UNIVERSIDADE FEDERAL DE MINAS GERAIS
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Contributions to the Study of Repairable System Reliability: Minimum Repair with Discrete Times, Optimization considering the Brown-Proschan Model with Discrete Times, Mixing Distributions for Imperfect Repair.

Belo Horizonte
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Aos 07 dias do mês de março de 2023, às 14h00, em reunião pública virtual 79 (conforme orientações para a atividade de defesa de tese durante a vigência da Portaria PRPG nº 1819) OU na sala virtual do Instituto de Ciências Exatas da UFMG, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Estatística, para julgar a defesa de tese do aluno **DANILO GILBERTO DE OLIVEIRA VALADARES**, nº de matrícula 2019665080, intitulada: "*Contributions to the Study of Repairable System Reliability: Minimum Repair with Discrete Times, Optimization considering the Brown-Proschan Model with Discrete Times, Mixing Distributions for Imperfect Repair*", requisito final para obtenção do Grau de doutor em Estatística. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Roberto da Costa Quinino, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação:

(X) Aprovada.
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O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 07 de março de 2023.

Observações:

1. No caso de aprovação da tese, a banca pode solicitar modificações a serem feitas na versão final do texto. Neste caso, o texto final deve ser aprovado pelo orientador da tese. O pedido de expedição do diploma do candidato fica condicionado à submissão e aprovação, pelo orientador, da versão final do texto.
2. No caso de reprovação da tese com resubmissão do texto, o candidato deve submeter o novo texto dentro do prazo estipulado pela banca, que deve ser de no máximo 6 (seis) meses. O novo texto deve ser avaliado por todos os membros da banca que então decidirão pela aprovação ou reprovação da tese.
3. No caso de reprovação da tese com resubmissão do texto e nova defesa, o candidato deve submeter o novo texto com a antecedência à nova defesa que o orientador julgar adequada. A nova defesa, mediante todos os membros da banca, deve ser realizada dentro do prazo estipulado pela banca, que deve ser de no máximo 6 (seis) meses. O novo texto deve ser avaliado por todos os membros da banca. Baseada no novo texto e na nova defesa, a banca decidirá pela aprovação ou reprovação da tese.

Acknowledgments

Apparently, unplanned variations at a given point in our lives have unimaginable consequences, which is how I regard the completion of this work. I thus want to express my gratitude to everyone who contributed to this.

I thank my family, despite my distance over this time, for their gratuitous and spontaneous support.

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"*If you want to have good ideas you must have many ideas. Most of them will be wrong, and what you have to learn is which ones to throw away.*"

(Linus Pauling)

Resumo

Reparos podem ser classificados de acordo com a eficiência e em situações extremas são classificados como: “Reparos Perfeitos”, ou seja, o sistema retorna à uma condição de novo após cada reparo, ou, “Mínimo Reparo”, ou seja, o sistema retorna à uma condição similar anterior à falha.

O mínimo reparo é bastante aplicável e um dos modelos mais utilizados é o processo de lei das potências. Nesse modelo, o tempo de falha do equipamento é uma variável aleatória contínua, assumindo uma alta acurácia na ferramenta de medida. Entretanto, em muitas situações práticas, as falhas são observadas e anotadas como um número inteiro, indicando um processo discreto. Assim, a distribuição Weibull discreta foi utilizada em substituição da distribuição Weibull contínua para a primeira falha. Os dois modelos têm complexidades similares e alguns benefícios foram observados ao utilizar a distribuição discreta: menor desvio padrão para o parâmetro relacionado à degradação do sistema e menor AIC. Os valores foram comparados utilizando um banco de dados real e bancos de dados reportados na literatura. Uma cadeia de Markov com estados e tempos discretos foi proposta para a obtenção do número médio de falhas em um dado intervalo e uma política ótima de manutenção foi incluída na análise.

Uma outra classificação para o reparo é uma situação intermediária entre o mínimo reparo e entre o reparo perfeito, conhecido como “Reparo Imperfeito”. O modelo de Brown-Proschan define que uma unidade recebe um reparo perfeito com probabilidade p, ou, recebe um mínimo reparo com probabilidade $1 - p$. Se, além disso, a suposição de tempos discretos for válida, pode-se encontrar o valor exato do custo médio por unidade de tempo para uma política de manutenção através do uso de uma cadeia de Markov com tempos e estados discretos.

Quando o tempo de falha é considerado uma variável aleatória contínua e um reparo imperfeito é executado, os modelos Arithmetic Reduction of Age, Arithmetic Reduction of Intensity e Quase-Renovação podem ser considerados na modelagem. Um novo modelo com uma distribuição de mistura é proposto para o reparo imperfeito e esse ajustou-se melhor que as metodologias citadas para um banco de dados real, tornando-se uma opção na escolha para o melhor ajuste dos dados.

Palavras-chave: sistemas reparáveis, mínimo reparo, reparo perfeito, reparo imperfeito, Weibull contínua, Weibull discreta, distribuições de mistura, cadeias de Markov.

Abstract

Repairs are generally classified according to their efficiency and, in extreme situations, they are classified as "Perfect Repairs", i.e., the system returns to a new condition after a repair, or "Minimum Repair", i.e., the repair leaves the system in the same condition as it was prior to the failure.

In many practical situations, minimal repair is quite applicable. Among the most widely used models is the Power Law Process. Additionally, it is common to consider equipment failure time as a continuous variable, assuming a high accuracy in the measurement tool. However, in other situations, failures are observed and noted as an integer, such as day number and time, thus indicating a discrete process. In this study, the discrete Weibull distribution was used to replace the continuous Weibull distribution for the first failure. The two models have similar complexities, and some benefits were observed when using the discrete distribution: lower standard deviation for the parameter related to system degradation and lower AIC, when compared using an actual database and a database (six cases) already reported in the literature. Moreover, the use of a Markov chain with discrete states was proposed to obtain the mean number of failures in a given interval and an optimal maintenance policy was included in the analysis.

When the system fails, another possibility is that it returns to an intermediate situation between minimal and perfect repair, known as "Imperfect Repair". The Brown-Proschan model for imperfect repairs states that a unit is perfectly repaired with a p probability, and the system receives a minimum repair if the said probability is $1 - p$. If we assume that the system follows a discrete Weibull distribution and by using a Markov chain with discrete states and times, one can find the exact value of the average cost per time unit for a maintenance policy, and thus obtain an optimal cost-based strategy for the Brown-Proschan model.

In other situations in which failure time is considered a continuous variable and an imperfect repair is performed, the Arithmetic Reduction of Age, Arithmetic Reduction of Intensity, and quasi-renewal processes can be considered for modeling. Using an actual database, a model with a mixture distribution is proposed for imperfect repair after failure. This new model is a better fit than the methodologies mentioned, becoming an option for choosing the best fit.

Keywords: repairable systems, minimum repair, perfect repair, imperfect repair, continuous Weibull, discrete Weibull, mixture distributions, Markov chain.

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Chapter 1

INTRODUCTION

A repairable system is every system that, when a failure occurs, can be restored to its operating condition after a repair process. Examples of repairable systems are cars, computers, and airplanes.

Probabilistic models for repairable systems should be able to describe the occurrences of failures over time. Moreover, these should consider how systems are serviced.

Two types of maintenance will be addressed in this work: 1. preventive maintenance (PM), in which a system receives maintenance at fixed and regular times preventively, aimed at reducing the probability of failure or degradation; 2. Corrective maintenance, in which a system receives maintenance after a failure has occurred and is returned to a condition of performing a required function.

For PM, an assumption widely used in the construction of models is that of "Perfect Repair" (PR), i.e., the system is brought to like-new condition again after repair. Now, for corrective maintenance, the assumption of "Minimal Repair" (MR) is usually made. This is a repair that leaves the system in the same condition as before the failure.

Regarding situations in which PR is valid after each failure, it is found that the times between failures are identically distributed. If, moreover, the assumption of independence is added, this process is known as renewal. DR and COLL (1972), Berman (1981), and Lin, Truong, and Fine (2013) studied this case.

Now, a model combining both PR and MR has been of interest to Barlow and Hunter (1960), Gertsbakh (1977), and Lai, Leung, Tao, and Wang (2001). Gilardoni and Colosimo (2007) performed a work motivated by a problem of maintenance of the energy transformers of a Brazilian company.

Another approach used in model failures and their respective maintenance would be to consider an intermediate situation between MR and PR, generically referred to as "Imperfect Repair" (IR). Brown and Proschan (1983) have defined a model in which the unit that failed can receive a PR with a probability p or an MR with a probability $1-p$, which will be referred to as (BP). An extension of the previous model was proposed by Block, Borges, and Savits (1985), allowing probabilities to be time-dependent.

Virtual age models were proposed by Kijima, Morimura, and Suzuki (1988). Consider that the system has a virtual age at an instant t, $V_{i-1}(t), V_{i-1}(t) < t$, the value immediately after $(i-1)-$th repair, X_i is the time between $i-$th failure and the $(i-1)-$th failure, w_r is the degree of repair after $i-$th failure, in which $0 < w_r < 1$. The model is constructed so that the additional age of the X_i system is reduced to $w_r X_i$, furthermore, the $i-$th repair does not solve issues prior to the $(i-1)-$th repair, implying that $V_i = V_{i-1} + w_r X_i$. Doyen and Gaudoin (2004) generalize the model proposed by Kijima, Morimura, and Suzuki (1988), creating a model class called Arithmetic Reduction of Age (ARA), in which the virtual age of the $V(t)$ system is a function that may depend on the entire failure history. In the paper published by Doyen (2011), the modeling considering virtual age and the optimization of maintenance policies are discussed simultaneously. Toledo, Freitas, Colosimo, and Gilardoni (2015) deeply explore these models and propose a graph for model adjustment verification. However, recently, Finkelstein and Cha (2021) have discussed the validity of assumptions of virtual age models. In addition to the ARA model class, Doyen and Gaudoin (2004) also proposed the Arithmetic Reduction of Intensity (ARI) class for IR. This class characterizes the efficiency of the repair by reducing the intensity function of the failure process, which will be further defined.

Hongzhou Wang and Pham (1996) proposed the model of "Quasi-Renewal" (QR). This model assumes that if a new system has a π failure time distribution, after the $i-th$ failure, the system will have a $w_r^{i-1}\pi$ distribution, where $0 < w_r < 1$ is constant. Smith and Leadbetter (1963) proposed a similar model to consider that the times between failures followed Weibull distribution, however, after a repair, the system was in a better condition than in its initial state, i.e., the system went through technological innovation. Rehmert and Nachlas (2008) studied the quasi-renewal process for different distributions and analyzed the performance of equipment through the availability function at a given time. Aydoğdu, Şenoğlu, and Kara (2010) studied the estimation of parameters for the quasi-renewal process with the assumption of the time of occurrence of the first event being described with a Weibull distribution. Wu, Peng, and Wu (2020) reviewed the model extensions, applications, and open challenges.

Overall, works regarding complex systems are divided into two steps. The first step consists in estimating the parameters of the system under study, and the second step consists in optimizing a maintenance policy.

Maintenance policies are intended to reduce the risks and costs caused by system failures. The consequences of accidents due to failures are production stops, damage, and social aspects difficult to measure. Consequently, several industries prioritize the development of an appropriate policy. However, the biggest difficulty lies in determining when and how to maintain a system before it fails. A common solution is the determination of the periodicity of PMs based on cost, i.e., minimizing the final average cost, which is a function of the parameters of the model proposed for the system.

Details of various types of maintenance policies are found in Meeker, Escobar, and Pascual (2021), Lin, Zuo, and Yam (2001), Nakagawa (2006), Nakagawa and Mizutani

(2009), and Rausand, Barros, and Høyland (2021).

1.1 Scope of Work

This work consists of three volumes in paper format. To ensure a better understanding for the reader, we will develop a new introduction which will be further directed toward the content of said chapter in each chapter.

In the first volume, described in Chapter 2, a model considering MR and failure times as discrete random variables is proposed. After a long literature review, all papers found worked with times as a continuous variable. Through an interaction between the University and a company, we received an actual database in a daily format which further characterizes a discrete distribution than a continuous one. This situation motivated us to think of situations in which discrete distributions could also be used. In several other situations like this, failures are observed and noted in time units, such as days and hours as observed in Nelson (1995), Rigdon and Basu (2000), Meeker, Escobar, and Pascual (2021), Rai, Chaturvedi, and Bolia (2020), Jelinski and Moranda (1972), and Kumar and Klefsjö (1992). The discrete Weibull (DW) distribution, proposed by Nakagawa and Osaki (1975), may be used to model the first failure of a system as well as continuous Weibull (CW) and has the same level of complexity. By adjusting the two distributions (discrete and continuous) for the actual database and other databases found in the literature, DW showed lower AIC values and standard errors (for the shape parameter), indicating that there may be no loss in the adjustments, at least in the tested data. Additionally, the determination of a cost-based maintenance policy has been adjusted. The optimal interval was obtained through a Markov chain, using discrete states. This methodology is extremely broad and can be extended to other discrete distributions.

In the second volume, described in Chapter 3, an optimal maintenance policy is obtained for the BP model through Markov chains and the use of DW. The maintenance cost reduction approach involves the knowledge of the average number of failures in a given interval. Obtaining this expectation is not trivial and researchers choose simulation to calculate it when using the CW. Throughout the text, the distribution proposed by Nakagawa and Osaki (1975) is used, but there is no impediment to the use of other discrete distributions.

Now, in the third volume, described in Chapter 4, we consider failure times as a continuous random variable. Motivated by data observed in a practical case, the chapter proposes an approach to IR in which the distributions of failure times would be modeled as a mixture of two distributions: a mixture term is a distribution considering the failure time as conditioned to MR with a weight w_{MR}, the other term of the mixture considers that the failure time will be conditioned to a PR with weight $w_{PR} = 1 - w_{MR}$. In this text, it is considered that the time distribution of the $(i+1)-th$ failure, known the time of the $i-th$ failure, will be a mixture of distributions considering MR and PR. In other words, the analysis of the distribution of subsequent failures will always be dependent on the occurrence time of the last failure. The work consisted in estimating the parameters of the system under study and optimizing a maintenance policy. This new model is a better fit than the ARA, ARI, and QR methodologies, rendering it an option in selecting the best fit.

Chapter 2

REPAIRABLE SYSTEM ANALYSIS USING THE DISCRETE WEIBULL DISTRIBUTION

2.1 Introduction

Complex systems, such as automobiles, concrete mixer trucks, airplanes, medical diagnostic systems, and locomotives, are repaired and not replaced when failures occur. However, if the number of failures is large, the costs of repair is high. Therefore, there is an interest in determining the reliability and other performance characteristics of a proper preventive maintenance policy. Beyond the economic issue, the reliability is determinant as an accident prevention. This subject has gained increasing importance, and a wide range of problems have been addressed in the literature. Two appointments are relevant to the reliability of equipment: the time at which the failure occurs, and the type of repair the equipment requires.

In many practical situations, when the failures occur, the systems are repaired to a similar condition as they were before the failure. Such repairs are named as "Minimum Repairs" (MR) or '*as bad as old*' (ABAO). For such cases, one of the most used model is the Power Law Process (PLP). However, the Weibull distribution addresses all first failures and the PLP model addresses each successive failure for a repairable system. We can also model by looking at successive failure times according to a truncated Weibull distribution when the last failure time is noted.

A model that considers the minimal repair policy is generally easy to use and interpret, as it provides useful and practical results in real application cases. Contributions are abundant in the literature addressing such an approach; for example Barlow and Hunter (1960), Gertsbakh (1977), Lai, Leung, Tao, and Wang (2001), Baker (2001), Rigdon and

Basu (2000), and Tadj, Ouali, Yacout, and Ait-Kadi (2011), to list a few. Gilardoni and Colosimo (2007) presented a study motivated by a Brazilian power transformer maintenance problem. They set the interval for a perfect repair conditioned to a minimal repair between perfect repairs (renovation).

The assumption of perfect repair after a maintenance implies that the system returns to like-new condition. In the literature, this situation is defined as '*as good as new*' (AGAN). From a practical point of view, AGAN is a valid assumption when replacement or major maintenance is performed on a given system.

Another situation would be an intermediate repair condition between ABAO and AGAN, usually referred to as "Imperfect repair" (IR) by Pham and Wang (1996). More recently, some contributions, such as Ma, Wu, Li, and Kang (2018), Doyen, Drouilhet, and Brenière (2019), Liu, Finkelstein, Vatn, and Dijoux (2020), and Omshi and Grall (2021), have dealt with imperfect repair.

When the failure times are correlated with each other, copula and frailty models have been widely used Goethals, Janssen, and Duchateau (2008). The works of Yaping Wang and Pham (2011), Zhang (2018), and Slimacek and Lindqvist (2016) and Brito, Tomazella, and Ferreira (2022) address this problem specifically in the area of reliability.

In most common situations, the reliability analysis are based on having complete information about the system. However, in other situations, failure times are missing or incorrectly noted. The papers of Yu, Tian, and Tang (2008) and Balakrishnan and Stehlık (2015) and Si, Yang, Monplaisir, and Chen (2018) proposed models and tests to overcome this circumstance. The influence of common cause failures on the reliability is addressed by Shao, Yang, Bian, and Gou (2020).

The modeling of the failure times mentioned in all the previous references assumes random times as continuous variables, indicating a high degree of accuracy in the measurement system, and coherently using a probability density function, such as the Weibull distribution.

However, data was collected by an interaction between the University and a company in the construction sector, in which those responsible for maintenance wanted to minimize the maintenance costs of mixer trucks. The received fault log was archived in daily format. One concern of the maintenance sector is that for short times, like a month, the average number of failures is similar (slightly higher) to the expected value considering a continuous Weibull distribution for the first failures and a PLP model for successive failures. However, for longer periods, such as six months, the average number of failures was lower than the theoretical value. The company used the repairable system analysis from the Minitab software for their evaluation. Such circumstances motivated us to evaluate the performance of the discrete Weibull as a model for the first failure and the application of a minimum repair policy in the equipment.

In practical situations as above, failures are observed and recorded as time units, such as the number of days and hours related to a discrete process; or are naturally noted as discrete, such as the number of cycles a system operates before its failure. Burnham and Anderson (2002) stated that from a philosophical point of view all random variables are discrete—their possible values are incremented by some minimum step size, δ. In addition, they accept only a countable number of possible values. In this scenario, it is important to evaluate a model that considers a discrete distribution instead of a continuous distribution. To the best of our knowledge, an approach dealing reliability problems considering discrete times in repairable systems has not been explored in the literature.

Nakagawa and Osaki (1975) proposed the probability function for Weibull distribution in the discrete context that will be examined in this study. Similar to the continuous version, the discrete version proposed by Nakagawa and Osaki (1975) had the same number of parameters. The adoption of time as discrete does not change the complexity of the modeling, and a procedure to evaluate/compare the models would be based on some metrics as their AIC values, as discussed by Buckland, Burnham, and Augustin (1997). Fitting the two distributions (continuous and discrete) in a real case related to mixer trucks and other data sets found in the literature, we noted smaller AIC values (for the model) and standard errors (for shape parameters) in the proposed discrete Weibull distribution, indicating that the discrete approach does not cause adjustment losses.

Furthermore, in practical situations, the determination of a proper maintenance policy is required, and an optimal interval for perfect maintenance is presented in which the average number of failures is obtained through a Markov chain approach (using discrete states). This result was easily understood by the engineers of the case study company. Moreover, such an approach also allows us to obtain the probability function of the frequency of failures for a chosen time t. This information can be useful for assessing the possibility that the maintenance department will receive a large amount of equipment for minimal repair.

The remainder of this chapter is organized as follows. The discrete Weibull distribution (DW) and its application to repairable systems with minimal repairs are presented in Section 2.2. The application to a real data set and other data sets found in the literature is presented in Section 2.3. The optimal interval for maintenance using the Markov chain approach (to obtain the average time to failure) is discussed in Section 2.4. Finally, Section 2.5 concludes the article.

2.2 Discrete Weibull Distribution and Its Application in Repairable Systems with Minimum Repair

This section consists of two subsections. In Subsection 2.2.1, a brief review of the DW introduced by Nakagawa and Osaki (1975) and detailed in Khan, Khalique, and Abouammoh (1989) and Almalki and Nadarajah (2014) is presented. In Subsection 2.2.2, we discuss the employment of DW in the context of repairable systems with minimum repair. Moreover, we develop a likelihood function considering truncation by failure and time.

2.2.1 Discrete Weibull Distribution and its Properties

Let T be a random variable related to the first failure following a DW, $T \sim \mathrm{DW}(\beta, q)$, in which $\beta > 0$ (shape parameter) and $0 < q < 1$. The probability function (PF) of T is expressed as

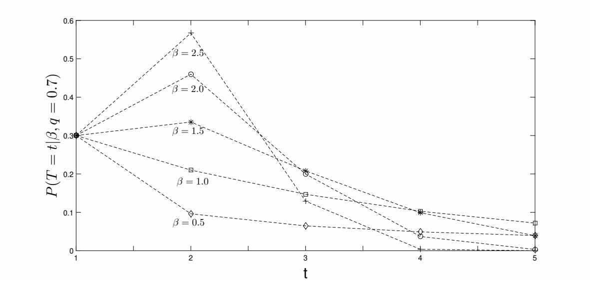
$$P(T = t|\beta, q) = q^{(t-1)^\beta} - q^{t^\beta}, t = 1, 2, 3, \ldots, \tag{2.1}$$

in which the mean and variance are given respectively by:

$$\begin{aligned} \mathbb{E}(T) &= \mu = \sum_{t=1}^{\infty} t \left[q^{(t-1)^\beta} - q^{t^\beta}\right] = \sum_{t=0}^{\infty} q^{t^\beta} \text{ and} \\ \mathbb{V}(T) &= \sigma^2 = \sum_{t=1}^{\infty} 2tq^{t^\beta} + \mathbb{E}(T) - \mathbb{E}(T)^2 \end{aligned} \tag{2.2}$$

The distribution function of T can be expressed as $P(T \leq t|\beta, q) = 1 - q^{t^\beta}$. In Figure 2.1 the plots of the probability function for the DW, for $q = 0.7$ and $\beta = 0.5, 1.0, 1.5, 2.0,$ and 2.5, are shown. This distribution is flexible and includes increasing and decreasing ratio failures, such as the continuous Weibull (CW) distribution. The dashed lines indicate the trend, whereas the markers indicate the discrete values.

Figure 2.1: Discrete Weibull probability function for $q = 0.7$ versus β.



Source: Elaborated by the author.

2.2.2 Using the Discrete Weibull in Repairable Systems

Consider $T_0, T_1, T_2, \ldots, T_n; (T_0 = 0)$, random variables related to the n failure times of a system since its beginning. After each failure, a minimum repair was made, and the time spent on such a task was very small and not relevant to the conclusions of the study.

Let $F(t) \equiv P(T_i \leq t)$ represent the failure time distribution and $\bar{F}(t) \equiv P(T_i > t)$ be the survival function.

Tadj, Ouali, Yacout, and Ait-Kadi (2011) define mathematically that the minimum repair is needed if:

$$P\{T_i > t | T_{i-1} = s\} = \frac{\bar{F}(t)}{\bar{F}(s)}, \; s < t \tag{2.3}$$

Under a DW, Equation (2.3) can be written as:

$$P\{T_i > t | T_{i-1} = s\} = \frac{q^{t^\beta}}{q^{s^\beta}}, \; s < t \tag{2.4}$$

Thus, the conditional probability function is expressed by Equation (2.5),

$$P\{T_i = t | T_{i-1} = s\} = \pi_{\text{MR}}(t_i | t_{i-1}) = \frac{q^{(t-1)^\beta} - q^{t^\beta}}{q^{s^\beta}}, \; s < t \tag{2.5}$$

Note that the distribution of the first failure follows a DW, but the successive failure distribution is a function of the previous failure.

Because $P[T_1 = t_1 \cap T_2 = t_2, \cap \cdots \cap T_n = t_n]$ can be written as $P[T_1 = t_1] \times P[T_2 = t_2 | t_1] \times \cdots \times P[T_n = t_n | t_{n-1}]$, using Equation (2.5), the likelihood function can be obtained with the observed failure times and by assuming a DW.

Assuming $(j = 1, \ldots, k)$ identical and independent systems are observed until the occurrence of n_j-th failure, the likelihood function can be expressed as

$$L\left(\beta, q|t_{1,1}, \ldots, t_{k,n_k}\right) = \prod_{j=1}^{k}\prod_{i=1}^{n_j} \pi_{\text{MR}}(t_{j,i}|t_{j,i-1}). \tag{2.6}$$

In a circumstance in which k independent systems are truncated at times T_j, the likelihood function is written as

$$L\left(\beta, q|t_{1,1}, \ldots, t_{k,n_k}, T_1, \ldots, T_k\right) = \prod_{j=1}^{k}\left[\frac{q^{T_j{}^{\beta}}}{q^{t_{j,n_j}^{\beta}}}\right] \times \prod_{i=1}^{n_j}\left[\pi_{\text{MR}}\left(t_{j,i}|t_{j,i-1}\right)\right] \tag{2.7}$$

In cases that include truncation by failure and/or time, those truncated by failure follow the condition $T_j = t_{j,n_j}$ and the likelihood function can be obtained using Equations (2.6) and (2.7).

2.3 Numerical Examples and Discussions

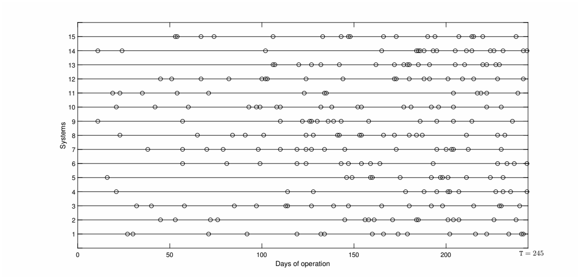
This section contains two subsections. In Subsection 2.3.1, a numerical example related to the maintenance problem of concrete mixer trucks is presented. The DW and CW models were fitted to the dataset. In Subsection 2.3.2 both Weibull distributions are fitted to several datasets found in the literature. The results are discussed in the second subsection. The traditional CW is denoted by $T \sim CW\left(\beta_c, \theta\right)$, where $\beta_c > 0$ (shape parameter) and $\theta > 0$ (scale parameter).

2.3.1 Numerical Example - Concrete Mixer Trucks

The present study was motivated by a real situation involving maintenance problems in mixer trucks owned by a Brazilian company. An unexpected failure of this equipment is costly and can compromise the delivery of concrete to the final customer. The repairs carried out for the faults are considered minimal repairs and are generally completed in a short time.

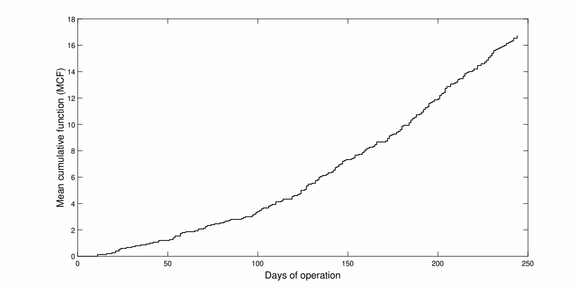
The data set consists of failure records for a sample of 15 mixer trucks from the company fleet. Data were collected from February to October 2021, when 251 failures were observed, followed by minimal repairs. Figure 2.2 shows the events (failures) vs. the operation time (in days), where each line corresponds to a sample unit, and each symbol "o" represents a failure time. The data for the 15 trucks were time-truncated for 245 days. Figure 2.3 presents the mean cumulative number of failures (non-parametric) versus time and shows a concave (curving up) pattern, which indicates that the time between failures decreases over time, that is, the system reliability deteriorates.

Figure 2.2: Failure times in days of operation for each truck (horizontal lines are the trucks and "o" are failures)



Source: Elaborated by the author.

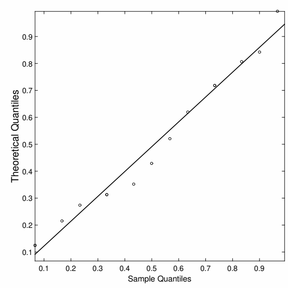
Figure 2.3: Mean Cumulative Number of Failures Versus Time



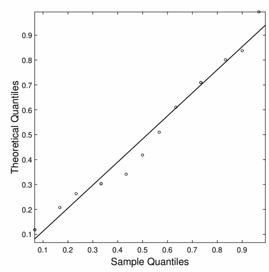
Source: Elaborated by the author.

As indicated in Subsection 2.2.2, the distribution of the first failure must be a DW. This requirement is similar for the use of the PLP model in the continuous case when the distribution of the first failure follows CW. Thus, to compare DW with CW, the condition of the distribution of the first failure needs to be satisfied. Meeker, Escobar, and Pascual (2021) recommend to draw QQ-plots and consider the value of AIC to evaluate the adherence of the possible distributions. QQ-plots of the first failure of the 15 items are shown in Figures 2.5a and 2.5b for DW and CW, respectively. The observed linear tendency in both plots indicates that the null hypothesis of DW or CW was not rejected. To draw the plots in Figure 2.4, the parameters of both distributions (DW and CW) were estimated using the maximum likelihood method with the first failure of the 15 systems. Notice in Figure 2.4 that the QQ-plots (drawn to evaluate the adherence of DW and CW distributions) are very similar and it is not possible to decide which distribution is the most adequate. The values of AIC for DW and CW (using the 15 failure times) are 133.41 and 133.59, respectively, which are also very close and in accordance with the results shown in the QQ-plots. In this sense, an initial (naive) assessment of the most adequate distribution (DW or CW) based on the first failure times is unsuccessful, and the decision should consider all failure events and their respective repairs.

Figure 2.4: QQ-plots under Discrete and Continuous Weibull distributions.



(a) QQ-plot for Discrete Weibull



(b) QQ-plot for Continuous Weibull

Source: Elaborated by the author.

A program in R was developed to estimate the parameters of DW. For the continuous case, we consider the likelihood function under the PLP described in Rigdon and Basu (2000). This code is available from the authors upon request for educational and research purposes. Table 2.1 presents the estimates of the parameters with their standard errors in parentheses ($\hat{\beta}$ and $\hat{q}$ for DW and $\hat{\beta}_c$ and $\hat{\theta}$ for CW), the value of the log-likelihood function $\left(\hat{l}\right)$, and Akaike's information criterion (AIC), which value is obtained as

$$\text{AIC} = -2\hat{l} + 2p, \tag{2.8}$$

where p is the number of parameters. The AIC is a criterion for the best model fit. According to Burnham and Anderson (2002), the AIC value can be viewed as an estimate of the expected relative distance between the fitted model and the unknown true mechanism that generated the observed data. Their values must be used in relative terms over the set of candidate models, that is, the difference between the AIC values is particularly important. According to Millar (2011), if the difference in AIC between the two models is greater than two, then the model with the smaller AIC is strongly preferred.

Table 2.1: Estimates of the parameters, $\hat{l}$ and AIC under Discrete and Continuous Weibull

Model	Estimates and s.e.	$\hat{l}$	AIC
DW	$\hat{\beta} = 1.69$ (0.10)	-886.16	1776.32
	$\hat{q} = 0.9984$ (0.0009)		
CW	$\hat{\beta}_c = 1.68$ (0.11)	-895.88	1795.77
	$\hat{\theta} = 45.91$ (1.49)		

Source: Elaborated by the author.

In the case of DW, the value of the likelihood function in (2.7) is used to obtain $\hat{l}$. As previously mentioned, for the continuous case, we consider the likelihood function under the PLP described in Rigdon and Basu (2000). We highlight that because the number of parameters of the DW and CW models is identical, the best model is the model with the highest value of $\hat{l}$.

Note that, in Table 2.1, the standard error of the shape parameter of the DW is smaller than that of the CW. Moreover, the DW model provides a lower Akaike's information criterion, AIC (with a difference of 19.45); therefore, this distribution can be an alternative model in the inference process. In repairable systems, the use of AIC in model selection has been practiced, for example, in Doyen (2011), Toledo, Freitas, Colosimo, and Gilardoni (2015), Zantek, Hanson, Damien, and Popova (2015), Mullor, Mulero, and Trottini (2019), Yaping Wang and Pham (2011), Yang, Zhang, and Hong (2013), and Cui, Lin, Chen, and Zhu (2021).

2.3.2 Numerical examples using data sets available in the literature

In Subsection 2.3.1, we present a numerical example where the failure times were recorded as discrete values. This alternative model can be as efficient as a continuous one without increasing the complexity of the analysis. In Subsection 2.3.2 we re-analyze six datasets found in the literature considering DW and CW. The criteria for the choice of these examples were based on data availability. The aim is to verify the discrepancies/advantages of the analyses considering these distributions. Next, we briefly describe the datasets used in this subsection.

1. Case A: Data presented in Nelson (1995). It is related to the times (in days of service) at which valve seats were replaced on 41 diesel engines in a service fleet (a

total of 48 failures is observed).

2. Case B: An example shown in Rigdon and Basu (2000) related to failure times of four systems; one truncated in time and three in failure (a total of 26 failures times).

3. Case C: Data set in Meeker, Escobar, and Pascual (2021), associated with the failure times (in days) of problems observed in ten networked microcomputers. Most of the trouble reports were easy to address (like replacing a defective mouse, rebooting the computer, remaking the computer's file system from the server, removing stuck floppy disks, tightening loose connector, etc.).

4. Case D: Data set in Rai, Chaturvedi, and Bolia (2020), which contains the failure times of 18 aero engines with time between overhauls of 550 hours.

5. Case E: The failure times of a software system described in Jelinski and Moranda (1972). We employed the first twenty records; the last value is assumed truncated on time.

6. Case F: The failure times of hydraulic systems of LHD machines described in Kumar and Klefsjö (1992). We focus on LHD 11 since it is the machine with the most failure times of 28. However, similar results can be obtained for other machines.

The estimates of the shape parameter (the standard error is in parentheses) under the two models, the log likelihood $\hat{l}$ and AIC are listed in Table 2.2. Analyzing Table 2.2, we observe similar performance, not showing significant discrepancies. However, lower standard errors and AIC values were observed for the DW. Similar values of AIC are observed for the cases A, B, D, and F indicating that the observed failures may be generated (equally likely) by the models DW or CW. However, the AIC values under the DW model for cases C and E are considerably lower (the differences are larger than two units), signifying that the observed failures are most likely generated by the DW model. Therefore, we suggest that the researcher can fit both distributions (DW and CW) when the failure times are recorded as discrete processes and calculate the AIC values (under DW and CW). If the difference between the values of AIC is at least two units, then choose the model with the lowest AIC; otherwise, the maintenance team can opt for the most familiar approach (DW or CW). Independent of the model selection, diagnostic analysis is always necessary, as shown in Figure 2.5. Considering the current computational resources, such tasks are relatively simple and may help select the best option.

Table 2.2: Estimates of the shape parameter, s.e. (in parenthesis), loglikelihood $\hat{l}$ and AIC

Case	Model	Shape Param.	$\hat{l}$	AIC
A	DW	1.395 (0.14)	−346.48	696.96
	CW	1.399 (0.20)	−346.49	696.98
B	DW	0.66 (0.128)	−137.67	279.35
	CW	0.67 (0.129)	−137.96	279.92
C	DW	1.32 (0.138)	−302.59	609.18
	CW	1.337 (0.141)	−306.35	616.70
D	DW	1.55 (0.163)	−184.11	364.23
	CW	1.56 (0.230)	−184.13	364.25
E	DW	1.61 (0.35)	−47.72	99.44
	CW	1.59 (0.37)	−49.73	103.46
F	DW	1.32 (0.21)	−156.95	317.90
	CW	1.32 (0.25)	−157.09	318.17

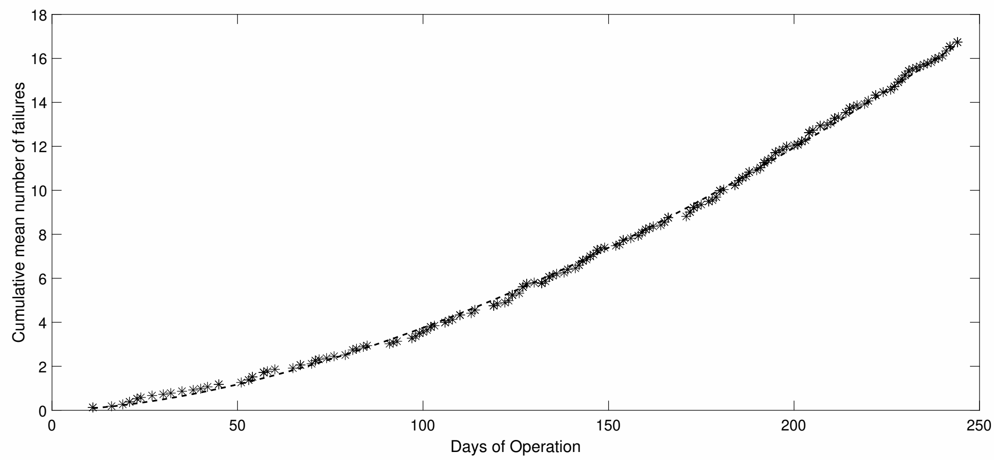
Source: Elaborated by the author.

2.4 The optimum time for the maintenance and the average number of failures by Markov chain

In general, the determination of a preventive maintenance interval that minimizes the average cost related to the repairs is essential in a study of repairable systems of industrial engineering sector. Usually, this step is performed after the selection of the most adequate model and its respective inference on the parameters. In several studies, such features have been explored; for example, in Gilardoni and Colosimo (2007) and Gilardoni, Toledo, Freitas, and Colosimo (2016), and in several examples discussed in Wang, Pham, et al. (2006) and Nakagawa (2006).

To state the optimum preventive maintenance policy, a model to predict the average number of the failures is needed. In Subsection 2.3.1 we conclude that DW is more appropriate for the concrete mixer truck example because it yields a lower AIC than CW does. This only signifies that the observations are most likely generated by a DW, considering the two options (DW and CW); thus, a diagnostic analysis of the fitness of the chosen model is necessary. Figure 2.5 shows the observed cumulative average failures of the 15 concrete truck mixers (marked by the symbol '*') and the average number of failures obtained using a Markov chain approach (dashed line). We observed (see Figure 2.5) that the cumulative means under the DW model are very similar to the empirical cumulative means, which indicates a goodness of fit. In addition, the DW model yielded an annual predicted mean value of failures lower than that of CW. This output is considered by the company team to be closer to what they had already observed in the previously observed data. The approach to calculating the average number of failures using a Markov chain is the subject of this section.

Figure 2.5: Diagnostic plots: comparison between the empirical and estimated MCFs using DW.



Source: Elaborated by the author.

In this study, we consider the failure times to be discrete; thus, the average number of failures until a specific time T_f (an integer), denoted by $\Lambda_{MR}(T_f)$, is obtained by the Markov chain, employing discrete values for the states and time. Let $\boldsymbol{P}$ be the transition probability matrix of a Markov chain described by the states $E = (s; j)$ $s = 0, \ldots, T_f$, $j = 0, \ldots, T_f$ and $s \leq j$. The variable s indicates the number of failures, and variable j is the elapsed time (in discrete units). For example, state $E = (3; 5)$ represents a total of 3 failures at time 5. State $E = (0; 0)$ symbolizes the beginning of the process. Based on the properties of arithmetic progression, the total number of states for a fixed value of T_f is $N = \dfrac{(T_f + 2) \times (T_f + 1)}{2}$. After each failure, a minimum repair is performed, and the conditional transition probabilities is expressed as

$$P[E_{i+1}(s_2; j_2)|E_i(s_1; j_1)] = \begin{cases} \dfrac{q^{(j_2-1)^\beta} - q^{j_2^\beta}}{q^{j_1^\beta}}, \\ \quad \text{if } (s_2 = s_1 + 1; j_2 = j_1 + 1; j_1 < T_f) \\ 1 - \dfrac{q^{(j_2-1)^\beta} - q^{j_2^\beta}}{q^{j_1^\beta}}, \\ \quad \text{if } (s_2 = s_1; j_2 = j_1 + 1; j_1 < T_f) \\ 1, \quad \text{if } (s_2 = s_1; j_2 = j_1; j_1 = T_f) \\ 0, \quad \text{otherwise} \end{cases} \tag{2.9}$$

The vector of probabilities $\boldsymbol{f}$ related to the states $(s;j)$, $s = 0,\ldots,T_f$; $j = 0,\ldots,T_f$, after T_f transitions is obtained as $\boldsymbol{f} = \boldsymbol{v} \times \boldsymbol{P}^{T_f}$, where $\boldsymbol{v} = (1,0,0,\ldots,0)$, a row vector of size N indicating that the process starts at state $E = (0,0)$. Note that the probabilities of vector $\boldsymbol{f}$ are positive (> 0) if state j is equal to T_f, which is the absorbent state. In these terms, we can express $\Lambda_{MR}(T_f)$ in Equation (2.10) as

$$\Lambda_{MR}(T_f) = \boldsymbol{f} \times \boldsymbol{k} \tag{2.10}$$

with $\boldsymbol{k}$, a column vector of size N, which $\left(iT_f + \dfrac{i(3-i)}{2}\right)$-th element is equal to $(i-1)$ for $i = 1,\ldots,(T_f+1)$; the others elements are equal to zero.

Note that this approach, using a Markov chain for the number of failures, yields a probability function for possible failures.

We developed a numerical example by considering $q = 0.5$, $\beta = 1.5$, $T_f = 2$ to obtain $\Lambda_{MR}(T_f)$ using a Markov chain. Using these parameters, the space of states is $E = \{(0;0),(0;1),(0;2),(1;1),(1;2),(2;2)\}$, the transition probability matrix *boldsymbolP* is given by Equation (2.11), and $\boldsymbol{v} = (1,0,0,0,0,0)$. Therefore, $\boldsymbol{f} = \boldsymbol{v} \times \boldsymbol{P}^2$=(0, 0, 0.141, 0, 0.5, 0.359), only three ($T_f + 1 = 3$) elements of the vector $\boldsymbol{f}$ are > zero, the first ($i = 1$) is at the $\left(1 \times 2 + \frac{1(3-1)}{2}\right)$=3rd position, the second ($i = 2$) is at the $\left(2 \times 2 + \frac{2(3-2)}{2}\right) = 5$-th position, and the third ($i = 3$) is the $\left(3 \times 2 + \frac{3(3-3)}{2}\right) = 6$-th element. In this case, the corresponding vector $\boldsymbol{k}^t = (0,0,0,0,1,2)$ results in $\Lambda_{MR}(T_f)$=$\boldsymbol{f} \times \boldsymbol{k} = 1.218429$. In other words, at time $T_f = 2$, an average of approximately 1.22 failures is expected.

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.282 & 0 & 0.718 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.282 & 0.718 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.11}$$

Now, consider a repairable system that will operate during T_f units of time starting at $t_0 = 0$, as in Valdez-Flores and Feldman (1989) and Gilardoni and Colosimo (2007), assuming the following conditions:

i) Preventive maintenance (PM) check points are scheduled after every T_f units of time;

ii) at each PM check point, a repair action of fixed cost C_{PM} is executed, which instantly returns the system to a like-new condition (perfect PM);

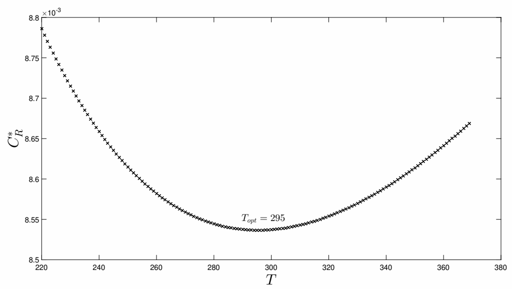
iii) between successive PM check points, a minimal repair (MR) is done after each failure and the expected cost for each is C_{MR}. That is, for each period defined by successive PM check points, the expected total cost is equal to the expected cost per failure times the expected number of failures;

iv) repair costs and failure times are independent;

v) repair times are neglected.

Using the previous conditions (i) to (v), we can express $C(T_f) = C_{\text{PM}} + C_{\text{MR}} \times \Lambda_{MR}(T_f)$. The value of $\Lambda_{MR}(T_f)$ was calculated using Equation (2.10). For large values of T_f, $C(T_f)$ is also large. Therefore, it is reasonable to work with the expected cost per unit of time $C^*(T_f) = \dfrac{C(T_f)}{T_f}$. Regarding the discreteness of time T_f, we can calculate the value of T_f that minimizes $C^*(T_f)$ using an exhaustive search. However, in many practical situations, companies do not afford the values of the costs C_{PM} and C_{MR}; only the ratio $R = \dfrac{C_{\text{PM}}}{C_{\text{MR}}}$, generally with $C_{\text{PM}} > C_{\text{MR}}$. In these cases, determining the optimum value of T_f that minimizes $C^*(T_f)$, is equivalent to determining the value of T_f that minimizes $C^*_R(T_f) = \dfrac{C^*(T_f)}{C_{\text{PM}}} = \dfrac{\Lambda_{MR}(T_f)}{(R \times T_f)} + \dfrac{1}{T_f}$. By choosing $R = 15$ in the numerical example discussed in Subsection 2.3.1, the optimum value of T_f that minimizes $C^*_R(T_f)$ is 295 days. That is, every 295 days, perfect preventive maintenance must be performed. Figure 2.6 illustrates the profile of $C^*_R(T_f)$ versus T_f indicating the optimal value at $T_{\text{opt}} = 295$.

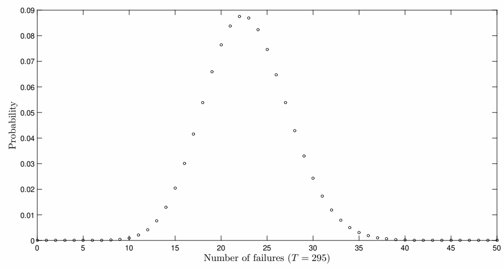
Figure 2.6: Optimal PM Times



Source: Elaborated by the author.

Figure 2.7 presents the probability distribution for the number of failures at $T_f = 295$ after calculating the vector $\boldsymbol{f}$. Adopting the optimum policy, an estimate of the likely number of failures was in the range [10, 34] with a probability of 0.99.

Figure 2.7: Distribution of the Number of Failures ($T_f = 295$)



Source: Elaborated by the author.

2.5 Final Remarks

In this Chapter, the problem related to the specification of a PM policy for repairable systems was studied, primarily focusing on predicting the reliability of the systems using a well-fitted model, assuming that the failure times can be discrete values. A model employing DW was fitted and compared to the CW model. We considered the failures of the concrete mixer trucks of a Brazilian company to illustrate the comparison between the two approaches. In addition, some datasets of failure times found in the literature were used. Thus, we can conclude that the two approaches are similar. However, the DW approach yielded a lower AIC and standard errors for the shape parameter. We believe that the researcher may fit the data using both alternatives, as computational resources and free software (such as R, Python, Octave, etc.) are available, and decide in each situation, the most recommended approach for the case.

One relevant characteristic (using discrete times) is the possibility of obtaining the average number of failures using the Markov chain approach with sets of discrete state space and time. Consequently, the probability distribution of the number of failures may be useful for the allocation of man-hours necessary to realize the maintenance policy. The use of a Markov chain with discrete state space proved to be understandable and prompted engineers to want to improve the model for use in other company situations, such as imperfect repair.

In future work, an expansion of the approach discussed in this article (DW) is suggested to deal with cases in which the repair can be considered an imperfect repair (IR), meaning that the system returns to an intermediate state between minimum repair (MR) and PM. We believe that even complex structures of imperfect repair may be approached by a Markov chain with discrete space states, and the average number of failures can be calculated by avoiding the use of Monte Carlo simulation (or parametric bootstrap), as in Gilardoni, Toledo, Freitas, and Colosimo (2016).

Chapter 3

OPTIMAL BROWN-PROSCHAN REPAIR POLICY FOR REPAIRABLE SYSTEMS USING THE DISCRETE WEIBULL FAILURE DISTRIBUTION AND MARKOV CHAINS

3.1 Introduction

When failures occur in complex systems (e.g., trucks, aircraft, medical diagnostic equipment, and trains), they can be repaired instead of being replaced. According to Sharma and Rai (2021), analyzing the reliability of repairable systems has always been challenging for industries, resulting in the emergence of a current and comprehensive research field. Thus, the study of repairable systems requires an analysis of the time at which each failure occurs and the appropriate type of repair suitable for each failure.

A repair action that returns a system to its previous condition is known as Minimal Repair (MR) or As Bad As Old (ABAO). This model is easily applied and widely described, as in Barlow and Hunter (1960), Gertsbakh (1977), Lai, Leung, Tao, and Wang (2001), Baker (2001), Gilardoni and Colosimo (2007), and Tadj, Ouali, Yacout, and Ait-Kadi (2011). Conversely, an extreme repair action that returns the system to a brand-new condition is known as Perfect Repair (PR) or As Good As New (AGAN). Practically, PRs involve more detailed maintenance or complete replacements. Cha and Finkelstein (2020) recently discussed the validity of this assumption.

An intermediate repair action between PR and MR is imperfect repair (IR), as outlined in Pham and Wang (1996). There are several IR models. Brown and Proschan

(1983) proposed a model where a unit can receive a PR at a p probability or an MR at $1-p$; we will refer to this as the Brown-Proschan (BP) model. Block, Borges, and Savits (1985) extended the BP model by allowing time-dependent probabilities. Kijima, Morimura, and Suzuki (1988) proposed virtual age models, while Kijima, Morimura, and Suzuki (1988) and Doyen and Gaudoin (2004) generalized these models, creating classes of models called Arithmetic Reduction of Age (ARA) and Arithmetic Reduction of Intensity (ARI). Toledo, Freitas, Colosimo, and Gilardoni (2015) studied real applications and verified that the ARA and ARI approaches were more feasible than MR. Liu, Finkelstein, Vatn, and Dijoux (2020) discussed relevant asymptotic properties of the ARA and the BP models. Sharma and Rai (2021) considered a censored data analysis method (right and multiply) based on failure modes for repairable systems supported by virtual age models and likelihood functions.

The study of maintenance policies for repairable systems has been an active research topic in recent studies. Notable studies include Wang, Pham, et al. (2006), Nakagawa (2006), and Nakagawa et al. (2014), which presented several models for decreasing expected maintenance costs. Mahdavi, Javadi, and Catalão (2023), Zhao, Cai, Mizutani, and Nakagawa (2021), Hamdan, Tavangar, and Asadi (2021), and Sheu, Liu, and Zhang (2019). Asadi, Hashemi, and Balakrishnan (2022) presented an overview of some classical models and discussed signature-based models of preventive maintenance (PM).

An interaction between the university and a big mining company uncovered an issue with every characteristic of a BP repairable system that we could not find a complete discussion of in the existing literature. Syamsundar, Naikan, and Wu (2021) observed that a defined repair policy is essential for studying the maintenance policy. The heads of the maintenance department informed us that they schedule PM every month in a set of similar systems that are returned to brand-new conditions after each PM. Accordingly, a third-party company performs the PM; however, the service is only possible monthly, as the mining area is located in a remote region. Even with a defined PM schedule, failures between PMs require repair. There is a daily failure report as part of the repair policy. In the event of a failure, the maintenance department can perform a PR or MR in each system, but there is insufficient time to decide which repair will result in future savings. A comprehensive adherence assessment showed that the Discrete Weibull Distribution (DW) proposed by Nakagawa and Osaki (1975) is suitable for the distribution of the first failures owing to its discrete nature. As a parsimonious factor and considering a large data set, we can consider the known DW parameters.

Using DW to model the times of failure required a cautious explanation once the use of continuous distributions was predominant. The time of failure models previously mentioned considers random times as continuously scaled variables, indicating the high accuracy of the measurement system and using a probability density function (e.g., the continuous Weibull distribution [CW]). However, failure reports consider time units (e.g., the number of days and hours) to be related to a discrete process or naturally discrete (e.g., the number of operation cycles before a failure). Burnham and Anderson (2002) posited that all random variables are discrete from a philosophical standpoint, i.e., their values increase by a minimum step (δ). Furthermore, they only accept countable values. From this perspective, we should evaluate a model considering a discrete, not continuous, distribution, as defended by the head of maintenance that introduced the mining problem. Moreover, recently Valadares, Quinino, Cruz, and Ho (2023) affirmed that the use of DW can be useful in repairable systems, being an alternative model when the data's nature is discrete.

The present study aims to determine the p value of the BP policy that provides the lowest cost for a PM with the DW for the first failure time. Lee and Lee (1999) and Cui, Kuo, Loh, and Xie (2004) also used such an approach but considered a continuous model (CW). Determining the optimal p^* value for the lowest expected costs requires obtaining the average of the total number of failures as well as the average of the total number of PRs and MRs once each type has its own average cost. We can use the non-trivial results from Lim and Park (1999) whenever the first failure distribution is CW. The solutions employed to find the value of p that minimizes the expected cost use numerical approximations or Monte Carlo simulations. Assuming a DW distribution for the first failure does not simplify the average of the total number of failures (or the proportion between PR and MR). However, the discrete distribution allows us to use a Markov chain with discrete states and times to determine the exact average of the total number of failures until PM is performed. This study describes the details of this methodology.

The remainder of this chapter is organized as follows. Section 3.2 describes the DW and its application to repairable systems. Section 3.3 presents the use of a Markov chain to obtain the expected number of failures. Section 3.4 provides a numerical example, while Section 3.5 discusses the optimization process. Finally, Section 3.6 presents the final remarks.

3.2 Probabilistic Models

This section is composed of two subsections: Subsection 3.2.1 reviews the DW introduced by Nakagawa and Osaki (1975) and detailed by Khan, Khalique, and Abouammoh (1989); Subsection 3.2.2 discusses the application of this distribution in repairable systems under three assumptions: MR, PR, and the BP model.

This article only considers the assumptions that failures in the observed systems are related to a discrete process and that the first failure follows a DW. However, the proposed methods can be extended to other discrete distributions.

3.2.1 The Discrete Weibull Distribution

Let T be a random variable related to the first failure in a DW, $T \sim \text{DW}(\beta, q)$, where $\beta > 0$ (shape parameter) and $0 < q < 1$. The probability function (PF) of T is expressed as follows:

$$P(T = t|\beta, q) = \pi(t|\beta, q) = q^{(t-1)^\beta} - q^{t^\beta}, t = \{1, 2, 3, \ldots\} \tag{3.1}$$

where the mean and variance are respectively given by:

$$\begin{aligned} &\mathbb{E}(T) = \mu = \sum_{t=1}^{\infty} t\left[q^{(t-1)^\beta} - q^{t^\beta}\right] = \sum_{t=0}^{\infty} q^{t^\beta} \text{ and} \\ &\mathbb{V}(T) = \sigma^2 = \sum_{t=1}^{\infty} 2tq^{t^\beta} + \mathbb{E}(T) - \mathbb{E}(T)^2 \end{aligned} \tag{3.2}$$

The distribution function for T is $P(T \leq t|\beta, q) = 1 - q^{t^\beta}$. It is a flexible distribution that includes increasing and decreasing ratios of failures like the continuous Weibull distribution.

3.2.2 Using the Discrete Weibull in Repairable Systems

Consider $T_0, T_1, T_2, \ldots, (T_0 = 0)$ as random variables related to the discrete times at which failures occur at the beginning of system operations. $X_i = T_i - T_{i-1}, (i = 1, 2, \ldots)$ is the time between the $i-th$ and $(i-1)-th$ failures. The repair times were too short and insignificant to draw conclusions.

$F_{T_i}(t) \equiv P(T_i \leq t)$ is the time of the failure distribution function and $\bar{F}_{T_i}(t) \equiv P(T_i > t)$ is the survival function. Tadj, Ouali, Yacout, and Ait-Kadi (2011) considers that if an MR occurs at the time s, then:

$$P\{T_i > t|T_{i-1} = s\} = \frac{\bar{F}_{T_i}(t)}{\bar{F}_{T_{i-1}}(s)}, s < t. \tag{3.3}$$

With DW modeling for the first time of failure, Equation (3.3) becomes:

$$P\{T_i > t|T_{i-1} = s\} = \frac{q^{t^\beta}}{q^{s^\beta}},\ s < t. \tag{3.4}$$

The PF for the occurrence of a failure at time t, conditioned by an MR at time s and expressed by $\pi_{\mathrm{MR}}(t_i|t_{i-1})$, is:

$$\pi_{\mathrm{MR}}(t_i|t_{i-1}) = P\left\{T_i = t|T_{i-1} = s\right\} = \frac{q^{(t-1)^\beta} - q^{t^\beta}}{q^{s^\beta}},\ s < t. \tag{3.5}$$

Consider that the system is observed until a fixed value T_f (the system is truncated in time) and MR adoption after failures. The average number of failures, denoted by $\Lambda_{MR}\left(T_f\right)$, is given by:

$$\Lambda_{MR}\left(T_f\right) = \sum_{i=1}^{T_f} P\left\{T_i = i|T_{i-1} = i-1\right\} = \sum_{i=1}^{T_f} \frac{q^{(i-1)^\beta} - q^{i^\beta}}{q^{(i-1)^\beta}}. \tag{3.6}$$

To calculate the mean number of failures when the PRs are performed owing to failures, we define $F_{X_i}\left(x\right) \equiv P\left(X_i \leq x\right)$ as the distribution function for times between failures and $\bar{F}_{X_i}\left(x\right) \equiv P\left(X_i > x\right)$ as the survival function for those times. If PR occurs after a failure, the distribution between failures is given by $F_{X_i} = F_X, \forall i$. Thus, the PF of a DW conditioned to a failure at time s, expressed as $\pi_{\mathrm{PR}}(t_i|t_{i-1})$, is:

$$\pi_{\mathrm{PR}}(t_i|t_{i-1}) = P\left\{T_i = t|T_{i-1} = s\right\} = q^{(t-s-1)^\beta} - q^{(t-s)^\beta},\ s < t. \tag{3.7}$$

Let $\left\{N_{PR}\left(T_f\right), T_f \geq 0\right\}$ be the process for counting failures with $N_{PR}\left(0\right) = 0$. $N_{PR}\left(T_f\right)$ is a random variable representing the number of failures/repairs that the system undergoes in the $(0, T_f]$ interval with a PR occurring immediately after a failure.

Therefore, the average number of failures with a PR, denoted by $\Lambda_{PR}\left(T_f\right)$, is:

$$\Lambda_{PR}\left(T_f\right) = \sum_{k=0}^{T_f} k \times P\left[N_{PR}(T_f) = k\right] \tag{3.8}$$

where k and $0 \leq k \leq T_f$ are the possible number of observed failures until time T_f.

Obtaining $\Lambda_{PR}\left(T_f\right)$ involves identifying the distribution of the sum of times between the failures once $P\left[N_{PR}\left(T_f\right) \geq k\right] = P\left(\sum_{i=1}^{k} X_i \leq T_f\right)$. We cannot express it through a closed form of the DW; thus, considerable computing effort is required as T_f increases. Instead, we use a Markov chain for a simpler solution. In Section 3.3, we describe the Markov chain approach to calculate $\Lambda_{PR}\left(T_f\right)$.

Now, consider the BP model proposed by Brown and Proschan (1983) as an IR model. Upon failure, the system can receive a PR at probability p or an MR at probability $1-p$. The distribution of the times between failures considers two subsequent PRs. This model can accommodate extreme cases: $p = 0$ results in an MR and $p = 1$ results in a PR. Calculating the mean number of failures for a BP or PR model is a complex task; we must also calculate the number of MRs between two PRs. Typically, we can calculate the expected number of failures through a Monte Carlo simulation, for instance, when using CW for BP Lim and Park (1999). However, as we assume DW in this study, we adopt the

Markov chain approach by using discrete states and times to calculate the mean expected number of failures, as detailed in Section 3.3.

3.3 The average number of failures using a Markov chain

This section is divided into three subsections, each explains the calculation of the average number of failures up to the fixed value T_f, for each type of repair, i.e., for MR, PR, and BP. Although the repair BP model is a general case that includes MR and PR as particular cases, we opted to present them separately to facilitate the reader's understanding.

3.3.1 The Model I: MR

Let $\boldsymbol{P}$ be the transition probability matrix of a Markov chain described by the states $E = (s;j)$, $s = 0,\ldots,T_f$, $j = 0,\ldots,T_f$, $s \leq j$. The variable s indicates the number of failures, and variable j is the elapsed time (in discrete units). The total number of states for a fixed value of T_f is defined as N and for the MR model, based on the properties of arithmetical progression, $N = \dfrac{(T_f+2)\times(T_f+1)}{2}$. The set of states is $\boldsymbol{E} = \{(0,0),(0,1),(0,2),\ldots,(0,T_f),(1,1),(1,2),\ldots,(1,T_f),\ldots,(T_f-1,T_f-1),(T_f-1,T_f),(T_f,T_f)\}$. For example, state $E = (3;5)$ indicates a total of three failures at time 5, and state $E = (0;0)$ symbolizes the beginning of the process.

After each failure, a minimum repair is performed and the conditional transition probabilities are expressed by Equation (3.9).

$$P[E_{i+1}(s_2;j_2)|E_i(s_1;j_1)] = \begin{cases} \dfrac{q^{(j_2-1)^\beta} - q^{j_2^\beta}}{q^{j_1^\beta}}, \\ \qquad \text{if } (s_2 = s_1+1; j_2 = j_1+1; j_1 < T_f) \\ 1 - \dfrac{q^{(j_2-1)^\beta} - q^{j_2^\beta}}{q^{j_1^\beta}}, \\ \qquad \text{if } (s_2 = s_1; j_2 = j_1+1; j_1 < T_f) \\ 1, \quad \text{if } (s_2 = s_1; j_2 = j_1; j_1 = T_f) \\ 0, \quad \text{otherwise} \end{cases} \tag{3.9}$$

where (s_1, s_2, j_1, j_2) assume values in $\{0, 1, \ldots, T_f\}$.

The vector $\boldsymbol{f} = (f^1, f^2, \ldots, f^N)$ related to the probabilities of the states $\boldsymbol{E} = \{(0,0), (0,1), \ldots, (0,T_f), (1,1), (1,2), \ldots, (1,T_f), \ldots, (T_f-1, T_f-1), (T_f-1, T_f), (T_f, T_f)\}$ after T_f transitions is obtained as $\boldsymbol{f} = \boldsymbol{v} \times \boldsymbol{P}^{T_f}$, with $\boldsymbol{v} = (1, 0, 0, \ldots, 0)$, a row vector of dimension N indicating that the process starts at state $E = (0,0)$. The probabilities of the vector $\boldsymbol{f}$ are positive (> 0) for the states $E = (s; T_f)$, which are absorbent states. In these terms, we can express $\Lambda_{MR}(T_f)$ in Equation (3.10) as:

$$\Lambda_{MR}(T_f) = \boldsymbol{f} \times \boldsymbol{k} \tag{3.10}$$

where $\boldsymbol{k} = (k^1, k^2, \ldots, k^N)$ is a column vector that contains the value of state s related to each state of set $\boldsymbol{E}$. Specifically, $\boldsymbol{k} = (0, \ldots, 0, 1, \ldots, 1, \ldots, T_f - 1, T_f - 1, T_f)$, namely, the first $T_f + 1$ elements are equal to zero, the next T_f elements are equal to one, and so on, until the antepenultimate and penultimate are both equal to $T_f - 1$ and the $N-$th is equal to T_f.

To illustrate, let us consider an example of a system observed until time $T_f = 2$ assuming a DW for the first failure time of parameters $\beta = 1.5$ and $q = 0.5$. In this case, there are six states in $\boldsymbol{E} = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$ and the elements of the transition matrix $\boldsymbol{P}$ are

$$\boldsymbol{P} = \begin{array}{c|cccccc|} & (0,0) & (0,1) & (0,2) & (1,1) & (1,2) & (2,2) \\ (0,0) & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ (0,1) & 0 & 0 & 0.281571 & 0 & 0.718429 & 0 \\ (0,2) & 0 & 0 & 1 & 0 & 0 & 0 \\ (1,1) & 0 & 0 & 0 & 0 & 0.281571 & 0.718429 \\ (1,2) & 0 & 0 & 0 & 0 & 1 & 0 \\ (2,2) & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

which yields the vector $\boldsymbol{f} = \{0, 0, 0.1408, 0, 0.5, 0.3592\}$ which is multiplied by the vector $\boldsymbol{k}^t = \{0, 0, 0, 1, 1, 2\}$, resulting in an average $\Lambda_{MR}(T_f) = 1.218$ failures.

3.3.2 The Model II: PR

When a PR is promoted after a failure, additional information is required in the transition probability. Let be the state $E = (s; j; l)$, with: $s = 0, \ldots, T_f$; $j = 0, \ldots, T_f$. The s and j variables have the same definitions described in Subsection 3.3.1, and l indicates the last time that an overhaul occurred; for example, the state $E = (3; 5; 4)$ indicates three failures, until time 5 with the last overhaul at time 4. The timings of the first and second failures are not required for the transition probabilities of the Markov chain. The state $E = (0; 0; 0)$ describes the beginning of the process. Now, considering that the system is evaluated until $T_f = 10$ and is currently in the state $E_i = (2; 4; 2)$, then the system can reach either state $E_{i+1} = (2; 5; 2)$ (absence of failure) or state $E_{i+1} = (3; 5; 5)$ (failure and overhaul). The total number of states for a fixed value of T_f is $N = (T_f + 1) + \sum_{i=0}^{T_f-1} \dfrac{(T_f - i)(T_f - i + 1)}{2}$ in the PR model. Explicitly the set of states is

$$\boldsymbol{E} = \Big\{(0,0,0), \ldots, (0,T_f,0), (1,1,1), \ldots, (1,T_f,1), (1,2,2), \ldots, (1,T_f,2), \ldots, (1,T_f,T_f),\\ (2,2,2), \ldots, (2,T_f,2), \ldots, (T_f-1,T_f-1,T_f-1), (T_f-1,T_f,T_f), (T_f,T_f,T_f)\Big\} \tag{3.11}$$

The transition matrix can be described by the following equation in (3.12):

$$P[E_{i+1}(s_2; j_2; l_2)|E_i(s_1; j_1; l_1)] = \begin{cases} \dfrac{q^{(j_2-l_1-1)^\beta} - q^{(j_2-l_1)^\beta}}{q^{(j_1-l_1)^\beta}}, \\ \quad \text{if } (s_2 = s_1 + 1; j_2 = j_1 + 1; l_2 = j_2; j_1 < T_f) \\ 1 - \dfrac{q^{(j_2-l_1-1)^\beta} - q^{(j_2-l_1)^\beta}}{q^{(j_1-l_1)^\beta}}, \\ \quad \text{if } (s_2 = s_1; j_2 = j_1 + 1; l_2 = l_1; j_1 < T_f) \\ 1, \quad \text{if } (s_2 = s_1; j_2 = j_1; l_2 = l_1; j_1 = T_f) \\ 0, \quad \text{otherwise} \end{cases} \tag{3.12}$$

where $(s_1, s_2, j_1, j_2, l_1, l_2)$ assume values in $\{0, 1, \ldots, T_f\}$. The vector $\boldsymbol{f} = (f^1, f^2, \ldots, f^N)$ related to the probabilities of the states $\boldsymbol{E}$ after T_f transitions, is obtained as $\boldsymbol{f} = \boldsymbol{v} \times \boldsymbol{P}^{T_f}$, with $\boldsymbol{v} = (1, 0, 0, \ldots, 0)$, a row vector of size N indicating that the process starts at state $E^1 = (0, 0, 0)$. Again, we can express $\Lambda_{PR}(T_f)$ using Equation (3.10) with $\boldsymbol{k} = (k^1, k^2, \ldots, k^N)$, a column vector that contains the value of state s related to each state of set $\boldsymbol{E}$.

The same inputs of the example used in Subsection 3.3.1 are considered here for a PR ($T_f = 2$). In this case, the process is described by seven states: $\boldsymbol{E} = \{(0,0,0), (0,1,0), (0,2,0), (1,1,1), (1,2,1), (1,2,2), (2,2,2)\}$, the elements of the transition matrix $\boldsymbol{P}$ are:

$$\begin{array}{c} \\ (0,0,0) \\ (0,1,0) \\ (0,2,0) \\ (1,1,1) \\ (1,2,1) \\ (1,2,2) \\ (2,2,2) \end{array} \begin{array}{c} \begin{array}{ccccccc} (0,0,0) & (0,1,0) & (0,2,0) & (1,1,1) & (1,2,1) & (1,2,2) & (2,2,2) \end{array} \\ \left(\begin{array}{ccccccc} 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.2816 & 0 & 0 & 0.7184 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

which yields the vector $\boldsymbol{f} = \{0, 0, 0.1408, 0, 0.25, 0.3592, 0.25\}$ which is multiplied by the vector $\boldsymbol{k}^t = \{0, 0, 0, 1, 1, 1, 2\}$, resulting in an average $\Lambda_{PR}(T_f) = 1.109$ failures.

3.3.3 The Model III: BP

Whenever the repaired system accepts the BP model, the state $E = (s; j; l)$ has the same definitions as those described in subsection 3.3.2. However, once $s = 0, \dots, T_f$; $j = 0, \dots, T_f$; $l = 0, \dots, T_f$, $s \leq j$, $l \leq j$, $l \geq 0$, this chain has more states than those used with PR.

For example, a system observed until time $T_f = 10$ is in state $E = (3; 5; 0)$. This state indicates that the system had three failures, was at time five, and none of them was a PR. Therefore, this system can reach the following states (and their respective probabilities) in the next stage.

1. $E_{i+1}(3; 6; 0)$, $P[E_{i+1}(3; 6; 0)|E_i(3; 5; 0)] = 1 - \frac{q^{(5)^\beta} - q^{(6)^\beta}}{q^{(5)^\beta}}$, if no failures occurred between times 5 and 6.

2. $E_{i+1}(4; 6; 0)$, $P[E_{i+1}(4; 6; 0)|E_i(3; 5; 0)] = \frac{q^{(5)^\beta} - q^{(6)^\beta}}{q^{(5)^\beta}} \times (1 - p)$, if the system fails and undergoes an MR;

3. $E_{i+1}(4; 6; 6)$, $P[E_{i+1}(4; 6; 6)|E_i(3; 5; 0)] = \frac{q^{(5)^\beta} - q^{(6)^\beta}}{q^{(5)^\beta}} \times p$, if the system fails and undergoes a PR;

The total number of states for a fixed value of T_f is N and that for the BP model is

$$N = 2(T_f + 1) + \sum_{j=0}^{T_f - 2} \frac{(2T_f - j + 1)(2 + j)}{2}.$$

Specifically, the set of states is

$$\begin{aligned}\boldsymbol{E} = \Big\{&(0,0,0),\dots,(0,T_f,0),(1,1,0),(1,1,1),(1,2,0),(1,2,1),\\ &(1,2,2),\dots,(1,T_f,0),\dots,(1,T_f,T_f),(2,2,0),(2,2,1),(2,2,2),\\ &(2,3,0),\dots,(2,T_f,0),\dots,(2,T_f,T_f),\dots,(T_f-1,T_f-1,0),\dots,\\ &(T_f-1,T_f-1,T_f-1),(T_f,T_f,0),\dots,(T_f,T_f,T_f)\Big\}\end{aligned} \tag{3.13}$$

The equations in (3.14) describe a complete transition matrix.

$$P[E_{i+1}(s_2;j_2;l_2)|E_i(s_1;j_1;l_1)] = \begin{cases} \dfrac{q^{(j_2-l_1-1)^\beta} - q^{(j_2-l_1)^\beta}}{q^{(j_1-l_1)^\beta}} \times p, \\ \quad \text{if } (s_2 = s_1+1; j_2 = j_1+1; l_2 = j_2; j_1 < T_f) \\ \dfrac{q^{(j_2-l_1-1)^\beta} - q^{(j_2-l_1)^\beta}}{q^{(j_1-l_1)^\beta}} \times (1-p), \\ \quad \text{if } (s_2 = s_1+1; j_2 = j_1+1; l_2 = l_1; j_1 < T_f) \\ 1 - \dfrac{q^{(j_2-l_1-1)^\beta} - q^{(j_2-l_1)^\beta}}{q^{(j_1-l_1)^\beta}}, \\ \quad \text{if } (s_2 = s_1; j_2 = j_1+1; l_2 = l_1); j_1 < T_f) \\ 1, \quad \text{if } (s_2 = s_1; j_2 = j_1; l_2 = l_1; j_1 = T_f) \\ 0, \quad \text{otherwise} \end{cases} \tag{3.14}$$

After obtaining the transition matrix $\boldsymbol{P}$, we follow the steps described in Subsection 3.3.2 to calculate $\Lambda_{BP}(T_f)$.

Note that the transition matrix described by the equations (3.14) comprises the cases shown in Subsections 3.3.1 and 3.3.2, considering $p = (0;1)$. The BP model generates more states and is thus computationally more complex. If the repair system is solely on MR, we recommend using Model I or Equation (3.6). Similarly, we recommend using Model II if the system uses only PR.

If we update the example used in Subsections 3.3.1–3.3.2 for a BP and adopt the probability $p = 0.3$ for PR, then the process is described by 11 states, namely, $\boldsymbol{E} = \{(0,0,0),(0,1,0),(0,2,0),(1,1,0),(1,1,1),(1,2,0),(1,2,1),(1,2,2),(2,2,0),$ $(2,2,1),(2,2,2)\}$, and the elements of the transition matrix $\boldsymbol{P}$ are as follows:

	(0,0,0)	(0,1,0)	(0,2,0)	(1,1,0)	(1,1,1)	(1,2,0)	(1,2,1)	(1,2,2)	(2,2,0)	(2,2,1)	(2,2,2)
(0,0,0)	0	0.5	0	0.35	0.15	0	0	0	0	0	0
(0,1,0)	0	0	0.2816	0	0	0.5029	0	0.21553	0	0	0
(0,2,0)	0	0	1	0	0	0	0	0	0	0	0
(1,1,0)	0	0	0	0	0	0.2816	0	0	0.5029	0	0.21553
(1,1,1)	0	0	0	0	0	0	0.5	0	0	0.35	0.15
(1,2,0)	0	0	0	0	0	1	0	0	0	0	0
(1,2,1)	0	0	0	0	0	0	1	0	0	0	0
(1,2,2)	0	0	0	0	0	0	0	1	0	0	0
(2,2,0)	0	0	0	0	0	0	0	0	1	0	0
(2,2,1)	0	0	0	0	0	0	0	0	0	1	0
(2,2,2)	0	0	0	0	0	0	0	0	0	0	1

which outcomes the vector $\boldsymbol{f} = \{0, 0, 0.1408, 0, 0, 0.35, 0.075, 01078, 0.17601, 0.0525, 0.097935\}$ and multiplied by the vector $\boldsymbol{k}^t = \{0, 0, 0, 1, 1, 1, 2, 2, 2\}$ results an average $\Lambda_{BP}(T_f) = 1.18566$ failures.

3.4 A Numerical Example

In this section, three DW distributions demonstrate the capacity to calculate the mean number of failures $[\Lambda(T_f)]$ using the Markov chain method for the three models discussed in Section 3.3. We compared the results with simulated data, and each scenario used 300.000 trials.

The operation time is $T_f = 30$, and the scale parameter is $q = 0.9$, which is fixed in all scenarios. We define two shape parameters: $\beta = (0.5; 1.5)$. Note that when the system has $\beta = 1$, there is no difference between the MR and PR models, both of which yield $\Lambda(30) = 3$. Table 3.1 lists the simulated and calculated results for the mean number of failures in the MR and PR models.

Table 3.1: Mean number of failures in the MR and PR models through a simulation and a Markov chain.

	MR		**PR**	
β	**Simulation**	**Markov chain**	**Simulation**	**Markov chain**
0.5	0.5657	0.5671	0.6671	0.6693
1.5	12.7878	12.7880	6.3878	6.3878

Source: Elaborated by the author.

Table 3.2: Mean number of failures in the BP model through a simulation and a Markov chain, $\beta = (0.5; 1.5)$

	0.50		**1.50**	
p	**Simulation**	**Markov chain**	**Simulation**	**Markov chain**
0.20	0.5846	0.5864	9.5797	9.5753
0.40	0.6051	0.6062	8.1887	8.1855
0.60	0.6251	0.6267	7.3774	7.3696
0.80	0.6502	0.6477	6.8114	6.8085

Source: Elaborated by the author.

The mean number of failures in the BP model was evaluated for $p = (0.2; 0.4; 0.6; 0.8)$. Table 3.2 lists the results.

Notably, the Markov chain approach and simulation results are very similar for all repair models (MR, PR, and BP) and the DW distribution parameters. There is no evidence to reject computational implementation with a Markov chain as incorrect. In addition, note that the system is rejuvenating in cases where $\beta < 1$, and more MR actions will result in a lower mean number of failures. When $\beta > 1$, indicating deterioration, more MRs result in more significant failures.

To assess the average cost of failures, it is necessary to determine the mean number of failures that have undergone PR and MR until time T_f. We define $N_{BP}^{PR}(T_f)$ and $N_{BP}^{MR}(T_f)$ as the number of PRs and MRs during the interval $(0, T_f]$, $T_f \geq 0$, assuming a BP model such that $N_{BP}(T_f) = N_{BP}^{PR}(T_f) + N_{BP}^{MR}(T_f)$ represents the total number of failures/repairs within the interval. The mean total number of failures is given by $\Lambda_{BP}(T_f)$ wherein $p \times 100\%$ percent of this value indicates the mean number of PRs, denoted by $\Lambda_{BP}^{PR}(T_f)$, and consequently $(1-p) \times 100\%$, the mean number of MRs, denoted by $\Lambda_{BP}^{MR}(T_f)$. In mathematical terms, we can easily obtain the mean number $\Lambda_{BP}^{PR}(T_f)$ for BP, once the Markov chain yields $\Lambda_{BP}(T_f)$. Thus, $\Lambda_{BP}^{PR}(T_f)$ is given by Equation (3.15).

$$\begin{aligned}\Lambda_{BP}^{PR}(T_f) &= \mathbb{E}\left\{\mathbb{E}\left(\sum_{i=1}^{N_{BP}(T_f)} I_i | N_{BP}(T_f), p\right)\right\} \\ &= \mathbb{E}\left\{N_{BP}(T_f) \times p\right\} = p \times \Lambda_{BP}(T_f)\end{aligned} \tag{3.15}$$

where I_i indicates the occurrence of a PR after $i-th$ failure. Similarly, we obtain Λ_{BP}^{MR} or its equivalent $\Lambda_{BP}^{MR}(T_f) = \Lambda_{BP}(T_f) - \Lambda_{BP}^{PR}(T_f)$.

3.5 Optimization

The optimization problem described in this section is taken from a real case. A Brazilian mining company seeks to determine a corrective maintenance policy. They currently perform a monthly preventive maintenance service (PM) on trucks used for mineral exploiting, extracting, and processing. A third-party company performs the preventive maintenance every 30 days. More frequent service is unfeasible due to the remote localization and costs. The mining company performs any repairs between PMs. Whenever a truck presents a failure, the local team can perform a PR at a p probability or an MR at

$(1-p)$. The BP model is considered valid, and the type of repair is subject to a random policy.

Let $Y_0, Y_1, \ldots, (Y_0 = 0)$ represent the PM occurrences from the system start and $Z_k = Y_k - Y_{k-1}$ is the time between the $k-th$ and the $(k-1)-th$ PM.

The contract establishes a fixed Z_k, and the study aims to obtain an optimal p^* for the lowest mean cost of repairs over 30 days. The system follows a DW for the first failure, as described in Section 3.1. We assume the following:

- The PMs will occur every $Z = 30$ units of time (days), equally spaced, i.e., $Z_k = Y_k - Y_{k-1} = Z, \forall k$.
- C_{PR} is the cost of a PR, C_{MR} is the cost of an MR, and C_{PM} is the cost of a PM.
- A PM returns a system to an as good as new condition.
- Two consecutive PMs establish an overhaul cycle.

Based on previous conditions, the cost between PMs $(k-1, k)$ can be written as $C(Z) = C_{MR} N_{BP}^{MR}(Z) + C_{PR} N_{BP}^{PR}(Z) + C_{PM}$. Higher Z values result in higher $C(Z)$; therefore, it makes sense to estimate the cost per time unit: $H(Z) = C(Z)/Z$. The mean cost per desired time unit is:

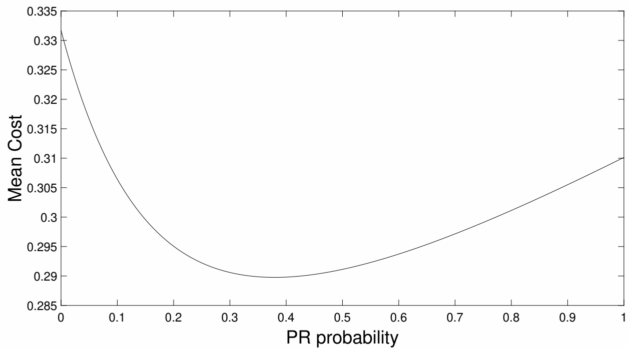
$$\begin{aligned}\mathbb{E}\left[H\left(Z\right)\right] =& \frac{\mathbb{E}\left[C\left(Z\right)\right]}{Z} \\ =& \frac{C_{MR} \times \Lambda_{BP}\left(Z\right) \times (1-p) + C_{PR} \times \Lambda_{BP}\left(Z\right) \times p + C_{PM}}{Z}\end{aligned} \tag{3.16}$$

The optimal maintenance policy is given by the p^* value that provides the lowest mean expected cost $\mathbb{E}\left[H\left(Z\right)\right]$ from the Equation (3.16).

The parameters of DW are $q = 0.9$, $\beta = 1.5$, the costs: $C_{PR} = 1.30$, $C_{MR} = 0.70$, $C_{PM} = 1.00$, and $Z = 30$. An MR seeks only to get the truck up and running, while a PR also considers other criteria. The PM and PR return the system to AGAN condition. However, a PR results in an unplanned stop of work. The economic values reflect the real percentual difference adopted by the mining company.

Figure 3.1 shows the mean cost of repairs between PMs as the probability function for a PR occurrence. This optimization shows that an optimal corrective maintenance to reach the lowest costs indicated by Equation (3.16) is $p^* = 0.379$, based on the initial conditions. Exhaustive research provided the optimization, discretizing the p value by 0.001 increments in the $0-1$ range. Each value underwent Equation (3.16), and we chose the p^* value that reached the lowest mean expected cost.

Figure 3.1: Mean cost of repairs between PMs as the probability function of a PR occurrence (p).



Source: Elaborated by the author.

3.6 Final Remarks

Optimization based on the lowest expected cost using the BP model requires knowledge about the mean number of failures within an interval between PMs. However, obtaining the expected value with a CW distribution for first failure is not simple, leading past researchers to choose methods such as the Monte Carlo simulation that return approximate results. The DW distribution for first failures can, if adherent, obtain the exact value of the expected number of failures.

This article proposes calculating the mean number of failures using a Markov chain for three repair models: MR, PR, and BP. The time of failures is considered discrete and the first failure follows DW. The method can be extended to other discrete distributions without complicating the application. Results from the Markov chain were close to the simulated results.

We applied our method to a Brazilian mining company that sought the lowest repair costs between PRs and fit the BP model. We found an optimal corrective maintenance policy indicating that the PR proportion should be 37.9% of the executed truck maintenance, while the MR proportion should be 62.1%.

We implemented our developed methodology into the R software; the macros are available upon direct request. An extension of this study with a Markov chain could consider that the probability p discussed in this article depends on time, as per Block, Borges, and Savits (1985).

Chapter 4

ANALYSIS OF REPAIRABLE SYSTEMS WITH IMPERFECT REPAIR USING A MIXTURE DISTRIBUTION

4.1 Introduction

More complex systems, such as trucks, planes, medical diagnostic systems, and locomotives, among others, are repaired and not replaced when they fail. As noted in Sharma and Rai (2021), the reliability analysis of repairable systems has always been a challenging task for industries, especially when the collected data is censored and the number of failures cannot be directly observable and must be estimated. A high failure rate implies a high cost. Therefore, determining the reliability and other performance characteristics is important for an adequate preventive maintenance policy and corrective maintenance evaluations as noted in Syamsundar, Naikan, and Wu (2021).

Generally, repairs can be classified as to their efficiency, and in extreme situations they can be classified as "Perfect Repair" (PR) or '*As Good As New*' (AGAN), that is, the system returns to the new condition after repair or "Minimal Repair" or '*as bad as old*' (ABAO), that is, the repair leaves the system in the same condition as before the failure.

It is quite common to work with a periodic maintenance policy called Preventive Maintenance (PM), in which a PR is performed at each fixed time interval and failures observed between the PRs are repaired according to the MR. This approach is generally easy to use and understand, and provides useful and practical results for many real world applications. Several studies use this approach, for example Barlow and Hunter (1960), Gertsbakh (1977), and Lai, Leung, Tao, and Wang (2001). Gilardoni and Colosimo (2007) were motivated by a problem of maintenance of energy transformers of a Brazilian company and defined the interval of PR conditioned on the MR between the PRs (renewal).

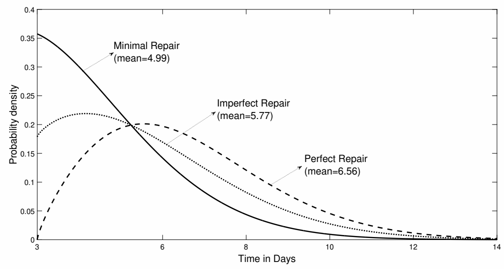
While the approach combining the policies of MR and PR is suitable in several practical situations, in others, the repair performed may be imperfect, having an intermediate efficiency between AGAN and ABAO, generically called "Imperfect Repair" (IR). IR is discussed in several studies.

Brown and Proschan (1983) defined a model where the failed unit can receive a PR with a probability p or a MR with a probability of $1-p$. This model was extended by Block, Borges, and Savits (1985), now allowing probabilities to be time dependent. Virtual age models were proposed by Kijima, Morimura, and Suzuki (1988). Doyen and Gaudoin (2004) generalized the model proposed by Kijima, Morimura, and Suzuki (1988) and created a class of models called Arithmetic Reduction of Age (ARA) and Arithmetic Reduction of Intensity (ARI). Toledo, Freitas, Colosimo, and Gilardoni (2015) studied a real application in which the ARA and ARI approaches were more suitable than the use of MR. Hongzhou Wang and Pham (1996) proposed the model of "Quasi-Renewal" (QR). This model assumes that if a new system has a π failure time distribution, after the $k-th$ failure, the system will have a $w_R^{k-1}\pi$ distribution, where $w_R > 0$. Besides these approaches, other ways to model IR include: "Improvement Factor", proposed by Malik (1979), "Shock Model", proposed by Kijima and Nakagawa (1991), among others. Pham and Wang (1996) makes an excellent review of the proposed IR models.

Using data observed in the field, this article proposes an approach to IR in which the failure time distributions are a mixture of two distributions: a term of the mixture is a distribution that considers that the failure time is conditioned to the MR with a weight w_{MR} and the other term considers that the failure time is conditioned to an PR with weight $w_{PR} = 1 - w_{MR}$. The formal definition of mixture distribution will be presented in the next section. Note that we are not claiming that a PR or MR will be performed as in Brown and Proschan (1983). The repairs can all be imperfect, and we assume that the failure time of the next failure after the IR will depend on the mixture.

For a better understanding of the proposal, we assume that the first failure (in days) for an equipment can be modeled according to a continuous Weibull (CW) distribution with shape parameter $\beta_c = 1.8$ and scale parameter $\theta = 4$. If the first failure occurs within 3 days then the time of the next failure will depend on the type of repair performed: MR, PR, or IR. The average time using PR will be the longest (occurring at 6.56 days on average), while the average time using MR will be the shortest (occurring at 4.99 days on average). Using IR, in turn, will result in a shortest average time compared to PR and a longest average time compared to MR. If the weight of each component in the mixture is equal, then the average is 5.77 days. In this study, we assume that the distribution of the time of the $(i+1)-th$ failure once the time of the $i-th$ failure is known will be a mixture of the distributions considering MR and PR. Figure 4.1 illustrates the distribution of the time of the second failure conditioned to the time of the first failure occurred in 3 days, using PR, MR, and IR. As the distribution of the first failure in this example is CW, the conditional distribution of the second failure conditioned on the time of the first failure (3 days) will be Weibull truncated at time 3 days when using MR (MR and Weibull with a threshold parameter equal to 3 days). The analysis of the distribution of subsequent failures has a rationale like that discussed here and will always depend on the time of occurrence of the last failure. In this scenario, we can construct the likelihood function for an observed set of failures by assuming that the distributions of failure times follow the assumption of IR.

Figure 4.1: Distribution of time of second failure conditioned to MR, PR, and IR



Source: Elaborated by the author.

Furthermore, studies on complex systems are usually divided into two stages. The first stage comprises estimating the parameters of the system under study, and the second stage comprises optimizing a maintenance policy. Maintenance policies aim at reducing the risks and costs caused by system failures. Details on various types of maintenance policies are found in Meeker, Escobar, and Pascual (2021), Lin, Zuo, and Yam (2001), Nakagawa (2006), Nakagawa and Mizutani (2009), and Rausand and Hoyland (2003). This work will focus on planned PM.

The remainder of this chapter is organized as follows: The mixture model is presented in Section 4.2, and other repair models are introduced. The studied maintenance policy is described in Section 4.3. Application to a real data set is presented in Section 4.4. Final remarks are outlined in Section 4.5.

4.2 Mixture, Minimal Repair, and Perfect Repair Statistical Models

Before getting into mathematical modeling, some definitions should be made. Consider $T_0, T_1, T_2, ..., (T_0 = 0)$ random variables referring to the failure times in relation to the start of the system. Also consider $X_i = T_i - T_{i-1}, (i = 1, 2, ...)$, as the time between $i-th$ failure and the $(i-1)-th$ failure. Repair times will be considered small and not significant.

Set the failure counting process $\{N(t), t \geq 0\}$, $N(0) = 0$. $N(t)$ is a random variable representing the number of failures/repairs the system has undergone in the range $(0, t]$, so that:

$$N(t) = sup\left\{n : \sum_{i}^{n} X_i \leq t\right\} \tag{4.1}$$

Let $F_{X_i}(x) \equiv P(X_i \leq x)$ as the function distribution of times between failures, and, $\bar{F}_{X_i}(x) \equiv P(X_i > x)$ as the survival time between failures function. If we perform PRs, the distribution between failures will always be given by $F_{X_i} = F_X, \forall i$. The probability density function is given by $\pi_{X_i}(x) = {}^{\partial F_{X_i}(x)}/_{\partial x}$. We can also write the probability density function as a function of operating time and it will be denoted by $\pi_{PR}(t_i|t_{i-1})$, that is, $\pi_{X_i}(x)$ considering a threshold parameter equal to t_{i-1}.

Set $F_{T_i}(t) \equiv P(T_i \leq t)$, as the distribution of failure times, and, $\bar{F}_{T_i}(t) \equiv P(T_i > t)$, as the failure time survival function. Tadj, Ouali, Yacout, and Ait-Kadi (2011) mathematically defines that a MR occurs on failure if:

$$P\{T_i > t|T_{i-1} = s\} = \frac{\bar{F}_{T_i}(t)}{\bar{F}_{T_{i-1}}(s)}, s < t \tag{4.2}$$

The probability density function for MR will be given by $\pi_{MR}(t_i|t_{i-1}) = {}^{\partial F(t_i)}/_{\partial t}/\bar{F}(t_{i-1})$. Thus, the probability density function of the proposed mixture for the $i-th$ failure is given by:

$$\pi_{IR}(t_i|t_{i-1}) = w_{MR}\pi_{MR}(t_i|t_{i-1}) + w_{PR}\pi_{PR}(t_i|t_{i-1}) \tag{4.3}$$

Where w_{MR}, refers to the weight relative to the minimum repair, and, w_{PR} refers to the PR, where $w_{MR} + w_{PR} = 1$.

Set $\boldsymbol{\mu}$ as the vector of unknown parameters of the model distributions. Then, considering that $j = (1, ..., k)$ identical and independent systems are analyzed, and we observe up to the $n_j - th$ failure for each system. The likelihood function can be given by:

$$L(\boldsymbol{\mu}|t_{1,1}, \ldots, t_{k,n_k}) = \prod_{j=1}^{k}\prod_{i=1}^{n_j}[w_{MR} \times \pi_{MR}(t_{j,i}|t_{j,i-1}) + w_{PR} \times \pi_{PR}(t_{j,i}|t_{j,i-1})] \tag{4.4}$$

For the situation where the k independent systems are truncated at the times T_j, the likelihood is given by:

$$L\left(\boldsymbol{\mu}|t_{1,1},\ldots,t_{k,n_k},T_1,\ldots,T_k\right)=\prod_{j=1}^{k}\left[w_{MR}\times\bar{F}_{MR}\left(T_j|t_{j,n_j}\right)+w_{PR}\times\bar{F}_{PR}\left(T_j|t_{j,n_j}\right)\right]\times\prod_{i=1}^{n_j}\left[w_{MR}\times\pi_{MR}\left(t_{j,i}|t_{j,i-1}\right)+w_{PR}\times\pi_{PR}\left(t_{j,i}|t_{j,i-1}\right)\right] \quad (4.5)$$

In cases which include truncation by failure and/or time, those truncated by failures follow the condition: $T_j = t_{j,n_j}$ and the likelihood function can be easily obtained using simultaneously Equations (4.4) and (4.5).

Despite the multiple models for mixture distributions, in the present work, the mixture will be modeled through the CW, one of the most used distributions in the reliability field. First, because this distribution is related to the PLP, and second, because in many practical applications, the times between failures $X_1, X_2, ...$ follow the CW after a PR.

That said, note that the first failure distribution must follow the CW, and the distributions for the mixture model considering the next failures are conditioned to the time of the last failure.

Assuming that $T \sim CW\left(\beta_c,\theta\right)$, $\beta_c > 0$ is the shape parameter and $\theta > 0$ e is the scale parameter, the parameterization chosen in this study to represent T was:

$$\pi\left(t|\beta_c,\theta\right)=\frac{\beta_c}{\theta}\left(\frac{t}{\theta}\right)^{\beta_c-1}exp^{-(t/\theta)^{\beta_c}} \quad (4.6)$$

The substitution of (4.6) in (4.4) or (4.5) will represent the likelihood function for the evaluation of failures in systems where the first failure can be modeled by the CW and the subsequent failures can be modeled by the distribution using IR defined in (4.3).

4.2.1 Other Imperfect Repair Models

For describing the other models compared with the model proposed in this article, define the conditional failure intensity function of $T_{N(t)+1}$ by:

$$\rho_{T_{N(t)+1}|H_t}\left(t\right)=\lim_{\Delta t\to 0}\frac{P\left[N\left(t+\Delta t\right)-N\left(t\right)=1|H_t\right]}{\Delta t} \quad (4.7)$$

Where H_t refers to all available system information in the range $(0, t]$.

Moreover, the cumulative failure function conditioned to H_t is:

$$\Lambda_{T_{N(t)+1}|H_t}(t) = \int_0^t \rho_{T_{N(t)+1}|H_t}(x)\, d(x) \tag{4.8}$$

Before the first failure occurs, the intensity function given by Equation (4.7) is a deterministic and continuous function in time $\varphi(t)$. A well-known way of representing this function is through the PLP, such that, $\varphi(t) = {}^{\beta_c}/_{\theta}\left({}^{t}/_{\theta}\right)^{\beta_c - 1}$. The PLP will be used to evaluate the numerical example.

Doyen and Gaudoin (2004) proposed the *ARA* and *ARI* model classes. The model class *ARA* assumes that the conditional failure intensity function at the time t is equal to the intensity function of a system that has never failed with an equivalent age $V(t)$, known as the virtual age. Where, $V(t) \leq t$, and is defined by:

$$V(t) = V\left(t; N(t); T_1, ..., T_{N(t)}\right) \tag{4.9}$$

If we consider a single type of maintenance, the class of ARA models is defined by a memory m, ARA_m, in which the function of the failure rate of a system in t is:

$$\rho_{N(t)+1|H_t}(t) = \varphi\left(t - (1 - w_R)\sum_{i=0}^{min\{m-1;N(t)-1\}} w_R^i T_{N(t)-i}\right) \tag{4.10}$$

In which the maintenance effect is represented by w_R $(0 \leq w_R \leq 1)$,and includes MR and PR if $(w_R = 1; w_R = 0)$, respectively.

Another alternative is the ARI class of models, which implies that each repair reduces the intensity function of the system. The ARI_m model assumes that each repair reduces the increment in the intensity function considering the last m failures. The intensity function of this model is:

$$\rho_{N(t)+1|H_t}(t) = \varphi(t) - (1 - w_R)\sum_{i=0}^{min\{m-1;N(t)-1\}} w_R^i \times \varphi\left(T_{N(t)-i}\right) \tag{4.11}$$

The *ARA* and *ARI* models with a PLP initial intensity function imply that the first failure of each equipment follows the CW.

Generally, for these classes of models, and considering the existence of k independent systems, the likelihood is given by:

$$L(\boldsymbol{\mu}|H_t) = \prod_{j=1}^{k} e^{-\Lambda(T_j)+\Lambda\left(t_{j,n_j}\right)}\left[\prod_{i=1}^{n_j}\rho\left(t_{j,i}\right)\right] \times e^{-\sum_{i=1}^{n_j}\Lambda(t_{j,i})-\Lambda(t_{j,i-1})} \tag{4.12}$$

Pham and Wang (1996) proposed for IRs the model known as QR. Assuming that X, positive random variable, represents the failure time of a repairable system, immediately after maintenance, the failure time is reduced by a fraction w_R, where $0 < w_R < 1$. That is:

$$X_i = w_R^{i-1} X \tag{4.13}$$

For this specific case, considering that $X \sim CW(\beta_c, \theta)$, the likelihood for the QR model is:

$$L(\boldsymbol{\mu}|H_t) = \prod_{j=1}^{k} e^{\left[\frac{-\left(T_j - t_{j,n_j}\right)}{\theta \times w^{n_j - 1}}\right]^{\beta_c}} \left[\prod_{i=1}^{n_j} \frac{\beta_c \left(t_{j,i} - t_{j,i-1}\right)^{\beta_c - 1}}{\left(\theta \times w^{i-1}\right)^{\beta_c}} \times e^{-\left[\frac{t_{j,i} - t_{j,i-1}}{\theta \times w^{i-1}}\right]^{\beta_c}}\right] \tag{4.14}$$

Similarly, according to Equation (4.5), in cases which include truncation by failure and/or time, those truncated by failures follow the condition: $T_j = t_{j,n_j}$ and the likelihood function of the Equations (4.12) and (4.14) can be easily adjusted.

4.3 Optimal Preventive Maintenance Policy

Assuming that Y_k are the times of the PM in relation to the start of the system and $Z_k = Y_k - Y_{k-1}$ represents the time elapsed between the $k - th$ PM and the $(k-1) - th$ PM.

The PM policy presented in this article is like the one described in Nakagawa and Mizutani (2009). The objective is to minimize the overall average cost of maintenance. The following aspects will be considered when constructing the model:

- The PMs will occur every Z time units, equally spaced, i.e., $Z_k = Y_k - Y_{k-1} = Z, \forall k$.
- C_{IR} is the cost of IR after a failure, C_{PM} is the cost of PM (PR).
- $N(Z)$ is the number of failures observed within the time Z.
- Repairs after failures will follow the studied mixture distribution and, after each PM, AGAN can be used.

Based on the above conditions, the cost between successive PMs can be defined as $C(Z) = C_{IR} N(Z) + C_{PM}$. As large values of Z imply a large value of $C(Z)$, it makes sense to estimate the cost per unit of time: $H(Z) = C(Z)/Z$. Then, the average cost per unit of time desired, will be:

$$\mathbb{E}[H(Z)] = \mathbb{E}[C(Z)]/Z \tag{4.15}$$

The optimal maintenance policy will be given by the value of Z^* that minimizes the result of Equation (4.15). The difficulty in applying this policy lies in the calculation of $\mathbb{E}\left[N\left(Z\right)\right]$. Thus, we chose to use Monte Carlo simulation to calculate the average number of failures. For this, the following procedure was followed:

1. Use the maximum likelihood estimates $\hat{\beta}_c, \hat{\theta}, \hat{w}_{MR}$ to generate a large number K of systems under the condition of IR mixture.

2. Define a time interval for the analysis of the average cost of PM and discretize this interval.

3. Generate the failures for each system considering the truncation in Z, defined in the previous item.

4. Calculate the cost for each system in Z.

5. Using Equation (4.15), calculate the average cost per unit of time, given the values calculated in the previous item.

6. Repeat the previous procedures for each Z of the vector generated in item 2.

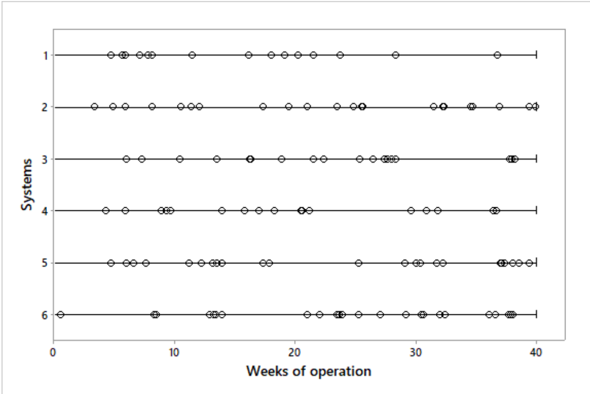
7. Considering the different values of Z, we calculated the value of Z^*, which has the lowest average cost $\mathbb{E}\left[H\left(Z^*\right)\right]$, calculated in item 5.

4.4 Numerical Example

Figure 4.2 schematically represents the failures and repairs in the period from 2021/02 to 2021/12 for concrete mixer trucks used to transport concrete, belonging to a Brazilian company. In this figure, each line corresponds to a sample unit, and each "o" symbol represents a failure time. Thus, six units of equipment were considered, totaling 121 observed failures. Reported times are represented in weeks, and the experiment ended at week 40 for all equipment. This database is different from that presented in Chapter 2.

The fit of the database obtained by the mixture model will be compared with other types of IRs presented in this study. The estimation method chosen will be the one with maximum likelihood, and the Akaike Information Criterion (AIC) will be used for comparing the adjustments of the models. According to Millar (2011), if the difference in AIC between two models is two or more, the model with the smaller AIC is strongly preferred. Note that as all IR models studied: ARA, ARI, QR, and Mixture have the same number of unknown parameters, the selection of models can be based on the highest estimate of the log- likelihood $\left(\hat{l}\right)$. We will also present the results for the model based on the minimum repair.

Figure 4.2: Failure times in weeks of operation for each truck (horizontal lines are trucks and "o" are failures)



Source: Elaborated by the author.

Initially, we evaluated the sample at the time of the first failure of the six equipment. The Anderson-Darling adherence test was used in this case and indicated that we should not reject the hypothesis that the sample comes from a CW, $p - value = 0.654$. Thus, we assume that the CW can be used to model the times of first failures.

Table 4.1 reports the point estimates for the model parameters, the log-likelihood estimates, and the respective AIC values. The values were obtained using macros built in the R software to obtain the maximum likelihood estimators using the Equation (4.5), Equation (4.12), and Equation (4.14) and are available to readers by direct request to the authors. Considering that model classes ARA and ARI are defined by memory m, the table shows the model with the highest log-likelihood within each class. Coincidentally, they both have $m = 7$.

Table 4.1: Point Estimates for several models

	QR	MR	ARA_7	ARI_7	Mixture
$\hat{\beta}_c$	0.949	1.275	1.539	1.621	1.651
$\hat{\theta}$	2.280	3.792	4.112	3.804	4.296
w_R/w_{MR}	0.983	-	0.777	0.833	0.694
$\hat{l}$	-202.950	-200.565	-199.059	-197.745	-195.343
AIC	399.90	397.13	392.12	389.49	384.69

Source: Elaborated by the author.

The AIC criterion indicates that the best fit of the data was achieved using the mixture model. In this model, the standard errors and respective confidence intervals were not obtained via asymptotic theory. As the maintenance effect parameter is limited to the range $[0, 1]$ and the shape (β_c) and scale (θ) parameters are positive and there is no certainty if the sample size is enough to use the asymptotic theory, confidence intervals of the mixture parameters were obtained through the parametric bootstrap. A bootstrap sample can be obtained as follows:

1. Use the maximum likelihood estimates $\hat{\beta}_c, \hat{\theta}, \hat{w}_{MR}$ to generate K systems under the IR condition with a mixture of distributions, keeping the same truncation structure as the original system;

2. From the generated data, obtain the maximum likelihood estimators and denote it by $\bar{\beta}_c^{(b)}, \bar{\theta}^{(b)}, \bar{w}_{MR}^{(b)}$.

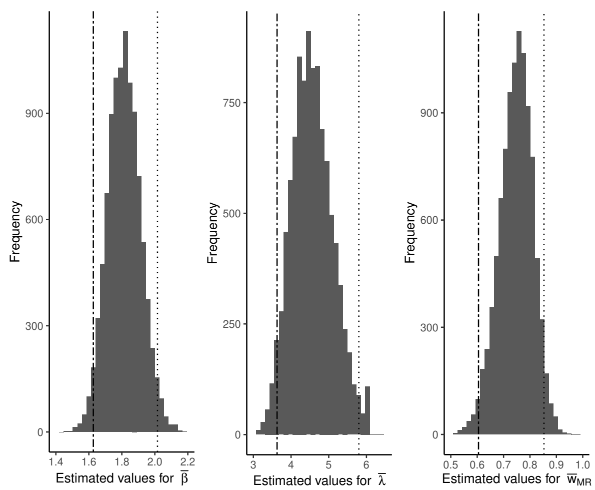
3. Repeat the previous procedures B times, so that, $b = 1, ..., B$.

4. Build the Bias-corrected and accelerated confidence intervals (BC_a) for β_c, θ and w_{MR}. Details of the BC_a procedure can be found in DiCiccio and Efron (1996).

Specifically, the maximum likelihood point estimates and 95% bootstrap confidence intervals estimated for the mixture are: $\hat{\beta}_c = 1.651\,(1.627 - 2.017)$, $\hat{\theta} = 4.296\,(3.629 - 5.799)$ and $\hat{w}_{MR} = 0.694\,(0.605 - 0.853)$.

The Anderson-Darling adherence test to test the normality hypothesis of the bootstrap samples distribution rejects the normality hypothesis for all parameters ($p - value < 0.01$). See Stephens (2017) for details. Figure 4.3 shows a histogram of the Bootstrap sample parameter distributions, with the dashed lines indicating the lower and upper bounds of the bootstrap 95% confidence intervals. Thus, using bootstrap to obtain the confidence intervals is more adequate than using the asymptotic properties of the maximum likelihood estimators.

Figure 4.3: Representation of the distribution of parameters for Bootstrap samples

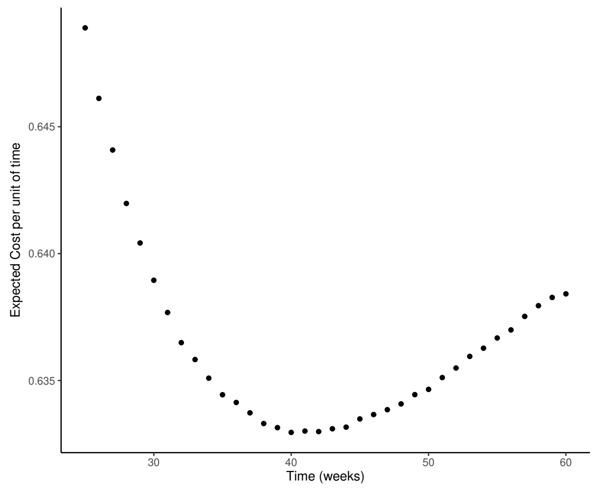


Source: Elaborated by the author.

The estimated parameters $\hat{\beta}_c = 1.651$ and $\hat{w}_{MR} = 0.694$ indicate aging of the studied systems and increasingly frequent failures will occur over time. A PM policy is then recommended to minimize expected costs.

According to data reported by the company, $C_{IR}/C_{PM} = 1/5$. Using the algorithm presented at the end of Section 4.3 with a daily discretization, we obtained the optimal value Z^* equal to 40.7 weeks (approximately 285 days) with $H(Z^*) = 0.6330$. Thus, the optimal PM $H(Z^*)$ indicates that an PR must be performed on each device every $Z^* = 285$ days. Figure 4.4 shows the behavior of $H(Z)$ as a function of time Z.

Figure 4.4: Optimal Preventive Maintenance Policy at Equally Spaced Times



Source: Elaborated by the author.

4.5 Final Remarks

There are several models for evaluating IR systems available in the literature, with an emphasis on ARA, ARI, and QR. This article proposed a new model based on distribution mixtures using a real database.

The parameters of the proposed mixture model were estimated by the maximum likelihood method. Using the AIC criterion, the proposed model was superior to the ARA and ARI models and the QR model.

Considering the difficulty of ensuring that the sample size is sufficient to guarantee good results using the asymptotic theory of maximum likelihood estimators, we chose to build confidence intervals for the parameters of the mixture model using the parametric bootstrap method. The bootstrap distribution did not show normality and thus it is better than the asymptotic results of the maximum likelihood estimator.

The data transfer company wanted maintenance costs to be reduced and PM to be planned. Thus, we indicated that an optimal maintenance policy would be with a PR every 40.7 weeks.

The developed methodology was implemented using the R software and the macros are available to the readers upon direct request to the authors. The general conclusion is that the mixture model is one of the possibilities that should be evaluated by the researcher when choosing the most appropriate methodology. Future studies should determine a single optimal maintenance policy considering that each piece of equipment has a different failure adjustment mechanism such as ARA, ARI, mixture, etc. We understand that in many cases, this approach would be more realistic.

Chapter 5

CONCLUSION

In Chapter 2, a new minimum repair model was proposed, considering that failure times are discrete random variables and follow DW for the first failure. The model using DW was compared with the traditional CW model through an actual database and six cases reported in the literature. It was concluded that the two approaches are similar, however, the new model produced lower AIC values and smaller standard deviations for the shape parameter. Additionally, an optimal maintenance policy was obtained through Markov's chain approach using discrete states and times. This chapter has been published in the journal *IEEE Transactions on Reliability (in press)*.

Optimizing a maintenance policy based on minimizing maintenance costs requires the mean number of failures in a given interval. Obtaining this expectation for models that consider the CW for the first failure is not trivial and requires numerical approximations or assessments through Monte Carlo simulation. Motivated by these difficulties, Chapter 3 introduced a method for calculating the mean number of failures through the Markov chain approach for the models: MR, PR, and BP, considering failure times as discrete variables and DW for the first failure. The point that facilitated the exact calculation of the mean number of failures is that, by fixing a truncation time, the number of failures observed in a system is finite. Seeking possible misconceptions in the methodology, we compared several cases using the Markov chain approach and Monte Carlo simulation study. The results were similar and close, and the results of the Monte Carlo simulation with new replicas fluctuated around the result obtained using the Markov chain. Therefore, we did not find evidence to reject that the proposed methodology is correctly implemented. The method was applied to a database of a Brazilian mining company that wanted to reduce the costs of the sector, including the BP model for its equipment.

In Chapter 4, motivated by an actual database, a new modeling proposal was made based on mixing distributions considering IR for repairable systems. The parameters of the proposed mixing model were estimated by the maximum likelihood method. Using the AIC criterion, the proposed model was better than the ARA, ARI, and QR models. Confidence intervals for the parameters of the mixture model were calculated using the parametric bootstrap method. The overall conclusion is that the mixing model is one of the possibilities that should be assessed by the researcher when choosing the most appropriate methodology.

Some possibilities for continuing with this research are shown:

- to assess the extent of the use of DW by considering other types of IR.
- for the mixing model, in particular:
    1. checking the adjustment of the model to other databases.
    2. study of the statistical properties of model estimators.
    3. to obtain the mean cost per time unit through the mathematical differentiation method $H(.)$.
    4. for the assessment of other maintenance policies considering multiple objectives and different types of failures in such a way as to comprise situations that may occur in practice.

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