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Igor André Ramos Almeida

On two problems in quaternionic hyperbolic geometry

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Igor André Ramos Almeida

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On two problems in quaternionic hyperbolic geometry

IGOR ANDRÉ RAMOS ALMEIDA

Tese defendida e aprovada pela banca examinadora constituída por:

Prof. Nikolai Alexandrovitch Goussevskii
UFMG

Prof. Claudio Gorodski
USP

Prof. Ezequiel Rodrigues Barbosa
UFMG

Prof. Maurício Barros Corrêa Júnior
UFMG

Prof. Vyacheslav Futorny
USP

Belo Horizonte, 17 de setembro de 2021.

Av. Antônio Carlos, 6627 – Campus Pampulha - Caixa Postal: 702
CEP-31270-901 - Belo Horizonte – Minas Gerais - Fone (31) 3409-5963
e-mail: pgmat@mat.ufmg.br - home page: http://www.mat.ufmg.br/pgmat

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Não, não tenho caminho novo,
o que tenho de novo
é o jeito de caminhar.

Aprendi:
(o caminho me ensinou)
a caminhar cantando
como convém
a mim
e aos que vão comigo
pois já não vou mais sozinho.

Thiago de Mello

Resumo

Neste trabalho consideramos dois problemas em geometria hiperbólica quaterniônica. O primeiro problema é descrever módulos de configurações finitas de pontos no espaço projetivo quaterniônico com relação a ação diagonal do grupo de isometrias do espaço hiperbólico quarterniônico $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Para resolver este problema introduzimos alguns invariantes de triplas de pontos da geometria hiperbólica quarternionica. Em particular, definimos análogos quarterniônicos para os invariantes de Goldman para configurações mistas, introduzidos por ele em geometria hiperbólica complexa. O segundo problema está relacionado com a geometria de bissetores no espaço hiperbólico quaterniônico $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Desenvolvemos a teoria básica de bissetores em geometria hiperbólica quarterniônica. Em particular, mostramos que bissetores possuem várias decomposições em subvariedades totalmente geodésicas. Em contraste com a geometria hiperbólica complexa, onde bissetores admitem apenas dois tipos de decomposição (descritas por Mostow e Goldman), mostraremos que no caso quarteniônico a geometria de bissetores é bem mais rica. Descrevemos uma família infinita de diferentes decomposições de bissetores em $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ por subvariedades totalmente geodésicas de $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isométricas ao espaço hiperbólico complexo $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.

Palavras-chave: Espaço hiperbólico quaterniônico; Invariante de Cartan; Matriz de Gram; Invariante de forma de Brehm; Bissetores.

Abstract

In this work, we consider two problems in quaternionic hyperbolic geometry. The first problem is to describe the moduli of finite configurations of points in quaternionic projective space relative to the action of the isometry group of quaternionic hypebolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. To solve this problem, we introduce some basic invariants of triples of points in quaternionic hyperbolic geometry. In particular, we define quaternionic analogues of the Goldman invariants for mixed configurations of points introduced by him in complex hyperbolic geometry. The second problem is related to geometry of bisectors in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. We develop some of the basic theory of bisectors in quaternionic hyperbolic geometry. In particular, we show that bisectors enjoy various decompositions by totally geodesic submanifolds. In contrast to complex hyperbolic geometry, where bisectors admit only two types of decomposition (described by Mostow and Goldman), we show that in the quaternionic case geometry of bisectors is more rich. We describe an infinite family of different decompositions of bisectors in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ by totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.

Keywords: Quaternionic hyperbolic space; Cartan's invariant; Gram matrix; Brehm shape invariant; Bisectors.

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Introduction

In this work, we consider two problems in quaternionic hyperbolic geometry.

In the first part, we define important numerical invariants associated to an ordered triple of points in quaternionic projective space. These invariants describe the equivalence classes of such triples relative to the action of the isometry group of quaternionic hypebolic space $\mathrm{H}^n_{\mathbb{Q}}$. We give a construction of the quaternionic angular invariant, an analogue of the Cartan invariant in complex hyperbolic geometry, see [8], which parametrizes triples of isotropic points. Also, we represent a quaternionic analogue of Brehm's shape invariant, see [6], in complex hyperbolic geometry, which is used to parametrize triples of points in $\mathrm{H}^n_{\mathbb{Q}}$. Then we define a quaternionic analogue of the Goldman η- invariant for mixed configurations of points introduced by him in complex hyperbolic geometry to study the intersection of bisectors, see [14]. Using these invariants, we describe the moduli of the corresponding triples relative to the action of the isometry group of quaternionic hypebolic space $\mathrm{H}^n_{\mathbb{Q}}$. In order to solve the congruence problems, we use the methods related to Gram matrices of configurations of points developed in complex hyperbolic geometry in [6], [5], [10], [11], [12], [16], [17].

The second part of this work develops the theory of bisectors in quaternionic hyperbolic space. It is well known that the rank one symmetric spaces of non-compact type are real, complex and quaternionic hyperbolic spaces, or a Cayley hyperbolic plane. In order to construct discrete groups of isometries in these geometries, one needs an appropriate notion of polyhedra, which becomes nontrivial in the absence totally geodesic hypersurfaces from which to form a polyhedron's faces in spaces with non-constant sectional curvature. This happens in complex and quaternionic hyperbolic geometries, and in geometry of the Cayley hyperbolic plane. Since the faces of Dirichlet fundamental polyhedra are in bisectors, it is natural to use bisectors as the building blocks for polyhedra in these geometries. Therefore, it is necessary to understand the geometric structure of such hypersurfaces. In complex hyperbolic geometry, it was done by Mostow [19] and Goldman [14]. They showed that bisectors in complex hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$ admit a decomposition into complex hyperplanes and into totally real totally geodesic submanifolds (meridian decomposition). Goldman also proved that these decompositions are unique. We will show that some basic results from complex hyperbolic geometry carry over to the quaternionic case. But, it will be shown that the geometry of bisectors in quaternionic hyperbolic space is more rich. First, we prove an analogue of the Mostow decomposition of bisectors in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then we show that bisectors in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ admit non-singular foliations by totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. More exactly, we will show that any bisector in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is the total space of a fiber bundle over a bisector in a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ whose fibers are totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ as well. We will show that such decompositions are not unique. Finally, we will show that any bisector in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is a union of totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ intersecting in a common point. In some sense, this is an analogue of the Goldman meridian decomposition. We call such decompositions the fan decompositions. The existence of fan decompositions implies, in particular, that any bisector in quaternionic hyperbolic space is starlike with respect to any point in its real spine.

The work is organized as follows. In section I, we summarize some basic results about quaternions and geometry of quaternionic hyperbolic space. In section II, we describe the moduli of triples of points in quaternionic hyperbolic geometry. In section III, we study geometry of bisectors in quaternionic hyperbolic space.

Section 1

Preliminaries

In this section, we recall some basic results related to quarternions and geometry of projective and hyperbolic spaces.

1.1 Quaternions

First, we recall some basic facts about the quaternions we need. The quaternions $\mathbb{Q}$ are the $\mathbb{R}$-algebra generated by the symbols i, j, k with the relations

$$l^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = 1, \quad ki = -ik = j.$$

So, $\mathbb{Q}$ is a skew field and a 4-dimensional division algebra over the reals.

Let $a \in \mathbb{Q}$. We write $a = a_0 + a_1 i + a_2 j + a_3 k$, $a_i \in \mathbb{R}$, then by definition

$$\bar{a} = a_0 - a_1 i - a_2 j - a_3 k, \quad \mathrm{Re}\, a = a_0, \quad \mathrm{Im}\, a = a_1 i + a_2 j + a_3 k.$$

Note that, in contrast with the complex numbers, $\mathrm{Im}\, a$ is not a real number (if $a_i \neq 0$ for some $i = 1, 2, 3$), and that conjugation obeys the rule

$$\overline{ab} = \bar{b}\bar{a}.$$

Also, we define $|a| = \sqrt{a\bar{a}}$. We have that if $a \neq 0$ then

$$a^{-1} = \bar{a}/|a|^2.$$

In what follows, we will identify the reals numbers with $\mathbb{R}1$ and, usually use the subfield $\mathbb{C} \subset \mathbb{Q}$ generated by 1 and i.

Two quaternions a and b are called *similar* if there exists $\lambda \neq 0$ such that

$$a = \lambda b \lambda^{-1}.$$

By replacing λ by $\tau = \lambda/|\lambda|$, we may always assume λ to be unitary.

The following proposition was proved in [6].

Proposition 1.1. *Two quaternions a and b are similar if and only if* $\mathrm{Re}\ a = \mathrm{Re}\ b$ *and* $|a| = |b|$. *Moreover, every similarity class contains a complex number, unique up to conjugation.*

Corollary 1.2. *Any quaternion a is similar to a unique complex number $b = b_0 + b_1 i$ such that* $b_1 \geq 0$.

Also, this proposition implies that every quaternion is similar to its conjugate.

Example: $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}$.

We say that $a \in \mathbb{Q}$ is *imaginary* if $\mathrm{Re}(a) = 0$. Let us suppose that a is imaginary and $|a| = 1$. Then $a^2 = -1$. This implies that the real span of 1 and a is a subfield of $\mathbb{Q}$ isomorphic to the field of complex numbers. We denote this subfield by $\mathbb{C}(a)$. It is easy to prove that any subfield of $\mathbb{Q}$ containing real numbers and isomorphic to the field of complex numbers is of the form $\mathbb{C}(a)$ for some a imaginary with $|a| = 1$.

The following was proved in [9].

Proposition 1.3. *Let a be as above. Then the centralizer of a is* $\mathbb{C}(a)$.

More generally, for any $\lambda \in \mathbb{Q}$, let $\mathbb{C}(\lambda)$ also denote the real span of 1 and λ.

Proposition 1.4. *Let $\lambda \in \mathbb{Q} \setminus \mathbb{R}$. Then the centralizer of λ is* $\mathbb{C}(\lambda)$.

1.2 Hyperbolic spaces

In this section we discuss two models for the hyperbolic spaces, its isometry group and totally geodesics submanifolds. The references for this section is [9] and [20].

1.2.1 Projective Model

We denote by $\mathbb{F}$ one of the real division algebras $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$. Let us write $\mathbb{F}^{n+1}$ for a right $\mathbb{F}$ -vector space of dimension $n+1$. The $\mathbb{F}$- projective space $\mathbb{P}\mathbb{F}^n$ is the manifold of right $\mathbb{F}$- lines in $\mathbb{F}^{n+1}$. Let π denote a natural projection from $\mathbb{F}^{n+1} \setminus \{0\}$ to the projective space $\mathbb{P}\mathbb{F}^n$.

Let $\mathbb{F}^{n,1}$ denote a $(n+1)$-dimensional $\mathbb{F}$-vector space equipped with a Hermitian form $\Psi = \langle -, - \rangle$ of signature $(n, 1)$. Then there exists a (right) basis in $\mathbb{F}^{n+1}$ such that the Hermitian product is given by $\langle v, w \rangle = v^* J_{n+1} w$, where v^* is the Hermitian transpose of v and $J_{n+1} = (a_{ij})$ is the $(n+1) \times (n+1)$-matrix with $a_{ij} = 0$ for all $i \neq j$, $a_{ii} = 1$ for all $i = 1, \dots, n$, and $a_{ii} = -1$ when $i = n+1$.

That is,

$$\langle v, w \rangle = \bar{v}_1 w_1 + \ldots + \bar{v}_n w_n - \bar{v}_{n+1} w_{n+1},$$

where v_i and w_i are coordinates of v and w in this basis. We call such a basis in $\mathbb{F}^{n,1}$ an *orthogonal basis* defined by a Hermitian form $\Psi = \langle -, - \rangle$.

Let V_-, V_0, V_+ be the subsets of $\mathbb{F}^{n,1} \setminus \{0\}$ consisting of vectors where $\langle v, v \rangle$ is negative, zero, or positive respectively. Vectors in V_0 are called *null* or *isotropic*, vectors in V_- are called *negative*, an vectors in V_+ are called *positive*. Their projections to $\mathbb{P}\mathbb{F}^n$ are called *isotropic*, *negative*, and *positive* points respectively.

The projective model of *hyperbolic space* $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ is the set of negative points in $\mathbb{P}\mathbb{F}^n$, that is, $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}} = \pi(V_-)$.

We will consider $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ equipped with the Bergman metric [9]:

$$d(p, q) = \cosh^{-1}\{|\Psi(v, w)|[\Psi(v, v)\Psi(w, w)]^{-1/2}\},$$

where $p, q \in \mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$, and $\pi(v) = p, \pi(w) = q$.

The boundary $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{F}} = \pi(V_0)$ of $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ is the sphere formed by all isotropic points.

Let $\mathrm{U}(n, 1; \mathbb{F})$ be the unitary group corresponding to this Hermitian form Φ. If $g \in \mathrm{U}(n, 1; \mathbb{F})$, then $g(V_-) = V_-$ and $g(v\lambda) = (g(v))\lambda$, for all $\lambda \in \mathbb{F}$. Therefore $\mathrm{U}(n, 1; \mathbb{F})$ acts in $\mathbb{P}\mathbb{F}^n$, leaving $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ invariant.

The group $\mathrm{U}(n, 1; \mathbb{F})$ does not act effectively in $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$. The kernel of this action is the center $\mathrm{Z}(n, 1; \mathbb{F})$. Thus the projective group $\mathrm{PU}(n, 1; \mathbb{F}) = \mathrm{U}(n, 1; \mathbb{F})/\mathrm{Z}(n, 1; \mathbb{F})$ acts effectively. In spite of this fact we shall find it convenient to deal with $\mathrm{U}(n, 1; \mathbb{F})$ rather than $\mathrm{PU}(n, 1; \mathbb{F})$. The center

$\mathbb{Z}(n,1,\mathbb{F})$ in $\mathrm{U}(n,1;\mathbb{F})$ is $\{\pm\mathrm{E}\}$ if $\mathbb{F}=\mathbb{R}$ or $\mathbb{Q}$, and is the circle group $\{\lambda\mathrm{E}:|\lambda|=1\}$ if $\mathbb{F}=\mathbb{C}$. Here E is the identity transformation of $\mathbb{F}^{n,1}$.

It is well-known, see for instance [9], that $\mathrm{PU}(n,1;\mathbb{F})$ acts transitively in $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ and doubly transitively on $\partial\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$.

We remark that

- if $\mathbb{F}=\mathbb{R}$ then $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ is a real hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$,
- if $\mathbb{F}=\mathbb{C}$ then $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ is a complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$,
- if $\mathbb{F}=\mathbb{Q}$ then $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ is a quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

It is easy to show [9] that $\mathrm{H}^{1}_{\mathbb{Q}}$ is isometric to $\mathrm{H}^{4}_{\mathbb{R}}$.

1.2.2 The ball model

In this section, we consider the space $\mathbb{F}^{n,1}$ equipped by an orthogonal basis

$$e=\{e_1,\ldots,e_n,e_{n+1}\}.$$

For any $v\in\mathbb{F}^{n,1}$, we write $v=(z_1,\ldots,z_n,z_{n+1})$, where z_i, $i=1,\ldots,n+1$ are coordinates of v in this basis.

If $v=(z_1,\ldots,z_n,z_{n+1})\in V_-$, the condition $\langle v,v\rangle<0$ implies that $z_{n+1}\neq 0$. Therefore, we may define a set of coordinates $w=(w_1,\ldots,w_n)$ in $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ by $w_i(\pi(z))=z_iz_{n+1}^{-1}$. In this way $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ becomes identified with the ball

$$B=B(F)=\{w=(w_1,\ldots,w_n)\in\mathbb{F}^n:\Sigma_{i=1}^{n}|w_i|^2<1\}.$$

With this identification the map $\pi:V_-\to\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ has the coordinate representation $\pi(z)=w$, where $w_i=z_iz_{n+1}^{-1}$.

1.2.3 Totally geodesic submanifolds

We will need the following result, see [9], which describes all totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$.

Let F be a subfield of $\mathbb{F}$. An F-*unitary subspace* of $\mathbb{F}^{n,1}$ is an F-subspace of $\mathbb{F}^{n+1}$ in which the Hermitian form Φ is F-valued. An F-*hyperbolic subspace* of $\mathbb{F}^{n,1}$ is an F-unitary subspace in which the Hermitian form Φ is non-degenerate and indefinite.

Proposition 1.5. *Let M be a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$. Then either*

(a) M is the intersection of the projectivization of an F-hyperbolic subspace of $\mathbb{F}^{n,1}$ with $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ for some subfield F of $\mathbb{F}$, or

(b) $\mathbb{F} = \mathbb{Q}$, and M is a 3-dimensional totally geodesic submanifold of a totally geodesic quaternionic line $\mathrm{H}^1_{\mathbb{Q}}$ in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

From the last proposition follows that

- in the real hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$ any totally geodesic submanifold is isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{R}},\ k = 1,\dots,n$,
- in the complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ any totally geodesic submanifod is isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{C}},\ k = 1,\dots,n$, or to $\mathrm{H}^{\mathrm{k}}_{\mathbb{R}},\ k = 1,\dots,n$,
- in the quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ any totally geodesic submanifold is isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{Q}},\ k = 1,\dots,n$, or to $\mathrm{H}^{\mathrm{k}}_{\mathbb{C}},\ k = 1,\dots,n$, or to $\mathrm{H}^{\mathrm{k}}_{\mathbb{R}},\ k = 1,\dots,n$, or to a 3-dimensional totally geodesic submanifold of a totally geodesic quaternionic line $\mathrm{H}^1_{\mathbb{Q}}$.

In what follows we will use the following terminology:

- A totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^1_{\mathbb{Q}}$ is called a *quaternionic geodesic*.
- A totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^1_{\mathbb{C}}$ is called a *complex geodesic*.
- A totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^2_{\mathbb{R}}$ is called a *real plane*.

A basic property of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is that two distinct points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \cup \partial \mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ span a unique quaternionic geodesic. We also remark that any 2-dimensional totally geodesic submanifold of a totally geodesic quaternionic line $\mathrm{H}^1_{\mathbb{Q}}$ is isometric to $\mathrm{H}^1_{\mathbb{C}}$.

Proposition 1.6. *Let V be a subspace of $\mathbb{F}^{n,1}$. Then each linear isometry of V into $\mathbb{F}^{n,1}$ can be extended to an element of $\mathrm{U}(n,1;\mathbb{F})$.*

This is a particular case of the Witt theorem, see [20].

Corollary 1.7. *Let $S \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ be a totally geodesic submanifold. Then each linear isometry of S into $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ can be extended to an element of the isometry group of $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$.*

An interesting class of totally geodesic submanifolds of the quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ are submanifods which we call totally geodesic submanifolds of complex type, or simply, submanifolds of complex type. Their construction is the following. Let $\mathbb{C}^{n+1}(a) \subset \mathbb{Q}^{n+1}$ be the subset of vectors in $\mathbb{Q}^{n+1}$ with coordinates in $\mathbb{C}(a)$, where a is a imaginary quaternion, $|a| = 1$. Then $\mathbb{C}^{n+1}(a)$ is a vector space over the field $\mathbb{C}(a)$. The projectivization of $\mathbb{C}^{n+1}(a)$, denoted by $\mathrm{M}^n(\mathbb{C}(a))$, is a submanifold of $\mathbb{P}\mathbb{Q}^n$ of real dimension $2n$. We call this submanifold $\mathrm{M}^n(\mathbb{C}(a))$ a *submanifold of complex type of maximal dimension*. It is clear that the space $\mathbb{C}^{n+1}(a)$ is indefinite. The intersection $\mathrm{M}^n(\mathbb{C}(a))$ with $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, called a *totally geodesic submanifold of complex type of maximal dimension*. It was proven in [9] that all these submanifolds are isometric, and, moreover, they are globally equivalent with respect to the isometry group of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, that is, for any two such submanifolds M and N there exists an element $g \in \mathrm{PU}(n,1;\mathbb{Q})$ such that $\mathrm{M} = \mathrm{g}(\mathrm{N})$. In particular, all of them are globally equivalent with respect to $\mathrm{PU}(n,1;\mathbb{Q})$ to the *canonical totally geodesic complex submanifold* $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ defined by $\mathbb{C}^{n+1} \subset \mathbb{Q}^{n+1}$. This corresponds to the canonical subfield of complex numbers $\mathbb{C} = \mathbb{C}(i) \subset \mathbb{Q}$ in the above.

If $V^{k+1} \subseteq \mathbb{C}^{n+1}(a)$ is a subspace of complex dimension $k+1$, then its projectivization W is called a *submanifold of complex type of complex dimension* k. When $V^{k+1} \subseteq \mathbb{C}^{n+1}$, then its projectivization W is called a *canonical submanifold of complex type of complex dimension* k In this case, we will denote W as $\mathbb{P}\mathbb{C}^k$.

If $V^{k+1} \subseteq \mathbb{C}^{n+1}(a)$ is indefinite, then the intersection of its projectivization with $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. We call this submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ a *totally geodesic submanifold of complex type of complex dimension* k. When $V^{k+1} \subseteq \mathbb{C}^{n+1}$, we call this totally geodesic submanifold a *canonical totally geodesic submanifold of complex type of complex dimension* k, or a *canonical complex hyperbolic submanifold of dimension* k of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. In this case, we will denote this submanifold as $\mathrm{H}^{\mathrm{k}}_{\mathbb{C}}$.

1.2.4 Stabilizators of totally geodesic submanifolds

Let M be a totally geodesic submanifold in $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$. Let $I(M)$ denote the subgroup of $\mathrm{PU}(n,1;\mathbb{F})$ which leaves M invariant.

If F is a subfield of $\mathbb{F}$, we let $\mathcal{N}^+(F,\mathbb{F}) = \{\lambda \in \mathbb{F}^+; \lambda F \lambda^{-1} = F\}$, where $\mathbb{F}^+$ denotes the subgroup in $\mathbb{F}$ of elements with norm one.

The following propositions can be found in [9], pp.74.

Proposition 1.8. *Let M be a totally geodesic submanifold, such that M is the intersection of the projectivization of an F-hyperbolic subspace of $\mathbb{F}^{n,1}$ with $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ for some subfield F of $\mathbb{F}$. Then the elements $g \in I(M)$ are of the form:*

$$g = \begin{pmatrix} A\lambda & 0 \\ 0 & B \end{pmatrix},$$

where $A \in U(m,1;F)$, $\lambda \in \mathcal{N}^+(F,\mathbb{F})$ and $B \in U(n-m;\mathbb{F})$.

Proposition 1.9. *Let M be a 3-dimensional totally geodesic submanifold of a totally geodesic quaternionic line $\mathrm{H}^1_{\mathbb{Q}}$ in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then the elements $g \in I(M)$ are of the form:*

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \text{ where } A = \begin{pmatrix} a & b \\ -\varepsilon b & \varepsilon a \end{pmatrix} \in U(1,1;\mathbb{Q}),$$

$\varepsilon = \pm 1$ *and* $B \in U(n-1;\mathbb{Q})$.

1.2.5 A little more about the isometry group of the quaternionic hyperbolic space

Let us consider the complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. It has a natural complex structure related to its isometry group, and its isometry group is generated by the holomorphic isometry group, which is the projective group $\mathrm{PU}(n,1;\mathbb{C})$ and the anti-holomorphic isometry σ induced by complex conjugation in $\mathbb{C}^{n+1}$. This anti-holomorphic isometry corresponds to the unique non-trivial automorphism of the field of complex numbers.

Quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ has no complex structure related to its isometry group. Its isometry group sometimes is denoted by $\mathrm{PSp}(n,1)$ because in the orthogonal basis of $\mathbb{Q}^{n,1}$ the group $\mathrm{U}(n,1;\mathbb{Q})$ can be considered as a group of symplectic matrices preserving the Hermitian form $\Psi = \langle -,- \rangle$ of signature $(n,1)$, that is, $\mathrm{Sp}(n,1)$

We recall that if $f : \mathbb{Q} \to \mathbb{Q}$ is an automorphism of $\mathbb{Q}$, then f is an inner automorphism of $\mathbb{Q}$, that is, $f(q) = aqa^{-1}$ for some $a \in \mathbb{Q}$.

It follows from the fundamental theorem of projective geometry (see Theorem 2.26 in [2]) that each projective map $L : \mathbb{P}\mathbb{Q}^n \to \mathbb{P}\mathbb{Q}^n$ is induced by a semilinear (linear) map $\tilde{L} : \mathbb{Q}^{n+1} \to \mathbb{Q}^{n+1}$.

It is easy to see that if a projective map $L : \mathbb{P}\mathbb{Q}^n \to \mathbb{P}\mathbb{Q}^n$ is induced by a semilinear map $\tilde{L} : \mathbb{Q}^{n+1} \to \mathbb{Q}^{n+1}$, $\tilde{L}(v) = ava^{-1}$, $v \in \mathbb{Q}^{n+1}$, $a \in \mathbb{Q}$, then it is induced also by a linear map $v \mapsto av$. Therefore, the projective group of $\mathbb{P}\mathbb{Q}^n$ is the projectivization of the linear group of $\mathbb{Q}^{n+1}$.

It follows from the above that if $L : \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \to \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is an isometry, then L is induced by a linear isometry $\tilde{L}$ in $\mathrm{Sp}(n,1)$, which is the projectivization of the linear group $\mathrm{U}(n,1;\mathbb{Q}) = \mathrm{Sp}(n,1)$.

Next we consider a curious map, which is an isometry of the quaternionic hyperbolic space, that has no analogue in geometries over commutative fields. Let $\tilde{L_a} : v \mapsto av$, $v \in \mathbb{Q}^{n+1}$, $a \in \mathbb{Q}$, a is not real. The projectivization of this linear map defines a non-trivial map $L_a : \mathbb{P}\mathbb{Q}^n \to \mathbb{P}\mathbb{Q}^n$. We remark that in projective spaces over commutative fields, like $\mathbb{R}$ and $\mathbb{C}$, this map L_a is identity. It easy to see that $\tilde{L_a} \in \mathrm{U}(n,1;\mathbb{Q})$ if and only if $|a| = 1$.

Proposition 1.10. *The fixed point set of L_a is a totally geodesic submanifold of complex type of maximal dimension in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. This submanifold is globally equivalent to the canonical complex hyperbolic submanifold $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.*

Proof. The proof follows from Proposition 1.4. □

Section 2

Moduli of triples in quaternionic hyperbolic geometry

In this section, we describe numerical invariants associated to an ordered triple of points in $\mathbb{PQ}^n$ which define the equivalence class of a triple relative to the diagonal action of $\mathrm{PU}(n,1;\mathbb{Q})$.

2.1 The Gram matrix

Let $p = (p_1, \dots, p_m)$ be an ordered m-tuple of distinct points in $\mathbb{PQ}^n$ of quaternionic dimension $n \geq 1$. Then we consider a Hermitian quaternionic $m \times m$-matrix

$$G = G(p, v) = (g_{ij}) = (\langle v_i, v_j \rangle),$$

where $v = (v_1, \dots, v_m)$, $v_i \in \mathbb{Q}^{n,1}$, $\pi(v_i) = p_i$, is a lift of p.

We call G a *Gram matrix* associated to a m-tuple p defined by v. Of course, G depends on the chosen lifts v_i. When replacing v_i by $v_i\lambda_i$, $\lambda_i \in \mathbb{Q}$, $\lambda_i \neq 0$, we get $\tilde{G} = D^* \, GD$, where D is a diagonal quaternionic matrix, $D = diag(\lambda_1, \dots, \lambda_m)$,

We say that two Hermitian quaternionic $m \times m$ - matrices H and $\tilde{H}$ are *equivalent* if there exists a diagonal quaternionic matrix $D = diag(\lambda_1, \dots, \lambda_m)$, $\lambda_i \neq 0$, such that $\tilde{H} = D^* \; H \; D$.

Thus, to each ordered m-tuple p of distinct points in $\mathbb{PQ}^n$ is associated an equivalence class of Hermitian quaternionic $m \times m$ - matrices.

Proposition 2.1. *Let $p = (p_1, \cdots, p_m)$ be an ordered m-tuple of distinct negative points in $\mathbb{PQ}^n$. Then the equivalence class of Gram matrices associated to p contains a matrix $G = (g_{ij})$ such that $g_{ii} = -1$ and $g_{1j} = r_{1j}$ are real positive numbers for $j = 2, \dots, m$.*

Proof. Let $v = (v_1, \ldots, v_m)$ be a lift of p. Since the vectors v_i are negative, we have that $g_{ij} \neq 0$ for all $i, j = 1, \ldots, m$, see, for instance, [20]. First, by appropriate re-scaling, we may assume that $g_{ii} = \langle v_i, v_i \rangle = -1$. Indeed, since $\langle v_i, v_i \rangle < 0$, then $\lambda_i = 1/\sqrt{-\langle v_i, v_i \rangle}$ is well defined. Since $\lambda_i \in \mathbb{R}$, we have that

$$\langle v_i\lambda_i, v_i\lambda_i \rangle = \lambda_i^2 \langle v_i, v_i \rangle = \langle v_i, v_i \rangle / |\langle v_i, v_i \rangle| = -1.$$

Then we get the result we need by replacing the vectors v_i, $i = 2, \ldots, m$, if necessarily, by $v_i\lambda_i$, where

$$\lambda_i = \overline{\langle v_1, v_i \rangle} / |\langle v_1, v_i \rangle|.$$

Indeed, since $|\lambda_i| = 1$, we have that $\langle v_i\lambda_i, v_i\lambda_i \rangle = -1$, $i = 2, \ldots, m$. On the other hand, for all $i > 1$

$$\langle v_1, v_i\lambda_i \rangle = \langle v_1, v_i \rangle \lambda_i = |\langle v_1, v_i \rangle| > 0.$$

□

Let p and q be two points in $\mathbb{PQ}^n$. We say that p and q are *orthogonal* if $\langle v, w \rangle = 0$ for some lifts v and w of p and q respectively. It is clear that if p and q are *orthogonal*, then $\langle v, w \rangle = 0$ for all lifts v and w of p.

Let $p = (p_1, \ldots, p_m)$ be an ordered m-tuple of distinct points in $\mathbb{PQ}^n$. We call p *generic* if p_i and p_j are not orthogonal for all $i, j = 1, \ldots, m$.

Let $G = (g_{ij})$ be a Gram matrix associated to p. Then p is generic if and only if $g_{ij} \neq 0$ for all $i, j = 1, \ldots, m$.

Proposition 2.2. *Let $p = (p_1, \cdots, p_m)$ be an ordered generic m-tuple of distinct positive points in $\mathbb{PQ}^n$. Then the equivalence class of Gram matrices associated to p contains a matrix $G = (g_{ij})$ such that $g_{ii} = 1$ and $g_{1j} = r_{1j}$ are real positive numbers for $j = 2, \ldots, m$.*

Proof. The proof is a slight modification of the proof of Proposition 2.1 □

It is easy to see that a matrix $G = (g_{ij})$ defined in Propositions 2.1 and 2.2 is unique. We call this matrix G a *normal form* of the associated Gram matrix. Also, we call G the *normalized* Gram matrix.

We recall that a subspace $V \subset \mathbb{F}^{n,1}$ is called *singular* or *degenerate* if it contains at least one non-zero vector that is orthogonal to all vectors in V. Otherwise, V is called *regular*.

Remark 2.3. It is easy to see that if V is singular then V contains at least one isotropic vector and does not contain negative vectors.

Lemma 2.4. *Let $V = \{v_1, \dots, v_m\}$ and $W = \{w_1, \dots, w_m\}$ be two subspaces of $\mathbb{Q}^{n,1}$ spanned by v_i and w_i. Suppose that V and W are regular, and $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle$, for all $i, j = 1, \dots m$. Then the correspondence $v_i \mapsto w_i$ can be extended to an isometry of $\mathbb{Q}^{n,1}$.*

Proof. The proof follows from Theorem 1 in [15]. □

Proposition 2.5. *Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered m-tuples of distinct negative points in $\mathbb{PQ}^n$. Then p and p' are congruent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if their associated Gram matrices are equivalent.*

Proof. Let V and V' be the subspaces spanned by v_i and v'_i, $i = 1, \dots, m$. Then it is clear that V and V' are regular. Since all the points p_i are distinct, Lemma 2.4 implies that the map defined by $v \mapsto v'$ extends to a linear isometry of $\mathbb{Q}^{n,1}$. The projectivization of this isometry maps p in p'. □

Corollary 2.6. *Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered m-tuples of distinct negative points in $\mathbb{PQ}^n$. Then p and p' are congruent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if their normalized Gram matrices are equal.*

By applying the similar arguments, we get the following

Proposition 2.7. *Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered generic m-tuples of distinct positive points in $\mathbb{PQ}^n$ such that the subspaces V and V' spanned by some lifts of p and p' are regular. Then p and p' are congruent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if only if their associated Gram matrices are equivalent.*

Corollary 2.8. *Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered generic m-tuples of distinct positive points in $\mathbb{PQ}^n$ such that the subspaces V and V' spanned by some lifts of p and p' are regular. Then p and p' are congruent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if their normalized Gram matrices are equal.*

Remark 2.9. It is easy to see that a subspace V in $\mathbb{Q}^{n,1}$ is singular if and only if its projectivization is a projective submanifold of $\mathbb{PQ}^n$ tangent to $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ at an unique isotropic point p, lying, except this point p, in the positive part of $\mathbb{PQ}^n$.

2.2 Invariants of triangles in quaternionic hyperbolic geometry

In this section, we define some invariants of ordered triples of points in $\mathbb{P}\mathbb{Q}^n$ which generalize Cartan's angular invariant and Brehm's shape invariants in complex hyperbolic geometry to quaternionic hyperbolic geometry.

2.2.1 Quaternionic Cartan's angular invariant

First, we recall the definition of Cartan's angular invariant in complex hyperbolic geometry.

Let $p = (p_1, p_2, p_3)$ be an ordered triple of points in $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then Cartan´s invariant $\mathbb{A}(p)$ of p is defined as

$$\mathbb{A}(p) = \arg(-\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle),$$

where v_i is a lift of p_i.

It is easy to see that $\mathbb{A}(p)$ is well-defined, that is, it is independent of the chosen lifts, and it satisfies the inequality

$$-\pi/2 \leq A(p) \leq \pi/2.$$

The inequalities above follow from the following proposition, see [14].

Proposition 2.10. *Let $v, w, u \in \mathbb{C}^{2,1}$ be isotropic or negative vectors, then*

$$\mathrm{Re}(\langle v, w\rangle\langle w, u\rangle\langle u, v\rangle) \leq 0.$$

Remark 2.11. It is possible to extend the Cartan invariant of triples of isotropic points to triples of points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}} \cup \partial \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Indeed, no difficulty arises in the above definition because $\langle v, w\rangle \neq 0$ for any $v, w \in V_0 \cup V_-$.

Cartan's invariant is the only invariant of an ordered triple of isotropic points in the following sense:

Proposition 2.12. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ be two ordered triples of distinct points in $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then p and p' are congruent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{C})$ if and only if $\mathbb{A}(p) = \mathbb{A}(p')$.*

The Cartan angular invariant $\mathbb{A}$ enjoys also the following properties, see [14]:

1. If σ is a permutation, then

$$\mathbb{A}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = \text{sign}(\sigma)\mathbb{A}(p_1, p_2, p_3),$$

2. Let $p = (p_1, p_2, p_3)$ be an ordered triple of points in $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then p lies in the boundary of a complex geodesic in $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ if and only if $\mathbb{A}(p) = \pm\pi/2$,

3. Let $p = (p_1, p_2, p_3)$ be an ordered triple of points in $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then p lies in the boundary of a real plane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ if and only if $\mathbb{A}(p) = 0$,

4. *Cocycle property*. Let p_1, p_2, p_3, p_4 be points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}} \cup \partial \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then

$$\mathbb{A}(p_1, p_2, p_3) + \mathbb{A}(p_1, p_3, p_4) = \mathbb{A}(p_1, p_2, p_4) + \mathbb{A}(p_2, p_3, p_4).$$

5. If $g \in \mathrm{PU}(n, 1; \mathbb{C})$ is a holomorphic isometry, then $\mathbb{A}(g(p)) = \mathbb{A}(p)$, and if g is an anti-holomorphic isometry, then $\mathbb{A}(g(p)) = -\mathbb{A}(p)$.

Now we define Cartan's angular invariant in quaternionic hyperbolic geometry.

Let $v = (v_1, v_2, v_3)$ be an ordered triple of vectors in $\mathbb{Q}^{n,1}$. Then

$$H(v_1, v_2, v_3) = \langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \langle v_3, v_1 \rangle$$

is called the *Hermitian triple product*.

An easy computation gives the following.

Lemma 2.13. *Let $w_i = v_i\lambda_i$, $\lambda_i \in \mathbb{Q}$, $\lambda_i \neq 0$, then*

$$H(w_1, w_2, w_3) = \langle w_1, w_2, w_3 \rangle = \overline{\lambda}_1 H(v_1, v_2, v_3)\lambda_1 |\lambda_2|^2 |\lambda_3|^2 =$$

$$\frac{\overline{\lambda}_1}{|\lambda_1|} H(v_1, v_2, v_3) \frac{\lambda_1}{|\lambda_1|} |\lambda_1|^2 |\lambda_2|^2 |\lambda_3|^2.$$

Corollary 2.14. *Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points, $p_i \in \mathbb{PQ}^n$. Then there exists a lift $v = (v_1, v_2, v_3)$ of $p = (p_1, p_2, p_3)$ such that $H(v_1, v_2, v_3)$ is a complex number.*

Proof. The proof follows by applying Lemma 2.13, Proposition 1.1 and the equality $\frac{\bar{\lambda}}{|\lambda|}\frac{\lambda}{|\lambda|} = 1$. □

The historically first definition of Cartan's angular invariant in quaternionic hyperbolic geometry was given in [1]. In this paper, the authors defined the quaternionic Cartan angular invariant $\mathbb{A}(p) = \mathbb{A}(p_1, p_2, p_3)$ of an ordered triple $p = (p_1, p_2, p_3)$ of distinct points, $p_i \in \mathrm{H}^n_{\mathbb{Q}} \cup \partial \mathrm{H}^n_{\mathbb{Q}}$, to be the angle between the quaternion $H(v_1, v_2, v_3)$ and the real line $\mathbb{R} \subset \mathbb{Q}$, where v_i is a lift of p_i.

They proved that $\mathbb{A}(p)$ does not depend on the chosen lifts, and it is the only invariant of a triple of isotropic points in the above sense.

Next, we represent a convenient formula to compute $\mathbb{A}(p)$.

Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points, $p_i \in \mathrm{H}^n_{\mathbb{Q}} \cup \partial \mathrm{H}^n_{\mathbb{Q}}$, and $v = (v_1, v_2, v_3)$ be a lift of $p = (p_1, p_2, p_3)$. Then it follows from Proposition 1.1 and Lemma 2.13 that

$$\mathbb{A}^*(p) = \arccos\left(-\frac{\mathrm{Re}(\ H(v_1, v_2, v_3))}{|H(v_1, v_2, v_3)|}\right)$$

does not depend on the chosen lifts v_i.

Proposition 2.15. $\mathbb{A}(p) = \mathbb{A}^*(p)$.

Proof. The proof follows from an easy computation, see also [7]. □

The quaternionic Cartan angular invariant $\mathbb{A}(p) = \mathbb{A}(p_1, p_2, p_3)$ introduced above enjoys the following properties, see [1]:

1. $0 \leq A(p) \leq \pi/2$,
2. If σ is a permutation, then

$$\mathbb{A}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = \mathbb{A}(p_1, p_2, p_3),$$

3. Let $p = (p_1, p_2, p_3)$ be an ordered triple of points in $\partial \mathrm{H}^n_{\mathbb{Q}}$. Then p lies in the boundary of a complex geodesic in $\mathrm{H}^n_{\mathbb{Q}}$ if and only if $\mathbb{A}(p) = \pi/2$,
4. Let $p = (p_1, p_2, p_3)$ be an ordered triple of points in $\partial \mathrm{H}^n_{\mathbb{Q}}$. Then p lies in the boundary of a real plane in $\mathrm{H}^n_{\mathbb{Q}}$ if and only if $\mathbb{A}(p) = 0$.

It is seen that this quaternionic Cartan angular invariant, in contrast to the Cartan angular invariant in complex hyperbolic geometry, is non-negative, symmetric, and one can show that it does not enjoy the cocycle property. We think that the definition of the Cartan angular invariant in quaternionic hyperbolic geometry given above does not explain well its relation with the classical Cartan angular invariant in complex hyperbolic geometry. In what follows, we discuss another possible definitions of quaternionic Cartan´s angular invariant and explain why the quaternionic Cartan angular invariant must be non-negative and symmetric.

We start with the following simple fact which has a far reaching consequence for the construction of invariants of triples in quaternionic hyperbolic geometry. We think that this may also help for defining of invariants of triples in other hyperbolic geometries, for instance, in the hyperbolic octonionic plane.

Proposition 2.16. *Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points, $p_i \in \mathbb{PQ}^n$. Then there exists a submanifold $W \subset \mathbb{PQ}^n$ of complex type of complex dimension 2 passing through the points p_i, that is, $p_i \in W$, $i = 1, 2, 3$. Moreover, this submanifold W can be chosen, up to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$, to be the canonical complex submanifold $\mathbb{PC}^2 \subset \mathbb{PQ}^n$.*

Proof. Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points, $p_i \in \mathbb{PQ}^n$, and $v = (v_1, v_2, v_3)$ be a lift of $p = (p_1, p_2, p_3)$.

First, let us suppose that p_i and p_j are not orthogonal, for all $i \neq j$. Consider

$$H(v_1, v_2, v_3) = \langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \langle v_3, v_1 \rangle.$$

It follows from Corollary 2.14 and Lemma 2.13 that there exists λ_1 such that $H(v_1\lambda_1, v_2, v_3)$ is a complex number. Let us fix this λ_1, and let $w_1 = v_1\lambda_1$. Then it follows from Lemma 2.13 that $H(w_1, v_2, v_3)$ is a complex number for any lifts of p_2 and p_3. Let now $\lambda_2 = \langle v_2, w_1 \rangle$, $w_2 = v_2\lambda_2$. We have that $\langle w_1, w_2 \rangle$ is real. Setting $\lambda_3 = \langle v_3, w_1 \rangle$ and $w_3 = v_3\lambda_3$, we have that $\langle w_3, w_1 \rangle$ is real. Since $H(w_1, w_2, w_3) \in \mathbb{C}$, it follows that $\langle w_2, w_3 \rangle \in \mathbb{C}$. Therefore for this normalizarion all Hermitian products are complex. This implies that the complex span of w_1, w_2, w_3 is $\mathbb{C}$- unitary subspace of $\mathbb{Q}^{n,1}$ of dimension 3, see Section 1.2.3, and, therefore, the points p_1, p_2, p_3 lie in a submanifold $W \subset \mathbb{PQ}^n$ of complex type of complex dimension 2.

Now let us suppose that the set $\{p_1, p_2, p_3\}$ contains orthogonal points. Assume, for example, that p_1 and p_2 are orthogonal, that is, $\langle v_1, v_2 \rangle = 0$, where v_1 and v_2 are lifts of p_1 and p_2. Let v_3 be a lift of p_3. Setting $\lambda_1 = \langle v_1, v_3 \rangle$, $\lambda_2 = \langle v_2, v_3 \rangle$, and $w_1 = v_1\lambda_1$, $w_2 = v_2\lambda_2$, $w_3 = v_3$, we have that all

Hermitian products are real. It follows that the complex span of w_1, w_2, w_3 is $\mathbb{C}$- unitary subspace of $\mathbb{Q}^{n,1}$ of dimension 3, and, therefore, the points p_1, p_2, p_3 lie in a submanifold $W \subset \mathbb{P}\mathbb{Q}^n$ of complex type of complex dimension 2.

The rest follows from the results of Section 1.2.3. □

Corollary 2.17. *Let $p = (p_1, p_2, p_3)$ be a triple of distinct negative points, $p_i \in \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then p lies in a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ of complex type of complex dimension 2.*

Corollary 2.18. *Let $p = (p_1, p_2, p_3)$ be a triple of distinct isotropic points, $p_i \in \partial\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then p lies in the boundary of a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ of complex type of complex dimension 2.*

These results show that geometry of triples of points in $\mathbb{P}\mathbb{Q}^n$ is, in fact, geometry of triples of points in $\mathbb{P}\mathbb{C}^2$. Therefore, all the invariants of triples of points in $\mathbb{P}\mathbb{Q}^n$ relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ can be constructed using 2-dimensional complex hyperbolic geometry.

First, we give a new definition of the Cartan angular invariant in quaternionic hyperbolic geometry.

Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct isotropic points, $p_i \in \partial\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. By Corollary 2.18, we know that p lies in the boundary of a totally geodesic submanifold M of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ of complex type of complex dimension 2.

We know that M is the projectivization of negative vectors in a 3-dimensional complex subspace V^3 of $\mathbb{C}^{n,1}$, $\mathbb{C}^{n,1} \subset \mathbb{Q}^{n,1}$, and the boundary of $\mathrm{H}^2_{\mathbb{C}}$ is the projectivization of isotropic vectors in V^3.

Let $v = (v_1, v_2, v_3)$ be a lift of $p = (p_1, p_2, p_3)$. Then $v_i \in V^3$. We define

$$\mathbb{A}^{**} = \mathbb{A}^{**}(p) = \arg(-\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle).$$

Now we briefly explain how to use this invariant to classify ordered triples of isotropic points relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$.

Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ be two ordered triples of distinct points in $\partial\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Suppose that $\mathbb{A}^{**}(p) = \mathbb{A}^{**}(p')$. We will show that p and p' are equivalent relative to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$.

By Corollary 2.18, we have that p is contained in the boundary of a totally geodesic submanifold $M(p)$ of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ of complex type of complex dimension 2. Also the same is true for p', where $p' \in \partial M(p')$. We know, see Section 1.2.3, that all such submanifolds are equivalent relative to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$, therefore, there exists an element $f \in \mathrm{PU}(n, 1; \mathbb{Q})$ such that $f(M(p)) = M(p')$. So, we can assume without loss of generality that p and p' are in the boundary of the same submanifold $M = \mathrm{H}^2_{\mathbb{C}} \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then, by applying the classical result of Cartan, we have that there exists a complex hyperbolic isometry g of M, $g \in \mathrm{PU}(2, 1; \mathbb{C})$, such that $g(p) = p'$. This isometry g can be extended to an element of $\mathrm{PU}(n, 1; \mathbb{Q})$ by the Witt theorem. This proves that p and p' are equivalent relative the action of $\mathrm{PU}(n, 1; \mathbb{Q})$.

Next we show why it is more convenient to consider quaternionic Cartan's angular invariant to be symmetric and non-negative (non-positive)

We need the following lemma, see Lemma 7.1.7 in [14].

Lemma 2.19. *Let $p = (p_1, p_2, p_3)$ be an ordered triples of distinct points in $\partial \mathrm{H}^2_{\mathbb{C}}$. Then there exists a real plane $P \subset \mathrm{H}^2_{\mathbb{C}}$ such that inversion (reflection) i_p in P satisfies*

$$i_P(p_1) = p_2, \quad i_P(p_2) = p_1, \quad i_P(p_3) = p_3.$$

Remark 2.20. We recall that i_P is an anti-holomorphic isometry of $\mathrm{H}^2_{\mathbb{C}}$ and

$$\mathbb{A}(p_2, p_1 p_3) = -\mathbb{A}(p_1, p_2, p_3).$$

Also, as it is easy to see that any anti-holomorphic isometry of $\mathrm{H}^2_{\mathbb{C}}$ is a composition of an element of $\mathrm{PU}(2, 1; \mathbb{C})$ and an anti-holomorphic reflection.

Proposition 2.21. *Let $p = (p_1, p_2, p_3)$ be an ordered triples of distinct points in $\partial \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, $n > 1$. Then there exists an element $f \in \mathrm{PU}(n, 1; \mathbb{Q})$ such that*

$$f(p_1) = p_2, \quad f(p_2) = p_1, \quad f(p_3) = p_3.$$

Proof. Repeating the arguments above, we can assume that p is in the boundary of the $M = \mathrm{H}^2_{\mathbb{C}} \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Let $P \subset M = \mathrm{H}^2_{\mathbb{C}}$ be a real plane and i_P be the reflection in P acting in M as in Lemma 2.19. We will show that this map can be extended to an isometry of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Notice that i_P as a map of M is not induced by a linear map, it is induced by a semilinear map, therefore, we cannot apply the Witt theorem in this case.

Let K be a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^2_{\mathbb{Q}}$ which contains M.

An easy argument shows that there exists a totally geodesic submanifold N of complex type of complex dimension 2 in K intersecting M orthogonally along P (see also Theorem 3.19) . Let i_N be the geodesic reflection in N. We have that i_N is an element of the isometry group of K, isomorphic to $\mathrm{PU}(2,1;\mathbb{Q})$, whose fixed point set is N. We notice that i_N is induced by a linear map, therefore, it follows from the Witt theorem that i_N can be extended to an isometry f in $\mathrm{PU}(n,1;\mathbb{Q})$. Note that by construction f leaves M invariant and its restriction to M coincides with i_P. Therefore, $f(p_1,p_2,p_3) = (p_2,p_1,p_3)$. It is easy to see that the fixed point set of f in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is a totally geodesic submanifold of complex type of maximal dimension in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. This submanifold is globally equivalent to the canonical submanifold $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}} \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. □

Corollary 2.22. *Let $p = (p_1,p_2,p_3)$ and $p' = (p'_1,p'_2,p'_3)$ be two ordered triples of distinct points in $\partial\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Suppose that $\mathbb{A}^{**}(p) = -\mathbb{A}^{**}(p')$. Then p and p' are equivalent relative to the diagonal action of* $\mathrm{PU}(n,1;\mathbb{Q})$.

Remark 2.23. This imply that if for two ordered triples of distinct isotropic points $p = (p_1,p_2,p_3)$ and $p' = (p'_1,p'_2,p'_3)$ we have that $|\mathbb{A}^{**}(p)| = |\mathbb{A}^{**}(p')|$, then p is equivalent to p' relative to the diagonal action of group $\mathrm{PU}(n,1;\mathbb{Q})$. Therefore, it is natural to consider instead of $\mathbb{A}^{**}$ its absolute value. Then the invariant $|\mathbb{A}^{**}|$ lies in the interval $[0,\pi/2]$ and is symmetric.

Corollary 2.24. $|\mathbb{A}^{**}| = \mathbb{A}^{*}$.

2.2.2 Quaternionic Brehm's invariants

First, we recall the definition of the Brehm shape invariants in complex hyperbolic geometry.

Let $p = (p_1,p_2,p_3)$ be an ordered triple of distinct points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ and $v = (v_1,v_2,v_3)$ be a lift of $p = (p_1,p_2,p_3)$.

Brehm [6] defined the invariant which he called the *shape invariant*, or σ - *invariant*:

$$\sigma(p) = \mathrm{Re}\frac{\langle v_1,v_2\rangle\langle v_2,v_3\rangle\langle v_3,v_1\rangle}{\langle v_1,v_1\rangle\langle v_2,v_2\rangle\langle v_3,v_3\rangle}.$$

It is easy to check that $\sigma(p)$ is well- defined, that is, it does not depend on the chosen lifts, and it is invariant relative the diagonal action of the full isometry group of $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.

We consider $\{p_1, p_2, p_3\}$ as the vertices of a triangle in hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$. Brehm [6] showed that the side lengths and the shape invariant are independent and characterize the triangle up to isometry.

Now let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points in $\mathrm{H}^n_{\mathbb{Q}}$, and $v = (v_1, v_2, v_3)$ be a lift of $p = (p_1, p_2, p_3)$.

It is easy to check that if $w_i = v_i\lambda_i$, then

$$\langle w_1, w_2\rangle\langle w_2, w_3\rangle\langle w_3, w_1\rangle[\langle w_1, w_1\rangle\langle w_2, w_2\rangle\langle w_3, w_3\rangle]^{-1} =$$

$$\frac{\overline{\lambda}_1}{|\lambda_1|}\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle[\langle v_1, v_1\rangle\langle v_2, v_2\rangle\langle v_3, v_3\rangle]^{-1}\frac{\lambda_1}{|\lambda_1|}.$$

This formula and Proposition 1.1 imply that

$$\sigma^*(p) = -\mathrm{Re}(\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle[\langle v_1, v_1\rangle\langle v_2, v_2\rangle\langle v_3, v_3\rangle]^{-1})$$

is independent of the chosen lifts. Also, it is clear, that $\sigma^*(p)$ is invariant relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$.

We call this number $\sigma^*(p)$ the *quaternionic σ - shape invariant.*

Proposition 2.25. $\sigma^*(p) \geq 0$.

Proof. By applying Corollary 2.17, we can assume that p lies in $M = \mathrm{H}^2_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{Q}}$. Then the result follows from Proposition 2.10. □

As the first application of the results above, we have the following.

Theorem 2.26. *A triangle in $\mathrm{H}^n_{\mathbb{Q}}$ is determined uniquely up to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$ by its three side lengths and its quaternionic σ- shape invariant σ^*.*

Proof. Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points in $\mathrm{H}^n_{\mathbb{Q}}$. By applying Corollary 2.17, we may assume that p lies in $M = \mathrm{H}^2_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{Q}}$. Then the result follows from Proposition 2.5 and results in [6]. □

In [5], Brehm and Et-Taoui introduced another invariant in complex hyperbolic geometry which they called the *direct shape invariant*, or, *τ - invariant.* Below, we recall the definition of this invariant.

Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points in $\mathrm{H}^n_{\mathbb{C}}$ and $v = (v_1, v_2, v_3)$ be a lift of $p = (p_1, p_2, p_3)$. Then the *direct shape invariant* is defined to be

$$\tau = \tau(p) = \frac{H(v_1, v_2, v_3)}{|H(v_1, v_2, v_3)|}.$$

It is easy to check that $\tau(p)$ is independent of the chosen lifts. Also, it was proved in [5] that two triangles $\mathrm{H}^n_{\mathbb{C}}$ are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{C})$ if and only if the three corresponding edge lengths and the direct shape invariant τ of the two triangles coincide.

Remark 2.27. Note that the σ-shape invariant is symmetric, but for the τ-shape invariant we have that $\tau(p_2, p_1, p_3) = \overline{\tau(p_1, p_2, p_3)}$. This implies that the σ-shape invariant (with the side lengths) describes triangles up to the full isometry group of complex hyperbolic space (which includes anti-holomorphic isometries), but τ-shape invariant (with the side lengths) describes triangles up to the group of holomorphic isometries $\mathrm{PU}(n, 1; \mathbb{C})$.

Now we define an analogue of τ-shape invariant in quaternionic hyperbolic geometry. We start with the following lemma whose proof is based on a direct computation.

Lemma 2.28. *Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points in $\mathrm{H}^n_{\mathbb{Q}}$ and $v = (v_1, v_2, v_3)$ be a lift of $p = (p_1, p_2, p_3)$. Let $w_i = v_i\lambda_i$, $\lambda_i \in \mathbb{Q}$, $\lambda_i \neq 0$, then*

$$H(w_1, w_2, w_3)|H(w_1, w_2, w_3)|^{-1} = \frac{\overline{\lambda}_1}{|\lambda_1|} H(v_1, v_2, v_3)|H(v_1, v_2, v_3)|^{-1} \frac{\lambda_1}{|\lambda_1|}.$$

It is easy to see that $H(w_1, w_2, w_3)|H(w_1, w_2, w_3)|^{-1}$ is similar to $H(v_1, v_2, v_3)|H(v_1, v_2, v_3)|^{-1}$ for any $\lambda_i \in \mathbb{Q}$, $\lambda_i \neq 0$. Moreover, this similarity class contains a complex number, unique up to conjugation, see Proposition 1.1. Let $\tau^*(p)$ denote a unique complex number with non-negative imaginary part in this similarity class.

We define the *quaternionic τ-shape invariant* to be $\tau^* = \tau^*(p)$. It is clear that $\tau^*(p)$ does not depend on the chosen lifts.

Proposition 2.29. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ be two ordered triples of distinct points in $\mathrm{H}^n_{\mathbb{Q}}$. Then these two triangles are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if the three corresponding edge lengths and the quaternionic τ-shape invariant τ^* of the two triangles coincide.*

Proof. By applying Corollary 2.17, we may assume that p and p' are in $M = \mathrm{H}^2_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{Q}}$ Then the result follows from Proposition 2.5. □

2.3 Moduli of triples of positive points

In this section, we describe the invariants associated to an ordered triple of positive points in $\mathbb{PQ}^n$ which define the equivalence class of the triple relative to the diagonal action of $\mathrm{PU}(n,1;\mathbb{Q})$.

First, we show how positive points in $\mathbb{PQ}^n$ are related to totally geodesic submanifolds in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n-1}}_{\mathbb{Q}}$. We call such submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ *totally geodesic quaternionic hyperplanes* in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

We recall that π denotes a natural projection from $\mathbb{Q}^{n+1} \setminus \{0\}$ to the projective space $\mathbb{PQ}^n$.

If $H \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is a totally geodesic quaternionic hyperplane, then $H = \pi(\tilde{H}) \cap \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, where $\tilde{H} \subset \mathbb{Q}^{n,1}$ is a quaternionic linear hyperplane. Let $\tilde{H}^{\perp}$ denote the orthogonal complement of $\tilde{H}$ in $\mathbb{Q}^{n,1}$ with respect to the Hermitian form Φ. Then $\tilde{H}^{\perp}$ is a positive quaternionic line, and $\pi(\tilde{H}^{\perp})$ is a positive point in $\mathbb{PQ}^n$. Thus the totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ bijectively correspond to positive points. We call $p = \pi(\tilde{H}^{\perp})$ the *polar point* of a totally geodesic quaternionic hyperplane H. So, the invariants associated to an ordered triple of positive points in $\mathbb{PQ}^n$ are invariants of an ordered triple of totally geodesic quaternionic hyperplane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Let H_1 and H_2 be distinct totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ and p_1, p_2 be their polar points. Let v_1 and v_2 in $\mathbb{Q}^{n,1}$ be their lifts. Then we define

$$d(H_1, H_2) = d(p_1, p_2) = \frac{\langle v_1, v_2 \rangle \langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle \langle v_2, v_2 \rangle}.$$

It is easy to see that $d(H_1, H_2)$ is independent of the chosen lifts of p_1, p_2, and that $d(H_1, H_2)$ is invariant with respect to the diagonal action of $\mathrm{PU}(n,1;\mathbb{Q})$.

There is no accepted name for this invariant in the literature. It is not difficult to show using standard arguments that the distance or the angle between H_1 and H_2 is given in terms of $d(H_1, H_2)$, (see, for instance, the Goldman [14] for the case of complex hyperbolic geometry). So, we will call this invariant $d(H_1, H_2)$ the *distance-angular invariant* or, simply, *d-invariant.*

Also, it is easy to see that:

- H_1 and H_2 are concurrent if and only if $d(H_1, H_2) < 1$,
- H_1 and H_2 are asymptotic if and only if $d(H_1, H_2) = 1$,
- H_1 and H_2 are ultra-parallel if and only if $d(H_1, H_2) > 1$.

Moreover, $d(H_1, H_2)$ is the only invariant of an ordered pair of totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. We have also that the angle θ between H_1 and H_2 (in the case $d(H_1, H_2) < 1$) is given by $\cos^2(\theta) = d(H_1, H_2)$, and the distance ρ between H_1 and H_2 (in the case $d(H_1, H_2) \geq 1$) is given by $\cosh^2(\rho) = d(H_1, H_2)$. We remark that $0 < \theta \leq \pi/2$. We say that H_1 and H_2 are *orthogonal* if $\theta = \pi/2$, this is equivalent to the equality $d(H_1, H_2) = 0$.

Now let (H_1, H_2, H_3) be an ordered triple of distinct totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Let p_1, p_2, p_3 be the polar points of H_1, H_2, H_3 and v_1, v_2, v_3 be their lifts in $\mathbb{Q}^{n,1}$. Then, by applying Proposition 2.16, we can assume that p_1, p_2, p_3 lie in a submanifold $W \subset \mathbb{P}\mathbb{Q}^n$ of complex type of complex dimension 2 passing through the points p_i. Moreover, this submanifold W can be chosen, up to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$, to be the canonical complex submanifold $\mathbb{P}\mathbb{C}^2 \subset \mathbb{P}\mathbb{Q}^n$. Therefore, we can assume without loss of generality that the coordinates of the vectors v_1, v_2, v_3 are complex numbers.

Let $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ be the Gram matrix associated to the points p_1, p_2, p_3 defined by the chosen vectors v_1, v_2, v_3 as above. Then it follows from Proposition 2.2 and the proof of Proposition 2.16 that $g_{ii} = 1$, $g_{1j} = r_{1j} \geq 0$, and $g_{23} = r_{23}e^{\alpha}$. We call such a matrix G a *complex normal form* of the associated Gram matrix. Also, we call G the *complex normalized* Gram matrix.

Next we construct the moduli space of ordered triples of distinct totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. We consider only the regular case, that is, when for all triples in question the spaces spanned by lifts of their polar points are regular, see Corollary 2.8. It is easy to see that in non-regular case, the totally geodesic quaternionic hyperplanes H_1, H_2, H_3 are all asymptotic, that is, $d(H_i, H_j) = 1$, $i \neq j$. It was shown in [10] that a similar problem, the congruence problem for triples of complex geodesic in complex hyperbolic plane, cannot be solved by using Hermitian invariants.

An ordered triple $H = (H_1, H_2, H_3)$ of totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is said to be *generic* if H_i and H_j are not orthogonal for all $i, j = 1, 2, 3$. It is clear that H is generic if and only if the corresponding triple of polar points is generic.

We start with the following proposition:

Proposition 2.30. *Let $H = (H_1, H_2, H_3)$ and $H' = (H'_1, H'_2, H'_3)$ be two ordered generic triples of distinct totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ be their polar points. Let $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ be their lifts in $\mathbb{Q}^{n,1}$ such that the Gram matrices $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ and $G' = (g'_{ij}) = (\langle v'_i, v'_j \rangle)$ are complex normalized. Suppose that the spaces spanned by v_1, v_2, v_3 and v'_1, v'_2, v'_3 are regular. Then H and H' are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if $G = G'$ or $\overline{G} = G'$.*

Proof. If $G = G'$, then it follows from Corollary 2.8 that there exists a linear isometry $L : \mathbb{Q}^{n,1} \longrightarrow \mathbb{Q}^{n,1}$ such that $L(v_i) = v'_i$. Let us suppose that $\overline{G} = G'$. Then we have that $g_{1j} = g'_{1j}$ because g_{1j} and g'_{1j} are real. Also, if $g_{23} = r_{23}e^{\alpha}$, then $g'_{23} = r_{23}e^{-\alpha}$. Let us consider the semi-linear map $L_j : \mathbb{Q}^{n,1} \longrightarrow \mathbb{Q}^{n,1}$ defined by the rule $L_j(v) = jvj^{-1}$. Then we have that

$$\langle L_j(v_k), L_j(v_l)\rangle = \langle jv_kj^{-1}, jv_lj^{-1}\rangle = \overline{j^{-1}}\langle jv_k, jv_l\rangle j^{-1} = -\bar{j}\langle v_k, v_l\rangle j^{-1} = j\langle v_k, v_l\rangle j^{-1} = \overline{\langle v_k, v_l\rangle}.$$

Here we have used that $jzj^{-1} = \overline{z}$ for any complex number z and that $j^{-1} = -j$.

It follows that the Gram matrix of the vectors $L_j(v_k)$, $k = 1, 2, 3$, is equal to $\overline{G}$. Therefore, using the first part of the proof, we get that the triples v and v' are equivalent relative to the diagonal action of $\mathrm{U}(n, 1; \mathbb{Q})$. This implies that H and H' are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$. □

Let $p = (p_1, p_2, p_3)$ be an ordered generic triple of distinct positive points in $\mathbb{P}\mathbb{Q}^n$. Let $v = (v_1, v_2, v_3)$ be their lifts in $\mathbb{Q}^{n,1}$. Let $H(v_1, v_2, v_3) = (\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle) \in \mathbb{Q}$.

Proposition 2.30 justifies the following definition.

We define the *angular invariant* $A = A(p)$ of an ordered generic triple of distinct positive points $p = (p_1, p_2, p_3)$ in $\mathbb{P}\mathbb{Q}^n$ to be the argument of the unique complex number $b = b_0 + b_1 i$ with $b_1 \geq 0$ in the similarity class of

$$\tau(v_1, v_2, v_3) = H(v_1, v_2, v_3)|H(v_1, v_2, v_3)|^{-1}.$$

It follows from Lemma 2.28 and Corollary 1.2 that $A = A(p)$ is defined uniquely by the real part of $\tau(v_1, v_2, v_3)$ which does not depend on the chosen lifts v_1, v_2, v_3.

It is clear from the construction that $A = A(p)$ is invariant with respect to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$. Also, $0 \leq A(p) \leq \pi$.

By applying the proof of Proposition 2.16, we can chose lifts $v = (v_1, v_2, v_3)$ of $p = (p_1, p_2, p_3)$ such that the Gram matrix associated to $p = (p_1, p_2, p_3)$ defined by $v = (v_1, v_2, v_3)$ is the complex normalized Gram matrix of $p = (p_1, p_2, p_3)$, that is, $g_{ii} = 1$, $g_{1j} = r_{1j} > 0$, and $g_{23} = r_{23}e^{\alpha}$, $r_{23} > 0$. It is clear that for these lifts $A(p) = \alpha$.

Now we are ready to describe the moduli space of ordered triples of distinct regular generic totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Theorem 2.31. *Let* $H = (H_1, H_2, H_3)$ *and* $H' = (H'_1, H'_2, H'_3)$ *be two ordered distinct regular generic totally geodesic quaternionic hyperplanes in* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. *Then* $H = (H_1, H_2, H_3)$ *and* $H' = (H'_1, H'_2, H'_3)$ *are equivalent relative to the diagonal action of* $\mathrm{PU}(n, 1; \mathbb{Q})$ *if and only if* $d(H_i, H_j) = d(H'_i, H'_j)$ *for all* $i < j$, $i, j = 1, 2, 3$, *and* $A(p) = A(p')$, *where* $p = (p_1, p_2, p_3)$ *and* $p = (p'_1, p'_2, p'_3)$ *are the polar points of* H) *and* H'.

Proof. We can choose lifts $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ of $p = (p_1, p_2, p_3)$ and $p = (p'_1, p'_2, p'_3)$ such that the Gram matrices G and G' associated to p and $p\prime$ defined by v and v' are complex normalized. Then it follows from the definition of the Gram matrix that $d_{ij} = d(p_i, p_j) = g_{ij}$, $g_{ji} = |g_{ij}|^2$, and that

$$A(p) = A(p_1, p_2, p_3) = \arg(g_{12}\, g_{23}\, g_{31}) = \arg(r_{12}\, g_{23j}\, r_{31}).$$

The first equality implies that $|g_{ij}| = \sqrt{d_{ij}}$. Since H is generic, we have that $r_{1j} > 0$ for all $j > 1$, and that $g_{23} \neq 0$. Therefore, the second equality implies that $A(p) = \arg(g_{23})$. Thus, all the entries of the complex normalized Gram matrix $G(p)$ of p are recovered uniquely in terms of the invariants d_{ij} and $A(p)$ above. Now the proposition follows from Corollary 2.8. $\square$

Next, as a corollary of Theorem 2.31, we give an explicit description of the moduli space of regular generic triples of totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. First of all, it follows from Theorem 2.31 that $\mathrm{PU}(n, 1; \mathbb{Q})$-congruence class of an ordered regular generic triple H of distinct generic totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is described uniquely by three d-invariants d_{12}, d_{13}, d_{23} and the angular invariant $\alpha = \mathbb{A}(p_1, p_2, p_3)$. Now, let $G = (g_{ij})$ be the complex normalized Gram matrix of H. Then $g_{ii} = 1,\ g_{1j} = r_{1j} > 0$, and $\arg(g_{23}) = A(p_1, p_2, p_3)$. We have that $d_{1j} = r^2_{1j}$ and $d_{23} = r^2_{23}$. Also, $g_{23} = r_{23}\, e^{i\alpha} = r_{23}(\cos\alpha + i sen\,\alpha)$. A straightforward computation shows that

$$\det G = 1 - (r^2_{12} + r^2_{13} + r^2_{23}) + 2 r_{12} r_{13} r_{23} \cos\alpha.$$

Using the lexicographic order, we define $r_1 = \sqrt{d}_{12},\ r_2 = \sqrt{d}_{13},\ r_3 = \sqrt{d}_{23}$.

It follows from the Silvester criterium that $\det G \leq 0$, therefore, we have

Corollary 2.32. *The moduli space* $\mathcal{M}_0(3)$ *of regular generic totally geodesic quaternionic hyperplanes in* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ *is homeomorphic to the set*

$$\mathbb{M}_0(3) = \{(r_1, r_2, r_3, \alpha) \in \mathbb{R}^4 : r_i > 0,\ \alpha \in (o, \pi],\ 1 - (r^2_1 + r^2_1 + r^2_3) + 2 r_1 r_2 r_3 \cos\alpha \leq 0\}.$$

Remark 2.33. If the triple $H = (H_1, H_2, H_3)$ is not generic, we need less invariants to describe the equivalence class of H. For instance, if H_2 and H_3 are both orthogonal to H_1, we need only $d(H_2, H_3)$.

2.4 Moduli of triples of points in $\mathbb{P}\mathbb{Q}^n$: mixed configurations

As we know from Section 2.2.2, that, up to isometries of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, triples of points of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, that is, triangles, are characterized by the side lengths and the Brehm shape invariant. Also, triples of points in $\partial\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, that is, ideal triangles, are characterized by the Cartan angular invariant, see Section 2.2.1. The description of the moduli of triples of positive points was given in Section 2.3.

In this section, we describe the invariants for mixtures of the three types of points in $\mathbb{P}\mathbb{Q}^n$ relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$. Using this, we describe the moduli of the corresponding configuration.

First, we give a list of all the triples of points in $\mathbb{P}\mathbb{Q}^n$.

Let $p = (p_1, p_2, p_3)$ be a triple of points in $\mathbb{P}\mathbb{Q}^n$. Then we have the following possible configurations (up to permutation).

1. Ideal triangles: p_i is isotropic for $i = 1, 2, 3$.
2. Triangles: p_i is negative, that is, p_i in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ for $i = 1, 2, 3$.
3. Triangles of totally geodesic quaternionic hyperplanes in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.
4. Triangles with two ideal vertices: p_1 and p_2 are isotropic, and (a) p_3 is negative, (b) p_3 is positive.
5. Triangles with one ideal vertex: p_1 is isotropic, and (a) p_2, p_3 are negative, (b) p_2, p_3 are positive.
6. Triangles with one negative vertex and two positive vertices: p_1 is negative, and p_2, p_3 are positive.
7. Triangles with one positive vertex and two negative vertices: p_1 is positive, and p_2, p_3 are negative.

The first three cases have been already established in Sections 2.2.1, 2.2.2, and 2.3. Below, we describe invariants of mixed configurations 4-7.

First, we recall the definition of so called η-invariant [14] in complex hyperbolic geometry.

Let (p_1, p_2, q) be an ordered triple of points in $\mathbb{PC}^n$. We suppose that p_1 and p_2 are isotropic and q is positive. Let (v_1, v_2, w) be a lift of (p_1, p_2, p_3). Then it is easy to check that the complex number

$$\eta(v_1, v_2, w) = \frac{\langle v_1, w\rangle\langle w, v_2\rangle}{\langle v_1, v_2\rangle\langle w, w\rangle}$$

is independent of the chosen lifts and will be denoted by $\eta(p_1, p_2, q)$.

We call the number $\eta(p_1, p_2, q)$ the Goldman η*-invariant.* This invariant was introduced by Goldman [14] to study the intersections of bisectors in complex hyperbolic space. Later, Hakim and Sandler [16] generalized the Goldman construction for more general triples of points.

The aim of this section is to define analogous invariants in quaternionic hyperbolic geometry and to prove the congruence theorems for each triples in question.

2.4.1 Triangles with two ideal vertices: p_1, p_2 are isotropic, (a) p_3 is positive, (b) p_3 is negative

First we consider the case when p_1 and p_2 are isotropic and p_3 is positive. This configuration was considered by Goldman [14] in complex hyperbolic geometry. Geometrically it can be considered as two points in the boundary of quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ and a totally geodesic quaternionic hyperplane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Let v_1, v_2 be isotropic vectors representing p_1, p_2 and v_3 a positive vector representing p_3. Consider the following quaternion:

$$\eta(v_1, v_2, v_3) = \langle v_1, v_3\rangle\langle v_3, v_2\rangle\langle v_1, v_2\rangle^{-1}\langle v_3, v_3\rangle^{-1}.$$

Now let us take another lifts of $p = (p_1, p_2, p_3)$: $v_1' = v_1\lambda_1$, $v_2' = v_2\lambda_2$, $v_3' = v_3\lambda_3$.

It is easy to check that

$$\eta(v_1\lambda_1, v_2\lambda_2, v_3\lambda_3) = \frac{\bar{\lambda}_1}{|\lambda_1|}\eta(v_1, v_2, v_3)\frac{\lambda_1}{|\lambda_1|}.$$

Therefore, this implies that $\eta(v_1, v_2, v_3)$ is independent of the choices of lifts of p_2 and p_3, and if we change a lift of p_1, we get a similar quaternion.

By applying Proposition 2.16, we can assume that p_1, p_2, p_3 lie in a submanifold $W \subset \mathbb{PQ}^n$ of complex type of complex dimension 2 passing through the points p_i. Moreover, this submanifold W

can be chosen, up to the action of $\mathrm{PU}(n,1;\mathbb{Q})$, to be the canonical complex submanifold $\mathbb{PC}^2 \subset \mathbb{PQ}^n$. Therefore, we can assume without loss of generality that the coordinates of the vectors v_1, v_2, v_3 are complex numbers.

Let $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ be the Gram matrix associated to the points p_1, p_2, p_3. Then $g_{11} = 0$, $g_{22} = 0$. Since $g_{33} = \langle v_3, v_3 \rangle > 0$, replacing v_3 by $v_3 a$, where $a = 1/\sqrt{g_{33}}$, we may assume that $g_{33} = 1$. Replacing v_2 by $v_2 b$, where $b = 1/\langle v_1, v_2 \rangle$, we may assume that $g_{12} = 1$. We keep the same notations for the modified vectors. After that, replacing v_1 by $v_1 b$, where $b = 1/\langle v_3, v_1 \rangle$ and v_2 by $v_2 c$, where $c = \langle v_1, v_3 \rangle$, we get $g_{12} = g_{13} = 1$. So, after this re-scaling we get that $G = (g_{ij}$ has the following entrances: $g_{11} = g_{22} = 0, g_{12} = g_{13} = 1, g_{33} = 1$, g_{23} is an arbitrary complex number.

We call such a matrix G a *complex normal form* of Gram matrix associated to p_1, p_2, p_3 . Also, we call G the *complex normalized* Gram matrix.

It is clear that for p_1, p_2, p_3 any vectors v_1, v_2 and v_3 which represent p_1, p_2, and p_3 generate a regular space in $\mathbb{Q}^{n,1}$. Therefore, repeating almost word for word the proof of Proposition 2.30, we get

Proposition 2.34. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ as above. Let $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ be their lifts in $\mathbb{Q}^{n,1}$ such that the Gram matrices $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ and $G' = (g'_{ij}) = (\langle v'_i, v'_j \rangle)$ associated to p and p' are complex normalized. Then p and $'$ are equivalent relative to the diagonal action of $\mathrm{PU}(n,1;\mathbb{Q})$ if and only if $G = G'$ or $\overline{G} = G'$.*

Again this justifies the following definition.

Let $p = (p_1, p_2, p_3)$ be a triple of points as above. Then the *quaternionic η-invariant*, $\eta = \eta(p)$, is defined to be the unique complex number $b = b_0 + b_1$ with $b_1 \geq 0$, in the similarity class of $\eta(v_1, v_2, v_3)$.

Theorem 2.35. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ as above. Then p and p' are equivalent relative to the diagonal action of $\mathrm{PU}(n,1;\mathbb{Q})$ if and only if $\eta(p) = \eta(p')$.*

Proof. It follows from the above that we can choose lifts $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ of $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ such that the Gram matrices G and G' associated to p and p' defined by v and v' are complex normalized. Then it follows from the definition of the Gram matrix that $g_{23} = \overline{\eta(p)}$. This implies that $G = G'$. Now the proof follows from Lemma 2.1. □

The case when p_3 is negative is similar. The η-invariant is defined by the same formula and the proof of the congruence theorem is a slight modification of Theorem 2.35.

2.4.2 Triangles with one ideal vertex: p_1 is isotropic, (a) p_2, p_3 are negative, (b) p_2, p_3 are positive

Now we consider a configuration when p_1 is isotropic and p_2, p_3 are negative. So, in this case p_1 and p_2 and p_3 represent a triangle in $\mathrm{H}^n_{\mathbb{Q}}$ with one ideal vertex.

Let v_2, v_3 be negative vectors representing p_2, p_3 and v_1 an isotropic vector representing p_1.

Consider the following quaternion:

$$\eta(v_1, v_2, v_3) = \langle v_1, v_3 \rangle \langle v_3, v_2 \rangle \langle v_1, v_2 \rangle^{-1} \langle v_3, v_3 \rangle^{-1}.$$

It is easy to check that

$$\eta(v_1\lambda_1, v_2\lambda_2, v_3\lambda_3) = \frac{\bar{\lambda}_1}{|\lambda_1|} \eta(v_1, v_2, v_3) \frac{\lambda_1}{|\lambda_1|}.$$

Therefore, this implies that $\eta(v_1, v_2, v_3)$ is independent of the choices of lifts of p_2 and p_3, and if we change a lift of p_1, we get a similar quaternion.

By applying Proposition 2.16, we can assume that p_1, p_2, p_3 lie in a submanifold $W \subset \mathbb{P}\mathbb{Q}^n$ of complex type of complex dimension 2 passing through the points p_i. Moreover, this submanifold W can be chosen, up to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$, to be the canonical complex submanifold $\mathbb{P}\mathbb{C}^2 \subset \mathbb{P}\mathbb{Q}^n$. Therefore, we can assume without loss of generality that the coordinates of the vectors v_1, v_2, v_3 are complex numbers.

Let $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ be the Gram matrix associated to the points p_1, p_2, p_3. Then $g_{11} = 0$, $g_{22} < 0$, and $g_{33} < 0$. It is not difficult to show that by appropriate re-scaling we may assume that $g_{11} = 0$, $g_{22} = -1$, $g_{33} = -1$, $g_{12} = 1$, $g_{23} = r_{23} > 0$, and g_{13} is an arbitrary complex number.

We call such a matrix G a *complex normal form* of Gram matrix associated to p_1, p_2, p_3 . Also, we call G the *complex normalized* Gram matrix.

It is clear that for p_1, p_2, p_3 any vectors v_1, v_2 and v_3 which represent p_1, p_2, and p_3 generate a regular space in $\mathbb{Q}^{n,1}$. Therefore, repeating again almost word for word the proof of Proposition 2.30, we get

Proposition 2.36. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ as above. Let $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ be their lifts in $\mathbb{Q}^{n,1}$ such that the Gram matrices $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ and $G' = (g'_{ij}) = (\langle v'_i, v'_j \rangle)$ associated to p and p' are complex normalized. Then p and $'$ are equivalent relative to the diagonal action of* $\mathrm{PU}(n, 1; \mathbb{Q})$ *if and only if $G = G'$ or $\overline{G} = G'$.*

Again this justifies the following definition.

Let $p = (p_1, p_2, p_3)$ be a triple of points as above. Then the *quaternionic η-invariant*, $\eta = \eta(p)$, is defined to be the unique complex number $b = b_0 + b_1$ with $b_1 \geq 0$, in the similarity class of $\eta(v_1, v_2, v_3)$.

Theorem 2.37. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ as above. Then p and p' are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if $\eta(p) = \eta(p')$ and $d(p_2, p_3) = d(p'_2, p'_3)$.*

Proof. It follows from the above that we can choose lifts $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ of $p = (p_1, p_2, p_3)$ and $p = (p'_1, p'_2, p'_3)$ such that the Gram matrices G and G' associated to p and $p\prime$ defined by v and v' are complex normalized. Then it follows from the definition of the Gram matrix that $\eta(p) = g_{13}$ and $r_{23} = \sqrt{d(p_2, p_3)}$. This implies that $G = G'$. Now the proof follows from Lemma 2.4. □

It follows from this theorem that the congruence class relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ of $p = (p_1, p_2, p_3)$ is defined $\eta(p)$ and the distance between p_2 and p_3.

The case when p_2, p_3 are positive is similar. The η-invariant is defined by the same formula and the proof of the congruence theorem is a slight modification of Theorem 2.37. We only remark that geometrically this configuration is equivalent to one isotropic point and two totally geodesic quaternionic hyperplane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

2.4.3 Triangles with one negative vertex and two positive vertices: p_1 is negative, p_2, p_3 are positive

Now we consider a configuration when p_1 is negative and p_2, p_3 are positive. So, in this case p_1 represents a point in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, and p_2 and p_3 represent two totally geodesic quaternionic hyperplane H_2 and H_3 in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Let v_1 a negative vector representing p_1 and v_2, v_3 be positive vectors representing p_2, p_3.

Let $\eta(v_1, v_2, v_3)$ be following quaternion:

$$\eta(v_1, v_2, v_3) = \langle v_1, v_3 \rangle \langle v_3, v_2 \rangle \langle v_1, v_2 \rangle^{-1} \langle v_3, v_3 \rangle^{-1}.$$

We see that $\eta(v_1, v_2, v_3)$ is not well-defined when the points p_1 and p_2 are orthogonal, that is, $\langle v_1, v_2 \rangle = 0$. It follow from the definition of polar points that p_1 is orthogonal to p_2 if and only if

$p_1 \in H_2$. Also we see that $\eta(v_1, v_2, v_3) = 0$ when p_1 is orthogonal to p_3 or p_2 is orthogonal to p_3. We analyze all these special configurations in the end of this section and show that in all these cases we need less invariants to describe the congruence class then in generic case.

So, we say that a triple $p = (p_1, p_2, p_3)$ as above is *generic* if and only if all the pairs (p_i, p_j) are not orthogonal, $i \neq j$.

In what follows, let $p = (p_1, p_2, p_3)$ be a generic triple as above. It is easy to check that

$$\eta(v_1\lambda_1, v_2\lambda_2, v_3\lambda_3) = \frac{\bar{\lambda}_1}{|\lambda_1|}\eta(v_1, v_2, v_3)\frac{\lambda_1}{|\lambda_1|}.$$

Therefore, this implies that $\eta(v_1, v_2, v_3)$ is independent of the choices of lifts of p_2 and p_3, and if we change a lift of p_1, we get a similar quaternion.

By applying Proposition 2.16 again, we can assume that p_1, p_2, p_3 lie in a submanifold $W \subset \mathbb{PQ}^n$ of complex type of complex dimension 2. Moreover, this submanifold W can be chosen, up to the action of $\mathrm{PU}(n, 1; \mathbb{Q})$, to be the canonical complex submanifold $\mathbb{PC}^2 \subset \mathbb{PQ}^n$. Therefore, we can assume without loss of generality that the coordinates of the vectors v_1, v_2, v_3 are complex numbers.

Let $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ be the Gram matrix associated to the triple of points (p_1, p_2, p_3). Then $g_{11} < 0$, $g_{22} > 0$, and $g_{33} > 0$. It is not difficult to show that by appropriate re-scaling we may assume that $g_{11} = -1$, $g_{22} = 1$, $g_{33} = 1$, $g_{12} = r_{12} > 0$, $g_{13} = r_{13} > 0$, and g_{23} is an arbitrary complex number.

We call such a matrix G a *complex normal form* of Gram matrix associated to (p_1, p_2, p_3). Also, we call G the *complex normalized* Gram matrix.

Let $p = (p_1, p_2, p_3)$ be a triple of points as above. Then the *quaternionic η-invariant*, $\eta = \eta(p)$, is defined to be the unique complex number $b = b_0 + b_1 i$ with $b_1 \geq 0$, in the similarity class of $\eta(v_1, v_2, v_3)$. As before we have

Proposition 2.38. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ as above. Let $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ be their lifts in $\mathbb{Q}^{n,1}$ such that the Gram matrices $G = (g_{ij}) = (\langle v_i, v_j \rangle)$ and $G' = (g'_{ij}) = (\langle v'_i, v'_j \rangle)$ associated to p and p' are complex normalized. Then p and p' are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if $G = G'$ or $\overline{G} = G'$.*

Now we define one more invariant. Let q_1 be a negative point and q_2 be a positive point. Let w_1 be a vector representing q_1 and w_2 be a vector representing q_2. Then we define

$$d(q_1, q_2) = d(w_1, w_2) = \frac{\langle w_1, w_2\rangle\langle w_2, w_1\rangle}{\langle w_1, w_1\rangle\langle w_2, w_2\rangle}.$$

It is easy to see that $d(q_1, q_2)$ is independent of the chosen lifts w_1, w_2, and that $d(q_1, q_2)$ is invariant with respect to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$.

We call $d(q_1, q_2)$ the *mixed distant invariant* associated to the points q_1, q_2. By applying the standard arguments it is easy to prove that this invariant defines the distance ρ between the point $q_1 \in \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ and the totally geodesic quaternionic hyperplane H in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ whose polar point is q_2, namely,

$$\sinh^2(\rho(q_1, H)) = -d(q_1, q_2).$$

Theorem 2.39. *Let $p = (p_1, p_2, p_3)$ and $p' = (p'_1, p'_2, p'_3)$ as above. Then p and p' are equivalent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ if and only if $\eta(p) = \eta(p')$, $d(p_1, p_2) = d(p'_1,\ p'_2)$, and $d(p_1, p_3) = d(p'_1,\ p'_3)$.*

Proof. It follows from the above that we can choose lifts $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ of $p = (p_1, p_2, p_3)$ and $p = (p'_1, p'_2, p'_3)$ such that the Gram matrices G and G' associated to p and p' defined by v and v' are complex normalized. Then it follows from the definition of the Gram matrix that $r_{12} = \sqrt{d(p_1, p_2)}$, $r_{13} = \sqrt{d(p_1, p_3)}$, and $g_{23} = \overline{\eta(p)}(r_{12}/r_{13})$. This implies that $G = G'$. Now the proof follows from Lemma 2.4. □

As a corollary of this theorem we have

Theorem 2.40. *Let $p_1 \in \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ and let H_1, H_2 be two totally geodesic quaternionic hyperplane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ whose polar points are p_2 and p_3, respectively. Then the congruence class of the triple (p_1, H_1, H_2) relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$ is defined uniquely by two distances $\rho(p_1, H_1)$, $\rho(p_1, H_2)$, the d-invariant $d(H_1, H_2)$, and the angular invariant of the triple (p_1, p_2, p_3).*

Now we consider some special configurations of triples $p = (p_1, p_2, p_3)$. Let, for instance, p be a configuration where all the pairs (p_i, p_j), $i \neq j$, are orthogonal. In this case, the totally geodesic quaternionic hyperplanes H_1 and H_2 with polar points p_2 and p_3 intersect orthogonally and p_1 lies in their intersection. It is clear then that all such configurations are congruent relative to the diagonal action of $\mathrm{PU}(n, 1; \mathbb{Q})$. So, the moduli space in this case is trivial. Another example: suppose that p_1 is orthogonal to p_2, and p_1 is orthogonal to p_3. This implies that H_1 and H_2 intersect, and p_1 lies in their intersection. This implies that in order to describe the congruence class of such configuration we need only the d-invariant of H_1 and H_2, the angle between H_1 and H_2. Another special configuration may be analyzed easily in a similar way.

2.4.4 Triangles with one positive vertex and two negative vertices: p_1 is positive, p_2, p_3 are negative

In this section, we consider an ordered triple (p_1, p_2, p_3), where p_1 is positive and p_2, p_3 are negative.

Geometrically, this corresponds to a totally geodesic quaternionic hyperplane H in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ and two points p_2 and p_3 in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Theorem 2.41. *Let H be a totally geodesic quaternionic hyperplane H in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ with polar point p_1, and p_2 and p_3 in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Suppose that $p = (p_1, p_2, p_3)$ is generic. Then the congruence class of the triple (H, p_1, p_2) is defined uniquely by two distances $\rho(p_1, H)$, $\rho(p_1, H)$, and $\eta(p) = \eta(p_1, p_2, p_3)$.*

Proof. The proof of this theorem is slight modification of the proof of Therem 2.39. □

Section 3

Bisectors in quaternionic hyperbolic space

In $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$, as it follows from Proposition 1.5, totally geodesic (real) hypersurfaces exist only when $\mathbb{F} = \mathbb{R}$. Therefore, when $\mathbb{F} \neq \mathbb{R}$, a reasonable substitute are the bisectors which are as closed as possible to being totally geodesic. A comprehensive study of bisectors in complex hyperbolic space the reader can find in the Goldman book [14].

Giraud [13] and Mostow [19] described the structure of bisectors in complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ in terms of a foliation by totally geodesic complex submanifolds of maximal dimension. Goldman [14] proved that bisectors enjoy another decomposition into totally real, totally geodesic submanifolds, which is called the *meridian decomposition*. It is important that these decompositions are unique.

In this section, we will describe various decompositions of bisectors in quaternionic hyperbolic space. We show that geometry of bisectors in quaternionic hyperbolic space is more rich than in complex hyperbolic geometry. First, we prove an analogue of the Mostow decomposition of bisectors in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then we show that bisectors in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ admit non-singular foliations by totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. More exactly, we will show that any bisector in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is the total space of a fiber bundle over a bisector in a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ whose fibers are totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ as well. We will show that such decompositions are not unique. Finally, we will show that any bisector in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is the union of totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ intersecting in a common point. In some sense, this is an analogue of the Goldman meridian decomposition. We call such decompositions the *fan decompositions*. The existence of fan decompositions implies, in particular, that any bisector in quaternionic hyperbolic space is starlike with respect to any point in its real spine.

The proof of these results are based on the classification of reflective submanifolds in quaternionic hyperbolic space [18].

3.1 Bisectors in Hyperbolic spaces

Let p_1, p_2 be two distinct points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$. The *bisector equidistant from* p_1 *and* p_2, or the *bisector of* $\{p_1, p_2\}$ is defined as

$$B(p_1, p_2) = \{p \in \mathrm{H}^{\mathrm{n}}_{\mathbb{F}} : d(p_1, p) = d(p_2, p)\}.$$

An equidistant hypersurface or *bisector* in $\mathrm{H}^{\mathrm{n}}_{\mathbb{F}}$ is a subset $B = B(p_1, p_2)$ for some pair of points p_1, p_2. In this case, we will say that B is equidistant from p_1 (or equidistant from p_2).

Next we consider the following two cases: (1) $\mathbb{F} = \mathbb{C}$, and (2) $\mathbb{F} = \mathbb{Q}$.

3.1.1 Bisectors in complex hyperbolic space

In this section, we recall some basic results on bisectors in complex hyperbolic space.

Let $\{p_1, p_2\}$ be as above. Then there exists the unique complex geodesic $\Sigma \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ spanned by p_1 and p_2. Following Mostow [19], we call Σ the *complex spine* with respect to the pair $\{p_1, p_2\}$. The *spine* of B (with respect to $\{p_1, p_2\}$) equals

$$\sigma\{p_1, p_2\} = B(p_1, p_2) \cap \Sigma = \{p \in \Sigma : d(p_1, p) = d(p_2, p)\}$$

that is, the orthogonal bisector of the geodesic segment joining p_1 and p_2 in Σ.

The following result is due to Giraud [13] and Mostow [19], see also [14].

Theorem 3.1. *Let* B, Σ *and* σ *be as above. Let* $\Pi_\Sigma : \mathrm{H}^{\mathrm{n}}_{\mathbb{C}} \to \Sigma$ *be orthogonal projection onto* Σ. *Then*

$$B = \Pi_\Sigma^{-1}(\sigma) = \bigcup_{s \in \sigma} \Pi_\Sigma^{-1}(s)$$

.

The complex hyperplanes $\Pi_\Sigma^{-1}(s)$, for $s \in \sigma$, are called the *slices* of B (with respect to $\{p_1, p_2\}$), and the decomposition of a bisector into slices is called the *slice decomposition*, or the *Mostow decomposition*.

Since orthogonal projection $\Pi_\Sigma : \mathrm{H}^n_{\mathbb{C}} \to \Sigma$ is a real analytic fibration, we obtain that a bisector is a real analytic hypersurface in $\mathrm{H}^n_{\mathbb{C}}$ diffeomorphic to $\mathbb{R}^{2n-1}$.

It was shown by Goldman [14] that the slices, the spine and complex spine of a bisector B depend intrinsically on the hypersurface B, and not on the defining pair $\{p_1, p_2\}$.

Theorem 3.2. *Suppose that $\{p_1, p_2\}$ and $\{p'_1, p'_2\}$ are two pairs of distinct points in complex hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$ such that $B = B(p_1, p_2)$ and $B' = B(p'_1, p'_2)$ are equal. Then the slices (respectively spine, complex spine) of B with respect to $\{p_1, p_2\}$ equal the slices (respectively spine, complex spine) of B' with respect to $\{p'_1, p'_2\}$.*

Goldman's proof of this theorem is based on the following: the complex hyperbolic space is a complex analytic manifold; a bisector is Levi-flat and its maximal holomorphic submanifolds are its slices.

The endpoints of the spine of a bisector B are called the *vertices* of B. Since a geodesic $\sigma \in \mathrm{H}^n_{\mathbb{C}}$ is completely determined by the unordered pair $\partial\sigma \subset \partial\mathrm{H}^n_{\mathbb{C}}$ consisting of its endpoints, bisectors are completely parametrized by unordered pairs of distinct points in $\partial\mathrm{H}^n_{\mathbb{C}}$. Therefore, there is a duality between bisectors and geodesics. We associate to every bisector a geodesic, its spine σ. Conversely, if $\sigma \subset \mathrm{H}^n_{\mathbb{C}}$ is a geodesic, then there exists a unique bisector $B = B_\sigma$ with the spine σ. Given a geodesic σ, there exists a unique complex geodesic $\Sigma \supset \sigma$. Let $R_\sigma : \Sigma \to \Sigma$ be the unique reflection whose fixed-point set is σ. Take an arbitrary point $p_1 \in \Sigma \setminus \sigma$ and let $p_2 = R_\sigma(p_1)$. Then

$$B = B(p_1, p_2) = \Pi_\Sigma^{-1}(\sigma)$$

is a bisector having σ as its spine. Therefore, we have

Theorem 3.3. *There is a natural bijective correspondence between bisectors in $\mathrm{H}^n_{\mathbb{C}}$ and geodesics in $\mathrm{H}^n_{\mathbb{C}}$.*

Next, we recall so called the *meridianal decomposition* of bisectors in complex hyperbolic space invented by Goldman [14].

The following result shows that bisectors decompose into totally real geodesic submanifolds as well as into complex totally geodesic submanifolds of complex dimension $n - 1$.

Theorem 3.4. *Let $\sigma \in \mathrm{H}^n_{\mathbb{C}}$ be a real geodesic. For each $k \geq 2$, the bisector B having spine σ is the union of all $\mathbb{R}^k$-planes containing σ.*

Following Goldman, we refer to the $\mathbb{R}^n$-planes containing the spine as the *meridians* of the bisector B.

We remark that in contrast to the Mostow decomposition, which is a non-singular foliation, the meridianal decomposition is a singular foliation having the geodesic σ as its singular set.

Goldman's proof of Theorem 3.4 is based on a sufficiently complicate computation. As a corollary of the existence of the meridianal decomposition, we have the following.

Theorem 3.5. *Bisectors in complex hyperbolic space are starlike with respect to any point in its real spine.*

3.2 Bisectors in quaternionic hyperbolic space

In this section, we will study bisectors in n- dimensional quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

As we have seen in the previous section, the orthogonal projections play an important role in the study of bisectors. The first problem we consider when the orthogonal projections onto totally geodesic submanifolds have totally geodesic fibers.

3.2.1 Orthogonal projections and reflective submanifolds

It is elementary that in real hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$, the orthogonal projections onto totally geodesic submanifolds have totally geodesic fibers. It is not true, in general, in hyperbolic spaces with non-constant sectional curvature.

For instance, if M is a totally geodesic submanifold of real dimension one, that is, a real geodesic, in complex hyperbolic space $\mathrm{H}^2_{\mathbb{C}}$, then the fibers of the orthogonal projection onto M are not totally geodesic, because in this case the fibers have real dimension 3, but we know that any totally geodesic submanifold in $\mathrm{H}^2_{\mathbb{C}}$ has real dimension 1 or 2.

Another example: let M be a totally real totally geodesic plane in $\mathrm{H}^3_{\mathbb{C}}$, that is, M is isometric to $\mathrm{H}^2_{\mathbb{R}}$. Then, using again the classification of totally geodesic submanifolds in $\mathrm{H}^3_{\mathbb{C}}$, it is easy to prove that the fibers of the orthogonal projection onto M are not totally geodesic.

One more example: let M be a totally geodesic submanifold in quaternionic hyperbolic space $\mathrm{H}^2_{\mathbb{Q}}$ of complex type of real dimension 2, that is, M is isometric to $\mathrm{H}^1_{\mathbb{C}}$. Then it can be shown that the fibers of the orthogonal projection onto M are not totally geodesic.

We will show that this problem is related to so-called reflective submanifolds of hyperbolic spaces.

First, we recall the definition of reflective submanifolds.

Let N be a Riemannian manifold and M a submanifold of N. Then M is called *reflective* if the geodesic reflection of N in M is a globally well-defined isometry of M.

Since any reflective submanifold is a connected component of the fixed point set of an isometry, it is totally geodesic.

Also, we recall the definition of symmetric submanifolds.

A submanifold M of a Riemannian manifold N is called *symmetric* if for each point p in M there exists an isometry I_p of N such that

$$I_p(p) = p, \quad I_p(M) = M, \quad (I_p)_*X = -X, \quad (I_p)_*Y = Y$$

for all $X \in t, Y \in \nu_p M$.

Here we denote by T_pM the tangent space of M at p, by $\nu_p M$ the normal space of M at p, and by $(I_p)_*$ the differential of I_p.

For symmetric spaces there is the following useful criterion.

Proposition 3.6. *A totally geodesic submanifold of a simply connected Riemannian symmetric space is symmetric if and only if it is reflective.*

As a reflective submanifold is symmetric, at each point there exists a totally geodesic submanifold normal to it. In symmetric spaces this normal submanifold is also reflective. Since all hyperbolic spaces are symmetric this result holds for any hyperbolic space.

Let M be a reflective submanifold in a symmetric simply connected Riemannian space N, and $p \in M$. Let $M_p^\perp$ denote its *orthogonal complement* at p, that is, the connected, complete, totally geodesic submanifold of N with $TM_p^\perp = \nu_p M$.

The following theorems describe all reflective submanifolds in complex and quaternionic hyperbolic spaces, see [18].

Theorem 3.7. *Let M be a reflective submanifold in complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then M is either isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{C}}$, $k = 1, \dots, n-1$, or to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$.*

Theorem 3.8. *Let M be a reflective submanifold in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then M is either isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{Q}}$, $k = 1, \dots, n-1$, or to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.*

Corollary 3.9. *Let M be a reflective submanifold in complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$, and $p \in M$. Then $M_p^{\perp}$ is reflective, and it is isometric to $\mathrm{H}^{\mathrm{n-k}}_{\mathbb{C}}$ if M is isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{C}}$, and $M_p^{\perp}$ is isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$ if M is isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$.*

Corollary 3.10. *Let M be a reflective submanifold in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then $M_p^{\perp}$ is reflective, and it is isometric to $\mathrm{H}^{\mathrm{n-k}}_{\mathbb{Q}}$ if M is isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{Q}}$, and $M_p^{\perp}$ is isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ if M is isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.*

All this implies the following results.

Theorem 3.11. *Let M be a totally geodesic submanifold in complex hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. Then the fibers of the orthogonal projection onto M are totally geodesic if and only if M is either isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{C}}$, $k = 1, \dots, n-1$, or to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$.*

Theorem 3.12. *Let M be a totally geodesic submanifold in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Then the fibers of the orthogonal projection onto M are totally geodesic if and only if M is either isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{Q}}$, $k = 1, \dots, n-1$, or to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.*

3.2.2 Mostow decomposition of bisectors in quaternionic hyperbolic space

In this section, we describe the Mostow decomposition of bisectors in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Let p_1, p_2 be two distinct points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, and let $B(p_1, p_2)$ be the bisector of $\{p_1, p_2\}$, that is,

$$B(p_1, p_2) = \{p \in \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} : d(p_1, p) = d(p_2, p)\}.$$

Then it follows from projective geometry that there exists the unique quaternionic geodesic $\Sigma \subset \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ spanned by p_1 and p_2. We call Σ the *quaternionic spine* with respect to the pair $\{p_1, p_2\}$. The *real spine* of B (with respect to $\{p_1, p_2\}$) equals

$$\sigma\{p_1, p_2\} = B(p_1, p_2) \cap \Sigma = \{p \in \Sigma : d(p_1, p) = d(p_2, p)\}$$

that is, the orthogonal bisector of the geodesic segment joining p_1 and p_2 in Σ. Since Σ is isometric to $\mathrm{H}^{4}_{\mathbb{R}}$, it follows that σ is isometric to $\mathrm{H}^{3}_{\mathbb{R}}$.

Let $\Pi_\Sigma : \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \to \Sigma$ be orthogonal projection onto Σ. It follows from Theorem 3.12 that the fibers of Π are totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}-1}_{\mathbb{Q}}$.

First, we prove the following proposition.

Proposition 3.13. *For any $p \in \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \backslash \Sigma$, and $q \in \Sigma$, the geodesics from $\Pi_{\Sigma}(p)$ to p and q are orthogonal and span a totally real totally geodesic 2-plane.*

Proof. To prove this, we will use the ball model for $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, see Section 1.2.2. In what follows, we identify $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ with the unit ball $\mathbb{D}$ in $\mathbb{Q}^n$,

$$\mathbb{D} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{Q}^n : |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 < 1\}.$$

We write $(z_1, z_2, \ldots, z_n) = (z_1, z')$, where $z' = (z_2, \ldots, z_n) \in \mathbb{Q}^{n-1}$. If $z_2 = z_3 = \ldots = z_n = 0$, we denote z' by $0'$.

Since the isometry group of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ acts transitively on the set of quaternionic geodesics in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, we can assume without loss of generality that $\Sigma = \{(z, 0') \in \mathbb{D} : z \in \mathbb{Q}\}$. Therefore, in this case, if $p = (z, z') \in \mathbb{D}$, we have that $\Pi_{\Sigma}(p) = (z, 0')$.

Let $p = (z, z') \in \mathbb{D} \setminus \Sigma$, $p' = \Pi_{\Sigma}(p) = (z, 0')$, and $q = (w, 0') \in \Sigma, w \in \mathbb{Q}$. Let $P = (z, z', 1)^t$, $P' = (z, 0', 1)^t$, $Q = (w, 0', 1)^t$ be vectors in $\mathbb{Q}^{n,1}$ representing the points p, p', q, respectively.

Now we compute the Hermitian triple product $\langle P, P', Q \rangle$. We have

$$\langle P, P', Q \rangle = \langle P, P' \rangle \langle P', Q \rangle \langle Q, P \rangle = (\bar{z}z - 1)(\bar{z}w - 1)(\bar{w}z - 1) = (|z|^2 - 1)|\bar{z}w - 1|^2.$$

So, the Hermitian triple product $\langle P, P', Q \rangle$ is real. Therefore, the points $p, p', q,$ lie in a totally real geodesic plane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. □

Also we need so-called the Pythagorean theorem in hyperbolic plane $\mathrm{H}^2_{\mathbb{R}}$, see [3].

Theorem 3.14. *Let a, b, c be vertices of a right triangle with right angle at a. Then*

$$\cosh d(b, c) = \cosh d(a, b) \cosh d(a, c).$$

Now we are ready to prove the Mostow decomposition theorem in quaternionic hyperbolic geometry.

Theorem 3.15. *Let B, Σ and σ be as above. Let $\Pi_{\Sigma} : \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \to \Sigma$ be orthogonal projection onto Σ. Then*

$$B = \Pi_{\Sigma}^{-1}(\sigma) = \bigcup_{s \in \sigma} \Pi_{\Sigma}^{-1}(s)$$

.

Proof. Let $p \in \Pi_\Sigma^{-1}(\sigma)$. First, we have that $d(\Pi_\Sigma(p), p_1) = d(\Pi_\Sigma(p), p_2)$. By applying Proposition 3.13, we have that the points $p,\ \Pi_\Sigma(p),\ p_1$ lie in a totally real geodesic plane in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. The same is true for the points $p,\ \Pi_\Sigma(p),\ p_2$. Then it follows from the Pythagorean theorem applied to the right triangles with vertices $p,\ \Pi_\Sigma(p),\ p_1$ and $p,\ \Pi_\Sigma(p),\ p_2$ that $d(p_1, p) = d(p_2, p)$, that is $p \in B$.

□

We call the quaternionic hyperplanes $\Pi_\Sigma^{-1}(p)$, for $p \in \sigma$, the *slices* of the bisector B (with respect to $\{p_1, p_2\}$). It is clear that any two distinct slices of B are ultraparallel.

Since orthogonal projection $\Pi_\Sigma : \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \to \Sigma$ is a real analytic fibration, we have:

Corollary 3.16. *A quaternionic bisector is a real analytic hypersurface in* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ *diffeomorphic to* $\mathbb{R}^{4n-1}$.

Now we show that the quaternionic spine Σ and the real spine σ of B depend intrinsically on B, and not on the pair $\{p_1, p_2\}$ used to define B. As was remarked above, Goldman's proof of the complex hyperbolic analogue of this fact is based on the following: the complex hyperbolic space is a complex analytic manifold; a bisector is Levi-flat and its maximal holomorphic submanifolds are its slices. Since quaternionic hyperbolic space has no natural complex structure, this proof does not work in the quaternioic case. We prove this result using some elementary facts in quaternionic projective geometry.

Theorem 3.17. *Let us suppose that* $\{p_1, p_2\}$ *and* $\{p_1', p_2'\}$ *be two pairs of distinct points in* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ *such that the bisectors* $B = B(p_1, p_2)$ *and* $B' = B'(p_1', p_2')$ *are equal. Then the slices (respectively, quaternioic spine, real spine) of* B *with respect to* $\{p_1, p_2\}$ *equal the slices (respectively, quaternioic spine, real spine) of* B' *with respect to* $\{p_1', p_2'\}$.

Proof. First we show that the slices of B coincide with the slices of B'.

We need the following facts from projective geometry: any two different projective submanifolds in quaternionic projective space $\mathbb{PQ}^n$, $n > 1$, of quaternionic dimension $n-1$ are transversal along their intersection; every intersection of projective submanifolds in $\mathbb{PQ}^n$ is a projective submanifold in $\mathbb{PQ}^n$. Also, we recall that any totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{k}}_{\mathbb{Q}}$, $k = 1, \dots, n-1$, is the intersection of a projective submanifold in $\mathbb{PQ}^n$ of dimension k with $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Let s' be a slice of B'. Since $B = B'$, then s' is either a slice of B, or s' intersects a slice s of B transversally. Suppose that $s \cap s' \neq \emptyset$ and $s \neq s'$. Since $s \subset B$ and $s' \subset B$, and B is a smooth submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ of real dimension $4n-1$, it follows from transversality of s and

s' along their intersection that $\dim_{\mathbb{R}} s + \dim_{\mathbb{R}} s' - (4n-1) = \dim_{\mathbb{R}}(s \cap s')$. This implies that $\dim_{\mathbb{R}}(s \cap s') = 4n - 7$. This is impossible, since $s \cap s'$ is the intersection of a projective submanifold in $\mathbb{PQ}^n$ with $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Therefore, we get that $s = s'$. So, the slices of B' are slices of B. Since each slice is orthogonal to Σ, any pair of distinct slices of bisector are ultraparallel and their unique common orthogonal quaternionic geodesic is equal to Σ.

Therefore, B completely determines the quaternionic spine Σ. Since $\sigma = B \cap \Sigma$, the bisector B uniquely determines its real spine. $\square$

3.2.3 Automorphism of bisectors

If N is a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^3_{\mathbb{R}}$, then there exists the unique quaternionic geodesic M which contains N. Then the pair $\{M, N\}$ defines a bisector B whose quaternionic spine equals to M and real spine equals to N. Indeed, take a point p in $M \setminus N$. Let γ be the unique real geodesic in M which contains p and which is orthogonal to N. Let $p' \in \gamma$ be the point symmetric to p. It is clear that M is the quaternionic spine of the bisector $B = B(p, p')$, and N is the real spine of B.

Since the group of isometries $PU(n, 1, \mathbb{Q})$ of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ acts transitively on such pairs $\{M, N\}$, we get that $PU(n, 1, \mathbb{Q})$ acts transitively on bisectors. Since the quaternionic geodesic M containing N is unique, we have that a bisector in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is defined uniquely by its real spine. Furthermore, the stabilizer of a bisector in $PU(n, 1, \mathbb{Q})$ equals the stabilizer of its real spine. This group is described in Proposition 1.9.

3.2.4 Orthogonality of totally geodesic submanifolds in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$

Suppose that M and N are totally geodesic submanifolds in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.

Let I_M and I_N denote the geodesic reflections in M and N respectively. Since M and N are reflective, I_M and I_N are isometries of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. The fixed point set of I_M equals to M, and the fixed point set of I_N equals to N.

The proof of the following theorem is standard.

Theorem 3.18. *The following conditions are equivalent:*

1. *I_M and I_N commute,*

2. $I_M(N) = N$,

3. $I_N(M) = M$,

4. M *and* N *intersect orthogonally in* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

We recall that $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ has a totally geodesic submanifold isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$.

Theorem 3.19. *Let* M *be a totally geodesic submanifold of* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ *isometric to* $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$*. Then for any totally geodesic submanifold* $S \subset M$ *isometric to* $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$ *there exists a totally geodesic submanifold* N *isometric to* $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ *such that* $S = M \cap N$ *and* N *is orthogonal to* M *along* S.

Proof. To prove this, we will use the ball model for $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, see Section 1.2.2. In what follows, we identify $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ with the unit ball $\mathbb{D}$ in $\mathbb{Q}^n$,

$$\mathbb{D} = \{(q_1, q_2, \ldots, q_n) \in \mathbb{Q}^n : |q_1|^2 + |q_2|^2 + \ldots + |q_n|^2 < 1\}.$$

Let us consider the following subsets of $\mathbb{D}$:

$$\mathbb{D}_{\mathbb{C}} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 < 1\},$$

$$\mathbb{D}_{\mathbb{R}} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 < 1\},$$

where $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$ and $\mathbb{R}$ is the standard subfield of real numbers of $\mathbb{Q}$.

Then $\mathbb{D}_{\mathbb{C}}$ is a totally geodesic submanifold of $\mathbb{D}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$, and $\mathbb{D}_{\mathbb{R}}$ is a totally geodesic submanifold of $\mathbb{D}_{\mathbb{C}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$.

Since the group of isometries of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ acts transitively on the set of pairs of the form (M, S) in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, we can assume without loss of generality that $M = \mathbb{D}_{\mathbb{C}}$ and $S = \mathbb{D}_{\mathbb{R}}$.

Let $a \in \mathbb{Q}$ be purely imaginary, $|a| = 1$. Then $a^2 = -1$. Let $\mathbb{C}(a)$ be the subfield of $\mathbb{Q}$ spanned by 1 and a. Denote by $\mathbb{D}_a$ the following set in $\mathbb{D}$:

$$\mathbb{D}_a = \{(a_1, a_2, \ldots, a_n) \in \mathbb{C}(a)^n : |a_1|^2 + |a_2|^2 + \ldots + |a_n|^2 < 1\}.$$

We know, see Section 1.2.3, that $\mathbb{D}_a$ is a totally geodesic submanifold of $\mathbb{D}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. We remark that $\mathbb{D}_a = \mathbb{D}_{\mathbb{C}}$ if $a = i$. It is clear that $\mathbb{D}_{\mathbb{C}} \cap \mathbb{D}_a = \mathbb{D}_{\mathbb{R}}$, provided that $a \neq i$.

Let $V_{\mathbb{C}}$, V_a, $V_{\mathbb{R}}$ denote the sets of vectors in $\mathbb{Q}^{n,1}$ with coordinates in $\mathbb{C}$, $\mathbb{C}(a)$, and $\mathbb{R}$ respectively. Then we have that $\mathbb{D}_{\mathbb{C}}$, $\mathbb{D}_a$, $\mathbb{D}_{\mathbb{R}}$ are projectivizations of $V_{\mathbb{C}}$, V_a, and $V_{\mathbb{R}}$ respectively.

Consider the following semi-linear map $L_a : \mathbb{Q}^{n,1} \longrightarrow \mathbb{Q}^{n,1}$ given by $L_a(v) = ava^{-1}$, $v \in \mathbb{Q}^{n,1}$. Then it follows that $L_a(v) = v$ for all $v \in V_a$, see Section 1.2.5. In particular, $L_i(v) = v$ for all $v \in V_i$, that is, $L_i(v) = v$ for all $v \in V_{\mathbb{C}}$. We know that projectivization of this semi-linear map L is equal to projectivization of the linear map $L'(v) = av$. We have that $L' \circ L' = -I$, where I is the identity map of $\mathbb{Q}^{n,1}$. This implies that projectivization of L'_a is an isometric involution of $\mathbb{D}$ whose fixed point set equals to $\mathbb{D}_a$.

Since $ik = -ki$, we have that $L'_i \circ L'_k = -L'_k \circ L'_i$. It follows that projectivizations of L'_i and L'_k are commuting involutions whose fixed points sets are $\mathbb{D}_i = \mathbb{D}_{\mathbb{C}}$ and $\mathbb{D}_k$.

Let again $M = \mathbb{D}_{\mathbb{C}}$ and $S = \mathbb{D}_{\mathbb{R}}$. We define N to be $\mathbb{D}_k$. Then $M \cap N = S$. Also, let I_M be projectivization of L'_i, and let I_N be projectivization of L'_k. Therefore, we get that I_M and I_N are geodesic reflections in M and N respectively such that $I_M(N) = N$, $I_N(M) = M$, I_M and I_N commute. □

We remark that I_M acts on N as a geodesic reflection in $S \subset N$, and I_N acts on M as a geodesic reflection in $S \subset M$.

3.2.5 Complex type decomposition of bisectors in quaternionic hyperbolic space

In this section, we show that bisectors in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ enjoy a decomposition into totally geodesic submanifolds isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$.

Let p_1, p_2 be two distinct points in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, and let $B = B(p_1, p_2)$ be the bisector of $\{p_1, p_2\}$.

Let $\Sigma_{\mathbb{C}}$ be a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ containing the points p_1 and p_2. Let $\Pi_{\Sigma_{\mathbb{C}}} : \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \to \Sigma_{\mathbb{C}}$ denote the orthogonal projection onto $\Sigma_{\mathbb{C}}$.

Since $\Sigma_{\mathbb{C}}$ is a totally geodesic reflective submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$, $\Pi_{\Sigma_{\mathbb{C}}}$ has totally geodesic fibers. They are isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ as well.

Let $\sigma_{\mathbb{C}} = B \cap \Sigma_{\mathbb{C}}$. It is clear that $\sigma_{\mathbb{C}}$ is the bisector in $\Sigma_{\mathbb{C}}$ defined by $\{p_1, p_2\}$, that is,

$$\sigma_{\mathbb{C}} = \{p \in \Sigma_{\mathbb{C}} : d(p_1, p) = d(p_2, p)\}.$$

Theorem 3.20.
$$B = \Pi^{-1}_{\Sigma_{\mathbb{C}}}(\sigma_{\mathbb{C}}) = \bigcup_{p\in\sigma_{\mathbb{C}}} \Pi^{-1}_{\Sigma_{\mathbb{C}}}(p)$$
.

Proof. Let $\alpha \subset \Sigma_{\mathbb{C}}$ be the real geodesic passing through p_1 and p_2 and $c \subset \Sigma_{\mathbb{C}}$ be the complex spine of the bisector $\sigma_{\mathbb{C}}$. Let $o \in \alpha$ be the midpoint of the geodesic segment of α defined by p_1 and p_2. So, $o \in \sigma_{\mathbb{C}}$. Let $\beta \subset c$ be the real geodesic passing through o orthogonal to α. We see that β is the real spine of the bisector $\sigma_{\mathbb{C}}$.

Let $p \in \sigma_{\mathbb{C}}$, and let l_p be a point in $\Pi^{-1}_{\Sigma_{\mathbb{C}}}(p)$. We will show that $l_p \in B$.

Since $p \in \sigma_{\mathbb{C}}$, it follows from Goldman's meridianal decomposition that p lies in a totally geodesic submanifold $S \subset \sigma_{\mathbb{C}} \subset \Sigma_{\mathbb{C}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$ containing β, S is a meridian of $\sigma_{\mathbb{C}}$.

Then, by applying Theorem 3.19, we get that there exists a unique totally geodesic submanifold N isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ such that $S = M \cap N$ and N is orthogonal to $\Sigma_{\mathbb{C}}$ along S. It is easy to see that the point l_p lies in N. Let I_N be reflection in N. We have that $\Sigma_{\mathbb{C}}$ is invariant with respect to I_N and I_N acts in $\Sigma_{\mathbb{C}}$ as reflection in S. All this implies that $I_N(l_p) = l_p$ and $I_N(p_1) = p_2$. Therefore, I_N maps triangle (l_p, p, p_1) onto triangle (l_p, p, p_2). This implies that $d(l_p, p_1) = d(l_p, p_2)$, that is, $l_p \in B$. □

We call the constructed decomposition of the bisector B a *complex type decomposition* of B. It is easy to see that $\Pi_{\Sigma_{\mathbb{C}}} : B \to \sigma_{\mathbb{C}}$ defines a fiber bundle over $\sigma_{\mathbb{C}}$ whose fibres are totally geodesic submanifolds of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$. We remark that, in contrast to the Mostow decomposition, the base of this fiber bundle, $\sigma_{\mathbb{C}}$, is not totally geodesic. Another difference with the Mostow decomposition is that the complex type decomposition of B is not unique: we can start with any totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ passing through the points p_1 and p_2. Also, one sees that all these decompositions are equivalent with respect to the stabilizer of the set $\{p_1, p_2\}$ in the isometry group of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

3.2.6 Fan decomposition of bisectors in quaternionic hyperbolic space

In this section, we show that any bisector in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ is a union of totally geodesic submanifolds in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ passing through a point in its real spine. This decomposition in somewhat is similar to the meridional decomposition of bisectors in complex hyperbolic space.

We call a such decomposition *fan decomposition* of a bisector in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$.

Then it follows from this result that bisectors in quaternionic hyperbolic space $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ are starlike with respect to any point in its real spine.

Let B be a bisector in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. Let Σ and σ be its quaternionic and real spine respectively. Also, let $\Pi_{\Sigma} : \mathrm{H}^{\mathrm{n}}_{\mathbb{Q}} \to \Sigma$ be orthogonal projection onto Σ.

Take a point $o \in \sigma$ and let γ be the unique real geodesic in Σ containing o orthogonal to σ. Let p_1, p_2 be distinct points in γ symmetric with respect to o. Then it is clear that $B = B(p_1, p_2)$.

Theorem 3.21. *Let B and $o \in \sigma$ as above. Then B is a union of totally geodesic submanifolds in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ passing through the point o.*

Proof. Let M be a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ containing γ. Then $B_M = M \cap B$ is a bisector in M. We denote by Σ_M the complex spine of B_M and by σ_M the real spine of B_M. Then Σ_M is a totally geodesic submanifold of Σ isometric to $\mathrm{H}^{1}_{\mathbb{C}}$ containing γ, and σ_M is a real geodesic in Σ_M passing through the point o orthogonal to γ. By applying Goldman's meridianal decomposition, we have that there exists a totally geodesic submanifold $S \subset B_M$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$ containing σ_M, S is a meridian of B_M. Let us choose any such S and fix it. It follows from Theorem 3.19 that there exists a unique totally geodesic submanifold N isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ such that $S = M \cap N$ and N is orthogonal to M along S.

Let I_N denote geodesic reflection in N. Then it follows Theorem 3.19 that M is invariant with respect to I_N, and, moreover, $I_N(p_1) = p_2$, $I_N(p_2) = p_1$.

We first show that $N \subset B$. It suffices to show that for any $p \in N$ we have that $d(p, p_1) = d(p, p_2)$. But this is clear because

$$d(p, p_1) = d(I_N(p), (I_N(p_1)) = d(p, p_2).$$

Conversely, we will show that each $p \in B$ lies in a totally geodesic submanifold N constructed above for some M isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ containing the points p_1 and p_1.

Let $p \in B$ and $p' = \Pi_{\Sigma}(p)$. Then it follows from Theorem 3.15 that $p' \in \sigma$. Let β be the unique real geodesic passing through o and p'. We have that $\beta \subset \sigma$ because σ is a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$. It follows from Proposition 3.13 that there exists a totally geodesic submanifold S of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{R}}$ containing the point p and the geodesic β. Let c_1 be a totally geodesic submanifold isometric to $\mathrm{H}^{1}_{\mathbb{C}}$ containing β and p_1, p_2. Let c_2 be a totally geodesic submanifold isometric to $\mathrm{H}^{1}_{\mathbb{C}}$

containing the points p and p'. Let M be a totally geodesic submanifold of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ containing c_1 and c_2. Then M contains S and the points p_1, p_2. By applying Theorem 3.19, we get there exists a totally geodesic submanifold N of $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{C}}$ orthogonal to M along S. It follows from the above that $N \subset B$. Since, by construction, $p \in N$, the result follows.

□

As a corollary of this theorem we have the following:

Theorem 3.22. *Bisectors in quaternionic hyperbolic space* $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ *are starlike with respect to any point in its real spine.*

Proof. Let B be a bisector in $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ and o be a point in σ. Let p be an arbitrary point in B. By applying Theorem 3.21, we have that there exists a totally geodesic submanifold $M \subset B$ isometric to $\mathrm{H}^{\mathrm{n}}_{\mathbb{Q}}$ containing o and p. This implies that the geodesic passing through o and p lies in B. □

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