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Phase transition on a randomly horizontally stretched square lattice

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"A persistência é o caminho do êxito".
Charles Chaplin

Resumo

Nesta tese consideramos um modelo de percolação de elos na rede quadrada horizontalmente esticada, que é obtida alterando a distância entre as colunas de $\mathbb{Z}_+^2$ de acordo com uma família de cópias independentes e identicamente distribuídas (i.i.d.) de uma variável aleatória positiva ξ, onde ξ satisfaz $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) < \infty$, para toda constante $c > 64$. Declaramos independentemente cada elo vertical aberto com probabilidade p e cada elo horizontal aberto com probabilidade $p^{|e|}$, onde $|e|$ é o comprimento do elo e. Construimos um esquema de renormalização multiescala e o usamos para mostrar a existência de percolação construindo um cluster aberto infinito. Com isso, mostramos que o processo que percolação considerado exibe uma transição de fase não trivial.

Palavras chave: percolação; rede horizontalmente esticada; transição de fase; ambientes aleatórios; renormalização multiescala.

Abstract

In this thesis we consider a bond percolation model on a horizontally stretched square lattice which is obtained by stretching the distance between the columns of $\mathbb{Z}_+^2$ according to a collection of independent and identically distributed (i.i.d.) copies of a positive random variable ξ, where ξ satisfies $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) < \infty$, for all constant $c > 64$. We independently declare each vertical edge open with probability p and each horizontal edge open with probability $p^{|e|}$, where $|e|$ is the length of the edge e. We construct a multiscale renormalization scheme and use it to show the existence of percolation, by constructing an infinite open cluster. This shows that the percolation process we are considering here exhibits a non-trivial phase transition.

Keywords: percolation; horizontally stretched lattice; phase transition; random environments; multiscale renormalization.

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Introduction

Let G be an infinite, locally finite and connected graph and denote by $E(G)$ and $V(G)$ its set of edges and vertices, respectively. A bond percolation configuration is an element $\omega = (\omega(e) : e \in E(G))$ of the sample space $\Omega = \{0,1\}^{E(G)}$. An edge $e \in E(G)$ is said open if $\omega(e) = 1$ and it is said closed if $\omega(e) = 0$. Let Σ be the σ-algebra generated by the cylinder sets, that is, events which only depend on the state of a finite number of edges. For each edge $e \in E(G)$, let $p_e \in [0,1]$ be the probability that e being open. When $p_e = p$ for every edge $e \in E(G)$ and for some $p \in [0,1]$, the model is said to be homogeneous. Otherwise, it is said to be inhomogeneous.

The homogeneous percolation model is known as a Bernoulli bond percolation model, where each edge is open with probability $p \in [0,1]$, and closed with probability $1-p$, independently of the states of other edges. This means that $\{\omega(e) : e \in E(G)\}$ is a family of independent Bernoulli random variables with mean $p \in [0,1]$. Denote by $\mathbb{P}_p$ the law that concerns this Bernoulli bond percolation model. This model is the simplest example of percolation model and it was introduced by Broadbent and Hammersley in 1957 [4], when they modeled mathematically the flow of a fluid through a porous medium.

A set of edges of $E(G)$, $\{e_1, e_2, \ldots, e_n\}$, $n \geq 1$, where $e_i = \langle x_i, y_i \rangle$, with $x_i, y_i \in V(G)$, for all $i = 1, 2, \ldots, n$, is called a path if $x_1, x_2, \ldots, x_n$ are distinct and $y_i = x_{i+1}$, for all $i = 1, 2, \ldots, n-1$. A path is said to be open if all its edges are open, that is, if $\omega(e_i) = 1$, for all $i = 1, 2, \ldots, n$. We say that two vertices $x, y \in V(G)$ are connected if there is an open path $\{e_1, e_2, \ldots, e_n\}$ with $x_1 = x$ and $y_n = y$ and we will denote this by $x \leftrightarrow y$. Given a vertex $v \in V(G)$, we define the cluster of v in a percolation configuration ω as the set

$$C_v(\omega) = \{u \in V(G) : u \leftrightarrow v\}.$$

Let $\{v \leftrightarrow \infty\}$ be the event that v is connected to infinity given by

$$\{v \leftrightarrow \infty\} = \{\omega \in \Omega : |C_v(\omega)| = \infty\}.$$

A fundamental question in percolation theory is to determine whether the event $\{v \leftrightarrow \infty\}$ has a strictly positive probability when the configuration ω is randomly sampled. The critical point of G is defined by

$$p_c(G) = \sup\{p \in [0,1] : \mathbb{P}_p(v \leftrightarrow \infty) = 0\}.$$

Notice that the value of $\mathbb{P}_p(v \leftrightarrow \infty)$ changes from 0, when $p < p_c(G)$, to positive, when $p > p_c(G)$. For this reason, when we have $0 < p_c(G) < 1$ we say that the percolation model undergoes a non-trivial phase transition. For $d \geq 2$, Broadbent and Hammersley [4] proved that the critical point of Bernoulli bond percolation model satisfies $0 < p_c(\mathbb{Z}^d) < 1$. In the case of the square lattice, Harris [8], in 1960, showed that $\mathbb{P}_{\frac{1}{2}}\big((0,0) \leftrightarrow \infty\big) = 0$ and Kesten [11], twenty years later, showed that $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

Our goal in this work is to investigate the effect of introducing inhomogeneities on the lattice, specifically understanding how they modify the phase transition and the critical points in a percolation model. In some percolation models, inhomogeneities arise through the introduction of environments that specify the rules for assigning probabilities p_e to each edge e of the graph.

We will consider here the square lattice, $\mathbb{Z}^2$. One way of introducing inhomogeneities into it involves fixing certain columns, which will form what we will call the environment. In these columns, the edges will be open with probability p, while the remaining edges will be open with probability q, for some $p, q \in [0,1]$. Formally, given a subset $\Lambda \subset \mathbb{Z}$, we define the set

$$\mathrm{E}_{\text{vert}} = \{\langle (x,y),(x,y+1)\rangle : x \in \Lambda, y \in \mathbb{Z}\},$$

which contains the edges of those columns that project into Λ. Given $p, q \in [0,1]$, an edge $e \in E(\mathbb{Z}^2)$ will be open with probability

$$p_e = \begin{cases} p, & \text{if } e \in \mathrm{E}_{\text{vert}} \\ q, & \text{if } e \notin \mathrm{E}_{\text{vert}} \end{cases}.$$

We will denote by $\mathbb{P}^{\Lambda}_{p,q}(\cdot)$ the probability law in $\{0,1\}^{E(\mathbb{Z}^2)}$ which governs this percolation model.

In the special case where $\Lambda = \{0\}$, Zhang [20] showed that $\mathbb{P}^{\{0\}}_{p,q}((0,0) \leftrightarrow \infty) = 0$, for any $p \in [0,1)$ and $q \leq p_c(\mathbb{Z}^2) = \frac{1}{2}$. He used arguments similar to those of Harris in [8], involving the construction of dual circuits around the origin, together with the Russo, Seymour and Welsh [17, 18] techniques. On the other extreme, when $\Lambda = \mathbb{Z}$, Kesten showed that $\mathbb{P}^{\mathbb{Z}}_{p,q}((0,0) \leftrightarrow \infty) > 0$ if and only if $p + q > 1$ and $p, q \in (0,1)$ (see Section 11.9 in [7]). In the case where Λ consists only of bounded gaps, namely, there exists $k \in \mathbb{Z}_+$ such that for every $l \in \mathbb{Z}$, $\Lambda \cap [l, l+k] \neq \emptyset$, a classic argument due to Aizenman and Grimmett [1] ensures that for any $\epsilon > 0$ there is $\delta = \delta(k,\epsilon) > 0$ such that $\mathbb{P}^{\Lambda}_{p_c+\epsilon, p_c-\delta}((0,0) \leftrightarrow \infty) > 0$, where $p_c = p_c(\mathbb{Z}^2)$.

In all examples above the environments Λ are deterministic. Next, we will mention some percolation models where the environment Λ is randomly determined.

For each $\rho \in [0,1]$, let ν_ρ be the probability measure on subsets of $\mathbb{Z}$ in which the events $\{i \in \Lambda\}_{i \in \mathbb{Z}}$ are independent, each with probability ρ. Duminil-Copin, Hilário, Kozma and Sidoravicius [5] showed that for any $\epsilon > 0$ and $\rho > 0$ there is a $\delta = \delta(\rho,\epsilon) > 0$ such that $\mathbb{P}^{\Lambda}_{p_c+\epsilon, p_c-\delta}((0,0) \leftrightarrow \infty) > 0$, for ν_ρ-almost everywhere environment, where $p_c = p_c(\mathbb{Z}^2)$.

In [3], Bramson, Durrett and Schonmann showed that for any $\rho \in [0,1)$, there exists a sufficiently large $p < 1$ such that $\mathbb{P}_{0,p}((0,0) \leftrightarrow \infty) > 0$ for ν_ρ-almost every environment Λ. This means that if edges of columns are deleted according to Bernoulli trials with mean ρ, the phase transition is present. Hoffman [10] examined a similar case, where both rows and columns are independently deleted with the same probability ρ and proved that the model still undergoes a non-trivial phase transition.

Hilário, Sá, Sanchis and Teixeira [9] considered a random environment Λ obtained by stretching horizontally the square lattice $\mathbb{Z}_+^2$ according to a positive random variable ξ. They considered independent and identically distribute copies of a positive random variable ξ to stretch the distance between the columns of $\mathbb{Z}_+^2$, obtaining a horizontally stretched square lattice. Once the environment Λ of columns was given, they declared in the resulting lattice independently each vertical edge open with probability p and closed with probability $1-p$, and each horizontal edge e open with probability $p^{|e|}$ and closed with probability $1-p^{|e|}$, where $p \in [0,1]$ and $|e|$ is the length of e. By using a multiscale renormalization scheme, they proved that a non-trivial phase transition occurs when $\mathbb{E}(\xi^{1+\eta}) < \infty$, for some $\eta > 0$.

In our work, motivated by Hilário, Sá, Sanchis and Teixeira [9], instead of their moment condition, we assume a better one (see Remark 1.1), namely, $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) < \infty$, for all constant $c > 64$. Using a multiscale renormalization scheme, we show that the model still undergoes a non-trivial phase transition.

This text is organized as follows. In Chapter 1, we will introduce the bond percolation model on the horizontally stretched square lattice and we will also state our main result, namely, Theorem 1.1. In Chapter 2, we will present some results about renewal processes that we will use to prove the decoupling inequality in Lemma 2.6 which is very important in our multiscale scheme which will described in Chapter 3. This scheme is divided into two parts: in Section 3.1 we will describe the environments and show, in Lemma 3.2, that the probability of an interval being bad decays exponentially with its length, while in Section 3.2, in Lemmas 3.4 and 3.5, we will control the probabilities of horizontal and vertical crossings, respectively. We will use these two lemmas to construct the infinite open cluster in the proof of Theorem 1.1. Finally, in Chapter 4, we make some concluding remarks and some comments about further works.

Chapter 1

The Model and Results

Let $\mathbb{Z}_+$ be the set of all nonnegative integers and denote $\mathbb{Z}_+^* = \mathbb{Z}_+ \backslash \{0\}$. We will consider percolation in the lattice obtained from $\mathbb{Z}_+^2$ randomly stretching its horizontal edges. Formally speaking, let ξ be a positive random variable and $\{\xi_i\}_{i\in\mathbb{Z}_+}$ a sequence of i.i.d. copies of ξ and consider

$$\Lambda = \left\{ \sum_{1\leq i\leq k} \xi_i : k \in \mathbb{Z}_+^* \right\},$$

which is called an environment. Also, Λ can be seen as an increasing sequence

$$\Lambda = \{x_k \in \mathbb{R} : x_0 = 0 \text{ and } x_k = x_{k-1} + \xi_k, \text{ for } k \in \mathbb{Z}_+^*\}. \tag{1.1}$$

This sequence is called a renewal process with interarrival distribution ξ. We will denote by $\mu_\xi(\cdot)$ the probability measure that governs this renewal process.

Given a realization of an environment Λ we can define the lattice $\mathcal{L}_\Lambda = (V_\Lambda, E_\Lambda)$ where the vertex and edge sets are given, respectively, by

$$V_\Lambda = \Lambda \times \mathbb{Z}_+ = \{(x, y) \in \mathbb{R}^2 : x \in \Lambda, y \in \mathbb{Z}_+\};$$

$$E_\Lambda = \{\langle (x_i, n), (x_j, m)\rangle : |i - j| + |n - m| = 1, \text{ with } x_i, x_j \in \Lambda \text{ and } n, m \in \mathbb{Z}_+\}.$$

Notice that this lattice $\mathcal{L}_\Lambda$ can be seen as the lattice obtained by $\mathbb{Z}_+^2$ by stretching or contracting the horizontal edges in such a way that ξ_i gives the random separation between the i-th and $(i+1)$-th column in the stretched lattice.

We will also consider a bond percolation process in $\mathcal{L}_\Lambda$ as follows. For each $p \in [0, 1]$, denote by $\mathbb{P}_p^\Lambda(\cdot)$ the probability measure on $\{0, 1\}^{E_\Lambda}$ under which the random variables $\{\nu(e)\}_{e\in E_\Lambda}$ are independent Bernoulli random variables with mean

$$p_e = p^{|e|}, \tag{1.2}$$

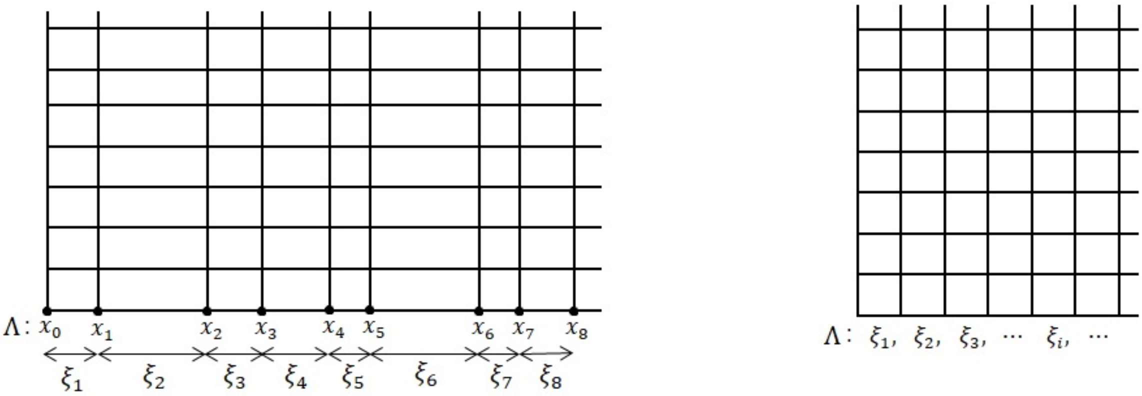
where, for each $e = \langle v_1, v_2\rangle \in E_\Lambda$, $|e| = ||v_1 - v_2||$ denotes the length of e, with $|| \cdot ||$

meaning the Euclidean norm in $\mathbb{R}^2$. We say that an edge $e \in E_\Lambda$ is open if $\omega(e) = 1$, and closed otherwise. We write $\{(0,0) \leftrightarrow \infty\}$ to represent the event that there is an infinite open path starting at $(0,0)$.

An equivalent formulation for the bond percolation model on $\mathcal{L}_\Lambda$ defined by (1.2) is the following: consider the first quadrant of the square lattice $\mathbb{Z}_+^2$ and, conditional on $\xi_1, \xi_2, \ldots$, declare each edge $e \in E(\mathbb{Z}_+^2)$ open independently with probability

$$p_e = \begin{cases} p, \text{ if } e = \langle (i,j), (i,j+1) \rangle \text{ for some } i,j \\ p^{\xi_i}, \text{ if } e = \langle (i,j), (i+1,j) \rangle \text{ for some } i,j \end{cases} . \tag{1.3}$$

Figure 1.1: Illustration of the lattice $\mathcal{L}_\Lambda$ (on the left) and the alternative formulation on $\mathbb{Z}_+^2$ (on the right). The environment Λ is given by x_k's (left) or by $\xi_k = x_k - x_{k-1}$ (right).



Source: Compiled by the author.

In the case that ξ is a positive and integer-valued random variable, the percolation model defined on $\mathcal{L}_\Lambda$ with parameters given by (1.2) can be mapped to yet another equivalent model on $\mathbb{Z}_+^2$ as follows. Consider the environment $\Lambda \subseteq \mathbb{Z}_+$ distributed as μ_ξ and define the set of edges

$$E_{\text{vert}}(\Lambda^C) = \{\langle (x,y), (x,y+1) \rangle \in E(\mathbb{Z}_+^2) : x \notin \Lambda, \ y \in \mathbb{Z}_+\}.$$

Declare each edge $e \in \mathbb{E}(\mathbb{Z}_+^2)$ open independently with probability

$$p_e = \begin{cases} 0, \text{ if } e \in \mathbb{E}_{\text{vert}}(\Lambda^C) \\ p, \text{ if } e \notin E_{\text{vert}}(\Lambda^C) \end{cases} , \tag{1.4}$$

and closed otherwise. This formulation is also a stretched square lattice obtained from $\mathbb{Z}_+^2$ by removing the edges lying in vertical columns that project to Λ^C while preserving all other edges. Each one of these remaining edges is open independently with probability p. Notice that the resulting graph is similar to the stretched lattice $\mathcal{L}_\Lambda$ defined above, however, the edges are now split into unit length segments. We can recover the original formulation on $\mathcal{L}_\Lambda$ by declaring an edge open if all the corresponding unitary edges in $\mathbb{Z}_+^2$ are open in the new formulation.

Since all the formulations given above are equivalent, we will make a slight abuse of notation by also denoting the probability law of all versions by $\mathbb{P}_p^\Lambda(\cdot)$.

The following result implies that under certain conditions the resulting percolation process exhibits a non-trivial phase transition.

Theorem 1.1. *Let c be a constant satisfying $c > 64$ and ξ be a positive random variable that satisfies $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) < \infty$. Then there is a critical point $p_c \in (0,1)$, depending on the law of ξ only, such that for $p < p_c$, we have*

$$\mathbb{P}_p^\Lambda((0,0) \leftrightarrow \infty) = 0, \text{ for } \mu_\xi\text{-almost all } \Lambda, \tag{1.5}$$

and, for $p > p_c$, we have

$$\mathbb{P}_p^\Lambda((0,0) \leftrightarrow \infty) > 0, \text{ for } \mu_\xi\text{-almost all } \Lambda. \tag{1.6}$$

The proof of Theorem 1.1, which will be done in Section 3.3, is delicate and is based on controlling the environment and the crossing events in the resulting stretched lattice through multiscale analysis, that will be developed in Chapter 3.

Remark 1.1. *It is worth to mention that the moment condition $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) < \infty$ of Theorem 1.1 is better than $\mathbb{E}(\xi^{1+\eta}) < \infty$, for all $\eta > 0$, which appears in Theorem 1.1 of [9]. In fact, if $\xi > e^{c^2}$, then $1 - c(\log \xi)^{-1/2} > 0$, and so, $\exp\big(-\eta(\log \xi)(1 - c(\log \xi)^{-1/2})\big) \leq 1$. Thus,*

$$e^{c(\log \xi)^{1/2}} = \xi^\eta \exp\big(c(\log \xi)^{1/2} - \eta \log \xi\big) \leq \xi^\eta,$$

hence,

$$\begin{aligned}\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) &= \mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{1 \leq \xi \leq e^{c^2}\}}\big) + \mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi > e^{c^2}\}}\big) \\ &\leq \max_{1 \leq \xi \leq e^{c^2}} \big\{\xi \; e^{c(\log \xi)^{1/2}}\big\} + \mathbb{E}(\xi^{1+\eta}) \\ &\leq e^{2c^2} + \mathbb{E}(\xi^{1+\eta}) \\ &< \infty.\end{aligned}$$

Chapter 2

The Decoupling Inequality

The main result of this chapter is the decoupling inequality given in Lemma 2.6 of Section 2.3, which will be essential in the multiscale scheme given in Chapter 3. For completeness, we will present some definitions and results about Markov chains and renewal processes in Sections 2.1 and 2.2, respectively. For more details about Markov chains see [15] and about renewal processes see [9, 13]. Those who are familiar with these subjects can skip reading Sections 2.1 and 2.2.

2.1 Some Basics Facts about Discrete Markov Chain

Given a countable set I, we say that $\lambda = (\lambda_i : i \in I)$ is a measure on I if $0 \leq \lambda_i < \infty$, for all $i \in I$. If in addition we have $\sum_{i \in I} \lambda_i = 1$, λ is called a probability distribution.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to I$ a random variable. Define $\lambda = (\lambda_i : i \in I)$ by

$$\lambda_i = \mathbb{P}(X = i) = \mathbb{P}\big(\{\omega : X(\omega) = i\}\big),$$

and call λ the distribution of X.

A sequence of random variables $(X_n)_{n \in \mathbb{Z}_+}$ is a Markov chain with initial distribution λ if the following conditions hold:

- X_0 has distribution λ, that is, $\mathbb{P}(X_0 = i) = \lambda_i$, for all $i \in I$;
- for $n > 0$ and for all $i_0, \ldots, i_{n-2}, i, j \in I$, we have

$$\mathbb{P}(X_n = j | X_0 = i_0, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i) = \mathbb{P}(X_n = j | X_{n-1} = i) = p_{ij}.$$

The matrix $P = (p_{ij} : i, j \in I)$ is called the transition matrix of the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ and each element p_{ij} represents the probability of making a transition from state i to state j in a single step. The second condition above is called the **memoryless property** or also **Markov property**.

The probability of making a transition from state i to state j over n steps, for any $n \in \mathbb{Z}_+$, is given by

$$\mathbb{P}(X_{m+n} = j | X_m = i) = (P^n)_{ij}. \tag{2.1}$$

We will say that a distribution λ on I is **stationary** for the Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ if it satisfies, for all $i \in I$

$$\lambda_i = \sum_{j\in I} \lambda_j \cdot p_{ij},$$

that is,

$$\lambda P = \lambda. \tag{2.2}$$

Notice that, if $(X_n)_{n\in\mathbb{Z}_+}$ is a Markov chain such that its initial state has a stationary distribution λ then, by (2.1) and (2.2), we conclude that $\mathbb{P}(X_n = i) = \lambda_i$, for all $n \in \mathbb{Z}_+$.

A Markov chain $(X_n)_{n\in\mathbb{Z}}$ with transition probability P is said **irreducible** if for any two states $i, j \in I$ there is a positive integer $n \in \mathbb{Z}_+^*$ such that $(P^n)_{ij} > 0$. This ensures that any state will reach any other state with positive probability. We say that a state $i \in I$ is **aperiodic** if

$$\gcd\{n \in \mathbb{Z}_+^* : (P^n)_{ii} > 0\} = 1.$$

A Markov chain is **aperiodic** if the state i is aperiodic for each $i \in I$.

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a Markov chain with transition matrix P. We say that a state $i \in I$ is **recurrent** if

$$\mathbb{P}(X_n = i \text{ for infinitely many } n | X_0 = i) = 1,$$

otherwise the state $i \in I$ is called **transient**. In other words, a recurring state is one in which we constantly return and a transient state is one that we leave forever. We define the **first passage time** to state $i \in I$ as the random variable T_i given by

$$T_i = \inf\{n \in \mathbb{Z}_+^* : X_n = i\},$$

where $\inf \emptyset = \infty$. A recurrent state $i \in I$ is called **positive recurrent** if it satisfies $\mathbb{E}(T_i | X_0 = i) < \infty$, otherwise it is called **null recurrent**.

Proposition 2.1 (Theorem 1.7.7 of [15]). *Let $(X_n)_{n\in\mathbb{Z}_+}$ be an irreducible Markov chain. Then the following statements are equivalent:*

(*i*) *every state is positive recurrent;*

(*ii*) *some state i is positive recurrent;*

(*iii*) *$(X_n)_{n\in\mathbb{Z}_+}$ has a stationary distribution.*

2.2 Some Definitions and Results on Renewal Processes

We will begin this section with some definitions and notations about renewal processes which will be used in the proof of the decoupling inequality in Section 2.3. First we will define a renewal process and then consider two other processes related to it. One of them will be regarded as a Markov chain and we will prove some results involving its memoryless property.

Let ξ be a positive integer-valued random variable, called interarrival time, and χ a non-negative integer-valued random variable, called delay. Consider, as before, $\{\xi_i\}_{i\in\mathbb{Z}_+^*}$ i.i.d. copies of ξ and also independent of χ. We define recursively the renewal process $X = X(\xi,\chi) = \{X_i\}_{i\in\mathbb{Z}_+}$ as

$$X_0 = \chi, \text{ and } X_i = X_{i-1} + \xi_i, \text{ for all } i \in \mathbb{Z}_+^*. \tag{2.3}$$

We say that the i-th renewal occurs at time t if $X_{i-1} = t$. We will denote by $\mu_\xi^\chi(\cdot)$ the probability law that governs the renewal process X, regarded as a random element on a probability space supporting χ and the i.i.d. copies of ξ.

It is suitable to define two other processes related to X, namely

$$Y = Y(\xi,\chi) = \{Y_n\}_{n\in\mathbb{Z}_+} \text{ and } Z = Z(\xi,\chi) = \{Z_n\}_{n\in\mathbb{Z}_+},$$

respectively as,

$$Y_n = \begin{cases} 1, \text{ if a renewal of } X \text{ occurs at time } n \\ 0, \text{ otherwise} \end{cases} \tag{2.4}$$

and

$$Z_n = \min\{X_i - n : i \in \mathbb{Z}_+ \text{ and } X_i - n \geq 0\}. \tag{2.5}$$

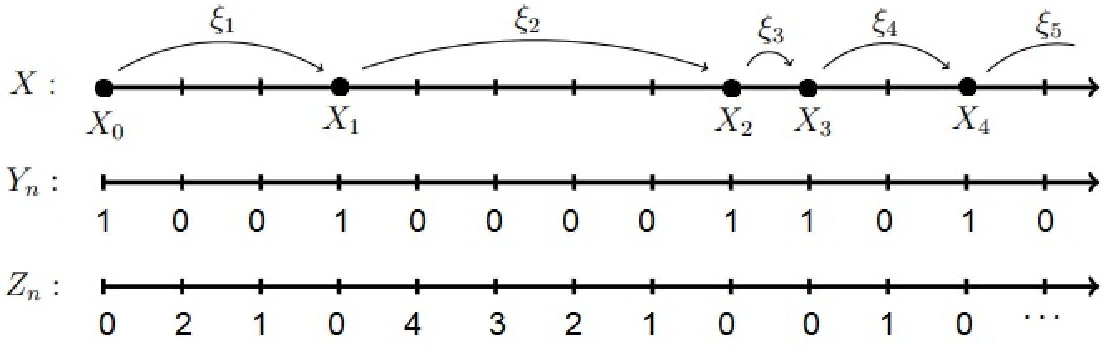
Notice that Z_n returns us how long until the next renewal. Furthermore, if we know one of the processes X, Y or Z, then we are able to determine the other two (see Figure 2.1). Thus, with an abuse of notation, we will also refer to the processes Y and Z as renewal processes with interarrival time ξ and delay time χ. Additionally, we will use $\mu_\xi^\chi(\cdot)$ to denote the probability law of the renewal processes Y and Z.

It is important to notice that Z is a Markov chain with transition probability given by

$$\mathbb{P}(Z_n = i | Z_{n-1} = j) = \begin{cases} \mathbb{P}(\xi = i+1), \text{ if } j = 0 \\ 1, \text{ if } j = i+1 \\ 0, \text{ otherwise} \end{cases}, \tag{2.6}$$

for all $n \in \mathbb{Z}_+^*$ and $i, j \in \mathbb{Z}_+$.

Figure 2.1: Example to illustrate the processes X, Y and Z. In this example we have $\xi_1 = 3$, $\xi_2 = 5$, $\xi_3 = 1$ and $\xi_4 = 2$.



Source: Compiled by the author.

For $m \in \mathbb{Z}_+$ consider the left shift operator, $\theta_m : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ given by

$$\theta_m(x_0, x_1, \dots) = (x_m, x_{m+1}, \dots).$$

It is desirable that the renewal process Z be invariant under θ_m, that is

$$\theta_m Z \overset{d}{=} Z, \text{ for any } m \in \mathbb{Z}_+^*, \tag{2.7}$$

where $\overset{d}{=}$ means equality in distribution. When $\mathbb{E}(\xi) < \infty$, we can define a random variable $\rho = \rho(\xi)$ with distribution

$$\rho_k = \mathbb{P}(\rho = k) := \frac{1}{\mathbb{E}(\xi)} \sum_{i=k+1}^{\infty} \mathbb{P}(\xi = i), \text{ for all } k \in \mathbb{Z}_+, \tag{2.8}$$

independent of everything else. Consider the renewal process $Z(\xi, \rho)$ obtained by using ρ as the delay time and defined as in (2.5). By definition, we have that $Z_0 \overset{d}{=} \rho$. Furthermore, for all $k \in \mathbb{Z}_+$, we can make the following decomposition

$$\mathbb{P}(Z_1 = k) = \sum_{i \in \mathbb{Z}_+} \mathbb{P}(Z_1 = k \mid Z_0 = i)\mathbb{P}(Z_0 = i). \tag{2.9}$$

But, using the transition probability given in (2.6), we see that only two transition probabilities are non-zero, namely $\mathbb{P}(Z_1 = k | Z_0 = i)$ when $i = 0$ and $i = k + 1$, so we have

$$\begin{aligned}\mathbb{P}(Z_1 = k) &= \mathbb{P}(Z_1 = k|Z_0 = 0)\mathbb{P}(Z_0 = 0) + \mathbb{P}(Z_1 = k|Z_0 = k+1)\mathbb{P}(Z_0 = k+1)\\ &= \mathbb{P}(\xi = k+1)\cdot \rho_0 + 1\cdot \rho_{k+1}\\ &= \mathbb{P}(\xi = k+1)\,\frac{1}{\mathbb{E}(\xi)} + \frac{1}{\mathbb{E}(\xi)}\sum_{i=k+2}^{\infty}\mathbb{P}(\xi = i)\\ &= \frac{1}{\mathbb{E}(\xi)}\sum_{i=k+1}^{\infty}\mathbb{P}(\xi = i) = \rho_k.\end{aligned}$$

It means that $Z_1 \overset{d}{=} \rho$. Now, suppose that for some $n \in \mathbb{Z}_+$ we have $Z_n \overset{d}{=} \rho$, that is, $\mathbb{P}(Z_n = k) = \rho_k$, for all $k \in \mathbb{Z}_+$. Using a decomposition analogous to the one above and (2.6), again we have only two non-zero transition probabilities, $\mathbb{P}(Z_{n+1} = k|Z_n = 0)$ and $\mathbb{P}(Z_{n+1} = k|Z_n = k+1)$. Hence we can write

$$\begin{aligned}\mathbb{P}(Z_{n+1} = k) &= \mathbb{P}(Z_{n+1} = k|Z_n = 0)\mathbb{P}(Z_n = 0) + \mathbb{P}(Z_{n+1} = k|Z_n = k+1)\mathbb{P}(Z_n = k+1)\\ &= \mathbb{P}(\xi = k+1)\cdot \rho_0 + 1\cdot \rho_{k+1}\\ &= \mathbb{P}(\xi = k+1)\,\frac{1}{\mathbb{E}(\xi)} + \frac{1}{\mathbb{E}(\xi)}\sum_{i=k+2}\mathbb{P}(\xi = i)\\ &= \sum_{i=k+1}\mathbb{P}(\xi = i) = \rho_k.\end{aligned}$$

It means that $Z_{n+1} \overset{d}{=} \rho$. Thus, by finite induction, we can conclude that

$$Z_n \overset{d}{=} Z_0 \overset{d}{=} \rho, \text{ for all } n \in \mathbb{Z}_+. \tag{2.10}$$

Since all Z_n, $n \in \mathbb{Z}_+$, have the same distribution, it follows that

$$\theta_m Z = (Z_m, Z_{m+1}, \dots) \overset{d}{=} (Z_0, Z_1, \dots) = Z.$$

Consequently, since the processes X, Y and Z are related and one determines the other two completely, we also have that $\theta_m Y \overset{d}{=} Y$, for $Y = Y(\xi, \rho)$. For this reason, for a fixed ξ, the random variable ρ defined above is called stationary delay.

Lemma 2.1. *Let c be a constant satisfying* $c > 64$. *If* ξ *is a positive integer-valued random variable satisfying* $\mathbb{E}\big(\xi e^{c(\log\xi)^{1/2}}\mathbb{1}_{\{\xi\geq 1\}}\big) < \infty$, *then the stationary delay* $\rho = \rho(\xi)$, *given by* (2.8), *satisfies* $\mathbb{E}\big(e^{c(\log\rho)^{1/2}}\mathbb{1}_{\{\rho\geq 1\}}\big) < \infty$.

Proof: Notice that

$$\mathbb{E}\big(e^{c(\log\rho)^{1/2}}\mathbb{1}_{\{\rho\geq 1\}}\big) = \sum_{k=1}^{\infty} e^{c(\log k)^{1/2}}\rho_k.$$

By (2.8) we can write

$$
\begin{aligned}
\mathbb{E}\big(e^{c(\log\rho)^{1/2}}\mathbb{1}_{\{\rho\geq 1\}}\big) &= \sum_{k=1}^{\infty} e^{c(\log k)^{1/2}}\frac{1}{\mathbb{E}(\xi)}\sum_{i=k+1}^{\infty}\mathbb{P}(\xi=i)\\
&\leq \sum_{k=1}^{\infty} e^{c(\log k)^{1/2}}\frac{1}{\mathbb{E}(\xi)}\sum_{i=k}^{\infty}\mathbb{P}(\xi=i)\\
&= \frac{1}{\mathbb{E}(\xi)}\sum_{k=1}^{\infty} e^{c(\log k)^{1/2}}\sum_{i=1}^{\infty}\mathbb{1}_{\{i\geq k\}}\mathbb{P}(\xi=i)\\
&= \frac{1}{\mathbb{E}(\xi)}\sum_{i=1}^{\infty}\mathbb{P}(\xi=i)\sum_{k=1}^{\infty}\mathbb{1}_{\{k\leq i\}}e^{c(\log k)^{1/2}}\\
&= \frac{1}{\mathbb{E}(\xi)}\sum_{i=1}^{\infty}\mathbb{P}(\xi=i)\sum_{k=1}^{i}e^{c(\log k)^{1/2}}. \qquad (2.11)
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{k=1}^{i} e^{c(\log k)^{1/2}} &= e^{c(\log 1)^{1/2}} + e^{c(\log 2)^{1/2}} + \cdots + e^{c(\log i)^{1/2}}\\
&= e^{c(\log i)^{1/2}}\left(\frac{e^{c(\log 1)^{1/2}}}{e^{c(\log i)^{1/2}}} + \frac{e^{c(\log 2)^{1/2}}}{e^{c(\log i)^{1/2}}} + \cdots + 1\right)\\
&\leq e^{c(\log i)^{1/2}}\cdot i, \qquad (2.12)
\end{aligned}
$$

substituting (2.12) in (2.11), we have

$$
\mathbb{E}\big(e^{c(\log\rho)^{1/2}}\mathbb{1}_{\{\rho\geq 1\}}\big) \leq \frac{1}{\mathbb{E}(\xi)}\sum_{i=1}^{\infty} i\cdot e^{c(\log i)^{1/2}}\ \mathbb{P}(\xi=i) = \frac{1}{\mathbb{E}(\xi)}\mathbb{E}\big(\xi e^{c(\log\xi)^{1/2}}\mathbb{1}_{\{\xi\geq 1\}}\big),
$$

which is finite, since $\mathbb{E}\big(\xi e^{c(\log\xi)^{1/2}}\mathbb{1}_{\{\xi\geq 1\}}\big) < \infty$. ■

Next we will show how the renewal process $Z = \{Z_n\}_{n\in\mathbb{Z}_+}$ forgets its initial state when n goes to ∞. To do this, we will need some definitions.

We say that a random variable ξ is aperiodic if

$$
\gcd\{k\in\mathbb{Z}_+^* : \mathbb{P}(\xi=k)>0\} = 1.
$$

In what follows we will assume that ξ is aperiodic and $\mathbb{E}(\xi)<\infty$. Let $X = X(\xi,\chi)$ and $X' = X'(\xi,\chi')$ be two independent renewal processes with interarrival time ξ and delays χ and χ', respectively. Consider the respectively sequences Y and Y' given as in (2.4) and define

$$
T = \min\{k\in\mathbb{Z}_+^* : Y_k = Y'_k = 1\}, \qquad (2.13)
$$

as the coupling time of X and X'. We will denote by $\nu_\xi^{\chi,\chi'}(\cdot)$ the product measure $\nu_\xi^{\chi}\otimes\nu_\xi^{\chi'}$.

The next lemma will guarantee that the processes X and X' meet a.s..

Lemma 2.2. *If ξ is aperiodic and $\mathbb{E}(\xi) < \infty$, then*

$$\mu_\xi^{\chi,\chi'}(T < \infty) = 1. \tag{2.14}$$

Proof: Let $\widetilde{Y} = \widetilde{Y}(\xi, \chi, \chi') = \{\widetilde{Y}_n\}_{n \in \mathbb{Z}_+}$ be the renewal process defined by

$$\widetilde{Y}_n = Y_n \cdot Y'_n, \text{ for all } n \in \mathbb{Z}_+.$$

Denote by $\widetilde{\xi}$ its interarrival time. Let us show that this interarrival time does not take the infinite value, that is,

$$\mathbb{P}(\widetilde{\xi} = \infty) = 0. \tag{2.15}$$

If we would have $\mathbb{P}(\widetilde{\xi} = \infty) > 0$, then the process $\widetilde{Y}$ had an infinite jump, that is, from a moment the processes Y and Y' would not meet anymore, which would imply that

$$\lim_{n \to +\infty} \mu_\xi^{\rho,\rho}(\widetilde{Y}_n = 1) = 0. \tag{2.16}$$

But, observe that, by stationarity of ρ, for all $n \in \mathbb{Z}_+$, we have,

$$\begin{aligned} \mu_\xi^{\rho,\rho}(\widetilde{Y}_n = 1) &= \mu_\xi^{\rho}(Y_n = 1)\ \mu_\xi^{\rho}(Y'_n = 1) \\ &= \mu_\xi^{\rho}(Z_n = 0)\ \mu_\xi^{\rho}(Z'_n = 0) \\ &= \rho_0^2 = \left(\frac{1}{\mathbb{E}(\xi)}\right)^2 > 0, \end{aligned}$$

which contradicts (2.16). Thus, we have $\mathbb{P}(\widetilde{\xi} = \infty) = 0$. This means that the process $\widetilde{Y}$ must have infinitely many jumps of finite size each, that is

$$\mu_\xi^{\delta_0,\delta_0}(\widetilde{Y}_n = 1 \text{ i.o.}) = 1, \tag{2.17}$$

where δ_x denotes the delta distribution concentrated in $x \in \mathbb{Z}$.

Since ξ is aperiodic, it follows that there is $n_0 \in \mathbb{Z}_+$ such that

$$\mu_\xi^{\delta_0}(Y_n = 1) > 0, \text{ for all } n \geq n_0.$$

(This result can be seen in [12], Proposition 1.7). Thus, using (2.17) and the classical equality $\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^C)\mathbb{P}(B^C)$, for any events A and B, we have, for $m \in \mathbb{Z}$,

$$\begin{aligned} 1 &= \mu_\xi^{\delta_0,\delta_0}(\widetilde{Y}_n = 1 \text{ i.o. }) \\ &= \mu_\xi^{\delta_0,\delta_0}(\widetilde{Y}_n = 1 \text{ i.o. } |(Y_{n_0}, Y'_{n_0+m}) = (1,1))\ \mu_\xi^{\delta_0,\delta_0}((Y_{n_0}, Y'_{n_0+m}) = (1,1)) \\ &\quad + \mu_\xi^{\delta_0,\delta_0}(\widetilde{Y}_n = 1 \text{ i.o. } |(Y_{n_0}, Y'_{n_0+m}) \neq (1,1))\ \mu_\xi^{\delta_0,\delta_0}((Y_{n_0}, Y'_{n_0+m}) \neq (1,1)). \end{aligned} \tag{2.18}$$

Notice that if

$$xz + y(1-z) = 1, \text{ for } x, y \in [0,1] \text{ and } z \in (0,1),$$

then, we should have $x = y = 1$. In fact, $xz + y(1-z)$ is a convex combination of the points x and y, and since $z \in (0,1)$, the only way this combination results in 1 is when the points x and y coincide and are both equal to 1.

So, for $m \in \mathbb{Z}$, from (2.18) we get

$$1 = \mu_\xi^{\delta_0,\delta_0}(\widetilde{Y}_n = 1 \text{ i.o. } |(Y_{n_0}, Y'_{n_0+m}) = (1,1)) = \mu_\xi^{\delta_0,\delta_m}(\widetilde{Y} = 1 \text{ i.o.}).$$

Hence, for any $i, j \in \mathbb{Z}_+$,

$$\mu_\xi^{\delta_i,\delta_j}(\widetilde{Y} = 1 \text{ i.o.}) = \mu_\xi^{\delta_0,\delta_{|i-j|}}(\widetilde{Y} = 1 \text{ i.o.}) = 1. \tag{2.19}$$

Notice that $\{\widetilde{Y}_n = 1 \text{ i.o.}\} \subseteq \{\exists\, k_0 \in \mathbb{Z}_+ \,:\, Y_{k_0} = Y'_{k_0} = 1\} \subseteq \{T < \infty\}$. Therefore,

$$\begin{aligned}\mu_\xi^{\chi,\chi'}(T < \infty) &\geq \mu_\xi^{\chi,\chi'}(\widetilde{Y}_n = 1 \text{ i.o.}) \\ &= \sum_{i\in\mathbb{Z}_+}\sum_{j\in\mathbb{Z}_+} \mu_\xi^{\delta_i,\delta_j}(\widetilde{Y}_n = 1 \text{ i.o.})\mathbb{P}(\chi = i)\mathbb{P}(\chi' = j) \\ &= \sum_{i\in\mathbb{Z}_+}\sum_{j\in\mathbb{Z}_+} \mathbb{P}(\chi = i)\mathbb{P}(\chi' = j) \\ &= 1,\end{aligned}$$

where the second equality comes from (2.19). ∎

The next lemma shows that the renewal process forgets the delay.

Lemma 2.3. *If ξ is aperiodic and $\mathbb{E}(\xi) < \infty$, then for every event A, we have*

$$\lim_{n\to\infty}\left|\mu_\xi^{\chi}(\theta_n Y \in A) - \mu_\xi^{\chi'}(\theta_n Y' \in A)\right| = 0, \tag{2.20}$$

for any delays χ and χ'. Moreover,

$$\lim_{n\to\infty}\mu_\xi^{\chi}(Z_n = k) = \rho_k, \tag{2.21}$$

for all $k \in \mathbb{Z}_+$ and for any delay χ.

Proof: Let A be any event. Notice that

$$\mu_\xi^{\chi}(\theta_n Y \in A) = \mu_\xi^{\chi,\chi'}(\theta_n Y \in A \mid T \leq n)\mu_\xi^{\chi,\chi'}(T \leq n) + \mu_\xi^{\chi,\chi'}(\theta_n Y \in A \mid T > n)\mu_\xi^{\chi,\chi'}(T > n)$$

and

$$\mu_\xi^{\chi'}(\theta_n Y' \in A) = \mu_\xi^{\chi,\chi'}(\theta_n Y' \in A \mid T \leq n)\mu_\xi^{\chi,\chi'}(T \leq n) + \mu_\xi^{\chi,\chi'}(\theta_n Y' \in A \mid T > n)\mu_\xi^{\chi,\chi'}(T > n).$$

Thus,

$$\begin{aligned}
&\left|\mu_\xi^\chi(\theta_n Y\in A)-\mu_\xi^{\chi'}(\theta_n Y'\in A)\right| \\
&\le \left|\mu_\xi^{\chi,\chi'}(\theta_n Y\in A\mid T\le n)-\mu_\xi^{\chi,\chi'}(\theta_n Y'\in A\mid T\le n)\right|\mu_\xi^{\chi,\chi'}(T\le n) \\
&+\left|\mu_\xi^{\chi,\chi'}(\theta_n Y\in A\mid T> n)-\mu_\xi^{\chi,\chi'}(\theta_n Y'\in A\mid T> n)\right|\mu_\xi^{\chi,\chi'}(T> n). \qquad (2.22)
\end{aligned}$$

The condition $\{T\le n\}$ means that Y and Y' met before time n. By the loss of memory of the renewal processes Y and Y', and the fact that they have the same interarrival time, we have that $\mu_\xi^{\chi,\chi'}(\theta_n Y\in A\mid T\le n)=\mu_\xi^{\chi,\chi'}(\theta_n Y'\in A\mid T\le n)$. Since $\left|\mu_\xi^{\chi,\chi'}(\theta_n Y\in A\mid T> n)-\mu_\xi^{\chi,\chi'}(\theta_n Y'\in A\mid T> n)\right|\le 1$, it follows from (2.22) that

$$\left|\mu_\xi^\chi(\theta_n Y\in A)-\mu_\xi^{\chi'}(\theta_n Y'\in A)\right|\le \mu_\xi^{\chi,\chi'}(T>n). \qquad (2.23)$$

Taking the limit as $n\to\infty$ on both sides of (2.23) we have

$$\begin{aligned}
\lim_{n\to\infty}\left|\mu_\xi^\chi(\theta_n Y\in A)-\mu_\xi^{\chi'}(\theta_n Y'\in A)\right| &\le \lim_{n\to\infty}\mu_\xi^{\chi,\chi'}(T>n) \\
&=\mu_\xi^{\chi,\chi'}(T=\infty)=0,
\end{aligned}$$

where the last equality follows from (2.14). This finishes (2.20).

Now, in order to prove (2.21), take $\chi'=\rho$ in (2.20). Thus, we have, for $k\in\mathbb{Z}_+$,

$$\left|\lim_{n\to\infty}\left[\mu_\xi^\chi(Y_{n+k}=1)-\mu_\xi^\rho(Y_{n+k}=1)\right]\right|=0,$$

which implies

$$\lim_{n\to\infty}\mu_\xi^\chi(Y_{n+k}=1)=\lim_{n\to\infty}\mu_\xi^\rho(Y_{n+k}=1).$$

Since $\{Z_n=k\}=\{Y_{n+k}=1\}$, we have

$$\lim_{n\to\infty}\mu_\xi^\chi(Z_n=k)=\lim_{n\to\infty}\mu_\xi^\rho(Z_n=k)=\mu_\xi^\rho(Z_0=k)=\rho_k,$$

where the second equality follows from (2.10). ■

2.3 Decoupling Inequality

The purpose of this section is to establish the decoupling inequality for stationary renewal processes given in Lemma 2.6. First, in Lemma 2.4, we will show the bounded subadditivity property of the function $e^{c(\log x)^{1/2}}$, for $x\in\left(e^{c^2/4},\infty\right)$. This subadditivity will be used in Theorem 2.1 to prove that $\mathbb{E}_\xi^{\chi,\chi'}\left(e^{c(\log T)^{1/2}}\mathbb{1}_{\{T\ge 1\}}\right)<\infty$. This together with the Markov inequality will be useful to show $\mu_\xi^{\chi,\chi'}(T>n)\le c_0\, e^{-c(\log n)^{1/2}}$, where c_0 is a constant.

Lemma 2.4. *Let* $G(x) = e^{c(\log x)^{1/2}}$ *for* $x \geq 1$. *Then*

$$G(x+y) \leq M\big(G(x)+G(y)\big), \tag{2.24}$$

where

$$M = \begin{cases} e^{\frac{c}{\sqrt{\log 2}}}, & \textit{if } x, y \leq e^{c^2/4} \\ 1, & \textit{if } x, y \geq e^{c^2/4} \\ 4, & \textit{otherwise.} \end{cases} \tag{2.25}$$

Proof: Let $x, y \geq 1$ and without loss of generality assume that $x \geq y$. In this case $x \geq \frac{x+y}{2}$. Since G is increasing, we obtain

$$\frac{G(x+y)}{G(x)+G(y)} \leq \frac{G(x+y)}{G((x+y)/2)} = \exp\left(\frac{c\,\log 2}{\sqrt{\log(x+y)}+\sqrt{\log((x+y)/2)}}\right), \tag{2.26}$$

and since $x+y \geq 2$, (2.26) implies that

$$\frac{G(x+y)}{G(x)+G(y)} \leq e^{\frac{c}{\sqrt{\log 2}}}. \tag{2.27}$$

If $x \geq e^{c^2/4}$, then $x+y \geq e^{c^2/4}$ and from (2.26), we have

$$\frac{G(x+y)}{G(x)+G(y)} \leq 4. \tag{2.28}$$

On the other hand, if $x, y \geq e^{c^2/4}$, then $x+y, \dfrac{x+y}{2} \geq e^{c^2/4}$ and from (2.26), we have

$$\frac{G(x+y)}{G(x)+G(y)} \leq 2. \tag{2.29}$$

By using a different argument, in the last upper bound we can replace 2 with 1 by noticing that for $x > e^{c^2/4}$ the function $\frac{G(x)}{x}$ is decreasing and for $x, y \geq e^{c^2/4}$ this implies that

$$G(x+y) = x\,\frac{G(x+y)}{x+y} + y\,\frac{G(x+y)}{x+y} \leq x\,\frac{G(x)}{x} + y\,\frac{G(y)}{y} = G(x)+G(y).$$

■

The next lemma will give us some moment estimates for the process Z_n, which will be used next. It is a minor adaptation of Lemma 4.1 of [13].

Lemma 2.5. *Let* $c > 64$ *be a constant and* ξ *be a random variable satisfying* $\mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big) < \infty$. *Consider the renewal process* $X = X(\delta_0, \xi)$ *given as in* (2.3). *Then, for all* $n \geq 0$, *we have*

(i) $\mathbb{E}\big(e^{c(\log Z_n)^{1/2}}\big) \leq \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big) \cdot n$, *and*

(ii) for each $\eta > 0$ *there is a constant* $K_0 = K_0(\eta)$ *such that* $\mathbb{E}(Z_n) \leq K_0 + \eta \cdot n$.

Proof: Notice that, for all $k \geq 0$, we have

$$\begin{aligned}
\mathbb{P}(Z_n = k) &= \sum_{j\geq 0} \mathbb{P}(Z_n = k,\, X_j < n,\, X_{j+1} \geq n) \\
&= \sum_{j\geq 0}\sum_{i=0}^{n-1} \mathbb{P}(X_j = i,,\; Z_n = k,\, X_{j+1} \geq n) \\
&= \sum_{j\geq 0}\sum_{i=0}^{n-1} \mathbb{P}(X_j = i,\, \xi_{j+1} = n + k - i) \\
&= \sum_{j\geq 0}\sum_{i=0}^{n-1} \mathbb{P}(X_j = i)\mathbb{P}(\xi_{j+1} = n + k - i) \\
&= \sum_{i=0}^{n-1} \mathbb{P}(\xi = n + k - i) \sum_{j\geq 0} \mathbb{P}(X_j = i) \\
&\leq \sum_{i=0}^{n-1} \mathbb{P}(\xi = n + k - i),
\end{aligned}$$

where the third and fourth equalities come from the fact that $\{Z_n = k\} = \{Y_{n+k} = 1\}$ and independence, respectively, and the inequality follows since $\sum_{j\geq 0} \mathbb{P}(X_j = i)$ represents the probability of there being a renewal in i, which we can bounded above by 1. Thus, we have

$$\begin{aligned}
\mathbb{E}\Big(e^{c(\log Z_n)^{1/2}}\Big) &= \sum_{k\geq 0} e^{c(\log k)^{1/2}} \mathbb{P}(Z_n = k) \\
&= \sum_{k\geq 0} e^{c(\log k)^{1/2}} \sum_{i=0}^{n-1} \mathbb{P}(\xi = n + k - i) \\
&\leq \sum_{k\geq 0}\sum_{i=0}^{n-1} e^{c[\log(n+k-i)]^{1/2}}\, \mathbb{P}(\xi = n + k - i) \\
&= \sum_{i=0}^{n-1}\sum_{k\geq 0} e^{c[\log(n+k-i)]^{1/2}} \mathbb{P}(\xi = n + k - i) \\
&\leq \sum_{i=0}^{n-1} \mathbb{E}\Big(e^{c(\log \xi)^{1/2}}\Big) \\
&= \mathbb{E}\Big(e^{c(\log \xi)^{1/2}}\Big) \cdot n,
\end{aligned}$$

which proves (i). Now, for each $n \in \mathbb{Z}_+^*$, define the random variable ψ_n as the number of renewals in $\{0, 1, 2, \ldots, n-1\}$. By definition of the process Z_n, we have that $n + Z_n = X_{\psi_n}$. Thus, using the Wald's equation [19], $\mathbb{E}(\xi_1 + \cdots + \xi_N) = \mathbb{E}(N)\mathbb{E}(\xi)$ where $\xi_1, \ldots, \xi_N$ are i.i.d. random variables and N is a positive and integer-valued random variable that is independent of ξ_i's, we get

$$n + \mathbb{E}(Z_n) = \mathbb{E}(\psi_n)\mathbb{E}(\xi). \tag{2.30}$$

By definition of the process Y_n we have that $\mathbb{E}(\psi_n) = \sum_{i=0}^{n-1} \mathbb{P}(Y_i = 1)$. Hence, item (ii) follows from (2.30) by noting that the Renewal Theorem implies that $\frac{1}{n}\mathbb{E}(\psi_n)$ goes to $\frac{1}{\mathbb{E}(\xi)}$ when n goes to ∞. ■

The next result establishes sufficient conditions on the interarrival time and on the delays in order that $\mathbb{E}_\xi^{\chi,\chi'}\big(e^{c(\log T)^{1/2}}\mathbb{1}_{\{T\geq 1\}}\big) < \infty$. It is also a minor adaptation of Theorem 4.2 of [13].

Theorem 2.1. *Let c be a constant satisfying $c > 64$. Suppose that ξ is an aperiodic and integer-valued random variable taking values greater that $e^{c^2/4}$ and satisfies $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}}\big) < \infty$. Also suppose that χ, χ' are non-negative integer-valued random variables with $\mathbb{E}\big(e^{c(\log \chi)^{1/2}}\mathbb{1}_{\{\chi\geq 1\}}\big)$ and $\mathbb{E}\big(e^{c(\log \chi')^{1/2}}\mathbb{1}_{\{\chi'\geq 1\}}\big)$ finite. Then, $\mathbb{E}_\xi^{\chi,\chi'}\big(e^{c(\log T)^{1/2}}\mathbb{1}_{\{T\geq 1\}}\big) < \infty$, where T is given by (2.13).*

Proof: Denote by $\mathbb{E}_\xi^{\chi,\chi'}(\cdot)$ the expected value with respect to the measure $\mu_\xi^{\chi,\chi'}$. Notice that

$$\begin{aligned}
\mathbb{E}_\xi^{\chi,\chi'}(e^{c(\log T)^{1/2}}) &= \mathbb{E}_\xi^{\chi,\chi'}\left(e^{c\left[\log\left(\min\{\chi,\chi'\}+T-\min\{\chi,\chi'\}\right)\right]^{1/2}}\right)\\
&\leq \mathbb{E}_\xi^{\chi,\chi'}\left(M\left(e^{c\left[\log\left(\min\{\chi,\chi'\}\right)\right]^{1/2}} + e^{c\left[\log\left(T-\min\{\chi,\chi'\}\right)\right]^{1/2}}\right)\right)\\
&= M\cdot\mathbb{E}_\xi^{\chi,\chi'}\left(e^{c\left[\log\left(\min\{\chi,\chi'\}\right)\right]^{1/2}}\right) + M\cdot\mathbb{E}_\xi^{\chi,\chi'}\left(e^{c\left[\log\left(T-\min\{\chi,\chi'\}\right)\right]^{1/2}}\right)\\
&\leq M\cdot\mathbb{E}_\xi^{\chi,\chi'}\left(e^{c\left[\log\left(\min\{\chi,\chi'\}\right)\right]^{1/2}}\right) + M\cdot\mathbb{E}_\xi^{\delta_0,\varphi}\left(e^{c(\log T)^{1/2}}\right),
\end{aligned}$$

where M is given in (2.25), the first inequality follows from (2.24) and the second inequality comes from the change of variable $\varphi = \max\{\chi,\chi'\} - \min\{\chi,\chi'\}$ together with the fact that the function $e^{c(\log x)^{1/2}}$ is increasing. It follows by the hypothesis that $\mathbb{E}_\xi^{\chi,\chi'}\left(e^{c(\log\min\{\chi,\chi'\})^{1/2}}\right) < \infty$, hence it is enough to show that $\mathbb{E}_\xi^{\delta_0,\varphi}\left(e^{c(\log T)^{1/2}}\right) < \infty$.

Let

$$Y = Y(\xi,\delta_0) \text{ and } Y' = Y'(\xi,\chi')$$

be independently of each other (as well as X and X'). The approach we will use next is to introduce random variables in order to make a sequence of attempts for processes Y and Y' to meet, where each attempt will have a positive probability of occurrence. Better yet, each attempt has a uniform lower bound which is obtained by taking $n_0 \in \mathbb{Z}_+$ and $\sigma > 0$ such that

$$\mu_\xi^{\delta_0}(Z_n = 0) \geq \sigma, \text{ for all } n \geq n_0. \tag{2.31}$$

This happens because from Lemma 2.3 we have

$$\lim_{n\to\infty}\mu_\xi^{\delta_0}(Z_n = 0) = \frac{1}{\mathbb{E}(\xi)}.$$

Define the random variables F_n, H_n and v_n, for $n \in \mathbb{Z}_+^*$, inductively, by $F_0 = 0$ and

$$\begin{aligned} H_{2n} &= \min\{X'_i - F_{2n} \ : \ X'_i - F_{2n} \geq 0\} = X'_{v_{2n}} - F_{2n}, \\ F_{2n+1} &= X'_{v_{2n}+n_0}, \\ H_{2n+1} &= \min\{X_i - F_{2n+1} \ : \ X_i - F_{2n+1} \geq 0\} = X_{v_{2n+1}} - F_{2n+1}, \\ F_{2n+2} &= X_{v_{2n+1}+n_0}. \end{aligned}$$

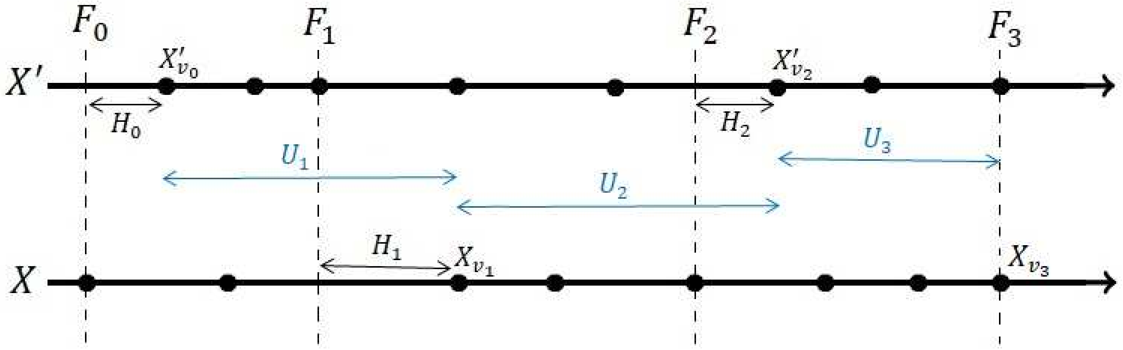
Observe that $H_n = 0$ correspond to a success, that is, the processes Y and Y' meet each other (see Figure 2.2 for an illustration of these definitions). The reason for adding n_0 steps to the variables $X_{v_1}, X'_{v_2}, \dots$ is that the probability of success is at least equal to σ. Now, define

$$\tau = \min\{k \in \mathbb{Z}_+ \ : \ H_k = 0\}$$

and, for $n \in \mathbb{Z}_+^*$,

$$U_{2n+1} = X_{v_{2n+1}} - X'_{v_{2n}} \text{ and } U_{2n+2} = X'_{v_{2n+2}} - X_{v_{2n+1}}.$$

Figure 2.2: Illustration of the above definitions when $n_0 = 2$, $H_0 > 0$, $H_1 > 0$, $H_2 > 0$ and $H_3 = 0$. Hence, $\tau = 3$ and we have a successful coupling, that is, the processes X and X' meet.



Source: Compiled by the author.

Consider also, the filtration $\{\mathcal{N}_i\}_{i \in \mathbb{Z}_+^*}$, where $\mathcal{N}_i$ is generated by the variables

$$X_j, X'_k, \text{ where } j \leq v_i \text{ and } k \leq v_{i-1} + n_0, \text{ when } i \text{ is odd,}$$

or

$$X_j, X'_k, \text{ where } k \leq v_i \text{ and } j \leq v_{i-1} + n_0, \text{ when } i \text{ is even.}$$

Notice that

$$T \leq X'_{v_0} + \sum_{i=1}^{\tau} U_i = \chi' + \sum_{i=1}^{\tau} U_i.$$

Since each U_i contains at least n_0 jumps of ξ and each $\xi > e^{c^2/4}$, we have that $U_i > e^{c^2/4}$. Thus, using (2.24), we get

$$\begin{aligned} e^{c(\log T)^{1/2}} &\le Me^{c(\log \chi')^{1/2}} + Me^{c\left(\log \sum_{i=1}^{\tau} U_i\right)^{1/2}} \\ &\le Me^{c(\log \chi')^{1/2}} + M\sum_{i=1}^{\tau} e^{c(\log U_i)^{1/2}} \\ &= Me^{c(\log \chi')^{1/2}} + M\sum_{i\ge 1} e^{c(\log U_i)^{1/2}}\mathbb{1}_{\{\tau\ge i\}}, \end{aligned}$$

where $\mathbb{1}_S$ denotes the indicator function of the set S. Hence,

$$\mathbb{E}^{\delta_0,\chi'}\left(e^{c(\log T)^{1/2}}\right) \le M\cdot\mathbb{E}\left(e^{c(\log\chi')^{1/2}}\right) + M\cdot\sum_{i\ge 1}\mathbb{E}\left(e^{c(\log U_i)^{1/2}}\mathbb{1}_{\{\tau\ge i\}}\right). \tag{2.32}$$

By the properties of conditional expectation, it follows that

$$\begin{aligned} \mathbb{E}\left(e^{c(\log U_i)^{1/2}}\mathbb{1}_{\{\tau\ge i\}}\right) &= \mathbb{E}\left(\mathbb{E}\left(e^{c(\log U_i)^{1/2}}\mathbb{1}_{\{\tau\ge i\}} \mid \mathcal{N}_{i-1}\right)\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(e^{c(\log U_i)^{1/2}} \mid \mathcal{N}_{i-1}\right)\mathbb{1}_{\{\tau\ge i\}}\right). \end{aligned} \tag{2.33}$$

Let us estimate the value of $\mathbb{E}\left(e^{c(\log U_i)^{1/2}} \mid \mathcal{N}_{i-1}\right)$. Notice that

$$\begin{aligned} \mathbb{E}\left(e^{c(\log U_i)^{1/2}} \mid \mathcal{N}_{i-1}\right) &= \mathbb{E}\left(e^{c(\log U_i)^{1/2}} \mid H_{i-1}\right) \\ &= \mathbb{E}\left(e^{c\left(\log(X_{v_{i-1}+n_0} - X_{v_{i-1}} + H_i)\right)^{1/2}} \,\middle|\, H_{i-1}\right). \end{aligned}$$

By adding and subtracting terms, we have the identity

$$X_{v_{i-1}+n_0} - X_{v_{i-1}} + H_i = X_{v_{i-1}+n_0} - X_{v_{i-1}+n_0-1} + X_{v_{i-1}+n_0-1} + \cdots + X_{v_{i-1}+1} - X_{v_{i-1}} + H_i.$$

Once again we can use (2.24) with the constant $M = 1$, because the difference between two consecutive X_i represents a jump of ξ that is greater than $e^{c^2/4}$. Thus, we get

$$\begin{aligned} &\mathbb{E}\left(e^{c(\log U_i)^{1/2}} \mid \mathcal{N}_{i-1}\right) \\ &\le \mathbb{E}\left(e^{c\left(\log(X_{v_{i-1}+n_0} - X_{v_{i-1}+n_0-1})\right)^{1/2}} \,\middle|\, H_{i-1}\right) + \cdots + \mathbb{E}\left(e^{c\left(\log(X_{v_{i-1}+1} - X_{v_{i-1}})\right)^{1/2}} \,\middle|\, H_{i-1}\right) \\ &\quad + M\,\mathbb{E}\left(e^{c(\log H_i)^{1/2}} \,\middle|\, H_{i-1}\right) \\ &= n_0\,\mathbb{E}\left(e^{c(\log\xi)^{1/2}}\right) + M\,\mathbb{E}\left(e^{c(\log H_i)^{1/2}} \mid H_{i-1}\right). \end{aligned} \tag{2.34}$$

By definitions of H_i and Z_n we have the following identity

$$H_i = Z_{H_{i-1}+\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}},$$

and thus,

$$\begin{aligned}
\mathbb{E}\Big(e^{c(\log H_i)^{1/2}} \;\Big|\; H_{i-1}\Big) &= \mathbb{E}\Big(e^{c\big(\log Z_{H_{i-1}+\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}}\big)^{1/2}} \;\Big|\; H_{i-1}\Big)\\
&= \mathbb{E}\Big(e^{c\big(\log Z_{H_{i-1}+\sum_{j\geq 0} j\mathbb{1}_{\{\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}=j\}}}\big)^{1/2}} \;\Big|\; H_{i-1}\Big)\\
&= \mathbb{E}\Big(\sum_{j\geq 0} e^{c\big(\log Z_{H_{i-1}+j}\big)^{1/2}} \mathbb{1}_{\{\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}=j\}} \;\Big|\; H_{i-1}\Big)\\
&= \sum_{j\geq 0} \mathbb{E}\Big(e^{c\big(\log Z_{H_{i-1}+j}\big)^{1/2}} \;\Big|\; H_{i-1}\Big)\mathbb{P}(\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}=j)\\
&\leq \sum_{j\geq 0} \mathbb{E}\Big((H_{i-1}+j)\mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big) \;\Big|\; H_{i-1}\Big)\mathbb{P}(\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}=j)\\
&= \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big)H_{i-1} + \sum_{j\geq 0} j\mathbb{P}(\xi_{l_1}+\cdots+\xi_{l_1+n_0-1}=j)\\
&= \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big)H_{i-1} + \mathbb{E}(\xi_{l_1}+\cdots+\xi_{l_1+n_0-1})\\
&= \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big)H_{i-1} + n_0\mathbb{E}(\xi),
\end{aligned}$$

where the inequality comes from (i) of Lemma 2.5. Substituting this in (2.34), we have

$$\begin{aligned}
\mathbb{E}\big(e^{c(\log U_i)^{1/2}} \mid \mathcal{N}_{i-1}\big) &\leq n_0\, \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big) + n_0\mathbb{E}(\xi) + \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big)H_{i-1}\\
&= K_1 + K_2\, H_{i-1},
\end{aligned}$$

where $K_1 = n_0\, \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big) + n_0\mathbb{E}(\xi)$ and $K_2 = \mathbb{E}\big(e^{c(\log \xi)^{1/2}}\big)$ are constants since $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}}\big) < \infty$. Hence, from (2.33), we obtain

$$\begin{aligned}
\mathbb{E}\big(e^{c(\log U_i)^{1/2}}\mathbb{1}_{\{\tau\geq i\}}\big) &\leq K_1\, \mathbb{P}(\tau \geq i) + K_2\, \mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i\}})\\
&\leq K_1\, (1-\sigma)^i + K_2\, \mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i\}}), \qquad (2.35)
\end{aligned}$$

where the last inequality follows because by the definition of τ we have that $\{\tau \geq i\}$ means that $Z_n \neq 0$, for all $n \leq i$.

Let us use (ii) of Lemma 2.5 with $\eta = \frac{1}{2}$ to estimate the value of $\mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i\}})$. Notice that, for $i \geq 2$, we have

$$\begin{aligned}
\mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i\}}) &\leq \mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i-1\}})\\
&= \mathbb{E}\big(\mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i-1\}} \mid \mathcal{N}_{i-2})\big)\\
&= \mathbb{E}\big(\mathbb{E}(H_{i-1} \mid \mathcal{N}_{i-2})\mathbb{1}_{\{\tau\geq i-1\}}\big)\\
&= \mathbb{E}\big(\mathbb{E}(H_{i-1} \mid H_{i-2})\mathbb{1}_{\{\tau\geq i-1\}}\big)\\
&= \mathbb{E}\left(\left(K_0 + \frac{1}{2}H_{i-2}\right)\mathbb{1}_{\{\tau\geq i-1\}}\right)\\
&\leq K_0\,(1-\sigma)^{i-1} + \frac{1}{2}\mathbb{E}(H_{i-2}\mathbb{1}_{\{\tau\geq i-1\}}).
\end{aligned}$$

By repeating this process $i-1$ times, we get

$$\begin{aligned}\mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i\}}) &\leq K_0\,(1-\sigma)^{i-1}+K_0\left(\frac{1}{2}\right)(1-\sigma)^{i-2}+\cdots+K_0\left(\frac{1}{2}\right)^{i-1}\\ &\leq K_0\,i\left(\max\left\{\frac{1}{2},1-\sigma\right\}\right)^{i-1}. \end{aligned} \tag{2.36}$$

Therefore, from (2.32), (2.35) and (2.36), we conclude that

$$\begin{aligned}\mathbb{E}^{\delta_0,\mathcal{X}'}\big(e^{c(\log T)^{1/2}}\big) &\leq M\,\mathbb{E}\big(e^{c(\log \mathcal{X}')^{1/2}}\big)+M\sum_{i\geq 1}\Big(K_1\,(1-\sigma)^i+K_2\,\mathbb{E}(H_{i-1}\mathbb{1}_{\{\tau\geq i\}})\Big)\\ &\leq M\,\mathbb{E}\big(e^{c(\log \mathcal{X}')^{1/2}}\big)+M\sum_{i\geq 1}\left(K_1\,(1-\sigma)^i+K_0\,K_2\,i\Big(\max\left\{\frac{1}{2},1-\sigma\right\}\Big)^{i-1}\right)\\ &\leq M\,\mathbb{E}\big(e^{c(\log \mathcal{X}')^{1/2}}\big)+M\,K_3\sum_{i\geq 1}(i+1)\Big(\max\left\{\frac{1}{2},1-\sigma\right\}\Big)^{i-1}\\ &<\infty,\end{aligned}$$

where $K_3=\max\{K_1,K_0\,K_2\}$. ■

The next lemma gives us the desired decoupling inequality for stationary renewals.

Lemma 2.6 (Decoupling Inequality). *Let c be a constant satisfying $c>64$ and ξ be an aperiodic and integer-valued random variable taking values greater than $\xi>e^{c^2/4}$. Suppose that $\mathbb{E}\big(\xi e^{c(\log\xi)^{1/2}}\big)<\infty$. Consider the associated renewal process Y defined in (2.4). Then, there exists $c_1=c_1(\xi,c)\in(0,+\infty)$ such that for all $n,m\in\mathbb{Z}_+$ and for every pair of events A and B, with*

$$A\in\sigma(Y_i:0\leq i\leq m)\text{ and }B\in\sigma(Y_i:i\geq m+2n),$$

we have

$$\mu_\xi^\rho(A\cap B)\leq\mu_\xi^\rho(A)\mu_\xi^\rho(B)+c_1\cdot e^{-c(\log n)^{1/2}}. \tag{2.37}$$

Proof: If $\mu_\xi^\rho(A)=0$, there is nothing to be proved, then suppose that $\mu_\xi^\rho(A)>0$. Using the definition of the renewal process Z given in (2.5), we have

$$\begin{aligned}\mu_\xi^\rho(A\cap B)&=\mu_\xi^\rho(A\cap B\cap\{Z_m>n\})+\mu_\xi^\rho(A\cap B\cap\{Z_m\leq n)\\ &\leq\mu_\xi^\rho(Z_m>n)+\mu_\xi^\rho(A\cap B\cap\{Z_m\leq n\})\\ &=\mu_\xi^\rho(Z_m>n)+\mu_\xi^\rho(A)\mu_\xi^\rho(B\cap\{Z_m\leq n\}|A).\end{aligned}$$

So, making a decomposition of the event $\{Z_m\leq n\}$, we have

$$\begin{aligned}
\mu_\xi^\rho(A\cap B) &\le \mu_\xi^\rho(Z_m>n)+\mu_\xi^\rho(A)\sum_{\substack{0\le i\le n\\ \mu_\xi^\rho(Z_m=i|A)>0}}\mu_\xi^\rho(B|\{Z_m=i\})\mu_\xi^\rho(Z_m=i|A)\\
&\le \mu_\xi^\rho(Z_m>n)+\mu_\xi^\rho(A)\max_{0\le i\le n}\mu_\xi^{\delta_{m+i}}(B)\sum_{\substack{0\le i\le n\\ \mu_\xi^\rho(Z_m=i|A)>0}}\mu_\xi^\rho(Z_m=i|A)\\
&= \mu_\xi^\rho(Z_m>n)+\mu_\xi^\rho(A)\max_{0\le i\le n}\mu_\xi^{\delta_{m+i}}(B)\;\mu_\xi^\rho(Z_m\le n|A)\\
&\le \mu_\xi^\rho(Z_m>n)+\mu_\xi^\rho(A)\max_{0\le i\le n}\mu_\xi^{\delta_{m+i}}(B). \qquad (2.38)
\end{aligned}$$

Now, we need compare the measures $\mu_\xi^{\delta_{m+i}}$ and μ_ξ^ρ, when $0\le i\le n$. For this, notice that $\mu_\xi^{\delta_{m+i}}(B)=\mu_\xi^{\delta_0}(\theta_{m+i}(B))$ and that we have given a space of size $2n-i$ for the renewal processes with delays δ_0 and ρ to couple. So, by the stationarity of ρ, we have

$$\begin{aligned}
|\mu_\xi^{\delta_{m+i}}(B)-\mu_\xi^\rho(B)| &= |\mu_\xi^{\delta_0}(\theta_{m+i}(B))-\mu_\xi^\rho(\theta_{m+i}(B))|\\
&= \big|\mu_\xi^{\delta_0}\big(\theta_{2n-i}Y\in\theta_{m+i}(B)\big)-\mu_\xi^\rho\big(\theta_{2n-i}Y\in\theta_{m+i}(B)\big)\big|\\
&\le \mu_\xi^{\delta_0,\rho}(T>2n-i)\\
&\le \mu_\xi^{\delta_0,\rho}(T>n),
\end{aligned}$$

where the inequalities follow since $0\le i\le n$ and from (2.23), respectively.

Thus, for all $0\le i\le n$, we get

$$\mu_\xi^{\delta_{m+i}}(B)\le\mu_\xi^\rho(B)+\mu_\xi^{\delta_0,\rho}(T>n).$$

Replacing this into (2.38) and using that $Z_m\overset{d}{=}Z_0\overset{d}{=}\rho$, we have

$$\begin{aligned}
\mu_\xi^\rho(A\cap B) &\le \mu_\xi^\rho(\rho>n)+\mu_\xi^\rho(A)\mu_\xi^\rho(B)+\mu_\xi^\rho(A)\mu_\xi^{\delta_0,\rho}(T>n)\\
&\le \mu_\xi^\rho(A)\mu_\xi^\rho(B)+\mu_\xi^\rho(\rho>n)+\mu_\xi^{\delta_0,\rho}(T>n)\\
&\le \mu_\xi^\rho(A)\mu_\xi^\rho(B)+e^{-c(\log n)^{1/2}}\mathbb{E}(e^{c(\log\rho)^{1/2}})+e^{-c(\log n)^{1/2}}\mathbb{E}_\xi^{\delta_0,\rho}(e^{c(\log T)^{1/2}}),
\end{aligned}$$

where the last inequality follows from the Markov inequality for $e^{c(\log\rho)^{1/2}}$ and $e^{c(\log T)^{1/2}}$. The proof of the lemma finishes by taking $c_1=\mathbb{E}\big(e^{c(\log\rho)^{1/2}}\big)+\mathbb{E}_\xi^{\delta_0,\rho}\big(e^{c(\log T)^{1/2}}\big)$, which is finite by Lemma 2.1 and Theorem 2.1. ■

Chapter 3

The Multiscale Scheme

In this chapter, we will present a multiscale scheme, which consists of two main parts: environments and crossing events. In Section 3.1, we will define an increasing sequence of numbers L_k, $k \geq 0$, called scales, and use them to partition the set $\mathbb{Z}_+$ into intervals of length L_k, called k-intervals. We will find some integer k_0, which will satisfy some necessary conditions that will appear throughout the text, and we will label the k-intervals as good or bad, recursively, for all $k \geq k_0$. Next, we will prove that the bad k-intervals are rare and that their probability is summable in k. This will show that, with strictly positive probability, there is an environment that has the following property: the first two k-intervals are good, for all $k \geq k_0$. In Section 3.2, we will construct horizontal and vertical crossing of rectangles whose sides depend on L_k and prove that, within k-good intervals, such crossings have a very high probability to occur. This allows us to build an infinite open cluster in the proof of Theorem 1.1.

3.1 Environments

As we will see, in Section 3.2, the scale L_0 must be chosen sufficiently large such that it satisfies the conditions below

$$L_0 > \sqrt[1-\mu]{\frac{2}{1-\mu}} \tag{3.1}$$

$$2\log L_0 - L_0^{1-\mu} \leq -3\log 2 \tag{3.2}$$

$$L_0 > \sqrt[1-\beta]{\frac{2}{1-\beta}} \tag{3.3}$$

$$L_0 \leq e^{\frac{1}{2}L_0^{1-\beta}} \tag{3.4}$$

$$L_0 \geq 96^{\frac{2}{\mu}} \tag{3.5}$$

$$L_0 \geq \sqrt[1-\mu]{76\log 2 + 48} \tag{3.6}$$

$$L_0 \geq \exp\left(\frac{8}{(1-\mu)^2}\right) \tag{3.7}$$

where $\mu < \beta$ are parameters in the interval $\big(0,1\big)$.

The next lemma gives us a lower bound for L_0 uniform on μ and β when we reduce the range in which these parameters belong, in order to simplify the above conditions.

Lemma 3.1. *Let* $\frac{1}{4} \leq \mu < \beta \leq \frac{3}{4}$. *If* $L_0 > e^{128}$, *then the conditions* (3.1)-(3.7) *above are satisfied.*

Proof: Consider the function $f(x) = \left(\frac{2}{1-x}\right)^{\frac{1}{1-x}}$, with $x \in (0,1)$. Notice that

$$f'(x) = \frac{1}{(1-x)^2}\left[1 + \log\left(\frac{2}{1-x}\right)\right] f(x) > 0, \text{ for all } x \in (0,1),$$

which implies that $f(x)$ is an increasing function in $(0,1)$. In particular, for $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$ we conclude that

$$f(x) \leq f\left(\frac{3}{4}\right) = 4096 < e^{128}.$$

Therefore, the conditions (3.1) and (3.3) hold for $\mu, \beta \in \left[\frac{1}{4}, \frac{3}{4}\right]$.

On the other hand, if $g(\mu) = 96^{\frac{2}{\mu}}$, for $\mu \in (0,1)$, then

$$g'(\mu) = -\frac{2\log 96}{\mu^2}\, g(\mu) < 0, \text{ for all } \mu \in (0,1).$$

In particular, $g(\mu)$ is decreasing in $\mu \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and in this interval we have

$$g(\mu) \leq g\left(\frac{1}{4}\right) \leq 96^8 < e^{128},$$

and so, the condition (3.5) is satisfied.

Now, consider the function $h(\mu) = \exp\left(\frac{8}{(1-\mu)^2}\right)$, if $\mu \in (0,1)$. Then,

$$h'(\mu) = \frac{16}{(1-\mu)^3}\, h(\mu) > 0.$$

In particular $h(\mu)$ is an increasing function in $\left[\frac{1}{4}, \frac{3}{4}\right]$ and in this interval, we have

$$h(\mu) \leq h\left(\frac{3}{4}\right) = \exp\left(\frac{8}{\frac{1}{16}}\right) = e^{128}$$

and this shows that the condition (3.7) holds.

Let $f(\mu) = (76\log 2 + 48)^{\frac{1}{1-\mu}}$, where $\mu \in (0,1)$, then

$$f'(\mu) = \frac{1}{(1-\mu)^2}(76\log 2 + 48)f(\mu) > 0,$$

and so, $f(\mu)$ is an increasing function in $\left[\frac{1}{4}, \frac{3}{4}\right]$. Therefore, for $\mu \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have

$$f(\mu) \leq f\left(\frac{3}{4}\right) = (76\log 2 + 48)^4 < 110^4 < e^{128},$$

and the condition (3.6) holds if $L_0 > e^{128}$.

Let $g(L_0, \mu) = L_0^{1-\mu} - 2\log L_0 - 3\log 2$, where $L_0 > 1$ and $\mu \in (0,1)$. Notice that

$$\frac{\partial}{\partial \mu} g(L_0, \mu) = -L_0^{1-\mu}\log L_0 < 0,$$

then, $g(L_0, \mu)$ is decreasing in μ. In particular, for $\mu \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have

$$g(L_0, \mu) \geq g\left(L_0, \frac{3}{4}\right) = L_0^{1/4} - 2\log L_0 - 3\log 2 := h(L_0). \tag{3.8}$$

On the other hand, if $L_0 > 8^4$, we have

$$h'(L_0) = \frac{1}{4L_0^{3/4}} - \frac{2}{L_0} > 0,$$

since $h(10^4) > 0$, it follows that $h(L_0) > 0$, for all $L_0 > 10^4$. Therefore, for $\mu \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and $L_0 > 10^4$, we have

$$g(L_0, \mu) \geq g\left(L_0, \frac{3}{4}\right) = h(L_0) \geq h(10^4) > 0,$$

and condition (3.2) is valid.

Let $f(L_0, \beta) = e^{\frac{1}{2}L_0^{1-\beta}} - L_0$, where $L_0 > 1$ and $\beta \in (0,1)$, then

$$\frac{\partial}{\partial \beta} f(L_0, \beta) = -\frac{1}{2}L_0^{1-\beta} e^{\frac{1}{2}L_0^{1-\beta}} \ln L_0 < 0,$$

and so $f(L_0, \beta)$ is decreasing in β. In particular, for $\beta \in \left[\frac{1}{4}, \frac{3}{4}\right]$, we have

$$f(L_0, \beta) \geq f\left(L_0, \frac{3}{4}\right) = e^{\frac{1}{2}L_0^{1/4}} - L_0 := h(L_0).$$

Notice that

$$h'(L_0) = \frac{1}{8}L_0^{-\frac{3}{4}} e^{\frac{1}{2}L_0^{1/4}} - 1$$

and

$$h''(L_0) = \frac{1}{32} L_0^{-\frac{7}{4}} e^{\frac{1}{2}L_0^{1/4}} \left(\frac{1}{2} L_0^{\frac{1}{4}} - 3 \right).$$

In particular $h''(L_0) > 0$ for all $L_0 > 6^4$, since $h'(6^8) > 0$, this implies that h is an increasing function for all $L_0 > 6^8$. So, if $L_0 \geq e^{128}$, we have

$$f(L_0, \beta) \geq h(L_0) \geq h(e^{128}) > 0,$$

and condition (3.4) is satisfied. ■

Let us assume from now on that L_0 is fixed and is given by

$$L_0 = \lceil e^{128} \rceil. \tag{3.9}$$

Define the sequence of scales, $(L_k)_{k \in \mathbb{Z}_+}$, as follows

$$L_{k+1} = L_k^{\frac{2}{k+1}} L_k = L_k^{\frac{k+3}{k+1}}, \quad \text{for all } k \geq 0. \tag{3.10}$$

Next we will show, by induction, that

$$L_k = L_0^{\frac{(k+1)(k+2)}{2}}, \text{ for all } k \geq 0. \tag{3.11}$$

In fact, from (3.10), we have

$$L_0 = L_0^{\frac{(0+1)(0+2)}{2}},$$

which shows that (3.11) holds for $k = 0$. Now, suppose that (3.11) holds for some $k \geq 0$. Then from (3.10) and the induction hypothesis, respectively, it follows that

$$L_{k+1} = L_k^{\frac{k+3}{k+1}} = \left(L_0^{\frac{(k+1)(k+2)}{2}} \right)^{\frac{k+3}{k+1}} = L_0^{\frac{(k+2)(k+3)}{2}}$$

and this shows that (3.11) holds for $k+1$.

By definition, L_0 is an integer and the same happens with $\frac{(k+1)(k+2)}{2}$, for any $k \geq 1$. These facts together with (3.11) imply that $L_k \in \mathbb{Z}_+$ and $L_k^{\frac{2}{k+1}} \in \mathbb{Z}_+$ for all $k \in \mathbb{Z}_+$.

Let $c = c(L_0)$ be a positive constant that satisfies the following condition

$$c > 4\sqrt{2 \log L_0}. \tag{3.12}$$

This will be the constant that appears in the moment condition $\mathbb{E}\big(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}\big) < \infty$ of Theorem 1.1. The relation of c and L_0 given by (3.12) is necessary, as we will see, in the proof of Lemma (3.2). Notice that, using the lower bound $L_0 > e^{128}$ given by Lemma 3.1, we can also obtain a lower bound for c, namely,

$$c > 4\sqrt{2 \log L_0} > 4\sqrt{256} = 64.$$

Remark 3.1. *The constant c that we obtained here, given by* (3.12)*, is similar to the one found in Theorem* 1.1 *of [6]. In this paper, the authors consider a renewal contact process with interarrival distribution μ, which has the same moment condition to ours, and they prove that it dies out almost surely with positive probability for some positive infection rate.*

Let c_1 be the constant given in the decoupling inequality of Lemma 2.6, L_0 be given by (3.9) and c be a constant satisfying (3.12). Then there is a positive integer $k_0 = k_0(L_0, c, c_1)$ such that all $k \geq k_0$ the following inequalities hold:

$$c - \frac{4}{k+1}\left(\frac{(k+1)(k+2)}{2}\right)^{\frac{1}{2}}(\log L_0)^{\frac{1}{2}} - \frac{c}{2}\left(\frac{k+3}{k+1}\right)^{\frac{1}{2}} > \frac{1}{2}\left(\frac{c}{2} - 2\sqrt{2\log L_0}\right), \tag{3.13}$$

$$\exp\left[-\left(\frac{c}{4} - \sqrt{2\log L_0}\right)\left(\frac{(k+1)(k+2)}{2}\right)^{\frac{1}{2}}(\log L_0)^{\frac{1}{2}}\right] < \frac{1}{c_1+1}, \tag{3.14}$$

$$\exp\left(\frac{c\,(\log L_k)^{1/2}}{2}\right) > \mathbb{E}(e^{c(\log\rho)^{1/2}}\mathbb{1}_{\{\rho\geq 1\}}), \tag{3.15}$$

To show these inequalities just notice that the left hand sides of (3.13), (3.14) and (3.15) go to $\frac{c}{2} - 2\sqrt{2\log L_0}$, 0 and ∞, respectively, as k goes to ∞. The conditions above will be necessary in the proof of Lemma 3.2.

For every $k \in \mathbb{Z}_+$, we will partition the set $\mathbb{Z}_+$ into disjoint intervals of length L_k. The i-th interval of scale k, denoted by I_i^k, is defined by

$$I_i^k = [iL_k, (i+1)L_k), \text{ for } i \in \mathbb{Z}_+. \tag{3.16}$$

Notice that each interval of scale k can be partitioned into $L_{k-1}^{2/k} \in \mathbb{Z}_+$ sub-intervals of scale $k-1$, and we denote this as

$$I_i^k = \bigcup_{l\in\mathcal{I}_{k,i}} I_l^{k-1},$$

where $\mathcal{I}_{k,i} = \left\{iL_{k-1}^{2/k}, iL_{k-1}^{2/k}+1, \ldots, (i+1)L_{k-1}^{2/k}-1\right\}$ represents the set of indices for the sub-intervals of scale $k-1$ which are within I_i^k.

Fix an environment $\Lambda \subseteq \mathbb{Z}_+$ as in (1.1). Let us label intervals I_i^k, for all $k \geq k_0$, either as *good* or *bad*, recursively, as follows. For $k = k_0$, we declare $I_i^{k_0}$ good if $\Lambda \cap I_i^{k_0} \neq \emptyset$, that is, if there exists at least one column present; otherwise, we declare it bad. For $k > k_0$, assuming that all intervals at scale $k-1$ have been defined, we declare I_i^k bad if it has at least two non-consecutive bad intervals of the scale $k-1$; otherwise, we declare it as good. Notice that a good interval at scale k can have a maximum of two bad intervals on the scale $k-1$ and, in this case, these intervals must be adjacent because otherwise we will have two non-consecutive bad intervals.

For each $i, k \in \mathbb{Z}_+$, let A_i^k be the event

$$A_i^k = \{I_i^k \text{ is bad}\}.$$

Sometimes we will write $\{I_i^k \text{ is good}\}$ for the complement of A_i^k. Now, define

$$p_k := \mu_\xi^\rho(A_0^k) = \mu_\xi^\rho(A_i^k), \tag{3.17}$$

where the last inequality comes from the stationarity of ρ. Next we will relate p_{k+1} and p_k for $k \geq k_0$. Notice that

$$\begin{aligned} p_{k+1} &= \mu_\xi^\rho(\text{there is at least two non-consecutive bad intervals on the scale } k) \\ &\leq \binom{L_k^{\frac{2}{k+1}}}{2} \mu_\xi^\rho(A_0^k \cap A_2^k) \\ &\leq \binom{L_k^{\frac{2}{k+1}}}{2} \left(p_k^2 + \frac{c_1}{e^{c(\log L_k)^{1/2}}}\right) \\ &\leq \frac{1}{2}\left(L_k^{\frac{2}{k+1}}\right)^2 \left(p_k^2 + \frac{c_1}{e^{c(\log L_k)^{1/2}}}\right), \end{aligned} \tag{3.18}$$

where the equality follows from definition of p_{k+1}. In the first inequality the binomial part corresponds to an upper bound for the number of choices for pairs of bad intervals on the scale k, the second inequality holds by the decoupling inequality given in Lemma 2.6 and the third inequality comes follow an upper bound of the binomial coefficient.

The next result shows that there is a scale from which the probability of an interval being bad decreases exponentially fast in k.

Lemma 3.2. *Let $L_0 > e^{128}$ and k_0 be the positive integer satisfying conditions* (3.13)-(3.15). *Then,*

$$p_k \leq \exp\left(-\frac{c(\log L_k)^{1/2}}{2}\right), \text{ for all } k \geq k_0. \tag{3.19}$$

Proof: The proof of this lemma will be done by induction on k. For the first step of the induction, using (3.17), the stationarity of ρ, Markov inequality and (3.15), we get

$$\begin{aligned} p_{k_0} &= \mu_\xi^\rho(A_0^{k_0}) = \mu_\xi^\rho(Z_0 > L_{k_0}) = \mathbb{P}(\rho > L_{k_0}) = \mathbb{P}\big(e^{c(\log\rho)^{1/2}} > e^{c(\log L_{k_0})^{1/2}}\big) \\ &\leq \frac{\mathbb{E}(e^{c(\log\rho)^{1/2}}\mathbb{1}_{\{\rho\geq 1\}})}{e^{c(\log L_{k_0})^{1/2}}} \leq \exp\left(-\frac{c(\log L_{k_0})^{1/2}}{2}\right). \end{aligned}$$

Now suppose that, for some $k \geq k_0$, we have

$$p_k \leq \exp\left(-\frac{c(\log L_k)^{1/2}}{2}\right). \tag{3.20}$$

Then,

$$
\begin{aligned}
p_{k+1} &\leq \frac{1}{2}\left(L_k^{\frac{2}{k+1}}\right)^2 \left(p_k^2 + \frac{c_1}{e^{c(\log L_k)^{1/2}}}\right) \\
&\leq \frac{L_k^{\frac{4}{k+1}}}{2}\left(e^{-c(\log L_k)^{1/2}} + c_1 e^{-c(\log L_k)^{1/2}}\right) \\
&\leq \frac{c_1+1}{2}\, L_k^{\frac{4}{k+1}} e^{-c(\log L_k)^{1/2}},
\end{aligned}
$$

where the first inequality above follows from (3.18) and the second one from (3.20). Hence, using the upper bound for p_{k+1} obtained above, we get

$$
\begin{aligned}
&\frac{p_{k+1}}{\exp\left(-\frac{c(\log L_{k+1})^{1/2}}{2}\right)} \\
\leq\ & \frac{c_1+1}{2}\, L_k^{\frac{4}{k+1}} e^{-c(\log L_k)^{1/2}} \exp\left(\frac{c(\log L_{k+1})^{1/2}}{2}\right) \\
=\ & \frac{c_1+1}{2}\ \exp\left[\frac{4}{k+1}\log L_k - c(\log L_k)^{1/2} + \frac{c(\log L_{k+1})^{1/2}}{2}\right] \\
=\ & \frac{c_1+1}{2}\ \exp\left[\frac{4}{k+1}\log L_k - c(\log L_k)^{1/2} + \frac{c}{2}\left(\frac{k+3}{k+1}\right)^{\frac{1}{2}}(\log L_k)^{\frac{1}{2}}\right] \\
=\ & \frac{c_1+1}{2}\ \exp\left[-(\log L_k)^{1/2}\left(c - \frac{4}{k+1}(\log L_k)^{\frac{1}{2}} - \frac{c}{2}\left(\frac{k+3}{k+1}\right)^{\frac{1}{2}}\right)\right]. \qquad (3.21)
\end{aligned}
$$

Notice that

$$
\frac{4}{k+1}(\log L_k)^{\frac{1}{2}} = \frac{4}{k+1}\left(\frac{(k+1)(k+2)}{2}\right)^{\frac{1}{2}}(\log L_0)^{\frac{1}{2}}. \tag{3.22}
$$

Thus, substituting (3.22) in (3.21) and using inequalities (3.13) and (3.14), we have

$$
\begin{aligned}
&\frac{p_{k+1}}{\exp\left(-\frac{c(\log L_{k+1})^{1/2}}{2}\right)} \\
&\leq \frac{c_1+1}{2}\ \exp\left[-(\log L_k)^{1/2}\left(c - \frac{4}{k+1}\left(\frac{(k+1)(k+2)}{2}\right)^{\frac{1}{2}}(\log L_0)^{\frac{1}{2}}\right.\right. \\
&\qquad \left.\left. - \frac{c}{2}\left(\frac{k+3}{k+1}\right)^{\frac{1}{2}}\right)\right] \qquad (3.23) \\
&\leq \frac{c_1+1}{2}\exp\left[-\left(\frac{c}{4} - \sqrt{2\log L_0}\right)(\log L_k)^{1/2}\right] \\
&= \frac{c_1+1}{2}\exp\left[-\left(\frac{c}{4} - \sqrt{2\log L_0}\right)\left(\frac{(k+1)(k+2)}{2}\right)^{\frac{1}{2}}(\log L_0)^{1/2}\right] \qquad (3.24) \\
&\leq \frac{c_1+1}{2}\ \frac{1}{c_1+1} = \frac{1}{2} < 1,
\end{aligned}
$$

which concludes the proof of the lemma. ■

The next lemma shows that with strictly positive probability, the environment Λ has the property that the intervals I_0^k and I_1^k are good for all scale L_k, with $k \geq k_0$.

Lemma 3.3. *Consider I_i^k given by* (3.16) *and let k_0 be the positive integer satisfying conditions* (3.13)-(3.15)*. Then,*

$$\mu_\xi^\rho\left(\bigcap_{k\geq k_0}\{I_0^k \textit{ and } I_1^k \textit{ are good}\}\right) > 0.$$

Proof: Notice that

$$\begin{aligned}
\mu_\xi^\rho\left(\bigcap_{k\geq k_0}\{I_0^k \text{ and } I_1^k \text{ are good}\}\right) &= 1 - \mu_\xi^\rho\left(\bigcup_{k\geq k_0} A_0^k \cup A_1^k\right)\\
&\geq 1 - \sum_{k\geq k_0}\left[\mu_\xi^\rho(A_0^k) + \mu_\xi^\rho(A_1^k)\right]\\
&= 1 - \sum_{k\geq k_0} 2p_k, \qquad (3.25)
\end{aligned}$$

where the inequality above comes from the union bound. It follows from Lemma 3.2 that

$$\sum_{k\geq k_0} 2p_k \leq 2\sum_{k\geq k_0}\exp\left(-\frac{c(\log L_k)^{1/2}}{2}\right). \qquad (3.26)$$

Since

$$\log L_k = \frac{(k+2)(k+1)}{2}\log L_0 \geq \frac{k^2}{2}\log L_0,$$

we have,

$$(\log L_k)^{1/2} \geq \frac{k}{\sqrt{2}}(\log L_0)^{1/2} \geq \frac{k}{2}(\log L_0)^{1/2}. \qquad (3.27)$$

Substituting (3.27) in (3.26), we get

$$\begin{aligned}
\sum_{k\geq k_0} 2p_k &\leq 2\sum_{k\geq k_0}\exp\left(-\frac{c}{2}\cdot\frac{k(\log L_0)^{1/2}}{2}\right)\\
&= 2\sum_{k\geq k_0}\left[\exp\left(-\frac{c(\log L_0)^{1/2}}{4}\right)\right]^k\\
&\leq 2\sum_{k\geq 1}\left[\exp\left(-\frac{c(\log L_0)^{1/2}}{4}\right)\right]^k\\
&= \frac{2\exp\left(-\frac{c(\log L_0)^{1/2}}{4}\right)}{1-\exp\left(-\frac{c(\log L_0)^{1/2}}{4}\right)}. \qquad (3.28)
\end{aligned}$$

Consider the function $f(x) = \dfrac{2x}{1-x}$, for $x \in (0,1)$. Notice that

$$f'(x) = \frac{2}{(1-x)^2} > 0,$$

which implies that $f(x)$ is a increasing function for all $x \in (0,1)$. So, we have that $f(x) \leq f\left(\frac{1}{5}\right) = \dfrac{1}{2}$, for all $x \in \left(0, \frac{1}{5}\right]$. Since $L_0 > e^{128}$, it follows that

$$(\log L_0)^{1/2} = \sqrt{\log L_0} > \sqrt{\log e^{128}} = \sqrt{128} = 8\sqrt{2}. \tag{3.29}$$

Since $c > 64$, using (3.29), we have

$$\exp\left(-\frac{c(\log L_0)^{1/2}}{4}\right) \leq \exp\left(-\frac{c \cdot 8\sqrt{2}}{4}\right) < \exp\left(-128\sqrt{2}\right) < \frac{1}{5},$$

and so, putting $x = \exp\left(-\dfrac{c(\log L_0)^{1/2}}{2}\right)$ in (3.28), we have

$$\sum_{k \geq k_0} 2p_k \leq \frac{1}{2}.$$

Therefore, from (3.25), we conclude that

$$\mu_\xi^\rho\left(\bigcap_{k \geq k_0} \{I_0^k \text{ and } I_1^k \text{ are good}\}\right) \geq \frac{1}{2} > 0. \tag{3.30}$$

■

3.2 Crossing Events

In this section, we will study the probability of crossing events within special rectangles in $\mathbb{Z}_+^2$. The bases of these rectangles will be intervals on some scale k and their heights will be much greater than their bases. This elongated form will be important so that we have many chances to cross these rectangles horizontally. Then we will use these rectangle crossings to build an infinite cluster.

Before stating our results, we need to introduce some notation and define some crossing events. Let $a, b, c, d \in \mathbb{Z}_+$ with $a < b$ and $c < d$, let $[a, b] = \{i \in \mathbb{Z}_+ : a \leq i \leq b\}$ and $[c, d] = \{i \in \mathbb{Z}_+ : c \leq i \leq d\}$. We define the rectangle R, denoted by

$$R = R([a, b) \times [c, d)) \tag{3.31}$$

as the subgraph of $\mathbb{Z}_+^2$ whose vertex and edge sets are given, respectively, by

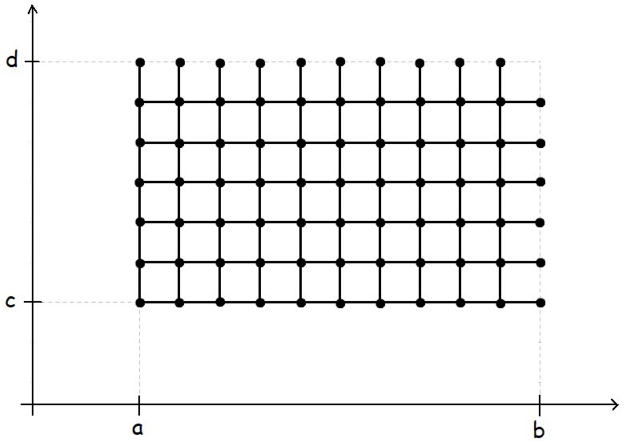
$$V(R) = [a, b] \times [c, d]$$

and

$$E(R) = \big\{\langle (x,y),(x+l,y+1-l)\rangle \ : \ (x,y) \in [a,b-1]\times[c,d-1],\ l\in\{0,1\}\big\}.$$

In other words, R is the rectangle $[a,b]\times[c,d]$ with the edges in the top and right sides removed, see Figure 3.1 for more details.

Figure 3.1: An illustration of the rectangle $[a,b]\times[c,d]$.



Source: Compiled by the author.

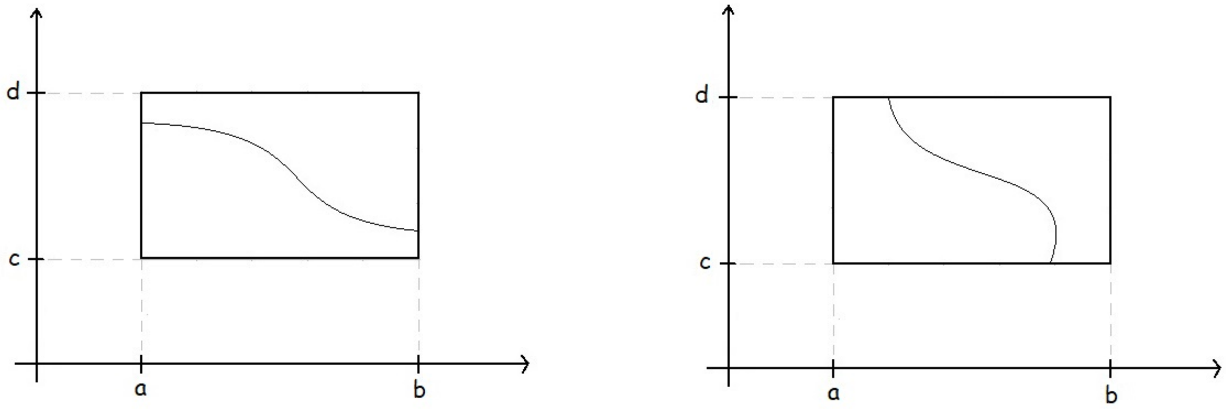
We define the horizontal and vertical crossing events in R, denoted, respectively, by $\mathcal{C}_h(R)$ and $\mathcal{C}_v(R)$, as

$$\mathcal{C}_h(R) = \big\{\{a\}\times[c,d] \leftrightarrow \{b\}\times[c,d] \text{ in } R\big\} \tag{3.32}$$

and

$$\mathcal{C}_v(R) = \big\{[a,b]\times\{c\} \leftrightarrow [a,b]\times\{d\} \text{ in } R\big\}. \tag{3.33}$$

Figure 3.2: An illustration of a horizontal crossing (on the left) and a vertical crossing (on the right) in the rectangle $[a,b]\times[c,d]$.



Source: Compiled by the author.

Let us proceed by defining the specific rectangles and crossings that are of interest to us. Consider a sequence of heights $H_0, H_1, \ldots,$. This sequence will be defined recursively as follows:

$$H_0 = 100 \text{ and } H_k = 2\left\lceil e^{L_k^{\left(1-\frac{\beta}{k+1}\right)}} \right\rceil H_{k-1}, \quad \text{for all } k \geq 1. \tag{3.34}$$

The choice of the initial height H_0 is really arbitrary and we could have chosen any other arbitrary positive integer.

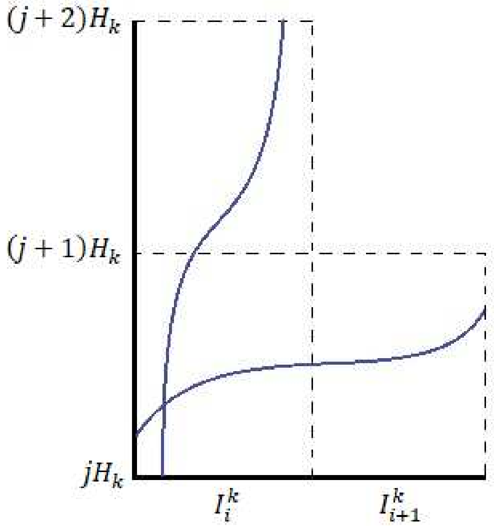
We will consider 2 types of rectangles: one whose base is formed by 2 consecutive intervals on some scale k and whose height is H_k and another whose base consists of one interval on some scale k and whose height is $2H_k$. For each $i, j, k \in \mathbb{Z}_+$, we will denote the horizontal and vertical crossings, at scale k, respectively as

$$H_{i,j}^k = \mathcal{C}_h\left((I_i^k \cup I_{i+1}^k) \times [jH_k, (j+1)H_k)\right) \tag{3.35}$$

and

$$V_{i,j}^k = \mathcal{C}_v\left(I_i^k \times [jH_k, (j+2)H_k)\right). \tag{3.36}$$

Figure 3.3: An illustration of occurrence of events $H_{i,j}^k$ and $V_{i,j}^k$, where the blue crossings are open paths.



Source: Compiled by the author.

Also, for every $i, j, k \in \mathbb{Z}_+$ and $p \in (0,1)$, define the probabilities

$$h_k(p; i, j) = \max_{\substack{\Lambda\,:\, I_i^k \text{ and } I_{i+1}^k \\ \text{are good}}} \mathbb{P}_p^\Lambda\left((H_{i,j}^k)^C\right)$$

and

$$v_k(p; i, j) = \max_{\substack{\Lambda\,:\, I_i^k \\ \text{is good}}} \mathbb{P}_p^\Lambda\left((V_{i,j}^k)^C\right),$$

where $(H^k_{i,j})^C$ and $(V^k_{i,j})^C$ denote the complementary event of $H^k_{i,j}$ and $V^k_{i,j}$, respectively. Also define

$$q_k(p;i.j) = \max\{h_k(p;i,j),\ v_k(p;i,j)\}. \tag{3.37}$$

Translation invariance allows us to write, for each $k \in \mathbb{Z}_+$,

$$q_k(p) := q_k(p;0,0) = q_k(p;i,j), \text{ for any } i,j \in \mathbb{Z}_+. \tag{3.38}$$

Now, we will show that, for p sufficiently close to 1, the sequence $q_k(p)$ decreases to zero fast in k.

Proposition 3.1. *Let $L_0 > e^{128}$ and k_0 be the positive integer satisfying conditions* (3.13)-(3.15)*. There is $p_0 = p_0(L_0, k_0, \beta, \mu) < 1$ such that, for all $k \geq k_0$ and $p \geq p_0$, we have*

$$q_k(p) \leq e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}. \tag{3.39}$$

The proof of Proposition 3.1 will be done by induction on k. For the induction step we will need of the following two lemmas:

Lemma 3.4. *Let $L_0 > e^{128}$, k_0 be the positive integer satisfying conditions* (3.13)-(3.15) *and $p > \frac{1}{2}$. For all $k \geq k_0$, if*

$$q_k(p) \leq e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}, \tag{3.40}$$

then

$$\mathbb{P}^\Lambda_p\left((H^{k+1}_{0,0})^C\right) \leq e^{-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}}, \tag{3.41}$$

for every environment $\Lambda \in \{I^{k+1}_0$ is good$\} \cap \{I^{k+1}_1$ is good$\}$.

The hypothesis $p > \frac{1}{2}$ is not important and was only adopted for convenience, to simplify the calculations in the proof.

Lemma 3.5. *Let $L_0 > e^{128}$ and k_0 be the positive integer satisfying conditions* (3.13)-(3.15)*. For all $k \geq k_0$, if*

$$q_k(p) \leq e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}, \tag{3.42}$$

then

$$\mathbb{P}^\Lambda_p\left((V^{k+1}_{0,0})^C\right) \leq e^{-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}}, \tag{3.43}$$

for every environment $\Lambda \in \{I^{k+1}_0$ is good$\}$.

Let us use the two lemmas above to show Proposition 3.1.

Proof of Proposition 3.1*:* This proof will be done by induction on k. Since $q_{k_0}(p)$ goes to 0 as p goes to 1, there is $p_0 = p_0(L_0, k_0, \beta, \mu) < 1$ such that, for all $p \geq p_0$,

$$q_{k_0}(p) \leq e^{-L_{k_0}^{1-\mu}}.$$

Now, suppose that for some $k \geq k_0$ we have $q_k(p) \leq e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}$. Thus, Lemmas 3.4 and 3.5 imply, respectively, that $\mathbb{P}_p^{\Lambda}\big((H_{0,0}^{k+1})^C\big) \leq e^{-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}}$ and $\mathbb{P}_p^{\Lambda}\big((V_{0,0}^{k+1})^C\big) \leq e^{-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}}$. Therefore, from (3.37), we conclude that $q_k(p) \leq e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}$, for all $k \geq k_0$. ■

Now we will prove Lemmas 3.4 and 3.5.

Proof of Lemma 3.4*:* Since $L_0 > e^{128}$ by Lemma 3.1 it satisfies the conditions (3.1)-(3.4). Notice that these conditions are respectively equivalent to

$$L_0 > \sqrt[1-\mu]{\frac{2}{1-\mu}}, \tag{3.44}$$

$$\exp\left(2\log L_0 - L_0^{1-\mu}\right) < \frac{1}{8}, \tag{3.45}$$

$$L_0 > \sqrt[1-\beta]{\frac{2}{1-\beta}}, \tag{3.46}$$

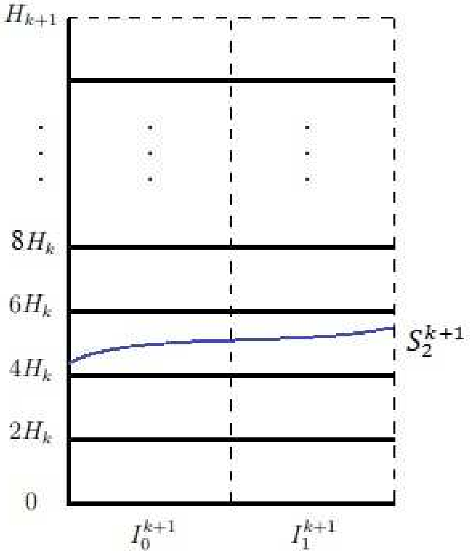
$$\exp\left(-e^{\frac{1}{2}L_0^{1-\beta}} + L_0\right) \leq 1. \tag{3.47}$$

Let $k \geq k_0$ and fix an environment Λ for which I_0^{k+1} and I_1^{k+1} are good intervals. By definition, both these intervals can contain at most two bad intervals at scale k and, in this case, they must be adjacent. Although the probability of crossing a bad interval on scale k is small, the exponential height of the rectangles guarantees that there will be a lot of attempts to do this. Indeed, let us divide the rectangle $R([0, 2L_{k+1}) \times [0, H_{k+1}))$ into strips of height $2H_k$ and verify whether or not these strips are crossed. For each $0 \leq j \leq \left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}} \right\rceil - 1$, define the events

$$S_j^{k+1} = \mathcal{C}_h\Big(R\big([0, 2L_{k+1}) \times [2jH_k, (2j+2)H_k)\big)\Big).$$

Notice if the event $H_{0,0}^{k+1}$ does not occur then none of the events S_j^{k+1} can occur, that is, $\left\{(H_{0,0}^{k+1})^C \subseteq \bigcap_j (S_j^{k+1})^C\right\}$, where $0 \leq j \leq \left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}} \right\rceil - 1$. See Figure 3.4.

Figure 3.4: An illustration of the occurrence of event S_2^{k+1}, that implies the occurrence of event $H_{0,0}^{k+1}$.



Source: Compiled by the author.

Therefore,

$$\mathbb{P}_p^\Lambda\big((H_{0,0}^{k+1})^C\big) \le \mathbb{P}_p^\Lambda\left(\bigcap_j (S_j^{k+1})^C\right) = \prod_j \mathbb{P}_p^\Lambda\big((S_j^{k+1})^C\big), \tag{3.48}$$

with j satisfying $0 \le j \le \left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}} \right\rceil - 1$. The inequality comes from inclusion of events and the equality holds since the events $(S_j^{k+1})^C$ are independent. Now, using the translation invariance of the events S_j^{k+1}'s in (3.48), we can write

$$\mathbb{P}_p^\Lambda\big((H_{0,0}^{k+1})^C\big) \le \Big(\mathbb{P}_p^\Lambda\big(S_0^{k+1}\big)^C\Big)^{\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}} \right\rceil} \le \Big(1-\mathbb{P}_p^\Lambda\big(S_0^{k+1}\big)\Big)^{e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}}. \tag{3.49}$$

In order to get an upper bound for the probability of S_0^{k+1}, we will build a horizontal crossing within the strip $R\big([0,2L_{k+1})\times[0,2H_k)\big)$ using the crossings events $H_{i,0}^k$ and $V_{i,0}^k$ in rectangles whose bases are good k-intervals of I_0^{k+1} and I_1^{k+1}, while, in the rectangles whose bases are bad k-intervals, we will open paths at its top as follows.

Notice that the base of the strip $R\big([0,2L_{k+1})\times[0,2H_k)\big)$ is divided into two parts of length L_{k+1} each one. Furthermore, each of these parts have $L_k^{\frac{2}{k+1}}$ intervals of length L_k. Since I_0^{k+1} and I_1^{k+1} are good intervals, among the $2L_k^{\frac{2}{k+1}}$ intervals in the base of the strip we can have at most two bad intervals L_k (and, in this case, they must be adjacent) in each part of length L_{k+1}.

For each $l \in \{0,1\}$, denote by j_l the lowest index of a bad k-interval within of I_l^{k+1} and consider the interval

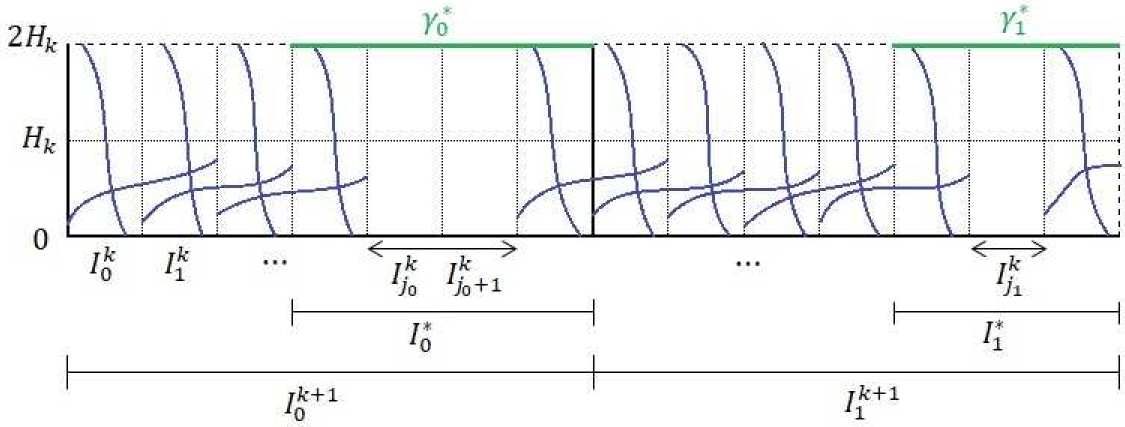
$$I_l^* = \left(I_{j_l-1}^k \cup I_{j_l}^k \cup I_{j_l+1}^k \cup I_{j_l+2}^k\right) \cap \left(I_0^{k+1} \cup I_1^{k+1}\right) \subseteq \mathbb{Z}_+,$$

that is, the interval formed by the first bad k-interval in I_l^{k+1}, one k-interval before and two k-intervals after it (as long as are contained in $I_0^{k+1} \cup I_1^{k+1}$). Also define, for each $l \in \{0,1\}$, the path γ_l^* formed by the edges of the form $\langle (m, 2H_k), (m+1, 2H_k)\rangle$, with $m \in I_l^*$. Thus, we have

$$\left(\bigcap_{\substack{i: I_i^k,\, I_{i+1}^k \\ \text{are good}}} H_{i,0}^k\right) \cap \left(\bigcap_{\substack{j: I_j^k \\ \text{is good}}} V_{j,0}^k\right) \cap \{\gamma_0^* \text{ and } \gamma_1^* \text{ are open paths}\} \subseteq S_0^{k+1}, \tag{3.50}$$

where $0 \le i, j \le 2L_k^{\frac{2}{k+1}} - 1$. See Figure 3.5 for more details.

Figure 3.5: An illustration of the occurrence of events $H_{i,0}^k$ and $V_{i,0}^k$ (blue crossings) and of two open paths γ_0^* and γ_1^* (green paths), which imply the occurrence of S_0^{k+1}. The intervals $I_{j_0}^k$, $I_{j_0+1}^k$ and $I_{j_1}^k$ correspond to bad k-intervals.



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Since I_0^* and I_1^* are disjoint intervals,

$$\begin{aligned}\mathbb{P}_p^\Lambda(\gamma_0^* \text{ and } \gamma_1^* \text{ are open paths}) &= \mathbb{P}_p^\Lambda(\gamma_0^* \text{ is an open path})\mathbb{P}_p^\Lambda(\gamma_1^* \text{ is an open path}) \\ &\ge \left(p^{4L_k}\right)^2 = p^{8L_k}. \end{aligned} \tag{3.51}$$

On other hand, by FKG inequality, independence of events H's and V's, and Bernoulli's inequality, we have

$$\begin{aligned}\mathbb{P}_p^\Lambda\left[\left(\bigcap_{\substack{i:I_i^k,\,I_{i+1}^k\\ \text{are good}}} H_{i,0}^k\right)\cap\left(\bigcap_{\substack{j:I_j^k\\ \text{is good}}} V_{j,0}^k\right)\right] &\geq \prod_{i=0}^{2L_k^{\frac{2}{k+1}}-1} \mathbb{P}_p^\Lambda(H_{i,0}^k)\,\mathbb{P}_p^\Lambda(V_{i,0}^k)\\ &\geq \left[\big(1-q_k(p)\big)^2\right]^{2L_k^{\frac{2}{k+1}}}\\ &\geq \big(1-4L_k^{\frac{2}{k+1}}\,q_k(p)\big). \end{aligned} \tag{3.52}$$

Hence, FKG inequality, (3.50), (3.51) and (3.52) imply that

$$\mathbb{P}_p^\Lambda(S_0^{k+1}) \geq \Big(1-4L_k^{\frac{2}{k+1}}\,q_k(p)\Big)\,p^{8L_k}. \tag{3.53}$$

Next we will show that

$$1-4L_k^{\frac{2}{k+1}}q_k(p) > \frac{1}{2}, \tag{3.54}$$

that is, $L_k^{\frac{2}{k+1}}q_k(p) < \frac{1}{8}$. Using (3.40), we get

$$\begin{aligned} L_k^{\frac{2}{k+1}}q_k(p) &\leq L_k^{\frac{2}{k+1}}\,e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}\\ &= \exp\left[\frac{2}{k+1}\log L_k - L_k^{\left(1-\frac{\mu}{k+1}\right)}\right]\\ &\leq \exp\big(2\log L_k - L_k^{1-\mu}\big), \end{aligned} \tag{3.55}$$

so, it is enough to show that

$$\exp\big(2\log L_k - L_k^{1-\mu}\big) < \frac{1}{8}.$$

For that, consider the function $f(x)=\exp\big(2\log x - x^{1-\mu}\big)$, for $x>0$. Thus, we have

$$f'(x) = \exp\big(2\log x - x^{1-\mu}\big)\left(\frac{2}{x}-(1-\mu)x^{-\mu}\right) < 0,$$

if $x > \sqrt[1-\mu]{\dfrac{2}{1-\mu}}$. So, $f(x)$ is decreasing for all $x > \sqrt[1-\mu]{\dfrac{2}{1-\mu}}$. It follows from (3.44) that $L_k \geq L_0 > \sqrt[1-\mu]{\dfrac{2}{1-\mu}}$. Thus, we get

$$f(L_k) \leq f(L_0) = \exp\big(2\log L_0 - L_0^{1-\mu}\big) < \frac{1}{8},$$

where the last inequality comes from (3.45), and so, (3.54) is proven. Since we are assuming $p > \frac{1}{2} > e^{-1}$, equation (3.53) implies that

$$\mathbb{P}_p^{\Lambda}(S_0^{k+1}) \geq \frac{1}{2}\, p^{8L_k} \geq e^{-1}\, e^{-8L_k} \geq e^{-9L_k}.$$

Thus, (3.49) implies

$$\begin{aligned}
\mathbb{P}_p^{\Lambda}\big((H_{0,0}^{k+1})^C\big) &\leq \Big[1 - \mathbb{P}_p^{\Lambda}(S_0^{k+1})\Big]^{e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}} \\
&\leq \Big(1 - e^{-9L_k}\Big)^{e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}} \\
&\leq \exp\left[-\exp\left(-9L_k + L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\right)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\mathbb{P}_p^{\Lambda}\big((H_{0,0}^{k+1})^C\big)}{\exp\left(-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right)} &\leq \exp\left[-\exp\left(-9L_k + L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\right) + L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right] \\
&= \exp\left[-\exp\left(-9L_{k+1}^{\left(1-\frac{2}{k+3}\right)} + L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\right) + L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right] \\
&= \exp\left[-\exp\left[L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\left(1 - 9L_{k+1}^{\left(-\frac{2}{k+3}+\frac{\beta}{k+2}\right)}\right)\right] + L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right] \\
&\leq \exp\left[-\exp\left[L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\left(1 - 9L_{k+1}^{\left(-\frac{2}{k+3}+\frac{\beta}{k+2}\right)}\right)\right] + L_{k+1}\right]. \qquad (3.56)
\end{aligned}$$

Notice that

$$\begin{aligned}
1 - 9L_{k+1}^{\left(-\frac{2}{k+3}+\frac{\beta}{k+2}\right)} &= 1 - 9L_{k+1}^{\left(\frac{(\beta-2)k+3\beta-4}{(k+2)(k+3)}\right)} \\
&= 1 - 9L_0^{\left(\frac{(\beta-2)k+3\beta-4}{2}\right)} \\
&\geq 1 - 9L_0^{\left(\frac{3\beta-4}{2}\right)} \\
&\geq 1 - 9L_0^{-\frac{1}{2}},
\end{aligned}$$

where the two inequalities come from the fact that $\beta < 1$. Since $L_0 > e^{128} > (18)^2$, then $L_0^{-\frac{1}{2}} \leq \frac{1}{18}$, which implies that $1 - 9L_0^{-\frac{1}{2}} \geq \frac{1}{2}$. Thus, we have

$$-\exp\left[L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\left(1 - 9L_{k+1}^{\left(-\frac{2}{k+3}+\frac{\beta}{k+2}\right)}\right)\right] \leq -\exp\left(\frac{1}{2}\, L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\right), \qquad (3.57)$$

and, substituting (3.57) in (3.56), we get

$$\begin{aligned}
\frac{\mathbb{P}_p^{\Lambda}\big((H_{0,0}^{k+1})^C\big)}{\exp\left(-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right)} &\leq \exp\left[-\exp\left(\frac{1}{2}\,L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}\right)+L_{k+1}\right] \\
&\leq \exp\left[-\exp\left(\frac{1}{2}\,L_{k+1}^{1-\beta}\right)+L_{k+1}\right] \\
&= \exp\left[-e^{\frac{1}{2}L_{k+1}^{1-\beta}}\left(1-\frac{L_{k+1}}{e^{\frac{1}{2}L_{k+1}^{1-\beta}}}\right)\right].
\end{aligned}$$

Next we will show that

$$\exp\left[-e^{\frac{1}{2}L_{k+1}^{1-\beta}}\left(1-\frac{L_{k+1}}{e^{\frac{1}{2}L_{k+1}^{1-\beta}}}\right)\right] \leq 1.$$

For that, consider the function $g(x)=1-\dfrac{x}{e^{\frac{1}{2}x^{1-\beta}}}$, where $x>0$. Since

$$g'(x)=e^{-\frac{1}{2}x^{1-\beta}}\left(\frac{1-\beta}{2}x^{1-\beta}-1\right)>0,$$

for all $x>\sqrt[1-\beta]{\dfrac{2}{1-\beta}}$, then $g(x)$ is increasing for all $x>\sqrt[1-\beta]{\dfrac{2}{1-\beta}}$. From (3.46), it follows that $L_k \geq L_0 > \sqrt[1-\beta]{\dfrac{2}{1-\beta}}$. Thus, $g(L_k)\geq g(L_0)$, for all $k\in\mathbb{Z}_+$. Hence,

$$\begin{aligned}
\exp\left[-e^{\frac{1}{2}L_{k+1}^{1-\beta}}\left(1-\frac{L_{k+1}}{e^{\frac{1}{2}L_{k+1}^{1-\beta}}}\right)\right] &\leq \exp\left[-e^{\frac{1}{2}L_{k+1}^{1-\beta}}\left(1-\frac{L_0}{e^{\frac{1}{2}L_0^{1-\beta}}}\right)\right] \\
&\leq \exp\left[-e^{\frac{1}{2}L_0^{1-\beta}}\left(1-\frac{L_0}{e^{\frac{1}{2}L_0^{1-\beta}}}\right)\right] \\
&= \exp\left(-e^{\frac{1}{2}L_0^{1-\beta}}+L_0\right) \\
&\leq 1,
\end{aligned}$$

where the last inequality comes from (3.47). Therefore, we can conclude that, for all $k\in\mathbb{Z}_+$, we have

$$\mathbb{P}_p^{\Lambda}\big((H_{0,0}^{k+1})^C\big)\leq e^{-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}}.$$

■

Proof of Lemma 3.5*:* Since $L_0>e^{128}$ by Lemma 3.1 it satisfies the conditions (3.5) and (3.6). Notice that these conditions are respectively equivalent to

$$\frac{1}{24}-2L_0^{-\frac{\mu}{2}}\geq\frac{1}{48} \tag{3.58}$$

and

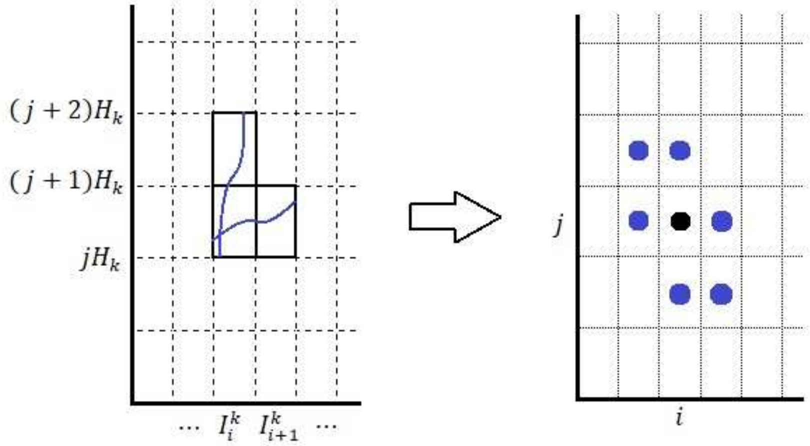
$$\frac{1}{48}L_0^{1-\mu} - \frac{19\log 2}{12} \geq 1. \tag{3.59}$$

Let $k \geq k_0$ and fix an environment Λ for which I_0^{k+1} is a good interval. By definition, this interval can contain at most two bad intervals on the scale k and, in this case they must be adjacent. In this way, either each I_i^k is good for every $i = 0, 1, \dots, \left\lfloor \frac{L_k^{\frac{2}{k+1}}}{2} \right\rfloor - 2$ or I_i^k is good for every $i = \left\lfloor \frac{L_k^{\frac{2}{k+1}}}{2} \right\rfloor + 1, \left\lfloor \frac{L_k^{\frac{2}{k+1}}}{2} \right\rfloor + 2, \dots, L_k^{\frac{2}{k+1}} - 1$. Assume, without loss of generality, that the first case holds, and let

$$M_k = \left\lfloor \frac{L_k^{\frac{2}{k+1}}}{2} \right\rfloor - 1. \tag{3.60}$$

In order to estimate the probability of the event $V_{0,0}^{k+1}$, we will consider the following rescaled lattice: each rectangle $R\big(I_i^k \times [jH_k, (j+1)H_k)\big)$ will correspond to a vertex (i, j), for all $i, j \in \mathbb{Z}_+$. Such vertex (i, j) is declared open if the event $H_{i,j}^k \cap V_{i,j}^k$ occurs in the original lattice. Notice that the resulting percolation process in this renormalized lattice is a dependent process, since the state of one vertex (i, j) depend on the state of other six vertices so that there is 6 vertices (i', j') such that the events $\{(i, j) \text{ is open}\}$ and $\{(i', j') \text{ is open}\}$ are dependent, see Figure 3.6 for more details.

Figure 3.6: On the left an illustration of the occurrence of event $H_{i,j}^k \cap V_{i,j}^k$ and on the right an illustration of the renormalized lattice with the vertex (i, j) corresponding to event $H_{i,j}^k \cap V_{i,j}^k$ in the original lattice. The vertex (i, j) is represented by a black ball and its state depends on the states of the six vertices represented by blue balls.



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Recall the definitions (3.31) and (3.33) and consider the rectangle

$$R_0^{k+1} = R\left([0, M_k) \times \left[0, 4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}} \right\rceil\right)\right),$$

and its vertical crossing event $\mathcal{C}_v(R_0^{k+1})$, where M_k is given by (3.60). Notice that

$$\mathbb{P}_p^{\Lambda}(V_{0,0}^{k+1}) \geq \mathbb{P}\big(\mathcal{C}_v(R_0^{k+1})\big). \tag{3.61}$$

Suppose that the the event $\mathcal{C}_v(R_0^{k+1})$ does not occur. Then there must be a sequence of distinct vertices, namely $(i_0, j_0), (i_1, j_1), \ldots, (i_n, j_n)$ in R_0^{k+1}, such that

(i) $\max\limits_{1 \leq l \leq n} \{|i_l - i_{l-1}|, |j_l - j_{l-1}|\} = 1$,

(ii) $(i_0, j_0) \in \{0\} \times \left[0, 4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\right\rceil\right]$ and $(i_n, j_n) \in \{M_k\} \times \left[0, 4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\right\rceil\right]$,

(iii) (i_l, j_l) is closed for every $l = 0, 1, \ldots, n$.

Notice that there are at most $4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\right\rceil 8^n$ such sequences with $n+1$ vertices that satisfies the conditions (i) and (ii) above. Moreover, the probability of a vertex in R_0^{k+1} to be declared closed is at most $2q_k(p)$. Also, by the dependence in the rescaled lattice, for every set with $n+1$ vertices, there exists at least $\frac{n}{6}$ vertices that have been declared open, independently of each other. Hence, we have

$$\begin{aligned}
\mathbb{P}_p^{\Lambda}(\mathcal{C}_v(R_0^{k+1})^C) &\leq \mathbb{P}\begin{pmatrix} \text{there is a sequence of } n+1 \text{ vertices in } R_0^{k+1} \\ \text{satisfying the conditions (i), (ii) and (iii)} \end{pmatrix} \\
&\leq \sum_{n \geq M_k} 4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\right\rceil 8^n \big(2q_k(p)\big)^{\frac{n}{6}} \\
&\leq 4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\right\rceil \sum_{n \geq M_k} 8^n \left(2\, e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}\right)^{\frac{n}{6}} \\
&\leq 4\left\lceil e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\right\rceil \sum_{n \geq M_k} 2^{\frac{19n}{6}} \exp\left(-\frac{n}{6}\, L_k^{\left(1-\frac{\mu}{k+1}\right)}\right).
\end{aligned}$$

Thus, from (3.60) and (3.61),

$$\begin{aligned}
\mathbb{P}_p^{\Lambda}\big((V_0^{k+1})^C\big) &\leq 4\, e^{L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)}}\, 2^{\frac{19}{12}\left(L_k^{\frac{2}{k+1}}-2\right)} \exp\left(-\frac{1}{6}\left(\frac{L_k^{\frac{2}{k+1}}}{2} - 1\right) L_k^{\left(1-\frac{\mu}{k+1}\right)}\right) \\
&\leq 4\, 2^{\frac{19}{12}L_k^{\frac{2}{k+1}}} \exp\left(L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)} + \frac{L_k^{\left(1-\frac{\mu}{k+1}\right)}}{6} - \frac{L_k^{\frac{2}{k+1}} L_k^{\left(1-\frac{\mu}{k+1}\right)}}{12}\right) \\
&\leq 4\, 2^{\frac{19}{12}L_k^{\frac{2}{k+1}}} \exp\left(L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)} + \frac{L_{k+1}^{\left(1-\frac{(2+\mu)}{k+3}\right)}}{6} - \frac{L_{k+1}^{\left(1-\frac{\mu}{k+3}\right)}}{12}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathbb{P}_p^{\Lambda}\big((V_0^{k+1})^C\big)}{\exp\left(-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right)} \\
&\leq 4\, 2^{\frac{19}{12}L_k^{\frac{2}{k+1}}} \exp\left(L_{k+1}^{\left(1-\frac{\beta}{k+2}\right)} + \frac{L_{k+1}^{\left(1-\frac{(2+\mu)}{k+3}\right)}}{6} - \frac{L_{k+1}^{\left(1-\frac{\mu}{k+3}\right)}}{12} + L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)} \right) \\
&= 4\, 2^{\frac{19}{12}L_k^{\frac{2}{k+1}}} \exp\left[-L_{k+1}^{\left(1-\frac{\mu}{k+3}\right)} \left(\frac{1}{12} - \frac{L_{k+1}^{-\frac{2}{k+3}}}{6} - L_{k+1}^{\left(-\frac{\beta}{k+2}+\frac{\mu}{k+3}\right)} - L_{k+1}^{\left(-\frac{\mu}{k+2}+\frac{\mu}{k+3}\right)} \right) \right]. \qquad (3.62)
\end{aligned}$$

Next we will show that

$$\frac{1}{12} - \frac{L_{k+1}^{-\frac{2}{k+3}}}{6} - L_{k+1}^{\left(-\frac{\beta}{k+2}+\frac{\mu}{k+3}\right)} - L_{k+1}^{\left(-\frac{\mu}{k+2}+\frac{\mu}{k+3}\right)} \geq \frac{1}{48}.$$

Notice that

$$\begin{aligned}
&\frac{1}{12} - \frac{L_{k+1}^{-\frac{2}{k+3}}}{6} - L_{k+1}^{\left(-\frac{\beta}{k+2}+\frac{\mu}{k+3}\right)} - L_{k+1}^{\left(-\frac{\mu}{k+2}+\frac{\mu}{k+3}\right)} \\
&= \frac{1}{12} - \frac{L_{k+1}^{-\frac{2}{k+3}}}{6} - L_{k+1}^{\left(\frac{-\beta(k+3)+\mu(k+2)}{(k+2)(k+3)}\right)} - L_{k+1}^{\left(\frac{-\mu(k+3)+\mu(k+2)}{(k+2)(k+3)}\right)} \\
&= \frac{1}{12} - L_0^{-(k+2)} - L_0^{\frac{(\mu-\beta)k+2\mu-3\beta}{2}} - L_0^{-\frac{\mu}{2}} \\
&\geq \frac{1}{12} - L_0^{-2} - 2L_0^{-\frac{\mu}{2}} \\
&\geq \frac{1}{24} - 2L_0^{-\frac{\mu}{2}} \\
&\geq \frac{1}{48}, \qquad (3.63)
\end{aligned}$$

where in the second equality we use the relation $L_{k+1} = L_0^{\frac{(k+2)(k+3)}{2}}$, the first inequality follows because $\mu < \beta$ implies that $(\mu-\beta)k + 2\mu - 3\beta < 2\mu - 3\mu = -\mu$, the third inequality follows since $L_0 > e^{128} > 2$ implies that $6L_0^2 > 24$ and the last inequality comes from (3.58). Hence, from (3.62) and (3.63), we have

$$\begin{aligned}
\frac{\mathbb{P}_p^{\Lambda}\big((V_0^{k+1})^C\big)}{\exp\left(-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}\right)} &\leq 4\cdot 2^{\frac{19}{12}\,L_k^{\frac{2}{k+1}}}\exp\left(-\frac{1}{48}\,L_{k+1}^{\left(1-\frac{\mu}{k+3}\right)}\right)\\
&= 4\ \exp\left(\frac{19\ \log 2}{12}L_k^{\frac{2}{k+1}}-\frac{1}{48}\,L_{k+1}^{\left(1-\frac{\mu}{k+3}\right)}\right)\\
&= 4\ \exp\left(\frac{19\ \log 2}{12}L_k^{\frac{2}{k+1}}-\frac{1}{48}\,\left(L_k^{\frac{k+3}{k+1}}\right)^{\left(\frac{k+3-\mu}{k+3}\right)}\right)\\
&= 4\ \exp\left(\frac{19\ \log 2}{12}L_k^{\frac{2}{k+1}}-\frac{1}{48}\,\left(L_k^{\frac{k+3-\mu}{k+1}}\right)\right)\\
&= 4\ \exp\left(\frac{19\ \log 2}{12}L_k^{\frac{2}{k+1}}-\frac{1}{48}\,L_k^{\frac{2}{k+1}}\,L_k^{\left(1-\frac{\mu}{k+1}\right)}\right)\\
&= 4\ \exp\left[-L_k^{\frac{2}{k+1}}\left(\frac{1}{48}\,L_k^{1-\frac{\mu}{k+1}}-\frac{19\ \log 2}{12}\right)\right]\\
&\leq 4\ \exp\left[-L_k^{\frac{2}{k+1}}\left(\frac{1}{48}\,L_k^{1-\mu}-\frac{19\ \log 2}{12}\right)\right]\\
&\leq 4\ \exp\left[-L_k^{\frac{2}{k+1}}\left(\frac{1}{48}\,L_0^{1-\mu}-\frac{19\ \log 2}{12}\right)\right]\\
&\leq 4\ \exp\left(-L_k^{\frac{2}{k+1}}\right), \qquad (3.64)
\end{aligned}$$

where in the fourth equality we use the identities $\frac{k+3-\mu}{k+1}=\frac{(k+1)+2-\mu}{k+1}=\frac{2}{k+1}+\big(1-\frac{\mu}{k+1}\big)$, in the second inequality we use that $\frac{\mu}{k+1}<\mu$, the third inequality holds since $L_k\geq L_0$ and the last inequality follows from (3.59). Observe that

$$\exp\left(-L_k^{\frac{2}{k+1}}\right)=\exp\left(-\left(L_0^{\frac{(k+1)(k+2)}{2}}\right)^{\frac{2}{k+1}}\right)=e^{-L_0^{k+2}}\leq e^{-L_0^2}.$$

Since $L_0>e^{128}>2$, we have $e^{-L_0^2}\leq\dfrac{1}{e^4}$. Hence,

$$4\,e^{-L_k^{\frac{2}{k+1}}}\leq 4\,e^{-L_0^2}\leq\frac{4}{e^4}\leq 1.$$

Therefore, by (3.64), we can conclude that, for all $k\in\mathbb{Z}_+$

$$\mathbb{P}_p^{\Lambda}\big((V_0^{k+1})^C\big)\leq e^{-L_{k+1}^{\left(1-\frac{\mu}{k+2}\right)}}.$$

■

3.3 Proof of Theorem 1.1

Now, we are in condition to prove Theorem 1.1, which gives us the non-trivial phase transition for our model. Using the results obtained in Sections 3.1 and 3.2, we will first show that there is a sufficiently large $p < 1$ such that, for almost all realizations of Λ, the event $\{(0,0) \leftrightarrow \infty\}$ occurs with strictly positive probability.

First, we will make an observation that will also be used in the proof.

Remark 3.2. *If* $0 \leq a_1, \ldots, a_n \leq 1$*, it follows by induction in* n *that*

$$\prod_{i=1}^{n}(1 - a_i) \geq 1 - \sum_{i=1}^{n} a_i. \tag{3.65}$$

From (3.71)*, if* L_0 *is as in Lemma* 3.1*, then for all* $k \in \mathbb{Z}_+$*, we have*

$$2L_k^{\frac{2}{k+1}} e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}} < L_0^{-(k+2)} < 1. \tag{3.66}$$

Also, from (3.66) *and* (3.65)*, we have*

$$\prod_{k \geq k_0} \left[1 - 2L_k^{\frac{2}{k+1}} \exp\left(-L_k^{\left(1-\frac{\mu}{k+1}\right)}\right)\right] \geq 1 - 2\sum_{k \geq k_0} L_k^{\frac{2}{k+1}} \exp\left(-L_k^{\left(1-\frac{\mu}{k+1}\right)}\right). \tag{3.67}$$

Proof of Theorem 1.1 (**percolation for p sufficiently large**)**:** The idea of the proof is to show percolation in the original model indirectly by considering a new model, which percolates, given by an aperiodic random variable $\widetilde{\xi}$ taking integer values greater than $e^{c^2/4}$ such that this model will be dominated by the original one.

Next we will define the new model described by the $\widetilde{\xi}$ variable. Let ξ be any positive random variable such that $\mathbb{E}(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}) < \infty$. Let a_0 and a_1 be two integers such that $a_0 < a_1$, $\mathbb{P}\big(\lceil \xi \rceil = a_0\big) > 0$ and $\mathbb{P}\big(\lceil \xi \rceil = a_1\big) > 0$. Consider $a_2 = \max\{a_1, \lceil e^{c^2/4} \rceil\} + 1$ and define a new random variable $\widetilde{\xi}$ by

$$\widetilde{\xi} = \begin{cases} \lceil \xi \rceil, \text{ if } \xi > a_2 \\ a_2, \text{ if } \xi \in (a_0, a_2] \\ a_2 - 1, \text{ if } \xi \leq a_0 \end{cases} . \tag{3.68}$$

Notice that $\widetilde{\xi}$ satisfies the properties we want, namely, $\widetilde{\xi}$ is an aperiodic, integer-valued random variable taking values greater than $e^{c^2/4}$ and $\{\widetilde{\xi}_i\}_{i \in \mathbb{Z}_+}$ dominates stochastically $\{\xi_i\}_{i \in \mathbb{Z}_+}$. Moreover, $\mathbb{E}(\widetilde{\xi} e^{c(\log \widetilde{\xi})^{1/2}}) < \infty$, since $\widetilde{\xi} e^{c(\log \widetilde{\xi})^{1/2}} \leq 2^{c+1} \xi e^{c(\log \xi)^{1/2}}$, for all $\xi \geq a_0$, this will be shown in Remark 3.3.

Suppose that Theorem 1.1 is valid for $\widetilde{\xi}$. So, by stochastic domination, the same will be true for the original model.

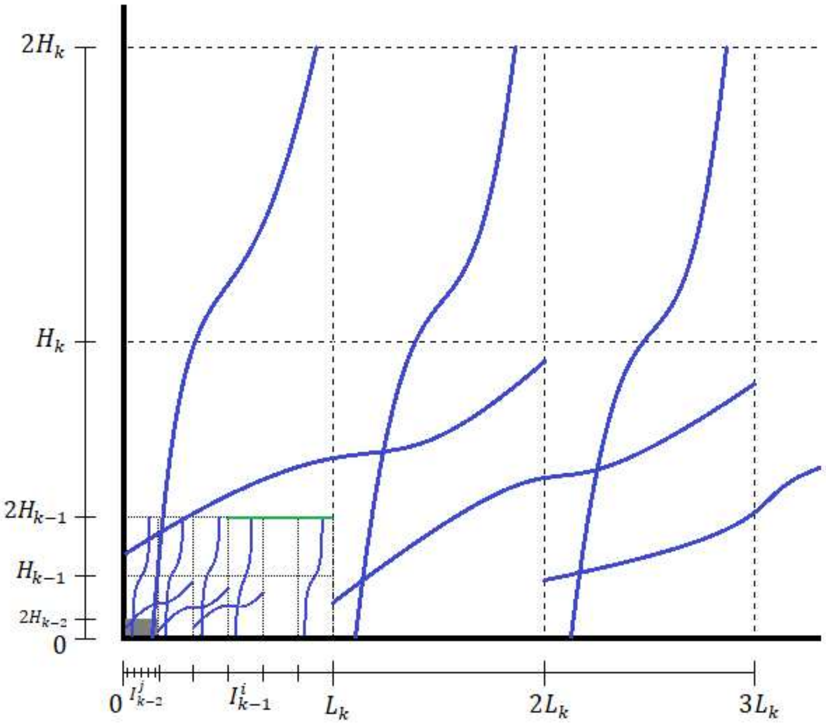
Next, we will prove Theorem 1.1 for the model given by the variable $\widetilde{\xi}$, which will be replaced with ξ, for the sake of notation. Being ξ an aperiodic and integer-valued random variable taking values greater than $e^{c^2/4}$, we can apply all the results previously obtained in Chapter 2.

Let β, μ and L_0 be given as in Lemma 3.1 and k_0 be the positive integer satisfying the conditions (3.13)-(3.15). In particular, L_0 satisfies the conditions (3.1)-(3.7). Notice that (3.30) implies that with strictly positive probability there is an environment Λ such that the intervals I_0^k and I_1^k are good for all scale $k \geq k_0$. So let Λ be some fixed environment with the above property.

According to the definition of the events $H_{i,j}^k$ and $V_{i,j}^k$, see Figure 3.7, we have that

$$\bigcap_{k \geq k_0} \bigcap_{i=0}^{L_k^{\frac{2}{k+1}}-2} (H_{i,0}^k \cap V_{i,0}^k) \subseteq \{\text{there is an infinite open cluster}\}. \tag{3.69}$$

Figure 3.7: An illustration of the intersections between the events $H_{i,0}^k$ and $V_{i,0}^k$ for different scales, showing the existence of the infinite cluster.



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By Proposition 3.1 there is p sufficiently close to 1, independently of Λ, such that $q_k(p) \leq e^{-L_k^{\left(1-\frac{\mu}{k+1}\right)}}$, for all $k \geq k_0$. Hence,

$$\begin{aligned}
\mathbb{P}_p^\Lambda\left(\bigcap_{k\geq k_0}\bigcap_{i=0}^{L_k^{\frac{2}{k+1}}-2}(H_{i,0}^k\cap V_{i,0}^k)\right) &\geq \prod_{k\geq k_0}\prod_{i=0}^{L_k^{\frac{2}{k+1}}-2}\mathbb{P}_p^\Lambda(H_{i,0}^k)\mathbb{P}_p^\Lambda(V_{i,0}^k)\\
&\geq \prod_{k\geq k_0}\left[\big(1-q_k(p)\big)^2\right]^{L_k^{\frac{2}{k+1}}-2}\\
&\geq \prod_{k\geq k_0}\big(1-q_k(p)\big)^{2L_k^{\frac{2}{k+1}}}\\
&\geq \prod_{k\geq k_0}\big(1-2L_k^{\frac{2}{k+1}}q_k(p)\big)\\
&\geq \prod_{k\geq k_0}\left[1-2L_k^{\frac{2}{k+1}}\exp\left(-L_k^{\left(1-\frac{\mu}{k+1}\right)}\right)\right]\\
&\geq 1-2\sum_{k\geq k_0}L_k^{\frac{2}{k+1}}\exp\left(-L_k^{\left(1-\frac{\mu}{k+1}\right)}\right), \qquad (3.70)
\end{aligned}$$

where the first, the second, the fifth, the sixth and the last inequalities follows from FKG inequality, (3.38), Bernoulli's inequality, (3.39) and (3.67), respectively.

Next we will show that for all $k\in\mathbb{Z}_+$, we have

$$L_k^{\frac{2}{k+1}}\exp\left(-L_k^{\left(1-\frac{\mu}{k+1}\right)}\right) < L_0^{-(k+2)}. \qquad (3.71)$$

Notice that

$$\begin{aligned}
L_k^{\frac{2}{k+1}}\exp\left(-L_k^{\left(1-\frac{\mu}{k+1}\right)}\right) &= L_0^{k+2}\ \exp\left(-L_0^{\frac{(k+1-\mu)(k+2)}{2}}\right)\\
&\leq L_0^{k+2}\ \exp\left(-L_0^{\frac{(1-\mu)}{2}(k+2)}\right)\\
&= \exp\left((k+2)\log L_0 - L_0^{\frac{(1-\mu)}{2}(k+2)}\right)\\
&= \exp\left((k+2)\log L_0 - e^{\frac{(1-\mu)(k+2)}{2}\log L_0}\right)\\
&\leq \exp\left((k+2)\log L_0 - \frac{(1-\mu)^2(k+2)^2(\log L_0)^2}{8}\right)\\
&= \exp\left[(k+2)\log L_0\left(1-\frac{(1-\mu)^2}{8}\ (k+2)\log L_0\right)\right]\\
&\leq \exp\left[-(k+1)(k+2)\log L_0\right]\\
&= L_0^{-(k+2)(k+1)} < L_0^{-(k+2)}, \qquad (3.72)
\end{aligned}$$

where the third inequality follows from (3.7), which implies that $\dfrac{(1-\mu)^2}{8}\log L_0 \geq 1$.

Hence, substituting (3.72) in (3.70), we obtain

$$\begin{aligned}
\mathbb{P}_p^{\Lambda}\left(\bigcap_{k\geq k_0}\bigcap_{i=0}^{L_k^{\frac{2}{k+1}}-2}(H_{i,0}^k\cap V_{i,0}^k)\right) &\geq 1-2\sum_{k\geq k_0}L_0^{-(k+2)} = 1-\frac{2}{L_0^2}\sum_{k\geq k_0}\left(\frac{1}{L_0}\right)^k\\
&\geq 1-\frac{2}{L_0^2}\sum_{k\geq 0}\left(\frac{1}{L_0}\right)^k = 1-\frac{2}{L_0^2}\,\frac{1}{1-\frac{1}{L_0}}\\
&= 1-\frac{2}{L_0(L_0-1)} > \frac{1}{2},
\end{aligned} \tag{3.73}$$

where the last inequality holds since $L_0 > e^{128} > 3$ implies that $\dfrac{2}{L_0(L_0-1)} < \dfrac{1}{2}$.

Therefore, from (3.69) and (3.73), we conclude that

$$\mathbb{P}_p^{\Lambda}(\text{there is an infinite open cluster}) > \frac{1}{2} > 0,$$

which proves the theorem. ■

Remark 3.3. *Now we will show that* $\mathbb{E}(\widetilde{\xi}e^{c(\log\widetilde{\xi})^{1/2}}) < \infty$*, where* $\widetilde{\xi}$ *is given by* (3.68)*. For that, let* $f(x) = xe^{c(\log x)^{1/2}}$*, where* $x\in(1,\infty)$*. Given* $k\in\mathbb{Z}_+$*, let* $\xi\in[a_0+k, a_0+k+1)$*. Since* $\dfrac{a_0+k+1}{a_0+k} = 1+\dfrac{1}{a_0+k} < 2$ *and* $\sqrt{\log(a_0+k+1)}+\sqrt{\log(a_0+k)}\big) > 1$*, it holds that*

$$\begin{aligned}
f(\widetilde{\xi}) = f(\lceil\xi\rceil) = f(a_0+k+1) &= \frac{f(a_0+k+1)}{f(a_0+k)}\,f(a_0+k)\\
&= \frac{a_0+k+1}{a_0+k}\,e^{c\left(\sqrt{\log(a_0+k+1)}-\sqrt{\log(a_0+k)}\right)}\,f(a_0+k)\\
&= \frac{a_0+k+1}{a_0+k}\,\exp\left(\frac{c\,\big(\log(a_0+k+1)-\log(a_0+k)\big)}{\sqrt{\log(a_0+k+1)}+\sqrt{\log(a_0+k)}}\right)f(a_0+k)\\
&= \frac{a_0+k+1}{a_0+k}\,\exp\left(\frac{c\,\log\left(\frac{a_0+k+1}{a_0+k}\right)}{\sqrt{\log(a_0+k+1)}+\sqrt{\log(a_0+k)}}\right)f(a_0+k)\\
&\leq 2\,e^{c\log 2}\,f(a_0+k) = 2^{c+1}f(a_0+k).
\end{aligned}$$

Therefore, since $f(\widetilde{\xi})\leq f(a_0)$*, for* $0\leq\xi\leq a_0$ *and* $\mathbb{E}\big(f(\xi)\mathbb{1}_{\{\xi\geq 1\}}\big) < \infty$*, we have*

$$\begin{aligned}
\mathbb{E}\big(f(\widetilde{\xi})\big) &= \mathbb{E}\big(f(\widetilde{\xi})\mathbb{1}_{\{0\leq\xi<a_0\}}\big)+\mathbb{E}\big(f(\widetilde{\xi})\mathbb{1}_{\{\xi\geq a_0\}}\big)\\
&\leq f(a_0)+2^{c+1}\mathbb{E}\big(f(\xi)\mathbb{1}_{\{\xi\geq a_0\}}\big)\\
&< \infty.
\end{aligned}$$

The absence of percolation for small values of p, given by (1.5) in Theorem 1.1, has been proved in [9], however, for pedagogical reasons we will reproduce it here.

In the following proof we will consider another lattice obtained from the original one by contracting the edges whose random variables ξ_i are small at a single vertex. Notice that percolation is facilitated in this new lattice, so to show the absence of percolation in the original lattice we simply need to show that the probability of the event $\{o \leftrightarrow \infty\}$ occurring in the new lattice is equal to 0.

Proof of Theorem 1.1 (**absence of percolation for p small enough**): Since ξ is a positive random variable, there is a value $0 < \lambda \leq 1$ such that $\mathbb{P}(\xi \geq \lambda) \geq \frac{1}{2}$. Fix this value of λ. For a fixed realization of i.i.d. copies $\{\xi_i\}_{i \in \mathbb{Z}_+^*}$ of ξ, define the sequence $\{J_k\}_{k \in \mathbb{Z}_+}$ by $J_0 = 0$ and, for all, $k \geq 1$,

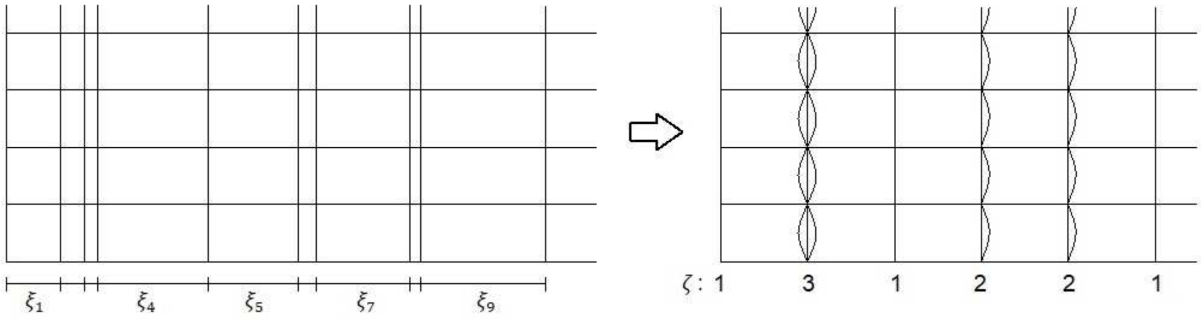
$$J_k = \min \left\{ n \ : \ \sum_{i=0}^{n} \mathbb{1}_{\{\xi_i \geq \lambda\}} \geq k \right\}.$$

Notice that J_k is counting the number of elements in the sequence $\{\xi_i\}_i$ that are necessary until k elements ξ_i appear whose value exceeds λ. Now, define another sequence of random variables $\{\zeta_k\}_{k \in \mathbb{Z}_+}$ by

$$\zeta_0 = 0 \text{ and } \zeta_k = J_k - J_{k-1}, \text{ for } k \geq 1. \tag{3.74}$$

Observe that ζ_k is counting how many elements of the sequence $\{\xi_i\}_i$ appeared between the $(k-1)$-th and k-th appearance of an element ξ_i whose value exceeds λ. See Figure 3.8 for more details.

Figure 3.8: Example of the definitions above where ξ_1, ξ_4, ξ_5, ξ_7 and ξ_9 are greater than λ and $\zeta_1 = 1$, $\zeta_2 = 3$, $\zeta_3 = 1$, $\zeta_4 = 2$, $\zeta_5 = 2$ and $\zeta_6 = 1$.



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Recall that, for a fixed environment Λ, the law of a bond percolation process in $\mathbb{Z}_+^2$ is denoted by $\mathbb{P}_p^\Lambda$ and given by (1.3). Now, we will realize three operations on the lattice $\mathcal{L}_\Lambda$ that will only increase the probability of the existence of an infinite cluster in the resulting lattice:

(1) when $\xi_i \geq \lambda$, we will replace ξ_i by λ (this means that all horizontal edges whose length was $\xi_i \geq \lambda$ will now be open with probability p^λ instead of p^{ξ_i});

(2) when $\xi_i < \lambda$, we will replace ξ_i by 0, that is, we will declare the horizontal edge $\langle (i,j),(i+1,j)\rangle$ open with probability 1 (this means that we can contract each horizontal edge $\langle (i,j),(i+1,j)\rangle$ into a single vertex);

(3) replace the probability of a vertical edge being open from p to p^λ.

Since the operation (2) connects ζ_k horizontal edges into a vertex, the probability of the new vertical edge (which is formed by a union of edges) being opened will be equal to $1-(1-p^\lambda)^{\zeta_k}$. Thus, these three operations lead to a model in $\mathbb{Z}^2_+$ in which each edge is independently declared open with probability

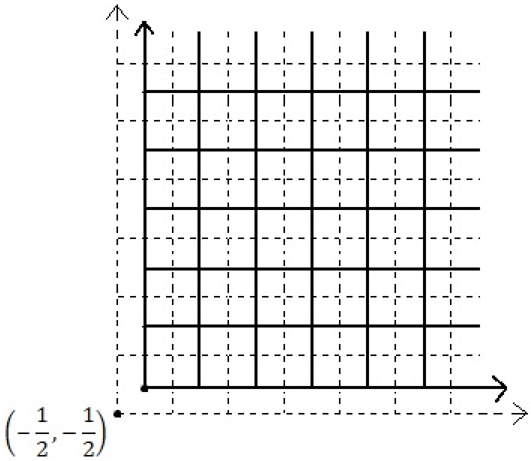
$$p_e = \begin{cases} 1-\left(1-p^\lambda\right)^{\zeta_i}, \text{ if } e = \langle (i,j),(i,j+1)\rangle \\ p^\lambda, \text{ if } e = \langle (i,j),(i+1,j)\rangle \end{cases} . \tag{3.75}$$

Therefore, we only need to show that in this new model the probability of the event $\{o \leftrightarrow \infty\}$ is equal to 0 to conclude that in the original model there is no infinite open cluster.

We will use a classical duality argument. Let us consider $(\mathbb{Z}^2_+)^\star := \mathbb{Z}^2_+ - \left(\frac{1}{2},\frac{1}{2}\right)$ as the lattice obtained of the original square lattice $\mathbb{Z}^2_+$ by shifting $\frac{1}{2}$ units left and down. Furthermore, $(\mathbb{Z}^2_+)^\star$ is not exactly the dual of $\mathbb{Z}^2_+$ (but a translation of it) because the edges along its leftmost vertical and lowermost horizontal semiaxis, $\left\{-\frac{1}{2}\right\} \times \left(\mathbb{Z}_+ - \frac{1}{2}\right)$ and $\left(\mathbb{Z}_+ - \frac{1}{2}\right) \times \left\{-\frac{1}{2}\right\}$ respectively, do not intercept any edge of $\mathbb{Z}^2_+$. See Figure 3.9 for more details.

Our goal is to show that there are infinitely many dual semicircuits in $(\mathbb{Z}^2_+)^\star$ connecting the leftmost vertical semiaxis and the lowest horizontal semiaxis of $(\mathbb{Z}^2_+)^\star$, which gives us that the origin of $\mathbb{Z}^2_+$ is trapped and cannot belong to an infinite open cluster in $\mathbb{Z}^2_+$. For this reason we need to consider $(\mathbb{Z}^2_+)^\star$ and not exactly the dual of $\mathbb{Z}^2_+$.

Figure 3.9: An illustration of the $(\mathbb{Z}^2_+)^\star$ lattice in dashed lines and the $\mathbb{Z}^2_+$ lattice in solid lines.



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We declare, for each edge $e^\star \in (\mathbb{Z}^2_+)^\star$ that intersects an edge $e \in \mathbb{Z}^2_+$ open (respectively closed) if e is closed (respectively open). Notice that, from (3.75), these probabilities are $1-p^\lambda$ and $(1-p^\lambda)^{\zeta_i}$ if $e^\star$ is a vertical or horizontal edge, respectively. The remaining edges $e^\star$ that do not intersect any edge of $\mathbb{Z}^2_+$ will have their states declared independently as follows: declare the vertical edges that lie on the leftmost semiaxis $\big\{-\frac{1}{2}\big\} \times \big(\mathbb{Z}_+ - \frac{1}{2}\big)$ open with probability $1-p^\lambda$ and closed with probability p^λ. Also, declare the horizontal edges of the type $\langle (i-\frac{1}{2}, -\frac{1}{2}), (i+\frac{1}{2}, -\frac{1}{2})\rangle$, that lie on the lowermost semiaxis, open with probability $(1-p^\lambda)^{\zeta_i}$ and closed with probability $1-(1-p^\lambda)^{\zeta_i}$. Hence, the edges $e^\star$ are open independently with probability

$$p_{e^\star} = \begin{cases} 1-p^\lambda, \text{ if } e^\star = \langle (i-\frac{1}{2}, j-\frac{1}{2}), (i-\frac{1}{2}, j+\frac{1}{2})\rangle \\ \big(1-p^\lambda\big)^{\zeta_i}, \text{ if } e^\star = \langle (i-\frac{1}{2}, j-\frac{1}{2}), (i+\frac{1}{2}, j-\frac{1}{2})\rangle. \end{cases} \tag{3.76}$$

Define, analogously to Λ in (1.1), Γ as the sequence of points whose increments are given by ζ_k's instead of ξ_i's, that is,

$$\Gamma = \left\{ x_k \in \mathbb{R} \;:\; x_0 = -\frac{1}{2} \text{ and } x_k = x_{k-1} + \zeta_k, \text{ for } k \in \mathbb{Z}^*_+ \right\}. \tag{3.77}$$

Comparing with (1.3), the probability law in $(\mathbb{Z}^2_+)^\star$ of a process given by (3.76) is, with an excusable abuse of notation, $\mathbb{P}^\Gamma_{1-p^\lambda}$.

Notice that, by our choice of λ, the random variables $\{\xi_k\}_{k\in\mathbb{Z}^*_+}$ are dominated by independent random variables with geometric distribution with parameter $\frac{1}{2}$, which have finite second moment. Thus, we have

$$\mathbb{E}(\zeta_k^2) < \infty. \tag{3.78}$$

Since the random variables ζ_k's have this moment condition, are positive and integer-valued, this allows us to find the corresponding constants α, β and μ along with the corresponding sequences of scales $L_k(\zeta_1, \alpha, \beta, \mu)$ and $H_k(L_0, \alpha, \beta, \mu)$ and a positive integer $k_0(L_0)$ of the multiscale renormalization done in this chapter.

Having H_k and L_k already defined, we can also define, in a similar way to (3.35) and (3.36), respectively, crossing events in the lattice $(\mathbb{Z}^2_+)^\star$, namely $H^{k,\star}_{i,j}$ and $V^{k,\star}_{i,j}$, with $i, j \in \mathbb{Z}^*_+$ and $k \in \mathbb{Z}_+$, as

$$H^{k,\star}_{i,j} = \mathcal{C}_h\left(\left(I^{k,\star}_i \cup I^{k,\star}_{i+1}\right) \times \left[(j+1)H_k - \frac{1}{2}, (j+2)H_k - \frac{1}{2}\right)\right)$$

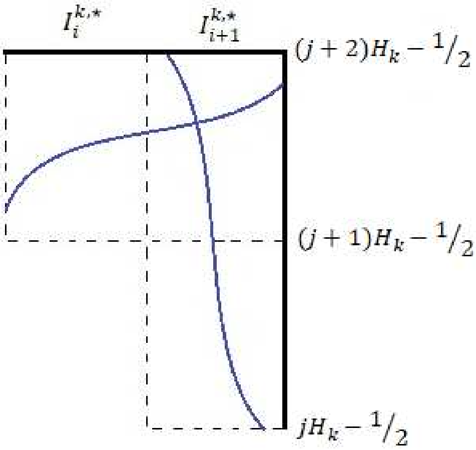
and

$$V^{k,\star}_{i,j} = \mathcal{C}_v\left(I^{k,\star}_{i+1} \times \left[jH_k - \frac{1}{2}, (j+2)H_k - \frac{1}{2}\right)\right),$$

where

$$I_i^{k,\star} = \left[iL_k - \frac{1}{2}, (i+1)L_k - \frac{1}{2}\right), \text{ for all } i \in \mathbb{Z}_+.$$

Figure 3.10: An illustration of the occurrence of events $H_{i,j}^{k,\star}$ and $V_{i,j}^{k,\star}$.



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Thus, we can use Lemmas 3.2 and 3.1 to ensure that there is p small enough, so that $1 - p^\lambda$ is sufficiently close to 1, such that

$$\begin{aligned} \mathbb{P}^{\Gamma}_{1-p^\lambda}(H_{0,0}^{k,\star} \cap V_{0,0}^{k,\star}) &\geq \left(1 - q_k(1 - p^\lambda)\right)^2 \\ &\geq 1 - 2q_k(1 - p^\lambda) \\ &\geq 1 - 2\exp\left(-L_k^{\left(1 - \frac{\mu}{k+1}\right)}\right) \\ &\to 1, \end{aligned} \tag{3.79}$$

as k goes to ∞. It follows that, almost certainly, the origin $(0,0) \in \mathbb{Z}_+^2$ is encompassed by an open dual semicircuit connecting the leftmost vertical and the lowermost horizontal semiaxis of $(\mathbb{Z}_+^2)^\star$. This semicircuit acts as a blocking structure for $(0,0)$ to percolate into the original lattice. Therefore, since $\{H_{0,0}^{k,\star} \cap V_{0,0}^{k,\star}\} \subseteq \{(0,0) \not\leftrightarrow \infty\}$, from (3.79), we can conclude that

$$\mathbb{P}^{\Lambda}_p((0,0) \leftrightarrow \infty) \to 0,$$

as k goes to ∞, which concludes the proof. ■

Chapter 4

Concluding Remarks and Further Works

Initially we had a result analogous to Theorem 1.1 with the moment condition $\mathbb{E}(\xi e^{(\log \xi)^\alpha} \mathbb{1}_{\{\xi \geq 1\}}) < \infty$, for $\alpha \in \left(\frac{1}{2}, 1\right)$. Our first idea was to use the techniques given in [16] to be able to include also the case $\alpha = \frac{1}{2}$ in this theorem. However, we have realized that replacing $(\log \xi)^\alpha$ with $c\,(\log \xi)^{1/2}$ in the moment condition above, our techniques would work with slight modifications. Nevertheless, as we saw, this constant is not just any constant, it depends on the initial scale, L_0. We believe that with the techniques developed in [16] it would be possible to generalize our result in the case $\alpha = \frac{1}{2}$ for any positive constant c.

It is also worthwhile pointing out that a natural continuation of our work, which would be very interesting to have, would be an analogous of Theorem 1.1 with the moment condition $\mathbb{E}(\xi) < \infty$, which is a problem that remained open in [9]. As we have seen, in Remark 1.1, our moment condition is a step further in this direction, however, the multiscale methods we know and the one we have used, seems not to work for this purpose.

In the course of our work, we realized that the multiscale technique with the geometry we have used fails when we have an exponential scale. We believe that perhaps with another type of geometry in the multiscale from the one we have used, it would be possible to handle this case; however, we were not able to do it.

The best scale we could find with our multiscale scheme, which was used in this work, is the one that asymptotically grows exponentially in k^2, see equation (3.11), and led us to the moment condition $\mathbb{E}(\xi e^{c(\log \xi)^{1/2}} \mathbb{1}_{\{\xi \geq 1\}}) < \infty$, for some constant $c > 64$. Similar moment condition is also found in [6] in the context of extinction of renewal contact process where they use an exponential scale. This raises the following question: is it possible to lower this condition or is it a limitation of the multiscale technique?

References

[1] Aizenman, M., Grimmett, G. *Strict monotonicity for critical points in percolation and ferromagnetic models*, Journal of Statistical Physics, **63(5-6)**, 817-835, (1991).

[2] Barsky, D. J., Grimmett, G. R., Newman, C. M. *Percolation in halfspaces: equality of critical densities and continuity of the percolation probability*, Probability Theory and Related Fields, **90(1)**, 111-148, (1991).

[3] Bramson, M., Durrett, R., Schonmann, R. H. *The contact processes in a random environment*, The Annals of Probability, 960-983, (1991).

[4] Broadbent, S. R., Hammersley, J. M. *Percolation processes: I. Crystals and Mazes*, Mathematical Proceedings of the Cambridge Philosophical Society **53**, 629–641, (1957).

[5] Duminil-Copin, H., Hilário, M. R., Kozma, G., Sidoravicius, V. *Brochette percolation*, Israel Journal of Mathematics, **225(1)**, 479-501, (2018).

[6] Fontes, L.R.G., Mountford, T.S., Ungaretti, D., Vares, M.E. *Renewal Contact Processes: phase transition and survival*, Stochastic Process. Appl. **161**, 102–136, (2023).

[7] Grimmett, G. R. *Percolation*, 2 ed, Grundlehren der Mathematischen Wissenschaften, **321**, Springer-Verlag, Berlin, (1999).

[8] Harris, T. E. *A lower bound for the critical probability in a certain percolation process*, Mathematical Proceedings of the Cambridge Philosophical Society, **56(1)**, 13-20, (1960).

[9] Hilario, M. R., Sá, M., Sanchis, R., Teixeira, A. *Phase transition for percolation on randomly stretched lattice*, The Annals of Applied Probability **33.4**, 3145–3168, (2023).

[10] Hoffman, C. *Phase transition in dependent percolation*, Communications in mathematical physics, **254(1)**, 1-22, (2005).

[11] Kesten, H. *The critical probability of bond percolation on the squarelattice equals 1/2*, Communications in Mathematical Physics, **74(1)**, 41-59, (1980).

[12] Levin, D. A., Peres, Y. *Markov chains and mixing times*, American Mathematical Society, Vol 107, (2017).

[13] Lindvall, T. *Lectures on the coupling method*, Dover Publications, Mineola, New York, (2002).

[14] Mitov, K. V., Omey, E. *Renewal Processes*, Springer Briefs in Statistics, Springer, Cham, (2014).

[15] Norris, J. R. *Markov Chain*, Cambridge University Press, Cambridge, (1997).

[16] Penrose, M. *Non-triviality of the vacancy phase transition for the Boolean model*, The Electronic Communications in Probability, **23**, (2018).

[17] Russo, L. *A note on percolation*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, **43(1)**, 39-48, (1978).

[18] Seymour, P. D., Welsh, D. J. *Percolation probabilities on the square lattice*, Annals of Discrete Mathematics, Elsevier 3, 227-245, (1978).

[19] Wald, A. *On cumulative sums of random variables*, Annals of Mathematical Statistics, **15(3)**, 283–296, (1944).

[20] Zhang, Y. *A note on inhomogeneous percolation*, The Annals of Probability, **22(2)**, 803-819, (1994).