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Entanglement entropy production in the Dynamical Casimir Effect

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Resumo

Neste trabalho nós investigamos a produção de entropia de emaranhamento no efeito Casimir dinâmico em ressonância paramétrica. Técnicas simpléticas aplicadas em sistemas bosônicos com estados inicias Gaussianos, nos permitem relacionar a produção assintótica de entropia de emaranhamento com os expoentes de Lyapunov. Nós estudamos o caso onde uma fronteira está fixa enquanto a outra oscila senoidalmente. Através de cálculos numéricos nós encontramos que em 1+1 dimensões os coeficientes de Lyapunov são zero e a produção de entropia é sublinear. Em 2+1 dimensões, os coeficientes de Lyapunov são diferentes de zero, levando a uma produção linear da entropia

Palavras-chave: efeito Casimir dinâmico, entropia de emaranhamento, técnicas simpléticas, teoria quântica de campos

Abstract

In this work, we investigate the production of entanglement entropy in the dynamical Casimir effect at parametric resonance. Symplectic techniques for the description of the time evolution of Gaussian states in bosonic systems allow us to relate the asymptotic production of entanglement entropy to the classical Lyapunov exponents. We consider the case where one boundary is static, and the other is oscillating sinusoidally. We find trough numerical computations that in 1+1 dimensions the Lyapunov exponents are zero and the entropy production is sublinear. In the 2+1 dimensions case, the Lyapunov exponents are nonzero, leading to a linear production of entropy.

Keywords: dynamical Casimir effect, entanglement entropy, symplectic techniques, quantum field theory.

Contents

1 Introduction

Quantum field theory (QFT) is one of the most successful theories ever conceived. Its predictions and the related validation experiments express the remotest limit of the accumulated knowledge in human history. What is a particle? What is the vacuum? What are the most fundamental building blocks of matter? The questions related to the foundations of QFT are the same as of the foundations of nature itself, often assuming an almost philosophical character. Quantum field theory emerged as the unification of quantum theory and Einstein's special relativity. Despite the noble accomplishments, it was not possible until this day to combine quantum mechanics and general relativity.

Notwithstanding, numerous interest, and yet unexplainable systems where these two theories must necessarily be combined insist on existing. Among them, black holes are perhaps the most intriguing example. While general relativity is required to describe the geometry of spacetime, one expects quantum effects near the singularity. This puzzle became even more attractive in 1974 when Stephen Hawking noted that QFT in curved spacetime predicts the creation of particles by black holes [1]. This discovery gave birth to a full field of research in the intersection between quantum field theory, general relativity, and thermodynamics.

Unfortunately (or not), there is a sophisticated mathematical structure in these theories, which make it hard to compute interesting observables in non-trivial scenarios. For this reason, several analog systems were used to gain a partial comprehension and, in this class, we have the dynamical Casimir effect. The static Casimir effect was foretold in 1948 by Hendrick Casimir[2], and it consists of the theoretical prediction that two conducting, neutral plates would attract each other. This prediction was then confirmed experimentally [3]. Later on, a generalization of this outcome, which was named as the dynamical Casimir effect (DCE), predicted that if the mirror is allowed to move, it is possible to create particles from the vacuum, with its spectrum entirely determined by the mirror trajectory [4]. As the studies around this new prediction improved, it was noticed that this system could be managed to reproduce the black hole's particle creation, as long as the mirror trajectory is chosen carefully [5], [6].

The main interests in the DCE varied over time. In the beginning, the main concern was to understand how the mirror trajectory would change the Casimir force between the mirrors, and then the particle spectrum became the focus of the efforts [7]. In this work, we concentrate on a third interest: the entanglement entropy production. Besides, we investigate a particular situation, which is the DCE at parametric resonance. The direct detection of the particle creation in the DCE requires a highly sophisticated experimental setup since the demanded energy is significant, and the maximal velocity of the boundary that could be performed in a laboratory is minimal compared to the speed of light. In fact, only one group have claimed to experimentally verify the dynamical Casimir effect, where a superconducting quantum interference device was employed as a moving mirror [8]. Therefore, the analysis of the DCE at resonance has the further importance of being potentially detectable.

Under the resonance condition, we show that it is possible to create an instability and, as a consequence, there is an exponential production of particles. Indeed, the very concept of instability attracts physicists attention from centuries [9]. For example, in the nineteenth century, the physicists were particularly concerned about understanding the solar system stability. This problem can be summarized into the following question: if a dynamical system has a family F of solution curves which fill up the entire phase space, and then is perturbed to a slightly modified system F', then is F' close to F in some topological sense? We describe how the quantum instability of the DCE at parametric resonance can be related to the classical notion of unstable dynamical systems.

With this purpose, a geometrical approach is necessary. In chapter two, we introduce the fundamental mathematical structure of symplectic geometry and its role in the formulation of classical mechanics. Chapter three is dedicated to describing the canonical quantization process, which will be applied to quantum field theory in flat and in curved spacetime in chapter four where we also describe two remarkable effects: the Unruh effect and Hawking radiation. Finally, in chapter five we introduce and apply symplectic techniques to the DCE at parametric resonance.

2 Symplectic geometry and Classical Mechanics

This chapter intends to provide a brief introduction to symplectic geometry and its applications in classical mechanics. The mathematical structure here established, will be essential to future chapters. We begin by introducing the symplectic algebra in the first section. Next, we generalize to symplectic manifolds and finally apply the formalism to the geometrical formulation of Hamiltonian mechanics. The contents of this chapter are mainly based in [10], [11], and [12].

2.1 Symplectic Algebra

2.1.1 Skew-Symmetric Bilinear Maps

We begin by defining a skew-symmetric bilinear map. Then we particularize to a symplectic skew-symmetric bilinear map, which is used to define a symplectic vector space.

Definition 2.1. *Let V be an m-dimensional vector space, and $\Omega : V \times V \longrightarrow \mathbb{R}$ a bilinear map. If $\Omega(u,v) = -\Omega(v,u)$, for all $u, v \in V$, then Ω is* ***skew-symmetric****.*

Theorem 2.1. *Let Ω be a skew-symmetric bilinear map on V. There is a basis*

$$u_1, ..., u_k, e_1, ..., e_n, f_1 ... f_n \in V,$$

such that

$$\Omega(u_i, v) = 0, \; \forall i \text{ and } \forall v \in V,$$
$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \; \forall i, j, \text{ and}$$
$$\Omega(e_i, f_j) = \delta_{ij}, \; \forall i, j.$$

With respect to this basis, we have:

$$\Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbb{I} \\ 0 & -\mathbb{I} & 0 \end{bmatrix}.$$

Proof

The proof of this theorem follows a modified version of the Gram-Schmidt process. Define $U := \{u \in U | \Omega(u, v) = 0, \forall v \in V\}$. Choose a basis $u_1, ..., u_k$ in U. Let W be the symplectic complement of V, such that:

$$V = U \bigoplus W.$$

Take any $e_1 \in W$ different from zero. There is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Under scalar multiplication we can set $\Omega(e_1, f_1) = 1$. Let W_1 be the subspace spanned by e_1, f_1 and W_1^Ω is its symplectic complement in W:

$$W = W_1 \bigoplus W_1^\Omega.$$

Take any $e_2 \in W_1^\Omega$, $e_2 \neq 0$. There is $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) = 1$, and a subspace W_2 spanned by these vectors. This process continues until we reach the dimension of V. Then we have:

$$V = U \bigoplus W_1 \bigoplus W_2 \bigoplus ... \bigoplus W_n,$$

where W_i has basis e_i, f_i with $\Omega(e_i, f_i) = 1$. By this construction, the dimension of the vector space V is $\dim V = k + 2n = m$.

□

2.1.2 Symplectic vector spaces

Definition 2.2. *The linear map $\tilde{\Omega} : V \longrightarrow V^*$ is defined by*

$$\tilde{\Omega}(v)(u) = \Omega(u, v). \tag{2.1}$$

From this definition it is clear that the kernel of $\tilde{\Omega}$ is the subspace U defined above. If $U = \{0\}$, it constitutes a bijection, and therefore an isomorphism.

Definition 2.3. *A skew-symmetric bilinear map Ω is* ***symplectic*** *if $\tilde{\Omega}$ is bijective. The map Ω is then called a linear symplectic structure on V, and the pair (V, Ω) a symplectic vector space.*

By the theorem 2.1, this symplectic vector space has a basis e_i, f_i satisfying:

$$\Omega(e_i, f_i) = \delta_{ij} \text{ and } \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \tag{2.2}$$

which is called a **symplectic basis (or Darboux basis)** of (V, Ω). Since $\dim U = 0$, the dimension of V is **even** and equals to $2n$. With respect to a symplectic basis, the linear symplectic structure becomes:

$$\Omega = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}. \tag{2.3}$$

2.1.3 Symplectic morphisms and symplectic groups

Definition 2.4. *Let* (V_1, Ω_1) *and* (V_2, Ω_2) *be two symplectic vectors spaces and* $\phi : V_1 \longrightarrow V_2$ *a linear map. We call* ϕ *symplectic when:*

$$\Omega_2\left(\phi(u), \phi(v)\right) = \Omega_1(u, v), \quad \forall u, v \in V_1. \tag{2.4}$$

Since Ω is non-degenerate, this symplectic morphism is necessarily injective. For $\dim V_1 = \dim V_2$, ϕ is an isomorphism and is called *symplectomorphisms*. When $(V_1, \Omega_1) = (V_2, \Omega_2) = (V, \Omega)$, ϕ is an automorphism of (V, Ω). The set of all symplectic automorphism forms a group denoted $Sp(V)$, and called *symplectic group*. Its elements $M \in Sp(V)$ can be represented as matrices from $GL(v)$ if a basis in V is chosen.

From equation (2.3), if a Darboux basis is determined, we can write:

$$\Omega(u, v) = u^T \Omega v, \tag{2.5}$$

where $u^T = (e_1, ..., e_n, f_1, ..., f_n)$. The operator M preserves the symplectic structure, that is,

$$\Omega(Mu, Mv) = \Omega(u, v), \tag{2.6}$$

exactly when its matrix satisfies

$$M^T \Omega M = \Omega. \tag{2.7}$$

Taking the determinant of equation (2.7) gives:

$$\begin{aligned} det(M^T \Omega M) &= det(\Omega) = 1 \\ det(M^T M) &= 1 \\ det(M) &= \pm 1. \end{aligned} \tag{2.8}$$

With the use of the Pfaffian is possible to show [13] that the **determinant of M is 1**.

2.1.4 Complex structures of vector spaces

Consider the vector space $\mathbb{R}^{2n}$. With the identification

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \Longleftrightarrow z = x + iy,$$

we have the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Definition 2.5. *Let* V *be a vector space. A* ***complex structure*** *on* V *is a linear map:*

$$J : V \longrightarrow V \ \textit{with}\ J^2 = -\mathbb{I}.$$

The pair (V, J) *is denoted a* ***complex vector space****.*

A complex structure J is equivalent to a structure of vector space over $\mathbb{C}$ if we identify the map J with the multiplication by $\sqrt{-1}$:

$$\sqrt{-1}v := Jv.$$

From $Jv = \lambda v$ we deduce that:

$$\begin{aligned} J^2 v = \lambda J v = \lambda^2 v \\ \lambda = \pm\sqrt{-1}. \end{aligned} \tag{2.9}$$

If V is symplectic with the form Ω, we call the complex structure *compatible* with Ω if

$$\Omega(Ju, Jv) = \Omega(u, v)\ \forall u, v \in V.$$

In that case, defining

$$G(u, v) := \Omega(u, Jv)\ \forall u, v \in V,$$

we have

$$G(Ju, v) = \Omega(u, v).$$

From the skew-symmetry of Ω and $J^2 = -\mathbb{I}$ we have:

$$G(u, v) = G(v, u) \text{ and } G(Ju, Jv) = G(u, v).$$

Therefore G is a symmetric bilinear form. When $G(u, u) \geq 0$ for all $u \in V$, we call G a *hermitian metric*, J a positive compatible complex structure and the triple (V, Ω, J) a **Kähler vector space**.

2.2 Symplectic Manifolds

Definition 2.6. *Let M be a smooth manifold of dimension m. The pair (M, ω) is called a symplectic manifold if there is defined on M a closed non-degenerate 2-form ω such that:*

- *$d\omega = 0$ (closed),*
- *on each tangent space T_pM, with $p \in M$, $\omega_p(X, Y) = 0$, $\forall Y \in T_pM$, only if $X = 0$ (non-degenerate).*

From these conditions, each tangent space T_pM is a symplectic vector space with even dimension $m = 2n$.

Example 2.1. *Consider $\mathbb{R}^{2n}$ with coordinates $x_1, ..., x_n, y_1, ..., y_n$. It is easy to see that the following 2-form is symplectic:*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i, \tag{2.10}$$

and that the basis

$$\left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n} \right\}$$

is a symplectic basis. The 2-form ω_0 is called the ***standard symplectic form****.*

Theorem 2.2. ***(Darboux's theorem):*** *To every point p of a symplectic manifold (M, ω) of dimension $2n$, there correspond an open neighbordood U of p and a smooth map*

$$F : U \longrightarrow \mathbb{R}^{2n} \text{ with } F^*\omega_0 = \omega|_U,$$

where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$ and F^* is the pullback.

Darboux's theorem says that, up to a symplectomorphism F, all symplectic manifolds of the same dimension are *locally the same*, and therefore symplectic geometry is a global theory. This fact raises the question of whether there are global distinguishing features on symplectic manifolds, and the answer is still subject of current research. We now move to the analysis of one significant example of a symplectic manifold: the cotangent bundle.

2.2.1 The cotangent bundle

Let M be an n-dimensional differentiable manifold. At any point $p \in M$ we can construct the tangent space to p, denoted by T_pM, and defined as the set of all tangent vectors at p. The disjoint union of the vector spaces from all the points in the manifold is called the *tangent bundle* TM. Similarly, we can consider the cotangent vector space of a point p, which is the dual of T_pM, denoted as T_p^*M. The disjoint union of all dual vector spaces in the manifold is denominated **cotanget bundle** and denoted T^*M.

The components $\xi_1, ..., \xi_n$ of a tangent vector $\xi \in T_pM$ are the values of the differential $dq_1, ..., dq_n$ on the vector ξ. These n 1-forms are linearly independent and therefore form a basis for the space of 1-forms on T_pM. In this way, every 1-form can be written:

$$\nu = p_1 dq_1 + ... + p_n dq_n, \tag{2.11}$$

where $p_1, ..., p_n$ are the coefficients of the expansion. A point of T^*M is a 1-form on the tangent space to M at some point of M. Together, the 2n numbers $x_1, ..., x_n$ and $p_1, ..., p_n$ form a collection of local coordinates for points in T^*M. Let us define the 2-form ω on T^*M as

$$\omega := d\nu = \sum_i^n dp_i \wedge dq_i. \tag{2.12}$$

This 2-form is clearly closed and non-degenerate, which shows that T^*M **is a symplectic manifold**. If M is the configuration space of a classical system, then T^*M is the corresponding classical phase space!

2.3 Classical Mechanics

2.3.1 The fundamental duality and Hamiltonian vector fields

Definition 2.7. *To every tangent vector ξ to a symplectic manifold (M^{2n}, ω) at a point p, we associate a 1-form ν on T_p^*M by the formula:*

$$\nu_\xi(\eta) = \omega(\eta, \xi), \quad \forall \eta \in T_pM.$$

The map induced by the symplectic structure is precisely the one introduced in equation (2.1), with $kernel = \{0\}$, and therefore constitutes an isomorphism between the vector spaces of vectors and 1-forms, which is called **the fundamental duality**.

Let H be a function on a symplectic manifold M. Then dH is a differential 1-form on M, and the fundamental duality associates to every point in the manifold a vector. In this way we obtain a vector field in M, which is called the **Hamiltonian vector field**. If (q,p) are the local coordinates, then this vector field assumes the form:

$$X_H = \sum_i^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{2.13}$$

2.3.2 Integral curves and Hamilton's equations

Definition 2.8. *Let X be a vector field in a manifold M. An integral curve for X with initial condition p_0 is the map $\gamma : I \longrightarrow M$, such that*

$$\dot\gamma(t) = X_{\gamma(t)}\,, \quad \forall t \in I, \text{ and } \gamma(0) = p_0\,,$$

where I is an open interval containing 0.

Intuitively, the integral curve of a vector field X with initial condition p_0 is a curve on the manifold M passing through p_0 and such that, for every point $p = \gamma(t)$ on this curve, the tangent vector to this curve at p, i.e., $\dot\gamma(t)$, coincides with the value, X_p, of the vector field X at p. For the Hamiltonian vector field X_H, the condition just expressed leads to:

$$\begin{aligned} \dot{\gamma(t)} = X_H(\gamma(t)) &= \sum_i^n \frac{\partial H}{\partial p_i}\frac{\partial \gamma(t)}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial \gamma(t)}{\partial p_i} \\ \sum_i^n \frac{\partial \gamma(t)}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \gamma(t)}{\partial p_i}\frac{dp_i}{dt} &= \sum_i^n \frac{\partial H}{\partial p_i}\frac{\partial \gamma(t)}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial \gamma(t)}{\partial p_i}, \end{aligned} \tag{2.14}$$

and from this last equation we obtain **Hamilton's equations**:

$$\frac{\partial H}{\partial p_i} = \dot q_i, \quad \frac{\partial H}{\partial q_i} = -\dot p_i. \tag{2.15}$$

From the Hamilton's equations, we can write the evolution equation for any observable $\mathcal{O}$ as:

$$\dot{\mathcal{O}}(q,p) = \frac{\partial \mathcal{O}}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial \mathcal{O}}{\partial p}\frac{\partial H}{\partial q} = \{\mathcal{O}, H\}, \tag{2.16}$$

where $\{\}$ is the *Poisson bracket*, defined as:

$$\{\mathcal{O}, \mathcal{P}\} = \frac{\partial \mathcal{O}}{\partial q}\frac{\partial \mathcal{P}}{\partial p} + \frac{\partial \mathcal{O}}{\partial p}\frac{\partial \mathcal{P}}{\partial q}, \tag{2.17}$$

for any observables $\mathcal{O}, \mathcal{P}$.

3 Canonical Quantization

In this chapter, we analyze the crucial subject of the canonical quantization process. For that purpose, the most natural approach is that of groups, algebras, and its representations. Group theory often appears as transformation groups, in the sense that they act as transformations of some particular object, and offers numerous applications in physics, ranging from uses in molecular spectroscopy to elementary particles. As an example, consider the group of rotations in the hydrogen atom. From the Hamiltonian, it is easy to see that the energy levels do not depend on any angles (since it describes the electron in a central potential $1/r$), and therefore the group of rotations constitutes a symmetry group and is called $SO(3)$. The study of symmetries in physical systems represents one of the main applications of group theory. In the given example, the transformations depend on a continuous parameter, namely, the rotated angle. This group is actually of a particular kind: the group of continuous transformations denoted Lie groups, which possess great importance in physics. In fact, at the beginning of quantum mechanics, group theory began to play such an influential role, that some reluctant physicists denoted the situation as "Gruppenpest" (group plague).

Our primary objective is to introduce only the necessary elements to construct precisely the canonical quantization process. Hence, we begin with a brief introduction to representation theory and Lie groups and algebras. Next, we focus on a particular kind of Lie algebra: the Heisenberg algebra, which lies at the core of quantum mechanics formulation. In the last section, we describe the canonical quantization formalism, as well as its accomplishments and inherent limitations. The contents of this chapter were mainly based on [14] and [15].

3.1 Groups and representations

Let us start by defining a group G, and its action of on a set M.

Definition 3.1. *A group G is a set with an associative multiplication, such that the set contains an identity element, and the multiplicative inverse of each element.*

Definition 3.2. *The action of a group G on a set M is the map*

$$(g, x) \in G \times M \longrightarrow g.x \in M,$$

that takes the ordered pair (g, x) of a group element $g \in G$ and an element $x \in M$ to another element $g.x \in M$ such that

$$g_1(g_2.x) = (g_1 g_2)x \tag{3.1}$$

and

$$e.x = x \tag{3.2}$$

where e is the identity element of G.

Example 3.1. *Consider the group $GL(2,\mathbb{R})$ of 2×2 invertible matrices with real entries, and the set of 2×1 column matrices. The action map is just the usual matrix multiplication. It is easy to see that conditions* (3.1) *and* (3.2) *are satisfied, where e is the identity matrix.*

Definition 3.3. *A representation (π, V) of a group G on a vector space V is a homomorphism*

$$\pi : g \in G \longrightarrow \pi(g) \in GL(V),$$

where $GL(V)$ is the group of invertible linear maps on V.

If V is finite dimensional, with dimension n, then there is an isomorphism between $GL(V)$ and the group of $n \times n$ invertible matrices $(GL(n,\mathbb{C}))$:

$$GL(V) \simeq GL(n,\mathbb{C}).$$

In quantum mechanics, we will often be interested in the case where our representation is unitary since the notion of probability is intrinsically connected with unitarity. Therefore we define:

Definition 3.4. *A representation (π, V) on a complex vector space V with Hermitian inner product $\langle , \rangle$ is unitary if it preserves the inner product, that is,*

$$\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle,$$

for al $g \in G$ and $v_1, v_2 \in V$.

The unitary matrices $\pi(g)$ belongs to a subgroup of $GL(n,\mathbb{C})$ denoted $U(n)$. It can be shown that a matrix M belongs to $U(n)$ if and only if

$$M^{-1} = M^{\dagger}, \tag{3.3}$$

where † is the conjugate transpose. The following definitions will be useful when we apply the theory to quantum mechanics, further in this chapter.

Definition 3.5. *A representation (π, V) is denoted irreducible if it has no sub-representations, that is, there is no $W \subset V$ such that $(\pi|_W, W)$ is a representation. Otherwise, the representation is called reducible.*

Definition 3.6. *Given representations π_1 and π_2 of dimensions n_1 and n_2, there is a representation of dimension $n_1 + n_2$ called the direct sum and denoted $\pi_1 \bigoplus \pi_2$, which is given by the homomorphism*

$$\left(\pi_1 \bigoplus \pi_2\right) : g \in G \longrightarrow \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}$$

Theorem 3.1. *Any unitary representation π can be writen as a direct sum*

$$\pi = \pi_1 \bigoplus \pi_2 \bigoplus \ldots \bigoplus \pi_n$$

where the π_j are irreducible.

3.2 Lie Groups and Algebras

Definition 3.7. *A Lie group is a smooth manifold equipped with a product satisfying the group properties and the additional condition that the group operations are differentiable.*

This definition is connected with the fifth of Hilbert's problems, which asks if the differentiability condition can be avoided. We could then consider the more general case of topological manifolds and continuous maps. However, the solution to this problem demonstrated that this generalization gives nothing new [14]. A very relevant class of Lie groups is formed by those which are subgroups of $GL(n)$. These are called **matrix Lie groups** and will be the main subject of the rest of the chapter. We now define the Lie algebras.

Definition 3.8. *A finite-dimensional real or complex Lie algebra is a finite-dimensional real or complex vector space $\mathfrak{g}$, together with a map $[,]$ (called **Lie bracket**) from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$, with the properties:*

1. *$[,]$ is bilinear.*
2. *$[,]$ is skew symmetric*

$$[X, Y] = -[Y, X], \quad \forall X, Y \in \mathfrak{g}.$$

3. *The Jacobi identity holds:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The above abstract definition is general. Our interest, however, will be in the particular case where the elements of $\mathfrak{g}$ are matrices. Therefore we can give a second, more concrete definition of Lie algebra, particular for this sub-algebra.

Definition 3.9. *For G a matrix Lie group of $n \times n$ invertible matrices, the Lie algebra of G, denoted $\mathfrak{g}$, is the space of $n \times n$ matrices X such that $e^{tX} \in G$ for $t \in \mathbb{R}$.*

In this case, the abstract Lie bracket is given by the commutator of matrices:

$$[,] : (X, Y) \in \mathfrak{g} \times \mathfrak{g} \longrightarrow [X, Y] = XY - YX \in \mathfrak{g}. \tag{3.4}$$

We will show that the elements of $\mathfrak{g}$, as defined above, satisfy the necessary conditions to be Lie algebra elements. However, before this demonstration, we would like to make explicit the geometrical relation between Lie groups and algebras. This relation is made clear by the following theorem.

Theorem 3.2. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then a matrix X is in $\mathfrak{g}$ if and only if there exists a smooth curve $\gamma(t) \in G$ for all t and such that $\gamma(0) = \mathbb{I}$ and*

$$\left.\frac{d\gamma(t)}{dt}\right|_{t=0} = X.$$

Thus, $\mathfrak{g}$ ***is the tangent space at the identity of*** G

<u>*Proof*</u>

Suppose $\gamma(t)$ is a smooth curve in G, with $\gamma(0) = \mathbb{I}$. It can be shown that, for all sufficiently small t, *log* $(\gamma(t)) \in \mathfrak{g}$ (see [15] for a proof of this statement). Consequently,

$$\frac{d\log(\gamma(t))}{dt} \in \mathfrak{g}$$

as well. We can write:

$$\log(\gamma(t)) = (\gamma(t) - \mathbb{I}) - \frac{(\gamma(t) - \mathbb{I})^2}{2} + ...$$

It is easy to see that, differentiating term by term with respect to t, all terms but the first will give zero at $t = 0$. Thus, we obtain that:

$$\left.\frac{d\log(\gamma(t))}{dt}\right|_{t=0} = \left.\frac{d\gamma(t)}{dt}\right|_{t=0} = X \in \mathfrak{g}.$$

□

Example 3.2. *In the last chapter, we saw that the classical phase space, which is the cotangent bundle of a configuration space, is a symplectic manifold, denoted T^*M. Let us call the set of smooth functions on T^*M as $C^\infty(T^*M)$. Then, the Poisson bracket is a map:*

$$\{,\} : C^\infty(T^*M) \times C^\infty(T^*M) \longrightarrow C^\infty(T^*M), \tag{3.5}$$

which satisfies the conditions (1)-(3) of the Lie algebra definition 3.8. The set $C^\infty(T^*M)$ corresponds to the space of classical observables in classical mechanics.

We defined the representation of a grup G as a homomorphism between G and $GL(n, \mathbb{C})$, the group of invertible $n \times n$ matrices. In a similar way, we can define a Lie algebra representation as follows.

Definition 3.10. *A complex Lie algebra representation* (ϕ, V) *of a Lie algebra* $\mathfrak{g}$ *on an* n*-dimensional complex vector space* V *is given by a real linear map*

$$\phi : X \in \mathfrak{g} \longrightarrow \phi(X) = M(n, \mathbb{C}),$$

satisfying

$$\phi\left([X, Y]\right) = \left[\phi(X), \phi(Y)\right],$$

and where $M(n, \mathbb{C})$ *is the space of* $n \times n$ *complex matrices.*

Therefore, if we choose a basis $X_1, ...X_n$ in a Lie algebra $\mathfrak{g}$ of dimension n, a representation is given by a choice of matrices $\phi(X_i)$ that satisfy the Lie bracket:

$$[\phi(X_i), \phi(X_j)] = \sum_{k=1}^{n} f_{ij}^{k} \phi(X_k), \tag{3.6}$$

where the f_{ij}^k are called *structure constants.*

Returning to definition 3.9, we will now show that the elements $X \in \mathfrak{g}$ obey the Lie brackets relations, and indeed $\mathfrak{g}$ constitutes a Lie algebra. The elements of the algebra $\mathfrak{g}$ can be exponentiated to generate the elements $g, h \in G$:

$$g = e^{\alpha^a X_a} \quad , h = e^{\beta^b X_b}, \tag{3.7}$$

where $\alpha, \beta \in \mathbb{R}$ are arbitrary parameters and $X_1...X_n$ are the elements of a basis in $\mathfrak{g}$. For sufficiently small α and β, the elements $g, h \in G$ can also be written as an exponential:

$$g.h = e^{\gamma^a X_a} = e^{\alpha^a X_a} e^{\beta^b X_b} \tag{3.8}$$

Taking the log, and adding and subtracting 1 inside it:

$$\gamma^a X_a = \ln\left[1 + e^{\alpha^a X_a} e^{\beta^b X_b} - 1\right] \equiv \ln\left[1 + Z\right], \tag{3.9}$$

where

$$\begin{aligned} Z \equiv e^{\alpha^a X_a} e^{\beta^b X_b} - 1 &\simeq \left(1 + \alpha^a X_a + \frac{(\alpha^a X_a)^2}{2}\right)\left(1 + \beta^b X_b + \frac{(\beta^b X_b)^2}{2}\right) - 1 \\ &\simeq \alpha^a X_a + \beta^b X_b + \alpha^a X_a \beta^b X_b + \frac{(\alpha^a X_a)^2}{2} + \frac{(\beta^b X_b)^2}{2} + ... \end{aligned} \tag{3.10}$$

Using that $\ln[1 + Z] \simeq Z - \dfrac{Z^2}{2} + ...$, we have:

$$\begin{aligned} \gamma^a X_a &\simeq \alpha^a X_a + \beta^b X_b + \alpha^a X_a \beta^b X_b + \frac{(\alpha^a X_a)^2}{2} + \frac{(\beta^b X_b)^2}{2} + \\ &- \left\{\frac{(\alpha^a X_a)^2}{2} + \frac{(\beta^b X_b)^2}{2} + \frac{\alpha^a X_a \beta^b X_b + \beta^b X_b \alpha^a X_a}{2}\right\} \\ &\simeq \alpha^a X_a + \beta^b X_b + \frac{\alpha^a X_a \beta^b X_b + \beta^b X_b \alpha^a X_a}{2}. \end{aligned} \tag{3.11}$$

Define $-2(\gamma^c - \alpha^c - \beta^c) \equiv \alpha^a \beta^b f^c_{ab}$. Thus, substituting in equation (3.11), we obtain:

$$[X_a, X_b] = f^c_{ab} X_c, \tag{3.12}$$

which is precisely a Lie algebra.

One significant example of Lie algebra is the Heisenberg algebra, which is the subject of our next section. This algebra, together with its *Schrödinger representation* contains the foundations of quantum mechanics.

3.3 The Heisenberg Algebra and the Schrödinger representation

Definition 3.11. *The Heisenberg algebra* $\mathfrak{h}_{2d+1}$ *is the vector space* $\mathbb{R}^{2d+1} = \mathbb{R}^{2d} \bigoplus \mathbb{R}$ *with the Lie bracket defined by its values on a basis*

$$X_i, Y_i, Z \text{ with } i = 1...d$$

by

$$[X_i, Y_j] = \delta_{ij} Z,\ [X_i, Z] = [Y_i, Z] = 0.$$

This is precisely the algebra satisfied by the operators Q_i and P_i in quantum mechanics, with the commutator as the Lie bracket and Z as the identity. After Jordan and Born introduced what is now known as the canonical commutation relations [16], Weyl was the first to recognize them as the relations of a Lie algebra.

We will be mainly interested in finding an infinite dimensional representation in $L^2(\mathbb{R})$, which is the space of square integrable functions, so we define:

Definition 3.12. *The Schrödinger representation of the Heisenberg algebra* $\mathfrak{h}_3$ *is the representation* $(\Gamma, L^2(\mathbb{R}))$ *satisfying*

1. $\Gamma(X)\Psi(q) = -iQ\Psi(q) = -iq\Psi(q)$.
2. $\Gamma(Y)\Psi(q) = -iP\Psi(q) = \dfrac{-d\Psi(q)}{dq}$.
3. $\Gamma(Z)\Psi(q) = -i\Psi(q)$.

3.4 Canonical Quantization

In chapter 2 we saw that the dynamical equations of any physical system could be written in terms of the Poisson brackets:

$$\{f, H\} = \frac{df}{dt}, \tag{3.13}$$

where f is some dynamical variable and H is the Hamiltonian. In addition we have the relations:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \text{ and } \{q_i, p_j\} = \delta_{ij}. \tag{3.14}$$

Similarly, in quantum mechanics, the dynamics of any variable O in the Heisenberg picture is:

$$\frac{dO(t)}{dt} = -i[O, H], \tag{3.15}$$

with

$$[Q_i, Q_j] = [P_i, P_j] = 0 \text{ and } [Q_i, P_j] = i\hbar\delta_{ij} \tag{3.16}$$

Following the definition of the last section, we see that equations (3.14) and (3.16) are nothing but representations of the same Heisenberg algebra, namely, the three dimensional Lie algebra of linear functions on phase space. This astonishing similarity raises the question of whether we can map classical observables into quantum observables in a systematic way. The first to propose such construction was Dirac [17], in the process now known as the canonical quantization.

In terms of representation theory, all the canonical quantization process relies only upon one step: the search for a unitary representation $(\pi', \mathbb{H})$ of the infinite dimensional Lie algebra of functions on phase space. The requirement that the representation must be unitary is related to the probabilistic notion in quantum mechanics since these are the transformations that preserve the inner product in the Hilbert space, which is postulated to give the probability amplitude. We now show that, only with the unitarity condition, we obtain self-adjoint operators.

Proposition 3.1. *Suppose G is a matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathbb{H}$ is a inner product space, Π is a representation of G acting on $\mathbb{H}$, and π is the associated representation of $\mathfrak{g}$. If Π is unitay, then $\pi(X)$ is skew self-adjoint for all $X \in \mathfrak{g}$.*

<u>*Proof*</u>

If Π is unitary, then for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$:

$$\left(e^{t\pi(X)}\right)^{\dagger} = \Pi\left(e^{tX}\right)^{\dagger} = \Pi\left(e^{tX}\right)^{-1} = e^{-t\pi(X)} \tag{3.17}$$

$$e^{t\pi(X)^{\dagger}} = e^{-t\pi(X)}. \tag{3.18}$$

Differentiating with respect to t at $t = 0$ gives

$$\pi(X) = -\pi(X)^{\dagger} \tag{3.19}$$

□

From a skew self-adjoint operator, we can easily obtain a self-adjoint operator, simply by multiplying by i. Let $\pi(f) = O_f$ be the self-adjoint representation operator of a function f on phase space, and $\pi'(f) = -iO_f$ be the skew self-adjoint representation. By the Lie algebra homomorphism property:

$$\pi'\left(\{f, g\}\right) = [\pi'(f), \pi'(g)] \tag{3.20}$$

we have

$$-iO_{\{f,g\}} = [-iO_f, -iO_g] = -\left[O_f, O_g\right] \tag{3.21}$$

$$O_{\{f,g\}} = -i\left[O_f, O_g\right]. \tag{3.22}$$

Therefore, we found a way to quantize a classical system by associating a function f on phase space with a self-adjoint operator O_f. The Schrödinger representation, as described in the previous section, provides a unitary representation of polynomial functions on phase space of degree at most one, by the rule:

$$O_1 = \mathbb{I}, O_q = Q, O_p = P, \tag{3.23}$$

and

$$\Gamma(1) = -i\mathbb{I}.\Gamma(q) = -iQ = -iq, \Gamma(p) = -iP = -\frac{d}{dq}. \tag{3.24}$$

Quadratic polynomials can also be quantized, as follows:

$$O_{\frac{p^2}{2}} = \frac{P^2}{2}, O_{\frac{q^2}{2}} = \frac{Q^2}{2}, O_{pq} = \frac{1}{2}\left(PQ + QP\right), \tag{3.25}$$

and

$$\Gamma\left(\frac{q^2}{2}\right) = -i\frac{Q^2}{2} = -i\frac{q^2}{2} \tag{3.26}$$

$$\Gamma\left(\frac{p^2}{2}\right) = -i\frac{P^2}{2} = \frac{i}{2}\frac{d^2}{dq^2} \tag{3.27}$$

$$\Gamma(pq) = -\frac{i}{2}\left(PQ + QP\right). \tag{3.28}$$

In consequence, the Schrödinger representation can be extended to all quadratic polynomials in phase space. However, a special care must be taken. For the function pq, for example, we can not write:

$$O_{pq} = PQ, \tag{3.29}$$

since PQ do not commute and is not self-adjoint. Nevertheless, as defined in (3.25) we have a self-adjoint combination.

At this point, one may ask: is it possible to define a different representation? Different representations result in different predictions of the theory? The answer to the first question is straightforward. We could, for example, define a function f which is a linear combination of q and p, in such a way that the Heisenberg algebra was preserved. The second question, though, is much more profound and the answer is enclosed in the following theorem, due to Stone and von Neumann (see [14]):

Theorem 3.3. ***(Stone-von Neumann's theorem)*** *Any irreducible representation Π of the Heisenberg group H_3 (with associated algebra $\mathfrak{h}_3$) on a Hilbert space, satisfying*

$$\Pi(Z) = -i\mathbb{I}$$

is unitarily equivalent to the Schrödinger representation.

This surprising theorem states that, up to unitary equivalence, all irreducible representations are the same. Notwithstanding, the situation drastically changes if we consider systems with infinitely many degrees of freedom, as in quantum field theory: the Stone-von Neumann theorem is not applicable and there exist uncountably many inequivalent representations [18]. This crucial point will be especially important when we consider curved space-time backgrounds, as will be seen in the next chapter.

Another fundamental theorem related to the quantization process is due to Gronevald and van Hove and is a no go theorem:

Theorem 3.4. ***(Gronevald-van Hove's theorem)*** *There is no map $f \longrightarrow O_f$ from polynomials on $\mathbb{R}^2$ to self-adjoint operators in $L^2(\mathbb{R})$ satisfying*

$$O_{\{f,g\}} = -i\,[O_f, O_g] \tag{3.30}$$

and

$$O_p = P, O_q = Q \tag{3.31}$$

for any Lie subalgebra of the functions on $\mathbb{R}^2$ for which the subalgebra of polynomials of degree less than of equal to two is a proper subalgebra, in the sense that satisfies (3.30) *and* (3.31)*.*

The above theorem asserts that it is not possible to quantize generic polynomials of degree higher than two. This result has severe implications in quantum field theory: we can only rigorously quantize free field theories in the canonical quantization approach.

4 Quantum field theory in curved spacetime

In quantum field theory in curved spacetime, we treat a quantum matter field propagating in curved, classical spacetime: this is a first approximation to a full quantum theory of matter interacting with the gravitational field. A complete theory would have to quantize spacetime itself and investigate the effects of coupling the gravitational field with a matter field. Nevertheless, we expect this first treatment to be valid in some regime. Of course, the range of validity will only be determined when the full theory is available, but we expect this approach to fail when the curvature is comparable to the Planck length($10^{-33}cm^2$). To see this, consider a highly localized particle of mass m, such that the fluctuations of the momentum are big, and can be writen as:

$$\Delta p \sim \frac{\hbar}{\Delta x}. \tag{4.1}$$

Then, from the energy/mass relation, we have:

$$mc \sim \frac{\hbar}{\Delta x} \tag{4.2}$$

$$\Delta x \sim \frac{\hbar}{mc} \tag{4.3}$$

In general relativity, we expect that quantum effects became important when the length scale, which we denote as l_p, is close to the Schwarzschild radius r_s:

$$l_p \sim r_s = \frac{2Gm}{c^2}. \tag{4.4}$$

Then, if we demand that $\Delta x = r_s$, we have that:

$$m^2 = \frac{c\hbar}{G}, \tag{4.5}$$

and substituting back into (4.4), we obtain the corresponding length:

$$l_p = \sqrt{\frac{G\hbar}{c^3}}, \tag{4.6}$$

which is precisely the Planck length.

Quantum field theory in curved spacetime has already accomplished many theoretical predictions and has guided us to great insights about fundamental questions in physics. Historically, the first study in this area was conducted by Parker [19] in 1966 with the aim of comprehending particle creation in the early Universe due to fast expansion. In 1974, with Hawking's great discovery [1], quantum field theory received enormous attention.

In this chapter, we apply the canonical quantization process to a real, scalar field and probe two notable examples: the Unruh and Hawking effects. The general procedure is the same as of the last chapter, with the difference that now we have infinite degrees of freedom. As will be shown, this feature will imply severe consequences, since the Stone-von Neumann theorem is no longer applicable. Furthermore, in general, curved spacetimes the outcomes are even more drastic, with the effect that we will only be able to define particles in some particular cases. The formulation presented in this chapter was originally developed in [20].

4.1 Quantum Field Theory in Flat Spacetime

The action of the Klein Gordon field ϕ in flat spacetime is:

$$S = -\frac{1}{2}\int d^4x \left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right). \tag{4.7}$$

With this action we derive the Klein Gordon equation, which is the classical equation of motion:

$$\partial_\mu\partial^\mu\phi - m^2\phi = 0. \tag{4.8}$$

We can foliate the Minkowski spacetime (see next section for a precise definition of this statement) by defining a set of spatial hypersurfaces Σ_t, each corresponding to a particular value of a global time t. Therefore, Σ_0 corresponds to the hypersurface at $t = 0$. We designate the phase space $\mathcal{M}$ as the set of pairs $\{\phi, \pi\}$, where ϕ and π are smooth functions of compact support on Σ_0, that is,

$$\mathcal{M} = \{\{\phi,\pi\}|\phi : \Sigma_0 \longrightarrow \mathbb{R}, \pi : \Sigma_0 \longrightarrow \mathbb{R}; \phi,\pi \in C_0^\infty(\Sigma_0)\}. \tag{4.9}$$

Thus, $\mathcal{M}$ is the set of initial data of the Klein Gordon equation. Consequently, every point of $\mathcal{M}$ uniquely determines a solution, and we define $\mathcal{S}$ to be the space of solutions which arise from initial data on $\mathcal{M}$. As usual, we can endow $\mathcal{M}$ with a symplectic structure:

$$\omega : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}, \tag{4.10}$$

$$\omega(\{\phi_1,\pi_1\},\{\phi_2,\pi_2\}) = \int_{\Sigma_0} (\pi_1\phi_2 - \pi_2\phi_1) d^3x. \tag{4.11}$$

With the symplectic structure, we can associate a function with each $\{\phi, \pi\}$:

$$\omega(\{\phi,\pi\}, .) : \mathcal{M} \longrightarrow \mathbb{R}, \tag{4.12}$$

in terms of which the Poisson brackets can be expressed in a basis independent manner as:

$$\{\omega(\{\phi_1,\pi_1\}, .), \omega(\{\phi_2,\pi_2\}, .)\} = -\omega(\{\phi_1,\pi_1\},\{\phi_2,\pi_2\}). \tag{4.13}$$

Since there is a direct correspondence between $\mathcal{M}$ and $\mathcal{S}$, the space of solutions is also endowed with a symplectic structure. Our goal is to find a representation of the Lie algebra (4.13) in terms of operators $\hat{\omega}(\psi, .)$ in a Hilbert space $\mathbb{H}$, for all $\psi \in \mathcal{S}$. The operators $\hat{\omega}(\psi, .)$ are in general linear combinations of P and Q. The next step is to construct this Hilbert space, departing from the space of solutions. The usual procedure is to complexify $\mathcal{S}$ and choose the subspace $\mathcal{S}^+$ of positive frequency solutions from the plane wave expansion. Then, using the symplectic form, we establish a positive definite inner product in $\mathcal{S}^+$, which then can be turned into a Hilbert space $\mathbb{H}$. However, we will not follow this scheme. In a general, curved, spacetime, it is not always possible to expand the solutions in terms of plane waves, and the positive frequency subspace is not achievable by this method. Therefore, we will pursue a more general approach, which can be generalized to curved spacetimes, and it goes as follows.

We first construct a Hilbert space $\mathcal{S}_\mu^{\mathbb{C}}$ that is the complexification of a completion of the space of solutions S under some metric μ. Then, we look for a subspace $\mathbb{H} \subset \mathcal{S}_\mu^{\mathbb{C}}$ which satisfies:

1. The extension of the inner product μ to $\mathcal{S}_\mu^{\mathbb{C}}$ is positive definite for all $\psi_1, \psi_2 \in \mathbb{H}$.
2. $\mathcal{S}_\mu^{\mathbb{C}}$ is equal to the span of $\mathbb{H}$ and its complex conjugate $\overline{\mathbb{H}}$, that is
$$\mathcal{S}_\mu^{\mathbb{C}} = \mathbb{H} \bigoplus \overline{\mathbb{H}}. \tag{4.14}$$
3. For all $\psi^+ \in \mathbb{H}$ and $\psi^- \in \overline{\mathbb{H}}$, we have
$$\mu(\psi^+, \psi^-) = 0. \tag{4.15}$$

Therefore, if we change μ we have different decompositions of $\mathcal{S}_\mu^{\mathbb{C}}$. The freedom in the choice of $\mathbb{H}$ can be encoded in the choice of a complex structure J. Let us define $J : \mathcal{S}_\mu \longrightarrow \mathcal{S}_\mu$ to be the complex structure for which $-\omega(\psi_1, J\psi_2)$ is positive definite, and

$$\omega(\psi_1, \psi_2) = \mu(\psi_1, J\psi_2). \tag{4.16}$$

In addition, we require that μ be such that J is norm preserving. Then, we extend the action of ω and J to $\mathcal{S}_\mu^{\mathbb{C}}$. In section 2.1.4 we saw that J has eigenvalues $\lambda = \pm i$, and therefore $\mathcal{S}_\mu^{\mathbb{C}}$ decomposes into two orthogonal eigensubspaces. Finally, we define our Hilbert space to be $\mathbb{H} \subset \mathcal{S}_\mu^{\mathbb{C}}$, the eigensubspace with $\lambda = +i$, and it follows directly that $\mathbb{H}$ satisfies conditions 1 to 3. From $\mathbb{H}$, the construction of the space of the quantum field theory is unequivocal: we take it to be the symmetric Fock space $\mathcal{F}_s(\mathbb{H})$, defined by

$$\mathcal{F}_s(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \left(\bigotimes_s^n \mathbb{H} \right), \tag{4.17}$$

where $\bigotimes_s^n$ is the symmetric tensor product(since we are considering a bosonic system) with $\bigotimes_s^0 = \mathbb{C}$. The only remaining step is to write the operators $\omega(\hat{\psi},)$ that are associated to functions $\omega(\psi,)$ in phase space. For every $\psi \in \mathcal{S}^{\mathbb{C}}$, we define the operator $\omega(\hat{\psi},)$ on $\mathcal{F}_s(\mathbb{H})$ to be:

$$\omega(\hat{\psi},) = ia(\overline{K\psi}) - ia^\dagger(k\psi) \tag{4.18}$$

where $a(\overline{k\psi})$ is the anihilation operator related to $K\psi \in \mathbb{H}$, and K is the projector:

$$K : \mathcal{S}_\mu^{\mathbb{C}} \longrightarrow \mathbb{H}. \tag{4.19}$$

In summary, we have found a representation of the Lie algebra (4.13) in terms of fundamental observables $\omega(\hat{\psi},)$ on a Fock space $\mathcal{F}_s(\mathbb{H})$. Notwithstanding, we should highlight that there remains an arbitrarity on our construction: the choice of the complex structure. In our construction, the choice of J was fundamental in the development of $\mathbb{H}$. Thus, different choices of J leads to different Fock space representations. If the system under consideration were finite dimensional, this would make no difference, since the Stone-von Neumann theorem guarantees that they are unitarily equivalent. However, in quantum field theory, this theorem is not applicable, and we have infinitely many inequivalent representations. Since the notion of particles is intrinsically related to the Fock space choice, we have infinite different notions of particles.

Nonetheless, among all representations, we have a natural, preferred choice, namely the one that is invariant under the Poincaré group. The Poincaré group is the group of symmetries in the Minkowski spacetime and is natural to ask the Hilbert space to be invariant under this group. It is possible to show that, with the canonical choice of complex structure, the corresponding Fock space is invariant under the Poincaré group. In a general, curved spacetime, this concept is much more restricted, since Poincaré invariance is lost. There are, however, some cases where the particle concept still holds. One of the most critical examples is the following: we start with a disperse distribution of matter so that the Minkowski metric is a good approximation. Then, the matter content collapses and forms a black hole. In the asymptotic future, a distant observer can also meaningfully define particles, and the unitary transformation that connects these two representations generates the Hawking radiation.

4.2 Quantum Field Theory in Curved Spacetime

In this section, we provide the formulation of quantum field theory in curved spacetime. In the previous part, we already introduced the majority of concepts and the required mathematical structures, and this section will be only an adaptation to the general scenario. We will follow the same steps of the flat spacetime quantization, emphasizing the essential distinctions.

We start by writing the action of a Klein Gordon field in curved spacetime, which is the same as before, except that now we replace the Minkowski metric by a general one, and the covariant derivative replaces the partial derivative:

$$S = -\frac{1}{2}\int (\nabla_a \nabla^a \phi + m^2\phi^2)\sqrt{-g}d^4x, \tag{4.20}$$

and the Klein Gordon equation:

$$\nabla^a \nabla_a \phi - m^2 \phi = 0. \tag{4.21}$$

The first important difference is that the existence and uniqueness of solutions of equation (4.21) are not guaranteed in general. We will restrict our treatment to the class of spacetimes where existence and uniqueness are warranted, and fortunately, there is a simple condition for this. The first requirement is that our spacetime (M, g_{ab}) must be time-orientable, that is, we need to be able to distinguish past and future. To make precise our construction, we perform the following definitions.

Definition 4.1. *Let $\lambda(t)$ be a future directed causal curve. The curve $\lambda(t)$ is said to be future inextendible if it has no future endpoints, that is, there is no $p \in M$ such that for every neighborhood O of p there exists t_0, such that $\lambda(t) \in O$ for all $t > t_0$. Similarly, a past inextendible curve is the one with no endpoints in the past.*

Definition 4.2. *Let $\Sigma \subset M$ be any closed set with no pair of points $p, q \in \Sigma$ that can be joined by a causal curve. The domain of dependence $D(\Sigma)$ is defined by*

$$D(\Sigma) = \{p \in M | \textit{every past and future inextendible causal curve through } p \textit{ intersects } \Sigma\}.$$

If $D(\Sigma) = M$, then Σ is called a Cauchy surface, and (M, g_{ab}) is said to be globally hyperbolic.

Intuitively, $D(\Sigma)$ is the collection of all events that can be causaly conectedd with events in Σ. The coming theorems are of fundamental importance for the initial value formulation of quantum field theory in curved spacetime.

Theorem 4.1. *If (M, g_{ab}) has a Cauchy surface Σ then M has topology $\mathbb{R} \times \Sigma$ and can be foliated by a one parameter family of smooth Cauchy surfaces Σ_t.*

Theorem 4.2. *Let (M, g_{ab}) be a globally hyperbolic spacetime with Cauchy surface Σ. Then the Klein Gordon equation has a well posed initial value formulation. Given any pair of smooth functions $(\phi_0, \dot{\phi}_0)$ on Σ, there exists a unique solution to the Klein Gordon equation, defined on all M.*

In essence, these two theorems assure that for a globally hyperbolic spacetime we can find a deterministic evolution of the Klein Gordon equation from initial conditions on Σ_0. Analogously to the flat spacetime case, we define the classical phase space M to be the set of points $\{\phi, \pi\}$, where ϕ and π are smooth and of compact support functions on the Cauchy surface Σ_0. By the theorems above, to every pair $\{\phi, \pi\}$ on Σ_0 corresponds a unique solution to equation (4.21). Again, we set $\mathcal{S}$ to be the space of these solutions. The symplectic structure ω is given by:

$$\omega(\{\phi_1, \pi_1\}, \{\phi_2, \pi_2\}) = \int_{\Sigma_0} (\pi_1 \phi_2 - \pi_2 \phi_1) d^3x. \tag{4.22}$$

From these definitions, there is no difficulty in following the same algorithm as before to construct the Fock space representation $\mathcal{F}_{\mathcal{S}}(\mathbb{H})$, with observables $\omega(\hat{\psi},)$ for every $\psi \in \mathcal{S}$.

4.2.1 Digression: The vacuum state

In this way, there is no mathematical obstacle in moving from Minkowski to globally hyperbolic spacetimes, except for a crucial aspect: there is no natural choice of representation in a general curved spacetime. Besides, since the choice of the vacuum state and the notion of particles are related to the Fock space and observable representations, there is no preferred notion of particles. This fact may sound like a massive problem in the formulation of the theory, but in fact, it is not. Quantum field theory is a theory of fields, not of particles. We will fall in inconsistencies if we call for a particle interpretation, but there is no *a priori* reason for demanding that. The general procedure in choosing a particular representation is to ask the vacuum state to carry the spacetime symmetries. This can be done by finding a unitary representation of the symmetry group and asking the vacuum state to be invariant under its action. For example, in the usual plane wave expansion in flat spacetime, the space of positive frequency solutions is invariant under the Poincaré transformations. In this sense, we have a natural, preferred choice of representation, and consequently of particles.

Another example is the de Sitter spacetime, where a preferred vacuum state can be defined [21]. However, in this geometry, the construction is not straightforward. Requiring that the representation carries the spacetime isometries is not sufficient to establish a unique vacuum state, and an additional condition must be added, namely, the Haddamard condition. From this further requisite, there is a natural choice of vacuum, called the Bunch-Davies vacuum, in the sense that it contains all the required symmetries [21]. In a generic spacetime, we usually do not have enough symmetries or conditions to define a preferred notion of vacuum, and consequently of particles.

The lack of a general notion of particles in general spacetimes has generated great commotion in the academic community, and also divergences. For some physicists like Wald and Davies, the particle concept is disposable, while for others, like Weinberg or the particle physics community, it is an indispensable idea. Indeed, we still do not have a completely satisfactory solution to this question. However, as was proposed in [22], we can still, in some sense, conceive particles in a general background. The basic idea is the following. The particle state described as an element of the Fock space, namely the eigenstates of the number operator, is a *global* state, in the sense that it is defined in all the spacetime and, as discussed, it is in general not unique. On the other hand, there is a *local* notion of particles, which is the one observed by finite size measurement apparatus, and can be defined even in general geometries. Therefore, these local particles are eigenstates of local operators. In this approach, the global particle state (when available) is an approximation to the local particle state. The exact way in which these two concepts resemble, however, is not straightforward.

A particular class of spacetime where the particle concept still holds are the stationary spacetimes. The rigorous construction of quantum field theory in stationary spacetimes where initially developed by Ashtekar and Magnon in [23], and now we give a brief description, with the goal to introduce the next sections properly. We begin by defining what a stationary spacetime is.

Let M be a manifold. A one-parameter group of diffeomorphisms ζ_t is a C^∞ map from $\mathbb{R} \times M \longrightarrow M$ such that for fixed $t \in \mathbb{R}$, $\zeta_t : M \longrightarrow M$ is a diffeomorphism and for all $t, s \in \mathbb{R}$, we have $\zeta_t \circ \zeta_s = \zeta_{t+s}$. For a fixed point $p \in M$, $\zeta_t(p) : \mathbb{R} \longrightarrow M$ is a curve, denoted the ***orbit*** of ζ_t, which passes through p at $t = 0$. Let v_p be the tangent vector to this curve at $t = 0$. Therefore, associated to the one-parameter transformations ζ_t we have a vector field v.

Definition 4.3. *A spacetime is said to be stationary if there exists a one-parameter group of isometries whose orbits are timelike curves.*

The isometries generated by ζ_t express the time translation symmetry of the spacetime. Equivalently, we can define a stationary spacetime as being one which possesses a timelike Killing vector field. Following the recipe given at the beginning of this chapter, we can use the parameter of this group of isometries to define positive and negative frequency states with respect to this parameter and then proceed to construct the Hilbert space representation.

Another class of interest manifolds is the one in which the metric is asymptotically flat. The precise definition of an asymptotically flat spacetime is not trivial, and we shall not present here. However, intuitively, an asymptotically flat spacetime is such that the metric resembles the Minkowski metric in a particular region. Therefore, in this region, the quantum field theory construction is precisely the same as in flat spacetime.

In summary, in this section, we discussed the nature of the particle concept and provided two examples of curved spacetime in which this concept is unambiguous: stationary and asymptotically flat spacetimes. In the next sections, we get deeper into these examples by analyzing two remarkable discoveries related to stationary and asymptotically flat spacetimes, respectively: the Unruh effect and the Hawking radiation!

4.3 Unruh effect

4.3.1 Accelerated reference frames and the notion of particles

In this section we shall derive the full computation of the Unruh effect [24]. This section is mainly based in [25] and in [26]. We begin by clarifying what we mean by an uniformly accelerated observer.

Consider two reference frames in Minkowski spacetime

$$S : (t, x) \text{ and } \tilde{S} : (\tilde{t}, \tilde{x}), \tag{4.23}$$

the second with velocity v with respect to the first. The acceleration in both frames is defined as:

$$a = \ddot{x} = \frac{d^2x}{dt^2}, \; \tilde{a} = \ddot{\tilde{x}} = \frac{d^2\tilde{x}}{d\tilde{t}^2}. \tag{4.24}$$

From the Lorentz transformations, the acceleration between the two reference frames are related by the equation:

$$\tilde{a} = \frac{(1-v^2)^{3/2}}{(1-uv)}a, \tag{4.25}$$

where u is the velocity in frame S. The inverse transformation is:

$$a = \tilde{a}\frac{(1-v^2)^{\frac{3}{2}}}{(1+\tilde{u}v)^3}, \tag{4.26}$$

where $\tilde{u}$ is the velocity vector in frame $\tilde{S}$. Therefore, given the acceleration of a test particle in frame $\tilde{S}$, equation (4.26) tell us what is the acceleration as seen from an observer in S. From that, we now give the definition of constant acceleration:

Definition 4.4. *The acceleration is called constant if it has the same value in any co moving frame; that is, at each instant of time, the acceleration in a frame traveling with the same velocity as the particle is constant.*

To illustrate this definition, suppose that a test particle has constant acceleration α. Let $\tilde{S}$ be the frame such that $\tilde{u} = 0$, that is, the reference frame moves at the same velocity as the particle. Then, $v = u$ and equation (4.26) becomes:

$$a = \alpha\,(1-u)^{\frac{3}{2}}. \tag{4.27}$$

If we integrate this equation twice, we obtain:

$$\frac{\left(x - x_0 + \frac{1}{\alpha}\right)^2}{\frac{1}{\alpha}} - \frac{(t-t_0)^2}{\frac{1}{\alpha}} = 1, \tag{4.28}$$

where x_0 and t_0 are initial values of position and time, respectively. Equation (4.28) has the form of the hyperbola equation, which allow us to conclude that a test particle at constant acceleration moves along hyperbolas.

In the last section we argued that in spacetimes with a timelike Killing vector field the quantum field theory construction is unambiguous and the particle content is well defined. We now investigate the Killing vector fields in a $1+1$ Minkowski spacetime. A Killing vector field is by definition the generator of isometries of a given manifold. Using the Lie derivative we can find those fields, that is:

$$\mathcal{L}_X \eta_{\mu\nu} = 0, \tag{4.29}$$

where X is the desired Killing vector field, and $\eta_{\mu\nu}$ is the Minkowski metric. This equation can be expressed in terms of coordinates as:

$$X^\alpha \partial_\alpha \eta_{\mu\nu} + \eta_{\mu\gamma}\partial_\nu X^\gamma + \eta_{\nu\gamma}\partial_\mu X^\gamma = 0. \tag{4.30}$$

Since $\eta_{\mu\nu}$ is coordinate independent, equation (4.30) becomes:

$$\partial_\mu X_\nu + \partial_\nu X_\mu = 0, \tag{4.31}$$

and taking its derivative with respect to ρ, we have:

$$\partial_\rho \partial_\mu X_\nu + \partial_\rho \partial_\nu X_\mu = 0. \tag{4.32}$$

Similarly, we can exchange the variables in equation (4.31) and write:

$$\partial_\nu \partial_\mu X_\rho + \partial_\nu \partial_\rho X_\mu = 0, \tag{4.33}$$

or

$$\partial_\mu \partial_\nu X_\rho + \partial_\mu \partial_\rho X_\nu = 0. \tag{4.34}$$

Summing equation (4.34) and (4.33), and subtracting (4.32), we obtain:

$$\partial_\mu \partial_\nu X_\rho = 0. \tag{4.35}$$

The solutions of this equation are of the form:

$$X_\rho = a_{\rho\alpha} X^\alpha + b_\rho, \tag{4.36}$$

for some constant b_ρ. If the first therm is zero and $X^\rho = b^\rho$, then we see that this field is the generator of translations. By the other hand, if the constant term is zero, the term $a_{\rho\alpha}$ is anti-symmetric and it has the form:

$$\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4.37}$$

for some constant α. Finally, the vector field becomes:

$$X = -\alpha \begin{pmatrix} x \\ t \end{pmatrix}, \tag{4.38}$$

which is the generator of boosts in Minkowski spacetime. Furthermore, if we request that this Killing vecotr field be timelike, then we are looking for solutions of the equation:

$$\eta_{\mu\nu}X^{\mu}X^{\nu} \geq 0 \tag{4.39}$$

$$\eta_{\mu\mu}X^{\mu}X^{\mu} \geq 0 \tag{4.40}$$

$$-\alpha^2(x^2 - t^2) \geq 0. \tag{4.41}$$

One possible solution is the hyperbola equation:

$$x^2 - t^2 = k, \tag{4.42}$$

where k is any positive constant.

From this last equation, we conclude that observers moving along hyperbolas are also traveling along integral curves of a timelike Killing vector field. Also, equation (4.28) showed that observers moving along hyperbolas are the ones with constant acceleration. The conclusion is that an accelerated observer can unambiguously define particles. In the following section, we compute how an accelerated observer sees the Minkowski vacuum.

4.3.2 Quantization in Rindler 1+1 spacetime

We now quantize a real, massless Klein Gordon field in an eternally accelerated frame of reference in $1+1$ dimensions. Let us start by defining the so called *light-cone coordinates*:

$$\overline{u} = t - x, \tag{4.43}$$

$$\overline{v} = t + x. \tag{4.44}$$

With this change of variables, the Minkowski metric becomes:

$$ds^2 = -d\overline{u}d\overline{v}. \tag{4.45}$$

The region where $\overline{u} < 0$ and $\overline{v} > 0$ is called the *right Rindler wedge(R)*, while the region where $\overline{u} > 0$ and $\overline{v} < 0$ is denoted the *left Rindler wedge(L)* (see image bellow).

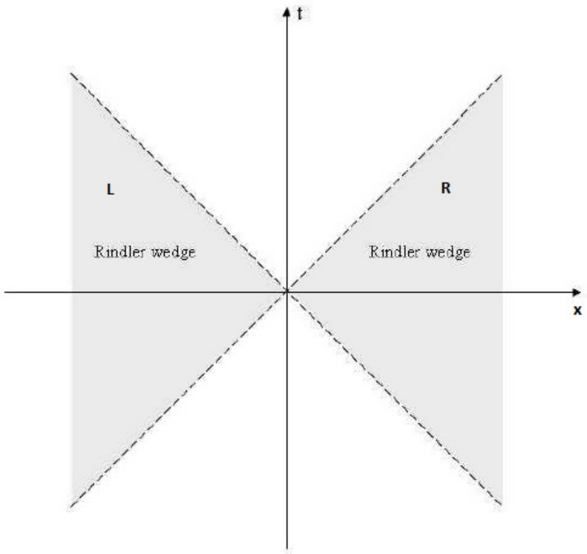

Figure 1 – Left and right Rindler wedges.

Let us define the coordinates in R to be:

$$t = a^{-1}e^{a\xi}\sinh a\eta, \tag{4.46}$$

$$x = a^{-1}e^{a\xi}\cosh a\eta. \tag{4.47}$$

In this new set o variables, the metric is now:

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2). \tag{4.48}$$

In the ordinary quantization in Minkowski spacetime we look for a solution to the Klein Gordon equation in terms of plane waves modes:

$$f_k = \frac{1}{\sqrt{4\pi\omega_k}}e^{-i\omega_k t + ikx}, \tag{4.49}$$

which, in terms of light cone coordinates turn into:

$$f_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega_k}}e^{-i\omega_k \overline{u}}, \text{right moving} \\ \frac{1}{\sqrt{4\pi\omega_k}}e^{-i\omega_k \overline{v}}, \text{left moving.} \end{cases} \tag{4.50}$$

Then, the field expansion is written:

$$\phi(x,t) = \sum_k a_k f_k(x,t) + a^\dagger f_k^*(x,t), \tag{4.51}$$

which defines the vacuum state $a_k|0_m\rangle = 0$, for all k.

A similar procedure can be made in region R, in order to quantize the Klein Gordon field in the frame of reference of a uniformly accelerated observer. In this region, the Klein Gordon equation becomes:

$$e^{-2a\xi}\left(-\partial_\eta^2+\partial_\xi^2\right)\phi(\eta,\xi)=0, \tag{4.52}$$

which admits planes wave solutions:

$$g_k^R=\frac{1}{\sqrt{4\pi\omega_k}}e^{-i\omega_k\eta+ik\xi}. \tag{4.53}$$

From these last two equations, we see that the variable η plays the role of time in the right Rindler wedge, and we can use it to split the space of solutions into positive and negative frequency solutions. From that, it follows the construction of the Hilbert space directly.

However, as explained in the last section, we need a valid solution in a Cauchy surface to make accurate the initial condition formulation. Thus, we define the global modes:

$$g_k^R=\begin{cases}\dfrac{1}{\sqrt{4\pi\omega_k}}e^{-i\omega_k\eta+ik\xi}, \text{ in R}\\ 0, \text{ in L}\end{cases} \tag{4.54}$$

and

$$g_k^L=\begin{cases}\dfrac{1}{\sqrt{4\pi\omega_k}}e^{i\omega_k\eta+ik\xi}, \text{ in L}\\ 0, \text{ in R.}\end{cases} \tag{4.55}$$

With this set of global modes, we make a field expansion which is valid in a Cauchy surface for the manifold:

$$\phi=\sum_k b_k g_k^R+c_k g_k^L+h.c \tag{4.56}$$

The fundamental question now is: How many particles does the eternally accelerated observer (in the R region) see in the Minkowski vacuum? Equivalently,

$$\langle 0_M|N_k^R|0_M\rangle=? \tag{4.57}$$

To answer that question we need the Bogoliubov coefficients:

$$g_\omega^R=\int_0^\infty d\omega'(\alpha_{\omega\omega'}f_{\omega'}+\beta_{\omega\omega'}f_{\omega'}^*), \tag{4.58}$$

where $f_{\omega'}=\dfrac{1}{\sqrt{4\pi\omega'}}e^{-i\omega'\overline{u}}$. The Fourier transform of g_ω^R is:

$$g_\omega^R(\overline{u})=\frac{1}{2\pi}\int_{-\infty}^\infty d\omega' e^{-i\omega'\overline{u}}\tilde{g}_\omega(\omega'). \tag{4.59}$$

Splitting this integral we obtain:

$$g_\omega^R(\overline{u})=\frac{1}{2\pi}\int_0^\infty d\omega' e^{-i\overline{u}\omega'}\tilde{g}_\omega(\omega')+\frac{1}{2\pi}\int_0^\infty d\omega' e^{i\overline{u}\omega'}g_\omega(-\omega'). \tag{4.60}$$

Comparing equations (4.58) and (4.60) we find that:

$$\alpha_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(\omega'), \text{ and } \beta_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(-\omega'). \tag{4.61}$$

We now make the following claim:

Proposition 4.1. $\tilde{g}_\omega(-\omega') = e^{-\frac{\pi\omega}{a}}\tilde{g}_\omega(\omega')$.

Proof

Let us define the null coordinates:

$$u = \eta - \xi, \tag{4.62}$$

$$v = \eta + \xi. \tag{4.63}$$

The relation between (u, v) and $(\overline{u}, \overline{v})$ is:

$$\overline{u} = -a^{-1}e^{-au}, \tag{4.64}$$

$$\overline{v} = a^{-1}e^{av}. \tag{4.65}$$

In terms of $(\overline{u}, \overline{v})$, the modes g^R_ω takes the form:

$$g^R_\omega = \begin{cases} \frac{1}{\sqrt{4\pi\omega_K}}e^{-i\frac{\omega}{a}\ln(-a\overline{u})}, \overline{u} < 0 \\ 0, \overline{u} > 0. \end{cases} \tag{4.66}$$

The Fourier transform of $g_\omega(-\omega')$ is:

$$\begin{aligned} \tilde{g}_\omega(-\omega') &= \int_{-\infty}^{\infty} d\overline{u}e^{-i\omega'\overline{u}}g_\omega(\overline{u}) \\ &= \frac{1}{\sqrt{4\pi\omega}}\int_{-\infty}^{0} d\overline{u}e^{-i\omega'\overline{u}}e^{-i\frac{\omega}{a}\ln(-a\overline{u})}. \end{aligned} \tag{4.67}$$

To evaluate this integral we move to the compelx plane. The point $z = 0$ is called a *branch point*, which we shall now define:

Definition 4.5. *The point z_0 is called a branch point if the value of $f(z)$ does not return to its initial value as a closed curve around the point is traced, in such a way that $f(z)$ varies continuously as the path is traced.*

As an example, consider the function $\log(z) = \ln R + i\theta$, and the two closed paths shown bellow:

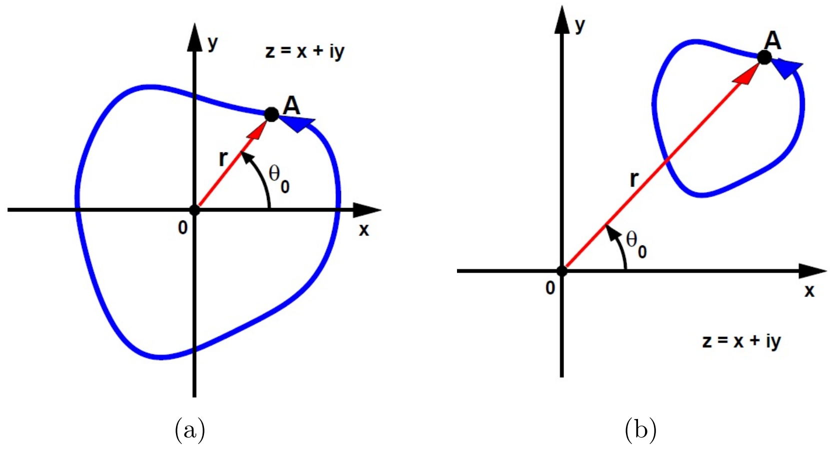

Figure 2 – Closed paths in the complex plane. In image (a), the path leads to a different values of the function $\log(z)$, while in image (b) it leads to the same value. Therefore, the point $z = 0$ is a branch point.

It is clear that the path (a) leads to a different value of the function $\ln z$ and path (b) leads to the same value. The reason for this is that the path (a) is around the branch point $z = 0$. Then, the function $\ln(-a\overline{u})$ is multivalued. To solve this problem we make the integration in a region where $\ln(-a\overline{u})$ is well defined. This region is:

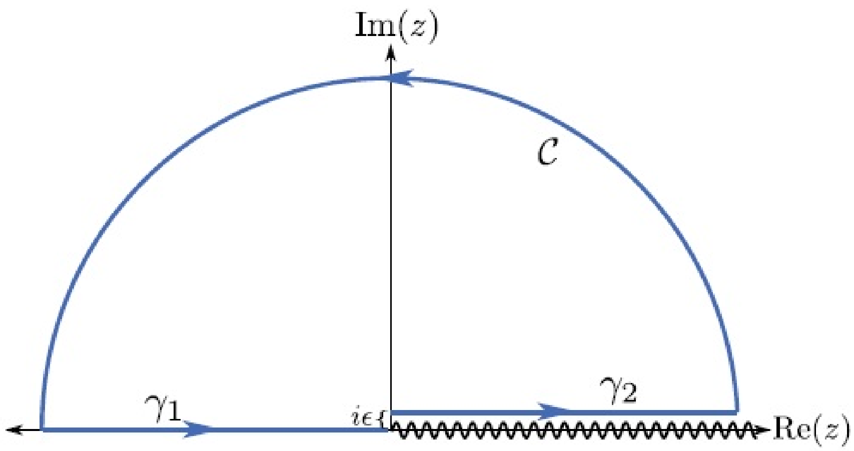

Figure 3 – Branch cut

Cauchy's integral theorem implies that:

$$\oint dz e^{-i\omega' z} e^{-i\frac{\omega}{a}\ln(-az)} = \left\{ \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right\} dz e^{-i\omega' z} e^{-i\frac{\omega}{a}\ln(-az)} = 0, \tag{4.68}$$

because there are no poles inside the countor. The integral over the arc C vanishs, so it follows that $\int_{\gamma_1} = -\int_{\gamma_2}$, that is,

$$\tilde{g}_\omega(-\omega') = -\frac{1}{\sqrt{4\pi\omega}} \int_{i\epsilon}^{\infty+i\epsilon} d\overline{u} e^{-i\omega'\overline{u}} e^{-i\frac{\omega}{a}\ln(-a\overline{u})}. \tag{4.69}$$

Making the variable change $\overline{u} \longrightarrow -\overline{u}$, the last equation becomes:

$$\tilde{g}_\omega(-\omega') = -\frac{1}{\sqrt{4\pi\omega}} \int_{-\infty-i\epsilon}^{-i\epsilon} d\overline{u} e^{i\omega'\overline{u}} e^{-i\frac{\omega}{a}\ln(a\overline{u})}. \tag{4.70}$$

We can write the argument of the logarithm as:

$$a\overline{u} = z = Re^{i\theta} = -Re^{i\theta}e^{-i\pi} = -Re^{-i\pi}, \tag{4.71}$$

then:

$$\ln(z) = \ln(-R) - i\pi. \tag{4.72}$$

Substituting back into equation (4.70) we have:

$$\tilde{g}_\omega(-\omega') = -\frac{1}{\sqrt{4\pi\omega}} \int_{-\infty}^{0} d\overline{u} e^{i\omega'\overline{u}} e^{-i\frac{\omega}{a}[\ln(-a\overline{u}) - i\pi]}, \tag{4.73}$$

and finally:

$$\tilde{g}_\omega(-\omega') = -e^{-\frac{\pi\omega}{a}} \tilde{g}_\omega(\omega'). \tag{4.74}$$

□

From the relation of this proposition, the final calculations are straightforward. Substituting relation (4.74) in (4.61) we have:

$$\alpha_{\omega\omega'} = e^{-\frac{\pi\omega}{a}} \beta_{\omega\omega'}. \tag{4.75}$$

From the normalization condition of the coefficients:

$$\sum_{\omega'} |\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 = 1, \tag{4.76}$$

and together with equation (4.74), we obtain, finally:

$$\sum_{\omega'} e^{-\frac{2\pi\omega}{a}} |\beta_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 = 1 \tag{4.77}$$

$$\sum_{\omega'} |\beta_{\omega\omega'}|^2 = N_\omega = \frac{1}{e^{-2\pi\frac{\omega}{a}} - 1}. \tag{4.78}$$

This final equation has a clear and extraordinary interpretation: a Rindler observer, that is, a uniformly accelerated observer, sees the Minkowski vacuum as a thermal bath with a Planck spectrum at temperature $T = \frac{a}{2\pi K_B}$. The conclusion is that the particle content is observer dependent, and this is our first example of the non-triviality of the vacuum definition. Since its proposal in 1976 [24], the Unruh effect was never confirmed experimentally, and it is not hard to comprehend why: from the above equation for the temperature, we notice that for an observer to see a thermal bath at $T = 1K$, the required acceleration is $a = 10^{21} m/s^2$! Nevertheless, there are several proposals to measure the Unruh effect. For a good review on the subject see [27].

4.4 Hawking Radiation

Before we begin the analyses of the Hawking radiation, we first introduce the so-called Penrose diagrams, since they will be beneficial in the following development. The contents of this section were mainly based in [25].

4.4.1 Penrose diagrams

The Penrose diagram is a useful way of representing the causal structure of an infinite spacetime in a finite diagram. To this purpose, we need to introduce the notion of conformal transformations of the metric.

Definition 4.6. *A conformal transformation is a map from a spacetime* (M, g) *to a spacetime* $(M, \tilde{g})$ *such that*

$$\tilde{g}_{\mu\nu}(x) = \Lambda^2(x) g_{\mu\nu}(x), \tag{4.79}$$

where $\Lambda(x)$ *is a smooth function of the spacetime coordinates and* $\Lambda(x) \neq 0$ *for all* $x \in M$.

It follows directly that:

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu > 0 &\Longleftrightarrow \tilde{g}_{\mu\nu}V^\mu V^\nu > 0 \\ g_{\mu\nu}V^\mu V^\nu = 0 &\Longleftrightarrow \tilde{g}_{\mu\nu}V^\mu V^\nu = 0 \\ g_{\mu\nu}V^\mu V^\nu < 0 &\Longleftrightarrow \tilde{g}_{\mu\nu}V^\mu V^\nu < 0. \end{aligned} \tag{4.80}$$

Hence, curves that are timelike/null/spacelike with respect to g remain timelike/null/spacelike with respect to $\tilde{g}$ and a null geodesics in g correspond o a null geodesic in $\tilde{g}$.

To construct a Penrose diagram we do the following steps:

1. We use a coordinate transformation on the spacetime (M, g) to bring infinity to a finite coordinate distance.
2. We perform a conformal transformation on g to a new metric $\tilde{g}$ that is regular on the edges.

With these two steps, we have $(M, \tilde{g})$ as a good representation of the original (M, g) in the sense that they have the same causal structure. It is conventional to add the points at infinity to construct a new spacetime $(\tilde{M}, \tilde{g})$ that is called the conformal compactification of (M, g). To illustrate clearly the above procedure, we construct a Penrose diagram in Minkowski spacetime.

Example 4.1. *The Minkowski metric in two dimensions is given by:*

$$ds^2 = -dt^2 + dx^2, \ \textit{with} \ -\infty < t, x < \infty. \tag{4.81}$$

Now we perform the two steps of the construction:

1. *Introducing the light-cone coordinates $u = t - x$ and $v = t + x$ the metric takes the form:*

$$ds^2 = -dudv. \tag{4.82}$$

We define now a new set of coordinates to bring infinity to a finite coordinate distance:

$$u = \tan\tilde{u} \text{ and } v = \tan\tilde{v}, \tag{4.83}$$

with $-\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2}$. Then, the metric becomes:

$$ds^2 = -\frac{1}{\cos\tilde{u}\cos\tilde{v}} d\tilde{u}d\tilde{v}, \tag{4.84}$$

which diverges as $\tilde{u}, \tilde{v} \longrightarrow \pm\frac{\pi}{2}$.

2. *Define a new metric through a conformal transformation on g:*

$$d\tilde{s}^2 = (\cos\tilde{u}\cos\tilde{v})^2 ds^2 = -d\tilde{u}d\tilde{v}. \tag{4.85}$$

The metric now is regular at infinity and we add the points $\pm\frac{\pi}{2}$ to construct the conformally compatified $(\tilde{M}, \tilde{g})$.

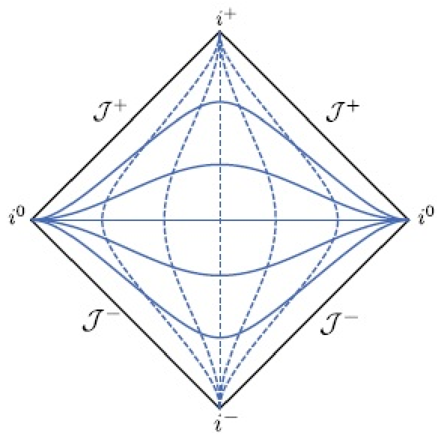

Figure 4 – Penrose diagram of the Minkowski spacetime. Lines with contant x are represented by dashed curves, while lines with contant t are represented by solid curves. We denote $i^{\pm}$ as the future/past timelike infinity, $\mathfrak{I}^{\pm}$ as the future/past null infinity, and i^0 as the spacelike infinity.

4.4.2 Particle creation by black holes

In 1974 Stephen Hawking made the incredible discovery that black holes spontaneously produce particles[1]. Nonetheless, it was already known that it is possible to extract energy of a Kerr black hole by a mechanism denoted *Penrose process* [28]. The idea of this process is the following: in the Kerr black hole, there is a region outside the event horizon, denoted *ergosphere*, where a falling test particle necessarily gains angular momentum. The

Penrose process consists of sending a particle in such a way that it decays in two particles inside the ergosphere, one of which falls into the black hole and the other scapes the ergoregion. The Killing vector ∂_t becomes spacelike inside the ergosphere, and therefore the energy of the infalling particle can be negative. Then, by energy conservation, the energy of the outside particle must be higher than the initial energy, which characterizes the extraction of the energy of the black hole. If we make a qualitative comparison with atoms, we can say that the Penrose process corresponds to a *stimulated emission*, while the Hawking radiation is comparable with a *spontaneous emission*.

Consider the spacetime of a spherically symmetric collapsing star, which consists of an initial sparse distribution of matter, such that we can approximately say that the metric is the Minkowski metric. This matter distribution then collapses to form a black hole, where the geometry in the far future becomes the Schwarzschild spacetime. This is a globally hyperbolic spacetime since it admits a Cauchy surface ($\mathfrak{I}^-$ for example). This spacetime is not stationary, but it is approximately stationary in the far past and future. Therefore, as discussed before, we can perform quantization in $\mathfrak{I}^-$ and in $\mathfrak{I}^+$ which has a meaningful notion of particles. The Penrose diagram of this spacetime is shown below. The wavy line in the top represents the singularity, the dashed line symbolizes the event horizon (H^+), and the solid line from i^- to the singularity represents the in-falling matter distribution.

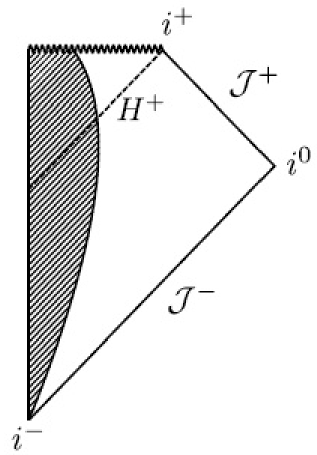

Figure 5 – Penrose diagram of a collapsing star

In $\mathfrak{I}^-$ we can define the in-vacuum by specifying a complete set of positive frequency modes. The same can not be done in $\mathfrak{I}^+$, since it is not a Cauchy surface. However, the surface $H^+ \bigcup \mathfrak{I}^+$ is a Cauchy surface so we can quantize the field in far future by specifying a complete set on $H^+ \bigcup \mathfrak{I}^+$. There will be three sets of modes:

1. f_i: positive frequency on $\mathfrak{I}^-$,

2. g_i: positive frequency on $\mathfrak{I}^+$ and zero on H^+,

3. h_i: positive frequency on H^+ and zero on $\mathfrak{I}^+$.

Actually, in H^+ we can not define positive frequency modes because ∂_t is null there. This detail, however, will not be important in defining particles in $\mathfrak{I}^+$.

The field can be expanded in two equivalent ways:

$$\phi(x) = \sum a_i f_i(x) + h.c = \sum b_i g_i(x) + c_i h_i(x) + h.c. \tag{4.86}$$

Since $\{f_i(x)\}$ forms a basis, we can write $g_i(x)$ in terms of $\{f_i\}$:

$$g_i(x) = \sum_j \alpha_{ij} f_j(x) + \beta_{ij} f_j^*(x). \tag{4.87}$$

Then, if we could solve the Klein Gordon equation exactly in Schwarzschild spacetime, we would only need to calculate the mean number of particles in $\mathfrak{I}^+$. However, the Klein Gordon equation in Schwarzschild is too complicated, which prevents us from finding analytical solutions. Instead, we ask: *if a solution to the Klein Gordon equation is positive frequency at $\mathfrak{I}^+$, then what is its form on $\mathfrak{I}^-$?*

In the tortoise coordinates (t, r^*, θ, ϕ), where $r^* = r + 2M \ln\left(\frac{r-2M}{2M}\right)$, the Schwarschild metric reads:

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2, \tag{4.88}$$

and the Klein Gordon equation:

$$\Box\phi(t, r^*, \theta, \phi) = 0, \text{ with } \Box = \nabla^\mu \nabla_\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\nu}\partial_\nu). \tag{4.89}$$

To find the solutions, we proced with a separation of variables. The solution of the angular part is given by the spherical harmonics $Y_{lm}(\theta, \phi)$, so we can write:

$$\phi(t, r^*, \theta, \phi) = \Xi_l(r^*, t) Y_{lm}(\theta, \phi), \tag{4.90}$$

where $\Xi(r^*, t)$ satisfies:

$$\left[\partial_t^2 - \partial_{r^*}^2 + V_l(r^*)\right]\Xi_l = 0, \tag{4.91}$$

and

$$V_l(r^*) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{4.92}$$

Let $\Xi_l(r^*, t) = e^{-i\omega t} R_{l\omega}(r^*)$. Then, equation (4.91) becomes:

$$\left(\partial_{r^*}^2 + \omega^2\right) R_{\omega l} = V_l R_{\omega l}. \tag{4.93}$$

The solutions to this last equation are very complicated, and we can not solve it analytically. Nevertheless, near $\mathfrak{I}^\pm$, r^* tends to infinity and $V_l(r^*)$ tends to zero. In this region, the solutions of the Klein Gordon equation are just planes waves:

$$\mathfrak{I}^- \begin{cases} f^+_{lm\omega} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \dfrac{Y_{lm}}{r}, \text{ outgoing} \\ f^-_{lm\omega} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \dfrac{Y_{lm}}{r}, \text{ ingoing} \end{cases} \tag{4.94}$$

$$\mathfrak{I}^+ \begin{cases} g^+_{lm\omega} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \dfrac{Y_{lm}}{r}, \text{ outgoing} \\ g^-_{lm\omega} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \dfrac{Y_{lm}}{r}, \text{ ingoing,} \end{cases} \tag{4.95}$$

where we wrote in terms of the null coordinates $u = t - r^*$ and $v = t + r^*$. Our main focus will be in the ingoing early modes and in the outgoing late modes, so we do the abbreviations:

$$\begin{aligned} f_\omega &\sim f^-_{lm\omega} \\ g_\omega &\sim g^+_{lm\omega}. \end{aligned} \tag{4.96}$$

The quantity that we are mainly interested to determine is the expectation value of the number of particles of a given frequency in $\mathfrak{I}^+$:

$$\langle in|N^{out}_\omega|in\rangle = \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2. \tag{4.97}$$

Because we have a definite frequency, there is an absolute uncertainty in time, which means that this quantity provides the number of particles with frequency ω emitted at any time. However, we are interested in counting the number of particles that hit $\mathfrak{I}^+$ near $u \longrightarrow \infty$, that is when the black hole has settled down to a stationary configuration. So we need to replace the wave type modes by wave packets. To find the necessary form of the modes in $\mathfrak{I}^-$ to form a wave packet in $\mathfrak{I}^+$, we trace this late time wave packet back in time. When traveling inwards from $\mathfrak{I}^+$, it will encounter the potential barrier. One part of the wave, $g^{(r)}_\omega$, will be reflected and end on $\mathfrak{I}^-$ with the same frequency. The remaining part, $g^{(t)}_\omega$, will enter the collapsing body. In that region the geometry of the spacetime is unknown. Notwithstanding, this packet will suffer a very high blueshift and will satisfy the geometric optics approximation, which means that we can trace it back by following a null geodesics γ.

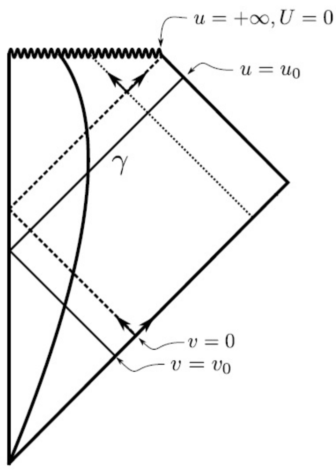

Figure 6 – We can trace back a wave packet in $\mathfrak{I}^+$, peaked at very high u_0, by following a null geodesic γ.

If we consider a packet in $\mathfrak{I}^+$ peaked at very high u_0, v_0 will be very small. Furthermore, in $\mathfrak{I}^+$ the Kruskal coordinate $U = -e^{-\kappa u}$(where $\kappa = \frac{1}{4M}$ is the surface gravity) will be very close to zero, so we can expand v in powers of U_0:

$$v = cU_0 + O(U_0^2). \tag{4.98}$$

Using $u = -\kappa^{-1}\ln(-U) = -\kappa^{-1}\ln(-cv)$, we can conclude that if a mode takes the form $g_\omega \sim e^{-i\omega u}$ in $\mathfrak{I}^+$, the transmitted part $g_\omega^{(t)}$ on $\mathfrak{I}$ will take the form:

$$g_\omega^{(t)} \sim \begin{cases} e^{i\frac{\omega}{\kappa}\ln(-v)}, \text{ v<0} \\ 0, \text{ v>0.} \end{cases} \tag{4.99}$$

We are now able to calculate the Bogoliubov coefficients. Writing $g_\omega^{(t)}$ as a superposition of $f_{\omega'}$:

$$g_\omega^{(t)} = \int_0^\infty d\omega' \left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f_{\omega'}^*\right), \tag{4.100}$$

where $f \sim e^{-i\omega' v}$. The Fourier transform of $g_\omega^{(t)}$ is:

$$\begin{aligned} g_\omega^{(t)}(v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{-i\omega' v}\tilde{g}_\omega(\omega') \\ &= \frac{1}{2\pi}\int_0^\infty d\omega' e^{-i\omega' v}\tilde{g}_\omega(\omega') + \int_0^\infty d\omega' e^{-i\omega' v}\tilde{g}_\omega(-\omega'). \end{aligned} \tag{4.101}$$

Comparing equations (4.100) and (4.101) we have that:

$$\alpha_{\omega\omega'} \sim \tilde{g}_\omega(\omega'), \text{ and } \beta_{\omega\omega'} \sim \tilde{g}_\omega(-\omega'). \tag{4.102}$$

From now on the calculation is very similar to the one conducted in the Unruh effect. Therefore, we do a similar claim:

Proposition 4.2.

$$\tilde{g}_\omega(-\omega') = e^{-\pi\frac{\omega}{\kappa}}\tilde{g}_\omega(\omega'),\ \omega' > 0. \tag{4.103}$$

The proof of this statement follows from the proof of propositon 4.1. This result, together with the normalization condition:

$$\sum_{\omega'} |\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}| = 1, \tag{4.104}$$

implies that

$$\langle N_\omega \rangle \sim \frac{1}{e^{\frac{\omega}{K_B T}} - 1}. \tag{4.105}$$

That is, a distant observer will see a thermal spectrum with temperature $T = \frac{\kappa}{2\pi K_B}$ coming from the black hole. Because the temperature is inversely proportional to the mass, the black hole will heat as it evaporates, and will eventually explode. The astounding fact that black holes emits particles, together with previous results, gave rise to a complete analogy between black hole physics and thermodynamics. The Hawking radiation, and also the Unruh effect, lie in a very intriguing interface between general relativity, quantum field theory, and thermodynamics. This intersection is widely explored nowadays in the pursuit to understand the most fundamental aspects of spacetime.

5 Dynamical Casimir Effect

5.1 Introduction

In a nutshell, the dynamical Casimir effect (DCE), or the non-stationary Casimir effect, consists of the quantum effects arising from one or more moving boundaries, where the field must vanish [7]. However, long before the formulation of quantum field theory, the problem of finding solutions to the classical wave equation with non-stationary boundary conditions was already in discussion. The first known article of this kind was written in 1921 and is due to Nicolai [29], where he finds exact solutions to the wave equation in a cavity whose length is varying according to.:

$$L(t) = L_0(\alpha t). \tag{5.1}$$

Just a few more works can be found about this subject around this time, and only forty years later, in the sixties, a significant effort was delivered once more. The Soviet scientists developed most of the articles regarding classical fields with moving boundaries at this time. In 1962, Askar'yan studied the field amplification in a cavity due to parametric resonance [30], and in 1967 Grinberg proposed a general method wich consists of expanding solutions in terms of instantaneous normal modes [31]. Furthermore, in 1985 Borisov investigated the solutions of the wave equation inside a membrane with a uniformly varying radius [32]. Meanwhile, in 1948, Hendrick Casimir made the non-classical theoretical prediction [2], later experimentally confirmed [3], that two uncharged conducting plates would feel an attractive force when sufficiently close. This effect was then known as the *Casimir effect*.

It was only in 1970 that quantum fields with moving boundaries came into consideration, with the seminal paper by Moore [33], which considered the quantum electromagnetic field inside a cavity in $1 + 1$ dimensions, and predicted the generation of photons from the vacuum. Five years later, DeWitt wrote a highly influential paper regarding general trajectories of accelerated conductors [34], followed in 1977 by an equally outstanding work by Fulling and Davies around the same subject, where the energy-momentum tensor in $1 + 1$ dimensions for a massless scalar field was obtained [35]. In 1989, Dodonov calculated the corrections to the Casimir force due to boundary motion [36], and in this same year, Yablanovitch proposed to simulate the Unruh effect by using a medium of rapidly decreasing refractive index inside a cavity [37]. Four years later, in a series of five far-reaching works([38], [39], [40], [41], and [42]), Schwinger tried to explain, using the dynamical Casimir effect, the phenomenon of *sonoluminescence*, which consists of light pulses emission by bubbles with high-frequency radii oscillations inside water due to acoustic pressure. It turned out that the DCE was not able to explain this phenomenon.

According to the DCE framework, it would be necessary a much larger time to accumulate enough energy to observe the emitted light than what was seen in experiments. In fact, the term *dynamical Casimir effect* was popularized by Schwinger and Yablanovitch. Schwinger wrote [38]: I interpret as a dynamical Casimir effect wherein dielectric media are accelerated and emit light. In the following years, much was accomplished in finding applications of the DCE. In 1997 Jaekel investigated the Brownian motion of the mirrors [43], in 2000 Nagatani analyzed the problem of backreaction [44], and Brevik applied the model to cosmology [45]. To introduce the dynamical Casimir effect in a cavity appropriately, we first look at the static Casimir effect, and the single mirror DCE.

5.1.1 The static Casimir effect

We make now a brief review of the Static Casimir Effect(CE). This section was mainly based in [46]. Throughout the rest of the chapther, we will only work with a real scalar field, which can be thought as a approximation of the electromagnetic field when polarisation effects are negligible.

The Klein-Gordon equation for a massless scalar field is:

$$(\partial_\mu \partial^\mu + m^2)\phi = 0. \tag{5.2}$$

Expanding in Fourier modes, we have that:

$$\hat{\phi}(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} [\hat{a}_k e^{-i\omega_k t + ikx} + \hat{a}_k^+ e^{i\omega_k t - ikx}]. \tag{5.3}$$

With the addition of the mirrors, say in $x = 0$ and $x = L$, we are imposing the boundary conditions to the field:

$$\hat{\phi}(0,t) = \hat{\phi}(L,t) = 0, \tag{5.4}$$

and the solution is now:

$$\hat{\phi}(x,t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} [\hat{a}_n e^{-i\omega_n t} + \hat{a}_n^+ e^{i\omega_n t}] \frac{sen(\omega_n x)}{\sqrt{2\omega_n}}, \tag{5.5}$$

where $\omega_n = n\pi/L$.

In terms of criation and anihilation operators, the Hamiltonian can be expressed as:

$$\hat{H} = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n [\hat{a}_n \hat{a}_n^+ + \hat{a}_n^+ \hat{a}_n]. \tag{5.6}$$

We are now able to compute the expected value of the vacuum energy:

$$E = \langle 0|\hat{H}|0\rangle = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n = \frac{\pi}{2L} \sum_{n=1}^{\infty} n = \infty \tag{5.7}$$

As seen from above, the average value of the energy of the vacuum state diverges. The regularization can be done from different methods. The divergent term runs with n^{-1} and we can use the Riemann zeta function to regularise, as follows. The Riemann zeta function:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \tag{5.8}$$

whith $x = -1$, becomes:

$$\zeta(-1) = \frac{-1}{12}. \tag{5.9}$$

Substituting into equation (5.7), it becomes:

$$E = \frac{-\pi}{24L}. \tag{5.10}$$

From this last equation, the force can now be calculated:

$$F = \frac{-dE}{dL} = \frac{-d(L\epsilon_0)}{dL} = \frac{-\pi}{24L^2}. \tag{5.11}$$

Even though we are treating a simplified case (real scalar field), the result carries two important aspects:

1. The force is attractive.
2. The dependence with the mirrors distance is L^{-2}.

A similar calculation in four-dimensional spacetime gives the force per unit area (f) between the plates as:

$$f = \frac{-\pi^2}{240L^4}. \tag{5.12}$$

5.1.2 The single mirror DCE

The single mirror model was deeply analyzed, and several analytical solutions where found [47]. Furthermore, this model is often used as a simplified analog of more complicated phenomena, like the Hawking and the Unruh radiation. In this chapter, our primary concern will be in cavities, rather than single mirrors. Nevertheless, we present now a brief description of this model. The single moving mirror in $1+1$ Minkowski spacetime consists of a perfectly reflecting moving boundary, where the massless scalar field, described by the Klein Gordon equation:

$$(\partial_t^2 - \partial_x^2)\phi(t,x) = 0, \tag{5.13}$$

is required to vanish at the mirror position $z(t)$:

$$\phi(t, x = z(t)) = 0. \tag{5.14}$$

In the null coordinates $u = x - t$ and $v = x + t$, the Klein Gordon equation for the modes ϕ_ω takes the simple form:

$$-\partial_u \partial_v \phi_\omega = 0, \tag{5.15}$$

with the general solution:

$$\phi_\omega = g(v) + h(u), \tag{5.16}$$

where $g(v)$ and $h(u)$ are arbitrary functions. In the presence of the mirror, the spacetime has a boundary, and we must consider separately the modes which are at the right or the left of the mirror. Let us consider the modes to the right of a mirror with a trajectory that starts at past timelike infinity, i^-, and ends in at future timelike infinity, i^+. In addition, let us demand that the mirror has an asymptotically inertial trajectory so that we can define particles in those regions. The mode functions that are positive frequency at $\mathfrak{I}^-$ we will call the *in* vacuum state, while the positive frequency modes at $\mathfrak{I}^+$ we will denote the *ou* vacuum state. The modes that are positive frequency at $\mathfrak{I}^-$ are:

$$\phi_\omega^{in} = \frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega p(u)}), \tag{5.17}$$

where $p(u)$ is a function of u called the *ray tracing function*. For these modes to vanish at the mirror we must have that $v = p(u)$ at the position of the mirror. Then, it is useful to introduce the functions :

$$u_m(t) = t - z(t), \text{ and} \tag{5.18}$$

$$v_m(t) = t + z(t), \tag{5.19}$$

which gives the value of u and v at the mirror location. Inverting the first equation to find $t(u)$ and then substituting in the second we find that:

$$p(u) = t(u) + z(t(u)). \tag{5.20}$$

Similarly, the modes that are positive frequency at $\mathfrak{I}^+$ are:

$$\phi_\omega^{out} = \frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega f(v)} - e^{-i\omega u}), \tag{5.21}$$

where $f(v)$ is defined in terms of $t(v)$. Therefore, for a chosen trajectory $z(t)$, the functions $p(u)$ and $f(v)$ are completely defined. It follows that to find the average number of particles at $\mathfrak{I}^+$, we only need to perform a Bogoliubov transformation between *in* and *out* modes. The function $p(u)$ completely characterize the mirror trajectory and is incorporated in several observables, like the two-point function, the energy flux, and the correlation functions. In their seminal work [5], Davies and Fulling computed the form of the stress-energy tensor in terms of the ray tracing function:

$$\langle T_{uu} \rangle = \frac{1}{24\pi}\left(\frac{3}{2}\left(\frac{p''}{p'}\right)^2 - \frac{p'''}{p'}\right), \tag{5.22}$$

where the prime indicates derivative with respect to u.

However, despite the general procedure being simple, its implementation is not easy for non-trivial trajectories. The inversions involved in finding the ray tracing function are not always possible. Nevertheless, there are many known trajectories which can be analytically solved and, among them, those which provide a Planck spectrum, similar to the black hole particle production. Davies and Fulling were the first to propose such a connection [35], but it is a consensus in the literature that their paper had some technical issues, like some questionable approximations(see [7]). Carlitz and Willey then proposed a well-formulated trajectory [6], with a Hawking-like particle production. Their strategy was to demand that the stress-energy tensor was that of constant energy flux, and then finding the corresponding ray tracing function.

5.2 Techniques

5.2.1 The Lock-Fuentes model

In this section, we explain the Lock-Fuentes model [48], which describes the evolution of scalar field for a finite period of time in terms of a Bogoliubov transformation. This model is crucial to the results that we will exhibit: if the field is initially in the vacuum state, it evolves into a squeezed state characterized by a symplectic transformation $M(t)$, which can be determined by the Lock-Fuentes model.

We consider a cavity containing a massless real scalar field ϕ, in 1+1 dimensional flat spacetime. The Klein-Gordon equation constrained by the stationary boundary conditions $\phi(t, x = x_1) = \phi(t, x = x_2) = 0$ admits the solutions:

$$\phi_m(t, x) = N_m e^{-i\omega_m t} sin[\omega_m(x - x_1)], \tag{5.23}$$

and their complex conjugate ϕ^*, where $N_m = 1/\sqrt{m\pi}$ is the normalization constant, $L = x_2 - x_1$ is the cavity length and $\omega_m = m\pi/L$ are the mode frequencies. The inner product is given by the Klein-Gordon inner product:

$$(\phi, \psi) = -i \int_{x_1}^{x_2} dx[\phi(\partial_t\psi^*) - \psi^*(\partial_t\phi))]. \tag{5.24}$$

The mode solutions are orthonormal: $(\phi_m, \phi_n) = \delta_{mn}, (\phi^*_m, \phi^*_n) = -\delta_{mn}$ and $(\phi_m, \phi^*_n) = 0$. The field operator is:

$$\Phi(t, x) = \sum_m [a_m\phi_m(t, x) + a^\dagger_m \phi^*_m(t, x)]. \tag{5.25}$$

The mode solutions are associated with particles via the annihilation and creation operators, and the Fock space is constructed from them.

We assume that the mirror is static for $t < 0$ and $t > T$. Therefore, we have solutions of the form (5.23) for $t < 0$ and $t > T$, which we call in- and out-solutions. A linear transformation from the set of in-solutions to the out-solutions is a Bogoliubov transformation, and we will use this transformation to describe the effect of the boundary motion. As a matrix equation, we can write this transformation as:

$$\tilde{\chi} = M\chi, \tag{5.26}$$

where $\chi = [\phi_1, \phi_2, ...\phi_1^*, \phi_2^*, ...]^T$ and $\tilde{\chi} = [\tilde{\phi_1}, \tilde{\phi_2}, ...\tilde{\phi_1^*}, \tilde{\phi_2^*}, ...]^T$ are column vectors and M is the Bogoliubov transformation matrix between these two sets of modes:

$$M = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix} \tag{5.27}$$

The $\alpha_{mn} = (\tilde{\phi_m}, \phi_n)$ and $\beta_{mn} = -(\tilde{\phi_m}, \phi_n^*)$ are the Bogoliubov coefficients.

The finite time Bogoliubov transformation

The basic assumption of the model is that: *for boundary motion much slower than the speed of light, the infinitesimal transformation of the modes is given by a pure spatial displacement combined with a pure phase evolution*:

$$\phi_m(t, x; x_1, x_2) \xrightarrow{e^{i\Omega(x_1,x_2)\delta t} M_\delta} \phi_m(t + \delta t, x; x_1 + \delta x_1, x_2 + \delta x_2), \tag{5.28}$$

with the matrix of frequencies defined as $\Omega \equiv \text{diag}(\omega_1, \omega_2, .., -\omega_1, -\omega_2, ...)$ and where the matrix M_δ gives the Bogoliubov transformation between modes spatially shifted,that is:

$$M_\delta(t; t')\phi_m = \sum_n (\tilde{\phi}_n, \phi_m)\tilde{\phi}_n, \tag{5.29}$$

where $t' = t + \delta t$, and

$$\tilde{\phi_n} = \phi_n(t + \delta t, x; x_1 + \delta x_1, x_2 + \delta x_2). \tag{5.30}$$

Considering motion for a finite time t, followed by an infinitesimal transformation, we have:

$$M(t + \delta t) = e^{i\Omega(x_1,x_2)\delta t} M_\delta(\delta x_1, \delta x_2) M(t). \tag{5.31}$$

From the equation above, we find the desired evolution equation:

$$\frac{dM(t)}{dt} = \left[i\Omega + Q^{(1)}\frac{dx_1}{dt} + Q^{(2)}\frac{dx_2}{dt} \right] M(t), \tag{5.32}$$

where:

$$Q^{(j)} = \begin{bmatrix} A^{(j)} & B^{(j)} \\ B^{(j)*} & A^{(j)*} \end{bmatrix}, A^{(j)}_{mn} = \left(\frac{\partial \phi_m}{\partial x_j}, \phi_n \right) \text{and } B^{(j)}_{mn} = -\left(\frac{\partial \phi_m}{\partial x_j}, \phi_n^* \right). \tag{5.33}$$

<u>*Proof:*</u>

The derivative of M is by definition:

$$M'(t) = \lim_{\delta t \to 0} \frac{M(t+\delta t) - M(t)}{\delta t} \tag{5.34}$$

Together with (5.31), it becomes:

$$\begin{aligned} M'(t) &= \lim_{\delta t \to 0} \frac{(e^{i\Omega \delta t} M_\delta - 1) M(t)}{\delta t} && (5.35) \\ &= \lim_{\delta t \to 0} \frac{(i\Omega + M'_\delta)\delta t}{\delta t} M(t) && (5.36) \\ &= (i\Omega + M'_\delta) M(t). && (5.37) \end{aligned}$$

Let us now determine the matrix elements of M'_δ. Considering that an infinitesimal displacement of the mirrors produces only a small perturbation on ϕ_m, we can write:

$$\tilde{\phi}_m = \phi_m + \frac{\partial \phi_m}{\partial x_1} \delta x_1 + \frac{\partial \phi_m}{\partial x_2} \delta x_2. \tag{5.38}$$

Substituting in (5.35):

$$\begin{aligned} [M'_\delta(t)]_{mn} &= \lim_{\delta t \to 0} \left\{ \frac{[M_\delta(t; t+\delta t) - M_\delta(t;t)]_{mn}}{\delta t} \right\} \\ &= \lim_{\delta t \to 0} \left\{ \frac{\left[(\phi_m, \phi_n) + \left(\frac{\partial \phi_m}{\partial x_1}, \phi_n \right) \delta x_1 + \left(\frac{\partial \phi_m}{\partial x_2}, \phi_n \right) \delta x_2 \right] - (\phi_m, \phi_n)}{\delta t} \right\} \\ &= \left(\frac{\partial \phi_m}{\partial x_1}, \phi_n \right) \frac{dx_1}{dt} + \left(\frac{\partial \phi_m}{\partial x_2}, \phi_n \right) \frac{dx_2}{dt}. \end{aligned} \tag{5.39}$$

This gives us the first block A in equation (5.131) and the other blocks are analogously determined. We have then the final result:

$$M'(t) = \left[i\Omega + Q^{(1)} \frac{dx_1}{dt} + Q^{(2)} \frac{dx_2}{dt} \right] M(t). \tag{5.40}$$

□

Integrating equation (5.32), we have the solution in terms of a time ordered exponential:

$$M(t) = \mathcal{T} e^{\int_0^t dt' \left(i\Omega + Q^{(1)} \frac{dx_1}{dt} + Q^{(2)} \frac{dx_2}{dt} \right)}, \tag{5.41}$$

which can also be written in terms of a Dyson series:

$$M(t) = \mathbb{I} + \int_0^t dt' \left(i\Omega + Q^{(1)} \frac{dx_1}{dt'} + Q^{(2)} \frac{dx_2}{dt'} \right) + ... \tag{5.42}$$

The above equation can be more easily solved in a manner analogous to the interaction picture of quantum mechanics, as follows. Let us define:

$$\begin{aligned} K_0 &= i\Omega, \\ V &= Q^1 \frac{dx_1}{dt} + Q^2 \frac{dx_2}{dt}, \\ V_I &= e^{-K_o t} V e^{K_o t}, \\ \xi_I(t) &= e^{-K_0 t} \xi(t), \end{aligned}$$

where $\xi(t)$ is the state in phase space, with evolution given by:

$$\xi(t) = M(t)\xi_0. \tag{5.43}$$

Deriving this equation with respect to time, we have:

$$\frac{d\xi(t)}{dt} = (K_0 + V)M(t)\xi_0 = (K_0 + V)\xi(t). \tag{5.44}$$

Analogously:

$$\begin{aligned} \frac{d\xi_I(t)}{dt} &= -K_0 e^{-K_0 t}\xi(t) + e^{-K_0 t}(K_0 + V)\xi(t) \\ &= e^{-K_0 t} V e^{K_0 t} e^{-K_0 t} \xi(t) \\ &= V_I \xi_I(t). \end{aligned} \tag{5.45}$$

For $t_0 = 0$, we have:

$$\xi_I(t) = e^{-K_0 t} M(t)\xi_0 = M_I(t)\xi_0, \tag{5.46}$$

with $M_I(t)$ defined as:

$$M_I(t) = e^{-K_0 t} M(t) \rightarrow M(t) = e^{K_0 t} M_I(t). \tag{5.47}$$

From the differential equation for $M(t)$, we obtain:

$$\begin{aligned} \frac{dM(t)}{dt} &= (K_0 + V)M(t) \\ K_0 e^{K_0 t} M_I(t) + e^{K_0 t} \frac{dM_I(t)}{dt} &= (K_0 + V) e^{K_0 t} M_I(t) \\ e^{-K_0 t} K_0 e^{K_0 t} M_I(t) + \frac{dM_I(t)}{dt} &= e^{-K_0 t} K_0 e^{K_o t} M_I(t) + e^{-K_o t} V e^{K_o t} M_I(t) \\ \frac{dM_I(t)}{dt} &= V_I M_I(t). \end{aligned} \tag{5.48}$$

Equation (5.48) generates the Dyson series for the interaction picture:

$$M_I(t) = \mathbb{I} + \int_0^t dt' V_I(t') + ... \tag{5.49}$$

which is related to the original Dyson series as:

$$M(t) = e^{K_0 t} \left(\mathbb{I} + \int_0^t dt' V_I(t') + ... \right). \tag{5.50}$$

If K_0 is not-time dependent, but commutes in diffent times, we replace:

$$e^{K_0 t} \Longrightarrow e^{\int dt' K_0(t')}.$$

Then, defining $\Theta(t) = \int_o^t dt'\Omega(t')$,equation (5.50) becomes:

$$\begin{aligned} M(t) &= e^{i\Theta(t)}\mathcal{T}e^{\int_0^t dt' V_I} \\ M(t) &= e^{i\Theta(t)}\mathcal{T}e^{\int_0^t \sum_{j=1}^2 e^{-i\Theta(t')}Q^{(j)}e^{i\Theta(t')}\frac{dx_j}{dt'}}. \end{aligned} \tag{5.51}$$

5.2.2 Entanglement entropy and the complex structure

In the last chapter we made clear the fundamental role that the complex structure (introduced in chapter 2) has in the choice of representation in quantum field theory. In this section we present a technique in which we are able to compute the entanglement entropy of a given subsystem directly from the choice of a complex structure [49]. This procedure is applicable to squeezed states (see definition below) in finite dimensional bosonic systems.

Let $V = \mathbb{R}^{2N}$ be a symplectic vector space. Let us decompose V in terms of direct sums of symplectic subspace V_i:

$$V = \bigoplus_i V_i. \tag{5.52}$$

Then, we choose:

$$\omega_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad j_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.53}$$

to be compatible symplectic and complex structures in each subspace, such that:

$$\Omega_0 = \bigoplus_i \omega_0 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \qquad J_0 = \bigoplus_i j_0 = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{5.54}$$

are compatible structures in V. These reference structures are associated with a reference vacuum, which we squeeze with a symplectic transformation.

Let us denote the *isotropy group* $\mathrm{Isot}(J_0)$ as the subgroup of $Sp(2N,\mathbb{R})$ that leaves the reference complex structure invariant under conjugation, that is:

$$\mathrm{Isot}(J_0) = \{M \in Sp(2N,\mathbb{R}) | MJ_0M^{-1} = J_0\}. \tag{5.55}$$

Because the isotropy group is a subgroup, we can construct the *coset space* of $\mathrm{Isot}(J_0)$, which is called the *squeeze* set and is defined as:

$$\mathrm{Squeeze}(J_0) = Sp(2N,\mathbb{R})/\mathrm{Isot}(J_0), \tag{5.56}$$

where / denotes the left coset. It can be shown that there is no intersection between the isotopry group and $\mathrm{Squeeze}(J_0)$. Thus,the elements of the coset are those which do not leave the complex structure invariant under conjugation. The polar decomposition allows us to write every symplectic matrix M as:

$$M = TR, \tag{5.57}$$

where $R \in$Isot(J_0) and $T \in$Squeeze(J_0) is symmetric and positive. Therefore, if we denote the space of all compatible complex structures on V as $\mathfrak{J}(V,\omega)$, we have the natural identification:

$$\mathfrak{J}(V,\omega) \sim \text{Squeeze}(J_0). \tag{5.58}$$

We can identify (V,Ω_0) as being the classical phase space. The Poisson brackets, defined in terms of the complex structure:

$$\{f,g\} = \Omega_0^{ab}\partial_a f \partial_b g \tag{5.59}$$

endows the space of smooth function with a Lie algebra, for all $f,g \in C^\infty(\mathbb{R}^{2N})$. When we canonicaly quantize the system, we will look for a representation of this algebra in a Hilbert space $\mathbb{H}_\mathbb{V}$, and also a unitary representation $U(M)$ of the symplectic transformation M. It can be shown that a quadratic Hermitian operator

$$\frac{1}{2}K_{ab}\xi^a\xi^b, \text{ where } K \in \text{Sym}(2N,\mathbb{R}) \tag{5.60}$$

generates the unitary transformation representing a symplectic transformation M with generator $\Omega_0 K$, that is:

$$U(M) = e^{-i\frac{1}{2}K_{ab}\xi^a\xi^b}, \text{ where } M = e^{\Omega_0 K}. \tag{5.61}$$

Now, we are able to enunciate the main result of [49], in which the entanglement entropy is given in terms of the complex structure. Let our Hilbert space be decomposed as:

$$\mathbb{H}_V = \mathbb{H}_A \bigotimes \mathbb{H}_B. \tag{5.62}$$

Then, the entanglemet entropy $S_A(|J\rangle)$ of a squeezed vacuum restricted to the subsystem $\mathbb{H}_A$ is:

$$S_A(|J\rangle) = Tr\left(P_A\frac{\mathbb{I}-iJ}{2}P_A \log\left|P_A\frac{\mathbb{I}-iJ}{2}P_a\right|\right), \tag{5.63}$$

where P_A is the projector to the phase space A of the subsystem, and $J = MJ_0M^{-1}$ is the squeezed complex structure.

The entanglement entropy

Given that our Hilbert space is decomposed as in equation (5.62), the definition of entanglement entropy is as follows.

First, we define the *density operator* ρ to be:

$$\rho \equiv |\psi\rangle\langle\psi|, \tag{5.64}$$

where $|\psi\rangle \in \mathbb{H}$. The *reduced density operators* ρ_A and ρ_B follow by taking the partial trace in the subspace A or B:

$$\rho_A \equiv Tr_B \rho \tag{5.65}$$

$$\rho_B \equiv Tr_A \rho. \tag{5.66}$$

Definition 5.1. *If the state* $|\psi\rangle \in \mathbb{H}$ *can be written as*

$$|\psi\rangle = |u\rangle_A \otimes |v\rangle_B, \tag{5.67}$$

where $|u\rangle_A \in \mathbb{H}_A$ *and* $|v\rangle_B \in \mathbb{H}_B$*, the state is called* ***separable****. Otherwise, it is called* ***entangled****.*

It is straightforward to notice that:

$$\text{If} \begin{cases} |\psi\rangle \text{ is separable, then } \rho_A \text{ and } \rho_B \text{ are pure} \\ |\psi\rangle \text{ is entangled, then } \rho_A \text{ and } \rho_B \text{ are mixed.} \end{cases}$$

The *von Neumann entropy* $S(\rho)$ is then defined as:

$$S(\rho) \equiv -Tr(\rho \ln \rho). \tag{5.68}$$

From this definition, the physical interpretation of this entropy is clear. The von Neumann entropy is set to quantify the uncertainty in the system state: if the system is sure to be in a given state, the state is pure, and the entropy is zero. If there is an uncertainty in the system state, the state is mixed, and the entropy will be nonzero. Given this preliminaries, we can now define the entanglement entropy.

Definition 5.2. *The entanglement entropy is defined to be the von Neumann entropy of the reduced states, that is,* $S(\rho_A)$ *and* $S(\rho_B)$*.*

Hence, the conclusion is that $S(\rho_A)$ and $S(\rho_B)$ measure how the subsystems are entangled, that is, they quantify the correlations between the subsystems $\mathbb{H}_A$ and $\mathbb{H}_B$.

5.2.3 Linear growth of the entanglement entropy

The primary objective of this section is to state, and to give a brief proof, of the main theorem derived in [50], in which the long time behavior of the entanglement entropy for quadratic Hamiltonians and initial Gaussian states is presented. We begin by studying the concept of instabilities in classical Hamiltonian systems, and by introducing the Lyapunov exponents. Then, we state the principal theorem and provide the main ingredients of the proof.

Consider a classical dynamical system with N degrees of freedom, and with phase space $V = \mathbb{R}^{2N}$. Observables on this space are described by smooth functions of $2N$ variables denoted ξ^a:

$$\begin{aligned} \mathcal{O} :& \Re^{2N} \to \Re \\ & \xi^a \to \mathcal{O}(\xi). \end{aligned} \tag{5.69}$$

Linear observables in this phase space are writen as:

$$q_i = q_{ia}\xi^a, \tag{5.70}$$

while quadratic observables are:

$$\mathcal{O} = \frac{1}{2} h_{ab}\xi^a\xi^b. \tag{5.71}$$

The dynamical equation for these observables are given by:

$$\dot{\mathcal{O}}(t) = \{\mathcal{O}(t), H(t)\} + \frac{\partial \mathcal{O}(t)}{\partial t}, \tag{5.72}$$

where the Poisson brakcets can be written in terms of the symplectic structure, as in equation (5.59). We call V^* the vector space formed by all linear observables, with elements v_a, and by w_a the elements of V. For the linear observables, the dynamical equation becomes:

$$\dot{\xi}^a(t) = \omega^{ab}\partial_b H(t). \tag{5.73}$$

In addition, if we assume that the Hamiltonian is quadratic, we can write:

$$\dot{\xi}^a(t) = K^a_b(t)\xi^b, \tag{5.74}$$

where the matrix K^a_b is defined as:

$$K^a_b(t) = \omega^{ac}h_{cb}(t). \tag{5.75}$$

Let us assumes that the solution of this equation can be given by:

$$\xi^a(t) = M^a_b(t)\xi^b(0). \tag{5.76}$$

Then, the problem now is reduced to solve the differential equation for M:

$$\dot{M}^a_b(t) = K^a_c(t)M^c_b(t), \tag{5.77}$$

with the identity as the initial condition. The solution of this last equation is a Dyson series.

Our interest now is investigate the effects of a linear perturbation $\delta\xi^a(t)$ on a solution $\xi_0^a(t)$. Therefore, we substitute:

$$\xi^a(t) = \xi_0^a(t) + \delta\xi^a(t) \tag{5.78}$$

in equation (5.73) to obtain the dynamical equation of the perturbation:

$$\delta\dot{\xi}^a(t) = K_b^a(t)\delta\xi^b(t). \tag{5.79}$$

Then, the evolution of the perturbation is given by:

$$\delta\xi^a(t) = M_b^a(t)\xi^b(0). \tag{5.80}$$

In order to determine the stability of the system, we can measure how two initially close configurations separate after a period of time. For that purpose, we need to introduce a metric g_{ab} and the norm:

$$||\delta\xi^a|| = \sqrt{g_{ab}\delta\xi^a\xi^b}. \tag{5.81}$$

The exponential rate of separation of two sufficiently close classical solutions is given by the Lyapunov exponent, defined as:

$$\lambda \equiv \lim_{t\to\infty} \frac{1}{t} \log \frac{||\delta\xi(t)||}{||\delta\xi(0)||}. \tag{5.82}$$

From this definition, we can characterize the stability of the system depending on the value of λ, that is:

$$\text{If } \lambda \begin{cases} < 0, & \text{convergent solution} \\ = 0, & \text{stable solution} \\ > 0, & \text{unstable solution.} \end{cases} \tag{5.83}$$

Up to this point, we are only describing the stability of classical systems. However, the Lyapunov exponents of the classical system are closely related to properties of the associated quantum system. The following theorem describes a relation between entanglement production and classical instabilities for bosonic system with initial Gaussian states:

Theorem 5.1. *Given a quadratic time-dependent Hamiltonian $H(t)$ with Lyapunov exponents λ_i, the long-time behavior of the entanglement entropy of a generic subsystem A is:*

$$S_A(t) \sim \left(\sum_{i=1}^{2N_A} \lambda_i \right) t \tag{5.84}$$

for all initial Gaussian states and all generic subsystems with N_A degrees of freedom.

Proof

We now present a general overview of the proof of this theorem. For a detailed exposition see sections 5 and 6 of [50]. Let us split our phase space into complementary subspaces:

$$V = A \bigoplus B, \text{ and} V^* = A^* \bigoplus B^*, \tag{5.85}$$

with dimension $\dim A = N_A$, and $\dim B = N_B$ and $N_A + N_B = N$. In each subspace we can choose a Darboux basis. Let us denote

$$D_A = (\theta^1, ..., \theta^{2N_A}) \tag{5.86}$$

as the basis in A^*, and

$$D_{A^*} = (v_r, ..., v_{2N_A}) \tag{5.87}$$

the basis for its dual, satisfying $\theta^r_a v^a_s = \delta^r_s$. Given a Darboux basis of the subsystem, we define the *symplectic cube* as:

$$\mathcal{V}_A = \left\{ \sum_{r=1}^{2N_A} c_r \theta^r | 0 \leq c_i \leq 1 \right\} \subset A^*. \tag{5.88}$$

We will be interested in how the volume of this cube will evolve in time. To compute its volume, we need to restrict our metric to the subspace. This is achieved by doing the contraction:

$$[G]_A = (\theta^r_a G^{ab} \theta^s_b). \tag{5.89}$$

It follows that the volume of the cube is:

$$\text{Vol}_G(\mathcal{V}_A) = \sqrt{\det[G]_A} = \sqrt{\det(\theta^r_a G^{ab} \theta^s_b)}. \tag{5.90}$$

It allow us to define a generalized Lyapunov exponent Λ_A as:

$$\Lambda_A = \lim_{t \longrightarrow \infty} \frac{1}{t} \log \frac{\text{Vol}_G(M^T(t)\mathcal{V}_A)}{\text{Vol}(\mathcal{V}_A)}, \tag{5.91}$$

where the symplectic evolution is driven in A^* by M^T.

The next step of the proof is to relate the Renyi entropy with the volume in phase space. We will use the Renyi entropy as an instrumental step in calculating the entanglement entropy, and it goes as follows. The Renyi entropy of order n is defined as :

$$R^{(n)}_A(|\psi\rangle) = -\frac{1}{n-1} \log Tr_{\mathbb{H}_A}(\rho^n_A). \tag{5.92}$$

It can be shown that, for Gaussian states, the Renyi entropy of order 2 and the entanglement entropy can be computed directly from the eigenvalues v of $i[J]_A$, where $[J]_A = (\theta^r_a J^a_b v^b_s)$[50]:

$$R^{(2)}_A = \sum_{i=1}^{N_A} \log(v_i) \text{ and } S_A = \sum_{i=1}^{N_A} S(v_i), \text{ where } S(v) = \frac{v+1}{2} \log \frac{v+1}{2} - \frac{v-1}{2} \log \frac{v-1}{2}. \tag{5.93}$$

The complex structure can be written in terms of the metric as:

$$J^a_b = G^{ab}\omega_{cb}, \tag{5.94}$$

and it follows that:

$$[J]_A = -[G]_A\omega_A. \tag{5.95}$$

Taking the determinant of this last equation we obtain:

$$|\det[J]_A| = \det[G]_A \det\omega_A = (\mathrm{Vol}_G(\mathcal{V}_A)). \tag{5.96}$$

Comparing equations (5.93) and (5.96) we conclude that:

$$R^{(2)}_A(|J\rangle) = \log \mathrm{Vol}_G(\mathcal{V}_A). \tag{5.97}$$

The final step of the proof relies in the association of the Renyi and entanglement entropies. We can do this as follows. The time evolution of the Renyi entropy is:

$$R^{(2)}_A\,(U(t)|J_0\rangle) = \log \mathrm{Vol}_{G(t)}(\mathcal{V}_A). \tag{5.98}$$

Equivalently, we can fix the metric and let the volume of the cube evolve:

$$R^{(2)}_A\,(U(t)|J_0\rangle) = \log \mathrm{Vol}_{G_0}(M^T(t)\mathcal{V}_A). \tag{5.99}$$

The last ingredient that we need is the fact that the Renyi entropy bounds the entanglement entropy, as can be seen in the image below:

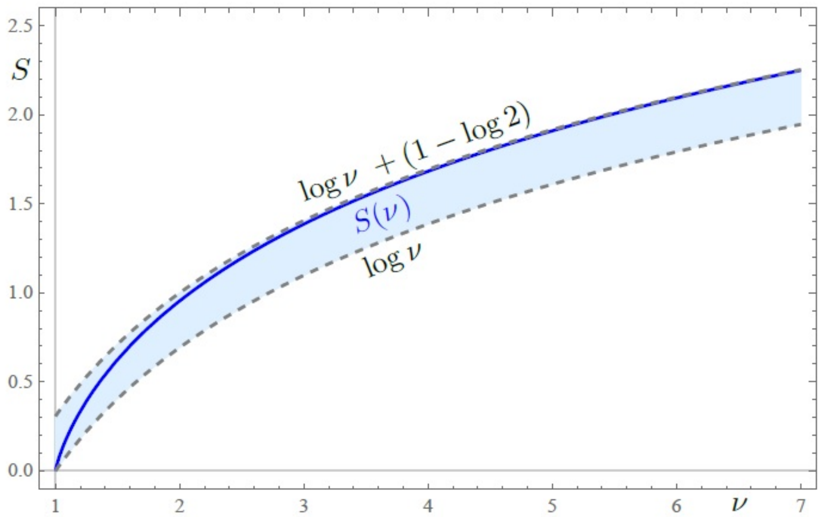

Figure 7 – Bounds on the entanglement entropy.

Therefore, since the difference between the entanglement entropy and the Renyi entropy is bounded by a state independent constant, we have that:

$$\lim_{t\longrightarrow\infty}\frac{1}{t}[S_A(U(t)|J_0\rangle) - R^{(2)}_A(U(t)|J_0\rangle)] = 0, \tag{5.100}$$

and finally:

$$\lim_{t\longrightarrow\infty}\frac{S_A(U(t)|J_0\rangle)}{t}=\lim_{t\longrightarrow\infty}\frac{1}{t}\log\mathrm{Vol}_{G_0}(M^T(t)\mathcal{V}_A). \tag{5.101}$$

□

5.2.4 Numerical Calculation

The main interest is to solve the differential equation (5.32) for the sinusoidal case. One strategy is to solve by expanding in a Dyson series, but this is too hard for the case of interest. Another approach is to solve (5.32) directly by numerical techniques.

The program used in the numerical solution was *Wolfram Mathematica 10.0*, with the function *NDSolve*, which finds a numerical solution to the ordinary differential equation for the matrix $M(t)$ with the independent variable t in the range t_{min} to t_{final}. The initial condition was set to be the identity for $t = 0$, that is, $M[0] = \mathbb{I}$:

```
sol = NDSolve[{B'[t] == VT[t].B[t], B[0] == IdentityMatrix[2*dim]}, B, {t, 0, tfinal}, WorkingPrecision → 20]

{{B → InterpolatingFunction[ Domain: {{0, 10000.000000000000000}} Output dimensions: {10, 10} ]}}
```

Figure 8 – Part of the code that solves equation (5.32). Here *dim* means half of the Bogoliubov matrix dimension. The integration is from 0 to t_{final}. The function *WorkingPrecision* specifies how many digits of precision should be maintained in internal computations

We notice that there are two relevant and unavoidable approximations:

Approximation 1: The first one is concerning solving equation (5.32): it is not possible to solve it exactly analytically nor numerically. To solve analytically, we need to expand the equation in a Dyson series, which represents an infinite set of integrals of increasing difficulty, and an adequate set of parameters must be chosen to make the first-orders a good approximation. The numerical solution has the advantage of resolving the equation to all orders, but necessarily introduces numerical errors.

Approximation 2: The second inevitable approximation regards the dimension: since we have an infinite number of modes, the Bogoliubov transformation is an infinite matrix. This problem is overcome in part if we stay in some regime where the relevant processes are restricted to a finite region of the Bogoliubov matrix. For example, if the oscillation frequency of the mirror is low, processes involving the production of high-frequency particles are very unlikely to occur.

As will be detailed, our strategy is to choose the amplitude and frequency of the mirror such that these two approximations are justified.

5.3 Sinusoidal oscillation

In this section we apply all the techniques presented in the previous sections to a specific system: the dynamical Casimir effect where one boundary is fixed, and the other is oscillating sinusoidally with amplitude A and period τ, that is:

$$\begin{cases} x_1(t) = 0. \\ x_2(t) = L_0 + A\sin(\nu t), \text{ with } \nu = \frac{2\pi}{\tau}. \end{cases} \tag{5.102}$$

In the next two subsections, we will focus on two different scenarios: the 1+1 and the 2+1 dimensional DCE. As will be shown, we were able to determine numerically the production of entropy in both cases. In the 1+1 case, the production is sublinear, while in the 2+1 the production is linear, with the rate determined by the classical Lyapunov coefficients. We begin with the 1+1 case, where we were also able to compute the Bogoliubov transformation analytically for a zero order expansion in the oscillation amplitude. Then, in the 2+1 case, we show that is possible to reduce the problem to a single oscillator at parametric resonance.

To the mirror trajectory of equation (5.102), the differential evolution equation becomes:

$$\frac{dM}{dt} = VM, \tag{5.103}$$

where

$$V = i\Omega + M^{(2)}\frac{dx_2}{dt} = i\Omega + M^{(2)}\frac{2\pi A}{\tau}\cos\left[\frac{2\pi t}{\tau}\right]. \tag{5.104}$$

5.3.1 1+1 dimensional DCE

First and second order analytical solutions

In this subsection, we obtain analytical results to equation (5.103) to first order in the Dyson series. Furthermore, we also assume that the oscillation amplitude is small enough so that we can say that the cavity length is constant. With these assumptions we recover the results presented in [48].

To this mirrors movement, and integrating in one period, equation (5.49) becomes, to first order:

$$M_I(\tau) = \left(\mathbb{I} + \int_0^\tau dt' e^{-i\Theta(t')}Q^{(2)}e^{i\Theta(t')}\frac{dx_2}{dt'}\right). \tag{5.105}$$

The integrand can be written:

$$\begin{aligned} \left[e^{-i\Theta(t')}Q^{(2)}e^{i\Theta(t')}\right]_{mn}\frac{dx_2}{dt'} &= \left[e^{-i\Theta(t')}\right]_{mk}Q^{(2)}_{kl}\left[e^{i\Theta(t')}\right]_{ln}\frac{dx_2}{dt'} \\ &= e^{-i\int_0^\tau dt'\omega_m(t')}Q^{(2)}_{mn}e^{i\int_0^\tau dt'\omega_m(t')}\frac{dx_2}{dt'}, \end{aligned} \tag{5.106}$$

resulting in:

$$[M_I(\tau)]_{mn} = \left(\mathbb{I}_{mn} + Q^{(2)}_{mn} \int_0^\tau dt' e^{-it\omega_m} A \frac{2\pi}{\tau} cos\left(\frac{2\pi t}{\tau}\right) e^{it\omega_n}\right). \tag{5.107}$$

Let us define the integral above as $I[y]$, where y stands for the indices m and n:

$$I[y] \equiv \int_0^\tau dt' \frac{2\pi A}{\tau} cos\left(\frac{2\pi t}{\tau}\right) e^{ity/L_0}. \tag{5.108}$$

We see that $I[y=0]=0$ if $m=n$, that is $y=0$.For $m \neq n$:

- For the upper left part of $M_I(\tau)$:

$$[M_I(\tau)]_{mn} = \left(\mathbb{I}_{mn} + A^{(2)}_{mn} I[-m+n]\right). \tag{5.109}$$

- For the upper right part of $M_I(\tau)$:

$$[M_I(\tau)]_{mn} = \left(\mathbb{I}_{mn} + B^{(2)}_{mn} I[-m-n]\right). \tag{5.110}$$

The next step is to compute the matrices $A^{(2)}$ and $B^{(2)}$ and the integral $I[y]$. The matrix elements calculations are developed in Appendix A and the results are:

$$A^{(2)}_{mn} = \begin{cases} 0, m=n \\ \frac{(-1)^{1+m+n}\sqrt{mn}}{L_0(m-n)}, m \neq n \end{cases} \tag{5.111}$$

$$B^{(2)}_{mn} = \frac{(-1)^{m+n}\sqrt{mn}}{L_0(m+n)} \tag{5.112}$$

$$I[y] = \frac{2iAL_0 y\tau \left(-1 + e^{\frac{iy\pi\tau}{L_0}}\right)}{4L_0^2 - y^2\tau^2} \tag{5.113}$$

Substituting these results in (5.107) and then in (5.50), it follows:

$$M(\tau) = e^{\tau K_0}(\mathbb{I} + Q^{(2)} I[y]). \tag{5.114}$$

Thus, the Bogoliubov coefficients can be extracted:

$$\alpha_{mn}(\tau) = \begin{cases} e^{i\tau\omega_m}, m=n \\ (-1)^{1+m+n} e^{i\omega_m\tau} \frac{2\pi A}{\tau L_0} \sqrt{\omega_m\omega_n} i \left\{\frac{1-e^{i(\omega_m-\omega_n)\tau}}{(\omega_m-\omega_n)^2 - (\frac{2\pi}{\tau})^2}\right\}, m \neq n \end{cases} \tag{5.115}$$

$$\beta_{mn}(\tau) = (-1)^{m+n} e^{-i\omega_m\tau} \frac{2\pi A}{\tau L_0} \sqrt{\omega_m\omega_n} i \left\{\frac{1-e^{i(\omega_m+\omega_n)\tau}}{(\omega_m+\omega_n)^2 - (\frac{2\pi}{\tau})^2}\right\} \tag{5.116}$$

To calculate the Bogoliubov transformation up to second order, we just add one more term in equation (5.50):

$$M(\tau) = e^{K_0\tau}\left(\mathbb{I} + \int_0^\tau dt' V_I(t') + \int_0^\tau dt' \int_0^{t'} dt'' V_I(t') V_I(t'')\right) \tag{5.117}$$

Let us call the second order integral $SI(\tau)$:

$$SI(\tau) \equiv \int_0^{\tau} dt' \int_0^{t'} dt'' V_I(t')V_I(t'') \tag{5.118}$$

Then:

$$\begin{aligned} SI(\tau) &= \int_0^{\tau} dt' \int_0^{t'} dt'' e^{-t'K_0} V(t') e^{t'K_0} e^{-t''K_0} V(t'') e^{t''K_0} \\ &= \int_0^{\tau} dt' \int_0^{t'} dt'' \left[e^{-tK_0}\right]_{mo} [V(t')]_{op} \left[e^{tK_0}\right]_{pq} \left[e^{-t''K_0}\right]_{qr} [V(t'')]_{rs} \left[e^{t''K_0}\right]_{sn}. \end{aligned} \tag{5.119}$$

All the terms with the exponential of K_0 are zero outside the diagonal. The terms that are nonzero are:

$$\begin{aligned} SI(\tau) &= \int_0^{\tau} dt' \int_0^{t'} dt'' \left[e^{-tK_0}\right]_{mm} [V(t')]_{mp} \left[e^{tK_0}\right]_{pp} \left[e^{-t''K_0}\right]_{qr} [V(t'')]_{pn} \left[e^{t''K_0}\right]_{nn}. \\ &= \int_0^{\tau} dt' \int_0^{t'} dt'' e^{-it\omega_m} [V(t')]_{mp} e^{it'\omega_p} e^{-it''\omega_p} [V(t'')]_{pn} e^{it''\omega_n}. \\ &= Q_{mp}Q_{pn} \int_0^{\tau} dt' \int_0^{t'} dt'' e^{-it'(\omega_m-\omega_p)} e^{-it''(\omega_p-\omega_n)} \left(\frac{2\pi A}{\tau}\right)^2 cos\left[\frac{2\pi t'}{\tau}\right] cos\left[\frac{2\pi t''}{\tau}\right]. \end{aligned} \tag{5.120}$$

The integrals above are very similar to (5.108). The final equation, despite simple, is very long and we shall not exhibit here.

Numerical results

Our main results come from the numerical analysis of equation (5.103). The general procedure to compute the entanglement entropy is this: the numerical integration will provide us with a Bogoliubov transformation $M(t)$ from $t = 0$ to any time t. With the matrix $M(t)$ we can compute the squeezed complex structure J, defined as:

$$J = MJ_0M^{-1}, \tag{5.121}$$

where J_0 is the reference complex structure defined in equation (5.54). Finally, with the squeezed complex structure we compute the entropy using:

$$S_A(|J\rangle) = Tr\left(P_A \frac{\mathbb{I} - iJ}{2} P_A \log\left|P_A \frac{\mathbb{I} - iJ}{2} P_a\right|\right). \tag{5.122}$$

To follow this scheme, the first thing we note is that the transformation matrices $M(t)$ in equations (5.32) and (5.122) are in different basis. The first is a transformation in the field modes, while the second is a transformation in the fields and in the conjugate momentum. Nevertheless, we can easily move from one basis to another with the block-diagonal matrix T:

$$T = \begin{pmatrix} \frac{1}{\sqrt{2\omega(m)}} & \frac{1}{\sqrt{2\omega(m)}} \\ -i\sqrt{\frac{\omega(m)}{2}} & i\sqrt{\frac{\omega(m)}{2}} \end{pmatrix}. \tag{5.123}$$

In the field/momentum basis the generator V is:

$$V' = TVT^{-1}. \tag{5.124}$$

The next step is to choose the subspace A of the Hilbert space $\mathbb{H}$. We choose the mirror frequency to be twice the frequency of the first mode, which is the mode with $m = 1$ and energy $\omega = \frac{\pi}{L_0}$. Our choice of A will be the subspace spanned by the resonant mode. Hence, we have all the necessary ingredients to compute the entanglement entropy between the subspace A and its complement.

The following images shows the production of the entanglement entropy in time, related to particle creation and mode mixing, with different sets of parameters. Since the desired mirror frequency is fixed by the resonant condition, the period τ is determined, and the only free parameters are the pair A, L_0. We notice that, for a fixed value of L_0, the energy delivered by the mirror is fully determined by the amplitude A. Since the period is already fixed, a higher value of A will imply in a higher velocity of the mirror. In the images bellow, we fixed the value of L_0 to be $L_0 = 10^4$, and varied the value of A.

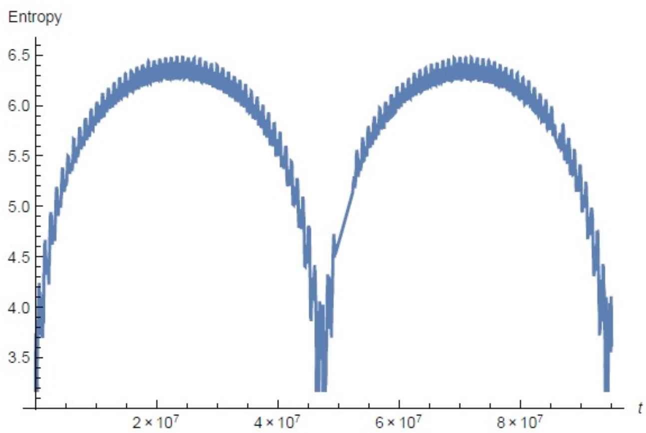

Figure 9 – Entanglement entropy production *versus* time for $dim = 5$, $A = 10^2$, and $L_0 = 10^4$.

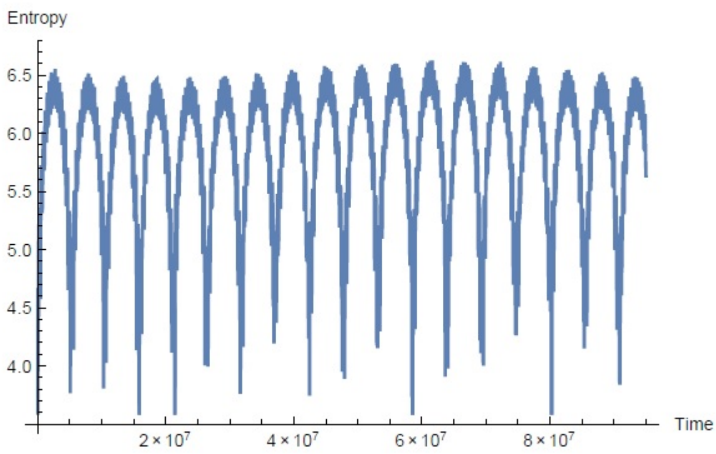

Figure 10 – Entanglement entropy production *versus* time for $dim = 5$, $A = 3 \times 10^2$, and $L_0 = 10^4$.

We see that in both plots the entropy is bounded. In the second image, the parameters we used correspond to a mirror with higher velocity in comparison to the first. We see that the oscillation period is smaller.

As discussed at the beginning of this chapter, the classical Lyapunov exponents can be used to characterize the long time behavior of the entropy production. Theorem 5.1 states that the sum of the first $2N_A$ positive Lyapunov coefficients, which we denote as Λ_A, gives the asymptotic rate of linear growth. Since our choice of subspace has dimension $\dim N_A = 1$, we need to sum over the first 2 positive coefficients. To compute the Lyapunov coefficients, we use the stroboscopic matrix S defined as:

$$S \equiv \frac{\log(M(\tau))}{\tau}. \tag{5.125}$$

The entropy evolution is modulated by oscillations. We are interested in how the evolution will be in full periods, that is, we will look at how the peaks of the entropy grow. Then, the Lyapunov exponents will be the eigenvalues of S, and the corresponding eigenvectors gives the direction of the exponential growth.

The following table presents the values of these eigenvalues for the different sets of parameters and working precisions. All of them are very close to zero, and we will only present the order of magnitude of the bigger eigenvalue in absolute value for each case.

A	L_0	WP=10	WP=15	WP=20	WP=25
10^{-3}	1	10^{-4}	10^{-7}	10^{-9}	10^{-13}
10^{-2}	1	10^{-3}	10^{-7}	10^{-9}	10^{-12}
10^{-2}	10	10^{-5}	10^{-8}	10^{-10}	10^{-12}
10^{-1}	10	10^{-4}	10^{-8}	10^{-10}	10^{-12}
1	10^{2}	10^{-6}	10^{-9}	10^{-11}	10^{-14}
10^{1}	10^{3}	10^{-7}	10^{-10}	10^{-13}	10^{-15}
10^{2}	10^{4}	10^{-7}	10^{-12}	10^{-14}	10^{-17}

Table 1 – Eigenvalues of the stroboscopic matrix for different sets of parameters.We see that, in all cases, the eigenvalues goes to zero as we increase the working precision WP.

Basically, we analyzed the eigenvalues for a wide range of parameters, and we see that, when augmenting the working precision, the eigenvalues approach zero in all cases.

Another fundamental aspect is the dependence of the above results with respect to the size of the computed matrix $M(t)$. The next table shows how the value of the sum of the Lyapunov coefficients is changing when we increase the size of $M(t)$. In this table, all results were computed using $A = 10^{-3}, L_0 = 1$.

$\dim M(t)$	Λ_A
5	10^{-9}
10	10^{-8}
15	10^{-7}
20	10^{-9}
25	10^{-8}
30	10^{-7}
35	10^{-6}
40	10^{-6}
45	10^{-5}

Table 2 – Behavior of the eigenvalues with an increasing size of the subsystem, where dim is the number of normal modes.

We see that the value of Λ_A is increasing when aumenting the size of $M(t)$. However, this occurs due to a higer acumulation of numerical error, since the matrix is bigger. Nevertheless, in each case, if we increase the working precision, then the eigenvalues will go to zero again.

Finally, we must estimate the error intrinsic in the numerical integration. One way to measure this is to see how close to a symplectic matrix the Bogoliubov transformation is along the time evolution. For a symplectic transformation, we would have:

$$M(t)\omega M(t)^T - \omega = 0 \text{ for all t.} \tag{5.126}$$

We can implement the above analysis by computing the norm of equation (5.126) for one period. The results are shown in the table bellow.

WP	Norm
10	10^{-4}
15	10^{-6}
20	10^{-7}
25	10^{-8}

Table 3 – Norm of equation (5.126) with increasing working precision.

This table is for a fixed value of A and L_0. However, this behavior is the same for any set of parameters.

Thus, we conclude that in the 1+1 dimensional DCE the Lyapunov exponents are zero, since the non-zero values obtained in the numerical calculation reduce to zero with an increasing precision. According to [51], the entropy must grow according to:

$$S_A(t) = \Lambda_A t + C_A \ln(t) + X_A(t), \tag{5.127}$$

where C_A is an integer, and $X_A(t)$ is a bounded function. We computed Λ_A and showed that it is zero. Therefore, the entanglement production must grow logarithmicaly or be bounded. This agrees with our numerical results for the entropy evolution, where we showed that it does not grow indefinitely. Further investigations are necessary to determine whether in the limit of large subsystems the entanglement entropy growth is logarithmic or bounded.

5.3.2 2+1 dimensional DCE

Now we proceed to the 2+1 dimensional DCE. As we will show, the relevant difference from the 1+1 case is that now the difference in energy between modes is no longer constant, and this will allow us to drastically simplify the evolution equation. We begin by extending the Lock-Fuentes model to 2+1 dimensions.

The formulation of the 2+1 dimensional DCE is very similar to the (1+1)D scenario. The Lock-Fuentes model, as described in their work's appendix, can easily be extrapolated to the N spatial dimensions case. The stationary solutions for a 2D rectangular cavity with dimensions L_x, L_y, and where the boundaries are located at $x = 0, x = x_1, y = 0, y_1$, is written as:

$$\phi_{\vec{m}} = N_{\vec{m}} e^{-i\omega_{\vec{m}} t} \sin[\omega_m (x - x_1)] \sin[\omega_n (y - y_1)], \tag{5.128}$$

where

$$\omega_{\vec{m}} = \left[\omega_m^2 + \omega_n^2\right]^{\frac{1}{2}} = \left[\left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2\right]^{\frac{1}{2}} \tag{5.129}$$

is determined by the two integers m, n, and $N_{\vec{m}}$ is a normalization constant. From these equations, the same arguments in building equation (5.32) are now valid, and the general equation reads:

$$\frac{dM(t)}{dt} = \left[i\Omega + Q_k^{(1)} \frac{dx_1^k}{dt} + Q_k^{(2)} \frac{dx_2^k}{dt} \right] M(t), \tag{5.130}$$

where the sum over k is implicit, and

$$Q_k^{(j)} = \begin{bmatrix} A_k^{(j)} & B_k^{(j)} \\ B_k^{(j)*} & A_k^{(j)*} \end{bmatrix}, (A_k^{(j)})_{mn} = \left(\frac{\partial \phi_m}{\partial x_j^k}, \phi_n \right) \text{and } (B_k^{(j)})_{mn} = -\left(\frac{\partial \phi_m}{\partial x_j^k}, \phi_n^* \right), \tag{5.131}$$

with $j = 1, 2$.

Let us consider the situation where L_x is moving sinusoidally according to (5.102) and L_y is fixed. This problem was analytically investigueted by Dodonov in [52]. He found that, at parametric resonance in (2+1)D, we can neglect the coupling between modes, the dominant contribution to the time evolution being the modulation of the frequencies. As a consequence, the terms $Q_k^{(j)}$ in (5.130), which are responsible for the mode mixing, can be neglected.

The evolution equation now has the simple form:

$$\frac{dM(t)}{dt} = i\Omega M(t), \tag{5.132}$$

and we have $N = 2\,\mathrm{dim}$ decoupled differential equations. Again, we use equation (5.122) to compute directly the entanglement entropy between the subspace spanned by the resonant mode, where $\omega_{m=1,n=1}$, and its complement. However, a special care must be taken. To compute the entropy via equation (5.122) we need to do a projection over the subspace of interest. Since distinct modes evolve independently, if we choose the subsystem to be a single mode, the entropy will be zero. In order for the entropy to be non-zero, we can consider *any* subspace that intersects more then one mode. As a simple example, we can consider a mixture of two modes. We first define a basis change of the kind:

$$\Phi_1 = \frac{\phi_1 + \phi_2}{\sqrt{2}}, \qquad \Phi_2 = \frac{\phi_1 - \phi_2}{\sqrt{2}}, \tag{5.133}$$

and

$$\Pi_1 = \frac{\pi_1 + \pi_2}{\sqrt{2}}, \qquad \Pi_2 = \frac{\pi_1 - \pi_2}{\sqrt{2}}. \tag{5.134}$$

The transformation matrix C that connects the new and the old basis is:

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{5.135}$$

We take as our subsystem the symplectic subspace spanned by Φ_1, Π_1.

Next, we look at the term $P_A \frac{\mathbb{I}-iJ}{2} P_A$ inside equation (5.122). This projected term is a 2×2 matrix. However, we first construct a 4×4 projection F_4, defined as the restriction of $\frac{\mathbb{I}-iJ}{2}$ to the subspace spanned by the first two modes. Then, using the transformation matrix C, we perform a change of basis in F_4, that is:

$$F'_4 = C F_4 C^{-1}. \tag{5.136}$$

Since now the subspaces are mixed due to the change of basis, we can perform the projection to the subspace spanned by Φ_1 and Π_1 and compute the entropy, which will not be zero.

The image bellow show the entanglement entropy production for $A = 5 \times 10^{-3}, L_0 = 1$.

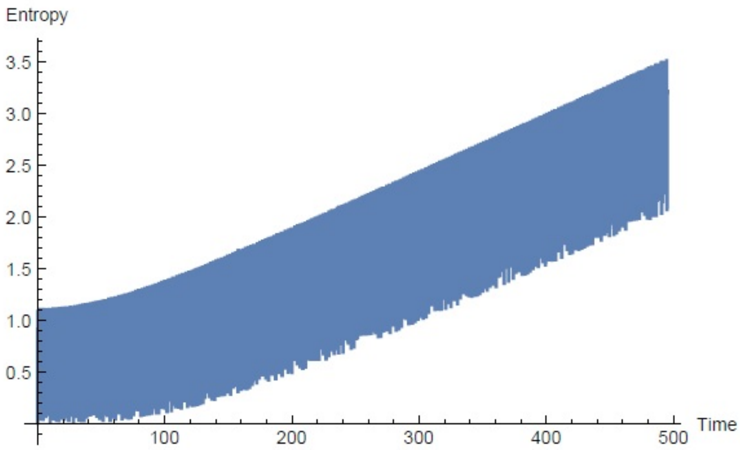

Figure 11 – Entanglement entropy production *versus* time for $dim = 5$, $A = 5 \times 10^{-3}$, and $L_0 = 1$.

Again, we can use the stroboscopic matrix to compute the Lyapunov exponents. The prediction of theorem 5.1 is that they will give the rate of entanglement production. For the parameters that we choose, the numerically computed value of of Λ_A is:

$$\Lambda_A = 0.0055. \tag{5.137}$$

In fact, if we increase the size of matrix $M(t)$ this value does not change. In the next image, we compare this rate with the entanglement evolution.

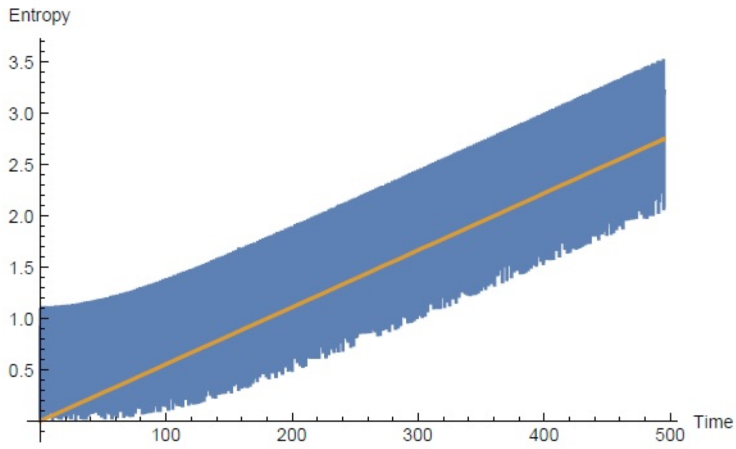

Figure 12 – Entanglement entropy production (in blue) and the Lyapunov exponent (in orange) *versus* time for $dim = 5$, $A = 5 \times 10^{-3}$, and $L_0 = 1$.

We can also compute Λ_A analytically for the (2+1)D scenario. A simple model for the parametric resonance is the Mathieu equation:

$$\ddot{x}(t) + \omega^2(t)x(t) = 0, \tag{5.138}$$

where $\omega^2(t) = (\omega_0^2 + \varepsilon \sin[\Omega t])$ is the natural frequency, which oscillates around ω_0 with frequency Ω and amplitude ε. It is a well known fact that the *Floquet* exponents μ can be computed directly from this equation as [50]:

$$\mu = \frac{\epsilon}{4\omega_0}. \tag{5.139}$$

Furthermore, the Floquet exponents are the Lyapunov exponents in this case for the stroboscopic evolution. Therefore, we can write equation (5.129) in this form and compute the Lyapunov exponents directly:

$$\omega_{\vec{m}} = \pi \left(\frac{1}{(L_0 + A\sin(2\omega_0))^2} + \frac{1}{L_0^2} \right)^{\frac{1}{2}}. \tag{5.140}$$

Doing a series expansion around $\epsilon = A/L_0 = 0$, we have that:

$$\omega_{\vec{m}} = \frac{2\pi}{L_0}\left(1 - \frac{\epsilon \sin(2\omega_0 t)}{2}\right) \tag{5.141}$$

$$\omega_{\vec{m}}^2 = \frac{2\pi^2}{L_0^2}(1 - \epsilon \sin(2\omega_0 t)). \tag{5.142}$$

Comparing this last equation with the Mathieu equation, we conclude that:

$$\mu = \Lambda_A = \frac{\sqrt{2}\pi\epsilon}{4L_0}. \tag{5.143}$$

Finally, substituting the values for A and L_0, we obtain:

$$\Lambda_A = 0.0055, \tag{5.144}$$

which agrees with equation (5.137).

With the evolution matrix, we can also compute the production of particles for every time, obtaining:

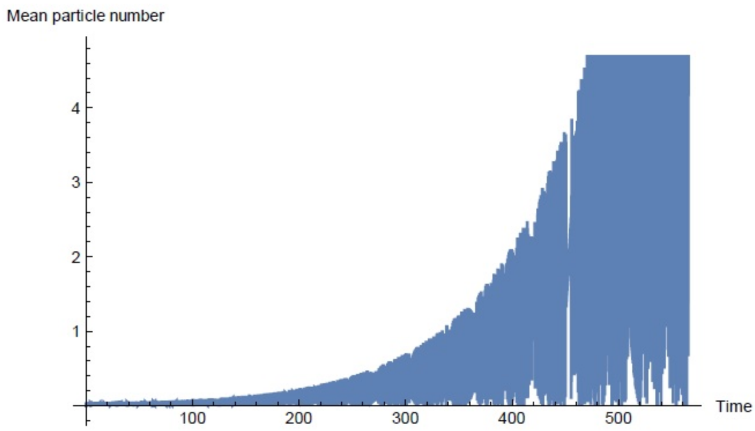

Figure 13 – Mean number of particles *versus* time for $dim = 5$, $A = 5 \times 10^{-3}$, and $L_0 = 1$.

Therefore, the production of particles in the (2+1)D dynamical Casimir effect grows exponentially in time.

6 Conclusion

In this work, our primary goal was to analyze the production of quantum correlations in the dynamical Casimir effect, which consists of the production of particles from the vacuum due to the presence of moving boundaries. This required the prior study of quantum field theory in time dependent backgrounds and symplectic techniques in classical mechanics and quantum information, used here to describe the dynamics of Gaussian states. In the presence of two mirrors, the system can display the phenomenon of parametric resonance, which leads to a pronounced amplification of the effect. We focused on the resonant case and studied the time evolution of the entanglement entropy numerically and analytically to quantify the production of correlations.

In chapter 2 we introduced the necessary mathematical structure of symplectic geometry that was fundamental in the subsequent chapters. Besides, we described the role of symplectic geometry in the formulation of classical mechanics. Chapter three is dedicated to defining the canonical quantization process, while in chapter four we apply this process to the quantization of a real, scalar field in flat and in curved spacetime. Furthermore, in this part, we discussed the definition and the non-triviality of the vacuum state, where we concluded that the particle notion is not always applicable. As an example of a system with a well-defined particle concept, we computed how a Rindler observer sees the Minkowski vacuum, finding a thermal bath as a result. We also studied the dynamics of a Klein-Gordon field in the curved spacetime of a gravitationally collapsing body. In this case, we computed the expected value of the mean number of particles for a distant observer, finding a Planck spectrum as a result, in a phenomenon that is now known as the Hawking radiation.

In chapter 5, we provided a complete description of the Lock-Fuentes model of the dynamical Casimir effect, which we then applied to this system at parametric resonance. We presented a full implementation of the model, with analytical and numerical calculations. As an outcome, we obtained a symplectic evolution matrix, known as the Bogoliubov transformation. Furthermore, we analyzed two other techniques that allowed us to compute the entanglement entropy evolution for this system. In the (1+1)D scenario, we showed that the Lyapunov exponents are zero, and the long-time behavior of the entropy is sublinear. In the (3+1)D, we showed that the entanglement entropy grows linearly, with the rate determined by the Lyapunov exponent. Moreover, the particle production, in this case, is exponential, which turns this system into a potentially measurable experiment.

A Matrix element determination

In this section, we compute explicitly the form of the elements of matrices $A^{(2)}$ and $B^{(2)}$. These results are general and can be applied to any mirror trajectory.

A.0.1 Matrix A$^{(2)}$

The matrix $A^{(2)}$ is defined as in equation (5.131) and, to determine the composing elements, we only need to do the Klein- Gordon inner product, defined in equation (5.24).

The approach to compute this inner product is the following: we define an instantaneous basis at t_0 to be:

$$\phi_m(t_0, x) = N_m \sin[\omega_m(x - x_1)], \tag{A.1}$$

$$\frac{\partial \phi_m}{\partial t}(t_0, x) = -i\omega_m N - M \sin[\omega_m(x - x_1)], \tag{A.2}$$

where $\omega_m = \frac{m\pi}{x_2 - x_1}$ is the frequency of the mode m and $N_m = \frac{1}{\sqrt{m\pi}}$ is the normalization constant. Hence, the computation of the inner product is straightforward, which is defined as:

$$A^{(2)}_{mn} = \left(\frac{\partial \phi_m}{\partial x_2}, \phi_n\right) = -i \int_{x_1}^{x_2} dx \left[\frac{\partial \phi_m}{\partial x_2} \partial_t \phi_n^* - \frac{\partial}{\partial t}\frac{\partial \phi_m}{\partial x_2} \phi_n^*\right]. \tag{A.3}$$

We begin by computing each individual term inside the integral:

$$\frac{\partial \phi}{\partial x_2} = -N_m \cos[\omega_m(x - x_1)](x - x_1)\frac{m\pi}{(x_2 - x_1)^2}, \tag{A.4}$$

$$\frac{\partial \phi_n^*}{\partial t} = -\omega_n N_n \sin[\omega_n(x - x_1)], \tag{A.5}$$

and:

$$\frac{\partial}{\partial t}\frac{\partial \phi_m}{\partial x_2} = \frac{\partial}{\partial x_2}\frac{\partial \phi_m}{\partial t} \tag{A.6}$$

$$= -iN_m \left\{-\frac{m\pi}{(x_2 - x_1)^2} \sin[\omega_m(x - x_1)] - \omega_m \cos[\omega_m(x - x_1)]\frac{(x - x_1)m\pi}{(x_2 - x_1)^2}\right\}. \tag{A.7}$$

Let us call the first term of the integral I_1. Then, we have:

$$I_1 = -i \int_{x_1}^{x_2} dx \frac{\partial \phi_m}{\partial x_2} \partial_t \phi_n^*. \tag{A.8}$$

$$I_1 = -\sqrt{\frac{m}{n}}\frac{\omega_n}{L^2} \int_{x_1}^{x_2} dx \cos[\omega_m(x - x_1)] \sin[\omega_n(x - x_1)(x - x_1)] \tag{A.9}$$

$$I_1 = \begin{cases} -\sqrt{\frac{m}{n}}\frac{\omega_n}{L^2}\frac{(-1)^{m+n}L^2 n}{(m^2 - n^2)\pi}, \; m \neq n \\ \sqrt{\frac{m}{n}}\frac{\omega_m}{L^2}\frac{L^2}{4m\pi}, \; m = n. \end{cases} \tag{A.10}$$

Similarly, the second therm of the integral ,I_2, gives:

$$I_2 = -i\int_{x_1}^{x_2} dx - \frac{\partial}{\partial t}\frac{\partial \phi_m}{\partial x_2}\phi_m^*$$
$$I_2 = -\sqrt{\frac{m}{n}}\frac{1}{L^2}\int_{x_1}^{x_2} dx\{\sin[\omega_m(x-x_1)]\sin[\omega_n(x-x_1)]+$$
$$\omega_m\cos[\omega_m(x-x_1)]\sin[\omega_n(x-x_1)](x-x_1)\} \tag{A.11}$$
$$I_2 = \begin{cases} -\sqrt{\frac{m}{}n}\frac{1}{L^2}\frac{\omega_m(-1)^{m+n}L^2 n}{(m^2-n^2)\pi}, \, m \neq n \\ -\sqrt{\frac{m}{}n}\frac{1}{L^2}\left(\frac{L}{2} - \frac{\omega_m L^2}{4m\pi}\right), \, m = n \end{cases}$$

Summing I_1 and I_2 we have, finally:

$$A_{mn}^{(2)} = \begin{cases} \frac{-(-1)^{m+n}\sqrt{mn}}{L(m-n)}, \, m \neq n \\ 0, \, m = n. \end{cases} \tag{A.12}$$

A.0.2 Matrix B$^{(2)}$

The computation and results for $B_{mn}^{(2)}$ are exactly the same as $A_{mn}^{(2)}$, except from a minus signal. This minus signal is due to the fact that the $B^{(2)}$ elements are defined with respect to the conjugate of the mode:

$$B_{mn}^{(2)} = -\left(\frac{\partial \phi_m}{\partial x_2}, \phi_n^*\right) = i\int_{x_1}^{x_2} dx\left[\frac{\partial \phi_m}{\partial x_2}\partial_t\phi_n - \frac{\partial}{\partial t}\frac{\partial \phi_m}{\partial x_2}\phi_n\right]. \tag{A.13}$$

In summary, the final result is:

$$A_{mn}^{(2)} = \begin{cases} \frac{(-1)^{1+m+n}\sqrt{\omega_m\omega_n}}{(x_2-x_1)(\omega_m-\omega_n)}, m \neq n \\ 0, m = n \end{cases} \quad \text{, and } B_{mn}^{(2)} = \frac{(-1)^{m+n}\sqrt{\omega_m\omega_n}}{(x_2-x_1)(\omega_m+\omega_n)}, \tag{A.14}$$

where x_2 and x_1 are general, time dependent, trajectories of the bounderies.

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