \subset \operatorname{int} B_{\varepsilon(x)}(x)$ and $d_\infty \in c_i(2) a_i^u$, then $s(x, a_i^s; x)$ intersects $u(y_i, c_i(2); y_i)$ in at least two points, and this contradicts the expansiveness.

Therefore $d_i(1) \in (a_i^s c_i(2) \setminus \{c_i(2)\}) \subset D_i^s(-)$ whenever $d(x, z_i)$ is small enough. Since $z_i \in D_i^s(+)$, $s(z_i, d_i(1); z_i)$ is an arc contained in $\operatorname{int} B_{\varepsilon(x)}(x)$, then $s(z_i, c_i(1); z_i) \cap u(y_i, c_1(2); y_i) \neq \emptyset$. In the same way, we obtain (3.13). This proves Claim 1.

For each $1 \leq i \leq p(x)$ fix $z_i \in u(x, a_i^u; x)$ satisfying (3.12) and (3.13). By the choice of $\varepsilon < \frac{c}{8}$, expansiveness ensures that each intersection of (3.12) and (3.13) is exactly one point, $w_i(-)$ and $w_i(+)$ respectively. Then we have that

$$J_i^s = s(x, y_i; x) \cup u(y_i, w_i(-); y_i) \cup s z_i, w_i(-); z_i) \cup u(x, z_i; x)$$

is a simple closed curve in D_i^s. Hence J_i^s bounds a disk R_i^s in D_i^s.



In the same way we obtain that

$$J^u_i = u(x, z_i; x) \cup s(z_i, w_i(+); z_i) \cup u(y_{i+1}, w_i(+); y_{i+1}) \cup s(x, y_{i+1}; x)$$

is a simple closed curve and bounds a disk R^u_i in D^u_i.

By the construction of J^s_i and J^u_i, it is clear that

$$\bigcup_{i=1}^{p(x)} (R^s_i \cup R^u_i)$$

is a closed disk containing x and its boundary is

$$\bigcup_{i=1}^{p(x)} (u(y_i, w_i(-); y_i) \cup s(z_i, w_i(-); z_i) \cup s(z_i, w_i(+); z_i) \cup u(y_{i+1}, w_i(+); y_{i+1})).$$

It follows that

$$U_x = \bigcup_{i=1}^{p(x)} (R^s_i \cup R^u_i) \setminus \partial \bigcup_{i=1}^{p(x)} (R^s_i \cup R^u_i)$$

is an open disk containing x. This is the U_x we want.

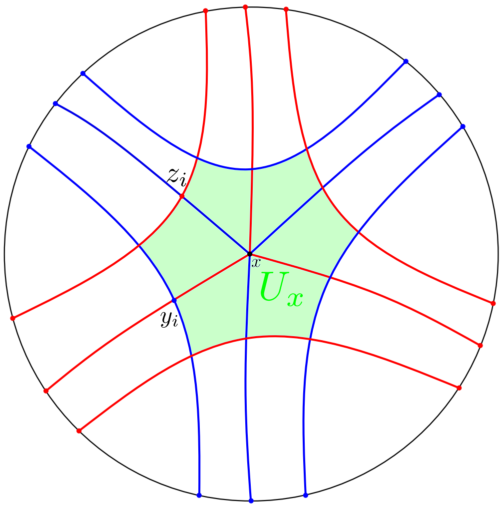


Figure 4 – U_x for x with $P^\sigma(x) = 3$.

3.5.2 Construction of $\varphi_x : U_x \to \mathbb{C}$

Before we construct φ_x, some definitions and observations will be necessary. We will see that for $p \geq 2$ there is a product structure with respect to $\mathcal{H}_p$ and $\mathcal{V}_p$ on special subsets of $\mathcal{D}_p$.

For $p \geq 2$, let $\mathcal{D}_p$, be as in 3.1 and recall that $\mathcal{H}_p$ and $\mathcal{V}_p$ are the lifting by $\pi_1 : \mathcal{D}_p \to \mathcal{D}_1$ of the horizontal and vertical foliations $\mathcal{H}_1$ and $\mathcal{V}_1$ on $\mathcal{D}_1$ respectively.

Let $R_\theta : \mathbb{C} \to \mathbb{C}$ denote the rotation that sends z to $e^{i\theta}z$ and for $1 \leq i \leq p$ we write

$$\mathcal{H}_p^i = R_{2\pi(i-1)/p}([0,1)) \quad \text{and} \quad \mathcal{V}_p^i = R_{\pi/p}(H_p^i),$$

and define

$$L_p^h = \bigcup_{i=1}^p \mathcal{H}_p^i \quad \text{and} \quad L_p^v = \bigcup_{i=1}^p \mathcal{V}_p^i.$$

Then these sets are union of arcs intersecting only at $0 \in \mathbb{C}$. Hence

$$L_p^h \quad \text{and} \quad L_p^v$$

are elements of $\mathcal{H}_p$ and $\mathcal{V}_p$ through $0 \in \mathbb{C}$ respectively.

Remark 3.33. *The letters h and v denote horizontal and vertical respectively.*

We denote by $\mathcal{D}_{p,i}^h$ the closed subset of $\mathcal{D}_p$ such that $\mathcal{H}_p^i$ and $\mathcal{V}_p^i$ are subsets of the boundary of $\mathcal{D}_{p,i}^h$. Similarly we define by $\mathcal{D}_{p,i}^h$ the closed subset of $\mathcal{D}_p$ which $\mathcal{V}_p^i$ and $\mathcal{H}_p^{i+1}$ are subsets of the boundary of $\mathcal{D}_{p,i}^h$.

By the definition of $\mathcal{D}_p$, $\mathcal{D}_{p,i}^h$ and $\mathcal{D}_{p,i}^h$, for $1 \leq i \leq p$ we have the following properties

- $\mathcal{D}_p = \bigcup_{i=1}^p (\mathcal{D}_{p,i}^h \cup \mathcal{D}_{p,i}^v)$,
- $\mathcal{D}_{p,i}^h \cap D_{p,i}^v = \mathcal{V}_p^i$,
- $\mathcal{D}_{p,i}^v \cap \mathcal{D}_{p,i+1}^h = \mathcal{H}_p^{i+1}$.

Let $(z_1, z_2) \in \mathcal{H}_p^i \times \mathcal{V}_p^i$. By the definition of vertical and horizontal foliations $\mathcal{V}_p$ and $\mathcal{H}_p$, the element of $\mathcal{V}_p$ through z_1 intersects the element of $\mathcal{H}_p$ through z_2 at exactly one point $\alpha_i^h(z_1, z_2) \in \mathcal{D}_{p,i}^h$. Similarly, for $(z_1', z_2') \in \mathcal{V}_p^i \times \mathcal{H}_p^{i+1}$ we find a single point $\alpha_i^v(z_1', z_2') \in \mathcal{D}_{p,i}^v$ which is the intersection of the element of $\mathcal{H}_p^{i+1}$ and of $\mathcal{V}_p^i$ through z_1' and z_2' respectively.

Therefore, for $1 \leq i \leq p$ we find homeomorphisms

$$\alpha_i^h : \mathcal{H}_p^i \times \mathcal{V}_p^i \to \mathcal{D}_{p,i}^h$$

and

$$\alpha_i^v : \mathcal{V}_p^i \times \mathcal{H}_p^{i+1} \to \mathcal{D}_{p,i}^v$$

defined as above.

For the construction of $\varphi : U_x \to \mathbb{C}$, we will see that in the construction of U_x we obtain a product structure in each R_i^σ, i.e, if $a, b \in R_i^\sigma$, then $R_i^\sigma \supset C_\varepsilon^s(a) \cap C_\varepsilon^u(b) \neq \emptyset$.

For $x \in S$, let U_x and $y \in s(x, y_i; x)$ as in construction of U_x. By the choice of z_i we have $c_y(1), c_y(2) \in C_\varepsilon^u(y)$ the first points of intersection of $C_\varepsilon^u(y)$ with $s(z_{i-1}, w_{i-1}(+); z_{i-1})$ and $s(z_i, w_i(-); z_i)$ respectively, i.e,

$$u(y, c_y(1); y) \subset R_{i-1}^u \quad \text{and} \quad u(y, c_y(2); y) \subset R_i^s,$$

$$u(y, c_y(1); y) \cap s(z_{i-1}, w_{i-1}(+); z_{i-1}) = \{c_y(1)\},$$
$$u(y, c_y(2); y) \cap s(z_i, w_i(-); z_i) = \{c_y(2)\}.$$

In the same way, for $z \in u(x, z_i; x)$ we find $d_z(k) \in C_\varepsilon^s(z)$ $(k = 1, 2)$ such that

$$s(z, d_z(1); z) \subset R_i^s \quad \text{and} \quad s(z, d_z(2); z) \subset R_i^u,$$

$$s(z, d_z(1); z) \cap u(y_i, w_i(-); y_i) = \{d_z(1)\},$$
$$s(z, d_z(2); z) \cap u(y_{i+1}, w_i(+); y_{i+1}) = \{d_z(2)\}.$$

Since z_i is sufficiently close to x, we see in the construction of U_x and by expansiveness that, for $(y, z) \in s(x, y_i; x) \times u(x, z_i; x)$ we have that $C_\varepsilon^u(y)$ intersects $C_\varepsilon^s(z)$ at a single point $\alpha_i^s(y, z)$, i.e,

$$C_\varepsilon^u(y) \cap C_\varepsilon^s(z) \subset u(y, c_y(2); y) \cap s(z, d_z(1); z) \neq \emptyset.$$

Furthermore, $\alpha_i^s(y, z)$ belongs to R_i^s, because z and $d_z(1)$ are in different connected components of $R_i^s \setminus (u(y, c_y(2); y))$. Therefore

$$\alpha_i^s : s(x, y_i; x) \times u(x, z_i; x) \to R_i^s \quad (1 \leq i \leq p(x))$$

are defined.

Note that $s(x, y_i; x)$ and $u(x, z_i; x)$ satisfies the hypothesis of Lemma 3.15, hence α_i^s is continuous. Moreover, by expansiveness we see that α_i^s is an injective map, because if there are $(y, z) \neq (y', z') \in s(x, y_i; x) \times u(x, z_i; x)$ with $\alpha_i^s((y, z)) = \alpha_i^s((y', z'))$, then $s(y, y'; x)$ intersects $C_\varepsilon^u(\alpha(y, z))$ twice, which contradicts expansiveness. Therefore α_i^s is continuous and injective.

By the α_i^s construction and the fact that it is continuous and injective, we have

$$\alpha_i^s(y, x) = y \quad \text{if} \quad y \in s(x, y_i; x),$$

and

$$\alpha_i^s(x, z) = z \quad \text{if} \quad z \in u(x, z_i; x).$$

Since $\alpha_i^s(y_i, z_i) = w_i(-)$, then

$$\alpha_i^s(s(x, y_i; x) \times \{z_i\} = s(z_i, w_i(-); z_i),$$

and

$$\alpha_i^s(\{y_i\} \times u(x, z_i; x)) = u(y_i, w_i(-); y_i),$$

hence α_i^s maps $s(x, y_i; x) \times u(x, z_i; x)$ onto the boundary of R_i^s. Since R_i^s is a disk and α_i^s is continuous, then α_i^s is surjective. Hence α_i^s is bijective. By the construction of α_i^s it is clear that $(\alpha_i^s)^{-1}$ is continuous. Therefore α_i^s is a homeomorphism.

For $1 \le i \le p(x)$, we write

$$E_i^s = (s(x, y_i; x) \setminus \{y_i\}) \times (u(x, z_i; x) \setminus \{z_i\})$$

and define

$$\beta_i^s = \alpha_i^s|_{E_i^s} : E_i^s \to U_i^s.$$

Since α_i^s maps $s(x, y_i; x) \times u(x, z_i; x)$ onto the boundary of R_i^s, then β_i^s is well defined and is a homeomorphism.

In the same way as above, for $(z, y) \in u(x, z_i; x) \times s(x, y_{i+1}; x)$ we have $C_\varepsilon^s(z) \cap C_\varepsilon^u(y) = \{\alpha_i^u(z, y)\} \subset R_i^u$ and therefore we obtain homeomorphisms

$$\alpha_i^u : u(x, z_i; x) \times s(x, y_{i+1}; x) \to R_i^u \quad ((1 \le i \le p(x))$$

such that

$$\alpha_i^u(z, x) = z \quad \text{if} \quad z \in u(x, z_i; x),$$

$$\alpha_i^u(x, y) = y \quad \text{if} \quad y \in s(x, y_{i+1}; x),$$

$$\alpha_i^u(u(x, z_i; x) \times \{y_{i+1}\}) = u(y_{i+1}, w_i(+); y_{i+1}),$$

$$\alpha_i^u(\{z_i\} \times s(x, y_{i+1}; x)) = s(z_i, w_i(+); z_i).$$

As above, we write

$$E_i^u = (u(x, z_i; x) \setminus \{z_i\}) \times (s(x, y_{i+1}; x) \setminus \{y_{i+1}\}),$$

and define

$$\beta_i^u = \alpha_i^u|_{E_i^u} : E_i^u \to U_i^u \quad (1 \le i \le p(x)).$$

Since $s(x, y_i; x) \setminus \{y_i\}$, $u(x, z_i; x) \setminus \{z_i\}$, $\mathcal{H}^i_{p(x)}$ and $\mathcal{V}^i_{p(x)}$ are homeomorphic to $[0, 1)$, for $1 \leq i \leq p(x)$ we find homeomorphisms

$$g^s_i : s(x, y_i; x) \setminus \{y_i\} \to \mathcal{H}^i_{p(x)} \quad \text{and} \quad g^u_i : u(x, z_i; x) \setminus \{z_i\} \to \mathcal{V}^i_{p(x)}. \tag{3.14}$$

Hence we define

$$r^s_i : U^s_i \to \mathcal{D}^h_{p(x),i} \quad \text{and} \quad r^u_i : U^u_i \to \mathcal{D}^v_{p(x),i)}$$

by

$$r^s_i = \alpha^h_i \circ (g^s_i \times g^u_i) \circ (\beta^s_i)^{-1},$$
$$r^u_i = \alpha^v_i \circ (g^u_i \times g^s_{i+1}) \circ (\beta^u_i)^{-1}.$$

Since all functions involved are homeomorphisms, it follows that r^s_i and r^u_i are homeomorphisms.

By expansiveness and the definition of U^σ_i we have the following properties:

$$r^s_i|_{U^s_i \cap U^u_i} = r^u_i|_{U^s_i \cap U^u_i} \quad \text{and} \quad r^u_i|_{U^u_i \cap U^s_{i+1}} = r^s_{i+1}|_{U^u_i \cap U^s_{i+1}}. \tag{3.15}$$

Therefore we define $\varphi_x : U_x \to \mathcal{D}_{p(x)}$ by

$$\varphi_x|_{U^\sigma_i} = r^\sigma_i \quad \text{for} \quad 1 \leq i \leq p(x),\ \sigma = s, u.$$

By (3.5.2) we have that φ_x is well defined and is a homeomorphism with $\varphi_x(x) = 0$. This completes the construction of φ_x.

3.5.3 Proof of (1), (2) and (3) in Proposition A

The first step is to show that for $x \in S$, under notations of construction of U_x and φ_x, the sets $\{W^s_\varepsilon(y) \cap U_x : y \in u(x, z_i; x\} \setminus \{z_i\}\}$ and $\{W^u_\varepsilon(z) \cap U_x : z \in s(x, y_i; x) \setminus \{y_i\}\}$ is a pair of foliations of U_x.

Let $x \in S$. We write

$$L^s(x, x) = \bigcup_{i=1}^{p(x)} (s(x, y_i; x))$$

and

$$L^u(x, x) = \bigcup_{i=1}^{p(x)} (u(x, z_i; x)).$$

By construction of φ_x and (3.5.2) we easily see that φ_x maps $s(x, y_i; x)$ and $u(x, z_i; x)$ onto $\mathcal{H}^i_{p(x)}$ and $\mathcal{V}^i_{p(x)}$ respectively. Therefore we have

$$\varphi_x(L^s(x, x)) = L^h_{p(x)} \quad \text{and} \quad \varphi_x(L^u(x, x)) = L^v_{p(x)}. \tag{3.16}$$

For $1 \leq i \leq p(x)$ and $z \in u(x, z_i; x) \setminus \{x, z_i\}$, we define

$$L^s(x, z) = \beta^s_i((s(x, y_i; x) \setminus \{y_i\}) \times \{z\}) \cup \beta^u_i(\{z\} \times (s(x, y_{i+1}; x) \setminus \{y_{i+1}\})),$$

.

Recall that for (z,y) in the domain of β_i^σ, then $\beta_i^\sigma(z,y) \in C_\varepsilon^{\sigma'}(z) \cap C_\varepsilon^\sigma(y)$, where $\sigma' = s$ if $\sigma = u$ and $\sigma' = u$ if $\sigma = s$. Hence we have that $L^s(x,z) \subset C_\varepsilon^s(z) \subset W_\varepsilon^s(x)$. By the construction of φ_x and since it is a homeomorphism, then φ_x sends $L^s(x,z)$ onto an element of $\mathcal{H}_{p(x)}$. Combining this fact, (3.5.3) and the facts that α_i^v is a homeomorphism for $1 \leq i \leq p(x)$ and $\mathcal{D}_{p(x)} = \bigcup_{i=1}^{p(x)}(D_{p(x),i}^h \cup D_{p(x),i}^v)$, we have that

$$\varphi_x(\{L^s(x,z) : z \in L^u(x,x)\}) = \mathcal{H}_{p(x)},$$

and by de definition of $\mathcal{H}_p(x)$ we obtain that

$$U_x = \bigcup_{z \in L^u(x,x)} L^s(x,z) \quad \text{(disjoint union)} . \tag{3.17}$$

We have that $L^s(x,z) \subset W_\varepsilon^s(z)$ is an arc contained in U_x and by expansiveness, (3.5.3) ensures that $W_\varepsilon^s(z)$ can not intersect U_x outside of $L^s(x,z)$, because if it is not true we find $z' \in W_\varepsilon^s(z)$ and $z' \in L^s(x,z'')$ for some $z'' \neq z$, and this contradicts expansiveness. Hence $L^s(x,z) = U_x \cap W_\varepsilon^s(z)$ for all $z \in L^u(x,x)$.

In the same way, for $1 \leq i \leq p(x)$ and $y \in s(x,y_i;x) \setminus \{x,y_i\}$ we write

$$L^u(x,y) = \beta_{i-1}^u((u(x,z_{i-1};x) \setminus \{z_{i-1}\}) \times \{y\}) \cup \beta_i^s(\{y\} \times (u(x,z_i;x) \setminus \{z_i\})).$$

Following the above argument we have that $L^u(x,y) \subset C_\varepsilon^u(y)$,

$$\varphi_x(\{L^u(x,y) : y \in L^s(x,x)\}) = \mathcal{V}_{p(x)},$$

$$U_x = \bigcup_{y \in L^s(x,x)} L^u(x,y) \quad \text{(disjoint union)} , \tag{3.18}$$

and hence $L^u(x,y) = U_x \cap W_\varepsilon^u(y)$ for all $y \in L^s(x,x)$.

Until now, we have shown that for $x \in S$, the local stable and unstable sets on U_x form a pair of foliations of U_x. To finish the proof, we must extend these foliations to the entire surface.

For $p \geq 2$, let $L_p = \{z \in \mathbb{C} : \operatorname{Im}(z^{p/2}) = 0\}$.

Claim 1. *For $x \in S$ and $\sigma = s,u$, there are $p \geq 2$ and an injective continuous map $j_x^\sigma : L_p \to S$ such that $j_x^\sigma(L_p) = W^\sigma(x)$.*

Proof of Claim 1. Since $\{U_x\}_{x \in S}$ is an open covering of S, by compactness we can take $\{U_a\}_{a \in A}$ a finite open covering of S for some finite subset A of S. Since $C_\varepsilon^s(x) \cap U_x$ has no branch points by Lemma 3.25 and $C_\varepsilon^s(y)$ are arcs for all $y \in U_x \setminus \{x\}$ and all $x \in S$ by Lemma 3.32, we have that $\operatorname{Sing}(f) \subset A$.

Let $0 < \rho < 2\varepsilon$ be a Lebesgue number of $\{U_a\}_{a \in A}$. For each $x \in S$, let $a(x)$ such that $B_\rho(x) \subset U_{a(x)}$. Since $\mathrm{Sing}(f) \subset A$, it is clear that $a(x) = x$ if $x \in \mathrm{Sing}(f)$.

Let $x \in S$ and, for $\sigma = s, u$, put

$$M^\sigma(a(x), x) = U_{a(x)} \cap W^\sigma_{2\varepsilon}(x).$$

Since $\rho < 2\varepsilon$ we have

$$W^\sigma_\rho(x) = U_{a(x)} \cap W^\sigma_\rho(x) \subset M^\sigma(a(x), x). \tag{3.19}$$

Since $x \in U_{a(x)}$, by (3.5.3) we find $w \in L^u(a(x), a(x))$ such that $x \in L^s(a(x), w)$.

Let n_0 given by (3.1) such that $f^N W^s_c(x) \subset W^s_\rho(f^N(x))$ and $f^{-N} W^u_c(x) \subset W^u_\rho(f^{-N}(x))$ for all $N \geq n_0$ and $x \in S$.

Let $M_{n_0} = \{kn_0 : k \in \mathbb{N}\}$ and fix $n \in M_{n_0}$. Since $M^s(a(x), x) \subset W^s_{2\varepsilon}(x)$, then

$$f^n(M^s(a(x), x)) \subset W^s_\rho(f^n(x)) \quad \text{for all} \quad x \in S. \tag{3.20}$$

Furthermore, we have that

$$W^s_\rho(f^n(x)) \subset U_{a(f^n(x))} \cap W^s_{2\varepsilon}(f^n(x) = M^s(a(f^n(x), f^n(x)),$$

therefore, by (3.20) we conclude that

$$M^s(a(x), x) \subset f^{-n}(M^s(a(f^n(x)), f^n(x))). \tag{3.21}$$

In the same way, for $m, n \in M_{n_0}$ with $m > n$, inductively we obtain

$$M^s(a(x), x) \subset M^s(f^n(a(x)), f^n(x)) \subset f^{n-m}(M^s(a(f^m(x)), f^m(x))). \tag{3.22}$$

For $n \in M_{n_0}$ we write

$$s_n(x) = f^{-n}(M^s(f^n(a(x)), f^n(x))),$$

and combining (3.21) and (3.22), we have that

$$s_0(x) \subset s_{n_0}(x) \subset s_{2n_0}(x) \subset \cdots . \tag{3.23}$$

Similarly, for $n \in M_{n_0}$ we define

$$u_n(x) = f^n(M^u(f^{-n}(a(x)), f^{-n}(x))),$$

in the same way as above we have that

$$u_0(x) \subset u_{n_0}(x) \subset u_{2n_0}(x) \subset \cdots . \tag{3.24}$$

Recall that 3.2 ensures that

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_\varepsilon(f^n(x))), \quad W^u(x) = \bigcup_{n\geq 0} f^{n}(W^s_\varepsilon(f^{-n}(x))),$$

combining this with the fact that $\{\sigma_n(x)\}_{n\in M_{n_0}}$ $(\sigma = s, u)$ are increasing, we obtain that

$$W^\sigma(x) = \bigcup_{n\in M_{n_0}} \sigma_n(x). \tag{3.25}$$

To finish the proof of Claim 1, note that for $y \in S$ one of the following cases holds,

1. $y \in W^\sigma(x)$ for some $x \in \operatorname{Sing}(f)$,
2. $y \in S \setminus (\bigcup_{x\in \operatorname{Sing}(f)} W^\sigma(x))$ for $\sigma = s$ or u.

Let $x \in \operatorname{Sing}(f)$ if the first item holds. Since $\operatorname{Sing}(f)$ is finite, we may assume that x is a fixed point of f^{n_0}. Furthermore, we have that $a(x) = x$, which implies that $M^\sigma(a(x), x) = L^\sigma(x, x)$. Hence $\sigma_n(x)$ is homeomorphic to $L_{p(x)}$ for all $n \in M_{n_0}$. Since $\sigma_n(x)$ are increasing and $W^\sigma(x) = \bigcup_{n\in M_{n_0}} \sigma_n(x)$, then we can construct a biective continuous map $j^\sigma_x : L_{p(x)} \to W^\sigma(x)$.

For $y \in W^\sigma(x) \setminus \{x\}$, we have that $W^\sigma(y) = W^\sigma(x)$. Hence a bijective continuous map $j^\sigma_x : L_{p(x)} \to W^\sigma(y)$ is obtained.

If the second item holds for $\sigma = s$ or u, it is checked that $M^\sigma(a(f^n(x)), f^n(x))$ is an open arc for all $n \in M_{n_0}$. Since $\sigma_n(y)$ are increasing, (3.25) ensures that we can construct a bijective map $j^\sigma_y : L_2 \to W^\sigma(y)$. This completes the proof of Claim 1.

Before we prove that $\mathcal{F}^\sigma_f$ are C^0 singular foliations, observe that if it is true, the above results ensures that (2) holds. Furthermore, the topology of $W^\sigma(x)$ induced by j^σ_x coincides with the leaf topology, which implies that $j^\sigma_x : L_p(x) \to W^\sigma(x)$ is a homeomorphism. As we saw above, φ_x sends $L^s(x, z)$ onto an element of $\mathcal{H}_{p(x)}$ and $L^u(x, z)$ onto an element of $\mathcal{V}_{p(x)}$. Since all envolved maps are homeomorphisms and $\mathcal{H}_{p(x)}$ is transverse to $\mathcal{V}_{p(x)}$, then $\mathcal{F}^s_f$ is transverse to $\mathcal{F}^u_f$. Therefore (2) and (3) of Proposition A hold.

Remark 3.34. *Note that since L_p $(p \geq 2)$ are path connected, so are $W^\sigma(x)$ $(\sigma = s, u)$.*

It remains to prove (1), i.e, $\mathcal{F}^\sigma_f$ are C^0 singular foliations for $\sigma = s, u$. To do this, it is enough to show that for $x, y \in S$ every connected component of $W^\sigma(y) \cap U_x$ is of form $L^\sigma(x, z)$. Indeed, since we have already shown that for each $x \in S$, $W^\sigma(x) \in \mathcal{F}^\sigma_f$ $(\sigma = s, u)$ are path connected and there exists $\varphi_x : U_x \to \mathbb{C}$ a homeomorphism satisfying that $\varphi_x(x) = 0$ and $\varphi_x(U_x) = \mathcal{D}_{p(x)}$, then it remains to prove that, if $W^\sigma(y) \cap U_x \neq \emptyset$, then φ_x sends each connected component of $W^\sigma(y) \cap U_x$ onto some element of $\mathcal{H}_{p(x)}$ or $\mathcal{V}_{p(x)}$ if $\sigma = s$ or $\sigma = u$, respectively. If $W^\sigma(y) \cap U_x$ is of form $L^\sigma(x, z)$, by the construction of φ_x, this last condition is satisfied.

We give the proof of (1) for $\sigma = s$ and for $\sigma = u$ is obtained in a similar way. Let $x, y \in S$ and let M_{n_0} be as in Claim 1. For each $w \in W^s(y) \cap U_x$, by (3.5.3) we find $z \in L^u(x, x)$ such that $w \in L^s(x, z)$. Since $L^s(x, z) \subset W^s_\varepsilon(z)$, by 3.1 there is $n_1 \in M_{n_0}$ such that

$$f^{n_1}(L^s(x, z)) \subset W^s_{\frac{\rho}{2}}(f^{n_1}(w)).$$

Since $w \in W^s(y)$, we may assume that $n_1 \in M_{n_0}$ is taken large enough such that $f^{n_1}(w) \in W^s_{\frac{\rho}{2}}(f^{n_1}(y))$. Then we have

$$f^{n_1}(L^s(x, z)) \subset W^s_\rho(f^{n_1}(y)).$$

Then 3.21 ensures that

$$f^{n_1}(L^s(x, z)) \subset W^s_\rho(f^{n_1}(y)) \subset M^s(a(f^{n_1}(y), f^{n_1}(y)).$$

By this fact and since $s_{n_1}(y)$ is defined as the pre image of $M^s(a(f^{n_1}(y)), f^{n_1}(y))$ by f^{n_1}, we have $L^s(x, z) \subset s_{n_1}(y)$ and then $L^s(x, z) \subset W^s(y)$, because $s_{n_1}(y) \subset W^s(y)$. This shows that there is a subset $\{z_\lambda\}_{\lambda\in\Lambda}$ of $L^u(x, x)$ such that

$$W^s(y) \cap U_x = \bigcup_{\lambda\in\Lambda} L^s(x, z_\lambda).$$

Since $L^s(x, z_\lambda)$ is homeomorphic to $L^h_{p(x)}$ for some $p(x) \geq 2$ and j^s_y is a homeomorphism, then $(j^s_y)^{-1}(L^s(x, z_\lambda))$ is open in $L_p(y)$. Furthermore, if $w, w' \in W^s(y) \cap U_x$, by (3.5.3) we have that either w, w' are contained in the same $L^s(x, z_\lambda)$ or in disjoints $L^s(x, z_\lambda)$, $L^s(x, z'_\lambda)$, hence we can assume that $L^s(x, z_\lambda)$ are mutually disjoint. Combining these facts with the fact that every collection of mutually disjoint open subsets of $L^h_{p(x)}$ is at most countable, we obtain that $\{z_\lambda\}_{\lambda\in\Lambda}$ is at most countable, which implies that each $L^s(x, z_\lambda)$ is a connected component of $W^(y) \cap U_x$.

Therefore $\mathcal{F}^\sigma_f$ $(\sigma = s, u)$ are C^0 singular foliations and Sing(f) is the set of all singular points of $\mathcal{F}^\sigma_f$. Combining this with the previous results, items (1), (2), and (3) of Proposition A are proven.

3.5.4 Proof of (4) in Proposition A

Lemma 3.35. *Let $\mathcal{F}$ be a C^0 singular orientable foliation in S such that all leaves of $\mathcal{F}$ is homeomorphic to $\mathbb{R}$ and consider a pair of arcs T and Γ transverse to $\mathcal{F}$ with $T \cap \Gamma = \emptyset$ and Γ compact. Suppose that the interval $[x_1, x_2] \subset T$ satisfy:*

1. $L^+_{\mathcal{F}}(x) \cap \Gamma \neq \emptyset$ *for all* $x \in [x_1, x_2)$;

2. $L^+_{\mathcal{F}}(x_2) \cap \Gamma = \emptyset$.

Denote by $\gamma : [x_1, x_2) \to \Gamma$ *the holonomy map, i.e; for all* $x \in [x_1, x_2)$*, we have that* $\gamma(x) \in \Gamma$ *satisfy* $(x, \gamma(x)]_{\mathcal{F}} = \{\gamma(x)\}$*. Then* γ *exnteds to the closed interval* $[x_1, x_2] \subset T$ *continuosly in* Γ*.*

Proof. It is easy to see that γ is continuous and locally injective. By contradiction, assume that γ does not extend to the closed interval. By compactness of Γ, $[x_1, x_2) \subset T$ must cover Γ infinitely many times under action of γ. Then we can consider a decomposition

$$[x_1, x_2) = [y_1, y_2) \cup \cdots \cup [y_i, y_{i+1}) \cup \cdots$$

such that $x_1 = y_1$ and $\gamma|_{[y_i, y_{i+1})} : [y_i, y_{i+1}) \to \Gamma$ is bijective for all $i \in \mathbb{N}$.

Note that, since $\gamma(y_i) = \gamma(x_1)$ for all $i \in \mathbb{N}$, then $y_i \in L^-_{\mathcal{F}}(\gamma(x_1))$ for all $i \in \mathbb{N}$.

Let $y_{i_0} \in [x_1, x_2)$ be the maximum of $\{y_i\}_{i \in \mathbb{N}}$ in $L^-_{\mathcal{F}}(\gamma(x_1))$ such that $(y_{i_0}, \gamma(x_1)]_{\mathcal{F}} \cap [x_1, x_2) = \emptyset$.

We claim that, for all $x \in [y_{i_0}, y_{i_0+1})$, we have that $[x, \gamma(x)]_{\mathcal{F}} \cap [x_1, x_2) = \emptyset$. Indeed, consider the set $\mathcal{A} = \{x \in [y_{i_0}, y_{i_0+1}] : (x, \gamma(x)]_{\mathcal{F}} \cap [x_1, x_2) \neq \emptyset\}$. To conclude the claim, it is sufficient to show that $\mathcal{A} = \emptyset$. By contradiction, we suppose that $\mathcal{A} \neq \emptyset$. Then there exists $\omega = \inf(\mathcal{A})$ and it is easy to see that $\omega \in \mathcal{A} \subset [y_{i_0}, y_{i_0+1})$, then $(\omega.\gamma(\omega)]_{\mathcal{F}} \cap [x_1, x_2) \neq \emptyset$, which implies that $\omega \neq y_{i_0}$. Furthermore, is easy to see that $x_1 \notin (\omega, \gamma(\omega)]_{\mathcal{F}}$. Since T and Γ are transversal to $\mathcal{F}$, then we have a neighborhood K of ω in $[y_{i_0}, y_{i_0+1})$ such that $(x, \gamma(x)]_{\mathcal{F}} \cap [x_1, x_2) \neq \emptyset$ for all $x \in K$. This contradicts that $\omega = \inf(\mathcal{A})$, because $\omega \neq y_{i_0}$. Therefore the claim holds.

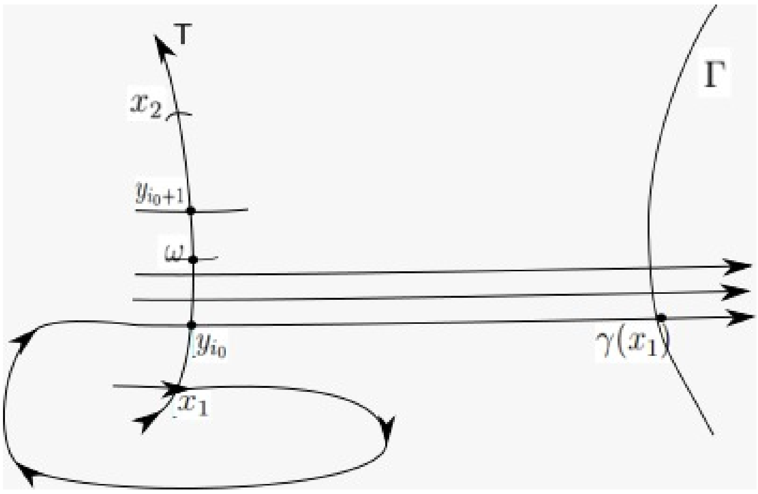


Figure 5

By the above claim, for all $x \in [y_{i_0}, y_{i_0+1})$, the arc $(x, \gamma(x)]_{\mathcal{F}}$ does not intersect $[x_1, x_2)$ before intersecting Γ, then we can define a first return map $\alpha : [y_{i_0}, x_2) \subset T \to [y_{i_0}, x_2)$ such that for all $x \in [y_{i_0}, x_2)$, we have that $\alpha(x) \in L^+_{\mathcal{F}}(x)$ and $(x, \alpha(x)]_{\mathcal{F}} \cap [y_{i_0}, x_2) = \{\alpha(x)\}$. It is easy to see that α is continuous for sufficiently large $i_1 \geq i_0 + 1$.

Since α is locally injective, then $\alpha|_{[y_{i_1}, x_2)}$ is an embedding, then we extend to $\bar{\alpha} : [y_{i_1}, x_2] \to [y_{i_0}, x_2]$.

$$\begin{tikzcd}
{[y_{i_1}, x_2)} \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\circlearrowright"] & {[y_{i_0}, x_2)} \arrow[dl, "\gamma"] \\
& \Gamma &
\end{tikzcd}$$

Figure 6

Since the diagram commutes and Γ is covered infinitely many times by $[y_{i_1}, x_2)$, then $\bar{\alpha}(x_2) = x_2$, which implies that $[x_2, \bar{\alpha}(x_2)]_{\mathcal{F}}$ is a compact leaf of $\mathcal{F}$ in $S \setminus \operatorname{Sing}(\mathcal{F})$. This contradicts that every leaf of $\mathcal{F}$ is homeomorphic to $\mathbb{R}$. $\square$

Consider $\mathcal{F}$ and $\mathcal{G}$ a pair of C^0 singular foliations on S such that $\operatorname{Sing}(\mathcal{F}) = \operatorname{Sing} = \operatorname{Sing}(\mathcal{G})$, and $\mathcal{RF}$ and $\mathcal{RG}$ under $S \setminus \operatorname{Sing}$ are transversal foliations.

Definition 3.36. *Let S be a compact surface and $\mathcal{F}$ and $\mathcal{G}$ a pair of foliations as above. A bi-foliated chart under open sets of $S \setminus \operatorname{Sing}$ is a pair (U_α, ϕ_α), where $U_\alpha \subset S \setminus \operatorname{Sing}$ is an open set and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^2$ is a homeomorphism such that*

1. *$\phi_\alpha(U_\alpha)$ is a rectangle as $(a, b) \times (c, d) \subset \mathbb{R}^2$;*
2. *$\phi_\alpha^{-1}((a, b) \times \{y\}))$ and $\phi_\alpha^{-1}(\{x\} \times (c, d))$ are leafs of $\mathcal{F}$ and $\mathcal{G}$, respectively.*

A bi-foliated atlas is an atlas such that all charts are bi-foliated charts.

Lemma 3.37. *Let $\mathcal{F}$ and $\mathcal{G}$ as above and let $V \subset S \setminus \operatorname{Sing}$ a retangular region, i.e; a simply connected region such that ∂V is formed by a pair of arcs in $\mathcal{F}$ and a pair of arcs in $\mathcal{G}$ arranged alternately. Then there exists a homeomorphism $j : V \to (0, 1) \times (0, 1)$ such that (V, j) is a bi-foliated chart.*

Lemma 3.38. *Let I and I' be intervals in leaves of $\mathcal{RF}^s$ and let $\bar{I}$ and $\bar{I}'$ denote the closure of I and I' in the leaves of mcF^s respectively. Suppose that $\bar{I}$ is compact. If $h : I \to I'$ is a map which sends $x \in I$ to $h(x) \in L^u_+(x)$ such that $(x, h(x)]_u \cap I' = \{h(x)\}$ and h is a homomorphism, then $\bar{I}'$ is compact and there is a bi-foliated chart*

$$H : [0, 1] \times [0, 1] \to S$$

such that

1. *$H([0, 1] \times \{0\}) = \bar{I}$ and $H([0, 1] \times \{1\}) = \bar{I}'$;*
2. *For all $z \in S$, $H^{-1}(W^s(z)) = [0, 1] \times A$ for some $A \subset [0, 1]$;*
3. *For all $z \in S$, $H^{-1}(W^u(z)) = B \times [0, 1]$ for some $B \subset [0, 1]$.*

The same is valid exchanging s and u.

Proof. See Lemma 6.2 in [14]. □

Lemma 3.39. *If Γ is a closed transversal of $\mathcal{RF}^s$ (resp. $\mathcal{RF}^u$), then Γ intersects every leaf of $\mathcal{RF}^s$ (resp. $\mathcal{RF}^u$) at least once.*

Proof. See Lemma 6.1 in [14]. □

Proof of (4). Let $\pi' : N \to S \setminus \mathrm{Sing}$ be a finite cover such that

1. There exists a lift $\hat{f}$ of f by π';
2. The lifts $\hat{\mathcal{F}}^\sigma_f (\sigma = s, u)$ of $\mathcal{RF}^\sigma_f$ by π' are orientable.

Consider $\pi : \bar{S} \to S$ be the branched cover induced from π'. It is easy to see that the lifts $\bar{\mathcal{F}}^\sigma_f$ by π are orientable, then we can take a lift $\bar{f} : \bar{S} \to \bar{S}$ of f by π.

Let $\bar{W}^\sigma_\varepsilon(x)$ denote the local stable and unstable sets for $\bar{f}$. If $\varepsilon > 0$ is small enough, then for all $x \in \bar{S}$ we have

$$\pi(\bar{W}^\sigma_\varepsilon(x)) = W^\sigma_\varepsilon(\pi(x)),$$

which implies that $\bar{W}^s_\varepsilon(x) \cap \bar{W}^u_\varepsilon(x) = \{x\}$ for all $x \in \bar{S}$. Hence $\bar{f}$ is expansive. Therefore $\mathcal{F}^\sigma_{\bar{f}} = \bar{\mathcal{F}}^\sigma_f$ for $\sigma = s, u$.

To conclude the proposition, let l be an arc in a leaf of $\mathcal{R}\bar{\mathcal{F}}^u_f$. Since $\bar{\mathcal{F}}^u_f = \mathcal{F}^u_{\bar{f}}$, then $\bar{f}^n(l)$ has the recurrent property for enough small $n < 0$. Using this fact, since the endpoints of $\bar{f}^n(l)$ are sufficiently close to each other, we can deform $\bar{f}^n(l)$ along the leaves of $\mathcal{R}\bar{\mathcal{F}}^s_f$ to construct a closed transversal Γ of $\mathcal{R}\bar{\mathcal{F}}^s_f$. By the previous Lemma, Γ intersects every leaf of $\mathcal{R}\bar{\mathcal{F}}^s_f$ in at least one point, and hence so does $\bar{f}^n(l)$. Therefore l intersects every leaf of $\mathcal{R}\bar{\mathcal{F}}^s_f$. Since l is arbitrary, we see that $\mathcal{F}^s_{\bar{f}}$ is minimal, which implies that $\mathcal{F}^s_f$ is minimal. In the same way, we obtain for $\sigma = u$. This finishes the proof.

3.5.5 Proposition B

The proof of Proposition B deviates from the objectives of this dissertation, therefore it will only be stated. For detailed proof, see [14].

Proposition B. *Let $f : S \to S$ be a homeomorphism and let $\mathcal{F}^s$ and $\mathcal{F}^u$ be transverse C^0 singular foliations on S. If $f(\mathcal{F}^\sigma) = \mathcal{F}^\sigma$ and $\mathcal{F}^\sigma$ is minimal for $\sigma = s, u$, then there are a constant $\lambda > 0$ and transverse invariant measures μ^σ for $\mathcal{F}^\sigma$ with every finite Borel measure of μ^σ non-atomic and positive on all non-empty open sets such that $f_*(\mu^s) = \lambda^{-1}\mu^s$ and $f_*(\mu^u) = \lambda\mu^u$.*

Proof. See Section 7 of [14]. □

3.5.6 Conclusion

We recall that $f : S \to S$ is a pseudo-Anosov homeomorphism if there exists a constant $\gamma > 1$ and a pair of transverse measures C^0 foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that the number of separatrices at each singular point is greater than 2, and f preserves the transverse C^0 singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, contracts all arcs in the leaves of $\mathcal{F}^s$ and expands arcs in the leaves of $\mathcal{F}^u$. It is easy to see that Propositions 3.1 and 3.1 ensures that f is pseudo-Anosov, therefore the main theorem is complete.

Theorem 3.40. *Every expansive homeomorphism of a compact surface is pseudo-Anosov.*

Proof. It follows directly from Propositions 3.1 and 3.1. ☐

4 Continuum-wise hyperbolicity on surfaces

4.1 Introduction and statement of results

In the classification of expansive homeomorphisms on surfaces, Lewowicz and Hiraide [18, 14] prove that expansiveness implies the homeomorphism is pseudo-Anosov. The hypothesis in the dynamics is only expansiveness, the space is a surface, and from these a complete understanding of the structure of local stable/unstable sets is obtained: they are subsets of C^0 transversal singular foliations with a finite number of singularities. A classification of topologically hyperbolic homeomorphisms on surfaces was also obtained [15]. Expansive surface homeomorphisms satisfying the shadowing property are conjugate to an Anosov diffeomorphism of the Torus $\mathbb{T}^2$. Expansiveness and shadowing together define a local product structure that is continuous and rule out the presence of singularities on local stable/unstable sets.

With the study of dynamics beyond topological hyperbolicity (see [1, 10, 5, 6, 8, 9]), which is being developed as the understanding of generalizations of hyperbolicity from a topological perspective, we could try to extend the reach of these techniques to distinct and more general scenarios. This idea was first explored by Artigue, Pacífico and Vieitez [4] in the study of N-expansive homeomorphisms on surfaces. They exhibit an example of a 2-expansive homeomorphism on the bitorus that is not expansive, and prove that non-wandering 2-expansive homeomorphisms on surfaces are, indeed, expansive. The techniques of Lewowicz/Hiraide are applied to study the structure of bi-assymptotic sectors, which are discs bounded by a local stable and a local unstable arc. Bi-asymptotic sectors naturally appear in the dynamics of 2-expansive homeomorphisms, and it is proved in [4] that these sectors should be contained in the wandering part of the system. The techniques seem to be restricted to the case of 2-expansive homeomorphisms and extend them to 3-expansiveness seems complicated. In particular, use them to understand the dynamics of more general continuum-wise expansive homeomorphisms, introduced by Kato in [17], seems difficult.

However, in the study of cw-expansive homeomorphisms a recent work of Artigue, Carvalho, Cordeiro and Vieitez [6] discussed the continuum-wise hyperbolicity, assuming that local stable and unstable continua of sufficiently close points of the space intersect (see Definition 4.2). Cw-hyperbolic systems share several important properties with the topologically hyperbolic ones, such as the L-shadowing property [5] and a spectral decomposition theorem [6], but a few important differences are noted on the pseudo-Anosov diffeomorphism of $\mathbb{S}^2$: the existence of stable/unstable spines, bi-asymptotic sectors, cantor sets in arbitrarily small dynamical balls, and a cantor set of distinct arcs in local stable/unstable sets.

In this paper we start to adapt the techniques of Lewowicz/Hiraide to the study of cw-hyperbolic homeomorphisms on surfaces. One important hypothesis in our results is cw_F-expansiveness, that asks for a finite number of intersections between any pair of local stable and local unstable continua (see Definition 4.2). We do not know an example of a cw-hyperbolic surface homeomorphism that is not cw_F-expansive, so the study of cw_F-hyperbolicity on surfaces seems to be the perfect first step in the theory. In our first result, we prove that local stable/unstable continua of cw_F-hyperbolic surface homeomorphisms are arcs (see Proposition 4.13). Cw_F-hyperbolicity is important in this step since in [2] there is a cw-expansive surface homeomorphism with non locally connected local stable continua. But even in the case they are locally connected, they are only assured to be contained in dendritations as proved in [3]. Section 2 is devoted to prove that they are arcs and this is done in a few important steps that are based in the ideas of Lewowicz and Hiraide.

In our second result we relate bi-asymptotic sectors and spines in a way that every regular sector contains a single spine and every spine is contained in a regular bi-asymptotic sector. The notion of regular bi-asymptotic sector is defined and pictures of non-regular sectors are presented. We give a complete description of the structure of local stable and local unstable continua inside a regular bi-asymptotic sector. We also prove that every bi-asymptotic sector contains a regular bi-asymptotic sector and, hence, a spine, and that non-regular sectors contain at least two distinct spines. All these results on sectors and spines are proved in Section 3 and allow us to conclude that the spines of a cw_F-hyperbolic surface homeomorphism are isolated from other spines and, hence, we obtain that there is at most a countable number of them.

In Section 4 we prove our main result using all the techniques developed in the previous sections. The hypothesis of the existence of at most a finite number of spines is important and will be discussed. We do not know examples of cw-hyperbolic surface homeomorphisms with an infinite number of spines, so this hypothesis also seems reasonable. The following is the main result of this article:

Theorem 4.1. *If a cw_F-hyperbolic surface homeomorphism has only a finite number of spines, then it is cw2-hyperbolic. Cw3-hyperbolic surface homeomorphisms have at most a finite number of spines, and are cw2-hyperbolic.*

We note that in [6] it is proved that the product of n copies of the pseudo-Anosov diffeomorphism of $\mathbb{S}^2$ is cw_{2^n}-hyperbolic but is not cw_{2^n-1}-expansive. Thus, the hypothesis on the space being a surface is important for our main result. We state the following questions that follow naturally from our results:

Question 1. *Does there exist a cw-hyperbolic surface homeomorphism that is not* cw_2*-hyperbolic?*

Question 2. *Does* cw_F*-hyperbolicity on surfaces imply finiteness on the number of spines and, hence,* cw_2*-hyperbolicity?*

Question 3. *Does cw-hyperbolicity on surfaces imply local connectedness of* $C^s_\varepsilon(x)$*?*

Question 4. *Can we adapt the techniques of this paper to prove that 3-expansive surface homeomorphisms are 2-expansive?*

4.2 Local stable/unstable continua are arcs

We begin this section with some precise definitions. Let (X, d) be a compact metric space and $f\colon X \to X$ be a homeomorphism. We consider the *c-stable set* of $x \in X$ as the set

$$W^s_c(x) := \{y \in X;\ d(f^k(y), f^k(x)) \leq c \ \text{ for every } \ k \geq 0\}$$

and the *c-unstable set* of x as the set

$$W^u_c(x) := \{y \in X;\ d(f^k(y), f^k(x)) \leq c \ \text{ for every } \ k \leq 0\}.$$

We consider the *stable set* of $x \in X$ as the set

$$W^s(x) := \{y \in X;\ d(f^k(y), f^k(x)) \to 0 \ \text{ when } \ k \to \infty\}$$

and the *unstable set* of x as the set

$$W^u(x) := \{y \in X;\ d(f^k(y), f^k(x)) \to 0 \ \text{ when } \ k \to -\infty\}.$$

We denote by $C^s_c(x)$ the c-stable continuum of x, that is the connected component of x on $W^s_c(x)$, and denote by $C^u_c(x)$ the c-unstable continuum of x, that is the connected component of x on $W^u_c(x)$.

Definition 4.2. *We say that f is cw-expansive if there exists $c > 0$ such that*

$$W^s_c(x) \cap W^u_c(x) \quad \text{is totally disconnected}$$

for every $x \in X$. A cw-expansive homeomorphism is said to be cw_F*-expansive if there exists $c > 0$ such that*

$$\#(C^s_c(x) \cap C^u_c(x)) < \infty \quad \text{for every} \quad x \in X.$$

Analogously, f is said to be cw_N*-expansive if there is $c > 0$ such that*

$$\#(C^s_c(x) \cap C^u_c(x)) \leq N \quad \text{for every} \quad x \in X.$$

We say that f satisfies the cw-local-product-structure if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$C^s_\varepsilon(x) \cap C^u_\varepsilon(y) \neq \emptyset \quad \textit{whenever} \quad \mathrm{d}(x,y) < \delta.$$

The cw-expansive homeomorphisms (resp. cw_F, cw_N) satisfying the cw-local-product-structure are called cw-hyperbolic (resp. cw_F, cw_N).

The main examples of cw-hyperbolic surface homeomorphism are the Anosov diffeomorphisms (or more generally the topologically hyperbolic homeomorphisms), and the pseudo-Anosov diffeomorphism of $\mathbb{S}^2$. The sphere $\mathbb{S}^2$ can be seen as the quotient of $\mathbb{T}^2$ by the antipodal map, and thus any 2x2 hyperbolic matrix A with integer coefficients and determinant one induces diffeomorphisms f_A on $\mathbb{T}^2$ and g_A on $\mathbb{S}^2$. The diffeomorphism f_A is Anosov and, hence, cw1-hyperbolic, while g_A is cw2-hyperbolic but not cw1-expansive (see [10, 6, 5] for more details). We recall some known properties of local stable/unstable continua for cw$_F$-hyperbolic homeomorphisms. Most of them hold assuming that X is a Peano continuum, that is a compact, connected, and locally connected metric space, but we assume from now on that $f\colon S \to S$ is a cw$_F$-hyperbolic homeomorphism of a closed surface S. We will use the symbol σ to denote both s and u. Since this will appear several times in what follows, we will not write in all of them that some statement holds for $\sigma = s$ and $\sigma = u$, we will simply say that it holds for σ.

Lemma 4.3 ([16] Thm. 1.6)**.** *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\operatorname{diam}(C^\sigma_\varepsilon(x)) > \delta \quad \textit{for every} \quad x \in S,$$

where $\operatorname{diam}(A) = \sup\{d(a,b); a, b \in A\}$ *denotes the diameter of the set A.*

Corollary 4.4. $\operatorname{int} C^\sigma_\varepsilon(x) = \emptyset$ *for every $x \in S$ and $\varepsilon \in (0, \frac{c}{2})$.*

Proof. The following is the proof for $\sigma = s$, but for $\sigma = u$ the proof is analogous. By contradiction, assume that $y \in \operatorname{int} C^s_\varepsilon(x) \neq \emptyset$ for some $x \in X$ and $\varepsilon < \frac{c}{2}$. Since $\operatorname{diam} C^u_\varepsilon(y) > \delta$ for some $\delta > 0$, then $C^s_\varepsilon(x) \cap C^u_\varepsilon(y)$ contains a non-trivial continuum. By the choice of ε, we have that $C^u_\varepsilon(y) \subset C^u_c(y)$, then $C^s_c(x) \cap C^u_c(x)$ contains a non-trivial continuum, which contradicts the *cw*-expansiveness. □

Let $\mathcal{C}$ denote the space of all sub-continua of S and $\mathcal{C}_\delta$ denote the set of all sub-continua of S with diameter smaller than δ. Let $\mathcal{C}^s$ and $\mathcal{C}^u$ denote the set of all stable and unstable continua of f, respectively. More precisely,

$$\mathcal{C}^s = \{C \in \mathcal{C};\ \operatorname{diam}(f^n(C)) \to 0,\ n \to \infty\}$$

$$\mathcal{C}^u = \{C \in \mathcal{C};\ \operatorname{diam}(f^n(C)) \to 0,\ n \to -\infty\}.$$

The following lemma is actually a characterization of cw-expansiveness with respect to local stable/unstable continua.

Lemma 4.5 ([3] Prop. 2.3.1)**.**

1. *There exists* $\varepsilon^* > 0$ *such that* $C^\sigma_{\varepsilon^*} \subset C^\sigma$*,*

2. *For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $C^\sigma \cap C_\delta \subset C^\sigma_\varepsilon$*.*

The following lemma is similar to Lemma 4.1 in [14] but assuming cw-expansiveness. Let $B_\delta(x)$ denote the ball of radius δ centered at x, that is the set of points whose distance to x is less than δ.

Lemma 4.6. *For each* $0 < \varepsilon < \frac{\varepsilon^*}{4}$ *there exists* $\delta \in (0, \varepsilon)$ *such that*

$$C^\sigma_\varepsilon(x) \cap B_\delta(x) = C^\sigma_{2\varepsilon}(x) \cap B_\delta(x)$$

for every $x \in S$*.*

Proof. For each $0 < \varepsilon < \frac{\varepsilon^*}{4}$ let $\delta^* \in (0, \varepsilon)$ be given by Lemma 4.5 and $\delta = \frac{\delta^*}{2}$. Since $\varepsilon < \frac{\varepsilon^*}{4}$, it follows that

$$C^\sigma_\varepsilon(x) \subset C^\sigma_{2\varepsilon}(x) \in C^\sigma.$$

The choice of δ ensures that $C^\sigma_{2\varepsilon}(x) \cap B_\delta(x) \subset C^\sigma_\varepsilon(x)$, and, hence,

$$C^\sigma_{2\varepsilon}(x) \cap B_\delta(x) = C^\sigma_\varepsilon(x) \cap B_\delta(x).$$

□

In the following lemma, the hypothesis of cw$_F$-expansiveness will be necessary. It was first proved in [3] for cw$_F$-expansive homeomorphisms, but we include a different proof using cw$_F$-hyperbolicity. In this article, an arc is a subset of S homeomorphic to $[0, 1]$.

Lemma 4.7 ([3] Thm. 6.7.1)**.** *There exists* $\varepsilon > 0$ *such that* $C^\sigma_\varepsilon(x)$ *is locally connected for every* $x \in S$*.*

Proof. We prove for $\sigma = s$ but the proof for $\sigma = u$ is similar. Let $0 < \varepsilon < \frac{c}{4}$ and by contradiction assume that $C^s_\varepsilon(x)$ is not locally connected for some $x \in S$. Then we can consider a sequence of arcs $(P_n)_{n\in\mathbb{N}} \subset C^s_\varepsilon(x)$ (as in the proof of Proposition 3.1 in [14]) such that $P_i \cap P_j = \emptyset$ if $i \neq j$ and $\mathrm{d}(P_n, P^*) \to 0$ for some non-trivial arc $P^* \subset C^s_\varepsilon(x)$ (here we also denote by d the Hausdorff distance on the space of continua). Let $y \in P^*$ be an interior point of P^* and consider $r \in (0, \varepsilon)$ such that

1. P^* separates $B_r(y)$, and

2. P_n separates $B_r(y)$ for every $n > n_0$ and for some $n_0 \in \mathbb{N}$.

We can assume that every P_n is contained in the same component A of $B_r(y) \setminus P^*$ taking a sub-sequence if necessary. The cw-local-product-structure ensures the existence of $\delta \in (0, \frac{r}{4})$ such that

$$C^s_{\frac{r}{4}}(a) \cap C^u_{\frac{r}{4}}(b) \neq \emptyset \quad \text{whenever} \quad \mathrm{d}(a,b) < \delta.$$

Since $\operatorname{int} C^s_{2\varepsilon}(y) = \emptyset$, there exists $z \in A \cap B_\delta(y)$ such that $z \notin C^s_{2\varepsilon}(y)$, and, hence,

$$C^s_r(z) \cap C^s_\varepsilon(y) = \emptyset.$$

In particular, $C^s_r(z) \cap P_n = \emptyset$ for every $n \in \mathbb{N}$. Choose $n_1 > n_0$ such that $P_{n_1} \cap B_r(y)$ separates $P^* \cap B_r(y)$ and $C^s_{\frac{r}{4}}(z)$. The choice of δ ensures that $C^u_{\frac{r}{4}}(y) \cap C^s_{\frac{r}{4}}(z) \neq \emptyset$. In particular, $C^u_{\frac{r}{4}}(y) \cap P_{n_1} \neq \emptyset$, and, hence, $C^u_\varepsilon(y)$ intersects an infinite number of distinct $P_n's$ (see Figure 7). This contradicts cw$_F$-expansiveness and finishes the proof.

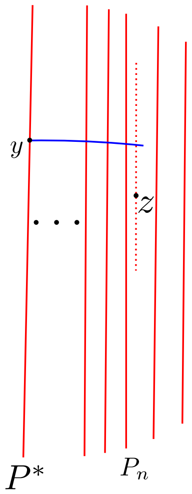


Figure 7

□

The following corollary is a consequence of this result. A subset $A \subset S$ is arcwise connected if for each pair of distinct points $x, y \in A$ there exists an arc $h\colon [0,1] \to A$ such that $h(0) = x$ and $h(1) = y$.

Corollary 4.8. *There exists $\varepsilon > 0$ such that $C^\sigma_\varepsilon(x)$ is arcwise connected and locally arcwise connected for every $x \in S$. Moreover, for each pair of distinct points $y, z \in C^\sigma_\varepsilon(x)$, there is a unique arc $\sigma(y,z;x)$ in $C^\sigma_\varepsilon(x)$ connecting y and z.*

Proof. From Lemma 4.7 and Theorem 5.9 of [12], it follows that $C^\sigma_\varepsilon(x)$ is a Peano space. Hence, Theorem 6.29 of [12] ensures that $C^\sigma_\varepsilon(x)$ is arcwise connected and locally arcwise connected. The uniqueness comes from the observation that two distinct arcs connecting y and z would create an open set bounded by a local stable (in the case $\sigma = s$) or local unstable (in the case $\sigma = u$) curve, which contradicts cw-expansiveness on surfaces. □

From now on we choose $\varepsilon > 0$ given by Corollary 4.8 and $\delta \in (0, \varepsilon)$ satisfying the Lemmas 4.3, 4.5, and 4.6. Following the steps of Hiraide in [14], we define an equivalence relation in the set of arcs starting on x and contained in $C^\sigma_\varepsilon(x)$.

Definition 4.9. *Let $x \in S$, and $y, z \in C^\sigma_\varepsilon(x)$. We write $y \sim z$ if*

$$\sigma(x, y; x) \cap \sigma(x, z; x) \supsetneq \{x\}.$$

We define the number of stable/unstable separatrices at x as

$$P^\sigma(x) = \#(C^\sigma_\varepsilon(x)/\sim).$$

Lemma 4.6 ensures that the number of separatrices at x does not depend on the choice of $\varepsilon < \frac{\varepsilon^*}{4}$. This explains the notation $P^\sigma(x)$ without ε being mentioned.

The following lemma follows an idea present in the works of Lewowicz/Hiraide: stable and unstable separatrices must be alternated as in Figure 8. In this step, the cw$_F$-hyperbolicity will also be necessary. Let $\partial B_\delta(x)$ denote the boundary of the ball $B_\delta(x)$, that is the set of points whose distance to x equals δ.

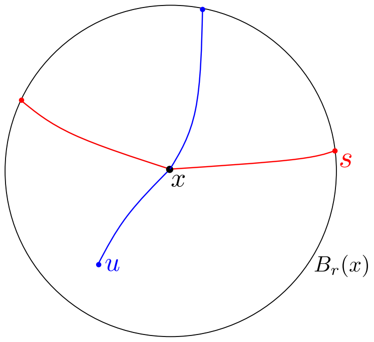


Figure 8 – Lemma 4.10

Lemma 4.10. *For each $x \in S$, there exists $r_0 > 0$ such that if $r \in (0, r_0)$, $y, z \in \partial B_r(x)$ are in different classes of $\sim$, $s(x, y; x)$ and $s(x, z; x)$ are arcs intersecting $\partial B_r(x)$ at one point and A is a component of $B_r(x) \setminus (s(x, y; x) \cup s(x, z; x))$, then there is $a \in C^u_\varepsilon(x) \cap A$ such that $u(x, a; x) \subset A$.*

Proof. If $C^s_\varepsilon(x) \cap C^u_\varepsilon(x) = \{x\}$, let $r_0 = \varepsilon$, otherwise,

$$C^s_\varepsilon(x) \cap C^u_\varepsilon(x) \supsetneq \{x\},$$

and we let

$$r_0 = \mathrm{d}(x, (C^s_\varepsilon(x) \cap C^u_\varepsilon(x)) \setminus \{x\}),$$

which is a positive number by cw_F-expansiveness. Let x, r, y, z, and A as above, and suppose there is no unstable arc $u(x, a; x) \subset A$. Choose $\delta_r \in (0, \frac{r}{4})$, given by the cw-local-product-structure, such that

$$C_{\frac{r}{4}}(x) \cap C_{\frac{r}{4}}(y) \neq \emptyset \quad \text{whenever} \quad d(x, y) < \delta_r.$$

Lemma 4.4 assures the existence of

$$b \in A \setminus C^s_\varepsilon(x) \quad \text{with} \quad \mathrm{d}(b,x) < \frac{\delta_r}{2}.$$

It follows that

$$C^s_{\frac{r}{4}}(b) \subset A.$$

If $C^s_\varepsilon(x) \cap C^u_\varepsilon(x) = \{x\}$, then $C^u_{\frac{r}{4}}(x) \cap A = \emptyset$ since there is no unstable arc $u(x,a;x) \subset A$; otherwise, $C^s_r(x) \cap C^u_r(x) = \{x\}$ since $r < r_0$, and, hence, $C^u_{\frac{r}{4}}(x) \cap A = \emptyset$.

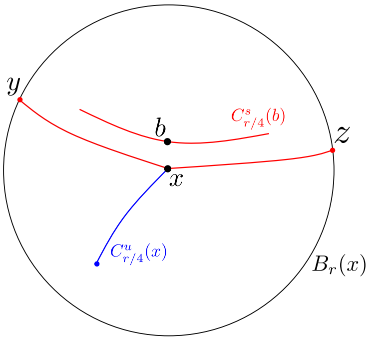


Figure 9

In both cases, we obtain

$$C^s_{\frac{r}{4}}(b) \cap C^u_{\frac{r}{4}}(x) = \emptyset.$$

This contradicts the choice of δ_r since $\mathrm{d}(b,x) < \frac{\delta_r}{2}$ (see Figure 9) and proves the existence of an unstable arc $u(x,a;x) \subset A$ and finishes the proof. □

The following corollary is a direct consequence of the previous Lemma.

Corollary 4.11. *$P^s(x) = P^u(x)$ for every $x \in S$.*

We recall that $C^\sigma_\varepsilon(x)$ is locally connected but the case $P^\sigma(x) = \infty$ is still not ruled out since we could have an infinite number of separatrices with diameter converging to zero. Also, in the expansive case, examples of pseudo-Anosov homeomorphisms with singularities containing a number of stable/unstable separatrices greater than two can be constructed. In the following lemma, we observe that these two scenarios do not occur in the case of cw_F-hyperbolic homeomorphisms. Indeed, the existence of bifurcation points contradicts cw_F-hyperbolicity.

Lemma 4.12. *$P^\sigma(x) \leq 2$ for every $x \in S$.*

Proof. Suppose, by contradiction, that there exists $x \in S$ with $P^\sigma(x) \geq 3$. Let $r > 0$ and x_1, x_2, x_3 be in different classes of $C^s_\varepsilon(x)$ such that $s(x,x_i;x) \subset B_r(x)$ and intersects $\partial B_r(x)$ only at x_i for every $i = 1,2,3$. Then $\bigcup_{i=1}^3 s(x,x_i;x)$ separates $B_r(x)$ in exactly three components, A_1, A_2, A_3 as in Figure 10.

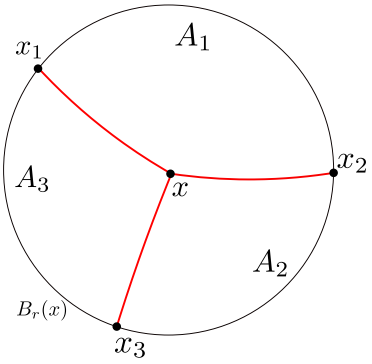


Figure 10

Using Lemma 4.10, we can find $y_i \in A_i$ such that $u(x, y_i; x) \subset A_i$. If $C^s_\varepsilon(x) \cap C^u_\varepsilon(x) = \{x\}$, let

$$r_0 = \min(\operatorname{diam} u(x, y_i; x))$$

otherwise, let

$$r_0 = \min(\mathrm{d}(x, (C^s_\varepsilon(x) \cap C^u_\varepsilon(x)) \setminus \{x\}), \min(\operatorname{diam} u(x, y_i; x))),$$

which is a positive number by cw_F-expansiveness. It follows that $\bigcup_{i=1}^3 u(x, z_i; x)$ divides $B_{r_0}(x)$ in three components, B_1, B_2, B_3, for some $z_i \in u(x, y_i; x)$ (see Figure 11).

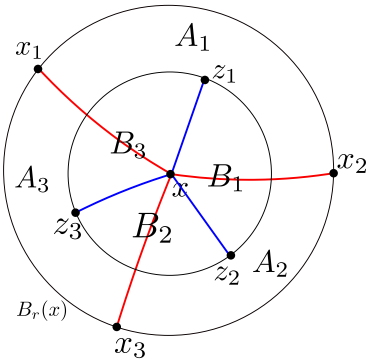


Figure 11

Since the stable and unstable arcs are alternating, then

$$A_i \cap B_j = \emptyset.$$

for some $i, j \in \{1, 2, 3\}$ (see Figure 12).

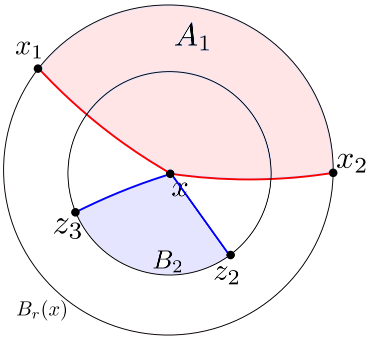


Figure 12

Choose $\delta_{r_0} \in (0, \frac{r_0}{4})$, given by cw-local-product-structure, such that

$$C^s_{\frac{r_0}{4}}(x) \cap C^u_{\frac{r_0}{4}}(y) \neq \emptyset \quad \text{whenever} \quad d(x,y) < \delta_{r_0}.$$

Lemma 4.4 assures the existence of

$$a \in A_i \setminus C^s_\varepsilon(x) \quad \text{with} \quad \mathrm{d}(a,x) < \frac{\delta_{r_0}}{2}$$

and

$$b \in B_j \setminus C^u_\varepsilon(x) \quad \text{with} \quad \mathrm{d}(b,x) < \frac{\delta_{r_0}}{2}.$$

It follows that

$$C^s_{\frac{r_0}{4}}(a) \subset A_i \quad \text{and} \quad C^u_{\frac{r_0}{4}}(b) \subset B_j,$$

and, hence, $C^s_{\frac{r_0}{4}}(a) \cap C^u_{\frac{r_0}{4}}(b) = \emptyset$ (see Figure 13).

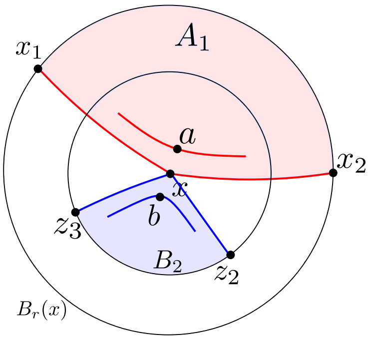


Figure 13

This contradicts the choice of δ_{r_0} and finishes the proof. □

There are two possible cases: either $P^\sigma(x) = 1$ and x is said to be a spine, or $P^\sigma(x) = 2$ and x is said to be a regular point. Let $\mathrm{Spin}(f)$ denote the set of all spines of f. The following proposition gathers all results we obtained so far. We prove that $C^\sigma_\varepsilon(x)$ is an arc for every $x \in S$.

Proposition 4.13. *If $x \in \mathrm{Spin}(f)$, then there is a homeomorphism $h^\sigma : [0,1] \to C^\sigma_\varepsilon(x)$ with $h^\sigma(0) = x$. Otherwise, there is a homeomorphism $h^\sigma : [-1,1] \to C^\sigma_\varepsilon(x)$ with $h^\sigma(0) = x$.*

Proof. Let $IC^\sigma(x)$ denote the union of all open arcs in $C^\sigma_\varepsilon(x)$ and

$$BC^\sigma(x) = C^\sigma_\varepsilon(x) \setminus IC^\sigma(x).$$

Note that $BC^\sigma(x)$ is always formed by two distinct points x_1 and x_2, since local connectedness of $C^\sigma_\varepsilon(x)$ ensures the existence of at least two points in $BC^\sigma(x)$ (as in Lemma 4.5 of [14]), and the existence of three distinct points in $BC^\sigma(x)$ would imply the existence of $y \in C^\sigma_\varepsilon(x)$ with $P^\sigma(y) \geq 3$, contradicting Lemma 4.12. If $x \in \mathrm{Spin}(f)$, then either $x = x_1$ or $x = x_2$, and the arc connecting x_1 to x_2 gives us a homeomorphism $h^\sigma\colon [0,1] \to C^\sigma_\varepsilon(x)$ such that $h^\sigma(0) = x$. If $x \notin \mathrm{Spin}(f)$, then $x \notin BC^\sigma(x)$ and the arc connecting x_1 to x_2 gives us a homeomorphism $h^\sigma : [-1,1] \to C^\sigma_\varepsilon(x)$ with $h^\sigma(0) = x$. □

Now we exhibit two important consequences of above results that will be important in the proofs of Section 3. In the first lemma, we prove that either a local stable/unstable continuum separates a small ball, or it contains a spine in this ball. The notation $c.c_x(A)$ is used to denote the connected component of x in the set A.

Lemma 4.14. *For each $0 < \varepsilon < \frac{\varepsilon^*}{4}$, there exists $\delta \in (0, \varepsilon)$ such that for each $x \in S$ one of the following holds:*

1. *$c.c_x(C^\sigma_\varepsilon(x) \cap B_\delta(x))$ separates $B_\delta(x)$,*
2. *$c.c_x(C^\sigma_\varepsilon(x) \cap B_\delta(x))$ contains a spine.*

Proof. If $x \in \mathrm{Spin}$, we are in case (2). Then we assume that $x \in S \setminus \mathrm{Spin}(f)$. For each $0 < \varepsilon < \frac{\varepsilon^*}{4}$, let $\delta \in (0, \varepsilon)$ be given by Lemma 4.6 such that

$$C^\sigma_\varepsilon(x) \cap B_\delta(x) = C^\sigma_{2\varepsilon}(x) \cap B_\delta(x).$$

Let $h^\sigma : [-1,1] \to C^\sigma_\varepsilon(x)$ be a homeomorphism as in Proposition 4.13 with $h^\sigma(0) = x$. Lemma 4.3 ensures the existence of $z \in C^\sigma_\varepsilon(x) \cap \partial B_\delta(x)$. Without loss of generality, we can assume that $z \sim h(-1)$. We assume item (1) is false and prove item (2). Suppose that $c.c_x(C^\sigma_\varepsilon(x) \cap B_\delta(x))$ does not separate $B_\delta(x)$. Since $z \in C^\sigma_\varepsilon(x) \cap \partial B_\delta(x)$ and $z \sim h(-1)$, then $\sigma(x, y; x) \subset \operatorname{int} B_\delta(x)$, where $y = h(1)$. Since $y \in C^\sigma_\varepsilon(x)$, then $C^\sigma_\varepsilon(y) \subset C^\sigma_{2\varepsilon}(x)$ and, hence,

$$C^\sigma_\varepsilon(y) \cap B_\delta(x) \subset C^\sigma_{2\varepsilon}(x) \cap B_\delta(x) = C^\sigma_\varepsilon(x) \cap B_\delta(x).$$

Thus, $y \in c.c_x(C^\sigma_\varepsilon(x) \cap B_\delta(x))$ and $P(y) = 1$, that is, $y \in \mathrm{Spin}(f)$.

□

In the last result of this section, we observe that local stable/unstable sets intersect transversely. First, we state a precise definition for topological transversality.

Definition 4.15. *Let α, β be arcs in S meeting at x. We say that α is topologically transversal to β at x if there exists a disk D such that*

1. *$\alpha \cap \beta \cap D = \{x\}$,*
2. *β separates D, and*
3. *the connected components of $(\alpha \setminus \beta) \cap D$ are in different components of $D \setminus \beta$ (See Fig. 14).*

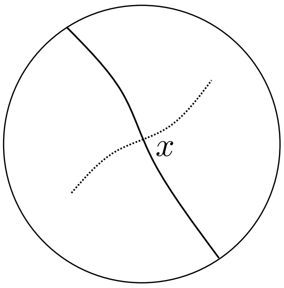


Figure 14 – Two arcs topologically transverse at the point x.

Lemma 4.16. *If $x, y \in S$ and $z \in C^s_\varepsilon(x) \cap C^u_\varepsilon(y) \setminus (Spin(f))$, then $C^s_\varepsilon(x)$ intersects $C^u_\varepsilon(y)$ transversely at z.*

Proof. Since $z \notin Spin(f)$, then $P^s(z) = P^u(z) = 2$. If the intersection is not transversal we find $a_1, a_2 \in C^s_{2\varepsilon}(z)$ and a disk D around z such that $s(a_1, a_2; z)$ separates D and there is a component of $D \setminus s(a_1, a_2; z)$ containing $b_1 \not\sim b_2 \in C^u_\varepsilon(z)$. This is a contradiction with 4.10 because $P_\varepsilon(z) = 2$. □

4.3 Bi-asymptotic sectors and spines

Bi-asymptotic sectors were introduced in [4] for N-expansive homeomorphisms on surfaces. These sectors were defined as being a disk bounded by the union of a local stable and a local unstable arc (see Figure 15a). In the case of 2-expansive homeomorphisms, a consequence of the arguments of [4] is that both intersections a_1, a_2 of a bi-asymptotic sector are not only transversal, but point outside the disk (see Figure 15a). Indeed, 2-expansiveness and non-existence of wandering points imply the non-existence of spines inside by-asymptotic sectors (see Proposition 3.5 in [4]), and this ensure the existence of a third intersection between the stable and unstable arcs bounding the sector if the intersection points inward.

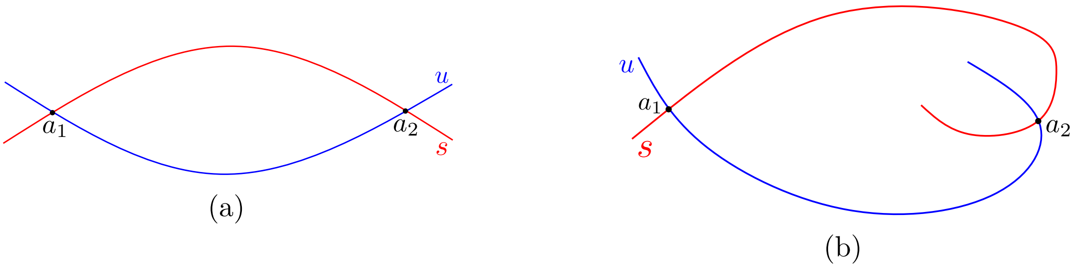


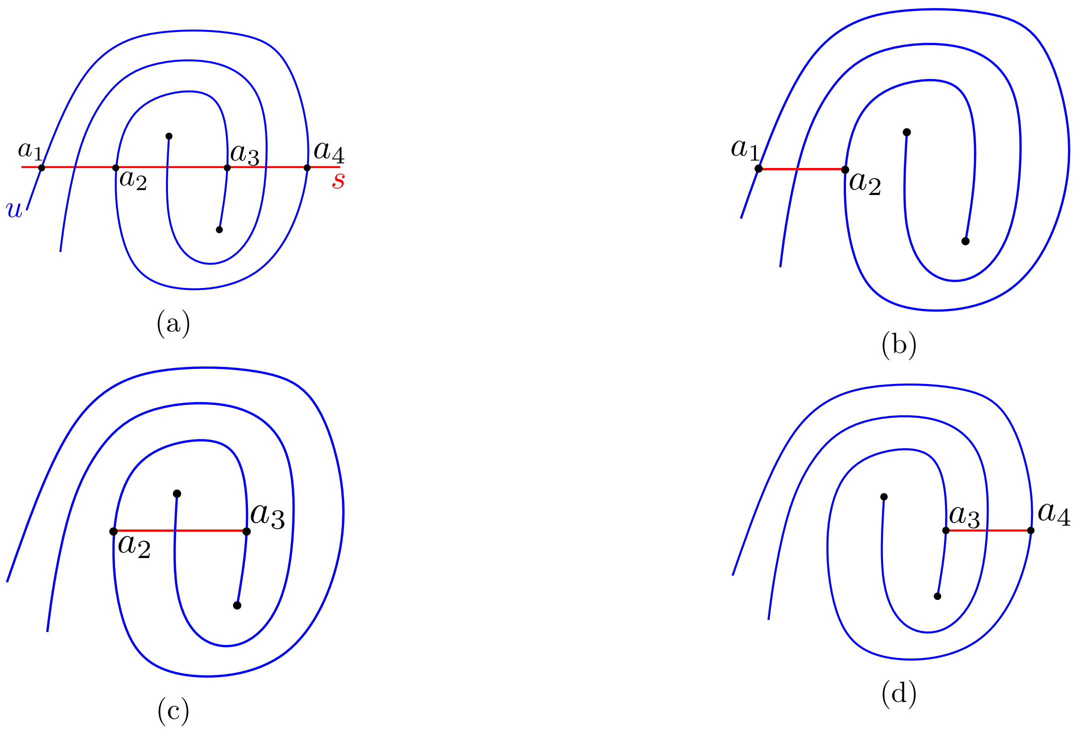
Figure 15

For cw_F-hyperbolic homeomorphisms, it is not possible to ensure the intersections points outward the disk. First, because there will be spines inside the sector, but also because this case allows more intersections between local stable/unstable arcs. The goal is to prove that every bi-asymptotic sector contains a spine and that every spine is contained in a bi-asymptotic sector. To prove this, we first understand the case of bi-asymptotic sectors with intersections pointing outward the sector (these sectors will be called regular). We will characterize the structure of stable/unstable arcs inside a regular bi-asymptotic sector obtaining a single spine inside it.

Definition 4.17. *We say that* $C^s_\varepsilon(x)$ *and* $C^u_\varepsilon(x)$ *form a bi-asymptotic sector if there exists a pair of sub-arcs* a^s, a^u *contained in* $C^s_\varepsilon(x)$ *and* $C^u_\varepsilon(x)$, *respectively, such that* $a^s \cup a^u$ *bounds a disk* D. *In this case,* D *is called the bi-asymptotic sector. Let* a_1 *and* a_2 *be the end points of* a^s *and* a^u. *A bi-asymptotic sector* D *is said to be regular if it satisfies the following:*

(Regularity condition) There exists neighborhoods V_{a_1} *of* a_1 *and* V_{a_2} *of* a_2 *such that* $C^\sigma_\varepsilon(x) \cap V_{a_1} \cap \operatorname{int} D = \emptyset$ *and* $C^\sigma_\varepsilon(x) \cap V_{a_2} \cap \operatorname{int} D = \emptyset$.

Without the regularity condition, the bi-asymptotic sectors can contain more than one spine, and, hence, a more complicated structure of stable/unstable arcs, as in Figure 16a, 16b and 16c.



(a)

(b)

(c)

(d)

Figure 16

Note that both the stable and unstable continua enter the disk passing through a_2, that is, for every neighborhood V of a_2 we have $C^\sigma_\varepsilon(a_2) \cap V \cap \operatorname{int} D \neq \emptyset$. Thus, the sector in Figure 16b formed considering the stable arc from a_1 and a_2 does not satisfy the regularity condition. The same happens with the sector bounded by the stable and unstable arcs connecting a_3 and a_4. Also, note that the sectors formed by the stable arc from a_1 to a_4 and from a_2 to a_3 satisfy the regularity condition. Inside these sectors, the structure of stable/unstable arcs is the same: there is a single spine and all stable/unstable arcs turn around this spine. We prove in this section that this is exactly the structure of stable/unstable arcs inside any regular bi-asymptotic sector. Let D be a regular bi-asymptotic sector bounded by a^s and a^u with $\operatorname{diam} D < \delta$ (given by Lemma 4.14), and let a_1 and a_2 be the end points of a^s and a^u. For $p \in D$, define $C^u_D(p)$ and $C^s_D(p)$ as the connected component of $C^u(p) \cap D$ and $C^s(p) \cap D$ containing p respectively. We remark that Lemma 4.14 also holds changing the ball $B_\delta(x)$ for the sector D, that is, for each $p \in D$, either $C^\sigma_D(p)$ separates D or $C^\sigma_D(p)$ contains a spine. The hypothesis of regularity is important to ensure the following result.

Lemma 4.18. *$C^\sigma_D(p) = a^\sigma$ for every $p \in a^\sigma$.*

Proof. It is clear that $C^\sigma_D(p) \supset a^\sigma$. By contradiction, assume that there exist $p \in a^\sigma$ and $y \in C^\sigma_D(p) \setminus a^\sigma$. This means that either $\sigma(a_1, y; x)$ or $\sigma(a_2, y; x)$ is contained in $\operatorname{int} D$ and contradicts the regularity condition. ☐

Note that in Figure 16b, the arc a^u connects a_1 to a_2, while $C^u_D(a_1)$ contains a^u and also an arc from a_2 to a spine in the interior of D. Also, a^s is the stable arc from a_1 to a_2, while $C^s_D(a_1)$ contains the arc connecting a_2 to a_4 (see Figure 16a). The following lemma is a consequence of the previous lemma and the transversality explained in Lemma 4.16. The following results also hold changing the roles of s and u but we will not use σ as before since it would make the presentation more complicated.

Lemma 4.19. *$1 \leq \#(C^s_D(p) \cap a^u) \leq 2$ for every $p \in D$. If $\#(C^s_D(p) \cap a^u) = 1$, then $C^s_D(p)$ contains a spine.*

Proof. If $p \in a^s$, then $C^s_D(p) = a^s$ since D is regular (see Lemma 4.18). This implies that $\#(C^s_D(p) \cap a^u) = \#(a^s \cap a^u) = 2$ (See Fig. 15a). If $p \in D \setminus a^s$, then $C^s_D(p) \cap a^s = \emptyset$, by Lemma 4.12, which implies that $C^s_D(p) \cap a^u \neq \emptyset$ since Lemma 4.3 ensures that $C^s_D(p) \cap (a^s \cup a^u) \neq \emptyset$. Therefore $\#(C^s_D(p) \cap a^u) \geq 1$. Note that every intersection between $C^s_D(p)$ and a^u is transversal and this together with Lemma 4.12 ensure that $\#(C^s_D(p) \cap a^u) \leq 2$. The last part of the statement is obtained using Lemma 4.14, since $\#(C^s_D(p) \cap a^u) = 1$ implies that $C^s_D(p)$ does not separate D, and, hence, contains a spine. ☐

Note that in the non-regular sectors of Figure 16a, while $a^s \cap a^u = \{a_1, a_2\}$, we have $C^s_D(p) \cap a^u = \{a_1, a_2, a_3\}$. Following [4], we define an order in the set $\mathcal{F}^s = \{C^s_D(x) : x \in D\}$ as follows: $C^s_D(x) < C^s_D(y)$ if a^s and $C^s_D(y)$ are separated by $C^s_D(x)$, i.e. , a^s and $C^s_D(y)$

are in different components of $D \setminus C^s_D(x)$. Note that a^s is a minimal element for the order and if $y \in \operatorname{int} D \cap \operatorname{Spin}$, then $C^s_D(y)$ does not separate D and $C^s_D(y)$ cannot be smaller than $C^s_D(z)$ for any $z \neq y$ in D. The following lemma is based on Lemma 3.2 in [4]. The regularity condition on the sector allows us to basically follow the original proof.

Lemma 4.20. *The order $<$ in $\mathcal{F}^s$ is total.*

Proof. Let $C^s_D(x)$ and $C^s_D(y)$ be different elements of $\mathcal{F}^s$ and suppose by contradiction that neither $C^s_D(x) < C^s_D(y)$ nor $C^s_D(y) < C^s_D(x)$. We assume that x and y are spines since in the other cases we obtain the result similarly. Consider $\gamma_1, \gamma_2, \gamma_3 \subset a^u$ sub-arcs as in Figure 17.

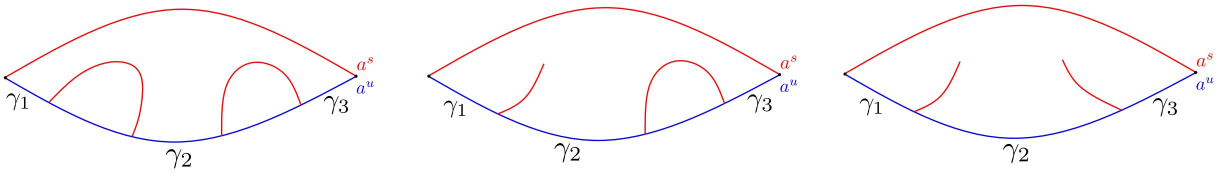


Figure 17 – Non-comparable stable arcs.

Since x and y are spines, it follows that $E = D \setminus (C^s_D(x) \cup C^s_D(y))$ is connected. For $1 \leq i < j \leq 3$, define

$$A_{ij} = \{x \in E : C^s_D(x) \cap \gamma_i \neq \emptyset, C^s_D(x) \cap \gamma_j \neq \emptyset\}.$$

By the definition of the subarcs, we have that A_{ij} is non-empty for all $1 \leq i < j \leq 3$. In addition, these sets are closed and cover E. Since E is connected, we can find z that belongs to all of them. Hence $\#(C^s_D(z) \cap a^u) \geq 3$ and this contradicts Lemma 4.19. In the other cases we just need to choose the appropriate arcs γ_i and change the definition of the set E accordingly. Figure 18 illustrates these choices.

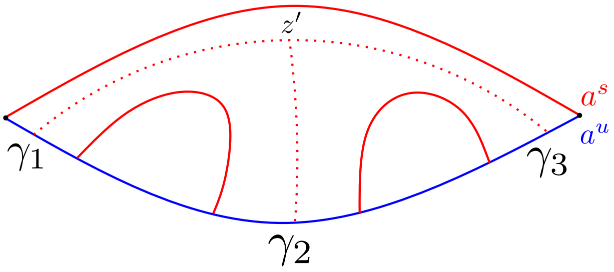


Figure 18

□

Note that the regularity condition is important to conclude the order is total since non-regular sectors can contain points $z \in \operatorname{int} D$ such that $\#(C^s_D(z) \cap a^u) \geq 3$, so the existence of a point in the intersection of A_{ij} would not imply a contradiction.

Lemma 4.20 ensures that inside a regular bi-asymptotic sector there is at most one spine, but does not necessarily prove the existence of a spine. This will be consequence of the following result that proves continuity for the variation of the arcs inside a regular bi-asymptotic sector. It is based on Lemma 3.3 in [4]. Lemma 4.19 ensures that $\#(C^s_D(x)\cap a^u)\leq 2$ for every $x\in a^u$. Then we consider a map $g: a^u\to a^u$ (see Figure 19) defined as

$$C^s_D(x)\cap a^u=\{x,g(x)\}.$$

Note that if $C^s_D(x)\cap a^u=\{x\}$, then $g(x)=x$ and Lemma 4.19 ensures that $C^s_D(x)$ contains a spine.

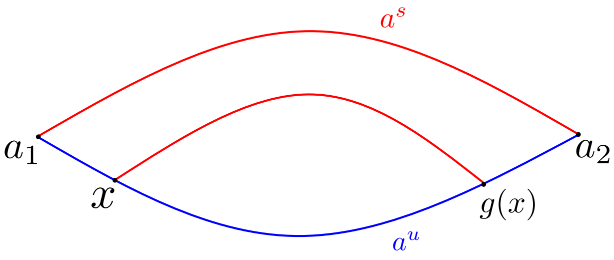


Figure 19

Note that we could have problems to define g in sectors that are not regular since in these cases $C^s_D(a_1)$ may not coincide with a^s and intersect a^u in three different points.

Lemma 4.21. *The map $g: a^u\to a^u$ is continuous.*

Proof. As a^u is homeomorphic to $[0,1]$, we can induce an ordering in a^u such that $a_1<a_2$. We prove that g is decreasing with this order, and since g is bijective, we conclude continuity. Suppose, by contradiction, that g is not decreasing, so there exist $x<y$ such that $g(x)<g(y)$. Note that $g(x)\neq x$ since $x<y$ and $x=g(x)<g(y)$ ensure the arcs $s(x,g(x);x)$ and $s(y,g(y);y)$ are not comparable, contradicting Lemma 4.20. The same reason ensures that $g(y)\neq y$. If $x<y<g(x)<g(y)$, then there is an intersection between $s(x,g(x);x)$ and $s(y,g(y);y)$, contradicting Lemma 4.12. If $x<g(x)<y<g(y)$ the arcs $s(x,g(x);x)$ and $s(y,g(y);y)$ are not comparable, contradicting Lemma 4.20. Other cases are obtained from these cases interchanging x and $g(x)$ or y and $g(y)$, leading to the same contradictions. ☐

The following proposition gathers all results we obtained so far about bi-asymptotic sectors. It is one of the directions of the equivalence between the existence of sectors and spines that we want to prove.

Proposition 4.22. *If D is a bi-asymptotic sector with diameter less than δ, then* $\operatorname{int} D\cap \operatorname{Spin}(f)\neq\emptyset$. *Moreover, if D is regular, then* $\#(\operatorname{int} D\cap\operatorname{Spin}(f))=1$.

Proof. First, we note that every regular bi-asymptotic sector D contains a unique spine in its interior. Since a^u is homeomorphic to $[0,1]$ and $g\colon a^u\to a^u$ is continuous, it follows that g has a fixed point, and since g is decreasing, this fixed point is unique. Clearly, a_1 and a_2 are not fixed points of g. This and the transversality proved in Lemma 4.16 ensure that this single spine is contained in $\operatorname{int} D$.

Now let D be a non-regular bi-asymptotic sector. Without loss of generality, let us assume that D does not satisfy the regularity condition at a_2, that is, $C^s_\varepsilon(x)$ enters D through a_2. If $C^s_\varepsilon(x)$ does not intersect the open arc $a^u \setminus \{a_1, a_2\}$, then the connected component of $C^s_\varepsilon(x)$ containing a_2 does not separate D, and by Lemma 4.14 we find a spine in $\operatorname{int} D$. If $C^s_\varepsilon(x)$ intersects $a^u \setminus \{a_1, a_2\}$ at a point y, then the transversality of the intersection ensures that

$$s(a_2, y; x) \cup u(a_2, y; x)$$

bounds a regular bi-asymptotic sector contained in D (see Figure 20). Thus, there exists a spine in $\operatorname{int} D$.

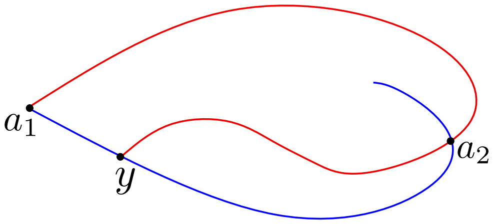


Figure 20

□

Now we prove the other direction of the equivalence.

Lemma 4.23. *If* $x \in \operatorname{Spin}(f)$*, then there exists* D_x *a regular bi-asymptotic sector such that* $x \in \operatorname{int} D_x$*.*

Proof. We begin choosing $y \in C^s_\varepsilon(x) \cap B_{\frac{\delta}{2}}(x)$ such that $s(x, y; x) \subset B_{\frac{\delta}{2}}(x)$. Since $C^u_{\frac{\varepsilon}{2}}(y)$ is an arc transversal to $C^s_\varepsilon(x)$ at y and $y \notin \operatorname{Spin}(f)$, we can choose a small neighborhood U of $C^s_\varepsilon(x)$ and $t_1, t_2 \in C^u_{\frac{\varepsilon}{2}}(y) \cap \partial U$ satisfying:

1. $t_1 \not\sim t_2$,
2. $u(t_1, t_2; y)$ intersects ∂U only at t_1 and t_2,
3. $u(t_1, t_2; y) \cap C^s_\varepsilon(x) = \{y\}$.

Since $u(t_1, t_2; y)$ is transversal to $C^s_\varepsilon(x)$ at y, and $u(t_1, t_2; y) \cap C^s_\varepsilon(x) = \{y\}$, then $u(t_1, t_2; y)$ divides both U and $C^s_\varepsilon(x)$ in exactly two components (see Figure 21).

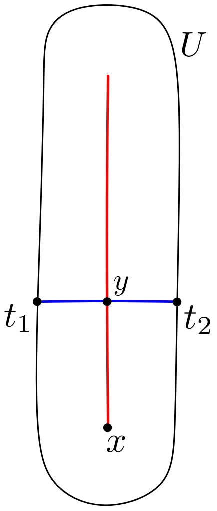


Figure 21

The semi-continuity of the map $a \to C^s_{\frac{\varepsilon}{2}}(a)$ (see page 15 and Theorem 6.7.1 of [3]) allows us to choose a disk V centered at x such that

$$C^s_{\frac{\varepsilon}{2}}(z) \subset U \quad \text{for every} \quad z \in V$$

and $C^s_{\frac{\varepsilon}{2}}(x) \cap \partial V = \{\tau\}$. In particular, $\partial V \setminus \{\tau\}$ is connected. For $i = 1, 2$, let $c_i = u(y, t_i, y)$ and

$$\mathcal{E}_i = \{z \in \partial V \setminus \{\tau\} : C^s_{\frac{\varepsilon}{2}}(z) \cap c_i \neq \emptyset\}.$$

Since $\mathcal{E}_i$ is closed and $\partial V \setminus \{\tau\}$ is connected, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are not empty, then $\mathcal{E}_1 \cap \mathcal{E}_2 \neq \emptyset$. Hence, by Lemma 4.12 there is $z \in \partial V \setminus \{\tau\}$ such that $C^s_{\frac{\varepsilon}{2}}(z)$ intersects $c_1 \setminus \{y\}$ and $c_2 \setminus \{y\}$. Choose intersections a_1, a_2, respectively, such that $s(z, a_1; z) \setminus \{a_1\}$ and $s(z, a_2; z) \setminus \{a_2\}$ are contained in the component of x in $U \setminus u(t_1, t_2; y)$. Hence these arcs bound a disk D with

$$\partial D = s(z, a_1; z) \cup s(z, a_2; z) \cup u(a_1, a_2; y).$$

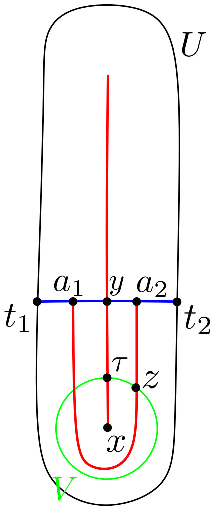


Figure 22

Since $a_1, a_2 \in C^s_{\frac{\varepsilon}{2}}(z)$, it follows that $a_2 \in C^s_{\varepsilon}(a_1)$. Also, $a_1, a_2 \in C^u_{\frac{\varepsilon}{2}}(y)$ ensures that $a_2 \in C^u_{\varepsilon}(a_1)$. Thus,

$$a_2 \in C^s_{\varepsilon}(a_1) \cap C^u_{\varepsilon}(a_1).$$

The regularity condition follows from the fact that $\operatorname{int} D$ is contained in one connected component of $U \setminus u(t_1, t_2; y)$ and the intersections in a_1 and a_2 being transversal ensure that $C^s_{\varepsilon}(z)$ enters the other component of $U \setminus u(t_1, t_2; y)$ through a_1 and a_2. This proves that D is a regular bi-asymptotic sector and $x \in \operatorname{int} D$.

Now assume that $\mathcal{E}_1 \neq \emptyset$ and $\mathcal{E}_2 = \emptyset$. In this case, $\partial V \setminus \{\tau\} \subset \mathcal{E}_1$. Choose

$$y' \in C^s_{\varepsilon}(x) \cap \operatorname{int} V$$

and $u(t_1', t_2'; y')$ a sub-arc of $C^u_{\frac{\varepsilon}{2}}(y')$ such that $t_1' \nsim t_2'$ and $u(t_1', t_2'; y')$ is contained in $\operatorname{int} U$ except, possibly, at t_1' and t_2'. Since $\operatorname{diam}(C^u_{\frac{\varepsilon}{2}}(y')) \geq \delta$, we can assume that either t_1' or t_2' belongs to ∂U. Let us assume that $t_1' \in \partial U$. Then we choose a sequence $(z_n)_{n \in \mathbb{N}} \subset \partial V \setminus \{\tau\}$ such that

1. $z_n \to \tau$ as $n \to \infty$,
2. z_n and c_1 are in different components of
$$U \setminus (u(y', t_1'; y') \cup s(y', y; x) \cup u(y, t_2; y))$$
for every $n \in \mathbb{N}$, and
3. $C^s_{\frac{\varepsilon}{2}}(z_n) \cap c_1 \neq \emptyset$ for every $n \in \mathbb{N}$.

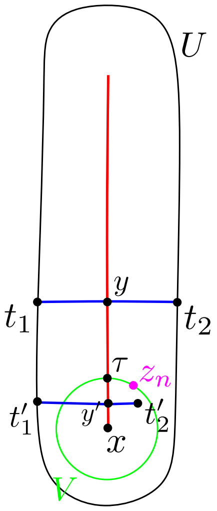


Figure 23

In particular, (3) implies that

$$C^s_{\frac{\varepsilon}{2}}(z_n) \cap u(y', t_1'; y') \neq \emptyset \quad \text{for every} \quad n \in \mathbb{N}.$$

This ensures the existence of $n_0 \in \mathbb{N}$ such that

$$C^s_{\frac{\varepsilon}{2}}(z_n) \cap u(y', t_2'; y') \neq \emptyset \quad \text{for every} \quad n \geq n_0,$$

since the intersection between $C^s_\varepsilon(x)$ and $u(t_1', t_2'; y)$ is transversal at y', and the semi-continuity again ensure that $C^s_{\frac{\varepsilon}{2}}(z_n)$ converge to a subset of $C^s_\varepsilon(x)$. As in the previous case, the regularity condition is assured by the transversality of these intersections and we obtain a regular bi-asymptotic sector D with $x \in \operatorname{int} D$.

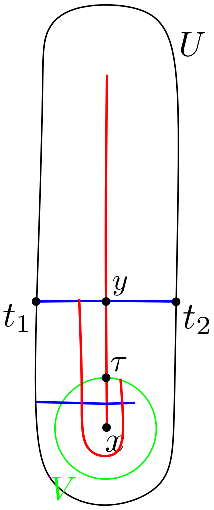


Figure 24

$\square$

A direct consequence of the previous lemma and Proposition 4.22 is the following:

Corollary 4.24. *Every spine is isolated from other spines, and* $\operatorname{Spin}(f)$ *is at most countable.*

In the proof of Lemma 4.23, the diameter of the sector D containing the spine x depends on the spine and is not necessarily uniform over all spines, i.e., for a sequence of distinct spines the diameters of the respective sectors could become arbitrarily small. The following Lemma ensures the existence of bi-asymptotic sectors with uniform diameter close to the local stable/unstable continua of every spine. However, these sectors may not contain the associated spines in its interior.

Lemma 4.25. *Let* $x \in \operatorname{Spin}(f)$ *and* $y \in C^s_{\frac{\varepsilon}{2}}(x)$ *with* $0 < \operatorname{diam}(s(x, y; x)) < \frac{\delta}{4}$. *If* $u(a_1, a_2; y)$ *is a sub-arc of* $C^u_\varepsilon(y)$ *containing* y *in its interior, then there exists a neighborhood* V *of* x *and* $z \in \partial V$ *such that* $\#(C^s_\varepsilon(z) \cap u(a_1, a_2; y)) \geq 2$.

Proof. Let U be a small neighborhood of $C^s_\varepsilon(x)$ and $u(a_1', a_2'; y)$ be a sub-arc of $u(a_1, a_2; y)$ contained in U such that

1. $u(a_1', a_2'; y) \cap C^s_\varepsilon(x) = \{y\}$,

2. $u(a_1', a_2'; y) \cap \partial U = \{a_1', a_2'\}$,
3. $\mathrm{d}(x, u(a_1', a_2'; y)) < \frac{\delta}{2}$.

The existence of this sub-arc is ensured by cw_F-expansiveness. Thus, $u(a_1', a_2'; y)$ separates U in two components and the diameter of the component W of x is less than δ. As in the proof of Lemma 4.23, let V be an open disk centered at x such that $C^s_\varepsilon(z) \subset U$ for all $z \in V$, and let being τ be the first intersection of ∂V with $C^s_\varepsilon(x)$. There exists $r > 0$ such that if $z \in \partial V \setminus \{\tau\}$ is r-close to τ, then $C^s_{\frac{\varepsilon}{2}}(z)$ intersects $u(a_1', a_2'; y)$. Since $\mathrm{Spin}(f)$ is at most countable (see Corollary 4.24), there exists

$$z \in B_r(\tau) \cap \partial V \setminus \{\tau\}$$

such that $C^s_\varepsilon(z)$ does not contain spines. Since $\mathrm{diam}(W) < \delta$, Lemma 4.14 ensures that $C^s_\varepsilon(z)$ separates W. Since $C^s_\varepsilon(z) \subset U$, it follows that

$$\#(C^s_{\frac{\varepsilon}{2}}(z) \cap u(a_1', a_2'; y)) \geq 2.$$

$\square$

Figure 25 – Long bi-asymptotic sectors close to $C^s_\varepsilon(x)$.

4.4 Cw_F-hyperbolicty and cw_2-hyperbolicty

In this section we prove Theorem 4.1. We first prove that a cw_F-hyperbolic surface homeomorphism that is not cw_2-expansive must contain infinitely many spines, and then we prove the finiteness of the set of spines for a cw_3-hyperbolic homeomorphism using the long bi-asymptotic sectors we constructed at the end of Section 3. These two results ensure that cw_3-hyperbolicity implies cw_2-hyperbolicity on surfaces. If a cw_F-hyperbolic surface homeomorphism is not cw_1-hyperbolic, then there exists arbitrarily small bi-asymptotic sectors, but this does not necessarily imply the existence of an infinite number of spines, since all these sectors can converge to the same single spine, as in the pseudo-Anosov diffeomorphism of $\mathbb{S}^2$. If the homeomorphism is not cw_2-expansive, then for each $\varepsilon > 0$ there exists $x \in S$ such that $\#(C^s_\varepsilon(x) \cap C^u_\varepsilon(x)) \geq 3$. We note below that either there exists two bi-asymptotic sectors with disjoint interiors and connected by their boundaries (see Figure 26) or there exists a non-regular bi-asymptotic sector. In the first case, Lemma 4.22 ensures that each of these sectors must contain a spine, and in the second case we will prove that inside any non-regular sector there exist at least two distinct spines. This ensures the existence of an infinite number of spines, since these sectors cannot accumulate in a pair of two spines when ε converges to zero.

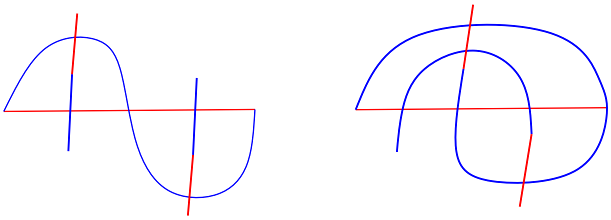


Figure 26 – Examples of bi-asymptotic sectors and their spines.

Proposition 4.26. *If a^s and a^u bound a non-regular bi-asymptotic sector D with* $\operatorname{diam}(D) \leq \delta$*, then there exist at least two distinct spines in* $\operatorname{int} D$*.*

Proof. Let a_1 and a_2 be the end points of a^s and a^u and assume that

$$C^s_D(a_1) \cap \operatorname{int} D \neq \emptyset \quad \text{and} \quad C^u_D(a_1) \cap \operatorname{int} D \neq \emptyset.$$

Lemma 4.14 ensures that both $C^s_D(a_1) \setminus \{a^s\}$ and $C^u_D(a_1) \setminus \{a^u\}$ either separate D or contain a spine in $\operatorname{int} D$. If both separate D, then there exist two bi-asymptotic sectors with disjoint interiors and connected by their boundaries, and Lemma 4.22 ensures the existence of two distinct spines in $\operatorname{int} D$ (see Figure 27).

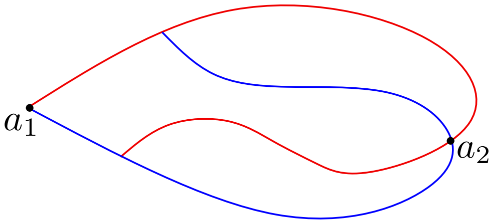


Figure 27

This figure illustrates the case where $C^s_D(a_1) \setminus \{a^s\}$ does not intersect $C^u_D(a_1) \setminus \{a^u\}$. In the case they intersect, in only a finite number of points by cw$_F$-expansive- ness, then the existence of an intersection a_n such that $s(a_2, a_n; a_1)$ and $u(a_2, a_n; a_1)$ bound a regular sector, ensures the existence of two sectors with disjoint interiors and connected by their boundaries. Indeed, if $C^s_D(a_1) \setminus \{a^s \cup s(a_2, a_n; a_1)\}$ does not intersect $u(a_2, a_n; a_1)$, then

$$C^s_D(a_1) \setminus \{a^s \cup s(a_2, a_n; a_1)\} \quad \text{and} \quad C^u_D(a_1) \setminus \{a^u \cup u(a_2, a_n; a_1)\}$$

bound a sector with interior disjoint from the first one (see Figure 28).

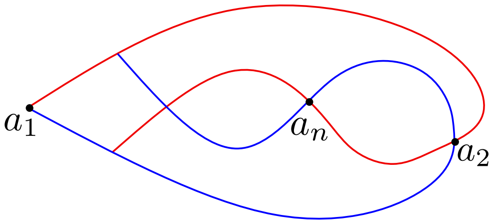


Figure 28

If $C^s_D(a_1) \setminus \{a^s \cup s(a_2, a_n; a_1)\}$ intersects $u(a_2, a_n; a_1)$ at a_m, then it also forms a sector with disjoint interior from the first one since the regular intersection in a_n ensures that $s(a_n, a_m; a_1)$ is outside the interior of the first sector. In this case, the sector bounded by $s(a_2, a_m; a_1)$ and $u(a_2, a_m; a_1)$ is not regular at a_m (see Figure 29). Since both $C^s_D(a_1) \setminus \{a^s\}$ and $C^u_D(a_1) \setminus \{a^u\}$ separate D, there exists an intersection a_j such that $s(a_2, a_j; a_1)$ and $u(a_2, a_j; a_1)$ bound a regular sector. Indeed, if a_m is a non-regular intersection as above, then $C^s_D(a_1) \setminus \{a^s \cup s(a_2, a_m; a_1)\}$ intersects $u(a_2, a_m; a_1)$ at a_j, and the sector formed by $s(a_2, a_j; a_1)$ and $u(a_2, a_j; a_1)$ is regular since the intersection at a_j comes from inside the non-regular sector bounded by $s(a_2, a_m; a_1)$ and $u(a_2, a_m; a_1)$ (see Figure 29).

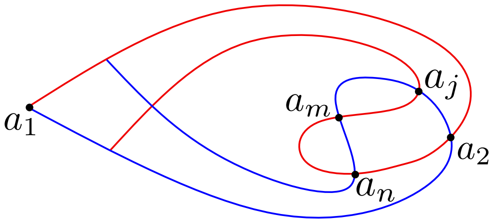


Figure 29

If both $C^s_D(a_1) \setminus \{a^s\}$ and $C^u_D(a_1) \setminus \{a^u\}$ do not separate D, then both end in spines. If these spines are actually the same spine and $C^s_D(a_1) \setminus \{a^s\}$ and $C^u_D(a_1) \setminus \{a^u\}$ do not intersect before the spine, then they bound a bi-asymptotic sector with this spine as one of the end points of the sector, so Lemma 4.22 ensures the existence of a spine in the interior of this sector that is, hence, distinct from the first spine (see Figure 30).

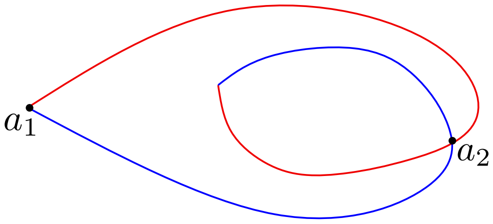


Figure 30

If $C^s_D(a_1) \setminus \{a^s\}$ and $C^u_D(a_1) \setminus \{a^u\}$ intersect before the spine in the end, then either there is an intersection bounding a regular sector and we argument as in the case above to create two sectors with disjoint interiors, or there are only non-regular intersections and we create a sector between the last one before the spine and the spine. In both cases a second spine appears. Now assume that $C^s_D(a_1)\setminus\{a^s\}$ separates D but $C^u_D(a_1)\setminus\{a^u\}$ does not. If $C^s_D(a_1)\setminus\{a^s\}$ does not intersect $C^u_D(a_1)\setminus\{a^u\}$ in $\operatorname{int} D$, then $C^s_D(a_1)\setminus\{a^s\}$ forms a bi-asymptotic sector with a sub-arc of a^u that does not contain the spine in $C^u_D(a_1)\setminus\{a^u\}$. Then Lemma 4.22 ensures the existence of a spine in this sector that is, hence, distinct from the first spine (see Figure 20). If $C^s_D(a_1)\setminus\{a^s\}$ intersects $C^u_D(a_1)\setminus\{a^u\}$ at $a_i\in\operatorname{int} D$, and $s(a_2,a_i,a_1)$ and $u(a_2,a_i,a_1)$ bound a regular sector, then there is a spine at the interior of this sector that is distinct from the spine in $C^u_D(a_1)\setminus\{a^u\}$ (see Figure 16a). If $s(a_2,a_i,a_1)$ and $u(a_2,a_i,a_1)$ bound a non-regular sector, then the spine in $C^u_D(a_1)\setminus\{a^u\}$ is inside this sector, but since $C^s_D(a_1)\setminus\{a^s\}$ separates D, it must intersect $C^u_D(a_1)\setminus\{a^u\}$ an additional time creating a sector that does not contain the spine in $C^u_D(a_1)\setminus\{a^u\}$ in its interior (see Figure 31).

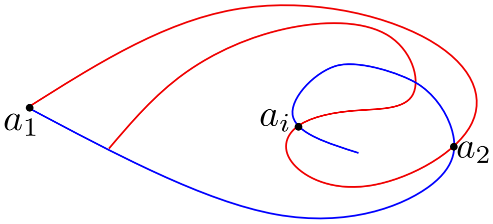


Figure 31

□

We are ready to prove our main theorem.

Proof of Theorem 4.1. Assume that f is a cw_F-hyperbolic homeomorphism with a finite number of spines. If f is not cw_2-expansive, then for each $\alpha\in(0,\delta)$ there exists $x\in S$ such that $\#(C^s_\alpha(x)\cap C^u_\alpha(x))\geq 3$. Using an order $<$ in $C^s_\alpha(x)$ we can choose three consecutive points $a_1,a_2,a_3\in C^s_\alpha(x)\cap C^u_\alpha(x)$, that is, $a_1<a_2<a_3$ and there are no points of $C^s_\alpha(x)\cap C^u_\alpha(x)$ in (a_1,a_2) and (a_2,a_3). This ensures that the stable/unstable arcs connecting a_1 to a_2, and also a_2 to a_3, form bi-asymptotic sectors (this is not true in the case there are intersections in (a_1,a_2) or (a_2,a_3) as in Figure 29). If the by-asymptotic sector formed by the stable/unstable arcs from a_1 to a_2 is regular in a_2, then the stable and unstable arcs from a_2 to a_3 form a bi-asymptotic sector with interior disjoint from the interior of the sector from a_1 to a_2. This ensures the existence of two distinct spines α-close. If the intersection in a_2 is not regular, then Proposition 4.26 ensures the existence of two distinct spines inside the non-regular sector. This proves that for each $\alpha\in(0,\delta)$ there exist two distinct spines α-close, and, hence, we obtain an infinite number of distinct spines for f, contradicting the assumption. This proves that f is cw_2-hyperbolic.

Now we prove that cw_3-hyperbolicity implies finiteness of the number of spines. This is the only step of the proof that we do not know how to prove assuming cw_F-hyperbolicity. Let f be a cw_3-hyperbolic homeomorphism and assume the existence of an infinite number of distinct spines. For each $\alpha \in (0, \varepsilon)$ choose $\delta_\alpha \in (0, \alpha)$ satisfying the Lemmas 4.3, 4.5, and 4.6. Consider x_1 and x_2 spines such that there exists $y \in C^s_\alpha(x_1) \cap C^u_\alpha(x_2) \neq \emptyset$ with

$$\operatorname{diam} s(x_1, y; x_1) < \frac{\delta_\alpha}{4} \quad \text{and} \quad \operatorname{diam} u(x_2, y; x_2) < \frac{\delta_\alpha}{4}.$$

Note that y is not a spine, since it is contained in the local stable continua of a spine. Lemma 4.25 ensures the existence of long bi-asymptotic sectors close to $C^s_\varepsilon(x_1)$ and $C^u_\varepsilon(x_2)$ intersecting in four distinct points (see Figure 32). Since this can be done for any $\alpha > 0$, it follows that f is not cw$_3$-hyperbolic.

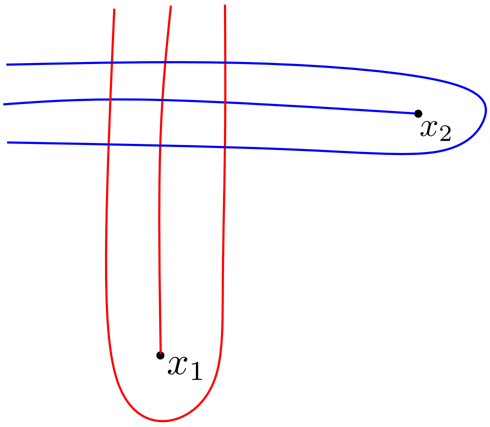


Figure 32

□

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