# Symmetries as a guide to regularization ambiguities: Standard and Beyond Standard Model examples

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*Dedico esta tese àquela que sempre me faz ir mais longe do que pensava ser capaz. Obrigado por sempre me apoiar, incondicionalmente, Marceli!*

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# Abstract

This thesis is about regularization of ultraviolet divergences appearing in the perturbative expansion of quantum field theories (QFT's). We present a general view of the ambiguities that may arise in the regularization program and develop a systematic approach to extract them from a general Feynman amplitude. We show that they are intimately related with the breaking of the symmetries of the underlying quantum field theory. We also show they are related to the violation of momentum routing invariance (MRI) of Feynman diagrams, allowing us to regard MRI as a symmetry to be preserved in the perturbative expansion of QFT's. We apply our formalism to a variety of theories and examples such as: the Higgs decay to two photons, the photon-photon scattering, and N=1 supersymmetric electrodynamics. In all cases we verify that regularization ambiguities can be consistently extracted being related to the breaking of the symmetries of the underlying QFT. We also study the role played by quadratic divergences on a variety of examples showing its connection (or not) to the ambiguities aforementioned.

Keywords: Renormalization; Electromagnetic decays; Effective Models in Quantum Field theory; Supersymmetric Models

# Declaration

This thesis is the result of research carried out between March 2011 and July 2014. The work presented in this thesis has not been submitted in fulfillment of any other degree or professional qualification.

No claim of originality is made for chapter 1 and 2. Chapter 3 arose from collaboration between myself, Dr. Marcos Sampaio, Dra. Maria Carolina Nemes, Dra. Brigitte Hiller and Luellerson Ferreira resulting in the publication (doi: 10.1103/PhysRevD.86.025016). Chapter 4 came from a collaboration between myself, Dr. Marcos Sampaio, Dra. Maria Carolina Nemes and Dr. Luis Cabral (doi: 10.1103/PhysRevD.87.065011). Chapter 5 was written in collaboration with Dr. Marcos Sampaio, Dra. Brigitte Hiller, Alexandre Vieira and Dr. Antonio Scarpelli (submitted for publication). Chapter 6 arose from a collaboration between myself, Dr. Marcos Sampaio and Dr. Perez-Victoria during my stay at Universidad de Granada. Appendix A contains two publications which are related to the material presented in this thesis. The first is a collaboration between Dr. G. Gazzola, Dr. Luis Cabral, Dra. Maria Carolina Nemes, Dr. Marcos Sampaio and myself while the second is a collaboration between Jean Felipe, Alexandre Vieira, Dr. Antonio Scarpelli, Dr. Marcos Sampaio and myself.

# Contents

# Chapter 1

# Introduction

The Standard Model of Elementary Particles (SM) is one of the major achievements of the human mind. Relying on aesthetic and simple concepts like symmetries, physicists developed a consistent theory capable of describing three of the four fundamental interactions found in Nature as well as its particle content. Recently, one of the lacking fundamental blocks of the theory (the Higgs boson) was finally found corroborating this journey toward answering a question that mankind has been asking since the Greeks: what are we and Nature itself made of? At present, the answer could be summarized in the table below [1]:

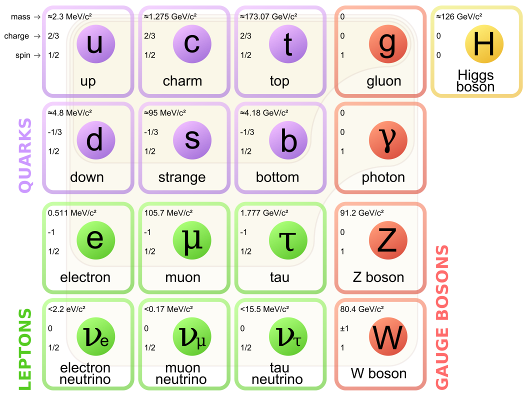



Figure 1.1: Particle content of the Standard Model

More important than presenting all the fundamental particles, it is to know how they actually interact, a feature that is summarized in the equation below

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi + \bar{\psi}_i y_{ij}\psi_j\phi + \text{h.c.} + |D_\mu\phi|^2 - V(\phi). \tag{1.1}$$

From a theoretical point of view, once one has the Lagrangian of a theory, all the techniques and strategies systematized in the subject of Quantum Field Theory can be promptly applied. Pragmatically, one has the following

$\mathcal{L} \rightarrow$ Feynman rules $\rightarrow$ Feynman diagrams $\rightarrow$ compute S-matrix elements

In other words, by using (1.1) one is, in principle, capable of predicting any elementary particle process.

This situation is particular interesting when one intends to obtain the magnetic moment of the muon, for instance. The first point to be emphasized is that there is a quantum mechanical anomaly, which means that the value of the magnetic moment of the muon, after quantization of the classical theory, is different from the one predicted by the classical theory. In the particular example of the muon one has

$$\mu = \frac{e\hbar}{2m_\mu}(1 + a_\mu), \tag{1.2}$$

where $a_\mu$ is the anomalous term. At present, this is one of the most accurate measured quantities in Physics whose value we quote below [2]

$$a_\mu^{\text{exp}} = 116\ 592\ 089(54)_{\text{stat}}(33)_{\text{syst}}(63)_{\text{tot}} \times 10^{-11}\ (\pm 0.54\text{ppm}). \tag{1.3}$$

Such accurate result motivates, from the theoretical side, an equally accurate prediction. The state of art, at present, takes into account QED contributions up to five loops, electroweak (EW) contributions up to two loops and lead with hadronic vacuum polarization loops (HVP) as well as hadronic light-by-light scattering (HLbyL). A recent evaluation is given in [3]

$$a_\mu^{\text{SM}} = 116\ 591\ 802(42)_{\text{HVP}}(26)_{\text{HLbyL}}(2)_{\text{QED+EW}}(49)_{\text{tot}} \times 10^{-11}\ (\pm 0.42\text{ppm}). \tag{1.4}$$

One should notice the incredibly agreement between these two results, which intimately shows the consistency of the Standard Model itself. As mentioned, to obtain such precise theoretical result one has to resort to multiloop calculations, which motivates the development of techniques to deal properly with such involved calculation. To properly discuss the subject of multiloop computations, one has to take a step backward and discuss the regularization/renormalization program itself.

As is well known, in Quantum Field Theory (QFT) one in general has to deal with infinities. They may appear in any calculation even at first order in perturbation theory, which urged a method to make some sense of them.[1].It is in this context that the renormalization program appeared which allows one to group all the infinities in the bare parameters of the theory we are dealing with. If this procedure is always possible, one says that the theory is renormalizable. In the past, renormalizability was a feature that all good theory should share, along with Lorentz invariance, for instance. Nowadays, the perspective has shifted a little, especially after the introduction of the concept of effective field theories (and consistent techniques to deal with them). In any case, even if the requirement of renormalizability is not as strong as it was in the past (it was one of the criterion used in the construction of the Standard Model, for instance), one is not excused away of the task of dealing with infinities which requires the introduction of some regularization method. At present, we have a variety of techniques, all with their advantages and disadvantages. However, all of them share one specific behavior: by making sense of the divergent integral (in other words, evaluating it to some value) one is implicit preserving (or not) some symmetries of the theory at hand. For instance, Dimensional Regularization is known to preserve automatically gauge invariance while introducing a cutoff is known to break such symmetry. In any case, it would be advantageous if, instead of automatically preserving (or not) a symmetry, one had the possibly to identify the ambiguities introduced by the regularization procedure. Therefore, the ambiguities would be fixed by imposing the symmetries (or experimental input) of the model. This feature is particular interesting in the case of quantum mechanical anomalies, for instance, in the ABJ chiral anomaly that we are going to discuss in detail

[1] It should be emphasized that this behavior is not a characteristic of the method of (approximate) computation, but rather from QFT itself since one has to consider interactions as local and excitations of the fields as point particles.

in chapter 3. As we will comment there, the classical model posses a gauge and chiral symmetry, one of which is broken upon quantization. In our point of view, both symmetries are equally important and there is no reason *a priori* to decide which one should be preserved by the regularization method, this choice should be left to experiments.

This is the point of view we are going to develop and systematize through out this thesis. We intend to present a consistent way to deal with regularization ambiguities, more specifically, how to identify and express them even in multiloop calculations. Once identified, we proceed to fix them by imposing the symmetries of theory which can (or not) be always enough to produce a definite answer. In the examples we are going to present in this thesis, which encompass Standard and Beyond Standard Model processes, we will always obtain a definite answer, the only exception being the ABJ chiral anomaly we had already commented about. However, this may not be always the case as discussed in [4] where an ambiguous result is obtained.

## Overview of the thesis

This thesis will be presented in cumulative form. Each subsequent chapter will contain the results of a work already published or submitted.

- Chapter 2 mostly contains the technical part of the thesis. We present the Implicit Regularization technique in its most general form, applicable to multiloop Feynman diagrams. We also present how regularization dependent terms (called surface terms) can be identified and extracted from a general amplitude.

- In chapter 3 we attach a physical meaning for the surface terms introduced in the previous chapter by showing how they are connected to the momentum routing ambiguity one has at depicting any Feynman diagram. We show, for an abelian gauge theory, that the conditions to implement momentum routing invariance (MRI) in the diagrams are exactly the same to demand gauge invariance. Thus, we conjecture that MRI may be a more fundamental condition that all theories should comply with. Finally, we study a scalar theory and shows that demanding MRI is crucial in order to obtain the two-loop universal coefficients of the beta function.

- In chapter 4 we apply the discussion carried out in the two previous chapters in two specific examples: the Higgs decay to two photons and the two photon scattering. In both cases we give an answer for a debate in the literature about the regularization dependent character of finite quantum corrections. We show how the many results found in the literature can be explained in our formalism and discuss, once again, the interplay between MRI and gauge invariance.

- Chapter 5 contains a extended discussion about quadratic divergences. We present how they can be properly treated in our formalism by introducing a general parametrization for divergences. Therefore, we can add a cutoff to our theory but not losing control of regularization dependent terms expressed by surface terms. We revisit some problems by this new perspective such as 1-loop renormalization QCD and the Higgs decay to two photons presented in the previous chapter. We also briefly discuss the importance of quadratic divergences for effective models of QCD (Nambu-Jona-Lasinio model in particular). Finally we present how the hierarchy problem can be translated in our formalism, showing that it is related to ambiguities coming from quadratic divergences.

- Chapter 6 contains a discussion on the background field method in the context of Supersymmetric theories. We use SQED as a probe to study a controversy related to the existence (or not) of higher order divergences in the diagrams needed for the computation of the SQED beta functions. We find that different background field methods furnish different results in this matter. We also find that this problem is unrelated to regularization ambiguities. Surface terms only appears as violating gauge terms as happened before, showing that the results of chapter 3 can be also extended to a supersymmetric theory

# Chapter 2

# Technicalities: the regularization program and how to deal with regularization ambiguities

## Introduction

A consistent renormalization program in QFT appeared after the work of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) [5, 6, 7, 8, 9, 10] in which a prescription to extract recursively the divergences of a multi-loop Feynman graph complying with unitarity, locality and Lorentz invariance was presented. The BPHZ program generalized the Dyson's subtraction to general overlapping diagrams to arbitrary loop order leading to the concept of renormalizable quantum field theory. Such program systematizes, according to the topology of the graph, the subtraction necessary to render the corresponding amplitude finite through the forest formula. The proof of finitude provided by the latter is by construction regularization independent. However, for concrete predictions such as scattering amplitudes in collision processes of elementary particles, the method of Dimensional Regularization (DReg) and minimal subtraction [11], [12] combined with Zimmerman's forest formula has proven to be an efficient and successful calculational tool particularly for gauge theories. The forest formula can be casted into a counterterm language by means of Bogoliubov's recursion formula [13], complying with locality, Lorentz invariance, unitarity and causality.

To calculate S-matrix elements in a symmetry preserving fashion in a quantum field theoretical model sensitive to dimensional continuation on the space-time, the problem is more subtle. The construction of an invariant regularization framework is aesthetically more appealing but this is not the main motivation. Although in one hand imposing constraint equations derived from Ward identities order by order in perturbation theory obliterates the need of an invariant regularization, on the other hand it renders the calculation more involved from the calculational viewpoint. Besides, if quantum symmetry breakings occur in perturbation theory, an invariant scheme is essential to judge it as physical or spurious. Supersymmetric gauge theories are conspicuous examples of models in which regularization and renormalization play a fundamental role especially as new accurate experimental evidence, viz. electroweak precision observables [14, 15, 16, 17], demands consistent theoretical calculations higher than one loop order to understand physics beyond the Standard Model.

Therefore, the construction of an invariant regularization is justified and, in order to be as reliable as DReg (wherever DReg can be applied), it must be shown to comply with locality, Lorentz invariance, unitarity and causality. Recently, an invariant regularization framework (IReg) has been developed and shown to be consistent and symmetry preserving in several instances [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. The essence of the method is to write the divergences in terms of loop integrals in one internal momentum which do not need to be explicitly evaluated. Moreover it acts in the physical dimension of the theory and gauge invariance is controlled by regularization dependent surface terms which when set to zero define a constrained version of IReg (CIReg) and deliver gauge invariant amplitudes automatically. Therefore it is in principle applicable to all physical relevant quantum field theories, supersymmetric gauge theories included. A non trivial question is whether we can generalize this program to arbitrary loop order in consonance with locality, unitarity and Lorentz invariance, especially when overlapping divergences occur. This was the main subject of [34] in which we used the simplest renormalizable field theoretical model to show how to implement IReg in such a way that it displays all the terms to be subtracted by Bogoliubov's recursion formula automatically. All the other physical theories can be treated within the same strategy after space time and internal algebra are performed.

## 2.1 The rules of implicit regularization

We restrict ourselves to massless theories and power counting infrared safe integrals. The first restriction is justified because, as we showed in [34], to implement a mass independent renormalization scheme in IReg we need only the massless basic divergent integrals that we present below. When infrared divergences do appear, a dual version of IReg operating in coordinate space displays infrared divergences as basic divergent integrals as well, in a way that infrared and ultraviolet degrees of freedom are clearly distinguished [35, 36, 37].

Given the amplitude of a $n$-loop Feynman graph with $L$ external legs, the basic strategy of IReg is to free all divergences of external momenta and express them in terms of basic divergent integrals in one loop momentum only. To achieve this purpose, we need to perform $(n-1)$ integrations, but the order in which they are performed is not clear a priori. In [34] we presented a systematic way to choose the order of integration which, as a byproduct, displays the counterterms to be subtracted by Bogoliubov's recursion formula. Considering that we made this choice, we can redefine the internal momenta in such a way that the integral in $k_l$ is the l-th we are going to deal with and it is typically of the form

$$I^{\nu_1\dots\nu_m} = \int_{k_l} \frac{A^{\nu_1\dots\nu_m}(k_l,q_i)}{\prod_i[(k_l-q_i)^2-\mu^2]} \ln^{l-1}\left(-\frac{k_l^2-\mu^2}{\lambda^2}\right), \tag{2.1}$$

where $l = 1\cdots n$. In the above equation, $q_i$ is an element (or combination of elements) of the set $\{p_1,\dots,p_L,k_{l+1},\dots,k_n\}$, $\int_{k_l} \equiv \int d^dk_l/(2\pi)^d$ and $\mu^2$ is an infrared regulator.

Since the original integral is infrared safe, the limit $\mu^2 \to 0$ is well-defined and must be taken in the end of the calculation. The logarithmical dependence appears because this is the characteristic behaviour of the finite part of massless amplitudes [38]. $\lambda$ is an arbitrary non-vanishing parameter with dimension of mass which parametrizes the freedom one has to subtract the divergences (renormalization group scale). It appears at one loop level and survives to higher orders through a regularization independent mathematical identity (eq. 2.11) as we show in the end of this section. The function $A^{\nu_1\cdots\nu_m}(k_l, q_i)$ may contain constants and all possible combinations of $k_l$ and $q_i$ compatible with the Lorentz structure. Care must be exercised when it contains a term like $(k_l - q_i)^2$. In this case, we must cancel it against one of the denominators because, as we are dealing with divergent integrals, symmetric integration is a forbidden operation [23], [39]. In chapter 4 we will study a particular example in which this prescription is vital to obtain a well defined, gauge invariant result.

Now, we apply the rules of IReg. Assuming that a regulator $\Lambda$ is implicit in the integral, we can use the following mathematical identity in the denominators:

$$\frac{1}{(k_l-q_i)^2-\mu^2} = \sum_{j=0}^{n_i^{(k_l)}-1} \frac{(-1)^j (q_i^2 - 2q_i\cdot k_l)^j}{(k_l^2-\mu^2)^{j+1}} + \frac{(-1)^{n_i^{(k_l)}}(q_i^2-2q_i\cdot k_l)^{n_i^{(k_l)}}}{(k_l^2-\mu^2)^{n_i^{(k_l)}}\left[(k_l-q_i)^2-\mu^2\right]}. \tag{2.2}$$

The values of $n_i^{(k_l)}$ are chosen such that all divergent integrals have a denominator free of $q_i$.

After the use of (2.2), the divergent integrals can be casted as a combination of

$$I_{log}^{(l)}(\mu^2) \equiv \int_{k_l}^{\Lambda} \frac{1}{(k_l^2-\mu^2)^{d/2}} \ln^{l-1}\left(-\frac{k_l^2-\mu^2}{\lambda^2}\right), \tag{2.3}$$

$$I_{log}^{(l)\nu_1\cdots\nu_r}(\mu^2) \equiv \int_{k_l}^{\Lambda} \frac{k_l^{\nu_1}\cdots k_l^{\nu_r}}{(k_l^2-\mu^2)^{\beta}} \ln^{l-1}\left(-\frac{k_l^2-\mu^2}{\lambda^2}\right), \tag{2.4}$$

or

$$I_{quad}^{(l)}(\mu^2) \equiv \int\limits_{k_l}^{\Lambda} \frac{1}{(k_l^2-\mu^2)^{\frac{d-2}{2}}} \ln^{l-1}\left(-\frac{k_l^2-\mu^2}{\lambda^2}\right), \tag{2.5}$$

$$I_{quad}^{(l)\nu_1\cdots\nu_{r+2}}(\mu^2) \equiv \int\limits_{k_l}^{\Lambda} \frac{k_l^{\nu_1}\cdots k_l^{\nu_{r+2}}}{(k_l^2-\mu^2)^{\beta}} \ln^{l-1}\left(-\frac{k_l^2-\mu^2}{\lambda^2}\right), \tag{2.6}$$

where $r = 2\beta - d$. It is important to note that only these type of divergences appear because linear BDI's always vanish[1]. Although we have already reduced the divergences to basic divergent integrals free of external momenta, we can show that the integrals defined above are related. For example, in a case with two Lorentz indices we have

$$I_{log}^{(l)\,\mu\nu}(\mu^2) = \sum_{j=1}^{l}\left(\frac{2}{d}\right)^j \frac{(l-1)!}{(l-j)!}\left\{\frac{g^{\mu\nu}}{2} I_{log}^{(l-j+1)}(\mu^2) - \frac{1}{2}\Upsilon_0^{(l)\,\mu\nu}\right\}, \tag{2.7}$$

where $\Upsilon_0^{(l)\,\mu\nu} \equiv \int\limits_k \frac{\partial}{\partial k^\mu}\left[\frac{k^\nu}{(k^2-\mu^2)^{d/2}} \ln^{l-j}\left(-\frac{k^2-\mu^2}{\lambda^2}\right)\right]$, and

$$I_{quad}^{(l)\,\mu\nu}(\mu^2) = \sum_{j=1}^{l}\left(\frac{2}{d-2}\right)^j \frac{(l-1)!}{(l-j)!}\left\{\frac{g^{\mu\nu}}{2} I_{quad}^{(l-j+1)}(\mu^2) - \frac{1}{2}\Upsilon_2^{(l)\,\mu\nu}\right\}, \tag{2.8}$$

where $\Upsilon_2^{(l)\,\mu\nu} \equiv \int\limits_k \frac{\partial}{\partial k^\mu}\left[\frac{k^\nu}{(k^2-\mu^2)^{\frac{d-2}{2}}} \ln^{l-j}\left(-\frac{k^2-\mu^2}{\lambda^2}\right)\right]$.

In the previous equations, $\Upsilon_0^{(l)\,\mu\nu}$, and $\Upsilon_2^{(l)\,\mu\nu}$ are (arbitrary) regularization dependent surface terms. In a more general case they are given by

$$\Upsilon_i^{(l)\nu_1\cdots\nu_j} \equiv \int \frac{d^dk}{(2\pi)^d} \frac{\partial}{\partial k_{\nu_1}} \frac{k^{\nu_2}\cdots k^{\nu_j}}{(k^2-\mu^2)^{\frac{d+j-2-i}{2}}} \ln^{l-1}\left[-\frac{(k^2-\mu^2)}{\lambda^2}\right]. \tag{2.9}$$

[1] We are considering only theories with even dimensions. For odd dimensions, an equivalent prescription can be carried out.

An equivalent definition in terms of Lorentz scalar objects $\Gamma_i^{(l,j)}$ is

$$g^{\{\nu_1\cdots\nu_j\}}\Gamma_i^{(l,j)} \equiv \Upsilon_i^{(l)\nu_1\cdots\nu_j}, \tag{2.10}$$

where $g^{\{\nu_1\cdots\nu_j\}} \equiv g^{\nu_1\nu 2}\cdots g^{\nu_{j-1}\nu_j}+$ symmetric combinations.

Finally the divergences can be written in terms of (2.3) and (2.5). However from (2.3) we see that this integral is ultraviolet and infrared divergent as $\mu^2 \to 0$. In order to separate these divergences and define a genuine ultraviolet divergent object we use the regularization independent relation

$$I_{log}^{(l)}(\mu^2) = I_{log}^{(l)}(\lambda^2) - \frac{b_d}{l}\ln^l\left(\frac{\mu^2}{\lambda^2}\right) + b_d\sum_{k=1}^{A}\binom{A}{k}\sum_{j=1}^{l-1}\frac{(-1)^k}{k^j}\frac{(l-1)!}{(l-j)!}\ln^{l-j}\left(\frac{\mu^2}{\lambda^2}\right), \tag{2.11}$$

$$\text{where} \quad \lambda^2 \neq 0, \;\; A \equiv \frac{(d-2)}{2}, \;\; b_d \equiv \frac{i}{(4\pi)^{d/2}}\frac{(-1)^{d/2}}{\Gamma(d/2)}. \tag{2.12}$$

For infrared safe models the infrared divergence must cancel in the amplitude as a whole. This in fact occurs because, as we use identity (2.2), the finite part of the amplitude will also have a logarithmical dependence in $\mu^2$ which exactly cancels the infrared divergence coming from the use of the scale relation (2.11).

This procedure was shown to comply with unitarity, locality and Lorentz invariance in [34]. The whole program is compatible with overlapping divergences through the Bogoliubov's recursion formula which means that the divergent content of an arbitrary Feynman graph can always be cast as a basic divergent integral.

# Chapter 3

# Momentum Routing Invariance in Feynman Diagrams and Quantum Symmetry Breakings

## Introduction

In the 1972 seminal paper by 't Hooft and Veltman [12] on dimensional regularization (DReg), they emphasized that besides respecting unitarity and causality, DReg also allowed for shifts in integration variables (loop momenta) of Feynman amplitudes. When no quantum symmetry breakings occurred, Ward identities were automatically satisfied crowning DReg as the ideal framework to handle ultraviolet divergences in perturbation theory of gauge theories. Indeed it is well known that the possibility of shifts in the integration variable is an important ingredient for diagrammatic proofs of gauge invariance in quantum electrodynamics.

In the early eighties, motivated by the construction of a framework applicable to models which are incompatible with dimensional continuation on the space-time dimension (for instance supersymmetric, chiral and topological quantum field theories), Elias, McKeon, Mann and collaborators [40] brought back the problem on loop momentum routing ambiguities. The latter stem from shift of integration variable surface terms which appear in the integer dimension but not in dimensionally continued space-times. Such ambiguities were used by the authors to warrant the validity of Ward-Slavnov-Taylor identities in some model calculations at one-loop level. In other words, this approach, called Preregularization, used integration variable ambiguities in four dimensional loop integrals to parametrize the divergent amplitudes in a way that Ward identities were preserved by a suitable choice of the routing labels in the loop of a diagram. Anomalies, such as the well known Adler-Bardeen-Bell-Jackiw triangle chiral anomaly, appear in this approach when the ambiguities proved themselves insufficient to preserve the full set of symmetry identities valid at classical level.

Of course Dimensional Reduction (DRed) is the most popular tool to perform Feynman diagram calculations in supersymmetric gauge theories and other dimension specific models which require modifications in gauge symmetry preserving dimensional regularization. Generalizations of DRed which ensure invariance to two-loop order in models of phenomenological importance have been done [41], but it is still unknown to what extent it preserves supersymmetry. An *invariant* regularization framework , which avoids the task of computing symmetry restoring counterterms order by order in perturbation theory for dimensional sensitive theories is hitherto unknown. In other words, the construction of an invariant regularization scheme to compute with supersymmetric gauge theories is justified on practical and theoretical grounds.

The main purpose of this contribution is to argue that any regularization framework that operates on the physical dimension of the theory is invariant (that is to say, does not lead to quantum symmetry breakings of classical symmetries) if momentum routing invariance (MRI) in the Feynman diagrams is respected. The reverse of the coin is that anomalies manifest themselves as breaking of momentum routing invariance, which has been conjectured before by Jackiw in [42]. Moreover we argue that Implicit Regularization (IReg) is the ideal arena to formulate and illustrate this idea to arbitrary loop order, and therefore can be an useful tool for Feynman diagram calculations when dimensional methods become dodgy.

## 3.1 Momentum Routing Invariance and Surface Terms at $n$-loop order

In this section we study the conditions which guarantee MRI to an arbitrary multi-loop Feynman diagram. We find that the only condition needed to preserve such symmetry is to set surface terms to zero.

As it is well known, if $f(k)$ is an arbitrary function, then

$$\begin{aligned} f(k+a) &= f(k) + a_\sigma \frac{\partial}{\partial k_\sigma} f(k) + \frac{a_\sigma a_\rho}{2!} \frac{\partial}{\partial k_\sigma k_\rho} f(k) + \cdots \\ &= exp\left(a_\sigma \frac{\partial}{\partial k_\sigma}\right) f(k). \end{aligned} \tag{3.1}$$

We now consider an arbitrary graph at one-loop order. Setting $k$ as its internal momentum, and $q_i$ as the external momenta we will have in general, for any theory, a vertex of the type depicted in figure 3.1.

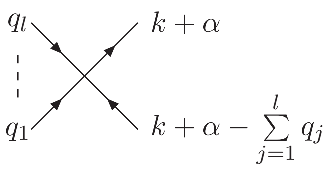



Figure 3.1: Generic vertex with arbitrary momentum routing $\alpha$.

Therefore, the amplitude of this graph can be expressed as

$$A \equiv \int \frac{d^dk}{(2\pi)^d} f(k+\alpha, q_i), \tag{3.2}$$

where for simplicity we consider a scalar amplitude, although the generalization for amplitudes with an arbitrary number of Lorentz indices is straightforward. We now present the cornerstone of our argument: if we have momentum routing invariance then

$$\int \frac{d^dk}{(2\pi)^d} f(k+\alpha, q_i) - \int \frac{d^dk}{(2\pi)^d} f(k+\beta, q_i) = 0, \tag{3.3}$$

must be satisfied for arbitrary $\alpha$ and $\beta$. Using identity (3.1) it reduces to

$$\int \frac{d^dk}{(2\pi)^d}\left[exp\left(\alpha_\sigma \frac{\partial}{\partial k_\sigma}\right)\right] f(k,q_i) - \int \frac{d^dk}{(2\pi)^d}\left[exp\left(\beta_\sigma \frac{\partial}{\partial k_\sigma}\right)\right] f(k,q_i) = 0. \quad (3.4)$$

For simplicity, we will consider $f(k, q_i)$ as a linear divergent integral. Therefore, by applying a regularization (for example, IReg) one obtains

$$\begin{aligned}
&\int \frac{d^dk}{(2\pi)^d}\left[f_{lin}(k,q_i) + f_{log}(k,q_i) + f_{fin}(k,q_i) + \alpha_\sigma \frac{\partial}{\partial k_\sigma} f_{lin}(k,q_i)\right] \\
&- \int \frac{d^dk}{(2\pi)^d}\left[f_{lin}(k,q_i) + f_{log}(k,q_i) + f_{fin}(k,q_i) + \beta_\sigma \frac{\partial}{\partial k_\sigma} f_{lin}(k,q_i)\right] = 0, \\
&(\alpha_\sigma - \beta_\sigma)\int \frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma} f_{lin}(k,q_i) = 0 \qquad (3.5)
\end{aligned}$$

since if $B \equiv \int \frac{d^dk}{(2\pi)^d}\frac{\partial f(k)}{\partial k^\sigma}$ and $f(k)$ is finite or logarithmic divergent, then by Gauss' theorem $B$ is null.

This would be the condition to implement MRI for linear divergent integrals. To proceed and make explicit the connection with surface terms, we must analyze further the function $f(k, q_i)$. After space-time and internal group algebra are performed, $f(k, q_i)$ is given by

$$f(k,q_i) = \frac{g(k,q_i)}{\prod_{j=1}^{L}\left[(k + l_j(q_i))^2 - \mu^2\right]} \qquad (3.6)$$

where $g(k, q_i)$ and $l_j(q_i)$ are polynomials in the momenta and $\mu^2$ is an infrared regulator. The divergence of such integral is controlled by the dimension of the theory ($d$), the number of internal lines ($L$), and the degree in $k$ of the polynomial $g(k, q_i)$ which we define as $m$.

Evidently, if $d + m - 2L \leq 0$ condition (3.4) is automatically satisfied as logarithmic divergent graphs are always momentum routing invariant. We proceed to linear divergent integrals which, after using the identity of IReg in all propagators one time, furnishes:

$$f(k,q_i) = f_{lin}(k,q_i) + f_{log}(k,q_i) + f_{fin}(k,q_i). \qquad (3.7)$$

In view of the comments above only the first term is of interest to us. Its general form is given by

$$\int \frac{d^d k}{(2\pi)^d} f_{lin}(k, q_i) = \int \frac{d^d k}{(2\pi)^d} \frac{\prod_i (k \cdot q_i)^{b_i} \prod_k (q_i \cdot q_k)^{c_{ik}}}{[k^2 - \mu^2]^L}, \tag{3.8}$$

where $d + s - 2L = 1$, $s \equiv \sum_i b_i$, and we canceled powers of $k^2$ in the numerator against propagators. This cancellation must always be performed because, as we are dealing with divergent integrals, symmetric integration is forbidden [23], [39].

In this case, condition (3.4) is satisfied only if

$$(\alpha_\sigma - \beta_\sigma) \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k_\sigma} f_{lin}(k, q_i) = (\alpha_\sigma - \beta_\sigma) h_{\nu_1 \cdots \nu_s}(q_i) \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k_\sigma} \frac{k^{\nu_1} \cdots k^{\nu_s}}{[k^2 - \mu^2]^L} = 0. \tag{3.9}$$

Since $\alpha$, $\beta$ are arbitrary, and $h_{\nu_1 \cdots \nu_s}(q_i)$ is a polynomial in $q_i$, we notice that the condition above is equivalent to the statement (see equation (2.9))

$$\Upsilon_0^{(1)\sigma\nu_1 \cdots \nu_s} = 0. \tag{3.10}$$

In other words, momentum routing invariance is guaranteed for linearly divergent graphs only if surface terms are systematically set to zero. We consider now quadratically divergent integrals which, after using identity (2.2) in all propagators two times, gives:

$$f(k, q_i) = f_{quad}(k, q_i) + f_{lin}(k, q_i) + f_{log}(k, q_i) + f_{fin}(k, q_i). \tag{3.11}$$

Since linear, logarithmical and finite cases were already analysed, we only need to deal with the first term which is

$$\int \frac{d^d k}{(2\pi)^d} f_{quad}(k, q_i) = \int \frac{d^d k}{(2\pi)^d} \frac{\prod_i (k \cdot q_i)^{b_i} \prod_k (q_i \cdot q_k)^{c_{ik}}}{[k^2 - \mu^2]^L}, \tag{3.12}$$

where $d + s - 2L = 2$, and $s \equiv \sum_i b_i$.

Now, condition (3.4) is satisfied only if

$$(\alpha_\sigma-\beta_\sigma)\int\frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma}f_{quad}(k,q_i)=(\alpha_\sigma-\beta_\sigma)h_{\nu_1\cdots\nu_s}(q_i)\int\frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma}\frac{k^{\nu_1}\cdots k^{\nu_s}}{[k^2-\mu^2]^L}=0, \tag{3.13}$$

and

$$(\alpha_\sigma-\beta_\sigma)(\alpha_\rho-\beta_\rho)\int\frac{d^dk}{(2\pi)^d}\frac{\partial^2}{\partial k_\sigma\partial k_\rho}f_{quad}(k,q_i)=(\alpha_\sigma-\beta_\sigma)(\alpha_\rho-\beta_\rho)h_{\nu_1\cdots\nu_s}(q_i)\times$$
$$\times\Bigg[g^{\nu_1\rho}\int\frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma}\frac{k^{\nu_2}\cdots k^{\nu_s}}{[k^2-\mu^2]^L}+\sum_{j=2}^{s-1}g^{\nu_j\rho}\int\frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma}\frac{k^{\nu_1}\cdots k^{\nu_{j-1}}k^{\nu_{j+1}}\cdots k^{\nu_s}}{[k^2-\mu^2]^L}+$$
$$+g^{\nu_s\rho}\int\frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma}\frac{k^{\nu_1}\cdots k^{\nu_{s-1}}}{[k^2-\mu^2]^L}-2L\int\frac{d^dk}{(2\pi)^d}\frac{\partial}{\partial k_\sigma}\frac{k^{\nu_1}\cdots k^{\nu_s}k^\rho}{[k^2-\mu^2]^{L+1}}\Bigg]=0. \tag{3.14}$$

The relations above are equivalent to the statements

$$\Upsilon_1^{(1)\sigma\nu_1\cdots\nu s}=0,$$
$$\Upsilon_0^{(1)\sigma\nu_2\cdots\nu_s}=0,\quad \Upsilon_0^{(1)\sigma\nu_1\cdots\nu_{j-1}\nu_{j+1}\cdots\nu_s}=0,\quad \Upsilon_0^{(1)\sigma\nu_1\cdots\nu_{s-1}}=0\quad\text{and}\quad \Upsilon_0^{(1)\sigma\nu_1\cdots\nu_s\rho}=0. \tag{3.15}$$

Therefore, we conclude that momentum routing invariance is guaranteed for quadratically divergent graphs only if surface terms are systematically set to zero. This result can be generalized to graphs with any kind of divergence proving that momentum routing invariance is verified only if we set all surface terms to zero. Although our results are restricted to one-loop order in perturbation theory, a general proof for arbitrary Feynman diagrams can also be developed. We present in the following the demonstration for two-loop graphs and state how it can be further generalized to an arbitrary number of loops.

Assume that the amplitude of a two-loop process is given by,

$$A^{(2)}\equiv\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}f(k_1+\alpha_1,k_2+\alpha_2,q_i), \tag{3.16}$$

in which space-time and internal group algebra have already been performed, and $\alpha_1$, $\alpha_2$ are arbitrary momentum routings that depend only on the external momenta $q_i$. Once again, momentum routing invariance is guaranteed by the condition

$$\int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \left[ \prod_{j=1}^{2} exp\left( \alpha_j^{\sigma_j} \frac{\partial}{\partial k_j^{\sigma_j}} \right) - \prod_{j=1}^{2} exp\left( \beta_j^{\sigma_j} \frac{\partial}{\partial k_j^{\sigma_j}} \right) \right] f(k_1, k_2, q_i) = 0. \tag{3.17}$$

At this point we must use the rules of IReg in one of the integrals, however, which one must be evaluated first is not clear *a priori*. Solving this problem was the main purpose of [34] in which we presented a prescription that systematizes the order of integration for multi-loop Feynman diagrams. Therefore, using this prescription, amplitude $A^{(2)}$ can be decomposed in three cases:

$$A^{(2)} = A_{k_1} + A_{k_2} + A_{fin}, \tag{3.18}$$

where in $A_{k_i}$ the integration over $k_i$ must be performed first, and $A_{fin}$ contains only finite terms which do not contribute. Once the order of integration has been determined we notice that condition (3.17) reduces to

$$\int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \prod_{j=1}^{2} \left( \frac{\partial}{\partial k_j^{\sigma_j}} \right)^{m_j} [A_{k_1} + A_{k_2}] = 0, \quad \forall\, m_j \in \mathbb{N}. \tag{3.19}$$

Since the proof for $A_{k_1}$ is essentially the same for $A_{k_2}$, we just consider the latter in the following. Remembering that $A_{k_2}$ is a function of $k_1$, $k_2$, and $q_i$ we notice that it can be rewritten as

$$A_{k_2} = \underbrace{\frac{1}{\prod_r [(k_1 + l_r(q_i))^2 - \mu^2]}}_{A_{k_2}^{(k_1)}} \underbrace{\frac{g(k_2, k_1, q_i)}{\prod_j [(k_2 + l_j(k_1, q_i))^2 - \mu^2]}}_{A_{k_2}^{(k_1,k_2)}}. \tag{3.20}$$

Therefore, we just have to prove that the condition below always holds

$$\int \frac{d^d k_1}{(2\pi)^d} \left( \frac{\partial}{\partial k_1^{\sigma_1}} \right)^{m_1} A_{k_2}^{(k_1)} \int \frac{d^d k_2}{(2\pi)^d} \left( \frac{\partial}{\partial k_2^{\sigma_2}} \right)^{m_2} A_{k_2}^{(k_1,k_2)} = 0. \tag{3.21}$$

We have two cases: $m_2 \neq 0$, and $m_2 = 0$. In the first one we can use the one-loop results to obtain terms of the type

$$\int \frac{d^d k_1}{(2\pi)^d} \left(\frac{\partial}{\partial k_1^{\sigma_1}}\right)^{m_1} \frac{1}{\prod\limits_r [(k_1 + l_r(q_i))^2 - \mu^2]} \times h_{\nu_1 \cdots \nu_s}(k_1, q_i) \Upsilon_j^{(1)\nu_1 \cdots \nu_s} = 0, \quad (3.22)$$

which is satisfied if all one-loop order surface terms are systematically set to zero. In the second case, we must use the rules of IReg in the $k_2$ integral to obtain

$$\int \frac{d^d k_1}{(2\pi)^d} \left(\frac{\partial}{\partial k_1^{\sigma_j}}\right)^{m_1} \frac{1}{\prod\limits_r [(k_1 + l_r(q_i))^2 - \mu^2]} \times h(k_1, q_i) \left[\text{BDI's} + \Upsilon_j^{(1)\nu_1 \cdots \nu_s}\right] +$$
$$+ \int \frac{d^d k_1}{(2\pi)^d} \left(\frac{\partial}{\partial k_1^{\sigma_j}}\right)^{m_1} \frac{1}{\prod\limits_r [(k_1 + l_r(q_i))^2 - \mu^2]} \times \left[\sum_{s=1}^{2} a_s(k_1, q_i) \ln^{s-1}(k_1, q_i)\right] = 0, \quad (3.23)$$

where $a_s(k_1, q_i)$ is a polynomial. Since $m_1 \neq 0$ we may use a similar analysis done in the one-loop case to show that the first integral is proportional to one-loop surface terms whereas the second is proportional to one-loop and two-loop ones. Therefore, we achieve our major goal: all terms involved in (3.17) are proportional to surface terms of one-loop and two-loop order, showing that momentum routing is guaranteed only if surface terms are systematically set to zero. This conclusion is not restricted to two-loop oder, since a similar demonstration can be performed to graphs with an arbitrary number of loops. Thus, we may state that the condition to implement momentum routing invariance is to set surface terms of all orders to zero.

## 3.2 Momentum Routing Invariance and QED Ward Identities

In the following we study the consequences of imposing MRI by setting surface terms to zero. Our starting point is a diagrammatic proof of abelian gauge invariance to arbitrary order in perturbation theory. As is well known, such proof lies on the possibility of performing shifts in the internal momenta of divergent integrals [43]. Here we show that this is ultimately connected to MRI in the

Feynman diagrams. To this end we use IReg in the integrals needed in the diagrammatic proof, and study under which conditions the Ward Identities are respected. In what follows we use a pictoric representation (figure 3.2) of the Ward Identities as in [43], namely:

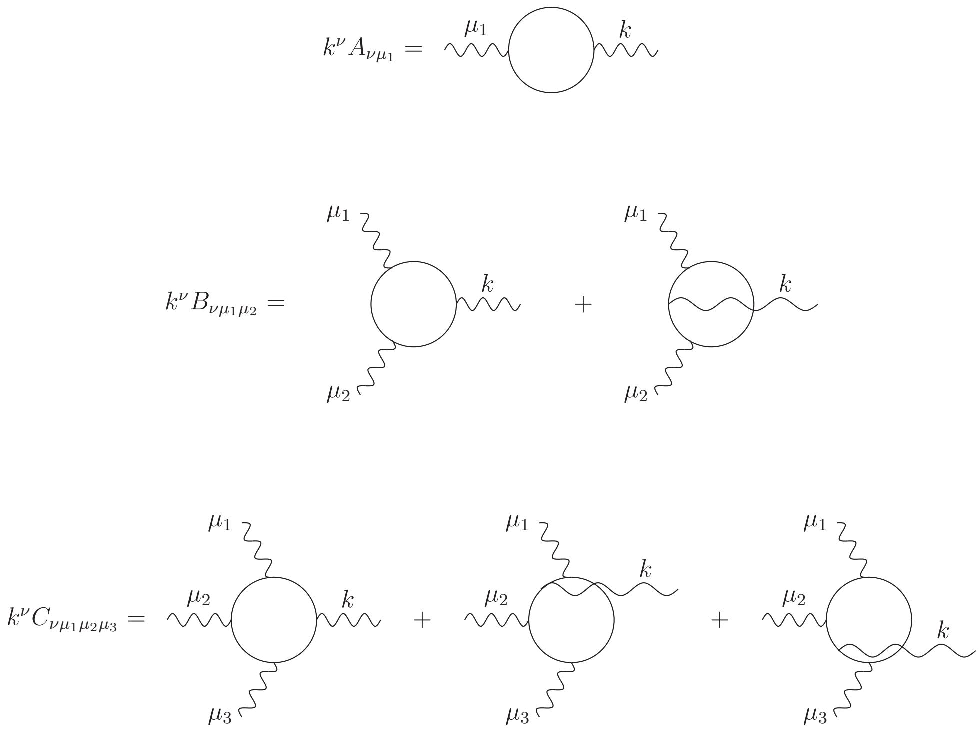



Figure 3.2: Pictoric representation of QED Ward identities $k^\nu A_{\nu\mu_1} = 0$, $k^\nu B_{\nu\mu_1\mu_2} = 0$, and $k^\nu C_{\nu\mu_1\mu_2\mu_3} = 0$. Diagrams with more than four external legs are finite and shifts are obviously allowed.

Explicitly,

$$k^\nu A_{\nu\mu_1} = Tr\int_p \gamma_{\mu_1}\left(\frac{1}{\not p + \not\alpha + \not k}\right)\not k\left(\frac{1}{\not p + \not\alpha}\right), \tag{3.24}$$

$$\begin{aligned} k^\nu B_{\nu\mu_1\mu_2} &= Tr\int_p \gamma_{\mu_2}\left(\frac{1}{\not p + \not\beta + \not k + \not q}\right)\gamma_{\mu_1}\left(\frac{1}{\not p + \not\beta + \not k}\right)\not k\left(\frac{1}{\not p + \not\beta}\right) \\ &+ Tr\int_p \gamma_{\mu_2}\left(\frac{1}{\not p + \not\beta + \not k + \not q}\right)\not k\left(\frac{1}{\not p + \not\beta + \not q}\right)\gamma_{\mu_1}\left(\frac{1}{\not p + \not\beta}\right), \end{aligned} \tag{3.25}$$

$$\begin{aligned} k^\nu C_{\nu\mu_1\mu_2\mu_3} &= Tr\int_p \gamma_{\mu_3}\left(\frac{1}{\not P + \not k + \not Q}\right)\gamma_{\mu_2}\left(\frac{1}{\not P + \not k + \not q_1}\right)\gamma_{\mu_1}\left(\frac{1}{\not P + \not k}\right)\not k\left(\frac{1}{\not P}\right) \\ &+ Tr\int_p \gamma_{\mu_3}\left(\frac{1}{\not P + \not k + \not Q}\right)\gamma_{\mu_2}\left(\frac{1}{\not P + \not k + \not q_1}\right)\not k\left(\frac{1}{\not P + \not q_1}\right)\gamma_{\mu_1}\left(\frac{1}{\not P}\right) \\ &+ Tr\int_p \gamma_{\mu_3}\left(\frac{1}{\not P + \not k + \not Q}\right)\not k\left(\frac{1}{\not P + \not Q}\right)\gamma_{\mu_2}\left(\frac{1}{\not P + \not q_1}\right)\gamma_{\mu_1}\left(\frac{1}{\not P}\right), \end{aligned} \tag{3.26}$$

where $P \equiv p + \delta$, $\alpha$, $\beta$, and $\delta$ are arbitrary routings and $\not Q \equiv \not q_1 + \not q_2$. By using IReg we finally obtain:

$$k^\nu A_{\nu\mu_1} = -4\Gamma_2^{(1,2)}k_{\mu_1} + 4\left(\Gamma_0^{(1,2)} - 4\Gamma_0^{(1,4)}\right)\left[k\cdot(k+2\alpha)\alpha_{\mu_1} + (k+\alpha)^2 k_{\mu_1}\right], \tag{3.27}$$

$$\begin{aligned} k^\nu B_{\nu\mu_1\mu_2} = -4\left(\Gamma_0^{(1,2)} - 4\Gamma_0^{(1,4)}\right)\big[&g_{\mu_1\mu_2}k\cdot(k+q+2\beta) + k_{\mu_1}(k+q+2\beta)_{\mu_2} + \\ &+ k_{\mu_2}(k+q+2\beta)_{\mu_1}\big], \end{aligned} \tag{3.28}$$

$$k^\nu C_{\nu\mu_1\mu_2\mu_3} = 4\left(\Gamma_0^{(1,2)} - 4\Gamma_0^{(1,4)}\right)\left[g_{\mu_1\mu_2}k_{\mu_3} + g_{\mu_1\mu_3}k_{\mu_2} + g_{\mu_2\mu_3}k_{\mu_1}\right]. \tag{3.29}$$

with surface terms defined in eq. (2.10). We notice that Ward identities are fulfilled provided

$$\Gamma_2^{(1,2)} = \Gamma_0^{(1,2)} = \Gamma_0^{(1,4)} = 0. \tag{3.30}$$

Thus by adopting an abelian gauge invariant regularization (setting surface terms to zero) one automatically preserves MRI.

Conversely, we could study the conditions under which the diagrams involved in the diagrammatic proof [43] of gauge invariance respect MRI

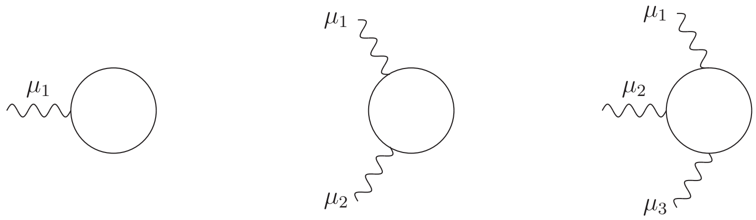



Figure 3.3: Diagrams needed in the diagramatic proof of abelian gauge invariance.

Performing the calculation one notices that MRI is respected if

$$\Gamma_2^{(1,2)} = \Gamma_0^{(1,2)} = \Gamma_0^{(1,4)} = 0, \tag{3.31}$$

which are the *same* conditions to preserve the Ward identities. Therefore we conclude that MRI is a necessary and sufficient condition to preserve abelian gauge symmetry at arbitrary loop order. This should be emphasized that the previous conclusion regards a theory free of chiral couplings (proportional to $\gamma_5$). This feature can be easily seen in the Adler-Bardeen-Bell-Jackiw (ABJ) chiral anomaly in which the axial Ward identity must be violated if momentum routing invariance is to be respected.

The ABJ chiral anomaly was already studied using IReg [19]. Here we would like to revisit the problem by a different point of view, in consonance with an ongoing work about anomalies in CPT-violating extensions of the Standard Model [44]. As pointed out in [45], chiral theories, if not properly treated, can furnish ambiguous results. Using the IReg framework has some advantages in this aspect since, independently of the procedure used to deal with the $\gamma_5$ matrix (whose algebra is the source of the ambiguities), all ambiguous/regularization dependent terms can be parametrized as surfaces terms. However, some caution should be exercised if we want to connect surfaces terms with momentum routing ambiguities. In such case, a defined prescription should be followed. In the particular case of the ABJ chiral anomaly, the trace involving the $\gamma_5$ matrix should be symmetrized, as already mentioned in [46]. Explicitly we have

$$
\begin{aligned}
\frac{1}{4}\mathrm{Tr}\left[\gamma_5\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\rho\gamma_\lambda\right] &= \epsilon_{\alpha\mu\beta\nu}g_{\rho\lambda} - \epsilon_{\alpha\mu\beta\rho}g_{\nu\lambda} + \epsilon_{\alpha\mu\nu\rho}g_{\beta\lambda} - \epsilon_{\alpha\beta\nu\rho}g_{\mu\lambda}+ \\
&+\epsilon_{\mu\beta\nu\rho}g_{\alpha\lambda} - \epsilon_{\lambda\alpha\mu\beta}g_{\rho\nu} + \epsilon_{\lambda\alpha\mu\nu}g_{\rho\beta} - \epsilon_{\lambda\alpha\beta\nu}g_{\rho\mu} + \epsilon_{\lambda\mu\beta\nu}g_{\rho\alpha} - \epsilon_{\lambda\rho\alpha\mu}g_{\nu\beta} + \\
&+\epsilon_{\lambda\rho\alpha\beta}g_{\nu\mu} - \epsilon_{\lambda\rho\mu\beta}g_{\nu\alpha} + \epsilon_{\lambda\rho\nu\alpha}g_{\mu\beta} - \epsilon_{\lambda\rho\nu\mu}g_{\alpha\beta} + \epsilon_{\lambda\rho\nu\beta}g_{\alpha\mu}.
\end{aligned}
\tag{3.32}
$$

After this comment, we proceed to the evaluation of the anomaly itself. The relevant diagrams are depicted in figure 3.4, whose amplitudes are given by

$$
T^{VVA}_{\mu\nu\alpha} = -\,Tr\int_k \gamma_\nu\left(\frac{1}{\not{k}+\not{k}_1}\right)\gamma_\mu\left(\frac{1}{\not{k}+\not{k}_3}\right)\gamma_\alpha\gamma_5\left(\frac{1}{\not{k}+\not{k}_2}\right) + (\mu\leftrightarrow\nu, p\leftrightarrow q),
\tag{3.33}
$$

where

$$
k_1 \equiv \alpha p + (\beta-1)q, \quad k_2 \equiv \alpha p + \beta q, \quad k_3 \equiv (\alpha-1)p + (\beta-1)q,
\tag{3.34}
$$

and $\alpha$, $\beta$ parametrize arbitrary momentum routings.

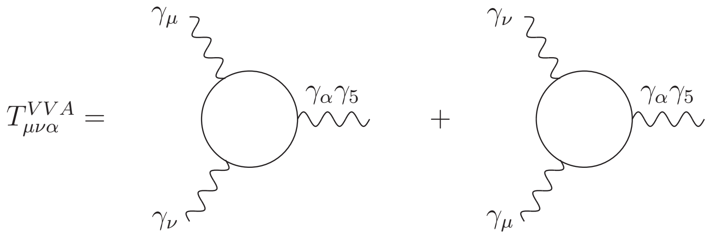



Figure 3.4: Diagrams of the Adler-Bardeen-Bell-Jackiw chiral anomaly with an arbitrary momentum routing.

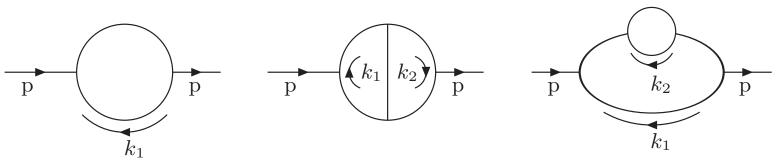
Using now eq. (3.32) and the rules of IReg one arrives at the vector and axial Ward identities, which cannot be satisfied simultaneously

$$
\begin{aligned}
p^\mu T^{VVA}_{\mu\nu\alpha} &= 4i(\alpha-\beta-1)\Gamma_0^{(1,2)}\epsilon_{\mu\nu\alpha\beta}p^\mu q^\beta \\
q^\nu T^{VVA}_{\mu\nu\alpha} &= -4i(\alpha-\beta-1)\Gamma_0^{(1,2)}\epsilon_{\mu\nu\alpha\beta}q^\nu p^\beta \\
(p+q)^\alpha T^{VVA}_{\mu\nu\alpha} &= 8i(\alpha-\beta-1)\Gamma_0^{(1,2)}\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta - \frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta.
\end{aligned}
\tag{3.35}
$$

A comment is in order to clarify the interplay between surface terms and MRI in the presence of anomalies. We notice that by setting $\Gamma_0^{(1,2)}$ to zero one implements MRI since the Ward identities will be independent of $\alpha$ and $\beta$. However, this choice for $\Gamma_0^{(1,2)}$ strikingly spoils the democracy that the calculation scheme should preserve between the vector and axial Ward identities which must be fixed on physical grounds [42]. In other words, the surface term should be left arbitrary and a new constraint should be imposed on the theory. In the case of the ABJ chiral anomaly, the pion decay into two photons (experimental data) requires the conservation of the vector current. Therefore, the surface term should be null which represents preservation of MRI. Once again, we notice the connection between abelian gauge symmetry (represented by the vector Ward identity in this case) and MRI.

## 3.3 Momentum Routing Invariance in the context of scalar field theories

Having demonstrated that the imposition of momentum routing invariance automatically preserves abelian gauge symmetry to all-orders,, we wonder which would be the consequences of MRI breaking in a theory with low symmetry content. We study massless $\phi_6^3$ theory by calculating its $\beta$-function at two-loop order [34]. The Feynman diagrams we will need are depicted in figure 3.5.




Since we want to stress the connection between surface terms and MRI, we will use an arbitrary momentum routing in the diagrams above. However, following the calculation outlined at [34], one can easily see that only the momentum routing of the subgraph of the third and sixth diagrams will affect the coefficients of the $\beta$-function at two-loop order. Thus, we will use the following convention

Adopting a minimal subtraction scheme which in IReg corresponds to subtracting BDI's only, one obtains the following two-loop corrections for propagator

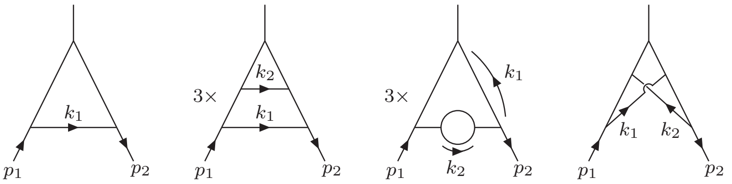



Figure 3.5: Diagrams contributing up to two-loop order of $\phi_6^3$ theory. $\Xi$ stands for two-point functions and $\Delta$ stands for three-point ones.

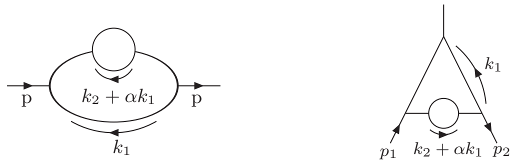



Figure 3.6: Diagrams whose choice of momentum routing may affect the coefficients of the $\beta$-function at two-loop order.

and vertex

$$\Xi = -\, g^2 \left[ \frac{p^2}{6} I_{log}^{(1)}(\lambda^2) \right] + i g^4 \Bigg\{ \Bigg[ - \left( \frac{29 b_6}{54} - \frac{(3\alpha^2 - 3\alpha + 1)\Gamma_0^{(1,2)}}{9} \right) I_{log}^{(1)}(\lambda^2) + \\ + \frac{5 b_6}{18} I_{log}^{(2)}(\lambda^2) \Bigg] p^2 \Bigg\} \\ + \text{terms that do not contribute to the } \beta\text{-function,} \tag{3.36}$$

$$\Delta = -\, g^3 \left[ I_{log}^{(1)}(\lambda^2) \right] + i g^5 \left[ \frac{5 b_6}{2} I_{log}^{(2)}(\lambda^2) - \left( \frac{17 b_6}{3} - (3\alpha^2 - 3\alpha + 1)\Gamma_0^{(1,2)} \right) I_{log}^{(1)}(\lambda^2) \right] \\ + \text{terms that do not contribute to the } \beta\text{-function,} \tag{3.37}$$

where $\quad b_6 \equiv -\dfrac{1}{2} \dfrac{i}{(4\pi)^3}$.

With these results, the $\beta$-function can be calculated yielding

$$\beta = -\frac{3 g^3}{4 (4\pi)^3} - \frac{125 g^5}{144 (4\pi)^6} + \frac{i g^5 \left[ 12(\alpha^2 - \alpha) + 5 \right] \Gamma_0^{(1,2)}}{6 (4\pi)^3} + O(g^6). \tag{3.38}$$

As is well known the coefficients of the $\beta$-function up to two-loop order in a mass independent scheme are universal. We notice that we obtain the universal values of the $\beta$-function only if we set $\Gamma_0^{(1,2)} = 0$, which is the same condition to preserve MRI. On the other hand it becomes also clear from our expression for the beta function that should the surface term be non-vanishing the two loop universal beta function coefficient would be momentum routing dependent (in our case would depend on $\alpha$). Therefore, we conclude that even in a theory with poor symmetry content such as $\phi_6^3$, momentum routing invariance is important since it is responsible for the preservation of the universal values of the $\beta$-function.

# Concluding Remarks

Momentum routing invariance (MRI) is clearly a symmetry of any Feynman graph, as allowed by energy-momentum conservation in the vertices. However, divergencies which typically appear in loop calculations may simultaneously break MRI and important symmetries of the underlying model. We demonstrated for abelian gauge theories that the offending terms are multiplied by surface terms which once set to zero automatically render the regularization invariant besides implementing MRI. Therefore, we may conjecture that an invariant four dimensional regularization framework should comply with MRI. We derived the formal expressions of surface terms to arbitrary order in perturbation theory. In presence of anomalous processes we use the case of the Adler-Bardeen-Bell-Jackiw (ABJ) chiral anomaly to illustrate that the same procedure can be applied to isolate the anomaly specific MRI violating terms, which can be fixed either to conserve the vector or the axial-vector current, in a manifestly "democratic" way. For theories with poor symmetry content, MRI manifests itself as an important ingredient in the calculation of the universal coefficients of the $\beta$-function. Any regularization scheme should comply with MRI. We propose Implicit Regularization as an ideal framework to implement such program in the physical dimension of the quantum field theoretical model, applicable to models where dimensional methods may fail.

# Chapter 4

# (Un)determined finite regularization dependent quantum corrections: the Higgs decay into two photons and the two photon scattering examples

## Introduction

On 4th July 2012 a new boson was announced using its decay into two photon as one of the main channels of discovery [47, 48]. The immediate question that arose was whether this new boson corresponds to the one predicted by the Standard Model (Higgs boson) or not. To help answering this question theoretical predicitions (loop corrections) on such decays must be set on consistent grounds.

Some time ago the W loop calculation of the Higgs decay into two photons was performed in the unitary gauge and the result obtained [49] contradicted previous ones found in the literature [50, 51, 52]. The reason pointed by the authors was the use of Dimensional Regularization. Soon after many authors performed calculations in the framework of Dimensional Regularization [53], lattice [54] and Loop Regularization [55]. In all cases the old results were recovered shedding many doubts on the statements presented on [49]. Other authors used Cutoff Regularization [56, 57] obtaining the same result of [49] thus concluding that such regularization is unpredictive if one works on the unitary gauge. Other works were devoted to the discussion of the decoupling theorem [58, 59] questioning the reliability of the predictions made on [49].

Contemporary to Gastmans et al. work, another paper questioned an old-stabilished result in the literature: the cross section of the two photon scattering [60]. Once again, doubts were raised against the use of regularization. A work followed in which this issue was explained [61] in the framework of Dimensional Regularization and Paulli-Villars recovering the old results found in the literature [62, 63].

The aim of the present work is to revise these two calculations with the purpose of illustrating that *a priori* undefined quantum corrections in Feynman diagram calculations, which entail regularization scheme dependence, are the common denominator of such discussion. Such arbitrarinesses must not be mistaken by finite parameters related to the freedom of defining renormalization constants to be fixed by renormalization conditions (i.e. the choice of a renormalization point). We propose the Implicit Regularization (IReg) framework to handle such ambiguities which acts on the physical dimension of the theory thus being particularly useful to dimensional specific models. In this context such arbitrariness are expressed by differences between divergent loop integrals with the same degree of divergence and independent of external momenta with the purpose of bringing about its physical interpretation namely their relation to momentum routing invariance (MRI) in an arbitrary Feynman diagram. Some regularizations may break MRI, an inevitable consequence of energy-momentum conservation at the vertices of Feynman diagrams. The striking connection between momentum routing invariance and preservation of gauge symmetry was realized long ago by t' Hooft and Veltman [12], by Jackiw in [64] as well as by Elias et al. in [65]. In the last chapter [66] we established the interplay between the vanishing of such arbitrary parameters expressed by surface terms and Abelian gauge invariance in the context of IReg. As explained in chapter 2, in this four-dimensional method, regularization dependent terms (surface terms) can be extracted out in a consistent way allowing a clear discussion of the ambiguities involved in the manipulation of divergent integrals. Therefore, instead of just adding the result of a different method to the literature we intend to show that the discussions presented in [53, 54, 55, 56, 57] can all be explained using just one framework.

## 4.1 A general view of regularization dependent integrals

In this section we discuss on general grounds the issue of regularization dependent integrals leaving the physical calculations of the Higgs decay as well as of the two photon scattering to subsequent sections. Proceeding this way we hope to set the subject, both from a conceptual and technical point of view, in a consistent and self-contained way allowing a clearer discussion of the examples just cited.

As is well known Quantum Field Theoretical Feynman diagram calculations involve integration in the momentum loops which must be regularized due to ultraviolet and sometimes infrared divergences. The renormalization program consistently redefines physical degrees of freedom order by order in perturbation theory. Symmetry requirements may either be ensured by an invariant regularization or imposed as constraint equations as dictated by Ward-Slavnov-Taylor identities order by order in the loops. Yet a little calculational tedious, the latter procedure is perfectly sound for both anomaly free theories and models in which the quantum symmetry breaking mechanism is well known.

A plethora of regularization schemes have been constructed to be used where gauge invariant Dimensional Regularization may fail, namely in the so called dimensional specific theories among which supersymmetric, chiral and topological quantum field theories figure in. A natural question would be which basic properties should a method that does not recourse to analytical continuation in the space-time dimension should retain in order to be invariant. We start by illustrating with simple examples following [67]. Let $\Delta$ be the superficial degree of divergence of a $n$ loop integral where the momentum $k_n$ runs. Consider the following 1-loop $\Delta = 2$ integrals,

$$A = \int_k \frac{k^2}{(k^2 - m^2)^2}, \tag{4.1}$$

and

$$B = I_{quad}(m^2) + m^2 I_{log}(m^2), \tag{4.2}$$

where $\int_k \equiv \int d^4k/(2\pi)^4$ and we recover the standard notation in Implicit Regularization

$$I_{log}(m^2) \equiv \int_k \frac{1}{(k^2-m^2)^2}, \quad \text{and} \quad I_{quad}(m^2) \equiv \int_k \frac{1}{(k^2-m^2)}. \tag{4.3}$$

We expect $A = B$ be guaranteed by any regularization procedure. However this is not the case. Proper-time regularization [68] for instance, introduces a cutoff $\Lambda$ after Wick rotation via the following identity at the level of propagators

$$\frac{\Gamma(n)}{(k^2+m^2)^n} = \int_0^\infty d\tau \tau^{n-1} e^{-\tau(k^2+m^2)} \rightarrow \int_{1/\Lambda^2}^\infty d\tau \tau^{n-1} e^{-\tau(k^2+m^2)}. \tag{4.4}$$

Thus it is trivial to obtain within the proper-time method that $A \neq B$ since

$$A_\Lambda = \frac{-2i}{(4\pi)^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right], \tag{4.5}$$

whereas

$$B_\Lambda = \frac{-2i}{(4\pi)^2}\left[\frac{\Lambda^2}{2} - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right]. \tag{4.6}$$

On the other hand it is straightforward to show that standard Dimensional Regularization leads to $A = B$. To circumvent such discrepancy the authors of [67] define a $n$-dimensional integral

$$I(\alpha,\beta) = \int_k^n \frac{1}{(\alpha k^2 + \beta m^2)}, \tag{4.7}$$

for $\alpha$ and $\beta$ arbitrary in order to write

$$A = -\frac{\partial}{\partial\alpha} I(\alpha,\beta)\Big|_{\alpha=\beta=1, n=4}, \tag{4.8}$$

and

$$B = I(\alpha,\beta)\Big|_{\alpha=\beta=1,n=4} + \frac{\partial}{\partial\beta} I(\alpha,\beta)\Big|_{\alpha=\beta=1,n=4}. \tag{4.9}$$

Then resorting to proper time regularization one gets

$$I(\alpha,\beta)_\Lambda = \alpha^{-n/2}\int_k^n \frac{1}{(k^2-\beta m^2)} = \frac{-\alpha^{n/2} i}{(4\pi)^2}\left[\Lambda^2 - \beta m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right], \tag{4.10}$$

from which is obtained

$$A_\Lambda^n = \frac{-i}{(4\pi)^2}\left[\frac{n}{2}\Lambda^2 - \frac{n}{2} m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right], \tag{4.11}$$

and

$$B_\Lambda^n = \frac{-i}{(4\pi)^2}\left[\Lambda^2 - 2m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right]. \tag{4.12}$$

Whilst keeping $n = 4$ violates $A = B$, the choices $n = 4$ in the term $\propto \ln\Lambda^2$ and $n = 2$ in the term $\propto \Lambda^2$ lead $A$ to coincide with $B$ at regularized level. Yet arbitrary the authors consider such prescription, which is generalizable to other integrals in Feynman amplitudes, a concrete realization for a four dimensional regularization. They claim that Veltman in [69] already notices that quadratic divergences are associated with $n = 2$ whereas logarithmic divergences have to be treated in $n = 4$ in Dimensional Regularization. Other authors have used a similar approach [70, 71, 72].

Let us now consider another related example. Consider the effect of a shift in the integration variable in a four dimensional integral. As well known such shifts accompany surface terms in more than logarithmically divergent integrals. Their value is highly regularization dependent. For instance take the difference between two linearly divergent integrals for $\omega = 2$

$$\Delta_1 = \int_k^{2\omega} \frac{k_\mu}{[(k-p)^2 - m^2]^2} - \int_k^{2\omega} \frac{(k+p)_\mu}{[k^2 - m^2]^2}. \tag{4.13}$$

Clearly $\Delta_1 = 0$ in Dimensional Regularization because no surface terms accompany shifts in the integration variable. In [65] the authors generalize the procedure adopted by Jauch and Rohrlich in [73] to evaluate $\Delta_1$ for $\omega$ exactly equal to 2. Their purpose was founded on the physical motivation of constructing four dimensional regularizations with properties compatible with Dimensional Regularization. By defining

$$I^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} = \int_k^{2\omega} \frac{\prod_{j=1}^{2n+1} k_{\mu_j}}{[(k-p)^2-m^2]^r}, \quad \text{and} \quad J^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} = \int_k^{2\omega} \frac{\prod_{j=1}^{2n+1}(k+p)_{\mu_j}}{[k^2-m^2]^r}, \tag{4.14}$$

in [65] is shown that whilst $I = J$ for $2\omega+2n+1-2r < 1$, if $2 > 2\omega+2n+1-2r > 1$ then

$$I^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} - J^{2n+1,r}_{\mu_1\cdots\mu_{2n+1}} = \frac{-i(2\pi)^4\pi^\omega G_{n,2n+1}(p)}{\Gamma(\omega)}\delta_{r,\omega+n}, \tag{4.15}$$

with

$$G_{n,2n+1}(p) = \frac{g_{\mu_{j_1}\mu_{j_2}}\cdots g_{\mu_{j_{2n-1}}\mu_{j_{2n}}} p_{\mu_{j_{2n+1}}} \sigma^{j_1\cdots j_{2n+1}}}{\Gamma(\omega)^{-1}\Gamma(\omega+n+1)n!2^{2n}}, \tag{4.16}$$

and

$$\sigma^{j_1\cdots j_{2n+1}} = \epsilon^{j_1\cdots j_{2n+1}}(-)^{sign(\epsilon)}. \tag{4.17}$$

For $n=0$ we immediately obtain

$$\Delta_1 = \frac{-i\pi^2(2\pi)^4}{2}\delta_{\omega 2}p_\mu. \tag{4.18}$$

A similar expression may be obtained for more than linearly divergent variable shifted integrals. It is immediate from above that the kronecker delta signs a discontinuity in the dimensionality $\omega$. The authors use these results to back up an integer dimensional regularization called Preregularization where the freedom of momentum routing in the loops is chosen to cancel out some surface terms in order to preserve Ward identities in chiral anomalies or supersymmetry [40, 74, 75]. A relevant question, given that shifts of integration variables are regularization dependent, would be to verify whether the argument could be turned the other way around, namely to exploit the consequences of momentum routing invariance over regularization schemes. Some technicalities deserve attention. Symmetric integration in $n$ (integer) dimensions, namely $k_\mu k_\nu \to g_{\mu\nu}k^2/n$ under integration in $k$ for divergent integrals does *not* hold in general and has been a source of disagreements in loop calculations as well discussed in [39] in the context of CPT violation in quantum field theory and used in [49] to study Higgs' s decay in two photons. In particular symmetric integration was used in [73] to evaluate $\Delta_1$.

As explained in chapter 2, we propose the Implicit Regularization framework to deal with these ambiguities. As showed there, there is the appearance of the following objects

$$\Upsilon_0^{\mu\nu} \equiv \int_k^d \frac{\partial}{\partial k_\mu} \frac{k^\nu}{(k^2 - m^2)^{\frac{d}{2}}} = d\left[\frac{g^{\mu\nu}}{d} I_{log}(m^2) - I_{log}^{\mu\nu}(m^2)\right], \tag{4.19}$$

and

$$\Upsilon_2^{\mu\nu} \equiv \int_k^d \frac{\partial}{\partial k_\mu} \frac{k^\nu}{(k^2 - m^2)^{\frac{d-2}{2}}} = (d-2)\left[\frac{g^{\mu\nu}}{(d-2)} I_{quad}(m^2) - I_{quad}^{\mu\nu}(m^2)\right]. \tag{4.20}$$

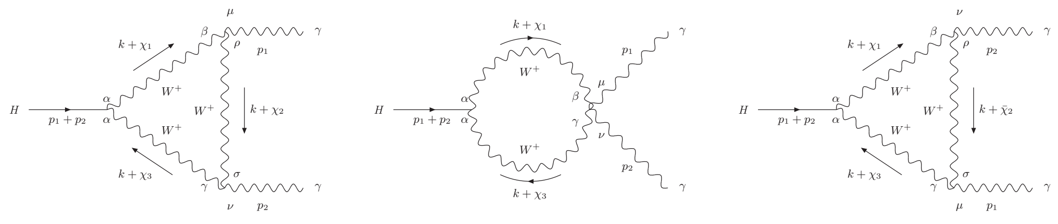
The surface terms $\Upsilon$'s are regularization dependent terms which however can be shown to be physical meaningful and therefore be fixed. In the process of reducing the set of loop integrals to basic divergent integrals it can be shown that the vanishing of surface terms expressed by the $\Upsilon$'s reflects momentum routing invariance in the loops of a Feynman diagram [18, 66]. Attributing spurious values to such surface terms is the root of quantum symmetry breakings by regularizations. Once we attach a physical meaning to them, as it is proposed in the Implicit Regularization program we may regularize infinities in a regularization independent fashion because the renormalization constants can be defined in terms of basic divergent integrals themselves.

As for the examples we presented earlier, it is immediate that $A = B$ within our approach because summing and subtracting $m^2$ in the numerator of $A$ leads to $B$. Whenever even powers of internal momenta appear in the numerator, one can always make use of such artifice to avoid ambiguous symmetric integration [23]. As for $\Delta_1$ in equation (4.13) one obtains within Implicit Regularization

$$\Delta_1^{IR} = \Upsilon_0^{\mu\nu} p_\nu. \tag{4.21}$$

## 4.2 Higgs decay into two photons

In this section we will study the W loop contributions to the Higgs decay into two photons. Using the unitary gauge we have only three Feynman diagrams to evaluate (fig. 5.3). Notice we are not choosing a specific routing for the diagrams



$M_1$ $M_2$ $M_3$

Figure 4.1: Diagrams with arbitrary momentum routing $\chi$

since we want to study how the final amplitude depends on it.

The sum of the three diagrams can be symplified to the expression (the strategy is to group the terms of the integrand in order of $M_w^{-n}$ and consider that the external photons are onshell $p_i^2 = 0$ and $p_1^2 + p_2^2 = M_h^2$) [1]

$$M = ie^2 g M_w \Big[M^{(a)}_{\mu\nu} + M^{(b)}_{\mu\nu} + M^{(c)}_{\mu\nu}\Big](\epsilon_1{}^{\mu})^*(\epsilon_2{}^{\nu})^* + (p_1 \leftrightarrow p_2, \mu \leftrightarrow \nu), \tag{4.22}$$

$$\begin{aligned} M^{(a)}_{\mu\nu} &= -\frac{4}{M_w^2}\Big[g_{\mu\nu}(p_1)^\alpha (p_2)^\beta I^{(3)}_{\alpha\beta} + (p_1 \cdot p_2) I^{(3)}_{\mu\nu} - (p_1)_\nu (p_2)^\alpha I^{(3)}_{\mu\alpha} - (p_2)_\mu (p_1)^\alpha I^{(3)}_{\nu\alpha}\Big] + \\ &\quad + \frac{2}{M_w^2}\Big[g_{\mu\nu}(p_1 \cdot p_2) - (p_2)_\mu (p_1)_\nu\Big] I^{(3)}_2, \end{aligned} \tag{4.23}$$

$$M^{(b)}_{\mu\nu} = \int\limits_k \frac{3(g_{\mu\nu}k^2 - 4k_\mu k_\nu)}{(q_1^2 - M_w^2)(q_2^2 - M_w^2)(q_3^2 - M_w^2)}, \tag{4.24}$$

$$M^{(c)}_{\mu\nu} = 6g_{\mu\nu}\Big[(p_1 \cdot p_2) I^{(3)}_0 - (p_1)^\alpha I^{(3)}_\alpha - \frac{M_w^2}{2} I^{(3)}_0\Big] + 6\Big[2(p_1)_\nu I^{(3)}_\mu - (p_2)_\mu (p_1)_\nu I^{(3)}_0\Big], \tag{4.25}$$

$$I^{(3)}_{0,2,\mu,\mu\nu} = \int\limits_k \frac{1, k^2, k_\mu, k_\mu k_\nu}{(q_1^2 - M_w^2)(q_2^2 - M_w^2)(q_3^2 - M_w^2)}. \tag{4.26}$$

As one may notice, only $M^{(a)}_{\mu\nu}$ and $M^{(b)}_{\mu\nu}$ contain divergent terms. At this point we must choose a regularization in order to deal properly with such terms. We employ IReg since all regularization-dependent objects (surface terms) can be consistently treated allowing a clear discussion about ambiguities as will be seen

[1] We define $q_i = k + \chi_i$, $\bar{q}_i = k + \bar{\chi}_i$ and $\int\limits_k = \int \frac{d^4k}{(2\pi)^4}$

below. Therefore the first term is given by

$$M^{(a)}_{\mu\nu} = \frac{\left[(p_2)_\mu (p_1)_\nu - g_{\mu\nu}(p_1 \cdot p_2)\right]}{M_w^2}\left[\frac{i}{16\pi^2} - 2\Gamma_0^{(1,2)}\right]. \tag{4.27}$$

The first point to be noticed is that this term is gauge invariant and, in general, ambiguous since it depends on a surface term. Another feature is that it is proportional[1] to $\tau^0$ which furnishes us a clue that it may be the term missing on [49]. In fact, if one performs a symmetric regularization in four-dimensions (by replacing $k_\mu k_\nu \to g_{\mu\nu}k^2/4$) it will be null. In other words a 4-dimensional regularization that resorts to such substitution is evaluating the surface term to a precise value in this case $i/32\pi^2$. On the other hand, if one uses Dimensional Regularization the surface term will vanish which furnishes a **non-null** amplitude in the limit $\tau^{-1} \to 0$. In the framework of IReg there is no reason a priori to favour one of these two values since we are dealing with ambiguous objects in nature. From our perspective physical conditions, other than the regularization method, should constrain the value the surface term should assume. In general, one such condition is to impose gauge invariance, however, since this term is **already** gauge invariant, this consideration will not fix it. Therefore, we should leave it arbitrary and proceed with the calculation of the amplitude for now. The sum of the two last terms is[2]

$$\begin{aligned} M^{(b)}_{\mu\nu} + M^{(c)}_{\mu\nu} = {} & \frac{i}{16\pi^2 M_w^2}\left[(p_2)_\mu (p_1)_\nu - g_{\mu\nu}(p_1 \cdot p_2)\right]\left[\frac{3\tau^{-1}}{2} + \frac{3(2\tau^{-1} - \tau^{-2})f(\tau)}{2}\right] \\ & + g_{\mu\nu}(p_1 \cdot p_2)\left(\frac{3\tau^{-1}}{2M_w^2}\Gamma_0^{(1,2)}\right). \end{aligned} \tag{4.28}$$

[1] We define $\tau = \frac{M_h^2}{4M_w^2}$

[2] Where we define

$$f(\tau) = \begin{cases} \arcsin^2(\sqrt{\tau}) & \text{for} \quad \tau \leq 1\,, \\ -\dfrac{1}{4}\left[\ln\dfrac{1+\sqrt{1-\tau^{-1}}}{1-\sqrt{1-\tau^{-1}}} - i\pi\right]^2 & \text{for} \quad \tau > 1\,. \end{cases}$$

Readily one may notice the appearance of another surface term due to $M^{(b)}_{\mu\nu}$ which explicitly breaks gauge invariance. Since there are no other terms to consider, one should impose gauge invariance as a physical condition that the whole amplitude should fulfill. Thus the otherwise arbitrary surface term must assume a precise value which in our case is null. This choice also fixes the surface term appearing in (5.54) since in the framework of IReg there is no distinction between surface terms coming from integrals with the same degree of divergence and the same Lorentz structure. This approach is different from the one found in [56] where a cutoff scheme is used and it is claimed that the ambiguities can be parametrized by different boundary conditions for the integrals appearing in (5.54) and (5.55).

After all these considerations we obtain the amplitude for the Higgs decay into two photons in the framework of IReg

$$M = -\frac{e^2 g}{16\pi^2 M_w}\left[(p_2)_\mu (p_1)_\nu - g_{\mu\nu}(p_1\cdot p_2)\right]\left[2+\frac{3}{\tau}+\frac{3}{\tau}\left(2-\frac{1}{\tau}\right)f(\tau)\right](\epsilon_1{}^{\mu})^*(\epsilon_2{}^{\nu})^*, \tag{4.29}$$

which agrees with previous one found in the literature [50, 51, 52].

In the time this work was been written, another paper devoted to this decay appeared [76]. Their authors have a point of view similar to ours in the sense that ambiguities should be fixed on physical grounds [1]. They use the equivalence theorem as well as the conservation of charge as inputs that their amplitude must fulfill. Since these are consequences of gauge invariance there is no surprise that just the imposition of such requirement gives us an unambiguous result.

Another interesting point discussed there is the role played by momentum routing freedom. From their point of view the loop-momentum of the three diagrams must be chosen in a particular way as to reduce the superficial degree of divergence of the amplitude to a logarithmic one.[2] However, from our point of

[1]It should be emphasized that their definition of the ambiguity is more closely related to the one found in [42]. We, on the other hand, define it by (4.19) which is more closely related to the preservation of Abelian gauge invariance [66]

[2]They find that all three diagrams must contain the same momentum routing. Therefore it is no surprise that our result before regularization (5.53) contains at most logarithmic divergent integrals since we also adopted the same momentum routing for all three diagrams ($\chi_1$).

view momentum routing invariance (MRI) is a symmetry that must be respected since it is connected with Abelian gauge invariance as well as supersymmetry preservation [66]. The importance of this statement is particular clear if, instead of considering the calculation of the whole amplitude, one evaluates each diagram **individually**. Following the reasoning of [66] one founds out that momentum routing dependent terms will arise always multiplied by arbitrary-valued objects (surface terms). Therefore, since individual diagrams are not supposed to be gauge invariant, the only symmetry left in order to fix the ambiguities is demanding momentum routing invariance. As can be seen, we could have adopted this approach from the beginning of our work avoiding completely the discussion of gauge invariance (since the two symmetries are connected it is not a surprise that the surface terms must be null in both cases). However, in order to make contact with the literature we performed the calculation of the whole amplitude with the same routing for all three diagrams which evidently is not the more general situation. Therefore, it is not a surprise that our result is independent of the momentum routing $\chi_1$ even though we still have an ambiguity expressed by $\Gamma_0^{(1,2)}$.

## 4.3 Two photon scattering

In this brief section we would like to comment on the result found on [60]. As in the case just analized, the problem lies on divergent integrals which appear as intermediate steps of the calculation. Explicitly we have [61]

$$\begin{aligned} A^{\mu\nu\rho\sigma} = & \int_k \frac{m^4 S_1^{\mu\nu\rho\sigma} + 2m^2 \left(2S_2^{\mu\nu\rho\sigma} - k^2 S_1^{\mu\nu\rho\sigma}\right)}{\left(k^2-m^2\right)^4} \\ & + \int_k \frac{24k^\mu k^\nu k^\rho k^\sigma + \left(k^2\right)^2 S_1^{\mu\nu\rho\sigma} - 4k^2 S_2^{\mu\nu\rho\sigma}}{\left(k^2-m^2\right)^4}, \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} S_1^{\mu\nu\rho\sigma} =& g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\rho\nu}, \\ S_2^{\mu\nu\rho\sigma} =& g^{\mu\nu}k^\rho k^\sigma + g^{\mu\rho}k^\nu k^\sigma + g^{\mu\sigma}k^\rho k^\nu + g^{\rho\nu}k^\mu k^\sigma + g^{\sigma\nu}k^\rho k^\mu + g^{\rho\sigma}k^\mu k^\nu. \end{aligned}$$

As can be readily seen, the integral above is divergent, thus ambiguous. Such statement is particularly clear in the framework of IReg since it is evaluated to $(\Gamma_0^{(1,2)} - 4\Gamma_0^{(1,4)})S_1^{\mu\nu\rho\sigma}$ where $\Gamma_0^{(1,i)}$ is a surface term coming from an integral with Lorentz structure $k^{\nu_1}\cdots k^{\nu_i}$ as defined in chapter 2. Therefore, there is no preferred value this integral should assume, it should be left arbitrary being fixed by the imposition of physical conditions. As discussed in [61], a non-null value for $A^{\mu\nu\rho\sigma}$ implies the breaking of gauge invariance which means the surface terms must obey $\Gamma_0^{(1,2)} = 4\Gamma_0^{(1,4)}$ in order to respect such symmetry. Thus, there is no ambiguity left on the final amplitude which as expected agrees with previous results found in the literature [61]. In summary, as in the case of [49], the authors of [60] performed a symmetric regularization on the integral above which in turn gave a precise value to the surface terms ($A^{\mu\nu\rho\sigma} = (i/96\pi^2)S_1^{\mu\nu\rho\sigma}$). Such choice resulted in a different cross section for the two photon scattering than the one found previously in the literature [62, 63]. However, since the integral is ambiguous in nature there is no reason to assume a precise value for the surface terms which must be fixed on physical grounds.

## Concluding remarks

In this chapter we studied the decay of the Higgs boson into two photons as well as the two photon scattering amplitude. Both processes must have only finite corrections since the photon does not couple with the Higgs boson neither with itself. However, in the intermediate steps of the calculation one may encounter divergent integrals and the issue of regularization is particularly important in order to give a meaningful result. To discuss the ambiguities inherent in such process we used the framework of Implicit Regularization which can consistently separate the divergent, finite and ambiguous part of any integral. We found out that although the divergent parts cancel as expected there are some ambiguities left (parametrized as surface terms). These should not be fixed by the regularization scheme *a priori*, but should be left arbitrary been determined by physical conditions. In the cases studied here, the condition used was the gauge invariance of the final result which univocally fixed the surface terms thus recovering the amplitude for the Higgs decay as well as the cross section of the two photon scattering found previously in the literature.

# Chapter 5

# Guises and Disguises of Quadratic Divergences

## Introduction

The Higgs boson discovery at the LHC [47, 48] ($m_H \approx 125\,GeV$) as well as the lack of data supporting on low energy extensions to the standard model (SM) such as supersymmetry (SuSy) has renewed the interest in possible explanations for both the electroweak hierarchy and the naturalness problem. These issues are related to a certain extent to how we interpret quadratic divergences in field theories. Two kinds of hierarchy problems arise in the SM [77]. The most commonly referred one, which we will simply call hierarchy problem, is related to the large radiative corrections to the Higgs mass stemming from quadratic divergences in the cutoff which supposedly cancel against the tree level value to a very high precision at the weak scale. Consequently the Higgs mass becomes quadratically sensitive to a cutoff scale $\Lambda$. On the other hand the gauge hierarchy problem has to do with logarithmic divergences which determine the running of the coupling constants: if on one level the scale that characterizes the symmetry breaking of the GUT which unifies quantum chromodynamics and electroweak theory [78] is $10^{14}GeV$, on the other level the electroweak symmetry breaking scale is about $10^{2}GeV$. Explaining this gap is known as gauge hierarchy problem [79]. The solution of the hierarchy problem involves how one bypasses the quadratic divergences which, unlike other divergences of a renormalizable theory which are multiplicatively renormalized, lead to a subtractive renormalization of the Higgs boson mass. Thus the hierarchy problem is reduced to the naturalness of such subtraction. SuSy avoids such subtractive renormalization and would solve the technical naturalness of the SM should SuSy particles be sufficiently light.

It is important to remark that contrarily to the interpretation in ultraviolet (UV) complete theories, quadratic divergences cannot be excused away as an artifact of the regularization procedure by simply adopting dimensional regularization, for instance. Taking the SM as the low energy limit of a more complete theory, a cutoff must be introduced to set a landmark in which new degrees of freedom appear. Notice that the meaning of a cutoff $\Lambda$ is twofold: it can play the role of the UV cutoff of an UV complete theory ($\Lambda \to \infty$) or a cutoff in an effective theory at which new degrees of freedom appear (merging scale). For example, for low energy models of QCD, $\Lambda \approx 1\,GeV$, as quarks and gluons are not well defined degrees of freedom in this region [70]. Evidently, for both the ultraviolet complete and the effective theory, naive subtraction of quadratic divergences has no effect upon the low energy dynamics. However in drawing conclusions about new physics, such subtraction becomes a subtle and relevant issue.

However, as pointed out in [80], the absence of quadratic divergences does not fully solve the hierarchy problem. The SM must also be UV completed at the scale $\Lambda$ by a theory without quadratic divergences. Thus the problem also passes by at which scale a complete theory (say, SuSy) appears. That is because a matching of the parameters of the high energy and low energy physics ought to guarantee light Higgs mass parameters at the merging scale. Such fine tuning would be avoided only if $\Lambda \approx 10^3 GeV$ [80].

Different constructions, differing by their level of sophistication, have appeared in the literature to explain away the role played by quadratic divergences in the naturalness problem, given that new physics has not been found at LHC with $\sqrt{s} = 8\,TeV$. It is worthwhile to discuss some proposals to give a panorama on the subject. For instance, naturalness without SuSy was proposed and studied by Jack and Jones in [81, 82], whereas in [83] it was constructed a non-SuSy hypothetical theory which has the same particle content as softly broken minimal supersymmetric QED. It was shown that such theory was gauge invariant and free of quadratic divergences up to two loop order.

The oldest and widely discussed proposal is the Veltman condition [69], by which in the SM

$$C_V = \frac{3}{2}m_W^2 + \frac{3}{4}m_Z^2 + \frac{3}{4}m_H^2 - \sum_f n_f m_f^2, \tag{5.1}$$

$n_f = 3$ for quarks and $n_f = 1$ for leptons, would make the coefficient of the $\Lambda^2$ contribution to $m_H^2$ vanish if $C_V = 0$. However, this leads to $m_H = 316 GeV$.

In [84], within the scotogenic model of neutrino masses, the introduction of two scalar doublets, distinguished by $Z_2$ symmetry, leads to two Veltman conditions which, in principle, could satisfy the vanishing of quadratic divergences without narrowing the Higgs mass as much. An alternative to the Veltman condition would be the compositeness of the Higgs particle by a strong infrared dynamics, forming a fermionic bound state, which at high energy breaks into its elementary fermionic constituents and, hence, quadratic divergences would be absent [85].

On the grounds that neither the Veltman condition is satisfied for the measured value of the Higgs mass (electroweak scale) nor SuSy has been found at the LHC energies, reference [80] supposed that Veltman condition could be satisfied at some large energy $\mu_V$ where SuSy dominates (see also [86]). Whereas simply imposing $C_V = 0$ leads to $m_H =\approx 316\, GeV$, at odds with the current value, in terms of physical masses and couplings (5.1) is supposed to be renormalization group invariant. Thus Veltman condition (5.1) at one loop order can be written in terms of running couplings as [80]

$$C_V^\hbar(\mu) = 6\lambda(\mu) + \frac{9}{4}g^2(\mu) + \frac{3}{4}g'^2(\mu) - 6y_t^2(\mu), \tag{5.2}$$

where $\mu$ is the renormalization scale, $\lambda$ is the Higgs potential self coupling, $y_t$ is the top Yukawa coupling and $g$, $g'$ are the electroweak gauge couplings. Setting $C_V^\hbar(\mu) = 0$ allows us to infer at which scale the Veltman condition is fulfilled (higher loop order corrections leads only to a small modification in $\mu_{V_\hbar}$ [87],[88]). A NNLO calculation for the running couplings [80], using as inputs $m_H = 126\, GeV$, $\bar{m}_t(m_t) = 161.5\, GeV$ and $\alpha_3(m_Z) = 0.1196$ for the strong coupling constant at the $Z$ boson mass, leads to $C_V^\hbar(\mu) = 0$ at $\mu$ slightly larger than the Planck scale, which means that the SM is fine tuned up to this scale. Another adjustment in the parameters of the high energy fundamental theory must be performed in order to keep the Higgs and other singlets masses light at the merging scale. Such fine tuning is however unrelated to quadratic divergences. The appealing feature of this construction is that it puts off the solution of the hierarchy problem to the high energy complete UV theory.

In [89] it was introduced new degrees of freedom through adding other contributions to Higgs boson wave function renormalization. Effectively, those new degrees of freedom change the Higgs boson coupling and, guided by naturalness, the authors construct a weak scale effective theory in which the new extra scalar fields cancel the quadratic divergences. They also argue that the parameter space of their "natural theories" can be tested to percent level precision through Higgs boson coupling measurements at LHC.

It is noteworthy that there have been claims which establish the Higgs lightness due to huge cancelations because an anthropic selection destroyed naturalness [90]. Along similar lines, some authors claim a finite naturalness scenario in the sense that quadratic divergences are simply put aside (ignoring uncomputable power divergences) so that the Higgs mass is naturally small at least until there are no heavier particles [91]. They verify that finite naturalness is satisfied by the SM whilst for its extensions it remains valid only if the new physics is not much above the weak scale.

An interesting analysis on naturalness of the SM and extensions based on Bayesian statistics was performed in [92]. ATLAS and CMS [47, 48, 93] operates in $20/fb$ with center of mass energy in the range $\sqrt{s} = 7$ to $8\,TeV$ and will continue searching for SuSy to $13\,TeV$. Moreover a $\sqrt{s} = 100\,TeV$ Very Large Hadron Collider (VLHC) may be constructed. Roughly speaking, Bayesian statistics is a numerical estimate of belief in a proposition (model), given the experimental data. Such an estimation is weighed by the Bayes-factor $B$. Unsurprisingly the evidence for the SM without quadratic divergences over SM with quadratic divergences, given both the $m_Z$ and $m_H$ measures, is huge ($B \approx 10^{30}$, given that $B = 150$ is considered very strong in the Jeffrey's scale [92]). A comparison between the likelihood of the SM and the constrained minimal supersymmetric SM (CMSSM) [94, 95, 96] indicated, using as inputs the measured values of $m_H$, $m_Z$

and LHC at $20/fb$, that the Bayes factor favors the CMSSM over the SM with quadratic divergences by $\approx 10^{30}$, whereas SM without quadratic divergences is favored over the CMSSM by $\approx 700$. Before the LHC measurements, this factor would be only $\approx 2$. This is related to the "fine tuning price". They conclude their paper arguing that natural models are most probable and naturalness is not simply an aesthetic principle. Moreover, the fine tuning price of null results from the VLHC ($\approx 400$) would be slightly less than that of LHC ($\approx 500$).

In this contribution, we point out another perspective on the role and control of quadratic divergences in quantum field theory, in general, and in the Higgs naturalness problem, in particular. Our viewpoint is consonant with the works of Fujikawa [97] and Aoki and Iso [77], but justifies them at a prior regularized level led by symmetry constraints.

We illustrate our proposal with well-known examples such as the gluon vacuum self energy of QCD and the Higgs decay in two photons within this approach. We also discuss frameworks in effective low-energy QCD models, where quadratic divergences are indeed fundamental.

Our discussion is essentially based on an approach where UV divergences are parameterized, after being reduced to basic divergent integrals (BDI) in one internal momentum, as functions of a cutoff and a renormalization group scale $\lambda$ [66, 98]. This construction takes a step forward from the one presented in chapter 2 in the sense that it is possible to introduce a cutoff, which is crucial in order to address the hierarchy problem. It should be noted that all the rules of IReg are still to be followed, the only difference being that BDI's should now be replaced by general parametrizations in terms of $\Lambda$. To define these relations, equations involving derivatives of BDI's were used since they are regularization independent and must be satisfied by any explicit regularization.

Arbitrary regularization dependent terms, which may be responsible for symmetry breaking in the underlying model, will be systematically displayed as surface terms (ST) as before. The general parametrization we construct for each BDI clearly displays the ST undetermined character, which is fixed by symmetry requirements. That is because each BDI itself contains undetermined and regularization dependent parameters. In this case, usually they can be hidden in the arbitrariness of defining a renormalization constant. However, as we shall see in the case of quadratic divergences in the hierarchy problem, they may break symmetries as well. It should be emphasized that with this approach, we have control of the regularization dependent terms which may break gauge, for instance, even though we work with an explicit cutoff.

This work is about arbitrary regularization dependent parameters, more specifically in the case of quadratic divergences, and how symmetry constraints which fix such parameters shed light on issues such as the hierarchy problem. In the latter, the scaling argument of Bardeen [99] and conformal anomaly can be used to fix a undetermined parameter in the isolated quadratic divergence which contribute to the Higgs boson mass. A generalization of this strategy to higher loops is presented. This strategy follows Jackiw proposal in [42] by which undetermined regularization dependent parameters must be fixed via symmetry and/or phenomenology constraints. In this way, we show that his strategy is accomplished not only to finite models, in which finite quantum corrections cannot be excused away by renormalization group conditions, but also to renormalizable and effective models, notably in studying quantum symmetry breakings.

To gain insight on how we deal with arbitrary parameters in a regularization independent way, as well as to give a general overview on quadratic divergences in QFT, we discuss the appearance of quadratic divergences in QCD and in the electroweak Higgs decay in two photons. Finally we comment on the importance of quadratic divergences in low-energy QCD effective models.

## 5.1 Basic divergent integrals, regularization ambiguities and parametrizations

In this section, we discuss regularization ambiguities and general parameterizations of regularization dependent quantities. As explained in chapter 2 all divergences can be expressed in terms of basic divergent integrals as below (we restrict ourselves to four dimensions)

$$I_{log}^{\mu_1 \cdots \mu_{2n}}(m^2) \equiv \int_k \frac{k^{\mu_1} \cdots k^{\mu_{2n}}}{(k^2 - m^2)^{2+n}} \tag{5.3}$$

and

$$I^{\mu_1\cdots\mu_{2n}}_{quad}(m^2) \equiv \int_k \frac{k^{\mu_1}\cdots k^{\mu_{2n}}}{(k^2-m^2)^{1+n}}, \tag{5.4}$$

where $\int_k \equiv \int d^4k/(2\pi)^4$. The basic divergent integrals with Lorentz indices can be expressed in terms of the ones without indices throw surface terms (ST). Such local *regularization dependent* surface terms are intrinsically arbitrarily valued. Let us take one loop BDI as examples. If the integrals are $d$-dimensional, it is straightforward to show that

$$\Upsilon^{\mu\nu}_0 \equiv \int^d_k \frac{\partial}{\partial k_\mu}\frac{k^\nu}{(k^2-m^2)^{\frac{d}{2}}} = d\left[\frac{g^{\mu\nu}}{d}I_{log}(m^2) - I^{\mu\nu}_{log}(m^2)\right], \tag{5.5}$$

and

$$\Upsilon^{\mu\nu}_2 \equiv \int^d_k \frac{\partial}{\partial k_\mu}\frac{k^\nu}{(k^2-m^2)^{\frac{d-2}{2}}} = (d-2)\left[\frac{g^{\mu\nu}}{(d-2)}I_{quad}(m^2) - I^{\mu\nu}_{quad}(m^2)\right]. \tag{5.6}$$

Such arbitrary surface terms are physical meaningful. The vanishing of ST's expressed by the $\Upsilon$'s reflects momentum routing invariance in the loops of a Feynman diagram [18, 66]. Spurious evaluations of such ST's are at the heart of quantum symmetry breaking by regularizations.

We shall now construct general parameterizations for loop integrals which incorporate explicitly arbitrary regularization dependent terms which will be fixed on physical grounds. Consider the regularization independent relations satisfied by the following logarithmically basic divergent integrals in $d$ (integer) dimensional spacetime:

$$\begin{aligned} \frac{dI_{log}(m^2)}{dm^2} &= -\frac{b_d}{m^2}, \\ \frac{dI^{\mu\nu}_{log}(m^2)}{dm^2} &= -\frac{g^{\mu\nu}}{d}\frac{b_d}{m^2}, \end{aligned} \tag{5.7}$$

where

$$b_d = \frac{i}{(4\pi)^{d/2}}\frac{(-)^{d/2}}{\Gamma(d/2)}. \tag{5.8}$$

A general parametrization involving a cutoff $\Lambda \to \infty$ that obeys the relations above is

$$
\begin{aligned}
I_{log}(m^2) &= b_d \ln\left(\frac{\Lambda^2}{m^2}\right) + \alpha_1, \\
I^{\mu\nu}_{log}(m^2) &= \frac{g^{\mu\nu}}{d}\left[b_d \ln\left(\frac{\Lambda^2}{m^2}\right) + \alpha_1'\right],
\end{aligned}
\tag{5.9}
$$

where $\alpha_1$, $\alpha_1'$ are arbitrary dimensionless regularization dependent constants. Similarly

$$
\begin{aligned}
\frac{dI_{quad}(m^2)}{dm^2} &= \frac{(d-2)}{2}\, I_{log}(m^2), \\
\frac{dI^{\mu\nu}_{quad}(m^2)}{dm^2} &= \left(\frac{d}{2}\right) I^{\mu\nu}_{log}(m^2),
\end{aligned}
\tag{5.10}
$$

which leads to the general parameterizations

$$
\begin{aligned}
I_{quad}(m^2) =& \frac{(d-2)}{2}\left[\alpha_2\Lambda^2 + b_d m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \alpha_3 m^2\right], \\
I^{\mu\nu}_{quad}(m^2) =& \frac{g^{\mu\nu}}{2}\left[\alpha_2'\Lambda^2 + b_d m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \alpha_3' m^2\right],
\end{aligned}
\tag{5.11}
$$

in which all regularization dependence is encoded in the $\alpha$'s.

Now, if we use these parameterizations in the surface terms of equations (5.5) and (5.6), we get

$$
\Upsilon^{\mu\nu}_0 \propto g^{\mu\nu}(\alpha_1 - \alpha_1'),
\tag{5.12}
$$

and

$$
\Upsilon^{\mu\nu}_2 \propto g^{\mu\nu}[(\alpha_2 - \alpha_2')\Lambda^2 + (\alpha_3 - \alpha_3')m^2].
\tag{5.13}
$$

These results exhibit the regularization dependence of the ST. For instance, in the four-dimensional case $\Upsilon^{\mu\nu}_0 = g^{\mu\nu}[i/8(4\pi)^2]$ and $\Upsilon^{\mu\nu}_2 = g^{\mu\nu}\Lambda^2[i/4(4\pi)^2]$ in sharp cutoff regularization, while they are both zero in DReg. As mentioned before the BDI's are the barebones of the amplitude UV behavior and can be absorbed in the definition of renormalization constants as they stand. In order to define a mass independent scheme we may trade $m^2$ for $\lambda^2 \neq 0$ and write the following regularization independent relation

$$I_{log}(m^2) = I_{log}(\lambda^2) + b\ln\left(\frac{\lambda^2}{m^2}\right), \tag{5.14}$$

where $\lambda$ plays the role of renormalization group scale (see [66] and references therein). However,

$$I_{quad}(m^2) = I_{quad}(\lambda^2) + m^2 I_{log}(m^2) - \lambda^2 I_{log}(\lambda^2) + b(m^2 - \lambda^2), \tag{5.15}$$

where we note that the RHS is not completely written in terms of the renormalization scale $\lambda$. That is because, as discussed in [97], quadratic divergences, in contrast with logarithmic, must be subtractively renormalized. Renormalization group flow is essentially described by a scale engendered by logarithmic divergences. For the sake of completeness we write out the explicit parametrization for logarithmic BDI's to arbitrary loop order. After subtraction of subdivergences according to BPHZ formalism, we may define the divergence of $n^{th}$ loop order in terms of basic divergent integrals for both massive and massless theories [34] in the form

$$I^{(n)}_{log}(m^2) \equiv \int_k \frac{1}{(k^2-m^2)^2}\ln^{n-1}\left(-\frac{(k^2-m^2)}{\lambda^2}\right), \tag{5.16}$$

which obeys

$$I^{(n+1)}_{log}(m^2) = I^{(n+1)}_{log}(\lambda^2) - b\sum_{i=1}^{n+1}\frac{n!}{i!}\ln^i\left(\frac{m^2}{\lambda^2}\right). \tag{5.17}$$

Likewise

$$\begin{aligned}\frac{dI^{(n)}_{log}(\lambda^2)}{d\lambda^2} &= -\frac{(n-1)}{\lambda^2}I^{(n-1)}_{log}(\lambda^2) + \frac{b_d}{\lambda^2}A^{(n)},\\ \frac{dI^{(n)\,\mu\nu}_{log}(\lambda^2)}{d\lambda^2} &= -\frac{(n-1)}{\lambda^2}I^{(n-1)\,\mu\nu}_{log}(\lambda^2) + \frac{g^{\mu\nu}}{2}\frac{b_d}{\lambda^2}B^{(n)}.\end{aligned} \tag{5.18}$$

After some algebra, one can demonstrate that the parametrization below respects (5.18)

$$I^{(n)}_{log}(\lambda^2) = \sum_{i=1}^{n}\frac{(n-1)!}{(i-1)!}\left[\frac{(-b_d)A^{(i)}}{(n-i+1)!}\ln^{n-i+1}\left(\frac{\Lambda^2}{\lambda^2}\right) + \sum_{j=0}^{n-i}\frac{a_{n-j-i+1}}{j!(n-j-i)!}\ln^j\left(\frac{\Lambda^2}{\lambda^2}\right)\right]$$

and

$$I_{log}^{(n)\,\mu\nu}(\lambda^2) = \frac{g^{\mu\nu}}{2}\sum_{i=1}^{n}\frac{(n-1)!}{(i-1)!}\left[\frac{(-b_d)B^{(i)}}{(n-i+1)!}\ln^{n-i+1}\left(\frac{\Lambda^2}{\lambda^2}\right)+\sum_{j=0}^{n-i}\frac{a'_{n-j-i+1}}{j!(n-j-i)!}\ln^{j}\left(\frac{\Lambda^2}{\lambda^2}\right)\right], \tag{5.19}$$

where

$$\begin{aligned} A^{(i)} \equiv \Gamma(d/2)\lim_{\delta\to 0}\Bigg[ &-(i-1)\sum_{l=0}^{i-2}\binom{i-2}{l}\frac{(-1)^{1+l}}{\delta^{i-2}}\frac{\Gamma(1-\delta(i-2-l))}{\Gamma(d/2+1-\delta(i-2-l))} \\ &+\left(\frac{d}{2}\right)\sum_{l=0}^{i-1}\binom{i-1}{l}\frac{(-1)^{1+l}}{\delta^{i-1}}\frac{\Gamma(1-\delta(i-1-l))}{\Gamma(d/2+1-\delta(i-1-l))}\Bigg], \\ B^{(i)} \equiv \Gamma(d/2)\lim_{\delta\to 0}\Bigg[ &-(i-1)\sum_{l=0}^{i-2}\binom{i-2}{l}\frac{(-1)^{1+l}}{\delta^{i-2}}\frac{\Gamma(1-\delta(i-2-l))}{\Gamma(d/2+2-\delta(i-2-l))} \\ &+\left(\frac{d+2}{2}\right)\sum_{l=0}^{i-1}\binom{i-1}{l}\frac{(-1)^{1+l}}{\delta^{i-1}}\frac{\Gamma(1-\delta(i-1-l))}{\Gamma(d/2+2-\delta(i-1-l))}\Bigg], \end{aligned} \tag{5.20}$$

and $a_i$, $a'_i$ are arbitrary constants. We have, for instance, the surface terms

$$\frac{1}{2}\sum_{j=1}^{n}\left(\frac{2}{d}\right)^{j}\frac{(n-1)!}{(n-j)!}\Upsilon_0^{(n)\mu\nu} = -I_{log}^{(n)\,\mu\nu}(\lambda^2)+\frac{g^{\mu\nu}}{2}\sum_{j=1}^{n}\left(\frac{2}{d}\right)^{j}\frac{(n-1)!}{(n-j)!}I_{log}^{(l-j+1)}(\lambda^2). \tag{5.21}$$

Generalization to an arbitrary number of Lorentz indices can be obtained in a similar fashion. For use in the next section, we explicitly write the ST's at one loop order up to four Lorentz indices:

$$\Upsilon_2^{\mu\nu} \equiv g^{\mu\nu}I_{quad}(m^2)-2I_{quad}^{\mu\nu}(m^2)=\Gamma_2^{(1,2)}\upsilon_1 g^{\mu\nu}\,, \tag{5.22}$$

$$\Upsilon_0^{\mu\nu} \equiv g^{\mu\nu}I_{log}(m^2)-4I_{log}^{\mu\nu}(m^2)=\Gamma_0^{(1,2)}\upsilon_2 g^{\mu\nu}, \tag{5.23}$$

$$\Upsilon_2^{\mu\nu\alpha\beta} \equiv g^{\{\mu\nu\alpha\beta\}}I_{quad}(m^2)-8I_{quad}^{\mu\nu\alpha\beta}(m^2)=\Gamma_2^{(1,4)}\upsilon_3 g^{\{\mu\nu\alpha\beta\}}\,, \tag{5.24}$$

$$\Upsilon_0^{\mu\nu\alpha\beta} \equiv g^{\{\mu\nu\alpha\beta\}} I_{log}(m^2) - 24 I_{log}^{\mu\nu\alpha\beta}(m^2) = \Gamma_0^{(1,4)} \upsilon_4 g^{\{\mu\nu} g^{\alpha\beta\}}. \tag{5.25}$$

## 5.2 Example: Cancelation of Quadratic Divergences and Renormalization of QCD at one loop

In this section, we show that quadratic divergences that appear in gluon self energies cancel out as they should, since they organize themselves into quadratic surface terms which are set to zero on gauge invariance grounds. We take the opportunity to evaluate the beta function of QCD using a different approach from the one presented in [22]. In the present case, we will show, relying on the parametrization of BDI's just presented in the last section, how a cutoff can be introduced while respecting gauge invariance.

For completeness, we present the bare QCD Lagrangian,

$$\mathcal{L}_0 = \frac{1}{4}(F^a_{0\mu\nu})^2 - \frac{1}{2\alpha}(\partial^\mu A^a_{0\mu})^2 + \bar{\psi}^i_0(i\gamma_\mu D^{ij}_\mu - m_0\delta^{ij})\psi^j_0 + +i(\partial^\mu \bar{c}^a_0) D^{ab}_\mu c^b_0, \tag{5.26}$$

and the definition of the counterterms in function of the renormalization constants

$$A^a_{0\mu} = Z_3^{1/2} A^a_\mu \ , \ \ c^a_0 = \tilde{Z}_3^{1/2} c^a \ , \ \ \psi_0 = Z_2^{1/2}\psi \ \ , g_0 = Z_g g \ , \ \ m_0 = Z_m m \, . \tag{5.27}$$

Thus, $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{ct}$, where $\mathcal{L}$ is equal to $\mathcal{L}_0$, except that it is written in terms of the renormalized variables, whereas $\mathcal{L}_{ct}$ is the counterterm Lagrangian, which reads

$$\begin{aligned}
\mathcal{L}_{ct} &= (Z_3 - 1)\frac{1}{2} A^\mu_a \delta^{ab}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu_b + (\tilde{Z}_3 - 1)\bar{c}^a\delta_{ab}(-i\partial^2)c^b \\
&+ (Z_2 - 1)\bar{\psi}^i(i\gamma^\mu\partial_\mu - m)\psi^i - (Z_2 Z_m - 1)m\bar{\psi}^i\psi^i \\
&- (Z_1 - 1)\frac{1}{2} g f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^\mu_b A^\nu_c - (\tilde{Z}_1 - 1) i g f^{abc}(\partial^\mu \bar{c}^a)c^b A^c_\mu \\
&- (Z_4 - 1)\frac{1}{4} g^2 f^{abe} f^{cde} A^a_\mu A^b_\nu A^\mu_c A^\nu_d + (Z_{1F} - 1) g \bar{\psi}^i t^a_{ij}\gamma^\mu\psi^j A^a_\mu \, ,
\end{aligned} \tag{5.28}$$

where we have defined

$$Z_1 \equiv Z_g Z_3^{3/2} \ , \ \ Z_4 \equiv Z_g^2 Z_3^2 \ , \tilde{Z}_1 \equiv Z_g \tilde{Z}_3 Z_3^{1/2} \ , \ \ Z_{1F} \equiv Z_g Z_2 Z_3^{1/2} \, .$$

The equality of $Z_g$ for all the couplings leads to the Slavnov-Taylor identities:

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} = \frac{Z_4}{Z_1} \, . \tag{5.29}$$

The Feynman rules for QCD can be found in any textbook. We work in the Feynman gauge, where $\alpha = 1$. We will focus mainly on the gluon self-energy and the three-gluon vertex, from which the beta function at one loop order can be computed. Further details can be found in [22].

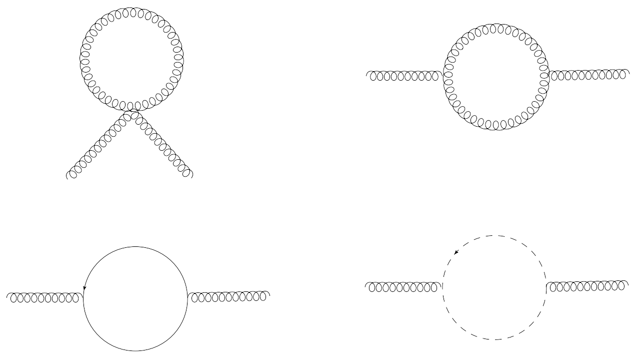


Figure 5.1: One-loop gluon self-energy

The gluon self-energy is composed of four contributions, as depicted in figure 5.1,

$$\Pi^{ab}_{\mu\nu} = \Pi^{ab}_{\mu\nu}(1) + \Pi^{ab}_{\mu\nu}(2) + \Pi^{ab}_{\mu\nu}(3) + \Pi^{ab}_{\mu\nu}(4), \tag{5.30}$$

where $\Pi^{ab}_{\mu\nu}(1)$, $\Pi^{ab}_{\mu\nu}(2)$, $\Pi^{ab}_{\mu\nu}(3)$ and $\Pi^{ab}_{\mu\nu}(4)$ represent the gluon tadpole, the gluon loop, the ghost loop and the quark loop, respectively. It is purely transversal as required by the Slavnov-Taylor identities and thus it does not admit a mass term and there should be no mass renormalization. Hence, the quadratic divergences which appear in $\Pi^{ab}_{\mu\nu}$ should cancel out.

We begin with the gluon tadpole,

$$\Pi^{ab}_{\mu\nu}(1) = -g^2 C_2(G)\delta^{ab} 3 \int_k \frac{g_{\mu\nu}}{k^2-\mu^2} = -3g^2 g_{\mu\nu} C_2(G)\delta^{ab} I_{quad}(\mu^2), \tag{5.31}$$

in which $\mu$ is fictitious mass which should be set to zero in the end. At this point, one may argue that $I_{quad}(\mu^2) = 0$ as $\mu \to 0$, but in an general calculation this may not be the case. We shall carry the quadratic divergences until the end, so that all regularization dependent parameters are ultimately fixed by symmetry. This approach is adequate for interpreting the role of quadratic divergences in the examples we exploit in the next sections.

The gluon loop amplitude reads

$$\Pi^{ab}_{\mu\nu}(2) = \frac{(-i)^2}{2} \int_k g^2 f^{acd} f^{bcd} N_{\mu\nu} \frac{1}{[k^2-\mu^2]} \frac{1}{[(k+p)^2-\mu^2]} \,, \tag{5.32}$$

where

$$N_{\mu\nu} = 2p_\mu p_\nu - 5(p_\mu k_\nu + p_\nu k_\mu) - 10 k_\mu k_\nu - g_{\mu\nu}[(p-k)^2 + (k+2p)^2] \,, \tag{5.33}$$

which yields

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(2) = & -\frac{1}{2} g^2 C_2(G)\delta^{ab}[(2p_\mu p_\nu - 4p^2 g_{\mu\nu}) J(p^2,\mu^2) - g_{\mu\nu}(2I_{quad}(\mu^2) + p^\alpha p^\beta \Upsilon^0_{\alpha\beta}) \\ & - 10(\, p_\nu J_\mu(p^2,\mu^2) + J_{\mu\nu}(p^2,\mu^2)\,)] \,. \end{aligned} \tag{5.34}$$

Integrals $J_{\mu\nu}(p^2,\mu^2)$, $J_\mu(p^2,\mu^2)$ and $J(p^2,\mu^2)$ are defined in appendix B.

As for the ghost loop, we have

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(3) &= -g^2 f^{dac} f^{cbd} \int_k \frac{i^2}{k^2-\mu^2} \frac{(p+k)_\mu k_\nu}{[(k+p)^2-\mu^2]} \\ &= -g^2 \delta^{ab} C_2(G) \left[ p_\nu J_\mu(p^2,\mu^2) + J_{\mu\nu}(p^2,\mu^2) \right] . \end{aligned} \tag{5.35}$$

The fermion loop contribution to the gluon self energy is identical to the vacuum polarization tensor of $QED$, except for the colour and number of fermions

$(n_f)$ factors. It has been computed within IReg [20] and reads

$$\Pi^{ab}_{\mu\nu}(4) = \frac{4}{3}g^2 C(r) n_f \delta^{ab}\Bigg\{\Big(p_\mu p_\nu - p^2 g_{\mu\nu}\Big)\Bigg[I_{log}(\mu^2) - b\Bigg(\ln\Big(-\frac{p^2}{\mu^2}\Big) - \frac{5}{3}\Bigg)\Bigg] + \Upsilon^{(2)}_{\mu\nu}(\mu^2) + p^2\Upsilon^{(0)}_{\mu\nu} + p^\alpha p^\beta \Upsilon^{(0)}_{\mu\nu\alpha\beta} + p^\alpha p_\mu \Upsilon^{(0)}_{\nu\alpha} + p^\beta p_\nu \Upsilon^{(0)}_{\mu\beta} + p^\alpha p^\beta g_{\mu\nu}\Upsilon^{(0)}_{\alpha\beta}\Bigg\}, \tag{5.36}$$

where we are using the following notation for the surface terms $g_{\mu\nu} g_{\alpha\beta}\Upsilon^{\mu\beta}_i \equiv \Upsilon^{(i)}_{\nu\alpha}$, and so forth. Altogether, $\Pi^{ab}_{\mu\nu} = \sum_{i=1}^{4} \Pi^{ab}_{\mu\nu}(i)$ reads

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p^2,\lambda^2) = -\frac{b}{9}g^2(p^2 g_{\mu\nu} - p_\mu p_\nu)\delta^{ab}\Bigg\{ i\Big[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\Big] I_{\log}(\lambda^2) \\ + \Big(15C(r) - 6n_f\Big)\ln\Big(\frac{\lambda^2}{p^2}\Big) - 2C(r) + 2n_f\Bigg\} \\ + g^2\delta^{ab}\Big(C_2(G) + \frac{4}{3}C(r)n_f\Big)\Bigg\{\Upsilon^{(2)}_{\mu\nu}(\lambda^2) - \lambda^2\Upsilon^{(0)}_{\mu\nu} + p^2\Upsilon^{(0)}_{\mu\nu} + p^\alpha p^\beta\Upsilon^{(0)}_{\mu\nu\alpha\beta} \\ + p^\alpha p_\mu \Upsilon^{(0)}_{\nu\alpha} + p^\beta p_\nu \Upsilon^{(0)}_{\mu\beta} + p^\alpha p^\beta g_{\mu\nu}\Upsilon^{(0)}_{\alpha\beta}\Bigg\},\end{aligned} \tag{5.37}$$

where we used relation (5.14) in order to introduce a renormalization group scale $\lambda$. Notice that the infrared divergences, as $\mu \to 0$, cancel out as they should and only the quadratic surface term has a dependence on $\lambda^2$ (we use, for simplicity $\Upsilon^{(0)}_{\alpha\beta}(\mu^2) \equiv \Upsilon^{(0)}_{\alpha\beta}$). We also have used the relation

$$\Upsilon^{(2)}_{\alpha\beta}(\mu^2) = \Upsilon^{(2)}_{\alpha\beta}(\lambda^2 \neq 0) + (\mu^2 - \lambda^2)\Upsilon^{(0)}_{\alpha\beta}. \tag{5.38}$$

Setting the surface terms to zero or, accordingly, making $\Gamma^{(l,j)}_i = 0$ in (5.22)-(5.25) renders the total amplitude transverse, as required by gauge invariance. This amounts to exercising a constrained gauge invariant version of IReg (CIReg) [66]. Notice that the quadratic divergences organized themselves as surface terms. If evaluated in dimensional regularization, they yield zero because the latter is a gauge invariant framework. We shall systematically set the surface terms to zero and express the BDI's as a function of $\lambda$ until the end of this section.

We define the counterterm for the amplitude (5.37) by minimally subtracting the BDI expressed by $I_{log}(\lambda^2)$:

$$Z_3 = 1 - i\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] I_{\log}(\lambda^2) g^2 + O(g^3). \tag{5.39}$$

The class of one loop three-gluon vertex graphs, from which we shall define $Z_1$, are shown in fig. 5.2.

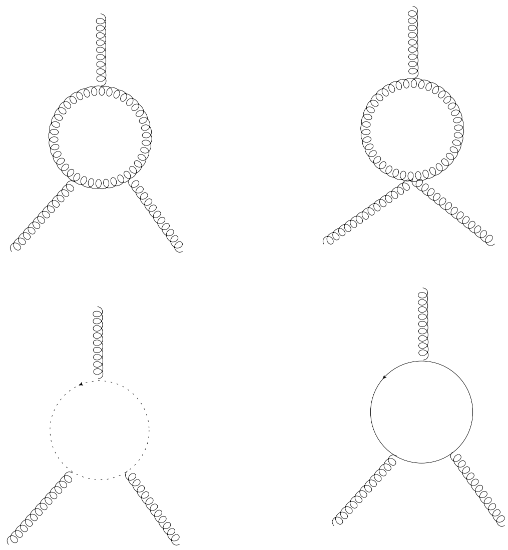


Figure 5.2: One-loop three-gluon vertex

For the sake of brevity, we shall present only the result here. Let $p$ and $q$ be the external momenta. Then

$$\begin{aligned}\Lambda^{abc}_{\mu\nu\lambda}(p,q) =& - igf^{abc}V_{\mu\nu\lambda}(p,q,p+q)\left[ig^2\left(\frac{2}{3}C_2(G) - \frac{4}{3}C(r)n_f\right)I_{log}(\lambda^2) + Z_1 - 1\right] \\ &+ \widetilde{\Lambda}^{abc}_{\mu\nu\lambda}(p,q)\,,\end{aligned} \tag{5.40}$$

where $V_{\mu\nu\lambda}(p,q,p+q) = (p-q)_\lambda g_{\mu\nu} - p_\mu g_{\nu\lambda} + q_\nu g_{\mu\lambda}$, and $\widetilde{\Lambda}^{abc}_{\mu\nu\lambda}(p,q)$ represents the finite part of the amplitude. We then define

$$Z_1 = 1 + ig^2\bigg( -\frac{2}{3}C_2(G) + \frac{4}{3}C(r)n_f\bigg)I_{log}(\lambda^2)\,. \tag{5.41}$$

Since we are mainly interested in the computation of the beta function, we will not present the other renormalization constants. However, as explicitly showed in [22], all of them obey Slavnov-Taylor identities expressed by (5.29):

$$\begin{aligned}\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} = \frac{Z_4}{Z_1} =& 1 + ig^2C_2(G)I_{log}(\lambda^2) \\ =& 1 + ig^2C_2(G)\bigg[b\ln\left(\frac{\Lambda^2}{\lambda^2}\right) + \alpha_1\bigg]. \end{aligned} \tag{5.42}$$

In the last line, we make use of the parametrization of BDI's introduced in the last section. Since the Slavnov-Taylor identities are manifestations of gauge invariance, we are intimately showing how a cutoff can be introduced without breaking gauge symmetry. In other words, after we disentangle BDI's and surface terms, we can safely introduce a cutoff in the former by using the parametrization, since all symmetry breaking terms are encoded in the latter.

To conclude this section, we compute the beta function of QCD at one-loop level. As explained before, we have introduced the parameter $\lambda$, which will play the role of renormalization group scale. Thus, we have

$$\beta(g) = \lambda\frac{\partial g}{\partial \lambda}. \tag{5.43}$$

Recall (5.27): $g_0 = Z_g g$, $Z_g = Z_1 Z_3^{-3/2}$. Hence,

$$2\lambda^2\frac{\partial}{\partial\lambda^2}\Big(Z_g g\Big) = 0 \Longrightarrow \beta(g) = -2g\lambda^2\frac{\partial \ln Z_g}{\partial\lambda^2}\,. \tag{5.44}$$

Now, using that

$$\lambda^2\frac{\partial}{\partial\lambda^2}I_{log}(\lambda^2) = -b \tag{5.45}$$

in the equation above yields, after some simple algebra,

$$\beta = -\frac{g^3}{3(4\pi)^2}\Big(11C_2(G) - 4C(r)n_f\Big) + O(g^5)\,. \tag{5.46}$$

In a similar fashion, we can work with an explicit cutoff, in which case the renormalization group scale would be $\Lambda$, on the basis of a cutoff parameter independence of the Green's functions in the Wilsonian renormalization group [100, 101]. In this case, the renormalization constants will explicitly depend on $\Lambda$, for instance,

$$Z_1 = 1 + ig^2\Bigg(-\frac{2}{3}C_2(G) + \frac{4}{3}C(r)n_f\Bigg)\Bigg[b\ln\left(\frac{\Lambda^2}{\lambda^2}\right) + \alpha_1\Bigg], \tag{5.47}$$

which furnishes the following result

$$\lambda^2\frac{\partial}{\partial\lambda^2}\ln Z_g(g,\Lambda^2/\lambda^2) = -\Lambda^2\frac{\partial}{\partial\Lambda^2}\ln Z_g(g,\Lambda^2/\lambda^2). \tag{5.48}$$

Thus, after simple algebra, we obtain the same result for the beta function.

In summary, an explicit cutoff can always be introduced once we have correctly identified the regularization dependent terms. The surface terms would boil down to the arbitrary terms in the parametrization of $I_{log}$'s and $I_{quad}$'s and would be fixed by gauge symmetry as well. The advantage of working with basic divergent integrals is that we can neatly identify regularization dependent terms as surface terms. Finally, this procedure can be worked out to general loop order [34].

## 5.3 Higgs decay to two photons

In this section we will discuss how to fix arbitrariness involved in the calculation of the Higgs decay to two photons. This decay was a subject of discussion in the recent literature ([102] and references therein) as presented in chapter 4. In order to make the discussion more pleasant for the reader, we will present again some of the results already discussed in the last chapter. Once again we will consider only the W boson loop, since it already contains all relevant aspects regarding arbitrariness we intend to discuss in the following. The diagrams in the unitary gauge that contribute are shown in figure 5.3.

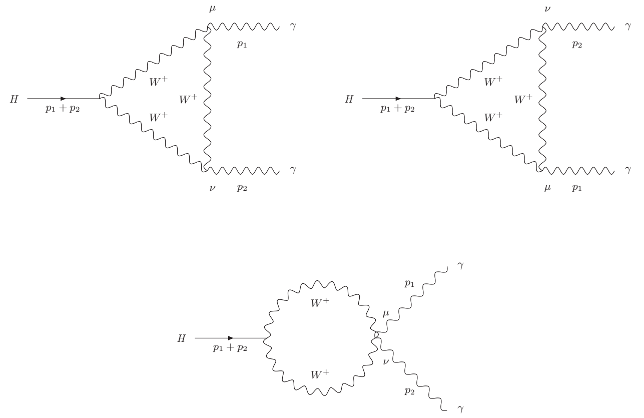



Figure 5.3: Diagrams that contribute to the Higgs decay to two photons

The contributions can be simplified to

$$M = ie^2 g M_w \Big[ M^{(a)}_{\mu\nu} + M^{(b)}_{\mu\nu} + M^{(c)}_{\mu\nu} \Big] ({\epsilon_1}^{\mu})^* ({\epsilon_2}^{\nu})^* + (p_1 \leftrightarrow p_2, \mu \leftrightarrow \nu), \tag{5.49}$$

with

$$\begin{aligned} M^{(a)}_{\mu\nu} &= -\frac{4}{M_w^2} \Big[ g_{\mu\nu} (p_1)^\alpha (p_2)^\beta I^{(3)}_{\alpha\beta} + (p_1 \cdot p_2) I^{(3)}_{\mu\nu} - (p_1)_\nu (p_2)^\alpha I^{(3)}_{\mu\alpha} - (p_2)_\mu (p_1)^\alpha I^{(3)}_{\nu\alpha} \Big] \\ &\quad + \frac{2}{M_w^2} \Big[ g_{\mu\nu} (p_1 \cdot p_2) - (p_2)_\mu (p_1)_\nu \Big] I^{(3)}_2, \end{aligned} \tag{5.50}$$

$$M^{(b)}_{\mu\nu} = \int_k \frac{3(g_{\mu\nu} k^2 - 4 k_\mu k_\nu)}{(q_1^2 - M_w^2)(q_2^2 - M_w^2)(q_3^2 - M_w^2)}, \tag{5.51}$$

$$\begin{aligned} M^{(c)}_{\mu\nu} &= 6 g_{\mu\nu} \Big[ (p_1 \cdot p_2) I^{(3)}_0 - (p_1)^\alpha I^{(3)}_\alpha - \frac{M_w^2}{2} I^{(3)}_0 \Big] \\ &\quad + 6 \Big[ 2 (p_1)_\nu I^{(3)}_\mu - (p_2)_\mu (p_1)_\nu I^{(3)}_0 \Big], \end{aligned} \tag{5.52}$$

$$I^{(3)}_{0,2,\mu,\mu\nu} = \int_k \frac{1, k^2, k_\mu, k_\mu k_\nu}{(q_1^2 - M_w^2)(q_2^2 - M_w^2)(q_3^2 - M_w^2)}. \tag{5.53}$$

There is an inherent arbitrariness in the expressions above which will present itself as a surface term in our framework. Explicitly, we have[1]

$$M^{(a)}_{\mu\nu} = \frac{\big[(p_2)_\mu(p_1)_\nu - g_{\mu\nu}(p_1\cdot p_2)\big]}{M_w^2}\left[\frac{i}{16\pi^2} - 2\Gamma_0^{(1,2)}\right], \tag{5.54}$$

$$\begin{aligned} M^{(b)}_{\mu\nu} + M^{(c)}_{\mu\nu} &= \frac{i}{16\pi^2 M_w^2}\big[(p_2)_\mu(p_1)_\nu - g_{\mu\nu}(p_1\cdot p_2)\big]\left[\frac{3\tau^{-1}}{2} + \frac{3(2\tau^{-1}-\tau^{-2})f(\tau)}{2}\right] \\ &+ g_{\mu\nu}(p_1\cdot p_2)\left(\frac{3\tau^{-1}}{2M_w^2}\Gamma_0^{(1,2)}\right). \end{aligned} \tag{5.55}$$

As one can immediately notice, the surface term in the second expression breaks gauge invariance. Therefore, as in the last section, the imposition of such symmetry will fix the ambiguity to a precise value (in the present case, it will be null), furnishing the well-established value for this decay.

Now we would like to perform the same analysis again, but from a different point of view, which will clarify the role played by gauge symmetry and quadratic divergences. We begin by defining the amplitude

$$M_{\mu\nu} = ie^2 g M_w\Big[M^{(a)}_{\mu\nu} + M^{(b)}_{\mu\nu} + M^{(c)}_{\mu\nu}\Big] + (p_1 \leftrightarrow p_2, \mu \leftrightarrow \nu), \tag{5.56}$$

which, if gauge invariance is to be guaranteed, must satisfy

$$M_{\mu\nu}p_1^\mu p_2^\nu = 0. \tag{5.57}$$

[1]We define $\tau = \frac{M_h^2}{4M_w^2}$ and

$$f(\tau) = \begin{cases} \arcsin^2(\sqrt{\tau}) & \text{for} \quad \tau \le 1\,, \\ -\dfrac{1}{4}\left[\ln\dfrac{1+\sqrt{1-\tau^{-1}}}{1-\sqrt{1-\tau^{-1}}} - i\pi\right]^2 & \text{for} \quad \tau > 1\,. \end{cases}$$

By using expressions (5.50), (5.51) and (5.52) one obtains

$$M_{\mu\nu}p_1^\mu p_2^\nu = ie^2 g M_w \left[ \int_k \frac{3}{(k-p_1)^2 - M_w^2} - \int_k \frac{3}{k^2 - M_w^2} \right.$$
$$\left. + \int_k \frac{3}{(k-p_2)^2 - M_w^2} - \int_k \frac{3}{(k-p_1-p_2)^2 - M_w^2} \right] \quad (5.58)$$

This is the main result of this section. Firstly, we notice the appearance of quadratic divergent integrals which could indicate that the arbitrariness stemming in the Higgs decay to two photons shares a common origin with the hierarchy problem. However, this is not the case. To demonstrate this, one may resort to the general parametrization of quadratic divergences presented in eq. (5.11). As can be easily seen, there is an arbitrary parameter ($\alpha_2$) multiplying a cutoff $\Lambda^2$ which is in the root of the hierarchy problem as will be explained in a latter section. For the present case, however, this arbitrariness will play no role, since we have a difference between quadratic divergent integrals, which results in the cancellation of such coefficient. Therefore, the ambiguity in the present case must have a different origin.

Secondly, we notice that the expression above can be related to the following tadpole

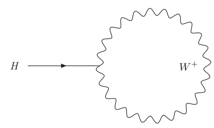



Figure 5.4: Tadpole

whose analytical expression is given by

$$T = -\frac{g}{M_w} \int_k \left[ 1 - \frac{3M_w^2}{k^2 - M_w^2} \right]. \quad (5.59)$$

The first term is a quartic divergent integral which can, in principle, be added and subtracted to expression (5.58) in order to reproduce the tadpole exactly. However, since such integrals do not depend on any physical scale, they are completely unphysical and should be discarded. Therefore, one can immediately notice that the result (5.58) is a difference between tadpoles with different momentum routing. As explained in [66], the condition to implement momentum routing is just to demand that the difference between the same Feynman diagram with different momentum routing is null. Therefore, one can easily notice that the condition to have a gauge invariant result for the Higgs decay is just to demand momentum routing invariance of the tadpole depicted in fig 5.4. In other words, by demanding this tadpole to be momentum routing invariant, gauge symmetry will be automatically respected.

To conclude, we will demonstrate that the gauge breaking term is exactly the same we had before in eq. (5.55). We begin using the following expansion

$$f(k+a) = f(k) + a_\sigma \frac{\partial}{\partial k_\sigma} f(k) + \frac{a_\sigma a_\rho}{2!} \frac{\partial^2}{\partial k_\sigma k_\rho} f(k) + \cdots , \tag{5.60}$$

which in our case is given by

$$\frac{1}{(k+a)^2 - M_w^2} = \frac{1}{k^2 - M_w^2} - 2a_\sigma \frac{k_\sigma}{(k^2 - M_w^2)^2} - a_\sigma a_\rho \frac{\partial}{\partial k_\rho} \frac{k_\sigma}{(k^2 - M_w^2)^2} + \cdots . \tag{5.61}$$

Thus, following the reasoning of [66]

$$\int_k \frac{1}{(k+a)^2 - M_w^2} - \int_k \frac{1}{k^2 - M_w^2} = -a^2 \Gamma_0^{(1,2)}. \tag{5.62}$$

Replacing this result into eq. (5.58), we finally obtain

$$\begin{aligned} M_{\mu\nu} p_1^\mu p_2^\nu &= ie^2 g M_w 6 (p_1 \cdot p_2) \Gamma_0^{(1,2)} \\ &= ie^2 g M_w (p_1 \cdot p_2)^2 \left( \frac{3\tau^{-1}}{M_w^2} \Gamma_0^{(1,2)} \right) \end{aligned} \tag{5.63}$$

which is exactly what the imposition of the Ward identity to eq. (5.55) would furnish[1].

[1]There is a extra two factor due to the sum of the crossed diagram.

In summary, we clarified that the Higgs decay to two photons is not connected to quadratic ambiguities. The arbitrariness inherent of this calculation comes from the subtraction of logarithmic divergent integrals and is fixed by gauge invariance. We would like also to stress that, since in our framework regularization dependent terms can be consistently identified, it is possible to introduce a cutoff without breaking gauge invariance. In other words, after the identification of the surface terms (which control the symmetry breaking), any divergent amplitude will be written in terms of BDI's which, by means of the general parametrizations presented in section 5.1, will depend explicitly on a cutoff $\Lambda^2$.

## 5.4 Quadratic Divergences and Effective Theories: Nambu-Jona-Lasinio model

Quadratic divergences play a vital role in the description of dynamical chiral symmetry breaking in models of low energy QCD, such as the Nambu–Jona-Lasinio model [103, 104]. This model belongs to the class of non-renormalizable Lagrangians and the regulator, usually expressable in terms of a cutoff $\Lambda$ for the UV divergent one-loop quark integrals appearing at leading order of $N_c$, is characteristic of the scale at which spontaneous breakdown of chiral symmetry occurs, typically of the order of 1 GeV. We illustrate the role of the quadratic divergence in terms of the original 2 flavor NJL model applied to the light quarks with $N_c = 3$

$$L_{NJL} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m_c)\psi(x) + \frac{G}{2}[(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)i\gamma_5\tau_i\psi(x))^2], \quad (5.64)$$

with $G > 0$. There are two relevant quantities needed to be considered to set the scale of chiral symmetry breaking, one related to the weak decay constant of the pion $f_\pi \sim 93$ MeV, which is a logarithmically divergent integral and the other with the gap equation, which is quadratically divergent. We will consider for simplicity the chiral limit, $m_c = 0$. The integrals are regulated using a Pauli-Villars regularization with two subtractions in a form which is equivalent to the sharp Euclidean cutoff when scalar integrals are considered. Other regularizations have been discussed in [70], all displaying a quadratic divergence for the gap equation, and leading to similar conclusions. For comparison we will also use the general parametrization of eqs. (5.9) and (5.11). While $f_\pi$ depends explicitly only on the constituent quark mass $M$ and $\Lambda$,

$$\begin{aligned} f_\pi^2 &= -4N_c M^2 i I_{log}^{\Lambda}(M^2) \\ &= \frac{N_c M^2}{(2\pi)^2}\Big( \ln(1+\tilde{\lambda}^2) - \frac{\tilde{\lambda}^2}{1+\tilde{\lambda}^2}\Big) \end{aligned} \tag{5.65}$$

where $\tilde{\lambda}^2 = \frac{\Lambda^2}{M^2}$, the gap equation also depends on the coupling strength of the four quark interaction $G$,

$$M - m_c = M\frac{N_c G M^2}{(2\pi)^2}(\tilde{\lambda}^2 - ln(1+\tilde{\lambda}^2)) \tag{5.66}$$

which has to reach a critical value $G_{cr}$ for the phase transition from the Wigner-Weyl phase ($M = 0$ in the chiral limit) to the asymmetric phase to occur.

The solution of the gap equation corresponds to the minimum of the effective potential $V(\sigma, \pi)$ calculated to lowest order in $N_c$ counting, with $\sigma = \bar{\psi}(x)\psi(x), \pi_i = \bar{\psi}(x) i\gamma_5 \tau_i \psi(x)$ the standard auxiliary bosonic variables (see e.g. [105])

$$\begin{aligned} V(\sigma, \pi_i) = \frac{\sigma^2+\pi_i^2}{2G}\Big(1 - \frac{N_c G \Lambda^2}{4\pi^2}\Big) + \frac{N_c}{8\pi^2}\Big[(\sigma^2+\pi_i^2)^2 \ln\Big(1+\frac{\Lambda^2}{\sigma^2+\pi_i^2}\Big) \\ -\Lambda^4 \ln\Big(1+\frac{\sigma^2+\pi_i^2}{\Lambda^2}\Big)\Big] \end{aligned} \tag{5.67}$$

Taking the expectation value in the vacuum $<\pi_i>=0$, the minimum is localized at $<\sigma>= M \neq 0$, if the curvature

$$C = \partial_\sigma^2 V(\sigma, \pi_i)_{|\sigma=0,\pi_i=0} < 0$$

corresponding to the onset of a mexican hat shaped potential. The latter condition leads to $G > G_{cr} = \frac{2\pi^2}{N_c \Lambda^2}$.

This critical value is however constrained by the empirical value of $f_\pi$, which sets a minimal value for the cutoff $\Lambda_{cr} \sim .72$ GeV, obtained by solving eq. (5.65). This result has been shown long ago in [106]: below this critical value there are no solutions for the given value of $f_\pi$ and above this value two branches emerge as functions of $(M, \Lambda)$, representing asymptotically a strong coupling regime (branch 1) where the quark mass goes faster to infinity than the cutoff, $\tilde{\lambda}^2 = \frac{\Lambda^2}{M^2} \to 0$, and a weak coupling regime (branch 2) with $\tilde{\lambda}^2 = \frac{\Lambda^2}{M^2} \to \infty$. Both regimes are in the phase of spontaneous breakdown of chiral symmetry, but empirical values of the light constituent quark masses $200 \prec M \prec 400$ MeV rule out branch 1. Branch 2 is characterized by a coupling $N_c G \Lambda^2 \sim \mathcal{O}(1)$. This is the result of the gap equation with the $f_\pi$ constraint and evidences a quadratic divergence, while for branch 1 one would obtain asymptotically $G f_\pi^2 \sim \mathcal{O}(1)$, [106].

One can thus infer that the quadratic divergence of the gap equation is necessary to ensure a sensible solution for the values of the constituent quark masses together with the empirical value of $f_\pi$ in the phase of spontaneously broken chiral symmetry, at leading order of $N_c$ counting. Furthermore the condition $\Lambda \geq \Lambda_{cr}$ must be fulfilled, whereby $\Lambda_{cr}$ is uniquely determined by the logarithmic divergence associated with $f_\pi$.

The meson mass spectrum, in this case the Goldstone pion and the $\sigma$-meson with $m_\sigma = 2M$, emerge as consequence of dynamical chiral symmetry breaking with the cutoff dependence completely absorbed in the value of the constituent quark mass.

Further light can be shed on the relevance of the quadratic divergence: in a recent extended version of the NJL model, which contemplates the most general combinations of spin zero multiquark interactions relevant at the scale of chiral symmetry breaking, including a complete set of explicit symmetry breaking interactions [107],[108], it is shown that $\Lambda$ associated with the quadratic divergences of the gap equation can be used to establish a counting scheme which allows to classify all relevant interactions (i.e. which survive in the limit of $\Lambda \to \infty$) in the phase of spontaneous symmetry breaking. This counting scheme is in consonance with the large $N_c$ counting scheme and requires the $\Lambda^2$ behavior of the gap equation.

If instead one would use the parametrizations (5.9) and (5.11) for the logarithmic and quadratic divergence one would obtain

$$f_\pi^2 = \frac{N_c M^2}{(2\pi)^2}(\ln(\tilde{\lambda}^2) - \alpha_1) \tag{5.68}$$

and for the gap equation

$$M - m_c = M\frac{N_c G M^2}{(2\pi^2)}(\alpha_2 \tilde{\lambda}^2 - \ln(\tilde{\lambda}^2) + \alpha_3) \tag{5.69}$$

which for $\tilde{\lambda}^2 \gg 1$, $\alpha_2 = \alpha_1 = 1, \alpha_3 = 0$ reduces to the result using Pauli-Villars regularization.

Knowing that the curvature of the effective potential is in this case given by

$$C = \partial_\sigma^2 V(\sigma, \pi_i)_{|\sigma=0,\pi_i=0} = \frac{1}{2G} - \frac{N_c \alpha_2 \Lambda^2}{4\pi^2},$$

the absence of the quadratic divergence in the gap equation, obtained by chosing $\alpha_2 = 0$, leads to $C = \frac{1}{2G}$; since $G > 0$ one deduces immediately that only the symmetric phase is described in this case, independently of any parameters of the model. The Implicit Regularization thus corroborates the fact that the presence of quadratic divergences is essential to be able to reach the phase of spontaneously broken chiral symmetry.

## 5.5 Hierarchy problem

As discussed in the introduction, many proposals have been devised to interpret the role and fate of quadratic divergences in the hierarchy problem. It is necessary to introduce a cutoff to serve as a merging scale when we study the SM as an effective theory. A general parametrization for ultraviolet divergences with an explicit scale $\Lambda$ much greater than the characteristic masses of the model can be constructed, as we have demonstrated in earlier sections. The parametrization of quadratic divergence embodies a regularization dependent coefficient multiplying $\Lambda^2$. As a result of negative searches for SuSy at the TeV scale, its original motivation of solving the hierarchy problem by canceling out the quadratic divergences becomes questionable. This is because the stability at quantum level of the hierarchy EW scale $\ll M_{Planck}$ becomes more difficult to respect, although some extensions in MSSM have been envisaged [109]. The natural question, following the examples we presented in previous sessions, is whether or not it can be fixed on symmetry grounds [42].

Because of its chiral nature, the Lagrangian of the standard model possesses conformal invariance, except for the Higgs mass term, which is related to the hierarchy problem. Bardeen [99] has argued that, once the classical conformal invariance and its minimal violation by quantum anomalies are imposed on the SM, it can be freed from quadratic divergences (and hence the hierarchy problem) and one can, in principle, directly interpolate the electroweak scale and the Planck scale. Such idea has been taken forward to envisage extensions to the SM with a flat Higgs potential at the Planck scale [110].

It remains to establish this hypothesis into a calculational framework. This is exactly where the general parametrization for basic divergent integrals is useful. We use the Bardeen's hypothesis as a symmetry guide to fix an arbitrary parameter multiplying $\Lambda^2$ to zero. This has also an aesthetic appeal, since we would be left with logarithmic divergences which can be multiplicatively renormalized.

Weinberg was the first to examine the complications caused by quadratic divergences in a mass independent renormalization scheme [111]. Fujikawa in [97] addressed this problem by introducing a counterterm independent of the scalar mass to subtract the quadratic divergent contribution. This was done in a similar fashion as Callan avoided the quadratic divergence by a mass insertion technique in his Callan-Symanzik equation [112, 113]. This in turn is closely related to the scaling argument of Bardeen [99] [1].

Moreover, as explained by Aoki and Iso in [77], at classical level the Higgs mass term breaks scale invariance of the SM, which would have an increase of symmetry should the mass term vanish. Such increase of symmetry has no role in controlling divergences, since scale invariance is broken by the logarithmic runnings of the couplings. Once quadratic divergences are subtracted, the trace of the energy momentum tensor becomes proportional to $\Delta m^2 H^\dagger H + \beta_{g_i} \mathcal{O}_i$, in which $\Delta m^2 \propto m^2$ and not $\Lambda^2$. The anomalous term and the mass term are soft breaking

[1] As we will see, this is exactly what dimensional regularization does in terms of subtracting the quadratic divergence, that is setting the arbitrary parameter multiplying $\Lambda^2$ to zero.

terms, since they do not generate quadratic divergences. Still according to [77], in the Wilsonian renormalization group, quadratic divergences determine a position of the critical surface of the theory, and the scaling behavior around such critical surface is determined by logarithmic divergences. The subtraction of quadratic divergences, according to [77], then amounts to a coordinate transformation in the theory space and, thus, such divergences have no role whatsoever in the fine tuning problem. As we will see, these conclusions can be reached already at regularized level using the symmetry argument proposed in [99] to fix arbitrary regularization dependent parameters.

Consider the Higgs sector of the SM Lagrangian,

$$\mathcal{L}_H(x) = [D^\mu\Phi(x)]^\dagger[D_\mu\Phi(x)] - \mu^2\Phi(x)^\dagger\Phi(x) - \lambda[\Phi(x)^\dagger\Phi(x)]^2, \quad (5.70)$$

where $\Phi(x) = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ is the Higgs field, $D_\mu$ is the covariant derivative that couples it with the gauge fields and $\mu^2 < 0$. The scale transformations

$$x' = e^{-\alpha}x, \quad \text{and} \quad \phi'(x) = e^{-\alpha d}\phi(e^{-\alpha}x), \quad (5.71)$$

leave (5.70) unchanged for $\mu^2 = 0$, where $\alpha$ here is a scale parameter and $d$ is the scale dimension of the field. The mass term breaks that classical conservation law because it is the only one in the Lagrangian which does not possess scale dimension equal to four, i. e. the mass did not transform according to the rules (5.71). Therefore,

$$\Theta^\mu_\mu = m^2\Phi(x)^\dagger\Phi(x), \quad (5.72)$$

where $m^2 = -2\mu^2$ is the tree level Higgs mass.

Since quantum corrections make the couplings depend on the renormalization group scale, the Lagrangian (5.70) changes due to the scale transformation

$$x' = e^{-\alpha}x \to \Lambda' = e^{\alpha}\Lambda, \quad (5.73)$$

which leads to

$$\delta\mathcal{L}_H(m_H(\Lambda), \lambda(\Lambda)) = \alpha\{m_H^2(\Lambda)\gamma\Phi(x)^\dagger\Phi(x) + \beta_\lambda[\Phi(x)^\dagger\Phi(x)]^2\}, \quad (5.74)$$

where $m_H^2(\Lambda)$ and $\lambda(\Lambda)$ are the renormalized Higgs mass and self-coupling, respectively, and

$$\gamma = \frac{\Lambda^2}{m_H^2(\Lambda^2)} \frac{\partial m_H^2(\Lambda^2)}{\partial \Lambda^2} \tag{5.75}$$

is the renormalization group gamma function. Hence, the complete violation of the dilatation current, due to the mass term and quantum corrections, is given by

$$\Theta_\mu^\mu = (m^2 + m_H^2(\Lambda)\gamma)\Phi(x)^\dagger\Phi(x) - \beta_\lambda[\Phi(x)^\dagger\Phi(x)]^2. \tag{5.76}$$

We may now write, for the Higgs renormalized mass, according to equation (5.11) [114],

$$m_H^2(\Lambda) \quad = \quad m^2 - \frac{3\alpha_2}{8\pi^2\upsilon^2}[m_Z^2 + 2m_W^2 + m^2 - 4m_t^2]\Lambda^2 + O\left(\ln\frac{\Lambda}{m}\right). \tag{5.77}$$

Using (5.77) in (5.75), we get

$$m_H^2(\Lambda)\gamma = -\frac{3\alpha_2}{8\pi^2\upsilon^2}[m_Z^2 + 2m_W^2 + m^2 - 4m_t^2]\Lambda^2 + O(m^2). \tag{5.78}$$

Now if we try to restore the classical limit, taking $m \to 0$ and $\beta_\lambda \to 0$ in equation (5.76), the only term that spoils the recovery of the dilatation current conservation is

$$\Theta_\mu^\mu = \frac{-3\alpha_2}{8\pi^2\upsilon^2}[m_Z^2 + 2m_W^2 - 4m_t^2]\Lambda^2\Phi(x)^\dagger\Phi(x). \tag{5.79}$$

In a non-supersymmetric scenario in order to restore the classical limit ($\Theta_\mu^\mu = 0$), we have to choose $\alpha_2 = 0$.

Furthermore, with this choice we can assure that quadratic divergences are not present in the beta function. After spontaneous symmetry breaking, we find the relation between the Higgs mass and its self-coupling given by $m^2 = 2\lambda\upsilon^2$, where $\upsilon$ is its vacuum expectation value. This relation varies with the renormalization group scale, i. e. $m_H^2(\Lambda) = 2\lambda(\Lambda)\upsilon^2$. Differentiating both sides of this equation with respect to $\Lambda$, we have

$$m_H^2(\Lambda)\gamma = \beta_\lambda v^2. \tag{5.80}$$

Therefore, if the quadratic term in the cutoff exists, the beta function would depend on it, as we can see if we replace equation (5.78) in equation (5.80).

Let us consider some numerical implications of our results. We can estimate in which scale the fine-tuning starts and perturbation theory breaks by asking where $|\delta m^2| = O(m^2)$. Considering $\alpha_2 = -1$ (obtained in a sharp cutoff regularization), the experimental data for the masses ($m_t = 173 GeV$, $m_W = 80.2 GeV$, $m_Z = 91.2 GeV$ and $m = 126\ GeV$) and the VEV value ($v = 246\ GeV$) that scale would be $\Lambda \approx 0.5\ TeV$. It means that the SM model as an effective theory should be reliable up to this scale and new physics should appear beyond it. However, as mentioned before, this new physics has not been found with $\sqrt{s} = 8\ TeV$.

Nevertheless, we can choose $\alpha_2 = 0$ considering the arguments above, which leave us with the logarithmic correction to the Higgs mass given by [114],

$$\delta m^2 = \frac{3m^2}{16\pi^2 v^2}[2m_t^2 + 2m_W^2 + m^2 - m_Z^2] \ln \frac{\Lambda^2}{m_H^2}. \tag{5.81}$$

The estimate for the SM cutoff now is extremely large. We have $|\delta m^2| = O(m^2)$ when $\Lambda \approx 10^7\ TeV$. We conclude that consistency of scale symmetry breaking avoids the fine-tuning and makes perturbation theory ($|\delta m^2| << O(m^2)$) reliable up to this scale.

We end this section making connection with dimensional regularization. Let us write basic quadratic divergent integral $I_{quad}(m^2)$

$$I_{quad}(m^2) = \lim_{\mu \to 0} \int_k \frac{1}{(k^2 - m^2 - \mu^2)}. \tag{5.82}$$

We can write (5.82) as

$$\int_k \frac{1}{(k^2 - m^2 - \mu^2)} = I_{quad}(\mu^2) + m^2 I_{log}(\mu^2) + m^4 \int_k \frac{1}{(k^2 - \mu^2)^2 (k^2 - m^2 - \mu^2)}. \tag{5.83}$$

Using (5.14) and

$$\int_k \frac{1}{(k^2-\mu^2)^2(k^2-m^2-\mu^2)} = -b\left(1+\frac{\mu^2}{m^2}\right)\ln\left[\frac{m^2}{\mu^2}\left(1+\frac{\mu^2}{m^2}\right)\right] + \frac{b}{m^2}, \quad (5.84)$$

we obtain

$$I_{quad}(m^2) = \lim_{\mu\to 0} I_{quad}(\mu^2) + m^2 I_{log}(m^2) + \frac{i}{(4\pi)^2}m^2. \quad (5.85)$$

In dimensional regularization it is well known that

$$\lim_{\mu\to 0} I_{quad}(\mu^2) = 0, \quad (5.86)$$

which ultimately implies that $\alpha_2 = 0$, taking our general parameterizations for $I_{quad}$ and $I_{log}$ into account.

As a final comment, we would like to emphasize that the procedure adopted in this paper is applicable to higher order calculations. Should we take our parametrization to two loop order, it is not difficult to show that [88]

$$\int_k\int_q \frac{1}{k^2q^2(k+q)^2} = a_1\Lambda^2 + a_2\mu^2 + a_3\mu^2\ln\frac{\Lambda^2}{\mu^2}, \quad (5.87)$$

where $\mu$ is a mass infrared regulator and $a_1$, $a_2$ and $a_3$ are arbitrary finite constants, which are combinations of constants of integrations. The equation above is an example of a general parametrization of a typical leading quadratic divergence of two-loop order. As we can see, it is possible in our approach to adjust arbitrary finite constants in higher order calculations so as to have a null contribution from quadratic divergences.

# Concluding remarks

In this work, it was carried out a discussion on the role of quadratic divergences in quantum field theory. This discussion was based in a general parametrization of basic divergent integrals. These basic divergent integrals, obtained in the context of Implicit Regularization, embodies all the divergent content of a given amplitude. The parametrization we adopted embodies all possible results coming from different regularization procedures. Arbitrary constants which naturally appear in the procedure can be adjusted so as to enforce symmetries of the model or experimental results.

After presenting some examples in which the presence or the cancelation of quadratic divergences plays primordial roles, we discuss the hierarchy problem. It is shown that the classical scaling argument of Bardeen and conformal anomaly can be used as a symmetry guide to fix the arbitrary regularization dependent parameter in the isolated quadratic divergence which contributes to the Higgs mass.

# Chapter 6

# Controversy on the background field method: massless $N = 1$ supersymmetric electrodynamics as an example

## Introduction

In Particle Physics symmetries have always been used as a guide in order to construct theories to describe Nature, a idea that culminated in the Standard Model itself. Although it has passed many experimental tests, it must be viewed as an effective theory since it does not incorporate all fundamental interactions by excluding gravity. Therefore, extensions to it have been proposed being supersymmetry one of the most appealing. The reason lies on the elegance of its construction since, by considering the Poincaré group as the guiding symmetry, it emerges naturally. Much effort have been dedicated to the subject after it was first proposed in the seventies [115, 116, 117].

After the works of Ferrara and Piguet [118, 119, 120, 121], it was found that the $U_R(1)$ current (related to the chiral anomaly) and the trace of the energy-momentum tensor (related to the beta function) belong to the same supermultiplet. According to Adler-Bardeen theorem [122], the chiral anomaly is exhausted at one-loop order in perturbation theory. Since the chiral anomaly and the beta function belong to the same supermultiplet is was expected that the beta function would also not receive higher order correction. However, since the work of Novikov, Shifman, Vainshtein e Zakharov (NSVZ) [123], which obtained an exact expression for the beta function of $N = 1$ Super Yang-Mills, many other works [124, 125, 126, 127, 128, 129, 130, 131] followed in which different regularization were applied and in all cases higher order corrections for the beta functions were found. This controversy regarding the existence (or not) of higher order corrections to the beta function of supersymmetric theories became known as the 'Anomaly Puzzle".

In this work we intend to study, up to two-loop level, the existence (or not) of corrections to the beta function of SQED. We would like also to understand the apparent paradox found in [131] where the two-loop coefficient of the SYM beta function was computed using the background field method (covariant derivative formalism [132]). Due to the non-abelian character of the theory, this method is urged to be applied which results in a huge simplification in the number of diagrams. It was found that there is no two-loop divergence which, in a first view, could indicate the absence of higher loop corrections to the beta function. However, the renormalized two-point Green function still depends on the renormalization scale introduced at one-loop level, allowing the computation of the two-loop coefficient for the beta function, which is shown to be non-null. Therefore, in the present work we will use a simpler theory (SQED) as a probe to study if the behavior found for SYM occurs in general. For this purpose, we will compute the two-loop coefficient of the SQED beta function using two different approaches: the standard background field method [133] and the one based on the covariant derivative formalism [132]. We will find that in the first case there is a two-loop divergence, allowing the computation of the beta function coefficient by standard renormalization constants. It is also possible to perform the computation using the renormalized two-point Green function, furnishing the same result as before as expected. In the second case, we obtain the same result of [131]: there is no two-loop divergence, even though the renormalized two-point Green function depends on a renormalization scale furnishing a non-null value for the two-loop beta function coefficient. Therefore, we conjecture that the paradox found in [131] has its origin in the covariant supergraph formalism itself being, possibly, an artifact of the rescaling anomaly.

## 6.1 Massless $N = 1$ supersymmetric electrodynamics

In this section we present the theory we are going to work with: massless $N = 1$ supersymmetric electrodynamics. In the superfield formalism the classical action is given by [134]

$$S = \int d^4xd^2\theta\, W^2 + \int d^4xd^4\theta\, \bar{\Phi}_+ e^{gV}\Phi_+ + \int d^4xd^4\theta\, \bar{\Phi}_- e^{gV}\Phi_-, \tag{6.1}$$

where $\Phi$ is a chiral field that express the matter part of the action and $V$ is a real scalar superfield that contais the gauge field $A_\mu$ of QED as one of its components (therefore, it is the supersymmetric generalization to the gauge field). Finally, $W$ is the supersymmetric generalization to the stress tensor of QED. In terms of the superfield $V$ one has

$$S = \int d^4xd^2\theta\, W^2 = \frac{1}{2}\int d^4xd^4\theta\, VD^\beta\bar{D}^2D_\beta V. \tag{6.2}$$

The following step would be to perform the quantization of the classical theory, however, since we want to use the background field method, we have to introduce this new field at this point. For the abelian case we will have a linear quantum-background splitting as below [135]

$$V \to V + B, \tag{6.3}$$

where B is the background gauge field. We may now perform the quantization as usual, introducing a gauge fixing term for the quantum gauge field $V$ as well as sources [133]. The relevant fact to be noticed is that, by construction, the action will be gauge invariant in the background gauge field, which must remain valid even after renormalization. Thus, the renormalization constant for the background field $Z_B$ will be related to the one for the gauge coupling $Z_g$ as follows[1]

$$Z_g Z_B^{1/2} = 1. \tag{6.4}$$

Thus, in order to obtain the beta function of the theory we need only to compute the two-point functions with background fields as external legs.

As stated in the introduction, we intend to compute the two-loop corrections of the SQED beta function. The Feynman rules can be derived from the action [135], the relevant ones for our computation are expressed below

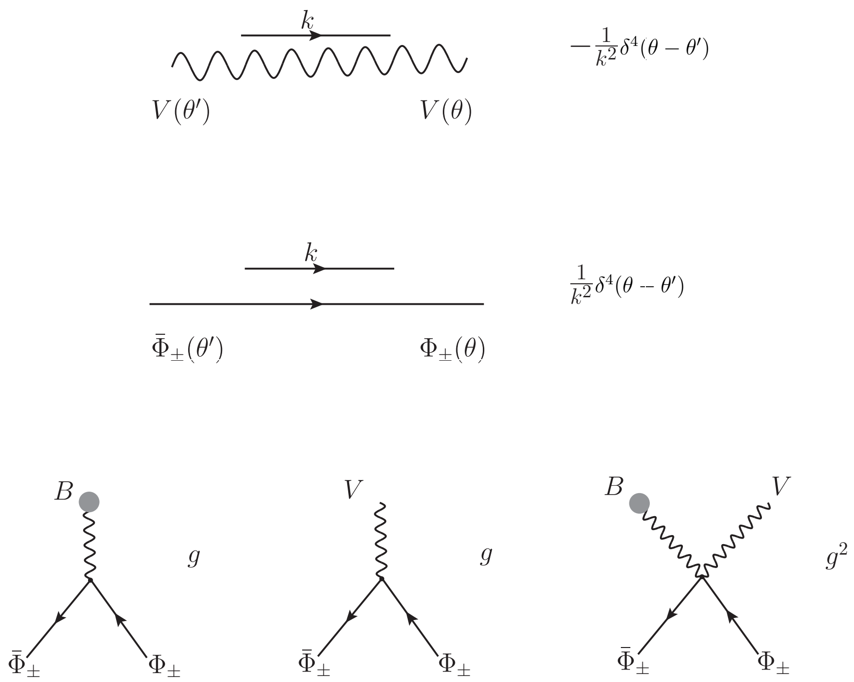



Figure 6.1: Feynman rules needed for the evaluation of two point functions in the background field up to two-loop order.

We may now begin the calculation starting by the one-loop contribution whose diagram[2], using the background field method, is depicted below

[1]We are using the definitions $g_0 = Z_g g$ and $B_0 = Z_B^{1/2} B$, where $g_0$, $B_0$ and $g$, $B$ are bare and regularized functions respectively.

[2]We do not include a tadpole diagram since, as we are dealing with a massless theory, it can be promptly set to zero in the IReg formalism as discussed in the end of chapter 5. However, even if such diagram was included, it would cancel in the sum. Thus, in order to simplify the discussion, we opt to omit hereafter all the tadpoles diagrams.

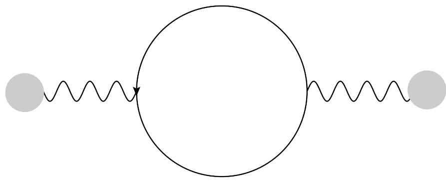


Figure 6.2: 1-loop diagram.

furnishing the following effective action

$$\Lambda^{(1)} \equiv 2\frac{g^2}{2}\int_{p,\theta} B(-p,\theta)\int_k \left[\frac{D^2\bar{D}^2 - k^{\beta\dot{\alpha}}D_\beta\bar{D}_{\dot{\alpha}} - k^2}{k^2(k+p)^2}\right] B(p,\theta), \tag{6.5}$$

where we have already performed the D-algebra manipulations, $\int_k$ stands for $\int \frac{d^4k}{(2\pi)^4}$, $\int_{p,\theta}$ for $\int \frac{d^4p}{(2\pi)^4}\int d^4\theta$, and $B(p,\theta)$ is the background gauge field. The two factor accounts contributions from chiral fields $\Phi$ with different signs. Regarding supersymmetric definitions and conventions we are following the ones found in [135].

To proceed, we need to resort to some regularization technique. We will choose the IReg formalism (already described in chapter 2) which, by not resorting to any kind of dimensional extension can be promptly applied in supersymmetric theories [25, 66, 131]. After some D-algebra manipulation the final result is

$$\begin{aligned}\Lambda^{(1)} =&(-i)\frac{g^2}{2}\int_{p,\theta} B(-p,\theta)D^\beta\bar{D}^2D_\beta B(p,\theta)\left[I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) + 2b\right] + \\ &(+i)\frac{g^2}{2}\int_{p,\theta} B(-p,\theta)\left[p^{\beta\dot{\alpha}}D_\beta\bar{D}_{\dot{\alpha}} + 2k^2\right]\Gamma_0^{(1,2)}B(p,\theta). \end{aligned}\tag{6.6}$$

Some comments are in order: notice that, apart from the surface term $\Gamma_0^{(1,2)}$, we have a gauge invariant result. This was expected since, as we are working with the background field method, gauge invariance is explicitly maintained being broken just by regularization dependent terms. Thus we verify that the condition to preserve gauge invariance, even in supersymmetric theories, is to set surface terms to zero as discussed in chapter 3. Another aspect to be mentioned is the appearance of a divergent integral parametrized as a $I_{log}(\lambda^2)$. Such term must be renormalized as usual and it will contribute to the renormalization constant $Z_B$

as we are going to show in the end of this section. Notice also the appearance of the renormalization scale $\lambda^2$ in the finite part as well.

We proceed now to the two loop contributions which are depicted below

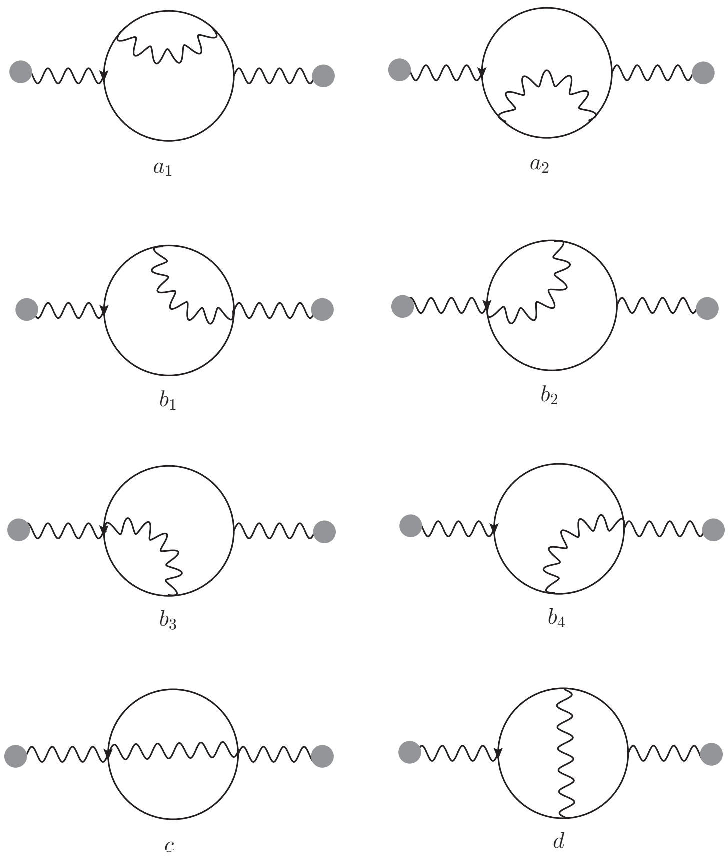



Figure 6.3: 2-loop diagrams.

The first diagram furnishes

$$\Lambda_{a_1}^{(2)} \equiv 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\int_{k,l}\left[\frac{\bar{D}^2D^2-(-k-p)^{\dot{\alpha}\beta}\bar{D}_{\dot{\alpha}}D_\beta-(-k-p)^2}{k^2(k+p)^2l^2(l-k-p)^2}\right]B(p,\theta). \tag{6.7}$$

To proceed, we will apply the IReg framework to multiloop diagrams [34] which states that the integral in $l$ must be treated first. In fact, such integral corresponds to a one-loop subdiagram indicating how, for this particular example, the IReg formalism implements automatically the subtractions to be performed by

Bogoliubov's recursion formula. After the calculation we obtain

$$\begin{aligned}\Lambda^{(2)}_{a_1} =& 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\int_k \left[\frac{\bar{D}^2D^2 + k^{\dot{\alpha}\beta}\bar{D}_{\dot{\alpha}}D_\beta - k^2}{k^2(k-p)^2}\right](ib)\left[\ln\left(-\frac{p^2}{\lambda^2}\right) - 2\right]B(p,\theta)\\ &+ 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\int_k \left[\frac{\bar{D}^2D^2 + k^{\dot{\alpha}\beta}\bar{D}_{\dot{\alpha}}D_\beta - k^2}{k^2(k-p)^2}\right](-i)I_{log}(\lambda^2)B(p,\theta),\end{aligned} \tag{6.8}$$

where the second line is a subdivergence which is be to canceled by a one-loop counterterm. Instead of evaluation the first line further we proceed to the next diagram (the reason will be apparent shortly). The procedure to be followed is very similar to the one explained above, thus we just quote the result

$$\begin{aligned}\Lambda^{(2)}_{a_2} =& 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\int_k \left[\frac{D^2\bar{D}^2 - k^{\beta\dot{\alpha}}D_\beta\bar{D}_{\dot{\alpha}} - k^2}{k^2(k+p)^2}\right](ib)\left[\ln\left(-\frac{p^2}{\lambda^2}\right) - 2\right]B(p,\theta)\\ &+ 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\int_k \left[\frac{D^2\bar{D}^2 - k^{\beta\dot{\alpha}}D_\beta\bar{D}_{\dot{\alpha}} - k^2}{k^2(k+p)^2}\right](-i)I_{log}(\lambda^2)B(p,\theta).\end{aligned} \tag{6.9}$$

Once again, the last line will be subtracted by applying Bogoliubov's recursion formula.

The reason why we have not performed the computation of the integral in $k$ will become apparent now. After performing the D-algebra on the diagrams $b_1 \cdots b_4$ we will see that they all can be expressed in terms of $\Lambda^{(2)}_{a_1}$ and $\Lambda^{(2)}_{a_2}$ as below

$$\Lambda^{(2)}_{b_1} = \Lambda^{(2)}_{b_2} = -\Lambda^{(2)}_{a_1}, \qquad \Lambda^{(2)}_{b_3} = \Lambda^{(2)}_{b_4} = -\Lambda^{(2)}_{a_2}. \tag{6.10}$$

We proceed to diagram $c$ which is given by

$$\Lambda^{(2)}_c = -2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\int_{k,l}\frac{1}{k^2l^2(l-k-p)^2}B(p,\theta). \tag{6.11}$$

No further analysis should be taken, since the last diagram, $d$, can be written in terms of $\Lambda_{a_1}^{(2)}$, $\Lambda_{a_2}^{(2)}$, and $\Lambda_c^{(2)}$

$$\begin{aligned}\Lambda_c^{(2)} &= \Lambda_{a_1}^{(2)} + \Lambda_{a_2}^{(2)} - \Lambda_c^{(2)} - 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)p^2\left[(I_{log}(\lambda^2)+b)\Gamma_0^{(1,2)} - b\Gamma_0^{(2,2)}\right]B(p,\theta)\\ &- 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\left[p^{\beta\dot{\alpha}}\bar{D}_{\dot{\alpha}}D_\beta(-b\Gamma_0^{(2,2)} + 2b\Gamma_0^{(1,2)})\right]B(p,\theta)\\ &- 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)D^\beta\bar{D}^2D_\beta B(p,\theta)b\left[I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) - \Gamma_0^{(1,2)}\right]B(p,\theta).\end{aligned} \tag{6.12}$$

Therefore, the final two-loop correction for the effective action is given by

$$\begin{aligned}\Lambda^{(2)} &= -2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\left[p^2 I_{log}(\lambda^2)\Gamma_0^{(1,2)} + bp^2(\Gamma_0^{(1,2)} - \Gamma_0^{(2)})\right]B(p,\theta)\\ &- 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)\left[p^{\beta\dot{\alpha}}\bar{D}_{\dot{\alpha}}D_\beta(-b\Gamma_0^{(2,2)} + 2b\Gamma_0^{(1,2)})\right]B(p,\theta)\\ &- 2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)D^\beta\bar{D}^2D_\beta B(p,\theta)b\left[I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) - \Gamma_0^{(1,2)}\right]B(p,\theta).\end{aligned} \tag{6.13}$$

Notice the appearance of one and two-loop surface terms ($\Gamma_0^{(1,2)}$, and $\Gamma_0^{(2,2)}$ respectively) whose presence violate gauge invariance. As already discussed, to obtain a one-loop gauge invariant result, $\Gamma_0^{(1,2)}$ was set to zero. However, even if we consider this condition, we will not have a two-loop invariant result, since two-loop surface terms will still be present. This indicates that surface terms, order by order, must be set to zero in order to preserve the symmetry of the theory. A curios aspect is the appearance of a divergent integral multiplied by a surface term of one-loop order. This presents no problem to our method since the surface term was **already** set to zero at one-loop order. After these considerations, our result simplifies to

$$\Lambda^{(2)} = -2\frac{g^4}{2}\int_{p,\theta} B(-p,\theta)D^\beta\bar{D}^2D_\beta B(p,\theta)b\left[I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right)\right]B(p,\theta). \tag{6.14}$$

Finally, notice that we have a divergent result parametrized by $I_{log}(\lambda^2)$. A curios fact is that, although we are working at two-loop level, only the BDI of one order appears. Our next task is to be perform the renormalization of the theory whose bare action is

$$S_0 = \int d^4x d^4\theta \, \left[ \frac{1}{2} B_0 D^\beta \bar{D}^2 D_\beta B_0 + \bar{\Phi}_{0+} e^{g_0 B_0} \Phi_{0+} + \bar{\Phi}_{0-} e^{g_0 B_0} \Phi_{0-} \right] + \text{gauge fixing terms} + \text{terms on } V_0. \tag{6.15}$$

Performing a multiplicative renormalization defined by $B_0 = Z_B^{1/2} B$, $g_0 = Z_g g$, and $\Phi_{0\pm} = Z_{\Phi_\pm} \Phi_\pm$, one can easily find that the counterterm for the two-point function in the background field is given by $A \equiv Z_B - 1$. As already mentioned, in the background field method the following relation holds $Z_g Z_B^{1/2} = 1$, therefore, the beta function can be calculated through the two-point function renormalization constant as below

$$\beta \equiv \lambda \frac{\partial}{\partial \lambda} g = -g \lambda \frac{\partial}{\partial \lambda} \ln Z_g = -g \lambda \frac{\partial}{\partial \lambda} \ln Z_B^{-1/2}. \tag{6.16}$$

Supposing that both the countertem A as well as the beta function have an expansion in the coupling constant $g$ we arrive at the expressions below

$$\beta_1 = \frac{1}{2} \lambda \frac{\partial}{\partial \lambda} A_1 \tag{6.17}$$

$$\beta_2 = A_1 \beta_1 + \frac{1}{2} \lambda \frac{\partial}{\partial \lambda} \left( A_2 - \frac{A_1^2}{2} \right) \tag{6.18}$$

where $\beta_i$ is the $i$-loop coefficient of the beta function, $A_i$ is the $i$-loop two point function counterterm, and $\lambda$ is the renormalization group scale. Using the one and two-loop corrections we obtained (equations (6.6) and (6.14) respectively), and adopting a minimal subtraction scheme (which in the IReg framework amounts to the subtraction of BDI's only) we have

$$A_1 = i I_{log}(\lambda^2), \qquad A_2 = 2b I_{log}(\lambda^2). \tag{6.19}$$

Finally, by using the relations

$$\lambda\frac{\partial}{\partial\lambda}I_{log}(\lambda^2) = 2\lambda^2\frac{\partial}{\partial\lambda^2}I_{log}(\lambda^2) = -2b, \qquad b = \frac{i}{(4\pi)^2}, \tag{6.20}$$

we obtain the corrections for the beta function of SQED up to two-loop level in the IReg formalism

$$\beta = \frac{1}{(4\pi)^2}g^3 + \frac{1}{8(4\pi^2)^2}g^5 + \mathcal{O}(g^7), \tag{6.21}$$

which agrees with previous ones found in the literature [136, 137, 138, 123].[1]

It should be emphasized that we performed the above computation using only the renormalization constant for the background field $Z_B$. In other words, we just had to take into account the divergent part of the one and two-loop results. An analogous computation can be performed, using the renormalization group equation, in which only the renormalized effective action must be considered. It should be expected that both ways furnish the same results, thus, as a check for our method, we also compute the two-loop coefficients of the SQED beta function using just the renormalized effective action.

As can be found in [43], the renormalization group equation reads

$$\left[\lambda\frac{\partial}{\partial\lambda} + \beta\frac{\partial}{\partial g} - \gamma\right] G^{(2)}_{ren}(g,\lambda) = 0, \tag{6.22}$$

where

$$\beta \equiv \lambda\frac{\partial}{\partial\lambda}g = -g\lambda\frac{\partial}{\partial\lambda}\ln Z_g, \qquad \gamma \equiv \lambda\frac{\partial}{\partial\lambda}\ln Z_B. \tag{6.23}$$

Since in the background field method $Z_g Z_B^{1/2} = 1$, we find that $\gamma = 2\frac{\beta}{g}$. Thus,

$$\left[\lambda\frac{\partial}{\partial\lambda} + \beta\left(\frac{\partial}{\partial g} - \frac{2}{g}\right)\right] G^{(2)}_{ren}(g,\lambda) = 0. \tag{6.24}$$

From equations (6.6) and (6.14) we obtain the renormalized two-point function

[1] To obtain the results of the last three references, it should be taken into account that our definition for the coupling constant [135] differs from the usual one by a factor $\sqrt{2}$.

as below

$$G^{(2)}_{ren}(g,\lambda)=\frac{1}{2}\int_{p,\theta}B(-p,\theta)D^{\beta}\bar{D}^2D_{\beta}B(p,\theta)\left\{1+ib\left[\ln\left(-\frac{p^2}{\lambda^2}\right)-2\right]g^2\right.$$
$$\left.+2b^2\ln\left(-\frac{p^2}{\lambda^2}\right)g^4\right\}. \quad (6.25)$$

Replacing the expression above in the renormalization group equation we finally obtain

$$\beta=\frac{1}{(4\pi)^2}g^3+\frac{1}{8(4\pi^2)^2}g^5+\mathcal{O}(g^7), \quad (6.26)$$

which is the same result as before.

We would like now to perform the whole calculation again, by using a different approach: the background field method based on the covariant supergraph formalism [132]. The reason is the following: in [131], in which the aforementioned formalism was applied, the Super Yang-Mills theory was studied and, particularly, the beta functions coefficients were computed up to two-loop order. It was found there that **no** two-loop divergence appeared, what would indicate a null two-loop coefficient. However, the renormalized effective action at two-loop order stills carried a dependence on the renormalization scale $\lambda$ allowing the computation of the beta function from the renormalization group equation, furnishing a non-null result for the two-loop coefficient. Therefore, it seems that there is an inconsistency, since both approaches should be equivalent. It was conjectured there that this difference should have its origin in the rescaling anomaly [139]. Therefore, we would like now, using SQED as a probe, to study if the same behavior occurs in this case.

A complete description of the background field method based on the covariant supergraph formalism can be found in [132, 135]. The main idea is to take a step backward and work, from the beginning, with an action that depends only on background covariant derivatives. This way, all dependence on the background field will only appears implicitly. The main gain on this approach is the reduction on the number of diagrams (for instance, the two-loop correction we are going to compute requires knowledge of only three diagrams, not eight as in the previous case). Explicitly, the quadratic part of action in the gauge fields we are going to work with is given by

$$S = -\int d^4xd^4\theta\nabla^\alpha\bar{\nabla}^2\nabla_\alpha, \tag{6.27}$$

where $\nabla$ is a covariant derivative in the unsplit gauge field $(V+B)$. The splitting can be carried out, in the quantum-chiral but background-vector representation, as

$$\nabla_\alpha = e^{-V}\boldsymbol{\nabla}_\alpha e^V, \qquad \bar{\nabla}_{\dot{\alpha}} = \bar{\boldsymbol{\nabla}}_{\dot{\alpha}}, \tag{6.28}$$

being $\boldsymbol{\nabla}$ background covariant derivatives. From this point, the quantization procedure should be adopted (adding gauge fixing and source terms as usual). Chiral fields should also be included to define SQED properly (this fields must also be written in the background covariant representation). After all these considerations we obtain the covariant Feynman rules [135] which, applied in our case, furnishes the following one-loop effective action (the diagram depicted is the same of fig. 6.1)

$$\mathcal{A}^{(1)} = (-ig^2)\int_{p,\theta}\mathbf{W}^\alpha(p)\boldsymbol{\Gamma}_\alpha(-p)\int_k\frac{1}{k^2}\frac{1}{(k+p)^2}. \tag{6.29}$$

Notice that our result has no explicit dependence on the background field $B$, it appears only through the field strength $\mathbf{W}^\alpha$ and the spinor connection $\boldsymbol{\Gamma}_\alpha$. This is a feature of the method since background covariance, by construction, is always maintained. Notice also the appearance of a divergent integral in $k$, which can be promptly dealt in the IReg formalism furnishing

$$\mathcal{A}^{(1)} = (-ig^2)\int_{p,\theta}\mathbf{W}^\alpha(p)\boldsymbol{\Gamma}_\alpha(-p)\left[I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) + 2b\right]. \tag{6.30}$$

To conclude the one-loop calculation, we would like to write our result in terms of an explicit background gauge field. For this purpose, we have to recover the definitions of $\mathbf{W}^\alpha$ and $\boldsymbol{\Gamma}_\alpha$ found in [140]

$$\boldsymbol{\Gamma}_\alpha = iD_\alpha\frac{B}{2}, \quad \mathbf{W}_\alpha = i\bar{D}^2D_\alpha B. \tag{6.31}$$

Therefore, our final one-loop result reads

$$\mathcal{A}^{(1)} = (-i)\frac{g^2}{2}\int_{p,\theta} B(-p,\theta)D^\beta \bar{D}^2 D_\beta B(p,\theta)\left[I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) + 2b\right]. \tag{6.32}$$

Comparing with our previous expression, eq. (6.6), we notice that there is **no** dependence on a surface term this time. We conjecture that this feature may be a consequence of the method which, by maintaining background covariance from the beginning, have automatically canceled all gauge breaking terms that could occur. We also remark that this is the only difference between the two results.

We proceed now to the two-loop contribution whose diagrams are depicted below [141]

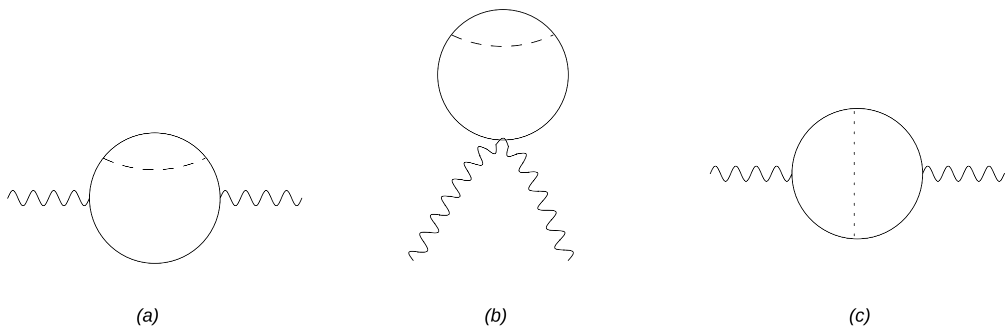



Figure 6.4: 2-loop diagrams in the background covariant approach.

As in the one-loop case, the effective action will only have an implicit dependence on the background gauge field, through the field strength $\mathbf{W}^\alpha$ and the vector connection $\mathbf{\Gamma}^{\underline{a}}$ as the expressions for the diagrams above reveal

$$\mathcal{A}_a^{(2)} = 4g^4 \int_{p,\theta} \Big[ 2\mathbf{W}^{\alpha}(-p,\theta)\bar{\mathbf{W}}^{\dot{\alpha}}(p,\theta)\,(I_1)_{\alpha\dot{\alpha}} + \frac{1}{2}\boldsymbol{\nabla}^{\alpha}\mathbf{W}_{\alpha}(-p,\theta)\boldsymbol{\nabla}^{\beta}\mathbf{W}_{\beta}(p,\theta)I_2 \\ -\frac{1}{2}\mathbf{\Gamma}^{\underline{a}}(-p,\theta)\mathbf{\Gamma}^{\underline{b}}(p,\theta)\,(I_3)_{\underline{ab}} \Big] \tag{6.33}$$

$$\mathcal{A}_b^{(2)} = 4g^4 \int_{p,\theta} \left[ \frac{1}{2}\mathbf{\Gamma}^{\underline{a}}(-p,\theta)\Gamma_{\underline{a}}(p,\theta)I_4 \right] \tag{6.34}$$

$$\mathcal{A}_c^{(2)} = 4g^4 \int_{p,\theta} \left[ \frac{1}{4}\mathbf{\Gamma}^{\underline{a}}(-p,\theta)\mathbf{\Gamma}^{\underline{b}}(p,\theta)\,(I_5)_{\underline{ab}} + \frac{1}{4}\boldsymbol{\nabla}^{\alpha}\mathbf{W}_{\alpha}(-p,\theta)\boldsymbol{\nabla}^{\beta}\mathbf{W}_{\beta}(p,\theta)I_6 \right] \tag{6.35}$$

where $I_i$ are the following integrals

$$(I_1)_{\alpha\dot{\alpha}} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\,(I_1)_{\mu} = \sigma^{\mu}_{\alpha\dot{\alpha}}(-i)^2 \int_{q,k} \frac{(p-k)_{\mu}}{q^2(q+k)^2k^4(k-p)^2}, \tag{6.36}$$

$$I_2 \equiv (-i)^2 \int_{q,k} \frac{1}{q^2(q+k)^2k^4(k-p)^2}, \tag{6.37}$$

$$(I_3)_{\underline{ab}} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}\,(I_3)_{\mu\nu} = \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}(-i)^2 \int_{q,k} \frac{4k_{\mu}k_{\nu} - 2p_{\mu}k_{\nu} - 2k_{\mu}p_{\nu} + p_{\mu}p_{\nu}}{q^2(q+k)^2k^4(k-p)^2}, \tag{6.38}$$

$$I_4 \equiv (-i)^2 \int_{q,k} \frac{1}{q^2(q+k)^2k^4}, \tag{6.39}$$

$$(I_5)_{\underline{ab}} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}\,(I_5)_{\mu\nu} = \sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}}(-i)^2 \int_{q,k} \frac{4k_{\mu}q_{\nu} - 2k_{\mu}p_{\nu} + 2p_{\mu}q_{\nu} - p_{\mu}p_{\nu}}{q^2(q+k)^2k^2(k+p)^2(q-p)^2}, \tag{6.40}$$

$$I_6 \equiv (-i)^2 \int_{q,k} \frac{1}{q^2(q+k)^2k^2(k+p)^2(q-p)^2}. \tag{6.41}$$

To proceed we must adopt a regularization method, which, in our case, will be the IReg framework. We start with the first integral which, after applying the rules of IReg for multiloop diagrams [34], furnishes

$$(I_1)_{\mu} = (-i)^2 \int_k \frac{(p-k)_{\mu}}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]} \left[ I_{log}(\lambda^2) - b\ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right) + 2b \right]. \tag{6.42}$$

Notice the inclusion of the fictitious mass $\mu^2$ which must be added in order to regularize the infrared divergence. There is also an UV divergence parametrized as a $I_{log}(\lambda^2)$ which is just an one-loop subdivergence that is going to be canceled by application of Bogoliubov's recursion formula. In other words, after the subtraction of the subdivergence one obtains

$$(I_1)_\mu = (-i)^2 \int_k \frac{(p-k)_\mu}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]} \left[-b \ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right) + 2b\right], \tag{6.43}$$

which is a finite UV integral. Adopting the following definitions

$$U_\mu \equiv \lim_{\mu^2\to 0} \int_k \frac{k_\mu}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]}, \tag{6.44}$$

$$U_\mu^{(2)} \equiv \lim_{\mu^2\to 0} \int_k \frac{k_\mu}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]} \left[\ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right)\right], \tag{6.45}$$

$$F \equiv \lim_{\mu^2\to 0} \int_k \frac{1}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]} \left[-b \ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right) + 2b\right], \tag{6.46}$$

our result is written as

$$(I_1)_\mu = (-i)^2 \left[p_\mu F + bU_\mu^{(2)} - 2bU_\mu\right]. \tag{6.47}$$

The next two integrals can be treated in the same way furnishing

$$I_2 =(-i)^2 F, \tag{6.48}$$

$$(I_3)_{\mu\nu} =(-i)^2 \left[p_\mu p_\nu F + 2b(p_\mu U_\nu^{(2)} - 2p_\mu U_\nu + p_\nu U_\mu^{(2)} - 2p_\nu U_\mu - 2U_{\mu\nu}^{(2)} + 4U_{\mu\nu})\right], \tag{6.49}$$

where we defined

$$U_{\mu\nu} \equiv \lim_{\mu^2\to 0} \int_k \frac{k_\mu k_\nu}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]}, \tag{6.50}$$

$$U_{\mu\nu}^{(2)} \equiv \lim_{\mu^2\to 0} \int_k \frac{k_\mu k_\nu}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]} \left[\ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right)\right]. \tag{6.51}$$

Notice that only $(I_3)_{\mu\nu}$ contains a UV divergent part (contained in $U_{\mu\nu}^{(2)}$, and $U_{\mu\nu}$).

We proceed to the tadpole diagram whose contribution, after the subtraction of subdivergences, is given by

$$
\begin{aligned}
I_4 &= (-i)^2 \int_k \frac{1}{(k^2-\mu^2)^2}\left[-b\ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right)+2b\right], \\
&= (-i)^2 \int_k \frac{[(k-p)^2-\mu^2]}{(k^2-\mu^2)^2[(k-p)^2-\mu^2]}\left[-b\ln\left(-\frac{(k^2-\mu^2)}{\lambda^2}\right)+2b\right], \\
&= (-i)^2\left[-bI^{(2)}_{log}(\lambda^2)+2bI_{log}(\lambda^2)+\frac{b^2}{2}\ln^2\left(-\frac{p^2}{\lambda^2}\right)-b^2\ln\left(-\frac{p^2}{\lambda^2}\right)+p^2F\right].
\end{aligned}
\tag{6.52}
$$

Finally, we deal with the third graph whose integrals can be expressed as

$$
(I_5)_{\mu\nu} = (-i)^2\left[4I^{\mathcal{O}_2}_{\mu\nu}-2p_\nu I^{\mathcal{O}}_\mu+2p_\mu \bar{I}^{\mathcal{O}}_\nu-p_\mu p_\nu I^{\mathcal{O}}\right], \tag{6.53}
$$

$$
I_6 = I^{\mathcal{O}}, \tag{6.54}
$$

where

$$
I^{\mathcal{O}} \equiv \int_{q,k}\frac{1}{q^2(q+k)^2k^2(k+p)^2(q-p)^2}, \tag{6.55}
$$

$$
I^{\mathcal{O}}_\mu \equiv \int_{q,k}\frac{k_\mu}{q^2(q+k)^2k^2(k+p)^2(q-p)^2}, \tag{6.56}
$$

$$
\bar{I}^{\mathcal{O}}_\nu \equiv \int_{q,k}\frac{q_\nu}{q^2(q+k)^2k^2(k+p)^2(q-p)^2}, \tag{6.57}
$$

$$
I^{\mathcal{O}_2}_{\mu\nu} \equiv \int_{q,k}\frac{k_\mu q_\nu}{q^2(q+k)^2k^2(k+p)^2(q-p)^2}. \tag{6.58}
$$

Apart from the integral $F$, which encodes an IR divergence, all the others can be treated in the usual IReg formalism (explained in chapter 2). The results can be found in appendix B. To proper treat the IR divergence, one should resort to the IReg generalization presented in [37], however, in the present case, the integral $F$ will cancel out in the calculation not requiring any further treatment. We now proceed noticing that in the effective actions $\mathcal{A}^{(2)}_i$ we have different structures in terms of the field strength and vector connection. For reasons that are going to be apparent soon we choose to group all the contributions proportional to the vector connection obtaining

$$\mathcal{A}_\Gamma^{(2)} = 4g^4 \int_{p,\theta} \mathbf{\Gamma}^{\underline{a}}(-p,\theta)\mathbf{\Gamma}^{\underline{b}}(p,\theta) \left[ -\frac{1}{2}(I_3)_{\underline{ab}} + \frac{1}{2}(g_{\underline{ab}} I_4) + \frac{1}{4}(I_5)_{\underline{ab}} \right]. \tag{6.59}$$

Replacing the values of the integrals found in appendix B we have

$$\begin{aligned} \mathcal{A}_\Gamma^{(2)} = 4g^4 \int_{p,\theta} \mathbf{\Gamma}^{\underline{a}}(-p,\theta)\mathbf{\Gamma}^{\underline{b}}(p,\theta) \left( \frac{p_{\underline{a}} p_{\underline{b}}}{p^2} - g_{\underline{ab}} \right) & \left[ b^2 \ln\left( -\frac{p^2}{\lambda^2} \right) + \frac{b^2\pi^2}{36} \right. \\ & \left. + \frac{b^2\zeta(3)}{2} - \frac{8b^2}{3} + \frac{Fp^2}{2} \right] \\ + 4g^4 \int_{p,\theta} \mathbf{\Gamma}^{\underline{a}}(-p,\theta)\mathbf{\Gamma}^{\underline{b}}(p,\theta) g_{\underline{ab}} & \left[ \frac{b}{2} \left( \Gamma_0^{(2,2)} - 2\Gamma_0^{(1,2)} \right) \right]. \end{aligned} \tag{6.60}$$

First notice that, although all the integrals above are UV divergent, the net result is UV finite. Notice also the appearance of one an two-loop surfaces term ($\Gamma_0^{(2,2)}$, and $\Gamma_0^{(1,2)}$), which break gauge invariance. Differently from the standard background method computation, we do not need to set all surfaces terms to zero in order to have a gauge invariant result. We just need to require them to obey the relation $\Gamma_0^{(2,2)} = 2\Gamma_0^{(1,2)}$. After this consideration we obtain a gauge invariant result which obeys the following relation

$$\int d^4\theta \mathbf{\Gamma}^{\underline{a}}(-p,\theta)\mathbf{\Gamma}^{\underline{b}}(p,\theta) \left( \frac{p_{\underline{a}} p_{\underline{b}}}{p^2} - g_{\underline{ab}} \right) = \frac{3}{2} \int d^2\theta \mathbf{W}^\alpha \mathbf{W}_\alpha. \tag{6.61}$$

Since we have also the relations below

$$\int d^4\theta \mathbf{W}^\alpha(-p,\theta) p_{\alpha\dot{\alpha}} \bar{\mathbf{W}}^{\dot{\alpha}}(p,\theta) = \frac{1}{2} \int d^2\theta \mathbf{W}^\alpha(-p,\theta)\mathbf{W}_\alpha(p,\theta), \tag{6.62}$$

$$\int d^4\theta \boldsymbol{\nabla}^\alpha \mathbf{W}_\alpha(-p,\theta) \boldsymbol{\nabla}^\beta \mathbf{W}_\beta(p,\theta) = -\frac{1}{2} \int d^2\theta \mathbf{W}^\alpha(-p,\theta)\mathbf{W}_\alpha(p,\theta), \tag{6.63}$$

and $(I_1)_{\alpha\dot{\alpha}} \propto p_{\alpha\dot{\alpha}}$, we finally obtain the following two-loop effective action

$$\mathcal{A}^{(2)} = 4g^4 \int_p d^2\theta \mathbf{W}^\alpha(-p,\theta)\mathbf{W}_\alpha(p,\theta) b^2 \left[ \frac{1}{2} \ln\left( -\frac{p^2}{\lambda^2} \right) + \frac{\pi^2}{24} + \frac{3\zeta(3)}{2} - 2 \right], \tag{6.64}$$

which, written in terms of the background field by means of eq. (6.31), is given by

$$\mathcal{A}^{(2)} = \frac{g^4}{2}\int_{p,\theta} B(-p,\theta)D^\beta\bar{D}^2D_\beta B(p,\theta)b^2\left[2\ln\left(-\frac{p^2}{\lambda^2}\right)+\frac{\pi^2}{6}+6\zeta(3)-8\right]. \tag{6.65}$$

Some comments are in order. First, notice the disappearance of the IR divergent integral F which furnishes us a perfectly IR finite result. Next, one should compare the result above with the one expressed by eq. (6.14) which was obtained using the standard background field method. The first notorious difference is the disappearance of the UV divergent integral. This means that there isn't a two-loop contribution for the renormalization constant $Z_B$, which indicates that the two-loop coefficient of the SQED beta function computed by means of eq. (6.16) would be null. However, one could also compute the beta function using the renormalization group equation. For this purpose, one would need the following renormalized two-point function

$$\begin{aligned} G^{(2)}_{ren}(g,\lambda) = \frac{1}{2}\int_{p,\theta} B(-p,\theta)D^\beta\bar{D}^2D_\beta B(p,\theta)\left\{1+ib\left[\ln\left(-\frac{p^2}{\lambda^2}\right)-2\right]g^2\right. \\ \left. +2b^2\left[\ln\left(-\frac{p^2}{\lambda^2}\right)+\frac{\pi^2}{12}+3\zeta(3)-4\right]g^4\right\}. \end{aligned} \tag{6.66}$$

By comparison with eq. (6.25), one notice that in both methods (standard and covariant derivative background field method) the dependence of the renormalized two-point function in the renormalization scale $\lambda$ is the same. Since this is the only relevant part for the computation of the SQED beta function, we find that in both cases there is a non-null two-loop beta function coefficient given by

$$\beta_2 = \frac{1}{8(4\pi^2)^2}g^5. \tag{6.67}$$

## 6.2 Discussion of the results and perspectives

In this chapter we have studied massless SQED up to two loop order. Our main motivation was the study of the anomaly puzzle which, as explained in the introduction, is related to the existence (or not) of higher order correction to beta functions of supersymmetric theories. Therefore, we intended to compute in perturbation theory corrections up to two-loop order of the beta function of SQED. We have chosen to use the IReg framework since it operates in the physical dimension of the theory (respecting supersymmetry) as well as displays in a clear way divergences (UV and IR) and regularization dependent terms (surface terms). We have found that the beta function receives corrections up to two-loop order although there are some subtleties. The first point is the application of the background field method which simplifies the computation by restricting the computation of the beta function to the knowledge of two-point functions in the background field. As explained before there are two roads to be followed: apply the standard background field method technique [133] or use the covariant derivative one [132]. In the first case, we obtained that the one and two-loop effective action contains a divergence. Therefore, the beta function could be computed in the usual way, by defining a renormalization constant in the background field, and we found the following values for the SQED beta function

$$\beta = \frac{1}{(4\pi)^2}g^3 + \frac{1}{8(4\pi^2)^2}g^5 + \mathcal{O}(g^7), \tag{6.68}$$

where are the same obtained before in the literature [136, 137, 138, 123]. As usual, the beta function could be computed by using the renormalization group equation, in which case only the renormalized two-point function in the background field is needed. The results are the same as before.

On the other hand, by using the covariant derivative background field method, we obtain that only the one-loop effective action contains a divergence. This could imply that the beta function does not receive higher order corrections, however, the renormalized two-point function still depends on the renormalization scale, which allows us to obtain the following beta function

$$\beta = \frac{1}{(4\pi)^2}g^3 + \frac{1}{8(4\pi^2)^2}g^5 + \mathcal{O}(g^7), \tag{6.69}$$

which is the same as before. This controversy was already obtained in the context of SYM theory [131]. Therefore, we found out with our computation that the above behavior is not characteristic of the SYM theory, being shared by the SQED theory as well. This allows us to conjecture that the reason may lie on the rescaling anomaly [139], being inherent to the definition of the covariant derivative background field method itself.

As perspectives we should include the study of how exactly the rescaling anomaly manifests itself in the covariant derivative background field method. Thus, one expect to be able to introduce some modifications in the usual multiplicative renormalization in order to solve this controversy in the computation of the beta function.

# Chapter 7

# Conclusion and perspectives

The Standard Model of Elementary Particles is one of the major achievements of the human mind. In order to build such theory, Quantum Field Theory has to be used, which inevitably lead to the appearance of infinities. This is a intrinsic feature of the theory by considering interactions as local and excitations of the fields as point particles. Therefore, it must not be seen as an artifact of the approximation methods used to tackle the subject (perturbation theory) being just expressed through them (Feynman diagrams). In this context, it is a necessity to develop a consistent program to deal with infinities, which is encoded in the regularization/renormalization program. Even though, nowadays, renormalization is not seen as a fundamental criterion a theory should fulfill, applying some regularization method is still needed in order to give sense to the infinities that appear. Many regularization techniques can be found in the literature sharing an aspect: by giving meaning to a divergent integral (evaluating it), one may preserve (or break) some symmetry of the theory. In this sense, it would be advantageous to have a regularization that do not impose a symmetry *a priori*, but instead allow it to be fixed by phenomenology, for instance.

In this thesis we have systematized the second point of view, by identifying in a precise and consistent way all ambiguities that may appear in the regularization program, at arbitrary order in perturbation theory. We have shown that these terms can always be related to the momentum routing invariance (MRI) of the Feynman diagram involved in the process one is dealing with, allowing us to conjecture that this is an even more fundamental symmetry that a quantum field theoretical result should fulfill. We have illustrated our method in the case of an abelian gauge theory such as Quantum Electrodynamics, showing the connection between MRI and gauge symmetry. We have also studied a scalar theory such as $\phi^3$, showing that MRI must be imposed in order to have a consistent result for the beta function. We have also applied our conclusions to solve a debate in the literature about the decay rate of the Higgs decay to two photons as well as the amplitude of the two photon scattering process. It should be emphasized that, although finite, both processes possess intermediate divergences urging the application of a regularization to obtain a definite result. We have also developed a new perspective on the divergent integrals themselves allowing us to introduce a cutoff even though no integral is explicitly evaluated. This allowed us to study the role of quadratic divergences in many examples, the hierarchy problem of the Standard Model included. Finally, we departed from Standard Model calculations and discussed ambiguities in the context of supersymmetric theories. As before, we were capable to identify all ambiguities fixing them by imposing gauge symmetry only. We also discussed a controversy related to the existence (or not) of higher order divergences in the diagrams needed for the computation of the supersymmetric beta functions. In the case of SuperQED we found that different background field methods furnish different results in this matter.

# Appendix A

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# Conductivity of Coulomb interacting massless Dirac particles in graphene: Regularization-dependent parameters and symmetry constraints

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**Abstract** – We compute the Coulomb correction $\mathcal{C}$ to the ac conductivity of interacting massless Dirac particles in graphene in the collisionless limit using the polarization tensor approach in a regularization-independent framework. Arbitrary parameters stemming from differences between logarithmically divergent integrals are fixed on physical grounds exploiting only the spatial $O(2)$ rotational invariance of the model which amounts to the transversality of the polarization tensor. Consequently $\mathcal{C}$ is unequivocally determined to be $(19-6\pi)/12$ within this effective model. We compare our result with explicit regularizations and discuss the origin of other results for $\mathcal{C}$ found in the literature.



**Introduction.** – The value of graphene ac conductivity corrected by a Coulomb interaction has been a matter of debate in the recent literature. Its general form for low frequencies can be obtained in the context of renormalization group techniques [1–3] based on scale relations valid near the quantum critical point:

$$\sigma(\omega) = \sigma_0 \left(1 + \mathcal{C}\frac{e^2}{v_F + \frac{e^2}{4}\ln\frac{\Lambda}{\omega}}\right). \tag{1}$$

In the equation above $\sigma_0 = e^2/4\hbar$ is known as "minimum conductivity" (in the absence of interactions), measured in [4,5] as $\sigma_0 = (1,01 \pm 0,04)e^2/4\hbar$, $v_F$ is the Dirac fermion velocity, which is renormalized by $\frac{e^2}{4}\ln\frac{\Lambda}{\omega}$ due to the Coulomb interaction and $\Lambda$ is an upper cutoff [6]. This is the case of the coefficient $\mathcal{C}$ which can be calculated within a model that takes Coulomb interaction into account. In order to obtain $\mathcal{C}$, the majority of the calculations relies on a perturbative analysis of effective theories based on disorder-free Coulomb interacting massless Dirac electrons. However, a complete eletronic structure calculation based on a realistic tight-binding Hamiltonian has been performed in [7]. In diagrammatic perturbation theory to first order in the electron-electron interaction, Feynman diagrams that contribute to the density-density and current-current response function (expressed together by a polarization tensor) can be drawn in analogy to field theoretical quantum electrodynamics (QED) [8]. The general structure of the polarization tensor can be established on geometric grounds as well as a continuity equation $\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$ which expresses the density-density response function in terms of a current-current response function. Although the geometry of the graphene honeycomb lattice is $C_6$ symmetric, the low effective model to be treated here is spatially $O(2)$ symmetric. Thus, while rotational invariance leads to transversality of the polarization tensor, the continuity equation constrains the form of the Coulomb vertex function leading to a Ward-Takahashi–like identity similar to QED, namely a relation between the vertex function and the electron self-energy [9,10]. It was claimed, however, that while the continuity equation holds at noninteracting level, it fails when electron-electron interactions are taken into account [8] because of ultraviolet infinities which demand a rigid momentum cutoff.

The main results available in the literature within effective models are obtained using the Kubo formula [1,9–11], the electron polarization operator [9,11] and kinetic equation [3,11] yielding different results for $\mathcal{C}$ depending on how intermediate divergent integrals are handled in each method.

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The *ab initio* tight-binding method of [7] claims that the ambiguity characterizing the various approaches is related to a chiral anomaly in the system in consequence of an unclear separation between infrared and ultraviolet physics and thus the regularization of an effective model is a crucial and important issue. The authors in [7] also point out that their result is compatible with [9] namely $\mathcal{C} = \frac{11-3\pi}{6}$, where dimensional regularization was employed even though it is known to be problematic in $2+1$ anomalous theories and needs to be handled with care when describing chiral anomalies [12,13]. It is well known that in the presence of anomalies such as the Adler-Bardeen-Bell-Jackiw (ABJ) chiral anomaly [14] a regularization should play no role in picking a particular axial or vector Ward identity but rather should display the anomaly in a manifestly "democratic way" to distinguish spurious from physical anomalies.

In [9] the authors used the polarization operator approach and dimensional regularization to handle intermediate divergent integrals in the calculation of the conductivity. They claimed that their previous result presented in [1] lacked a consistent dimensional continuation of Pauli matrices and thus violated a Ward identity based on the continuity equation. Moreover, a missing term (absent in the physical dimension) related to space-time continuation of the sigma matrix algebra was argued to account for the discrepancy between the results for $\mathcal{C}$, $\frac{25-6\pi}{12}$ in [1] and $\frac{11-3\pi}{6}$ in [9].

The purpose of the present contribution is to shed some light on this controversy employing the polarization operator method. Our strategy is based on a regularization independent approach namely implicit regularization (IR) that operates in the physical dimension of the model. An important feature of IR which makes it an ideal arena to calculate the conductivity is that it clearly parametrizes finite regularization-dependent terms which in our approach stem from differences between logarithmically divergent integrals (surface terms, ST) containing one integration variable only (loop momentum). This appears to be at the heart of controversial results for $\mathcal{C}$. Such arbitrary regularization-dependent ST should be fixed on physical grounds employing symmetry requirements of the underlying model or phenomenology. This is especially relevant when we work with radiative corrections that are finite although intermediate divergent integrals appear because the original amplitude is superficially divergent. This is the case of the effective models used to compute the conductivity.

As we shall demonstrate in this paper, we obtain for the ac conductivity using the current-current correlator,

$$\mathcal{C} = 2\pi\alpha + \frac{19-6\pi}{12}, \qquad (2)$$

where $\alpha$ is an arbitrary finite constant (ST) which parametrizes the regularization dependence and whose expression we derive in the next sections. In the present case we will demonstrate that $\alpha$ is unambiguously fixed to zero only by the general transversal structure of the polarization tensor inferred from spatial $O(2)$ rotational invariance. Consequently we have $\mathcal{C} = \frac{19-6\pi}{12}$. Two features are noteworthy. Firstly the continuity equation (whose validity is controversial when interactions are considered [8]) which leads to a Ward-Takahashi–like identity for the vertex function in [10] plays no role in determining $\alpha$. Secondly $\alpha$ turns out to be zero in dimensional regularization which means that, in principle, such regularization agrees with our result as well, yet this is not the value obtained in [9]. From the experimental point of view the best test ground for $\mathcal{C}$, whose controversial values differ by one order of magnitude, resides in the optical transparency of graphene as the transmittance is related to the conductivity in the optical regime as $t(\omega) = (1 + 2\pi\sigma(\omega)/c)^{-2}$. Experimental data on the optical transmission [5] suggest a negligible correction due to a Coulomb interaction compatible with the value $\mathcal{C} = \frac{19-6\pi}{12} \approx 0.01$. Furthermore, according to [5] the optical properties of graphene lie mainly on its two-dimensional structure and gapless eletronic spectrum and does not involve the chirality of the charge carriers [7]. Moreover, the model remains predictive because transversality derives from a very general symmetry property of the model which remains valid in the presence of interactions. We also map the different values of $\mathcal{C}$ to different evaluations of the regularization-dependent parameter $\alpha$ which is *per se* arbitrary and should be fixed on symmetry requirements.

**The formalism: electrical conductivity from imaginary time correlation function.** – The Hamiltonian of monolayer graphene is described by two-dimensional massless Dirac (quasi)particles with the speed of light replaced by $v_F$ and the pseudo-spin corresponding to sublattice indices [15],

$$\begin{aligned}\hat{H} &= \int \mathrm{d}^2\vec{r}\,\psi^\dagger(\vec{r}) v_F \vec{\sigma}\cdot\vec{p}\psi(\vec{r}) \\ &+ e^2 \int \mathrm{d}^2\vec{r}\mathrm{d}^2\vec{r'}\frac{\psi^\dagger(\vec{r})\psi(\vec{r})\psi^\dagger(\vec{r'})\psi(\vec{r'})}{|\vec{r}-\vec{r'}|}, \end{aligned} \qquad (3)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y)$ are Pauli matrices and we included a two-body Coulomb interaction between electrons.

Quantum electrodynamics in $2+1$ dimensions can be used to describe planar fermions in two spatial dimensions to calculate, for instance, the polarization operator defined through the effective action of fermions in the presence of electromagnetic fields. The polarization tensor can be interpreted in terms of the conductivity of graphene [16] as well as for the description of other interesting phenomena such as the Hall and Faraday effects, the light absorption rate by graphene sheets and the Casimir interaction of graphene [17,18].

Based on the fluctuation dissipation theorem [19], the expectation value of the electrical current density operator for real time $j(\vec{r},t)$ can be related to the imaginary time polarization tensor $\Pi_{\mu\nu}(\tau,\vec{r}) = \langle T[j_\mu(\tau,\vec{r})j_\nu(0,0)]\rangle$

with $j_\mu(\tau,\vec{r}) = (\psi^\dagger(\tau,\vec{r})\psi(\tau,\vec{r}), v_F\psi^\dagger(\tau,\vec{r})\vec{\sigma}\psi(\tau,\vec{r}))$, $\mu = 0,1,2$ and $\psi(\tau,\vec{r})$ is the two-component massless field.

The correlation function in the reciprocal space $\Pi_{\mu\nu}(q_\mu)$, $q_\mu = (i\Omega,\vec{q})$ and $\vec{q} = (q_1,q_2)$, will be calculated within an expansion in the coupling constant to order $e^2$ just as in [10], namely

$$\Pi_{\mu\nu}(q_\mu) = \Pi^0_{\mu\nu}(q_\mu) + \delta\Pi_{\mu\nu}(q_\mu, V_{\vec{k}}), \tag{4}$$

where $\Pi^0_{\mu\nu}$ represents the noninteracting contribution which characterizes the minimum conductivity, $\delta\Pi_{\mu\nu}$ is the first correction due to a Coulomb interaction of the quasiparticles and $V_{\vec{k}} = \frac{2\pi e^2}{|\vec{k}|}$ is the Fourier transform of the Coulomb potential. The leading-order term in (4) reads

$$\Pi^0_{\mu\nu}(q_\mu) = -4\int_\omega\int_{\vec{k}}\mathrm{Tr}[G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k+q}\sigma_\nu] \tag{5}$$

in which $\int_\omega \equiv \int_{-\infty}^{+\infty}\mathrm{d}\omega/(2\pi)$, $\int_{\vec{k}} \equiv \int \mathrm{d}^2k/(2\pi)^2$, $\sigma_\mu = (1_{2\times2},\sigma_x,\sigma_y)$, 4 is the number of two-component fermionic fields copies for the graphene and $G_k = \frac{i\omega+\vec{\sigma}\cdot\vec{k}}{\omega^2+\vec{k}^2}$ is the fermion propagator. In turn, the correction term to the correlation function can be written as

$$\delta\Pi_{\mu\nu}(q_\mu) = 4\int_{\vec{k},\vec{p},\omega,\omega'} V_{\vec{k}-\vec{p}}\mathrm{Tr}[\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3], \tag{6}$$

in which

$$\begin{aligned}\mathcal{A}_1 &\equiv G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k+q}G_{\omega'+\Omega,p+q}\sigma_\nu G_{\omega',p},\\ \mathcal{A}_2 &\equiv G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k+q}\sigma_\nu G_{\omega,k}G_{\omega',p} \quad \text{and}\\ \mathcal{A}_3 &\equiv G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k+q}G_{\omega'+\Omega,p+q}G_{\omega+\Omega,k+q}\sigma_\nu.\end{aligned}$$

The correction to the conductivity may be derived taking the spatial component of the polarization tensor (current-current approach),

$$\sigma(\Omega,|q|) = \frac{e^2}{\hbar}\frac{i\Omega}{q^2-\Omega^2}\Pi_{xx}(\Omega+i0,|q|), \tag{7}$$

the ac conductivity being given by $\lim_{q\to0}\sigma(\Omega,|q|)$ [9,11].

**Implicit regularization and the parametrization of (un)determined perturbative corrections.** – Implicit regularization (IR) is a momentum space framework that operates on the physical space-time dimension of the underlying model. It assumes an implicit regularization operating in the physical space-time dimension for a general $n$-loop Feynman amplitude only to allow the application of a mathematical identity at the level of propagators which displays its divergent content as basic divergent integrals (BDIs) that are written in terms of one internal momentum only. We suggest refs. [20,21] for a complete account of IR and discuss here some basic features related to leading-order calculations in two dimensions.

For instance, in the case in which the ultraviolet behavior is logarithmical, a one-loop Feynman amplitude is cast as a finite function of external momenta, a BDI (say $I_{log}(\lambda^2)$) and surface terms expressed by integrals of a total derivative in momentum space. The origin of these surface terms is that logarithmically divergent loop integrals $I^{\mu\nu\cdots}_{log}(\lambda^2)$ which contain in the integrand a product of internal momenta carrying Lorentz indices $\mu,\nu,\ldots$ can be expressed in a precise way as a product of metric tensors symmetrized in the Lorentz indices and $I_{log}(\lambda^2)$ plus a surface term. Such local, regularization-dependent surface terms are intrinsically arbitrarily valued. Within IR, regularization-dependent terms (surface terms) can be extracted out in a consistent way allowing for a clear discussion of the ambiguities involved in the manipulation of divergent integrals. Because it acts on the physical dimension of the theory, IR is particularly useful to dimensional specific models. In the latter, dimensional regularization methods flaw because of ambiguities in the analytic continuation on the space-time dimension.

To illustrate this method in connection with our calculation of the ac conductivity in graphene, we discuss the quantum mass generation for photons in quantum electrodynamics in $1+1$ Minkowski space-time dimension (Schwinger Model). We consider massless fermions for simplicity. This is a good example as in the calculation of graphene ac conductivity, after integrating in the frequencies, we end up with momentum integrals in 2-dimensional Euclidean space. Moreover, the photon mass generation is evaluated through the calculation of the vacuum polarization tensor

$$\Pi^{\mu\nu}_S(p) = i\mathrm{Tr}\int_k \gamma^\mu\frac{i}{\not{k}}\gamma^\nu\frac{i}{\not{k}+\not{p}}, \tag{8}$$

which turns out to be finite even though it is superficially (logarithmically) divergent. This happens to be the case in the polarization tensor and vertex function for the calculation of the conductivity as well.

We evaluate this amplitude using IR. After performing the trace algebra, we separate the divergent content in terms of the loop momentum using the identity

$$\frac{1}{[(k+p)^2-\mu^2]} = \frac{1}{(k^2-\mu^2)}\left[1-\frac{(p^2+2p\cdot k)}{(k+p)^2-\mu^2}\right], \tag{9}$$

$\mu$ being a fictitious mass for the electron which will be set to zero in the end. Then it is easy to show that [20]

$$\Pi^{\mu\nu}_S(p) = \frac{1}{\pi}\Big(\frac{\alpha+2}{2}g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big), \tag{10}$$

in which the arbitrary *regularization-dependent* parameter $\alpha$ is the difference between two logarithmically divergent integrals,

$$\begin{aligned}\alpha g_{\mu\nu} &\equiv g_{\mu\nu}I_{log}(\mu^2) - 2I_{log\,\mu\nu}(\mu^2)\\ &= \int_k\frac{g_{\mu\nu}}{(k^2-\mu^2)} - 2\int_k\frac{k_\mu k_\nu}{(k^2-\mu^2)^2}\\ &= \int_k\frac{\partial}{\partial k^\nu}\Big(\frac{k_\mu}{(k^2-\mu^2)}\Big) \equiv \Xi_{\mu\nu}.\end{aligned} \tag{11}$$

If explicitly calculated in dimensional or Pauli-Villars regularization it turns out to be zero in consonance with the transversal character of the polarization tensor required by gauge invariance, whereas $\alpha = 1/(4\pi)$ in sharp cut-off. The resulting radiatively generated photon mass is $m_\gamma^2 = e^2/\pi$. However, following Jackiw in [13] we can consider $\alpha$ as an arbitrary undetermined parameter. This is a good example of radiative corrections that are finite. As such they should be fixed on symmetry or phenomenological grounds. Thus, in this case, we can assign a vanishing value for $\alpha$ based either on transversality or on the "phenomenological" value of the generated photon mass, should it exist at all. However, in its chiral version, the chiral Schwinger model exhibits an anomalous nonsimultaneous conservation of the vector and chiral current [22] similar to the famous ABJ triangle anomaly [14]. The democratic display of the anomaly between the two Ward identities can only be achieved if $\alpha$ is left arbitrary. The calculation by itself should not decide which Ward identity is to be satisfied.

An arbitrary positive (renormalization group) mass scale $\lambda$ appears via a regularization-independent identity which enables us to write a BDI as a function of $\lambda$ only plus logarithmic functions of $\mu/\lambda$. At one-loop order it reads

$$I_{log}(\mu^2) = I_{log}(\lambda^2) + \frac{i}{4\pi}\ln\left(\frac{\mu^2}{\lambda^2}\right). \tag{12}$$

For the sake of comparison with other regularizations, we can also construct an explicit general parametrization for basic divergent integrals exhibiting explicitly all arbitrary regularization-dependent parameters and revealing the divergent behavior through a cutoff. In order to construct such parametrization in 1+1 space-time dimensions, let us consider the regularization-independent derivatives with respect to $\mu^2$,

$$\frac{\mathrm{d}I_{log}(\mu^2)}{\mathrm{d}\mu^2} = \frac{i}{4\pi\mu^2}, \qquad \frac{\mathrm{d}I_{log}^{\mu\nu}(\mu^2)}{\mathrm{d}\mu^2} = \frac{g^{\mu\nu}}{2}\frac{i}{4\pi\mu^2}. \tag{13}$$

A general parametrization which obeys the relations above is given by

$$\begin{aligned} I_{log}(\mu^2) &= -\frac{i}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right) + c, \\ I_{log}^{\mu\nu}(\mu^2) &= \frac{g^{\mu\nu}}{2}\left[-\frac{i}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right) + c'\right], \end{aligned} \tag{14}$$

in which $c$, $c'$ are also arbitrary dimensionless regularization-dependent constants and $\Lambda \to \infty$. The arbitrariness of the surface term defined in (11) becomes evident since $\Xi^{\mu\nu} = \alpha g^{\mu\nu} = (c - c')g^{\mu\nu}$.

**Calculation of ac conductivity.** – Hereafter we shall work in Euclidian space. The minimum conductivity can be calculated from eq. (5) after subtracting the zero-frequence mode, namely $\Pi_{\mu\nu}^{0,\Omega}(q_\mu) \to \Pi_{\mu\nu}^{0,\Omega=0}(q_\mu) - \Pi_{\mu\nu}^{0,\Omega}(q_\mu)$ which eliminates a spurious linear divergence $I_{lin}(\mu^2) = \int_{\vec{k}}(k^2+\mu^2)^{-1/2}$. After some algebra we obtain

$$\begin{aligned} \Pi_{\mu\nu}^{0}(q_\mu) = &-\frac{\sqrt{\Omega^2+q^2}}{16}\mathrm{Tr}\Big[\sigma_\mu\sigma_\nu - 2\delta_{\mu0}\sigma_\nu \\ &- \frac{(i\Omega + \vec{\sigma}\cdot\vec{q})\sigma_\mu(i\Omega+\vec{\sigma}\cdot\vec{q})\sigma_\nu}{\Omega^2+q^2}\Big] + \frac{\sqrt{q^2}}{16} \\ &\times \mathrm{Tr}\Big[\sigma_\mu\sigma_\nu - 4\delta_{\mu0}\sigma_\nu - \frac{\vec{\sigma}\cdot\vec{q}\,\sigma_\mu\,\vec{\sigma}\cdot\vec{q}\,\sigma_\nu}{q^2}\Big] \end{aligned} \tag{15}$$

which gives from (7) the well-known universal value of the conductivity $\sigma_0 = \frac{1}{4}\frac{e^2}{\hbar}$. The Coulomb correction to the conductivity can be calculated from eq. (6), which we separate into two parts namely the self-energy correction $\delta\Pi_{\mu\nu}^a$ and the vertex correction $\delta\Pi_{\mu\nu}^b$,

$$\delta\Pi_{\mu\nu}(i\Omega,0) = \delta\Pi_{\mu\nu}^a(i\Omega,0) + \delta\Pi_{\mu\nu}^b(i\Omega,0), \tag{16}$$

where we have already taken the limit $q \to 0$. Explicitly we have

$$\begin{aligned} \delta\Pi_{\mu\nu}^a = 8\int_{\omega,\omega',\vec{k},\vec{p}} V_{\vec{k}-\vec{p}}\mathrm{Tr}\Big[&G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k}\sigma_\nu \\ &\times G_{\omega,k}G_{\omega',p}\Big] \quad \text{and} \end{aligned} \tag{17}$$

$$\begin{aligned} \delta\Pi_{\mu\nu}^b = 4\int_{\omega,\omega',\vec{k},\vec{p}} V_{\vec{k}-\vec{p}}\mathrm{Tr}\Big[&G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k} \\ &\times G_{\omega'+\Omega,p}\sigma_\nu G_{\omega',p}\Big]. \end{aligned} \tag{18}$$

The integrals above can be simplified if we already take $\mu = \nu = x$, the diagonal spatial component of the Coulomb interaction correction $\delta\Pi_{xx}(i\Omega,0)$, to calculate the conductivity within the current-current approach. After subtracting the zero mode just as we did for the noninteracting case, we obtain, after tedious yet straightforward calculation, that $\delta\Pi_{xx}^a$ and $\delta\Pi_{xx}^b$ contribute to the conductivity with

$$\begin{aligned} \frac{\sigma_a}{\sigma_0 e^2} = -\Big(&\pi I_{log}(\lambda^2) + \frac{1}{4}\ln\lambda^2 - \pi\alpha \\ &+ \frac{3}{2}\ln 2 - \frac{1}{4} - \frac{1}{2}\ln\Omega\Big) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \frac{\sigma_b}{\sigma_0 e^2} = \Big(&\pi I_{log}(\lambda^2) + \frac{1}{4}\ln\lambda^2 + \pi\alpha + \frac{3}{2}\ln 2 \\ &-\frac{1}{2}\ln\Omega - \frac{\pi}{2} + \frac{1}{12}(4+3\pi) + \frac{4-\pi}{4}\Big), \end{aligned} \tag{20}$$

respectively. Thus, the correction to the conductivity is given by adding (19) to (20) to yield

$$\mathcal{C} = \left(2\pi\alpha + \frac{19-6\pi}{12}\right). \tag{21}$$

Some comments are in order. Firstly some remarkable (regularization-independent) cancellations take place. Note that the logarithmic divergences $I_{log}(\lambda^2)$ and the renormalization scale dependence expressed by $\ln(\lambda^2)$ cancel out as they should since the model is finite. The arbitrary regularization-dependent finite parameter $\alpha$ appeared as the Euclidian version of (11) in the calculation of $\delta\Pi_{\mu\nu}$ and should be understood as a free parameter of the model.

**Symmetry constraints.** – As discussed in [13], in theories where radiative corrections are finite, arbitrary parameters stemming from cancellation of divergent integrals should be fixed by symmetries of the underlying model or phenomenology. This is exactly the situation we are confronted with in the calculation of the Coulomb correction for the graphene ac conductivity. Spatial $O(2)$ rotational symmetry being preserved in both noninteracting and Coulomb interacting models leads to the following general structure for the polarization tensor [9,23]:

$$\Pi_{\mu\nu}(q) = \Pi_A(q_\mu)A_{\mu\nu} + \Pi_B(q_\mu)B_{\mu\nu}, \tag{22}$$

with

$$B_{\mu\nu} = \delta_{\mu i}\Big(\delta_{ij} - \frac{q_i q_j}{\vec{q}^2}\Big)\delta_{j\nu} \text{ and} \tag{23}$$

$$A_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{\vec{q}^2} - B_{\mu\nu}, \tag{24}$$

$\Pi_B(q_\mu)$ being its spatially transverse component. Hence $\Pi_{\mu\nu}(q)$ is transverse $q_\mu\Pi_{\mu\nu}(q) = \Pi_{\mu\nu}(q)q_\nu = 0$. Notice that $\Pi^0_{\mu\nu}(q)$ in eq. (15) is clearly transverse. Let us study the transversality properties of its $O(e^2)$ correction $\delta\Pi_{\mu\nu}(q)$. For the sake of convenience and comparison with the literature we define

$$\Sigma_{\Omega,p,q} \equiv \int_{\vec{k},\omega} V_{\vec{k}-\vec{p}}G_{\omega+\Omega,k+q}, \tag{25}$$

$\Sigma_{\Omega,p,0}$ being usually called electron self-energy in analogy to QED. Using this definition we can formally write

$$\begin{aligned} q^\mu\delta\Pi_{\mu\nu}(q_\mu) = 4\int_{\vec{k},\omega} &\mathrm{Tr}[G_{\omega,k}\sigma_\nu G_{\omega,k}\Sigma_{p,0,0}] \\ &- \mathrm{Tr}[G_{\omega+\Omega,k+q}\sigma_\nu G_{\omega+\Omega,k+q}\Sigma_{\Omega,p,q}], \end{aligned} \tag{26}$$

where we have used the straightforward identity

$$\begin{aligned} q^\mu\sigma_\mu &= -i\Omega\mathbf{1}_{2\times2} + \vec{q}\cdot\vec{\sigma} \\ &= G^{-1}_{\omega+\Omega,k+q} - G^{-1}_{\omega,k}. \end{aligned} \tag{27}$$

Because $\int_\omega (G_{\omega+\Omega,k+q})^2 = 0$ and using the cyclic property of the trace, the study of the transversality of $\delta\Pi_{\mu\nu}$ amounts to investigating the commutator $[G_{\omega+\Omega,k+q}, \Sigma_{\Omega,p,q}]$. For this purpose let us further develop the expression for $\Sigma_{\Omega,p,q}$. From (25), after a Feynman parametrization [24] for completing the square in the integration variable in the denominator and some lengthy yet direct algebra, we can write

$$\Sigma_{p,q} = \Sigma_{p+q,0} + e^2\pi\alpha\vec{\sigma}\cdot\vec{q}, \tag{28}$$

$\alpha$ being the same arbitrariness that appeared in the computation of $\mathcal{C}$ given by the (Euclidian version of) (11). Furthermore, following the rules described before we can demonstrate that $\Sigma_{p+q,0} \propto \vec{\sigma}\cdot(\vec{p}+\vec{q})$, namely

$$\begin{aligned} \frac{\Sigma_{p+q,0}}{e^2} = \frac{\vec{\sigma}\cdot(\vec{p}+\vec{q})}{8}\Big[&4\pi I_{log}(\lambda^2) - \ln\Big(\frac{(\vec{p}+\vec{q})^2}{\lambda^2}\Big) \\ &+ 4\ln 2 - 4\pi\alpha\Big]. \end{aligned} \tag{29}$$

We are now in the position to discuss the transversality of $\delta\Pi_{\mu\nu}$. Because of (29) it is evident that

$$[G_{\omega+\Omega,k+q}, \Sigma_{p+q,0}] = 0. \tag{30}$$

Therefore, according to (28),

$$[G_{\omega+\Omega,k+q}, \Sigma_{p,q}] = e^2\pi\alpha[G_{\omega+\Omega,k+q}, \vec{\sigma}\cdot\vec{q}]. \tag{31}$$

As the commutator on the RHS of the equation above does not vanish in general, we are led to conclude that transversality of the Coulomb correction of the polarization tensor implies $\alpha = 0$ which, in turn, fixes $\mathcal{C} = \frac{19-6\pi}{12}$. Notice that this is the same argument we illustrated for $\mathrm{QED}_2$, where gauge invariance fixes the value of the arbitrary parameter to vanish as well.

**Discussion.** – We proceed to discuss some results for $\mathcal{C}$ that appeared in the literature. It is straightforward to demonstrate that $\alpha$ turns out to be zero, should we employ dimensional regularization to evaluate (11), which, in principle, indicates that dimensional regularization also leads to $\mathcal{C} = \frac{19-6\pi}{12}$, contrarily to the result of [9]. Direct evaluation of (11) in sharp cutoff yields $1/(4\pi)$ which, following (21), yields $\mathcal{C} = \frac{25-6\pi}{12}$ but strikingly violates transversality as the commutator (31) does not vanish.

Finally it was pointed out in [9] that a Ward-Takahashi identity similar to the one that appears in field theoretical quantum electrodynamics could be derived defining the "vertex function" $\Lambda_\mu$ derived from the Fourier transform of the four-point matrix function $\pi_\mu(\vec{r_1}-\vec{r}, t_1-t, \vec{r}-\vec{r_2}, t-t_2) = \langle T_t j_\mu(t,\vec{r})\psi(t_1,\vec{r_1})\psi^\dagger(t_2,\vec{r_2})\rangle$ such that

$$\pi^\mu_{\omega,k;\omega+\Omega,k+q} = G_{\omega,k}\Lambda^\mu_{\omega,k;\omega+\Omega,k+q}G_{\omega+\Omega,k+q} \tag{32}$$

leading to

$$q_\mu\Lambda^\mu_{\omega,k;\omega+\Omega,k+q} = \Sigma_{\omega+\Omega,k+q,0} - \Sigma_{\omega,k,0}. \tag{33}$$

To derive (33) it is important to notice that the continuity equation was used in its naive form $\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$, which, according to [8], is no longer satisfied in an interacting system. Let us assume that (33) is valid for the Coulomb vertex function to the first order in the coupling constant,

$$\delta\Lambda_\mu(\Omega,p,q) = -\int_{\vec{k},\omega}\frac{2\pi e^2}{|\vec{p}-\vec{k}|}G_{\omega,k}\sigma_\mu G_{\omega+\Omega,k+q} \tag{34}$$

which, according to (33), should obey

$$q^\mu\delta\Lambda_\mu(\Omega,p,q) = \Sigma_{\omega+\Omega,p+q,0} - \Sigma_{\omega,p,0}. \tag{35}$$

Let us explicitly verify under which conditions this identity is satisfied evaluating separately both sides of (35) using the IR framework . After some lengthy yet perspicuous algebra we get, for the RHS

$$\Sigma_{p+q,0} - \Sigma_{p,0} = e^2\vec{\sigma}\cdot\vec{p}\left\{-\frac{1}{8}\ln\left[\frac{(\vec{p}+\vec{q})^2}{p^2}\right]\right\}$$
$$+ e^2\vec{\sigma}\cdot\vec{q}\left\{\frac{\pi}{2}I_{log}(\lambda^2) + \frac{1}{2}\ln 2 - \frac{1}{8}\ln\frac{(\vec{p}+\vec{q})^2}{\lambda^2}\right\}$$
$$+ e^2\vec{\sigma}\cdot\vec{q}\left(\frac{\pi\alpha}{2}\right), \qquad (36)$$

whereas the LHS of (35) yields

$$q^\mu\delta\Lambda_\mu = e^2\vec{\sigma}\cdot\vec{p}\left\{-\frac{1}{8}\ln\left[\frac{(\vec{p}+\vec{q})^2}{p^2}\right]\right\} + ie^2\Omega\left(\frac{1}{8}-\frac{\pi\alpha}{2}\right)$$
$$+ e^2\vec{\sigma}\cdot\vec{q}\left\{\frac{\pi}{2}I_{log}(\lambda^2) + \frac{1}{2}\ln 2 - \frac{1}{8}\ln\frac{(\vec{p}+\vec{q})^2}{\lambda^2}\right\}$$
$$+ e^2\vec{\sigma}\cdot\vec{q}\left(\frac{1}{8}\right), \qquad (37)$$

where $\alpha$ is exactly the same ST that appeared both in the computation of the $\mathcal{C}$ and in the study of the transversality of $\delta\Pi_{\mu\nu}$. Clearly the Ward-Takahashi identity (35) is only fulfilled if $\alpha = 1/(4\pi)$. The latter, however, is incompatible with the transversality of $\delta\Pi_{\mu\nu}$ which requires $\alpha = 0$. Curiously enough the value $\alpha = 1/(4\pi)$ which satisfies (35) leads to $\mathcal{C} = \frac{25-6\pi}{12}$.

**Conclusions.** – We employ a regularization-independent framework which manifestly preserves arbitrary regularization-dependent parameters which should be fixed by symmetries of the underlying model. It is therefore especially tailored to handle the graphene conductivity calculation which involves contributions that are separately divergent but whose sum is finite and regularization dependent. Based on spatial $O(2)$ symmetry only which is translated into transversality of the polarization tensor, a free parameter $\alpha$ is fixed to zero. This unambiguously results in $\mathcal{C} = \frac{19-6\pi}{12} \approx 0.01$ which is in agreement with experimental findings so far [5] and agrees with dimensional regularization because $\alpha$ turns out to be zero if explicitly evaluated within this regularization (see also [25]). On the other hand, if evaluated with a sharp cutoff, for instance, $\alpha = 1/(4\pi)$ breaking transversality and leading to $\mathcal{C} = \frac{25-6\pi}{12} \approx 0.51$. It is important to mention that no recourse to the vertex Ward-Takahashi identity, whose validity in the presence of interactions has been contested [8], was made: transversality is enough to fix the only free parameter of the model.

* * *

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# Arbitrariness in the gravitational Chern-Simons-like term induced radiatively

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The induction of a Lorentz- and CPT-violating Chern-Simons-like term in a fermionic theory embedded in linearized quantum gravity is reassessed. We explicitly show that gauge symmetry on underlying Feynman diagrams does not fix the arbitrariness inherent to such induced term at one loop order. We present the calculation in a nonperturbative expansion in the Lorentz-violating parameter $b_\mu$ and within a framework which, besides operating in the physical dimension, judiciously parametrizes regularization dependent arbitrary parameters usually fixed by symmetries.

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## I. INTRODUCTION

In the standard model of particle physics, Lorentz and CPT are regarded as fundamental symmetries. However, since the early 1990's possible violations of such symmetries have been studied [1–33]. The first model, introduced by Sean M. Carroll *et al.* [6], considered the theoretical and phenomenological consequences of adding to QED a Chern-Simons (CS)-like term proportional to a constant four-vector. They found out that such model predicts vacuum birefringence. However, astrophysical data establish stringent bounds to this kind of deviation from Lorentz and CPT symmetries [6,7]. Such small effects would come from spontaneous symmetry breaking of Lorentz symmetry in a more complete theory such as string theory [1].

One interesting aspect which has been vastly investigated is whether this CS-like term can be radiatively produced. One example of this mechanism occurs in extended QED with a Lorentz violating axial term, in which the CS-like term appears when we consider radiative corrections to the photon propagator. However, different results for the CS-like coefficient have been found (see, for example, [3,8–17]). The coefficient of the induced term, coming from the cancellations of divergences, is in fact regularization dependent. Using the Pauli-Villars method, for instance, this coefficient is found to be zero [3,12], while the result using dimensional regularization depends on how the dimensional continuation of the $\gamma_5$ matrix is carried out (see [34,35] and references therein).

Following the idea of an induced CS-like term in extended QED, it has also been discussed if a gravitational CS-like term can be radiatively induced in a fermionic theory in curved space-time. Phenomenologically, the existence of such term would imply that gravitational waves possess two degrees of polarization instead of four [18]. Nevertheless, the coefficient of such induced term turns out to depend on details involving the regularization of intermediate divergences as well [19,20].

In this work, we compute the 1-loop correction to the graviton propagator in the weak field approximation, using a more general approach called implicit regularization. Since it does not specify any particular regularization technique, allowing the reproduction of other results by choosing the method at the end of the calculation, it permits us to identify the sources of ambiguities. We find that the induced gravitational CS-like term depends on a set of surface terms which, coming from differences of divergent integrals, are arbitrary.

Following [12], arbitrary parameters that appear in finite radiative corrections must be fixed either by phenomenology or symmetries of the underlying model. By demanding gauge invariance of the action, which enforces transversality of the graviton self-energy, we find that the dependence in one of the surface terms remains in the final amplitude. This is the same result obtained in the case of extended QED in flat space. In such case, requiring transversality of the final amplitude does not determine the coefficient of the Carroll-Field-Jackiw (CFJ) term.

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The paper is organized as follows: in Sec. II, we carry out, with a pedagogical purpose, a review of the calculation of the induced CS-like term in extended QED in flat space. In Sec. III, we turn our attention to fermions in linearized quantum gravity with a Lorentz violating extension. We compute the 1-loop correction to the graviton propagator with implicit regularization to study the induced CS-like term in this case. In Sec. IV, we conclude and leave details of the integrals that appear in this work to the Appendix.

## II. REVISITING THE INDUCTION OF A CS-LIKE TERM IN EXTENDED QED

In order to motivate our line of reasoning, we revisit the induction of the Chern-Simons-like term (also called Carroll-Field-Jackiw term) in extended QED, whose action reads

$$S^{\rm ext}_{\rm QED} = \int d^4x \bar{\psi}(i\partial\!\!\!/ - A\!\!\!/ - m - b\!\!\!/\gamma_5)\psi. \tag{1}$$

The coefficient of the induced CFJ term is well known to be ambiguous and many different methods have been applied, furnishing various results. Here, we take as an example the calculation of [17], in which the implicit regularization scheme has been used. For simplicity, we treat the massless case. If the fermion is nonmassive, its propagator can be decomposed as [15]

$$\frac{i}{k\!\!\!/ - b\!\!\!/\gamma_5} = \frac{i}{k\!\!\!/ - b\!\!\!/}P_L + \frac{i}{k\!\!\!/ + b\!\!\!/}P_R, \tag{2}$$

where we are using the chiral projectors

$$P_{R,L} = \frac{1 \pm \gamma_5}{2}. \tag{3}$$

Note that with this decomposition, it is simple to perform the complete one loop calculation, without necessity of expanding the propagator. So, it is really a nonperturbative calculation in $b_\mu$ and the problem reduces to the calculation of just one Feynman graph. Here, we carry out the calculation with an arbitrary loop routing. The full one-loop photon self-energy is given by

$$\Pi^{\mu\nu} = \frac{1}{2}\{\Pi^{\mu\nu}_{+} + \Pi^{\mu\nu}_{-} + \Pi^{\mu\nu}_{5+} + \Pi^{\mu\nu}_{5-}\}, \tag{4}$$

with

$$\Pi^{\mu\nu}_{\pm}(p, \alpha p \pm b) = \int_k^\Lambda \mathrm{tr}\left\{\frac{\gamma^\nu(k\!\!\!/ + \alpha p\!\!\!/ \pm b\!\!\!/)\gamma^\mu[k\!\!\!/ + (\alpha+1)p\!\!\!/ \pm b\!\!\!/]}{(k + \alpha p \pm b)^2[k + (\alpha+1)p \pm b]^2}\right\} \tag{5}$$

and

$$\Pi^{\mu\nu}_{5\pm}(p, \alpha p \pm b) = \pm\int_k^\Lambda \mathrm{tr}\left\{\frac{\gamma^\nu(k\!\!\!/ + \alpha p\!\!\!/ \pm b\!\!\!/)\gamma^\mu[k\!\!\!/ + (\alpha+1)p\!\!\!/ \pm b\!\!\!/]\gamma_5}{(k + \alpha p \pm b)^2[k + (\alpha+1)p \pm b]^2}\right\}, \tag{6}$$

where $\int_k \equiv \int \frac{d^4k}{(2\pi)^4}$ and the superscript $\Lambda$ is used to indicate that some four dimensional regularization has been applied (say a cutoff) just to justify algebraic operations at the level of the integrands. Since the regularization was not specified yet, we can maintain, for a while, the dependence on the parameter $\alpha$. For a particular momentum routing in the loop, a variable $\alpha$ is fixed. This is just illustrative, since the dependence on $\alpha$ cannot be disentangled from the choice of the regularization procedure.

The induction of the CS-term comes from the $\Pi^{\mu\nu}_{5\pm}$ parts, so that we have

$$\begin{aligned}\Pi^{\mu\nu}_5 &= \frac{1}{2}[\Pi^{\mu\nu}_{5+}(p, \alpha p + b) + \Pi^{\mu\nu}_{5-}(p, \alpha p - b)] \\ &= \frac{1}{2}[\Pi^{\mu\nu}_{5+}(p, b_1) - \Pi^{\mu\nu}_{5+}(p, b_2)],\end{aligned} \tag{7}$$

with $b_1 = \alpha p + b$ and $b_2 = \alpha p - b$. So, let us calculate $\Pi^{\mu\nu}_{5+}(p, b)$, which, after Dirac algebra, can be written as

$$\begin{aligned}\Pi^{\mu\nu}_{5+}(p, b) &= 4ip_\beta\epsilon^{\nu\alpha\mu\beta}\int_k^\Lambda \frac{(b+k)_\alpha}{(k+b)^2(k+p+b)^2} \\ &= 4ip_\beta\epsilon^{\nu\alpha\mu\beta}(b_\alpha I + I_\alpha),\end{aligned} \tag{8}$$

with

$$I, I_\alpha = \int_k^\Lambda \frac{1, k_\alpha}{(k+b)^2(k+p+b)^2}. \tag{9}$$

We apply the implicit regularization framework [36] to treat these integrals. Let us make a brief review of the method. In this scheme, we assume the existence of an implicit regulator ($\Lambda$) in order to judiciously use the following identity to separate UV divergent basic integrals from the finite part:

$$\int_k \frac{1}{(k+p)^2 - m^2} = \int_k \frac{1}{k^2 - m^2} - \int_k \frac{(p^2 + 2p\cdot k)}{(k^2 - m^2)[(k+p)^2 - m^2]}, \tag{10}$$

where we have introduced a fictitious mass in the propagators. This is necessary because, although the present integrals are infrared safe, the above expression without mass will break the original integral in two infrared divergent parts. The limit $m^2 \to 0$ is taken in the end. In this process a renormalization scale $\lambda \neq 0$ is introduced. In general, besides a finite part in the UV limit, we get basic divergent integrals which are defined as

$$I_{\log}^{\mu_1\cdots\mu_{2n}}(m^2) \equiv \int_k \frac{k^{\mu_1}\cdots k^{\mu_{2n}}}{(k^2-m^2)^{2+n}} \tag{11}$$

and

$$I_{\text{quad}}^{\mu_1\cdots\mu_{2n}}(m^2) \equiv \int_k \frac{k^{\mu_1}\cdots k^{\mu_{2n}}}{(k^2-m^2)^{1+n}}. \tag{12}$$

The basic divergences with Lorentz indices can be judiciously combined as differences between integrals with the same superficial degree of divergence, according to the equations below, which define surface terms[1]:

$$\Upsilon_{2w}^{\mu\nu} = \eta^{\mu\nu} I_{2w}(m^2) - 2(2-w) I_{2w}^{\mu\nu}(m^2) \equiv v_{2w}\eta^{\mu\nu}, \tag{13}$$

$$\begin{aligned}\Xi_{2w}^{\mu\nu\alpha\beta} &= \eta^{\{\mu\nu}\eta^{\alpha\beta\}} I_{2w}(m^2) - 4(3-w)(2-w) I_{2w}^{\mu\nu\alpha\beta}(m^2)\\ &\equiv \xi_{2w}\eta^{\{\mu\nu}\eta^{\alpha\beta\}},\end{aligned} \tag{14}$$

$$\begin{aligned}\Sigma_{2w}^{\mu\nu\alpha\beta\gamma\delta} &= \eta^{\{\mu\nu}\eta^{\alpha\beta}\eta^{\gamma\delta\}} I_{2w}(m^2)\\ &\quad - 8(4-w)(3-w)(2-w) I_{2w}^{\mu\nu\alpha\beta\gamma\delta}(m^2)\\ &\equiv \sigma_{2w}\eta^{\{\mu\nu}\eta^{\alpha\beta}\eta^{\gamma\delta\}}.\end{aligned} \tag{15}$$

In the expressions above, $2w$ is the degree of divergence of the integrals and for the sake of brevity, we substitute the subscripts *log* and *quad* by 0 and 2, respectively. Surface terms can be conveniently written as integrals of total derivatives, namely

$$v_{2w}\eta^{\mu\nu} = \int_k \frac{\partial}{\partial k_\nu} \frac{k^\mu}{(k^2-m^2)^{2-w}}, \tag{16}$$

$$(\xi_{2w} - v_{2w})\eta^{\{\mu\nu}\eta^{\alpha\beta\}} = \int_k \frac{\partial}{\partial k_\nu} \frac{2(2-w)k^\mu k^\alpha k^\beta}{(k^2-m^2)^{3-w}}, \tag{17}$$

and

$$\begin{aligned}&(\sigma_{2w} - \xi_{2w})\eta^{\{\mu\nu}\eta^{\alpha\beta}\eta^{\gamma\delta\}}\\ &\quad = \int_k \frac{\partial}{\partial k_\nu} \frac{4(3-w)(2-w)k^\mu k^\alpha k^\beta k^\gamma k^\delta}{(k^2-m^2)^{4-w}}.\end{aligned} \tag{18}$$

We see that Eqs. (13)–(15) are undetermined because they are differences between divergent quantities. Each regularization scheme gives a different value for these terms. However, as physics should not depend on the schemes applied, we leave these terms to be arbitrary until the end of the calculation, fixing them by symmetry constraints or phenomenology, when it applies.

Concerning the surface terms, a comment is in order. As is well known, to perform shifts in integrals with degrees of divergence which are at least linear, it is necessary to compensate with surface terms. For this reason, in a 4D procedure as implicit regularization, which preserves until the end the surface terms, the final amplitude will depend on the routing in the loop momentum. This dependence appears in the coefficients of the surface terms. Nevertheless, in the implicit regularization scheme, the parameters defined in Eqs. (13)–(15) are adjusted in order to fix symmetries.

Returning to our calculations, the results of the integrals (9) in the implicit regularization framework are given by

$$I = I_{\log}(\lambda^2) - \frac{i}{16\pi^2}\left[\ln\left(-\frac{p^2}{\lambda^2}\right) - 2\right] \tag{19}$$

and

$$\begin{aligned}I_\alpha &= -\frac{(p+2b)_\alpha}{2}\left\{I_{\log}(\lambda^2)\right.\\ &\quad \left. - \frac{i}{16\pi^2}\left[\ln\left(-\frac{p^2}{\lambda^2}\right) - 2\right] - v_0\right\}\\ &= -\frac{(p+2b)_\alpha}{2}(I - v_0).\end{aligned} \tag{20}$$

Substituting these results in Eq. (8), we get

$$\Pi_{5+}^{\mu\nu}(p,b) = 4iv_0 b_\alpha p_\beta \epsilon^{\nu\alpha\mu\beta}. \tag{21}$$

So, we obtain

$$\begin{aligned}\Pi_5^{\mu\nu} &= \frac{1}{2}[4iv_0(\alpha p + b)_\alpha p_\beta \epsilon^{\nu\alpha\mu\beta} - 4iv_0(\alpha p - b)_\alpha p_\beta \epsilon^{\nu\alpha\mu\beta}]\\ &= 4iv_0 b_\alpha p_\beta \epsilon^{\nu\alpha\mu\beta}.\end{aligned} \tag{22}$$

The induced coefficient of the Carroll-Field-Jackiw term will then be given by

$$\Delta c_\mu = 2iv_0 b_\mu. \tag{23}$$

We see that the coefficient of the induced CS-type term is proportional to the undetermined parameter $v_0$. In [8] a definite result for $\Delta c_\mu$ was obtained in the nonperturbative approach. For this, a procedure was used in the calculation of the surface terms. Actually, these terms are dependent on the procedure adopted. In our result, this is expressed in the dependence on $v_0$.

In [15], it was shown that the procedure of [8] has as a consequence the violation of gauge symmetry at second order in $b_\mu$. However, in the follow-up paper [16], the author has shown that the use of an adequate Pauli-Villars

[1]The Lorentz indices between brackets stand for symmetrization of the tensor, i.e., $A^{\{\alpha_1\cdots\alpha_n}B^{\beta_1\cdots\beta_n\}} = A^{\alpha_1\cdots\alpha_n}B^{\beta_1\cdots\beta_n}$ + sum over permutations between the two sets of indices $\alpha_1\cdots\alpha_n$ and $\beta_1\cdots\beta_n$.

regulator in the calculation preserves gauge symmetry in second order in $b_\mu$ even in the nonperturbative approach. Enforcing this result, in [17], the complete one-loop calculation was performed with implicit regularization. The results for the zeroth and second order terms in $b_\mu$ are given below:

$$\Pi_0^{\mu\nu} = \Pi(p^2)(p^\mu p^\nu - p^2\eta^{\mu\nu}) - 4v_2\eta^{\mu\nu} - \frac{4}{3}\{v_0(p^\mu p^\nu - p^2\eta^{\mu\nu}) + (2p^\mu p^\nu + p^2\eta^{\mu\nu})(\xi_0 - 2v_0)\} \tag{24}$$

and

$$\frac{1}{2}(\Pi_{bb-}^{\mu\nu} + \Pi_{bb+}^{\mu\nu}) = -4\{(b^2\eta^{\mu\nu} + 2b^\mu b^\nu)(\xi_0 - 2v_0)\}. \tag{25}$$

If one uses symmetric integration when calculating $v_0$ and $\xi_0$, such that $k^\mu k^\nu \to \eta^{\mu\nu}k^2/4$ and $k^\mu k^\nu k^\alpha k^\beta \to \eta^{\{\mu\nu}\eta^{\alpha\beta\}}k^4/24$, one obtains

$$v_0 = \frac{i}{32\pi^2} \quad \text{and} \quad \xi_0 = \frac{5i}{96\pi^2}, \tag{26}$$

so that

$$\frac{1}{2}(\Pi_{bb-}^{\mu\nu} + \Pi_{bb+}^{\mu\nu}) = \frac{i}{24\pi^2}(b^2\eta^{\mu\nu} + 2b^\mu b^\nu), \tag{27}$$

as in [15]. However, this will cause gauge symmetry violation even in the zeroth order term. The condition for transversality of the photon self-energy for all orders in $b_\mu$ is $\xi_0 = 2v_0$ and $v_2 = 0$. A gauge invariant procedure will respect these conditions, as, for example, the Pauli-Villars regulator used in [16]. Since the $v_0$ parameter cannot be fixed, the coefficient of the Chern-Simons-like term is really regularization dependent.

## III. ARBITRARINESS IN THE INDUCED CS GRAVITY TERM

We consider a massless fermionic theory in a gravitational background with a CPT-violating term,

$$S = \int d^4x \left(\frac{i}{2} e\, e_a^\mu \bar{\psi}\gamma^a \overleftrightarrow{D}_\mu \psi - e\, e_a^\mu b_\mu \bar{\psi}\gamma^a\gamma_5\psi\right), \tag{28}$$

where $e_a^\mu$ is the tetrad, $e = \det e_a^\mu$ and $b_\mu$ is a constant four-vector.

In Eq. (28), in order to couple fermions with the gravitational field, we need to define the covariant derivative,

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega_{\mu ab}\sigma^{ab}\psi, \tag{29}$$

where $\omega_{\mu ab}$ is the spin connection, which depends on the tetrad, and $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$.

In the weak field approximation, we use the following expansions for the metric and the tetrad:

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} \tag{30}$$

and

$$e_{\mu a} = \eta_{\mu a} + \frac{1}{2}\kappa h_{\mu a}. \tag{31}$$

Therefore, the action (28) can be reexpressed as

$$S = \int d^4x \left\{\frac{1}{2} i\bar{\psi}\overleftrightarrow{\partial}\!\!\!/\,\psi + \frac{1}{2} i\kappa\left[h\bar{\psi}\overleftrightarrow{\partial}\!\!\!/\,\psi - \frac{1}{2}h_a^\mu\gamma^a\bar{\psi}\overleftrightarrow{\partial}\!\!\!/\,\psi + \frac{1}{4}\bar{\psi}\partial_b h_{ca}\gamma^{\{a}\gamma^b\gamma^{c\}}\psi - \frac{1}{4}\bar{\psi}\partial_c h_{ba}\gamma^{\{a}\gamma^b\gamma^{c\}}\psi + \bar{\psi}h_a^\mu b_\mu\gamma^a\gamma_5\psi + \frac{1}{2}\bar{\psi}h b\!\!\!/\gamma_5\psi\right] - \bar{\psi}b\!\!\!/\gamma_5\psi\right\} + O(\kappa^2). \tag{32}$$

Feynman rules, shown in Fig. 1, can be readily derived from Eq. (32).

In order to obtain the induced Chern-Simons-like term, we have to compute the linear part in $b_\mu$ of the one-loop correction for the graviton propagator. We opt to use the complete propagator rather than treat the axial term as a interaction. Figure 2 shows the two diagrams that contribute.

Their amplitudes read

$$\Pi_{(a)}^{\mu\nu\alpha\beta}(p) = i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[V^{\mu\nu}(k+p,k)S(k+p) \times V^{\alpha\beta}(k,k+p)S(k)] \tag{33}$$

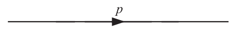
and

$$\Pi_{(b)}^{\mu\nu\alpha\beta}(p) = i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[S(k)V^{\alpha\beta\mu\nu}(p,p,k)]. \tag{34}$$

We write the following expansion for the fermion propagator

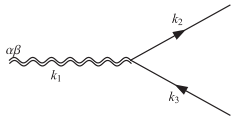
$$\frac{i}{k\!\!\!/ - b\!\!\!/\gamma_5} = \sum_{n=0}^{\infty}\frac{i}{k\!\!\!/}\left\{-ib\!\!\!/\gamma_5\frac{i}{k\!\!\!/}\right\}^n = \sum_{n=0}^{\infty}S_n(k). \tag{35}$$

Since the CS-like term we are interested in is linear in $b_\mu$, we can write

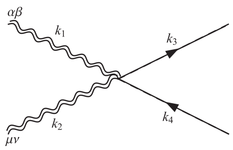



$$= S(p) = \frac{i}{\not{p} - \not{b}\gamma_5},$$

(a)




$$= V^{\alpha\beta}(k_2,k_3) = \frac{i\kappa}{8}[2\eta^{\alpha\beta}(\not{k}_2+\not{k}_3) - \gamma^{\alpha}(k_2+k_3)^{\beta} - \gamma^{\beta}(k_2+k_3)^{\alpha}]$$




$$\begin{aligned} &= V^{\alpha\beta\mu\nu}(k_1,k_2,k_4) = i\kappa^2\Big[\frac{5}{16}(\not{k}_1+\not{k}_2)\left(\frac{2}{5}\eta^{\alpha\beta}\eta^{\mu\nu} + \frac{1}{4}\gamma^{\{\alpha}\eta^{\beta\}\{\mu}\gamma^{\nu\}} + \frac{1}{4}\gamma^{\{\mu}\eta^{\nu\}\{\alpha}\gamma^{\beta\}}\right) \\ &+ \frac{1}{32}\gamma^{\{\alpha}\eta^{\beta\}\{\mu}(3k_4+4k_2-3k_1)^{\nu\}} - \frac{1}{16}\gamma^{\{\alpha}(2k_4+k_1)^{\beta\}}\eta^{\mu\nu} \\ &+ \frac{1}{16}\gamma^{\{\mu}\eta^{\nu\}\{\alpha}(3k_4+4k_1-3k_2)^{\beta\}} - \frac{1}{16}\gamma^{\{\mu}(2k_4+k_2)^{\nu\}}\eta^{\alpha\beta} \\ &- \frac{1}{4}\not{k}_4(\eta^{\mu\{\alpha}\eta^{\beta\}\nu} - \eta^{\mu\nu}\eta^{\alpha\beta})\Big]. \end{aligned}$$

(b)

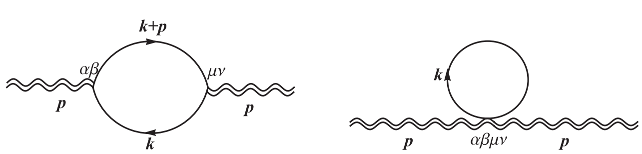
FIG. 1. Feynman rules for a fermionic theory in linearized quantum gravity with a Lorentz violating extension. (a) Fermion propagator. (b) Graviton-fermion vertices.

$$\begin{aligned} \Pi^{\mu\nu\alpha\beta}_{(a)\mathrm{CS}}(p) = i\int \frac{d^4k}{(2\pi)^4}&\mathrm{Tr}[V^{\mu\nu}(k+p,k)S_0(k+p) \\ &\times V^{\alpha\beta}(k,k+p)S_0(k)\not{b}\gamma_5 S_0(k)] \\ &+ i\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}[V^{\alpha\beta}(k,k+p)S_0(k) \\ &\times V^{\mu\nu}(k+p,k)S_0(k+p)\not{b}\gamma_5 S_0(k+p)] \end{aligned} \qquad (36)$$

and

$$\Pi^{\mu\nu\alpha\beta}_{(b)\mathrm{CS}}(p) = i\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}[S_0(k)\not{b}\gamma_5 S_0(k)V^{\alpha\beta\mu\nu}(p,p,k)]. \qquad (37)$$

These amplitudes are symmetric under the exchange $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$ as they should be. The amplitude $\Pi^{\mu\nu\alpha\beta}_{(b)\mathrm{CS}}(p)$ is null after the trace operation. The amplitude $\Pi^{\mu\nu\alpha\beta}_{(a)\mathrm{CS}}(p)$ is superficially cubically divergent.




(a) (b)

FIG. 2. 1-loop corrections for the graviton propagator. The double waved line and the solid line stand for the graviton and the fermion, respectively.

The result of the implicitly regularized amplitude $\Pi^{\mu\nu\alpha\beta}_{(a)\mathrm{CS}}(p)$ is given by (see a list of results of integrals in the Appendix)

$$\begin{aligned} \Pi^{\mu\nu\alpha\beta}_{(a)\mathrm{CS}}(p) = \frac{-i}{8}\kappa^2\Bigg[&\left(\frac{i}{48\pi^2} - 64\sigma_0 - 4\upsilon_0 + 4\xi_0\right)p^{\alpha}p^{\nu} \\ &- \left(\frac{i}{48\pi^2} + 32\sigma_0\right)\eta^{\alpha\nu}p^2\Bigg]\epsilon^{\lambda\rho\beta\mu}b_{\lambda}p_{\rho} \\ &+ (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu) + (\alpha\leftrightarrow\beta,\mu\leftrightarrow\nu). \end{aligned} \qquad (38)$$

Obviously this result contains arbitrariness expressed by surface terms. To try to fix them, we demand gauge invariance of the action, expressed by the transversality of the final amplitude. Explicitly, we have

$$\begin{aligned} p_{\alpha}\Pi^{\mu\nu\alpha\beta}_{(a)\mathrm{CS}}(p) = \frac{-i}{8}&\kappa^2(4\xi_0 - 4\upsilon_0 - 96\sigma_0) \\ &\times(\epsilon^{\lambda\rho\beta\mu}p^{\nu} + \epsilon^{\lambda\rho\beta\nu}p^{\mu})p^2 b_{\lambda}p_{\rho} = 0. \end{aligned} \qquad (39)$$

In order to satisfy Eq. (39), we must have $\xi_0 = \upsilon_0 = \sigma_0 = 0$ or $\xi_0 - \upsilon_0 = 24\sigma_0$. The former condition determines the CS-like term and the latter does not. If we replace this expression in Eq. (38) the result is

$$\begin{aligned} \Pi^{\mu\nu\alpha\beta}_{(a)\mathrm{CS}}(p) = \frac{-i}{24}\kappa^2\epsilon^{\lambda\rho\beta\mu}b_{\lambda}p_{\rho}&\left(\frac{i}{16\pi^2} + 96\sigma_0\right)(p^{\alpha}p^{\nu} - \eta^{\alpha\nu}p^2) \\ &+ (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu) + (\alpha\leftrightarrow\beta,\mu\leftrightarrow\nu). \end{aligned} \qquad (40)$$

We see that transversality is not sufficient to fix all surface terms leaving us an arbitrary result. Depending on the choice of the arbitrary term $\sigma_0$, we can either recover other results found in the literature [19–21] or even get zero.

The four terms of Eq. (40) assure the symmetry of the amplitude under the change $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. The consequent CS-like effective action is

$$\mathcal{L}_{\rm CS} = \left(\frac{1}{96\pi^2} - 16i\sigma_0\right)\kappa^2 b^\lambda h^{\mu\nu}\epsilon_{\alpha\mu\lambda\rho}\partial^\rho(\partial^2 h^\alpha_\nu - \partial_\nu\partial_\gamma h^{\gamma\alpha}). \tag{41}$$

If we set $\sigma_0 = 0$, this result agrees with the one of Ref. [19] where dimensional regularization was employed. Such behavior should be expected since the surface terms are zero if explicitly evaluated by this technique.

One more comment is in order. In the case of the extended QED in flat space-time, the transversality of the vacuum polarization tensor is trivially respected by the CFJ term, because of the presence of only one antisymmetric Lévi-Cività tensor contracted with the external momentum. In that case, this symmetry was not an alternative to try to determine the remaining surface term. The case of the Lorentz-violating model in a gravitational background is different, since the satisfaction of this symmetry is not trivial. It was necessary to enforce a relation among three parameters so as to satisfy it.

## IV. CONCLUDING REMARKS

In this work, we study the induction of a CS-like term by radiative corrections for a massless Lorentz- and CPT-violating fermionic theory embedded in a curved space-time. We adopt the framework of implicit regularization, which clearly parametrizes regularization dependent terms. Besides, we carry out the calculations in the nonperturbative approach in the Lorentz-violating parameter $b_\mu$. We imposed transversality of the amplitude as an attempt to fix the coefficient of the induced Lorentz-violating term. However, after enforcing this symmetry, the relations to be satisfied by the surface terms are not sufficient to determine the coefficient of the induced CS gravity term, leaving a free parameter.

This result should be compared with the one of the induction of a CFJ term in the extended QED in flat space. In that case, the satisfaction of transversality of the amplitude is trivial due to products involving symmetric and antisymmetric tensors. This is not the case here, since it was necessary to enforce a relation among three parameters so as to satisfy this symmetry.

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## APPENDIX

The result of the regularized integrals, after taking the trace, are

$$\int_k \frac{k^2}{k^4(k+p)^2} = I_{\log}(\lambda^2) + 2\tilde{b} - \tilde{b}\ln\left(-\frac{p^2}{\lambda^2}\right), \tag{A1}$$

$$\int_k \frac{k^2 k^\alpha}{k^4(k+p)^2} = \frac{1}{2}p^\alpha\left[-I_{\log}(\lambda^2) + \upsilon_0 - 2\tilde{b} + \tilde{b}\ln\left(-\frac{p^2}{\lambda^2}\right)\right], \tag{A2}$$

$$\int_k \frac{k^\alpha k^\beta}{k^4(k+p)^2} = \frac{1}{4}\eta^{\alpha\beta}\left[I_{\log}(\lambda^2) - \upsilon_0 + 2\tilde{b} - \tilde{b}\ln\left(-\frac{p^2}{\lambda^2}\right)\right] + \frac{1}{2}\tilde{b}\frac{p^\alpha p^\beta}{p^2}, \tag{A3}$$

$$\int_k \frac{k^\mu k^\alpha k^\beta}{k^4(k+p)^2} = \frac{1}{12}p^{\{\mu}\eta^{\alpha\beta\}}\left[-I_{\log}(\lambda^2) + \xi_0 + \tilde{b}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{5}{3}\tilde{b}\right] - \frac{1}{3}\tilde{b}\frac{p^\mu p^\alpha p^\beta}{p^2}, \tag{A4}$$

$$\begin{aligned}\int_k \frac{k^2 k^\alpha k^\beta}{k^4(k+p)^2} = &-\frac{1}{4}\eta^{\alpha\beta}p^2[I_{\log}(\lambda^2) - \upsilon_0] + \frac{1}{6}(p^2\eta^{\alpha\beta} + 2p^\alpha p^\beta)[I_{\log}(\lambda^2) - \xi_0] \\ &+ \frac{1}{2}\tilde{b}p^2\eta^{\alpha\beta}\left[\frac{1}{6}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{4}{9}\right] - \tilde{b}p^\alpha p^\beta\left[\frac{1}{3}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{13}{18}\right],\end{aligned} \tag{A5}$$

$$\int_k \frac{k^\mu k^\nu k^\alpha k^\beta}{k^4(k+p)^2} = -\frac{1}{24}\eta^{\{\mu\nu}\eta^{\alpha\beta\}}p^2[I_{\log}(\lambda^2)-\xi_0] + \frac{1}{48}(p^2\eta^{\{\alpha\beta}\eta^{\mu\nu\}} + p^{\{\alpha}p^\beta\eta^{\mu\nu\}})[I_{\log}(\lambda^2)-\xi_0-24\sigma_0]$$
$$+\frac{1}{8}\tilde{b}p^2\eta^{\{\mu\nu}\eta^{\alpha\beta\}}\left[\frac{1}{6}\ln\left(-\frac{p^2}{\lambda^2}\right)-\frac{4}{9}\right] - \frac{1}{72}\tilde{b}p^{\{\alpha}p^\beta\eta^{\mu\nu\}}\left[\frac{3}{2}\ln\left(-\frac{p^2}{\lambda^2}\right)-\frac{5}{2}\right] + \frac{1}{4}\tilde{b}\frac{p^\alpha p^\beta p^\mu p^\nu}{p^2}, \quad (A6)$$

$$\int_k \frac{k^2 k^\mu k^\alpha k^\beta}{k^4(k+p)^2} = \frac{1}{6}p^{\{\mu}\eta^{\alpha\beta\}}p^2[I_{\log}(\lambda^2)-\xi_0] - \frac{1}{8}(p^2p^{\{\mu}\eta^{\alpha\beta\}} + 2p^\alpha p^\beta p^\mu)[I_{\log}(\lambda^2)-\xi_0-24\sigma_0]$$
$$-\frac{1}{4}\tilde{b}p^2p^{\{\mu}\eta^{\alpha\beta\}}\left[\frac{1}{6}\ln\left(-\frac{p^2}{\lambda^2}\right)-\frac{4}{9}\right] + \tilde{b}p^\alpha p^\beta p^\mu\left[\frac{1}{4}\ln\left(-\frac{p^2}{\lambda^2}\right)-\frac{7}{12}\right], \quad (A7)$$

where $\lambda$ is mass scale and $\tilde{b} \equiv \frac{i}{(4\pi)^2}$.

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# Appendix B

In this appendix we collect the results in the IReg formalism of the majority of the integrals used in this thesis:

$$J(p^2,\mu^2) = \int_k \frac{1}{[k^2-\mu^2][(k+p)^2-\mu^2]} = I_{log}(\mu^2) - b\ln\left(-\frac{p^2}{\mu^2}\right) + 2b \quad (1)$$

$$J^{\mu}(p^2,\mu^2) = \int_k \frac{k^\mu}{[k^2-\mu^2][(k+p)^2-\mu^2]} = \frac{p^\mu}{2}\left[I_{log}(\mu^2) - b\ln\left(-\frac{p^2}{\mu^2}\right) + 2b\right] - \frac{1}{2}p_\alpha \Upsilon_0^{\mu\alpha} \quad (2)$$

$$J^{\mu\nu}(p^2,\mu^2) = \int_k \frac{k^\mu k^\nu}{[k^2-\mu^2][(k+p)^2-\mu^2]} = \frac{1}{3}p_\mu p_\nu\left[I_{log}(\mu^2) - b\ln\left(-\frac{p^2}{\mu^2}\right) + \frac{11}{6}b\right] \quad (3)$$

$$-\frac{1}{12}g_{\mu\nu}p^2\left[I_{log}(\mu^2) - b\ln\left(-\frac{p^2}{\mu^2}\right) + \frac{4}{3}b\right] + \frac{g^{\mu\nu}}{2}I_{quad}(\mu^2) - \frac{\Upsilon_2^{\mu\nu}}{2} + \frac{1}{4}\Upsilon_0^{\mu\nu}p^2 - \frac{1}{6}p_\alpha p_\beta \Upsilon_0^{\mu\nu\alpha\beta} \quad (4)$$

$$\int_k \frac{k^2}{k^4(k+p)^2} = I_{log}(\lambda^2) + 2b - b\ln(-\frac{p^2}{\lambda^2}), \tag{5}$$

$$\int_k \frac{k^2 k^\alpha}{k^4(k+p)^2} = \frac{1}{2}p^\alpha \left[-I_{log}(\lambda^2) + \upsilon_0 - 2b + b\ln\left(-\frac{p^2}{\lambda^2}\right)\right], \tag{6}$$

$$\int_k \frac{k^\alpha k^\beta}{k^4(k+p)^2} = \frac{1}{4}\eta^{\alpha\beta}\left[I_{log}(\lambda^2) - \upsilon_0 + 2b - b\ln\left(-\frac{p^2}{\lambda^2}\right)\right] + \frac{1}{2}\tilde{b}\frac{p^\alpha p^\beta}{p^2}, \tag{7}$$

$$\int_k \frac{k^\mu k^\alpha k^\beta}{k^4(k+p)^2} = \frac{1}{12}p^{\{\mu}\eta^{\alpha\beta\}}\left[-I_{log}(\lambda^2) + \xi_0 + b\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{5}{3}\tilde{b}\right] - \frac{1}{3}\tilde{b}\frac{p^\mu p^\alpha p^\beta}{p^2}, \tag{8}$$

$$\begin{aligned}\int_k \frac{k^2 k^\alpha k^\beta}{k^4(k+p)^2} =& -\frac{1}{4}\eta^{\alpha\beta}p^2[I_{log}(\lambda^2) - \upsilon_0] + \frac{1}{6}(p^2\eta^{\alpha\beta} + 2p^\alpha p^\beta)[I_{log}(\lambda^2) - \xi_0] + \\ &+ \frac{1}{2}bp^2\eta^{\alpha\beta}\left[\frac{1}{6}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{4}{9}\right] - bp^\alpha p^\beta\left[\frac{1}{3}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{13}{18}\right],\end{aligned} \tag{9}$$

$$\begin{aligned}\int_k \frac{k^\mu k^\nu k^\alpha k^\beta}{k^4(k+p)^2} =& -\frac{1}{24}\eta^{\{\mu\nu}\eta^{\alpha\beta\}}p^2[I_{log}(\lambda^2) - \xi_0] + \frac{1}{8}bp^2\eta^{\{\mu\nu}\eta^{\alpha\beta\}}\left[\frac{1}{6}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{4}{9}\right] \\ &+ \frac{1}{48}(p^2\eta^{\{\alpha\beta}\eta^{\mu\nu\}} + p^{\{\alpha}p^\beta\eta^{\mu\nu\}})[I_{log}(\lambda^2) - \xi_0 - 24\sigma_0] \\ &- \frac{1}{72}bp^{\{\alpha}p^\beta\eta^{\mu\nu\}}\left[\frac{3}{2}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{5}{2}\right] + \frac{1}{4}\tilde{b}\frac{p^\alpha p^\beta p^\mu p^\nu}{p^2},\end{aligned} \tag{10}$$

$$\begin{aligned}\int_k \frac{k^2 k^\mu k^\alpha k^\beta}{k^4(k+p)^2} =& \frac{1}{6}p^{\{\mu}\eta^{\alpha\beta\}}p^2[I_{log}(\lambda^2) - \xi_0] - \frac{1}{4}bp^2p^{\{\mu}\eta^{\alpha\beta\}}\left[\frac{1}{6}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{4}{9}\right] \\ &- \frac{1}{8}\left(p^2p^{\{\mu}\eta^{\alpha\beta\}} + 2p^\alpha p^\beta p^\mu\right)[I_{log}(\lambda^2) - \xi_0 - 24\sigma_0] \\ &+ bp^\alpha p^\beta p^\mu\left[\frac{1}{4}\ln\left(-\frac{p^2}{\lambda^2}\right) - \frac{7}{12}\right],\end{aligned} \tag{11}$$

where the Lorentz indices between brackets stand for symmetrization of the tensor, i.e. $A^{\{\alpha_1\cdots\alpha_n}B^{\beta_1\cdots\beta_n\}} = A^{\alpha_1\cdots\alpha_n}B^{\beta_1\cdots\beta_n}$ + sum over permutations between the two sets of indices $\alpha_1\cdots\alpha_n$ and $\beta_1\cdots\beta_n$.

We also have

$$U_\mu = \int_k \frac{k_\mu}{k^4(k-p)^2} = b\frac{p_\mu}{p^2}, \tag{12}$$

$$U_\mu^{(2)} = \int_k \frac{k_\mu}{k^4(k-p)^2} \ln\left(-\frac{k^2}{\lambda^2}\right) = b\frac{p^\mu}{p^2} \ln\left(-\frac{p^2}{\lambda^2}\right), \tag{13}$$

$$U_{\mu\nu} = \int_k \frac{k_\mu k_\nu}{k^4(k-p)^2} = \frac{g_{\mu\nu}}{4}\left(I_{log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) + 2b\right) + \frac{b}{2}\frac{p_\mu p_\nu}{p^2}, \tag{14}$$

$$\begin{aligned} U_{\mu\nu}^{(2)} = \int_k \frac{k_\mu k_\nu}{k^4(k-p)^2} \ln\left(-\frac{k^2}{\lambda^2}\right) = \frac{g_{\mu\nu}}{8}\left[2I_{log}^{(2)}(\lambda^2) + I_{log}(\lambda^2) - b\ln^2\left(-\frac{p^2}{\lambda^2}\right) + \right. \\ \left. + b\ln\left(-\frac{p^2}{\lambda^2}\right) + b\right] + \frac{p_\mu p_\nu}{p^2}\left[\frac{b}{4} + \frac{b}{2}\ln\left(-\frac{p^2}{\lambda^2}\right)\right], \end{aligned} \tag{15}$$

$$I^{\mathcal{O}} \equiv \int_{q,k} \frac{1}{q^2(q+k)^2k^2(k+p)^2(q-p)^2} = \frac{6\zeta(3)b^2}{p^2}, \tag{16}$$

$$I_\mu^{\mathcal{O}} \equiv \int_{q,k} \frac{k_\mu}{q^2(q+k)^2k^2(k+p)^2(q-p)^2} = -\frac{p_\mu}{2}I^{\mathcal{O}}, \tag{17}$$

$$\bar{I}_\nu^{\mathcal{O}} \equiv \int_{q,k} \frac{q_\nu}{q^2(q+k)^2k^2(k+p)^2(q-p)^2} = \frac{p_\mu}{2}I^{\mathcal{O}}, \tag{18}$$

$$\begin{aligned} I_{\mu\nu}^{\mathcal{O}_2} \equiv \int_{q,k} \frac{k_\mu q_\nu}{q^2(q+k)^2k^2(k+p)^2(q-p)^2} = -g_{\mu\nu}\left[\frac{b}{4}I_{log}(\lambda^2) - \frac{b^2}{4}\ln\left(-\frac{p^2}{\lambda^2}\right)\right. \\ \left. -\frac{p^2}{12}I^{\mathcal{O}} + \frac{11}{12}b^2 - b^2\frac{\pi^2}{36}\right] - \frac{p_\mu p_\nu}{p^2}\left[\frac{p^2}{3}I^{\mathcal{O}} - \frac{1}{6}b^2 + b^2\frac{\pi^2}{36}\right] \end{aligned} \tag{19}$$

where $\lambda$ is the renormalization scale and $b \equiv \frac{i}{(4\pi)^2}$.

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