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**DATA-DRIVEN STATIC OUTPUT-FEEDBACK CONTROL FOR TIME-DELAY
SYSTEMS**

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**DATA-DRIVEN STATIC OUTPUT-FEEDBACK CONTROL FOR
TIME-DELAY SYSTEMS**

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*«Sans travail toute vie pourrit.
Mais sous un travail sans âme,
la vie étouffe et meurt.»
(Albert Camus)*

RESUMO

Esta dissertação apresenta estruturas de controle por realimentação estática de saída baseadas em dados a tempo discreto para sistemas lineares invariantes no tempo e sistemas com retardo no tempo, sujeitos a retardos de entrada variáveis no tempo. O projeto do controlador proposto é baseado em dados de entrada-estado-saída e formulado por meio de condições LMI dependentes do retardo no tempo, derivadas da teoria de estabilidade de Lyapunov-Krasovskii. Uma vantagem fundamental da metodologia é sua capacidade de realizar a síntese de controle sem exigir conhecimento prévio do modelo matemático do sistema. Isso é alcançado evitando transformações de variáveis, restrições de igualdade ou algoritmos iterativos, graças a uma nova representação baseada em dados para sistemas com retardo no tempo em tempo discreto e à aplicação estratégica do Lema de Finsler. Além disso, são desenvolvidos esquemas de rastreamento de referência baseados em dados para sistemas lineares invariantes no tempo com retardos no tempo no estado e na entrada, garantindo a estabilidade assintótica do erro de rastreamento para referências constantes por partes. A eficácia das abordagens propostas é validada por meio de simulações numéricas e implementação experimental em um sistema de rotor-duplo para rastreamento do ângulo de arfagem, demonstrando desempenho adequado tanto em cenários teóricos quanto práticos.

Palavras-chave: controle baseado em dados; realimentação estática de saída; sistemas com retardo no tempo; rastreamento de referência; desigualdades matriciais lineares.

ABSTRACT

This dissertation presents data-driven static output-feedback (SOF) control frameworks for discrete-time linear time-invariant and time-delay systems with time-varying input delays. The proposed controller design is based on input-state-output data and formulated using delay-dependent LMI conditions derived via Lyapunov-Krasovskii stability theory. A key advantage of the methodology is its ability to achieve control synthesis without requiring prior knowledge of the system dynamics. This is accomplished by avoiding variable transformations, equality constraints, or iterative algorithms, thanks to a novel data-based representation of discrete-time delay systems and the strategic application of Finsler's Lemma. Additionally, data-driven reference-tracking control schemes are developed for both linear time-invariant and time-delay systems, ensuring asymptotic stability of the tracking error for piecewise-constant reference trajectories. The effectiveness of the proposed approaches are validated through numerical simulations and experimental implementation on a twin-rotor system for pitch angle tracking, demonstrating adequate performance in both theoretical and practical settings.

Keywords: data-driven control; static output-feedback; time-delay systems; reference tracking; linear matrix inequalities.

LIST OF FIGURES

Figure 2.1 – Modified block diagram for SOF with reference tracking blocks T_x and T_u , representing the state command matrix and the feedforward matrix, respectively.	26
Figure 2.2 – VTOL system open-loop response.	29
Figure 2.3 – VTOL system with stabilizing SOF control (2.28).	29
Figure 2.4 – TRMS control system schematic [68].	30
Figure 2.5 – TRMS control system.	31
Figure 2.6 – TRMS system with SOF control (2.29) combined with the tracking scheme from Figure 2.1, considering the piecewise constant reference (2.31).. . . .	31
Figure 3.1 – Modified block diagram for SOF with reference tracking blocks T_x and T_u , representing the state command matrix and the feedforward matrix, respectively.	44
Figure 3.2 – System (3.32) open loop response.	47
Figure 3.3 – System (3.32) with SOF control in (3.33).	47
Figure 3.4 – VTOL system SOF control (3.35) combined with the tracking scheme from Figure 3.1.	49
Figure 3.5 – System (2.27) with SOF control in (3.37).	49
Figure 3.6 – TRMS system with SOF control (3.38) combined with the tracking scheme from Figure 3.1.	50

LIST OF ABBREVIATIONS

SOF	Static output-feedback
CPS	Cyber-physical system
LPV	Linear parameter-varying
PID	Proportional integral derivative
TDS	Time-delay system
LKF	Lyapunov-Krasovskii functional
LMI	Linear matrix inequality
LTI	Linear time-invariant
VTOL	Vertical Take-off and Landing
TRMS	Twin Rotor MIMO System

LIST OF SYMBOLS

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The n -dimensional Euclidean space
$\mathbb{R}^{m \times n}$	The set of real matrices of order m by n
$\mathbb{N}$	The set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$
$\mathbb{N}_{\leq k}$	The set $\{1, 2, \dots, k\}$ for a given $k \in \mathbb{N}$
$\mathbb{Z}^+$	The set of positive integers
$\mathbb{S}_+^n$	The set of $n \times n$ symmetric positive definite matrices
$x^\top$ or $X^\top$	Transpose of a vector x or a matrix X
$X^\dagger$	Pseudo-inverse of a matrix X
$\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ or $\begin{bmatrix} A & \star \\ B^\top & C \end{bmatrix}$	Shorthand notation for $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$
$\text{He}(X)$	Shorthand notation for $X + X^\top$, where X is any matrix expression
$P > 0$ (≥ 0)	P is a symmetric positive (semi-)definite matrix
$\text{diag}(X_1, \dots, X_n)$	The block-diagonal matrix whose entries are $X_1, \dots, X_n$, $n \in \mathbb{N}$
I_n	Identity matrix of order n . If the order is clear in the context, it is omitted
$0_{n \times m}, 0_n$	Null matrix of order n by m . If the order is clear in the context, it is omitted
$\mathcal{C}^n$	Space of the functions mapping the set $\{-\max(\bar{h}, \bar{\tau}), \dots, 0\}$ into $\mathbb{R}^n$

CONTENTS

1	INTRODUCTION	14
1.1	Objectives	17
1.2	Dissertation outline and contributions	17
2	DATA-DRIVEN SOF CONTROL FOR LINEAR SYSTEMS SUBJECT TO INPUT DELAY	19
2.1	Problem formulation	19
2.2	Main results	21
2.2.1	Reference tracking scheme	26
2.3	Examples	28
2.3.1	Example 1: VTOL aircraft model	28
2.3.2	Example 2: Pitch angle control of a twin-rotor aerodynamic system – Practical experiment	30
2.4	Final remarks	32
3	DATA-DRIVEN SOF CONTROL FOR LINEAR SYSTEMS WITH STATE AND INPUT DELAYS	33
3.1	Problem formulation	33
3.2	Main results	36
3.2.1	Reference tracking scheme	43
3.3	Examples	45
3.3.1	Example 1: Stabilization	46
3.3.2	Example 2: VTOL aircraft model	47
3.3.3	Example 3: Pitch angle control of a dual-rotor aerodynamic system – Practical experiment	49
3.4	Final remarks	50
4	CONCLUDING REMARKS	51
4.1	Future research	51
4.2	Submitted conference and journal papers	52
	BIBLIOGRAPHY	53
	Appendix	59
	APPENDIX A USEFUL LEMMAS	60

1 INTRODUCTION

The large scale access to computers and communication networks induced a significant change in industrial processes, logistic, and transportation systems, giving birth to what we refer to as Cyber-Physical Systems (CPSs), defined by the integration of the dynamics of physical systems with those among computers and communication networks [1]. Along with the emergence of new technology, the scientific community has devoted significant effort to provide a better understanding of the advantages and limitations of the new frameworks and how to successfully apply control techniques on such systems.

The rise of CPS introduced relevant transformation in the modeling of industrial processes. Driven by the increased flexibility of CPSs, the growth in scale and complexity has made developing reliable models based on first principles and system identification rather unpractical or even impossible, in some cases significantly more time consuming than the controller design itself [2]. On the other hand, the interconnection of sensors, actuators, and controllers through a communication network is responsible for the generation and storage of large amounts of data, providing valuable state information about the process operation and equipment [3]. Data-driven techniques arise as a way to use this abundance of data to monitor critical variables, detect anomalies [4, 5] and diagnose faults [6, 7], make decisions [8, 9], or even analyze stability and design controllers directly from data.

In the field of systems and control, mathematical modeling plays a fundamental role in analysis and controller design. As mentioned above, obtaining a suitable model, *i.e.*, one that correctly shapes the behavior of a dynamical system, is not an easy task. One can possibly derive models from first principles, for instance, from Newton's laws of motion, Kirchhoff's circuit laws, and Euler's equations of motion. However, it requires high expertise as systems become increasingly complex [2]. Also, in some cases of interest, there is no clear physics underlying the system for them to be modeled from principles, such as the stock market [10], social influence and opinion dynamics [11], epidemic spreading [12] and human decision making [13].

In the absence of a reliable system model, this missing information on dynamics must be taken into account by something else [14]. In this dissertation, we refer to substitute the knowledge of the system dynamics by the specific knowledge of structure of the system and process data, where by data, we mean state, input and output measurements from the dynamical system of interest under some specific assumptions on the 'richness' of the data. Among the previously mentioned techniques that involve the use of process data, the object of interest of this dissertation is the data-driven control theory, which is defined by the set of theories and methods in which data collected from the system of interest is leveraged to eliminate the need for explicit model knowledge or an intermediate identification step in system analysis and controller design.

Despite the fact that the literature on data-driven control is still incipient, the first investigations into data-driven control techniques for linear time-invariant systems date back to the 20th century. A notable example is the Ziegler-Nichols method for PID controller tuning [15]. Further early investigation includes relay auto-tuning of PID controllers [16–18], iterative feedback tuning [19], virtual reference feedback tuning [20,21] and model-free adaptive control [22,23].

The renewed interest in data-driven control research is motivated by a result in a seminal paper [24] by Willems *et al.*, widely known as the Willems' Fundamental Lemma. In this work, Willems and his coauthors state that if the input signal of a linear system is persistently exciting, then a window of data from this signal can generate the entire system behavior. In other words, a sufficiently long and rich (*i.e.*, generated by persistently exciting input) trajectory of a linear system can be used to parameterize the entire behavior of the data-generating system. Based on this cornerstone, several automatic control problems were tackled from a data-driven point of view, ranging from optimal control [25–29], predictive control [30–32], control of time-delay systems [33] and control of linear parameter-varying (LPV) systems [34–36]. Refer to [37] for a survey on data-driven control, which inspired by [24], introduces a data-driven parametrization for linear systems via input-state data and also several other feedback control design conditions.

Although the parameterization proposed in [37] offers advantages for the indirect design of controllers, the works of Van Waarde *et al.* [38–40] expanded the possibilities for direct synthesis in broader scenarios. In [38], the authors introduced the concept of data informativity, which refers to conditions to determine whether the data contain sufficient information for the design of specific controllers [14]. In [39,41], two new generalizations of the S-lemma are proposed that, along with the data informativity framework of analysis, enable data-driven state-feedback design for linear systems subject to process disturbance in a broader way. Based on the matrix-constrained relaxation proposed by [41], [42,43] deals with several scenarios of stabilization subject to input saturation. On the other hand, using the S-lemma [39], [44–46] proposes several event-triggered control design strategies.

In recent decades, significant research has also been dedicated to the analysis and stabilization of time-delay systems (TDSs). Time delays, which are inherent in many control theory applications (*e.g.*, communication networks, industrial automation, chemical processes, rolling mills, and hydraulic and pneumatic systems), can severely impact system performance and stability [47]. Epidemics can also be modeled as time-delay systems, as the latest data considered also include delays both in diagnosis and reporting. Derived from the Lyapunov stability theory and its direct method, there are two main methods used for the time-domain stability analysis of time-delay systems: the Krasovskii method of Lyapunov functionals¹ [48] and the Razumikhin method of Lyapunov functions [49]. Both approaches can lead to finite-

¹ A Lyapunov functional (not to be confused with function) accounts for the system's history and its state derivative, rather than just the present state, enabling the stability analysis of time-delay systems.

dimensional LMI conditions for stability analysis and controller design; however, the Krasovskii method can be applied to a wider range of problems and also leads to less conservative delay-dependent results compared to the Razumikhin method [50].

The stability analysis of TDSs via the Lyapunov-Krasovskii theory, which is the one considered in this work, is done by guaranteeing the negative-definiteness of the forward difference of a positive definite Lyapunov-Krasovskii functional (LKF) candidate. The construction of LKFs and the choice of summation inequalities to bound the LKF difference are still an open problem, leading to only sufficient LMI conditions.

Considerable amount of the research effort towards TDSs is aimed towards obtaining tighter upper bounds for the difference of LKFs. In the discrete-time case, functionals based on double- and triple-summation terms were developed to reduce conservativeness. Naturally, it entails the necessity to develop discrete-time variations of bounding techniques for those terms, such as the Jensen inequality [51], Wirtinger-based inequality [52], Bessel-Legendre inequalities [53] and the multiple auxiliary functions-based inequalities [54]. The latter being an extension of Jensen inequality destined to bound double-summations that arise from the forward difference of triple-summation terms of LKFs.

Furthermore, it is well known that delay-independent conditions cannot be used for the stability analysis of unstable (in the sense that A , $A + A_d$ or $A - A_d$ are not Hurwitz) plants by a feedback with delay [50] (see [50, 55] for further details regarding delay-independent stability). In this spirit, the forward difference of double and triple-summation terms of the LKF results in a linear combination of positive functions with inverses of convex delay parameters. As a direct consequence, the development of delay-dependent LMI conditions for feedback stabilization of TDSs relies on the selection of reciprocally convex inequalities to obtain an upper bound of the latter functional terms, namely the reciprocally convex inequality [56], Moon inequality [57], and the delay-dependent reciprocally convex inequality [58].

Although significant attention has been paid to model-based stabilization of time-delay systems, data-driven control of TDSs remains an area that lacks substantial contributions in the literature. In this context, [59], [60], and [61] address different scenarios of data-driven co-design of event-triggered mechanisms and control under network-induced delays. However, despite their relevance to sampled-data systems, none of these works explicitly accounts for the presence of time-delays during the data collection stage. On the other hand, [33] proposes state-feedback design conditions for stabilization, guaranteed cost, and $\mathcal{H}_\infty$ control, along with an algorithm to estimate constant time delays. The state-feedback $\mathcal{H}_\infty$ control of input-delayed systems is covered by [62]. In particular, unlike the previously mentioned works, both [33, 62] are based on Lyapunov-Krasovskii arguments and explicitly consider constant time delays during the open-loop data collection stage.

To the best of the author's knowledge, there are no records in the literature addressing the problem of data-driven static output-feedback design for linear systems subject to time-varying

delays. In practical control systems applications, obtaining complete state information is not always possible. The static output-feedback control provides an elegant, yet easily implemented, method in practical scenarios to tackle the lack of closed-loop full-state information [63]. Meanwhile, data-driven control design based solely on open-loop input-output trajectories remains an open problem. Therefore, like every data-driven control work mentioned in this chapter, the results presented in this dissertation rely on input-state-output data for controller design. However, unlike the approaches previously discussed, this work proposes a static output-feedback strategy where this assumption of state data availability is not required during the closed-loop operation.

1.1 Objectives

This work is concerned with the data-driven static output-feedback stabilization of linear discrete-time delay systems. Hence, motivated by the above discussions, the main objectives of this dissertation are summarized as follows:

1. Using input-state-output data, propose a novel data-driven SOF controller synthesis condition for unknown linear systems with time-varying delays in the control input.
2. Provide a reference tracking scheme for linear systems based on open-loop collected data to guarantee piecewise constant reference tracking.
3. Use input-state-output data to propose a novel data-driven LMI-based delay-dependent condition to design the SOF controller for unknown linear time-delay systems with time-varying input delay.
4. To provide a novel reference tracking scheme for time-delay systems based on open-loop collected data, ensuring piecewise constant reference tracking in the presence of time-varying delays.

1.2 Dissertation outline and contributions

The Dissertation's organization and the related contributions of each chapter are the following:

Chapter 2 concerns the problem of static output-feedback control for discrete-time linear systems subject to time-varying input delay. Open-loop input-state-output data are used to derive Delay-dependent LMI conditions by using the Krasovskii method of Lyapunov functionals along with the application of the Wirtinger-based summation inequality and the delay-dependent Moon inequality, which does not require any prior knowledge of the system dynamics. Moreover, reference-tracking blocks are designed from data to address piecewise-constant reference tracking and guarantee the asymptotic stability of the tracking error. An

application of the results for the pitch angle tracking of a twin-rotor system is conducted to demonstrate the effectiveness of the proposed conditions in real-life scenarios. The results presented in this chapter have been accepted for publication [64].

Chapter 3 provides synthesis conditions for data-driven static output-feedback control of discrete-time delay systems subject to time-varying input delay. The inclusion of triple-summation terms on the Lyapunov-Krasovskii functional, bounded by the Wirtinger-based inequality, multiple auxiliary functions-based inequalities and Moon's inequality, allow obtaining less conservative results for the stabilization of systems subject to input-delays, relying on open-loop input-state-output data to account for the unknown system's dynamics. Such LMI conditions are novel not only in the context of data-driven control, but also in time-delay systems, as they simultaneously address multiple independent time delays. In addition, a novel data-driven reference-tracking input scheme is developed for time-delay systems that addresses piecewise-constant reference trajectories. An experimental implementation of the proposed results on a twin-rotor system for pitch angle tracking depicts robust performance in practical settings. The results presented in this chapter have been submitted to [65].

Finally, Chapter 4 presents concluding remarks and suggestions for future work.

2 DATA-DRIVEN SOF CONTROL FOR LINEAR SYSTEMS SUBJECT TO INPUT DELAY

This chapter proposes a novel data-driven static output-feedback (SOF) control approach for discrete-time linear systems with time-varying input delay. The controller design relies on the open-loop system's input-state-output data along with delay-dependent Linear Matrix Inequality-based conditions derived from the arguments of the Lyapunov-Krasovskii stability theory. A notable distinction of the proposed approach is that the controller gains are obtained without prior knowledge of the system dynamics. Furthermore, a reference-tracking input scheme is designed from data to address piecewise-constant reference tracking and guarantee the asymptotic stability of the tracking error. Simulation and experimental examples are conducted to illustrate the effectiveness of the proposed conditions. The results presented in this chapter have been accepted for publication at [\[64\]](#).

2.1 Problem formulation

In this chapter, discrete-time linear systems subject to time-varying delay in the control input are considered. This class of systems is described as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_{k-h_k}, \quad \forall k \in \mathbb{Z}^+ \\ y_k &= Cx_k, \\ x_k &= \phi_k, \quad k \in \{-\bar{h}, \dots, 0\}, \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output, and $\phi \in \mathbb{C}^n$ represents the initial conditions. The system matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$ are constant (*i.e.*, the system is time-invariant) but unknown. The function h_k denotes the time-varying delay, which satisfies $\underline{h} \leq h_k \leq \bar{h}$, where the lower and upper bounds of the delay ($\underline{h}$ and $\bar{h}$, respectively) are known positive integers.

The measurements obtained from the system (2.1) can be rearranged into a dataset $\mathcal{D} = \{X_1, X_0, U_{\bar{h}}, Y_0\}$ as follows:

$$\begin{aligned} X_1 &= [x(1) \ x(2) \ \cdots \ x(T)], \\ X_0 &= [x(0) \ x(1) \ \cdots \ x(T-1)], \\ U_{\bar{h}} &= [u(-\bar{h}) \ u(-\bar{h}+1) \ \cdots \ u(T-\bar{h}-1)], \\ Y_0 &= [y(0) \ y(1) \ \cdots \ y(T-1)]. \end{aligned} \tag{2.2}$$

The analysis is conducted under the following assumptions about the system:

Assumption 2.1

There exists a sequence of input-state-output measurements from the system (2.1), defined respectively as $\{u_k\}_{k=-\bar{h}}^{T-\bar{h}-1}$, $\{x_k\}_{k=0}^T$, and $\{y_k\}_{k=0}^{T-1}$, where T represents the number of collected samples, and h_k remains constant during the window in which the data is collected.

Remark 2.1

The assumption 2.1 is inspired by [33] and is reasonable in the context of networked control. Considering a scenario where data are obtained from controlled open-loop experiments, it is possible to collect and later transmit the data for processing, maintaining a constant delay.

Assumption 2.2

Considering the dataset $\mathcal{D} = \{X_1, X_0, U_{\bar{h}}, Y_0\}$, first described in (2.2), the following rank condition is satisfied:

$$\text{rank} \begin{bmatrix} X_0 \\ U_{\bar{h}} \end{bmatrix} = n_x + n_u.$$

Remark 2.2

The rank condition presented in Assumption 2.2, which arises from Lemma A.2, is essential to obtain the data-based representation and to achieve it is sufficient to ensure that the input signal $\{u_k\}_{k=-\bar{h}}^{T-\bar{h}-1}$ is persistently exciting of order $n_x + 1$.

Remark 2.3: Duration of the experiment

From the rank condition in Assumption 2.2, it follows that the minimum window requirement T is such that $T \geq n_x + n_u$. However, in this case, the number T refers to the number of columns of each matrix in the dataset $\mathcal{D}$, separately. Due to the mismatch between the collection interval of x_k , and u_{k-h_k} , the minimum duration of the experiment is $T_d \geq n_x + n_u + \bar{h}$.

Given the available dataset $\mathcal{D}$, the following Lemma establishes a data-based representation of the system (2.1).

Lemma 2.1: Data-based representation of LTI systems with input delay

Consider the available dataset $\mathcal{D}$ under Assumption 2.2. The system (2.1) admits a data-based representation defined by

$$\begin{aligned} x_{k+1} &= \hat{X} \begin{bmatrix} x_k \\ u_{k-h_k} \end{bmatrix}, \\ y_k &= \hat{Y} x_k, \end{aligned} \tag{2.3}$$

where

$$\hat{X} = X_1 \begin{bmatrix} X_0 \\ U_{\bar{h}} \end{bmatrix}^\dagger, \quad \hat{Y} = Y_0 X_0^\dagger.$$

Proof. Proof of Lemma 2.1 follows the same rationale presented in [37, Theorem 1]. $\square$

Consider the following static output-feedback control law (SOF):

$$u_k = G^{-1} L y_k, \quad (2.4)$$

where $G \in \mathbb{R}^{n_u \times n_u}$ and $L \in \mathbb{R}^{n_u \times n_y}$ are the gains to be designed. The data-based representation (2.3) and the control law (2.4) allow us to rewrite the closed-loop dynamics as

$$x_{k+1} = \hat{X} \begin{bmatrix} I_{n_x} & 0_{n_x} \\ 0_{n_x} & G^{-1} L \hat{Y} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-h_k} \end{bmatrix}. \quad (2.5)$$

Problem 2.1

Consider the dataset $\mathcal{D}$ under Assumption 2.2 collected from (2.1), design a static output-feedback controller (2.4), being $K = G^{-1} L$ the control gain, such that the origin of the closed-loop system

$$x_{k+1} = A x_k + B K C x_{k-h_k}, \quad (2.6)$$

under Assumption 2.1, is asymptotically stable.

In addition to ensuring the asymptotic stability of the closed-loop system's origin, it is of interest to design a controller capable of guaranteeing the tracking of a piecewise constant reference. The proposed reference tracking method, adapted from [66, Section 8.4.1], is employed to address the static output feedback problem in place of state feedback. Its core idea is to compute a state command matrix, which maps the reference input to a corresponding state reference, along with a feedforward term proportional to the reference that represents the steady-state control signal. Therefore, the problem 2.2 is formally stated as follows.

Problem 2.2

Given a dataset $\mathcal{D}$ collected from (2.1), a static output-feedback law (2.4) and the closed-loop system (2.5), design a reference tracking structure such that the tracking error of the output with respect to a piecewise constant reference $r_k \in \mathbb{R}^{n_y}$ asymptotically converges to zero.

2.2 Main results

In the following Theorem, a sufficient condition for the design of (2.4) is presented.

Theorem 2.1: [64]

Let the positive integers $\underline{h}$ and $\bar{h}$, where $\underline{h} < \bar{h}$, the dataset $\mathcal{D} = \{X_1, X_0, U_{\bar{h}}, Y_0\}$ and a scalar $\epsilon \in \mathbb{R}^+$. If there exist matrices $G \in \mathbb{R}^{n_u \times n_u}$, $L \in \mathbb{R}^{n_u \times n_y}$, $M_1, M_2, M_3 \in \mathbb{R}^{n_x \times n_x}$, $M_4 \in \mathbb{R}^{n_y \times n_x}$, $E_1, E_2, E_3, E_5 \in \mathbb{R}^{n_u \times n_y}$, $E_4 \in \mathbb{R}^{n_y \times n_y}$, $Y_1, Y_2 \in \mathbb{R}^{(8n_x + n_y + n_u) \times 2n_x}$ and positive definite matrices $\mathcal{P} \in \mathbb{S}_+^{3n_x}$, $R_1, R_2, Z_1, Z_2 \in \mathbb{S}_+^{n_x}$, such that the following inequalities hold

$$\begin{bmatrix} \Phi(\underline{h}) & Y_2 \\ \star & -\tilde{Z}_2 \end{bmatrix} < 0, \quad \begin{bmatrix} \Phi(\bar{h}) & Y_1 \\ \star & -\tilde{Z}_2 \end{bmatrix} < 0, \quad (2.7)$$

with

$$\begin{aligned} \Phi(\underline{h}) = & F_1^\top (\mathcal{P} + \Pi_1) F_1 - F_2^\top \mathcal{P} F_2 - H_1^\top \tilde{Z}_1 H_1 + \tilde{R} \\ & + \text{He} \left\{ \Gamma^\top(\underline{h}) \mathcal{P} F_1 \right\} - \text{He} \left\{ \Gamma^\top(\underline{h}) \mathcal{P} F_2 \right\} \\ & - \text{He} \{ Y_1 H_2 + Y_2 H_3 \} + \text{He} \{ \mathcal{U} \mathcal{B} \}, \end{aligned} \quad (2.8)$$

$$\begin{aligned} \Phi(\bar{h}) = & F_1^\top (\mathcal{P} + \Pi_1) F_1 - F_2^\top \mathcal{P} F_2 - H_1^\top \tilde{Z}_1 H_1 + \tilde{R} \\ & + \text{He} \left\{ \Gamma^\top(\bar{h}) \mathcal{P} F_1 \right\} - \text{He} \left\{ \Gamma^\top(\bar{h}) \mathcal{P} F_2 \right\} \\ & - \text{He} \{ Y_1 H_2 + Y_2 H_3 \} + \text{He} \{ \mathcal{U} \mathcal{B} \}, \end{aligned} \quad (2.9)$$

$$\tilde{Z}_1 = \text{diag} (Z_1, 3\gamma(\underline{h}) Z_1), \quad \tilde{Z}_2 = \text{diag} (Z_2, 3Z_2),$$

$$\Pi_1 = \text{diag} (\underline{h}^2 Z_1 + \tilde{h}^2 Z_2, 0_{2n_x}),$$

$$\tilde{R} = \text{diag} (0_{n_x}, R_1, R_2 - R_1, 0_{n_x}, -R_2, 0_{3n_x + n_y + n_u}),$$

$$v_i = \begin{bmatrix} 0_{n_x \times (i-1)n_x} & \mathbf{I}_{n_x} & 0_{n_x \times (8-i)n_x} & 0_{n_x \times (n_y + n_u)} \end{bmatrix},$$

$$\bar{v} = \begin{bmatrix} 0_{n_u \times (8n_x + n_y)} & \mathbf{I}_{n_u} \end{bmatrix},$$

$$\hat{v} = \begin{bmatrix} 0_{n_y \times 8n_x} & \mathbf{I}_{n_y} & 0_{n_y \times n_u} \end{bmatrix},$$

$$\Gamma(\bar{h}) = \begin{bmatrix} v_2 \\ \underline{h}v_6 \\ \tilde{h}v_7 \end{bmatrix}, \quad \Gamma(\underline{h}) = \begin{bmatrix} v_2 \\ \underline{h}v_6 \\ \tilde{h}v_8 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} v_1 - v_2 \\ v_6 - v_3 \\ v_7 + v_8 - v_4 - v_5 \end{bmatrix}, \quad H_3 = \begin{bmatrix} v_4 - v_5 \\ v_4 + v_5 - 2v_8 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0_{n_x \times (8n_x + n_y + n_u)} \\ v_6 - v_2 \\ v_7 + v_8 - v_3 - v_4 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} v_2 - v_3 \\ v_2 + v_3 - 2v_6 \end{bmatrix}, \quad H_2 = \begin{bmatrix} v_3 - v_4 \\ v_3 + v_4 - 2v_7 \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} \hat{X} \begin{bmatrix} v_2^\top & \bar{v}^\top \end{bmatrix}^\top - v_1 \\ \hat{Y} v_4 - \hat{v} \\ L\hat{v} - G\bar{v} \end{bmatrix},$$

$$\mathcal{U} = \begin{bmatrix} M_1 & \Psi E_1 & \Psi \\ M_2 & \Psi E_2 & \epsilon \Psi \\ 0_{n_x \times n_x} & 0_{n_x \times n_y} & 0_{n_x \times n_u} \\ M_3 & \Psi E_3 & \epsilon \Psi \\ 0_{4n_x \times n_x} & 0_{4n_x \times n_y} & 0_{4n_x \times n_u} \\ M_4 & E_4 & 0_{n_y \times n_u} \\ 0_{n_u \times n_x} & E_5 & I_{n_u} \end{bmatrix}, \quad \Psi = \hat{X} \begin{bmatrix} 0_{n_x \times n_u} \\ I_{n_u} \end{bmatrix},$$

where $i \in \mathbb{N}_{\leq 8}$, $\tilde{h} \triangleq \bar{h} - \underline{h}$ and $\gamma(\underline{h}) = 1$ (if $\underline{h} = 1$) otherwise $\gamma(\underline{h}) = (\underline{h} + 1) / (\underline{h} - 1)$ (if $\underline{h} > 1$).

Then, the origin of the closed-loop system (2.5) is asymptotically stable for any integer time-varying delay $h_k \in \{\underline{h}, \dots, \bar{h}\}$ via the static output-feedback controller:

$$u_k = G^{-1} L y_k.$$

Proof. Consider the Lyapunov–Krasovskii Functional (LKF) candidate given by

$$V(\phi_k) = \sum_{i=1}^3 V_i(\phi_k), \quad (2.10)$$

where

$$\begin{aligned} V_1 &= \varpi_k^\top \mathcal{P} \varpi_k, \\ V_2 &= \sum_{i=k-\underline{h}}^{k-1} x_i^\top R_1 x_i + \sum_{i=k-\bar{h}}^{k-\underline{h}-1} x_i^\top R_2 x_i, \\ V_3 &= \underline{h} \sum_{i=-\underline{h}+1}^0 \sum_{j=k+i}^k \eta_j^\top Z_1 \eta_j + \tilde{h} \sum_{i=-\bar{h}+1}^{-\underline{h}} \sum_{j=k+i}^k \eta_j^\top Z_2 \eta_j, \end{aligned}$$

being

$$\varpi_k = \left(x_k, \sum_{i=k-\underline{h}}^{k-1} x_i, \sum_{i=k-\bar{h}}^{k-\underline{h}-1} x_i \right),$$

and $\eta_i = x_i - x_{i-1}$. Notice that (2.10) is positive definite, since $\mathcal{P} > 0$, $R_1 > 0$, $R_2 > 0$, $Z_1 > 0$ e $Z_2 > 0$.

We define

$$\zeta_k = (x_{k+1}, x_k, x_{k-\underline{h}}, x_{k-h_k}, x_{k-\bar{h}}, \nu_{1_k}, \nu_{2_k}, \nu_{3_k}, y_{k-h_k}, u_{k-h_k}), \quad (2.11)$$

with

$$\begin{aligned}\nu_{1_k} &= \frac{1}{\underline{h}+1} \sum_{i=k-\underline{h}}^k x_i, \quad \nu_{2_k} = \frac{1}{h_k-\underline{h}+1} \sum_{i=k-h_k}^{k-\underline{h}} x_i, \\ \nu_{3_k} &= \frac{1}{\bar{h}-h_k+1} \sum_{i=k-\bar{h}}^{k-h_k} x_i.\end{aligned}$$

Considering the fact that

$$\begin{aligned}\varpi_k &= (F_2 + \Gamma(h_k)) \zeta_k, \\ \varpi_{k+1} &= (F_1 + \Gamma(h_k)) \zeta_k,\end{aligned}$$

with

$$\Gamma(h_k) = \begin{bmatrix} v_2 \\ \underline{h}v_6 \\ (h_k - \underline{h})v_7 + (\bar{h} - h_k)v_8 \end{bmatrix},$$

then $\Delta V_1(\phi_k)$ can be rewritten as

$$\begin{aligned}\Delta V_1(\phi_k) &= \zeta_k^\top \left[(F_1 + \Gamma(h_k))^\top \mathcal{P} (F_1 + \Gamma(h_k)) \right. \\ &\quad \left. - (F_2 + \Gamma(h_k))^\top \mathcal{P} (F_2 + \Gamma(h_k)) \right] \zeta_k.\end{aligned}$$

Notice that the variation of V_2 leads to:

$$\begin{aligned}\Delta V_2(\phi_k) &= x_k^\top R_1 x_k + x_{k-\underline{h}}^\top (R_2 - R_1) x_{k-\underline{h}} \\ &\quad - x_{k-\bar{h}}^\top R_2 x_{k-\bar{h}}\end{aligned}$$

which can be rewritten as:

$$\Delta V_2(\phi_k) = \zeta_k^\top \tilde{R} \zeta_k.$$

Furthermore, the variation of V_3 results in:

$$\begin{aligned}\Delta V_3(\phi_k) &= \eta_{k+1}^\top \left(\underline{h}^2 Z_1 + \tilde{h}^2 Z_2 \right) \eta_{k+1} \\ &\quad - \underline{h} \sum_{i=k-\underline{h}+1}^k \eta_i^\top Z_1 \eta_i - \tilde{h} \sum_{i=k-\bar{h}+1}^{k-\underline{h}} \eta_i^\top Z_2 \eta_i,\end{aligned} \tag{2.12}$$

that can be rewritten as

$$\begin{aligned}\Delta V_3(\phi_k) &= \eta_{k+1}^\top \left(\underline{h}^2 Z_1 + \tilde{h}^2 Z_2 \right) \eta_{k+1} - \underline{h} \sum_{i=k-\underline{h}+1}^k \eta_i^\top Z_1 \eta_i \\ &\quad - \tilde{h} \sum_{i=k-\bar{h}+1}^{k-h_k} \eta_i^\top Z_2 \eta_i - \tilde{h} \sum_{i=k-h_k+1}^{k-\underline{h}} \eta_i^\top Z_2 \eta_i.\end{aligned} \tag{2.13}$$

Based on Lemma A.4, an upper bound for $\Delta V(\phi_k)$ is obtained:

$$\begin{aligned} \Delta V(\phi_k) &\leq \zeta_k^\top \left[(F_1 + \Gamma(h_k))^\top \mathcal{P} (F_1 + \Gamma(h_k)) \right. \\ &\quad - (F_2 + \Gamma(h_k))^\top \mathcal{P} (F_2 + \Gamma(h_k)) + \tilde{R} \\ &\quad \left. + F_1^\top \Pi_1 F_1 - H_1^\top \tilde{Z}_1 H_1 \right] \zeta_k \\ &\quad - \zeta_k^\top \begin{bmatrix} H_2 \\ H_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\tilde{h}}{h_k - \underline{h}} \tilde{Z}_2 & 0_{2n_x} \\ 0_{2n_x} & \frac{\tilde{h}}{\bar{h} - h_k} \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} H_2 \\ H_3 \end{bmatrix} \zeta_k. \end{aligned} \quad (2.14)$$

Thereafter, by applying Lemma A.3, with $\varrho = \frac{(h_k - \underline{h})}{\bar{h}}$, we obtain:

$$\Delta V(\phi_k) \leq \zeta_k^\top \Xi(h_k) \zeta_k, \quad (2.15)$$

with

$$\begin{aligned} \Xi(h_k) &= F_1^\top (\mathcal{P} + \Pi_1) F_1 + \text{He} \{ \Gamma^\top(h_k) \mathcal{P} F_1 \} - F_2^\top \mathcal{P} F_2 \\ &\quad - \text{He} \{ \Gamma^\top(h_k) \mathcal{P} F_2 \} + \tilde{R} - \text{He} \{ Y_1 H_2 + Y_2 H_3 \} \\ &\quad - H_1^\top \tilde{Z}_1 H_1 + \varrho Y_1 \tilde{Z}_2^{-1} Y_1^\top + (1 - \varrho) Y_2 \tilde{Z}_2^{-1} Y_2^\top. \end{aligned} \quad (2.16)$$

Considering

$$\tilde{\mathcal{B}} = \begin{bmatrix} \hat{X} \begin{bmatrix} v_2^\top & \bar{v}^\top \end{bmatrix}^\top - v_1 \\ \hat{Y} v_4 - \hat{v} \\ G^{-1} L \hat{v} - \bar{v} \end{bmatrix}, \quad (2.17)$$

which yields $\tilde{\mathcal{B}} \zeta_k = 0$ and

$$\tilde{\mathcal{U}} = \begin{bmatrix} M_1 & \Psi E_1 & \Psi G \\ M_2 & \Psi E_2 & \epsilon \Psi G \\ 0_{n_x \times n_x} & 0_{n_x \times n_y} & 0_{n_x \times n_u} \\ M_3 & \Psi E_3 & \epsilon \Psi G \\ 0_{4n_x \times n_x} & 0_{4n_x \times n_y} & 0_{4n_x \times n_u} \\ M_4 & E_4 & 0_{n_y \times n_u} \\ 0_{n_u \times n_x} & E_5 & G \end{bmatrix}, \quad (2.18)$$

by the arguments of Lemma A.6, the negative definiteness of (2.15) is ensured if

$$\Xi(h_k) + \tilde{\mathcal{U}} \tilde{\mathcal{B}} + \tilde{\mathcal{B}}^\top \tilde{\mathcal{U}}^\top < 0. \quad (2.19)$$

By applying Schur Complement in (2.19), one obtain (2.7), which is affine and, consequently, convex relative to h_k . Therefore, the feasibility of the LMI condition in (2.7) implies that the time-difference of the LKF (2.15) is negative definite for all $h_k \in \{\underline{h}, \dots, \bar{h}\}$. $\square$

2.2.1 Reference tracking scheme

Integral control remains the predominant paradigm for eliminating steady-state tracking errors in control systems. Nevertheless, this approach introduces additional constraints that might complicate the attainment of feasible solutions, particularly in scenarios requiring high-performance bandwidth. Integral control inherently modifies system dynamics, often amplifying design complexity and imposing inherent limitations on achievable bandwidth. Therefore, a viable alternative is to use feedforward compensation strategies [66] as an alternative to integral control, using their ability to improve transient performance and overcome stability challenges associated with traditional error-integration methods.

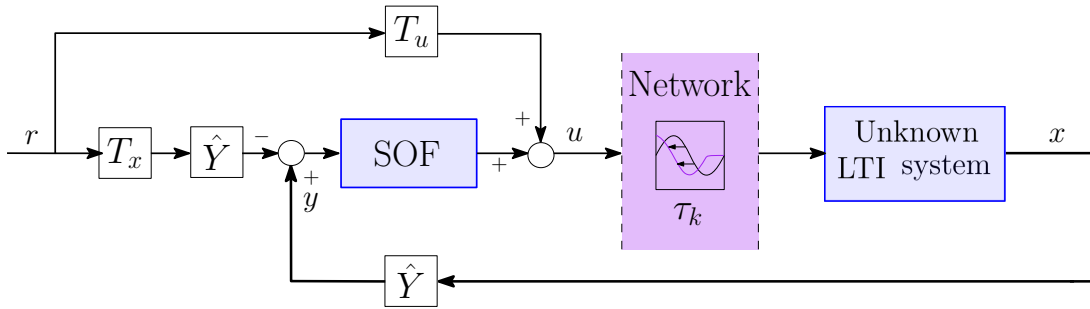


Figure 2.1 – Modified block diagram for SOF with reference tracking blocks T_x and T_u , representing the state command matrix and the feedforward matrix, respectively.

Figure 2.1 illustrates a schematic of the combination of an unknown linear system with communication delay controlled by the static output-feedback (2.4) obtained via Theorem 2.1 with the reference tracking structure. The resulting closed-loop dynamics of this schematic are given by

$$x_{k+1} = \hat{X} \begin{bmatrix} I_{n_x} & 0_{n_x} & 0_{n_x \times n_y} \\ 0_{n_u \times n_x} & \hat{J} & (T_u - \hat{J}T_x) \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-h_k} \\ r_{k-h_k} \end{bmatrix} \quad (2.20)$$

where $\hat{J} = G^{-1}LY$, and $T_x \in \mathbb{R}^{n_x \times n_y}$ and $T_u \in \mathbb{R}^{n_u \times n_y}$ are, respectively, the state command matrix and the feedforward matrix of the reference tracking blocks. The design conditions for these gains are presented in the proposition below.

Proposition 2.1

Consider the system (2.3) together with the static output-feedback law given in (2.4). Given a dataset $\mathcal{D} = \{X_1, X_0, U_h, Y_0\}$, the inclusion of matrices T_x and T_u as depicted in Figure 2.1, obtained through:

$$\begin{bmatrix} T_x \\ T_u \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 0_{n_x \times n_y} \\ I_{n_y} \end{bmatrix}, \quad (2.21)$$

where

$$\Theta = \begin{bmatrix} \hat{X} - [I_{n_x} & 0_{n_x \times n_u}] \\ \hat{Y} & 0_{n_y \times n_u} \end{bmatrix},$$

and with the resulting form in (2.20), ensures that the tracking error for a piecewise constant reference will asymptotically converge to zero if $\Theta \in \mathbb{R}^{(n_x+n_y) \times (n_x+n_u)}$ is non-singular.

Proof. From the schematic in Figure 2.1, $u = K\hat{Y}(x - T_x r) + T_u r$. The following constraint is imposed:

$$\hat{Y}T_x r = r \implies \hat{Y}T_x = I_{n_y}, \quad (2.22)$$

so that it is possible to rewrite $u = K(y - r) + T_u r$. T_x and T_u are designed so that when $k \rightarrow \infty$, the output satisfies $y \rightarrow r$. This can be achieved with an u_{eq} which, through the state reference defined as $x_r = T_x r$, drives the state to $x_{eq} = x_r$. Furthermore, ensuring that the tracking error $y - r = e \rightarrow 0$, the asymptotic stability of Theorem 2.1 guarantees that $u \rightarrow u_{eq}$ is satisfied, even in the presence of time-delay. Thus, the following equalities will hold:

$$\begin{aligned} x_{eq} &= x_r = T_x r, \\ u_{eq} &= K(y_{eq} - r) + T_u r \implies u_{eq} = T_u r. \end{aligned} \quad (2.23)$$

Moreover, considering (2.3) in steady state, we have

$$x_{eq} = \hat{X} \begin{bmatrix} x_{eq} \\ u_{eq} \end{bmatrix} \implies 0 = (\hat{X} - [I_{n_x} \quad 0_{n_x \times n_u}]) \begin{bmatrix} x_{eq} \\ u_{eq} \end{bmatrix}. \quad (2.24)$$

Assuming $r \neq 0$ and substituting x_{eq} and u_{eq} as in (2.23), we have

$$0 = (\hat{X} - [I_{n_x} \quad 0_{n_x \times n_u}]) \begin{bmatrix} T_x \\ T_u \end{bmatrix}. \quad (2.25)$$

Finally, combining (2.22) and (2.25), we obtain the following system of linear equations:

$$\begin{bmatrix} \hat{X} - [I_{n_x} & 0_{n_x \times n_u}] \\ \hat{Y} & 0_{n_y \times n_u} \end{bmatrix} \begin{bmatrix} T_x \\ T_u \end{bmatrix} = \begin{bmatrix} 0_{n_x \times n_y} \\ I_{n_y} \end{bmatrix}, \quad (2.26)$$

whose solution is (2.21). $\square$

Remark 2.4

In the case where Θ is singular and a unique solution to (2.25) does not exist, the validity of Theorem 2.1 remains unaffected. Under such conditions, one may resort to alternative tracking control schemes that exploit the system's stabilization properties to ensure acceptable performance.

2.3 Examples

In this section, examples are provided to emphasize the effectiveness of the proposed static output feedback approaches. Example 1 consists of the stabilization of a VTOL (Vertical Take-Off and Landing) aircraft model. In Example 2, tests were conducted on a real dual-rotor system, a simplified helicopter model.

We emphasize that the system matrices for the experiments conducted in this section are not used for controller design itself, but only for simulations to acquire data and validate the design. All design conditions were implemented in MATLAB R2023b using the YALMIP parser and the MOSEK solver.

2.3.1 Example 1: VTOL aircraft model

Consider the following linearized model of a VTOL helicopter from [67]. The state space model, discretized for a sampling period of $0.1s$, is given by:

$$A = \begin{bmatrix} 0.9963 & 0.0027 & 0.0019 & -0.0048 \\ 0.0048 & 0.8990 & 0.0002 & -0.4021 \\ 0.0100 & 0.3505 & 0.9293 & 0.1420 \\ 0 & 0 & 0.1000 & 1 \end{bmatrix}, \quad (2.27)$$

$$B = \begin{bmatrix} 0.0442 & 0.0176 \\ 0.0318 & -0.7592 \\ -0.5520 & 0.4490 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

with x_1, x_2, x_3 , and x_4 representing, respectively, the horizontal and vertical velocities (in knots), the pitch rate (degrees/s) and the pitch angle (degrees). The inputs u_1 and u_2 represent the collective and longitudinal cyclic controls, respectively.

To satisfy the condition in Assumption 2.2, which enables the use of the data-based representation in Lemma 2.1, 8 samples ($T = 8$) of input-state-output data were collected from the system, considering a constant input delay of $\bar{h} = 3$, which is the maximum upper bound supported by the system, and applying white noise as input.

In this example, the goal is to design a static output feedback (SOF) controller (2.4) that stabilizes the system (2.27) for a maximum time delay. Considering $\underline{h} = 1$ with $\epsilon = 0.1$,

the static output feedback controller designed using Theorem 2.1 was able to stabilize the system for $\bar{h} = 3$. The gains for the SOF control in the form of (2.4) were obtained as follows:

$$G = \begin{bmatrix} 1.4338 & 1.6481 \\ 0.1614 & 1.7958 \end{bmatrix}, \quad L = \begin{bmatrix} -0.2147 & 1.5694 \\ -0.2286 & 0.9376 \end{bmatrix}. \quad (2.28)$$

Figures 2.2 and 2.3 illustrate, respectively, the open- and closed-loop system response with the designed controller considering the initial conditions $\phi_0 = x_0 = [0.32 \ 0.42 \ 0.50 \ 0.09]^\top$ and a time-varying delay $h_k = \text{round}(2 + \sin(\frac{k\pi}{5}))$. As expected, the controller ensures the stability of the closed-loop system.

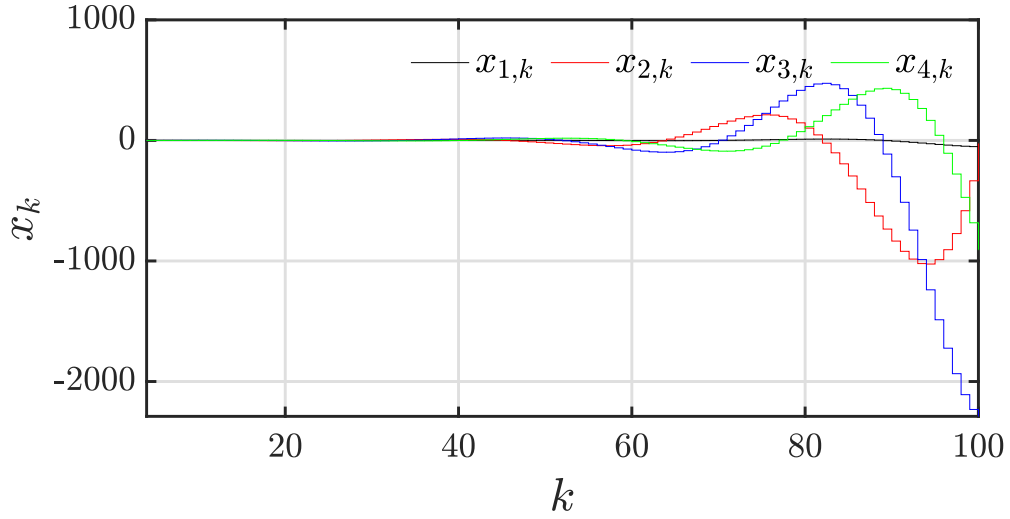


Figure 2.2 – VTOL system open-loop response.

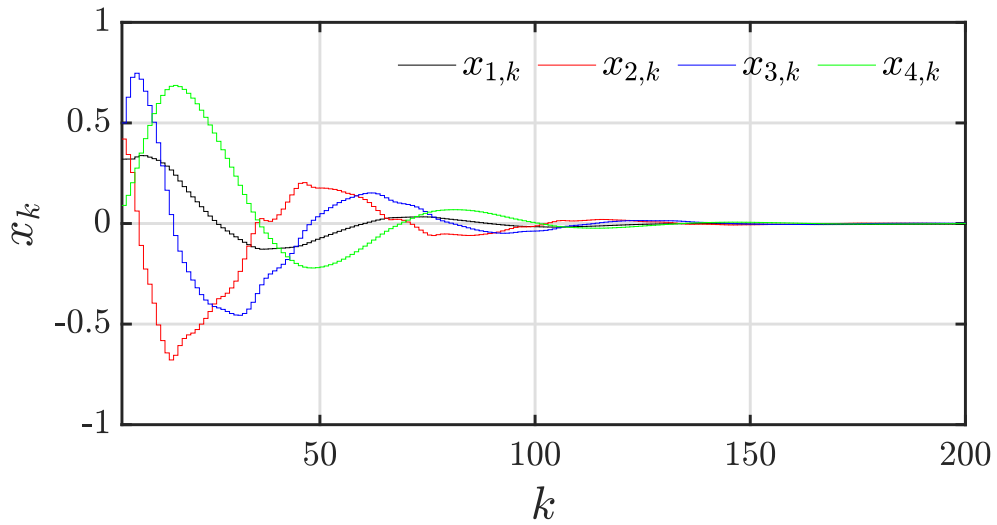


Figure 2.3 – VTOL system with stabilizing SOF control (2.28).

2.3.2 Example 2: Pitch angle control of a twin-rotor aerodynamic system – Practical experiment

The Twin Rotor MIMO System (TRMS), manufactured by Feedback Instruments Ltd. [68], is an educational plant designed to model the dynamics of a helicopter in a simplified form. It consists of two rotors mounted on a tower with a counterweight, allowing the control of pitch and yaw angles by varying the rotational speeds of the rotors. The TRMS captures the essence of helicopter dynamics, with the key difference being that, in a real helicopter, the attitude is controlled by varying the blade angles rather than the rotor speeds.

The pitch and yaw angles are measured using digital encoders and transmitted to a local PC via a data acquisition board. Using Simulink, the control signal is calculated and sent to the TRMS as a voltage signal applied to the terminals of the DC motors driving the rotors. A schematic provided by the manufacturer that illustrates the control system in question is presented in Figure 2.4. Figure 2.5 depicts the twin-rotor setup employed in this example.

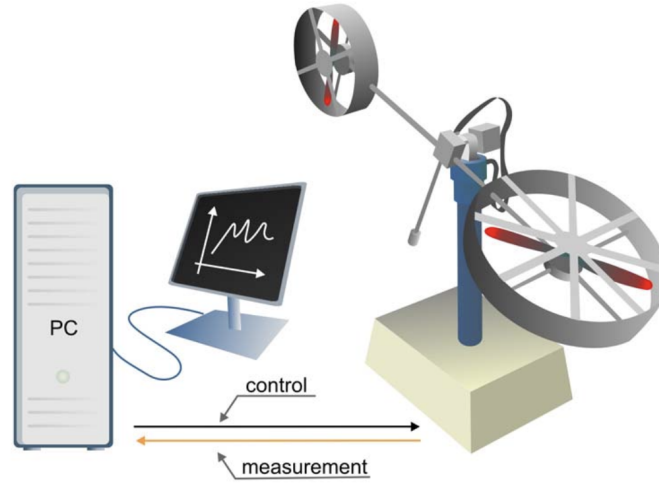


Figure 2.4 – TRMS control system schematic [68].

Considering the pitch angle tracking problem for a sampling period $T_s = 0.001s$, with a constant delay of $\bar{h} = 2000$, ($T = 6000$) input-state-output data samples were collected while operating the system in open-loop with an input $u_k = 0.6U_k + \omega_k$ rad, with $\omega_k \sim \mathcal{N}(0, 0.01)$ and U_k a unit step. The larger number of collected samples allow us to capture the essence of the dynamics under disturbances and measurement noise. The choice of the input amplitude and noise covariance was motivated by the idea of oscillating the nonlinear system close to a region of interest, where the pitch rotor is horizontal. From Theorem 2.1, considering $\underline{h} = 1$, $\bar{h} = 2000$, and $\epsilon = 0.001$, the gains for the SOF control in the form of (2.4) are given by:

$$G = 25.0157, \quad L = -0.0727. \quad (2.29)$$

For pitch angle tracking, Proposition 2.1 provides:

$$T_x = \begin{bmatrix} 1 & 0.0198 & 2.6490 \end{bmatrix}^\top, \quad T_u = 2.7219. \quad (2.30)$$

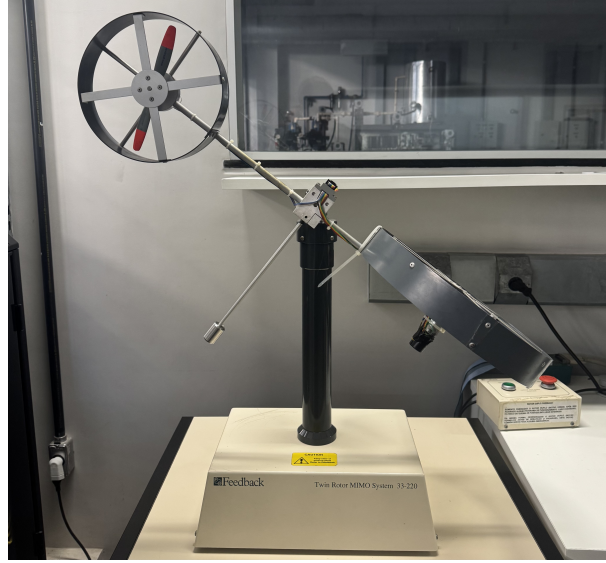


Figure 2.5 – TRMS control system.

Figure 2.6 illustrates the tracking performance with the reference tracking block scheme for the piecewise constant signal:

$$r_k = 0.4U_k + 0.2U(k - 4 \cdot 10^4) \text{ rad.} \quad (2.31)$$

The presence of measurement noise, disturbances, and the nonlinear nature of the TRMS necessitate collecting a larger number of samples to achieve satisfactory results. Nevertheless, a tiny tracking error remains in the closed-loop response.

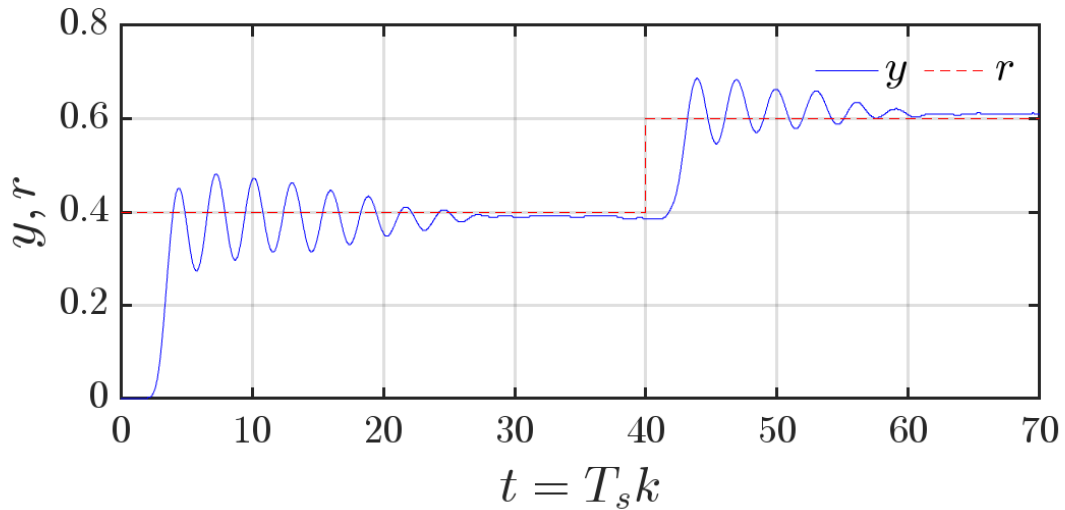


Figure 2.6 – TRMS system with SOF control (2.29) combined with the tracking scheme from Figure 2.1, considering the piecewise constant reference (2.31)..

2.4 Final remarks

This chapter proposed a data-driven synthesis of static output feedback controllers for discrete-time linear systems with time-varying input delay. Additionally, a tracking block structure was proposed to ensure the tracking of piecewise constant references. The main distinction of the proposed methods lies in achieving stabilization in the presence of time delay in the input signal without requiring knowledge of the system matrices. To illustrate the potential and effectiveness of the proposed conditions, two case studies were presented: a numerical example involving the stabilization of a VTOL aircraft model and a real-world application of reference tracking for pitch angle control in a dual-rotor aerodynamic system.

3 DATA-DRIVEN SOF CONTROL FOR LINEAR SYSTEMS WITH STATE AND INPUT DELAYS

This chapter introduces a data-driven static output-feedback (SOF) control framework for discrete-time linear systems subject to time-varying state and input delays. The proposed controller synthesis leverages input-state-output data acquired from the open-loop system, integrated with delay-dependent data-based Linear Matrix Inequality (LMI) conditions derived using Lyapunov-Krasovskii stability theory. A salient feature of this methodology is its ability to compute controller gains without prior knowledge of the system dynamics, thereby eliminating the dependence on explicit mathematical models. Furthermore, a data-driven reference-tracking input scheme is developed for time-delay systems, addressing piecewise-constant reference trajectories while guaranteeing asymptotic stability of the tracking error. The efficacy of the proposed approach is rigorously validated through numerical simulations and also experimental implementation on a twin-rotor system for pitch angle tracking, demonstrating robust performance in both theoretical and practical settings. The results presented in this chapter have been submitted to [65].

3.1 Problem formulation

Consider the linear discrete-time system with state and input time-varying delay. Such class of systems is described by:

$$\begin{aligned} x_{k+1} &= Ax_k + A_d x_{k-h_k} + Bu_{k-\tau_k}, \quad \forall k \in \mathbb{Z}^+ \\ y_k &= Cx_k, \\ x_k &= \phi_k, \quad k \in \{-\max(\bar{h}, \bar{\tau}), \dots, 0\}, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input vector, $y \in \mathbb{R}^{n_y}$ is the output vector and ϕ is the initial condition. The system matrices $A \in \mathbb{R}^{n_x \times n_x}$, $A_d \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$ are unknown, but constant (i.e., the system is time-invariant). The functions τ_k, h_k denote time-varying delays that satisfy $\underline{h} \leq h_k \leq \bar{h}$ and $\underline{\tau} \leq \tau_k \leq \bar{\tau}$, where the lower bounds $(\underline{h}, \underline{\tau})$ and the upper bounds $(\bar{h}, \bar{\tau})$ are known positive integers. Notice that an upper bound $\max(\bar{h}, \bar{\tau}) \in \mathbb{Z}_+$ is given for state and input delays h_k and τ_k , respectively.

The analysis is conducted under the following hypothesis regarding the system:

Assumption 3.1

There exists a sequence of input-state-output measurements from (3.1), defined respectively as $\{u_k\}_{k=-\bar{\tau}}^{T-\bar{\tau}-1}$, $\{x_k\}_{k=-\bar{h}}^T$, and $\{y_k\}_{k=0}^{T-1}$ where T represents the number of collected samples and both h_k and τ_k remain constant during the data collection window.

Remark 3.1

Similarly to Assumption 2.1, Assumption 3.1 is inspired by [33] and is reasonable when it comes to Networked Control Systems, since considering a scenario in which data are obtained in open-loop controlled experiments, it is completely possible to collect samples and transmit them later for processing, forcing a constant transmission delay on purpose.

The measurements obtained from the system (3.1) can be rearranged in a dataset $\tilde{\mathcal{D}} = \{X_1, X_0, X_{\bar{h}}, U_{\bar{\tau}}, Y_0\}$, namely:

$$\begin{aligned} X_1 &= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}, \\ X_0 &= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \\ X_{\bar{h}} &= \begin{bmatrix} x(-\bar{h}) & x(-\bar{h}+1) & \cdots & x(T-\bar{h}-1) \end{bmatrix}, \\ U_{\bar{\tau}} &= \begin{bmatrix} u(-\bar{\tau}) & u(-\bar{\tau}+1) & \cdots & u(T-\bar{\tau}-1) \end{bmatrix}, \\ Y_0 &= \begin{bmatrix} y(0) & y(1) & \cdots & y(T-1) \end{bmatrix}. \end{aligned} \quad (3.2)$$

Assumption 3.2

Considering the dataset $\tilde{\mathcal{D}} = \{X_1, X_0, X_{\bar{h}}, U_{\bar{\tau}}, Y_0\}$, first described in (3.2), the following rank condition is satisfied:

$$\text{rank} \begin{bmatrix} X_0 \\ X_{\bar{h}} \\ U_{\bar{\tau}} \end{bmatrix} = 2n_x + n_u.$$

Remark 3.2

The rank condition shown in Assumption 3.2, which also arises from Willems' Fundamental Lemma, referenced as Lemma A.2, is essential for the data-driven representation and to achieve it is sufficient to ensure that the input signal $\{u_k\}_{k=-\bar{\tau}}^{T-\bar{\tau}-1}$ is persistently exciting of order $2n_x + 1$.

Remark 3.3: Duration of the experiment

From the rank condition in Assumption 3.2, it follows that the minimum window requirement T is such that $T \geq 2n_x + n_u$. However, in this context, the number T refers to the number of columns of each matrix in the dataset $\tilde{\mathcal{D}}$, individually. When considering x_{k+1} , x_k , and x_{k-h_k} as a single entity, due to the mismatch between the collection interval, the minimum number of samples collected from the state vector x_k will be $T_x \geq 2n_x + n_u + \min(\bar{h}, T)$ and the minimum duration of the experiment $T_d \geq 2n_x + n_u + \max(\bar{h}, \bar{\tau})$.

Given the available dataset $\tilde{\mathcal{D}}$, the following Lemma establishes a new data-based representation of the system (3.1).

Lemma 3.1: Data-based representation of time-delay systems with input delay

Consider the dataset $\tilde{\mathcal{D}}$ under Assumption 3.2. The system (3.1) admits a data-based representation defined by

$$\begin{aligned} x_{k+1} &= \hat{X} \begin{bmatrix} x_k \\ x_{k-h_k} \\ u_{k-\tau_k} \end{bmatrix}, \\ y_k &= \hat{Y} x_k, \end{aligned} \quad (3.3)$$

where

$$\hat{X} = X_1 \begin{bmatrix} X_0 \\ X_{\bar{h}} \\ U_{\bar{\tau}} \end{bmatrix}^\dagger, \quad \hat{Y} = Y_0 X_0^\dagger.$$

Proof. Proof of Lemma 3.1 follows the same rationale presented in [37, Theorem 1]. $\square$

Consider the following static output-feedback control law (SOF):

$$u_k = G^{-1} L y_k, \quad (3.4)$$

where $G \in \mathbb{R}^{n_u \times n_u}$ and $L \in \mathbb{R}^{n_u \times n_y}$ are the control gains to be designed. The data-based representation (3.3) and control law (3.4) allow us to rewrite the closed-loop dynamics as

$$x_{k+1} = \hat{X} \begin{bmatrix} I_{n_x} & 0_{n_x} & 0_{n_x} \\ 0_{n_x} & I_{n_x} & 0_{n_x} \\ 0_{n_u \times n_x} & 0_{n_u \times n_x} & G^{-1} L \hat{Y} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-h_k} \\ x_{k-\tau_k} \end{bmatrix}. \quad (3.5)$$

Based on the previous definitions, the first problem addressed in this work is stated as follows.

Problem 3.1

Consider the dataset $\tilde{\mathcal{D}}$ under Assumption 3.2 collected from (3.1), design a static output-feedback controller (3.4) being $K = G^{-1} L$ the control gain, such that the origin of the closed-loop system

$$x_{k+1} = A x_k + A_d x_{k-h_k} + B K C x_{k-\tau_k}, \quad (3.6)$$

under Assumption 3.1, is asymptotically stable.

In addition to guaranteeing the asymptotic stability of the closed-loop system's origin, it is also desirable to design a control law that ensures the tracking of a piecewise constant reference. The proposed reference tracking method consists of an adaptation of the framework proposed by [66, Section 8.4.1] to time-delay systems. Thus, the second problem addressed in this work is formulated as follows.

Problem 3.2

Given a dataset $\tilde{\mathcal{D}}$ collected from (3.1), a static output-feedback law (3.4) and the closed-loop system (3.5), design a reference tracking structure such that the output tracking error with respect to a piecewise constant reference $r_k \in \mathbb{R}^{n_y}$ asymptotically converges to zero.

3.2 Main results

It is instructive to contextualize the foundation upon which the stabilization conditions for time-delay systems are built. Time-delay systems belong to a class of functional differential equations that are infinite-dimensional [50], and reflecting the nature of time-delay systems, the stability analysis of this class of systems, obtained through the Lyapunov-Krasovskii functional method, also leads to an infinite-dimensional optimization problem. To render this problem tractable, a set of bounding techniques, such as the Wirtinger-based inequality [69], multiple auxiliary functions-based inequality [54], and Moon's inequality [57] are employed in this work to approximate the time-difference of the LKF, ultimately leading to a finite set of delay-dependent LMIs, which can be efficiently solved by off-the-shelf solvers. An adequate choice of the candidate Lyapunov-Krasovskii functional and bounding techniques is of utmost importance in reducing the conservativeness of the stabilization conditions.

Based on this foundation and the data-based representation provided by Lemma 3.1, the following Theorem establishes a sufficient condition for the design of a stabilizing SOF control law (3.4) from data.

Theorem 3.1

Let the dataset $\tilde{\mathcal{D}} = \{X_1, X_0, X_{\bar{h}}, U_{\bar{\tau}}, Y_0\}$, positive integers $\underline{\tau}$, $\bar{\tau}$, $\underline{h}$ and $\bar{h}$, where $\underline{h} < \bar{h}$ and $\underline{\tau} < \bar{\tau}$, and a scalar $\epsilon \in \mathbb{R}^+$ be given. If there exist matrices $G \in \mathbb{R}^{n_u \times n_u}$, $L \in \mathbb{R}^{n_u \times n_y}$, $M_1, M_2, M_3, M_4 \in \mathbb{R}^{n_x \times n_x}$, $M_5 \in \mathbb{R}^{n_y \times n_x}$, $E_1, E_2, E_3, E_4, E_6 \in \mathbb{R}^{n_u \times n_y}$, $E_5 \in \mathbb{R}^{n_y \times n_y}$, $Y_1, Y_2, \bar{Y}_1, \bar{Y}_2 \in \mathbb{R}^{(17n_x + n_y + n_u) \times 2n_x}$ and positive definite matrices $\mathcal{P} \in \mathbb{S}_+^{5n_x}$, $R_1, R_2, Z_1, Z_2, \bar{R}_1, \bar{R}_2, \bar{Z}_1, \bar{Z}_2, \bar{S}_1, \bar{S}_2, \bar{Q}_1, \bar{Q}_2 \in \mathbb{S}_+^{n_x}$, such that the following inequalities hold

$$\begin{aligned} & \begin{bmatrix} \Phi(\underline{h}, \underline{\tau}) & \bar{Y}_2 & Y_2 \\ \star & -\hat{S}_1 - \hat{Z}_2 & 0_{2n_x} \\ \star & \star & -\tilde{Z}_2 \end{bmatrix} < 0, & \begin{bmatrix} \Phi(\underline{h}, \bar{\tau}) & \bar{Y}_1 & Y_2 \\ \star & -\hat{S}_2 - \hat{Z}_2 & 0_{2n_x} \\ \star & \star & -\tilde{Z}_2 \end{bmatrix} < 0, \\ & \begin{bmatrix} \Phi(\bar{h}, \underline{\tau}) & \bar{Y}_2 & Y_1 \\ \star & -\hat{S}_1 - \hat{Z}_2 & 0_{2n_x} \\ \star & \star & -\tilde{Z}_2 \end{bmatrix} < 0, & \begin{bmatrix} \Phi(\bar{h}, \bar{\tau}) & \bar{Y}_1 & Y_1 \\ \star & -\hat{S}_2 - \hat{Z}_2 & 0_{2n_x} \\ \star & \star & -\tilde{Z}_2 \end{bmatrix} < 0, \end{aligned} \quad (3.7)$$

with

$$\begin{aligned}
\Phi(h_k, \tau_k) = & F_1^\top (\mathcal{P} + \Pi) F_1 - F_2^\top \mathcal{P} F_2 + \text{He}\{\Gamma^\top(h, \tau) \mathcal{P} F_1\} - \text{He}\{\Gamma^\top(h, \tau) \mathcal{P} F_2\} \\
& - H_1^\top \tilde{Z}_1 H_1 - \text{He}\{Y_1 H_2 + Y_2 H_3\} + \tilde{R} \\
& - \bar{H}_1^\top \hat{Z}_1 \bar{H}_1 - \text{He}\{\bar{Y}_1 \bar{H}_2 + \bar{Y}_2 \bar{H}_3\} + \bar{H}_2^\top \hat{S}_2 \bar{H}_2 + \bar{H}_3^\top \hat{S}_1 \bar{H}_3 \\
& - \bar{H}_4^\top \tilde{S}_1 \bar{H}_4 - \bar{H}_5^\top \tilde{S}_1 \bar{H}_5 - \bar{H}_6^\top \tilde{S}_2 \bar{H}_6 - \bar{H}_7^\top \tilde{S}_2 \bar{H}_7 \\
& - \bar{H}_8^\top \tilde{Q}_1 \bar{H}_8 - \bar{H}_9^\top \tilde{Q}_2 \bar{H}_9 + \text{He}\{\mathcal{UB}\}, \quad \forall (h_k, \tau_k) \in [\underline{h}, \bar{h}] \times [\underline{\tau}, \bar{\tau}],
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\tilde{Z}_1 &= \text{diag}(Z_1, 3\gamma(\underline{h}) Z_1), \quad \tilde{Z}_2 = \text{diag}(Z_2, 3Z_2), \\
\hat{Z}_1 &= \text{diag}(\bar{Z}_1, 3\sigma(\underline{\tau}) \bar{Z}_1), \quad \hat{Z}_2 = \text{diag}(\bar{Z}_2, 3\bar{Z}_2), \\
\tilde{S}_1 &= \text{diag}(2\bar{S}_1, 4\bar{S}_1), \quad \tilde{S}_2 = \text{diag}(2\bar{S}_2, 4\bar{S}_2), \\
\hat{S}_1 &= \text{diag}(\bar{S}_1, 3\bar{S}_1), \quad \hat{S}_2 = \text{diag}(\bar{S}_2, 3\bar{S}_2), \\
\tilde{Q}_1 &= \text{diag}(2\bar{Q}_1, 4\bar{Q}_1), \quad \tilde{Q}_2 = \text{diag}(2\bar{Q}_2, 4\bar{Q}_2), \\
\Pi &= \text{diag}\left(\underline{h}^2 Z_1 + \tilde{h}^2 Z_2 + \underline{\tau}^2 \bar{Z}_1 + \tilde{\tau}^2 \bar{Z}_2 + \frac{1}{2} \tilde{\tau}(\tilde{\tau} + 1)(\bar{S}_1 + \bar{S}_2) \right. \\
&\quad \left. + \frac{1}{2} \underline{\tau}(\underline{\tau} + 1)(\bar{Q}_1 + \bar{Q}_2), 0_{4n_x}\right),
\end{aligned}$$

$$\begin{aligned}
\tilde{R} &= \text{diag}(0_{n_x}, R_1 + \bar{R}_1, R_2 - R_1, 0_{n_x}, -R_2, \\
&\quad \bar{R}_2 - \bar{R}_1, 0_{n_x}, -\bar{R}_2, 0_{6n_x+n_y+n_u}),
\end{aligned}$$

$$v_i = \begin{bmatrix} 0_{n_x \times (i-1)n_x} & \mathbf{I}_{n_x} & 0_{n_x \times (17-i)n_x} & 0_{n_x \times (n_y+n_u)} \end{bmatrix},$$

$$\bar{v} = \begin{bmatrix} 0_{n_u \times (17n_x+n_y)} & \mathbf{I}_{n_u} \end{bmatrix},$$

$$\hat{v} = \begin{bmatrix} 0_{n_y \times 17n_x} & \mathbf{I}_{n_y} & 0_{n_y \times n_u} \end{bmatrix},$$

$$\Gamma(h_k, \tau_k) = \begin{bmatrix} v_2 \\ \underline{h} v_9 \\ (h_k - \underline{h}) v_{10} + (\bar{h} - h_k) v_{11} \\ \underline{\tau} v_{12} \\ (\tau_k - \underline{\tau}) v_{13} + (\bar{\tau} - \tau_k) v_{14} \end{bmatrix}, \quad \forall (h_k, \tau_k) \in [\underline{h}, \bar{h}] \times [\underline{\tau}, \bar{\tau}]$$

$$F_1 = \begin{bmatrix} v_1 - v_2 \\ v_9 - v_3 \\ v_{10} + v_{11} - v_4 - v_5 \\ v_{12} - v_6 \\ v_{13} + v_{14} - v_7 - v_8 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0_{n_x \times (17n_x+n_y+n_u)} \\ v_9 - v_2 \\ v_{10} + v_{11} - v_3 - v_4 \\ v_{12} - v_2 \\ v_{13} + v_{14} - v_6 - v_7 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} v_2 - v_3 \\ v_2 + v_3 - 2v_9 \end{bmatrix}, \quad H_2 = \begin{bmatrix} v_3 - v_4 \\ v_3 + v_4 - 2v_{10} \end{bmatrix},$$

$$\begin{aligned}
H_3 &= \begin{bmatrix} v_4 - v_5 \\ v_4 + v_5 - 2v_{11} \end{bmatrix}, \quad \bar{H}_1 = \begin{bmatrix} v_2 - v_6 \\ v_2 + v_6 - 2v_{12} \end{bmatrix}, \\
\bar{H}_2 &= \begin{bmatrix} v_6 - v_7 \\ v_6 + v_7 - 2v_{13} \end{bmatrix}, \quad \bar{H}_3 = \begin{bmatrix} v_7 - v_8 \\ v_7 + v_8 - 2v_{14} \end{bmatrix}, \\
\bar{H}_4 &= \begin{bmatrix} v_8 - v_{14} \\ v_8 - 4v_{14} + 6v_{16} \end{bmatrix}, \quad \bar{H}_5 = \begin{bmatrix} v_7 - v_{13} \\ v_7 - 4v_{13} + 6v_{15} \end{bmatrix}, \\
\bar{H}_6 &= \begin{bmatrix} v_7 - v_{14} \\ v_7 + 2v_{14} - 6v_{16} \end{bmatrix}, \quad \bar{H}_7 = \begin{bmatrix} v_6 - v_{13} \\ v_6 + 2v_{13} - 6v_{15} \end{bmatrix}, \\
\bar{H}_8 &= \begin{bmatrix} v_2 - v_{12} \\ v_2 + 2v_{12} - 6v_{17} \end{bmatrix}, \quad \bar{H}_9 = \begin{bmatrix} v_6 - v_{12} \\ v_6 - 4v_{12} + 6v_{17} \end{bmatrix}, \\
\mathcal{B} &= \begin{bmatrix} \hat{X} \begin{bmatrix} v_2^\top & v_4^\top & \bar{v}^\top \end{bmatrix}^\top - v_1 \\ \hat{Y}v_7 - \hat{v} \\ L\hat{v} - G\bar{v} \end{bmatrix}, \\
\mathcal{U} &= \begin{bmatrix} M_1 & \Psi E_1 & \Psi \\ M_2 & \Psi E_2 & \epsilon \Psi \\ 0_{n_x \times n_x} & 0_{n_x \times n_y} & 0_{n_x \times n_u} \\ M_3 & \Psi E_3 & \epsilon \Psi \\ 0_{2n_x \times n_x} & 0_{2n_x \times n_y} & 0_{2n_x \times n_u} \\ M_4 & E_4 & \epsilon \Psi \\ 0_{7n_x \times n_x} & 0_{7n_x \times n_y} & 0_{7n_x \times n_u} \\ M_5 & E_5 & 0_{n_y \times n_u} \\ 0_{n_u \times n_x} & E_6 & I_{n_u} \end{bmatrix}, \quad \Psi = \hat{X} \begin{bmatrix} 0_{2n_x \times n_u} \\ I_{n_u} \end{bmatrix},
\end{aligned}$$

where $i \in \mathbb{N}_{\leq 17}$, $\tilde{h} \triangleq \bar{h} - \underline{h}$, $\tilde{\tau} \triangleq \bar{\tau} - \underline{\tau}$, $\gamma(\underline{h}) = 1$ (if $\underline{h} = 1$) otherwise $\gamma(\underline{h}) = (\underline{h} + 1) / (\underline{h} - 1)$ (if $\underline{h} > 1$) and $\sigma(\underline{\tau}) = 1$ (if $\underline{\tau} = 1$) otherwise $\sigma(\underline{\tau}) = (\underline{\tau} + 1) / (\underline{\tau} - 1)$ (if $\underline{\tau} > 1$).

Then, the origin of the closed-loop system (3.5) is asymptotically stable for any integer time-varying delays $h_k \in \{\underline{h}, \dots, \bar{h}\}$ and $\tau_k \in \{\underline{\tau}, \dots, \bar{\tau}\}$ via the static output-feedback controller:

$$u_k = G^{-1}Ly_k.$$

Proof. Firstly, notice that the bottom-right matrix block in $\Phi(h_k, \tau_k)$ (i.e. the matrix block (19,19)) guarantees that $\text{He}(-G) < 0$, which ensures that the matrix G is non-singular. Consider the Lyapunov–Krasovskii Functional (LKF) candidate given by

$$V(\phi_k) = \sum_{i=1}^7 V_i(\phi_k), \quad (3.9)$$

with

$$\begin{aligned}
V_1 &= \varpi_k^\top \mathcal{P} \varpi_k, \\
V_2 &= \sum_{i=k-\underline{h}}^{k-1} x_i^\top R_1 x_i + \sum_{i=k-\bar{h}}^{k-\underline{h}-1} x_i^\top R_2 x_i, \\
V_3 &= \underline{h} \sum_{i=-\underline{h}+1}^0 \sum_{j=k+i}^k \eta_j^\top Z_1 \eta_j + \tilde{h} \sum_{i=-\bar{h}+1}^{-\underline{h}} \sum_{j=k+i}^k \eta_j^\top Z_2 \eta_j, \\
V_4 &= \sum_{i=k-\underline{\tau}}^{k-1} x_i^\top \bar{R}_1 x_i + \sum_{i=k-\bar{\tau}}^{k-\underline{\tau}-1} x_i^\top \bar{R}_2 x_i, \\
V_5 &= \underline{\tau} \sum_{i=-\underline{\tau}+1}^0 \sum_{j=k+i}^k \eta_j^\top \bar{Z}_1 \eta_j + \tilde{\tau} \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{j=k+i}^k \eta_j^\top \bar{Z}_2 \eta_j, \\
V_6 &= \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{l=-\bar{\tau}+1}^i \sum_{j=k+l}^k \eta_j^\top \bar{S}_1 \eta_j + \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{l=i}^{-\underline{\tau}} \sum_{j=k+l}^k \eta_j^\top \bar{S}_2 \eta_j, \\
V_7 &= \sum_{i=-\underline{\tau}+1}^0 \sum_{l=i}^0 \sum_{j=k+l}^k \eta_j^\top \bar{Q}_1 \eta_j + \sum_{i=-\underline{\tau}+1}^0 \sum_{l=-\underline{\tau}+1}^i \sum_{j=k+l}^k \eta_j^\top \bar{Q}_2 \eta_j,
\end{aligned}$$

where

$$\varpi_k = \left(x_k, \sum_{i=k-\underline{h}}^{k-1} x_i, \sum_{i=k-\bar{h}}^{k-\underline{h}-1} x_i, \sum_{i=k-\underline{\tau}}^{k-1} x_i, \sum_{i=k-\bar{\tau}}^{k-\underline{\tau}-1} x_i \right),$$

and $\eta_i = x_i - x_{i-1}$. First, notice that the LKF in (3.9) is positive definite, since $\mathcal{P}, R_1, R_2, Z_1, Z_2, \bar{R}_1, \bar{R}_2, \bar{Z}_1, \bar{Z}_2, \bar{S}_1, \bar{S}_2, \bar{Q}_1, \bar{Q}_2 \in \mathbb{S}_+$.

In order to obtain an upper-bound for the time-difference of the LKF, $\Delta V(\phi_k) = V(\phi_{k+1}) - V(\phi_k)$, we derive the time-difference of each element of the LKF in terms of the following extended state vector:

$$\zeta_k = (x_{k+1}, x_k, \delta_k, \rho_k, \nu_k, \mu_k, y_{k-h_k}, u_{k-h_k}), \quad (3.10)$$

where

$$\begin{aligned}
\delta_k &= (x_{k-\underline{h}}, x_{k-h_k}, x_{k-\bar{h}}), \\
\rho_k &= (x_{k-\underline{\tau}}, x_{k-\tau_k}, x_{k-\bar{\tau}}), \\
\nu_k &= (\nu_{1_k}, \nu_{2_k}, \nu_{3_k}), \\
\mu_k &= (\mu_{1_k}, \mu_{2_k}, \mu_{3_k}, \mu_{4_k}, \mu_{5_k}, \mu_{6_k}),
\end{aligned}$$

with

$$\begin{aligned}
\nu_{1k} &= \frac{1}{\underline{h}+1} \sum_{i=k-\underline{h}}^k x_i, \quad \nu_{2k} = \frac{1}{h_k-\underline{h}+1} \sum_{i=k-h_k}^{k-\underline{h}} x_i, \quad \nu_{3k} = \frac{1}{\bar{h}-h_k+1} \sum_{i=k-\bar{h}}^{k-h_k} x_i, \\
\mu_{1k} &= \frac{1}{\underline{\tau}+1} \sum_{i=k-\underline{\tau}}^k x_i, \quad \mu_{2k} = \frac{1}{\tau_k-\underline{\tau}+1} \sum_{i=k-\tau_k}^{k-\underline{\tau}} x_i, \quad \mu_{3k} = \frac{1}{\bar{\tau}-h_k+1} \sum_{i=k-\bar{\tau}}^{k-\tau_k} x_i, \\
\mu_{4k} &= \frac{1}{(\tau_k-\underline{\tau}+2)(\tau_k-\underline{\tau}+1)} \sum_{j=-\tau_k}^{-\underline{\tau}} \sum_{i=k+j}^{k-\underline{\tau}} x_i, \quad \mu_{5k} = \frac{1}{(\bar{\tau}-\tau_k+2)(\bar{\tau}-\tau_k+1)} \sum_{j=-\bar{\tau}}^{-\tau_k} \sum_{i=k+j}^{k-\tau_k} x_i, \\
\mu_{6k} &= \frac{1}{(\underline{\tau}+2)(\underline{\tau}+1)} \sum_{j=-\underline{\tau}}^0 \sum_{i=k+j}^k x_i.
\end{aligned}$$

Considering the fact that

$$\begin{aligned}
\varpi_k &= (F_2 + \Gamma(h_k, \tau_k)) \zeta_k, \\
\varpi_{k+1} &= (F_1 + \Gamma(h_k, \tau_k)) \zeta_k,
\end{aligned}$$

then $\Delta V_1(\phi_k)$ can be rewritten as

$$\begin{aligned}
\Delta V_1(\phi_k) &= \zeta_k^\top \left[(F_1 + \Gamma(h_k, \tau_k))^\top \mathcal{P} (F_1 + \Gamma(h_k, \tau_k)) \right. \\
&\quad \left. - (F_2 + \Gamma(h_k, \tau_k))^\top \mathcal{P} (F_2 + \Gamma(h_k, \tau_k)) \right] \zeta_k.
\end{aligned}$$

Notice that the variation of $V_2 + V_4$ results in:

$$\begin{aligned}
\Delta(V_2(\phi_k) + V_4(\phi_k)) &= x_k^\top (R_1 + \bar{R}_1) x_k + x_{k-\underline{h}}^\top (R_2 - R_1) x_{k-\underline{h}} \\
&\quad - x_{k-\bar{h}}^\top R_2 x_{k-\bar{h}} + x_{k-\underline{\tau}}^\top (\bar{R}_2 - \bar{R}_1) x_{k-\underline{\tau}} \\
&\quad - x_{k-\bar{\tau}}^\top \bar{R}_2 x_{k-\bar{\tau}}, \\
&= \zeta_k^\top \tilde{R} \zeta_k.
\end{aligned}$$

The variation of $V_3 + V_5$ leads to:

$$\begin{aligned}
\Delta(V_3(\phi_k) + V_5(\phi_k)) &= \eta_{k+1}^\top \left(\underline{h}^2 Z_1 + \tilde{h}^2 Z_2 + \underline{\tau}^2 \bar{Z}_1 + \tilde{\tau}^2 \bar{Z}_2 \right) \eta_{k+1} \\
&\quad - \underline{h} \sum_{i=k-\underline{h}+1}^k \eta_i^\top Z_1 \eta_i - \tilde{h} \sum_{i=k-\bar{h}+1}^{k-h_k} \eta_i^\top Z_2 \eta_i - \tilde{h} \sum_{i=k-h_k+1}^{k-\underline{h}} \eta_i^\top Z_2 \eta_i \\
&\quad - \underline{\tau} \sum_{i=k-\underline{\tau}+1}^k \eta_i^\top \bar{Z}_1 \eta_i - \tilde{\tau} \sum_{i=k-\bar{\tau}+1}^{k-\tau_k} \eta_i^\top \bar{Z}_2 \eta_i - \tilde{\tau} \sum_{i=k-\tau_k+1}^{k-\underline{\tau}} \eta_i^\top \bar{Z}_2 \eta_i.
\end{aligned} \tag{3.11}$$

Applying Lemma A.4, it follows that:

$$\begin{aligned}
\Delta(V_3(\phi_k) + V_5(\phi_k)) &\leq \zeta_k^\top \left(F_1^\top \Pi_1 F_1 - H_1^\top \tilde{Z}_1 H_1 - \bar{H}_1^\top \hat{Z}_1 \bar{H}_1 \right. \\
&\quad - \begin{bmatrix} H_2 \\ H_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\tilde{h}}{h_k-\underline{h}} \tilde{Z}_2 & 0_{2n_x} \\ 0_{2n_x} & \frac{\tilde{h}}{\bar{h}-h_k} \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} H_2 \\ H_3 \end{bmatrix} \\
&\quad \left. - \begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\tilde{\tau}}{\tau_k-\underline{\tau}} \hat{Z}_2 & 0_{2n_x} \\ 0_{2n_x} & \frac{\tilde{\tau}}{\bar{\tau}-\tau_k} \hat{Z}_2 \end{bmatrix} \begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix} \right) \zeta_k.
\end{aligned} \tag{3.12}$$

where $\Pi_1 = \text{diag}(\underline{h}^2 Z_1 + \tilde{h}^2 Z_2 + \underline{\tau}^2 \bar{Z}_1 + \tilde{\tau}^2 \bar{Z}_2, 0_{4n_x})$. The variation of $V_6(\phi_k)$ results in:

$$\begin{aligned} \Delta V_6(\phi_k) = & \frac{1}{2} \tilde{\tau}(\tilde{\tau} + 1) \eta_{k+1}^\top (\bar{S}_1 + \bar{S}_2) \eta_{k+1} - \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{j=k-\bar{\tau}+1}^{k+i} \eta_j^\top \bar{S}_1 \eta_j \\ & - \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{j=k+i}^{k-\underline{\tau}} \eta_j^\top \bar{S}_2 \eta_j. \end{aligned} \quad (3.13)$$

Then, we can separate the first summation of (3.13) into the following three terms:

$$\begin{aligned} \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{j=k-\bar{\tau}+1}^{k+i} \eta_j^\top \bar{S}_1 \eta_j = & \sum_{i=-\bar{\tau}+1}^{-\tau_k} \sum_{j=k-\bar{\tau}+1}^{k+i} \eta_j^\top \bar{S}_1 \eta_j + \sum_{i=-\tau_k+1}^{-\underline{\tau}} \sum_{j=k-\tau_k+1}^{k+i} \eta_j^\top \bar{S}_1 \eta_j \\ & + (\tau_k - \underline{\tau}) \sum_{j=k-\bar{\tau}+1}^{k-\tau_k} \eta_j^\top \bar{S}_1 \eta_j. \end{aligned} \quad (3.14)$$

Now, splitting the last sum of (3.13), one has

$$\begin{aligned} \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{j=k+i}^{k-\underline{\tau}} \eta_j^\top \bar{S}_2 \eta_j = & \sum_{i=-\bar{\tau}+1}^{-\tau_k} \sum_{j=k+i}^{k-\tau_k} \eta_j^\top \bar{S}_2 \eta_j + \sum_{i=-\tau_k+1}^{-\underline{\tau}} \sum_{j=k+i}^{k-\underline{\tau}} \eta_j^\top \bar{S}_2 \eta_j \\ & + (\bar{\tau} - \tau_k) \sum_{j=k-\tau_k+1}^{k-\underline{\tau}} \eta_j^\top \bar{S}_2 \eta_j. \end{aligned} \quad (3.15)$$

Applying Lemmas A.4 and A.5 in (3.14) and (3.15), one can write $\Delta V_6(\phi_k)$ as

$$\begin{aligned} \Delta V_6 \leq & \zeta_k^\top \left[F_1^\top \Pi_2 F_1 - \bar{H}_4^\top \tilde{S}_1 \bar{H}_4 - \bar{H}_5^\top \tilde{S}_1 \bar{H}_5 - \bar{H}_6^\top \tilde{S}_2 \bar{H}_6 - \bar{H}_7^\top \tilde{S}_2 \bar{H}_7 \right] \zeta_k \\ & - \zeta_k^\top \left(\begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix}^\top \begin{bmatrix} \left(\frac{\tilde{\tau}}{\tau_k - \underline{\tau}} - 1 \right) \hat{S}_2 & 0_{2n_x} \\ 0_{2n_x} & \left(\frac{\tilde{\tau}}{\bar{\tau} - \tau_k} - 1 \right) \hat{S}_1 \end{bmatrix} \begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix} \right) \zeta_k. \end{aligned} \quad (3.16)$$

where $\Pi_2 = \text{diag}(\frac{1}{2} \tilde{\tau}(\tilde{\tau} + 1)(\bar{S}_1 + \bar{S}_2), 0_{4n_x})$. Therefore, one has

$$\begin{aligned} \Delta V_6 \leq & \zeta_k^\top \left[F_1^\top \Pi_2 F_1 - \bar{H}_4^\top \tilde{S}_1 \bar{H}_4 - \bar{H}_5^\top \tilde{S}_1 \bar{H}_5 - \bar{H}_6^\top \tilde{S}_2 \bar{H}_6 - \bar{H}_7^\top \tilde{S}_2 \bar{H}_7 \right. \\ & \left. + \bar{H}_2^\top \hat{S}_2 \bar{H}_2 + \bar{H}_3^\top \hat{S}_1 \bar{H}_3 \right] \zeta_k \\ & - \zeta_k^\top \left(\begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\tilde{\tau}}{\tau_k - \underline{\tau}} \hat{S}_2 & 0_{2n_x} \\ 0_{2n_x} & \frac{\tilde{\tau}}{\bar{\tau} - \tau_k} \hat{S}_1 \end{bmatrix} \begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix} \right) \zeta_k. \end{aligned} \quad (3.17)$$

Finally, the variation of $V_7(\phi_k)$ results in the following:

$$\begin{aligned} \Delta V_7(\phi_k) = & \frac{1}{2} \underline{\tau}(\underline{\tau} + 1) \eta_{k+1}^\top (\bar{Q}_1 + \bar{Q}_2) \eta_{k+1} - \sum_{i=-\underline{\tau}+1}^0 \sum_{j=k+i}^k \eta_j^\top \bar{Q}_1 \eta_j \\ & - \sum_{i=-\underline{\tau}+1}^0 \sum_{j=k-\underline{\tau}+1}^{k+i} \eta_j^\top \bar{Q}_2 \eta_j. \end{aligned}$$

$$\Delta V_7(\phi_k) \leq \zeta_k^\top \left(F_1^\top \Pi_3 F_1 - \bar{H}_8^\top \tilde{Q}_1 \bar{H}_8 - \bar{H}_9^\top \tilde{Q}_2 \bar{H}_9 \right) \zeta_k. \quad (3.18)$$

where $\Pi_3 = \text{diag}\left(\frac{1}{2}\mathcal{T}(\mathcal{T}+1)(\bar{Q}_1 + \bar{Q}_2), 0_{4n_x}\right)$.

Thus, by summing the variations of the terms in the LKF, we arrive at an upper bound for $\Delta V(\phi_k)$:

$$\begin{aligned} \Delta V(\phi_k) \leq & \zeta_k^\top \left[(F_1 + \Gamma(h_k, \tau_k))^\top \mathcal{P} (F_1 + \Gamma(h_k, \tau_k)) \right. \\ & - (F_2 + \Gamma(h_k, \tau_k))^\top \mathcal{P} (F_2 + \Gamma(h_k, \tau_k)) + \tilde{R} \\ & + F_1^\top \Pi F_1 - H_1^\top \tilde{Z}_1 H_1 - \bar{H}_1^\top \hat{Z}_1 \bar{H}_1 + \bar{H}_2^\top \hat{S}_2 \bar{H}_2 \\ & + \bar{H}_3^\top \hat{S}_1 \bar{H}_3 - \bar{H}_4^\top \tilde{S}_1 \bar{H}_4 - \bar{H}_5^\top \tilde{S}_1 \bar{H}_5 - \bar{H}_6^\top \tilde{S}_2 \bar{H}_6 \\ & - \bar{H}_7^\top \tilde{S}_2 \bar{H}_7 - \bar{H}_8^\top \tilde{Q}_1 \bar{H}_8 - \bar{H}_9^\top \tilde{Q}_2 \bar{H}_9 \left. \right] \zeta_k \\ & - \zeta_k^\top \left(\begin{bmatrix} H_2 \\ H_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\bar{h}}{h_k - \bar{h}} \tilde{Z}_2 & 0_{2n_x} \\ 0_{2n_x} & \frac{\bar{h}}{\bar{h} - h_k} \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} H_2 \\ H_3 \end{bmatrix} \right. \\ & \left. - \begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\bar{\tau}}{\tau_k - \bar{\tau}} (\hat{Z}_2 + \hat{S}_2) & 0_{2n_x} \\ 0_{2n_x} & \frac{\bar{\tau}}{\bar{\tau} - \tau_k} (\hat{Z}_2 + \hat{S}_1) \end{bmatrix} \begin{bmatrix} \bar{H}_2 \\ \bar{H}_3 \end{bmatrix} \right) \zeta_k. \end{aligned} \quad (3.19)$$

Hence, applying Lemma A.3 twice, with $\varrho = \frac{(h_k - \bar{h})}{\bar{h}}$ and $\sigma = \frac{(\tau_k - \bar{\tau})}{\bar{\tau}}$, we obtain the following result:

$$\Delta V(\phi_k) \leq \zeta_k^\top \Xi(h_k, \tau_k) \zeta_k, \quad (3.20)$$

with

$$\begin{aligned} \Xi(h_k, \tau_k) = & F_1^\top (\mathcal{P} + \Pi) F_1 + \text{He} \left\{ \Gamma^\top(h_k, \tau_k) \mathcal{P} F_1 \right\} - F_2^\top \mathcal{P} F_2 \\ & - \text{He} \left\{ \Gamma^\top(h_k, \tau_k) \mathcal{P} F_2 \right\} + \tilde{R} - \text{He} \{ Y_1 H_2 + Y_2 H_3 \} \\ & - H_1^\top \tilde{Z}_1 H_1 - \bar{H}_1^\top \hat{Z}_1 \bar{H}_1 + \bar{H}_2^\top \hat{S}_2 \bar{H}_2 + \bar{H}_3^\top \hat{S}_1 \bar{H}_3 \\ & - \bar{H}_4^\top \tilde{S}_1 \bar{H}_4 - \bar{H}_5^\top \tilde{S}_1 \bar{H}_5 - \bar{H}_6^\top \tilde{S}_2 \bar{H}_6 - \bar{H}_7^\top \tilde{S}_2 \bar{H}_7 \\ & - \bar{H}_8^\top \tilde{Q}_1 \bar{H}_8 - \bar{H}_9^\top \tilde{Q}_2 \bar{H}_9 + \varrho Y_1 \tilde{Z}_2^{-1} Y_1^\top + (1 - \varrho) Y_2 \tilde{Z}_2^{-1} Y_2^\top \\ & + \sigma \bar{Y}_1 (\hat{Z}_2 + \hat{S}_2)^{-1} \bar{Y}_1^\top + (1 - \sigma) \bar{Y}_2 (\hat{Z}_2 + \hat{S}_1)^{-1} \bar{Y}_2^\top. \end{aligned} \quad (3.21)$$

Considering

$$\tilde{\mathcal{B}} = \begin{bmatrix} \hat{X} \begin{bmatrix} v_2^\top & v_4^\top & \bar{v}^\top \end{bmatrix}^\top - v_1 \\ \hat{Y} v_7 - \hat{v} \\ G^{-1} L \hat{v} - \bar{v} \end{bmatrix}, \quad (3.22)$$

which yields $\tilde{\mathcal{B}}_{\zeta_k} = 0$ and

$$\tilde{\mathcal{U}} = \begin{bmatrix} M_1 & \Psi E_1 & \Psi G \\ M_2 & \Psi E_2 & \epsilon \Psi G \\ 0_{n_x \times n_x} & 0_{n_x \times n_y} & 0_{n_x \times n_u} \\ M_3 & \Psi E_3 & \epsilon \Psi G \\ 0_{2n_x \times n_x} & 0_{2n_x \times n_y} & 0_{2n_x \times n_u} \\ M_4 & E_4 & \epsilon \Psi G \\ 0_{7n_x \times n_x} & 0_{7n_x \times n_y} & 0_{7n_x \times n_u} \\ M_5 & E_5 & 0_{n_y \times n_u} \\ 0_{n_u \times n_x} & E_6 & G \end{bmatrix}, \quad (3.23)$$

by the arguments of Lemma A.6, the negative definiteness of (3.20) is ensured if

$$\Xi(h_k, \tau_k) + \tilde{\mathcal{U}}\tilde{\mathcal{B}} + \tilde{\mathcal{B}}^\top \tilde{\mathcal{U}}^\top < 0. \quad (3.24)$$

Knowing that (3.24) is affine in h_k and τ_k , the condition is satisfied for each h_k and τ_k within $\{\underline{h}, \dots, \bar{h}\}$ and $\{\underline{\tau}, \dots, \bar{\tau}\}$ if it is ensured at the vertices of both intervals. By applying the Schur Complement twice in (3.24), we obtain (3.7), which is convex relative to h_k and τ_k . Therefore, the feasibility of the LMI conditions in (3.7) implies that the time-difference of the LKF (3.20) is negative definite for all $h_k \in \{\underline{h}, \dots, \bar{h}\}$ and $\tau_k \in \{\underline{\tau}, \dots, \bar{\tau}\}$. Consequently, the origin of the closed-loop system is stable. This concludes the proof. $\square$

Remark 3.4

The chosen Lyapunov-Krasovskii functional (3.9) is an adaptation of the approach presented in [70] for the linear time-invariant case. To further reduce conservativeness in stabilization under input delay, additional triple summation terms V_6 and V_7 were incorporated relative to previous LKF (2.10).

3.2.1 Reference tracking scheme

Although the result in Theorem 3.1 ensures the asymptotic stability of the closed-loop system (3.5), it is also interesting to ensure tracking of piecewise-constant reference inputs. As an alternative to integral control, similar to what was presented in Section 2.2.1 and [66, Section 8.4.1], in this section, we introduce a novel data-driven feedforward compensation strategy for time-delay systems subject to input delay.

Figure 3.1 depicts a schematic of the combination of an unknown time-delay linear system controlled by the proposed data-driven static output feedback approach with the reference tracking structure, and communication delays. To ensure tracking of piecewise constant reference inputs, the reference tracking blocks must be coupled with the system as

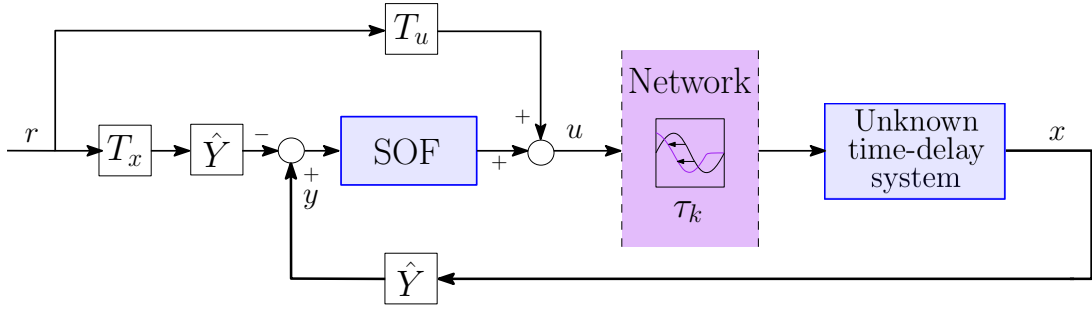


Figure 3.1 – Modified block diagram for SOF with reference tracking blocks T_x and T_u , representing the state command matrix and the feedforward matrix, respectively.

illustrated in Figure 3.1. The resulting closed-loop dynamics of this schematic are given by

$$x_{k+1} = \hat{X} \begin{bmatrix} I_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x \times n_y} \\ 0_{n_x} & I_{n_x} & 0_{n_x} & 0_{n_x \times n_y} \\ 0_{n_u \times n_x} & 0_{n_u \times n_x} & \hat{J} & (T_u - \hat{J}T_x) \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-h_k} \\ x_{k-\tau_k} \\ r_{k-\tau_k} \end{bmatrix} \quad (3.25)$$

where $\hat{J} = G^{-1}L\hat{Y}$ and $T_x \in \mathbb{R}^{n_x \times n_y}$ and $T_u \in \mathbb{R}^{n_u \times n_y}$ are the state command matrix and the feedforward matrix of the reference tracking blocks, respectively. The design conditions for these gains are presented in the proposition below.

Proposition 3.1

Consider the system (3.3) together with the static output feedback given in (3.4) that guarantees asymptotic stability as stated in Theorem 3.1. Given a dataset $\tilde{\mathcal{D}} = \{X_1, X_0, X_{\bar{h}}, U_{\bar{\tau}}, Y_0\}$, the inclusion of matrices $T_x \in \mathbb{R}^{n_x \times n_y}$ and $T_u \in \mathbb{R}^{n_u \times n_y}$ as depicted in Figure 3.1, with closed-loop dynamics given by (3.25), obtained through:

$$\begin{bmatrix} T_x \\ T_u \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 0_{n_x \times n_y} \\ I_{n_y} \end{bmatrix}, \quad (3.26)$$

where

$$\Theta = \begin{bmatrix} \hat{X}\Lambda - [I_{n_x} & 0_{n_x \times n_u}] \\ \hat{Y} & 0_{n_y \times n_u} \end{bmatrix},$$

with

$$\Lambda = \begin{bmatrix} I_{n_x} & 0_{n_x} \\ I_{n_x} & 0_{n_x} \\ 0_{n_u \times n_x} & I_{n_u} \end{bmatrix},$$

ensures that the tracking error for a piecewise constant reference will asymptotically converge to zero if $\Theta \in \mathbb{R}^{(n_x+n_y) \times (n_x+n_u)}$ is non-singular.

Proof. From the schematic in Figure 3.1, $u = K\hat{Y}(x - T_x r) + T_u r$. The following constraint

is imposed:

$$\hat{Y}T_x r = r \implies \hat{Y}T_x = I_{n_y}, \quad (3.27)$$

so that it is possible to rewrite $u = K(y - r) + T_u r$. T_x and T_u are designed so that when $k \rightarrow \infty$, the output satisfies $y \rightarrow r$. This can be achieved with an input u_{eq} which, through the state reference defined as $x_r = T_x r$, drives the system state to $x_{eq} = x_r$. Furthermore, ensuring that the tracking error $y - r = e \rightarrow 0$, the asymptotic stability of Theorem 3.1 guarantees that $u \rightarrow u_{eq}$ is satisfied, even in the presence of time-delay. Thus, the following equalities will hold:

$$\begin{aligned} x_{eq} &= x_r = T_x r, \\ u_{eq} &= K(y_{eq} - r) + T_u r \implies u_{eq} = T_u r. \end{aligned} \quad (3.28)$$

Moreover, note that $x_{k-h_k} = x_{eq}$ when the tracking error is null. Thus, considering (3.3) in steady state, we have

$$x_{eq} = \hat{X}\Lambda \begin{bmatrix} x_{eq} \\ u_{eq} \end{bmatrix} \implies 0 = (\hat{X}\Lambda - \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} \end{bmatrix}) \begin{bmatrix} x_{eq} \\ u_{eq} \end{bmatrix}. \quad (3.29)$$

Assuming $r \neq 0$ and substituting x_{eq} and u_{eq} as in (3.28), we have

$$0 = (\hat{X}\Lambda - \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} \end{bmatrix}) \begin{bmatrix} T_x \\ T_u \end{bmatrix}. \quad (3.30)$$

Finally, combining (3.27) and (3.30), we obtain the following system of linear equations:

$$\begin{bmatrix} \hat{X}\Lambda - \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} \end{bmatrix} \\ \hat{Y} \end{bmatrix} \begin{bmatrix} T_x \\ T_u \end{bmatrix} = \begin{bmatrix} 0_{n_x \times n_y} \\ I_{n_y} \end{bmatrix}, \quad (3.31)$$

whose solution is (3.26). This concludes the proof. $\square$

Remark 3.5

For a singular Θ , the nonexistence of a unique solution for (3.30) does not compromise the fundamental result of Theorem 3.1. In such cases, it is naturally possible to utilize alternative tracking strategies based on the underlying stabilization condition.

3.3 Examples

This section illustrates the efficacy of the proposed data-driven static output-feedback (SOF) control framework through illustrative numerical and experimental examples. In the first example, stabilization of a generic linear system was performed with a time-varying delay in the dynamics and input. Example 2 consists of an application of the reference tracking scheme for velocity control of a Vertical Take-Off and Landing (VTOL) aircraft model. In Example 3, pitch angle tracking tests were conducted on a testbed twin-rotor system plant,

a simplified helicopter model. Each simulation was implemented considering both input and state time-varying delays described by the following function

$$d_k = \text{round} \{ \underline{d}/2 + \bar{d}/2 + (-\underline{d}/2 + \bar{d}/2) \sin(k\pi/2) \}.$$

We emphasize that the system matrices for the experiments conducted in this section are not directly used for the controller design itself, but only for data acquisition and design validation. All design conditions were implemented in MATLAB using the parser YALMIP and the MOSEK solver.

3.3.1 Example 1: Stabilization

Consider the following LTI system with both state and input delay, adapted from [71]:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.25 & 0.1 \\ 0 & 0.121 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \quad (3.32)$$

In this example, our aim is to find a SOF controller that stabilizes the system (3.32) for maximum state and input delay. For the given values of $\underline{h} = 1, \bar{h} = 4949, \underline{\tau} = 1, \bar{\tau} = 21$, where $\bar{h}$ and $\bar{\tau}$ are, respectively, the maximum upper bounds for state and input delays, $\epsilon = 0.1$ and $T = 6$ samples of input-state-output data, the SOF control law obtained through Theorem 3.1 is given by the following gains:

$$G = \begin{bmatrix} 0.5553 & -0.9281 \\ -1.0171 & 2.0638 \end{bmatrix} \cdot 10^5, \quad L = \begin{bmatrix} 1.3254 \\ -0.8952 \end{bmatrix} \cdot 10^4. \quad (3.33)$$

Figure 3.2 presents the open-loop system response, while Figure 3.3 depicts the closed-loop system response with the designed stabilizing controller for initial conditions $\phi_0 = x_0 = [0.48 \ 0.99]^\top$. The proposed controller, designed without prior knowledge of the system's dynamics, ensures the stability of the closed-loop system, which is inherently open-loop unstable due to the presence of time delays.

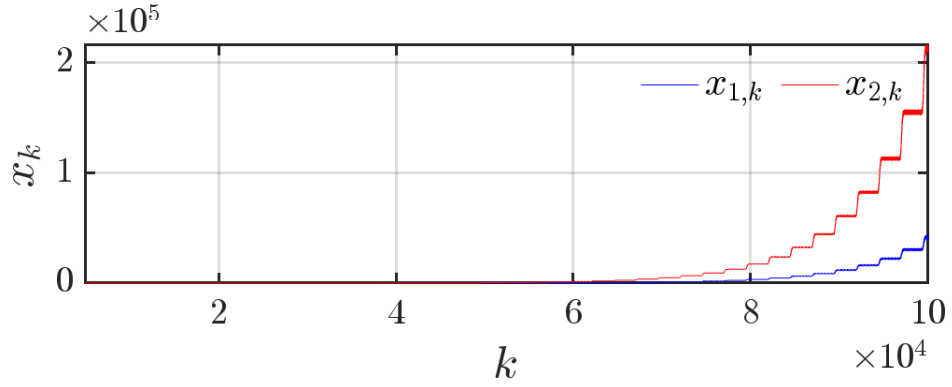


Figure 3.2 – System (3.32) open loop response.

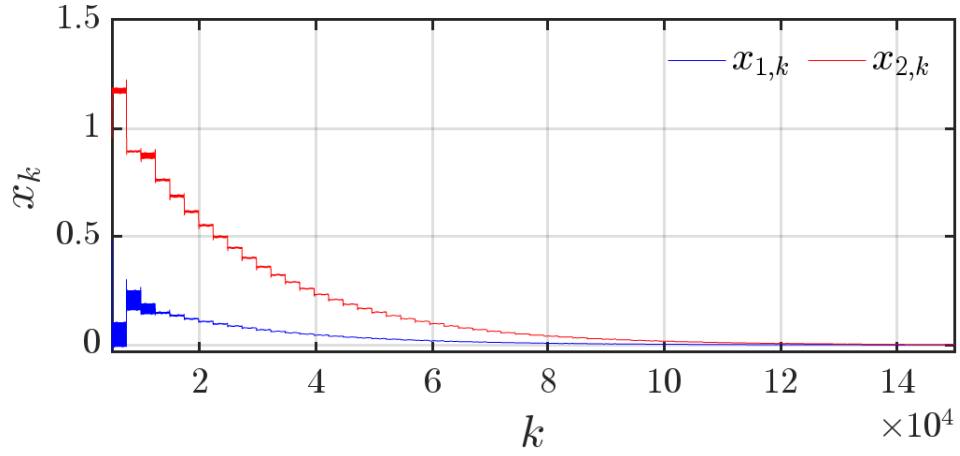


Figure 3.3 – System (3.32) with SOF control in (3.33).

3.3.2 Example 2: VTOL aircraft model

Consider the following linearized model of a VTOL Helicopter adapted from [67]. The state-space model, discretized for a sampling period of $0.1s$, is

$$\begin{aligned}
 A &= \begin{bmatrix} 0.9963 & 0.0027 & 0.0019 & -0.0048 \\ 0.0048 & 0.8990 & 0.0002 & -0.4021 \\ 0.0100 & 0.3505 & 0.9293 & 0.1420 \\ 0 & 0 & 0.1000 & 1 \end{bmatrix}, \\
 A_d &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0337 & 0.0035 \\ 0 & 0 & 0.0003 & 0.0062 \end{bmatrix}, \\
 B &= \begin{bmatrix} 0.0442 & 0.0176 \\ 0.0318 & -0.7592 \\ -0.5520 & 0.4490 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
 \end{aligned} \tag{3.34}$$

with x_1, x_2, x_3 , and x_4 representing, respectively, the horizontal and vertical velocities (in knots), the pitch rate (degrees/s), and the pitch angle (degrees). The inputs u_1 and u_2 represent the collective and longitudinal cyclic controls, respectively. The proposed A_d matrix implies a cross-dependency on the time-delay involving the aircraft's pitching dynamics.

Firstly, we intend to design a SOF controller (3.4) from the data set (3.2) that stabilizes the system (3.34) for a maximum time delay. Considering $\underline{h} = 1$ and $\underline{\tau} = 1$ with $\epsilon = 0.1$ and a time horizon of $T = 10$, the static output feedback controller designed by Theorem 3.1 was able to stabilize the system up to $\bar{h} = 72$ and $\bar{\tau} = 4$. The SOF control gains obtained in the form of (3.4) are:

$$G = \begin{bmatrix} 1229.90 & 686.99 \\ 174.85 & 1061.31 \end{bmatrix}, \quad L = \begin{bmatrix} -149.45 & 978.39 \\ -137.77 & 416.47 \end{bmatrix}. \quad (3.35)$$

Now considering the tracking problem of the horizontal and vertical velocities, Proposition 3.1 provides:

$$T_x = \begin{bmatrix} 1.0000 & 0 \\ 0 & 1.0000 \\ 0.0071 & -0.0704 \\ -0.1148 & 1.1392 \end{bmatrix}, \quad T_u = \begin{bmatrix} 0.0435 & 0.3530 \\ 0.0690 & -0.7216 \end{bmatrix}. \quad (3.36)$$

Figures 3.4 illustrates the closed-loop system response with the designed controller for initial conditions $\phi_0 = x_0 = [0.32 \ 0.86 \ 0.26 \ 0.88]^\top$ and a piecewise constant reference $r_k = [0.5 \ 0.5]$ for $k \leq 5000$ and $r_k = [1.0 \ 1.0]$ for $k > 5000$. As expected, the controller designed without the knowledge of the matrices (3.34), ensures the reference tracking of x_1 and x_2 .

Finally, considering system (2.27), where there is no state delay, we are interested in observing the reduction in conservatism for SOF control design subject to input delays. Considering again $\underline{\tau} = 1$ with $\epsilon = 0.1$ and a time horizon of $T = 10$, the static output feedback controller designed by Theorem 3.1 was able to stabilize the system up to $\bar{\tau} = 4$, as opposed to $\bar{\tau} = 3$ from Theorem 2.1. The SOF control gains obtained in the form of (3.4) are:

$$G = \begin{bmatrix} 4.9776 & 3.1207 \\ 0.7919 & 4.6231 \end{bmatrix}, \quad L = \begin{bmatrix} -0.7381 & 3.9490 \\ -0.7841 & 1.9539 \end{bmatrix}. \quad (3.37)$$

Figure 3.5 depicts the closed-loop system response with the designed stabilizing controller for initial conditions $\phi_0 = x_0 = [0.32 \ 0.86 \ 0.26 \ 0.88]^\top$. The proposed controller ensures the stability of the closed-loop system, even in the presence of time delays, with an approximate 33% increase in the upper bound.

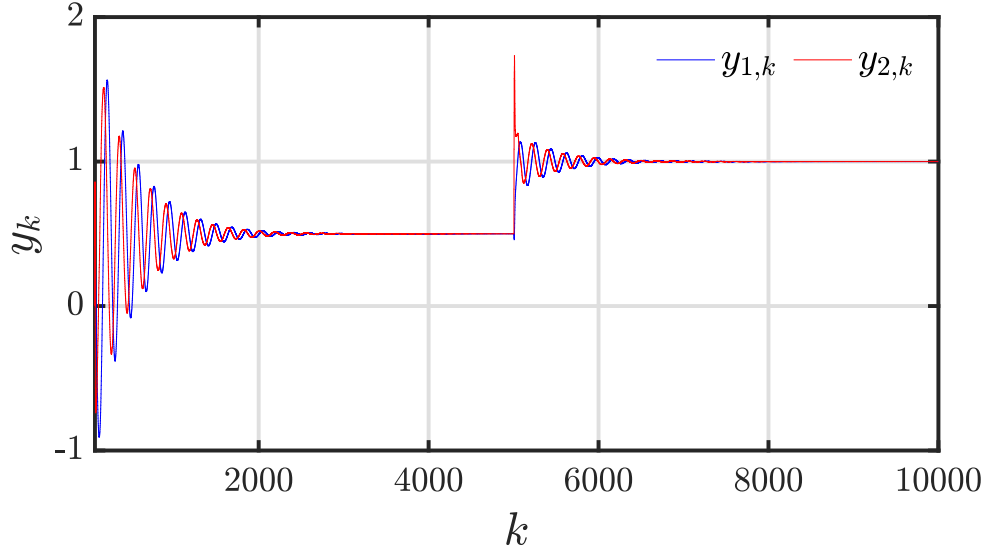


Figure 3.4 – VTOL system SOF control (3.35) combined with the tracking scheme from Figure 3.1.

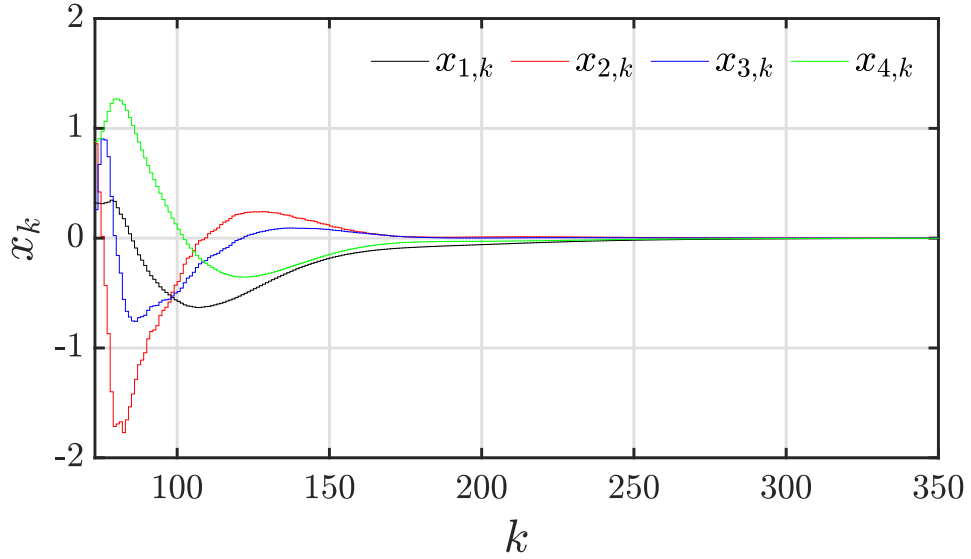


Figure 3.5 – System (2.27) with SOF control in (3.37).

3.3.3 Example 3: Pitch angle control of a dual-rotor aerodynamic system – Practical experiment

Consider the pitch angle regulation of the twin-rotor system described in Section 2.3.2. With a sampling period of $T_s = 0.001s$ and a constant delay of $\bar{h} = 2000$, then $T = 6000$ samples of input-state-output data were collected by operating the system in open-loop with an input $u_k = 0.6U_k + \omega_k$ rad, where $\omega_k \sim \mathcal{N}(0, 0.01)$ and U_k is a unit step. From Theorem 3.1, considering $\underline{\tau} = 1$, $\bar{\tau} = 2000$, and $\epsilon = 0.001$, the SOF control gains in the form of (3.4) are:

$$G = 15.5396, \quad L = -0.0726. \quad (3.38)$$

For pitch angle tracking, Proposition 3.1 provides:

$$T_x = \begin{bmatrix} 1 & 0.0189 & 2.6347 \end{bmatrix}^\top, \quad T_u = 2.6861. \quad (3.39)$$

The tracking performance with the reference tracking block scheme for the constant signal $r_k = 0.6U_k$ rad is depicted in Figure 3.6. The results obtained once again highlight the effectiveness of the proposed approach in designing the tracking scheme considering only available data. Moreover, it is observed that the presence of noise in the data and the nonlinear nature of TRMS require a larger number of collected samples to obtain satisfactory results for reference tracking in a practical scenario.

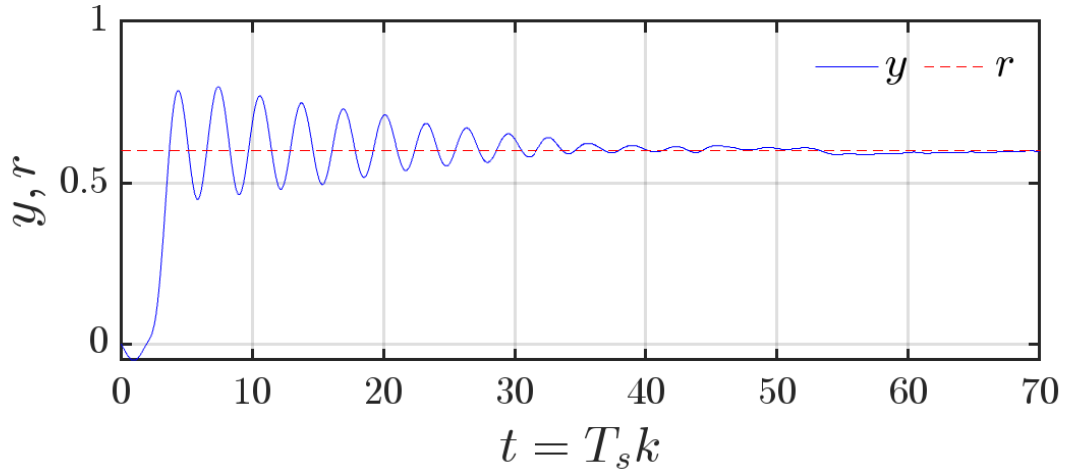


Figure 3.6 – TRMS system with SOF control (3.38) combined with the tracking scheme from Figure 3.1.

3.4 Final remarks

This chapter proposed a data-driven synthesis of static output feedback controllers for discrete-time linear systems with time-varying state and input delay. Additionally, a tracking block structure was proposed to ensure the tracking of piecewise constant references. The main distinction of the proposed methods lies in achieving stabilization, based on the Lyapunov-Krasovskii stability theory, in the presence of time delays in both input signal and the state dynamics, without requiring knowledge of the system matrices. Another significant contribution of this work is the development of a data-driven reference tracking framework specifically designed for time-delay systems. To illustrate the potential and effectiveness of the proposed conditions, three case studies were presented: a numerical example involving the stabilization of a generic time-delay linear system, another numerical example regarding the angular velocity tracking of a VTOL aircraft model and a real application of reference tracking for pitch angle control of a twin-rotor aerodynamic system.

4 CONCLUDING REMARKS

This dissertation has addressed the data-driven static output-feedback control problem for linear time-invariant and time-delay systems subject to time-varying input delay.

Chapter 2 focused on the data-driven static output-feedback control for discrete-time linear time-invariant systems with time-varying input delay. Based on the use of Wirtinger-based summation inequality and the delay-dependent Moon's inequality, LMI-based conditions have been obtained from input-state-output open-loop data. The stabilization conditions were applied along with a novel reference tracking block structure. As a result, the proposed methods achieved stabilization and piecewise constant reference tracking in the presence of time-varying input delay, without requiring knowledge of the system matrices.

Chapter 2 focused on the data-driven static output-feedback control for discrete-time linear time-delay systems subject to time-varying input delay. Starting from a Lyapunov-Krasovskii functional constructed with triple-summation terms, bounded by the Wirtinger-based inequality, multiple auxiliary functions-based inequalities and Moon's inequality, delay-dependent LMI-based conditions have been obtained, relying on open-loop input-state-output data to account for the unknown system's dynamics. Additionally, a novel data-driven reference tracking block structure for time-delay systems was proposed. Subsequently, the results were used in conjunction and achieved stabilization and tracking for piecewise-constant reference trajectories in the presence of input delays, without requiring knowledge of the system matrices.

In both Chapters 2 and 3, the effectiveness of the proposed conditions was illustrated with testbed systems. A numerical example involving the stabilization and angular velocity tracking of a VTOL aircraft model and a real application of reference tracking for pitch angle control of a twin-rotor aerodynamic system was presented.

4.1 Future research

Some suggestions of possible next steps following this master's research are discussed in this section. They are based on the further development of the main objective of this thesis: to propose data-driven control strategies for the stabilization of time-delay systems. The next steps are listed as follows:

- a) Robust conditions to deal with disturbances and measurement noise: The presence of measurement noise impedes the recovery of an exact representation of the actual system from the collected data. A significant contribution was made towards the extension of data-driven results to noisy data (see [39, 41, 72]), presenting conditions that stabilize a set of systems that are consistent with noisy data. However, the noise in the data is considered in the form of process disturbance. Furthermore, to fit

conditions into the proposed frameworks, it requires matrix manipulations that are typically only viable for some conditions that arise from congruence transformations. Further investigation in obtaining an extension of these conditions for genuine measurement noise and time-delay systems is recommended.

- b) Conservativeness reduction of the proposed stabilization subject to input delay: To provide satisfactory conditions for the stability of systems subject to input delay under the lens of Lyapunov-Krasovskii functionals has been proven to be significantly challenging. The application of Bessel-Legendre inequalities and LKFs with delay-product matrix polynomials to continuous-time systems has also been proven to lead to less conservative results [57, 58, 73]. However, the study of those functionals and inequalities in the context of discrete-time systems is still in the embryonic phase. Motivated by the problems mentioned above, design conditions can be derived considering polynomial LKF terms and Bessel-Legendre inequalities separately.

4.2 Submitted conference and journal papers

During the period in which this master's research was developed, contributions related to the topics of data-driven control have been attained. The conference and journal publications submitted related to the specific topic of this dissertation are listed below.

- a) **BRENAG, E. C.**; PESSIM, P. S. P.; de OLIVEIRA, P. M.; PALHARES, R. M. "Quasi-Data-Driven Static Output-feedback Control of Linear Systems with Input and State delays";
- b) **BRENAG, E. C.**; PESSIM, P. S. P.; de OLIVEIRA, P. M.; PALHARES, R. M. "Controle Baseado em Dados para Sistemas Lineares Sujeitos a Retardo no Tempo na Entrada". Accepted for publication at the XVII Brazilian Symposium on Intelligent Automation (SBAI 2025).

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Appendix

APPENDIX A – USEFUL LEMMAS

This appendix presents useful technical lemmas used in the development of the data-driven representation and delay-dependent LMI proposed in this dissertation. The persistence of excitation condition is a sufficient condition necessary to guarantee that input data are rich enough in order to represent input-output trajectories of a linear system as a linear combination of input-output measurements. The following Lemma A.1 presents an important result that arises from the persistence of excitation.

Lemma A.1: Persistency of Excitation (Adapted from [37, Lemma 1])

Consider a linear time-invariant system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned} \tag{A.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input vector and $y \in \mathbb{R}^{n_y}$ is the output vector. If the input $\{u_k\}_0^{T-1}$ is persistently exciting of order $n_x + 1$, then the following rank condition is fulfilled:

$$\text{rank} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} = n_x + n_u,$$

where

$$\begin{aligned} X_0 &= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \\ U_0 &= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}. \end{aligned} \tag{A.2}$$

Proof. The proof is omitted and can be found in [37]. □

Moreover, we introduce Lemma A.2, also known as Willems' fundamental lemma, a cornerstone for data-driven control strategies. It states that if the response signal component of a linear system is persistently exciting (*i.e.*, Lemma A.1 holds), then a window of data from this signal can generate the entire behavior of the system [24].

Lemma A.2: Willems' Fundamental Lemma [37, Lemma 1]

Consider the system (A.1). Then, the following holds.

1. If the measured $\{u_d\}_0^{T-1}$, is persistently exciting of order $n + 1$, then any t -long

input-output trajectory of system (A.1) can be expressed as:

$$\begin{bmatrix} \{u\}_0^{t-1} \\ \{y\}_0^{t-1} \end{bmatrix} = \begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} g$$

where

$$\begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} := \begin{bmatrix} u_d(0) & u_d(1) & \cdots & u_d(T-t) \\ u_d(1) & u_d(2) & \cdots & u_d(T-t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u_d(t-1) & u_d(t) & \cdots & u_d(T-1) \\ y_d(0) & y_d(1) & \cdots & y_d(T-t) \\ y_d(1) & y_d(2) & \cdots & y_d(T-t+1) \\ \vdots & \vdots & \ddots & \vdots \\ y_d(t-1) & y_d(t) & \cdots & y_d(T-1) \end{bmatrix} \quad (\text{A.3})$$

is the corresponding Hankel matrix and $g \in \mathbb{R}^{T-t+1}$.

2. Any linear combination of the columns of the matrix in (A.3), that is,

$$\begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} g$$

is a t -long input/output trajectory of (A.1).

Proof. The proof is omitted and can be found in [37]. □

Next, we introduce a set of inequalities used to derive stabilization conditions for discrete-time delay systems. In particular, the inequality stated in Lemma A.3 is used to establish delay-dependent conditions that explicitly account for the time-varying delay. This approach reduces conservatism compared to the reciprocally convex inequality [56], albeit at the cost of additional decision variables. The inequality is formally stated in the following lemma.

Lemma A.3: Moon's inequality [57]

For any matrices $R_1 \in \mathbb{S}_+^n$, $R_2 \in \mathbb{S}_+^n$, $Y_1 \in \mathbb{R}^{2n \times n}$, and $Y_2 \in \mathbb{R}^{2n \times n}$, the following inequality holds:

$$\begin{bmatrix} \frac{1}{\vartheta} R_1 & 0 \\ 0 & \frac{1}{1-\vartheta} R_2 \end{bmatrix} \geq \Theta_M(\vartheta), \quad \forall \vartheta \in (0, 1)$$

where

$$\Theta_M(\vartheta) = \text{He} \left\{ Y_1 \begin{bmatrix} I_n & 0_n \end{bmatrix} + Y_2 \begin{bmatrix} 0_n & I_n \end{bmatrix} \right\} - \vartheta Y_1 R_1^{-1} Y_1^\top - (1 - \vartheta) Y_2 R_2^{-1} Y_2^\top.$$

Proof. The proof is omitted and can be found in [57]. $\square$

Furthermore, we present Lemmas A.4 and A.5, which introduce summation inequalities that, when combined with double and triple summation Lyapunov-Krasovskii functionals and Lemma A.3, contribute to deriving less conservative delay-dependent LMIs.

Lemma A.4: Wirtinger-based inequality [69]

For a given symmetric positive definite matrix $R \in \mathbb{S}_+^n$, integers $b > a$, any sequence of discrete-time variable $x : \mathbb{Z}[a, b] \rightarrow \mathbb{R}^n$, the following inequalities hold:

$$\begin{aligned} \sum_{i=a}^{b-1} \eta_i^\top R \eta_i &\geq \frac{1}{b-a} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}^\top \begin{bmatrix} R & 0 \\ 0 & 3 \frac{(b-a+1)}{(b-a-1)} R \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \\ &\geq \frac{1}{b-a} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}^\top \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \end{aligned}$$

where $\eta_i = x_i - x_{i-1}$, $\Theta_1 = x_b - x_a$, and $\Theta_2 = x_b + x_a - \frac{2}{b-a+1} \sum_{i=a}^b x_i$.

Proof. The proof is omitted and can be found in [69]. $\square$

Lemma A.5: [70]

For a given symmetric positive definite matrix $R \in \mathbb{S}_+^n$, three given non-negative integers a, b, k satisfying $a < b \leq k$, a vector function $x(\cdot) \in \mathbb{R}^n$, and denoting $\eta_k = x_{k+1} - x_k$, the following inequalities hold:

$$\begin{aligned} \sum_{s=-b}^{-a-1} \sum_{i=k+s}^{k-a-1} \eta_i^\top R \eta_i &\geq \begin{bmatrix} \Theta_3 \\ \Theta_4 \end{bmatrix}^\top \begin{bmatrix} 2R & 0 \\ 0 & 4R \end{bmatrix} \begin{bmatrix} \Theta_3 \\ \Theta_4 \end{bmatrix} \\ \sum_{s=-b}^{-a-1} \sum_{i=k-b}^{k+s} \eta_i^\top R \eta_i &\geq \begin{bmatrix} \Theta_5 \\ \Theta_6 \end{bmatrix}^\top \begin{bmatrix} 2R & 0 \\ 0 & 4R \end{bmatrix} \begin{bmatrix} \Theta_5 \\ \Theta_6 \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned}
 \Theta_3 &= x_{k-a} - \frac{1}{b-a+1} \sum_{i=k-b}^{k-a} x_i, \\
 \Theta_4 &= x_{k-a} + \frac{2}{b-a+1} \sum_{i=k-b}^{k-a} x_i - \frac{6}{(b-a+2)(b-a+1)} \sum_{s=-b}^{-a} \sum_{i=k+s}^{k-a} x_i, \\
 \Theta_5 &= x_{k-b} - \frac{1}{b-a+1} \sum_{i=k-b}^{k-a} x_i, \\
 \Theta_6 &= x_{k-b} - \frac{4}{b-a+1} \sum_{i=k-b}^{k-a} x_i + \frac{6}{(b-a+2)(b-a+1)} \sum_{s=-b}^{-a} \sum_{i=k+s}^{k-a} x_i.
 \end{aligned} \tag{A.4}$$

Proof. The proof is omitted and can be found in [54]. $\square$

Finally, we introduce Lemma A.6, which presents a set of inequalities known as Finsler's Lemma. The following Lemma provides equivalent conditions for the existence of a matrix that ensures the positive definiteness of a quadratic form.

Lemma A.6: Finsler [74]

If there exist $\eta \in \mathbb{R}^{n_1}$, $\mathcal{Q} \in \mathbb{R}^{n_1 \times n_1}$, and $\mathcal{B} \in \mathbb{R}^{n_2 \times n_1}$ with $\text{rank}(\mathcal{B}) < n_1$, being $\mathcal{B}_\perp$ a basis for the Nullspace of $\mathcal{B}$ (i.e., $\mathcal{B}\mathcal{B}_\perp = 0$), then the following conditions are equivalent

- i) $\eta^\top \mathcal{Q} \eta < 0, \forall \eta \neq 0 : \mathcal{B}\eta = 0$,
- ii) $\mathcal{B}_\perp^\top \mathcal{Q} \mathcal{B}_\perp < 0$,
- iii) $\exists \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^\top \mathcal{B} < 0$,
- iv) $\exists \mathcal{X} \in \mathbb{R}^{n_1 \times n_2} : \mathcal{Q} + \mathcal{X} \mathcal{B} + \mathcal{B}^\top \mathcal{X}^\top < 0$,

where $\mathcal{X}$ is a matrix of slack variables.

Proof. The proof is omitted and can be found in [74]. $\square$