

Universidade Federal de Minas Gerais
Escola de Engenharia
Programa de Pós-Graduação em Engenharia Elétrica
Laboratório de Modelagem, Análise e Controle de Sistemas Não-Lineares

ESTIMAÇÃO DE ESTADOS COM RESTRIÇÕES PARA SISTEMAS DINÂMICOS LINEARES E NÃO-LINEARES

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CONSTRAINED STATE ESTIMATION FOR LINEAR AND NONLINEAR DYNAMIC SYSTEMS

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To my mother Luzia, for her love and incentive.

A minha mãe Luzia, pelo amor e incentivo.

RESUMO

Estimadores de estados têm sido aplicados a distintas áreas, tais como em engenharia aeroespacial, econometria e geofísica, para inferir variáveis não-observadas (estados e, eventualmente, parâmetros) de um sistema dinâmico a partir de duas fontes de informação incertas: as medições e um modelo matemático. Sob as premissas de modelo linear e ruído Gaussiano, o filtro de Kalman é a solução ótima recursiva mais conhecida para o problema de estimação de estados, ao passo que o filtro de Kalman estendido e, mais recentemente, o filtro de Kalman *unscented* são as soluções aproximadas mais comumente empregadas para o caso não-linear.

Na prática, todavia, informação adicional sobre o sistema pode ser conhecida, e essa terceira fonte de informação pode ser útil para melhorar as estimativas de estado. Nesta tese, é considerado o cenário no qual a dinâmica e os distúrbios do sistema são tais que o vetor de estados satisfaz uma restrição de igualdade ou uma restrição de desigualdade, as quais são assumidamente conhecidas a priori. Estimação de estados com restrições tem recebido crescente atenção tanto no ambiente acadêmico quanto na indústria, especialmente, durante os últimos dez anos.

No presente trabalho, além de se apresentar uma ampla revisão do atual estado da arte em estimação de estados com restrições, são desenvolvidos métodos de filtragem de Kalman para impor restrições de igualdade ou de desigualdade na estimativa de estado. Tratam-se ambos os casos de sistemas lineares e não-lineares. Para o último caso, propõem-se algoritmos baseados no filtro de Kalman *unscented*. Ademais, é apresentada uma metodologia geral para estimação de estados com uma restrição de igualdade no ganho do estimador, visando a, indiretamente, impor propriedades especiais na estimativa de estado. Exemplos simulados e experimentais são usados para ilustrar os algoritmos estudados e propostos ao longo desta tese.

ABSTRACT

State estimators have been applied to disparate fields, such as in aerospace engineering, econometrics, and geophysics, to infer unobserved variables (states and, occasionally, parameters) of a dynamic system providing two uncertain sources of information, namely, the measurements and a mathematical model. Under linear model and Gaussian noise assumptions, the Kalman Filter is the well-known optimal recursive solution for the state-estimation problem, whereas the extended Kalman filter and, more recently, the unscented Kalman filter are the most commonly employed approximate solutions for the nonlinear case.

In practice, however, additional information about the system may be available, and this third source of information may be useful for improving state estimates. A scenario we have in mind is the case in which the dynamics and the disturbances are such that the state vector of the system satisfies an assumed known equality or inequality constraint. Constrained state estimation has been receiving increasing attention in both academia and industry, especially in the last ten years.

In addition to providing a wide overview of the current state of the art in constrained state estimation, the present work is concerned with the development of Kalman filtering methods for enforcing an equality constraint or inequality constraint on the state estimate. Both linear and nonlinear cases are considered. For the latter, algorithms based on the unscented Kalman filter are proposed. Furthermore, we also present a general framework for state estimation with an equality constraint on the estimator gain, aiming at indirectly enforcing special properties on the state estimate. Simulated and experimental examples are used to illustrate the applications of the algorithms studied and presented along this thesis.

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“Who, then, can separate us from the love of Christ? Can trouble do it, or hardship or persecution or hunger or poverty or danger or death? For I am certain that nothing can separate us from His love: neither death nor life, neither angels nor other heavenly rulers or powers, neither the present nor the future, neither the world above nor the world below – there is nothing in all creation that will ever be able to separate us from the love of God which is ours through Christ Jesus our Lord.”

Romans 8:35,38-39

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LIST OF SYMBOLS AND ACRONYMS

Symbols

$\mathbb{N}$	nonnegative integers;
$\mathbb{Z}$	integers;
$\mathbb{R}$	real numbers;
$\mathbb{R}_+$	nonnegative real numbers;
$\in$	is an element of;
$\triangleq$	equals by definition;
$I_{n \times n}$	$n \times n$ identity matrix;
$0_{n \times m}$	$n \times m$ zero matrix;
$1_{n \times m}$	$n \times m$ unit matrix;
A^T	transpose of A ;
A^{-1}	inverse of A ;
A^L	left inverse of A ;
A^R	right inverse of A ;
$A^{1/2}$	square-root of $A = A^{1/2}(A^{1/2})^T$;
$A_{(i,j)}$	(i,j) entry of A ;
$\text{col}_i(A)$	i th column of A ;
$\text{row}_i(A)$	i th row of A ;
$\det(A)$	determinant of A ;
$\text{diag}(A)$	diagonal of A ;
$\text{diag}(A_1, \dots, A_n)$	block-diagonal matrix;
$\text{tr}(A)$	trace of A ;
$\mathcal{N}(A)$	null space of A ;
$\mathcal{P}_{\mathcal{N}(A)}$	projector with range $\mathcal{N}(A)$, where $\mathcal{P}_{\mathcal{N}(A)}\mathcal{P}_{\mathcal{N}(A)} = \mathcal{P}_{\mathcal{N}(A)}$;
x_i	i th entry of x ;

$ x $	absolute value of x ;
$\ x\ _2$	Euclidian norm of x ;
$\sqrt{x}$	square root of x ;
$\ln(x)$	Neperian logarithm of x ;
$\log_{10}(x)$	logarithm with base 10 of x ;
$\delta(x)$	Dirac-delta function at x ;
$\max(x), \min(x)$	maximum and minimum values of x ;
$\arctan(x)$	arc tangent of x ;
$\sin(x), \cos(x), \tan(x)$	sine, cossine, and tangent of x ;
$\rho(x y)$	conditional probability density function of x given y ;
$E[\cdot]$	mathematical expectation;
$\bar{x}$	mean of the random variable x ;
σ_x	standard deviation of the random variable x ;
k	discrete-time index;
t	continuous-time index;
T_s	sampling interval;
f	dynamic or process nonlinear model in state space;
h	measurement map or observation nonlinear model;
g	nonlinear equality state constraint function;
l	nonlinear inequality state constraint function;
$A_{k-1}, B_{k-1}, G_{k-1}, C_k$	matrices of the linear model in state space;
D_{k-1} and d_{k-1}	matrix and vector of the linear equality state constraint;
E_{k-1} and e_{k-1}	matrix and vector of the linear inequality state constraint;
a_k and b_k	vectors of the interval constraint on the state vector;
$\hat{A}_{k-1}, \hat{B}_{k-1}, \hat{G}_{k-1}, \hat{C}_k$	Jacobians of the linear model in state space;
$x_k, x(t)$	state vector;
$x(kT_s)$	state vector at continuous-time sampled at time $t = kT_s$;
$u_{k-1}, u(t)$	input vector;
$w_{k-1}, w(t)$	process noise vector;
$y_k, y(kT_s)$	measurements or output vector;
$v_k, v(kT_s)$	measurement noise vector;
$\theta_k, \theta(t)$	parameter vector;

$\check{x}_k, \check{x}(t)$	joint state (and parameter) vector, that is, $\check{x}_k = \begin{bmatrix} x_k^T & \theta_k^T \end{bmatrix}^T$;
$\tilde{x}_k, \tilde{x}(t)$	augmented state vector, that is, $\tilde{x}_k = \begin{bmatrix} x_k^T & w_k^T & v_k^T \end{bmatrix}^T$;
$\tilde{y}_k, \tilde{y}(kT_s)$	augmented measurement vector, that is, $\tilde{y}_k = \begin{bmatrix} y_k^T & d_{k-1}^T \end{bmatrix}^T$;
$J(x)$	profit or cost function of x ;
$\Phi(x_1, \dots, x_n)$	functional of the sequence $x_1, \dots, x_n$;
W_k	weighting matrix;
M_k, N_k, O_k	matrices of the equality constraint on the estimator gain;
$\hat{x}_{k k-1}$	forecast state estimate;
$\hat{x}_{k k}$	data-assimilation state estimate;
$\hat{\theta}_{k k}$	data-assimilation parameter estimate;
$\hat{y}_{k k-1}$	forecast output estimate;
$\hat{x}_{k k}^c$	constrained state estimate;
$\hat{x}_{k k}^f, \hat{x}_{k k}^b$	forward and backward state estimates;
$\hat{x}_{k k}^s$	smoothed state estimate;
$\hat{x}_{k k}^p$	projected state estimate;
$\hat{x}_{k k}^g$	projected state estimate by gain projection;
$\hat{x}_{k k}^t$	truncated state estimate;
$\hat{\tilde{x}}_{k k}$	data-assimilation augmented state estimate;
$\hat{x}_{i,k k}$	i th entry of $\hat{x}_{k k}$;
$\hat{x}_{k k,i}$	data-assimilation state estimate at the i th iteration at time k ;
$e_{k k-1}$	forecast estimation error;
$e_{k k}$	data-assimilation estimation error;
$\nu_{k k}$	innovation;
$P_{k k-1}^{xx}$	forecast error-covariance matrix;
$P_{k k}^{xx}$	data-assimilation error-covariance matrix;
$P_{k k-1}^{yy}$	innovation covariance matrix;
$P_{k k-1}^{xy}$	cross-covariance matrix;
$P_{k k}^{xxc}$	constrained error-covariance matrix;
$P_{k k}^{xxf}, P_{k k}^{xxb}$	forward and backward error-covariance matrices;
$P_{k k}^{xxs}$	smoothed error-covariance matrix;
$P_{k k}^{xpp}$	projected error-covariance matrix;
$P_{k k}^{xpg}$	projected error-covariance matrix by gain pojection;

$P_{k k}^{xxt}$	truncated error-covariance matrix;
Q_{k-1}	process-noise covariance matrix;
R_k	measurement-noise covariance matrix;
$\tilde{R}_k$	augmented measurement-noise covariance matrix;
K_k	Kalman gain at time k ;
L_k	constrained Kalman gain at time k ;
$\mathcal{X}_{j,k k}$	j th column of the sigma-point matrix $\mathcal{X}_{k k}$;
γ_x, γ_P	weight vectors for calculating the state estimate and error covariance;
Ψ_{UT}	unscented transform function yielding $\mathcal{X}_{k k}$, γ_x , and γ_P ;
Ψ_{ICUT}	interval-constrained unscented transform function yielding $\mathcal{X}_{k k}$ and γ_k .

Acronyms

CDF	Central Difference Filter;
CEKF	Constrained Extended Kalman Filter;
CKF	Constrained Kalman Filter;
CIUKF	Constrained Interval Unscented Kalman Filter;
CPU	Central Processing Unit;
CUKF	Constrained Unscented Kalman Filter;
DDF	Divided Difference Filter;
ECKF	Equality-Constrained Kalman Filter;
ECUKF	Equality-Constrained Unscented Kalman Filter;
EKF	Extended Kalman Filter;
EKS	Extended Kalman Smoother;
EnKF	Ensemble Kalman Filter;
ERR	Error Reduction Error;
FPR	Flight Path Reconstruction;
GCKF	Gain-Constrained Kalman Filter;
GCKP	Gain-Constrained Kalman Predictor;
GCUKF	Gain-Constrained Unscented Kalman Filter;
GPS	Global Positioning System;
GSF	Gaussian Sum Filter;

ICUT	Inequality-Constrained Unscented Transform;
IEKF	Iterated Extended Kalman Filter;
IMU	Inertial Measurement Unit;
IUKF	Interval Unscented Kalman Filter;
KF	Kalman Filter;
KP	Kalman Predictor;
KS	Kalman Smoother;
LECUKF	Equality-Constrained Unscented Kalman Filter;
LEO	Low-Earth Orbit;
LUKF	Localized Unscented Kalman Filter;
MAEKF	Measurement-Augmentation Extended Kalman Filter;
MAKF	Measurement-Augmentation Kalman Filter;
MAEKF	Measurement-Augmentation Unscented Kalman Filter;
MEMS	Micro-Electro-Mechanical Systems;
MHD	Magnetohydrodynamics;
MHE	Moving Horizon Estimator;
MT	Mean Trace;
NARMA	Nonlinear Autoregressive Moving Average;
NED	North-East-Down;
NMHE	Nonlinear Moving Horizon Estimator;
ODE	Ordinary Differential Equation;
PDF	Probability Density Function;
PEKF-EP	Projected Extended Kalman Filter by Estimate Projection;
PF	Particle Filter;
PKF-EP	Projected Kalman Filter by Estimate Projection;
PKF-GP	Projected Kalman Filter by Gain Projection;
PKF-IC	Projected Kalman Filter for Inequality-Constrained Systems;
PKF-SP	Projected Kalman Filter by System Projection;
PLUKF	Localized Unscented Kalman Filter;
PUKF	Projected Unscented Kalman Filter;
PUKF-EP	Projected Unscented Kalman Filter by Estimate Projection;
RMSE	Root Mean Square Error;
RNDDR	Recursive Nonlinear Dynamics Data Reconciliation;

RTS	Rauch-Tung-Striebel;
SCKF	Spatially Constrained Kalman Filter;
SDEKBF	Sampled-Data Extended Kalman-Bucy Filter;
SDREF	State-Dependent Ricatti Equation Filter;
SDUKBF	Sampled-Data Unscented Kalman-Bucy Filter;
SIUKF	Sigma-Point Interval Unscented Kalman Filter;
SPKF	Sigma-Point Kalman Filters;
TKF	Truncated Kalman Filter;
TIUKF	Truncated Interval Unscented Kalman Filter;
TPUKF	Two-Step Projected Unscented Kalman Filter;
TUKF	Truncated Unscented Kalman Filter;
UKF	Unscented Kalman Filter;
UKS	Unscented Kalman Smoother;
UnbKF-UI	Unbiased Kalman Filter with Unknown Inputs;
URNDDR	Unscented Recursive Nonlinear Dynamics Data Reconciliation;
UT	Unscented Transform.

RESUMO ESTENDIDO

Apresenta-se a seguir um resumo estendido relativo ao projeto de pesquisa desenvolvido nesta tese. Primeiramente, introduz-se o problema estudado, motivando-o por meio de uma breve revisão bibliográfica e explicitando os objetivos da presente tese. Em seguida, discutem-se brevemente as contribuições desenvolvidas neste projeto, quer as de cunho prático, quer as de conteúdo teórico. Finalmente, as conclusões são apresentadas.

Introdução

Motivação

Estudos astronômicos, em especial no fim do século XVIII, serviram como um dos principais e pioneiros estímulos ao desenvolvimento da teoria de estimação. Visando a estimar os parâmetros das equações de movimento de planetas e cometas a partir de dados medidos por telescópios, Gauss propôs o método dos mínimos quadrados em 1795 (Sorenson, 1970). Pouco mais de um século e meio depois, incentivado pelo início da corrida espacial, em janeiro de 1959, Rudolf Kalman, por meio de “*um único, gigante e persistente exercício matemático*” (Kalman, 2003) inventou o algoritmo de filtragem (Kalman, 1960), o qual leva seu nome, em reconhecimento ao considerado, por muitos, até hoje, o principal trabalho científico sobre estimação de estados. Atualmente, o filtro de Kalman é solução de prateleira para muitos problemas do setor aeroespacial (Cipra, 1993; de Moraes et al., 2007). Além disso, sua aplicação estende-se do treinamento de redes neurais (Rios Neto, 1997) à previsão meteorológica (Carme et al., 2001).

Apesar do amplo uso do filtro de Kalman (KF), as premissas de linearidade e Gaussianidade que lhe garantem otimalidade impedem sua aplicação em sistemas não-lineares. Métodos de filtragem de partículas (PF) podem ser empregados para, de forma aproximada, tratar sistemas não-lineares e não-gaussianos (Gordon et al., 1993; Arulampalam et al., 2002).

No entanto, seu elevado custo computacional inviabiliza sua implementação em várias aplicações de tempo real (Daum, 2005).

Por outro lado, métodos de aproximação gaussiana contornam tal dificuldade pelo emprego de algoritmos baseados no KF. Desse grupo, o filtro de Kalman estendido (EKF) (Schmidt, 1970; Jazwinski, 1970), cujo cerne está na linearização analítica ou numérica do modelo do sistema, é a solução mais usada nas últimas quatro décadas. No entanto, EKF apresenta algumas limitações como sensibilidade às condições iniciais e à sintonia das matrizes de covariância de ruído, especialmente, para sistemas fortemente não-lineares. Nesse caso, pode-se observar divergência ou mesmo instabilidade do estimador (Reif et al., 1999). Alternativamente, o filtro de Kalman *unscented* (UKF) (Julier et al., 1995; Julier & Uhlmann, 2004), um algoritmo da família dos filtros de Kalman de pontos sigma (SPKF) (Lefebvre et al., 2002; van der Merwe et al., 2004), usa uma técnica de linearização estatística que emprega um número reduzido de amostras escolhidas deterministicamente (Julier et al., 2000). Vários trabalhos recentes (Julier et al., 2000; Haykin, 2001; van der Merwe et al., 2004; Lefebvre et al., 2004; Romanenko & Castro, 2004; Hovland et al., 2005; Choi et al., 2005; Crassidis, 2006) relatam o desempenho superior do UKF comparado ao EKF. Portanto, nesta tese, são desenvolvidos algoritmos baseados no UKF.

KF é o estimador de estados ótimo para sistemas lineares e Gaussianos que usa duas fontes de informação incertas, a saber: um modelo matemático (usado na etapa de predição) e as medições (usadas na etapa de assimilação de dados). Em alguns casos, contudo, informação adicional sobre o sistema pode ser conhecida, e essa terceira fonte de informação pode ser útil para melhorar as estimativas de estado. Nesta tese, é considerado o cenário no qual a dinâmica e os distúrbios do sistema são tais que o vetor de estados satisfaz uma restrição de desigualdade (Rao, 2000; Robertson & Lee, 2002; Goodwin et al., 2005) ou uma restrição de igualdade (Teixeira et al., 2007; Ko & Bitmead, 2007). Por exemplo, em reações químicas, as concentrações dos reagentes e produtos são não-negativas (Massicotte et al., 1995; Chaves & Sontag, 2002), ao passo que no problema de estimação de atitude com parametrização em quatérnios, o vetor de atitude deve ter norma unitária (Crassidis & Markley, 2003).

Estimação de estados com restrições tem recebido crescente atenção tanto no ambiente acadêmico quanto na indústria, especialmente, durante os últimos dez anos (Simon, 2008). No presente trabalho, deseja-se investigar e propor métodos de filtragem de Kalman para impor restrições de igualdade ou de desigualdade na estimativa de estado. Tratam-se ambos os casos de sistemas lineares e não-lineares, para os quais propõem-se algoritmos baseados no UKF.

Formulação do Problema

Para o sistema dinâmico estocástico não-linear de tempo discreto

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}, k-1), \quad (1)$$

$$y_k = h(x_k, v_k, k), \quad (2)$$

em que $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{N} \rightarrow \mathbb{R}^n$ e $h : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{N} \rightarrow \mathbb{R}^m$ são, respectivamente, os modelos de processo e de observação, os quais são assumidamente conhecidos, o problema de estimação de estados é definido a seguir; veja a Figura 1.1. Assuma que, para todo $k \geq 1$, são conhecidas as medições $y_k \in \mathbb{R}^m$, as entradas $u_{k-1} \in \mathbb{R}^p$, e as funções de densidade de probabilidade (PDF) $\rho_{x_0}(x_0)$, $\rho_{w_{k-1}}(w_{k-1})$ e $\rho_{v_k}(v_k)$, em que $x_0 \in \mathbb{R}^n$ é o vetor de estados inicial, $w_{k-1} \in \mathbb{R}^q$ é o ruído de processo, representando perturbações desconhecidas, e $v_k \in \mathbb{R}^r$ é o ruído de medição, o qual trata das incertezas nas observações. Defina a função de lucro

$$J(x_k) \triangleq \rho(x_k | (y_1, \dots, y_k)), \quad (3)$$

em que $\rho(x_k | (y_1, \dots, y_k))$ é a PDF condicional *a posteriori* do vetor de estados $x_k \in \mathbb{R}^n$ condicionada às medições passadas $y_1, \dots, y_{k-1}$ e presente y_k . A solução completa do *problema de estimação de estados* é dada por $\rho(x_k | (y_1, \dots, y_k))$ e o maximizador $\hat{x}_{k|k} \in \mathbb{R}^n$ de J é a *estimativa de estados* ótima.

Conforme mencionado anteriormente, para sistemas lineares e Gaussianos, KF fornece a solução recursiva ótima para o problema de estimação de estados. Para o caso não-linear, todavia, a solução para esse problema é complicada devido ao fato de $\rho(x_k | (y_1, \dots, y_k))$ não ser completamente caracterizada por sua média e covariância (Daum, 2005). Dessa maneira, aproximações baseadas no KF são investigadas nesta tese.

Neste trabalho, considera-se que o vetor de estados está sujeito à restrição de igualdade

$$g(x_k, k-1) = d_{k-1}, \quad (4)$$

e/ou à restrição de desigualdade

$$l(x_k, k-1) \leq e_{k-1}, \quad (5)$$

em que se assumem conhecidos $g : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^s$, $d_{k-1} \in \mathbb{R}^s$, $l : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^t$ e $e_{k-1} \in \mathbb{R}^t$. O

maximizador $\hat{x}_{k|k}^c$ de (3) sujeito a (4) e/ou (5) é a *estimativa de estado restrita* ótima. Neste trabalho, também se estuda a restrição de intervalo

$$a_k \leq x_k \leq b_k, \quad (6)$$

em que $a_{i,k} < b_{i,k}$, $i = 1, \dots, n$, a qual é um caso especial de (5).

Objetivos

Esta tese trata do desenvolvimento de algoritmos de estimação de estados para sistemas lineares e não-lineares com o uso de conhecimento *a priori* na forma de restrições. Ademais, investigam-se aplicações de engenharia para esses algoritmos. Dessa maneira, cinco são os objetivos deste projeto de pesquisa, a saber:

1. Estudar e deduzir uma solução ótima para o problema de estimação de estados com restrição de igualdade no vetor de estados para sistemas lineares, bem como soluções aproximadas para sistemas não-lineares. Para o último caso, usa-se a abordagem *unscented*. Investigar os problemas de estimação de atitude com parametrização em quatérnios (Crassidis & Markley, 2003) e assimilação de dados em magneto-hidrodinâmica com divergência nula (Tóth, 2000).
2. Propor e comparar algoritmos sub-ótimos de estimação de estados baseados no UKF para impor restrições de intervalo nas estimativas de estado de sistemas não-lineares.
3. Desenvolver uma metodologia geral em estimação de estados com restrição de igualdade no ganho do estimador a fim de impor propriedades especiais nas estimativas de estado. Mostrar como outros algoritmos propostos na literatura (Kitanidis, 1987; Darouach et al., 2003; Palanthandalam-Madapusi et al., 2006; Gillijns & De Moor, 2007b; Chandrasekar et al., 2007; Gupta & Hauser, 2007) são casos especiais dessa metodologia, bem como estender o algoritmo obtido para o caso não-linear usando a abordagem *unscented*.
4. Ilustrar a aplicação de filtros de Kalman não-lineares para solução de dois problemas em engenharia aeroespacial: reconstrução de trajetórias de voo (Mulder et al., 1999; Fagundes, 2007; Mendonça et al., 2007) e estimação de trajetórias de espaçonaves (Tapley et al., 2004; de Moraes et al., 2007).
5. Realizar estimação de estados para sistemas não-lineares de tempo contínuo com dinâmica

caótica (Fiedler-Ferrara & Prado, 1994) usando modelos identificados de tempo discreto (Aguirre, 2007) e UKF. Além disso, deseja-se discutir o impacto do não conhecimento das equações dinâmicas exatas e do uso de modelos aproximados.

Estudos de Caso de Estimação de Estados Sem Restrições

Reconstrução de Trajetórias de Vôo

O problema de reconstrução de trajetórias de vôo (FPR) (Mulder et al., 1999; de Mendonça, 2005) é o primeiro passo no procedimento de se usar dados de ensaios de vôo para obter informação sobre os parâmetros do modelo de uma aeronave (Hemerly et al., 2003; Góes et al., 2006; Jategaonkar, 2006). Basicamente, essa reconstrução significa estimar a trajetória da aeronave a partir de dados de instrumentação nela instalada. Esse passo é crucial e fundamental, pois sensores reais apresentam polarização, a qual deve ser considerada nos estágios subsequentes de análise dos dados. Valores estimados para polarização podem ser obtidos pelo processo de FPR.

Nesta tese, o problema de reconstrução de trajetórias de vôo é abordado no contexto de métodos de filtragem de Kalman não-linear. Procura-se elucidar três questões quanto: (i) à escolha do algoritmo de filtragem de Kalman não-linear empregado – EKF ou UKF; (ii) ao uso de um algoritmo de suavização no lugar de um método de filtragem; e (iii) ao emprego de uma abordagem iterativa de filtragem para melhorar as estimativas.

Dados simulados da aeronave Beaver DHC-2 usando o pacote computacional FDC (Rauw, 2005) e experimentais de uma aeronave planadora Puchacz SZD 50-3, cujo sistema de instrumentação embarcada FDAS foi desenvolvido pelo Centro de Estudos Aeronáuticos (CEA/UFMG) (Iscold & da Silva, 2005), foram utilizados durante essa análise. Os resultados obtidos são mostrados nas Figuras 3.2, 3.4 e 3.5 e Tabelas 3.1 e 3.2. De uma forma geral, observou-se que: (i) UKF apresenta desempenho superior com relação à precisão na estimação de estados e termos de polarização da unidade de medição inercial (IMU) quando comparado ao EKF, bem como maior robustez à inicialização, nível de ruído, sintonia das matrizes de covariância e a problemas numéricos, sem a necessidade de cálculo de matrizes Jacobianas; (ii) os suavizadores de Kalman acarretam em estimativas significativamente mais precisas e mais informativas, principalmente, para as variáveis de posição, as quais são observadas pelo receptor de GPS a uma taxa de amostragem inferior às demais variáveis; e (iii) o uso do EKF iterativo, aparentemente, não melhorou a qualidade das estimativas para os dados testados

em termos de precisão.

Estimação de Trajetórias de Espaçonaves

O presente estudo de caso ilustra e compara filtros de Kalman não-lineares para o problema de estimação de trajetórias de espaçonaves (Tapley et al., 2004; Curtis, 2005). Diferentes problemas e formulações podem ser considerados para estimação de trajetória de espaçonaves com base no número e tipo de medições disponíveis, incluindo distância (Duong & Winn, 1973; Pisacane et al., 1974), variação de distância (Bizup & Brown, 2004) e ângulo (Fowler & Lee, 1985; Lee & Alfriend, 2007). O caso em que as medições são fornecidas por uma constelação de satélites é estudado em (Cicci & Ballard, 1994a,b; Chiaradia et al., 2003; de Moraes et al., 2007).

São utilizados algoritmos de filtragem de tempo contínuo com medições amostradas (de seis satélites com órbita equatorial de baixa altitude) baseados no EKF (Jazwinski, 1970; Gelb, 1974) e UKF (Särkkä, 2007) para rastrear a órbita de um satélite em órbita geossíncrona de elevada altitude.

Utilizam-se dados simulados na presente análise. Os resultados obtidos são exibidos nas Figuras 3.6, 3.7, 3.8, 3.9, 3.10 e 3.11 e Tabelas 3.3 e 3.4. A abordagem *unscented* mostrou-se menos sensível à inicialização e ao intervalo de amostragem das medições do que a abordagem estendida. No entanto, o método baseado no UKF apresenta um custo computacional cerca de duas vezes maior. Além disso, o algoritmo *unscented* produz estimativas mais precisas para a excentricidade da órbita, quando comparado ao algoritmo estendido. Finalmente, a fim de estimar a inclinação da órbita da espaçonave, o uso de medições de ângulo ajudam na convergência e melhoram o desempenho de ambos os algoritmos.

Uso de Modelos Identificados em Filtragem *Unscented*

Recentemente, a desafiadora tarefa de estimar estados e, eventualmente, alguns parâmetros de sistemas não-lineares tem recebido considerável atenção (Voss et al., 2004). Em particular, métodos de filtragem não-linear alternativos ao EKF, como os algoritmos da família de pontos sigma, destacam-se, sob vários aspectos, como ferramentas poderosas e promissoras no contexto de dinâmica não-linear e caos (Sitz et al., 2002, 2004; Bitencourt Jr. et al., 2004).

Neste trabalho, estuda-se o uso de modelos de tempo discreto identificados a partir de dados ruidosos em conjunto com UKF para estimar estados e parâmetros de sistemas não-lineares. Investiga-se o cenário no qual observações de todas as componentes do vetor

de estados estão disponíveis durante um curto intervalo de tempo para construir o modelo, enquanto somente uma componente do vetor de estados é medida para uso no algoritmo de filtragem. Para tal, utilizam-se dados simulados do sistema de Lorenz (Lorenz, 1963) e dados experimentais de um oscilador eletrônico conhecido como circuito de Chua (Chua, 1994; Tôres & Aguirre, 2000).

A partir dos testes realizados, cujos resultados são mostrados nas Figuras 3.12, 3.13, 3.15, 3.16 e 3.17 e Tabelas 3.5 e 3.6, relata-se que UKF é viável para estimar conjuntamente estados e parâmetros desses sistemas não-lineares caóticos a partir de modelos previamente identificados. Nesse cenário, a definição apropriada da matriz de covariância de ruído de processo parece ser fundamental na convergência do algoritmo de filtragem. Modelos ruins, geralmente, resultam em estimativas de estado pouco precisas ou em divergência do estimador, independente do valor atribuído a tal matriz, ao passo que o emprego de modelos mais precisos melhoram o desempenho das estimativas de estado, especialmente, para altos níveis de ruído nas medições.

Contribuições Teóricas em Estimação de Estados Com Restrições

Estimação de Estados com Restrições de Igualdade

Diversos métodos têm sido desenvolvidos para estimação de estados com restrições de igualdade. Uma das técnicas mais populares é o filtro de Kalman com medições aumentadas (MAKF), para o qual “medições” perfeitas das grandezas de restrição são anexadas às medições físicas (Porrill, 1988; Tahk & Speyer, 1990; Wen & Durrant-Whyte, 1992). Além disso, são apresentados os métodos de projeção das estimativas (Simon & Chia, 2002), projeção do sistema (Ko & Bitmead, 2007) e projeção do ganho (Gupta & Hauser, 2007).

As abordagens de medições aumentadas e projeção das estimativas são estendidas para o caso não-linear por meio do EKF (Alouani & Blair, 1993; Chen & Chiang, 1993; Simon & Chia, 2002; Gupta, 2007b). Alternativamente, um algoritmo de projeção de duas etapas para tratar restrições de igualdade não-lineares é apresentado em (Julier & LaViola Jr., 2007). Além disso, um método que usa multiplicadores de Lagrange é capaz de impor exatamente restrições de igualdade quadráticas e unidimensionais (Yang & Blasch, 2006). A Tabela 4.1 compara os principais métodos apresentados na literatura para impor restrições de igualdade lineares e não-lineares nas estimativas de estado.

Nesta tese, com relação ao problema de estimação de estados com restrições de igual-

dade, primeiramente considera-se o caso linear e Gaussiano. Nesse contexto, mostra-se como a existência de restrições de igualdade está vinculada à definição de ruído de processo e equações dinâmicas com propriedades especiais, as quais são resumidas nas equações (5.5)-(5.7). Assim, o filtro de Kalman com restrições de igualdade (ECKF) é derivado como a solução recursiva ótima (sob a ótica de máximo *a posteriori*) para o problema em questão; veja o Teorema 5.2.1. ECKF é comparado a quatro algoritmos alternativos apresentados na literatura. O algoritmo proposto e os quatro revisados são ilustrados por meio de dois exemplos: sistema de compartimentos para o qual observa-se conservação de massa (veja a Figura 5.3 e a Tabela 5.1) e um sistema de rastreamento de veículo terrestre com restrições cinemáticas (veja a Tabela 5.2).

Para o caso não-linear, três algoritmos sub-ótimos baseados no UKF são apresentados, a saber: filtro de Kalman *unscented* com restrições de igualdade (EUKF), filtro de Kalman *unscented* projetado por projeção das estimativas (PUKF-EP), e filtro de Kalman *unscented* com medições aumentadas (MAUKF). Dois exemplos são considerados: estimação de atitude com parametrização em quatérnios (veja a Figura 5.4 e a Tabela 5.3) e um pêndulo simples com conservação de energia mecânica, para o qual um modelo discretizado sem conservação de energia é usado (veja as Figuras 5.5 e 5.6 e as Tabelas 5.4 e 5.5).

A partir dos resultados obtidos para as quatro aplicações investigadas, observou-se que, além de impor uma restrição de igualdade conhecida *a priori*, os métodos propostos produzem estimativas de estado mais precisas e informativas que KF (para o caso linear) ou UKF (no caso não-linear). Além disso, todas as abordagens investigadas têm custo computacional semelhante e competitivo ao do KF (para algoritmos lineares) e UKF (caso não-linear).

Quanto à escolha do algoritmo, ECKF e os quatro métodos revisados produzem resultados de desempenho semelhante para o caso linear. No entanto, para o caso não-linear, com base nos exemplos investigados, recomenda-se testar, na seguinte ordem, EUKF, MAUKF e PUKF-EP para uma dada aplicação. Observe que, como tais métodos são aproximados, o desempenho dos mesmos depende da aplicação.

Estimação de Estados com Restrições de Intervalo

Algoritmos sub-ótimos também têm sido desenvolvidos para estimação de estados linear com restrições de desigualdade. Uma das técnicas mais populares é o estimador de horizonte móvel (MHE) (Rao et al., 2001), o qual formula a estimação de estados como um problema não-recursivo de programação quadrática. Alternativamente, métodos probabilísticos (Rotea & Lana, 2005) impõem a restrição de desigualdade por meio da adequada inicialização

do KF e apropriada sintonia de suas matrizes de covariância de ruído. O procedimento de truncamento (Shimada et al., 1998; Simon & Simon, 2007) remodela a PDF computada pelo KF, a qual é assumidamente Gaussiana e é dada pela estimativa de estados e pela covariância do erro de estimação, nas bordas da restrição de desigualdade. Finalmente, por meio da abordagem de projeção (Simon & Simon, 2006b), se a estimativa de estados não satisfaz a restrição de desigualdade, então a mesma é projetada no extremo da região de restrição.

Para sistemas não-lineares, algoritmos baseados no MHE têm sido empregados (Rao et al., 2003; Russo & Young, 1999). No entanto, como essas técnicas são não-recursivas, elas são computacionalmente inviáveis para algumas aplicações em tempo real (Vachhani et al., 2006). Para esses casos, o filtro de Kalman estendido restrito (CEKF) (Vachhani et al., 2006; Marcon et al., 2002), o qual é um caso especial do MHE com horizonte móvel unitário e é chamado de algoritmo recursivo de reconciliação de dados para sistemas dinâmicos não-lineares (RNDDR) em (Vachhani et al., 2005), é apresentado como um algoritmo de implementação mais simples e de menor custo computacional. Motivado pelo desempenho superior do UKF com relação ao EKF, o algoritmo *unscented* recursivo de reconciliação de dados para sistemas dinâmicos não-lineares (URNDDR), o qual, por conveniência, é chamado de filtro de Kalman *unscented* intervalar de pontos sigma (SIUKF) nesta tese, é apresentado em (Vachhani et al., 2006). Os algoritmos supracitados são comparados na Tabela 4.1.

Nesta tese, é estudado o problema de estimação de estados com restrições de intervalo para sistemas não-lineares. Pela combinação de uma de duas possíveis abordagens *unscented* para execução da etapa de predição e um de cinco possíveis candidatos para a etapa de assimilação de dados – as quais são usadas na literatura e listadas na Tabela 6.1, são propostos os seguintes algoritmos: filtro de Kalman *unscented* restrito (CUKF), filtro de Kalman *unscented* restrito intervalar (CIUKF), filtro de Kalman *unscented* intervalar (IUKF), filtro de Kalman *unscented* truncado (TUKF), filtro de Kalman *unscented* truncado intervalar (TIUKF), filtro de Kalman *unscented* projetado (PUKF).

Esses seis algoritmos são comparados ao UKF e SIUKF por meio de dois exemplos simulados de processos químicos, cujos estados são não-negativos. Os resultados obtidos são mostrados nas Figuras 6.1 e 6.2 e Tabelas 6.3 e 6.4. Verificou-se que, para uma boa inicialização, os oito algoritmos supracitados têm desempenho semelhante. No entanto, para o caso de inicialização ruim, as estimativas do UKF violam a restrição de intervalo. Nesse caso, o emprego de CUKF and PUKF resultam em uma pequena melhoria comparados ao UKF. Por outro lado, SIUKF, TUKF, and TIUKF produzem os melhores resultados, ao passo

que CIUKF e IUKF têm desempenho intermediário. Com relação ao custo computacional, IUKF, TIUKF, e TUKF são competitivos com UKF. Ademais, CUKF, CIUKF e PUKF são mais lentos que UKF, já que requerem a solução *online* de problemas de otimização. Por essa razão, SIUKF é o algoritmo mais lento.

Todavia, não é possível indicar qual método é mais adequado para um dado problema. Na verdade, parece que a escolha do método depende da aplicação. A partir dos dois exemplos investigados, considerando o compromisso entre precisão e complexidade computacional, TUKF e TIUKF parecem ser as melhores alternativas, seguidos por SIUKF, CIUKF, e IUKF.

Estimação de Estados com Restrição no Ganho

Em estimação de estados clássica, o ganho padrão do estimador é irrestrito no sentido de que todos os resíduos das medições são potencialmente usados para diretamente corrigir todas as componentes do vetor de estados. Semelhante ao caso de controle realimentado com saída estática (Syrmos et al., 1997), o qual pode ser visto como uma forma restrita do controle realimentado com vetor de estados completo, é possível restringir a forma do ganho do estimador. Fazendo isso, é possível impor propriedades especiais nas estimativas de estado.

Três distintas motivações para restringir o ganho do estimador são encontradas na literatura. Primeiramente, em (Kitanidis, 1987; Darouach et al., 2003; Palanthandalam-Madapusi et al., 2006; Gillijns & De Moor, 2007a,b; Gillijns, 2007), o ganho do estimador é restrito a fim de se obter estimativas não polarizadas apesar do fato de entradas exógenas desconhecidas (podendo ser determinísticas ou de média não-nula) serem presentes. Trata-se do filtro de Kalman não-polarizado com entradas desconhecidas (UnbKF-UI). De forma semelhante, em (Chandrasekar & Bernstein, 2006; Chandrasekar et al., 2007), o ganho é restrito para simplificar a estrutura do estimador e sua implementação com multi-processadores para aplicações envolvendo sistemas de dimensão elevada, tais como aqueles que envolvem equações diferenciais parciais discretizadas. A mesma idéia pode ser aplicada para tratar falha sensorial parcial ou completa. O algoritmo resultante é chamado de filtro de Kalman espacialmente restrito (SCKF). Finalmente, em (Gupta & Hauser, 2007), é apresentado o filtro de Kalman projetado com ganho projetado (PKF-GP), cujo ganho é restrito para produzir estimativas de estado satisfazendo uma restrição de igualdade linear. Portanto, desenvolve-se uma metodologia geral que inclui os métodos supracitados como casos especiais e que permite a futura proposição de novos estimadores de estados com ganho restrito.

Nesta tese, o filtro de Kalman com ganho restrito (GCKF) é derivado como solução

ótima de variância mínima para o problema de estimação de estados com restrição no ganho para sistemas lineares; veja a Proposição 7.1.2. Ademais, KF, UnbKF-UI, SCKF e PKF-GP são apresentados como casos especiais do GCKF. Também é derivado o preditor de Kalman com ganho restrito para o mesmo problema.

Além disso, usando a transformada *unscented*, o filtro de Kalman *unscented* com ganho restrito (GCUKF) é apresentado como extensão não-linear do GCKF. Embora esse algoritmo seja uma solução aproximada para o problema de estimação de estados não-linear, seu ganho satisfaz exatamente a restrição de igualdade.

Dois exemplos de sistemas não-lineares, a saber, o oscilador de van der Pol e um sistema pêndulo-carro, são usados para ilustrar uma aplicação do GCUKF na qual o vetor de entrada é desconhecido e deseja-se obter estimativas de estado não-polarizadas. Os resultados obtidos são mostrados nas Figuras 7.2, 7.3, 7.4 e 7.6 e Tabelas 7.1 e 7.2. Em ambos os casos, por meio do GCUKF, obtiveram-se estimativas de estados melhores do que aquelas obtidas pelo UKF com o vetor de entradas desconhecido.

Conclusões

Na presente tese, além de apresentar uma revisão do atual estado da arte em estimação de estados com restrições, são desenvolvidos métodos de filtragem de Kalman para impor restrições de igualdade e/ou de desigualdade na estimativa de estado. Ambos os casos de sistemas lineares e não-lineares são considerados. Para o último caso, propõem-se algoritmos baseados no filtro de Kalman *unscented*. Ademais, propõe-se uma metodologia geral para estimação de estados com uma restrição de igualdade no ganho do estimador, visando a impor indiretamente propriedades especiais na estimativa de estado. Exemplos simulados e experimentais são usados para ilustrar os algoritmos estudados e propostos ao longo desta tese.

A maioria do trabalho apresentado neste documento foi submetido ou publicado em periódicos internacionais (Aguirre, Teixeira & Tôrres, 2005; Teixeira, Chandrasekar, Tôrres, Aguirre & Bernstein, 2008a; Teixeira, Chandrasekar, Palanthandalam-Madapusi, Tôrres, Aguirre & Bernstein, 2008b; Teixeira, Tôrres, Iscold & Aguirre, 2008), revistas internacionais (Teixeira, 2008; Teixeira, Santillo, Erwin & Bernstein, 2008), anais de conferências internacionais (Teixeira, Tôrres, Iscold & Aguirre, 2005; Teixeira, Chandrasekar, Tôrres, Aguirre & Bernstein, 2007, 2008b; Teixeira, Ridley, Bernstein, Tôrres & Aguirre, 2008; Teixeira, Tôrres,

Aguirre & Bernstein, 2008) e relatórios internos (Teixeira, Tôrres, Aguirre & Iscold, 2006; Teixeira, Ridley, Bernstein, Tôrres & Aguirre, 2007).

INTRODUCTION

“If I have seen further, it is by standing on the shoulders of giants.”

Isaac Newton

1.1 Motivation

Astronomical studies provided the pioneer stimuli for the development of the estimation theory especially in the end of the 18th century. According to (Sorenson, 1970), Karl Friedrich Gauss was the leading protagonist of the birth of this theory, when he investigated the motion of planets and comets using data measured from telescopes. Aiming at estimating the parameters that characterize the equations of motion of these heavenly bodies, Gauss derived the well-known least-squares method in 1795.

About a century and a half later, the advent of the space race provided the main motivations for the development of the state-estimation theory.¹ Aerospace scientists were looking for a solution to the spacecraft trajectory estimation problem, for instance. Then, in the January of 1959, Rudolf Emil Kalman, by means of a “*single, gigantic, persistent mathematical exercise*” (Kalman, 2003) proposed the state-estimation algorithm (Kalman, 1960) that is named after him in recognition to the major breakthrough ever made in the field of state estimation. One of the first and most important applications of the Kalman filtering theory was in the on-board computer that guided the descent of the Apollo 11 lunar module to the moon (Schmidt, 1981). Currently, Kalman-like algorithms are the off-the-shelf technology to many applications in industry, including navigation systems (Cipra, 1993). At this point, it is important to recognize the previous outstanding theoretical contributions to the development of the state-estimation theory given by Fisher (1912), Wiener (1949), and Kolmogorov (1962), among others.

¹Recall that state-estimation algorithms are different from state observers (Luenberger, 1971) since only the former care about the existence of random disturbances in both dynamics and measurement map.

Similar to Gauss, Kalman also formulated his problem as an optimization problem that minimizes a functional of the square of the estimation error, but, unlike Wiener (1949), he used the state-space formulation. Kalman’s achievements in state estimation agree with the Albert Einstein’s quote that says that “*the mere formulation of a problem is far more often essential than its solution, which may be merely a matter of mathematical or experimental skill.*”

Nevertheless, the Gauss’ legacy was not the main source of inspiration for Kalman’s work. Kalman (2003) argues that his work was naturally inspired by the technological revolution created by Isaac Newton. In 1687, using only mathematical reasoning, Newton proved that the gravitational force causes the motion of the planets around the Sun, confirming thus the Kepler’s laws. According to Kalman, the revolutionary ideas inherited from Newton were the use of data to the detriment of physical assumptions to formulate a problem and the mathematization of the problem to build a solution. This approach is still responsible for many technological innovations nowadays. We mention, for instance, the increasing effort towards the use of data collected from onboard instrumentation during flight tests to determine the parameters of aircraft models (Jategaonkar, 2006; Góes, Hemerly, Maciel, Rios Neto, de Mendonça & Hoff, 2006).

1.1.1 Unconstrained State Estimation

The assumption of linear dynamics and linear measurement map, as well as the assumption of Gaussian distribution of the disturbances, restrict the use of the classical Kalman filter (KF) in some practical cases. Therefore, methods for the nonlinear scenario are necessary. The extended Kalman filter (EKF) is the straightforward extension of KF to the nonlinear case by means of analytical or numerical linearization of the system equations (Schmidt, 1970; Jazwinski, 1970). EKF has been the most widely used algorithm for nonlinear state estimation during the last four decades, covering applications from spacecraft trajectory estimation (Schmidt, 1981; de Moraes, da Silva & Kuga, 2007) and weather forecasting (Carme, Pham, & Verron, 2001) to training of neural networks (Rios Neto, 1997).

However, EKF is sensitive to initialization (Reif et al., 1999), and, for strongly nonlinear systems, its estimates may diverge (Reif, Günther, Yaz & Unbehauen, 1999; Julier & Uhlmann, 2004; van der Merwe, Wan & Julier, 2004; Cipra, 1993). Motivated by the EKF flaws, the unscented Kalman filter (UKF) was presented as an improved approximate nonlinear state estimator that propagates a reduced number of deterministically chosen ensembles

(Julier, Uhlmann & Durrant-Whyte, 1995; Julier & Uhlmann, 2004). UKF is based on “*the intuition that it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function or transformation*” (Julier, Uhlmann & Durrant-Whyte, 2000). Like UKF, other nonlinear methods (Ito & Xiong, 2000; Nogaard, Poulsen & Ravn, 2000) use the same principles and are categorized into the sigma-point Kalman filter (SPKF) framework (Lefebvre, Bruyninckx & De Schutter, 2002; van der Merwe, 2004).

Alternatively, since the decade of 1990, particle filtering methods have been proposed (Gordon, Salmon & Smith, 1993); see (Arulampalam, Maskell, Gordon & Clapp, 2002) for a helpful overview. Such methods are able to treat non-Gaussian distributions, as well as nonlinearity in the dynamics and measurement map. However, the high computational complexity of these methods inhibits their real-time application in some practical cases (Daum, 2005).

Along the last eight years, an impressive number of applications of SPKF algorithms has been reported, such as,

- in sensor fusion and navigation (Hall & Llinas, 2001; Wagner & Wienieke, 2003; Azizi & Houshargi, 2003; Shin & El-Sheimy, 2003; van der Merwe, Wan & Julier, 2004; Crassidis, 2006; Metzger, Wisotzky, Wendel & Trommer, 2005);
- target tracking (Farina, Ristic & Benvenuti, 2002; Ristic, Farina, Benvenuti & Arulampalam, 2003);
- spacecraft attitude estimation (Chen, Seereeram & Mehra, 2003; Crassidis & Markley, 2003; Markley, Crassidis & Cheng, 2005; Psiaki, 2005; VanDyke, Schwartz & Hall, 2004; Teixeira, Chandrasekar, Tórres, Aguirre & Bernstein, 2008b) and spacecraft trajectory estimation (Lee & Alfriend, 2007; Teixeira, Santillo, Erwin & Bernstein, 2008);
- flight path reconstruction (Teixeira, Tórres, Iscold & Aguirre, 2005; de Mendonça & Hemerly, 2007);
- aircraft failure detection (Brunke & Campbell, 2002, 2004);
- state-and-parameter estimation in nonlinear dynamics and chaos (Sitz, Schwarz, Kurths & Voss, 2002; Sitz, Schwarz & Kurths, 2004; Aguirre, Teixeira & Tórres, 2005);
- parameter estimation of continuous-time differential equations (Voss, Timmer & Kurths, 2004) and multilayer perceptron neural networks (Haykin, 2001; Wan, van der Merwe & Nelson, 2000; Feldkamp, Feldkamp & Prokhorov, 2001; Choi, Yeap, & Bouchard, 2004; Choi, Yeap & Bouchard, 2005);
- transmission of information (Bitencourt Jr., 2003; Bitencourt Jr., Tórres & Aguirre, 2004);

- data assimilation for large-scale systems (Chandrasekar, 2007; Chandrasekar, Ridley & Bernstein, 2007);
- state estimation for chemical processes (Hovland, von Hoff, Gallestey, Antoine, Farruggio & Paice, 2005; Vachhani, Narasimhan & Rengaswamy, 2006; Teixeira, Tôres, Aguirre & Bernstein, 2008), induction motors (Akin, Orguner & Ersak, 2003), and solar power plants (Hovland, von Hoff, Gallestey, Antoine, Farruggio & Paice, 2005).

Therefore, we believe that the sigma-point approach is the current state of the art in nonlinear state estimation. In this thesis, we focus on sigma-point algorithms for nonlinear state estimation.

1.1.2 Equality and Inequality State-Constrained State Estimation

The classical KF for linear systems provides optimal state estimates under standard noise and model assumptions. In practice, however, additional information about the system may be available, and this information may be useful for improving state estimates. A scenario we have in mind is the case in which the dynamics and the disturbances are such that the states of the system satisfy an equality (Teixeira, Chandrasekar, Tôres, Aguirre & Bernstein, 2007; Ko & Bitmead, 2007) or inequality constraint (Rao, 2000; Robertson & Lee, 2002; Goodwin, Seron & de Doná, 2005).

For example, in a chemical reaction, the species concentrations are nonnegative (Massicotte, Morawski & Barwicz, 1995; Chaves & Sontag, 2002). Also, additional examples of systems with inequality-constrained states arise in aeronautics (Rotea & Lana, 2005; Simon & Simon, 2006b). Regarding equality-constrained systems, in a compartmental model with zero net inflow (Bernstein & Hyland, 1993), mass is conserved. Likewise, in undamped mechanical systems, such as a system with Hamiltonian dynamics, conservation laws hold, while, the magnetic field satisfies the zero-divergence property in magnetohydrodynamics (Tóth, 2000; Brackbill & Barnes, 2002). Moreover, in the quaternion-based attitude estimation problem, the attitude vector must have unit norm (Crassidis & Markley, 2003; Kuang & Tan, 2002; Choukroun, Bar-Itzhack & Oshman, 2006). Additional examples of constrained systems arise in optimal control (de Oliveira & Biegler, 1994; Maciejowski, 2002; Goodwin, Seron & de Doná, 2005), parameter estimation (Chia, Chow & Chizeck, 1991; Aguirre, Donoso-Garcia & Santos Filho, 2000; Aguirre, Barroso, Saldanha & Mendes, 2004; Aguirre, Alves & Corrêa, 2007; Nepomuceno, Takahashi, Aguirre, Neto & Mendes, 2004; Walker, 2006), and navigation (Wen & Durrant-Whyte, 1992; De Geeter, van Brussel & De Schutter, 1997; Dissanayake,

Sukkarieh & Nebot, 2001; Alouani & Blair, 1993; Shen, Honga & Cong, 2006). In such cases, we wish to obtain state estimates that take advantage of prior knowledge of the states and use this information to obtain better estimates than those provided by KF in the absence of such information. Although it is difficult to make correspondingly precise statements in the case of nonlinear systems, the same principles and objectives apply.

Constrained state estimation has been receiving increasing attention in both academia and industry, especially in the last ten years (Simon, 2008). Various methods have been developed for equality-constrained state estimation. One of the most popular techniques is the measurement-augmentation Kalman filter, in which a perfect “measurement” of the constrained quantity is appended to the physical measurements (Porrill, 1988; Tahk & Speyer, 1990; Chia, Chow & Chizeck, 1991; Wen & Durrant-Whyte, 1992; Doran, 1992). In addition, estimate-projection (Simon & Chia, 2002), system-projection (Ko & Bitmead, 2007), and gain-projection (Gupta & Hauser, 2007) methods have been considered. Alternatively, the model reduction approach, in which the model parameterization is reduced by substituting the equality constraint into the dynamic model, can be employed (Wen & Durrant-Whyte, 1991; Simon, 2006).

The measurement-augmentation and estimate-projection approaches are also extended to the nonlinear case by means of EKF, yielding the approximate nonlinear equality-constrained algorithms: measurement-augmentation EKF (Alouani & Blair, 1993; Chen & Chiang, 1993; Walker, 2006) and projected EKF (Simon & Chia, 2002; Gupta, 2007b), respectively. A two-step projection algorithm for handling nonlinear equality constraints has also been presented (Julier & LaViola Jr., 2007). Also, a Lagrange-multiplier method for exactly enforcing a single quadratic nonlinear state constraint is given in (Yang & Blasch, 2006).

Suboptimal algorithms have also been developed for inequality-constrained linear state estimation. One of the most popular techniques is the moving horizon estimator (MHE) (Rao, Rawlings & Lee, 2001), which formulates the state-estimation problem as a non-recursive quadratic program. Alternatively, probabilistic methods (Rotea & Lana, 2005) enforce the inequality constraint by properly initializing the KF algorithm and tuning its noise covariances. The truncation procedure (Shimada, Shirai, Kuno & Miura, 1998; Simon, 2006; Simon & Simon, 2007) reshapes the probability density function computed by KF, which is assumed to be Gaussian and is given by the state estimate and the error covariance, at the inequality constraint edges. Finally, if the state estimate does not satisfy the inequality constraint, then it is projected onto the boundary of the constraint region by the projection approach (Simon,

2006; Simon & Simon, 2006b).

For nonlinear inequality-constrained systems, algorithms based on MHE are employed (Rao, Rawlings & Mayne, 2003; Haseltine & Rawlings, 2005; Russo & Young, 1999). However, since these techniques are non-recursive, they are computationally expensive and difficult to use in real-time applications (Vachhani et al., 2006). For such cases, the constrained EKF (Rao, Rawlings & Mayne, 2003; Vachhani, Narasimhan & Rengaswamy, 2006; Marcon, Trieiweiler & Secchi, 2002), which is a special case of MHE with unitary moving horizon and is called recursive nonlinear dynamics data reconciliation (RNDDR) in (Vachhani et al., 2005, 2006), is presented as a simpler and less computationally demanding algorithm. Motivated by the improved performance (Julier et al., 2000; Lefebvre et al., 2002, 2004; van der Merwe et al., 2004; Reif et al., 1999) of UKF over EKF, the unscented RNDDR, which is referred to as the sigma-point interval unscented Kalman filter in this thesis, is presented in (Vachhani et al., 2006).

1.1.3 Gain-Constrained State Estimation

State estimation is effectively a closed-loop problem, in which data are fed back to the plant. Like servo control, the state estimator acts based on an error signal, namely, the innovation, and uses statistical information to determine a “feedback” gain that can produce optimal or suboptimal state estimates. This data-injection process is analogous to full-state feedback control.

In classical state estimation, the standard data injection gain is unconstrained in the sense that all measurement residuals are potentially used to directly update all of the state estimates. Just as static output feedback control (Syrmos, Abdallah, Dorato & Grigoriadis, 1997) can be viewed as a restricted form of full-state feedback control, it is possible to restrict the form of the data-injection gain. In doing so it is possible to enforce special properties on the state estimate.

Three distinct practical motivations for constraining the data-injection gain can be found in the literature. First, in (Darouach, Zasadzinski & Boutayeb, 2003; Gillijns & De Moor, 2007a,b; Kitanidis, 1987; Palanthandalam-Madapusi, Gillijns, De Moor & Bernstein, 2006; Gillijns, 2007), the data-injection gain is restricted so that the state estimates are unbiased despite the fact that arbitrary (for example, deterministic or nonzero-mean) unknown exogenous inputs are present. Likewise, in (Chandrasekar, Bernstein, Barrero & De Moor, 2007; Chandrasekar & Bernstein, 2006; Chandrasekar, 2007), the data-injection gain is restric-

ted to simplify the estimator structure so as to facilitate multiprocessor implementation for applications involving large-scale systems such as discretized partial differential equations, as well as to handle partial or complete sensor outage. Finally, in (Gupta & Hauser, 2007), the data-injection gain is restricted to obtain state estimates satisfying a linear equality constraint. Therefore, we wish to develop a general framework that includes the aforementioned results as special cases, and that facilitates the proposition of new gain-constrained state estimators.

1.2 Problem Statement

For the stochastic nonlinear discrete-time dynamic system

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}, k-1), \quad (1.1)$$

$$y_k = h(x_k, v_k, k), \quad (1.2)$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{N} \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{N} \rightarrow \mathbb{R}^m$ are, respectively, the process and observation models, which are assumed to be known, the state estimation problem can be described as follows; see Figure 1.1. Assume that, for all $k \geq 1$, the known data are the measurements $y_k \in \mathbb{R}^m$, the inputs $u_{k-1} \in \mathbb{R}^p$, and the probability density functions (PDFs) $\rho_{x_0}(x_0)$, $\rho_{w_{k-1}}(w_{k-1})$, and $\rho_{v_k}(v_k)$, where $x_0 \in \mathbb{R}^n$ is the initial state vector, $w_{k-1} \in \mathbb{R}^q$ is the process noise, which represents unknown input disturbances, and $v_k \in \mathbb{R}^r$ is the measurement noise, concerning inaccuracies in the measurements. Next, define the profit function

$$J(x_k) \triangleq \rho(x_k | (y_1, \dots, y_k)), \quad (1.3)$$

where $\rho(x_k | (y_1, \dots, y_k))$ is the conditional *posterior PDF* of the state vector $x_k \in \mathbb{R}^n$ given the past and present measured data $y_1, \dots, y_k$.

Under the stated assumptions, the state-estimation problem is defined using the maximum-a-posteriori approach (Jazwinski, 1970; Maybeck, 1979; Thrun, Burgard & Fox, 2005).

Definition 1.2.1. $\rho(x_k | (y_1, \dots, y_k))$ is the complete solution to the *state-estimation problem*, while the maximizer $\hat{x}_{k|k} \in \mathbb{R}^n$ of J is the optimal *state estimate*.

Under Gaussian noise and linear model assumptions, KF provides the recursive optimal solution to the state-estimation problem. For nonlinear systems, however, the solution to

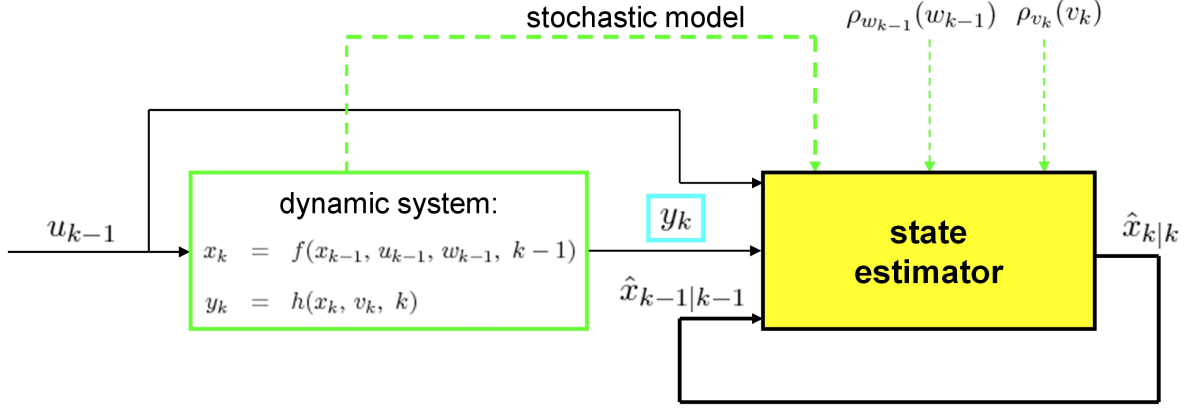


Figure 1.1: Diagram of the state-estimation problem under a statistical inference perspective. Our goal is to recursively estimate $\hat{x}_{k|k}$ using two uncertain sources of information, namely, the dynamic model and the measurements y_k . The inputs u_{k-1} and the statistical properties of the initial state vector $\rho_{x_0}(x_0)$, process noise $\rho_{w_{k-1}}(w_{k-1})$, and measurement noise $\rho_{v_k}(v_k)$ are assumed to be known.

the state-estimation problem is complicated by the fact that $\rho(x_k|(y_1, \dots, y_k))$ is not completely characterized by its mean and covariance (Daum, 2005). In this thesis, we thus investigate approximations based on the classical KF to provide suboptimal solutions to the nonlinear case.

Now, consider the equality constraint on the state vector

$$g(x_k, k-1) = d_{k-1}, \quad (1.4)$$

and the inequality constraint

$$l(x_k, k-1) \leq e_{k-1}, \quad (1.5)$$

where $g : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^s$, $d_{k-1} \in \mathbb{R}^s$, $l : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^t$, and $e_{k-1} \in \mathbb{R}^t$ are assumed to be known.² These constraints denote the feasible region of the state space in where x_k is assumed to be confined.

Definition 1.2.2. The maximizer of (1.3) subject to (1.4) and (1.5) is the optimal *constrained state estimate* $\hat{x}_{k|k}^c$.

In this thesis, we wish to take advantage of the prior knowledge provided by either (1.4) or (1.5) in order to find estimates satisfying one of these constraints. We also consider

²Note that the functions g and l are defined at time $k-1$ for the state vector x_k at time k . This feature is inherited from (5.8) in the context of equality-constrained linear systems; see Section 5.1.

the case in which the state vector satisfies both constraints. We address both the linear and nonlinear cases. Furthermore, we are interested in obtaining recursive solutions to the aforementioned problems.

1.3 Research Objectives

This research project is concerned with the development of linear and nonlinear state-estimation algorithms that enable the use of prior knowledge in the form of constraints. Furthermore, we are interested in investigating applications of nonlinear state-estimation algorithms. In order to address the general research goal, the following five specific objectives are identified:

1. Study the state-estimation problem for systems with an equality constraint on the state vector. First, our goal is to investigate how a linear equality constraint arises on the state vector of a linear dynamic system and then present an optimal Kalman filtering algorithm that enforces such constraint on the state estimates. Second, we wish to extend equality-constrained Kalman filters to the nonlinear case using the unscented approach. Third, we aim at applying such algorithms to solve problems, such as the quaternion-based attitude estimation problem and the zero-divergence data-assimilation problem for magnetohydrodynamics.
2. Present and compare suboptimal state-estimation algorithms based on UKF that enforce an interval constraint on the state estimates for nonlinear systems.³
3. Develop a general framework to state estimation with an equality constraint on the data-injection gain and show how other state-estimation algorithms are special cases of the resulting solution. We also want to extend such solution to the nonlinear scenario using the unscented approach.
4. Illustrate the application of nonlinear Kalman filters to solve the problems of flight path reconstruction and spacecraft trajectory estimation in the field of aerospace engineering.
5. Perform state estimation for continuous-time nonlinear systems with chaotic dynamics using data-driven discrete-time models and UKF. Moreover, we want to discuss the impact of not knowing the exact dynamic equations and using approximate models.

³An interval constraint is a type of inequality constraint (1.5) in which each component of the state vector is known to lie inside a given interval; see (4.73).

1.4 Thesis Outline

This thesis is organized in eight chapters (including this one) and three appendices. The main body of the text is divided into two parts. Part I deals with unconstrained state estimation and is composed by Chapters 2 and 3 and Appendix A. Part II addresses constrained state estimation and includes Chapters 4, 5, 6, and 7 and Appendices B and C.

Chapter 2 provides a literature review on unconstrained state estimation and sets the notation used along this document. KF is introduced as the optimal solution for linear and Gaussian systems. For the nonlinear scenario, EKF and UKF are reviewed. Furthermore, we define metrics and review concepts so as to assess the performance of state estimators.

Chapter 3 discusses three case studies of unconstrained state estimation. Both simulated and experimental results are presented. The first two studies are in the field of aerospace engineering, regarding the problems of flight path reconstruction and spacecraft trajectory estimation. Furthermore, we consider an application to nonlinear dynamics and chaos, in which data-driven discrete-time models are combined with unscented filtering to perform joint state-and-parameter estimation.

Chapter 4 presents a survey on constrained state estimation. Not only the key ideas but also the corresponding equations of the main methods that address this problem are reviewed and compared. Both equality-constrained and inequality-constrained cases are addressed, including both linear and nonlinear systems. Also, in order to illustrate how to fit inequality constraint information into the state-estimation problem, we recast the state-estimation problem from the perspective of quadratic programming.

The next three chapters present the main contributions of this thesis. In addition to theoretical developments, each chapter presents simulated examples to illustrate typical applications for the proposed algorithms.

Chapter 4 considers the state-estimation problem with an equality constraint on the state vector. We provide the conditions on the system dynamics for a linear system to be equality constrained. Then we derive the equality-constrained Kalman filter as the optimal solution for Gaussian and linear equality-constrained systems. We also compare this algorithm to alternative approaches. Finally, we develop three suboptimal algorithms based on UKF for equality-constrained state estimation for nonlinear systems.

The state-estimation problem for interval-constrained nonlinear systems is addressed in Chapter 6. We present six approximate solutions to this problem that are based on UKF.

Chapter 7 considers the state-estimation problem with a constraint on the data-assimilation gain rather than on the state vector. The gain-constrained Kalman filter is derived as the optimal solution to this problem for linear and Gaussian systems. In addition to discussing three special cases of practical interest of this algorithm, the aforementioned method is extended to the nonlinear case using the unscented approach.

Finally, concluding remarks and suggestions of continuity are discussed in Chapter 8.

Appendix A addresses three complementary topics on unconstrained state estimation, namely, joint state-and-parameter estimation, sampled-data continuous-time state estimation, and smoothing algorithms. Moreover, in this appendix, we differ three kinds of state estimators, namely, filters, predictors, and smoothers.

Appendix B considers an application of equality-constrained state estimation to magnetohydrodynamics. We address the problem of enforcing the zero-divergence constraint on the magnetic field in data assimilation for two-dimensional magnetohydrodynamic flow. Since this application uses specific reduced-order algorithms developed for large-scale systems and the obtained results are preliminary, this topic is presented in this appendix rather than in the main body of the text.

Appendix C extends the results of Chapter 7 by presenting the gain-constrained Kalman predictor as the predictor-like optimal solution to the gain-constrained state-estimation problem for linear and Gaussian systems.

1.5 Contributions of this Work

The present thesis covers six contributions in the field of state estimation. Three of them regard theoretical developments with illustrative applications, whereas the remaining three refer to case studies, in which state-estimation algorithms are applied to solve specific problems. We summarize each contribution and cite the corresponding thesis chapters and publications as follows:

1. Equality-constrained state estimation (Chapter 5):

- We investigate how a linear equality constraint arises on the state vector of a linear dynamic system and present necessary conditions on both the dynamics and process noise for the state vector to be equality constrained. In (Ko & Bitmead, 2007), this problem is stated in the opposite way, that is, given that a system satisfies an

equality constraint, the goal is to characterize the process noise. In these cases, we show that an equality-constrained linear system is not controllable in $\mathbb{R}^n$ from the process noise (but it is rather controllable in the subspace defined by the equality constraint) and that additional information regarding the initial condition provided by the equality constraint is useful for improving the classical KF estimates.

- We derive the equality-constrained Kalman filter (ECKF) as the solution to the equality-constrained state-estimation problem for linear systems. We discuss the connections of ECKF with four alternative approaches. We compare these algorithms by means of two examples, namely, a compartmental model in which the disturbances are constrained so that mass is conserved, and vehicle tracking with kinematic constraints.
- We develop and compare three suboptimal algorithms for equality-constrained state estimation for nonlinear systems. These methods, which extend algorithms developed for linear case, are based on the sigma-point approach. A quaternion-based attitude estimation problem is used to illustrate the application of these algorithms.
- We use equality-constrained Kalman filtering techniques to improve estimation when an approximate discretized model is used to represent a continuous-time process. According to (Rao, Rawlings & Mayne, 2003), constraints can also be used to correct model error. To illustrate this application, we consider a discretized model of an undamped single-degree-of-freedom pendulum without external disturbances. Although energy is conserved in the original, continuous-time system, the discretized model is approximate, and the energy constraint improves estimates of the discretized states.
- Publications related to this contribution: (Teixeira, Chandrasekar, Tôrres, Aguirre & Bernstein, 2007, 2008a,b).

2. Interval-constrained state estimation (Chapter 6):

- We investigate how combinations of one of two candidate unscented approaches for forecast and one of five candidate methods for data assimilation can be used to enforce an interval constraint on the state vector of nonlinear systems. Then six approximate solutions based on UKF to the aforementioned problem are presented.
- The presented algorithms are compared with relation to computational cost and with relation to whether or not the state estimates, as well as the error covariance,

are affected by the interval constraint in each step. We discuss two illustrative examples of chemical processes, namely, a batch reactor and a continuously-stirred-tank reactor.

- Publication related to this contribution: (Teixeira, Tôrres, Aguirre & Bernstein, 2008).

3. Gain-constrained state estimation (Chapter 7 and Appendix C):

- We develop a general framework to state estimation with an equality constraint on the data-injection gain, which includes the results of (Gupta & Hauser, 2007; Chandrasekar et al., 2007; Darouach et al., 2003; Kitanidis, 1987; Palanthandalam-Madapusi et al., 2006) as special cases. We first develop these results for linear systems. We consider both the one-step gain-constrained Kalman predictor, as well as the two-step gain-constrained Kalman filter (GCKF).
- We extend GCKF to the nonlinear case based on UKF, yielding the gain-constrained unscented Kalman filter. Two illustrative examples are discussed, namely, the van der Pol oscillator and a pendulum-cart system.
- Publication related to this contribution: (Teixeira, Chandrasekar, Palanthandalam-Madapusi, Tôrres, Aguirre & Bernstein, 2008b).

4. Case studies in aerospace engineering (Sections 3.1 and 3.2):

- We illustrate the application of the flight path reconstruction procedure to both simulated data and real data collected from a sailplane aircraft. By means of a comparative analysis, we shed some light on the choice of: (i) the nonlinear filtering algorithm – UKF is compared to EKF; (ii) the use of a Kalman smoother algorithm rather than a filtering method; and (iii) the employment of a iterated filtering approach to enhance the estimates.
- We present a tutorial illustrating and comparing two sample-data continuous-time nonlinear Kalman filters based on EKF and UKF to the problem of satellite trajectory estimation. We investigate the effect of poor initial information, as well as the ability of the filters to track changes in the eccentricity and inclination of the spacecraft orbit.
- Publications related to this contribution: (Teixeira, Tôrres, Iscold & Aguirre, 2005, 2008; Teixeira, Santillo, Erwin & Bernstein, 2008).

5. Application to nonlinear dynamics and chaos (Section 3.3):

- We address the problem of state estimation for continuous-time nonlinear systems using UKF with data-driven discrete-time models. We discuss the impact of not knowing the exact equations and using data-driven models in the context of state and joint state-and-parameter estimation. The procedure is illustrated by means of two examples that use simulated (Lorenz system) and measured (electronic oscillator) data.
- Publication related to this contribution: (Aguirre, Teixeira & Tôrres, 2005).

6. Application to magnetohydrodynamics (Appendix B):

- Since equality-constrained state-estimation algorithms have not been applied to magnetohydrodynamics, we present and implement two reduced-order techniques that enforce the zero-divergence constraint on the magnetic field.
- Publication related to this contribution: (Teixeira, Ridley, Bernstein, Tôrres & Aguirre, 2008).

Part I

UNCONSTRAINED STATE ESTIMATION

UNCONSTRAINED STATE ESTIMATION: A THEORETICAL BACKGROUND

“The discovery of the Kalman filter came about through a single, gigantic, persistent mathematical exercise. Just as Newton was lucky having timed his birth so as to have Kepler’s laws ready and waiting for him, I was lucky, too. The pieces of the slumbering Newtonian revolution, which I needed for my monster exercise, were available, scattered all around, partially forgotten but ready to be picked up and reassembled again.”

Rudolph E. Kalman (2003)

“Kalman filtering is not a triumph of applied probability theory: the theory has only a slight inheritance from probability theory while it has become an important pillar of systems theory.”

Rudolph E. Kalman (1994)

In this chapter, we review state estimators for unconstrained systems. Both the linear and nonlinear cases are addressed. As a byproduct, we set the notation used along this thesis.

First, by using the Bayesian approach, we review the recursive solution to the state-estimation problem. Then, in Section 2.2, under linear model and Gaussian noise assumptions, the Kalman filter is reviewed as the optimal solution to the state-estimation problem. For nonlinear systems, however, the solution to this problem is complicated, and thus approximate solutions are reviewed. We categorize these suboptimal solutions into two groups, namely, Gaussian approximate methods and particle filtering methods; see Section 2.3. In this thesis, we focus on the first group, whose algorithms are nonlinear extensions of the Kalman filter.

To compare the performance of state-estimation algorithms, we define metrics and review concepts in Section 2.4. In Sections 2.5 and 2.6, respectively, the extended Kalman

Filter and the sigma-point Kalman filters are reviewed as the most widely used Gaussian approximate algorithms for nonlinear state estimation. We discuss the key idea behind such methods and review results on their stability.

2.1 Recursive Bayesian Approach

We are interested in recursive solutions to the state-estimation problem. As k increases and new measurements become available, in order to be computationally feasible to obtain state estimates, we use a recursive Bayesian approach that allows the recursive processing of $\rho(x_k|(y_1, \dots, y_k))$.

First, define the Chapman-Kolmogorov equation (Papoulis & Pillai, 2001)

$$\rho(x_k|(y_1, \dots, y_{k-1})) \triangleq \int \rho(x_k|x_{k-1})\rho(x_{k-1}|(y_1, \dots, y_{k-1}))dx_{k-1}, \quad (2.1)$$

which determines the *prior PDF* $\rho(x_k|(y_1, \dots, y_{k-1}))$ from the posterior PDF $\rho(x_{k-1}|(y_1, \dots, y_{k-1}))$ evaluated at $k-1$, where the *state-transition prior PDF* $\rho(x_k|x_{k-1})$ is obtained from the process model (1.1) and $\rho_{w_{k-1}}(w_{k-1})$ as

$$\begin{aligned} \rho(x_k|x_{k-1}) &= \int \delta(w_{k-1}^*) \rho_{w_{k-1}}(w_{k-1})dw_{k-1} \\ &= \rho_{w_{k-1}}(w_{k-1}^*), \end{aligned} \quad (2.2)$$

where $\delta(\cdot)$ is the Dirac-delta function and w_{k-1}^* is obtained from the difference equation $x_k = f(x_{k-1}, u_{k-1}, w_{k-1}^*, k-1)$. We refer to (2.1)-(2.2) as the *forecast step*.

Likewise, the *observation likelihood PDF* $\rho(y_k|x_k)$ is obtained from the observation model (1.2) and $\rho_{v_k}(v_k)$ as

$$\begin{aligned} \rho(y_k|x_k) &= \int \delta(v_k^*) \rho_{v_k}(v_k)dv_k \\ &= \rho_{v_k}(v_k^*), \end{aligned} \quad (2.3)$$

where v_k^* is obtained from $y_k = h(x_k, v_k^*, k)$. Also, define the *normalization factor*

$$\rho(y_k|(y_1, \dots, y_{k-1})) \triangleq \int \rho(y_k|x_k)\rho(x_k|(y_1, \dots, y_{k-1}))dx_k, \quad (2.4)$$

which is a function of the measurements $y_1, \dots, y_k$. Note that (2.3)-(2.4) performs *data assimilation*.

lation onto x_k , that is, the information available in the current measured data y_k is assimilated into the prior PDF $\rho(x_k|(y_1, \dots, y_{k-1}))$ obtained during the forecast step.

The next result presents a recursive expression for the posterior PDF, which is the complete solution to the state-estimation problem; see Definition 1.2.1. That is, $\rho(x_k|(y_1, \dots, y_k))$ is presented as a function of $\rho(x_{k-1}|(y_1, \dots, y_{k-1}))$.

Proposition 2.1.1. The posterior PDF is recursively given by

$$\rho(x_k|(y_1, \dots, y_k)) = \frac{\rho(y_k|x_k)\rho(x_k|(y_1, \dots, y_{k-1}))}{\rho(y_k|(y_1, \dots, y_{k-1}))}, \quad (2.5)$$

where $\rho(y_k|x_k)$ is given by (2.3), $\rho(x_k|(y_1, \dots, y_{k-1}))$ is a function of $\rho(x_{k-1}|(y_1, \dots, y_{k-1}))$ and is given by (2.1), and $\rho(y_k|(y_1, \dots, y_{k-1}))$ is given by (2.4).

Proof. First, using Bayes' theorem (Papoulis & Pillai, 2001) in (1.3) yields

$$\begin{aligned} \rho(x_k|(y_1, \dots, y_k)) &= \frac{\rho((y_1, \dots, y_k)|x_k)\rho(x_k)}{\rho(y_1, \dots, y_k)} \\ &= \frac{\rho((y_1, \dots, y_{k-1}, y_k)|x_k)\rho(x_k)}{\rho(y_1, \dots, y_{k-1}, y_k)}. \end{aligned} \quad (2.6)$$

Applying the probabilistic relations $\rho((A_1, A_2)|B) = \rho(A_1|(A_2, B))\rho(A_2|B)$, and $\rho(A_1, A_2) = \rho(A_1|A_2)\rho(A_2)$ (Papoulis & Pillai, 2001) to (2.6) yields

$$\rho(x_k|(y_1, \dots, y_k)) = \frac{\rho(y_k|(y_1, \dots, y_{k-1}, x_k))\rho((y_1, \dots, y_{k-1})|x_k)\rho(x_k)}{\rho(y_k|(y_1, \dots, y_k))\rho(y_1, \dots, y_k)}.$$

Since, from (1.2), y_k is independent of $y_1, \dots, y_{k-1}$ given x_k and making use of Bayes' theorem again in $\rho((y_1, \dots, y_{k-1})|x_k)$, we have

$$\rho(x_k|(y_1, \dots, y_k)) = \frac{\rho(y_k|x_k)\rho(x_k|(y_1, \dots, y_{k-1}))\rho(y_1, \dots, y_{k-1})\rho(x_k)}{\rho(y_k|(y_1, \dots, y_{k-1}))\rho(y_1, \dots, y_{k-1})\rho(x_k)},$$

which confirms (2.5). □

Note that the recursion is introduced in (2.5) by (2.1).

2.2 State Estimation for Linear Systems

Consider the stochastic linear discrete-time dynamic system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1}, \quad (2.7)$$

$$y_k = C_k x_k + v_k, \quad (2.8)$$

where $A_{k-1} \in \mathbb{R}^{n \times n}$, $B_{k-1} \in \mathbb{R}^{n \times p}$, $G_{k-1} \in \mathbb{R}^{n \times q}$, and $C_k \in \mathbb{R}^{m \times n}$ are known matrices. Assume that, for all $k \geq 1$, the inputs $u_{k-1} \in \mathbb{R}^p$ are known and the outputs $y_k \in \mathbb{R}^m$ are measured. The process noise $w_{k-1} \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^m$ are assumed to be white, Gaussian, zero-mean, and mutually independent with known covariance matrices Q_{k-1} and R_k , respectively, that is, the PDFs of w_{k-1} and v_k are given by

$$\rho_{w_{k-1}}(w_{k-1}) = \frac{1}{\sqrt{2\pi \det(Q_{k-1})}} \exp \left[-w_{k-1}^T Q_{k-1}^{-1} w_{k-1} \right], \quad (2.9)$$

$$\rho_{v_k}(v_k) = \frac{1}{\sqrt{2\pi \det(R_k)}} \exp \left[-v_k^T R_k^{-1} v_k \right], \quad (2.10)$$

The initial state vector $x_0 \in \mathbb{R}^n$ is assumed to be Gaussian with initial estimate $\hat{x}_{0|0}$ and error-covariance $P_{0|0}^{xx} \triangleq E[(x_0 - \hat{x}_{0|0})(x_0 - \hat{x}_{0|0})^T]$, that is,

$$\rho_{x_0}(x_0) = \frac{1}{\sqrt{2\pi \det(P_{0|0}^{xx})}} \exp \left[-(x_0 - \hat{x}_{0|0})^T (P_{0|0}^{xx})^{-1} (x_0 - \hat{x}_{0|0}) \right], \quad (2.11)$$

and, for all $k \geq 0$, x_0 is assumed to be uncorrelated with w_k and v_k . Under the stated assumptions, (2.7)-(2.8) is named Gaussian-Markov model (Anderson & Moore, 1979; Papoulis & Pillai, 2001).

Consider the notation $\hat{x}_{k|k-1}$ indicating an estimate of x_k at time k based on information available up to and including time $k-1$. Likewise, $\hat{x}_{k|k}$ indicates an estimate of x_k at time k using information available up to and including time k .

Given the Gaussian and linear assumptions, the PDFs (2.1), (2.3), and (2.4) are also Gaussian and given by

$$\rho(x_k | (y_1, \dots, y_{k-1})) = \frac{1}{\sqrt{2\pi \det(P_{k|k-1}^{xx})}} \exp \left[-(x_k - \hat{x}_{k|k-1})^T (P_{k|k-1}^{xx})^{-1} (x_k - \hat{x}_{k|k-1}) \right], \quad (2.12)$$

$$\rho(y_k | x_k) = \frac{1}{\sqrt{2\pi \det(R_k)}} \exp \left[-(y_k - C_k x_k)^T R_k^{-1} (y_k - C_k x_k) \right], \quad (2.13)$$

$$\rho(y_k | (y_1, \dots, y_{k-1})) = \frac{1}{\sqrt{2\pi \det(P_{k|k-1}^{yy})}} \exp \left[-(y_k - \hat{y}_{k|k-1})^T (P_{k|k-1}^{yy})^{-1} (y_k - \hat{y}_{k|k-1}) \right], \quad (2.14)$$

where $\hat{x}_{k|k-1}$ is the *forecast state estimate*, $\hat{y}_{k|k-1}$ is the *forecast output estimate*, and the *forecast error covariance* $P_{k|k-1}^{xx}$, *innovation covariance* $P_{k|k-1}^{yy}$, and *cross covariance* $P_{k|k-1}^{xy}$

are defined by

$$P_{k|k-1}^{xx} \triangleq \mathbb{E} \left[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T \right], \quad (2.15)$$

$$P_{k|k-1}^{yy} \triangleq \mathbb{E} \left[(y_k - \hat{y}_{k|k-1})(y_k - \hat{y}_{k|k-1})^T \right], \quad (2.16)$$

$$P_{k|k-1}^{xy} \triangleq \mathbb{E} \left[(x_k - \hat{x}_{k|k-1})(y_k - \hat{y}_{k|k-1})^T \right]. \quad (2.17)$$

Also, from Definition 1.2.1, the complete solution to the state-estimation problem for linear systems is given by

$$\rho(x_k | (y_1, \dots, y_k)) = \frac{1}{\sqrt{2\pi \det(P_{k|k}^{xx})}} \exp \left[-(x_k - \hat{x}_{k|k})^T (P_{k|k}^{xx})^{-1} (x_k - \hat{x}_{k|k}) \right], \quad (2.18)$$

where the *data-assimilation state estimate* $\hat{x}_{k|k}$, which is the maximizer of (1.3), and the *data-assimilation error covariance*

$$P_{k|k}^{xx} \triangleq \mathbb{E} \left[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T \right] \quad (2.19)$$

fully characterize (2.18).

2.2.1 The Kalman Filter

In this section the state-estimation problem for linear systems is solved yielding the Kalman filter (KF) (Kalman, 1960).

Using (2.7) and (2.12), it follows that $\hat{x}_{k|k-1}$ and $P_{k|k-1}^{xx}$ are given by

$$\hat{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1|k-1} + B_{k-1} u_{k-1}, \quad (2.20)$$

$$P_{k|k-1}^{xx} = A_{k-1} P_{k-1|k-1}^{xx} A_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T. \quad (2.21)$$

Also, using (2.8) and (2.14), $\hat{y}_{k|k-1}$ and $P_{k|k-1}^{yy}$ are given by

$$\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1}, \quad (2.22)$$

$$P_{k|k-1}^{yy} = C_k P_{k|k-1}^{xx} C_k^T + R_k, \quad (2.23)$$

while the cross-covariance matrix $P_{k|k-1}^{xy}$ is given by

$$P_{k|k-1}^{xy} = P_{k|k-1}^{xx} C_k^T. \quad (2.24)$$

Define the Kalman gain $K_k \in \mathbb{R}^{n \times m}$ as

$$K_k \triangleq P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1}. \quad (2.25)$$

For details on (2.20)-(2.24), see (Thrun, Burgard & Fox, 2005). The next result recasts the state-estimation problem for the system (2.7)-(2.8) as a quadratic programming problem. In the sequel, this problem is solved yielding the KF algorithm.

Lemma 2.2.1 For the Gaussian-Markov system (2.7)-(2.8), $\hat{x}_{k|k}$ maximizes J given by (1.3) if and only if $\hat{x}_{k|k}$ minimizes

$$\mathcal{J}(x_k) \triangleq (x_k - \hat{x}_{k|k-1})^T (P_{k|k-1}^{xx})^{-1} (x_k - \hat{x}_{k|k-1}) + (y_k - C_k x_k)^T R_k^{-1} (y_k - C_k x_k). \quad (2.26)$$

Proof. Given (2.9)-(2.14) and using Proposition 2.1.1, we obtain (2.26). For details, see (Maybeck, 1979, pp. 234–235), (Jazwinski, 1970, pp. 207–208), or the proof of Lemma 5.2.1, which is similar to this proof. $\square$

Theorem 2.2.1. Let $\hat{x}_{k|k-1}$, $P_{k|k-1}^{xx}$, $\hat{y}_{k|k-1}$, $P_{k|k-1}^{yy}$, and $P_{k|k-1}^{xy}$ be, respectively, given by (2.20)-(2.24). Also, let K_k be given by (2.25). Then, from (2.18), $\hat{x}_{k|k}$ and $P_{k|k}^{xx}$ are given by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - \hat{y}_{k|k-1}), \quad (2.27)$$

$$P_{k|k}^{xx} = P_{k|k-1}^{xx} - K_k P_{k|k-1}^{yy} K_k^T. \quad (2.28)$$

Proof. Given (2.20)-(2.24) and using Lemma 2.2.1, we obtain (2.27)-(2.28). For details, see (Jazwinski, 1970) or the proof of Theorem 5.2.1, which is similar to this proof. $\square$

Remark 2.2.1. Note that KF is expressed in two steps, namely, the *forecast step* (2.20)-(2.24), where model information is used, and the *data-assimilation step* (2.25), (2.27)-(2.28), in which the measured data are injected into the estimates; see Figure 2.1.

In addition to being the maximum-a-posteriori optimal state estimator, KF has the following properties.

Theorem 2.2.2. (Gelb, 1974; Jazwinski, 1970) KF is the minimum-variance and maximum-likelihood optimal state estimator for linear systems with elements having Gaussian distribution.

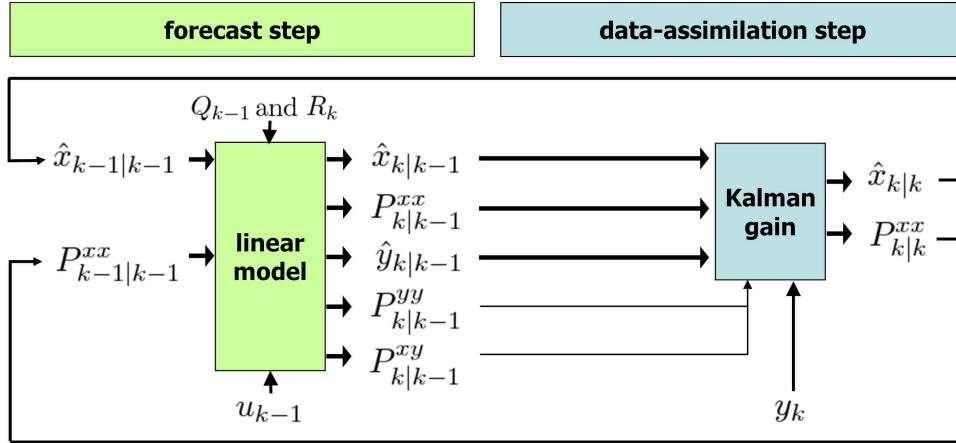


Figure 2.1: Diagram of the of the Kalman filter which is recursively expressed in two steps, namely, the forecast step and data-assimilation step.

Theorem 2.2.3. (Kalman, 1960; Jazwinski, 1970; van der Merwe, 2004) Regardless of the distribution of x_0 , w_{k-1} , and v_k , assuming that, for all $k \geq 1$, the values of the mean and covariance of such distributions are known, and assuming that $\hat{x}_{k|k}$ and $P_{k|k}^{xx}$ are given by (2.27) and (2.28), respectively, then KF is the minimum mean-square-error optimal linear state estimator .

Proposition 2.2.1. (Anderson & Moore, 1979; Gelb, 1974; Jazwinski, 1970) The KF state estimates are unbiased, that is, $E[x_k | (y_1, \dots, y_k)] - \hat{x}_{k|k} = 0_{n \times 1}$.

Proposition 2.2.2. (Crassidis & Junkis, 2004) The KF error-covariance asymptotically achieves the lower bound of the Cramér-Rao inequality.

Proposition 2.2.3. (Crassidis & Junkis, 2004, pp. 256–258) If the dynamics are time invariant, (C, A) is observable, $P_{0|0}$ is positive definite, Q_{k-1} is positive semidefinite, and R_k is positive definite, then, for all $k \geq 1$, the KF error estimates given by $x_k - \hat{x}_{k|k}$ are asymptotically stable.

For an introduction to Kalman filtering, including a list of references, we recommend (Rios Neto & Hemerly, 2007; Aguirre, 2007; Simon, 2006; Maybeck, 1979; Gelb, 1974), while a wide coverage is given by (Jazwinski, 1970; Kailath et al., 2000; Anderson & Moore, 1979; Maybeck, 1979; Simon, 2006).

2.3 State Estimation for Nonlinear Systems

Now consider the system (1.1)-(1.2). Let $\bar{x}_k$ and P_k , respectively, given by

$$\bar{x}_k \triangleq \mathbb{E}[x_k | (y_1, \dots, y_k)], \quad (2.29)$$

$$\bar{P}_k^{xx} \triangleq \mathbb{E}[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T | (y_1, \dots, y_k)], \quad (2.30)$$

denote the mean and covariance of $\rho(x_k | (y_1, \dots, y_k))$. In Section 1.2, we define the state-estimation problem for nonlinear systems. However, the solution to this problem is complicated by the fact that, for nonlinear systems, $\rho(x_k | (y_1, \dots, y_k))$ is not completely described by $\bar{x}_k$ and $\bar{P}_k^{xx}$ (Daum, 2005). Moreover, unlike the linear and Gaussian case shown in (2.18), the maximizer $\hat{x}_{k|k}$ of $\rho(x_k | (y_1, \dots, y_k))$ is not necessarily equal to $\bar{x}_k$.

In most cases, optimal solutions to this problem cannot be written in a closed form because they require the evaluation of the multidimensional integrals (2.1)-(2.4). One exception is the Daum filter (Daum, 1986), which is optimal for specific nonlinear dynamics and exponential distribution. Therefore, suboptimal algorithms are generally employed (Ristic, Arulampalam & Gordon, 2004; Daum, 2005). Henceforth, whenever a suboptimal algorithm is employed, $\hat{x}_{k|k}$ denotes the *pseudo-state estimate* and $P_{k|k}^{xx}$ denotes the *pseudo-error covariance*.

Two categories of suboptimal methods can be mentioned (van der Merwe, 2004), namely, the Gaussian approximate methods and the particle filtering methods, which are also known as sequential Monte Carlo methods. In this thesis, our focus is on the first group.

2.3.1 Gaussian Approximate Methods

The Gaussian approximate methods are nonlinear extensions of KF, which is the optimal linear state estimator that propagates only the first-order and second-order moments of the random vectors; see Theorem 2.2.3. Five methods can be mentioned here, namely, the extended Kalman filter (EKF) (Schmidt, 1970; Jazwinski, 1970; Gelb, 1974), the state-dependent Ricatti equation filter (SDREF) (Mracek, Cloutier & D'Souza, 1996; Chandrasekar, Ridley & Bernstein, 2005), the Gaussian sum filter (GSF) (Alspach & Sorenson, 1972; Anderson & Moore, 1979), the sigma-point Kalman filters (SPKF) (van der Merwe, Wan & Julier, 2004), and the ensemble Kalman filter (EnKF) (Evensen, 2003, 2006; Gillijns, Mendoza, Chandrasekar, De Moor, Bernstein & Ridley, 2006).

For the last four decades, EKF has been the most widely used algorithm for nonlinear state estimation. It uses the KF equations by analytically or numerically linearizing the dynamic model at the current state estimate. However, for strongly nonlinear dynamics, EKF may diverge and may yield inconsistent¹ estimates (Reif, Günther, Yaz & Unbehauen, 1999; Julier & Uhlmann, 2004; van der Merwe, Wan & Julier, 2004; Cipra, 1993). Moreover, EKF is sensitive to initialization (Reif, Günther, Yaz & Unbehauen, 1999).

SDREF also uses the KF equations. However, unlike EKF that uses Jacobians, which may not exist for nonsmooth systems, SDREF retains the full nonlinearity by rewriting the nonlinear equations as time-varying linear equations. The main drawback of this approach is that it requires a choice of time-varying linear dynamics and observation, which are not unique.

Alternatively, GSF treats both the state and output random vectors as the sum of Gaussian random vectors, and it parallelly employs either a KF or EKF algorithm for each one of the summed random vectors. Note that this idea is somewhat inspired by the central limit theorem (Papoulis & Pillai, 2001; Peebles, 1993). In doing so, GSF can approximate any type of distribution, including the multimodal ones.

Recently, SPKF are pointed out as promising algorithms due to the fact that, by means of the unscented transform, they propagate more accurately the mean and covariance of random vectors undergoing nonlinear transformations than does EKF (Julier, Uhlmann & Durrant-Whyte, 2000). Two interpretations are cited for the SPKF algorithms. On the one hand, they are viewed as methods that intrinsically perform statistical linearization of the dynamic model (Lefebvre, Bruyninckx & De Schutter, 2002; van der Merwe, 2004); on the other hand, SPKF are understood as particle filters that employ a reduced number of deterministically chosen ensembles (Julier, Uhlmann & Durrant-Whyte, 2000; Chandrasekar, Ridley & Bernstein, 2007). The pioneer and best known SPKF algorithm is the unscented Kalman filter (UKF) (Julier, Uhlmann & Durrant-Whyte, 1995; Julier & Uhlmann, 2004). Later, Ito & Xiong (2000) presented the central difference filter (CDF) and Nogaard, Poulsen & Ravn (2000) presented the divided difference filter (DDF). Both algorithms are based on Stirling's interpolation formula (Fröberg, 1970) to approximate the nonlinear model using a number of ensembles (van der Merwe, 2004). UKF, CDF, and DDF are presented in a unified framework in (Lefebvre et al., 2002; van der Merwe et al., 2004; van der Merwe, 2004).

Like SPKF, EnKF uses a reduced number of ensembles to propagate the mean and

¹The definitions of divergence and consistency are presented in Section 2.4.

covariance of random vectors undergoing nonlinear transformations. Because of that, EnKF is sometimes also understood as a particle filtering method (Gillijns et al., 2006). However, unlike SPKF, the number of ensembles is heuristically chosen. To perform data assimilation, EnKF applies the data-assimilation step of KF to each ensemble. Weather forecasting literature often employs EnKF to perform state estimation (Houtekamer & Mitchell, 1998; Evensen, 2006).

2.3.2 Particle Filtering Methods

Rather than employing some kind of linearization procedure and propagating only the first and second order moments of PDFs, the particle filters (PF) approximate the multidimensional integrals (2.1)-(2.4) using Monte Carlo sampling (Arulampalam, Maskell, Gordon & Clapp, 2002). The main strength of PF is their applicability to problems with any kind of distribution. However, to achieve good performance, the choice of distribution is crucial. On the other hand, the “curse of dimensionality” is the main drawback, that is, the computational complexity grows exponentially with the dimension of the state vector (Daum, 2005). Therefore, although the speed of computers has considerably grown since KF was presented, depending on the dimension of the state vector, the sampling-rate of the measurements, the degree of nonlinearity, and the type of distribution of the random vectors, the online implementation of PF might not be feasible.

2.4 Performance Assessment Criteria

To compare the performance of algorithms for state estimation, it is necessary to define metrics. This is especially important for the nonlinear case in which suboptimal algorithms are used.

We review four metrics that are commonly used in the literature, namely, *accuracy* (Teixeira, Santillo, Erwin & Bernstein, 2008; Teixeira, Chandrasekar, Tôrres, Aguirre & Bernstein, 2008a), *biasedness* (Teixeira et al., 2008), *information* (Lefebvre et al., 2004; Teixeira et al., 2008a), and *computational complexity*. We also present the concepts of *consistency* (Lefebvre, Bruyninckx & De Schutter, 2004; Farina, Ristic & Benvenuti, 2002), *efficiency* (Crassidis & Junkis, 2004), *divergence*, and *optimality* (Mehra, 1970; Anderson & Moore, 1979).

These indices are qualitatively and quantitatively referred along this thesis. Such metrics aim at assessing how representative the state estimate $\hat{x}_{k|k}$ and the error-covariance

$P_{k|k}^{xx}$ are with relation to $\bar{x}_k$ (2.29) and $\bar{P}_k^{xx}$ (2.30) over a c -run Monte Carlo simulation. Thus, let $\hat{x}_{i,k|k,j}$ denote the (pseudo-) state estimate of the i th component of the state vector x_k , $i = 1, \dots, n$, for the j th Monte Carlo simulation, where $j = 1, \dots, c$. Also, let $P_{k|k,j}^{xx}$ denote the (pseudo-) error covariance for the j th Monte Carlo simulation, $j = 1, \dots, c$.

First, the *accuracy* of the state estimate $\hat{x}_{i,k|k,j}$ over the time interval $[k_0, k_f]$ and over c Monte Carlo simulations is quantified by the root-mean-square error (RMSE) index

$$\text{RMSE}_i \triangleq \frac{1}{c} \sum_{j=1}^c \sqrt{\frac{1}{k_f - k_0 + 1} \sum_{k=k_0}^{k_f} (x_{i,k} - \hat{x}_{i,k|k,j})^2}, \quad i = 1, \dots, n, \quad (2.31)$$

where $x_{i,k}$ is the true value of the i th state component. An accurate state estimate $\hat{x}_{i,k|k,j}$ exhibits small spread around the true value $x_{i,k}$ over a given interval $[k_0, k_f]$.

If the (pseudo-) state estimate $\hat{x}_{k|k}$ satisfies $\mathbb{E}[x_k | (y_1, \dots, y_k)] - \hat{x}_{k|k} = 0_{n \times 1}$, then it is unbiased. We measure how *biased* the state estimate $\hat{x}_{i,k|k,j}$ is by evaluating the RMSE of the sample mean of the estimation error over $[k_0, k_f]$ and over c Monte Carlo simulations, that is,

$$\text{B}_i \triangleq \sqrt{\frac{1}{k_f - k_0 + 1} \sum_{k=k_0}^{k_f} \left[\frac{1}{c} \sum_{j=1}^c (x_{i,k} - \hat{x}_{i,k|k,j}) \right]^2}, \quad i = 1, \dots, n. \quad (2.32)$$

Simulation results (not included) show that the indices RMSE_i and B_i are similar. Hence, B_i is not considered further.

Note that, to calculate RMSE_i and B_i , $x_{i,k}$ must be known, and thus these indices are restricted to simulation studies and cannot be evaluated in practice.

Next, we assess how *informative* the state estimate $\hat{x}_{i,k|k,j}$ is by evaluating the mean trace (MT) of $P_{k|k,j}^{xx}$ over $[k_0, k_f]$ and over c Monte Carlo simulations, that is,

$$\text{MT} \triangleq \frac{1}{c} \sum_{j=1}^c \left[\frac{1}{k_f - k_0 + 1} \sum_{k=k_0}^{k_f} \text{tr}(P_{k|k,j}^{xx}) \right]. \quad (2.33)$$

MT quantifies the uncertainty in $\hat{x}_{k|k}$. The smaller the trace of $P_{k|k}^{xx}$, the more informative the state estimate $\hat{x}_{k|k}$. Note that, for the nonlinear state-estimation case, (2.33) is obtained from the pseudo-error covariance, and thus, smaller values of MT do not guarantee a smaller spread about the mean.

We quantify the *computational cost* T_{CPU} of a state estimator measuring the time

needed to compute $\hat{x}_{k|k}$ from $\hat{x}_{k-1|k-1}$. We present the average CPU time per time step over the time interval $[k_0, k_f]$.

Now assume that the posterior PDF $\rho(x_k|(y_1, \dots, y_k))$ is known. If the (pseudo-) state estimate $\hat{x}_{k|k}$ and the (pseudo-) error covariance $P_{k|k}^{xx}$ satisfy the inequality

$$P_{k|k}^{xx} - \mathbb{E}_{\rho(x_k|(y_1, \dots, y_k))} \left[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T \right] \geq 0_{n \times n}, \quad (2.34)$$

which indicates that the term on the left-hand side is positive semidefinite, then $\hat{x}_{k|k}$ is *consistent*. That is, the uncertainty quantified by $P_{k|k}^{xx}$ and associated to a consistent state estimate is not smaller than the expected-squared estimation error with respect to $\hat{x}_{k|k}$ under $\rho(x_k|(y_1, \dots, y_k))$. Otherwise, $P_{k|k}^{xx}$ is too small and no longer represents a reliable measure for the uncertainty on $\hat{x}_{k|k}$, leading to inconsistent subsequent state estimates. A numerical test for consistency based on the normalized squared estimation error is used in (Farina et al., 2002).

Next, assume that the state estimate $\hat{x}_{k|k}$ is unbiased and the posterior PDF $\rho(x_k|(y_1, \dots, y_k))$ is known. If $\hat{x}_{k|k}$ satisfies the Cramér-Rao inequality (Cramér, 1946)

$$\mathbb{E} \left[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T \right] - F_k^{-1} \geq 0_{n \times n}, \quad (2.35)$$

where the Fisher matrix F_k is given by

$$F_k \triangleq \mathbb{E} \left[\left(\frac{\partial}{\partial x_k} \ln \rho(x_k|(y_1, \dots, y_k)) \right) \left(\frac{\partial}{\partial x_k} \ln \rho(x_k|(y_1, \dots, y_k)) \right)^T \right], \quad (2.36)$$

then the estimator is *efficient*. The closer to the lower bound given by the Fisher matrix the expected squared estimation error is, the more efficient the estimator is. A recursive algorithm for calculating (2.36) is used in (Farina, Ristic & Benvenuti, 2002).

Note that, to evaluate (2.34) and (2.35), $\rho(x_k|(y_1, \dots, y_k))$ must be known, and thus these criteria are restricted to simulation studies in which $\rho(x_k|(y_1, \dots, y_k))$ can be at least approximated using Monte Carlo simulation. Also, observe that there is a tradeoff between the consistency and the efficiency of a state estimate.

If the estimation error $x_k - \hat{x}_{k|k}$ lies outside the confidence interval given by

$$-\alpha \sqrt{\text{diag}(P_{k|k}^{xx})} \leq x_k - \hat{x}_{k|k} \leq \alpha \sqrt{\text{diag}(P_{k|k}^{xx})}, \quad k \in [k_0, k_f], \quad (2.37)$$

over the time interval $[k_0, k_f]$, where $\alpha > 0$ is the confidence factor, then $\hat{x}_{k|k}$ is *divergent*. If $\rho(x_k | (y_1, \dots, y_k))$ is Gaussian, then $\alpha = 3$ indicates a confidence interval of 99.7%. Note that (2.37) can be understood as an alternative test for consistency.

Finally, define the innovation vector

$$\nu_{k|k-1} \triangleq y_k - \hat{y}_{k|k-1}, \quad (2.38)$$

where $\hat{y}_{k|k-1}$ is the output estimate. For KF, $\hat{y}_{k|k-1}$ given by (2.22). Under the KF assumptions (Section 2.2.1), $\hat{x}_{k|k}$ is *optimal* if and only if $\nu_{k|k-1}$ is white (Mehra, 1970), that is, if and only if, for all $k \geq 1$, $E[\nu_{k|k-1}] = 0_{m \times 1}$ and, for all $i \neq j$, $E[\nu_{i|i-1} \nu_{j|j-1}^T] = 0_{m \times m}$. An optimality test based on this innovation whiteness result is used in (Mehra, 1970; Lee & Alfriend, 2004) to assess suboptimal estimators based on KF. The closer to white noise the innovation sequence is, the better the state estimate is.

2.5 The Extended Kalman Filter

The extended Kalman filter (EKF) is the pioneer algorithm proposed for nonlinear state estimation (Schmidt, 1970; Jazwinski, 1970). The core of EKF is the analytical or numerical linearization of the nonlinear dynamics (1.1) and measurement map (1.2) to, then, employ the classical KF equations (2.20)-(2.25), (2.27)-(2.28). Next, we first discuss the assumptions made during the linearization, as well as the consequences of them not being satisfied. Then the EKF algorithm is presented and its asymptotic performance is evaluated.

2.5.1 Analytical Linearization by Taylor Series Expansion

Let x denote a random vector whose mean $\bar{x} \in \mathbb{R}^n$ and covariance $\bar{P}^{xx} \in \mathbb{R}^{n \times n}$ are assumed to be known. Also, let y be a random vector with mean $\bar{y} \in \mathbb{R}^m$ and covariance $\bar{P}^{yy} \in \mathbb{R}^{m \times m}$ obtained from the nonlinear transformation

$$y = h(x). \quad (2.39)$$

We want to obtain approximations $\hat{y}$ and $\hat{P}^{yy}$ to $\bar{y}$ and $\bar{P}^{yy}$, respectively.

Consider the Taylor series expansion of h at $\bar{x}$

$$y = h(\bar{x}) + \bar{H}(x - \bar{x}) + e^{\text{al}}, \quad (2.40)$$

where $\bar{H} \triangleq \frac{\partial h}{\partial x}|_{x=\bar{x}}$ is the Jacobian of h evaluated at $\bar{x}$ and e^{al} is the linearization error accounting for higher-order terms. The superscript “al” accounts for analytical linearization. Rewrite (2.40) as the first-order polynomial equation

$$y = A^{\text{al}}x + b^{\text{al}}, \quad (2.41)$$

where $A^{\text{al}} \triangleq \bar{H}$ and $b^{\text{al}} \triangleq (h(\bar{x}) - \bar{H}\bar{x}) + e^{\text{al}}$. Define

$$Q^{ee} \triangleq \mathbb{E}[e^{\text{al}}(e^{\text{al}})^{\text{T}}] + A^{\text{al}}\mathbb{E}[(x - \bar{x})(e^{\text{al}})^{\text{T}}] + \mathbb{E}[e^{\text{al}}(x - \bar{x})^{\text{T}}](A^{\text{al}})^{\text{T}}, \quad (2.42)$$

accounting for the uncertainty in the linearization error. Assume that

$$\mathbb{E}[e^{\text{al}}] \approx 0, \quad (2.43)$$

$$Q^{ee} \approx 0_{n \times n}. \quad (2.44)$$

Using (2.41), (2.43), and (2.44), $\hat{y}$ and $\hat{P}^{yy}$ are given by

$$\begin{aligned} \hat{y} &= \mathbb{E}[A^{\text{al}}x + b^{\text{al}}] \\ &= \bar{H}\mathbb{E}[x] + (h(\bar{x}) - \bar{H}\bar{x}) + \mathbb{E}[e^{\text{al}}] \\ &= h(\bar{x}), \end{aligned} \quad (2.45)$$

$$\begin{aligned} \hat{P}^{yy} &= \mathbb{E}[(A^{\text{al}}x + b^{\text{al}} - \hat{y})(A^{\text{al}}x + b^{\text{al}} - \hat{y})^{\text{T}}] \\ &= \mathbb{E}[(\bar{H}(x - \bar{x}))(\bar{H}(x - \bar{x}))^{\text{T}}] + Q^{ee} \\ &= \bar{H}\bar{P}^{xx}\bar{H}^{\text{T}}. \end{aligned} \quad (2.46)$$

Remark 2.5.1. If assumption (2.43) is not satisfied then $\hat{y}$ is biased. If assumption (2.44) is not verified then we cannot guarantee the consistency of $\hat{P}^{yy}$.

In (Lerro & Bar-Shalom, 1993) the effect of linearization errors on tracking applications is investigated.

2.5.2 Algorithm of the Extended Kalman Filter

Assume that the mean and covariance of $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are known and equal to zero and Q_{k-1} , R_k , respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated. Let $\hat{x}_{0|0}$ and $P_{0|0}^{xx}$ denote the initial values used by EKF.

We use (2.45) to obtain approximations to $\hat{x}_{k|k-1}$ and $\hat{y}_{k|k-1}$, which are given by (2.20) and (2.22) in KF, and (2.46) to obtain approximations to $P_{k|k-1}^{xx}$, $P_{k|k-1}^{yy}$, and $P_{k|k-1}^{xy}$, which are respectively given by (2.20), (2.23)-(2.24) in KF. In so doing, we obtain the forecast equations of EKF that are given by

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_{k-1}, 0_{q \times 1}, k-1), \quad (2.47)$$

$$P_{k|k-1}^{xx} = \hat{A}_{k-1} P_{k-1|k-1}^{xx} \hat{A}_{k-1}^T + \hat{G}_{k-1} Q_{k-1} \hat{G}_{k-1}^T, \quad (2.48)$$

$$\hat{y}_{k|k-1} = h(\hat{x}_{k|k-1}, 0_{r \times 1}, k), \quad (2.49)$$

$$P_{k|k-1}^{yy} = \hat{C}_k P_{k|k-1}^{xx} \hat{C}_k^T + \hat{H}_k R_k \hat{H}_k^T, \quad (2.50)$$

$$P_{k|k-1}^{xy} = P_{k|k-1}^{xx} \hat{C}_k^T, \quad (2.51)$$

where the Jacobians of f (1.1) and h (1.2) are evaluated at the current state estimate as

$$\hat{A}_{k-1} = \left. \frac{\partial f}{\partial x_{k-1}} \right|_{x_{k-1}=\hat{x}_{k-1|k-1}, u_{k-1}, 0_{q \times 1}, k-1}, \quad (2.52)$$

$$\hat{G}_{k-1} = \left. \frac{\partial f}{\partial w_{k-1}} \right|_{x_{k-1}=\hat{x}_{k-1|k-1}, u_{k-1}, 0_{q \times 1}, k-1}, \quad (2.53)$$

$$\hat{C}_k = \left. \frac{\partial h}{\partial x_k} \right|_{x_k=\hat{x}_{k|k-1}, 0_{r \times 1}, k}, \quad (2.54)$$

$$\hat{H}_k = \left. \frac{\partial h}{\partial v_k} \right|_{x_k=\hat{x}_{k|k-1}, 0_{r \times 1}, k}, \quad (2.55)$$

while the data-assimilation equations are given by (2.25), (2.27), and (2.28).

Remark 2.5.2. The uncertainty associated to $\hat{x}_{k-1|k-1}$ and $\hat{x}_{k|k-1}$ is not taken into account in the calculation of the Jacobians (2.52)-(2.55) (van der Merwe, 2004, pp. 35–43), and it is not compensated by Q_{k-1} in (2.48) and R_k in (2.50); see assumption (2.44).

Therefore we conjecture that the performance of EKF relies on two assumptions. First, from assumptions (2.43)-(2.44), we assume that f (1.1) and h (1.2) are not strongly nonlinear, that is, we assume that, for all $k \geq 1$, $\|e_{k-1}^{\text{al},f}\| \ll \|f(x_{k-1}, u_{k-1}, w_{k-1})\|$ and $\|e_k^{\text{al},h}\| \ll \|h(x_k, v_k)\|$, where $e_{k-1}^{\text{al},f}$ and $e_k^{\text{al},h}$ are the linearization errors associated to f and h , respectively.

The second assumption regards small biasedness of $\hat{x}_{k-1|k-1}$ and $\hat{x}_{k|k-1}$. That is, we assume in (2.52)-(2.55) that $\hat{x}_{k-1|k-1}$ given by (2.27) and $\hat{x}_{k|k-1}$ given by (2.20) are good approximations to the true means of $\rho(x_{k-1}|(y_1, \dots, y_{k-1}))$ (2.5) and $\rho(x_k|(y_1, \dots, y_{k-1}))$ (2.1),

respectively.

For systems with nondifferentiable dynamics and measurement map, an alternative EKF algorithm based on symmetric derivatives is presented in (Chandrasekar, Ridley & Bernstein, 2007). Also, a derivative-free algorithm based on ensemble propagation is presented in (Quine, 2006).

2.5.3 Asymptotic Performance Assessment

Proposition 2.2.3 presents the conditions under which the KF estimation error is stable. However, though EKF is a linearized time-varying extension of KF, it is not straightforward to prove its stability. Thus, we qualitatively discuss some works on the stability of EKF.

Ljung (1979) and Ursin (1980) investigate the asymptotic behavior of EKF in the particular case in which it is used to jointly estimate the state vector and the parameters of a time-varying linear system; see Section A.1. They show that the lack of coupling between the Kalman gain and the parameters leads to divergence of the estimates. Using the fact that Q_{k-1} and R_k play crucial roles on the convergence of KF, a modified EKF algorithm that forces the appearing of such coupling is presented.

In the context of deterministic dynamics, that is, assuming that, for all $k \geq 1$, $w_{k-1} = 0_{q \times 1}$ and $v_k = 0_{r \times 1}$, Boutayeb, Rafaralahy & Darouach (1997) present the necessary conditions for the local asymptotic stability of EKF. They show that the convergence of the algorithm is attained if the linearizations errors $e_{k-1}^{\text{al,f}}$ and $e_k^{\text{al,h}}$ are sufficiently small. Furthermore, two algorithm modifications are proposed, namely, the use of instrumental matrices to assess and tune the convergence of EKF and the use of a modified measurement-noise covariance to improve stability. In (Boutayeb & Aubry, 1999), a modified process-noise covariance is used to improve stability.

We conjecture that the aforementioned modifications are attempts to compensate the fact that the uncertainty associated to the linearization errors are not taken into account by EKF; see Remark 2.5.2.

The Lyapunov direct method and results on stochastic stability analysis are used in (Reif, Günther, Yaz & Unbehauen, 1999) to study stability of EKF. It is shown that if the system (1.1)-(1.2) satisfies the nonlinear observability rank condition, if the initial estimation error $x_0 - \hat{x}_{0|0}$ as well as the disturbing noise terms w_{k-1} and v_k are sufficiently small, then the error estimate $x_k - \hat{x}_{k|k}$ remains exponentially bounded in mean square (Reif, Günther,

Yaz & Unbehauen, 1999). The continuous-time case is addressed in (Reif, Günther, Yaz & Unbehauen, 2000).

2.5.4 Iterated Extended Filtering

In (2.49) for EKF, we assume that $\hat{x}_{k|k-1} \approx E[(x_k | (y_1, \dots, y_{k-1})]$ and the linearization of h is carried out at $\hat{x}_{k|k-1}$. Now, this assumption is still held but the linearization of h is iteratively performed at $\hat{x}_{k|k,i}$, for $i = 0, \dots, i_{\max}$, which is more informative than $\hat{x}_{k|k-1}$. In so doing, the linearized model assimilates the information provided by the current measurement y_k . We thus obtain the iterated extended Kalman filter (IEKF) (Gelb, 1974; Maybeck, 1979); see Figure 2.2.

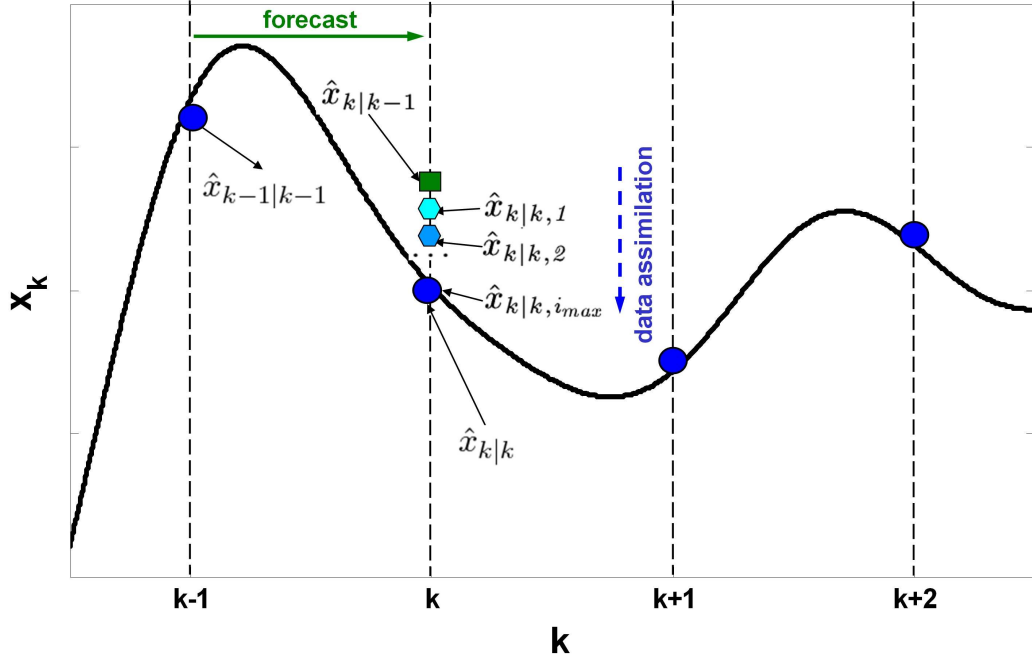


Figure 2.2: Diagram of the iterated filter using $i = 0, \dots, i_{\max}$ local iterations to yield an improved state estimate $\hat{x}_{k|k,i_{\max}}$.

Using the data-assimilation estimates from time $k-1$, we set $\hat{x}_{k-1|k-1} = \hat{x}_{k-1|k-1,i_{\max}}$ and $P_{k-1|k-1}^{xx} = P_{k-1|k-1,i_{\max}}^{xx}$. Also, we initialize the iterated estimate as $\hat{x}_{k|k,0} = \hat{x}_{k|k-1}$. Then, the forecast equations of IEKF are given by (2.47)-(2.48) together with

$$\hat{y}_{k|k,i} = h(\hat{x}_{k|k,i}, 0_{r \times 1}, k) + \hat{C}_{k,i}(\hat{x}_{k|k-1} - \hat{x}_{k|k,i}), \quad (2.56)$$

$$P_{k|k,i}^{yy} = \hat{C}_{k,i} P_{k|k-1}^{xx} \hat{C}_{k,i}^T + \hat{H}_{k,i} R_k \hat{H}_{k,i}^T, \quad (2.57)$$

$$P_{k|k,i}^{xy} = P_{k|k-1}^{xx} \hat{C}_{k,i}^T, \quad (2.58)$$

where

$$\hat{C}_{k,i} \triangleq \left. \frac{\partial h}{\partial x_k} \right|_{x_k = \hat{x}_{k|k,i}, 0_{r \times 1}, k}, \quad (2.59)$$

$$\hat{H}_{k,i} \triangleq \left. \frac{\partial h}{\partial v_k} \right|_{x_k = \hat{x}_{k|k,i}, 0_{r \times 1}, k}, \quad (2.60)$$

while the data-assimilation equations are given by

$$K_{k,i} = P_{k|k,i}^{xy} (P_{k|k,i}^{yy})^{-1}, \quad (2.61)$$

$$\hat{x}_{k|k,i+1} = \hat{x}_{k|k-1} + K_{k,i} (y_k - \hat{y}_{k|k,i}), \quad (2.62)$$

$$P_{k|k,i_{\max}}^{xx} = P_{k|k-1}^{xx} - K_{k,i_{\max}} P_{k|k,i_{\max}}^{yy} K_{k,i_{\max}}^T. \quad (2.63)$$

Note that, (2.56)-(2.62) are evaluated for each iteration i at time k . However, (2.63) is evaluated only for $i = i_{\max}$ at time k . Also, recall that, at $i = 0$, IEKF is similar to EKF.

Alternatively to the idea of employing iterations to improve the EKF estimates, a second-order EKF is presented in (Gelb, 1974). Such algorithm uses second-order Taylor series truncated models rather than first-order linearized models to propagate the mean and covariance of x_k and y_k . In this case, in addition to Jacobians, Hessians are calculated.

2.6 Sigma-Point Kalman Filters

Unlike EKF, the sigma-point Kalman filters (SPKF) (van der Merwe, Wan & Julier, 2004) do not rely on analytical or numerical linearization of the dynamical equations and measurement map in order to use the KF equations. Instead, SPKF are based on a statistical linearization technique called weighted statistical linear regression (Lefebvre et al., 2002), which is a numerical procedure that uses a reduced set of ensembles (known as *sigma points*) for approximating the posterior mean and covariance of a random vector undergoing a non-linear transformation. The unscented transform (UT) (Julier, Uhlmann & Durrant-Whyte, 2000) provides a deterministic way to generate such ensembles and to approximate the related posterior statistics.

UT is the core of the unscented Kalman filter (UKF) (Julier et al., 1995; Julier & Uhlmann, 2004), which is the pioneer and most commonly used SPKF algorithm. Therefore, in this thesis, we focus on algorithms based on UKF. To study other algorithms of the sigma-point family, we suggest the references (Ito & Xiong, 2000; Nogaard et al., 2000; van der Merwe

et al., 2004; van der Merwe, 2004).

2.6.1 Statistical Linearization by Weighted Statistical Linear Regression – Sigma-Point Approach

Let x denote a random vector whose mean $\bar{x} \in \mathbb{R}^n$ and covariance $\bar{P}^{xx} \in \mathbb{R}^{n \times n}$ are assumed to be known. Also, let y be a random vector with mean $\bar{y} \in \mathbb{R}^m$ and covariance $\bar{P}^{yy} \in \mathbb{R}^{m \times m}$ obtained from the nonlinear transformation (2.39). Similar to Section 2.5.1, our goal is to obtain approximations $\hat{y}$ and $\hat{P}^{yy}$ to $\bar{y}$ and $\bar{P}^{yy}$, respectively.

First, we review the *statistical linearization* procedure mentioned in (Gelb, 1974). Gelb (1974) claims that filters that use such technique provide improved results compared to EKF. Thus we approximate (2.39) as

$$y \approx A^{\text{sl}}x + b^{\text{sl}}, \quad (2.64)$$

where A^{sl} and b^{sl} are the minimizers of the expected-weighted-squared linearization error given by

$$J \triangleq \mathbb{E}[(e^{\text{sl}})^T W e^{\text{sl}}], \quad (2.65)$$

where $e^{\text{sl}} \triangleq h(x) - (A^{\text{sl}}x + b^{\text{sl}})$ and $W \in \mathbb{R}^{n \times n}$ is a positive-definite weighting matrix. The superscript “sl” accounts for statistical linearization. The necessary conditions for minimizers A^{sl} and b^{sl} are given by

$$\frac{\partial J}{\partial A^{\text{sl}}} = \mathbb{E}[W A^{\text{sl}}(x - \bar{x})(x - \bar{x})^T] + \mathbb{E}[W(h(x) - h(\bar{x}))(x - \bar{x})^T] = 0_{n \times n}, \quad (2.66)$$

$$\frac{\partial J}{\partial b^{\text{sl}}} = \mathbb{E}[W(h(x) - A^{\text{sl}}x - b^{\text{sl}})] = 0_{n \times 1}, \quad (2.67)$$

which, respectively, yield

$$A^{\text{sl}} = (\bar{P}^{xy})^T (\bar{P}^{xx})^{-1}, \quad (2.68)$$

$$b^{\text{sl}} = \bar{y} - A^{\text{sl}}\bar{x}, \quad (2.69)$$

where $\bar{P}^{xy} \triangleq \mathbb{E}[(h(x) - h(\bar{x}))(x - \bar{x})^T]$. Note that (2.68) and (2.69) are independent of W . Also, (2.68) and (2.69) assume that $\bar{y}$ and $\bar{P}^{xy}$ are known.

Weighted Statistical Linear Regression

Now, we use *weighted statistical linear regression* to yield approximations $\hat{y}$ and $\hat{P}^{xy}$ to $\bar{y}$ and $\bar{P}^{xy}$, which are used in (2.68) and (2.69) (Lefebvre et al., 2002; van der Merwe, 2004).

Let $\mathcal{X} \triangleq [\mathcal{X}_0 \ \mathcal{X}_1 \ \dots \ \mathcal{X}_{2n}] \in \mathbb{R}^{n \times (2n+1)}$ denote a set of sample vectors $\mathcal{X}_j \in \mathbb{R}^n$, $j = 0, \dots, 2n$, known as sigma points. Assume that the sigma points satisfy

$$\sum_{j=0}^{2n} \gamma_{j,x} \mathcal{X}_j = \bar{x}, \quad (2.70)$$

$$\sum_{j=0}^{2n} \gamma_{j,P} [\mathcal{X}_j - \bar{x}] [\mathcal{X}_j - \bar{x}]^T = \bar{P}^{xx}, \quad (2.71)$$

with weights satisfying

$$\sum_{j=0}^{2n} \gamma_{j,x} = 1, \quad (2.72)$$

$$\sum_{j=0}^{2n} \gamma_{j,P} = 1. \quad (2.73)$$

Propagating each sigma point through h (2.39) yields

$$\mathcal{Y}_j = h(\mathcal{X}_j), \quad j = 0, \dots, 2n. \quad (2.74)$$

We want to obtain approximations to A^{sl} (2.68) and b^{sl} (2.69) as the minimizers of

$$\hat{J} \triangleq \sum_{j=0}^{2n} \gamma_{j,x} \mathcal{E}_j^T \mathcal{E}_j, \quad (2.75)$$

where $\mathcal{E}_j \triangleq \mathcal{Y}_j - (A^{\text{sl}} \mathcal{X}_j + b^{\text{sl}})$, $j = 0, \dots, 2n$, is the linearization error at $\mathcal{X}_j$. Note that the cost function $\hat{J}$ is an approximation to (2.65). Define the following approximations to $\bar{y}$, $\bar{P}^{yy}$, and $\bar{P}^{xy}$, respectively,

$$\hat{y} \triangleq \sum_{j=0}^{2n} \gamma_{j,x} \mathcal{Y}_j, \quad (2.76)$$

$$\hat{P}^{yy} \triangleq \sum_{j=0}^{2n} \gamma_{j,P} [\mathcal{Y}_j - \hat{y}] [\mathcal{Y}_j - \hat{y}]^T, \quad (2.77)$$

$$\hat{P}^{xy} \triangleq \sum_{j=0}^{2n} \gamma_{j,P} [\mathcal{X}_j - \bar{x}] [\mathcal{Y}_j - \hat{y}]^T. \quad (2.78)$$

By substituting the approximations (2.76)-(2.78) of $\bar{y}$, $\bar{P}^{yy}$, and $\bar{P}^{xy}$ into (2.68)-(2.69), we obtain approximations to A^{sl} and b^{sl} .

Let

$$\hat{e}^{\text{sl}} \triangleq \sum_{j=0}^{2n} \gamma_{j,x} \mathcal{E}_j \quad (2.79)$$

and

$$\begin{aligned} \hat{P}^{ee} &\triangleq \sum_{j=0}^{2n} \gamma_{j,P} \mathcal{E}_j \mathcal{E}_j^T \\ &= \sum_{j=0}^{2n} \gamma_{j,P} [\mathcal{Y}_j - (A^{\text{sl}} \mathcal{X}_j + b^{\text{sl}})] [\mathcal{Y}_j - (A^{\text{sl}} \mathcal{X}_j + b^{\text{sl}})]^T \\ &= \sum_{j=0}^{2n} \gamma_{j,P} [(\mathcal{Y}_j - \hat{y}) - A^{\text{sl}}(\mathcal{X}_j - \bar{x})] [(\mathcal{Y}_j - \hat{y}) - A^{\text{sl}}(\mathcal{X}_j - \bar{x})]^T \\ &= \hat{P}^{yy} - A^{\text{sl}} \bar{P}^{xx} (A^{\text{sl}})^T \end{aligned} \quad (2.80)$$

denote approximations to $\bar{e}^{\text{sl}} \triangleq E[e^{\text{sl}}]$ and $\bar{P}^{ee} \triangleq E[(e^{\text{sl}} - \bar{e}^{\text{sl}})(e^{\text{sl}} - \bar{e}^{\text{sl}})^T]$, respectively, where (2.69) is used in (2.80). The more severe the nonlinearity of h is, the larger $\hat{e}^{\text{sl}}$ and $\hat{P}^{ee}$ are.

Remark 2.6.1. From (2.80), we have that

$$\hat{P}^{yy} = A^{\text{sl}} \bar{P}^{xx} (A^{\text{sl}})^T + \hat{P}^{ee}, \quad (2.81)$$

which indicates that, unlike (2.46) obtained from the analytical linearization procedure, (2.77) intrinsically accounts for the uncertainty of the linearization error. In other words, $\hat{P}^{ee}$ improves the consistency of $\hat{y}$ and $\hat{P}^{yy}$ in (2.76)-(2.77) compared to (2.45)-(2.46).

Further insight into the sigma-point approach can be obtained by relating it to a numerical integral evaluation technique called Gaussian quadrature (Daum, 2005; van der Merwe, 2004), which uses the Gauss-Hermite rule (Press, Teukolsky, Vetterling & Flannery, 1992). The idea behind the Gauss-Hermite rule is, like in the sigma-point approach, to approximate multi-dimensional integrals such as (2.1)-(2.4) by using a reduced set of deterministically chosen ensembles. The Gaussian-Hermite filter (Ito & Xiong, 2000) explicitly uses the Gauss-

Hermite rule to yield a suboptimal solution to the state-estimation problem.

The Unscented Transform

Next, we use the scaled UT (Julier, 2002) to obtain approximations $\hat{y}$ and $\hat{P}^{yy}$ to $\bar{y}$ and $\bar{P}^{yy}$. In addition to assuming that $\bar{x}$ and $\bar{P}^{xx}$ are known, UT assumes that the PDF of x is symmetrical. That is, the PDF of x is not necessarily Gaussian, but it can be Student-T, Cauchy, and even a multimodal symmetrical distribution.

By choosing the sigma-point matrix $\mathcal{X}$ as

$$\mathcal{X} \triangleq \bar{x}1_{1 \times (2n+1)} + \sqrt{n+\lambda} \begin{bmatrix} 0_{n \times 1} & (\bar{P}^{xx})^{1/2} & -(\bar{P}^{xx})^{1/2} \end{bmatrix}, \quad (2.82)$$

with weights

$$\gamma_{0,x} \triangleq \frac{\lambda}{n+\lambda}, \quad (2.83)$$

$$\gamma_{0,P} \triangleq \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta), \quad (2.84)$$

$$\gamma_{j,x} = \gamma_{j,P} \triangleq \frac{1}{2(n+\lambda)}, \quad j = 1, \dots, 2n, \quad (2.85)$$

where $(\bar{P}^{xx})^{1/2} \in \mathbb{R}^{n \times n}$ is the Cholesky square root, $\lambda \triangleq \alpha^2(n + \kappa) - n > -n$, $0 < \alpha \leq 1$, $\beta \geq 0$, and $\kappa \geq 0$, the conditions (2.70)-(2.73) are satisfied. For simplicity, henceforth, we refer to (2.82)-(2.85) as the function Ψ_{UT} , which is given by

$$[\gamma_x, \gamma_P, \mathcal{X}] = \Psi_{\text{UT}}(\bar{x}, \bar{P}^{xx}, n, \alpha, \beta, \kappa). \quad (2.86)$$

Propagating each sigma point through h yields (2.74), such that $\hat{y}$ is given by (2.76) and $\hat{P}^{yy}$ is given by (2.77).

The scaling factors λ , κ , and β affect the forth and higher-order moments of the sigma points. If x has Gaussian distribution, then $\beta = 2$ is the optimal value (Julier, 2002; Haykin, 2001). Also, α determines the spread of the sigma points around $\bar{x}$. The stronger the nonlinearity of h , the smaller α should be set (Julier, 2002). The default values for such parameters are $\alpha = 1$, $\beta = 0$, and $\lambda = \kappa = \max(0, 3 - n)$. In this case, we have the original UT (Julier et al., 2000).

Alternative schemes for choosing sigma points are discussed in (Julier, 2003; Julier & Uhlmann, 2002, 2004; Luca et al., 2006). Finally, we point out that CDF and DDF, which,

like UKF, are SPKF algorithms, employ different schemes to generate the sigma points (van der Merwe, 2004).

Proposition 2.6.1. UT yields the true mean $\hat{y} = \bar{y}$ and the true covariance $\hat{P}^{yy} = \bar{P}^{yy}$ if $h = h_1 + h_2$, where h_1 is linear and h_2 is quadratic (Julier et al., 2000, Theorem 2). Otherwise, $\hat{y}$ is a *pseudo mean* and $\hat{P}^{yy}$ is a *pseudo covariance*.

Remark 2.6.2. Since x is assumed symmetrical, errors are introduced in $\hat{y}$ and $\hat{P}^{yy}$ related to the forth and higher-order terms of the Taylor series expansion of h (Julier & Uhlmann, 1996).

Example 2.6.1 This example is adapted from (Bitencourt Jr., 2003). Let $x \in \mathbb{R}^2$ be a random vector with zero mean and covariance $\bar{P}^{xx} = I_{2 \times 2}$ undergoing the nonlinear transformation $y = \arctan(x + 1_{2 \times 1})$. We use the analytical linearization scheme of EKF, the unscented transform, and Monte Carlo sampling to obtain approximations to the mean and covariance of y . Figure 2.3 compares the results from these three methods. Compared to the Monte Carlo result, UT yields better results (regarding both mean and covariance) than the analytical linearization case. Also, note that the EKF approach yields a covariance whose area

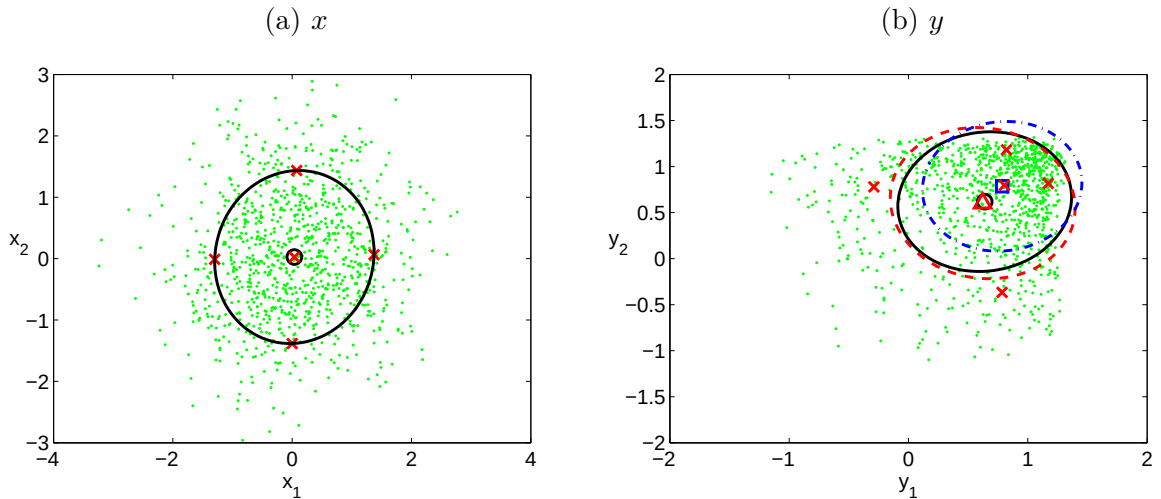


Figure 2.3: (a) 1000 samples (dots) of $x \in \mathbb{R}^2$ whose mean $\bar{x} = 0_2$ and covariance $\bar{P}^{xx} = I_{2 \times 2}$ are assumed to be known. The sigma points are represented by $(\times)$. (b) Comparison of the approximated mean $\hat{y}$ and covariance $\hat{P}^{yy}$ of $y = \arctan(x + 1_{2 \times 1})$ using: (i) Monte Carlo sampling (circle and solid ellipsoid), (ii) analytical linearization approach (square and dot-dashed ellipsoid), and (iii) unscented transform (triangle and dashed ellipsoid).

is smaller than the one using Monte Carlo sampling, that is, given that $\hat{P}^{ee}$ (2.80) is ignored by the EKF approach, $\hat{P}^{yy}$ is underestimated. $\square$

2.6.2 Algorithm of the Unscented Kalman Filter

Assume that the mean and covariance of $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are known and equal to zero and Q_{k-1} , R_k , respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated. Initialize UKF with $\hat{x}_{0|0}$ and $P_{0|0}^{xx}$. Also, the parameters of UT, α , β , and κ , are set.

We use (2.76) to obtain approximations to $\hat{x}_{k|k-1}$ and $\hat{y}_{k|k-1}$, which are given by (2.20) and (2.22) in KF, and (2.77)-(2.78) to yield approximations to $P_{k|k-1}^{xx}$, $P_{k|k-1}^{yy}$, and $P_{k|k-1}^{xy}$, which are respectively given by (2.20), (2.23)-(2.24) in KF.

Define the augmented state vector $\tilde{x}_{k-1} \in \mathbb{R}^{\tilde{n}}$ by

$$\tilde{x}_{k-1} \triangleq \begin{bmatrix} x_{k-1} \\ w_{k-1} \\ v_k \end{bmatrix}, \quad (2.87)$$

where $\tilde{n} \triangleq n + q + r$, and the augmented data-assimilation error covariance $P_{k-1|k-1}^{\tilde{x}\tilde{x}} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ as

$$P_{k-1|k-1}^{\tilde{x}\tilde{x}} \triangleq \begin{bmatrix} P_{k-1|k-1}^{xx} & 0_{n \times q} & 0_{n \times r} \\ 0_{q \times n} & Q_{k-1} & 0_{q \times r} \\ 0_{r \times n} & 0_{r \times q} & R_k \end{bmatrix}. \quad (2.88)$$

Let the weights be given by (2.83)-(2.85) replacing n by $\tilde{n}$.

In so doing, we obtain the forecast equations

$$[\gamma_x, \gamma_P, \tilde{\mathcal{X}}_{k-1|k-1}] = \Psi_{\text{UT}}(\hat{x}_{k-1|k-1}, P_{k-1|k-1}^{\tilde{x}\tilde{x}}, \tilde{n}, \alpha, \beta, \kappa), \quad (2.89)$$

$$\mathcal{X}_{j,k|k-1}^x = f(\mathcal{X}_{j,k-1|k-1}^x, u_{k-1}, \mathcal{X}_{j,k-1|k-1}^w, k-1), \quad j = 0, \dots, 2\tilde{n}, \quad (2.90)$$

$$\hat{x}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \mathcal{X}_{j,k|k-1}^x, \quad (2.91)$$

$$P_{k|k-1}^{xx} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{X}_{j,k|k-1}^x - \hat{x}_{k|k-1}] [\mathcal{X}_{j,k|k-1}^x - \hat{x}_{k|k-1}]^T, \quad (2.92)$$

$$\mathcal{Y}_{j,k|k-1} = h(\mathcal{X}_{j,k|k-1}^x, \mathcal{X}_{j,k|k-1}^v, k), \quad j = 0, \dots, 2\tilde{n}, \quad (2.93)$$

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \mathcal{Y}_{j,k|k-1}, \quad (2.94)$$

$$P_{k|k-1}^{yy} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}][\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^T, \quad (2.95)$$

$$P_{k|k-1}^{xy} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{X}_{j,k|k-1}^x - \hat{x}_{k|k-1}][\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^T, \quad (2.96)$$

where Ψ_{UT} is defined by (2.86),

$$\begin{bmatrix} \mathcal{X}_{k-1|k-1}^x \\ \mathcal{X}_{k-1|k-1}^w \\ \mathcal{X}_{k-1|k-1}^v \end{bmatrix} \triangleq \tilde{\mathcal{X}}_{k-1|k-1}, \quad (2.97)$$

$\mathcal{X}_{k-1|k-1}^x \in \mathbb{R}^{n \times (2\tilde{n}+1)}$, $\mathcal{X}_{k-1|k-1}^w \in \mathbb{R}^{q \times (2\tilde{n}+1)}$, and $\mathcal{X}_{k-1|k-1}^v \in \mathbb{R}^{r \times (2\tilde{n}+1)}$. Like KF and EKF, the UKF data-assimilation equations are given by (2.25), (2.27)-(2.28).

The calculation of the square root of $P_{k-1|k-1}^{\tilde{x}\tilde{x}}$ in (2.89) is the most expensive computational task of UKF. Efficient and stable methods, such as the Cholesky decomposition, should be used (Julier et al., 2000). Alternatively, an algorithm that propagates the square root of the error covariance can be used (van der Merwe & Wan, 2001; van der Merwe, 2004).

In the particular case of additive process and measurement noise, unlike (2.87)-(2.88), the state vector and error covariance need not to be augmented, and thus an alternative algorithm, which reduces the computational cost and increases numerical robustness, can be used (Haykin, 2001).

Based on simulations, UKF is often claimed to be less sensitive to initialization than EKF (Xiong, Zhang & Chan, 2006; van der Merwe, Wan & Julier, 2004; Teixeira, Santillo, Erwin & Bernstein, 2008; Lee & Alfriend, 2007). According to (Julier, Uhlmann & Durrant-Whyte, 1995), UKF yields estimates that are as accurate as those provided by the second-order version of EKF (Gelb, 1974), but at a reduced computational cost since it does not require the calculation of Jacobians and Hessians.

2.6.3 Asymptotic Performance Assessment

Using results from (Reif et al., 1999; Boutayeb & Aubry, 1999) in the context of EKF, Xiong, Zhang & Chan (2006) present the first results on stability of UKF. Actually, these authors analyse a modified UKF algorithm in which the process-noise covariance Q_{k-1}

is enlarged. For the particular case of nonlinear dynamics (1.1) and linear measurement map (2.8), it is shown that if Q_{k-1} is properly enlarged, then the estimation error remains exponentially bounded. Recall that, unlike in (Reif et al., 1999), no assumptions are made on the initial estimation error. However, there is a tradeoff between stability and efficiency of the estimator, since the enlargement of Q_{k-1} causes an increase in the error covariance $P_{k|k}^{xx}$.

The result above is further extended to other Gaussian approximate filters including EKF in addition to UKF (Wu, Hu & Hu, 2007). In (Xiong, Zhang & Chan, 2007), this result is rigorously extended to the general case of nonlinear dynamics (1.1) and nonlinear measurement map (1.2).

In Remark 2.6.1 we see that UKF intrinsically employs enlarged process and measurement noise covariances by means of $\hat{P}^{ee}$ (2.80). In accordance to (Boutayeb & Aubry, 1999; Xiong et al., 2006), note that this feature intrinsically improves the stability of UKF compared to EKF.

2.7 Chapter Summary

In this chapter, the Gaussian approximate estimation framework for nonlinear unconstrained state estimation is developed. First, we show how to obtain a recursive solution to the state-estimation problem by using a Bayesian approach. Then, the Kalman filter is introduced as the optimal solution for the linear and Gaussian case. The nonlinear case is addressed by means of nonlinear extensions of the Kalman filter, namely, the extended Kalman filter and the sigma-point Kalman filters. We indicate that the core of such methods is in the calculation of the statistics of a random vector undergoing a nonlinear transformation. While the former uses analytical or numerical linearization of the dynamics and measurement map (requiring, therefore, the calculation of Jacobians) to approximate the mean and covariance of the transformed random vector, the latter employs a statistical linearization technique that propagates a reduced set of ensembles. We discuss how the definition of the process noise and measurement-noise covariance matrices plays a crucial role on the stability of such methods. We emphasize the unscented Kalman filter as the most widely used algorithm of the sigma-point family.

We introduce four metrics to compare state-estimation algorithms, namely, accuracy, unbiasedness, information, and computational complexity. We also discuss the concepts of consistency, efficiency, divergence, and optimality. Such metrics and concepts are especially

important to compare suboptimal algorithms such as the Gaussian approximate ones used in this thesis.

CASE STUDIES OF UNCONSTRAINED NONLINEAR KALMAN FILTERING

“Experience does not err. Only your judgments err by expecting from her what is not in her power.”

Leonardo da Vinci

We present three case studies of unconstrained state estimation in this chapter. First, Section 3.1 discusses the application of nonlinear Kalman filtering and smoothing algorithms to the problem of flight path reconstruction. Then, sample-data continuous-time nonlinear filters are employed to solve the spacecraft trajectory estimation problem in Section 3.2. Finally, in Section 3.3, we investigate the use of data-driven discrete-time models in unscented filtering for continuous-time nonlinear systems with chaotic dynamics.

3.1 Case Study in Aeronautics – Flight Path Reconstruction

3.1.1 Introduction

Currently, there has been much effort towards the use of efficient algorithms to determine aircraft control and stability derivatives, not only from wind tunnel experiments or Computational Fluid Dynamics analysis, but also based on experimental data collected from onboard instrumentation during flight tests (Jategaonkar et al., 2004; Jategaonkar & Thielecke, 2002; Jategaonkar, 2006; Basappa & Jategaonkar, 2004; Curvo, 2000; de Mendonça, 2005; Maciel et al., 2005; Góes et al., 2006; Hemerly et al., 2003).

The problem of flight path reconstruction (FPR), which is also known as compatibility check (Klein & Schiess, 1977), is the first step in the procedure of using flight test data to obtain information about the parameters of the aircraft model (Mulder et al., 1999; Fagundes, 2007; de Mendonça, 2005; Mendonça et al., 2007). Roughly, this reconstruction means the estimation

of aircraft trajectory based on aircraft sensor data. This step is crucial and fundamental, because real sensors do present bias and sensitivity mismatch that must be accounted for in the subsequent phases of the analysis. Estimated values for the bias and sensitivity terms can be obtained from the procedure of flight path reconstruction.

Output-error and filter-error methods (Mulder et al., 1999; Fagundes, 2007; Mendonça et al., 2007; de Mendonça et al., 2004; Maciel et al., 2006; Curvo & Rios Neto, 2003; Jategaonkar et al., 2004) have been used to accomplish the sensor fusion, which leads to the estimation of sensor parameters, such as bias and sensitivity, as a byproduct of the trajectory reconstruction task. Whereas the former approach handles measurement errors only, the latter, which is considered in this thesis, is able to deal with process noise (present in the inertial measurement unit (IMU) measurements, for instance, and due to turbulence effects in flight tests) in addition to measurement noise.

Among the filter-error methods, EKF is the most commonly used algorithm; see Section 2.5. However, as discussed in Section 2.5.3, EKF can lead to unbounded estimation errors for highly nonlinear equations, such as for the kinematic analysis of six-degree-of-freedom rigid bodies. Recent work (van der Merwe et al., 2004; Romanenko & Castro, 2004; Hovland et al., 2005; Crassidis, 2006; Choi et al., 2005; Haykin, 2001; Julier et al., 2000; Lefebvre et al., 2002, 2004) illustrates the improved performance of UKF (see Section 2.6) over EKF. Moreover, a lot of work concerning the usage of UKF in aerospace and aeronautical applications has been published, such as the state-and-parameter estimation during aircraft failure (Brunke & Campbell, 2002, 2004), spacecraft attitude estimation (Crassidis & Markley, 2003; Markley et al., 2005; Chen et al., 2003; VanDyke et al., 2004), target tracking (Farina et al., 2002), and sensor-fusion and integrated navigation (Hall & Llinas, 2001; Wagner & Wieneke, 2003; Crassidis, 2006; van der Merwe et al., 2004; Metzger et al., 2005).

In this context, we illustrate the application of the FPR procedure to both simulated data and real data collected from a sailplane aircraft. The biases associated to each accelerometer and gyrometer in IMU are determined, together with high sampling-rate trajectory reconstruction from low frequency sampled Global Positioning System (GPS) data. By means of a comparative analysis, we shed some light on the choice of: (i) the nonlinear filtering algorithm – UKF is compared to EKF, see Sections 2.5.2 and 2.6.2; (ii) the use of a Kalman smoother algorithm (Section A.4) rather than a filtering method; and (iii) the employment of an iterated filtering approach (Section 2.5.4) to enhance the estimates. The results presented in this section are published in (Teixeira, Tôrres, Iscold & Aguirre, 2005, 2008) in further details.

3.1.2 Problem Statement

For the FPR problem, consider the process model (A.8) given by (omitting the argument t on the right-hand side)

$$\dot{x}(t) = \begin{bmatrix} (r_m - \Delta_r - w_r)v - (q_m - \Delta_q - w_q)w - g \sin \theta + (A_{x,m} - \Delta_{a_x} - w_{A_x}) \\ -(r_m - \Delta_r - w_r)u + (p_m - \Delta_p - w_p)w + g \cos \theta \sin \phi + (A_{y,m} - \Delta_{a_y} - w_{A_y}) \\ (q_m - \Delta_q - w_q)u - (p_m - \Delta_p - w_p)v + g \cos \theta \cos \phi + (A_{z,m} - \Delta_{a_z} - w_{A_z}) \\ (p_m - \Delta_p - w_p) + (q_m - \Delta_q - w_q) \sin \phi \tan \theta + (r_m - \Delta_r - w_r) \cos \phi \tan \theta \\ (q_m - \Delta_q - w_q) \cos \phi - (r_m - \Delta_r - w_r) \sin \phi \\ (q_m - \Delta_q - w_q) \sin \phi \sec \theta + (r_m - \Delta_r - w_r) \cos \phi \sec \theta \\ \text{row}_1(L_{B2NED}) [u \ v \ w]^T + W_{x_E} \\ \text{row}_2(L_{B2NED}) [u \ v \ w]^T + W_{y_E} \\ \text{row}_3(L_{B2NED}) [u \ v \ w]^T + W_{z_E} \end{bmatrix} + \begin{bmatrix} w_u \\ w_v \\ w_w \\ w_\phi \\ w_\theta \\ w_\psi \\ w_{x_E} \\ w_{y_E} \\ w_{z_E} \end{bmatrix}, \quad (3.1)$$

which represents the kinematics of a six-degree-of-freedom rigid body (Stevens & Lewis, 1992), where g is the gravity acceleration, as well as the observation model (A.9) given by (omitting the argument kT_s on the right-hand side, where T_s is the sampling interval) (Jategaonkar & Thielecke, 2002)

$$y(kT_s) = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \sqrt{u^2 + v^2 + w^2} \\ \arctan\left(\frac{w}{u}\right) \\ \arctan\left(\frac{v}{\sqrt{u^2 + w^2}}\right) \\ x_E \\ y_E \\ -z_E \end{bmatrix} + \begin{bmatrix} v_\phi \\ v_\theta \\ v_\psi \\ v_V \\ v_\alpha \\ v_\beta \\ v_{x_E} \\ v_{y_E} \\ v_H \end{bmatrix}. \quad (3.2)$$

The elements of (3.1)-(3.2) are determined as follows.

The state vector $x(t) \in \mathbb{R}^9$

$$x(t) = [u \ v \ w \ \phi \ \theta \ \psi \ x_E \ y_E \ z_E]^T, \quad (3.3)$$

is given by the translational velocity components u , v , and w along the rigid body; the Euler angles ϕ , θ , and ψ accounting for the rigid-body attitude with respect to the assumed earth-fixed north-east-down (NED) reference frame; and the position coordinates x_E , y_E , and z_E of

the aircraft's center of gravity relative to the NED reference.

The accelerations $A_{x,m}$, $A_{y,m}$, and $A_{z,m}$ measured by accelerometers and the angular rates p_m , q_m , and r_m measured from gyrometers (both localized in the onboard IMU in the aircraft's centre of gravity) play the role of inputs $u((k-1)T_s) \in \mathbb{R}^6$

$$u((k-1)T_s) = [A_{x,m} \ A_{y,m} \ A_{z,m} \ p_m \ q_m \ r_m]^T, \quad (3.4)$$

where the subscript “m” determines that the respective variable is measured by onboard instrumentation.

The observation vector $y(kT_s) \in \mathbb{R}^9$

$$y(kT_s) = [\phi_m \ \theta_m \ \psi_m \ V_{TAS,m} \ \alpha_m \ \beta_m \ x_{E,m} \ y_{E,m} \ H_m]^T, \quad (3.5)$$

is given by the Euler angles ϕ_m , θ_m , and ψ_m that are generally given by IMU; $V_{TAS,m}$ is the true airspeed measured directly by means of an annemometer or a Pitot tube system; α_m and β_m are respectively the attack and sideslip angles measured by vanes; $x_{E,m}$ and $y_{E,m}$ are the geographical coordinates provided by GPS receivers; and H_m refers to the altitude (or the negative of the position coordinate z_E in the NED frame) given by barometric sensors.

In (3.1), W_{x_E} , W_{y_E} , and W_{z_E} determine the components of the generally assumed constant atmospheric wind vector along the NED reference frame, while the orthogonal matrix L_{B2NED} relates the rigid body and the NED reference frames as follows,

$$L_{B2NED} = \begin{bmatrix} \cos \theta \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \cos \theta \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi \\ -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix}. \quad (3.6)$$

Also, the “measured” inputs (3.4) discount the respective bias (indicated as Δ_u , where u refers to all of the components of u_k) in (3.1). Process noise $w((k-1)T_s) \in \mathbb{R}^{15}$ includes both multiplicative $w((k-1)T_s)^{\text{mul}} = [w_{A_x} \ w_{A_y} \ w_{A_z} \ w_p \ w_q \ w_r]^T$ and additive terms $w((k-1)T_s)^{\text{add}} = [w_u \ w_v \ w_w \ w_\phi \ w_\theta \ w_\psi \ w_{x_E} \ w_{y_E} \ w_{z_E}]^T$, whereas measurement noise $v(kT_s) \in \mathbb{R}^9$ is assumed to be purely additive.

In our study only the bias variables given by the parameter vector $\theta(t) \in \mathbb{R}^6$

$$\theta(t) = [\Delta_{a_x} \ \Delta_{a_y} \ \Delta_{a_z} \ \Delta_p \ \Delta_q \ \Delta_r]^T \quad (3.7)$$

are jointly estimated with (3.3); see in Section A.1. Although, at first sight, this approach

may seem simplistic (since the sensitivity terms are not considered), it is successfully employed in (Mulder et al., 1999; van der Merwe et al., 2004). Furthermore, many parameters to be estimated may lead to nonreconstructible components (that is, that cannot be estimated) in the extended state vector $\check{x}(t) \in \mathbb{R}^{15}$ defined by (A.1).

To perform joint state-and-parameter estimation and, thus, solve the FPR problem, we use the EKF, UKF, IEKF, and UKS algorithms, which are reviewed in Sections 2.5.2, 2.6.2, 2.5.4, and A.4, respectively. Since the model (3.1) is represented in continuous time, we use the procedures discussed in Section A.2 to propagate the extended state vector.

3.1.3 Simulated Results

In this section we compare the performance of EKF, UKF, IEKF, and UKS in the FPR procedure by using simulated data obtained from the FDC toolbox (Rauw, 2005), which models a DHC-2 Beaver aircraft. Figure 3.1 shows the trajectory investigated. We set the elements of (3.7) to $\Delta_{a_x} = 0.01g \text{ m/s}^2$, $\Delta_{a_y} = -0.005g \text{ m/s}^2$, $\Delta_{a_z} = -0.01g \text{ m/s}^2$, $\Delta_p = 0.05 \text{ deg/s}$, $\Delta_q = -0.05 \text{ deg/s}$, and $\Delta_r = 0.05 \text{ deg/s}$. Parameter dynamics are modeled as a random-walk process, as discussed in Section A.1.

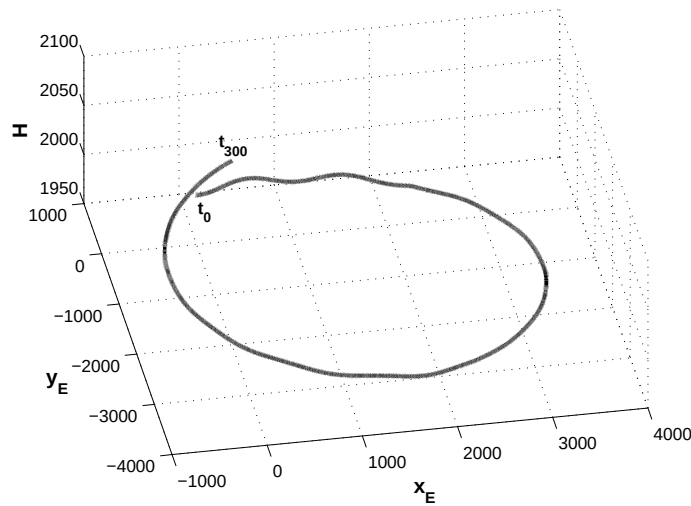


Figure 3.1: Simulated (—) and reconstructed (...) Beaver trajectories. These curves coincide. The former is obtained by integrating model equations at $\delta t = 0.01s$ during 300s. In the reconstruction we consider $T_s = 0.10 \text{ s}$ for all inputs and outputs, except for the GPS measurements ($T_{\text{GPS}} = 1.0 \text{ s}$).

To perform joint state-and-parameter estimation using the four algorithms mentioned above, we arbitrarily choose the initial state estimate $\hat{x}_{0|0} \neq x(0)$ as a feasible value con-

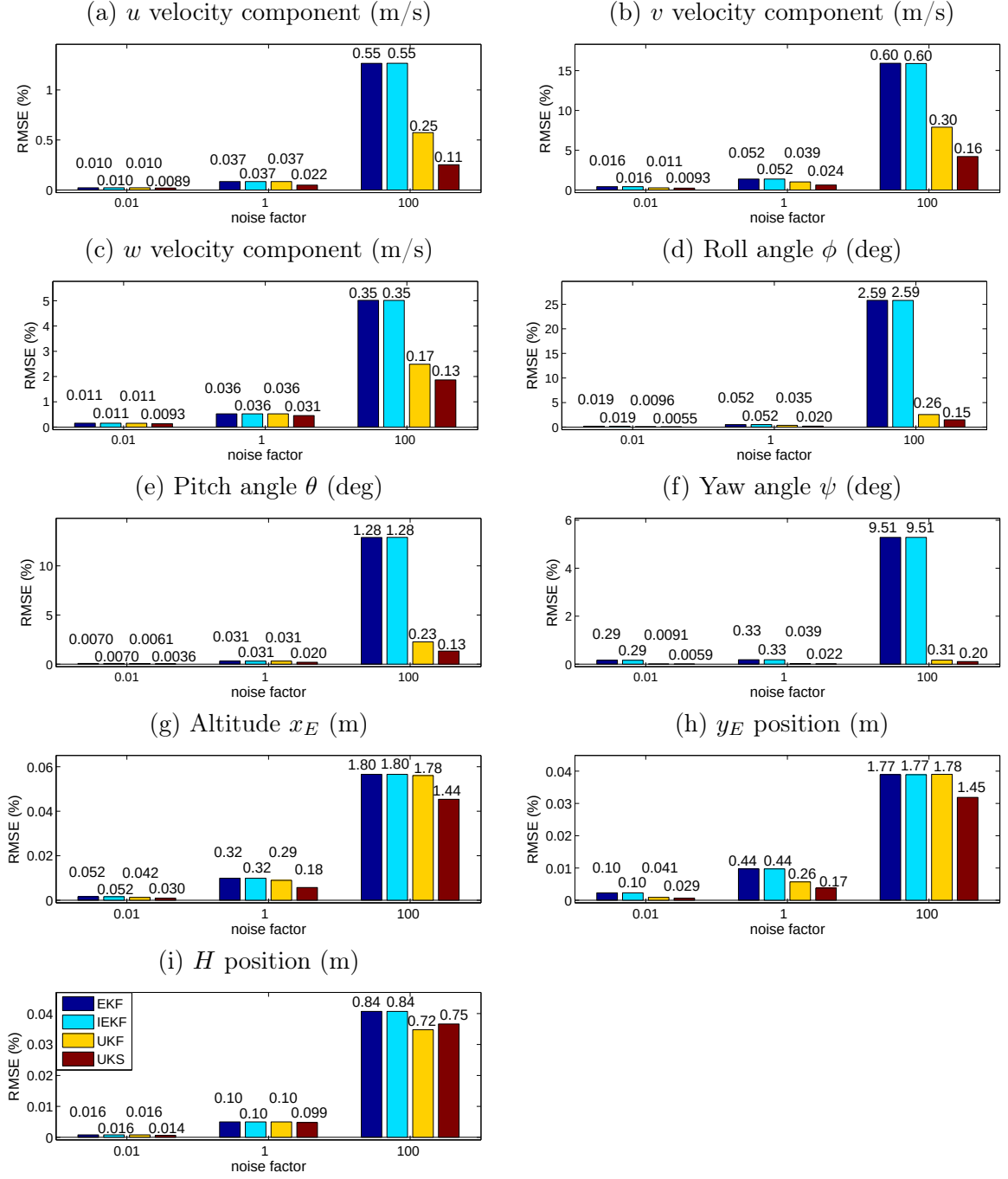


Figure 3.2: Normalized RMSE for the state estimates of the Beaver simulated data regarding three different noise factors (0.01, 1, 100) in the inputs and outputs. At each noise level, from the left to the right, the bars refer to EKF, IEKF, UKF, and UKS. The number above each bar accounts for RMSE (without normalization). Each index is the average of a 50-run Monte Carlo simulation.

cerning the range in which it can vary, and we set the diagonal of $P_{0|0}^{xx}$ based on the sample variance of typical time series of the components of (3.3). Moreover, we set $\hat{\theta}_{0|0} = 0_{6 \times 1}$ and

$P_{0|0}^{\theta\theta} = 100 \text{diag}(\theta(t))$. Note that, since the data are simulated, it is straightforward to set $Q_{k-1}^{\text{mul}} = 10^{-6} \text{diag}(1, 1, 1, 0.25, 0.25, 0.25)$ and $R_k = \text{diag}(2.5 \times 10^{-5} I_{3 \times 3}, 0.04, 2.5 \times 10^{-5} I_{3 \times 3}, 0.25, 1, 1)$ as the true values. Three noise levels are tested, that is, Q_{k-1}^{mul} and R_k are multiplied by the noise factors 0.01, 1, and 100. To help the convergence of the estimates obtained from EKF, UKF, UKS, and IEKF, we set $Q_{k-1}^{\theta} = 10^{-6} \text{diag}(I_{3 \times 3}, 0.03 I_{3 \times 3})$ and $Q_{k-1}^{\text{add}} = \text{diag}(10^{-4} I_{3 \times 3}, 10^{-10} I_{3 \times 3}, 10^{-4} I_{3 \times 3})$.

Regarding the state vector given by (3.3), UKF provides a performance close to the performance of EKF, especially at lower noise levels; see Figure 3.2. This result is in accordance with the results of (de Mendonça & Hemerly, 2007). However, UKF provides improved performance for either higher noise levels or some variables, such as v , ϕ , ψ , x_E , y_E , compared to EKF. In (van der Merwe et al., 2004), in the context of navigation systems using an UAV (helicopter) test platform, the improved estimation of the ψ angle using UKF compared to EKF is reported. Also, in accordance with the results presented in (Crassidis, 2006), UKF is less sensitive to initialization and noise covariance tuning. EKF is more prone to divergence than UKF, regarding the ψ estimates around $\pm\pi$ discontinuities due to a improper tuning of Q_{k-1}^{add} . Compared to UKF, UKS improves the accuracy of state estimates. Conversely, according to Figure 3.2, no significant improvement is provided by IEKF compared to EKF.

Table 3.1 compares the normalized RMSE index obtained from a 50-run Monte Carlo simulation using EKF, UKF, IEKF, and UKS. Note that UKF provides more accurate pa-

Table 3.1: Normalized RMSE (2.31) of estimated IMU bias for Beaver simulated data using a 50-run Monte Carlo simulation. We consider different noise levels in the model inputs and outputs. Indices close to 100 or larger indicate the divergence of the respective estimate.

Algorithm	Noise factor	Δ_{a_x} (%)	Δ_{a_y} (%)	Δ_{a_z} (%)	Δ_p (%)	Δ_q (%)	Δ_r (%)
EKF	0.01	1.04	2.27	6.32	0.70	1.30	2.11
IEKF	0.01	1.04	2.27	6.32	0.70	1.28	2.09
UKF	0.01	1.04	1.27	6.32	0.62	1.29	2.10
UKS	0.01	1.06	1.25	6.33	1.30	2.77	1.40
EKF	1	1.14	1.58	6.34	15.2	1.01	21.1
IEKF	1	1.13	1.59	6.34	15.2	0.93	20.6
UKF	1	0.98	0.84	6.35	5.21	2.41	6.12
UKS	1	0.96	0.98	6.31	4.44	9.97	2.87
EKF	100	72.3	772.7	14.0	5624.6	2743.9	50888.7
IEKF	100	72.4	773.7	14.1	5618.9	2740.7	50895.0
UKF	100	1.29	1.46	6.27	37.5	8.32	105.1
UKS	100	0.79	0.19	6.22	50.7	51.1	75.3

parameter estimates than EKF, especially at higher noise levels of w_{k-1}^{mul} and v_k . Also, EKF and IEKF exhibit similar performance. Furthermore, recall that both the EKF and IEKF estimates diverge when we set the noise factor to 100. Likewise, UKF and UKS results are similar; however, for the maneuver investigated, Δ_q is more accurately estimated by UKF than UKS, whereas the opposite occurs for Δ_r . This example is further discussed in (Teixeira et al., 2006, 2008).

3.1.4 Experimental Results

We now consider the data collected in a flight test conducted on a SZD 50-3 Puchacz sailplane (PZL-Bielsko, 2005), which is shown in Figure 3.3(a). The flight test data are obtained during a campaign performed by the Center for Aeronautic Studies (CEA/UFMG) in partnership with the Technological Institute of Aeronautics (ITA). The maneuver excites the aircraft longitudinal (phugoid and short period) and lateral-directional modes. The resulting trajectory is shown in Figure 3.3(b).

The flight data acquisition system (FDAS), that is, the onboard instrumentation developed in CEA/UFMG, comprises an airdata probe capable of measuring static and dynamic pressures, and angles of attack and sideslip; and accompanying sensors to measure outside air

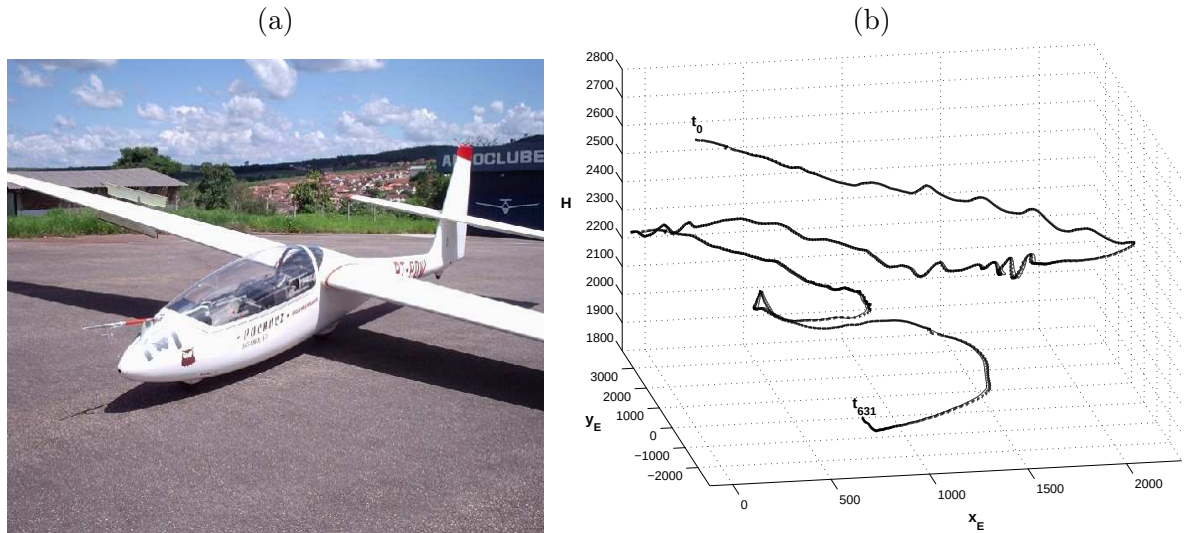


Figure 3.3: (a) SZD 50-3 Puchacz sailplane. This aircraft is manufactured in fiberglass, with 16.67m wingspan and empty weight of 3600N. (b) Measured (—) and UKF reconstructed (...) aircraft trajectories. Note that the solid and the dotted lines coincide most of the time. The flight path reconstruction procedure considers $T_s = 0.10$ s for all inputs and outputs, except for the GPS measurements ($T_{\text{GPS}} = 1.0$ s), during 631 s.

temperature, flight control positions, and aircraft accelerations using microelectromechanical devices (MEMS) accelerometers. The MicroStrain 3DM-GX1 IMU, which is connected to a portable computer, provides the angular velocities and translational accelerations. The inertial data collected from the commercial IMU is synchronized with the rest of the acquired data collected from the FDAS system by means of the MEMS accelerometer signals. Details on FDAS are found in (Iscold & da Silva, 2005).

To perform state-and-parameter estimation using EKF, UKF, IEKF, and UKS, we arbitrarily choose the initial state estimate $\hat{x}_{0|0} \neq x(0)$ as a feasible value concerning the range in which it can vary, while the diagonal of $P_{0|0}^{xx}$ is set based on the sample variance of time series of the components of the state vector. Also, like the simulated case, we set $\hat{\theta}_{0|0} = 0_{6 \times 1}$ and $P_{0|0}^{\theta\theta} = \text{diag}(4 I_{3 \times 3}, 10^{-4} I_{3 \times 3})$. The diagonals of Q_{k-1}^{mul} and R_k are defined from information provided by instrumentation specifications. Moreover, to help the convergence of the estimates, we set $Q_{k-1}^{\text{add}} = \text{diag}(0.04 I_{3 \times 3}, 2.5 \times 10^{-5} I_{3 \times 3}, 0.25, 4 I_{2 \times 2})$ and $Q_{k-1}^{\theta} = \text{diag}(4 \times 10^{-6} I_{3 \times 3}, 10^{-10} I_{3 \times 3})$.

In Table 2, experimental results obtained from the sailplane flight test data are evaluated. Note that, for the translational velocities, RMSE (2.31) is obtained considering that the variables V_T , α and β are perfectly measured, that is, the airdata probe is considered our calibration standard. Similarly, the attitude data provided by the commercial IMU, as well as the low frequency GPS data, are assumed to be perfect measurements. From Table 2, we see that EKF and UKF provide similar performance. Like in the simulation case, no significant improvement over EKF is achieved by IEKF, aiming at more properly dealing with observation nonlinearities. Conversely, as discussed in Section A.4 and illustrated in Figure 3.4, UKS improves the estimation of high frequency sampled data from low frequency sampled acquired

Table 3.2: RMSE (the normalized index is shown in *italics*) of state estimates for the sailplane data using the EKF, IEKF, UKF, and UKS algorithms. These indices consider only the state estimates after convergence (the first one third of the data window is neglected).

Alg.	u (m/s) (%)	v (m/s) (%)	w (m/s) (%)	ϕ (deg) (%)	θ (deg) (%)	ψ (deg) (%)	x_E (m) (%)	y_E (m) (%)	H (m) (%)
EKF	0.357 <i>0.99</i>	0.263 <i>2.52</i>	0.187 <i>1.78</i>	0.281 <i>0.98</i>	0.287 <i>0.98</i>	3.43 <i>1.90</i>	9.6 <i>0.47</i>	7.1 <i>0.21</i>	1.12 <i>0.040</i>
IEKF	0.365 <i>1.01</i>	0.264 <i>2.53</i>	0.190 <i>1.80</i>	0.281 <i>0.98</i>	0.287 <i>0.98</i>	3.43 <i>1.90</i>	9.6 <i>0.47</i>	7.2 <i>0.21</i>	1.11 <i>0.040</i>
UKF	0.357 <i>0.99</i>	0.263 <i>2.52</i>	0.188 <i>1.78</i>	0.281 <i>0.98</i>	0.287 <i>0.98</i>	3.88 <i>2.15</i>	9.6 <i>0.47</i>	7.2 <i>0.21</i>	1.10 <i>0.040</i>
UKS	0.216 <i>0.60</i>	0.139 <i>1.34</i>	0.106 <i>1.01</i>	0.126 <i>0.43</i>	0.086 <i>0.28</i>	3.51 <i>1.95</i>	0.64 <i>0.03</i>	0.73 <i>0.02</i>	0.10 <i>0.004</i>

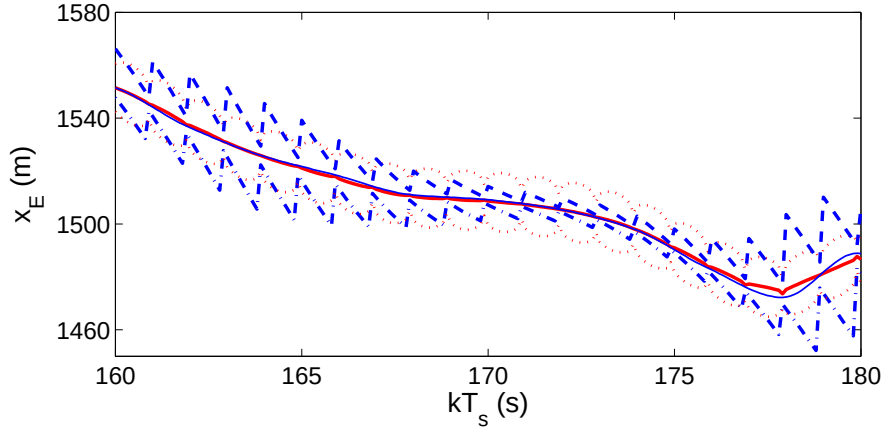


Figure 3.4: Window of data corresponding to the position x_E (thinner blue —) compared with the forward (— · —), backward (— — —), and smoothed (thicker red —) estimates. The dotted lines (···) indicate plus and minus three standard deviations around the smoothed estimate.

data (GPS measurements). Compared to UKF, UKS reduces RMSE by approximately 50% for the velocity and attitude variables and by a factor of ten for the position vector.

Based on the apparently small values observed for RMSE in Table 2, we claim that the results obtained for sensor biases estimation, which are shown in Figure 3.5, are good approximations for the true sensor parameters. However, we cannot validate this conclusion since there is no independent account for the sensor biases. Note that the parameters (3.7) estimated by EKF, UKF, and IEKF almost coincide. Also, according to Figure 3.5, their values are also consistent with those obtained from UKS.

3.1.5 Concluding Remarks

In this section, UKF is presented as a viable alternative to perform FPR without relying on the linearization of the kinematic equations accomplished by EKF. Due to this, UKF is more robust than EKF with relation to initialization, noise level, tuning of noise covariance matrices, and joint estimation of IMU bias terms.

In addition to the aforementioned algorithms, by combining past and future data in offline calculations to generate smoother high frequency sampled trajectory reconstruction from low frequency GPS data, UKS is used to substantially enhance the accuracy of the state estimates. However, significant performance improvements are not observed for IEKF compared to EKF for the flight data evaluated.

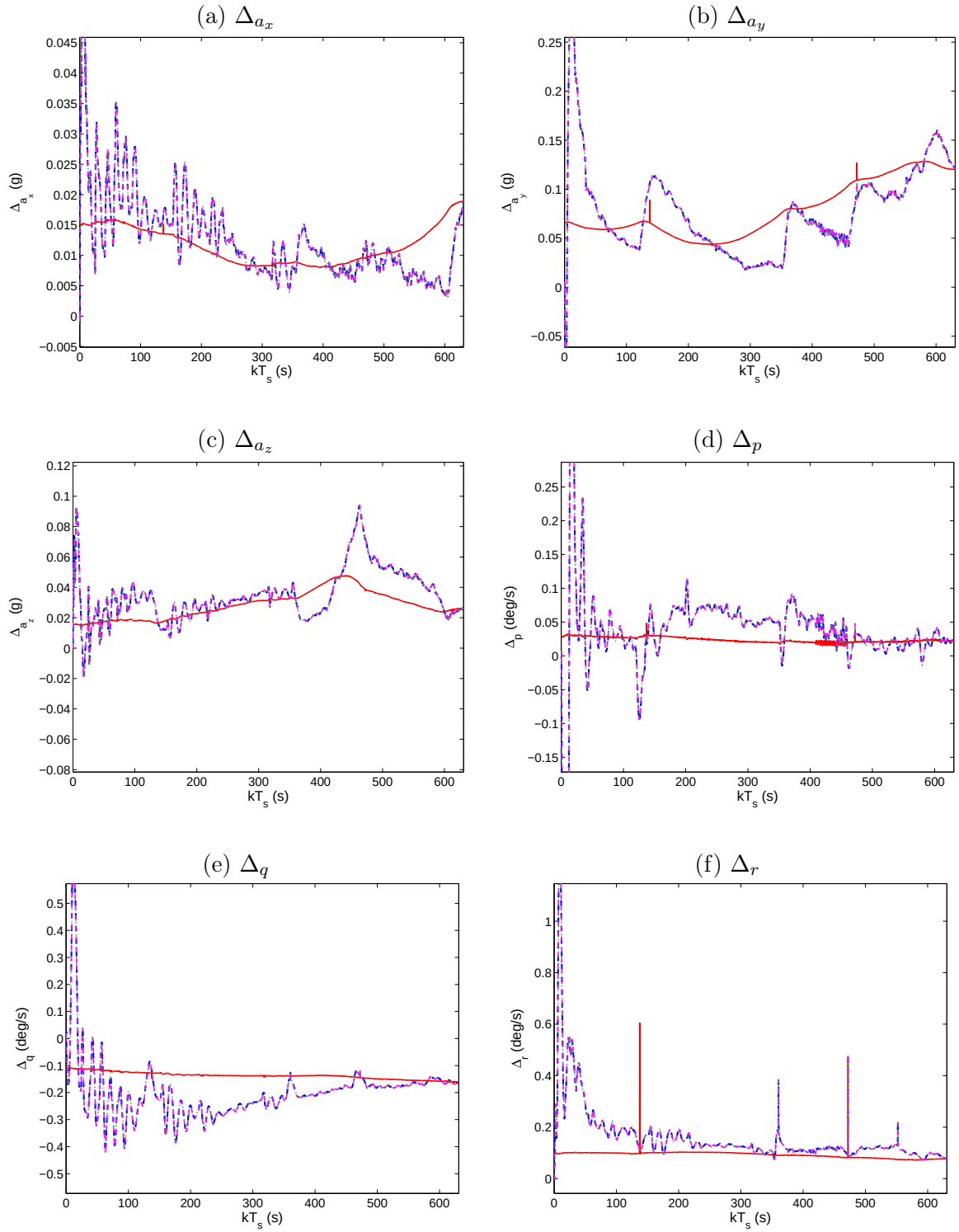


Figure 3.5: EKF (---), IEKF (⋯⋯), UKF (— · —), and UKS (—) joint estimated IMU sensor biases of the SZD 50-3 Puchacz sailplane. Note that the EKF, IEKF, and UKF parameter estimates almost coincide.

3.2 Case Study in Aerospace Engineering

– Spacecraft Trajectory Estimation

3.2.1 Introduction

The goal of the present section is to illustrate and compare nonlinear Kalman filters for the problem of satellite trajectory estimation, also known as orbit determination (Tapley, Schutz & Born, 2004). Various problems and alternative formulations can be considered for orbit determination based on the number and type of available measurements, including range, range rate, and angle (azimuth and elevation). The orbital dynamics of each body can be formulated either in terms of the six orbital parameters (Brouwer & Clemence, 1961, pp. 273–307) or the three instantaneous positions and velocities along the orbit. We adopt the latter formulation due to its simplicity in deriving and applying estimation methods formulated in terms of continuous-time differential equations.

Range-only orbit determination is considered in (Duong & Winn, 1973; Pisacane et al., 1974) using least-squares approaches and orbital element state representations. The use of angle-only data is considered in (Fowler & Lee, 1985). Orbit determination using range and angle measurements from a fixed-location radar tracking system is considered in (Lee & Alfriend, 2007), where UKF is found to have improved performance relative to EKF in relation to initialization. Issues that arise in the use of range-rate (doppler) measurements are discussed in (Bizup & Brown, 2004). Orbit determination with measurements provided by a constellation of satellites is considered in (Cicci & Ballard, 1994a,b; Chiaradia, Kuga & Prado, 2003; de Moraes, da Silva & Kuga, 2007).

Since the orbital dynamics and the measurement map are nonlinear, nonlinear estimation techniques are needed. In addition, since measurements are available with a specified sample interval, we consider the sampled-data extended Kalman-Bucy filter (SDEKBF), reviewed in Section A.2.1, and the sampled-data unscented Kalman-Bucy filter (SDUKBF), reviewed in Section A.2.2, as approximate solutions to the spacecraft trajectory estimation problem. In this thesis we illustrate and compare SDEKBF and SDUKBF by considering a constellation of six spacecraft in circular low-Earth orbit (LEO) that tracks a satellite in geosynchronous orbit. Unlike the GPS configuration, we assume that all six satellites are in equatorial orbits. This assumption renders the estimation problem more difficult due to loss of observability as the target satellite leaves its equatorial orbit due to a change-in-inclination maneuver. Although inclining the orbits of the observing satellites makes the problem easier

– and thus is the logical choice in practice – we consider the equatorial case because of the challenge it poses to nonlinear estimation algorithms. In any event, the principles and methods discussed here can readily be applied to alternative configurations of target and spacecraft.

We are particularly interested in the ability of the observing spacecraft to acquire and track the target satellite with sparse measurements, that is, with a large sample interval. This objective is motivated by the need for satellites to simultaneously track a large number of objects. We thus compare the performance of the estimators for a range of sample intervals.

We focus on three main problems. First, we investigate the ability of the constellation of observing satellites to acquire the target satellite under poor initial information. Next, we consider the ability to track the eccentricity of the target satellite’s orbit when it remains in an equatorial orbit. Finally, we consider the ability of the filters to track the target satellite when it changes its inclination away from the equatorial plane. Extensive numerical examples are given to analyze the performance of each filter for each problem. The results presented in this section are published in further details in the tutorial (Teixeira, Santillo, Erwin & Bernstein, 2008).

3.2.2 Problem Statement

For the spacecraft trajectory estimation problem, we consider a single body, called the *target*, orbiting the Earth. We assume a radially symmetric Earth. Consider the process model (A.8) given by (omitting the argument t on the right-hand side)

$$\dot{x}(t) = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu}{(\sqrt{x_C^2 + y_C^2 + z_C^2})^3} x_C + u_x \\ -\frac{\mu}{(\sqrt{x_C^2 + y_C^2 + z_C^2})^3} y_C + u_y \\ -\frac{\mu}{(\sqrt{x_C^2 + y_C^2 + z_C^2})^3} z_C + u_z \end{bmatrix}, \quad (3.8)$$

where the state vector $x(t) = [x_C \ y_C \ z_C \ v_x \ v_y \ v_z]^T \in \mathbb{R}^6$ is given by the target position coordinates x_C , y_C , and z_C relative to the center of the Earth and by the target velocity components v_x , v_y , and v_z taken with respect to an arbitrary inertial frame (Bernstein, 2008; Curtis, 2005, pp. 150–153); the input vector given by $u(t) = [u_x \ u_y \ u_z]^T \in \mathbb{R}^3$ accounts

for perturbing forces, such as thrusting, drag, and solar pressure, per unit mass acting on the target; and $\mu \triangleq 398,860 \text{ km}^3/\text{s}^2$ is the Earth's gravitational parameter.

When the target is moving along an orbit, such as a circle, ellipse, parabola, or hyperbola, it is often useful to represent the target motion in terms of the six orbital parameters given by the specific angular momentum h_a , the inclination i , the right ascension of the ascending node Ω , the eccentricity e , the argument of perigee ω , and the true anomaly $\nu(t)$. The nonlinear transformations that convert position and velocity into orbital elements are given in (Teixeira et al., 2008; Tapley et al., 2004; Curtis, 2005).

Regarding the measurement model, we consider the case in which uniformly spaced satellites in circular equatorial LEO are observing a target satellite in a geostationary orbit. All available satellite measurements are assumed to occur simultaneously at a fixed sample interval T_s . Measurements from the j th satellite are unavailable when the line-of-sight path between the j th satellite and the target is blocked by the Earth. To determine blockage, we employ the scheme used in (Teixeira et al., 2008). Note that the number s_k of satellites observing the target at time kT_s is a function of k due to blockage by the Earth.

We address two cases, namely, range-only trajectory estimation and range-and-angle trajectory estimation. For the latter, we assume that the azimuth A_j and elevation E_j angles relative to the j th satellite are measured in addition to the range. Thus, the observation model (A.9) regarding both range and angle measurements is given by (omitting the argument kT_s on the right-hand side)

$$y(kT_s) = \begin{bmatrix} \sqrt{(x_C - x_1)^2 + (y_C - y_1)^2 + (z_C - z_1)^2} \\ \vdots \\ \sqrt{(x_C - x_{s_k})^2 + (y_C - y_{s_k})^2 + (z_C - z_{s_k})^2} \\ \arctan\left(\frac{y_C - y_1}{x_C - x_1}\right) \\ \vdots \\ \arctan\left(\frac{y_C - y_{s_k}}{x_C - x_{s_k}}\right) \\ \arctan\left(\frac{z_C - z_1}{\sqrt{(x_C - x_1)^2 + (y_C - y_1)^2}}\right) \\ \vdots \\ \arctan\left(\frac{z_C - z_{s_k}}{\sqrt{(x_C - x_{s_k})^2 + (y_C - y_{s_k})^2}}\right) \end{bmatrix} + \begin{bmatrix} r_{1,r} \\ \vdots \\ r_{s_k,r} \\ r_{1,A} \\ \vdots \\ r_{s_k,A} \\ r_{1,E} \\ \vdots \\ r_{s_k,E} \end{bmatrix}, \quad (3.9)$$

where $y(kT_s) \in \mathbb{R}^{3s_k}$; x_j , y_j , and z_j are the position coordinates of the j th satellite, $j = 1, \dots, s_k$; and $r(kT_s) = [r_{1,r} \ \dots \ r_{s_k,r} \ r_{1,A} \ \dots \ r_{s_k,A} \ r_{1,E} \ \dots \ r_{s_k,E}]^T \in \mathbb{R}^{3s_k}$ denotes range,

azimuth, and elevation measurement noise, respectively. The first three rows of (3.9) refer to the case of range-only measurements.

3.2.3 Simulated Results

We consider the case in which 6 satellites in circular LEO at a radius of 6600 km are observing a target satellite in an equatorial geosynchronous orbit at a radius of 42,164 km. The number 6 represents the smallest number of satellites for which at least 3 satellites are always able to view the target.

Initialization

To obtain $\hat{x}(t)$ and $P^{xx}(t)$, we integrate (A.20), (A.21), (A.36), and (A.37) using the variable-step-size Runge-Kutta algorithm `ode45` of Matlab with tolerance set to 10^{-12} . This tolerance is needed by the pseudo-error covariance propagation (A.37) between measurements. We test various values of $\hat{x}(0)$ corresponding to values of the initial true-anomaly error. As discussed in Sections 2.5.3 and 2.6.3, to enhance the stability of the filters, we set $Q(t) = 10^{-2}I_{6 \times 6}$.

If we initialize $P^{xx}(0)$ as the diagonal matrix $P^{xx}(0) = \text{diag}(100, 100, 1, 1, 1, 0.1)$, then convergence is not attained for SDEKBF for initial true-anomaly errors larger than 90 deg. This difficulty is thus due to the lack of observability of the linearized dynamics. However, if we choose $P^{xx}(0)$ to be the nondiagonal matrix $P^{xx}(0) = \text{diag}(100, 100, 1, 1, 1, 0.1) + 10^{-2}1_{6 \times 6}$, then both filters exhibit global convergence, that is, convergence is attained for all initial true-anomaly errors between ± 180 deg. For the remainder of this study, we use this choice of $P^{xx}(0)$. Alternatively, we can overcome this convergence issue either by considering the case in which the geometry of the observing satellites is not entirely coplanar or by including angle (azimuth and elevation) measurements in addition to range data. Simulations (not included) show that, for these cases, no substantial improvement in the accuracy of the estimates is obtained over the case in which a nondiagonal $P^{xx}(0)$ is set.

Target Acquisition

We first consider the ability of SDEKBF and SDUKBF to acquire the target, that is, to locate the target despite initial position and velocity errors. We set the sample interval to be $T_s = 1$ s and introduce gaussian measurement noise with $R(kT_s) = 0.01I_{6 \times 6} \text{ km}^2$. For

initial estimates that are erroneous by 180 deg, Figure 3.6 shows that the SDUKBF estimates are more accurate than the SDEKBF estimates.

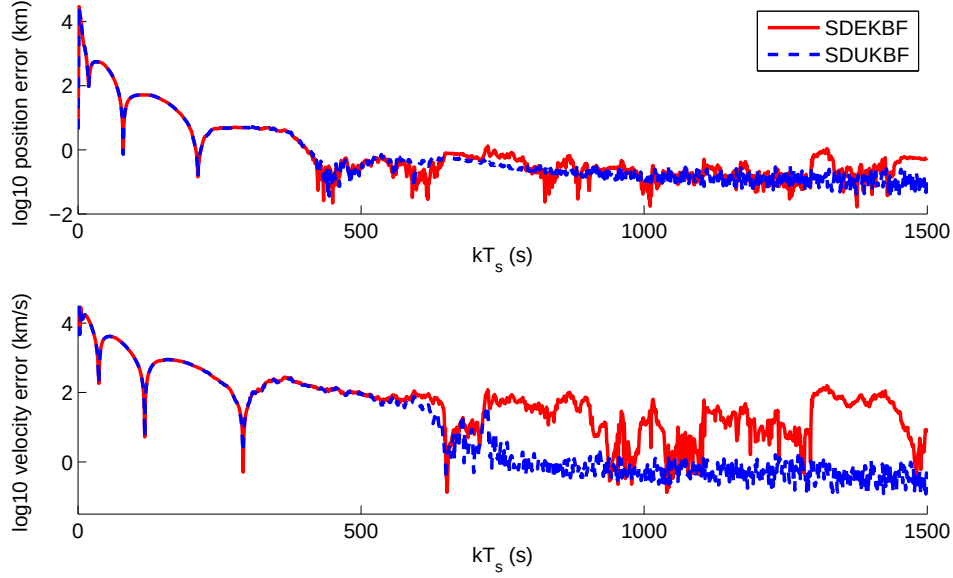


Figure 3.6: Target position-estimate (a) and velocity-estimate (b) errors with an initial true-anomaly error of 180 deg and nondiagonal $P^{xx}(0) = \text{diag}(100, 100, 1, 1, 1, 0.1) + 10^{-2} \mathbf{1}_{6 \times 6}$. The range data are measured with $T_s = 1$ s. After 600 s, the SDUKBF estimates are more accurate than the SDEKBF estimates. In particular, note that the SDUKBF velocity estimates are two orders of magnitude more accurate than the SDEKBF velocity estimates.

Table 3.3: RMSE (2.31), MT (2.33), and T_{CPU} (see Section 2.4) for $kT_s \in [500, 1500]$ s and for a 100-run Monte Carlo simulation using SDEKBF and SDUKBF. Range is measured with $T_s = 1$ s. All initial estimates are erroneous by -90 deg. $P_{(3,3)}^{xx}$ and $P_{(6,6)}^{xx}$ are not included in the calculation of MT because their values are much greater than the other diagonal entries.

	RMSE						MT	T_{CPU} (ms)
	x_C (km)	y_C (km)	z_C (km)	v_x (km/s)	v_y (km/s)	v_z (km/s)		
SDEKBF	0.1128	0.2996	63.93	0.0351	0.0841	0.5149	0.1661	19.5
SDUKBF	0.5525	0.3175	13.80	0.0958	0.0849	0.2450	0.5998	41.8

Based on a 100-run Monte Carlo simulation, Table 3.3 shows that SDUKBF yields more accurate estimates for z_C and v_z than does SDEKBF. However, the SDUKBF processing time T_{CPU} is about twice as large as the SDEKBF processing time. Moreover, although SDUKBF presents a larger value of MT (without accounting for the components z_C and v_z), which indicates less informative estimates, note that the SDEKBF estimates for the position

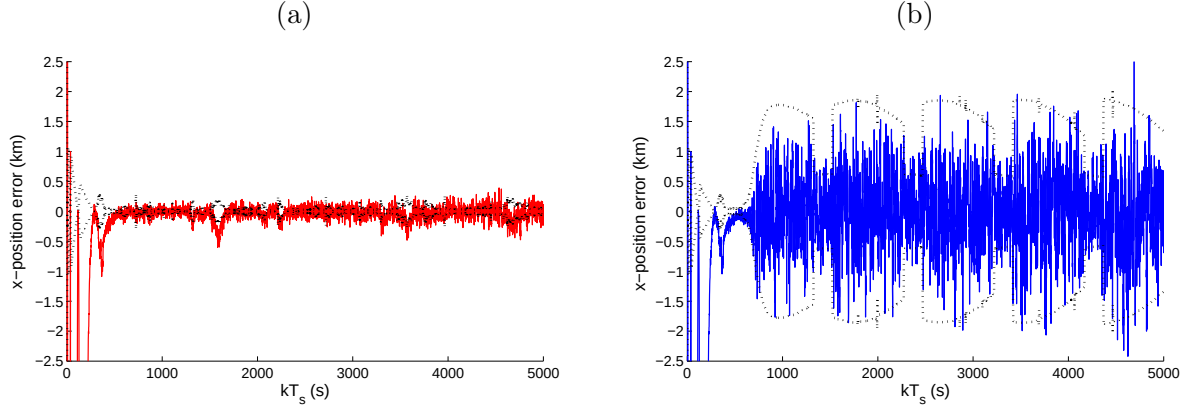


Figure 3.7: Target x_C -position-estimate errors (—) around the confidence interval ($\cdots$) defined by $\pm 3\sqrt{P_{(1,1)}^{xx}(kT_s)}$ using (a) SDEKBF and (b) SDUKBF. We consider an initial true-anomaly error of -90 deg. The range data are measured with $T_s = 1$ s.

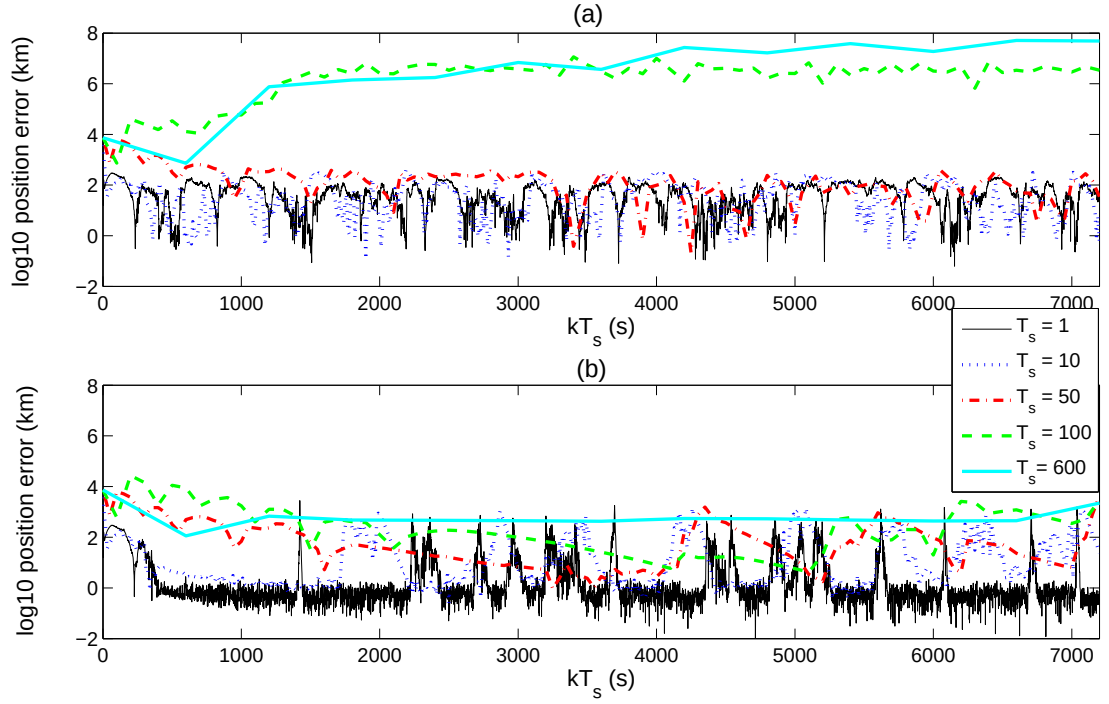


Figure 3.8: SDEKBF (a) and SDUKBF (b) target position-estimate errors for sample intervals $T_s = 1, 10, 50, 100, 600$ s with range-only measurements. The SDUKBF estimates are more accurate than the SDEKBF estimates for all sample intervals investigated. Also, SDEKBF does not converge for $T_s \geq 100$ s.

coordinate x do not remain inside the confidence interval defined by $\pm 3\sqrt{P_{(1,1)}^{xx}(kT_s)}$ (see Figure 3.7a), even with a larger value of $Q(t)$. Consequently, $P_{(1,1)}^{xx}(kT_s)$ of SDEKBF is not

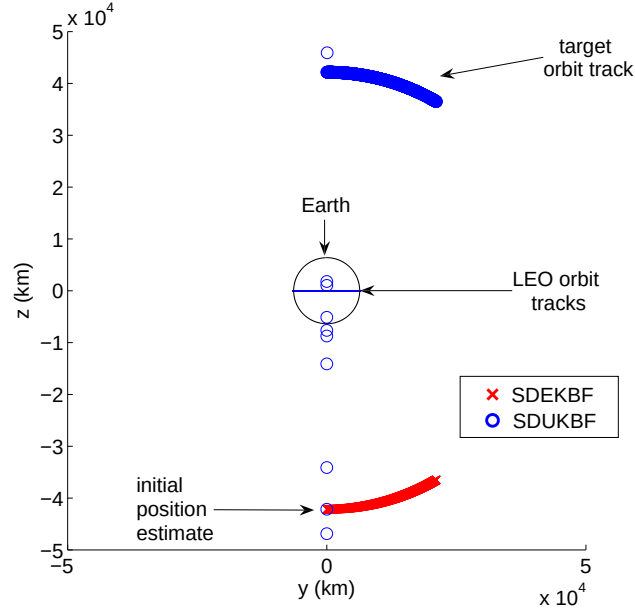


Figure 3.9: SDEKBF ‘x’ and SDUKBF ‘o’ target-position estimates. The initial location of the target is above the North Pole in a polar orbit, but we initialize both filters with an initial argument-of-perigee error of 180 deg, that is, over the South Pole. We set $P^{xx}(0) = \text{diag}(100, 100, 10^{10}, 1, 1, 0.1) + 10^{-2} \mathbf{1}_{6 \times 6}$. Range is measured with $T_s = 1$ s. SDUKBF approaches the vicinity of the target within about 30 s, while SDEKBF converges to the mirror image of the z -position state. The Earth and all LEO orbit locations are drawn to scale. We note that both filters fail to acquire the target when $P^{xx}(0) = \text{diag}(100, 100, 10^{10}, 1, 1, 0.1)$.

statistically consistent with the SDEKBF x -error estimates.

Next we consider the ability of the filters to acquire the target under time-sparse range-only measurements. For an initial true-anomaly error of 10 deg, Figure 3.8 shows the SDEKBF and SDUKBF position-estimate errors for $T_s = 1, 10, 50, 100, 600$ s. The SDUKBF estimates are more accurate than the SDEKBF estimates for all sample intervals investigated. Also, SDEKBF does not converge for $T_s \geq 100$ s, whereas SDUKBF converges for $T_s \leq 600$ s.

Finally, we consider the case in which 6 satellites in circular equatorial LEO are observing a target satellite in a polar orbit. We set the sample interval to be $T_s = 1$ s. For an initial estimate that is erroneous by 180 deg in the argument of perigee, that is, $\hat{x}_3(0) = -x_3(0)$ and $\hat{x}_6(0) = -x_6(0)$, we compare the performance of SDEKBF and SDUKBF. For this example, we set $P^{xx}(0) = \text{diag}(100, 100, 10^{10}, 1, 1, 0.1) + 10^{-2} \mathbf{1}_{6 \times 6}$, where $P_{(3,3)}^{xx}(0)$ is set to a large value to reflect the large initial uncertainty in z_C . Because of the polar orbit, the range-only output map from all of the observing spacecraft has even symmetry, and thus ambiguity can occur. Figure 3.9 shows that SDUKBF approaches the vicinity of the target

within about 30 s, while SDEKBF converges to the mirror image of the z_C -position component.

Eccentricity Estimation

We now consider the case in which the target performs an unknown thrust maneuver that changes the eccentricity of its orbit. In particular, the target is initially in a circular orbit as in the previous examples. At time $kT_s = 1000$ s, the target performs a 1-s burn that produces a specific thrust (that is, acceleration) $u_y = 0.5$ km/s², while, at time $kT_s = 1500$ s, the target performs a 1-s burn that produces a specific thrust $u_y = 0.3$ km/s². With an initial eccentricity of $e = 0$, corresponding to a circular orbit, the eccentricity after the first burn is $e \approx 0.35$, while the eccentricity after the second burn is $e \approx 0.59$. These target inputs are assumed to be unknown for state estimation. The initial true-anomaly error is set to -30 deg. Assuming measurement noise with $R(kT_s) = 0.01I_{6 \times 6}$ km², and range-only measurements available with $T_s = 10$ s, the SDEKBF and SDUKBF eccentricity estimates are shown in Figure 3.10.

For a 100-run Monte Carlo simulation, we obtain RMSE indices of 0.0862 and 0.0731, respectively, for the eccentricity estimates using SDEKBF and SDUKBF over $kT_s \in [500, 2500]$ s. These indices indicate that SDUKBF yields more accurate eccentricity estimates than does SDEKBF.

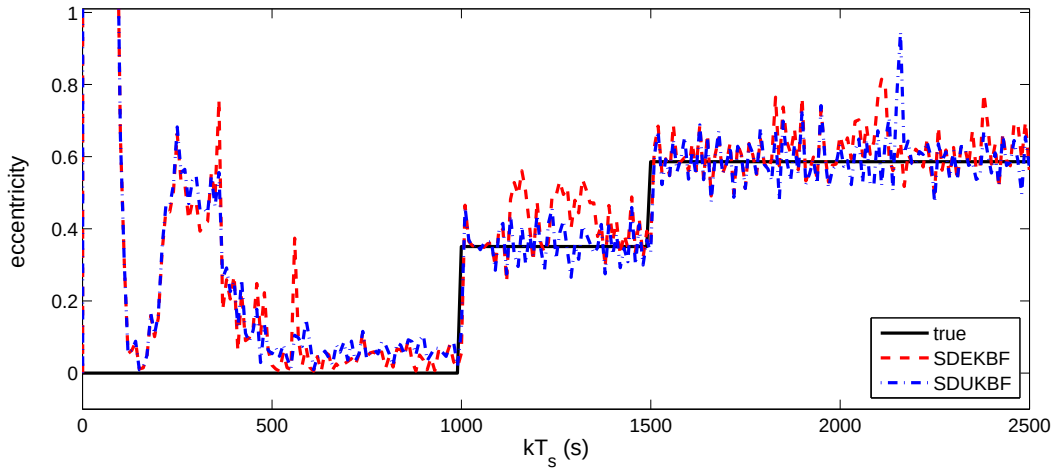


Figure 3.10: Estimated eccentricity with $T_s = 10$ s, with an initial true-anomaly error of -30 deg, and with range measurements. The target performs unknown 1-s burns at $kT_s = 1000$ s and $kT_s = 1500$ s in u_y . The initial eccentricity is $e = 0$, corresponding to the initial circular orbit, while the eccentricity after the first burn is $e \approx 0.35$, and the eccentricity after the second burn is $e \approx 0.59$. These results show the sensitivity of the eccentricity estimates to measurement noise.

Inclination Estimation

We now consider the case in which the target performs an unknown thrust maneuver that changes its inclination. The target is initially in a circular orbit with inclination $i = 0$ rad. At time $kT_s = 3000$ s, the target performs a 1-s burn that produces a specific thrust $u_z = 0.5$ km/s², while, at time $kT_s = 5000$ s, the target performs a 1-s burn that produces a specific thrust $u_z = -0.2$ km/s². The inclination after the first burn is $i \approx 0.16$ rad, whereas the inclination after the second burn is $i \approx 0.097$ rad. These target inputs are assumed to be unknown for state estimation. We investigate four cases as follows. In all cases, we set $T_s = 1$ s and we assume an initial true-anomaly error of -30 deg.

First, we assume range-only measurements with measurement-noise standard deviation of 0.032 km. For numerical conditioning, we set $R(kT_s) = 0.01I_{6 \times 6}$ km². This is case (i). The SDEKBF and SDUKBF inclination estimates are shown in Figure 3.11(a). After an initial transient, SDEKBF is able to track changes in the target's inclination. On the other hand, even though SDUKBF yields more accurate estimates over $kT_s \in [0, 3000]$ s, SDUKBF continues to yield erroneous inclination estimates close to zero after the target's maneuver. In this case, unlike SDEKBF, SDUKBF yields highly biased estimates for the position z_C for $kT_s > 3000$ s.

The inability of SDUKBF to detect changes in the inclination due to the target's maneuvers can be overcome by initializing the estimated inclination to a small nonzero value, specifically, -0.1 rad. However, if the estimate converges to zero, then the filter can fail to detect further changes in the target's inclination. Alternatively, we can slightly alter the inclination of the observing satellites so that the geometry is not entirely coplanar. We refer to it as case (ii). We thus change the orbit of two observing satellites by giving them an inclination of -0.1 rad and -0.2 rad, respectively. After an initial transient (from $kT_s = 3000$ s to $kT_s = 3200$ s), Figure 3.11(b) shows the ability of both filters to track the true inclination. Note that, according to Figure 3.11(b), SDEKBF yields less accurate inclination estimates than SDUKBF over $kT_s \in [0, 3000]$ s.

We now augment the range measurements with angle (azimuth and elevation) data. We assume range and angle measurement-noise standard deviations of 0.032 km and 0.032 rad, respectively. For numerical conditioning, we set $R(kT_s) = \text{diag}(0.01I_{6 \times 6} \text{ km}^2, 0.001I_{12 \times 12} \text{ rad}^2)$. This is case (iii). Figure 3.11(c) shows that, after the orbital maneuver, the filters track the inclination changes. The addition of angle measurements enables SDUKBF to detect changes

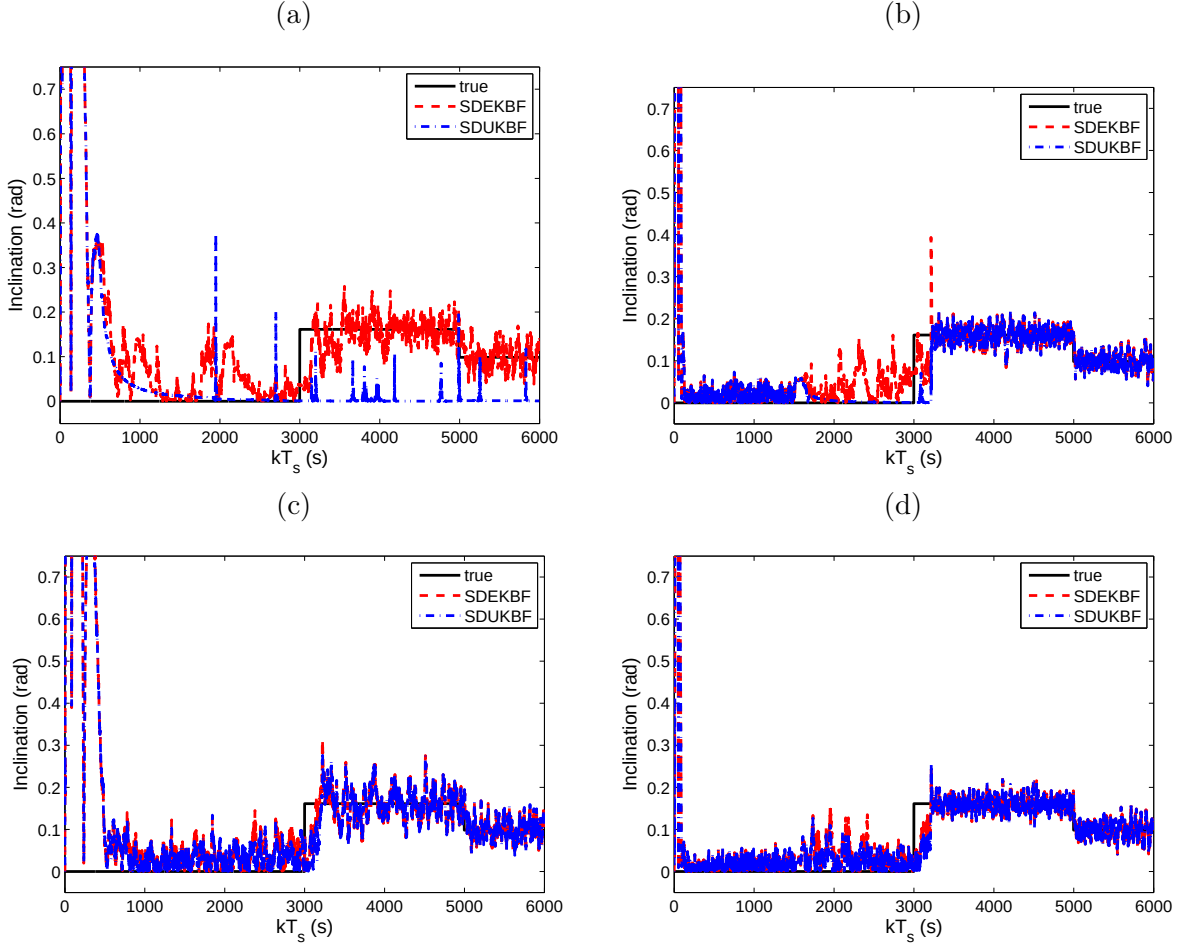


Figure 3.11: Estimated inclination with $T_s = 1$ s, with an initial true-anomaly error of -30 deg, and with (a)-(b) range-only measurements and (c)-(d) range and angle measurements. In (a), SDUKBF fails by getting stuck in estimating $i \approx 0$, while SDEKBF detects changes in the target's inclination. In (b), we slightly change the orbit of two observing satellites by giving them inclinations of -0.1 rad and -0.2 rad, respectively. After an initial transient, both filters provide improved estimates of the target's inclination. In (c) and (d), angle (azimuth and elevation) measurements in addition to range measurements from the observing satellites allow SDEKBF and SDUKBF to detect changes in the target's inclination. In (d), we slightly change the orbit of two observing satellites by giving them inclinations of -0.1 rad and -0.2 rad, respectively.

in the target's inclination. Moreover, Figure 3.11(d) shows the case (iv), when the geometry of the observing satellites is not entirely coplanar (as in Figure Figure 3.11(b)) and angle measurements are used in addition to range data. In this case, the inclination estimates are more accurate than the estimates shown in Figure 3.11(c).

Table 3.4 compares the performance of SDEKBF and SDUKBF for the four cases aforementioned over a 100-run Monte Carlo simulation. Comparing the indices RMSE, we observe that SDUKBF outperforms SDEKBF for z_C and v_z , SDEKBF outperforms SDUKBF

Table 3.4: RMSE (2.31), MT (excluding $P_{(3,3)}^{xx}$ and $P_{(6,6)}^{xx}$) (2.33), and T_{CPU} (see Section 2.4) for $kT_s \in [500, 6000]$ s and for a 100-run Monte Carlo simulation using SDEKBF and SDUKBF. The target is maneuvering such that its inclination changes, $T_s = 1$ s, and the initial true-anomaly error is -30 deg. We consider the following cases: (i) range-only measurements are used, (ii) range-only measurements are used with the geometry of the observing satellites not entirely coplanar, (iii) range and angle measurements are used, (iv) range and angle measurements are used with the geometry of the observing satellites not entirely coplanar.

	RMSE							MT	T_{CPU} (ms)
	x_C (km)	y_C (km)	z_C (km)	v_x (km/s)	v_y (km/s)	v_z (km/s)	i (rad)		
SDEKBF									
(i)	0.065	0.096	613.6	0.013	0.026	0.44	0.066	0.21	20.4
(ii)	0.065	0.116	22.8	0.013	0.026	0.15	0.039	0.26	20.5
(iii)	0.061	0.095	18.5	0.013	0.026	0.18	0.056	0.20	20.3
(iv)	0.063	0.115	12.6	0.013	0.026	0.12	0.037	0.26	20.9
SDUKBF									
(i)	0.257	0.131	605.7	0.034	0.028	0.34	0.108	0.72	38.6
(ii)	0.128	0.119	17.4	0.021	0.027	0.11	0.036	0.42	39.2
(iii)	0.090	0.097	14.3	0.011	0.026	0.16	0.049	0.22	38.5
(iv)	0.081	0.116	10.4	0.012	0.026	0.11	0.034	0.27	39.7

for x_C and v_x , and the filters have similar accuracy for y_C and v_y . Regarding the inclination estimates, except for case (i) for which SDUKBF fails to track the inclination changes, SDUKBF yields more accurate estimates than SDEKBF. Also, with the inclusion of angle measurements, both filters yield similar MT indices. Moreover, the SDUKBF processing time T_{CPU} is twice as long as the SDEKBF processing time.

3.2.4 Concluding Remarks

Under idealized assumptions on the astrodynamics of bodies orbiting the Earth, we apply SDEKBF and SDUKBF for range-only as well as range and angle observations provided by a constellation of 6 low-Earth orbiting satellites in circular, equatorial orbits. We study the ability of the filters to acquire and track a target satellite in geosynchronous orbit as a function of the sample interval, initial uncertainty, and type of available measurements.

For target acquisition, SDUKBF yields more accurate position and velocity estimates than SDEKBF. Moreover, the convergence of SDEKBF is sensitive to the initialization of the error covariance; in fact, a nondiagonal initial covariance is found to be more effective than a diagonal initial covariance. Like SDUKBF, by properly setting a nondiagonal initial error covariance, SDEKBF also exhibits global convergence, that is, convergence is attained for

all initial true-anomaly errors. Under time-sparse range-only measurements, SDEKBF is not able to track the target for a sample interval $T_s \geq 100$ s. On the other hand, the SDUKBF estimates converge for $T_s \leq 600$ s.

When the target is maneuvering such that its eccentricity changes, SDUKBF tracks the target's eccentricity more accurately than SDEKBF.

Unlike SDEKBF, SDUKBF is not able to detect changes in the target's inclination when the target is maneuvering. Nevertheless, either by having the observing satellites not entirely coplanar or by including angle measurements, convergence is attained for both filters. Furthermore, when angle measurements are also available, SDUKBF yields more accurate inclination estimates than SDEKBF. Finally, the SDUKBF processing time is about twice the SDEKBF processing time.

3.3 Case Study in Nonlinear Dynamics and Chaos

– Using Data-Driven Discrete-Time Models on Unscented Kalman Filtering

3.3.1 Introduction

The challenge of estimating states and occasionally some parameters of nonlinear systems (Monteiro, 2002) has received great attention recently (Voss, Timmer & Kurths, 2004). In particular, alternatives to EKF (Section 2.5) have been sought, such as UKF (Section 2.6), which has already attracted attention in the context of nonlinear dynamics (Sitz, Schwarz, Kurths & Voss, 2002; Sitz, Schwarz & Kurths, 2004; Bitencourt Jr., Tôres & Aguirre, 2004).

Given that UKF has great potential for real applications, we address UKF in a context that is somewhat more realistic than previously studied. In the former works, the dynamical equations, which govern the systems for which the states and occasionally parameters are to be estimated, are assumed to be known. In order to handle such differential equations in the context of the filter, numerical integration is necessary as discussed in Section A.2. In the present study, the models are not assumed to be known. Rather, the starting point is a set of noisy data from which a set of difference equations are built and which are used to implement UKF. As a by-product, because the models are in discrete time, no integration is required. In real-time applications this feature is most welcome. In the literature, one simulated (no noise added) example is found and it uses a feedforward neural network to model the Mackey-Glass chaotic time series (Haykin, 2001).

We illustrate the model-building and filtering stages using two examples, namely, the simulated Lorenz system and an electronic oscillator from which data are actually recorded. The results presented in this section are further discussed in (Aguirre, Teixeira & Tôres, 2005).

3.3.2 Problem Statement

Nonlinear Model Building from Data

In this study we use multivariable nonlinear autoregressive moving average (NARMA) polynomial models (Billings, Chen & Korenberg, 1989; Aguirre, 2007) to build a discrete-time model for the continuous-time dynamic system given by (A.8).

Once defined which mathematical representation to use, the identification of a model

consists roughly of two steps, namely, structure selection and parameter estimation. We use the orthogonal extended least-squares algorithm (Chen, Billings & Luo, 1989) to estimate the parameters of the NARMA polynomial models. As a by-product of this technique, we use the error reduction ratio (ERR) (Billings, Chen & Korenberg, 1989) as a criterion for structure selection.

Complementary procedures for structure selection using ERR are given in (Aguirre & Billings, 1995a; Lu et al., 2001). A discussion on the dynamical effects of inappropriate structure selection for polynomial models is found in (Aguirre & Billings, 1995a,b; Mendes & Billings, 1997). Alternative schemes to perform structure selection for multilayer perceptron neural networks and radial basis function models (Haykin, 1994; Braga et al., 2000) are given in (Judd & Mees, 1998; Chen et al., 2004; Teixeira et al., 2000; Aguirre et al., 2004).

Models for UKF Implementation

After the black-box identification stage briefly reviewed above, we use the identified model in the UKF algorithm to perform state-estimation and joint state-and-parameter estimation (as discussed in Section A.1). The dynamic model needed to implement UKF is used to perform two tasks, as seen in Section 2.6.2. First, the model is used to forecast the entire state vector $\hat{x}_{k|k}$ from the measurements y_k at time k and prior state estimate $\hat{x}_{k-1|k-1}$ provided by the filter at time $k - 1$. Second, the model is used to propagate the sigma points to estimate how not only the pseudo mean $\hat{x}_{k|k}$ but also the pseudo-error covariance $P_{k|k}^{xx}$ is affected by the system.

Having realized the different roles of the model within the filter, a couple of remarks are in order. From a dynamical point of view, the model should provide good predictions $\hat{x}_{k|k}$ of the full state vector at time k . Because the model need not to produce free-run predictions, the requirements on its dynamics are less exacting as, for instance, when free-runs are used to reconstruct attractor properties. Therefore, it is unnecessary to require for the model dynamics to be topologically equivalent to the original dynamics. Moreover the dynamical inaccuracy of the model is somewhat compensated by the dynamical information provided by the measurements that drive the filter.

From a static point of view, the model is used to propagate the sigma points. In this respect the dynamical features are not important but the nonlinearity of the model should be adequate. A poor approximation of the nonlinearity has the effect of mapping the sigma points incorrectly and this can alter significantly the pseudo-error covariance matrix and, in

turn, the filter stability. Therefore if the system cannot be correctly approximated by a given model representation, a different one should be tried.

3.3.3 Simulated Results – The Lorenz System

We now consider the Lorenz system (Lorenz, 1963; Fiedler-Ferrara & Prado, 1994)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\lambda_1 x_1(t) + \lambda_1 x_2(t) \\ \lambda_2 x_1(t) - x_2(t) - x_1(t)x_3(t) \\ -\lambda_3 x_3(t) + x_1(t)x_2(t) \end{bmatrix}, \quad (3.10)$$

with $\lambda_1 = 12$, $\lambda_2 = 40$, and $\lambda_3 = 4$. This system is also studied in the context of joint state-and-parameter estimation in (Sitz et al., 2002; Bitencourt Jr., 2003). We integrate (3.10) with $T_s = 20$ ms using the fourth-order Runge-Kutta algorithm with initial condition $x(0) = [0.1 \ 0.1 \ 0.1]^T$. Then we add Gaussian white noise with covariance $R^{\text{id}}(kT_s) = \text{diag}(1.34, 1.88, 1.72)$ to the integrated data and obtain the data shown in Figure 3.12(a). As discussed in Section 3.3.2, one thousand observations of these data are used to build the discrete-time model

$$\begin{aligned} x_{1,k} = & 0.861921218252 x_{1,k-1} + 0.234404677244 x_{2,k-1} \\ & -0.752478985861 \times 10^{-3} x_{2,k-1} x_{3,k-1} - 0.183251890379 \times 10^{-2} x_{1,k-1} x_{3,k-1}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} x_{2,k} = & 1.15961883382 x_{2,k-1} - 0.572159400169 \times 10^{-2} x_{2,k-1} x_{3,k-1} \\ & -0.142963657294 \times 10^{-1} x_{1,k-1} x_{3,k-1} + 0.577685873218 x_{1,k-1}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} x_{3,k} = & 0.962354730753 x_{3,k-1} + 0.553656929609 \times 10^{-2} x_{2,k-1}^2 \\ & +0.940412257934 \times 10^{-2} x_{1,k-1} x_{2,k-1} - 0.116546062702 \times 10^{-2} x_{3,k-1}^2 \\ & +0.399604189148 \times 10^{-2} x_{1,k-1}^2, \end{aligned} \quad (3.13)$$

where only the deterministic part of the NARMA model is shown. The stochastic part, used during parameter estimation in order to avoid bias, is not shown nor used by UKF. The attractor to which (3.11)-(3.13) settles under free-run simulation is shown in Figure 3.12(b).

We combine the identified model (3.11)-(3.13) with UKF equations to estimate the three components of the Lorenz attractor. Because only the state component $x_1(kT_s)$ of (3.10) corrupted with white Gaussian noise with covariance $R(kT_s)$ (see Table 3.5) is chosen to drive the filter, we use (2.8) as the observation model with $C_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Therefore

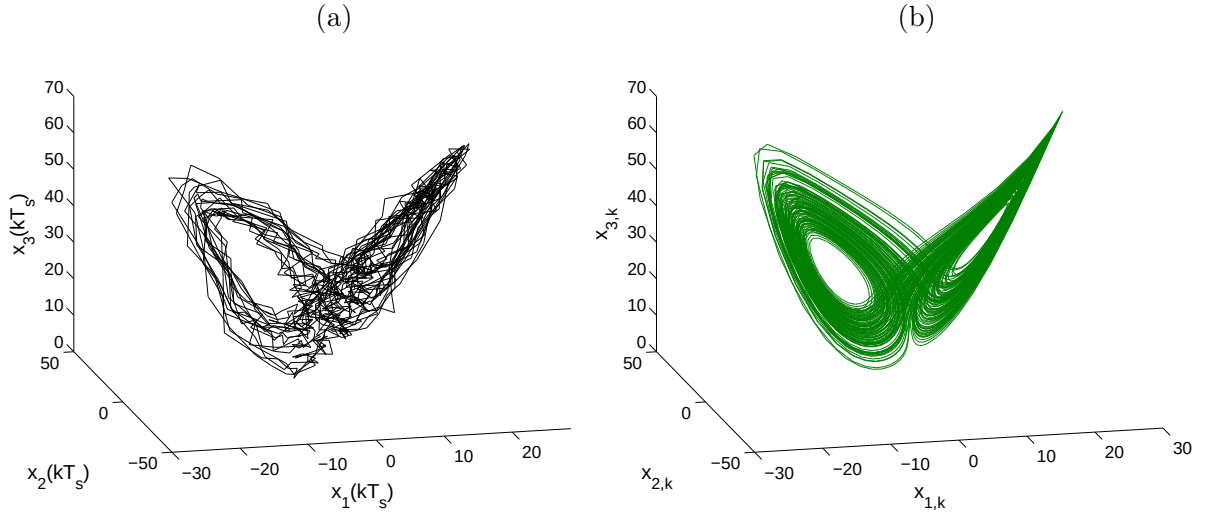


Figure 3.12: (a) Lorenz attractor obtained by numerical integration of (3.10) with $\lambda_1 = 12$, $\lambda_2 = 40$, and $\lambda_3 = 4$ and initial conditions $x(0) = [0.1 \ 0.1 \ 0.1]^T$. Gaussian white noise is added to each component. In (b), the attractor is obtained by free-run simulation of (3.11)-(3.13) with the same initial conditions.

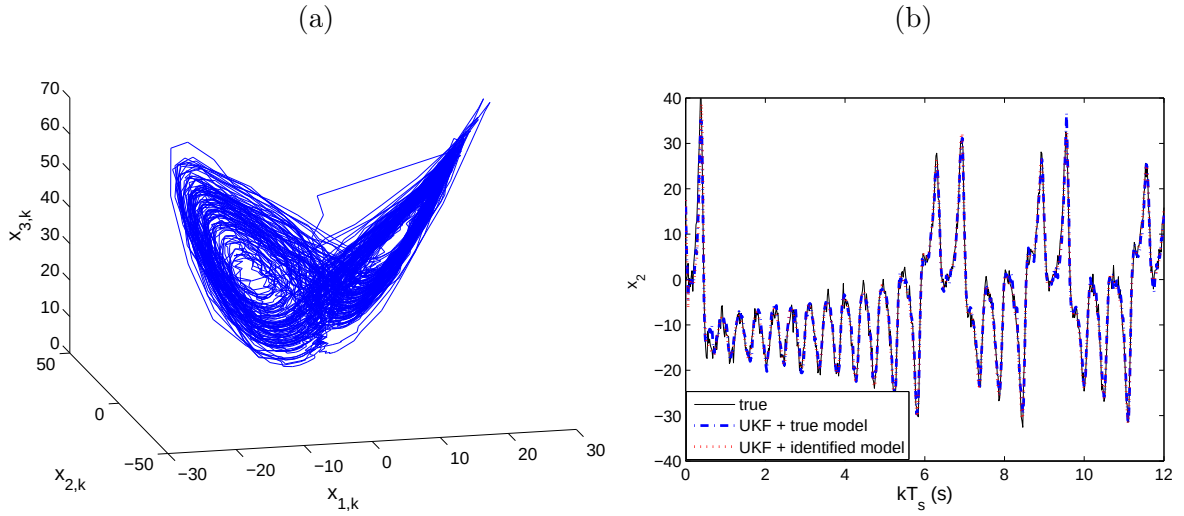


Figure 3.13: (a) Lorenz attractor obtained by UKF using (3.11)-(3.13) and x_1 as the single driver. The transient at the beginning is included in the plot to give an idea of how quickly the filter settles. In (b), it is shown an window of data corresponding to the x_2 component of the Lorenz attractor, where (—) indicates the true state obtained by the numerical integration of (3.10), (---) accounts for the state estimate using UKF with the identified model (3.11)-(3.13), and (····) indicates the state estimate using UKF with the true model (3.10). These curves coincide at most time steps.

the three components are only needed during the model-building phase. From then on, only one component is used to estimate the full state vector.

Whenever the identified model (3.11)-(3.13) is used, we set a nonzero process-noise covariance Q_{k-1} . Otherwise, the small model imperfections added to the high sensitivity to initial conditions of chaotic systems lead to bad tracking performance. Hence Q_{k-1} is set to be a diagonal matrix with elements equal to the variance of the one-step-ahead prediction errors obtained by the simulation of (3.11)-(3.13). In a sense, matrix Q_{k-1} thus obtained quantifies model uncertainties.

Figure 3.13(b) shows the true state $x_2(kT_s)$ obtained by integration of the Lorenz equations (3.10), as well as the state estimates $\hat{x}_{2,k|k}$ obtained by UKF using the theoretical (3.10) and identified (3.11)-(3.13) models. Table 3.5 summarizes the UKF performance for various noise levels $R(kT_s)$ and for both identified and true models.

Note that UKF with the identified model is quite competitive especially at high noise levels. However, note that, in practice, the true model is not available in general and some type of identified model is required. Furthermore, the estimation error of x_3 is significantly less than for x_1 and x_2 . Also, the errors related to such variables are rather similar being the error of x_1 slightly smaller than the error related to x_2 . This is in good agreement with the estimates of the (normalized) observability degrees $\delta_{x_3} = 1.00$, $\delta_{x_2} = 0.37$ and $\delta_{x_1} = 0.30$ (Letellier & Aguirre, 2002). In fact, such indices indicate that x_2 is slightly more observable than x_1 , but clearly show that x_3 is significantly more observable than the other two. An important remark is that it is well known that such indices have a local character, rather than global. This is consistent with the fact that the performance of the filter depends quite strongly on the “local” performance of the model.

Table 3.5: Normalized RMSE (2.31) for the Lorenz system corrupted with different levels of Gaussian white noise. The measurement noise levels correspond to noise with covariance $R(kT_s)$ added to the $x_1(kT_s)$ component which is used to drive UKF. Both the true (3.10) and the identified (3.11)-(3.13) models are tested.

Model	$R(kT_s)$	$x_{1,k}(\%)$	$x_{2,k}(\%)$	$x_{3,k}(\%)$
true	1.35	1.25	1.57	0.88
identified	1.35	3.31	4.23	2.52
true	8.41	3.53	4.41	2.41
identified	8.41	6.06	7.55	3.92
true	33.6	8.40	10.90	5.79
identified	33.6	9.91	12.36	6.28

3.3.4 Experimental Results – The Chua’s Circuit

We now consider an actual implementation of the electronic oscillator known as Chua’s circuit (Matsumoto, 1984), whose dynamic equations are given by (Chua, 1994)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_1}x_1(t) + \frac{1}{RC_1}x_2(t) - \frac{1}{C_1}i_d(x_1(t)) \\ \frac{1}{RC_2}x_1(t) - \frac{1}{RC_2}x_2(t) + \frac{1}{C_2}x_3(t) \\ -\frac{1}{L}x_2(t) \end{bmatrix}, \quad (3.14)$$

where R , C_1 , C_2 , L , m_0 , m_1 e B_p are the circuit parameters, and $i_d(x_1(t))$ accounting for the nonlinear behavior of the oscillator is given by

$$i_d(x_1(t)) = m_0x_1(t) + \frac{1}{2}(m_1 - m_0)(|x_1(t) + B_p| - |x_1(t) - B_p|). \quad (3.15)$$

In the gyrator-based implementation used (Tôrres & Aguirre, 1996; Tôrres, 2001), the frequency of the main peak in the spectral power density is around 1.6 Hz, enabling us to collect data with a sampling time of $T_s = 30$ ms (Tôrres & Aguirre, 2000).

According to the procedure briefly reviewed in Section 3.3.2, we use one thousand observations of each state variable, which are shown in Figure 3.14, to build the model

$$\begin{aligned} x_{1,k} = & 1.21036203206 x_{1,k-1} + 0.972047183733 \times 10^{+2} x_{3,k-1} + 0.598221283675 x_{2,k-1} \\ & - 0.171315924858 \times 10^{-1} x_{1,k-1}^3 + 0.398097703675 \times 10^{+5} x_{2,k-1} x_{3,k-1}^2 \\ & + 0.819593354856 \times 10^{+4} x_{1,k-1} x_{3,k-1}^2 - 0.111072563509 \times 10^{+3} x_{2,k-1}^2 x_{3,k-1}, \end{aligned} \quad (3.16)$$

$$\begin{aligned} x_{2,k} = & 0.895252009866 x_{2,k-1} - 0.223303681119 \times 10^{+6} x_{3,k-1}^3 + 0.0837143492947 x_{1,k-1} \\ & + 0.129325180333 \times 10^{+3} x_{3,k-1} - 0.677214817432 \times 10^{-3} x_{1,k-1}^3 \\ & + 0.181085268783 \times 10^{-3} x_{1,k-1}^2 - 0.204050446812 \times 10^{+1} x_{1,k-1} x_{2,k-1} x_{3,k-1}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} x_{3,k} = & 0.938265057186 x_{3,k-1} - 0.637485684220 \times 10^{-3} x_{2,k-1} \\ & - 0.362134020185 \times 10^{-4} x_{1,k-1} + 0.315363093575 \times 10^{-5} x_{1,k-1}^2 \\ & + 0.632868345441 \times 10^{-5} x_{2,k-1}^2 - 0.643318682312 \times 10^{-5} x_{1,k-1} x_{2,k-1} \\ & + 0.187619321537 \times 10^{-2} x_{1,k-1} x_{3,k-1}. \end{aligned} \quad (3.18)$$

As before, we show only the deterministic part of the model. As illustrated in Figure 3.15(a), the model (3.16)-(3.18) does not produce a mathematically symmetrical attractor because it includes even parity terms (Letellier, Gouesbet & Rulkov, 1996). When we impose symmetry

conditions during modeling, the new model does not settle unto the double scroll attractor as well, and, for the reasons discussed in Section 3.3.2, the state estimates are not improved.

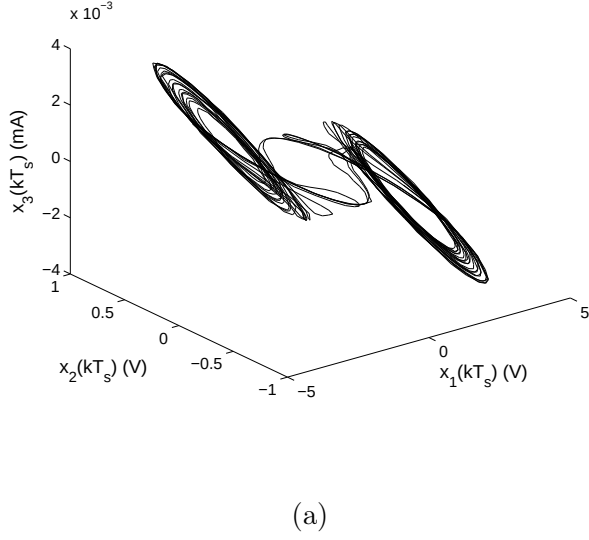


Figure 3.14: The measured double scroll attractor of the implemented electronic oscillator. These are the data used to build model (3.16)-(3.18).

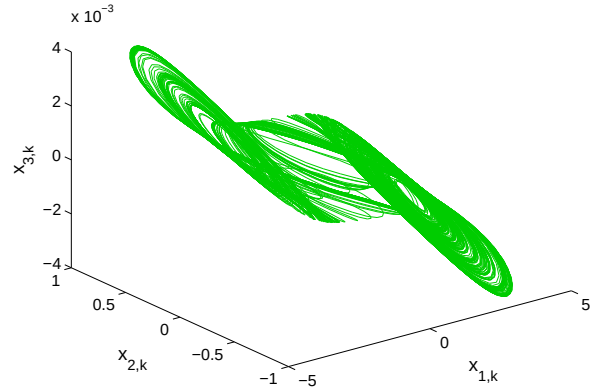
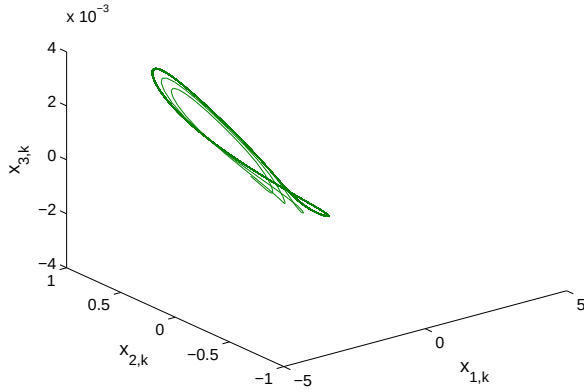


Figure 3.15: Attractors produced by the identified model (3.16)-(3.18) with initial conditions $x_0 = [3.2 \ 0.37 \ -0.009]^T$ (a) without and (b) with parameter perturbation.

We wish to verify if models with improved global dynamics outperform locally optimal models. Hence, changes are implemented in the identified model as follows. The attractor of (3.16)-(3.18) shown in Figure 3.15(a) seems to be close to a genuine solution of the system in the sense that it can be the “stable version” of one of the unstable periodic orbits that compose the original attractor. When this is the case, it has been argued that the model can be perturbed in order to become chaotic (Aguirre, Maquet & Letellier, 2002). In the present study, this is achieved by slightly increasing the underlined parameter in (3.17). For instance,

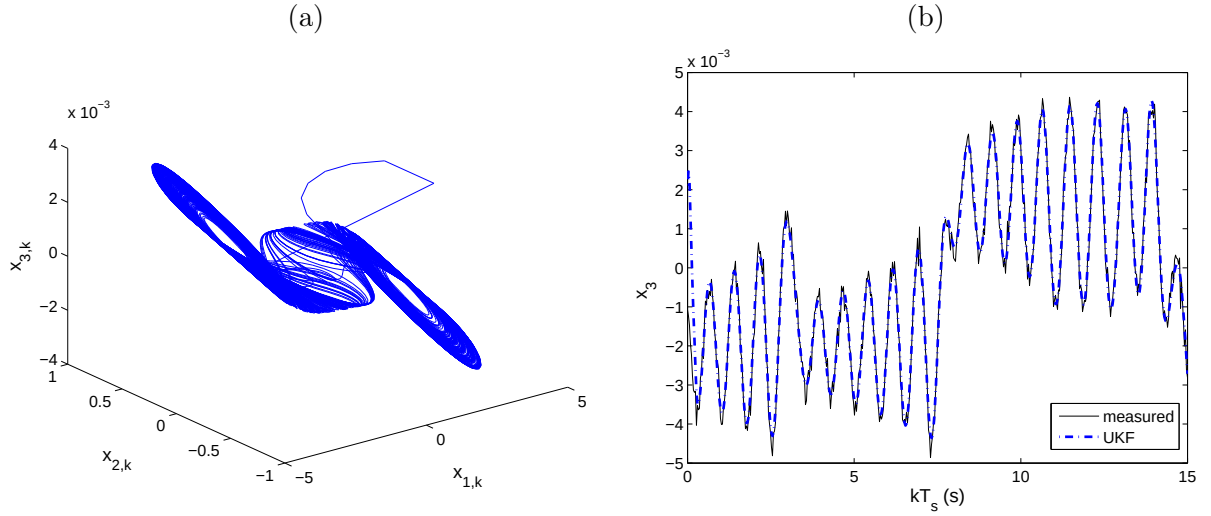


Figure 3.16: (a) The estimated double scroll attractor of the implemented electronic oscillator. We perform state estimation using UKF with model (3.16)-(3.18) and the $x_1(kT_s)$ component driving the filter. In (b), we show the detail of the x_3 component: measured data (—) and estimated component $\hat{x}_{3,k|k}$ (---). In the plot (a) the transient at the beginning is included to give an idea of how quickly the filter settles.

if 0.92427441 is used, then the perturbed identified model settles to the double-scroll attractor shown in Figure 3.15(b).¹

We combine UKF with (3.16)-(3.18) to estimate the full state vector of the electronic circuit using only the $x_1(kT_s)$ component to drive the filter. That is, the observation model is given by (2.8) with $C_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. We investigate the following cases: (i) using (3.16)-(3.18); (ii) slightly perturbing the underlined parameter of (3.17); and (iii) using (3.16)-(3.18) but jointly estimating the three parameters in *italic* (which correspond to the terms of each equation with higher ERR index; see Section 3.3.2) as discussed in Section A.1. In all cases, analogous to the Lorenz system example, Q_{k-1} is set as a diagonal matrix with elements equal to the one-step-ahead prediction error variance of the respective model, while the measurement-noise covariance is set to $R(kT_s) = 10^{-9}$.

Results are summarized in Table 3.6. The very low values of RMSE of the x_1 component are a consequence of low measurement noise in our experimental setup and due to the fact that the x_1 variable is used to drive the filter. Also, note that, although the perturbed model is, in a sense, closer to the original system (for instance, both have a positive Lyapunov

¹We tested other parameter values for perturbing (3.17) in order to obtain the double-scroll attractor. Nevertheless, the greater the disturbance (that is, the farther from the optimal set of parameters), the worse the UKF performance in estimating the states, although the model may display improved global features.

Table 3.6: Normalized RMSE (2.31) for the electronic system. We implement UKF with the identified model (3.16)-(3.18) in three contexts (see text).

situation investigated		x_1 (%)	x_2 (%)	x_3 (%)
(i)	identified model	2.80×10^{-7}	2.58	0.87
(ii)	perturbed identified model	3.75×10^{-7}	5.85	2.86
(iii)	joint parameter estimation	3.70×10^{-7}	3.03	1.24

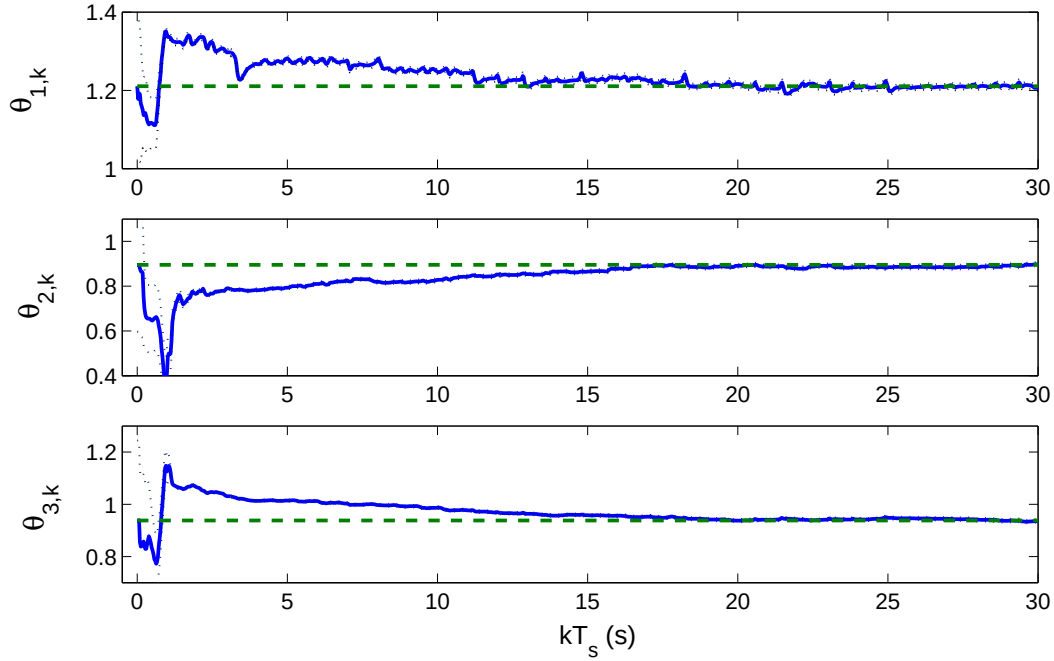


Figure 3.17: Parameter estimation for the Chua's circuit using extended least squares and UKF. The solid lines (—) show the results obtained from the joint estimation of the three parameters in italic in (3.16)-(3.18). The dotted lines ($\cdot \cdot \cdot$) indicate plus and minus one standard deviation ($\pm \sqrt{\text{diag}(P_{k|k}^{\theta\theta})}$) of the parameters estimated by UKF. The dashed lines (- - -) indicate the values for these parameters estimated by extended least squares during the modeling step.

exponent), the unperturbed model, which has a lower one-step-ahead prediction error, is better suited for UKF. When parameters are estimated in addition to the states, the estimated parameters converge to values close to those obtained during the modeling stage; see Figure 3.17. Consequently situations (i) and (iii) have close indices RMSE in Table 3.6.

3.3.5 Discussion and Concluding Remarks

We address the state-estimation problem for nonlinear systems, considering two examples: the Lorenz system (by simulation) and an implementation of the electronic oscillator

known as Chua’s circuit. The procedure followed in this study is innovative in two ways. First, UKF is implemented using discrete-time models such that no numerical integration is required in the forecast step. Second, the models are built from data only.

In both examples, we use one thousand observations of the three state variables to build the model, whereas only the variable x_1 is used to drive the filter. Attempts are made using components x_2 and x_3 to drive the filter. In the case of the Lorenz system, driving the filter with x_2 is also fine. However, due to symmetry, driving the system with x_3 yields estimates of x_1 and x_2 that can turn out to be the mirror image of the true variables. This is also true when the filter is driven by the x_2 variable for the electronic oscillator. This is due to the fact that observing such systems from such variables results in the superposition of fixed points (Letellier, Gouesbet & Rulkov, 1996). Finally, in the case of the electronic oscillator, if we drive the filter with x_3 , then no convergence is attained.

For the electronic oscillator, not only state estimation but also joint state-and-parameter estimation is performed. The parameter values estimated by UKF are very close to those estimated by the extended least squares estimator, thus confirming a fact known from theory.

Although the model quality does have a direct bearing on the filter performance, setting a proper process-noise covariance Q_{k-1} is very important. For instance, if we set an unsuitable process Q_{k-1} when a model that settles to a limit cycle rather than to a chaotic attractor is used by UKF, then the UKF estimates also converges to a limit cycle, even if the driving signal is chaotic. In fact, underestimating the trace of Q_{k-1} indicates to the filter that we are greatly confident in the model and consequently the filter settles to the “model-dictated” limit cycle. However, if Q_{k-1} is properly set (for instance, as a diagonal matrix with elements equal to the variance of the one-step-ahead prediction errors), then the filter settles to a chaotic attractor that very closely resembles the original one. The former discussion suggests that it is nice to have not only dynamical models for nonlinear systems but also some measure of uncertainty to define the process-noise covariance.

The use of discrete-time models built from data in connection with UKF to estimate states and parameters of nonlinear systems is viable. We consider the case in which the full state vector is available to build the model but only one variable is measured to drive the filter. Uncertainties and imperfections in the model can be compensated for, to some extent, by setting a suitable nonzero process-noise covariance. Poor models usually result in poor filter performance or even to the lack of convergence, regardless of the process-noise covariance matrix.

Part II

CONSTRAINED STATE ESTIMATION

CONSTRAINED STATE ESTIMATION: A THEORETICAL BACKGROUND

“There is nothing so practical as a good theory.”

Kurt Lewin (1951)

This chapter presents a survey on the current state of the art in the field of constrained state estimation. Using a unified notation, our goal is to review the key ideas and corresponding equations of the main methods so far proposed in the literature so that they can be understood and implemented.

We categorize these methods into four groups according to the type of constraint. First, we address the linear and nonlinear equality-constrained cases in Sections 4.1 and 4.2, respectively. Next, in Section 4.3, the constrained state estimation problem is recast from the perspective of quadratic programming. Then we review approximate methods for both linear and nonlinear inequality-constrained state estimation in Sections 4.4 and 4.5, respectively. Finally, in Section 4.6, we summarize and compare the main characteristics of the fourteen algorithms reviewed along this chapter.

4.1 State Estimation for Equality-Constrained Linear Systems

4.1.1 Problem Statement

We consider the linear system given by (2.7)-(2.8), whose equations are repeated here for convenience, that is,

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1}, \quad (4.1)$$

$$y_k = C_k x_k + v_k. \quad (4.2)$$

We assume that the state vector x_k is known to satisfy the equality constraint

$$D_{k-1}x_k = d_{k-1}, \quad (4.3)$$

where, for all $k \geq 1$, $D_{k-1} \in \mathbb{R}^{s \times n}$ and $d_{k-1} \in \mathbb{R}^s$ are assumed to be known.¹ Without loss of generality, we assume that $\text{rank}(D_{k-1}) = s$. Moreover, we assume that $\rho_{w_{k-1}}(w_{k-1})$, $\rho_{v_k}(v_k)$, and $\rho_{x_0}(x_0)$ are Gaussian and, respectively, given by (2.9)-(2.11).

The goal of the equality-constrained state-estimation problem is to obtain state estimates that satisfy the equality constraint (4.3). Next, we review four extensions of KF that solve this problem.

4.1.2 The Measurement-Augmentation Kalman Filter

The measurement-augmentation Kalman filter (MAKF) (Porrill, 1988; Tahk & Speyer, 1990; Chia et al., 1991; Wen & Durrant-Whyte, 1992; Doran, 1992; Alouani & Blair, 1993; Gupta, 2007b) treats the equality constraint (4.3) as perfect measurements and uses the KF equations. That is, we replace (4.2) by the augmented observation

$$\tilde{y}_k = \tilde{C}_k x_k + \tilde{v}_k, \quad (4.4)$$

where

$$\tilde{C}_k \triangleq \begin{bmatrix} C_k \\ D_{k-1} \end{bmatrix}, \quad (4.5)$$

$$\tilde{y}_k \triangleq \begin{bmatrix} y_k \\ d_{k-1} \end{bmatrix}, \quad \tilde{v}_k \triangleq \begin{bmatrix} v_k \\ 0_{s \times 1} \end{bmatrix}. \quad (4.6)$$

With (4.4), MAKF uses (2.20)-(2.21) together with the augmented forecast equations

$$\hat{\tilde{y}}_{k|k-1} = \tilde{C}_k \hat{x}_{k|k-1}, \quad (4.7)$$

$$P_{k|k-1}^{\tilde{y}\tilde{y}} = \tilde{C}_k P_{k|k-1}^{xx} \tilde{C}_k^T + \tilde{R}_k, \quad (4.8)$$

$$P_{k|k-1}^{x\tilde{y}} = P_{k|k-1}^{xx} \tilde{C}_k^T, \quad (4.9)$$

¹In Section 5.1 we investigate how a linear equality constraint arises on the state vector of a linear dynamic system and present necessary conditions on both the dynamics and process noise for the state to be equality constrained.

where

$$\tilde{R}_k = \begin{bmatrix} R_k & 0_{m \times s} \\ 0_{s \times m} & 0_{s \times s} \end{bmatrix}, \quad (4.10)$$

and the augmented data-assimilation equations given by

$$\tilde{K}_k = P_{k|k-1}^{xy} (P_{k|k-1}^{\tilde{y}\tilde{y}})^{-1}, \quad (4.11)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \tilde{K}_k (\tilde{y}_k - \hat{\tilde{y}}_{k|k-1}), \quad (4.12)$$

$$P_{k|k}^{xx} = P_{k|k-1}^{xx} - \tilde{K}_k P_{k|k-1}^{\tilde{y}\tilde{y}} \tilde{K}_k^T. \quad (4.13)$$

Remark 4.1.1. In practice, however, to avoid ill-conditioning problems with the inversion of $P_{k|k-1}^{\tilde{y}\tilde{y}}$ in (4.11), (4.10) is generally replaced by

$$\tilde{R}_k = \begin{bmatrix} R_k & 0_{m \times s} \\ 0_{s \times m} & \delta R_k \end{bmatrix}, \quad (4.14)$$

where we set $\delta R_k = \varepsilon_r I_{s \times s}$ and $\varepsilon_r > 0$ is a very small number compared to the entries of R_k . From our experience, we set $10^{-15} \leq \varepsilon_r \leq 10^{-9}$.

4.1.3 The Projected Kalman Filter by Estimate Projection

The projected Kalman filter by estimate projection (PKF-EP) (Simon & Chia, 2002; Simon, 2006) projects the state estimate $\hat{x}_{k|k}$ given by (2.27) onto the hyperplane defined by (4.3).

Define the cost function

$$J_p(x_k) \triangleq (x_k - \hat{x}_{k|k})^T W_k^{-1} (x_k - \hat{x}_{k|k}), \quad (4.15)$$

where $W_k \in \mathbb{R}^{n \times n}$ is assumed to be positive definite and is often set to $W_k = P_{k|k}^{xx}$, where $P_{k|k}^{xx}$ is given by (2.28). The next result presents the solution to the problem of minimizing J_p subject to (4.3).

Proposition 4.1.1. (Simon & Chia, 2002) Let $K_k^p \in \mathbb{R}^{m \times s}$ denoting the projected Kalman gain be given by

$$K_k^p \triangleq W_k D_{k-1}^T (D_{k-1} W_k D_{k-1}^T)^{-1}. \quad (4.16)$$

Then, $\hat{x}_{k|k}^p$ given by

$$\hat{x}_{k|k}^p = \hat{x}_{k|k} + K_k^p(d_{k-1} - D_{k-1}\hat{x}_{k|k}) \quad (4.17)$$

is the minimizer of (4.15) subject to (4.3) with projected error covariance $P_{k|k}^{xxp} \triangleq E[(x_k - \hat{x}_{k|k}^p)(x_k - \hat{x}_{k|k}^p)^T]$ given by

$$P_{k|k}^{xxp} = P_{k|k}^{xx} - K_k^p(D_{k-1}W_kD_{k-1}^T)(K_k^p)^T. \quad (4.18)$$

PKF-EP is formed by forecast (2.20)-(2.24), data assimilation (2.25), (2.27)-(2.28), and projection (4.16)-(4.17) steps; see Figure 4.1. Note that PKF-EP does not recursively feed the projected estimate $\hat{x}_{k|k}^p$ and the projected error covariance $P_{k|k}^{xxp}$ back in forecast in (2.20)-(2.21). Therefore, $P_{k|k}^{xxp}$ need not to be calculated in (4.18).

Furthermore, PKF-EP has the following properties.

Proposition 4.1.2. (Simon & Chia, 2002, Theorem 3) If $W_k = P_{k|k}^{xx}$, where $P_{k|k}^{xx}$ is given by (2.28), then $P_{k|k}^{xxp}$ given by (4.18) is the minimum error covariance.

Proposition 4.1.3. (Simon & Chia, 2002, Theorem 1) For any positive definite $W_k \in \mathbb{R}^{n \times n}$, the PKF-EP estimate $\hat{x}_{k|k}^p$ (4.17) for the system (4.1)-(4.2) subject to the constraint (4.3) is unbiased, that is, $E[x_k|(y_1, \dots, y_k)] - \hat{x}_{k|k}^p = 0_{n \times 1}$.

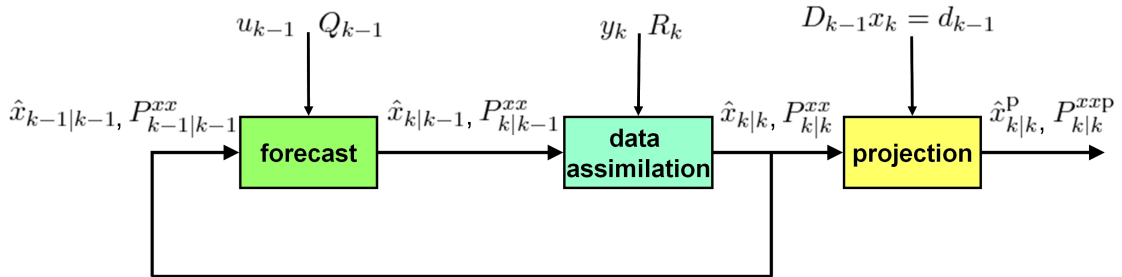


Figure 4.1: Diagram of the projected Kalman filter by estimate projection. The projection step is not connected by feedback recursion to the forecast step.

4.1.4 The Projected Kalman Filter by Gain Projection

The projected Kalman filter by gain projection (PKF-GP) (Gupta & Hauser, 2007; Gupta, 2007a) uses a modified Kalman gain $L_k \in \mathbb{R}^{n \times m}$ to enforce (4.3).

Define the cost function

$$J_g(L_k) \triangleq \text{tr} \left[(I_{n \times n} - L_k C_k) P_{k|k-1}^{xx} (I_{n \times n} - L_k C_k)^T + L_k R_k L_k^T \right], \quad (4.19)$$

where $P_{k|k-1}^{xx}$ is given by (2.21), and L_k is the gain used in

$$\hat{x}_{k|k}^g = \hat{x}_{k|k-1} + L_k (y_k - \hat{y}_{k|k-1}), \quad (4.20)$$

where $\hat{x}_{k|k-1}$ is given by (2.20) and $\hat{y}_{k|k-1}$ is given by (2.22). Note that (4.20) has the same form of (2.27), and that the term in the right-hand side of (4.19) can be written in the same form of $P_{k|k}^{xx}$ in (2.28). Moreover, recall that KF provides the minimizer K_k (2.25) for J_g ; see Theorem 2.2.2. The next result shows that PKF-GP provides the minimizer for J_g subject to

$$D_{k-1} \hat{x}_{k|k}^g = d_{k-1}. \quad (4.21)$$

Using (4.20), (4.21) is written as

$$D_{k-1} L_k (y_k - \hat{y}_{k|k-1}) = d_{k-1} - D_{k-1} \hat{x}_{k|k-1}. \quad (4.22)$$

Proposition 4.1.4. (Gupta & Hauser, 2007, pp. 7–11) The gain L_k given by

$$\begin{aligned} L_k = & K_k + D_{k-1}^T (D_{k-1} D_{k-1}^T)^{-1} (d_{k-1} - D_{k-1} \hat{x}_{k|k}) \\ & \times [(y_k - \hat{y}_{k|k-1})^T (P_{k|k-1}^{yy})^{-1} (y_k - \hat{y}_{k|k-1})]^{-1} (y_k - \hat{y}_{k|k-1})^T (P_{k|k-1}^{yy})^{-1}, \end{aligned} \quad (4.23)$$

where K_k is given by (2.25) and $\hat{x}_{k|k}$ is given (2.27), is the minimizer of J_g (4.19) subject to (4.22).

From (4.23), define

$$K_k^g \triangleq D_{k-1}^T (D_{k-1} D_{k-1}^T)^{-1}. \quad (4.24)$$

Using (2.27) and substituting (4.23) into (4.20) yields (Gupta & Hauser, 2007)

$$\hat{x}_{k|k}^g = \hat{x}_{k|k} + K_k^g (d_{k-1} - D_{k-1} \hat{x}_{k|k}), \quad (4.25)$$

with associated error covariance $P_{k|k}^{xxg} \triangleq \mathbb{E}[(x_k - \hat{x}_{k|k}^g)(x_k - \hat{x}_{k|k}^g)^T]$ given by

$$P_{k|k}^{xxg} = P_{k|k} - K_k^g (D_{k-1} D_{k-1}^T) (K_k^g)^T. \quad (4.26)$$

Therefore, PKF-GP is a three-step algorithm whose forecast step is given by

$$\hat{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1|k-1}^g + B_{k-1} u_{k-1}, \quad (4.27)$$

$$P_{k|k-1}^{xx} = A_{k-1} P_{k-1|k-1}^{xxg} A_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T, \quad (4.28)$$

together with (2.22)-(2.24), whose data-assimilation step is given by (2.25), (2.27)-(2.28), and whose projection step is given by (4.24)-(4.26).

From (4.21), recall that PKF-GP enforces the equality constraint on the state estimate $\hat{x}_{k|k}^g$ and not on the state vector x_k . Such case is called statistical constraint in (Chia, Simon & Chizeck, 2006).

Remark 4.1.2. If we set $W_k = I_{n \times n}$ in the equations of the projection step of PKF-EP given by (4.16)-(4.18), then we obtain the projection step of PKF-GP given by (4.24)-(4.26).

Remark 4.1.3. Unlike PKF-EP, PKF-GP recursively feeds the projected estimate $\hat{x}_{k|k}^g$ (4.25) and the error covariance $P_{k|k}^{xxg}$ given by (4.26) back in forecast (4.27)-(4.28) (Gupta, 2007a); see Figure 4.2.

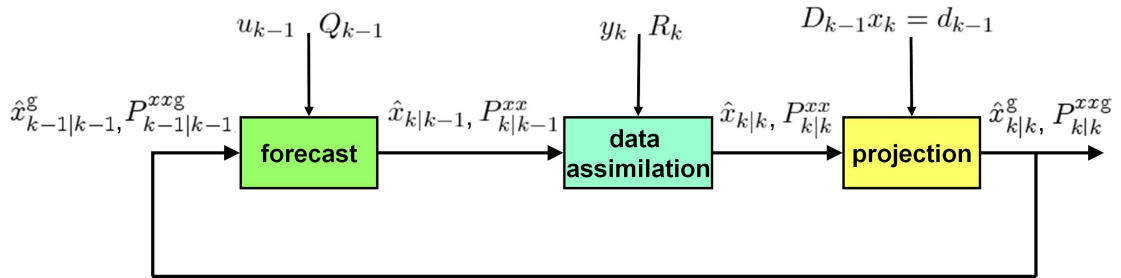


Figure 4.2: Diagram of the projected Kalman filter by gain projection. The projection step is connected by feedback recursion to the forecast step.

4.1.5 The Projected Kalman Filter by System Projection

Consider the time-invariant linear dynamic system

$$x_k = Ax_{k-1} + Bu_{k-1} + Gw_{k-1}, \quad (4.29)$$

$$y_k = Cx_k + v_k. \quad (4.30)$$

Assume that x_k satisfies the equality constraint

$$Dx_k = d, \quad (4.31)$$

such that the following conditions hold²

$$DG = 0_{s \times q}, \quad (4.32)$$

$$DA = D, \quad (4.33)$$

$$DBu_{k-1} = 0_{s \times 1}, \quad \text{for all } k \geq 1. \quad (4.34)$$

Note that, since (4.32) holds, Gw_{k-1} is constrained in the null space of D and $GQ_{k-1}G^T$ is a “constrained” singular covariance (Ko & Bitmead, 2007).

Under the stated assumptions, we consider the projected Kalman filter by system projection (PKF-SP) (Ko & Bitmead, 2005, 2007; Ko, 2005), which uses the KF equations (2.20)-(2.25), (2.27)-(2.28), but initialized with

$$\hat{x}_{0|0}^p = \mathcal{P}_{\mathcal{N}(D)}\hat{x}_{0|0} + D^T(DD^T)^{-1}d, \quad (4.35)$$

$$P_{0|0}^{xxp} = \mathcal{P}_{\mathcal{N}(D)}P_{0|0}^{xx}, \quad (4.36)$$

where the projector

$$\mathcal{P}_{\mathcal{N}(D)} = U_2U_2^T \quad (4.37)$$

with range given by the null space of D is obtained by the singular value decomposition

$$D^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_s \\ 0_{(n-s) \times s} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \quad (4.38)$$

²In (Ko & Bitmead, 2007), (4.32)-(4.34) are written as $Gw_{k-1} \in \mathcal{N}(D)$, $Ax_{k-1} \in \mathcal{N}(D)$, and $Bu_{k-1} \in \mathcal{N}(D)$, respectively, where $\mathcal{N}(D)$ denotes the null space of D .

where $U_2 \in \mathbb{R}^{n \times (n-s)}$. Figure 4.3 illustrates how the forecast, data-assimilation, and projection steps are connected for PKF-SP.

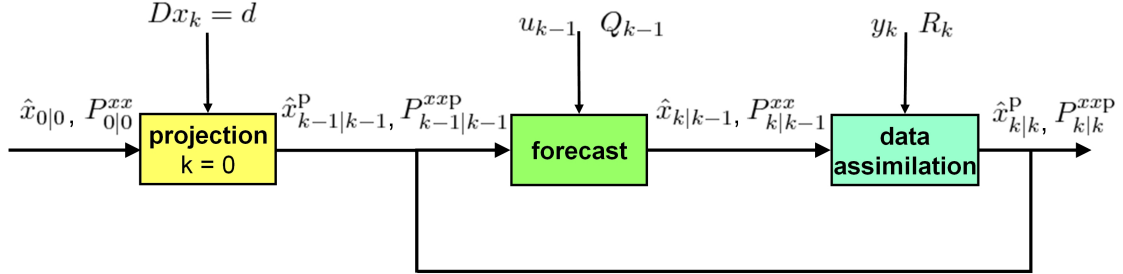


Figure 4.3: Diagram of the projected Kalman filter by system projection. The initial state estimate and associated error covariance carry the information provided by the equality constraint.

Remark 4.1.4. Note that, assuming time-invariant dynamics and providing that (4.32)-(4.34) hold, the projection step is performed only at $k = 0$ (initialization) to guarantee constraint satisfaction for all $k \geq 1$ (Ko & Bitmead, 2007; Teixeira et al., 2007).

4.2 State Estimation for Equality-Constrained Nonlinear Systems

4.2.1 Problem Statement

Now we consider the nonlinear system given by (1.1)-(1.2), whose equations are repeated here for convenience, that is,

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}, k-1), \quad (4.39)$$

$$y_k = h(x_k, v_k, k), \quad (4.40)$$

and whose state vector x_k is known to satisfy the equality constraint

$$g(x_k, k-1) = d_{k-1}, \quad (4.41)$$

where $g : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^s$ and $d_{k-1} \in \mathbb{R}^s$ are assumed to be known. We assume that the mean and covariance of $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are known and equal to zero and Q_{k-1} , R_k , respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated. Our goal is to obtain state estimates satisfying the equality constraint (4.41).

Next, using the analytical linearization approach discussed in Section 2.5.1, the algorithms PKF-EP and MAKF are extended to the nonlinear case. Furthermore, we review an extension of UKF to the equality-constrained case. These extended algorithms provide sub-optimal solutions to the equality-constrained state-estimation problem for nonlinear systems. Unlike the linear case (Section 4.1), these approaches do not guarantee that the nonlinear equality constraint (4.41) is exactly satisfied, but they provide approximate solutions.

4.2.2 The Measurement-Augmentation Extended Kalman Filter

To extend the MAKF algorithm to the nonlinear case (Alouani & Blair, 1993; Chen & Chiang, 1993; Walker, 2006), we replace (4.4) by the augmented observation

$$\tilde{y}_k = \tilde{h}(x_k, \tilde{v}_k, k), \quad (4.42)$$

where $\tilde{y}_k$ and $\tilde{v}_k$ are given by (4.6) and

$$\tilde{h}(x_k, \tilde{v}_k, k) \triangleq \begin{bmatrix} h(x_k, v_k, k) \\ g(x_k, k-1) \end{bmatrix}. \quad (4.43)$$

With (4.42), the measurement-augmentation extended Kalman filter (MAEKF) combines (2.47)-(2.48) with the augmented forecast equations

$$\hat{\tilde{y}}_{k|k-1} = \tilde{h}(\hat{x}_{k|k-1}, 0_{(r+s) \times 1}, k), \quad (4.44)$$

$$P_{k|k-1}^{\tilde{y}\tilde{y}} = \hat{\tilde{C}}_k P_{k|k-1}^{xx} \hat{\tilde{C}}_k^T + \tilde{R}_k, \quad (4.45)$$

$$P_{k|k-1}^{x\tilde{y}} = P_{k|k-1}^{xx} \hat{\tilde{C}}_k^T, \quad (4.46)$$

where $\tilde{R}_k$ is given by (4.10) and $\hat{\tilde{C}}_k$ and $\hat{\tilde{H}}_k$ are given by

$$\hat{\tilde{C}}_k = \left. \frac{\partial \tilde{h}}{\partial x_k} \right|_{x_k = \hat{x}_{k|k-1}, 0_{(r+s) \times 1}, k}, \quad (4.47)$$

$$\hat{\tilde{H}}_k = \left. \frac{\partial \tilde{h}}{\partial \tilde{v}_k} \right|_{x_k = \hat{x}_{k|k-1}, 0_{(r+s) \times 1}, k}. \quad (4.48)$$

Like MAKF, the data-assimilation equations of MAEKF are given by (4.11)-(4.13). Remark 4.1.1 is also applicable here.

Alternatively, the smoothly constrained extended Kalman filter (De Geeter et al.,

1997) can be used. Until a given threshold is achieved, this algorithm iteratively enforces the linearized version of

$$g(x_k, k-1) + v_k^d = d_{k-1} \quad (4.49)$$

with decreasing degree of uncertainty in the covariance of v_k^d . While (4.49) is referred to as *soft equality constraint*, (4.41) is known as *hard equality constraint* (De Geeter et al., 1997; Ko & Bitmead, 2005).

4.2.3 The Projected Extended Kalman Filter by Estimate Projection

The projected extended Kalman filter by estimate projection (PEKF-EP) (Simon & Chia, 2002; Gupta, 2007b) is the nonlinear extension of PKF-EP. Like MAEKF, instead of enforcing (4.41), PEKF-EP enforces the truncated first-order Taylor series expansion of (4.41) given by

$$\hat{D}_{k-1}x_k = \hat{d}_{k-1} \quad (4.50)$$

where

$$\hat{D}_{k-1} = \left. \frac{\partial g}{\partial x_k} \right|_{x_k = \hat{x}_{k|k}, k-1}, \quad (4.51)$$

$$\hat{d}_{k-1} \triangleq d_{k-1} + \hat{D}_{k-1}\hat{x}_{k|k} - g(\hat{x}_{k|k}, k-1), \quad (4.52)$$

in which the linearization is taken at the current state estimate $\hat{x}_{k|k}$ (2.27).

With (4.50), the projection equations of PEKF-EP are given by

$$K_k^p = W_k \hat{D}_{k-1}^T (\hat{D}_{k-1} W_k \hat{D}_{k-1}^T)^{-1}, \quad (4.53)$$

$$\hat{x}_{k|k}^p = \hat{x}_{k|k} + K_k^p (d_{k-1} - g(\hat{x}_{k|k}, k-1)), \quad (4.54)$$

$$P_{k|k}^{xxp} = P_{k|k}^{xx} - K_k^p (\hat{D}_{k-1} W_k \hat{D}_{k-1}^T) (K_k^p)^T. \quad (4.55)$$

Similar to the linear case, we set $W_k = P_{k|k}^{xx}$; see Proposition 4.1.2.

PEKF-EP is formed by forecast (2.47)-(2.51), data assimilation (2.25), (2.27)-(2.28), and projection (4.53)-(4.54) steps; see Figure 4.1. Note that, (4.55) need not to be implemented.

4.2.4 The Two-Step Projected Unscented Kalman Filter

In (Julier & LaViola Jr., 2007), a method that uses the projection approach twice is presented. In this thesis, we refer to this method as the two-step projected unscented Kalman filter (TPUKF). First, TPUKF enforces the equality constraint (4.41) on each sigma point during the forecast step. Then, after data assimilation, in the projection step, TPUKF projects the data-assimilation state estimate onto the hypersurface (4.41).

Let $\hat{x}_{k|k}^p = o(\hat{x}_{k|k}, k)$, where $o : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^n$, denote a projection function whose argument $\hat{x}_{k|k}$ is projected onto the hypersurface g (4.41) at time k yielding $\hat{x}_{k|k}^p$. In (Julier & LaViola Jr., 2007) it is not shown how to obtain a specific projection function o for a given equality constraint (4.41). The next example illustrates a case investigated in the aforementioned paper.

Example 4.2.1 Assume that $x_k \in \mathbb{R}^2$ is known to have unitary norm, that is, g (4.41) is given by $\sqrt{x_1^2 + x_2^2} = 1$. Then, the projection function o to enforce (4.41) on the state estimate can be chosen as $o(\hat{x}_{k|k}, k) = \frac{\hat{x}_{k|k}}{\sqrt{\hat{x}_{1,k|k}^2 + \hat{x}_{2,k|k}^2}}$, that is, $\hat{x}_{k|k}$ is normalized. $\square$

The forecast equations of TPUKF are given by

$$\begin{bmatrix} \gamma_x, \gamma_P, \tilde{\mathcal{X}}_{k-1|k-1} \end{bmatrix} = \Psi_{UT} \left(\hat{x}_{k-1|k-1}^p, P_{k-1|k-1}^{\tilde{x}\tilde{x}p}, \tilde{n}, \alpha, \beta, \kappa \right), \quad (4.56)$$

$$\mathcal{X}_{j,k-1|k-1}^{xp} = o(\mathcal{X}_{j,k-1|k-1}^x, k-1), \quad j = 0, \dots, 2\tilde{n}, \quad (4.57)$$

$$\mathcal{X}_{j,k|k-1}^x = f(\mathcal{X}_{j,k-1|k-1}^{xp}, u_{k-1}, \mathcal{X}_{j,k-1|k-1}^w, k-1), \quad j = 0, \dots, 2\tilde{n}, \quad (4.58)$$

together with (2.91)-(2.96), where Ψ_{UT} is defined by (2.86), and the augmented state vector and augmented error covariance are defined using the projected state estimate $\hat{x}_{k-1|k-1}^p$ and error covariance $P_{k-1|k-1}^{xxp}$ from the last iteration, that is,

$$\tilde{x}_{k-1}^p \triangleq \begin{bmatrix} x_{k-1}^p \\ w_{k-1} \\ v_k \end{bmatrix}, \quad P_{k-1|k-1}^{\tilde{x}\tilde{x}p} \triangleq \begin{bmatrix} P_{k-1|k-1}^{xxp} & 0_{n \times q} & 0_{n \times r} \\ 0_{q \times n} & Q_{k-1} & 0_{q \times r} \\ 0_{r \times n} & 0_{r \times q} & R_k \end{bmatrix}. \quad (4.59)$$

Like in UKF, the data-assimilation equations of TPUKF are given by (2.25), (2.27)-(2.28), while the projection step of TPUKF is given by

$$\hat{x}_{k|k}^p = o(\hat{x}_{k|k}, k), \quad (4.60)$$

$$P_{k|k}^{xxp} = P_{k|k}^{xx} + (\hat{x}_{k|k}^p - \hat{x}_{k|k})(\hat{x}_{k|k}^p - \hat{x}_{k|k})^T, \quad (4.61)$$

where $\hat{x}_{k|k}$ and $P_{k|k}^{xx}$ are obtained from the data-assimilation step.

Note that, unlike PEKF-EP, the projected step of TPUKF is projected by feedback recursion to the forecast step. Furthermore, note that each sigma point $\mathcal{X}_{j,k-1|k-1}^x$, $j = 0, \dots, 2\tilde{n}$, is projected onto g (4.41) in (4.57) such that both their sample mean and covariance are changed. Nonetheless, their sample mean (state estimate) does not satisfy (4.41). In (4.60), the equality constraint (4.41) is enforced directly on the state estimate. Depending on the choice of o , $\hat{x}_{k|k}^P$ may exactly satisfy (4.41); however, this information is only approximately assimilated in $P_{k|k}^{xxP}$ (4.61).

4.3 Quadratic Programming Approach

In Section 2.1 the state-estimation problem is developed using the recursive Bayesian approach. Now, we recast the state-estimation problem using a quadratic programming perspective in order to show how to enforce inequality constraints on the state estimates. Such perspective highlights the duality between state estimation and optimal control (Cox, 1964; Jazwinski, 1970). However, rather than employing the Hamilton-Jacobi equations or the Pontryagin's principle (Gelfand & Fomin, 1963; Kirk, 1970; Anderson & Moore, 1989) to find a solution to the state-estimation problem, we solely focus on the maximum-a-posteriori Bayesian criterion.

Define the functional

$$\Phi(x_1, \dots, x_T) \triangleq \rho((x_1, \dots, x_T)|(y_1, \dots, y_T)), \quad (4.62)$$

where $\rho((x_1, \dots, x_T)|(y_1, \dots, y_T))$ is the joint conditional posterior PDF of the trajectory of the state vector $x_1, \dots, x_T$ given the trajectory of measured data $y_1, \dots, y_T$, and T is the length of the trajectory. Recall that $\rho(x_k|(y_1, \dots, y_k))$ is a marginal PDF in (1.3) (Jazwinski, 1970).

Definition 4.3.1. The maximizer of Φ (4.62) is the optimal trajectory of state estimates $\hat{x}_{1|T}, \dots, \hat{x}_{T|T}$.

Using Bayes' theorem (Papoulis & Pillai, 2001) in (4.62) yields

$$\rho((x_1, \dots, x_T)|(y_1, \dots, y_T)) = \frac{\rho((y_1, \dots, y_T)|(x_1, \dots, x_T))\rho(x_1, \dots, x_T)}{\rho(y_1, \dots, y_T)}. \quad (4.63)$$

Note that only the terms in the numerator of (4.63) are proportional to $x_1, \dots, x_T$. Using the

Markovian property of (4.39) and the fact that $v_1, \dots, v_T$ are mutually independent, then such terms are given by

$$\rho((y_1, \dots, y_T)|(x_1, \dots, x_T)) = \prod_{k=1}^T \rho_{v_k}(v_k^*), \quad (4.64)$$

$$\begin{aligned} \rho(x_1, \dots, x_T) &= \rho_{x_0}(x_0) \prod_{k=1}^T \rho(x_k|x_{k-1}) \\ &= \rho_{x_0}(x_0) \prod_{k=1}^T \rho_{w_{k-1}}(w_{k-1}^*), \end{aligned} \quad (4.65)$$

where $\rho(x_k|x_{k-1})$ is the state-transition PDF given by (2.2), $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are, respectively, the process and measurement noise PDFs, $\rho_{x_0}(x_0)$ is the initial state PDF, w_{k-1}^* is obtained from the difference equation $x_k = f(x_{k-1}, u_{k-1}, w_{k-1}^*, k-1)$, and v_k^* is obtained from $y_k = h(x_k, v_k^*, k)$.

Lemma 4.3.1 For the system (4.39)-(4.40), $\hat{x}_{1|T}, \dots, \hat{x}_{T|T}$ maximize Φ given by (4.62), if and only if $\hat{x}_{1|T}, \dots, \hat{x}_{T|T}$ minimize

$$\Phi_1(x_1, \dots, x_T) \triangleq -\ln \rho_{x_0}(x_0) - \sum_{k=1}^T [\ln \rho_{w_{k-1}}(w_{k-1}^*) + \ln \rho_{v_k}(v_k^*)]. \quad (4.66)$$

Proof. Given that $-\ln$ is strictly decreasing and using (4.64)-(4.65), maximizing (4.62) is equivalent to minimizing the negative of the logarithm of the numerator of (4.63), that is, it is equivalent to minimizing (4.66). $\square$

For simplicity, we make the following assumptions for the remainder of this section. First, we consider the stochastic nonlinear discrete-time dynamic system with additive noise

$$x_k = f(x_{k-1}, u_{k-1}, k-1) + G_{k-1}w_{k-1}, \quad (4.67)$$

$$y_k = h(x_k, k) + v_k, \quad (4.68)$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{N} \rightarrow \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^m$, and $G_{k-1} \in \mathbb{R}^{n \times q}$. Second, we assume that $\rho_{w_{k-1}}(w_{k-1})$, $\rho_{v_k}(v_k)$, and $\rho_{x_0}(x_0)$ are Gaussian and, respectively, given by (2.9)-(2.11).

Lemma 4.3.2 For the system (4.67)-(4.68), $\hat{x}_{1|T}, \dots, \hat{x}_{T|T}$ maximize Φ given by (4.62),

if and only if $\hat{x}_{1|T}, \dots, \hat{x}_{T|T}$ minimize

$$\begin{aligned} \Phi_2(x_1, \dots, x_T) \triangleq & (x_0 - \hat{x}_{0|0})^T (P_{0|0}^{xx})^{-1} (x_0 - \hat{x}_{0|0}) + \sum_{k=1}^T \left[(y_k - h(x_k, k))^T R_k^{-1} (y_k - h(x_k, k)) \right. \\ & \left. + (x_k - f(x_k, u_{k-1}, k-1))^T \tilde{Q}_{k-1}^{-1} (x_k - f(x_k, u_{k-1}, k-1)) \right], \end{aligned} \quad (4.69)$$

where

$$\tilde{Q}_{k-1} \triangleq G_{k-1} Q_{k-1} G_{k-1}^T. \quad (4.70)$$

Proof. Using Lemma 4.3.1 and substituting (4.67)-(4.68) and (2.9)-(2.11) into (4.66) yields (4.69). $\square$

The minimization of Φ_2 (4.69) is a *quadratic programming problem*, while the sequence of state estimates $\hat{x}_{1|T}, \dots, \hat{x}_{T|T}$ is the minimizing trajectory of Φ_2 . Solving this problem, however, is computationally demanding, because its dimension grows with time as more measurements are processed. The moving horizon approach can be used to reduce the computational burden of the state-estimation problem by bounding its size (Muske, Rawlings & Lee, 1993; Michalska & Mayne, 1995). That is, the number of measurements to be processed is fixed to M , where $M \geq 1$, such that the old measurements are discarded from the estimation window in order to obtain $\hat{x}_{T-M+1|T}, \dots, \hat{x}_{T|T}$; see Figure 4.4. Recall that, in Section 2.1, we have $M = 1$.

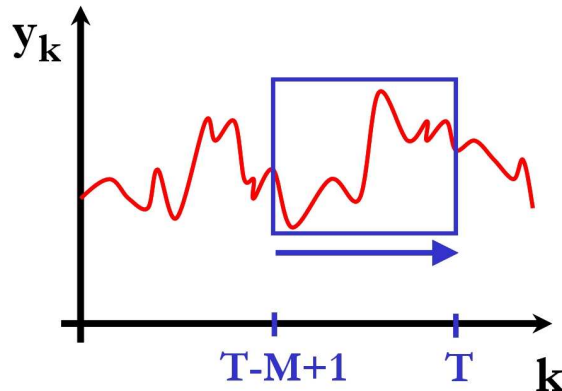


Figure 4.4: Diagram of the moving horizon approach. It is emphasized how the moving window of size M moves with time. Adapted from (Rao & Rawlings, 2001).

Let the function $\mathcal{Z}_{T-M}(x_{T-M})$ denote the *arrival cost* (Striebel, 1965) which summa-

izes the effect of the data $y_1, \dots, y_{T-M}$ on the state vector x_{T-M} .³

Lemma 4.3.3 For the system (4.67)-(4.68), $\hat{x}_{T-M+1|T}, \dots, \hat{x}_{T|T}$ maximize Φ given by (4.62), if and only if $\hat{x}_{T-M+1|T}, \dots, \hat{x}_{T|T}$ minimize

$$\begin{aligned} \Phi_3(x_{T-M+1}, \dots, x_T) \triangleq & \mathcal{Z}_{T-M}(x_{T-M}) + \sum_{k=T-M+1}^T \left[(y_k - h(x_k, k))^T R_k^{-1} (y_k - h(x_k, k)) \right. \\ & \left. + (x_k - f(x_k, u_{k-1}, k-1))^T \tilde{Q}_{k-1}^{-1} (x_k - f(x_k, u_{k-1}, k-1)) \right], \end{aligned} \quad (4.71)$$

where $\tilde{Q}_{k-1}$ is given by (4.70)

Proof. Using Lemma 4.3.2, we can rearrange Φ_2 (4.69) by breaking the time interval into two intervals $[1, T-M]$ and $[T-M+1, T]$. Then, using the optimality principle (Bertsekas, 2001a,b; Anderson & Moore, 1989), it can be shown that minimizing (4.69) is equivalent to minimizing (4.71). For details, see (Rao, 2000; Rao et al., 2003). $\square$

Since the quadratic programming problem mentioned above is optimization based, it is straightforward to handle constraints on the decision variables $x_{T-M+1}, \dots, x_T$, and even on the noise sequences $w_{T-M}, \dots, w_{T-1}$ and $v_{T-M+1}, \dots, v_T$ (Rao, 2000). In this thesis, however, we are concerned to enforce constraints only on the state vector. We define the constrained state-estimation problem from the quadratic programming perspective as follows.

Definition 4.3.2. The minimization of Φ_3 (4.71) subject to (1.4) and (1.5), for $k = T-M+1, \dots, T$, where $M \geq 1$, is the *constrained state-estimation problem*. The sequence of constrained state estimates $\hat{x}_{T-M+1|T}^c, \dots, \hat{x}_{T|T}^c$ is the constrained minimizing trajectory of Φ_3 .

4.4 State Estimation for Inequality-Constrained Linear Systems

4.4.1 Problem Statement

We consider the linear system given by (4.1)-(4.2), whose state vector is known to satisfy the inequality constraint

$$E_{k-1}x_k \leq e_{k-1}, \quad (4.72)$$

³ $\mathcal{Z}_{T-M}(x_{T-M})$ depends on $\rho(x_{T-M}|y_1, \dots, y_{T-M})$ and is proportional to the uncertainty of the estimate $\hat{x}_{T-M|T-M}$ with relation to the true value x_{T-M} (Rao, 2000).

where $E_{k-1} \in \mathbb{R}^{t \times n}$ and $e_{k-1} \in \mathbb{R}^t$ are assumed to be known. Moreover, we assume that $\rho_{w_{k-1}}(w_{k-1})$, $\rho_{v_k}(v_k)$, and $\rho_{x_0}(x_0)$ are Gaussian, uncorrelated, and, respectively, given by (2.9)-(2.11). Our goal is to obtain state estimates that satisfy the inequality constraint (4.72).

We also consider the case where x_k satisfies the interval constraint

$$a_k \leq x_k \leq b_k, \quad (4.73)$$

where $a_k \in \mathbb{R}^n$ and $b_k \in \mathbb{R}^n$ are assumed to be known and, for $j = 1, \dots, n$, $a_{j,k} < b_{j,k}$. Also, if $x_{j,k}$ is left-unbounded or right-unbounded, then we set $a_{j,k} = -\infty$ or $b_{j,k} = \infty$, respectively. Note that (4.73) is a special case of (4.72) with $E_{k-1} = \text{diag}(I_{n \times n}, -I_{n \times n})$ and $e_{k-1} = \begin{bmatrix} b_k \\ -a_k \end{bmatrix}$.

Remark 4.4.1. Note that Gaussian noise and inequality-constrained state vector are mutually exclusive assumptions even for linear systems (Rao et al., 2001, 2003; Goodwin et al., 2005). This is not true for equality-constrained systems (Teixeira et al., 2008a). Rao (2000, pp. 13-15) presents an illustrative example which shows that inequality constraints on the state vector may substantially alter the structure of the state-estimation problem. Therefore, unless an appropriate distribution is considered for the noise terms and state vector, the methods for inequality-constrained state estimation are approximate.

The four suboptimal methods reviewed in this section can be used to enforce the equality constraint (4.3) in addition to (4.72) and (4.73).

4.4.2 The Moving Horizon Estimator

The moving horizon estimator (MHE) (Muske, Rawlings & Lee, 1993; Robertson, Lee & Rawlings, 1996; Rao & Rawlings, 2001; Rao, Rawlings & Lee, 2001) provides a suboptimal solution to the inequality-constrained state-estimation problem for linear systems.

From Lemma 4.3.3, MHE solves the constrained quadratic programming problem given by

$$\hat{x}_{T-M+1|T}^c, \dots, \hat{x}_{T|T}^c = \arg \min_{\{x_k : E_{k-1}x_k \leq e_{k-1}, \quad k = T-M+1, \dots, T\}} \Phi_4(x_{T-M+1}, \dots, x_T), \quad (4.74)$$

where the functional Φ_4 is given by

$$\begin{aligned} \Phi_4(x_{T-M+1}, \dots, x_T) \triangleq & \hat{\mathcal{Z}}_{T-M}(x_{T-M}) + \sum_{k=T-M+1}^T \left[(y_k - C_k x_k)^T R_k^{-1} (y_k - C_k x_k) + \right. \\ & \left. [x_k - (A_{k-1} x_{k-1} + B_{k-1} u_{k-1})]^T \tilde{Q}_{k-1}^{-1} [x_k - (A_{k-1} x_{k-1} + B_{k-1} u_{k-1})] \right], \end{aligned} \quad (4.75)$$

where $\tilde{Q}_{k-1}$ is given by (4.70), and $\hat{\mathcal{Z}}_{T-M}(x_{T-M})$ is an approximation to the arrival cost $\mathcal{Z}_{T-M}(x_{T-M})$, which is shown in (4.71), and is given by

$$\hat{\mathcal{Z}}_{T-M}(x_{T-M}) = (x_{T-M} - \hat{x}_{T-M|T-M}^c)^T (P_{T-M|T-M}^{xx})^{-1} (x_{T-M} - \hat{x}_{T-M|T-M}^c), \quad (4.76)$$

where $\hat{x}_{T-M|T-M}^c$ is the state estimate obtained in the last iteration and $P_{T-M|T-M}^{xx}$ is approximated using the KF equations (2.21), (2.23)-(2.25), (2.28).

Note that $P_{T-M|T-M}^{xx}$ does not assimilate the additional information provided by (4.72). From (4.74), recall that MHE is a one-step algorithm.

The horizon size M is chosen to give a tradeoff between estimation accuracy and computational effort. Though the MHE formulation is attractive, its implementation requires a larger computational effort than most other inequality-constrained approaches even for small horizon lengths (Simon, 2008).

Results on the stability of MHE are found in (Rao et al., 2001).

4.4.3 The Constrained Kalman Filter

For the unconstrained linear case, MHE is equivalent to KF. In (Rao, 2000, Appendix D), by setting $M = 1$, it is obtained the analytical unconstrained solution to (4.74) yielding the KF equations (2.20)-(2.25), (2.27)-(2.28). Note that, in this case, Lemma 4.3.3 is similar to Lemma 2.2.1.

The constrained Kalman filter (CKF) is a special case of MHE with unitary horizon, that is, $M = 1$. Its forecast equations are given by

$$\hat{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1|k-1}^c + B_{k-1} u_{k-1}, \quad (4.77)$$

together with (2.21), (2.23)-(2.24). The data-assimilation equations of CKF are given by

$$\hat{x}_{k|k}^c = \arg \min_{\{x_k : E_{k-1}x_k \leq e_{k-1}\}} \mathcal{J}(x_k), \quad (4.78)$$

together with (2.25) and (2.28), where $\mathcal{J}(x_k)$ is given by (2.26).

Note that the inequality-constraint information is not assimilated by the error-covariance $P_{k|k}^{xx}$ in (2.28) such that, unlike KF, CKF is a suboptimal algorithm; see Remark 4.4.1.

Various methods can be used to solve (4.78) (Fletcher, 2000). Alternatively, an iterative solver to (4.78) is used in (Ungarala et al., 2007). In this case, with the use of slack variables, the inequality constraint (4.72) is iteratively replaced by equality constraints.

4.4.4 The Projected Kalman Filter for Inequality-Constrained Systems

PKF-EP can be extended to enforce the inequality constraint (4.72) yielding the projected Kalman filter for inequality-constrained systems (PKF-IC) (Simon & Simon, 2006a,b; Simon, 2006). In this case, instead of using (4.16)-(4.18), we solve online the constrained quadratic programming problem given by

$$\hat{x}_{k|k}^p = \arg \min_{\{x_k : E_{k-1}x_k \leq e_{k-1}\}} J_p(x_k), \quad (4.79)$$

where $J_p(x_k)$ is given by (4.15). The resulting algorithm is formed by forecast (2.20)-(2.24), data assimilation (2.25), (2.27)-(2.28), and projection (4.79) steps, which are connected as shown in Figure 4.1.

In Matlab, the standard quadratic programming routine **quadprog** can be readily used to solve (4.79). Simon (2006) suggests the use of an active set method (Fletcher, 1981) to solve (4.79). Such methods use the fact that only the constraints that are active at the solution of the optimization problem are significant in the optimality conditions. If the set of active constraints is known, then the solution to the problem (4.79) is equivalent to the minimizer of $J_p(x_k)$ subject to the equality constraint formed by the active constraints of (4.72). That is, the inequality-constrained problem is replaced by an equality-constrained problem.

Alternatively, in (Gupta & Hauser, 2007), PKF-GP is extended to the inequality-constrained case. In this case, the Kalman gain is required to satisfy an inequality constraint.

4.4.5 The Truncated Kalman Filter

In this section, we consider the case in which x_k is known to satisfy the interval constraint (4.73).

Let $\hat{x}_{k|k}$ given by (2.27) and $P_{k|k}^{xx}$ given by (2.28) be, respectively, the mean and covariance of $\rho(x_k|(y_1, \dots, y_k))$ (2.18) obtained from KF; see Section 2.2.1. We want to truncate $\rho(x_k|(y_1, \dots, y_k))$ at the n constraint edges given by the rows of (4.73) such that the mean $\hat{x}_{k|k}^t$ of the truncated PDF is an interval-constrained state estimate with truncated error covariance $P_{k|k}^{xxt}$. This procedure is called PDF truncation (Shimada et al., 1998; Simon, 2006; Simon & Simon, 2007) and is illustrated in the following example.

Example 4.4.1 Consider the case where, even though $\hat{x}_{k|k} = [1 \ 1]^T$ satisfies the interval constraint (4.73) with parameters $a_k = [0 \ -1]^T$ and $b_k = [3 \ 1.75]^T$, $P_{k|k}^{xx} = I_{2 \times 2}$ has significant area outside (4.73); as shown in Figure 4.5. Therefore, $\hat{x}_{k|k}^t = [1.23 \ 0.67]^T$ is obtained by shifting $\hat{x}_{k|k}$ towards the centroid of the truncated PDF and $P_{k|k}^{xxt} = \text{diag}(0.52, 0.45)$ is obtained by truncating $P_{k|k}^{xx}$ due to the prior knowledge provided by (4.73). $\square$

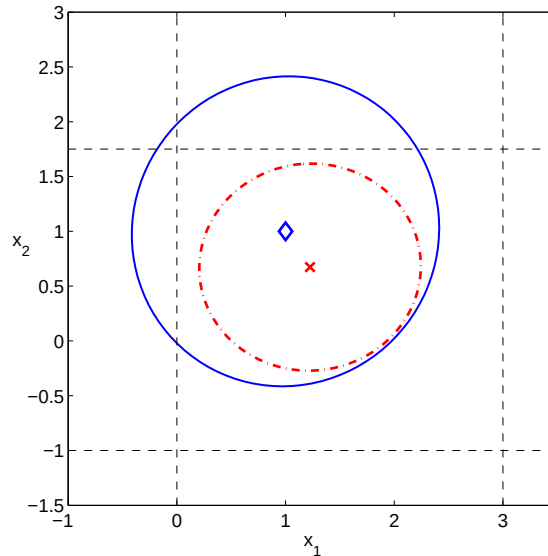


Figure 4.5: Illustrative example of the truncation approach. The unconstrained estimate $\hat{x}_{k|k}$ ($\diamond$) satisfying the interval constraint (4.73) ($- -$) and its covariance $P_{k|k}^{xx}$ ($—$) with significant area outside (4.73) compared to the truncated estimate $\hat{x}_{k|k}^t$ ($\times$) and covariance $P_{k|k}^{xxt}$ ($- \cdot -$).

The PDF-Truncation Procedure

The PDF-truncation procedure is performed in i steps, where $i = 1, \dots, n$. Let $\hat{x}_{k|k,i}^t$ and $P_{k|k,i}^{xxt}$ denote, respectively, the state estimate and the error covariance after enforcing the first $i - 1$ rows of (4.73).

First, at $i = 1$, initialize

$$\hat{x}_{k|k,1}^t = \hat{x}_{k|k}, \quad (4.80)$$

$$P_{k|k,1}^{xxt} = P_{k|k}^{xx}, \quad (4.81)$$

where $\hat{x}_{k|k}$ is given by (2.27) and $P_{k|k}^{xx}$ is given by (2.28). Then, for $i = 1, \dots, n$, perform the following four-step loop: (i) find the orthogonal matrix $S \in \mathbb{R}^{n \times n}$ and the diagonal matrix $D \in \mathbb{R}^{n \times n}$ from the Schur decomposition (Bernstein, 2005) of $P_{k|k,i}^{xxt}$ given by

$$SDS^T = P_{k|k,i}^{xxt}, \quad (4.82)$$

where $P_{k|k,i}^{xxt}$ is symmetric by definition; (ii) perform Gram-Schmidt orthogonalization to find the orthogonal matrix $\Theta \in \mathbb{R}^{n \times n}$ satisfying

$$\Theta D^{1/2} \text{col}_i(S^T) = \begin{bmatrix} \sqrt{P_{(i,i),k|k,i}^{xxt}} & 0_{1 \times (n-1)} \end{bmatrix}^T, \quad (4.83)$$

which is given by

$$\text{row}_l(\Theta) = \begin{cases} \frac{1}{\sqrt{P_{(i,i),k|k,i}^{xxt}}} \text{row}_i(S) D^{1/2}, & l = 1, \\ \left(e_l - \sum_{q=1}^{l-1} \left(e_l^T \text{col}_q(\Theta^T) \right) \text{col}_q(\Theta^T) \right)^T, & l = 2, \dots, n, \end{cases} \quad (4.84)$$

where $e_l \triangleq \text{col}_l(I_{n \times n})$, if $\text{row}_l(\Theta) = 0_{1 \times n}$, then set

$$\text{row}_l(\Theta) = \left(e_l - \sum_{q=1}^{l-1} \left(e_l^T \text{col}_q(\Theta^T) \right) \text{col}_q(\Theta^T) \right)^T; \quad (4.85)$$

finally, normalize

$$\text{row}_l(\Theta) = \frac{1}{\|\text{row}_l(\Theta)\|_2} \text{row}_l(\Theta), \quad l = 1, \dots, n; \quad (4.86)$$

(iii) find the parameters of the interval constraint $\alpha_{k,i} \leq z_{i,k,i} \leq \beta_{k,i}$, where $\alpha_{k,i} < \beta_{k,i} \in \mathbb{R}$ are given by

$$\alpha_{k,i} = \frac{1}{\sqrt{P_{(i,i),k|i,i}^{xxt}}}(a_{i,k} - x_{i,k|i,i}^t), \quad (4.87)$$

$$\beta_{k,i} = \frac{1}{\sqrt{P_{(i,i),k|i,i}^{xxt}}}(b_{i,k} - x_{i,k|i,i}^t), \quad (4.88)$$

and $z_{k,i} \triangleq \Theta D^{-1/2} S^T (x_k - \hat{x}_{k|i,i}^t) \in \mathbb{R}^n$ with mean

$$\hat{z}_{k,i} = \begin{bmatrix} \mu_i & 0_{1 \times (n-1)} \end{bmatrix}^T \quad (4.89)$$

and covariance

$$P_{k,i}^{zz} = \text{diag}(\sigma_i^2, 1_{1 \times (n-1)}), \quad (4.90)$$

where

$$\nu_i = \frac{\sqrt{2}}{\sqrt{\pi} \left[\text{erf} \left(\frac{\beta_{k,i}}{\sqrt{2}} \right) - \text{erf} \left(\frac{\alpha_{k,i}}{\sqrt{2}} \right) \right]}, \quad (4.91)$$

$$\mu_i = \nu_i \left[\exp \left(\frac{-\alpha_{k,i}^2}{2} \right) - \exp \left(\frac{-\beta_{k,i}^2}{2} \right) \right], \quad (4.92)$$

$$\sigma_i^2 = \nu_i \left[\exp \left(\frac{-\alpha_{k,i}^2}{2} \right) (\alpha_{k,i} - 2\mu_i) - \exp \left(\frac{-\beta_{k,i}^2}{2} \right) (\beta_{k,i} - 2\mu_i) \right] + \mu_i^2 + 1, \quad (4.93)$$

where the error function $\text{erf}(\cdot)$ is given by

$$\text{erf}(t) \triangleq \frac{2}{\sqrt{\pi}} \int_0^t \exp(-\tau^2) d\tau; \quad (4.94)$$

and (iv) perform the inverse transformation

$$\hat{x}_{k|i,i+1}^t = SD^{1/2} \Theta^T \hat{z}_{k,i} + \hat{x}_{k|i,i}^t, \quad (4.95)$$

$$P_{k|i,i+1}^{xxt} = SD^{1/2} \Theta^T P_{k,i}^{zz} \Theta D^{1/2} S^T. \quad (4.96)$$

Finally, at $i = n$, we set

$$\hat{x}_{k|k}^t = \hat{x}_{k|k,n+1}^t, \quad (4.97)$$

$$P_{k|k}^{xxt} = P_{k|k,n+1}^{xxt}. \quad (4.98)$$

Note that, since the components of $z_{k,i}$ are uncorrelated to each other – see (4.90) – and only the first component of $z_{k,i}$ is constrained by (4.87)-(4.88), we have a one-dimensional PKF truncation in (4.89)-(4.93). The truncation is performed by removing the portion of the assumed Gaussian PDF of $z_{k,i}$ that is outside of the constraint (4.87)-(4.88).

For simplicity, henceforth, we refer to (4.80)-(4.98) as the function Γ_{trunc} , which is defined by

$$\left[\hat{x}_{k|k}^t, P_{k|k}^{xxt} \right] = \Gamma_{\text{trunc}} \left(\hat{x}_{k|k}, P_{k|k}^{xx}, n, a_k, b_k \right). \quad (4.99)$$

The Algorithm of the Truncated Kalman Filter

Now we present the equations of the truncated Kalman filter (TKF) (Simon, 2006). TKF is obtained by appending the PDF-truncation procedure to the KF equations by feedback recursion; see Figure 4.6. That is, TKF is a three-step algorithm whose forecast step is given by

$$\hat{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1|k-1}^t + B_{k-1} u_{k-1}, \quad (4.100)$$

$$P_{k|k-1}^{xx} = A_{k-1} P_{k-1|k-1}^{txx} A_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T, \quad (4.101)$$

together with (2.22)-(2.24), whose data-assimilation step is given by (2.25), (2.27)-(2.28), and whose *truncation* step is given by (4.99). Note that TKF avoids the explicit online solution of a constrained optimization problem at each time step.

Remark 4.4.2. Unlike MHE, CKF, and PKF-IC, TKF assimilates the interval-constraint information in both the state estimate $\hat{x}_{k|k}^t$ and the error covariance $P_{k|k}^{xxt}$.

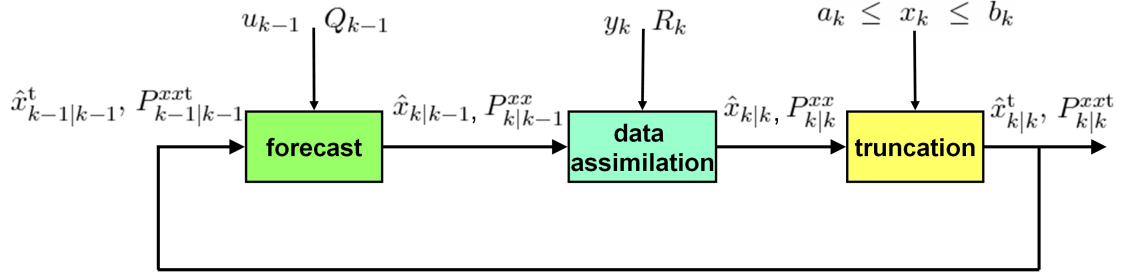


Figure 4.6: Diagram of the truncated Kalman filter. The truncation step is connected by feedback recursion to the forecast step.

4.5 State Estimation for Inequality-Constrained Nonlinear Systems

4.5.1 Problem Statement

In this section, we review three suboptimal algorithms for inequality-constrained state estimation for nonlinear systems, whose state vector is known to satisfy

$$l(x_k, k-1) \leq e_{k-1}, \quad (4.102)$$

where $l : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}^t$ and $e_{k-1} \in \mathbb{R}^t$ are assumed to be known. Since each method assumes specific kinds of dynamics and measurement map, we clarify which model we are considering before presenting each algorithm. In so doing, we introduce each algorithm in the most general fashion. Moreover, for all cases, we assume that the mean and covariance of $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are known and equal to zero and Q_{k-1} , R_k , respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated.

Our goal is to obtain state estimates that satisfy the inequality constraint (4.102). Using the three methods reviewed in this section, it is possible to enforce the equality constraint (4.41) in addition to (4.102).

4.5.2 The Nonlinear Moving Horizon Estimator

Consider the system (4.67)-(4.68) whose state vector is known to satisfy the inequality constraint (4.102). By combining the MHE equations (see Section 4.4.2) with the EKF equations (Section 2.5.2), we obtain the nonlinear moving horizon estimator (NMHE) (Rao et al., 2003) as a suboptimal solution to the inequality-constrained state-estimation problem

for nonlinear systems.

NMHE is a one-step algorithm that solves the constrained quadratic programming problem given by

$$\hat{x}_{T-M+1|T}^c, \dots, \hat{x}_{T|T}^c = \arg \min_{\{x_k : l(x_k, k-1) \leq e_{k-1}, \quad k = T-M+1, \dots, T\}} \Phi_3(x_{T-M+1}, \dots, x_T), \quad (4.103)$$

where the functional Φ_3 is given by (4.71) with arrival cost $\hat{Z}_{T-M}(x_{T-M})$ approximated by (4.76), where $P_{T-M|T-M}^{xx}$ is obtained from the EKF equations (2.48), (2.50)-(2.51), (2.25), (2.28).

Remark 4.5.1. Results on the stability of NMHE are found in (Rao et al., 2003; Rao, 2000). Roughly speaking, $\hat{Z}_{T-M}(x_{T-M}) \leq Z_{T-M}(x_{T-M})$ is one of the sufficient conditions for the stability of NMHE (Rao et al., 2003, Proposition 3.4). That is, $\hat{Z}_{T-M}(x_{T-M})$ cannot carry more information than the information of the true $Z_{T-M}(x_{T-M})$. From (4.76), this implies that, if the pseudo-error covariance $P_{T-M|T-M}^{xx}$ obtained from the EKF equations is underestimated (in other words, inconsistent – see (2.34)), then the estimation errors may not be bounded.

Most reported applications of NMHE are in the chemical industry, such as constrained state estimation for polymerization processes (Gesthuisen & Engell, 1998; Russo & Young, 1999).

4.5.3 The Constrained Extended Kalman Filter

Now, we consider the system (4.39), (4.68) with the state vector satisfying (4.102). The constrained extended Kalman filter (CEKF) is a special case of NMHE with unitary moving horizon. The forecast equations of CEKF are given by

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}^c, u_{k-1}, 0_{q \times 1}, k-1), \quad (4.104)$$

together with (2.48), (2.50)-(2.51), while the data-assimilation equations are given by

$$\hat{x}_{k|k}^c = \arg \min_{\{x_k : l(x_k, k-1) \leq e_{k-1}\}} \mathcal{J}_1(x_k), \quad (4.105)$$

together with (2.25) and (2.28), where $\mathcal{J}_1(x_k)$ is given by

$$\mathcal{J}_1(x_k) \triangleq (x_k - \hat{x}_{k|k-1})^T (P_{k|k-1}^{xx})^{-1} (x_k - \hat{x}_{k|k-1}) + (y_k - h(x_k, k))^T R_k^{-1} (y_k - h(x_k, k)). \quad (4.106)$$

Note that, like in CKF, the inequality-constraint information is not assimilated by the error-covariance $P_{k|k}^{xx}$ in (2.28).

CEKF is called recursive nonlinear dynamics data reconciliation (RNDDR) in (Vachhani et al., 2005, 2006). Applications of CEKF to petrochemical industrial processes are presented in (Marcon et al., 2002, 2004).

4.5.4 The Sigma-Point Inequality-Constrained Unscented Kalman Filter

The sigma-point interval unscented Kalman filter (SIUKF) (Vachhani, Narasimhan & Rengaswamy, 2006) is based on the idea of propagating sigma points that satisfy the inequality constraint rather than enforcing it only on the state estimate itself. It uses the interval-constrained unscented transform (ICUT) during the forecast step, while it solves $2n+1$ optimizations problems during the data-assimilation step. In so doing, note that, like TPUKF (Section 4.2.4), SIUKF enforces the constraint twice.

Next, we consider the system (4.39), (4.68) with the state vector satisfying (4.73) and (4.102).

The Interval-Constrained Unscented Transform

Consider the random vector $x_k \in \mathbb{R}^n$ with mean $\bar{x}_k$ and covariance $\bar{P}_k^{xx}$. Assume that x_k satisfy the interval constraint (4.73). In order to satisfy $a_k \leq \mathcal{X}_{j,k} \leq b_k$, $j = 0, \dots, 2n$, ICUT chooses the sigma points as

$$\begin{aligned} \mathcal{X}_k = \bar{x}_k 1_{1 \times (2n+1)} + & \begin{bmatrix} 0_{n \times 1} & \theta_{1,k} \text{col}_1[(\bar{P}_k^{xx})^{1/2}] & \dots & \theta_{n,k} \text{col}_n[(\bar{P}_k^{xx})^{1/2}] \\ & -\theta_{n+1,k} \text{col}_{n+1}[(\bar{P}_k^{xx})^{1/2}] & \dots & -\theta_{2n,k} \text{col}_{2n}[(\bar{P}_k^{xx})^{1/2}] \end{bmatrix}, \end{aligned} \quad (4.107)$$

where, for $i = 1, \dots, n$ and $j = 1, \dots, 2n$

$$\theta_{j,k} \triangleq \min(\text{col}_j(\Theta)), \quad (4.108)$$

$$\Theta_{(i,j)} \triangleq \begin{cases} \sqrt{n+\kappa}, & \text{if } S_{(i,j)} = 0, \\ \min\left(\sqrt{n+\kappa}, \frac{b_{i,k} - \bar{x}_{i,k}}{S_{(i,j)}}\right), & \text{if } S_{(i,j)} > 0, \\ \min\left(\sqrt{n+\kappa}, \frac{a_{i,k} - \bar{x}_{i,k}}{S_{(i,j)}}\right), & \text{if } S_{(i,j)} < 0, \end{cases} \quad (4.109)$$

$$S \triangleq \begin{bmatrix} (\bar{P}_k^{xx})^{1/2} & -(\bar{P}_k^{xx})^{1/2} \end{bmatrix}, \quad (4.110)$$

with weights, for $j = 1, \dots, 2n$,

$$\gamma_{0,k} \triangleq d_k, \quad (4.111)$$

$$\gamma_{j,k} \triangleq c_k \theta_{j,k} + d_k, \quad (4.112)$$

satisfying $\sum_{j=0}^{2n} \gamma_{j,k} = 1$, where

$$c_k \triangleq \frac{2\kappa - 1}{2(n + \kappa) \left(\sum_{j=1}^n \theta_{j,k} - (2n + 1)\sqrt{n + \kappa} \right)}, \quad (4.113)$$

$$d_k \triangleq \frac{1}{2(n + \kappa)} - \frac{2\kappa - 1}{(2\sqrt{n + \kappa}) \left(\sum_{j=1}^n \theta_{j,k} - (2n + 1)\sqrt{n + \kappa} \right)}. \quad (4.114)$$

For simplicity, henceforth, we refer to (4.107)-(4.114) as the function Ψ_{ICUT} , which is defined by

$$[\gamma_k, \mathcal{X}_k] = \Psi_{\text{ICUT}}(\bar{x}_k, \bar{P}_k^{xx}, a_k, b_k, n, \kappa). \quad (4.115)$$

Note that, for UT, by setting $\alpha = 1$ and $\beta = 0$, $\theta_{j,k} = \sqrt{n + \kappa}$, for all $j = 1, \dots, 2n$; see (2.89).

Example 4.5.1 Figure 4.7 illustrates how the sigma points of ICUT are chosen compared to UT. This example is adapted from (Vachhani et al., 2006). Note that whenever a sigma point violates (4.73), it is projected onto the nearest surface $\bar{x}_k = a_k$ or $\bar{x}_k = b_k$. In so doing, unlike UT, the sigma points are not necessarily symmetric around $\bar{x}_k$ such that their weighted sample mean and covariance may not capture $\bar{x}_k$ and $\bar{P}_k^x$, respectively. Figure 4.7 suggests that ICUT intrinsically performs a “truncation” of the PDF propagated by the sigma points (4.107). $\square$

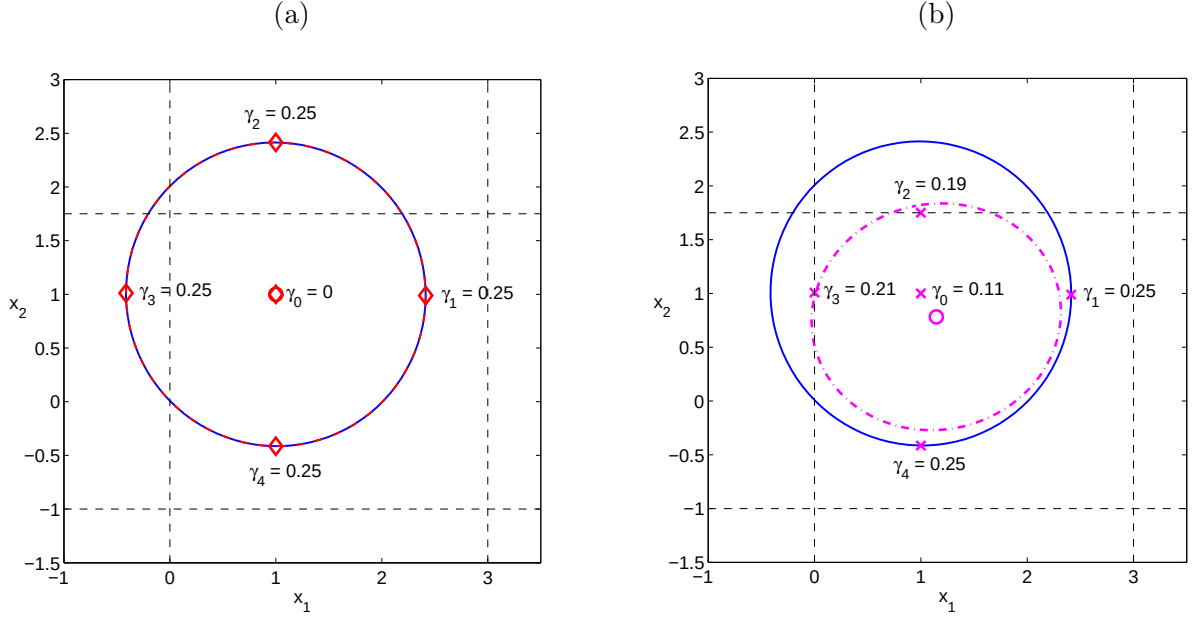


Figure 4.7: Illustrative example of the interval-constrained unscented transform compared to the original unscented transform. Sigma points of (a) UT ($\diamond$) in comparison with those of (b) ICUT ($\times$) and related weights for an example where $x_k \in \mathbb{R}^2$, $\bar{x}_k = [1 \ 1]^\top$, $\bar{P}_k^{xx} = I_{2 \times 2}$, $a_k = [0 \ -1]^\top$, $b_k = [3 \ 1.75]^\top$, and $\kappa = 0$. The circle ($\circ$) represents the weighted mean of sigma points, the dot-dashed line ($-\cdot-$) denotes the corresponding covariance, and the solid line ($---$) is the true covariance. For UT, these lines coincide.

Algorithm of the Sigma-Point Inequality-Constrained Unscented Kalman Filter

Consider the augmented state vector $\tilde{x}_{k-1}^c \in \mathbb{R}^{\tilde{n}}$ and the augmented error covariance $P_{k-1|k-1}^{\tilde{x}\tilde{x}c} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, respectively, given by

$$\tilde{x}_{k-1} \triangleq \begin{bmatrix} x_{k-1}^c \\ w_{k-1} \\ v_k \end{bmatrix}, \quad P_{k-1|k-1}^{\tilde{x}\tilde{x}c} \triangleq \begin{bmatrix} P_{k-1|k-1}^{xxc} & 0_{n \times q} & 0_{n \times r} \\ 0_{q \times n} & Q_{k-1} & 0_{q \times r} \\ 0_{r \times n} & 0_{r \times q} & R_k \end{bmatrix}, \quad (4.116)$$

where $\tilde{n} \triangleq n + q + r$. Thus we rewrite (4.73) as

$$\tilde{a}_k \leq \tilde{x}_k^c \leq \tilde{b}_k, \quad (4.117)$$

where,

$$\tilde{a}_k \triangleq \begin{bmatrix} a_k \\ a_k^w \\ a_k^v \end{bmatrix}, \quad \tilde{b}_k \triangleq \begin{bmatrix} b_k \\ b_k^w \\ b_k^v \end{bmatrix}. \quad (4.118)$$

In this thesis, we set $a_k^w = -\infty_{q \times 1}$, $a_k^v = -\infty_{r \times 1}$, $b_k^w = \infty_{r \times 1}$, and $b_k^v = \infty_{r \times 1}$.

SIUKF is a two-step estimator whose forecast step is given by

$$\begin{bmatrix} \gamma_{k-1}, \tilde{\mathcal{X}}_{k-1|k-1}^c \end{bmatrix} = \Psi_{\text{ICUT}} \left(\hat{x}_{k-1|k-1}^c, P_{k-1|k-1}^{\tilde{x}\tilde{x}c}, \tilde{a}_{k-1}, \tilde{b}_{k-1}, \tilde{n}, \kappa \right), \quad (4.119)$$

$$\mathcal{X}_{j,k|k-1} = f(\mathcal{X}_{j,k-1|k-1}^x, u_{k-1}, \mathcal{X}_{j,k-1|k-1}^w, k-1), \quad j = 0, \dots, 2n, \quad (4.120)$$

$$\hat{x}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} \mathcal{X}_{j,k|k-1}^x, \quad (4.121)$$

$$P_{k|k-1}^{xx} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} [\mathcal{X}_{j,k|k-1}^x - \hat{x}_{k|k-1}] [\mathcal{X}_{j,k|k-1}^x - \hat{x}_{k|k-1}]^T, \quad (4.122)$$

where Ψ_{ICUT} is defined by (4.115) and $\tilde{\mathcal{X}}_{k-1|k-1}^c$ is partitioned as (2.97), and whose data-assimilation step is given by

$$\hat{\mathcal{X}}_{j,k|k}^x = \arg \min_{\{\mathcal{X}_{j,k}^x : a_k \leq \mathcal{X}_{j,k}^x \leq b_k, \quad l(\mathcal{X}_{j,k}^x, k-1) \leq e_{k-1}\}} J_{\mathcal{X}}(\mathcal{X}_{j,k}^x), \quad j = 0, \dots, 2\tilde{n}, \quad (4.123)$$

$$\hat{x}_{k|k}^c = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} \hat{\mathcal{X}}_{j,k|k}^x, \quad (4.124)$$

$$P_{k|k}^{xxc} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} \left[\hat{\mathcal{X}}_{j,k|k}^x - \hat{x}_{k|k}^c \right] \left[\hat{\mathcal{X}}_{j,k|k}^x - \hat{x}_{k|k}^c \right]^T, \quad (4.125)$$

where

$$J_{\mathcal{X}}(\mathcal{X}_{j,k}^x) \triangleq \begin{bmatrix} (y_k - h_k(\mathcal{X}_{j,k}^x))^T R_k^{-1} (y_k - h_k(\mathcal{X}_{j,k}^x)) \\ + (\mathcal{X}_{j,k}^x - \mathcal{X}_{j,k|k-1}^x)^T (P_{k|k-1}^{xx})^{-1} (\mathcal{X}_{j,k}^x - \mathcal{X}_{j,k|k-1}^x) \end{bmatrix}, \quad (4.126)$$

and each entry of $\hat{\mathcal{X}}_{k|k}^x \triangleq [\hat{\mathcal{X}}_{0,k|k}^x \quad \hat{\mathcal{X}}_{1,k|k}^x \quad \dots \quad \hat{\mathcal{X}}_{2n,k|k}^x] \in \mathbb{R}^{n \times (2\tilde{n}+1)}$ is the solution of a nonlinear constrained optimization problem.

Remark 4.5.2. SIUKF enforces the interval constraint (4.73) during both the forecast and data-assimilation steps, while the nonlinear inequality constraint (4.102) is enforced

only during data assimilation. Moreover, not only $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$, but also $P_{k|k-1}^{xx}$ and $P_{k|k}^{xx}$, assimilate the interval-constraint information.

4.6 Algorithms: Summary of Characteristics

We compare the structure of the algorithms presented along this chapter. Table 4.1 indicates, for each algorithm, whether or not the (pseudo-) state estimates, as well as the (pseudo-) error covariance, are affected by the constraint information in each step of the estimator, namely, forecast, data assimilation, and constraint (projection or truncation) steps. We also indicate which type of constraint each algorithm can handle.

Furthermore, recall that, at least for the linear case, there exist closed solutions to the equality-constrained state-estimation problem, whereas methods for the inequality-constrained case are suboptimal and require the online solution of optimization problems. As stated in Remark 4.4.1, one of the reasons for which the inequality-constrained state-estimation problem is more complicated than the equality-constrained case is the fact that the existence of inequality constraints and the definition of Gaussian noise are mutually exclusive assumptions.

Table 4.1: Summary of characteristics of constrained state-estimation algorithms for linear (4.1)-(4.2) and nonlinear systems (4.39)-(4.40). It is shown whether or not the (pseudo-) state estimates and (pseudo-) error covariance assimilate the constraint information in each step of the estimator. Also, it is indicated which type of constraint each algorithm can handle, namely, linear equality constraint (4.3) (LE), nonlinear equality constraint (4.41) (NE), linear inequality constraint (4.72) (LI), interval constraint (I) (4.73), and nonlinear inequality constraint (4.102) (NI).

Algorithm	Section	forecast		data assimilation		constraint		type of constraint				
		$\hat{x}_{k k-1}$	$P_{k k-1}^{xx}$	$\hat{x}_{k k}$	$P_{k k}^{xx}$	$\hat{x}_{k k}^c$	$P_{k k}^{xxc}$	LE	NE	LI	I	NI
Linear system												
MAKF [‡]	4.1.2	no	no	yes	yes	–	–	yes	no	no	no	no
PKF-EP	4.1.3	no	no	no	no	yes	–	yes	no	no	no	no
PKF-GP	4.1.4	no	no	no	no	yes	yes	yes	no	no	no	no
PKF-SP	4.1.5	no	no	no	no	yes*	yes*	yes	no	no	no	no
MHE [‡]	4.4.2	–	–	yes	no	–	–	yes	no	yes	yes	no
CKF [‡]	4.4.3	no	no	yes	no	–	–	yes	no	yes	yes	no
PKF-IC	4.4.4	no	no	no	no	yes	–	yes	no	yes	yes	no
TKF	4.4.5	no	no	no	no	yes	yes	yes	no	no	yes	no
Nonlinear system												
MAEKF [‡]	4.2.2	no	no	yes	yes	–	–	yes	yes	no	no	no
PEKF-EP	4.2.3	no	no	no	no	yes	–	yes	yes	no	no	no
TPUKF	4.2.4	yes	yes	no	no	yes	yes	yes	yes	no	no	no
NMHE [‡]	4.5.2	–	–	yes	no	–	–	yes	yes	yes	yes	yes
CEKF [‡]	4.5.3	no	no	yes	no	–	–	yes	yes	yes	yes	yes
SIUKF [‡]	4.5.4	yes	yes	yes	yes	–	–	yes [†]	yes [†]	yes [†]	yes	yes [†]

* The projection step is performed only at initialization.

[†] The correspondent constraint is enforced only during the data-assimilation step.

[‡] The data-assimilation and constraint steps are jointly executed.

4.7 Chapter Summary

State estimators for linear and nonlinear systems use two uncertain sources of information, namely, the model and measured data, to obtain state estimates. In some cases, however, additional information about the system may be available, and this information may be useful for improving state estimates. In this chapter, we review methods that enforce equality and inequality constraints as a third source of information to improve the state estimates.

First, to consider equality-constrained systems, we present the measurement-augmentation approach in which a perfect “measurement” of the constrained quantity is appended to the physical measurements. Both linear and nonlinear cases are covered by this approach by means of MAKF and MAEKF, respectively. For the latter, however, the constraint is approximately satisfied by means of the analytical linearization procedure employed by EKF.

Next, projection methods are considered for equality-constrained systems. These methods project the state estimates onto the hyperplane (in the case of nonlinear systems, hypersurface) defined by the equality constraint. For linear systems, we consider PKF-EP, PKF-GP, and PKF-SP, while PEKF-EP and TPUKF address the nonlinear case. This idea is also extended to enforce inequality constraints by means of PKF-IC.

To enforce interval constraints, we present the truncation procedure that is used by TKF. In this case, the probability density function computed by KF, which is assumed to be Gaussian and is given by the state estimate and the error covariance, is reshaped at the inequality constraint edges.

After recasting the state-estimation problem from the perspective of quadratic programming, we present MHE and, for nonlinear systems, NMHE. At the cost of being computationally expensive, these algorithms can enforce both equality and inequality constraints by solving a quadratic programming optimization problem. We show that CKF and CEKF are special and simplified cases of the aforementioned algorithms with unitary moving horizon.

Finally, motivated by the improved performance of UKF over EKF for unconstrained systems, we present SIUKF. In this method, the sigma points are chosen to satisfy an interval constraint during the forecast step, while the nonlinear inequality constrained is enforced sigma-point-wise during data assimilation.

EQUALITY-CONSTRAINED STATE ESTIMATION: LINEAR DYNAMICS, OPTIMAL LINEAR SOLUTION, AND NONLINEAR EXTENSIONS

“Nothing in life is to be feared. It is only to be understood.”

Marie Curie

We now address the state-estimation problem with an equality constraint on the state vector. A literature review for this topic is presented in Sections 4.1 and 4.2. In Section 5.1, we investigate how a linear equality state constraint arises in a linear dynamic system, and present necessary conditions on both the dynamics and process noise for the state to be equality constrained. Then we derive the equality-constrained Kalman filter (ECKF) as the solution to the equality-constrained state-estimation problem for linear systems and present its connections to alternative approaches in Section 5.2. In Section 5.3 we develop and compare three suboptimal algorithms for equality-constrained state estimation for nonlinear systems based on the unscented Kalman filter. Finally, both linear and nonlinear examples are presented in Sections 5.4 and 5.5, respectively. The novel results presented in this chapter are published in (Teixeira, Chandrasekar, Tôrres, Aguirre & Bernstein, 2007, 2008a,b).

5.1 Equality-Constrained Linear Systems

Consider the stochastic linear discrete-time dynamic system (2.7)-(2.8), whose equations are repeated here for convenience, that is,

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1}, \quad (5.1)$$

$$y_k = C_k x_k + v_k. \quad (5.2)$$

Assume that, for all $k \geq 1$, the input u_{k-1} is known and the output y_k is measured. Moreover, as in Section 2.2, assume that $\rho_{w_{k-1}}(w_{k-1})$, $\rho_{v_k}(v_k)$, and $\rho_{x_0}(x_0)$ are Gaussian and, respectively, given by (2.9)-(2.11). Also, for all $k \geq 0$, x_0 is assumed to be uncorrelated with w_k and v_k .

In (5.1), assume that $\text{rank}(G_{k-1}) = q < n$, and define $t \triangleq n - q$, where $1 \leq t \leq n$. The case $t = n$ indicates that $G_{k-1}w_{k-1}$ is absent. Therefore, there exists $E_{k-1} \in \mathbb{R}^{t \times n}$ such that $\text{rank}(E_{k-1}) = t$ and

$$E_{k-1}G_{k-1} = 0_{t \times q}. \quad (5.3)$$

Let $T_{1,k-1} \in \mathbb{R}^{(n-t) \times n}$ be such that $T_{k-1} \triangleq \begin{bmatrix} T_{1,k-1} \\ E_{k-1} \end{bmatrix} \in \mathbb{R}^{n \times n}$ is invertible. For example, we can choose $T_{1,k-1} \triangleq G_{k-1}^T$. Multiplying (5.1) by T_{k-1} yields

$$\begin{bmatrix} T_{1,k-1} \\ E_{k-1} \end{bmatrix} x_k = \begin{bmatrix} T_{1,k-1}A_{k-1} \\ E_{k-1}A_{k-1} \end{bmatrix} x_{k-1} + \begin{bmatrix} T_{1,k-1}B_{k-1} \\ E_{k-1}B_{k-1} \end{bmatrix} u_{k-1} + \begin{bmatrix} T_{1,k-1}G_{k-1} \\ 0_{t \times q} \end{bmatrix} w_{k-1}.$$

Hence, for all $k \geq 1$,

$$E_{k-1}x_k = e_{k-1}, \quad (5.4)$$

where $e_{k-1} \triangleq E_{k-1}(A_{k-1}x_{k-1} + B_{k-1}u_{k-1})$. Note that e_{k-1} is not necessarily constant even if system (5.1)-(5.2) is time invariant. Also, since $\text{rank}(G_{k-1}) < n$, $G_{k-1}w_{k-1}$ has singular covariance $G_{k-1}Q_{k-1}G_{k-1}^T$ (Ko & Bitmead, 2007; Goodwin, Seron & de Doná, 2005).

Let s be an integer satisfying $1 \leq s \leq t$, and partition $E_{k-1} \triangleq \begin{bmatrix} E_{1,k-1} \\ D_{k-1} \end{bmatrix}$, where $E_{1,k-1} \in \mathbb{R}^{(t-s) \times n}$ and $D_{k-1} \in \mathbb{R}^{s \times n}$. It thus follows from (5.3) that

$$D_{k-1}G_{k-1} = 0_{s \times q}. \quad (5.5)$$

Proposition 5.1.1. Assume that (5.5) holds, as well as

$$D_{k-1}A_{k-1} = D_{k-1} \quad (5.6)$$

and

$$D_{k-1}B_{k-1}u_{k-1} = 0_{s \times 1} \quad \text{for all } k \geq 1. \quad (5.7)$$

Then, for all $k \geq 1$,

$$D_{k-1}x_k = d_{k-1}, \quad (5.8)$$

where

$$d_{k-1} \triangleq D_{k-1}x_{k-1}. \quad (5.9)$$

Proof. It follows from (5.4) and (5.5) that $D_{k-1}x_k = D_{k-1}(A_{k-1}x_{k-1} + B_{k-1}u_{k-1}) = D_{k-1}x_{k-1} = d_{k-1}$. $\square$

Corollary 5.1.1 If (5.1)-(5.2) is time invariant and (5.5)-(5.7) hold, then, for all $k \geq 1$,

$$Dx_k = d, \quad (5.10)$$

where $D \triangleq D_{k-1}$ and

$$d \triangleq Dx_0. \quad (5.11)$$

Note that, if $s = t = n$, then $x_k = D^{-1}d$. Hence this case is not of practical interest.

The next result shows that, if (5.1) is equality constrained, then it is not controllable in $\mathbb{R}^n$ from the process noise, but it is rather controllable in the subspace defined by (5.8).

Proposition 5.1.2. If (5.5)-(5.6) hold, then (A_{k-1}, G_{k-1}) is not controllable.

Proof. Multiplying the controllability matrix

$$\mathcal{K}(A_{k-1}, G_{k-1}) \triangleq \begin{bmatrix} G_{k-1} & A_{k-1}G_{k-1} & \cdots & A_{k-1}^{n-1}G_{k-1} \end{bmatrix}$$

by D_{k-1} yields

$$D_{k-1}\mathcal{K}(A_{k-1}, G_{k-1}) = \begin{bmatrix} D_{k-1}G_{k-1} & D_{k-1}A_{k-1}G_{k-1} & \cdots & D_{k-1}A_{k-1}^{n-1}G_{k-1} \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 0_{s \times q} & D_{k-1}G_{k-1} & \cdots & D_{k-1}A_{k-1}^{n-2}G \end{bmatrix} \\
&= 0_{s \times nq},
\end{aligned}$$

implying that the columns of $\mathcal{K}$ lie on the null space of D_{k-1} such that $\text{rank}(\mathcal{K}) < n$. $\square$

Definition 5.1.1. Assuming that, for all $k \geq 1$, $D_{k-1} \in \mathbb{R}^{s \times n}$ satisfying (5.5)-(5.7) and d_{k-1} defined by (5.9) are known, the objective of the *equality-constrained state-estimation* problem is to maximize (1.3) subject to (5.8).

5.2 The Equality-Constrained Kalman Filter

In this section we solve the equality-constrained state-estimation problem to obtain the equality-constrained Kalman filter (ECKF). Let $\hat{x}_{k|k}^p$ denote the solution of the equality-constrained state estimation problem.

Lemma 5.2.1 $\hat{x}_{k|k}^p$ maximizes J given by (1.3) subject to (5.8) if and only if $\hat{x}_{k|k}^p$ minimizes $\mathcal{J}$ given by (2.26) subject to (5.8), where $\hat{x}_{k|k-1}$ and $P_{k|k-1}^{xx}$ are given by (2.20) and (2.21), respectively.

Proof. From Proposition 2.1.1, maximizing (1.3) subject to (5.8) is equivalent to minimizing the negative of the numerator of (2.5) subject to (5.8), that is,

$$\begin{aligned}
\max_{\{x_k : D_{k-1}x_k = d_{k-1}\}} J(x_k) &= \min_{\{x_k : D_{k-1}x_k = d_{k-1}\}} \{-\ln[\rho(x_k|(y_1, \dots, y_{k-1}))\rho(y_k|x_k)]\} \\
&= \min_{\{x_k : D_{k-1}x_k = d_{k-1}\}} \{-\ln \rho(x_k|(y_1, \dots, y_{k-1})) - \ln \rho(y_k|x_k)\} \\
&= \min_{\{x_k : D_{k-1}x_k = d_{k-1}\}} \mathcal{J}(x_k),
\end{aligned}$$

where $\rho(x_k|(y_1, \dots, y_{k-1}))$ is given by (2.12), $\rho(y_k|x_k)$ is given by (2.13), and we have used the fact that $-\ln$ is strictly decreasing. $\square$

Theorem 5.2.1. Define the projected error covariance $P_k^{xsp} \triangleq E[(x_k - \hat{x}_{k|k}^p)(x_k -$

$\hat{x}_{k|k}^p)^T]$. Also let $\hat{x}_{k|k-1}$ and $P_{k|k-1}^{xx}$ be given by

$$\hat{x}_{k|k-1} \triangleq A_{k-1}\hat{x}_{k-1|k-1}^p + B_{k-1}u_{k-1}, \quad (5.12)$$

$$P_{k|k-1}^{xx} \triangleq A_{k-1}P_{k-1|k-1}^{xxp}A_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T. \quad (5.13)$$

Define

$$\hat{d}_{k-1|k} \triangleq D_{k-1}\hat{x}_{k|k}, \quad (5.14)$$

$$P_{k|k}^{dd} \triangleq D_{k-1}P_{k|k}^{xx}D_{k-1}^T, \quad (5.15)$$

$$P_{k|k}^{xd} \triangleq P_{k|k}^{xx}D_{k-1}^T, \quad (5.16)$$

$$K_k^p \triangleq P_{k|k}^{xd}(P_{k|k}^{dd})^{-1}, \quad (5.17)$$

where $\hat{x}_{k|k}$ is given by (2.27) and $P_{k|k}^{xx}$ is given by (2.28). Then $\hat{x}_{k|k}^p$ and $P_{k|k}^{xxp}$ are given by

$$\hat{x}_{k|k}^p = \hat{x}_{k|k} + K_k^p(d_{k-1} - \hat{d}_{k-1|k}), \quad (5.18)$$

$$P_{k|k}^{xxp} = P_{k|k}^{xx} - K_k^p P_{k|k}^{dd} (K_k^p)^T. \quad (5.19)$$

Proof. Using Lemma 5.2.1, let $\lambda \in \mathbb{R}^s$ and define the Lagrangian

$$\mathcal{L} \triangleq \mathcal{J}(x_k) + 2\lambda^T(D_{k-1}x_k - d_{k-1}),$$

where $\mathcal{J}(x_k)$ is given by (2.26). The necessary conditions for a minimizer $\hat{x}_{k|k}^p$ are given by

$$\frac{\partial \mathcal{L}}{\partial x_k} = 2((P_{k|k-1}^{xx})^{-1}(\hat{x}_{k|k}^p - \hat{x}_{k|k-1}) - C_k^T R_k^{-1}(y_k - C_k \hat{x}_{k|k}^p) + D_{k-1}^T \lambda) = 0_{n \times 1}, \quad (5.20)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = D_{k-1}\hat{x}_{k|k}^p - d_{k-1} = 0_{s \times 1}. \quad (5.21)$$

It follows from (5.20) that

$$((P_{k|k-1}^{xx})^{-1} + C_k^T R_k^{-1} C_k)(\hat{x}_{k|k}^p - \hat{x}_{k|k-1}) = C_k^T R_k^{-1}(y_k - C_k \hat{x}_{k|k-1}) - D_{k-1}^T \lambda. \quad (5.22)$$

From (2.28), using (2.23)-(2.25) and the matrix inversion lemma (Bernstein, 2005), we have

$$\begin{aligned} P_{k|k}^{xx} &= P_{k|k-1}^{xx} - K_k P_{k|k-1}^{yy} K_k^T \\ &= P_{k|k-1}^{xx} - P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} P_{k|k-1}^{yy} (P_{k|k-1}^{yy})^{-T} (P_{k|k-1}^{xy})^T \end{aligned}$$

$$\begin{aligned}
&= P_{k|k-1}^{xx} - P_{k|k-1}^{xx} C_k^T (C_k P_{k-1|k-1}^{xx} C_k^T + R_k)^{-1} C_k P_{k|k-1}^{xx} \\
&= ((P_{k|k-1}^{xx})^{-1} + C_k^T R_k^{-1} C_k)^{-1}.
\end{aligned} \tag{5.23}$$

Furthermore, from (2.25), using (2.23)-(2.24), we have

$$\begin{aligned}
K_k &= P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \\
&= P_{k|k-1}^{xx} C_k^T (C_k P_{k|k-1}^{xx} C_k^T + R_k)^{-1} \\
&= P_{k|k}^{xx} (P_{k|k}^{xx})^{-1} P_{k|k-1}^{xx} C_k^T (C_k P_{k|k-1}^{xx} C_k^T + R_k)^{-1} \\
&= P_{k|k}^{xx} (C_k^T R_k^{-1} C_k + (P_{k|k-1}^{xx})^{-1}) P_{k|k-1}^{xx} C_k^T (C_k P_{k|k-1}^{xx} C_k^T + R_k)^{-1} \\
&= P_{k|k}^{xx} (C_k^T R_k^{-1} C_k P_{k|k-1}^{xx} C_k^T + C_k^T R_k^{-1} R_k) (C_k P_{k|k-1}^{xx} C_k^T + R_k)^{-1} \\
&= P_{k|k}^{xx} C_k^T R_k^{-1} (C_k P_{k|k-1}^{xx} C_k^T + R_k) (C_k P_{k|k-1}^{xx} C_k^T + R_k)^{-1} \\
&= P_{k|k}^{xx} C_k^T R_k^{-1}.
\end{aligned} \tag{5.24}$$

Substituting (5.23) and (5.24) into (5.22) and multiplying by P_k^{xx} yields

$$\hat{x}_{k|k}^p = \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1}) - P_k^{xx} D_{k-1}^T \lambda. \tag{5.25}$$

Substituting (5.25) into (5.21) yields

$$d_{k-1} = D_{k-1} \hat{x}_{k|k-1} + D_{k-1} P_{k|k}^{xx} C_k^T R_k^{-1} (y_k - C_k \hat{x}_{k|k-1}) - D_{k-1} P_{k|k}^{xx} D_{k-1}^T \lambda,$$

which implies

$$\begin{aligned}
\lambda &= (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} (D_{k-1} \hat{x}_{k|k-1} - d_{k-1}) \\
&\quad + (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} D_{k-1} K_k (y_k - C_k \hat{x}_{k|k-1}).
\end{aligned} \tag{5.26}$$

Likewise, substituting (5.26) into (5.25) yields

$$\begin{aligned}
\hat{x}_{k|k}^p &= \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1}) - P_{k|k}^{xx} D_{k-1}^T (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} (D_{k-1} \hat{x}_{k|k-1} - d_{k-1}) \\
&\quad - P_{k|k}^{xx} D_{k-1}^T (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} D_{k-1} K_k (y_k - C_k \hat{x}_{k|k-1}) \\
&= \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1}) - P_{k|k}^{xx} D_{k-1}^T (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} \\
&\quad \times (D_{k-1} \hat{x}_{k|k-1} - d_{k-1} + D_{k-1} K_k y_k - D_{k-1} K_k C_k \hat{x}_{k|k-1}) \\
&= \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1})
\end{aligned}$$

$$+ P_{k|k}^{xx} D_{k-1}^T (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} [d_{k-1} - D_{k-1}(\hat{x}_{k|k-1} + K_k(y_k - C_k \hat{x}_{k|k-1}))].$$

Now using (5.14)-(5.17), (2.25)-(2.28), we obtain

$$\begin{aligned} \hat{x}_{k|k}^p &= \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}) \\ &\quad + P_{k|k}^{xx} D_{k-1}^T (D_{k-1} P_{k|k}^{xx} D_{k-1}^T)^{-1} [d_{k-1} - D_{k-1}(\hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}))] \\ &= \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}) + K_k^p(d_{k-1} - \hat{d}_{k-1|k}) \\ &= \hat{x}_{k|k} + K_k^p(d_{k-1} - \hat{d}_{k-1|k}), \end{aligned}$$

which proves (5.18).

Given the symmetry between (5.18) and (2.27) and recalling that $P_{k|k}^{xx}$ is given by (2.28), it follows that $P_{k|k}^{xkp}$ is given by (5.19). $\square$

Remark 5.2.1. Note that ECKF is expressed in three steps, namely, the forecast step (5.12)-(5.13), (2.22)-(2.24), the data-assimilation step (2.25), (2.27)-(2.28), and the projection step (5.14)-(5.19), in which the data-assimilation state estimate $\hat{x}_{k|k}$ is projected onto the hyperplane defined by the equality constraint (5.8). Figure 5.1 shows how these three steps are connected by ECKF.

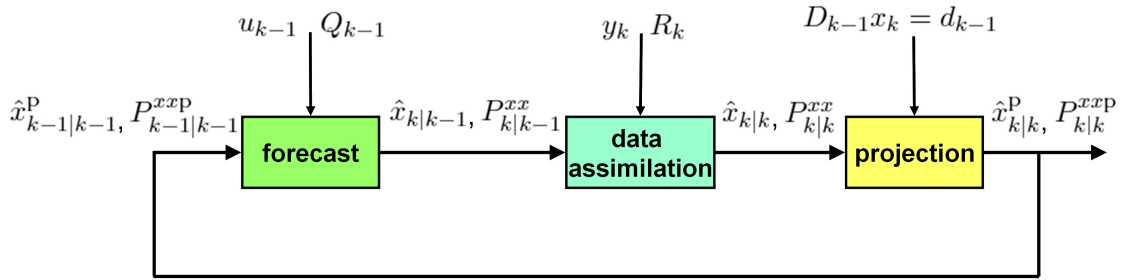


Figure 5.1: Diagram of the equality-constrained Kalman filter. The projection step is connected by feedback recursion to the forecast step.

Lemma 5.2.2 Let $\mathcal{N}(D_{k-1})$ denote the null space of D_{k-1} , let $W \in \mathbb{R}^{n \times n}$ be positive definite, and define

$$\mathcal{P}_{\mathcal{N}(D_{k-1})} \triangleq I_{n \times n} - W D_{k-1}^T (D_{k-1} W D_{k-1}^T)^{-1} D_{k-1}. \quad (5.27)$$

Then $\mathcal{P}_{\mathcal{N}(D_{k-1})}$ is an oblique projector (Bernstein, 2005) whose range is $\mathcal{N}(D_{k-1})$.

For the following two results, let $\hat{x}_{k|k}$ given by (2.27) and $P_{k|k}^{xx}$ given by (2.28) denote the data-assimilation state estimate and error covariance provided by ECKF. Also, let $\hat{x}_{k|k}^p$ given by (5.18) and $P_{k|k}^{xxp}$ given by (5.19) denote the projected estimate and associated error covariance of ECKF.

Proposition 5.2.1. Set $W = P_{k|k}^{xx}$ in (5.27). Then, the projection step (5.14)-(5.19) is equivalent to

$$\hat{x}_{k|k}^p = \mathcal{P}_{\mathcal{N}(D_{k-1})}\hat{x}_{k|k} + \bar{d}_{k-1}, \quad (5.28)$$

$$P_{k|k}^{xxp} = \mathcal{P}_{\mathcal{N}(D_{k-1})}P_{k|k}^{xx}, \quad (5.29)$$

where $\bar{d}_{k-1} \triangleq P_{k|k}^{xx}D_{k-1}^T(D_{k-1}P_{k|k}^{xx}D_{k-1}^T)^{-1}d_{k-1}$.

Proof. Using Lemma 5.2.2 and substituting (5.14)-(5.17) into (5.18) and (5.19) yields (5.28)-(5.29). $\square$

Proposition 5.2.2. Assume that (5.1)-(5.2) is time invariant, that is, consider the system (4.29)-(4.30). Also, assume that D in (5.10) satisfies (5.5)-(5.7). Furthermore, assume that, for a given $k-1$, $D\hat{x}_{k-1|k-1}^p = d$ and $DP_{k-1|k-1}^{xxp} = 0_{s \times n}$. Then $D\hat{x}_{k|k} = d$, $DP_{k|k}^{xx} = 0_{s \times n}$, $\hat{x}_{k|k}^p = \hat{x}_{k|k}$, and $P_{k|k}^{xxp} = P_{k|k}^{xx}$.

Proof. Multiplying (5.12)-(5.13) by D yields

$$D\hat{x}_{k|k-1} = DA\hat{x}_{k-1|k-1}^p + DBu_{k-1} = D\hat{x}_{k-1|k-1}^p + 0_{s \times 1} = d, \quad (5.30)$$

$$\begin{aligned} DP_{k|k-1}^{xx} &= DAP_{k-1|k-1}^{xxp}A^T + DGQ_{k-1}G^T \\ &= DP_{k-1|k-1}^{xxp}A^T + 0_{s \times q}Q_{k-1}G^T = 0_{s \times n}A^T = 0_{s \times n}. \end{aligned} \quad (5.31)$$

With (2.24) and (5.31), multiplying (2.25) by D yields

$$DK_k = DP_{k|k-1}^{xy}(P_{k|k-1}^{yy})^{-1} = DP_{k|k-1}^{xx}C^T(P_{k|k-1}^{yy})^{-1} = 0_{s \times n}C^T(P_{k|k-1}^{yy})^{-1} = 0_{s \times m}. \quad (5.32)$$

With (5.30) and (5.32), multiplying (2.27)-(2.28) by D yields

$$D\hat{x}_{k|k} = D\hat{x}_{k|k-1} + DK_k(y_k - \hat{y}_{k|k-1}) = d + 0_{s \times m}(y_k - \hat{y}_{k|k-1}) = d, \quad (5.33)$$

$$DP_{k|k}^{xx} = DP_{k|k-1}^{xx} - DK_kP_{k|k-1}^{yy}K_k^T = 0_{s \times n} + 0_{s \times m}P_{k|k-1}^{yy}K_k^T = 0_{s \times n}. \quad (5.34)$$

Given (5.33)-(5.34), from (5.28)-(5.29), we have $\hat{x}_{k|k}^p = \hat{x}_{k|k}$ and $P_{k|k}^{xxp} = P_{k|k}^{xx}$. $\square$

Corollary 5.2.1 Assume that $D\hat{x}_{1|1}^p = d$ and $DP_{1|1}^{xxp} = 0_{s \times n}$. Then, for all $k \geq 2$, $D\hat{x}_{k|k} = d$ and $DP_{k|k}^{xx} = 0_{s \times n}$.

Therefore, for the time-invariant system (4.29)-(4.30), whenever (5.5)-(5.7) hold, the projection step of ECKF given by (5.14)-(5.19) is required only at $k = 1$, so that, for all $k \geq 2$, the data-assimilation estimate $\hat{x}_{k|k}$ given by (2.27) satisfies $D\hat{x}_{k|k} = d$.

Remark 5.2.2. In practice, to avoid ill-conditioning problems in the inversion of (5.15) in (5.17), (5.15) is replaced by

$$P_{k|k}^{dd} = D_{k-1}P_{k|k}^{xx}D_{k-1}^T + \delta R_k, \quad (5.35)$$

where δR_k is set as in (4.14).

Remark 5.2.3. Note that, if D_{k-1} and d_{k-1} are uncertain, then (5.8) can be replaced by the soft equality constraint (De Geeter et al., 1997; Walker, 2006)

$$d_{k-1} = D_{k-1}x_k + v_k^d, \quad (5.36)$$

where $v_k^d \in \mathbb{R}^s$ is a white, Gaussian, zero mean noise with covariance R_k^d , and it is treated as an extra noisy measurement by MAKF (see Section 4.1.2) but with $\tilde{R}_k^d \triangleq \begin{bmatrix} R_k & 0_{m \times s} \\ 0_{s \times m} & R_k^d \end{bmatrix}$ replacing $\tilde{R}_k$.

5.2.1 Connections to Alternative Approaches

We now compare ECKF to the four equality-constrained state-estimation algorithms presented for linear systems in Section 4.1.

First we consider the measurement-augmentation Kalman filter (MAKF), which treats (5.8) as perfect measurements; see Section 4.1.2. In (Teixeira et al., 2007, 2008a, Appendix I), the MAKF and ECKF estimates are proved to be equal. Likewise, in (Gupta, 2007b, Section 3), the MAKF equations are written as the ECKF equations.

Now we compare ECKF to the projected Kalman filter by system projection (PKF-SP), which is presented in Section 4.1.5 in the context of time-invariant systems and assuming that (5.5)-(5.7) hold. With Corollary 5.2.1 and comparing (4.35)-(4.36) to (5.28)-(5.29), we

see that, similar to ECKF, which performs projection only at $k = 1$ to guarantee constraint satisfaction for all $k \geq 1$, PKF-SP performs projection in initialization, that is, only at $k = 0$.

In Section 4.1.3, we review the projected Kalman filter by estimate projection (PKF-EP) which projects $\hat{x}_{k|k}$ onto the hyperplane (5.8) for all $k \geq 1$. However, unlike ECKF, PKF-EP does not recursively feed the projected estimate $\hat{x}_{k|k}^p$ (4.17) and error covariance $P_{k|k}^{xsp}$ given by (4.18) back in forecast (2.20)-(2.21); see Figures 4.1 and 5.1. Therefore, the PKF-EP forecast estimate $\hat{x}_{k|k-1}$ (2.20) is different from its ECKF counterpart (5.12).

Finally, we consider the projected Kalman filter by gain projection (PKF-GP), which is reviewed in Section 4.1.4. Like in ECKF, PKF-GP recursively feed the projected estimate $\hat{x}_{k|k}^g$ (4.25) and error covariance $P_{k|k}^{xxg}$ given by (4.26) back in forecast; compare Figures 4.2 and 5.1. Moreover, if we replace $P_{k|k}^{xx}$ by $I_{n \times n}$ in the projection equations of ECKF given by (5.14)-(5.19), then we obtain the projection equations of PKF-GP given by (4.24)-(4.26). That is, from (5.27), PKF-GP uses an orthogonal projection $\mathcal{P}_{\mathcal{N}(D_{k-1})}$ to yield its state estimates.

Also, for the special case of unitary horizon, ECKF and the moving horizon estimator (MHE) (Section 4.4.2) minimize the same cost function, specifically, (2.26). However, ECKF provides the analytical solution to the equality-constrained optimization problem. Moreover, unlike MHE, ECKF enforces the constraint information on the error covariance in addition to the state estimate.

We also remark that, if $m = 0$, then the problem solved by ECKF in Theorem 5.2.1 is similar to the case in which KF is used as an iterative solver for systems of linear algebraic equations such as $Dx = d$, where $D \in \mathbb{R}^{s \times n}$, $d \in \mathbb{R}^s$, and $\text{rank}(D) = s$ (Pinto & Rios Neto, 1990, Theorem 2).

5.3 Unscented Filtering for Equality-Constrained Nonlinear Systems

Now we consider the nonlinear system given by (1.1)-(1.2), whose state vector x_k is known to satisfy the equality constraint (4.41), which is repeated here for convenience,

$$g(x_k, k-1) = d_{k-1}. \quad (5.37)$$

The assumptions on (1.1)-(1.2), (5.37) that are discussed in Section 1.2 are also valid here.

In this section, by using the unscented transform (UT) (Section 2.6.1), we extend the

algorithms PKF-EP, ECKF, and MAKF, which are presented in Section 4.1, to the nonlinear case. These unscented-based approaches provide suboptimal solutions to the equality-constrained state-estimation problem for nonlinear systems. Unlike the linear case addressed in Section 5.2, these approaches do not guarantee that the nonlinear equality constraint (5.37) is exactly satisfied, but they provide approximate solutions.

5.3.1 The Equality-Constrained Unscented Kalman Filter

The projection step of ECKF given by (5.14)-(5.19) is now extended to the nonlinear case by means of UT.

Using (2.82)-(2.85) to choose sigma points and weights associated to the data-assimilation state estimate $\hat{x}_{k|k}$ (2.27) and associated error covariance $P_{k|k}^{xx}$ (2.28), we have

$$\left[\gamma_x, \gamma_P, \mathcal{X}_{k|k}^x \right] = \Psi_{\text{UT}} \left(\hat{x}_{k|k}, P_{k|k}^{xx}, n, \alpha, \beta, \kappa \right), \quad (5.38)$$

where Ψ_{UT} is defined by (2.86). Then the sigma points $\mathcal{X}_{j,k|k}^x \in \mathbb{R}^n$, $j = 0, \dots, 2n$, are propagated through (5.37), producing

$$\mathcal{D}_{j,k|k} = g(\mathcal{X}_{j,k|k}^x, k-1), \quad j = 0, \dots, 2n, \quad (5.39)$$

such that $\hat{d}_{k-1|k}$, $P_{k|k}^{dd}$, and $P_{k|k}^{xd}$ are given by

$$\hat{d}_{k-1|k} = \sum_{j=0}^{2n} \gamma_{j,x} \mathcal{D}_{j,k|k}, \quad (5.40)$$

$$P_{k|k}^{dd} = \sum_{j=0}^{2n} \gamma_{j,P} [\mathcal{D}_{j,k|k} - \hat{d}_{k-1|k}] [\mathcal{D}_{j,k|k} - \hat{d}_{k-1|k}]^T + \delta R_k, \quad (5.41)$$

$$P_{k|k}^{xd} = \sum_{j=0}^{2n} \gamma_{j,P} [\mathcal{X}_{j,k|k}^x - \hat{x}_{k|k}] [\mathcal{D}_{j,k|k} - \hat{d}_{k-1|k}]^T, \quad (5.42)$$

where δR_k is set as in Remark 5.2.2, and K_k^p , $\hat{x}_{k|k}^p$, and $P_{k|k}^{xpp}$ are given by (5.17), (5.18), and (5.19), respectively.

Now define the augmented state vector $\tilde{x}_{k-1}^p \in \mathbb{R}^{\tilde{n}}$ and the augmented error covariance

$P_{k-1|k-1}^{\tilde{x}\tilde{x}p} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ as

$$\tilde{x}_{k-1}^p \triangleq \begin{bmatrix} x_{k-1}^p \\ w_{k-1} \\ v_k \end{bmatrix}, \quad P_{k-1|k-1}^{\tilde{x}\tilde{x}p} \triangleq \begin{bmatrix} P_{k-1|k-1}^{xxp} & 0_{n \times q} & 0_{n \times r} \\ 0_{q \times n} & Q_{k-1} & 0_{q \times r} \\ 0_{r \times n} & 0_{r \times q} & R_k \end{bmatrix}, \quad (5.43)$$

where $\tilde{n} \triangleq n + q + r$, such that the sigma-point matrix $\tilde{\mathcal{X}}_{k-1|k-1}$ in

$$\begin{bmatrix} \gamma_x, \gamma_P, \tilde{\mathcal{X}}_{k-1|k-1} \end{bmatrix} = \Psi_{\text{UT}} \left(\hat{\tilde{x}}_{k-1|k-1}^p, P_{k-1|k-1}^{\tilde{x}\tilde{x}p}, \tilde{n}, \alpha, \beta, \kappa \right), \quad (5.44)$$

is chosen based on $\hat{x}_{k-1|k-1}^p$ (5.18), and is partitioned as in (2.97). Then, by appending the projection step given by (5.38)-(5.42), (5.17)-(5.19) to the UKF forecast equations (5.44), (2.90)-(2.96), and to the UKF data-assimilation equations (2.25), (2.27)-(2.28), as illustrated in Figure 5.1, we obtain the *equality-constrained unscented Kalman filter* (ECUKF).

5.3.2 The Projected Unscented Kalman Filter by Estimate-Projection

If, unlike ECUKF, we append the projection step given by (5.38)-(5.42), (5.17)-(5.19) to the UKF equations given by (2.89)-(2.96), (2.25), (2.27)-(2.28) without feedback recursion as shown in Figure 4.1, we obtain the *projected unscented Kalman filter by estimate-projection* (PUKF-EP). Note that PUKF-EP is the nonlinear extension of PKF-EP, which is reviewed in Section 4.1.3.

5.3.3 The Measurement-Augmentation Unscented Kalman Filter

To extend MAKF presented in Section 4.1.2 to the nonlinear case, we replace (5.2) by the augmented observation (4.42). Also, define the augmented state vector $\tilde{x}_{k-1} \in \mathbb{R}^{\tilde{n}}$ and the augmented error covariance $P_{k-1|k-1}^{\tilde{x}\tilde{x}} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ as

$$\tilde{x}_{k-1} \triangleq \begin{bmatrix} x_{k-1} \\ w_{k-1} \\ \tilde{v}_k \end{bmatrix}, \quad P_{k-1|k-1}^{\tilde{x}\tilde{x}} \triangleq \begin{bmatrix} P_{k-1|k-1}^{xx} & 0_{n \times q} & 0_{n \times (r+s)} \\ 0_{q \times n} & Q_{k-1} & 0_{q \times (r+s)} \\ 0_{(r+s) \times n} & 0_{(r+s) \times q} & \tilde{R}_k \end{bmatrix}, \quad (5.45)$$

where $\tilde{n} \triangleq n + q + r + s$, $\tilde{v}_k$ is given by (4.6), and $\tilde{R}_k$ is given by (4.10). Remark 4.1.1 regarding $\tilde{R}_k$ is also applicable here.

With (4.42), the *measurement-augmentation unscented Kalman filter* (MAUKF) combines (2.89)-(2.92) with the augmented forecast equations

$$\tilde{\mathcal{Y}}_{j,k|k-1} = \tilde{h}(\mathcal{X}_{j,k|k-1}^x, \mathcal{X}_{j,k|k-1}^v, k), \quad j = 0, \dots, 2\tilde{n}, \quad (5.46)$$

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \tilde{\mathcal{Y}}_{j,k|k-1}, \quad (5.47)$$

$$P_{k|k-1}^{\tilde{y}\tilde{y}} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\tilde{\mathcal{Y}}_{j,k|k-1} - \hat{y}_{k|k-1}][\tilde{\mathcal{Y}}_{j,k|k-1} - \hat{y}_{k|k-1}]^T, \quad (5.48)$$

$$P_{k|k-1}^{x\tilde{y}} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{X}_{j,k|k-1}^x - \hat{x}_{k|k-1}][\tilde{\mathcal{Y}}_{j,k|k-1} - \hat{y}_{k|k-1}]^T, \quad (5.49)$$

while its data-assimilation equations given by (4.11)-(4.13).

5.4 Linear Simulated Examples

In this section, two examples are investigated using KF, ECKF, MAKf, PKF-EP, PKF-GP, and PKF-SP, which are, respectively, presented in Sections 2.2.1, 5.2, 4.1.2, 4.1.3, 4.1.4, and 4.1.5.

5.4.1 Compartmental System

Consider the linear discrete-time compartmental model (5.1)-(5.2) involving mass exchange among compartments (Bernstein & Hyland, 1993) whose matrices are given by

$$A = \begin{bmatrix} 0.94 & 0.028 & 0.019 \\ 0.038 & 0.95 & 0.001 \\ 0.022 & 0.022 & 0.98 \end{bmatrix}, \quad B = 0_{3 \times 1}, \quad G = \begin{bmatrix} 0.05 & -0.03 \\ -0.02 & 0.01 \\ -0.03 & 0.02 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (5.50)$$

with state vector $x_k \in \mathbb{R}^3$ composed by the amount of mass in each compartment, initial condition $x_0 = [1 \ 1 \ 1]^T$, and process noise and observation noise covariance matrices $Q_{k-1} = \sigma_w^2 I_{3 \times 3}$ and $R_k = \sigma_v^2 I_{2 \times 2}$. The free-run simulation of this system is shown in Figure 5.2 for $\sigma_w = 1.0$. Note that (5.5)-(5.7) hold for (5.50) such that the trajectory of $x_k \in \mathbb{R}^3$ lies on the plane (5.10), whose parameters are assumed to be known and are given by

$$D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad d = 3, \quad (5.51)$$

that is, conservation of mass is verified.

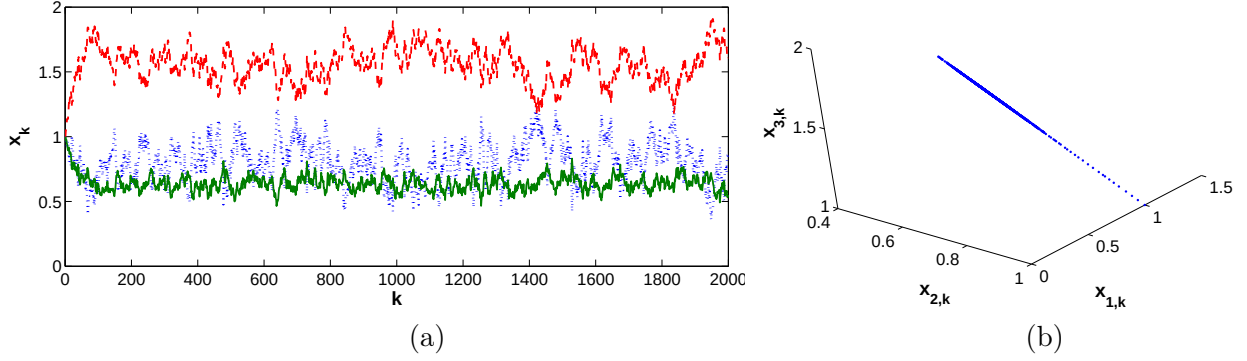


Figure 5.2: Free-run simulation of the compartmental model. In (a), the state components are shown evolving with time and, in (b), in state space. Note that, for all $k \geq 0$, $x_k \in \mathbb{R}^3$ lies on the plane $x_{1,k} + x_{2,k} + x_{3,k} = 3$.

For state estimation, the KF algorithm is initialized with

$$\hat{x}_{0|0} = [2 \quad 1 \quad 0]^T, \quad P_{0|0}^{xx} = I_{3 \times 3}. \quad (5.52)$$

Figure 5.3 shows that the KF estimates do not lie on the plane (5.10). Even if $\hat{x}_{0|0} = x_0$ or $\sigma_w = 0$, KF does not produce estimates satisfying (5.10).

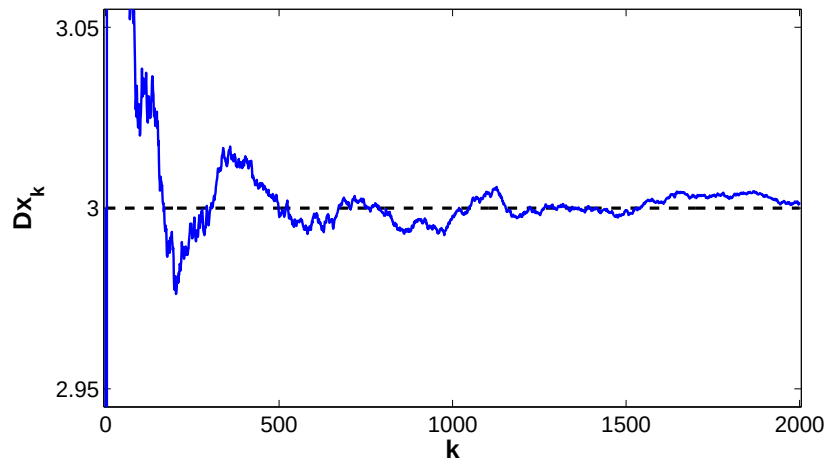


Figure 5.3: Estimate of the total mass (constraint) Dx_k in the conservative compartmental system (5.50) using KF (—) in comparison with the true value (---).

Next, we implement the ECKF algorithm. From a 100-run Monte Carlo simulation for each one of these process noise levels, namely, $\sigma_w = 0, 0.1, 0.5, 1.0$, and $\sigma_v = 0.01$, Table 5.1

shows that the ECKF estimates satisfy the equality constraint. In addition, these estimates are both more accurate and more informative than the KF estimates.

For MAKF, PKF-EP, PKF-GP, and PKF-SP, initialization is defined as in (5.52), except for PKF-SP; see (4.35)-(4.36). Table 5.1 summarizes the results. ECKF, MAKF, PKF-EP, PKF-GP, and PKF-SP guarantee that (5.10) is satisfied and yield improved estimates compared to KF. All equality-constrained methods produce similar results concerning RMSE and MT for this time-invariant system. This result is in accordance with (Ko & Bitmead, 2007, Theorem 2) regarding PKF-SP and PKF-EP. Recall that the numerical differences regarding the RMS constraint error in Table 5.1 are negligible for the equality-constrained algorithms. However, though not shown in Table 5.1, it is relevant to mention that PKF-EP produces less accurate and less informative forecast estimates $\hat{x}_{k|k-1}$ compared to the other constrained algorithms. This is expected because, as illustrated in Figure 4.1, PKF-EP does not use $\hat{x}_{k-1|k-1}^P$ to calculate $\hat{x}_{k|k-1}$.

Table 5.1: RMS constraint error, RMSE (2.31), and MT (2.33), from $k = 1500$ to 2000 , for 100-run Monte Carlo simulation for compartmental system, concerning process noise levels $\sigma_w = 0, 0.1, 0.5$, and 1.0 , and KF, ECKF, MAKF, PKF-EP, PKF-GP, and PKF-SP algorithms.

σ_w	KF	ECKF	MAKF	PKF-EP	PKF-GP	PKF-SP
	Percent RMS constraint error					
0	0.12	4.52×10^{-15}	4.24×10^{-11}	4.53×10^{-15}	3.39×10^{-13}	8.19×10^{-12}
0.1	0.22	4.52×10^{-15}	2.01×10^{-11}	4.52×10^{-15}	4.01×10^{-13}	8.19×10^{-12}
0.5	0.40	4.50×10^{-15}	0.88×10^{-11}	4.51×10^{-15}	3.92×10^{-13}	8.19×10^{-12}
1.0	0.62	4.53×10^{-15}	0.50×10^{-11}	4.51×10^{-15}	5.17×10^{-13}	8.19×10^{-12}
	RMSE $_i$, $i = 1, 2, 3$ ($\times 10^{-3}$)					
0	0.57, 0.36, 2.93	0.10, 0.16, 0.21				
0.1	6.26, 2.60, 7.34	6.25, 2.54, 4.19				
0.5	9.01, 4.58, 13.2	9.01, 4.55, 6.75				
1.0	9.35, 5.58, 19.7	9.35, 5.56, 8.07				
	MT ($\times 10^{-4}$)					
0	0.0996	0.0012				
0.1	1.0515	0.6352				
0.5	2.8057	1.4722				
1.0	5.4646	1.8387				

5.4.2 Tracking a Land-Based Vehicle

Consider a land vehicle whose linear dynamics are represented by (5.1)-(5.2), (5.10) with matrices given by

$$A = \begin{bmatrix} 1 & 0 & T_s & 0 \\ 0 & 1 & 0 & T_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ T_s \sin \theta \\ T_s \cos \theta \end{bmatrix}, G = \begin{bmatrix} -\sin \theta & 0 \\ -\cos \theta & 0 \\ 0 & \sin \theta \\ 0 & \cos \theta \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (5.53)$$

$$D = \begin{bmatrix} 1 & -\tan \theta & 0 & 0 \\ 0 & 0 & 1 & -\tan \theta \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (5.54)$$

the state vector $x_k \in \mathbb{R}^4$ compounded by northerly and easterly position and velocity components, $T_s = 2s$, $x_0 = [0 \ 0 \ 10 \tan \theta \ 10]^T$, $Q_{k-1} = \text{diag}(10, 10)$, and $R_k = \text{diag}(400, 10)$. This example is also investigated in (Simon & Chia, 2002; Ko & Bitmead, 2007; Simon, 2006). It is known that the vehicle is moving in a straight line with a heading of $\theta = 60$ deg. That is, an equality constraint (5.10) defines the subspace to which the vehicle trajectory is confined. Unlike (Simon & Chia, 2002; Simon, 2006), G given by (5.53) projects the process noise w_{k-1} onto (5.10). Recall that (5.5)-(5.7) are satisfied for (5.53)-(5.54). That is, (5.53)-(5.54) define an equality-constrained system such that a modified Q_{k-1} need not to be used by PKF-SP as in (Ko & Bitmead, 2007). The commanded acceleration u_{k-1} , which is assumed to be known, is alternatively set to $\pm 1 \text{ m/s}^2$, as if the vehicle was accelerating and decelerating in traffic.

State estimation is initialized with

$$\hat{x}_{0|0} = \begin{bmatrix} 500 & \frac{500}{\tan \theta} & 30 & \frac{30}{\tan \theta} \end{bmatrix}^T, \quad P_{0|0}^{xx} = \text{diag}(900, 900, 4, 4), \quad (5.55)$$

for KF, ECKF, MAKF, PKF-EP, PKF-GP, and PKF-SP algorithms, except for PKF-SP; see (4.35)-(4.36). Table 5.2 summarizes the results obtained from a 100-run Monte Carlo simulation. All equality-constrained methods guarantee that the constraint (5.54) is satisfied and yield more accurate and more informative estimates than KF. Note that ECKF, MAKF, PKF-EP, PKF-GP, and PKF-SP produce similar results concerning RMSE and MT.

Table 5.2: RMS constraint errors for (5.54), RMSE (2.31), and MT (2.33), from $k = 50$ to 500, for 100-run Monte Carlo simulation for land-based vehicle system for the KF, MAKF, PKF-EP, PKF-GP, PKF-SP, and ECKF algorithms.

KF	ECKF	PKF-EP	PKF-GP	PKF-SP	MAKF
RMS constraint error					
3.7	7.0×10^{-16}	4.4×10^{-8}	4.2×10^{-10}	2.0×10^{-12}	3.2×10^{-9}
5.1×10^{-2}	4.7×10^{-15}	7.1×10^{-9}	2.5×10^{-10}	1.7×10^{-13}	4.4×10^{-12}
RMSE _{<i>i</i>} (m), <i>i</i> = 1, 2, 3, 4					
5.93, 2.73, 3.53, 2.04	4.72, 2.72, 3.52, 2.04				
MT					
53.8	46.2				

5.5 Nonlinear Simulated Examples

In this section, two nonlinear examples are investigated using UKF, ECUKF, PUKF-EP, and MAUKF, which are, respectively, presented in Sections 2.6.2, 5.3.1, 5.3.2, and 5.3.3. Also, in Appendix B, we present an application of equality-constrained state estimation to magnetohydrodynamics.

5.5.1 Attitude Estimation

Consider an attitude estimation problem (Crassidis & Markley, 2003; Kuang & Tan, 2002; Choukroun et al., 2006), whose kinematics are modeled as

$$\dot{e}(t) = -\frac{1}{2}\Omega(t)e(t), \quad (5.56)$$

where the state vector is the quaternion vector $e(t) = [e_0(t) \ e_1(t) \ e_2(t) \ e_3(t)]^T$, the matrix $\Omega(t)$ is given by

$$\Omega(t) = \begin{bmatrix} 0 & r(t) & -q(t) & p(t) \\ -r(t) & 0 & p(t) & q(t) \\ q(t) & -p(t) & 0 & r(t) \\ -p(t) & -q(t) & -r(t) & 0 \end{bmatrix}, \quad (5.57)$$

and the angular velocity vector $u(t) = [p(t) \ q(t) \ r(t)]^T$ is a known input. Since $\|e(0)\|_2 = 1$ and $\Omega(t)$ is skew symmetric, it follows that, for all $t > 0$,

$$\|e(t)\|_2 = 1. \quad (5.58)$$

We set $e(0) = [0.9603 \ 0.1387 \ 0.1981 \ 0.1387]^T$ and

$$u(t) = \begin{bmatrix} 0.03 \sin\left(\frac{2\pi}{600}t\right) & 0.03 \sin\left(\frac{2\pi}{600}t - 300\right) & 0.03 \sin\left(\frac{2\pi}{600}t - 600\right) \end{bmatrix}^T. \quad (5.59)$$

To perform attitude estimation, we assume that

$$\check{u}_{k-1} = u((k-1)T_s) + \beta_{k-1} + w_{k-1}^u \quad (5.60)$$

is measured by rate gyros, where T_s is the discretization step, $w_{k-1}^u \in \mathbb{R}^3$ is zero mean, Gaussian noise, and $\beta_{k-1} \in \mathbb{R}^3$ is drift error. The discrete-time equivalent of (5.56) augmented by the gyro drift random-walk model (Crassidis & Markley, 2003) is given by

$$\begin{bmatrix} e_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} A_{k-1} & 0_{4 \times 3} \\ 0_{3 \times 4} & I_{3 \times 3} \end{bmatrix} \begin{bmatrix} e_{k-1} \\ \beta_{k-1} \end{bmatrix} + \begin{bmatrix} 0_{4 \times 1} \\ w_{k-1}^\beta \end{bmatrix}, \quad (5.61)$$

where $e_k \triangleq e(kT_s)$, $w_{k-1}^\beta \in \mathbb{R}^3$ is process noise associated with the drift-error model, $x_k \triangleq \begin{bmatrix} e_k \\ \beta_k \end{bmatrix} \in \mathbb{R}^7$ is the state vector, $w_{k-1} \triangleq \begin{bmatrix} w_{k-1}^u \\ w_{k-1}^\beta \end{bmatrix} \in \mathbb{R}^6$ is the process noise, and

$$A_{k-1} \triangleq \cos(s_{k-1})I_{4 \times 4} - \frac{1}{2} \frac{T_s \sin(s_{k-1})}{s_{k-1}} \Omega_{k-1}, \quad (5.62)$$

$$\Omega_{k-1} \triangleq \Omega((k-1)T_s), \quad (5.63)$$

$$s_{k-1} \triangleq \frac{T_s}{2} \|\check{u}_{k-1} - \beta_{k-1} - w_{k-1}^u\|_2. \quad (5.64)$$

The constraint (5.58) also holds for $t = kT_s$, that is,

$$x_{1,k}^2 + x_{2,k}^2 + x_{3,k}^2 + x_{4,k}^2 = 1. \quad (5.65)$$

We set $T_s = 0.1$ s, $\beta_{k-1} = [0.001 \ -0.001 \ 0.0005]^T$ rad/s, and $Q_{k-1} = \text{diag}(10^{-5}I_{3 \times 3}, 10^{-10}I_{3 \times 3})$. Attitude observations $y_k^{[i]} \in \mathbb{R}^3$ for a direction sensor are given by

(Crassidis & Markley, 2003)

$$y_k^{[i]} = C_k r^{[i]} + v_k^{[i]}, \quad (5.66)$$

where $r^{[i]} \in \mathbb{R}^3$ is a reference direction vector to a known point, and C_k is the rotation matrix from the reference to the body-fixed frame

$$C_k = \begin{bmatrix} x_{1,k}^2 - x_{2,k}^2 - x_{3,k}^2 + x_{4,k}^2 & 2(x_{1,k}x_{2,k} - x_{3,k}x_{4,k}) & 2(x_{1,k}x_{3,k} - x_{2,k}x_{4,k}) \\ 2(x_{1,k}x_{2,k} - x_{3,k}x_{4,k}) & -x_{1,k}^2 + x_{2,k}^2 - x_{3,k}^2 + x_{4,k}^2 & 2(x_{1,k}x_{4,k} + x_{2,k}x_{3,k}) \\ 2(x_{1,k}x_{3,k} + x_{2,k}x_{4,k}) & 2(-x_{1,k}x_{4,k} + x_{2,k}x_{3,k}) & -x_{1,k}^2 - x_{2,k}^2 + x_{3,k}^2 + x_{4,k}^2 \end{bmatrix}. \quad (5.67)$$

We assume that two directions are available (Crassidis & Markley, 2003; Lee et al., 2007), which can be obtained using either a star tracker or a combined three-axis accelerometer/three-axis magnetometer. We set $r^{[1]} = [1 \ 0 \ 0]^T$, $r^{[2]} = [0 \ 1 \ 0]^T$, and $R_k = 10^{-4}I_{6 \times 6}$. These direction measurements are assumed to be provided at a lower rate, specifically, at 1 Hz, which corresponds to a sample interval of $10T_s$ s.

We implement state estimation using UKF, ECUKF, PUKF-EP, and MAUKF with (5.61), (5.66), and constraint (5.65). We initialize these algorithms with

$$\hat{x}_{0|0} = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad P_{0|0}^{xx} = \text{diag}(0.5I_{4 \times 4} \ 0.01I_{3 \times 3}). \quad (5.68)$$

Table 5.3: RMS constraint error, RMSE (2.31), and MT (2.33), from $k = 600$ to 10000, for 100-run Monte Carlo simulation for attitude estimation using UKF, ECUKF, PUKF-EP, and MAUKF.

UKF	ECUKF	PUKF-EP	MAUKF
Percent RMS constraint error ($\times 10^{-4}$)			
240.8	6.49	8.46	6.50
RMSE _i for $e_0, e_1, e_2, e_3, \beta_1, \beta_2, \beta_3$ ($\times 10^{-3}$)			
1.336	1.401	1.334	1.401
1.382	1.452	1.382	1.452
1.355	1.421	1.355	1.421
1.345	1.406	1.344	1.406
0.124	0.129	0.124	0.129
0.123	0.128	0.123	0.129
0.123	0.129	0.124	0.129
MT for attitude ($\times 10^{-6}$)			
8.18	5.41	8.16	5.43
MT for drift ($\times 10^{-7}$)			
1.17	1.09	1.17	1.10

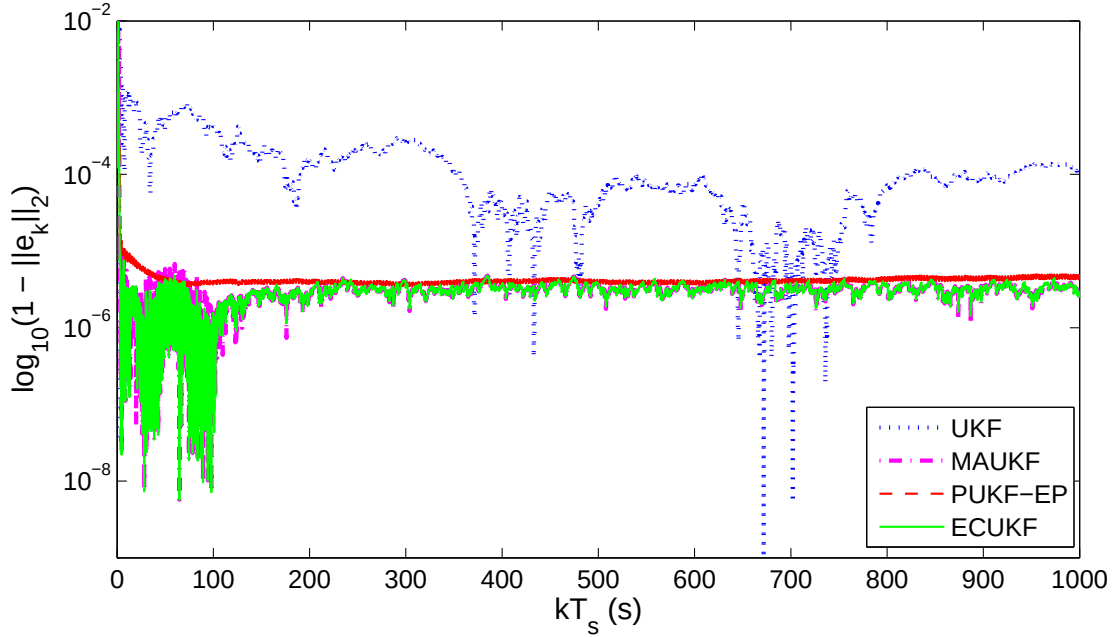


Figure 5.4: Estimation errors of the quaternion vector norm using UKF, ECUKF, PUKF-EP, and MAUKF algorithms. ECUKF and MAUKF estimates almost coincide.

Table 5.3 shows the results obtained from a 100-run Monte Carlo simulation. With the usage of prior knowledge, the values of MT show that more informative estimates are produced compared to the unconstrained estimates given by UKF. However, a slight increase in the RMS error is observed for algorithms ECUKF and MAUKF implying loss of accuracy around 5% compared to UKF. The equality constraint is more closely tracked whenever a constrained filter is employed; see Figure 5.4. Percent errors around 10^{-4} are obtained, and ECUKF exhibits the smallest error.

5.5.2 Simple Pendulum

We consider the continuous-time undamped and unforced pendulum

$$\ddot{\theta}(t) + \frac{g}{L} \sin \theta(t) = 0, \quad (5.69)$$

where $\theta(t)$ is the angular position such that $\theta = 0$ rad corresponds to the stable equilibrium position, g is the gravity acceleration, and L is the length. Given noisy measurements of angular position and/or angular velocity of the pendulum, we want to obtain estimates for these variables that satisfy the energy conservation property.

Using Euler discretization with time step T_s , such that $t = kT_s$, and defining $x_{1,k} = \theta(kT_s)$ and $x_{2,k} = \dot{\theta}(kT_s)$, we obtain the approximate discretized model

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k-1} + T_s x_{2,k-1} \\ x_{2,k-1} - T_s \frac{g}{L} \sin(x_{1,k-1}) \end{bmatrix} + \begin{bmatrix} w_{1,k} \\ w_{2,k} \end{bmatrix}, \quad (5.70)$$

with noisy measurements of the true (continuous-time model) values of angular position and velocity given by

$$\begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(kT_s) \\ x_2(kT_s) \end{bmatrix} + \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix}. \quad (5.71)$$

By conservation of energy in (5.69) we have

$$-mgL \cos(x_1(t)) + \frac{mL^2}{2} x_2^2(t) = E(0), \quad (5.72)$$

where $E(0)$ is the total mechanical energy and m is the pendulum mass. Next, we define the approximate energy E_k of the discrete-time model by

$$E_k \triangleq -mgL \cos(x_{1,k}) + \frac{mL^2}{2} x_{2,k}^2. \quad (5.73)$$

We use the 4-th order Runge-Kutta integration scheme to obtain $x(kT_s)$ for $L = 1$ m, $T_s = 10$ ms, and initial conditions $\theta(0) = \frac{3\pi}{4}$, $\dot{\theta}(0) = \frac{\pi}{50}$. We assume that $E(0)$ is known and we implement equality-constrained state estimation by constraining $E_k = E(0)$ for all $k \geq 1$. The state estimation is initialized with $Q_{k-1} = \sigma_w^2 I_{2 \times 2}$, $R_k = \sigma_v^2 I_{2 \times 2}$, $\hat{x}_{0|0} = [1 \ 1]^T$, and $P_{0|0}^{xx} = I_{2 \times 2}$, where three values of observation noise are tested, namely, $\sigma_v = 0.1, 0.25$, and 0.5 , and process noise with $\sigma_w = 0.007$ is set to help convergence of estimates as discussed in Section 2.6.3. A 100-run Monte Carlo simulation is performed for each σ_v .

Table 5.4 shows the percent RMS errors related to the equality constraint (5.73). Figure 5.5 compares the accuracy of the algorithms with relation to (5.73). It can be noticed, in this example, that the free-run simulation of the discretized model results in an unrealistic increasing energy E_k , that is, the model is unstable. UKF is not able to closely track the constraint. For higher observation noise levels, that is, $\sigma_v = 0.5$, RMS constraint errors between 4% and 7% are observed. Whenever equality-constrained filters are employed, these indices are substantially reduced. In addition to improvement in the accuracy of constraint

satisfaction, the usage of prior knowledge also results in more accurate and more informative estimates.

Table 5.4: RMS constraint error, RMSE (2.31), and MT (2.33), from $k = 3000$ to 4000 , for 100-run Monte Carlo simulation for the discretized pendulum, concerning different levels of observation noise $\sigma_v = 0.1, 0.25$, and 0.5 , and algorithms, namely, UKF, ECUKF, PUKF-EP, and MAUKF.

σ_v	UKF	ECUKF	PUKF-EP	MAUKF
Percent RMS constraint error ($\times 10^{-2}$)				
0.1	2.6×10^2	1.33	2.3	1.34
0.25	4.1×10^2	2.86	5.7	2.86
0.5	5.7×10^2	5.20	11.4	5.19
RMSE $_i, i = 1, 2$ ($\times 10^{-2}$)				
0.1	1.92, 2.48	0.848, 1.794	0.89, 1.90	0.849, 1.794
0.25	3.08, 4.96	1.265, 2.832	1.49, 3.44	1.266, 2.833
0.5	4.81, 8.69	1.738, 3.874	2.51, 5.50	1.739, 3.876
MT ($\times 10^{-3}$)				
0.1	1.45	0.697	0.73	0.697
0.25	4.37	1.836	2.22	1.837
0.5	11.0	3.778	5.59	3.779

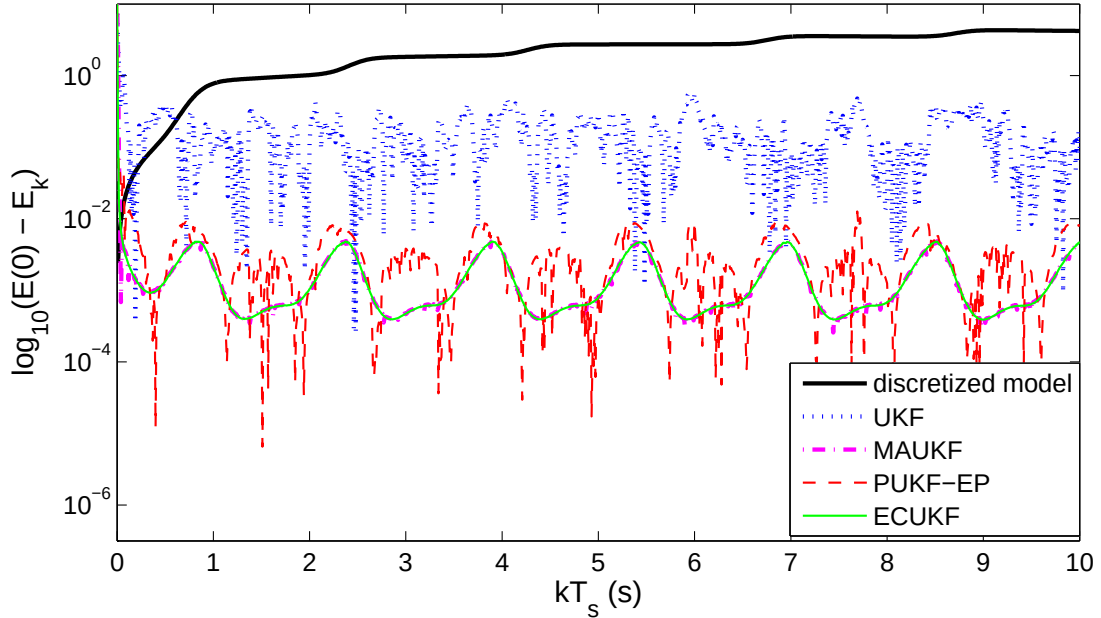


Figure 5.5: Estimation error of the total energy $E(0)$ for the discretized pendulum (5.70)-(5.71) for UKF, MAUKF, PUKF-EP, and ECUKF with $\sigma_v = 0.25$. The thick black solid line refers to the energy E_k calculated from free-run simulation of the discretized model (5.70). Energy estimates using MAUKF and ECUKF almost coincide.

Table 5.5: RMS constraint error, RMSE (2.31), and MT (2.33), from $k = 3000$ to 4000 , for 100-run Monte Carlo simulation for the single pendulum, concerning the continuous-time process model (5.69), $\sigma_v = 0.25$, algorithms UKF, ECUKF, PUKF-EP, and MAUKF, and different observation models. Whenever model (5.69) is used, we set $\sigma_w = 0.0003$ to help convergence of estimates.

y	UKF	ECUKF	PUKF-EP	MAUKF
Percent RMS constraint error ($\times 10^{-2}$)				
$x_1, x_2 + (5.70)$	4.1×10^2	2.86	5.7	2.86
x_1	6.5×10^2	5.70	10.8	5.68
x_2	4.8×10^2	3.46	9.3	3.47
x_1, x_2	0.3×10^2	0.12	0.9	0.12
RMSE $_i$, $i = 1, 2$ ($\times 10^{-2}$)				
$x_1, x_2 + (5.70)$	3.08, 4.96	1.265, 2.832	1.49, 3.44	1.266, 2.833
x_1	4.83, 11.8	1.868, 4.096	2.77, 6.64	1.866, 4.098
x_2	4.03, 5.73	1.338, 3.031	1.78, 3.81	1.340, 3.032
x_1, x_2	0.96, 2.11	0.247, 0.548	0.68, 1.49	0.247, 0.548
MT ($\times 10^{-3}$)				
$x_1, x_2 + (5.70)$	4.37	1.836	2.22	1.837
x_1	17.7	4.818	9.45	4.818
x_2	6.16	2.073	2.66	2.074
x_1, x_2	0.71	0.084	0.36	0.084

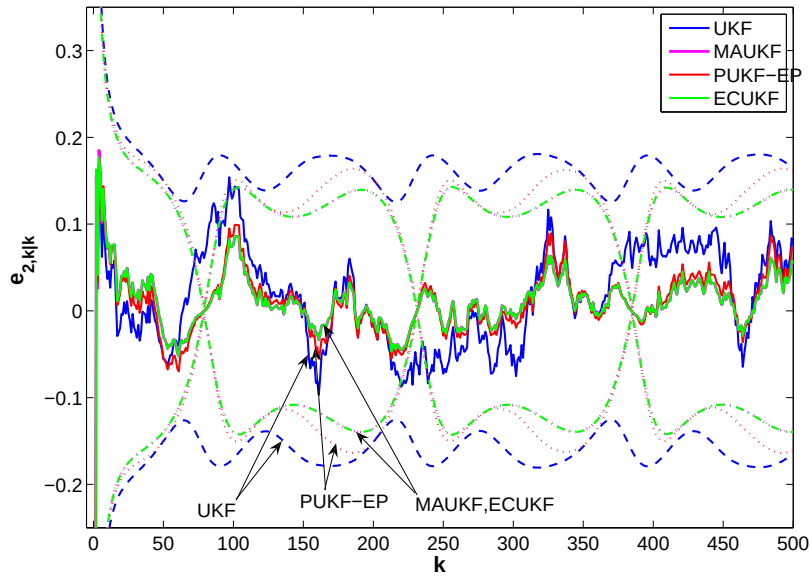


Figure 5.6: Comparison of estimation errors $e_{2,k|k} = x_2(kT_s) - \hat{x}_{2,k|k}^p$ (solid lines) of angular position for UKF, ECUKF, PUKF-EP, and MAUKF algorithms at $\sigma_v = 0.25$ and associated $\pm 3\sqrt{P_{(2,2),k|k}^{xxp}}$ limits (dashed, dot-dashed, dotted, and dot-dashed lines, respectively). The estimate error and the confidence intervals for ECUKF and MAUKF coincide.

According to Table 5.5, the same comparative analysis is applicable when the true continuous-time model (5.69) or when a different linear observation model is used. The use of a 4-th Runge-Kutta integration, as discussed in Section A.2, in place of Euler discretization yields better RMSE and MT indices, but with approximately 10 times larger RMSE constraint error and with approximately 6 times larger computational time.

This numerical example suggests that ECUKF and MAUKF are better suited to equality-constrained nonlinear filtering than PUKF-EP. For this nonlinear example, the performance of ECUKF and MAUKF are similar. Figure 5.6 illustrates how the estimation error for angular position $x_{1,k}$ and associated pseudo-error covariance evolve. Note that, whenever $x_{1,k} = 0$, a more informative estimate is observed for $x_{2,k}$.

5.6 Concluding Remarks

In this chapter we show that the problem of equality-constrained state estimation for linear systems arises from the definition of both process noise and dynamic equations with special properties, specifically, (5.5)-(5.7), such that the system is not controllable in $\mathbb{R}^n$ from the process noise. In this case, the classical Kalman filter (KF) does not guarantee that its estimates satisfy the equality constraint on the state vector.

Then we present the equality-constrained Kalman filter (ECKF) as the optimal solution to this problem. Moreover, we mention its equivalence to the measurement-augmentation Kalman filter (MAKF) and present its connections to the projection Kalman filter by system-projection (PKF-SP), the projection Kalman filter by estimate-projection (PKF-EP), and the projection Kalman filter by gain-projection (PKF-GP). We compare these five methods by means of two examples: a compartmental model with mass conservation and a vehicle tracking with kinematic constraints.

For the nonlinear case, three suboptimal algorithms based on the unscented Kalman filter (UKF) are presented, namely, the equality-constrained unscented Kalman filter (ECUKF), the projected unscented Kalman filter (PUKF-EP), and the measurement-augmented unscented Kalman filter (MAUKF). These methods are compared on two examples, including a quaternion-based attitude estimation problem, as well as a mechanical system involving conserved energy.

Numerical results suggest that, in addition to exactly (in the linear case) or very closely (nonlinear case) satisfying a known equality constraint of the system, the proposed

methods can be used to produce more accurate and more informative estimates than KF (in the linear case) or UKF (in nonlinear case). For the linear scenario, the five equality-constrained methods yield similar results. For nonlinear systems, among the algorithms investigated, ECUKF and MAUKF seem to be the most suitable for equality-constrained state-estimation applications. Moreover, all equality-constrained approaches require similar processing time, which is competitive to KF (for linear algorithms) and UKF (for nonlinear cases) processing time.

Finally, we address the case where an approximate discretized model is used to represent a continuous-time process in state estimation. Improved estimates are obtained when equality-constrained Kalman filtering algorithms are employed to enforce conserved quantities of the original continuous-time model, but without the higher computational burden required by more accurate integration schemes.

INTERVAL-CONSTRAINED STATE ESTIMATION: A COMPARISON OF APPROXIMATE SOLUTIONS

“You can never solve a problem on the level on which it was created.”

Albert Einstein

The present chapter addresses the state-estimation problem for interval-constrained nonlinear systems. First, Section 6.1 defines this problem, whose literature review is presented in Sections 4.4 and 4.5. Then, in Section 6.2, we present six approximate solutions to the aforementioned problem that are based on the unscented Kalman filter (UKF) (Section 2.6.2). Finally, in Section 6.3, these novel algorithms are compared to UKF and to the sigma-point interval unscented Kalman filter (SIUKF) (Section 4.5.4) in terms of accuracy and computational cost. This is accomplished by means of two illustrative examples: a batch reactor and a continuously stirred tank reactor. In both cases, our goal is to obtain nonnegative state estimates. The contributions presented in this chapter are partially published in (Teixeira, Tôrres, Aguirre & Bernstein, 2008).

6.1 State Estimation for Interval-Constrained Nonlinear Systems

The general state-estimation problem is defined in Section 1.2. Now we define the interval-constrained state-estimation problem. Thus, consider the stochastic nonlinear discrete-time dynamic system given by (1.1)-(1.2), whose equations are repeated here for convenience, that is,

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}, k-1), \quad (6.1)$$

$$y_k = h(x_k, v_k, k), \quad (6.2)$$

and whose state vector x_k is known to satisfy, for all $k \geq 0$, the interval constraint (4.73), that is,

$$a_k \leq x_k \leq b_k, \quad (6.3)$$

where $a_k \in \mathbb{R}^n$ and $b_k \in \mathbb{R}^n$ are assumed to be known and, for $j = 1, \dots, n$, $a_{j,k} < b_{j,k}$. Also, if $x_{j,k}$ is left-unbounded or right-unbounded, then we set $a_{j,k} = -\infty$ or $b_{j,k} = \infty$, respectively. Moreover, we assume that the mean and covariance of $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are known and equal to zero and Q_{k-1} , R_k , respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated.

Thus, the objective of the interval-constrained state-estimation problem is to maximize J in (1.3) subject to (6.3). That is, we look for the maximizer of J that satisfies (6.3). However, in addition to nonlinear dynamics (6.1)-(6.2), the solution to this problem is complicated due to the inclusion of the interval constraint (6.3). We thus extend approximate algorithms derived in the linear scenario (see Section 4.4) to provide suboptimal estimates in the nonlinear case.

6.2 Unscented Filtering for Interval-Constrained Nonlinear Systems

In this section we present suboptimal solutions based on UKF to the interval-constrained state-estimation problem. Such algorithms are combinations of the approaches reviewed in Sections 4.4 and 4.5; see Table 6.1.

We combine either the unscented transform (UT) given by (2.86) or the interval-constrained UT (ICUT) given by (4.115), which are used during the forecast step of UKF and

Table 6.1: Interval-constrained state estimators based on the unconstrained UKF. We make explicit the procedure used during the forecast step (column-wise), as well as the type of KF update used during data-assimilation (row-wise). Inside parentheses, we cite the section in which the method is either reviewed or presented.

data assimilation \ forecast	UT (2.6.1)	ICUT (4.5.4)
classical KF update (2.2.1)	UKF (2.6.2)	IUKF (6.2.3)
CEKF update (4.5.3)	CUKF (6.2.1)	CIUKF (6.2.2)
sigma-point constrained update (4.5.4)	–	SIUKF (4.5.4)
classical KF update plus truncation (4.4.5)	TUKF (6.2.4)	TIUKF (6.2.5)
classical KF update plus projection (4.4.4)	PUKF (6.2.6)	–

SIUKF, respectively, together with one of the following data-assimilation approaches, namely, (i) the classical KF update (2.25), (2.27)-(2.28), (ii) the constrained Kalman update of CEKF (4.105), (2.25), (2.28), (iii) the sigma-point constrained update of SIUKF (4.123)-(4.125), (iv) the classical KF update (2.25), (2.27)-(2.28) followed by either the truncation procedure (4.99), or (v) the projection approach (4.79). Then we obtain six algorithms, namely, the constrained UKF (CUKF), the constrained interval UKF (CIUKF), the interval UKF (IUKF), the truncated UKF (TUKF), the truncated interval UKF (TIUKF), and the projected UKF (PUKF), which are presented as follows.

6.2.1 The Constrained Unscented Kalman Filter

We now present the *constrained unscented Kalman filter* (CUKF). Combining UT for forecast and the inequality-constrained update of the constrained extended Kalman filter for data-assimilation, CUKF is the straightforward unscented-based extension of CEKF, which is reviewed in Section 4.5.3.

Similar to UKF, CUKF is a two-step estimator. The forecast step is given by

$$\left[\gamma_x, \gamma_P, \tilde{\mathcal{X}}_{k-1|k-1} \right] = \Psi_{\text{UT}} \left(\hat{x}_{k-1|k-1}^c, P_{k-1|k-1}^{\tilde{x}\tilde{x}}, \tilde{n}, \alpha, \beta, \kappa \right), \quad (6.4)$$

together with (2.90)-(2.96), where Ψ_{UT} is defined by (2.86), the sigma-point matrix $\tilde{\mathcal{X}}_{k-1|k-1}$ is partitioned as in (2.97), $P_{k-1|k-1}^{\tilde{x}\tilde{x}}$ is given by (2.88), and $\hat{x}_{k-1}^c$ is the augmented state vector given by

$$\hat{x}_{k-1}^c \triangleq \begin{bmatrix} x_{k-1}^c \\ w_{k-1} \\ v_k \end{bmatrix}, \quad (6.5)$$

where x_{k-1}^c is obtained from the data-assimilation step as follows. To enforce (6.3), we replace (2.27) of UKF by the constrained optimization problem

$$\hat{x}_{k|k}^c = \arg \min_{\{x_k : a_k \leq x_k \leq b_k\}} J_2(x_k) \quad (6.6)$$

where the cost function $J_2(x_k)$ is given by (4.106), such that the data-assimilation step of CUKF is given by (2.25), (6.6), and (2.28).

Note that the information provided by (6.3) is not assimilated into the error covariance

$P_{k|k}^{xx}$ in (2.28). Moreover, like SIUKF, it is also possible to enforce a general nonlinear inequality constraint (4.102) rather than the interval constraint (6.3) in the optimization problem (6.6).

6.2.2 The Constrained Interval Unscented Kalman Filter

The *constrained interval unscented Kalman filter* (CIUKF) is presented as a simplified version of SIUKF. CIUKF is a two-step estimator. Its forecast step is given by

$$\begin{bmatrix} \gamma_{k-1}, \tilde{\mathcal{X}}_{k-1|k-1} \end{bmatrix} = \Psi_{\text{ICUT}} \left(\hat{x}_{k-1|k-1}^c, P_{k-1|k-1}^{\tilde{x}\tilde{x}}, \tilde{a}_{k-1}, \tilde{b}_{k-1}, \tilde{n}, \kappa \right), \quad (6.7)$$

together with (4.120)-(4.122) and

$$\mathcal{Y}_{j,k|k-1} = h(\mathcal{X}_{j,k|k-1}^x, \mathcal{X}_{j,k|k-1}^v, k), \quad j = 0, \dots, 2\tilde{n}, \quad (6.8)$$

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} \mathcal{Y}_{j,k|k-1}, \quad (6.9)$$

$$P_{k|k-1}^{yy} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} [\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}] [\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^T, \quad (6.10)$$

$$P_{k|k-1}^{xy} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,k-1} [\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}] [\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^T, \quad (6.11)$$

where Ψ_{ICUT} is defined by (4.115), $\tilde{x}_{k-1}^c$ and $P_{k-1|k-1}^{\tilde{x}\tilde{x}}$ in (6.7) are, respectively, given by (6.5) and (2.88), and $\tilde{\mathcal{X}}_{k-1|k-1}$ is partitioned as in (2.97). The data-assimilation step of CIUKF is given by (2.25), (6.6), and (2.28).

Note that the forecast step of both CIUKF and SIUKF use ICUT, while the data-assimilation equations of CIUKF and CUKF (Section 6.2.1) are equal. That is, the data-assimilation step of CIUKF is a single sigma-point special case of the data-assimilation step of SIUKF. However, unlike SIUKF, $P_{k|k}^{xx}$ of CIUKF and CUKF are not affected by (6.3).

6.2.3 The Interval Unscented Kalman Filter

The *interval unscented Kalman filter* (IUKF), which is a simplified version of CIUKF (Section 6.2.2), is also a two-step estimator. Like CIUKF, its forecast step is given by

$$\begin{bmatrix} \gamma_{k-1}, \tilde{\mathcal{X}}_{k-1|k-1} \end{bmatrix} = \Psi_{\text{ICUT}} \left(\hat{x}_{k-1|k-1}^c, P_{k-1|k-1}^{\tilde{x}\tilde{x}}, \tilde{a}_{k-1}, \tilde{b}_{k-1}, \tilde{n}, \kappa \right), \quad (6.12)$$

together with (4.120)-(4.122), (6.8)-(6.11), while its data-assimilation step is given by (2.25), (2.27)-(2.28).

Note that the forecast equations of IUKF are equal to the forecast equations of CIUKF, while its data-assimilation equations are equal to the data-assimilation equations of UKF. Unlike in SIUKF and CIUKF, note that (6.3) is not enforced by IUKF during data assimilation.

6.2.4 The Truncated Unscented Kalman Filter

Now we present the *truncated unscented Kalman filter* (TUKF) as the unscented-based nonlinear extension of the truncated Kalman filter (TKF), which is reviewed in Section 4.4.5. TUKF is obtained by appending the PDF truncation procedure to the UKF equations by feedback recursion; see Figure 4.6. That is, TUKF is a three-step algorithm whose forecast step is given by

$$\left[\gamma_x, \gamma_P, \tilde{\mathcal{X}}_{k-1|k-1} \right] = \Psi_{\text{UT}} \left(\hat{x}_{k-1|k-1}^t, P_{k-1|k-1}^{\tilde{x}\tilde{x}t}, \tilde{n}, \alpha, \beta, \kappa \right), \quad (6.13)$$

together with (2.90)-(2.96), where Ψ_{UT} is given by (2.86), $\tilde{\mathcal{X}}_{k-1|k-1}$ is partitioned as in (2.97), $\hat{x}_{k-1}^t$ and $P_{k-1|k-1}^{\tilde{x}\tilde{x}t}$ are, respectively, given by

$$\tilde{x}_{k-1}^t \triangleq \begin{bmatrix} x_{k-1}^t \\ w_{k-1} \\ v_k \end{bmatrix}, \quad P_{k-1|k-1}^{\tilde{x}\tilde{x}t} \triangleq \begin{bmatrix} P_{k-1|k-1}^{xxt} & 0_{n \times q} & 0_{n \times r} \\ 0_{q \times n} & Q_{k-1} & 0_{q \times r} \\ 0_{r \times n} & 0_{r \times q} & R_k \end{bmatrix}. \quad (6.14)$$

Like UKF, the data-assimilation step of TUKF is given by (2.25), (2.27)-(2.28), while its truncation step is given by (4.99).

TUKF has two advantages. First, unlike CUKF and CIUKF, it avoids the explicit online solution of a constrained optimization problem at each time step. Second, TUKF assimilates the interval-constraint information in both the state estimate $\hat{x}_{k|k}^t$ and error covariance $P_{k|k}^{xxt}$.

6.2.5 The Truncated Interval Unscented Kalman Filter

We present the *truncated interval unscented Kalman filter* (TIUKF) obtained from the combination of IUKF (Section 6.2.3), which uses ICUT, and the PDF truncation approach of TUKF (Section 6.2.4).

TIUKF is a three-step estimator. Its forecast step is given by

$$\begin{bmatrix} \gamma_{k-1}, \tilde{\mathcal{X}}_{k-1|k-1} \end{bmatrix} = \Psi_{\text{ICUT}} \left(\hat{x}_{k-1|k-1}^t, P_{k-1|k-1}^{\tilde{x}\tilde{x}t}, \tilde{a}_{k-1}, \tilde{b}_{k-1}, \tilde{n}, \kappa \right), \quad (6.15)$$

together with (4.120)-(4.122), (6.8)-(6.11), where Ψ_{ICUT} is given by (4.115), $\tilde{\mathcal{X}}_{k-1|k-1}$ is partitioned as in (2.97), $\hat{x}_{k-1}^t$ and $P_{k-1|k-1}^{\tilde{x}\tilde{x}t}$ are given by (6.14). Like TUKF, the data-assimilation step of TIUKF is given by (2.25), (2.27)-(2.28), while the truncation step is given by (4.99).

Note that TIUKF enforces (6.3) during both the forecast step (in both $\hat{x}_{k|k-1}$ and $P_{k|k-1}^{xx}$) and truncation step (in both $\hat{x}_{k|k}^t$ and $P_{k|k}^{xxt}$).

6.2.6 The Projected Unscented Kalman Filter

The *projected unscented Kalman filter* (PUKF) is the unscented-based nonlinear extension of the projected Kalman filter for inequality-constrained systems (PKF-IC), which is presented in Section 4.4.4.

PUKF is obtained by appending the projection equation

$$\hat{x}_{k|k}^p = \arg \min_{\{x_k : a_k \leq x_k \leq b_k\}} J_p(x_k), \quad (6.16)$$

to UKF, where $J_p(x_k)$ is given by (4.15). Similar to PKF-IC, we set $W_k = P_{k|k}^{xx}$ in (4.15), where $P_{k|k}^{xx}$ is given by (2.28).

Thus, PUKF is a three-step algorithm whose forecast step is given by (2.89)-(2.96), whose data-assimilation step is given by (2.25), (2.27)-(2.28), and whose projection step is given by (6.16). Note that the projected estimate $\hat{x}_{k|k}^p$ is not recursively fed back in the forecast step at $k+1$; see Figure 4.1.

6.2.7 Algorithms: Summary of Characteristics

We now compare the structure of the UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF algorithms. Table 6.1 lists each algorithm with relation to the specific approaches used for the forecast and data-assimilation steps. Actually, TUKF, TIUKF, and PUKF are three-step algorithms that employ the classical KF update during the data-assimilation step and use a third step (truncation or projection) to enforce the interval constraint.

Table 6.2: Summary of characteristics of state-estimation algorithms for interval-constrained nonlinear systems. It is shown whether or not the (pseudo-) state estimates and the (pseudo-) error covariance assimilate the interval constraint information in each step of the estimator. Also, it is indicated how many constrained optimization problems (COP) are solved at each time step, as well as which type of constraint each algorithm can enforce in addition to the interval constraint (I) (6.3), namely, the nonlinear equality (NE) (4.41) and nonlinear inequality (NI) (4.102) constraints.

Algorithm	Section	forecast		data assimilation		constraint		# COP	type of constraint		
		$\hat{x}_{k k-1}$	$P_{k k-1}^{xx}$	$\hat{x}_{k k}$	$P_{k k}^{xx}$	$\hat{x}_{k k}^c$	$P_{k k}^{x^c}$		NE	I	NI
UKF	2.6.2	no	no	no	no	–	–	0	no	no	no
SIUKF [†]	4.5.4	yes	yes	yes	yes	–	–	$2n + 1$	yes [‡]	yes	yes [‡]
CUKF [†]	6.2.1	no	no	yes	no	–	–	1	yes [‡]	yes [‡]	yes [‡]
CIUKF [†]	6.2.2	yes	yes	yes	no	–	–	1	yes [‡]	yes	yes [‡]
IUKF	6.2.3	yes	yes	no	no	–	–	0	no	yes*	no
TUKF	6.2.4	no	no	no	no	yes	yes	0	no	yes [‡]	no
TIUKF	6.2.5	yes	yes	no	no	yes	yes	0	no	yes	no
PUKF	6.2.6	no	no	no	no	yes	–	1	yes [‡]	yes [‡]	yes [‡]

* The interval constraint is enforced only during the forecast step.

[†] The data-assimilation and constraint steps are jointly executed.

[‡] The constraint is enforced only during data-assimilation or constraint steps.

Table 6.2 indicates, for each algorithm, whether or not the (pseudo-) state estimates, as well as the (pseudo-) error covariance, are affected by the interval constraint in each step of the estimator. We also account for the number of constrained optimization problems that must be explicitly solved at each time step to enforce (6.3). Moreover, we indicate whether or not each algorithm can enforce the nonlinear equality constraint (4.41) and the nonlinear inequality constraint (4.102) in addition to the interval constraint (6.3).

6.3 Simulated Examples

6.3.1 Batch Reactor

We consider the gas-phase reaction



with reaction-rate proportion $k_1 = 0.16$, taking place in a well-mixed, constant-volume, isothermal batch reactor (Vachhani et al., 2006). Let the state vector $x(t) \in \mathbb{R}_+^2$ be given by the partial pressures of A and B in atm, whose dynamics are given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2k_1 x_1^2(t) \\ k_1 x_1^2(t) \end{bmatrix}. \quad (6.18)$$

We set $x(0) = [3 \ 1]^T$.

To perform state estimation using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF, as discussed in Section A.2, we integrate the process model (6.18) with $T_s = 0.1$ s using the 4th-order Runge-Kutta algorithm such that $x_k \triangleq x(kT_s)$. We set $Q_{k-1} = 10^{-6} I_{2 \times 2}$ to help convergence of estimates using UKF (see Section 2.6.3) with $\hat{x}_{0|0} = x(0)$. For uniformity, this value is used in the remaining cases. We assume that we measure the reactor pressure

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + v_k, \quad (6.19)$$

where $R_k = 0.01$ is the variance of $v_k \in \mathbb{R}$. We want to enforce the interval constraint (6.3), where $a_k = 0_{2 \times 1}$ and $b_k = \infty_{2 \times 1}$. First, we set a poor initialization given by $\hat{x}_{0|0} = [0.1 \ 4.5]^T$ and $P_{0|0}^{xx} = 36 I_{2 \times 2}$ and we refer to it as case 1. Case 1 is investigated in (Vachhani

et al., 2006) using UKF and SIUKF. We also investigate a second case (case 2) with good initialization given by $\hat{x}_{0|0} = [2.5 \ 0.5]^T$ and $P_{0|0}^{xx} = 0.25I_{2 \times 2}$. Whenever a constrained optimization problem is solved, since the measurement model (6.19) is linear, we use the function `quadprog` of Matlab, which implements a subspace trust region optimization method for quadratic programming.

Table 6.3 presents a performance comparison among the aforementioned algorithms for a 100-run Monte Carlo simulation regarding the performance indices RMSE (2.31), MT (2.33), and T_{CPU} ; see Section 2.4 for details. When applicable, we replace $\hat{x}_{k|k}$ by either $\hat{x}_{k|k}^t$ or $\hat{x}_{k|k}^p$ in (2.31) and $P_{k|k}^{xx}$ by either $P_{k|k}^{xxt}$ or $P_{k|k}^{xtp}$ in (2.33). Figure 6.1 shows the state estimates for $x_{2,k}$ and the associated standard deviation for a given simulation of case 1 using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF.

Table 6.3: RMSE (2.31), MT (2.33), and T_{CPU} (see Section 2.4) for a 100-run Monte Carlo simulation for the batch-reaction system using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF. Case-2 results are in italic. In each row, the numbers inside parentheses sort the performance indices in increasing order.

	UKF	IUKF	SIUKF	TUKF	TIUKF	CUKF	CIUKF	PUKF
	RMSE (atm)							
$x_{1,k}$	0.855 (8) <i>0.056</i> (4)	0.398 (4) <i>0.055</i> (1)	0.365 (3) <i>0.067</i> (8)	0.340 (2) <i>0.056</i> (5)	0.329 (1) <i>0.055</i> (2)	0.704 (6) <i>0.056</i> (6)	0.398 (5) <i>0.055</i> (3)	0.704 (7) <i>0.056</i> (7)
$x_{2,k}$	0.858 (8) <i>0.062</i> (4)	0.449 (4) <i>0.061</i> (1)	0.422 (3) <i>0.071</i> (8)	0.397 (2) <i>0.062</i> (5)	0.388 (1) <i>0.061</i> (2)	0.710 (6) <i>0.062</i> (6)	0.449 (5) <i>0.061</i> (3)	0.710 (7) <i>0.062</i> (7)
	MT (atm ²)							
	24.7 (6) <i>0.014</i> (5)	0.899 (4) <i>0.012</i> (2)	0.791 (1) <i>0.010</i> (1)	0.829 (3) <i>0.014</i> (6)	0.793 (2) <i>0.012</i> (3)	34.9 (7) <i>0.017</i> (7)	0.899 (5) <i>0.012</i> (4)	34.9 (8) <i>0.014</i> (8)
	T_{CPU} (ms)							
	10.8 (3) <i>9.3</i> (2)	7.5 (1) <i>7.9</i> (1)	160.4 (8) <i>128.5</i> (8)	11.5 (4) <i>11.9</i> (4)	10.6 (2) <i>11.1</i> (3)	55.9 (6) <i>30.9</i> (6)	37.0 (5) <i>30.9</i> (5)	57.0 (7) <i>31.6</i> (7)

According to Table 6.3, for case 1, UKF presents the worst performance. A slight improvement regarding RMSE is achieved by CUKF and PUKF. According to Figure 6.1(c), the insatisfactory performance of UKF, CUKF, and PUKF is due to the very slow convergence of the error covariance. Note that TIUKF, TUKF, SIUKF, CIUKF, and IUKF produce the best results concerning RMSE and MT.

For case 2, the performance of all algorithms is competitive. However, among the interval-constrained methods, CUKF and PUKF result in worse indices compared to the other algorithms; see Table 6.3.

In sum, we observe that methods that assimilate the interval-constraint information

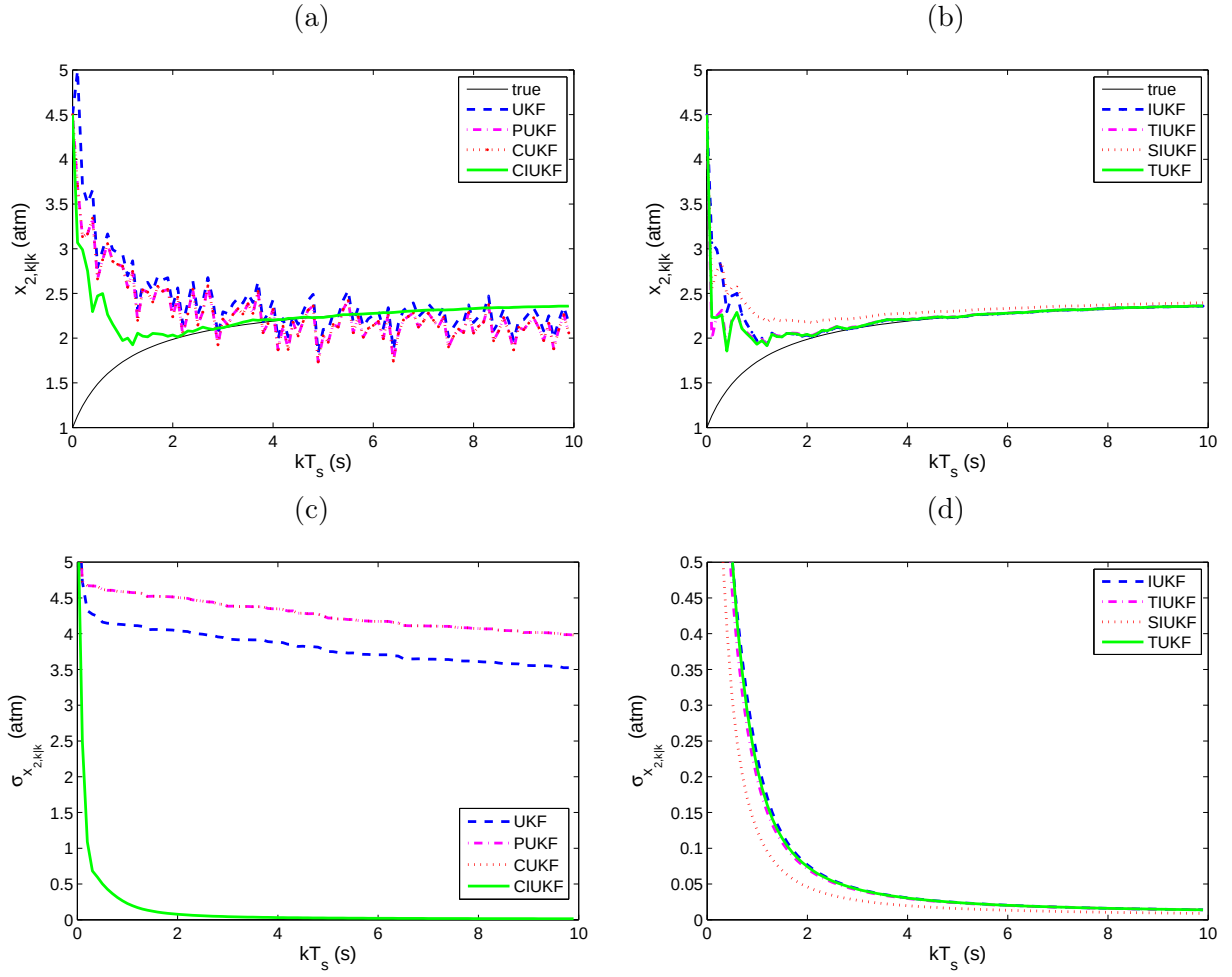


Figure 6.1: (a)-(b) Estimate of $x_{2,k}$ and (c)-(d) associated standard deviation $\sigma_{x_{2,k}} \triangleq \sqrt{P_{(2,2),k|k}^{xx}}$ for the poor initialization case using (a), (c) UKF, PUKF, CUKF, CIUKF, and (b), (d) IUKF, TIUKF, SIUKF, TUKF.

in both the state estimate and error covariance, namely, CIUKF, SIUKF, TUKF, TIUKF, and IUKF, have an improved performance compared to UKF.

When it comes to computational cost, according to Table 6.3, IUKF, TIUKF, and TUKF are quite competitive to UKF. Moreover, CUKF, CIUKF, and PUKF present intermediate processing time, whereas, for this example, SIUKF was more than thirteen times slower than UKF.

6.3.2 Continuously Stirred Tank Reactor

We consider the gas-phase, reversible reactions



with reaction-rate proportions $k_1 = 0.5$, $k_2 = 0.05$, $k_3 = 0.2$, $k_4 = 0.01$, stoichiometric matrix $S = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$, and reaction rates $r(t) = \begin{bmatrix} k_1 x_1(t) - k_2 x_2(t) x_3(t) \\ k_3 x_2^2(t) - k_4 x_3(t) \end{bmatrix}$ (Haseltine & Rawlings, 2005). Let the state vector $x(t) \in \mathbb{R}_+^3$ be given by the concentrations of A, B, and C in mol/l. We assume that these reactions take place in a well-mixed, isothermal continuously stirred tank reactor (CSTR), whose dynamics are given by

$$\dot{x}(t) = \frac{1}{3} \left(S^T r(t) + \frac{1}{V_R} \left(\begin{bmatrix} c_f & -x(t) \end{bmatrix} u(t) \right) \right), \quad (6.22)$$

where $V_R = 1001$ is the reactor volume, $c_f = [0.5 \ 0.05 \ 0]^T$ mol/l denotes inlet concentrations, $u(t) = \begin{bmatrix} q_f & q_o \end{bmatrix}^T$ is the input vector, and $q_f \geq 0$ and $q_o \geq 0$ are the volumetric inlet and effluent flow rates. We set $x(0) = [0.5 \ 0.05 \ 0]^T$ and $q_f = q_o = 1$

To perform state estimation using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF, we integrate the process model (6.22) with $T_s = 0.25$ s using the 4th-order Runge-Kutta algorithm, as discussed in Section A.2, such that $x_k \triangleq x(kT_s)$. To help convergence using UKF with $\hat{x}_{0|0} = x(0)$, we set $Q_{k-1} = 10^{-6} I_{3 \times 3}$. For uniformity, this value is used in the remaining cases. Also, we assume we measure the total pressure

$$y_k = \begin{bmatrix} R & R & R \end{bmatrix} x_k + v_k, \quad (6.23)$$

where $R = 32.84 \text{ atm} \times \text{l/mol}$ is a constant, and $R_k = 0.25^2$ is the variance of $v_k \in \mathbb{R}$. We want to enforce the interval constraint (6.3), where $a_k = 0_{3 \times 1}$ and $b_k = \infty_{3 \times 1}$. First, we set a poor initialization given by $\hat{x}_{0|0} = [0 \ 0 \ 3.5]^T$ and $P_{0|0}^{xx} = 4 I_{3 \times 3}$ and we refer to it as case 1. Case 1 is investigated in (Haseltine & Rawlings, 2005) using EKF (Section 2.5.2) and NMHE (Section 4.5.2). We also investigate a second case (case 2) with good initialization given by $\hat{x}_{0|0} = [0.6 \ 0.1 \ 0.05]^T$ and $P_{0|0}^{xx} = 0.5 I_{3 \times 3}$. Similar to the previous example, whenever a constrained optimization problem is solved, we use the function `quadprog` of Matlab.

Table 6.4 presents a performance comparison among the aforementioned algorithms for a 100-run Monte Carlo simulation regarding the performance indices RMSE (2.31), MT (2.33), and T_{CPU} . When applicable, we replace $\hat{x}_{k|k}$ by either $\hat{x}_{k|k}^t$ or $\hat{x}_{k|k}^p$ in (2.31) and $P_{k|k}^{xx}$ by either $P_{k|k}^{xxt}$ or $P_{k|k}^{xtp}$ in (2.33). Figure 6.2 shows the state estimate for $x_{3,k}$ and the associated standard deviation for a given simulation.

Table 6.4: RMSE (2.31), MT (2.33), and T_{CPU} (see Section 2.4) for a 100-run Monte Carlo simulation for the CSTR system using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF. Case-2 results are in italic. In each row, the numbers inside parentheses sort the performance indices in increasing order.

	UKF	IUKF	SIUKF	TUKF	TIUKF	CUKF	CIUKF	PUKF
	RMSE (mol/l)							
$x_{1,k}$	0.131 (8) <i>0.014</i> (6)	0.050 (4) <i>0.008</i> (1)	0.027 (1) <i>0.012</i> (5)	0.048 (3) <i>0.018</i> (8)	0.045 (2) <i>0.018</i> (7)	0.082 (6) <i>0.011</i> (3)	0.058 (5) <i>0.009</i> (2)	0.083 (7) <i>0.011</i> (4)
$x_{2,k}$	0.554 (8) <i>0.079</i> (8)	0.044 (4) <i>0.005</i> (1)	0.012 (2) <i>0.010</i> (4)	0.010 (1) <i>0.009</i> (3)	0.020 (3) <i>0.010</i> (5)	0.204 (6) <i>0.055</i> (6)	0.059 (5) <i>0.007</i> (2)	0.206 (7) <i>0.055</i> (7)
$x_{3,k}$	0.628 (8) <i>0.081</i> (8)	0.152 (4) <i>0.006</i> (1)	0.132 (1) <i>0.013</i> (3)	0.136 (2) <i>0.015</i> (4)	0.137 (3) <i>0.016</i> (5)	0.263 (6) <i>0.056</i> (6)	0.163 (5) <i>0.008</i> (2)	0.267 (7) <i>0.056</i> (7)
	MT (mol ² /l ²)							
	38.81 (8) <i>0.045</i> (8)	0.086 (5) <i>0.003</i> (4)	0.067 (1) <i>0.002</i> (1)	0.073 (3) <i>0.003</i> (3)	0.069 (2) <i>0.003</i> (2)	10.43 (7) <i>0.020</i> (6)	0.085 (4) <i>0.003</i> (5)	10.37 (6) <i>0.020</i> (7)
	T_{CPU} (ms)							
	9.6 (3) <i>6.3</i> (2)	6.0 (1) <i>5.7</i> (1)	189.6 (8) <i>192.5</i> (8)	11.1 (4) <i>9.1</i> (4)	8.8 (2) <i>8.8</i> (3)	38.0 (6) <i>33.2</i> (5)	33.4 (5) <i>33.5</i> (6)	38.5 (7) <i>33.7</i> (7)

As shown in Figure 6.2(a), for case 1, UKF does not converge, yielding estimates that do not satisfy (6.3). Note also that, although CUKF and PUKF slightly improve convergence, they still result in large error and covariance; see Figure 6.2(c). On the other hand, SIUKF and the truncation-based algorithms TUKF and TIUKF present the best performance indices; see Table 6.4. CIUKF and IUKF also provide a good performance, although slightly inferior compared to SIUKF, TUKF, and TIUKF.

For case 2, the performance indices of all algorithms studied are competitive. However, among the interval-constrained methods, CUKF and PUKF present the worst results. In sum, we observe that methods that assimilate the interval-constraint information in both the state estimate and pseudo error covariance, namely, CIUKF, SIUKF, TUKF, TIUKF, and IUKF, provide an improved performance compared to UKF.

When it comes to computational cost, IUKF, TIUKF, and TUKF are competitive to UKF; see Table 6.4. Furthermore, CUKF, CIUKF, and PUKF present intermediate processing time, whereas, in this example, SIUKF was more than twenty times slower than UKF.

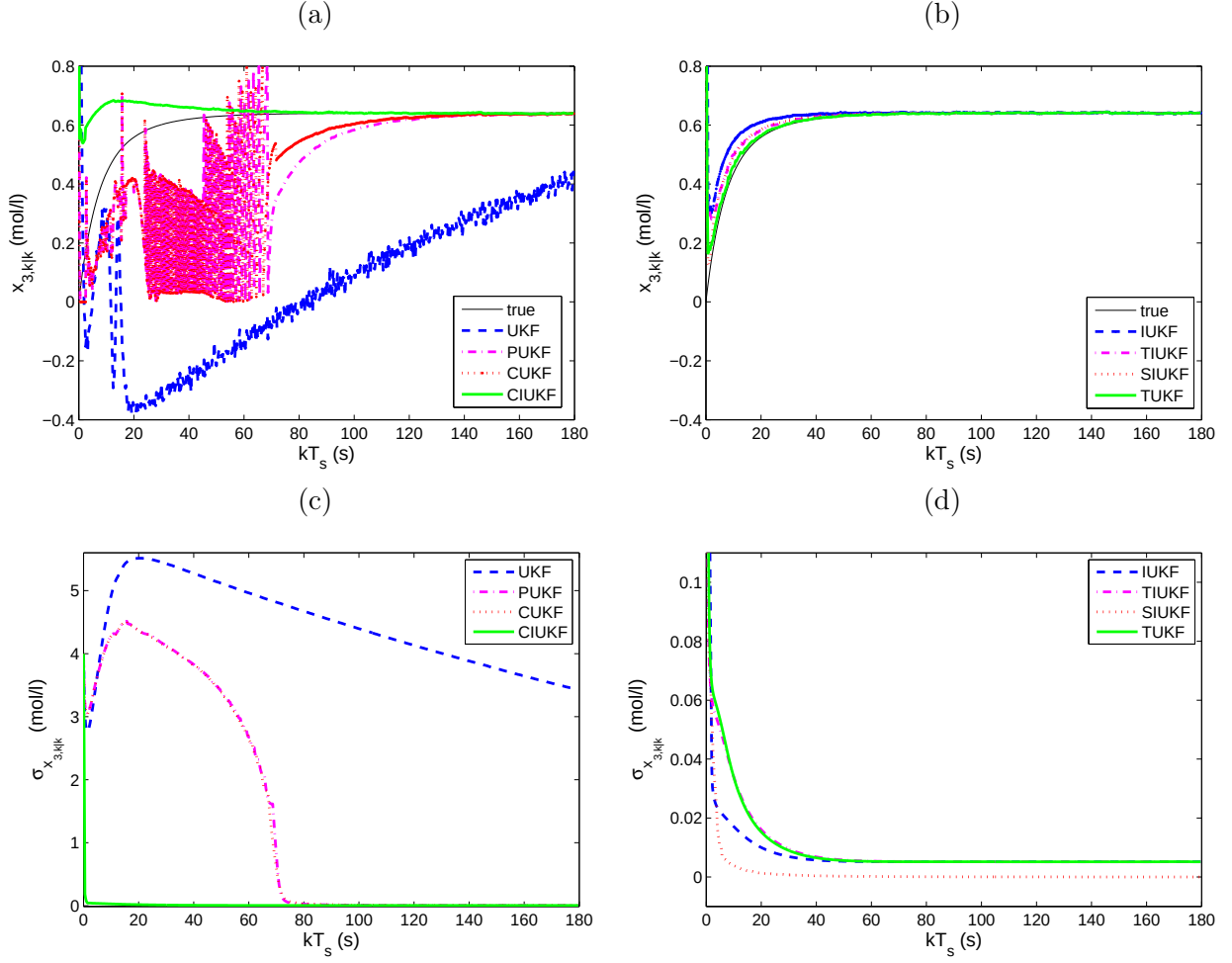


Figure 6.2: (a)-(b) Estimate of $x_{3,k}$ and (c)-(d) associated standard deviation $\sigma_{x_{3,k}} \triangleq \sqrt{P_{(3,3),k|k}^{xx}}$ for the poor initialization case using (a), (c) UKF, PUKF, CUKF, CIUKF, and (b), (d) IUUKF, TIUKF, SIUKF, TUKF.

6.4 Concluding Remarks

We address the interval-constrained state-estimation problem for nonlinear systems. As shown in see Table 6.1, we investigate how combinations of one of two candidate unscented approaches for forecast and one of five candidate methods for data assimilation can be used. In so doing, we review UKF and SIUKF and present CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF. These methods are compared with relation to whether or not the state estimates, as well as the pseudo error covariance, are affected by the interval constraint in each step; see Table 6.2.

We discuss two illustrative examples, whose state estimates are constrained to be

nonnegative. We observe that, for a good initialization of the state vector and error covariance, the performance indices of all algorithms are competitive. However, if a poor initialization is set, then the UKF estimates violate the interval constraint. In this case, only a slight improvement over UKF is observed using CUKF and PUKF. On the other hand, SIUKF and the truncation-based algorithms TUKF and TIUKF provide the best performance. CIUKF and IUKF also provide a good performance, although slightly inferior compared to SIUKF, TUKF, and TIUKF.

Regarding computational cost, IUKF, TIUKF, and TUKF are competitive with UKF. Furthermore, CUKF, CIUKF, and PUKF are three to five times slower than UKF, whereas SIUKF is thirteen to twenty times slower than UKF.

Since the methods investigated in this study are approximate, it is not clear to point out which one is the best method. Instead, it seems that the choice of a method depends on the application. However, for the cases investigated, considering the tradeoff between accuracy and computational cost, TUKF and TIUKF seem to be promising algorithms to enforce interval constraints in nonlinear systems.

GAIN-CONSTRAINED STATE ESTIMATION: OPTIMAL LINEAR SOLUTION, SPECIAL CASES, AND NONLINEAR EXTENSION

*“It is not knowledge, but the act of learning, not possession but
the act of getting there, which grants the greatest enjoyment.”*

Carl Friedrich Gauss

This chapter considers the state-estimation problem with a constraint on the data-assimilation gain. We derive the gain-constrained Kalman filter in Section 7.1. Special cases of this algorithm include the enforcing of a linear equality constraint on the state vector, the enforcing unbiased estimation for systems with unknown inputs, and simplification of the estimator structure for large-scale systems; see Section 7.2. In Section 7.3 the gain-constrained Kalman filter is extended to the nonlinear case, yielding the gain-constrained unscented Kalman filter. Finally, two illustrative examples are presented in Section 7.4. The new results presented in this chapter are published in (Teixeira, Chandrasekar, Palanthandalam-Madapusi, Tôrres, Aguirre & Bernstein, 2008b).

7.1 The Gain-Constrained Kalman Filter

Consider the stochastic linear discrete-time dynamic system given by (2.7)-(2.8), whose equations are repeated here for convenience, that is,

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1}, \quad (7.1)$$

$$y_k = C_k x_k + v_k. \quad (7.2)$$

Assume that, for all $k \geq 1$, the input u_{k-1} is known and the output y_k is measured. Moreover, assume that $\rho_{w_{k-1}}(w_{k-1})$, $\rho_{v_k}(v_k)$, and $\rho_{x_0}(x_0)$ are Gaussian and, respectively, given by (2.9)-(2.11). Also, for all $k \geq 0$, x_0 is assumed to be uncorrelated with w_k and v_k .

For the system (7.1)-(7.2), we consider a two-step filter with forecast step

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}, \quad (7.3)$$

$$\hat{y}_{k|k-1} = C_k\hat{x}_{k|k-1}, \quad (7.4)$$

and data-assimilation step

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - \hat{y}_{k|k-1}), \quad (7.5)$$

where the filter gain $L_k \in \mathbb{R}^{n \times m}$ minimizes the cost function

$$J_L(L_k) \triangleq \mathbb{E}[(x_k - \hat{x}_{k|k})^T W_k (x_k - \hat{x}_{k|k})] \quad (7.6)$$

subject to

$$M_k L_k N_k = O_k, \quad (7.7)$$

where, for all $k \geq 1$, the weighting matrix $W_k \in \mathbb{R}^{n \times n}$ is assumed to be positive definite, $M_k \in \mathbb{R}^{b \times n}$ is assumed to be right invertible, $N_k \in \mathbb{R}^{m \times c}$ is assumed to be left invertible, and $O_k \in \mathbb{R}^{b \times c}$. These conditions imply that $b \leq n$ and $c \leq m$, respectively. Recall that (7.3)-(7.4) are equal to the KF forecast equations (2.20) and (2.22), respectively, and are repeated for convenience. Also, note that (7.5) has the same form of the Kalman update equation (2.25). The constraint (7.7), which is absent in KF (Section 2.2.1), is used to enforce special properties on the state estimate $\hat{x}_{k|k}$.

Recall that, unlike KF that is presented using the maximum-a-posteriori approach in Theorem 2.2.1, we present the *gain-constrained Kalman filter* (GCKF) using the minimum-variance approach mentioned in Theorem 2.2.2.

Next, define the *forecast error* $e_{k|k-1}$

$$e_{k|k-1} \triangleq x_k - \hat{x}_{k|k-1}, \quad (7.8)$$

and let the innovation $\nu_{k|k-1}$ be given by (2.38). Also, let the forecast error covariance $P_{k|k-1}^{xx}$, innovation covariance $P_{k|k-1}^{yy}$, and cross covariance $P_{k|k-1}^{xy}$ be given by (2.15), (2.16), and (2.17), respectively. It follows from (7.1)-(7.4) that

$$e_{k|k-1} = A_{k-1}e_{k-1|k-1} + G_{k-1}w_{k-1}, \quad (7.9)$$

$$\nu_{k|k-1} = C_k e_{k|k-1} + v_k, \quad (7.10)$$

where $e_{k|k}$ is the *data-assimilation error* defined by

$$e_{k|k} \triangleq x_k - \hat{x}_{k|k}, \quad (7.11)$$

whose data-assimilation error covariance is given by (2.19). Finally, it follows from (7.4)-(7.5), (7.8), and (7.11) that

$$e_{k|k} = (I_{n \times n} - L_k C_k) e_{k|k-1} - L_k v_k. \quad (7.12)$$

The following lemma will be useful.

Lemma 7.1.1 The forecast error given by (7.8) satisfies

$$E[e_{k|k-1} v_k^T] = 0, \quad (7.13)$$

and the data-assimilation error (7.11) satisfies

$$E[e_{k|k} w_k^T] = 0. \quad (7.14)$$

Proof. Multiplying (7.9) by v_k^T and using that facts that w_{k-1} and v_k are uncorrelated and that $e_{k-1|k-1}$ and v_k are uncorrelated yield (7.13). Likewise, substituting (7.9) into (7.12) and multiplying the result by w_k^T yield (7.14). $\square$

Proposition 7.1.1. For the filter (7.3)-(7.5), the data-assimilation error covariance $P_{k|k}^{xx}$ is updated using

$$P_{k|k}^{xx} = P_{k|k-1}^{xx} - L_k (P_{k|k-1}^{xy})^T - P_{k|k-1}^{xy} L_k^T + L_k P_{k|k-1}^{yy} L_k^T, \quad (7.15)$$

where

$$P_{k|k-1}^{xx} = A_{k-1}P_{k-1|k-1}^{xx}A_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T, \quad (7.16)$$

$$P_{k|k-1}^{yy} = C_kP_{k|k-1}^{xx}C_k^T + R_k, \quad (7.17)$$

$$P_{k|k-1}^{xy} = P_{k|k-1}^{xx}C_k^T. \quad (7.18)$$

Proof. It follows from (7.9)-(7.10) and Lemma 7.1.1 that (2.15)-(2.17) are, respectively, given by (7.16)-(7.18). Moreover, using Lemma 7.1.1 again, (7.12) implies that (2.19) is given by

$$P_{k|k}^{xx} = (I_{n \times n} - L_k C_k)P_{k|k-1}^{xx}(I_{n \times n} - L_k C_k)^T + L_k R_k L_k^T,$$

which, by using (7.17)-(7.18), yields (7.15). $\square$

Next, using (7.11) and (2.19) in (7.6) yields

$$J_L(L_k) = \text{tr} \left(P_{k|k}^{xx} W_k \right). \quad (7.19)$$

Assume that, for all $k \geq 1$, $P_{k|k-1}^{yy}$ is positive definite. For convenience, we define

$$M_k^R \triangleq M_k^T (M_k M_k^T)^{-1}, \quad (7.20)$$

$$N_k^L \triangleq (N_k^T N_k)^{-1} N_k^T, \quad (7.21)$$

and

$$\Pi_k \triangleq W_k^{-1} M_k^T (M_k W_k^{-1} M_k^T)^{-1} M_k, \quad (7.22)$$

$$\Omega_k \triangleq N_k [N_k^T (P_{k|k-1}^{yy})^{-1} N_k]^{-1} N_k^T (P_{k|k-1}^{yy})^{-1}. \quad (7.23)$$

Also, let K_k be the classical Kalman gain given by (2.25). Note that Π_k and Ω_k are oblique projectors, that is, $\Pi_k^2 = \Pi_k$ and $\Omega_k^2 = \Omega_k$, but are not necessarily symmetric. If $W_k = I_{n \times n}$, then Π_k is an orthogonal projector. Note that, if M_k is square, then $\Pi_k = I_{n \times n}$; likewise, if N_k is square, then $\Omega_k = I_{m \times m}$. Also,

$$\Pi_k M_k^R = W_k^{-1} M_k^T (M_k W_k^{-1} M_k^T)^{-1}, \quad (7.24)$$

$$N_k^L \Omega_k = [N_k^T (P_{k|k-1}^{yy})^{-1} N_k]^{-1} N_k^T (P_{k|k-1}^{yy})^{-1}. \quad (7.25)$$

Proposition 7.1.2. The gain L_k that minimizes (7.19) and satisfies (7.7) is given by

$$L_k = K_k - \Pi_k (K_k - M_k^R O_k N_k^L) \Omega_k, \quad (7.26)$$

where K_k is given by (2.25) with $P_{k|k-1}^{xx}$, $P_{k|k-1}^{yy}$, and $P_{k|k-1}^{xy}$ given by (7.16), (7.17), and (7.18), respectively. Furthermore, $P_{k|k}^{xx}$ in (7.15) is given by the Riccati equation

$$\begin{aligned} P_{k|k}^{xx} = & P_{k|k-1}^{xx} - P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} (P_{k|k-1}^{xy})^T \\ & + (\Pi_k M_k^R O_k N_k^L \Omega_k) P_{k|k-1}^{yy} (\Pi_k M_k^R O_k N_k^L \Omega_k)^T \\ & + [\Pi_k P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \Omega_k] P_{k|k-1}^{yy} [\Pi_k P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \Omega_k]^T - \Pi_k (\Delta_k + \Delta_k^T) \Pi_k^T, \end{aligned} \quad (7.27)$$

where

$$\Delta_k \triangleq P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \Omega_k P_{k|k-1}^{yy} \Omega_k^T (M_k^R O_k N_k^L)^T. \quad (7.28)$$

Proof. Define the Lagrangian

$$\mathcal{L}_k(L_k) = J_L(L_k) + 2 \operatorname{tr}[(M_k L_k N_k - O_k) \Lambda^T], \quad (7.29)$$

where $\Lambda \in \mathbb{R}^{b \times c}$ is the Lagrange multiplier accounting for the bc constraints in (7.7). The necessary conditions for a minimizer L_k are given by

$$\frac{\partial \mathcal{L}_k}{\partial L_k} = 2(-W_k P_{k|k-1}^{xy} + W_k L_k P_{k|k-1}^{yy} + M_k^T \Lambda N_k^T) = 0_{n \times m}, \quad (7.30)$$

$$\frac{\partial \mathcal{L}_k}{\partial \Lambda} = 2(M_k L_k N_k - O_k) = 0_{b \times c}. \quad (7.31)$$

Note that (7.31) yields (7.7).

In (7.30), using the fact that W_k and $P_{k|k-1}^{yy}$ are positive definite yields

$$-P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} + L_k + W_k^{-1} M_k^T \Lambda N_k^T (P_{k|k-1}^{yy})^{-1} = 0_{n \times m}. \quad (7.32)$$

Pre-multiplying and post-multiplying (7.32) by M_k and N_k , respectively, and substituting (7.7) and (2.25) into the resulting expression yields

$$-M_k K_k N_k + O_k + M_k W_k^{-1} M_k^T \Lambda N_k^T (P_{k|k-1}^{yy})^{-1} N_k = 0_{b \times c},$$

which implies

$$\begin{aligned} \Lambda = & (M_k W_k^{-1} M_k^T)^{-1} M_k K_k N_k [N_k^T (P_{k|k-1}^{yy})^{-1} N_k]^{-1} \\ & - (M_k W_k^{-1} M_k^T)^{-1} O_k [N_k^T (P_{k|k-1}^{yy})^{-1} N_k]^{-1}. \end{aligned} \quad (7.33)$$

Using (7.20)-(7.25) and substituting (7.33) into (7.32) yields (7.26).

Finally, substituting (2.25) and (7.26) into (7.15) yields (7.27). $\square$

Proposition 7.1.3. L_k given by (7.26) is the unique global minimizer of $J_L(L_k)$ restricted to the convex constraint set defined by (7.7).

Proof. It follows from (Bernstein, 2005, p. 286) that, for all $0 < \alpha < 1$, $A_1, A_2 \in \mathbb{R}^{n \times m}$ such that $A_1 \neq A_2$, and positive-definite $B \in \mathbb{R}^{m \times m}$, $\text{tr} [\alpha(1 - \alpha)(A_1 - A_2)B(A_1 - A_2)^T] > 0$. Hence, for $A \in \mathbb{R}^{n \times n}$, the mapping $A \rightarrow \text{tr}(ABA^T)$ is strictly convex. It thus follows that $J_L(L_k)$ is strictly convex, and hence L_k is the unique global minimizer of $J_L(L_k)$ restricted to (7.7). $\square$

Remark 7.1.1. GCKF is expressed in two steps, namely, the forecast step given by (7.3), (7.16), (7.4), (7.17)-(7.18), and the data-assimilation step given by (7.20)-(7.23), (2.25), (7.26), (7.5), and (7.15).

Comparing Figure 7.1 to Figure 2.1, we see that the GCKF equations are identical to the classical KF equations (2.20)-(2.25), (2.27)-(2.28) with the exception of the gain

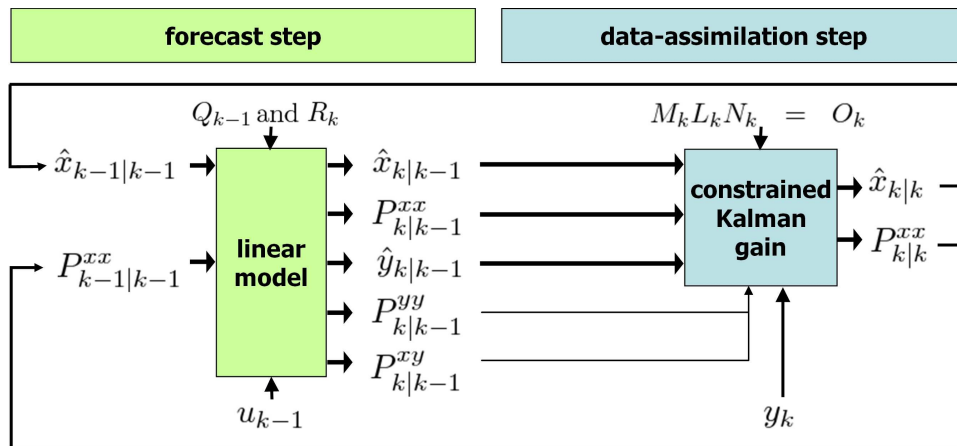


Figure 7.1: Diagram of the of the gain-constrained Kalman filter which is recursively expressed in two steps, namely, the forecast step and data-assimilation step, whose gain satisfy the constraint (7.7).

expression (7.26). Moreover, note that the first two terms on the right-hand side of (7.27) correspond to the data-assimilation error covariance of KF given by (2.28).

For brevity, the gain-constrained Kalman predictor (GCKP), which is a one-step algorithm, is derived in Appendix C.

7.2 Special Cases

7.2.1 The Kalman Filter

Assume that $M_k = I_{n \times n}$, $N_k = I_{m \times m}$, and $O_k = K_k$, where K_k is given by (2.25). In this case, the optimal gain L_k given by (7.26) that minimizes (7.19) subject to (7.7) is K_k , which is the classical Kalman gain, and (7.27) is equal to the Riccati equation of KF given by (2.25).

7.2.2 Condition $M_k = I_{n \times n}$

Assume that $M_k = I_{n \times n}$ such that the gain constraint (7.7) is expressed as

$$L_k N_k = O_k. \quad (7.34)$$

In this case, (7.22) yields $\Pi_k = I_{n \times n}$ and (7.20) yields $M_k^R = I_{n \times n}$. Hence, it follows from (7.26) that the optimal gain L_k that minimizes (7.19) and satisfies (7.34) is given by

$$L_k = K_k \Omega_{k\perp} + O_k N_k^L \Omega_k, \quad (7.35)$$

where N_k^L is given by (7.21), Ω_k is given by (7.23), the complementary oblique projector is given by

$$\Omega_{k\perp} \triangleq I_{m \times m} - \Omega_k, \quad (7.36)$$

and $P_{k|k}^{xx}$ in (7.15) is given by the Riccati equation

$$\begin{aligned} P_{k|k}^{xx} &= P_{k|k-1}^{xx} - P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} (P_{k|k-1}^{xy})^T \\ &\quad + (O_k N_k^L \Omega_k) P_{k|k-1}^{yy} (O_k N_k^L \Omega_k)^T \\ &\quad + [P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \Omega_k] P_{k|k-1}^{yy} [P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \Omega_k]^T - (\Delta_{1,k} + \Delta_{1,k}^T), \end{aligned} \quad (7.37)$$

where

$$\Delta_{1,k} \triangleq P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} \Omega_k P_{k|k-1}^{yy} \Omega_k^T (O_k N_k^L)^T. \quad (7.38)$$

Recall that, in this particular case, as well as in Section 7.2.1, the minimization of (7.19) is independent of W_k .

7.2.3 Condition $N_k = I_{m \times m}$

Assume that $N_k = I_{m \times m}$ such that the gain constraint (7.7) is expressed as

$$M_k L_k = O_k. \quad (7.39)$$

In this case, (7.23) yields $\Omega_k = I_{m \times m}$ and (7.21) yields $N_k^L = I_{m \times m}$. Hence, it follows from (7.26) that the optimal gain L_k that minimizes (7.19) and satisfies (7.39) is given by

$$L_k = \Pi_{k\perp} K_k + \Pi_k M_k^R O_k, \quad (7.40)$$

where M_k^R is given by (7.20), Π_k is given by (7.22), the complementary oblique projector is given by

$$\Pi_{k\perp} \triangleq I_{n \times n} - \Pi_k, \quad (7.41)$$

and $P_{k|k}^{xx}$ in (7.15) is given by

$$\begin{aligned} P_{k|k}^{xx} &= P_{k|k-1}^{xx} - P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} (P_{k|k-1}^{xy})^T \\ &\quad + (\Pi_k M_k^R O_k) P_{k|k-1}^{yy} (\Pi_k M_k^R O_k)^T + \Pi_k P_{k|k-1}^{xy} (P_{k|k-1}^{yy})^{-1} (P_{k|k-1}^{xy})^T \Pi_k^T \\ &\quad - \Pi_k \{ [P_{k|k-1}^{xy} (M_k^R O_k)^T] + [P_{k|k-1}^{xy} (M_k^R O_k)^T]^T \} \Pi_k^T. \end{aligned} \quad (7.42)$$

7.2.4 The Projected Kalman Filter by Gain Projection

For the system (7.1)-(7.2) whose state vector is known to satisfy the linear equality constraint (4.3), we consider the two-step estimator whose forecast step is given by (7.3)-(7.4) and whose data-assimilation step is given by (7.5). In (7.19) assume that

$$W_k = I_{n \times n}. \quad (7.43)$$

We look for a gain L_k that minimizes (7.19) subject to the gain constraint (4.22).

Note that (4.22) can be written in the form (7.7), where

$$M_k = D_{k-1}, \quad (7.44)$$

$$N_k = y_k - \hat{y}_{k|k-1}, \quad (7.45)$$

$$O_k = d_{k-1} - D_{k-1}\hat{x}_{k|k-1}, \quad (7.46)$$

where $\hat{x}_{k|k-1}$ is given by (7.3), $\hat{y}_{k|k-1}$ is given by (7.4), and D_{k-1} and d_{k-1} are the parameters of the equality constraint (4.3). Therefore, L_k is given by (7.26). Substituting (7.43)-(7.46) into (7.26), we obtain (4.23), which is the gain of the projected Kalman filter by gain projection (PKF-GP) presented in (Gupta & Hauser, 2007; Gupta, 2007a) and reviewed in Section 4.1.4.

7.2.5 The Unbiased Kalman Filter with Unknown Inputs

Consider the stochastic linear discrete-time dynamic system given by

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + F_{k-1}z_{k-1} + G_{k-1}w_{k-1}, \quad (7.47)$$

$$y_k = C_k x_k + H_k \tilde{z}_k + v_k, \quad (7.48)$$

where, for all $k \geq 1$, $F_{k-1} \in \mathbb{R}^{n \times s}$ and $H_k \in \mathbb{R}^{m \times t}$ are known matrices. No assumptions are made on the unknown input vectors $z_{k-1} \in \mathbb{R}^s$ and $\tilde{z}_k \in \mathbb{R}^t$. Note that, if we set $s = t$ and $z_k = \tilde{z}_k$, then we have the system with direct feedthrough studied in (Darouach, Zasadzinski & Boutayeb, 2003; Gillijns & De Moor, 2007b).

For the system (7.47)-(7.48), consider the two-step estimator whose forecast step is given by (7.3)-(7.4) and whose data-assimilation step is given by (7.5). In this case, $e_{k|k}$ defined by (7.12) is given by

$$\begin{aligned} e_{k|k} = & A_{k-1}(I_{n \times n} - L_k C_k)e_{k-1|k-1} + (I_{n \times n} - L_k C_k)G_{k-1}w_{k-1} \\ & - L_k v_k + (I_{n \times n} - L_k C_k)F_{k-1}z_{k-1} - L_k H_k \tilde{z}_k. \end{aligned} \quad (7.49)$$

If L_k satisfies

$$(I_{n \times n} - L_k C_k)F_{k-1} = 0_{n \times s}, \quad (7.50)$$

$$L_k H_k = 0_{n \times t}, \quad (7.51)$$

then $\hat{x}_{k|k}$ in (7.5) is *unbiased*, that is, $E[e_{k|k}] = 0_{n \times 1}$. Note that (7.50)-(7.51) can be written in the form (7.7), where

$$M_k = I_{n \times n}, \quad (7.52)$$

$$N_k = [C_k F_{k-1} \quad H_k], \quad (7.53)$$

$$O_k = [F_{k-1} \quad 0_{n \times t}]. \quad (7.54)$$

Substituting (7.52)-(7.54) into (7.26), we obtain the unbiased Kalman filter with unknown inputs (UnbKF-UI) (Darouach, Zasadzinski & Boutayeb, 2003; Gillijns & De Moor, 2007b), whose error covariance is given by (7.37).

We assume that $N_k \in \mathbb{R}^{m \times (s+t)}$ in (7.53) is left invertible, which is equivalent to the conditions (i) $\text{rank}(C_k F_{k-1}) = s$, (ii) $\text{rank}(H_k) = t$, and (iii) the columns of H_k are linearly independent of the columns of $C_k F_{k-1}$. These conditions are equivalent to $\text{rank}(N_k) = s + t$, which is shown to be a sufficient condition for the unbiasedness of the estimator (7.5) in (Darouach, Zasadzinski & Boutayeb, 2003, Lemma 2). Recall that $m \geq s + t$ is a necessary condition for (iii) to hold.

Condition $t = 0$

For the system (7.47)-(7.48), assume that $t = 0$. In this case, $e_{k|k}$ is given by

$$\begin{aligned} e_{k|k} = & A_{k-1}(I_{n \times n} - L_k C_k)e_{k-1|k-1} + (I_{n \times n} - L_k C_k)G_{k-1}w_{k-1} \\ & - L_k v_k + (I_{n \times n} - L_k C_k)F_{k-1}z_{k-1}, \end{aligned} \quad (7.55)$$

while, if (7.50) is satisfied, then $\hat{x}_{k|k}$ in (7.5) is unbiased. Thus, the gain that minimizes (7.19) subject to (7.7) is given by (7.35), where M_k is given by (7.52) and

$$N_k = C_k F_{k-1}, \quad (7.56)$$

$$O_k = F_{k-1}, \quad (7.57)$$

where, since $\text{rank}(C_k F_{k-1}) = s$, N_k given by (7.56) is left invertible. The error covariance is given by (7.37). The resulting estimator for this special case is presented in (Kitanidis, 1987).

Condition $s = 0$

For the system (7.47)-(7.48), consider the case $s = 0$. In this case, $e_{k|k}$ is given by

$$e_{k|k} = A_{k-1}(I_{n \times n} - L_k C_k)e_{k-1|k-1} + (I_{n \times n} - L_k C_k)G_{k-1}w_{k-1} - L_k v_k - L_k H_k \tilde{z}_k. \quad (7.58)$$

If (7.51) is satisfied, then $\hat{x}_{k|k}$ is unbiased. The gain that minimizes (7.19) and satisfies (7.7) is given by (7.35), where M_k is given by (7.52) and

$$N_k = H_k, \quad (7.59)$$

$$O_k = 0_{n \times t}, \quad (7.60)$$

where, since $\text{rank}(H_k) = t$, N_k given by (7.59) is left invertible. The error covariance is given by (7.37). The resulting estimator is presented in (Palanthandalam-Madapusi, Gillijns, De Moor & Bernstein, 2006).

7.2.6 The Kalman Filter with Constrained Output Injection

For the system (7.1)-(7.2), consider the two-step estimator whose forecast step is given by (7.3)-(7.4) and whose data-assimilation step is given by (7.5). We look for a gain L_k that minimizes (7.19) and constrains the estimator so that only estimates in the range of Γ_k are directly updated during data assimilation. Assume that $b < n$ and let $\Gamma_k \in \mathbb{R}^{n \times (n-b)}$ be a full column rank matrix. For example, Γ_k can have the form $\Gamma_k = \begin{bmatrix} I_{(n-b) \times (n-b)} \\ 0_{b \times (n-b)} \end{bmatrix}$. If $\Gamma_k = I_{n \times n}$, we have the classical KF. Also, let $\tilde{\Gamma}_k \in \mathbb{R}^{n \times b}$ be such that $\begin{bmatrix} \Gamma_k & \tilde{\Gamma}_k \end{bmatrix}$ is nonsingular and

$$\tilde{\Gamma}_k^T \Gamma_k = 0_{b \times (n-b)}. \quad (7.61)$$

That is, we look for a gain L_k that satisfies (7.7), where

$$M_k = \tilde{\Gamma}_k^T, \quad (7.62)$$

$$N_k = I_{m \times m}, \quad (7.63)$$

$$O_k = 0_{b \times m}. \quad (7.64)$$

Using (7.62)-(7.64), it follows from (7.40) that

$$L_k = [I_{n \times n} - W_k^{-1} \tilde{\Gamma}_k (\tilde{\Gamma}_k^T W_k^{-1} \tilde{\Gamma}_k)^{-1} \tilde{\Gamma}_k^T] K_k, \quad (7.65)$$

and $P_{k|k}^{xx}$ is given by (7.42). Given (7.61) we have that

$$\begin{bmatrix} \tilde{\Gamma}_k^T \\ \Gamma_k^T W_k \end{bmatrix} \left(W_k^{-1} \tilde{\Gamma}_k (\tilde{\Gamma}_k^T W_k^{-1} \tilde{\Gamma}_k)^{-1} \tilde{\Gamma}_k^T + \Gamma_k (\Gamma_k^T W_k \Gamma_k)^{-1} \Gamma_k^T W_k \right) = \begin{bmatrix} \tilde{\Gamma}_k^T \\ \Gamma_k^T W_k \end{bmatrix}. \quad (7.66)$$

Since $\begin{bmatrix} \tilde{\Gamma}_k^T \\ \Gamma_k^T W_k \end{bmatrix}$ is nonsingular, it follows that

$$W_k^{-1} \tilde{\Gamma}_k (\tilde{\Gamma}_k^T W_k^{-1} \tilde{\Gamma}_k)^{-1} \tilde{\Gamma}_k^T + \Gamma_k (\Gamma_k^T W_k \Gamma_k)^{-1} \Gamma_k^T W_k = I_{n \times n},$$

which implies that

$$I_{n \times n} - W_k^{-1} \tilde{\Gamma}_k (\tilde{\Gamma}_k^T W_k^{-1} \tilde{\Gamma}_k)^{-1} \tilde{\Gamma}_k^T = \Gamma_k (\Gamma_k^T W_k \Gamma_k)^{-1} \Gamma_k^T W_k. \quad (7.67)$$

Post-multiplying (7.67) by K_k and using (7.65), we have

$$L_k = \Gamma_k \tilde{L}_k, \quad (7.68)$$

where

$$\tilde{L}_k \triangleq (\Gamma_k^T W_k \Gamma_k)^{-1} \Gamma_k^T W_k K_k \quad (7.69)$$

is the gain of the spatially constrained Kalman filter presented in (Chandrasekar, Bernstein, Barrero & De Moor, 2007).

7.3 The Gain-Constrained Unscented Kalman Filter

Consider the stochastic nonlinear discrete-time dynamic system (1.1)-(1.2), whose equations are repeated here for convenience, that is,

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}, k-1), \quad (7.70)$$

$$y_k = h(x_k, v_k, k), \quad (7.71)$$

where, for all $k \geq 1$, the input u_{k-1} is assumed to be known and the output y_k is assumed to be measured. Also, like in Sections 2.5.2 and 2.6.2, we assume that the mean and covariance of $\rho_{w_{k-1}}(w_{k-1})$ and $\rho_{v_k}(v_k)$ are known and equal to zero and Q_{k-1} , R_k , respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated.

For the system (7.70)-(7.71), we consider a suboptimal filter with forecast step

$$\hat{x}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \mathcal{X}_{j,k|k-1}^x, \quad (7.72)$$

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \mathcal{Y}_{j,k|k-1}, \quad (7.73)$$

where, for $j = 0, 1, \dots, 2\tilde{n}$, $\mathcal{X}_{j,k|k-1}^x \in \mathbb{R}^{\tilde{n}}$ and $\mathcal{Y}_{j,k|k-1} \in \mathbb{R}^m$ are sets of sample vectors with weights $\gamma_{j,x} \in \mathbb{R}$, and data-assimilation step given by (7.5), where L_k is assumed to satisfy (7.7). We consider the unscented transform (UT) reviewed in Section 2.6.1 to obtain $\mathcal{X}_{j,k|k-1}^x$, $\mathcal{Y}_{j,k|k-1}$, and $\gamma_{j,x}$ for the augmented state vector (2.87) and augmented error covariance (2.88). Recall that (7.72)-(7.73) are equal to the UKF equations (2.91) and (2.94), respectively, and are repeated for convenience.

Therefore, we apply UT to the GCKF equations and obtain the *gain-constrained unscented Kalman filter* (GCUKF), whose forecast step is given by (2.89)-(2.90), (7.72), (2.92)-(2.93), (7.73), (2.95)-(2.96), and whose data-assimilation step is given by (7.26), (7.5), and (7.15).

Note that the GCUKF forecast equations are identical to the UKF forecast equations (2.89)-(2.96), while the GCUKF data-assimilation equations are identical to the GCKF data-assimilation equations (Section 7.1).

The next result shows that the gain (7.26) of GCUKF satisfies the constraint (7.7) with the pseudo-error covariance matrices $P_{k|k-1}^{yy}$ (2.95) and $P_{k|k-1}^{xy}$ (2.96).

Proposition 7.3.1. Let $P_{k|k-1}^{yy}$ and $P_{k|k-1}^{xy}$ be given, respectively, by (2.95) and (2.96), and let L_k be given by (7.26). Then (7.7) is satisfied.

Proof. Pre-multiplying (7.22) by M_k and post-multiplying (7.23) by N_k yields

$$\begin{aligned} M_k \Pi_k &= M_k W_k^{-1} M_k^T (M_k W_k^{-1} M_k^T)^{-1} M_k = M_k, \\ \Omega_k N_k &= N_k [N_k^T (P_{k|k-1}^{yy})^{-1} N_k]^{-1} N_k^T (P_{k|k-1}^{yy})^{-1} N_k = N_k. \end{aligned}$$

Also, pre-multiplying and post-multiplying (7.26) by M_k and N_k and using (7.20)-(7.21) yields

$$\begin{aligned} M_k L_k N_k &= M_k K_k N_k - M_k \Pi_k K_k \Omega_k N_k + M_k \Pi_k M_k^R O_k N_k^L \Omega_k N_k \\ &= M_k K_k N_k - M_k K_k N_k + M_k M_k^R O_k N_k^L N_k = O_k \end{aligned}$$

which confirms (7.7). □

7.4 Simulated Examples

7.4.1 Van der Pol Oscillator

We consider the Euler-discretized van der Pol oscillator given by

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k-1} + T_s x_{2,k-1} \\ -T_s x_{1,k-1} + (T_s + 1 - T_s x_{1,k-1}^2) x_{2,k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k-1}, \quad (7.74)$$

where $x_k \triangleq [x_{1,k} \ x_{2,k}]^T \in \mathbb{R}^2$, $u_{k-1} \in \mathbb{R}$ is an exogenous input, and T_s is the sampling interval, with noisy observation given by (7.2) with $C_k = \begin{bmatrix} 1 & 1 \end{bmatrix}$. For simulation, we set $x_0 = [1 \ 1]^T$, $T_s = 0.1$, $R_k = 0.04$, and

$$u_{k-1} = \begin{cases} T_s \sin(2kT_s), & \text{if } kT_s < 10 \text{ s or } kT_s \geq 30 \text{ s,} \\ T_s (\sin(2kT_s) + 0.5), & \text{if } 10 \text{ s} \leq kT_s < 20 \text{ s,} \\ T_s (\sin(2kT_s) - 0.5), & \text{if } 20 \text{ s} \leq kT_s < 30 \text{ s.} \end{cases} \quad (7.75)$$

Now, assuming that the input u_{k-1} is unknown, our goal is to use GCUKF for obtaining unbiased state estimates. For comparison, we perform state estimation using UKF in two distinct cases, namely, assuming u_{k-1} is known and assuming u_{k-1} is unknown. Whenever u_{k-1} is assumed to be unknown, we set $u_{k-1} = 0$ during the forecast step. We set

$\hat{x}_{0|0} = [0.5 \quad 1.5]^T$ and $P_{0|0}^{xx} = 0.5 I_{2 \times 2}$. As discussed in Section 2.6.3, to help convergence of UKF with u_{k-1} assumed to be known, we set $Q_{k-1} = 10^{-6} I_{2 \times 2}$. For uniformity, this value is used for the remaining two cases. Note that, in this example, u_{k-1} plays the same role of the unknown input z_{k-1} , which is defined in the context of linear systems in Section 7.2.5. Therefore we implement GCUKF to enforce the gain constraint (7.7) using M_k , N_k , and O_k set as (7.52), (7.56), and (7.57), respectively; see Section 7.2.5 for details. In this case, from (7.74), it follows that F_{k-1} used in (7.56)-(7.57) is given by $F_{k-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Figure 7.2 shows the true state components and the state estimates from the three cases mentioned above. Figure 7.3 shows the estimation error component $e_{2,k|k}$ around the confidence interval $\pm 3\sqrt{P_{(2,2),k|k}^{xx}}$, where $P_{k|k}^{xx}$ is the pseudo error-covariance given by (7.15). Observe that, when we assume u_{k-1} is unknown, $e_{2,k|k}$ given by UKF does not converge. On the other hand, the GCUKF estimates converge, but with a larger confidence interval compared to the case in which UKF is used with u_{k-1} assumed to be known. Likewise, if we set $Q_{k-1} = \text{diag}(10^{-6}, 0.05)$ using UKF with u_{k-1} assumed to be unknown, then $e_{2,k|k}$ converges with similar accuracy compared to GCUKF; see Figure 7.3. Such value of Q_{k-1} was heuristically chosen by increasing the original value until the estimation errors remain inside the confidence interval defined by $\pm 3\sqrt{\text{diag}(P_{k|k}^{xx})}$. Alternatively, statistical approaches to estimate Q_{k-1} offline are found in (Harvey, 2001).

Table 7.1 presents a performance comparison regarding RMSE (2.31) and MT (2.33) over a 100-run Monte Carlo simulation among the following cases: (i) UKF with known u_{k-1} , (ii) GCUKF with assumed unknown u_{k-1} , (iii) UKF with unknown u_{k-1} , and (iv) UKF with unknown u_{k-1} and a larger Q_{k-1} . Note that, regarding RMSE, case (ii) outperforms case (iii). Although case (iii) exhibits smaller MT than case (ii), as illustrated in Figure 7.3(c), the estimation error for case (iii) do not converge. Also, the indices RMSE and MT for cases (ii) and (iv) are close.

Finally, Figure 7.4 compares the input u_{k-1} (true value) to its estimate $\hat{u}_{k-1|k}$ given by (Palanthandalam-Madapusi et al., 2006; Gillijns & De Moor, 2007b),

$$\hat{u}_{k-1|k} = (F_{k-1}^T F_{k-1})^{-1} F_{k-1}^T L_k (y_k - \hat{y}_{k|k-1}), \quad (7.76)$$

where $\hat{y}_{k|k-1}$ is given by (7.73) and L_k is given by (7.26) using GCUKF. Alternatively, in (de Jong & Penzer, 1998; Penzer, 2007), it is proposed a least-squares approach for estimating

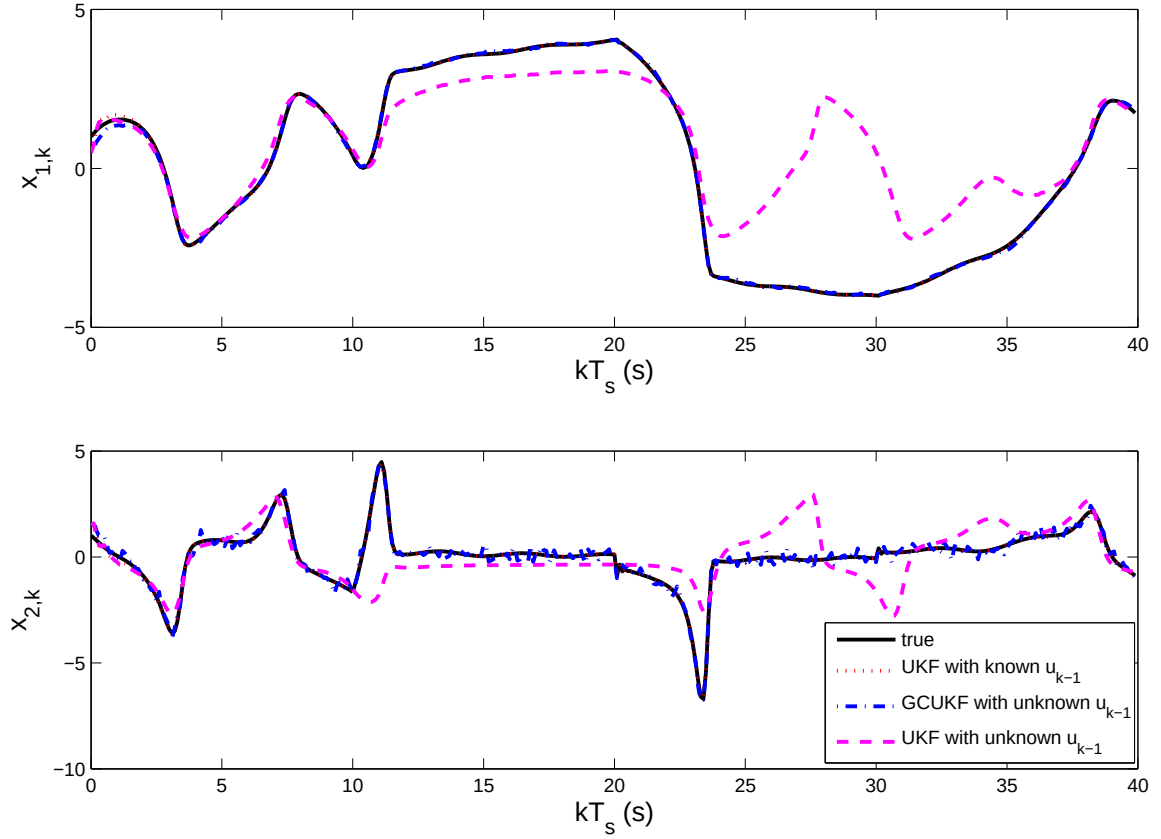


Figure 7.2: (i) Data-free simulation of the van der Pol oscillator (—) in comparison with estimated values using (ii) UKF with known u_{k-1} ($\cdots$), (iii) GCUKF with unknown u_{k-1} ($-\cdot-$), and (iv) UKF with unknown u_{k-1} ($--$). The cases (i), (ii), and (iii) coincide.

Table 7.1: RMSE (2.31) and MT (2.33) for a 100-run Monte Carlo simulation for the van der Pol oscillator using algorithms (i) UKF with known u_{k-1} , (ii) GCUKF with assumed unknown u_{k-1} , (iii) UKF with unknown u_{k-1} , and (iv) UKF with unknown u_{k-1} and a larger Q_{k-1} .

	case (i)	case (ii)	case (iii)	case (iv)
RMSE _{<i>i</i>} , <i>i</i> = 1, 2				
$x_{1,k}$	0.0487	0.0728	1.8936	0.0750
$x_{2,k}$	0.0596	0.2134	1.2996	0.2055
MT				
	0.0080	0.0424	0.0086	0.0382

unknown inputs of time series, and, in (Tapley & Ingram, 1971; Rios Neto & Cruz, 1985), a dynamical model compensation technique is used.

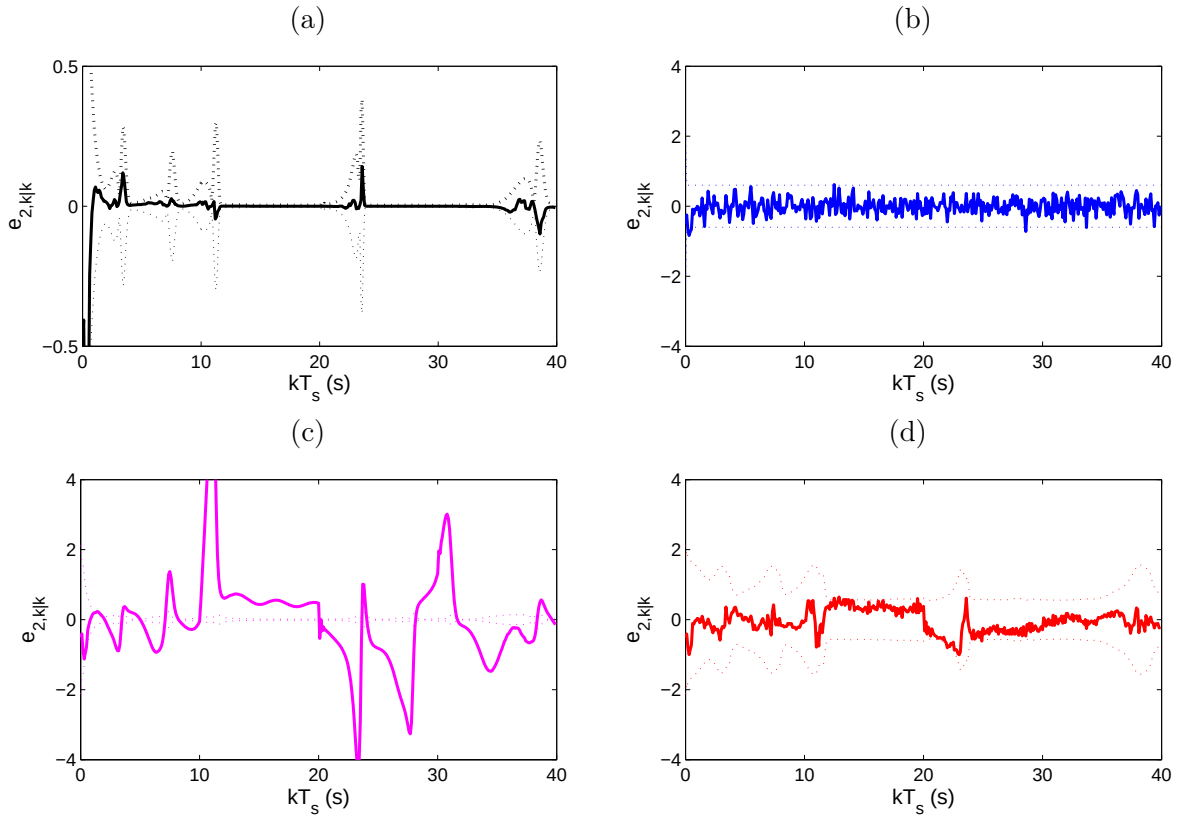


Figure 7.3: Comparison of estimation errors for x_2 using (a) UKF with u_{k-1} assumed to be known, (b) GCUKF with u_{k-1} assumed to be unknown, and (c) UKF with u_{k-1} assumed to be unknown. The case of UKF with u_{k-1} assumed to be unknown when a larger value of Q_{k-1} is used is denoted by (d). Dotted lines show the associated $\pm 3\sqrt{P_{(2,2),k|k}^{xx}}$ confidence limits. Observe that these limits for the cases (a) and (c) coincide.

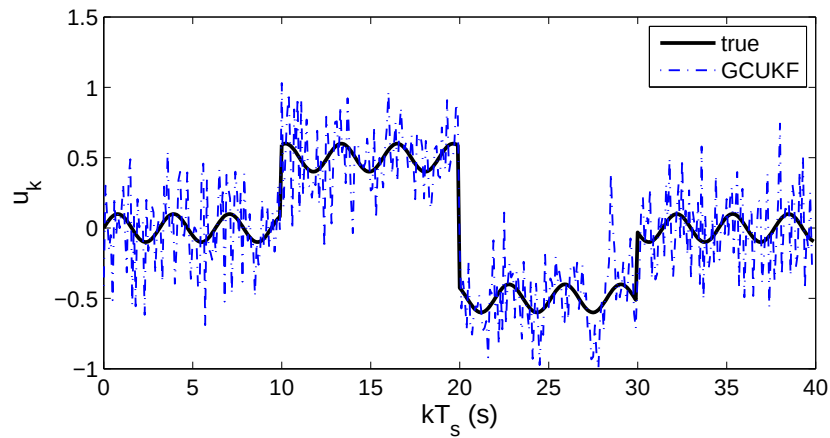


Figure 7.4: True (—) and estimated (---) unknown input u_{k-1} using GCUKF.

7.4.2 Pendulum-Cart System

We consider the pendulum-cart system illustrated in Figure 7.5 whose dynamics are given by

$$\dot{x}_1(t) = x_2(t), \quad (7.77)$$

$$\begin{aligned} \dot{x}_2(t) = & \frac{I + mL^2}{b(t)} \left(-cx_2(t) + mLx_4(t)^2 \sin(x_3(t)) \right. \\ & \left. - \frac{m^2 L^2 g}{I + mL^2} \cos(x_3(t)) \sin(x_3(t)) \right) + u_1(t), \end{aligned} \quad (7.78)$$

$$\dot{x}_3(t) = x_4(t), \quad (7.79)$$

$$\begin{aligned} \dot{x}_4(t) = & \frac{M + m}{b(t)} \left(-mLg \sin(x_3(t)) \right. \\ & \left. + \frac{mL}{M + m} \cos(x_3(t)) (mLx_4^2(t) \sin(x_3(t)) - cx_2(t)) \right) + u_2(t), \end{aligned} \quad (7.80)$$

where $x(t) \triangleq [z(t) \ \dot{z}(t) \ \theta(t) \ \dot{\theta}(t)]^T \in \mathbb{R}^4$, $z(t)$ is the cart position, $\theta(t)$ is the pendulum angular position, $F(t)$ is the force applied to the cart, M is the cart mass, m is the pendulum mass, L is the length to the pendulum center of mass, I is the pendulum inertia, c is viscous cart friction, $u(t) \triangleq [u_1(t) \ u_2(t)]^T$,

$$b(t) \triangleq (M + m)(I + mL^2) - m^2 L^2 \cos^2(x_3(t)), \quad (7.81)$$

$$u_1(t) \triangleq \frac{I + mL^2}{b(t)} F(t), \quad (7.82)$$

$$u_2(t) \triangleq \frac{mL \cos(x_3(t))}{b(t)} F(t). \quad (7.83)$$

For simulation, we set $x(0) = [0 \ 0 \ \pi/4 \ 0]^T$,

$$F(t) = \begin{cases} 0.25 \sin(t), & \text{if } t < 2.5 \text{ s or } t \geq 7.5 \text{ s,} \\ 1 + 0.25 \sin(t), & \text{if } 2.5 \text{ s} \leq t < 5.0 \text{ s,} \\ -1 + 0.25 \sin(t), & \text{if } 5.0 \text{ s} \leq t < 7.5 \text{ s,} \end{cases} \quad (7.84)$$

$M = 0.5$ Kg, $m = 0.5$ Kg, $L = 0.3$ m, $I = 0.006$ Kg-m², and $c = 0.1$ N/(m-s).

To perform state estimation, as discussed in Section A.2, we integrate the process model (7.77)-(7.80) with $T_s = 0.01$ s using the 4th-order Runge-Kutta algorithm such that $x_k \triangleq x(kT_s)$. We set $Q_{k-1} = 10^{-5} I_{4 \times 4}$ to help convergence of estimates obtained from UKF;

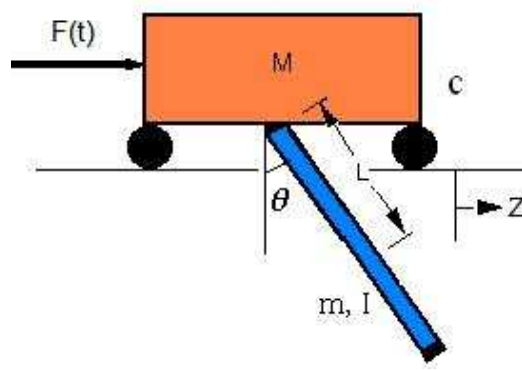


Figure 7.5: Pendulum-cart system.

see Section 2.6.3. The measurement model is given by (7.2) with $C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $R_k = 0.0005 I_{3 \times 3}$.

Assuming that the input $u(t)$ is unknown, our goal is to obtain unbiased state estimates. To accomplish that, we implement GCUKF to enforce the gain constraint (7.7) using M_k , N_k , and O_k set as (7.52), (7.56), and (7.57), respectively; see Section 7.2.5 for details. In this case, $u(t)$ plays the same role of the unknown input z_{k-1} used in the linear model (7.47) in Section 7.2.5. From (7.77)-(7.80), it follows that F_{k-1} used in (7.56)-(7.57) is given by $F_{k-1} = [0 \ 1 \ 0 \ 1]^T$.

For comparison, we investigate the following cases: (i) UKF with $u_{k-1} = u((k-1)T_s)$ assumed to be known, (ii) GCUKF with u_{k-1} assumed to be unknown, with matrices M_k , N_k , and O_k defined as above, (iii) UKF with u_{k-1} assumed to be unknown, and (iv) UKF with u_{k-1} assumed to be unknown but using $Q_{k-1} = 10^{-3} I_{4 \times 4}$ to account for the fact that u_{k-1} is unknown. For cases (ii) and (iii), Q_{k-1} is set as in case (i). Whenever u_{k-1} is assumed to be unknown, we set $u_{k-1} = 0_{2 \times 1}$ in the forecast step.

Figure 7.6 shows the estimation error components $e_{2,k|k}$ and $e_{3,k|k}$ around the confidence intervals, respectively, given by $\pm 3\sqrt{P_{(2,2),k|k}^{xx}}$ and $\pm 3\sqrt{P_{(3,3),k|k}^{xx}}$. Table 7.2 compares the RMSE index (2.31) for each state component regarding the four cases aforementioned in a 100-run Monte Carlo simulation.

Results from Table 7.2 show that GCUKF (case (ii)) outperforms UKF if the input is unknown (case (iii)) except for the cart position $x_{1,k}$. In Figure 7.6, observe that in the intervals where the mean of the input vector is nonzero (see $2.5 \text{ s} \leq kT_s \leq 7.5 \text{ s}$), the

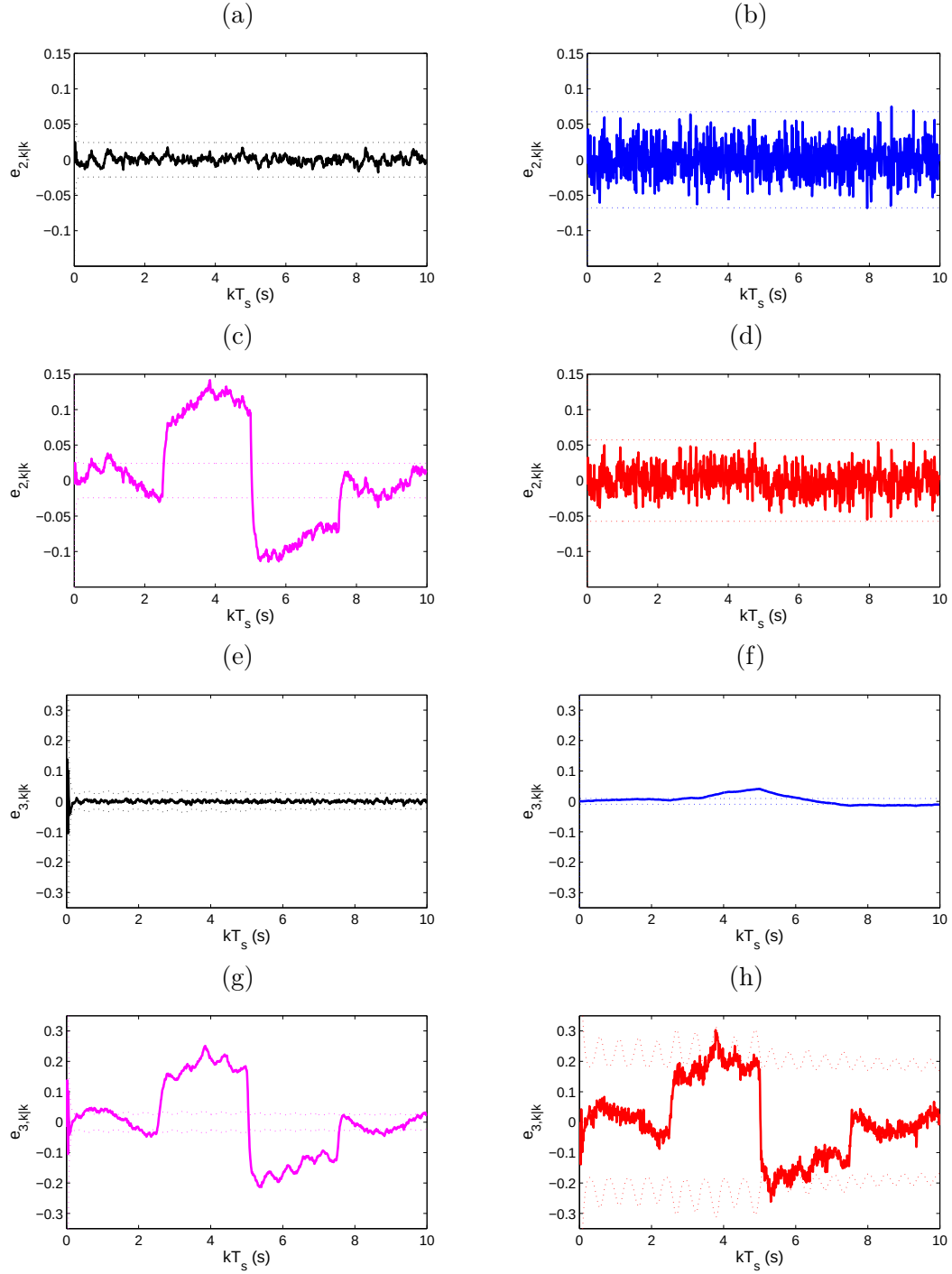


Figure 7.6: Comparison of estimation errors for (a)-(d) x_2 and (e)-(h) x_3 using (a), (e) UKF with u_{k-1} assumed to be known, (b), (f) GCUKF with u_{k-1} assumed to be unknown, and (c), (g) UKF with u_{k-1} assumed to be unknown. The case of UKF with u_{k-1} assumed to be unknown when a larger value of Q_{k-1} is used is denoted by (d) and (h). Dotted lines show the associated (a)-(d) $\pm 3\sqrt{P_{(2,2),k|k}^{xx}}$ and (e)-(h) $\pm 3\sqrt{P_{(3,3),k|k}^{xx}}$ confidence limits.

performance for GCUKF is better than case (iii). Actually, as shown in Figures 7.6(c) and 7.6(g), during this interval, the state estimates from case (iii) diverge. Moreover, as indicated in Table 7.2, by increasing Q_{k-1} , an improvement for case (iv) is observed for the velocity components when compared to the case (iii). Note that, according to Table 7.2, case (iv) is better than case (ii) for the velocity components, but with performance worse than the performance for case (iii) concerning the position variables.

Table 7.2: RMSE (2.31) for a 100-run Monte Carlo simulation for the pendulum-cart system using algorithms (i) UKF with known u_{k-1} , (ii) GCUKF with assumed unknown u_{k-1} , (iii) UKF with unknown u_{k-1} , and (iv) UKF with unknown u_{k-1} and a larger Q_{k-1} .

	case (i)	case (ii)	case (iii)	case (iv)
$x_{1,k}$	0.0061	0.0099	0.0066	0.0170
$x_{2,k}$	0.0059	0.0222	0.0699	0.0175
$x_{3,k}$	0.0117	0.0187	0.1199	0.1239
$x_{4,k}$	0.0099	0.0224	0.0246	0.0184

7.5 Concluding Remarks

This chapter derives the gain-constrained Kalman filter (GCKF) as the optimal solution to the problem of linear state estimation with constrained data-assimilation gain. Then, the classical Kalman filter (KF), the projected Kalman filter by gain projection (PKF-GP), the unbiased Kalman filter with unknown inputs (UnbKF-UI), and the spatially constrained Kalman filter (SCKF) are presented as special cases of GCKF. Table 7.3 summarizes how the gain constraint is set for all special cases of GCKF.

Table 7.3: Summary of characteristics of gain-constrained Kalman filtering algorithms for linear systems (2.7) -(2.8). It is specified how the gain constraint (7.7) is handled by each special case.

Algorithm	Section	M_k	N_k	O_k
KF	7.2.1	$I_{n \times n}$	$I_{m \times m}$	K_k
PKF-GP	7.2.4	D_{k-1}	$y_k - \hat{y}_{k k-1}$	$d_{k-1} - D_{k-1}\hat{x}_{k k-1}$
UnbKF-UI	7.2.5	$I_{n \times n}$	$[C_k F_{k-1} \quad H_k]$	$[F_{k-1} \quad 0_{n \times t}]$
UnbKF-UI ($t = 0$)	7.2.5	$I_{n \times n}$	$C_k F_{k-1}$	F_{k-1}
UnbKF-UI ($s = 0$)	7.2.5	$I_{n \times n}$	H_k	$0_{n \times t}$
SCKF	7.2.6	$\tilde{I}_k^T$	$I_{m \times m}$	$0_{b \times m}$

Using the unscented transform, the gain-constrained unscented Kalman filter (GCUKF) is presented as a nonlinear extension of GCKF. Although the resulting algorithm is an approximate solution to the state-estimation problem for nonlinear systems, its gain exactly satisfies the constraint.

Two examples of nonlinear systems, namely, a van der Pol oscillator and a pendulum-cart system, illustrate an application of GCUKF when the input vector is unknown. In both cases, improved estimates were obtained using GCUKF when compared to the case where the unscented Kalman filter is used with an assumed unknown input vector.

CONCLUSIONS AND FUTURE WORK

*“I was just guessing at numbers and figures, pulling your puzzles apart.
Questions of science, science and progress, do not speak as loud as my heart.
Nobody said it was easy, no one ever said it would be so hard.
I am going back to the start.”*

The Scientist - Coldplay (A Rush of Blood to the Head, 2002)

“The Lord is good; his love is eternal and his faithfulness lasts forever.”

The Psalms 100:5

8.1 Summary and Concluding Remarks

8.1.1 Unconstrained State Estimation

The first part of this thesis addresses the problem of unconstrained state estimation for both linear and nonlinear systems.

Chapter 2 presents a literature review on unconstrained state estimation. This problem is formulated from the perspective of recursive Bayesian estimation. Then the Kalman filter (KF) is introduced as the optimal state estimator in the maximum-a-posteriori sense for linear and Gaussian systems. Suboptimal solutions for the nonlinear case are categorized into two groups, namely, Gaussian approximate methods and particle filtering methods. We focus on the first group, reviewing two of its most known algorithms, namely, the extended Kalman filter (EKF) and the unscented Kalman filter (UKF). We discuss the key ideas behind such methods and review results on their stability. In sum, the former relies on the analytical or numerical linearization of the system equations, whereas the latter is based on a statistical linearization scheme that propagates a reduced number of deterministically chosen ensembles.

Recent theoretical and practical work illustrates the superior performance of the unscented approach over the extended method regarding asymptotic stability for strongly nonlinear systems, as well as sensitivity to initialization and to the tuning of the noise covariance matrices. Furthermore, we define metrics (accuracy, unbiasedness, information, and computational cost) and review concepts (consistency, efficiency, divergence, and optimality) so as to assess the performance of state estimators.

In Chapter 3, three case studies of unconstrained Kalman filtering algorithms are investigated. First, in order to perform flight path reconstruction, UKF is presented as a viable and improved alternative compared to EKF with relation to initialization, noise level, tuning of noise covariance matrices, and joint estimation of IMU bias terms. Moreover, a smoothing algorithm is used to substantially enhance the accuracy of the state estimates. We consider both simulated data and experimental data collected from a sailplane.

Second, we illustrate the use of sample-data continuous-time filters based on EKF and UKF to solve the problem of spacecraft trajectory estimation. The unscented approach is less sensitive than the extended approach with relation to initialization and time sparsity of measurements, but at a two times larger computational cost. Moreover, the unscented algorithm yields more accurate eccentricity estimates than the extended approach. Finally, when the goal is to track changes in the spacecraft inclination, we see that the use of angle measurements helps the convergence and improves the performance of both algorithms.

Third, we consider the use of discrete-time models built from data together with UKF to estimate states and parameters of nonlinear systems. We investigate the scenario in which observations of the full state vector are available only during a given period of time to build the model, while only one variable is measured to drive the filter. Both simulated (Lorenz system) and experimental (implementation of the Chua's circuit) examples are investigated. We observe that inaccuracies in the model can be compensated for, to some extent, by setting a suitable nonzero process-noise covariance. Nonetheless, we verify that, regardless of the value set to this matrix, poor models usually result in inaccurate state estimates or in divergence, whereas more accurate models pay off in state-estimation problems.

8.1.2 Constrained State Estimation

The second part of this thesis is concerned with the problem of constrained state estimation for linear and nonlinear systems. We consider the case in which we want to enforce equality or inequality constraints on the state vector, as well as the case where, in order

to enforce special properties on the state estimates, the estimator gain satisfies an equality constraint.

Chapter 4 provides a literature review on constrained state estimation for linear and nonlinear systems. Using a unified notation, we compare and list the corresponding equations of fourteen algorithms presented in the literature to enforce equality or inequality constraints on the state vector. Next, Chapters 5, 6, and 7 present contributions on constrained state estimation as follows.

The problems of linear and nonlinear state estimation with equality constraints on the state vector are addressed in Chapter 5. First, we show that the equality-constrained state-estimation problem for linear systems arises from the definition of both process noise and dynamic equations with special properties. Then, the equality-constrained Kalman filter (ECKF) is presented as the optimal solution in the maximum-a-posteriori sense for this problem, and it is compared to four alternative approaches. For the nonlinear case, three suboptimal algorithms based on UKF are presented, namely, the equality-constrained UKF, the projected UKF by estimate projection, and the measurement-augmented UKF. In addition to enforcing a known equality constraint of the system, the proposed methods yield more accurate and more informative estimates than KF (in the linear case) or UKF (in nonlinear case). Moreover, all equality-constrained approaches require similar processing time, which is competitive to KF (for linear algorithms) and UKF (for nonlinear cases) processing time. For the linear scenario, either ECKF or any of the four alternative algorithms investigated yield similar results. However, for the nonlinear case, considering the examples investigated, we recommend the user to test ECUKF, MAUKF, and PUKF-EP in this order for a given application. Recall that, since these methods are approximate, their performance depends on the application.

Chapter 6 addresses the state-estimation problem for interval-constrained nonlinear systems. By combining one of two candidate unscented approaches for forecast and one of five candidate methods for data assimilation – all proposed in the literature, we obtain the constrained UKF (CUKF), the constrained interval UKF (CIUKF), the interval UKF (IUKF), the truncated UKF (TUKF), the truncated interval UKF (TIUKF), and the projected UKF (PUKF). These six algorithms are compared to UKF and the sigma-point inequality-constrained UKF (SIUKF) by means of two illustrative examples. We verify that, for a good initialization, the eight aforementioned algorithms provide similar performance. However, if a poor initialization is set, then the UKF estimates may violate the interval constraint. In this

case, only a slight improvement over UKF is observed using CUKF and PUKF. On the other hand, SIUKF, TUKF, and TIUKF provide the best performance. Regarding computational cost, IUKF, TIUKF, and TUKF are competitive with UKF. Furthermore, CUKF, CIUKF, and PUKF are slower than UKF, while SIUKF is the slowest algorithm. However, it is not clear to point out which method is the best option for a given problem. Rather, it seems that the choice of a method depends on the application. From the examples investigated, considering the tradeoff between accuracy and computational complexity, we recommend the TUKF and TIUKF algorithms, followed by SIUKF, CIUKF, and IUKF.

Finally, Chapter 7 considers the state-estimation problem with a constraint on the estimation gain. We derive the gain-constrained Kalman filter (GCKF) as the minimum-variance optimal solution to the linear case. Then, the classical KF, the projected Kalman filter by gain projection, the unbiased Kalman filter with unknown inputs, and the spatially constrained Kalman filter are presented as special cases of GCKF. The one-step gain-constrained Kalman predictor is also derived for linear systems in Appendix C. Also, the gain-constrained UKF is presented as a nonlinear extension of GCKF by means of the unscented transform. Although the resulting algorithm is an approximate solution to the state-estimation problem for nonlinear systems, its gain exactly satisfies the constraint.

8.2 Publications

A large majority of the work presented in the Chapters 3, 5, 6, and 7 of this thesis has already been submitted or published in international journals, magazines, and conference proceedings. Here is a list of such publications:

Journal and Magazine Papers

- Aguirre, L. A., Teixeira, B. O. S., & Tôrres, L. A. B. (2005). Using data-driven discrete-time models and the unscented Kalman filter to estimate unobserved variables of nonlinear systems. *Physical Review E*, 72(2), 026226.
- Teixeira, B. O. S. (2008). Present and Future (Ask the Experts!). *IEEE Control Systems Magazine*, 28(2):16–18.
- Teixeira, B. O. S., Santillo, M. A., Erwin, R. S., & Bernstein, D. S. (2008). Spacecraft tracking using sample-data Kalman filters - extended and unscented filters for trajectory

estimation. *IEEE Control Systems Magazine*, 28, to appear.

- Teixeira, B. O. S., Chandrasekar, J., Palanthandalam-Madapusi, H. J., Tôrres, L. A. B., Aguirre, L. A., & Bernstein, D. S.. Gain-constrained Kalman filtering for linear and nonlinear systems. *IEEE Transactions on Signal Processing*, resubmitted.
- Teixeira, B. O. S., Chandrasekar, J., Tôrres, L. A. B., Aguirre, L. A., & Bernstein, D. S.. State estimation for linear and nonlinear equality-constrained systems. *International Journal of Control*, resubmitted.
- Teixeira, B. O. S., Tôrres, L. A. B., Iscold, P. H. A. O., & Aguirre, L. A.. Flight path reconstruction – a comparison of nonlinear Kalman filter and smoother algorithms. *Control Engineering Practice*, to be submitted.

International Conference Papers

- Teixeira, B. O. S., Tôrres, L. A. B., Iscold, P. H. A. O., & Aguirre, L. A. (2005). Flight path reconstruction using the unscented Kalman filter algorithm. In *18th International Congress of Mechanical Engineering*, (n. 0610), Ouro Preto – MG, Brasil. ABCM.
- Teixeira, B. O. S., Chandrasekar, J., Tôrres, L. A. B., Aguirre, L. A., & Bernstein, D. S. (2007). State estimation for equality-constrained linear systems. In *Proceedings of the 46th IEEE Conference on Decision and Control*, (pp. 6220–6225), New Orleans, USA.
- Teixeira, B. O. S., Chandrasekar, J., Tôrres, L. A. B., Aguirre, L. A., & Bernstein, D. S. (2008). Unscented filtering for equality-constrained nonlinear systems. In *Proceedings of the American Control Conference*, Seattle, USA, to appear.
- Teixeira, B. O. S., Ridley, A., Bernstein, D. S., Tôrres, L. A. B., & Aguirre, L. A. (2008). Data assimilation for magnetohydrodynamics with a zero-divergence constraint on the magnetic field. In *Proceedings of the American Control Conference*, Seattle, USA, to appear.
- Teixeira, B. O. S., Tôrres, L. A. B., Aguirre, L. A., & Bernstein, D. S. (2008). Unscented filtering for interval-constrained nonlinear systems. In *Proceedings of 47th IEEE Conference on Decision and Control*, Cancun, Mexico, submitted.

Internal Reports

- Teixeira, B. O. S., Tôrres, L. A. B., Aguirre, L. A., & Iscold, P. H. A. O. (2006). Flight path reconstruction – a comparison of nonlinear Kalman filter and smoother algorithms. *Technical report*, MACSIN/CPDEE/UFGM.
- Teixeira, B. O. S., Ridley, A., Bernstein, D. S., Tôrres, L. A. B., & Aguirre, L. A. (2007). Data assimilation for magnetohydrodynamics with a zero-divergence constraint on the magnetic field. *Technical report*, Department of Aerospace Engineering, University of Michigan – Ann Arbor.

8.3 Future Work

From the pursuit of the research objectives posed at the beginning of this thesis, relevant theoretical and practical issues for future investigation were identified. We summarize these suggestions for future work as follows:

1. A Lagrange-multiplier method for exactly enforcing a single quadratic nonlinear state equality constraint is given in (Yang & Blasch, 2006). We suggest the generalization of this result using the unscented approach to the case of multidimensional quadratic nonlinear equality constraint.
2. The use of constrained least-squares method is reported in (Aguirre et al., 2000, 2004, 2007; Nepomuceno et al., 2004; Walker, 2006) to estimate the parameters of nonlinear models in the context of grey-box system identification (Ljung, 1987). We suggest the use of the constrained Kalman filtering algorithms presented in Chapters 5 and 6 to address the same problem.
3. In Appendix B we investigate the problem of enforcing the zero-divergence constraint in data assimilation for magnetohydrodynamics. As further research, we suggest the investigation of an example whose closed solution is known such that the tradeoff between the accuracy of the estimates provided by the investigated algorithms and the accuracy of the satisfaction of the zero-divergence constraint can be evaluated.
4. It seems promising the incorporation of the unscented approach to the nonlinear moving horizon estimator, which is reviewed in Section 4.5.2, to propagate the error-covariance

matrix. In accordance to Remarks 2.6.1 and 4.5.1, we conjecture that the resulting algorithm may have improved stability properties.

5. In Section 5.1, we investigate how a linear equality constraint arises on the state vector of a linear dynamic system. To our best knowledge, an in-depth mathematical analysis of how a inequality constraint arises on the state vector of a linear system is still absent in the literature. Preliminary discussions are found in (Rao, 2000; Robertson & Lee, 2002; Goodwin et al., 2005); see Remark 4.4.1. Such investigation may lead to a precise formulation of the inequality-constrained state estimation problem and, consequently, to the derivation of the optimal solution, as well as the proposition of novel approximate algorithms.
6. Unbiased Kalman filtering algorithms for systems with unknown inputs are reviewed in the context of gain-constrained Kalman filtering in Section 7.2.5 and extended to the nonlinear case using the unscented approach in Section 7.3. Using the unscented approach, we suggest the extension to the nonlinear case of the improved algorithms presented in (Gillijns & De Moor, 2007a,b) for unbiased state estimation for systems with unknown inputs, adding emphasis on the input reconstruction problem. In so doing, applications in transmission of information (Tôrres, 2001) could be investigated, for instance.
7. In this thesis we investigate the incorporation of constraints in Kalman filtering algorithms. We recommend the use of constraints in particle filtering algorithms (Gordon et al., 1993; Arulampalam et al., 2002; Ristic et al., 2004), which are outside the scope of this thesis.
8. In (Pinto & Rios Neto, 1990) an algorithm based on KF is used to solve a system of linear algebraic equations. We suggest the investigation of the usage of algorithms based on UKF to solve a system of nonlinear algebraic equations.
9. Finally, apart from the constrained state-estimation problem, in addition to (Lee & Alfriend, 2004), further developments on adaptive Kalman filtering (Mehra, 1970, 1972; Myer & Tapley, 1976; Rios Neto & Kuga, 1985; Vargas & Hemerly, 2001; Powell, 2002) using the unscented approach are worthy of investigation. That is, we suggest the proposition of unscented filters that, in addition to the state vector, estimate the noise covariance matrices.

COMPLEMENTARY TOPICS ON UNCONSTRAINED STATE ESTIMATION

In this appendix we discuss four complementary topics on unconstrained state estimation. First, we show how to use state-estimation algorithms to estimate model parameters together with the state vector; see Section A.1. Second, in Section A.2, we address the topic of state estimation for systems with continuous-time dynamics and sampled measured data. Third, in Section A.3, we differ three kind of state estimators, namely, filters, predictors, and smoothers. Finally, in Section A.4, we present Kalman smoothers to yield improved state estimates using not only past and current measurements but also future data.

A.1 Joint State-and-Parameter Estimation

State estimators can be readily used to jointly estimate system parameters $\theta_k \in \mathbb{R}^{n_\theta}$ and the state vector x_k . Since such algorithms deal with nonlinear dynamical evolution, we treat a parameter as a “virtual” state to be estimated.

Define the *joint state vector* $\check{x}_k \in \mathbb{R}^{n+n_\theta}$ as

$$\check{x}_k \triangleq \begin{bmatrix} x_k \\ \theta_k \end{bmatrix}. \quad (\text{A.1})$$

Next, we rewrite (1.1)-(1.2) as

$$\check{x}_k = \check{f}(\check{x}_{k-1}, u_{k-1}, w_{k-1}, k-1) \quad (\text{A.2})$$

$$y_k = \check{h}(\check{x}_k, v_k, k), \quad (\text{A.3})$$

where the joint process noise $\check{w}_{k-1} \in \mathbb{R}^{q+n_\theta}$ is given by

$$\check{w}_{k-1} \triangleq \begin{bmatrix} w_{k-1} \\ w_{k-1}^\theta \end{bmatrix}, \quad (\text{A.4})$$

where $w_{k-1}^\theta \in \mathbb{R}^{n_\theta}$ is a white, Gaussian and zero-mean random vector with covariance Q_{k-1}^θ , and is uncorrelated to w_{k-1} . It is expected that θ_k either do not vary or that its variation is much slower than that of x_k . The larger the trace of Q_{k-1}^θ , the quicker θ_k varies with time. Hence the parameter dynamics is often modeled as the first-order Markov process (Mulder et al., 1999)

$$\theta_k = F_{k-1}^\theta \theta_{k-1} + w_{k-1}^\theta, \quad (\text{A.5})$$

where F_{k-1}^θ is a diagonal matrix whose entries satisfy $0 < F_{(i,i),k-1} \leq 1$, $i = 1, \dots, n_\theta$. If we set $F_{k-1}^\theta = I_{n_\theta \times n_\theta}$, then (A.5) models a random-walk process (Papoulis & Pillai, 2001). In this case, the nonzero-mean and non-stationary features of some sensors, such as some inertial measurement units, are addressed (van der Merwe et al., 2004). Using (1.1) and (A.5) we write (A.2) as

$$\begin{bmatrix} x_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} f(x_{k-1}, u_{k-1}, w_{k-1}, k-1) \\ F_{k-1}^\theta \theta_{k-1} \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ w_{k-1}^\theta \end{bmatrix}. \quad (\text{A.6})$$

Under the stated assumptions, the maximization of $\rho(\check{x}_k | (y_1, \dots, y_k))$ is the *joint state-and-parameter estimation problem*, while the *joint state estimate* $\hat{\check{x}}_{k|k}$ is its maximizer. Also, let

$$P_{k|k}^{\check{x}\check{x}} \triangleq \begin{bmatrix} P_{k|k}^{xx} & P_{k|k}^{x\theta} \\ P_{k|k}^{\theta x} & P_{k|k}^{\theta\theta} \end{bmatrix}. \quad (\text{A.7})$$

denote the *joint error-covariance matrix* associated to $\hat{\check{x}}_{k|k}$.

Similar to the importance of Q_{k-1} for the stability of EKF and UKF (see Sections 2.5.3 and 2.6.3), the tuning of Q_{k-1}^θ plays an important role on the asymptotic performance of the parameter estimate $\hat{\theta}_{k|k}$ (Haykin, 2001). The larger Q_{k-1}^θ is, the more assimilated the measurements are into $\hat{\theta}_{k|k}$. Different approaches, namely, constant or time-varying Q_{k-1}^θ , can be used to tune the state-estimator convergence rate and parameter tracking performance

(Haykin, 2001; van der Merwe, 2004). For the sake of simplicity, we set Q_{k-1}^θ to be constant.

Alternatively to (A.6), a *dual approach* is used in (Wan, van der Merwe & Nelson, 2000), in which, at each step k , two state estimators run sequentially. The first estimates the state vector $\hat{x}_{k|k}$ using the parameter estimate $\hat{\theta}_{k-1|k-1}$ from time $k-1$, while the second estimates the parameters $\hat{\theta}_{k|k}$ using the current states estimate $\hat{x}_{k|k}$. Note that the dual approach does not consider the coupling between $\hat{x}_{k|k}$ and $\hat{\theta}_{k|k}$ by means of $P_{k|k}^{x\theta}$ in (A.7). According to van der Merwe (2004), such cross covariance improves the estimation.

Finally, unlike the standard least-square estimator (Ljung, 1987; Aguirre, 2007), state-estimation algorithms are applicable to the case in which the dynamics is a nonlinear function of the parameters. This justifies the frequent application of state estimators to estimate parameters of neural networks (Rios Neto, 1997; Silva & Rios Neto, 1999; Nelson, 2000; Haykin, 2001; Feldkamp, Feldkamp & Prokhorov, 2001; Choi, Yeap, & Bouchard, 2004; Choi, Yeap & Bouchard, 2005; Zhan & Wan, 2006), radial basis functions (Burgmeier, 1995; Walker & Mees, 1998), and fuzzy systems (Simon, 2002, 2003). Nonlinear Kalman filtering algorithms are used to estimate parameters for chemical processes in (Maciel Filho, Rodrigues & Zaiat, 1999; Maciel Filho, Duarte & Ender, 2004) and for an application in electrical impedance tomography in (Trigo, 2005).

A.2 Nonlinear Kalman Filtering for Sampled-Data Continuous-Time Systems

Consider the stochastic nonlinear continuous-time dynamic system

$$\dot{x}(t) = f(x(t), u(t), w(t), t), \quad (\text{A.8})$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \rightarrow \mathbb{R}^n$, with discrete-time measurement map

$$y(kT_s) = h(x(kT_s), v(kT_s), k), \quad (\text{A.9})$$

where $h : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{N} \rightarrow \mathbb{R}^m$ and $T_s > 0$ is the fixed sampling rate. Assume that, for all $t \geq 0$, the input $u(t) \in \mathbb{R}^p$ is known and, for all $k \geq 1$, the output $y(kT_s) \in \mathbb{R}^m$ is measured. The process noise $w(t) \in \mathbb{R}^q$ and the measurement noise $v(kT_s) \in \mathbb{R}^r$ are assumed to have zero-mean and known covariance matrices $Q(t)$ and $R(kT_s)$, respectively, and are assumed to be mutually independent. Also, let $\hat{x}(0) \in \mathbb{R}^n$ and $P^{xx}(0)$ denote the initial state estimate

for the state vector $x(t) \in \mathbb{R}^n$ and the initial error-covariance, respectively. To solve the state-estimation problem for the system (A.8)-(A.9), we use two approaches.

In the first approach, the EKF and UKF algorithms are slightly modified to handle (A.8). For EKF, we replace (2.47) by

$$\hat{x}((k-1)T_s) = \hat{x}_{k-1|k-1}, \quad (\text{A.10})$$

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t), 0_{q \times 1}, t), \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.11})$$

$$\hat{x}_{k|k-1} = \hat{x}(kT_s), \quad (\text{A.12})$$

such that the forecast equations are given by (A.10)-(A.12), (2.48)-(2.51), whose discrete-time Jacobians $\hat{A}_{k-1}$ and $\hat{G}_{k-1}$ are obtained from discretizing the continuous-time Jacobians given by

$$\hat{A}((k-1)T_s) = \left. \frac{\partial f}{\partial x(t)} \right|_{x(t)=\hat{x}_{k-1|k-1}, u((k-1)T_s), 0_{q \times 1}, t}, \quad (\text{A.13})$$

$$\hat{G}((k-1)T_s) = \left. \frac{\partial f}{\partial w(t)} \right|_{x(t)=\hat{x}_{k-1|k-1}, u((k-1)T_s), 0_{q \times 1}, t}, \quad (\text{A.14})$$

and $\hat{C}_k$ and $\hat{H}_k$ are given by

$$\hat{C}_k = \left. \frac{\partial h}{\partial x(kT_s)} \right|_{x(kT_s)=\hat{x}_{k|k-1}, 0_{r \times 1}, kT_s}, \quad (\text{A.15})$$

$$\hat{H}_k = \left. \frac{\partial h}{\partial v(kT_s)} \right|_{x(kT_s)=\hat{x}_{k|k-1}, 0_{r \times 1}, kT_s}, \quad (\text{A.16})$$

while the data-assimilation equations are given by (2.25), (2.27), and (2.28).

For UKF, we replace (2.90) by

$$\mathcal{X}_j((k-1)T_s) = \mathcal{X}_{j,k-1|k-1}, \quad j = 0, \dots, 2\tilde{n}, \quad (\text{A.17})$$

$$\dot{\mathcal{X}}_j^x(t) = f(\mathcal{X}_j^x(t), u(t), \mathcal{X}_j^w(t), t), \quad j = 0, \dots, 2\tilde{n}, \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.18})$$

$$\mathcal{X}_{j,k|k-1} = \mathcal{X}_j(kT_s), \quad j = 0, \dots, 2\tilde{n}, \quad (\text{A.19})$$

such that the forecast equations are now given by (2.89), (A.17)-(A.19), (2.91)-(2.96) and the data-assimilation equations given by (2.25), (2.27)-(2.28).

Note that, the state estimate (A.11) and, in the case of UKF, the sigma points (A.18) are integrated over $[(k-1)T_s, kT_s]$; however, only the values at the right-hand endpoint of

$[(k-1)T_s, kT_s]$ are used. Also, the error covariance is propagated in discrete-time as in the original EKF and UKF algorithms. In (A.18), note that we assume that the process noise is constant over T_s s, implying that, actually, the process noise has band-limited spectral density; however, this assumption is plausible if T_s is much smaller than the main time constant of the system.

The second approach consists in the use of algorithms that, in the forecast step, propagate both the state estimate and the error-covariance in continuous time over the interval $[(k-1)T_s, kT_s]$ and perform data assimilation in discrete time at $t = kT_s$. The overall system can be viewed as a sampled-data system in which continuous-time dynamics are interrupted by instantaneous state jumps (Sun, Nagpal, Krishan & Khargonekar, 1993). Next, we present the sampled-data extended Kalman-Bucy filter (SDEKBF) (Jazwinski, 1970; Gelb, 1974) and the sampled-data unscented Kalman-Bucy filter (SDUKBF) (Särkkä, 2007). In both cases, let $\hat{x}(t)$ denote the forecast (pseudo-) state estimate and $P^{xx}(t) \triangleq \mathbb{E}[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T]$ denote the forecast (pseudo-) error covariance. Also, let $\hat{x}(kT_s+)$ denote the data-assimilation (pseudo-) state estimate and $P^{xx}(kT_s+) \triangleq \mathbb{E}[(x(kT_s) - \hat{x}(kT_s+))(x(kT_s) - \hat{x}(kT_s+))^T]$ denote the data-assimilation (pseudo-) error covariance. We use “-” and “+” to denote state estimates before and after data assimilation, respectively.

For systems with both dynamics and measurement map in continuous-time, the extended Kalman-Bucy filter (Kalman & Bucy, 1961; Jazwinski, 1970) and the unscented Kalman-Bucy filter (Särkkä, 2007) can be used.

A.2.1 The Sampled-Data Extended Kalman-Bucy Filter

Set $\hat{x}((k-1)T_s-) = \hat{x}((k-1)T_s+)$ and $P^{xx}((k-1)T_s-) = P^{xx}((k-1)T_s+)$. Then, the forecast step of SDEKBF is given by

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t), 0_{q \times 1}, t), \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.20})$$

$$\dot{P}^{xx}(t) = \hat{A}(t)P^{xx}(t) + P^{xx}(t)\hat{A}^T(t) + \hat{G}(t)Q(t)\hat{G}^T(t), \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.21})$$

$$\hat{y}(kT_s-) = h(\hat{x}(kT_s-), 0_{r \times 1}, k), \quad (\text{A.22})$$

$$P^{yy}(kT_s-) = \hat{C}(kT_s)P^{xx}(kT_s-)\hat{C}^T(kT_s) + \hat{H}(kT_s)R(kT_s)\hat{H}^T(kT_s), \quad (\text{A.23})$$

$$P^{xy}(kT_s-) = P^{xx}(kT_s-)\hat{C}^T(kT_s), \quad (\text{A.24})$$

where $\hat{x}(kT_s-)$ and $P^{xx}(kT_s-)$ denote the values of $\hat{x}(t)$ (A.20) and $P^{xx}(t)$ (A.21), respectively, at the right-hand endpoint of the interval $[(k-1)T_s, kT_s]$, and the Jacobians are evaluated along the trajectory of (A.20) as

$$\hat{A}(t) \triangleq \left. \frac{\partial f}{\partial x(t)} \right|_{x(t)=\hat{x}(t), u(t), 0_{q \times 1}, t}, \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.25})$$

$$\hat{G}(t) \triangleq \left. \frac{\partial f}{\partial w(t)} \right|_{x(t)=\hat{x}(t), u(t), 0_{q \times 1}, t}, \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.26})$$

$$(\text{A.27})$$

except

$$\hat{C}(kT_s) \triangleq \left. \frac{\partial h}{\partial x(kT_s)} \right|_{x(kT_s)=\hat{x}(kT_s-), 0_{r \times 1}, kT_s}, \quad (\text{A.28})$$

$$\hat{H}(kT_s) \triangleq \left. \frac{\partial h}{\partial v(kT_s)} \right|_{x(kT_s)=\hat{x}(kT_s-), 0_{r \times 1}, kT_s}. \quad (\text{A.29})$$

which are evaluated only at $t = kT_s$. Note that (A.20)-(A.21) are numerically integrated online from $(k-1)T_s$ to kT_s with initial values obtained from the data-assimilation step, which is given by

$$K(kT_s) = P^{xy}(kT_s-)(P^{yy}(kT_s-))^{-1}, \quad (\text{A.30})$$

$$\hat{x}(kT_s+) = \hat{x}(kT_s-) + K(kT_s)(y(kT_s) - \hat{y}(kT_s-)), \quad (\text{A.31})$$

$$P^{xx}(kT_s+) = P^{xx}(kT_s-) - K(kT_s)P^{yy}(kT_s-)K^T(kT_s). \quad (\text{A.32})$$

A.2.2 The Sampled-Data Unscented Kalman-Bucy Filter

Let the augmented state vector $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, where $\tilde{n} \triangleq n + q + r$, and the augmented data-assimilation error covariance $P^{\tilde{x}\tilde{x}}(t) \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ be given by

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ w(t) \\ v(kT_s) \end{bmatrix}, \quad P^{\tilde{x}\tilde{x}}(t) \triangleq \begin{bmatrix} P^{xx}(t) & 0_{n \times q} & 0_{n \times r} \\ 0_{q \times n} & Q(t) & 0_{q \times r} \\ 0_{r \times n} & 0_{r \times q} & R(kT_s) \end{bmatrix}. \quad (\text{A.33})$$

Furthermore, let the weights be given by (2.83)-(2.85) replacing n by $\tilde{n}$. Set $\hat{\tilde{x}}((k-1)T_s-) = \hat{\tilde{x}}((k-1)T_s+)$ and $P^{\tilde{x}\tilde{x}}((k-1)T_s-) = P^{\tilde{x}\tilde{x}}((k-1)T_s+)$. Then, the forecast equations of

SDUKBF are given by

$$\left[\gamma_x, \gamma_P, \tilde{\mathcal{X}}(t) \right] = \Psi_{\text{UT}} \left(\hat{x}(t), P^{\hat{x}\hat{x}}(t), \tilde{n}, \alpha, \beta, \kappa \right), \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.34})$$

$$\dot{\mathcal{X}}_j^x(t) = f(\mathcal{X}_j^x(t), u(t), \mathcal{X}_j^w(t), t), \quad j = 0, \dots, 2\tilde{n}, \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.35})$$

$$\dot{\hat{x}}(t) \triangleq \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \mathcal{X}_j^x(t), \quad t \in [(k-1)T_s, kT_s], \quad (\text{A.36})$$

$$\begin{aligned} \dot{P}^{xx}(t) &= \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{X}_j^x(t) - \hat{x}(t)] [f(\mathcal{X}_j^x(t), u(t), \mathcal{X}_j^w(t), t) - \dot{\hat{x}}(t)]^T \\ &\quad + \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [f(\mathcal{X}_j^x(t), u(t), \mathcal{X}_j^w(t), t) - \dot{\hat{x}}(t)] [\mathcal{X}_j^x(t) - \hat{x}(t)]^T, \\ &\quad t \in [(k-1)T_s, kT_s], \end{aligned} \quad (\text{A.37})$$

$$\mathcal{Y}_j(kT_s-) = h(\mathcal{X}_j^x(kT_s-), \mathcal{X}_j^v(kT_s-), k), \quad j = 0, \dots, 2\tilde{n}, \quad (\text{A.38})$$

$$\hat{y}(kT_s-) = \sum_{j=0}^{2\tilde{n}} \gamma_{j,x} \mathcal{Y}_j(kT_s-), \quad (\text{A.39})$$

$$P^{yy}(kT_s-) = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{Y}_j(kT_s-) - \hat{y}(kT_s-)] [\mathcal{Y}_j(kT_s-) - \hat{y}(kT_s-)]^T, \quad (\text{A.40})$$

$$P^{xy}(kT_s-) = \sum_{j=0}^{2\tilde{n}} \gamma_{j,P} [\mathcal{X}_j^x(kT_s-) - \hat{x}(kT_s-)] [\mathcal{Y}_j(kT_s-) - \hat{y}(kT_s-)]^T, \quad (\text{A.41})$$

where Ψ_{UT} is defined by (2.86), $\mathcal{X}_j^x(kT_s-)$, $j = 0, \dots, 2\tilde{n}$, and $P^{xx}(kT_s-)$ denote the values of $\mathcal{X}_j^x(t)$ (A.35) and $P^{xx}(t)$ (A.37), respectively, at the right-hand endpoint of the interval $[(k-1)T_s, kT_s]$. Note that the sigma-point matrix $\tilde{\mathcal{X}}(t)$ is continuously evaluated over $[(k-1)T_s, kT_s]$. Also, each sigma point $\tilde{\mathcal{X}}_j(t)$, $j = 0, \dots, 2\tilde{n}$, and the error covariance $P^{xx}(t)$ are numerically integrated online from $(k-1)T_s$ to kT_s . Like SDEKBF, the data-assimilation step of SDUKBF is given by (A.30)-(A.32).

A.3 Prediction, Filtering, and Smoothing

Recall that the state estimate $\hat{x}_{k|k}$ obtained from the posterior PDF $\rho(x_k | (y_1, \dots, y_k))$ (see Definition 1.2.1) assimilates the information provided not only by the past measured data $y_1, \dots, y_{k-1}$ but also by the current measurement y_k . In the literature, such estimator is called *filter*.

Nevertheless, the term filter is more commonly employed to refer to the process of recovering the signal s_k from $z_k = s_k + v_k$, which is corrupted by the noise signal v_k . In the

context of discrete-time systems, the goal of digital filters is to extract s_k from v_k . In this case, we assume that the desired signal s_k occupies a bandwidth different from the bandwidth of v_k . However, in the context of state estimation, we generally assume that v_k is white noise, which means that the bandwidth of v_k occupies the whole spectrum. Thus, filter-like state estimators are algorithms conceived in the time domain that use the deterministic information available in $y_1, \dots, y_k$ to yield $\hat{x}_{k|k}$.¹

Filters are built on the assumption that the time spent to process y_k is negligible compared to the sample interval in which the measurements are obtained. However, for large-scale applications, this assumption might not be feasible, and thus a *predictor* is employed to process only the past measured data $y_1, \dots, y_{k-1}$. Due to the absence of the knowledge of y_k , predictors are more sensitive to higher noise levels than filters. Also, its estimates are more uncertain than the filter estimates. Since there is some confusion in the literature concerning the Kalman predictor and filter algorithms, Teixeira (2008) clarifies the difference between these estimators.

For offline applications, in addition to past and present measurements $y_1, \dots, y_k$, future measured data $y_{k+1}, \dots, y_N$, $N > k$, can be used to yield less uncertain and more robust-to-noise estimates than the filter estimates. The reading process is often executed in this way. In order to understand a “noisy” handwritten word, we make use of the context, which is determined by words around the one to be “estimated” (Anderson & Moore, 1979). Such estimator is called *smoother*. It is also known as batch state estimator (Crassidis & Junkis, 2004).

A.4 Nonlinear Kalman Smoothing

We discuss in Section A.3 how a smoother yields more accurate and more informative estimates compared to a filter by using, in addition to past and present data $y_1, \dots, y_k$, future measurements $y_{k+1}, \dots, y_{k+N}$, where $N > k$ is size of the interval in which data is used. Now we present the extended Kalman smoother (EKS) (Gelb, 1974; Crassidis & Junkis, 2004) and the unscented Kalman smoother (UKS) (Haykin, 2001). Both algorithms are based on the fixed-interval smoothing approach. Other approaches are discussed in (Crassidis & Junkis, 2004).

¹Actually, the Wiener-Kolmogorov filtering theory (Wiener, 1949), which precedes the Kalman theory, was developed in the frequency domain. Like Kalman, Wiener used statistical techniques to establish his ideas. However, such frequency-domain approach, which was inherited from the aforementioned digital filters, has limited application because it requires stationarity of the random processes.

Both EKF and UKS combine the results of a *forward filter* and a *backward filter*; see Figure A.1. The former yields the state estimate $\hat{x}_{k|k}^f$ and error covariance $P_{k|k}^{xxf}$ given past and present measurements, while the latter yields $\hat{x}_{k|k}^b$ and $P_{k|k}^{xxb}$ given present and future measurements. These filters are run sequentially. Afterwards, during the *smoothing stage*, the smoother optimally combines the results obtained from the forward and backward stages, yielding the smoothed error covariance P_k^{xss} and the smoothed state estimate $\hat{x}_k^s$ as

$$(P_k^{xss})^{-1} = (P_{k|k}^{xxf})^{-1} + (P_{k|k}^{xxb})^{-1}, \quad (\text{A.42})$$

$$\hat{x}_k^s = P_k^{xss} \left[(P_{k|k}^{xxf})^{-1} \hat{x}_{k|k}^f + (P_{k|k}^{xxb})^{-1} \hat{x}_{k|k}^b \right]. \quad (\text{A.43})$$

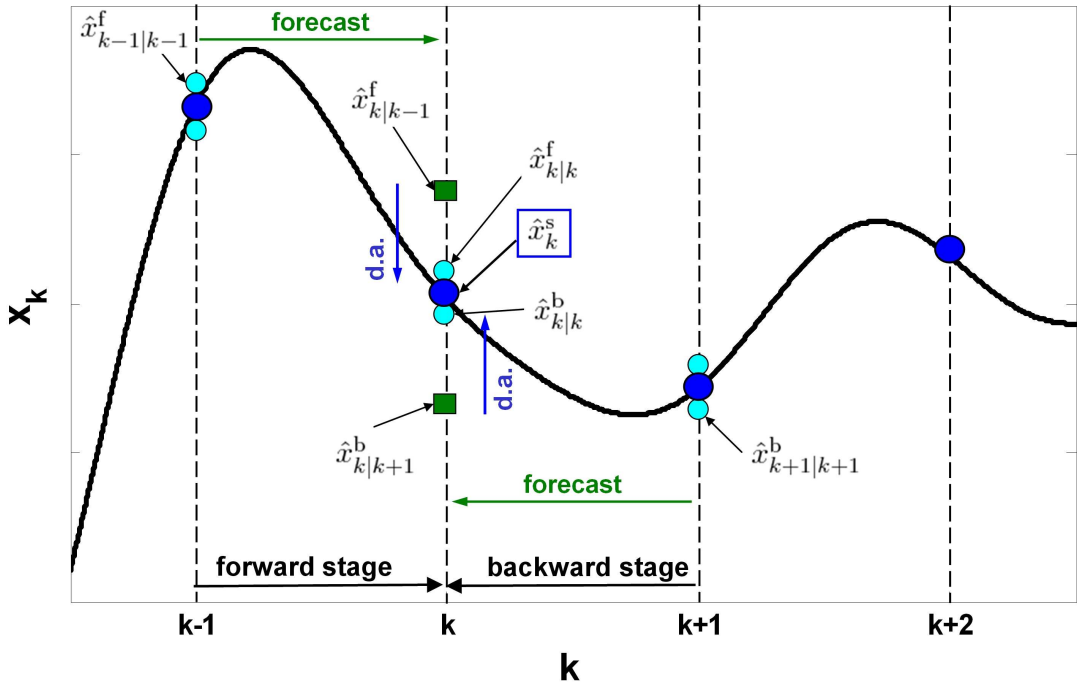


Figure A.1: Diagram of the Kalman smoother whose estimate $\hat{x}_k^s$ combines a forward estimate $\hat{x}_{k|k}^f$ (accounting for current and past measurements) and a backward estimate $\hat{x}_{k|k}^b$ (regarding future measurements).

Alternatively, a more computationally efficient EKS algorithm known as Rauch-Tung-Striebel (RTS) (Rauch, Tung & Striebel, 1965; Crassidis & Junkis, 2004) can be used. It reduces the computational cost by combining the backward and smoothing stages in a single stage.

Next we discuss how the forward and backward stages are performed. The forward

step is implemented using either EKF or UKF exactly as shown in Sections 2.5.2 and 2.6.2, respectively. However, in order to use the EKF and UKF algorithms to perform the backward step, we must have an inverse model f^{-1} (backward in time) for (1.1) with domain x_k and image x_{k-1} . There are at least three ways to circumvent this problem.

First, in some cases, we can obtain f^{-1} analytically. The following example illustrates this procedure.

Example A.4.1 Consider the nonlinear discrete-time dynamic system

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = f \left(\begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \end{bmatrix} \right) = \begin{bmatrix} x_{1,k-1}^2 + x_{2,k-1}^3 \\ \sqrt{x_{1,k-1}} \end{bmatrix}, \quad (\text{A.44})$$

whose inverse system (backward in time) can be obtained analytically as

$$\begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \end{bmatrix} = f^{-1} \left(\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \right) = \begin{bmatrix} x_{2,k}^2 \\ \sqrt[3]{x_{1,k} - x_{2,k}^4} \end{bmatrix}. \quad (\text{A.45})$$

□

If we have continuous-time dynamics (A.8), the inverse dynamical model is analytically given by

$$\dot{x}(-t) = -f(x(-t), u(-t), w(-t)), \quad (\text{A.46})$$

where $-t$ denotes time inversion for input and measurement data.

Second, in the case of EKS, we can obtain the inverse of the linearized model as

$$x_{k-1} \approx \hat{A}_{k-1}^{-1}(x_k - \hat{B}_{k-1}u_{k-1} - \hat{G}_{k-1}w_{k-1}), \quad (\text{A.47})$$

where $\hat{A}_{k-1}$ and $\hat{G}_{k-1}$ are, respectively, given by (2.52) and (2.53), with $x_{k-1} = \hat{x}_{k-1|k-1}^f$ and

$$\hat{B}_{k-1} = \left. \frac{\partial f}{\partial u_{k-1}} \right|_{x_{k-1}=\hat{x}_{k-1|k-1}^f, u_{k-1}, 0_{q \times 1}, k-1}. \quad (\text{A.48})$$

Then we use (A.47) to propagate both the state estimate (2.47) and error covariance (2.48).

Finally, the inverse model can be obtained by black-box identification and be used during the backward stage of both EKS and UKS. For example, inverting the time sequence of the training data, a neural network can be trained (Haykin, 2001; van der Merwe, 2004).

A.5 Appendix Summary

In this appendix, we show how to set state-estimators to jointly estimate the states and parameters of a dynamic system. In this case, the parameters are defined as “virtual” states and the state vector is augmented to take them into account. Since the results developed and discussed in this thesis are in discrete-time (see Chapter 2), we also show how to handle the hybrid case of continuous-time dynamics with sampled measured data. Though it is not the focus of this work, we briefly discuss how to use filtering algorithms to implement smoothers based on the extended and unscented Kalman filter algorithms.

APPLICATION TO MAGNETOHYDRODYNAMICS: DATA ASSIMILATION WITH ZERO-DIVERGENCE CONSTRAINT ON THE MAGNETIC FIELD

This appendix considers an application of equality-constrained state estimation to magnetohydrodynamics. We address the problem of enforcing the zero-divergence constraint on the magnetic field in data assimilation¹ for two-dimensional magnetohydrodynamic flow. A preliminary version of this study appears as (Teixeira, Ridley, Bernstein, Tôres & Aguirre, 2008), while a version with further results is reported as (Teixeira, Ridley, Bernstein, Tôres & Aguirre, 2007).

B.1 Introduction

Magnetohydrodynamics (MHD) describes plasma dynamics as a fluid moving under the influence of electromagnetic and pressure-gradient forces (Friedberg, 1987). MHD involves continuous-time coupled partial differential equations for which it is generally difficult to obtain closed-form solutions. Computational methods for MHD are based on finite-volume spatial and temporal discretization schemes (Leveque, 2002; Hirsch, 1990). In addition to difficulties associated with high nonlinearity and large dimension of the discretized equations, there is the concern about maintaining the zero-divergence property of the magnetic field (Brackbill & Barnes, 2002; Tóth, 2000; Dedner et al., 2002) as determined by the finite-difference approximation of the corresponding continuum quantity.

Three approaches are popular for handling the zero-divergence constraint in MHD simulation, namely, (i) the eight-wave formulation (Tóth, 2000), (ii) constrained transport

¹Data assimilation is the term commonly used by the weather forecast community to refer to state estimation, which is a term with origin in the control community.

and central difference discretizations (Tóth, 2000; Dedner et al., 2002), and (iii) the projection scheme (Brackbill & Barnes, 2002). The first approach employs a nonconservative formulation of the MHD equations, where terms proportional to the divergence are added, resulting in a more numerically robust discretized model. Constrained transport and central difference methods use special finite-difference discretizations to maintain the zero-divergence property. In the projection approach, the numerical solution of the nonzero-divergence finite-volume discretization scheme is projected onto the subspace of zero divergence.

Data assimilation for MHD is of interest for space weather forecasting applications (Sasaki & Goerss, 1982; Carton et al., 1999; Groth et al., 2000; Chandrasekar et al., 2007). Similar to the simulation problem, data assimilation for MHD (Barrero et al., 2005; Chandrasekar et al., 2007, 2004, 2007) may violate the zero-divergence constraint, even if the injected data and the model satisfy this equality constraint. However, zero-divergence data-assimilation algorithms have not been applied to MHD. The contribution of the present study is to demonstrate two data-assimilation techniques that enforce the zero-divergence constraint in the magnetic field.

To begin, we briefly review the field-interpolated central difference (Tóth, 2000) and projection (Tóth, 2000; Brackbill & Barnes, 2002) approaches for MHD simulation. Next, for data assimilation, to enforce the zero-divergence constraint while estimating the flow variables, we present the localized equality-constrained unscented Kalman filter (LECUKF), a reduced-order application of ECUKF (see Section 5.3.1), as well as the projected localized unscented Kalman filter (PLUKF), a reduced-order extension of the projected unscented Kalman filter (Section 5.3.2). Results are compared with the (unconstrained) localized unscented Kalman filter (LUKF) (Chandrasekar et al., 2007), whose estimates do not satisfy the zero-divergence constraint.

We use a two-dimensional numerical example to illustrate and compare the LUKF, LECUKF, and PLUKF algorithms. First, we briefly compare MHD simulated data regarding the zero-divergence property using the unconstrained second-order Russanov scheme (Hirsch, 1990) and the field-interpolated central difference and projection approaches. Then we use zero-divergence magnetic field data and the field-interpolated central difference model to perform data assimilation using LUKF and show that the estimates do not satisfy the zero-divergence property. We employ LECUKF to guarantee that the estimates in the localized region have zero divergence within numerical precision. Finally, we employ PLUKF to obtain estimates in the full grid satisfying zero divergence, but without feeding these constrained

estimates back into the next time step.

B.2 Magnetohydrodynamics Simulation

The ideal MHD equations are given by (Friedberg, 1987)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{v}), \quad (\text{B.1})$$

$$\frac{\partial \rho \vec{v}}{\partial t} = -\nabla \cdot \left[\rho \vec{v} \vec{v}^T + \left(p + \frac{1}{2} |\vec{B}|^2 \right) I_{3 \times 3} - \vec{B} \vec{B}^T \right], \quad (\text{B.2})$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \cdot \left(\rho \vec{v} \vec{B}^T - \vec{B} \vec{v}^T \right), \quad (\text{B.3})$$

$$\frac{\partial \vec{E}}{\partial t} = -\nabla \cdot \left[\left(E + p + \frac{1}{2} |\vec{B}|^2 \right) \vec{v} - \vec{B} (\vec{v} \cdot \vec{B}) \right], \quad (\text{B.4})$$

together with the zero-divergence constraint

$$\nabla \cdot \vec{B} = 0, \quad (\text{B.5})$$

where $\rho > 0$ is the mass density, $\vec{v} \triangleq [v_x \ v_y \ v_z]^T \in \mathbb{R}^3$ is the velocity, $\vec{m} \triangleq \rho \vec{v} \in \mathbb{R}^3$ is the momentum, $\vec{B} \triangleq [B_x \ B_y \ B_z]^T \in \mathbb{R}^3$ is the magnetic field, $E > 0$ is the total energy, and the hydrodynamic pressure $p \in \mathbb{R}$ is given by

$$p = (\gamma - 1) \left(E - \frac{1}{2} \rho |\vec{v}|^2 - \frac{1}{2} |\vec{B}|^2 \right), \quad (\text{B.6})$$

where γ is the specific heat ratio. For simplicity we consider only the two-dimensional case, where $v_z = 0$ and $B_z = 0$. However, the techniques that we employ can readily be extended to the three-dimensional case.

We apply finite-volume-based spatial and temporal discretization to (B.1)-(B.4) (Leveque, 2002). Assume that the channel consists of $n_x \times n_y$ identical cells; see Figure B.1. For all $i = 1, \dots, n_x$, $j = 1, \dots, n_y$, let $\rho^{[i,j]}$, $m_x^{[i,j]}$, $m_y^{[i,j]}$, $B_x^{[i,j]}$, $B_y^{[i,j]}$, and $E^{[i,j]}$ denote, respectively, the density, the momentum components in the x and y directions, the magnetic field components in the x and y directions, and the energy in cell $[i,j]$. Define $U^{[i,j]} \in \mathbb{R}^6$ by

$$U^{[i,j]} = \left[\rho^{[i,j]} \ m_x^{[i,j]} \ m_y^{[i,j]} \ B_x^{[i,j]} \ B_y^{[i,j]} \ E^{[i,j]} \right]^T. \quad (\text{B.7})$$

We use a second-order Rusanov scheme (Hirsch, 1990) to discretize (B.1)-(B.4) and

obtain a discrete-time model that enables us to update the flow variables at the center of each cell. The discrete-time state update equation (Leveque, 2002; Hirsch, 1990) is given by

$$U_k^{[i,j]} = U_{k-1}^{[i,j]} - \frac{T_s}{\Delta x} \left[\bar{F}_{\text{Rus},k-1}^{[i,j]} - \bar{F}_{\text{Rus},k-1}^{[i-1,j]} \right] - \frac{T_s}{\Delta y} \left[\bar{F}_{\text{Rus},k-1}^{[i,j]} - \bar{F}_{\text{Rus},k-1}^{[i,j-1]} \right], \quad (\text{B.8})$$

where $T_s > 0$ is the sampling time, Δx and Δy are the width and height of each cell, respectively, and $\bar{F}_{\text{Rus},k-1}^{[i,j]}$ is a nonlinear function of $U_{k-1}^{[i-1,j-1]}, \dots, U_{k-1}^{[i+2,j+2]}$; see (Leveque, 2002; Hirsch, 1990). Hence, $U_k^{[i,j]}$ depends on $U_{k-1}^{[i-2,j-2]}, \dots, U_{k-1}^{[i+2,j+2]}$, as expected for a spatially second-order scheme. Henceforth, we refer to (B.8) as the *base* scheme for MHD simulation. Likewise, for $i = 2, \dots, n_x - 1$, $j = 2, \dots, n_y - 1$, define

$$\nabla B_k^{[i,j]} \triangleq \frac{1}{\Delta x} \nabla B_{x,k}^{[i,j]} + \frac{1}{\Delta y} \nabla B_{y,k}^{[i,j]}, \quad (\text{B.9})$$

where

$$\nabla B_{x,k}^{[i,j]} \triangleq \frac{B_{x,k}^{[i+1,j]} - B_{x,k}^{[i-1,j]}}{2}, \quad (\text{B.10})$$

$$\nabla B_{y,k}^{[i,j]} \triangleq \frac{B_{y,k}^{[i,j+1]} - B_{y,k}^{[i,j-1]}}{2}, \quad (\text{B.11})$$

as an approximation for $\nabla \cdot \vec{B}$ in (B.5).

Next, define the state vector $x_k \in \mathbb{R}^n$, where $n \triangleq 6(n_x - 4)(n_y - 4)$, by

$$x_k \triangleq \left[(U_k^{[3,3]})^\top \dots (U_k^{[3,n_y-2]})^\top \dots (U_k^{[n_x-2,3]})^\top \dots (U_k^{[n_x-2,n_y-2]})^\top \right]^\top. \quad (\text{B.12})$$

Let the input vector $u_{k-1} \in \mathbb{R}^p$, where $p \triangleq 24(n_x + n_y - 4)$, denote the boundary conditions for the left, right, bottom, and right cells, that is,

$$u_{k-1} \triangleq \begin{bmatrix} u_{1,k-1} \\ u_{2,k-1} \end{bmatrix}, \quad (\text{B.13})$$

where $p_1 \triangleq 12(n_x + n_y - 2)$, $p_2 \triangleq 12(n_x + n_y - 6)$, and $u_{1,k-1} \in \mathbb{R}^{p_1}$, $u_{2,k-1} \in \mathbb{R}^{p_2}$ are given by

$$u_{1,k-1} \triangleq \left[(U_{k-1}^{[1,1:n_y]})^\top (U_{k-1}^{[n_x,1:n_y]})^\top (U_{k-1}^{[2:n_x-1,1]})^\top (U_{k-1}^{[2:n_x-1,n_y]})^\top \right]^\top, \quad (\text{B.14})$$

$$u_{2,k-1} \triangleq \left[(U_{k-1}^{[2,2:n_y-1]})^\top (U_{k-1}^{[n_x-1,2:n_y-1]})^\top (U_{k-1}^{[3:n_x-2,2]})^\top (U_{k-1}^{[3:n_x-2,n_y-1]})^\top \right]^\top. \quad (\text{B.15})$$

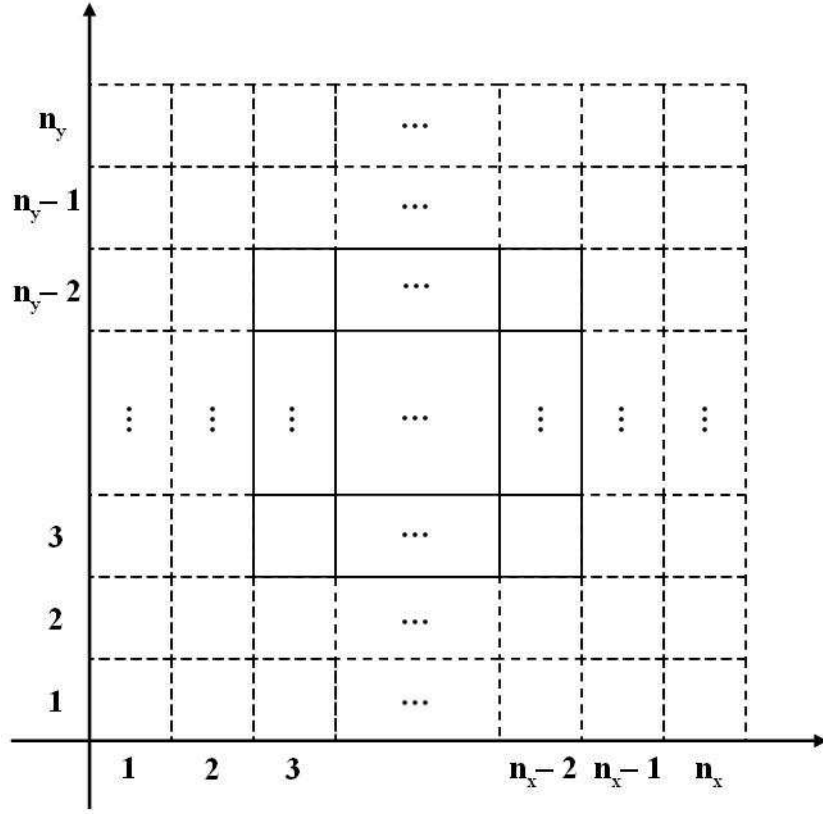


Figure B.1: Two-dimensional grid used in the finite-volume scheme.

We now rewrite (B.8) as the nonlinear discrete-time update model

$$x_k = f(x_{k-1}, u_{k-1}), \quad (\text{B.16})$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$. We assume zero-divergence boundary conditions, that is, for $i = 1, 2, n_x - 1, n_x$ and $j = 1, 2, n_y - 1, n_y$, we assume that

$$\nabla B_k^{[i,j]} = 0 \quad (\text{B.17})$$

is satisfied. However, the discretized model (B.16) does not guarantee that (B.17) is satisfied for all $k \geq 1$, $i = 3, \dots, n_x - 2$, and $j = 3, \dots, n_y - 2$.

To account for unknown disturbances $w_{k-1} \in \mathbb{R}^q$, let (B.16) be rewritten as the *truth model*

$$x_k = f(x_{k-1}, u_{k-1}) + G_{k-1} w_{k-1}, \quad (\text{B.18})$$

where $G_{k-1} \in \mathbb{R}^{n \times q}$ and w_{k-1} is a zero-mean, white, Gaussian process noise with covariance Q_{k-1} . Like (B.16), $G_{k-1}w_{k-1}$ in (B.18) can violate (B.17).

B.2.1 Zero-Divergence Simulation

The Field-Interpolated Central Difference Scheme

Let $\Omega_{z,k}^{[i,j]} \in \mathbb{R}$ denote the time-centered approximation of the z -component of the electric field at cell $[i,j]$,

$$\Omega_{z,k}^{[i,j]} = \text{row}_3 \left(-\frac{\vec{v}_{k-1}^{[i,j]} \times \vec{B}_{k-1}^{[i,j],\text{cd}} + \vec{v}_k^{[i,j]} \times \vec{B}_k^{[i,j]}}{2} \right), \quad (\text{B.19})$$

where $\vec{v}_k^{[i,j]} \triangleq \begin{bmatrix} v_{x,k}^{[i,j]} & v_{y,k}^{[i,j]} & 0 \end{bmatrix}^T$ and $\vec{B}_k^{[i,j]} \triangleq \begin{bmatrix} B_{x,k}^{[i,j]} & B_{y,k}^{[i,j]} & 0 \end{bmatrix}^T$ are given by the base scheme (B.8), and, for $i = 3, \dots, n_x - 2$ and $j = 3, \dots, n_y - 2$, the components of $\vec{B}_{k-1}^{[i,j],\text{cd}}$ are recursively updated as

$$B_{x,k}^{[i,j],\text{cd}} = B_{x,k-1}^{[i,j],\text{cd}} - \frac{T_s}{2\Delta y} \left(\Omega_{z,k}^{[i,j+1]} - \Omega_{z,k}^{[i,j-1]} \right), \quad (\text{B.20})$$

$$B_{y,k}^{[i,j],\text{cd}} = B_{y,k-1}^{[i,j],\text{cd}} + \frac{T_s}{2\Delta x} \left(\Omega_{z,k}^{[i+1,j]} - \Omega_{z,k}^{[i-1,j]} \right). \quad (\text{B.21})$$

Then, by appending (B.19)-(B.21) to (B.8), we have the field-interpolated *central-difference* scheme (CD) Tóth (2000), which, in order to satisfy (B.17), employs central differencing for the induction equation (B.4) on the original grid (Figure B.1).

The Projection Scheme

Let $B_k^{[i,j]} \triangleq \begin{bmatrix} B_{x,k}^{[i,j]} & B_{y,k}^{[i,j]} \end{bmatrix}$ and $B_k \triangleq \begin{bmatrix} B_k^{[2,2]} \dots B_k^{[2,n_x-1]} \dots B_k^{[n_x-1,1]} \dots B_k^{[n_x-1,n_y-1]} \end{bmatrix}^T \in \mathbb{R}^{n_B}$, where $n_B \triangleq (n + p_2)/6$, denotes the numerical values of the magnetic field provided by (B.8) plus the cells of the boundary conditions given by (B.15) at time k . Furthermore, rewrite (B.9) in the matrix form

$$\nabla B_k \triangleq E B_k, \quad (\text{B.22})$$

where $E \in \mathbb{R}^{(n/6) \times n_B}$.

The *projection* scheme (Brackbill & Barnes, 2002; Tóth, 2000) projects B_k orthogo-

nally onto the subspace of zero-divergence solutions by minimizing the cost function

$$J(B_k^p) \triangleq (B_k^p - B_k)^T (B_k^p - B_k) \quad (\text{B.23})$$

subject to

$$EB_k^p = 0_{(n/6) \times 1}. \quad (\text{B.24})$$

The solution is given by

$$B_k^p = B_k + K (0_{(n/6) \times 1} - EB_k) = \mathcal{P}B_k, \quad (\text{B.25})$$

where the gain matrix $K \in \mathbb{R}^{n_B \times (n/6)}$ and the orthogonal projector $\mathcal{P} \in \mathbb{R}^{n_B \times n_B}$ are given by

$$K \triangleq E^T (EE^T)^{-1}, \quad (\text{B.26})$$

$$\mathcal{P} \triangleq I_{n_B \times n_B} - KE. \quad (\text{B.27})$$

The projected magnetic field B_k^p is used at $k + 1$.

Note that K given by (B.26) and $\mathcal{P}$ given by (B.27) need to be computed only once. This direct approach (B.25) is restricted to moderate grid sizes due to large memory demand (Tóth, 2000). Also, density, moment components, and energy are not affected by the projection scheme at the current time step.

B.3 Magnetohydrodynamics Data Assimilation

For the process model (B.18), we assume that, for all $k \geq 1$, in addition to inputs $u_{k-1} \in \mathbb{R}^p$ given by (B.13), measurements $y_k \in \mathbb{R}^m$ of flow variables are available in certain cells and are given by (2.8). Also, rewrite the zero-divergence constraint (B.17) as

$$D \begin{bmatrix} x_k \\ u_{2,k} \end{bmatrix} = 0_{(n/6) \times 1}, \quad (\text{B.28})$$

where $D \in \mathbb{R}^{(n/6) \times (n+p_2)}$ and $u_{2,k}$ is given by (B.15). Assume that we know only the initial estimate $\hat{x}_{0|0}$ and the error-covariance $P_{0|0}^{xx}$ of the initial state vector x_0 , which is assumed to be Gaussian.

Under the stated assumptions, as discussed in Sections 5.1 and 5.3, the maximization of (1.3) subject to (B.28) is the equality-constrained state-estimation problem. The solution to this problem is complicated by the existence of the zero-divergence constraint and by the fact that (1.3) is not completely characterized by its first-order and second-order moments since (B.18) is nonlinear. Therefore, we use approximate unconstrained solutions based on UKF; see Section 2.6. Moreover, since the dimension of x_k for MHD applications is large, reduced-order algorithms are generally employed (Chandrasekar et al., 2007).

To obtain a reduced-order estimator, we partition $x_k \in \mathbb{R}^n$ as

$$x_k = \begin{bmatrix} x_{L,k} \\ x_{E,k} \end{bmatrix}, \quad (\text{B.29})$$

where $x_{L,k} \in \mathbb{R}^{n_L}$ accounts for the localized region where measurements are available, while $x_{E,k} \in \mathbb{R}^{n_E}$ accounts for the exterior part of the grid without measurements. In this case, (2.8) can be expressed as

$$y_k = C_{L,k}x_{L,k} + v_k, \quad (\text{B.30})$$

where $C_{L,k} \in \mathbb{R}^{m \times n_L}$ is formed by the columns of C_k associated with $x_{L,k}$. Similarly, from (B.28), we have

$$D_L x_{L,k} = 0_{(n_L/6) \times 1}, \quad (\text{B.31})$$

where $D_L \in \mathbb{R}^{(n_L/6) \times n_L}$. Let $n_{x_{L0}}$, $n_{x_{Lf}}$, $n_{y_{L0}}$, and $n_{y_{Lf}}$ denote the coordinates of the localized region of the grid shown in Figure B.1. The objective is to directly inject the measurement data y_k into only the states corresponding to the estimate of $x_{L,k}$. We thus use the localized unscented Kalman filter (LUKF) (Chandrasekar et al., 2007; Chandrasekar, 2007) to provide a suboptimal solution to the data-assimilation problem. Moreover, to enforce the zero-divergence constraint into the estimates of the localized region, we present the localized equality-constrained unscented Kalman filter (LECUKF). Also, we present the projected localized unscented Kalman filter (PLUKF) to enforce the zero-divergence constraint in the full grid.

B.3.1 The Localized Unscented Kalman Filter

Regarding system given by (B.18) and (B.30), LUKF is a two-step algorithm, whose forecast equations are given by

$$[\gamma_x, \gamma_P, \mathcal{X}_{L,k-1|k-1}] = \Psi_{UT} \left(\hat{x}_{L,k-1|k-1}, P_{L,k-1|k-1}^{xx}, n_L, \alpha, \beta, \kappa \right), \quad (\text{B.32})$$

$$\mathcal{X}_{k-1|k-1} = \begin{bmatrix} \mathcal{X}_{L,k-1|k-1} \\ \hat{x}_{E,k-1|k-1} \mathbf{1}_{1 \times (2n_L+1)} \end{bmatrix}, \quad (\text{B.33})$$

$$\mathcal{X}_{j,k|k-1} = f(\mathcal{X}_{j,k-1|k-1}, u_{k-1}), \quad j = 0, \dots, 2n_L, \quad (\text{B.34})$$

$$\hat{x}_{k|k-1} = \sum_{j=0}^{2n_L} \gamma_{j,x} \mathcal{X}_{j,k|k-1}, \quad (\text{B.35})$$

$$\begin{aligned} P_{L,k|k-1}^{xx} &= \sum_{j=0}^{2n_L} \gamma_{j,P} [\mathcal{X}_{L,j,k|k-1} - \hat{x}_{L,k|k-1}] [\mathcal{X}_{L,j,k|k-1} - \hat{x}_{L,k|k-1}]^T \\ &\quad + G_{L,k-1} Q_{L,k-1} G_{L,k-1}^T, \end{aligned} \quad (\text{B.36})$$

$$\hat{y}_{k|k-1} = C_{L,k} \hat{x}_{L,k|k-1}, \quad (\text{B.37})$$

$$P_{L,k|k-1}^{yy} = C_{L,k} P_{L,k|k-1}^{xx} C_{L,k}^T + R_k, \quad (\text{B.38})$$

$$P_{L,k|k-1}^{xy} = P_{L,k|k-1}^{xx} C_{L,k}^T, \quad (\text{B.39})$$

where Ψ_{UT} is defined by (2.86) with $\alpha = 0.6$, $\beta = 0$ (Chandrasekar et al., 2007), and $\kappa = 0$ (Haykin, 2001),

$$\begin{bmatrix} \mathcal{X}_{L,k|k-1} \\ \mathcal{X}_{E,k|k-1} \end{bmatrix} \triangleq \mathcal{X}_{k|k-1}, \quad (\text{B.40})$$

$$\begin{bmatrix} \hat{x}_{L,k|k-1} \\ \hat{x}_{E,k|k-1} \end{bmatrix} \triangleq \hat{x}_{k|k-1}, \quad (\text{B.41})$$

$P_{L,k-1|k-1}^{xx}$ is the localized data-assimilation error covariance, $P_{L,k|k-1}^{xx}$ is the localized forecast error covariance, $P_{L,k|k-1}^{yy}$ is the localized innovation covariance, and $P_{L,k|k-1}^{xy}$ is the localized cross covariance.

The data-assimilation equations are given by

$$K_{L,k} = P_{L,k|k-1}^{xy} (P_{L,k|k-1}^{yy})^{-1}, \quad (\text{B.42})$$

$$\hat{x}_{L,k|k} = \hat{x}_{L,k|k-1} + K_{L,k} (y_k - \hat{y}_{k|k-1}), \quad (\text{B.43})$$

$$\hat{x}_{E,k|k} = \hat{x}_{E,k|k-1}, \quad (\text{B.44})$$

$$P_{L,k|k}^{xx} = P_{L,k|k-1}^{xx} - K_{L,k} P_{L,k|k-1}^{yy} K_{L,k}^T, \quad (\text{B.45})$$

where $K_{L,k} \in \mathbb{R}^{n_L \times m}$ is the localized Kalman gain.

B.3.2 The Localized Equality-Constrained Unscented Kalman Filter

Now we consider LECUKF to provide a reduced-order suboptimal solution to the equality-constrained state-estimation problem. LECUKF is based on ECUKF, which is presented in Section 5.3.1. Regarding the system given by (B.18), (B.30), and (B.31), LECUKF is a three-step algorithm, whose forecast equations are given by

$$\begin{bmatrix} \gamma_x, \gamma_P, \mathcal{X}_{L,k-1|k-1}^P \end{bmatrix} = \Psi_{UT} \left(\hat{x}_{L,k-1|k-1}^P, P_{L,k-1|k-1}^{xxp}, n_L, \alpha, \beta, \kappa \right), \quad (\text{B.46})$$

$$\mathcal{X}_{k-1|k-1} = \begin{bmatrix} \mathcal{X}_{L,k-1|k-1}^P \\ \hat{x}_{E,k-1|k-1} \mathbf{1}_{1 \times (2n_L+1)} \end{bmatrix}, \quad (\text{B.47})$$

together with (B.34)-(B.39), where Ψ_{UT} is defined by (2.86), $\hat{x}_{L,k|k}^P \in \mathbb{R}^{n_L}$ is the projected localized state vector, and $P_{L,k|k}^{xxp}$ is the localized projected error covariance. The data-assimilation equations are given by (B.42)-(B.45). Finally, the zero-divergence constraint is enforced during the projection step whose equations are given by

$$K_{L,k}^P = P_{L,k|k}^{xx} D_L^T (D_L P_{L,k|k}^{xx} D_L^T)^{-1}, \quad (\text{B.48})$$

$$\mathcal{P}_{L,k} = (I_{n_L \times n_L} - K_{L,k}^P D_L), \quad (\text{B.49})$$

$$\hat{x}_{L,k|k}^P = \mathcal{P}_{L,k} \hat{x}_{L,k|k}, \quad (\text{B.50})$$

$$P_{L,k|k}^{xxp} = \mathcal{P}_{L,k} P_{L,k|k}^{xx}, \quad (\text{B.51})$$

where $\mathcal{P}_{L,k} \in \mathbb{R}^{n_L \times n_L}$ is an oblique projector whose range is the null space of D_L .

B.3.3 The Projected Localized Unscented Kalman Filter

PLUKF is based on PUKF-EP, which is presented in Section 5.3.2. Regarding the system given by (B.18), (B.30), and (B.28), PLUKF is a three-step algorithm, whose forecast equations are given by (B.32)-(B.39), whose data-assimilation equations are given by (B.42)-(B.45), and whose projection step is given by

$$K^P = D^T (D D^T)^{-1}, \quad (\text{B.52})$$

$$\mathcal{P} = (I_{n \times n} - K^p D), \quad (\text{B.53})$$

$$\hat{x}_{k|k}^p = \mathcal{P} \hat{x}_{k|k}. \quad (\text{B.54})$$

Note that (B.52)-(B.53) need to be computed only once.

Figure B.2 compares PLUKF and LECUKF. Note that, LECUKF enforces the zero-divergence constraint only in the localized region because it requires the calculation of the data-assimilation error covariance; see (B.48). On the other hand, PLUKF performs projection on the full grid. However, unlike LECUKF, PLUKF does not recursively feed the projected estimate $\hat{x}_{k|k}^p$ back in the forecast step.

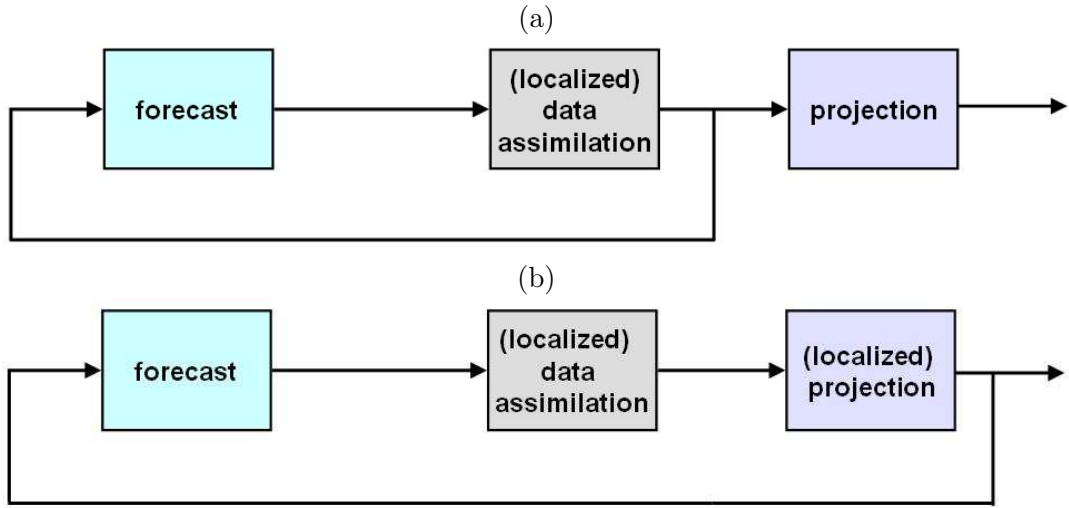


Figure B.2: Comparative diagram of (a) the localized equality-constrained unscented Kalman filter (LECUKF) and (b) the projected localized Kalman filter (PLUKF). In LECUKF, the projection step is localized and it is connected by feedback recursion.

B.4 Two-Dimensional Example

B.4.1 Simulation

We consider the two-dimensional grid shown in Figure B.1 with $n_x = 64$, $n_y = 24$, $\Delta x = 1$, $\Delta y = 1$, and $T_s = 0.01$ s. We set as initial conditions

$$U_0^{[i,j]} = [2 \quad 10 \quad 0 \quad 0 \quad 1 \quad 1]^T, \quad i = 1, \dots, n_x, \quad j = 1, \dots, n_y, \quad (\text{B.55})$$

which represent supersonic flow. For a given cell $[i, j]$, define

$$G^{[i,j]} \triangleq \text{diag}(0.01, 0.05, 0.05, 0, 0, 0.05). \quad (\text{B.56})$$

We consider the truth model (B.18), where we set $Q_{k-1} = I_{n \times n}$ and

$$G_{k-1} = \text{diag}\left(0_{6 \times 6}, \dots, G^{[14,30]}, \dots, 0_{6 \times 6}, \dots, G^{[14,32]}, \dots, \right. \\ \left. 0_{6 \times 6}, \dots, G^{[14,34]}, \dots, 0_{6 \times 6}, \dots, G^{[15,32]}, \dots, 0_{6 \times 6}\right). \quad (\text{B.57})$$

Moreover, u_{k-1} (B.13) is defined as follows. For all $k \geq 1$, we assume floating boundary conditions for the bottom and top cells, that is,

$$U_{k-1}^{[i,j]} = U_{k-1}^{[i,3]}, \quad i = 3, \dots, n_x - 2, \quad j = 1, 2, \quad (\text{B.58})$$

$$U_{k-1}^{[i,j]} = U_{k-1}^{[i,n_y-2]}, \quad i = 3, \dots, n_x - 2, \quad j = n_y - 1, n_y. \quad (\text{B.59})$$

For the left cells, we set constant boundary conditions

$$U_{k-1}^{[i,j]} = U_0^{[i,j]}, \quad i = 1, 2, \quad j = 1, \dots, n_y. \quad (\text{B.60})$$

Finally, for the right cells, we set floating boundary conditions

$$U_{k-1}^{[i,j]} = U_{k-1}^{[n_x-2,j]}, \quad i = n_x - 1, n_x, \quad j = 3, \dots, n_y - 2, \quad (\text{B.61})$$

except for

$$m_{x,k-1}^{[i,j]} = \begin{cases} -m_{x,k-1}^{[n_x-2,j]}, & \text{for } i = n_x - 1, n_x, \quad j = n_y/2 - 1, \dots, n_y/2 + 1, \\ m_{x,k-1}^{[n_x-2,j]}, & \text{for } i = n_x - 1, n_x, \quad j = 3, \dots, n_y/2, n_y/2 + 2, \dots, n_y - 2, \end{cases} \quad (\text{B.62})$$

for which reflective boundary conditions are defined to create a bowshock.

Figure B.3 shows the x and y components of the magnetic field at $kT_s = 15$ s for three simulated cases, namely, (i) using second-order Rusanov scheme as the base scheme (Base), (ii) combining the base scheme with the central difference scheme (CD), and (iii) combining the base scheme with the projection scheme (Proj). Though a bowshock wave is observed in all cases, the results are different, especially for case (iii). Note that, since the three numerical methods mentioned above provide approximate solutions to the MHD equations, we do not

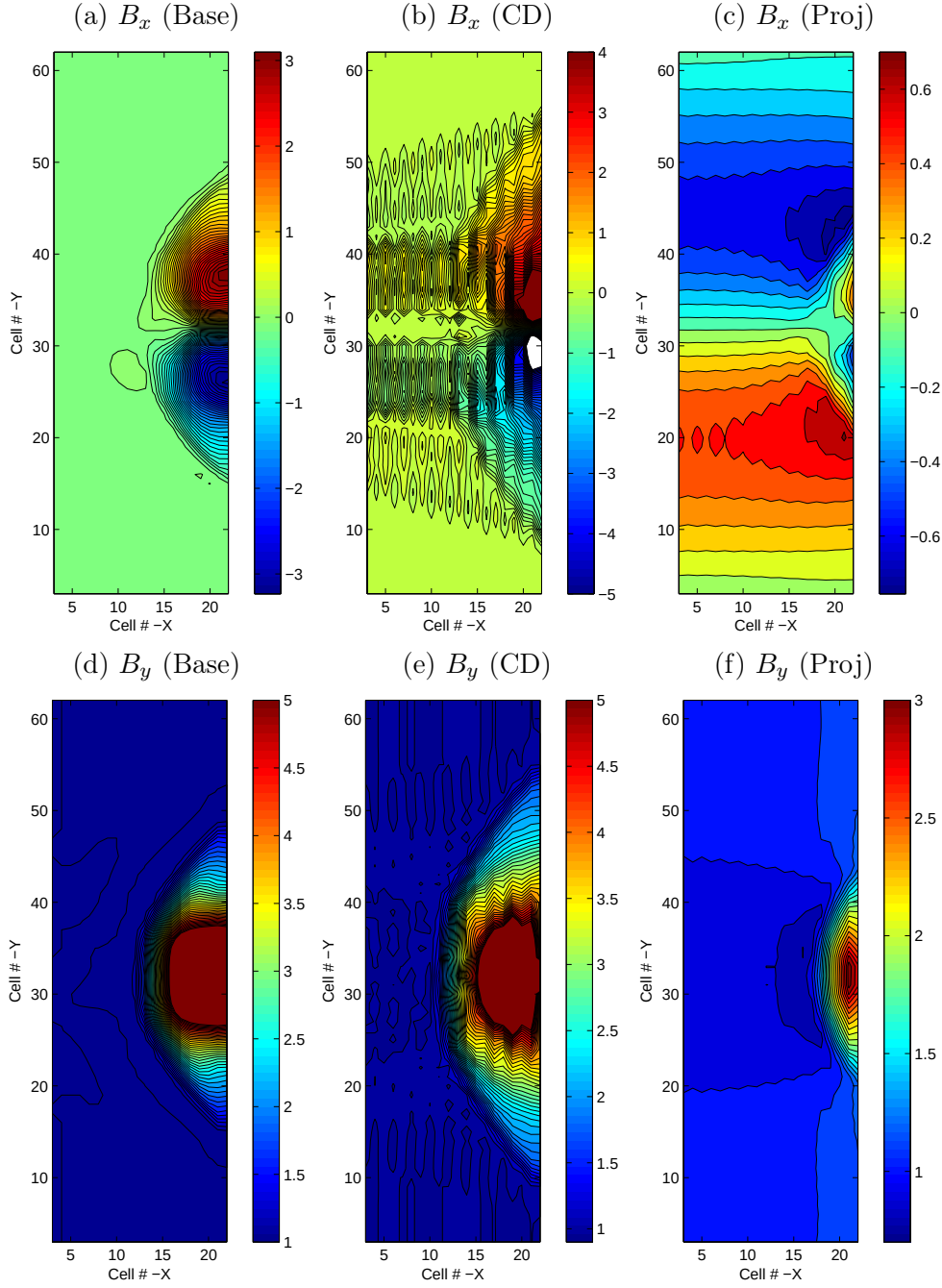


Figure B.3: B_x and B_y components of magnetic field at $kT = 15$ s obtained from two-dimensional MHD simulation using the base (Base), central difference (CD), and projection (Proj) approaches.

have a truth model for this example to compare our results with.

We use the index RMSE (2.31) to quantify the mean error associated with the zero-divergence constraint on the part of the grid indicated i_0, i_f, j_0, j_f . Figure B.4 shows RMSE

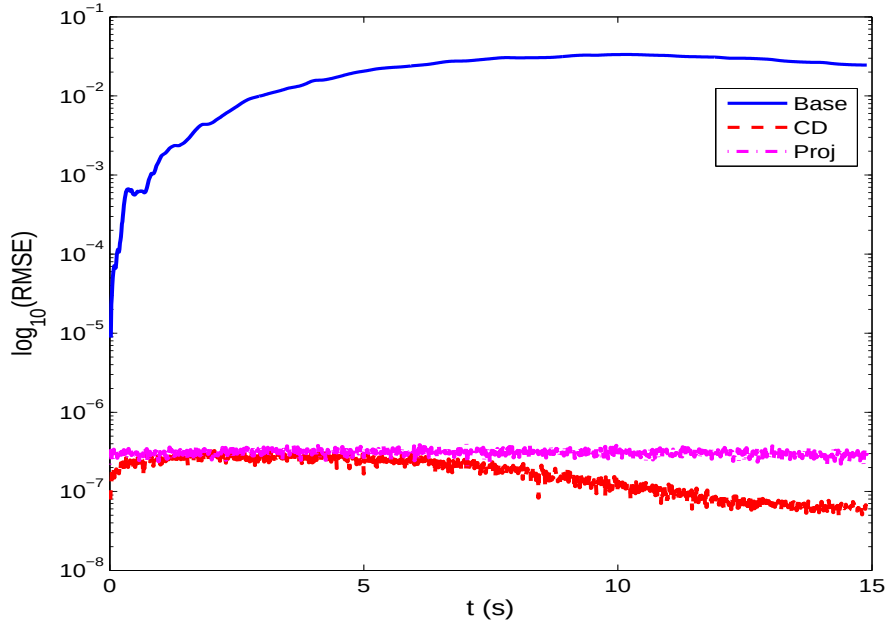


Figure B.4: RMSE index (2.31) for zero-divergence constraint evolving with time $t = kT_s$ for simulated data using the base (Base), central difference (CD), and projection (Proj) approaches.

as a function of time for $i_0 = 3$, $i_f = n_x - 2$, $j_0 = 3$, $j_f = n_y - 2$. While $\text{RMSE} = 0.03$ is observed for the base scheme, we see roundoff RMSE around 7.0×10^{-8} for the CD case and 3.0×10^{-7} for the projection case. Furthermore, Proj requires an increased processing time by about 20% compared to the CD case. Therefore, we use the CD scheme rather than the projection approach to provide zero-divergence measurement data for data-assimilation.

B.4.2 Data Assimilation

We consider the grid used in Section B.4.1 to study data assimilation for MHD. Moreover, we set $n_{x_{L0}} = 14$, $n_{x_{Lf}} = 19$, $n_{y_{L0}} = 22$, and $n_{y_{Lf}} = 42$. Next, define $C^{[i,j]} \in \mathbb{R}^{6 \times n}$ by

$$C^{[i,j]} \triangleq \begin{bmatrix} 0_{6 \times 6(n_y(i-1)+(j-1))} & I_{6 \times 6} & 0_{6 \times 6(n_y(i+1)+(n_y-j))} \end{bmatrix}. \quad (\text{B.63})$$

We assume that we have the observation model (2.8), where

$$C_k = \begin{bmatrix} C^{[17,30]} \\ C^{[17,34]} \end{bmatrix}, \quad (\text{B.64})$$

that is, measurements are obtained only from cells [17,30] and [17,34]. We set $R_k = 10^{-6}I_{m \times m}$. Both measured data from base scheme (Base) and central difference scheme (CD) are investigated; see Section B.4.1. Then, we implement LUKF and, to enforce the zero-divergence constraint (B.28), we implement LECUKF and PLUKF. Also, we set as initial conditions $\hat{x}_{0|0} = \hat{x}_{0|0}^P = x_0 + \delta x_0$ and $P_{L,0|0}^{xx} = P_{L,0|0}^{xyp} = 0.001I_{n \times n}$, where $x_0 \in \mathbb{R}^n$ is the true initial value used in Section B.4.1 and δx_0 is a zero-mean Gaussian random vector with covariance $0.0004I_{n \times n}$.

Figure B.6 shows the y components of magnetic field at $kT = 15$ s for eight different combinations of data (Base or CD), data-assimilation algorithm (LUKF, LECUKF, and PLUKF) and process model (Base or CD), namely, (i) data from base (Base) scheme with LUKF algorithm using Base model - Base + LUKF, (ii) Base data and LECUKF algorithm using Base model - Base + LECUKF, (iii) CD data and LUKF algorithm using Base model - CD + LUKF, (iv) CD data and LUKF algorithm using CD model - CD + LUKF (CD), (v) CD data and LECUKF algorithm using Base model - CD + LECUKF, (vi) CD data and LECUKF algorithm using CD model - CD + LECUKF (CD), (vii) CD data and PLUKF algorithm using Base model - CD + PLUKF, and (viii) CD data and PLUKF algorithm using CD model - CD + PLUKF (CD). Though a bowshock wave is created in all cases, the results are substantially different.

Figure B.7(a) shows RMSE as a function of time for $i_0 = n_{x_{L0}}$, $i_f = n_{x_{Lf}}$, $j_0 = n_{y_{L0}}$, $j_f = n_{y_{Lf}}$. While RMSE = 0.04 is observed for the three cases where LUKF is used, RMSE = 7×10^{-5} is observed for the cases where LECUKF is employed to enforce the zero-divergence constraint, and RMSE = 10^{-7} is obtained for the cases where PLUKF is used. Figure B.7(b) shows RMSE for $i_0 = 3$, $i_f = n_x - 2$, $j_0 = 3$, $j_f = n_y - 2$, that is, for the full grid. In this case, LECUKF yields RMSE values similar to those obtained when LUKF is used, whereas PLUKF guarantees zero divergence within numerical precision. Note that, although PLUKF provides magnetic-field estimates satisfying the zero-divergence constraint with the smallest RMSE, we cannot claim the same about the accuracy of its estimates because we do not have a closed solution of the MHD equations for the example under investigation to compare our results with. Since LECUKF and PLUKF are suboptimal estimators, they can provide less accurate estimates than does LUKF.

It is important to mention that LECUKF and PLUKF has processing time similar to the processing time of LUKF. Note that, even if the CD model is used during forecast and CD data is used during data assimilation, LUKF does not produce data-assimilation estimates

that satisfy the zero-divergence constraint because the Kalman gain given by (B.42) does not take the constraint into account.

B.5 Concluding Remarks

We investigate the problem of enforcing the zero-divergence constraint in both the simulation and the data assimilation of magnetohydrodynamics. This is accomplished by using either a field-interpolated central difference or a projection scheme for simulation. For data assimilation, we investigate the localized unscented Kalman filter (LUKF) and, to enforce the zero-divergence constraint, we present the localized equality-constrained unscented Kalman filter (LECUKF) and the projected localized unscented Kalman filter (PLUKF). A two-dimensional numerical example is used to illustrate the problems above.

Results show that, even if zero-divergence model and measured data are used for data-assimilation, the LUKF estimates do not satisfy the zero-divergence property. However, whenever LECUKF is used to enforce the zero-divergence constraint, the state estimates in the localized region satisfy this equality constraint within numerical precision. PLUKF enforces the zero-divergence constraint in the full grid but without recursively feeding the zero-divergence estimates back in the next estimation step.

As future work, we suggest the investigation of an example whose closed solution is known such that the tradeoff between the accuracy of the estimates provided by LUKF, LECUKF, and PLUKF and the accuracy of the satisfaction of the zero-divergence constraint can be evaluated.

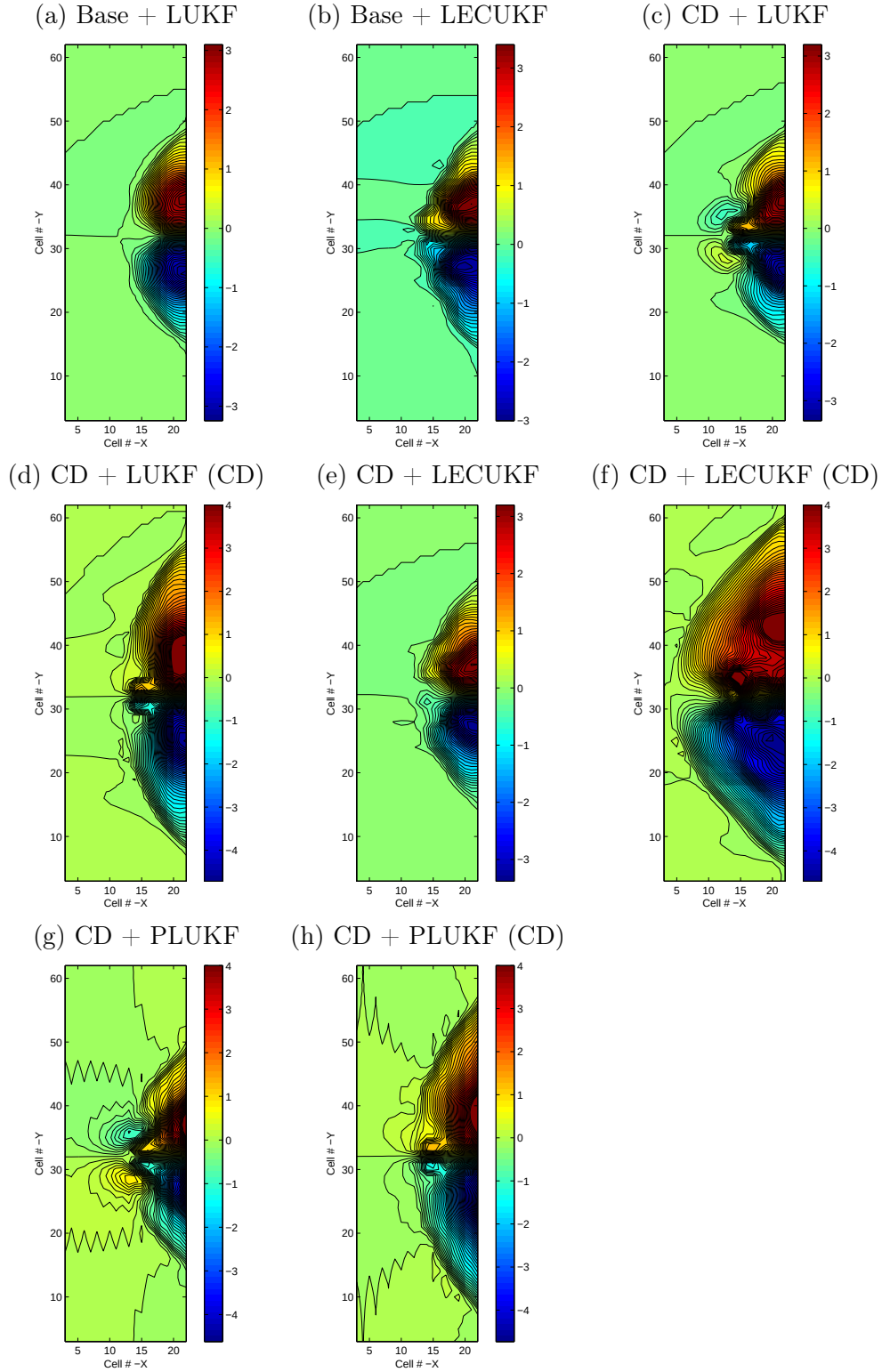


Figure B.5: B_x components of magnetic field at $kT_s = 15$ s obtained from two-dimensional MHD data assimilation by algorithms LUKF, LECUKF, and PLUKF using data obtained from the base (Base) and central difference (CD) approaches.

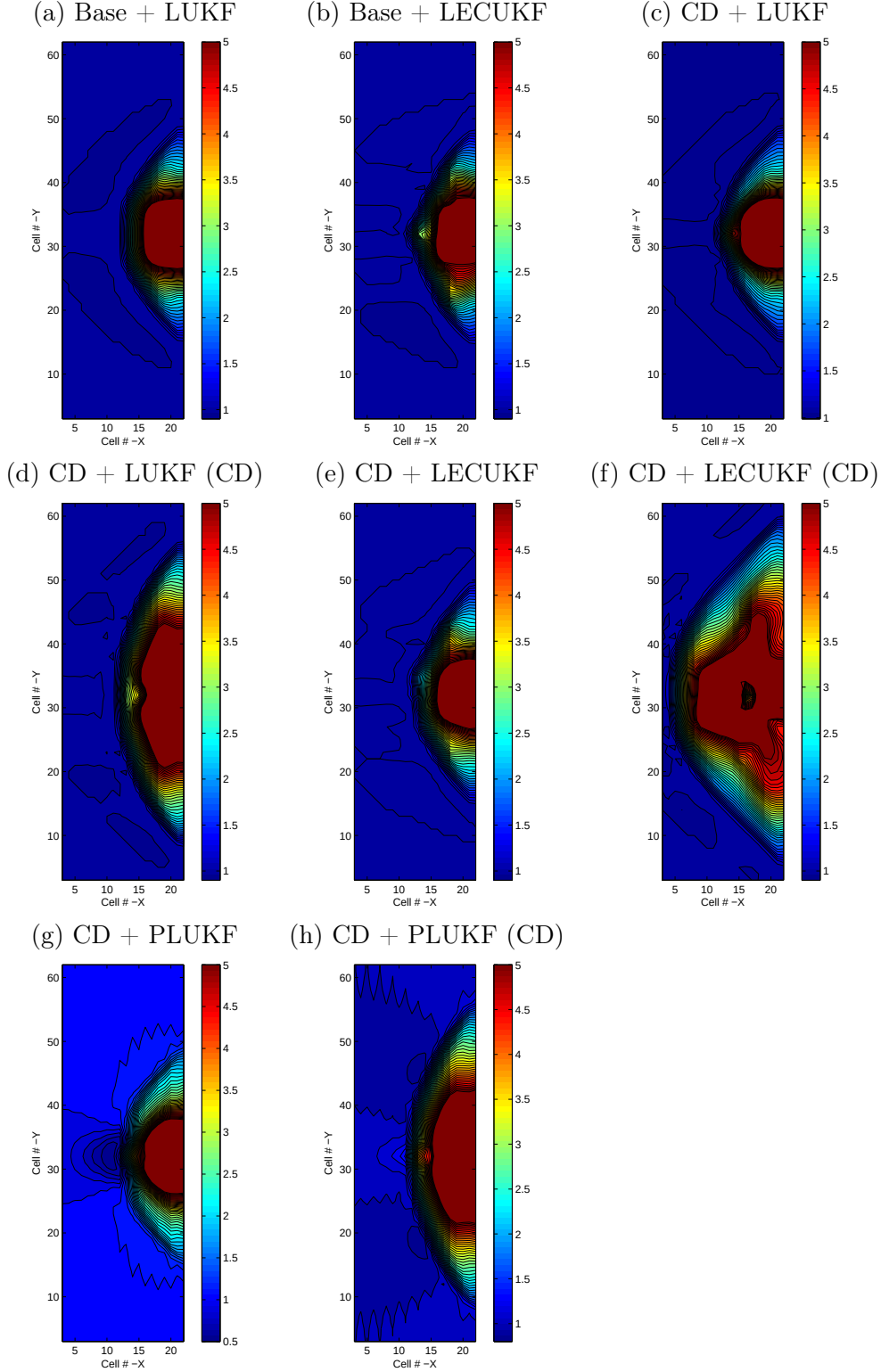
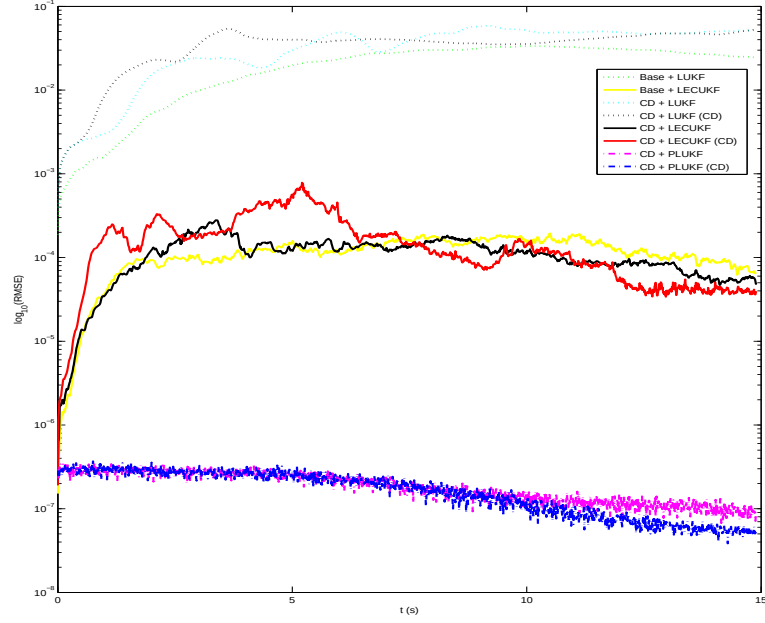


Figure B.6: B_y components of magnetic field at $kt_s = 15$ s obtained from two-dimensional MHD data assimilation by algorithms LUKF, LECUKF, and PLUKF using data obtained from the base (Base) and central difference (CD) approaches.

(a) localized region



(b) full grid

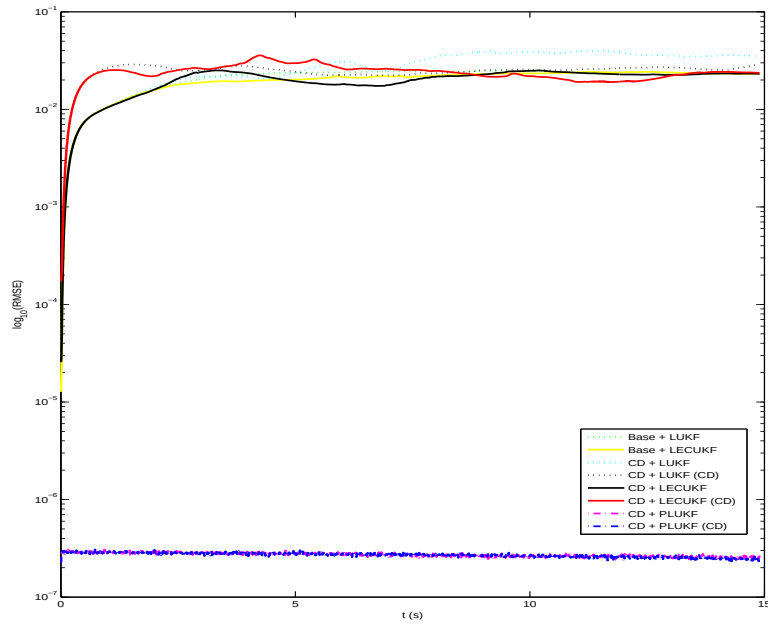


Figure B.7: RMSE index (2.31) for zero-divergence constraint evolving with time $t = kT_s$ for estimated data for six different combinations of data (Base or CD), data-assimilation algorithm (LUKF, LECUKF, PLUKF) plus model (Base or CD).

THE GAIN-CONSTRAINED KALMAN PREDICTOR

In Section 7.1 we present the equations of the two-step gain-constrained Kalman filter. In this appendix we derive the *gain-constrained Kalman predictor* (GCKP) for the system (7.1)-(7.2). Recall that Section A.3 and (Teixeira, 2008) discuss the differences between filters and predictors.

C.1 Derivation of the Gain-Constrained Kalman Predictor

We consider a one-step predictor of the form

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + L_{k-1}(y_{k-1} - \hat{y}_{k-1}), \quad (\text{C.1})$$

where

$$\hat{y}_{k-1} = C_{k-1}\hat{x}_{k-1}, \quad (\text{C.2})$$

and the predictor gain $L_{k-1} \in \mathbb{R}^{n \times m}$ minimizes

$$J_L(L_{k-1}) \triangleq \mathbb{E}[(x_{k-1} - \hat{x}_{k-1})^\top W_{k-1}(x_{k-1} - \hat{x}_{k-1})], \quad (\text{C.3})$$

subject to the constraint

$$M_{k-1}L_{k-1}N_{k-1} = O_{k-1}, \quad (\text{C.4})$$

where, for all $k \geq 1$, the weighting matrix $W_{k-1} \in \mathbb{R}^{n \times n}$ is assumed to be positive definite, $M_{k-1} \in \mathbb{R}^{b \times n}$ is assumed to be right invertible, $N_{k-1} \in \mathbb{R}^{m \times c}$ is assumed to be left invertible, and $O_{k-1} \in \mathbb{R}^{b \times c}$.

Next, define the *prediction error* e_k and the *innovation* ν_{k-1} by

$$e_k \triangleq x_k - \hat{x}_k, \quad (C.5)$$

$$\nu_{k-1} \triangleq y_{k-1} - \hat{y}_{k-1}, \quad (C.6)$$

and the *prediction error covariance* P_k^{xx} and the *innovation covariance* P_{k-1}^{yy} by

$$P_k^{xx} \triangleq E[e_k e_k^T], \quad (C.7)$$

$$P_{k-1}^{yy} \triangleq E[\nu_{k-1} \nu_{k-1}^T]. \quad (C.8)$$

It follows from (7.1)-(7.2) and (C.1)-(C.2) that

$$e_k = (A_{k-1} - L_{k-1}C_{k-1})e_{k-1} + G_{k-1}w_{k-1} - L_{k-1}v_{k-1}, \quad (C.9)$$

$$\nu_{k-1} = C_{k-1}e_{k-1} + v_{k-1}. \quad (C.10)$$

The following lemma will be useful.

Lemma C.1.1 The prediction error given by (C.5) satisfies

$$E[e_k w_k^T] = 0, \quad (C.11)$$

$$E[e_k v_k^T] = 0. \quad (C.12)$$

Proof. Given that w_k and v_k are mutually uncorrelated, we multiply (C.9) by either w_k^T or v_k^T , yielding (C.11) and (C.12), respectively. $\square$

Proposition C.1.1. For the predictor (C.1)-(C.2), the prediction error covariance P_k^{xx} is updated using

$$P_k^{xx} = A_{k-1}P_{k-1}^{xx}A_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T + L_{k-1}P_{k-1}^{yy}L_{k-1}^T - \tilde{P}_{k-1}L_{k-1}^T - L_{k-1}\tilde{P}_{k-1}^T, \quad (C.13)$$

where

$$P_{k-1}^{yy} = C_{k-1}P_{k-1}^{xx}C_{k-1}^T + R_{k-1} \quad (C.14)$$

and

$$\tilde{P}_{k-1} \triangleq A_{k-1}P_{k-1}^{xx}C_{k-1}^T. \quad (\text{C.15})$$

Proof. It follows from (C.10) and (C.12) that (C.8) is given by (C.14). Moreover, using (C.11), (C.9) implies that (C.7) is given by

$$P_k^{xx} = (A_{k-1} - L_{k-1}C_{k-1})P_{k-1}^{xx}(A_{k-1} - L_{k-1}C_{k-1})^T + G_{k-1}Q_{k-1}G_{k-1}^T + L_{k-1}R_{k-1}L_{k-1}^T,$$

which, by using (C.14)-(C.15), is equal to (C.13). $\square$

Next, using (C.5) and (C.7) in (C.3) yields

$$J_L(L_{k-1}) = \text{tr}(P_{k-1}^{xx}W_{k-1}). \quad (\text{C.16})$$

Assume that, for all $k \geq 1$, P_{k-1}^{yy} is positive definite. Then, for convenience, we define

$$\tilde{\Omega}_{k-1} \triangleq N_{k-1}[N_{k-1}^T(P_{k-1}^{yy})^{-1}N_{k-1}]^{-1}N_{k-1}^T(P_{k-1}^{yy})^{-1}, \quad (\text{C.17})$$

$$\tilde{K}_{k-1} \triangleq \tilde{P}_{k-1}(P_{k-1}^{yy})^{-1}. \quad (\text{C.18})$$

Also, let Π_{k-1} be given by (7.22), M_{k-1}^R be given by (7.20), and N_{k-1}^L be given by (7.21). Note that Π_{k-1} and $\tilde{\Omega}_{k-1}$ are oblique projectors, which satisfy (7.24) and

$$N_{k-1}^L\tilde{\Omega}_{k-1} = [N_{k-1}^T(P_{k-1}^{yy})^{-1}N_{k-1}]^{-1}N_{k-1}^T(P_{k-1}^{yy})^{-1}, \quad (\text{C.19})$$

respectively. Moreover, note that $\tilde{K}_{k-1}$ is the gain of the classical Kalman predictor (Jazwinski, 1970; Teixeira, 2008).

Proposition C.1.2. The gain L_{k-1} that minimizes (C.16) and satisfies (C.4) is given by

$$L_{k-1} = \tilde{K}_{k-1} - \Pi_{k-1}(\tilde{K}_{k-1} - M_{k-1}^R O_{k-1} N_{k-1}^L) \tilde{\Omega}_{k-1}, \quad (\text{C.20})$$

where the error covariance P_k^{xx} in (C.13) is updated using the Riccati equation

$$\begin{aligned} P_k^{xx} = & A_{k-1}P_{k-1}^{xx}A_{k-1}^T + Q_{k-1} - \tilde{P}_{k-1}(P_{k-1}^{yy})^{-1}(\tilde{P}_{k-1})^T \\ & + (\Pi_k D_k^R F_k E_k^L \tilde{\Omega}_k) P_{k-1}^{yy} (\Pi_k D_k^R F_k E_k^L \tilde{\Omega}_k)^T \\ & + [\Pi_k \tilde{P}_{k-1} (P_{k-1}^{yy})^{-1} \tilde{\Omega}_k] P_{k-1}^{yy} [\Pi_k \tilde{P}_{k-1} (P_{k-1}^{yy})^{-1} \tilde{\Omega}_k]^T - \Pi_k (\tilde{\Delta}_k + \tilde{\Delta}_k^T) \Pi_k^T, \end{aligned} \quad (\text{C.21})$$

where

$$\tilde{\Delta}_k \triangleq \tilde{P}_{k-1} (P_{k-1}^{yy})^{-1} \tilde{\Omega}_k P_{k-1}^{yy} \tilde{\Omega}_k^T (D_k^R F_k E_k^L)^T. \quad (\text{C.22})$$

Proof. Let $\mathcal{L}_{k-1}(L_{k-1})$ be the Lagrangian given by (7.29) and $\Lambda \in \mathbb{R}^{b \times c}$ be the Lagrange multiplier accounting for the bc constraints in (C.4). The necessary conditions for a minimizer L_{k-1} are given by.

$$\frac{\partial \mathcal{L}_{k-1}}{\partial L_{k-1}} = -W_{k-1} \tilde{P}_{k-1} + W_{k-1} L_{k-1} P_{k-1}^{yy} + M_{k-1}^T \Lambda N_{k-1}^T = 0_{n \times m}, \quad (\text{C.23})$$

$$\frac{\partial \mathcal{L}_{k-1}}{\partial \Lambda} = M_{k-1} L_{k-1} N_{k-1} - O_{k-1} = 0_{b \times c}. \quad (\text{C.24})$$

Note that (C.24) yields (C.4).

In (C.23), using the fact that W_{k-1} and P_{k-1}^{yy} are positive definite yields

$$-\tilde{P}_{k-1} (P_{k-1}^{yy})^{-1} + L_{k-1} + W_{k-1}^{-1} M_{k-1}^T \Lambda N_{k-1}^T (P_{k-1}^{yy})^{-1} = 0_{n \times m}. \quad (\text{C.25})$$

Pre-multiplying and post-multiplying (C.25) by M_{k-1} and N_{k-1} , respectively, and substituting (C.4) and (C.18) into the resulting expression yields

$$-M_{k-1} \tilde{K}_{k-1} N_{k-1} + O_{k-1} + M_{k-1} W_{k-1}^{-1} M_{k-1}^T \Lambda N_{k-1}^T (P_{k-1}^{yy})^{-1} N_{k-1} = 0_{b \times c}, \quad (\text{C.26})$$

which implies

$$\begin{aligned} \Lambda = & (M_{k-1} W_{k-1}^{-1} M_{k-1}^T)^{-1} M_{k-1} \tilde{K}_{k-1} N_{k-1} (N_{k-1}^T (P_{k-1}^{yy})^{-1} N_{k-1})^{-1} - \\ & (M_{k-1} W_{k-1}^{-1} M_{k-1}^T)^{-1} O_{k-1} (N_{k-1}^T (P_{k-1}^{yy})^{-1} N_{k-1})^{-1}. \end{aligned} \quad (\text{C.27})$$

Using definitions (7.20)-(7.22), (C.17)-(C.18) along with (7.24) and (C.19), and substituting (C.27) into (C.25) yields (C.20). Substituting (C.20) into (C.13) yields (C.21). $\square$

Proposition C.1.3. L_{k-1} given by (C.20) is the unique global minimizer of $J_L(L_{k-1})$ restricted to the convex constraint set defined by (C.4).

Proof. The proof is similar to the proof of Corollary 7.1.3 and is omitted. $\square$

GCKP is given by (C.2), (C.14)-(C.15), (7.20)-(7.22), (C.17)-(C.18), (C.20), (C.1), and (C.13). Note that the GCKP equations are identical to the classical Kalman predictor equations with the exception of the gain expression (C.20). It is straightforward to derive here the special cases shown in Section 7.2 in the context of GCKF. However, for the sake of brevity, we omit them.

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BIOGRAPHICAL NOTE

Bruno Otávio Soares Teixeira was born on September 7th, 1980, in Belo Horizonte, Minas Gerais, Brazil, where he lives since then. He concluded his high school at the Colégio Nossa Senhora do Monte Calvário in 1998. In 2003 he received the (first-class honors) bachelor degree in Control and Automation Engineering from the Federal University of Minas Gerais (UFMG). As an undergraduate student, from mid-1999 to 2002, he joined the MACSIN research group, where he worked on biomedical nonlinear signal processing under the supervision of Prof. Luis A. Aguirre. In 2003 he dealt with the modeling and control of the Usiminas' reheating furnace in his undergraduate degree project, being supervised by Prof. Fábio G. Jota.

He joined the Graduation Program in Electrical Engineering of UFMG in 2004 as a master student under the supervision of Prof. Leonardo A. B. Tôrres and Prof. Luis A. Aguirre. In June, 2005, his application to the doctorate program was accepted and he became a PhD student. From September, 2004 to July, 2006, he joined the Department of Electronic Engineering of UFMG as an assistant professor. From September, 2006 to November, 2007, he was a visiting scholar in the Aerospace Engineering Department at the University of Michigan, Ann Arbor, USA, under the supervision of Prof. Dennis S. Bernstein. His research interests are in state estimation, system identification, and control, with emphasis on aeronautical and aerospace applications.