Universidade Federal de Minas Gerais
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Polynomial bounds for automorphism groups of foliations

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Tese apresentada ao corpo docente de Pós Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais, como parte dos requisitos para obtenção do título de Doutor em Matemática.

Orientador:
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Fevereiro de 2016

ATA DA SEPTUAGÉSIMA PRIMEIRA DEFESA DE TESE DO ALUNO ALAN DO NASCIMENTO MUNIZ, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 25 DE FEVEREIRO DE 2016.

Aos vinte e cinco dias do mês de fevereiro de 2016, às 10h00, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Alan do Nascimento Muniz,** intitulada: *"Cotas polinomiais para grupos de automorfismos de folheações",* requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Maurício Barros Corrêa Junior, após dar conhecimento aos presentes o teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado, por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 25 de fevereiro de 2016.

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Abstract

This thesis concerns the problem of bounding polynomially orders of automorphism groups foliated surfaces $(X, \mathcal{F})$ of general type. After a brief recall of useful properties of foliated surfaces in the first chapter, the second one deals with the local behavior of such groups around a fixed point. Results are stated for regular points and reduced singularities. However, the approach is useful to describe the non-reduced case, as remarked at the end of the chapter. In the third chapter, foliations on the projective plane $\mathbb{P}^2$ are studied. It is proved that the automorphism group of a degree d foliation can be bounded quadratically on d. This bound is sharp, it is attained by the automorphism groups of the Jouanolou's examples. The fourth chapter concerns the case when X is geometrically ruled. Results toward the classification of foliations on such surfaces are stated and, under mild hypotheses, quadratic bounds are given. The last chapter is about foliations on surfaces that are not birationally ruled. Regular foliations of general type are analyzed. It is proved that they live on minimal surfaces of general type, hence linear bounds are trivially obtained. Next, singular foliations with ample canonical bundle are studied and cubic bounds are given for their automorphism groups under mild restrictions. Finally, it is shown that the technique developed so far provide bounds for the automorphism groups of general type foliations with a holomorphic first integral.

Resumo

Esta tese aborda o problema de limitar superiormente polinomialmente as ordens dos grupos de automorfismos de superfícies folheadas de tipo geral $(X, \mathcal{F})$. Depois de uma breve recordação de propriedades úteis de superfícies folheadas no primeiro capítulo, o segundo trata do comportamento local desses grupos em torno de um ponto fixo. Resultados são apresentados para pontos regulares e singularidades reduzidas. No entanto, a abordagem é útil para descrever o caso não-reduzido, como observado no final do capítulo. No terceiro capítulo, folheações no plano projetivo $\mathbb{P}^2$ são estudadas. É provado que o grupo de automorfismos de uma folheação de grau d pode ser limitada de forma quadrática em d. Esta cota é precisa, ela é atingida pelos grupos de automorfismos dos exemplos de Jouanolou. O quarto capítulo refere-se ao caso em que X é geometricamente regrada. Resultados em relação à classificação de folheações sobre essas superfícies são apresentados e, sob hipóteses fracas; cotas quadráticas são dadas. O último capítulo é sobre folheações em superfícies que não são birracionalmente regradas. São analisadas folheações regulares de tipo geral. É provado que elas vivem em superfícies minimais de tipo geral e, portanto, cotas lineares são trivialmente obtidas. Em seguida, folheações singulares com fibrado canônico amplo são estudadas e cotas cúbicas são apresentadas para seus grupos de automorfismos, sob restrições leves. Finalmente, mostra-se que a técnica desenvolvida até então fornece cotas para os grupos de automorfismos de folheações de tipo geral com uma integral primeira holomorfa.

O problema de cotar automorfismos começa no final do século XIX com os trabalhos de Klein, Hurwitz, Schwartz e outros. Klein estudou a quártica de equação $x^3y + y^3z + z^3x = 0$ em $\mathbb{P}^2$ e mostrou, em 1878, que seu grupo de automorfismos é isomorfo a $PSL(2, 7)$, que tem ordem 168, veja [16]. Esta curva ficou conhecida como a quártica de Klein. Foi provado por Schwartz, em 1879, que uma superfície de Riemann de gênero $g \geq 2$ tem um número finito de automorfismos, veja [35].

Hurwitz, em 1892, deu uma cota superior através da sua bem conhecida fórmula, veja [13]. Tais superfícies de Riemann podem ter no máximo $84(g-1)$ automorfismos. A quártica de Klein tem gênero três e atinge esta cota.

Todavia, cotas menores foram encontradas sob hipóteses mais fortes. Enunciaremos agora dois resultados que serão usados depois neste texto. Em 1895, Wiman estudou subgrupos cíclicos e provou o seguinte:

Teorema (Wiman [40])**.** *Seja C uma superfície de Riemann compacta de gênero $g \geq 2$ e seja $T \in Aut(C)$ cíclico. Então, a ordem de T é no máximo $4g+2$.*

Wiman também provou que esta cota é ótima. Para cada $g \geq 2$, a curva hiperelítica de equação $y^2 = x^{2g+1} - x$ tem o automorfismo $T(x,y) = (\epsilon^2 x, \epsilon y)$ onde ϵ é uma raiz da unidade primitiva de ordem $4g+2$. Mais tarde, em 1987, Nakajima estudou os grupos de automorfismos abelianos de curvas projetivas, sobre um corpo algebricamente fechado arbitrário. Ele provou o seguinte resultado:

Teorema (Nakajima [30])**.** *Seja C uma curva projetiva suave de gênero $g \geq 2$ sobre um corpo algebricamente fechado k, e seja G um subgrupo abeliano de $Aut(C)$. Então,*

(i) quando $\operatorname{char}(k) \neq 2$, temos que $|G| \leq 4g+4$;

(ii) quando $\operatorname{char}(k) = 2$, temos que $|G| \leq 4g+2$.

(iii) As estimativas de (i) e (ii) são as melhores possíveis. Isto é, Para infinitos inteiros $g \geq 2$, existem C e G que satisfazem $g(C) = g$ e $|G| = 4g+4$ (ou $|G| = 4g+2$) quando $\operatorname{char}(k) \neq 2$ (ou $\operatorname{char}(k) = 2$).

Para curvas complexas compactas (ou, equivalentemente, superfícies de Riemann compactas), a finitude do número de automorfismos é relacionada ao Teorema de Uniformização. Dizer que uma curva tem gênero pelo menos dois é o mesmo que dizer que seu recobrimento universal é o disco, o gênero determina esta classificação. Superfícies complexas compactas foram classificadas, a menos de bimeromorfismos, pelos trabalhos de Enriques, Kodaira e muitos outros. Esta classificação é baseada nos plurigêneros da superfície, mais precisamente no comportamento assintótico destes: a dimensão de Kodaira.

Seja X uma variedade complexa, compacta e suave, e denote por K_X seu fibrado canônico. O n-ésimo plurigênero de X é definido por $P_n(X) := \dim \mathrm{H}^0(X, K_X^{\otimes n})$. A sequência dos plurigêneros cresce polinomialmente e a dimensão de Kodaira de X, denotada por $\operatorname{Kod}(X)$, é, a grosso modo, o grau deste polinômio. Precisamente,

$$\operatorname{Kod}(X) := \limsup_{n \mapsto \infty} \frac{\log P_n(X)}{\log n}.$$

É conhecido que $\operatorname{Kod}(X) \leq \dim(X)$ e, quando vale a igualdade, X é chamada de uma variedade de tipo geral. Para curvas, podemos relacionar a dimensão de Kodaira com o gênero:

$\mathrm{Kod}(X)$	$g(X)$
$-\infty$	0
0	1
1	≥ 2

Então, as chamadas curvas de tipo geral são aquelas com um número finito de automorfismos. Analogamente, em 1950, Andreotti provou que uma superfície complexa de tipo geral tem um número finito de bimeromorfismos. Muitos autores estudaram estes grupos e ressaltamos um resultado que será útil no texto.

Teorema (Xiao [41]). *Seja X uma superfície minimal de tipo geral. Então,*

$$|Aut(X)| \leq (42K_X)^2$$

O conceito de superfícies minimais foi introduzido, até onde sabemos, por Castelnuovo e Enriques, em [6], e diz que X é minimal se para uma outra superfície Y e um mapa birracional $\Psi : Y \dashrightarrow X$, Ψ é de fato um morfismo. Zariski provou, em [42], que uma superfície com dimensão de Kodaira não negativa possui um modelo minimal único. Segue que se X é minimal então $\mathrm{Bim}(X) = \mathrm{Aut}(X)$.

A dimensão de Kodaira foi estendida às superfícies folheadas, isto é, às superfícies munidas de uma folheação. Superfícies folheadas de dimensão de Kodaira no máximo um foram classificadas, de forma independente, nos trabalhos de McQuillan [27] e Mendes [28], veja também [5]. Mais tarde, Pereira e Sanchez, em [31], provaram uma versão folheada do teorema de Andreotti: superfícies folheadas possuem de tipo geral possuem um número finito de bimeromorfismos. Em 2014, Corrêa e Fassarella deram uma cota exponencial para o número de automorfismos de uma superfície folheada de tipo geral, sob algumas hipóteses fracas. Eles provaram o seguinte:

Teorema ([7], Corollary 4.3). *Seja $\mathcal{F}$ uma folheação em uma superfície projetiva irredutível e não-racional X. Se $K_{\mathcal{F}}$ é amplo, então*

$$|Aut(\mathcal{F})| \leq \left((3m^2 + 2m)K_{\mathcal{F}}^2\right)^{(m^2K_{\mathcal{F}}^2+2)^2-1}$$

onde $m = (K_{\mathcal{F}} \cdot (K_X + 4K_{\mathcal{F}}) + 1)^2 + 3K_{\mathcal{F}}^2$.

O objetivo desta tese é prover cotas polinomiais para os grupos de automorfismos de superfícies folheadas. Após estabelecer alguns resultados básicos no primeiro capítulo, o segundo traz uma análise da situação local que será útil para determinar, no contexto global, o comportamento dos automorfismos ao redor de um ponto fixo. Para um grupo G finito de germes de biholomorfismos de $(\mathbb{C}^2, 0)$, vale:

- G é linearizável: $\varphi \in G \mapsto D\varphi_0 \in \mathrm{GL}(2, \mathbb{C})$ é injetivo.
- A ação de G sobe ao *blow-up* na origem $\pi : (X, E) \longrightarrow (\mathbb{C}^2, 0)$ e deixa o divisor excepcional E invariante.

As propriedades principais destes grupos para este trabalho são descritas nas seguintes proposições:

Proposição. *Seja $\mathcal{F}$ um germe de folheação regular em $(\mathbb{C}^2, 0)$ tal que a reta L tangente à folha que passa pela origem não é $\mathcal{F}$–invariante. Seja G um grupo finito de automorfismos que preservam $\mathcal{F}$. Então G é cíclico e existem coordenadas (x, y) onde $L = \{x = 0\}$ e G é gerado por um elemento da forma*

$$\varphi(x, y) = (l^{k+1}x, ly)$$

onde l é uma raiz da unidade e $k = tang(\mathcal{F}, L, 0) \geq 1$.

Proposição. *Seja $\mathcal{F}$ um germe de folheação em $(\mathbb{C}^2, 0)$ com uma singularidade reduzida na origem com autovalor λ. Se G é um grupo finito de automorfismos que preservam $\mathcal{F}$, então G tem um subgrupo normal abeliano diagonalizável H de índice no máximo 2. Além disso, se $\lambda \neq -1$, $G = H$.*

O terceiro capítulo é dedicado ao estudo de folheações no plano projetivo complexo, $\mathbb{P}^2$. Neste contexto, a descrição das folheações é bem estabelecida e os possíveis grupos de automorfismos são bem conhecidos. O resultado principal deste capítulo é o seguinte:

Teorema. *Seja $\mathcal{F}$ uma folheação em $\mathbb{P}^2$ de grau $d \geq 2$ tal que $Aut(\mathcal{F})$ é finito e deixa invariante a união de três retas em posição geral. Se estas retas não são $\mathcal{F}$–invariantes e contém no máximo singularidades reduzidas, então*

$$|Aut(\mathcal{F})| \leq 3(d^2 + d + 1)$$

exceto para $d = 2$ e $\mathcal{F}$ dada por

$$v = YZ\partial_X + XZ\partial_Y + XY\partial_Z.$$

Esta cota é ótima, sendo realizada pelos exemplos de Jouanolou, $\mathcal{J}_d$. De fato, para $d \geq 2$, $\mathcal{J}_d$ é definida por por

$$v = Z^d\partial_X + X^d\partial_Y + Y^d\partial_Z.$$

e o grupo de automorfismos de $\mathcal{J}_d$ é

$$\mathrm{Aut}(\mathcal{J}_d) \simeq \mathbb{Z}/(d^2 + d + 1)\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$$

gerado por

$$T(X : Y : Z) = (Y : Z : X)$$
$$\varphi(X : Y : Z) = (lX : l^dY : l^{d^2}Z)$$

onde $l^{d^2+d+1} = 1$ primitiva.

O quarto capítulo trata de folheações em superfícies geometricamente regradas, isto é, superfícies que admitem uma fibração $X \longrightarrow C$ sobre uma curva cujas fibras são isomorfas a $\mathbb{P}^1$. Algumas propriedades de folheações neste tipo de superfície são discutidas e os resultados principais deste capítulo são os seguintes:

Teorema. *Seja* $X \longrightarrow C$ *uma superfície regrada com invariante* e*. Seja* $\mathcal{F}$ *uma folheação em* X *de bigrau* (a,b) *tal que* $\mu(\mathcal{F},p) = \mathrm{Z}(\mathcal{F},F,p)$ *para cada* p *em uma fibra* $\mathcal{F}$*–invariante* F*. Então um dos seguintes é verdade:*

1. $\mathcal{F}$ *é tangente à regra, ou*
2. $a = 0$ *e* $\mathcal{F}$ *é de Riccati, ou*
3. $b = e = 0$*,* $a > 0$*,* C *é elítica e* $\mathcal{F}$ *é, a menos de um recobrimento, uma folhação regular em* $C \times \mathbb{P}^1$*, ou*
4. $\mathcal{F}$ *é uma folheação em* $\mathbb{P}^1 \times \mathbb{P}^1$ *com* $a > 0$*,* $b = 0$ *e toda singularidade está em fibras* $\mathcal{F}$*-invariantes. Em particular,* $\mathcal{F}$ *é de Riccati com respeito à outra projeção. Ou*
5. $e > 0$*,* $b = (a+1)e$ *e todas as singularidades estão em fibras* $\mathcal{F}$*-invariantes, ou*
6. $a > 0$*, e* $\mathcal{F}$ *tem ao menos uma singularidade que está em uma fibra não* $\mathcal{F}$*-invariante.*

Teorema. *Seja* $X \longrightarrow C$ *uma superfície regrada com invariante* e *sobre uma curva de gênero* $g \geq 1$*. Seja* $\mathcal{F}$ *uma folheação em* X *de bigrau* (a,b) *tal que:*

1. $Aut(\mathcal{F})$ *é finito;*
2. $\mathcal{F}$ *tem ao menos uma singularidade que está em uma fibra não* $\mathcal{F}$*-invariante.*
3. $\mu(\mathcal{F},p) = \mathrm{Z}(\mathcal{F},F,p)$ *se* p *está em uma fibra* F $\mathcal{F}$*–invariante.*

Então

- $|Aut(\mathcal{F}) \leq (8g+4)[(a+1)(2b-ae+2-2g)+2-2g]$ *se* $e < 0$
- *ou* $|Aut(\mathcal{F}) \leq a(4g+2)[(a+1)(2b-ae+2-2g)+2-2g]$ *se* $e \geq 0$*.*

O quinto, e último, capítulo versa sobre folheações em superfícies não-regradas, isto é, superfícies com dimensão de Kodaira não-negativa. Neste contexto, limitamos a ordem do grupo de automorfismos de uma folheação por um polinômio de grau três nos números de Chern da folheação e da superfície.

Teorema. *Seja* (X,F) *superfície folheada reduzida com fibrado canônico* $K_{\mathcal{F}}$ *amplo e* $Kod(X) \geq 0$*. Suponha que* $\mathcal{F}$ *tem uma singularidade reduzida* $p \in X$ *que não pertence a uma curva algébrica* $\mathcal{F}$*–invariante. Então*

$$\begin{aligned}|Aut(\mathcal{F})| \leq\ & 2(c_2(X) + (K_X + 4K_{\mathcal{F}})(5K_X + 12K_{\mathcal{F}})) \\ & \cdot [4(K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) + 6]c_2(T_X \otimes K_{\mathcal{F}})\end{aligned}$$

a menos que X *seja uma superfície de Dolgachev–Enriques.*

Uma superfície de Dolgachev–Enriques é definida como uma superfície cujo modelo minimal, X, possui $q(X) = p_g(X) = 0$. Na terminologia clássica, X é, de fato, uma superfície de Enriques quando $\mathrm{Kod}(X) = 0$ e X é uma superfície de Dolgachev quando é simplesmente conexa e $\mathrm{Kod}(X) = 1$.

Contents

Introduction

The problem of bounding automorphisms begins at the end of the 19th century with the works of Klein, Hurwitz, Schwartz and others. Klein studied the quartic $\{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2$ and showed in 1878 that its automorphism group is isomorphic to $PSL(2,7)$, which has order 168, see [16]. This curve is now known as the Klein quartic. It has been proved by Schwartz in 1879 that a compact Riemann surface of genus $g \geq 2$ has a finite number of automorphisms, see [35]. Hurwitz in 1892 gave a bound by means of his well known formula, see [13]. Such Riemann surface can have at most $84(g-1)$ automorphisms. The Klein quartic is a genus 3 curve that achieves this bound.

However, lower bounds were found with stronger assumptions. We state now two results that will be used later in this text. In 1895, Wiman studied cyclic subgroups and proved the following.

Theorem 0.1 (Wiman [40]). *Let C be a compact Riemann surface of genus $g \geq 2$ and let $T \in Aut(C)$ be cyclic. Then the order of T is at most $4g+2$.*

Wiman also proved that this bound is sharp. For each $g \geq 2$, the hyperelliptic curve of equation $y^2 = x^{2g+1} - x$ has the automorphism $T(x,y) = (\epsilon^2 x, \epsilon y)$ where ϵ is a primitive root of the unity of order $4g+2$.

Later, in 1987, Nakajima studied the abelian groups of automorphisms of projective curves over an arbitrary algebraically closed field. He proved the following result:

Theorem 0.2 (Nakajima [30]). *Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k, and let G be an abelian subgroup of $Aut(C)$. Then,*

(i) when $\operatorname{char}(k) \neq 2$, we have $|G| \leq 4g+4$;
(ii) when $\operatorname{char}(k) = 2$, we have $|G| \leq 4g+2$.
(iii) The estimates of (i) and (ii) are best possible. That is to say, for infinitely many integers $g \geq 2$, there exist C and G which satisfy $g(C) = g$ and $|G| = 4g+4$ (or $|G| = 4g+2$) when $\operatorname{char}(k) \neq 2$ (or $\operatorname{char}(k) = 2$).

For compact complex curves (or, equivalently, compact Riemann surfaces), the finiteness of the number of automorphisms is related to the Uniformization Theorem. To say that a curve has genus at least two is to say that its universal cover is the disc, the genus determines this classification. Compact complex surfaces have been classified, up to bimeromorphisms, by the work of Enriques, Kodaira and many others. This classification is based on the plurigenera of the surface, more precisely on their asymptotic behavior: the Kodaira dimension.

Let X be a compact complex manifold, and let K_X denote its canonical bundle. The n-th plurigenus of X is defined by $P_n(X) := \dim \mathrm{H}^0(X, K_X^{\otimes n})$. The sequence of the plurigenera grows polynomially and the Kodaira dimension of X, denoted by $\mathrm{Kod}(X)$, is, roughly speaking, the degree of this polynomial. Precisely,

$$\mathrm{Kod}(X) := \limsup_{n \mapsto \infty} \frac{\log\, P_n(X)}{\log\, n}$$

It is known that $\mathrm{Kod}(X) \leq \dim(X)$ and, when equality holds, X is called a general type manifold. For curves, we can relate the Kodaira dimension to the genus:

$\mathrm{Kod}(X)$	$g(X)$
$-\infty$	0
0	1
1	≥ 2

Then, the so called general type curves are the ones that have finite number of automorphisms. Analogously, in 1950 Andreotti proved that a general type surface has finite number of bimeromorphisms. Many authors studied these groups and we point out a result that will be useful in the text.

Theorem 0.3 (Xiao [41]). *Let X be a minimal surface of general type. Then,*

$$|Aut(X)| \leq (42K_X)^2$$

The concept of minimal surfaces was introduced, as far as we know, by Castelnuovo and Enriques, in [6], and says that X is minimal if for any other surface Y and birational map $\Psi : Y \dashrightarrow X$, Ψ is in fact a morphism. Zariski has proved in [42] that a surface with non-negative Kodaira has a unique minimal model. It follows that if X is minimal then $\mathrm{Bim}(X) = \mathrm{Aut}(X)$.

The Kodaira dimension has been extended to foliated surfaces, that is, to surfaces with a fixed foliation. Foliated surfaces of Kodaira dimension at most one were classified independently in the works of McQuillan [27] and Mendes [28], see also [5]. Later, Pereira and Sanchez, in [31], proved a foliated version of Andreotti's theorem: foliated surfaces of general type have a finite number of bimeromorphisms. In 2014, Corrêa and Fassarella gave an exponential bound to the number of automorphisms of a general type foliated surface under some mild assumptions. They proved the following:

Theorem 0.4 ([7], Corollary 4.3)**.** *Let* $\mathcal{F}$ *be a foliation on a non-rational smooth irreducible projective surface* X*. If* $K_{\mathcal{F}}$ *is ample then*

$$|Aut(\mathcal{F})| \leq \left((3m^2 + 2m)K_{\mathcal{F}}^2\right)^{(m^2K_{\mathcal{F}}^2+2)^2-1}$$

where $m = (K_{\mathcal{F}} \cdot (K_X + 4K_{\mathcal{F}}) + 1)^2 + 3K_{\mathcal{F}}^2$*.*

The objective of this thesis is to provide polynomial bounds for the automorphism groups of foliated surfaces. In chapter 2, we analyze the local situation that will be useful to determine the behavior of the automorphisms around a fixed point. These groups are locally linearizable and well-behaved under blow-ups of special points. We show that, in most cases, they are abelian.

In chapters 3 and 4 we deal with foliations on $\mathbb{P}^2$ and geometrically ruled surfaces, respectively. Among other results, we prove that a foliation on $\mathbb{P}^2$ of degree $d \gg 0$ has at most $3(d^2 + d + 1)$ automorphisms and the Jouanolou's examples show that this bound is sharp.

In chapter 5, we deal with foliations on surfaces of nonnegative Kodaira dimension. We prove that the automorphism group has order bounded by a cubic polynomial in the Chern numbers of the surface and the foliation.

Chapter 1

Preliminaries

1.1 Holomorphic Foliations on Surfaces

Let X be a smooth complex surface. A foliation $\mathcal{F}$ on X is given by an open covering $\mathcal{U} = \{U_i\}$ of X and 1-forms $\omega_i \in \Omega^1_X(U_i)$ satisfying the conditions:

1. For each non-empty intersection $U_i \cap U_j$ there exists a function $g_{ij} \in \mathcal{O}^*_{U_i \cap U_j}$ such that $\omega_i = g_{ij}\omega_j$;
2. For every i the zero set of ω_i is isolated.

The 1-forms $\{\omega_i\}$ patch together to form a global section

$$\omega = \{\omega_i\} \in \mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{L}),$$

where $\mathcal{L}$ is the line bundle over X determined by the cocycle $\{g_{ij}\}$. The singular set of $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is the zero set of the twisted 1-form ω.

Let T_X be the tangent sheaf of X. The tangent sheaf of $\mathcal{F}$, induced by a twisted 1-form $\omega \in \mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{L})$, is defined on each open set $U \subset X$ by

$$T_{\mathcal{F}}(U) = \{v \in T_X(U);\ i_v\omega = 0\}.$$

We can use vector fields instead of 1-forms. The foliation can be given by $v_i \in T_X(U_i)$ with codimension two zero set and satisfying $v_i = f_{ij}v_j$, where f_{ij} are holomorphic function on $U_i \cap U_j$ without zeros. The line bundle determined by the cocycle $\{f_{ij}\}$ is called the canonical bundle of $\mathcal{F}$ and is denoted by $K_{\mathcal{F}}$.

Frobenius Theorem implies that for every point in X_0, the complement of $\mathrm{Sing}(\mathcal{F})$, there exists a unique germ of smooth curve V invariant by $\mathcal{F}$, i.e., satisfying $i^*(\omega) = 0$ where $i : V \longrightarrow X$ is the inclusion. Analytic continuation of these subvarieties describes the leaves of $\mathcal{F}$. As usual, we will make abuse of notation by writing $(T_{\mathcal{F}})_x$ for the tangent space to the leaf passing through x or for the stalk of the sheaf $T_{\mathcal{F}}$ at x. In fact, the restriction of $T_{\mathcal{F}}$ to X_0 is invertible and, it is dual to the restriction of $K_{\mathcal{F}}$.

Definition 1.1. Let ω be a germ holomorphic 1-form on $(\mathbb{C}^2, 0)$ with isolated singularity. This singularity is said to be reduced if

- the linear part of ω is invertible and the quotient of its eigenvalues is not a negative rational number, or
- the linear part of ω has only one null eigenvalue.

Up to a linear change of coordinates, we can write

$$\omega = (\lambda y + \cdots)dx - (x + \cdots)dy$$

where $\lambda \notin \mathbb{Q}_{>0}$. λ is called the eigenvalue of $\mathcal{F}$. If $\lambda = 0$ it is called a saddle-node, otherwise it is called nondegenerate.

Theorem 1.2 (Seidenberg [36]). *If $\mathcal{F}$ is a foliation induced by ω with isolated singularity on a neighborhood U of $0 \in \mathbb{C}^2$ then, after a finite sequence of blow-ups $X \longrightarrow U$, $\mathcal{F}$ lifts to a foliation with only reduced singularities.*

Consequently, if $\mathcal{F}$ is a foliation on a compact complex surface, $\mathrm{Sing}(\mathcal{F})$ is finite and Seidenberg's theorem implies that after a finite sequence of blow-ups $\mathcal{F}$ lifts to a foliation with only reduced singularities.

1.2 Index Theorems

We recall some index formulae that will be useful throughout this thesis. For details and proofs see, for example, [5]. Firstly, fix a smooth compact complex surface X and a foliation $\mathcal{F}$ on X.

Let $p \in \mathrm{Sing}(\mathcal{F})$ and let $v = A(z,w)\frac{\partial}{\partial z} + B(z,w)\frac{\partial}{\partial w}$ be a holomorphic vector field that generates $\mathcal{F}$ in coordinates (z,w) around p. Then we define the residues

$$\mathrm{BB}(\mathcal{F},p) = \mathrm{Res}_{(0,0)}\left\{\frac{(\mathrm{tr}(Dv(z,w)))^2}{A(z,w)B(z,w)}dz \wedge dw\right\}$$
$$\mu(\mathcal{F},p) = \mathrm{Res}_{(0,0)}\left\{\frac{(\det(Dv(z,w)))^2}{A(z,w)B(z,w)}dz \wedge dw\right\}$$

where $\mathrm{Res}_{(0,0)}$ denotes the Grothendieck residue at the origin. If p is a regular point of the foliation, the residues vanish trivially. Therefore, we have that the sum of the residues represents characteristic numbers.

Proposition 1.3 (Baum–Bott). *Let $\mathcal{F}$ be a foliation on a surface X. Then*

$$\sum_{p\in\mathrm{Sing}(\mathcal{F})} \mathrm{BB}(\mathcal{F},p) = (K_X \otimes K_{\mathcal{F}}^{\vee})^2 = K_X^2 - 2K_{\mathcal{F}} \cdot K_X + K_{\mathcal{F}}^2$$
$$\sum_{p\in\mathrm{Sing}(\mathcal{F})} \mu(\mathcal{F},p) = c_2(T_X \otimes K_{\mathcal{F}}) = c_2(X) - K_{\mathcal{F}} \cdot K_X + K_{\mathcal{F}}^2$$

Observe that, since $\mathcal{F}$ is given by a section of $T_X \otimes K_{\mathcal{F}}$ with isolated zeros, the sum of the residues $\mu(\mathcal{F}, p)$ corresponds to the number of singularities of $\mathcal{F}$, counted with multiplicities. In fact, $\mu(\mathcal{F}, p)$ is the Poincaré-Hopf index at p of any local vector field generating $\mathcal{F}$, which is well defined and coincides with the Milnor number since the singularities are isolated.

Let $C \subset X$ be a curve such that none of its components is invariant by $\mathcal{F}$. For a point $p \in C$ let $\{f = 0\}$ be a reduced equation for C in some small neighborhood of p. If v is a local vector field with isolated zeros defining $\mathcal{F}$ around p then the tangency index between $\mathcal{F}$ and C is defined by

$$\operatorname{tang}(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,p}}{< f, v(f) >}.$$

Remark that $v(f)$ is not identically zero along C because C is not $\mathcal{F}$-invariant. In fact, the number tangencies between C and the leaves of $\mathcal{F}$ is finite. Their sum is called $\operatorname{tang}(\mathcal{F}, C)$ and can be calculated by the following:

Proposition 1.4 (Brunella)**.** *Let $\mathcal{F}$ be a foliation on a surface X and C be a compact curve, whose components are not invariant by $\mathcal{F}$. Then*

$$tang(\mathcal{F}, C) = C^2 + K_{\mathcal{F}} \cdot C$$

Let $C \subset X$ be a curve whose components are invariant by $\mathcal{F}$. For a point $p \in C$ let $\{f = 0\}$ be a reduced equation for C in some small neighborhood of p. If ω is a 1-form defining $\mathcal{F}$ around p, we can decompose ω as

$$g\omega = hdf + f\eta,$$

where η is a holomorphic 1-form, g and h are holomorphic functions with h and f relative prime. The function $\frac{h}{g}|_C$ is meromorphic and does not depend on the choice of g, h and η. We can define

$$\mathrm{Z}(\mathcal{F}, C, p) = \text{“vanishing order of } \left.\frac{h}{g}\right|_C \text{ at } p\text{”}.$$

If p is a smooth point of C, this recovers Gomez-Mont–Seade–Verjovsky index interpreted as the Poincaré–Hopf index of the restriction to C of a local vector field defining $\mathcal{F}$ at p.

Analogously, $\frac{1}{h}\eta|_C$ is a meromorphic 1-form, and its residue at p does not depend on the choice of the above decomposition. Then we set

$$\mathrm{CS}(\mathcal{F}, C, p) = \operatorname{Res}_p \left\{ \left. \frac{1}{h}\eta \right|_C \right\}.$$

This is the well-known Camacho–Sad index.

These indices are shown to be zero if p is a regular point for $\mathcal{F}$, hence their sums are finite and are calculated by the following:

Proposition 1.5 (Brunella)**.** *Let $\mathcal{F}$ be a foliation on a surface X and C a compact curve, each component of which is invariant by $\mathcal{F}$. Then*

$$\mathrm{Z}(\mathcal{F}, C) = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap C} \mathrm{Z}(\mathcal{F}, C, p) = \chi(C) + K_{\mathcal{F}} \cdot C$$

$$\mathrm{CS}(\mathcal{F}, C) = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap C} \mathrm{CS}(\mathcal{F}, C, p) = C^2$$

where $\chi(C) = -C^2 - K_X \cdot C$ is the virtual (arithmetic) Euler characteristic of C.

Let p be a reduced nondegenerate singularity of $\mathcal{F}$ with eigenvalue $\lambda \neq 0$.Then, there are exactly two germs of smooth separatrices passing through p, say C_1 and C_2. The indexes at p are shown in the following table:

$\mathrm{BB}(\mathcal{F}, p) = \lambda + \frac{1}{\lambda} + 2$	$\mathrm{CS}(\mathcal{F}, C_1, p) = \lambda$
$\mu(\mathcal{F}, p) = 1$	$\mathrm{CS}(\mathcal{F}, C_2, p) = \frac{1}{\lambda}$
$\mathrm{Z}(\mathcal{F}, C_i, p) = 1$, $i = 1, 2$	$\mathrm{CS}(\mathcal{F}, C_1 \cup C_2, p) = \lambda + \frac{1}{\lambda} + 2$
$\mathrm{Z}(\mathcal{F}, C_1 \cup C_2, p) = 0$	

1.3 Bimeromorphic Equivalence of Foliated Surfaces

Let X be a smooth compact complex surface and let $\mathcal{F}$ be a foliation on X. To simplify the notation we will refer to the pair $(X, \mathcal{F})$ as a foliated surface. A holomorphic (meromorphic) map $\psi : (Y, \mathcal{G}) \longrightarrow (X, \mathcal{F})$ is a holomorphic (resp. meromorphic) map from Y to X that sends leaves of $\mathcal{G}$ to leaves of $\mathcal{F}$. Precisely, if ω is a 1-form defining $\mathcal{F}$ in some open set $U \subset X$, then $\psi^*\omega$ defines $\mathcal{G}$ on $\psi^{-1}(U)$.

We say that $(Y, \mathcal{G})$ and $(X, \mathcal{F})$ are bimeromorphically equivalent if there exists a bimeromorphic map of foliated surfaces $\psi : (Y, \mathcal{G}) \longrightarrow (X, \mathcal{F})$.

Brunella, in [4], has introduced the concept of a minimal model for a foliated surface, in the sense of Castelnuovo–Enriques–Zariski.

Definition 1.6. Let $(X, \mathcal{F})$ be a reduced foliated surface, meaning that, $\mathcal{F}$ has only reduced singularities. We say that $(X, \mathcal{F})$ is minimal if and only if for any reduced foliated surface $(Y, \mathcal{G})$ and bimeromorphic map $\psi : (Y, \mathcal{G}) \longrightarrow (X, \mathcal{F})$, ψ is in fact a morphism.

Brunella has classified all types of foliations that does not have a unique minimal model. It turns out that they all live on ruled surfaces.

Theorem 1.7 (Brunella, [4])**.** *Let $\mathcal{F}$ be a holomorphic foliation on a projective surface S without minimal model. Then $\mathcal{F}$ is bimeromorphically equivalent to a foliation in the following list:*

1. *rational fibrations;*
2. *nontrivial Riccati foliations;*
3. *the very special foliation $\mathcal{H}$.*

A foliation is Riccati if it is totally transversal to a general fiber of a rational fibration.

The concept of Kodaira dimension for holomorphic foliations has been introduced independently by L. G. Mendes and M. McQuillan. It is the invariant that plays the major role in the birational classification of foliated surfaces.

Let X be a compact complex manifold and let L be a line bundle over X. The sequence $h^0(X, L^{\otimes n}) = \dim_{\mathbb{C}} \mathrm{H}^0(X, L^{\otimes n})$ grows polynomially for $n \gg 0$. The Itaka-Kodaira dimension of the pair (X, L), denoted by $\mathrm{Kod}(L)$, is defined by the degree of this polynomial, or $\mathrm{Kod}(L) = -\infty$ if $h^0(X, L^{\otimes n}) = 0$ for all $n > 0$.

If one replaces L by K_X, $\mathrm{Kod}(X) = \mathrm{Kod}(K_X)$ is called the Kodaira dimension of X. We always have that $\mathrm{Kod}(X) \leq \dim(X)$. When equality holds, X is called a general type manifold.

We could define the Kodaira dimension of $\mathcal{F}$ as the Itaka–Kodaira dimension of the line bundle $K_{\mathcal{F}}$, for short $\mathrm{Kod}(\mathcal{F})$. But this is not a birational invariant for $\mathcal{F}$. For example, the radial foliation (lines passing through a point) on $\mathbb{P}^2$ has canonical bundle $K_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2}(-1)$ but a suitable birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by polynomials of high degree transforms it into a foliation $\mathcal{G}$ with canonical bundle $K_{\mathcal{G}} = \mathcal{O}_{\mathbb{P}^2}(d-1)$. Fortunately, $\mathrm{Kod}(K_{\mathcal{F}})$ is a birational invariant for foliations having only reduced singularities. So if $\mathcal{F}$ is a foliation on the complex surface X and $\mathcal{G}$ is any reduced model of $\mathcal{F}$, the Kodaira dimension of $\mathcal{F}$ is defined as $\mathrm{Kod}(K_{\mathcal{G}})$.

Definition 1.8. Let $\mathcal{F}$ be a foliation on a compact complex surface X and let $\mathcal{G}$ be a reduced foliation on a surface Y bimeromorphically equivalent to $\mathcal{F}$. The Kodaira dimension of $\mathcal{F}$ is given by

$$\mathrm{Kod}(\mathcal{F}) = \limsup \frac{\log h^0(Y, K_{\mathcal{G}}^{\otimes n})}{\log n}$$

The birational classification of foliated surfaces is summarized in the following table (see [5] for details):

$\mathrm{Kod}(\mathcal{F})$	Description
$-\infty$	Rational fibrations Hilbert modular foliations
0	Quotient of a foliation generated by a global vector field
1	Riccati foliations Turbulent foliations Nonisotrivial elliptic fibrations Isotrivial higher genus fibrations
2	General type foliations

1.4 Automorphism of Foliations

Let $\mathcal{F}$ be a codimension k holomorphic foliation on a complex manifold X. For a bimeromorphic self map $\phi : X \longrightarrow X$ and a meromorphic k-form ω that defines $\mathcal{F}$, we say that ϕ preserves $\mathcal{F}$ if $\phi^*\omega$ also defines $\mathcal{F}$.

Definition 1.9. The bimeromorphism group of $\mathcal{F}$, denoted by $\mathrm{Bim}(\mathcal{F})$, is the maximal subgroup of $\mathrm{Bim}(X)$ generated by the bimeromorphic self maps that preserve $\mathcal{F}$. The automorphism group of $\mathcal{F}$, denoted by $\mathrm{Aut}(\mathcal{F})$, is the maximal subgroup of $\mathrm{Bim}(\mathcal{F})$ defined by the automorphisms that preserve $\mathcal{F}$.

If $(Y, \mathcal{G})$ is the minimal model of $(X, \mathcal{F})$, then $\mathrm{Bim}(\mathcal{F}) \simeq \mathrm{Aut}(\mathcal{G})$, by definition. Then we may work with automorphisms and minimal models, whenever they exist. However, in the following chapters we will assume slightly weaker hypotheses.

By Bochner-Montgomery Theorem (see [17, III – Theorem 1.1]), $\mathrm{Aut}(X)$ is a complex Lie group whose Lie algebra is the algebra of global vector fields, whenever X is compact. In [31], Pereira and Sánchez have proved that, in this situation, for any foliation $\mathcal{F}$, $\mathrm{Aut}(\mathcal{F})$ is a closed Lie subgroup of $\mathrm{Aut}(X)$. It is also important to see how these groups act on the vector bundles associated to the foliation.

A foliation $\mathcal{F}$ on a complex manifold X can given by a vector bundle morphism

$$\iota : E \longrightarrow TX$$

that is injective on $U = X \backslash \mathrm{Sing}(\mathcal{F})$. The group $\mathrm{Aut}(\mathcal{F})$ acts linearly on TX and preserves $\iota(E|_U)$. Then we have an action on E over U:

$$\begin{aligned} \rho : \mathrm{Aut}(\mathcal{F}) \times E|_U &\longrightarrow E|_U \\ (g, (p, v)) &\mapsto (g(p), \iota^{-1}(Dg_p \cdot \iota(v))) \end{aligned}$$

Since $\mathrm{Sing}(\mathcal{F})$ has codimension at least two, by Hartog's Theorem, ρ extends to an action

$$\rho : \mathrm{Aut}(\mathcal{F}) \times E \longrightarrow E$$

that is linear on the fibers by the Identity Principle. Dualizing, tensorizing and taking exterior powers preserves the linearity, hence we can define linear actions on the line bundles associated the foliation. In particular, for dimension two we have that the line bundles K_X, $K_{\mathcal{F}}$ and $N_{\mathcal{F}}$ have linear actions of $\mathrm{Aut}(\mathcal{F})$. This yields a linear action on their sheaves of sections, thus in the their global sections.

This will be useful in Chapter 5 in order to construct invariant pencils that will play an important role in bounding the order of the automorphism group.

Chapter 2

Automorphisms of Germs of Foliations

In this chapter we will describe some important properties of groups of germs of automorphisms at $(\mathbb{C}^2, 0)$ fixing a germ of foliation. Our interest here is to carry out useful descriptions of finite groups that will be linked to the stabilizer of some special points in the global situations that we will be dealt with in the following chapters.

2.1 Finite Groups of Automorphisms

The first, and maybe the most important, fact that we will state here is that every finite group of germs of holomorphic automorphisms is linearizable.

Lemma 2.1. *Let G be a finite subgroup of* $\mathrm{Diff}(\mathbb{C}^n, 0)$. *Then, the map $\varphi \mapsto D\varphi_0$ is a monomorphism from G to $GL(n, \mathbb{C})$.*

Proof. The map is a homomorphism, by the chain rule. Suppose that φ is in its kernel and is not the identity, that is $\varphi(x) = (x) + D^k\varphi_0(x) + \dots$ is the Taylor expansion, k the least order of a derivative that is not identically null. The mth power of φ is

$$\varphi^m(x) = (x) + mD^k\varphi_0(x) + \dots$$

But φ has finite order, hence $mD^k\varphi_0 \equiv 0$ for some m, which is a contradiction. Then, φ is the identity. □

This monomorphism can be given by an explicit change of coordinates:

$$g = \sum_{\varphi \in G} (D\varphi_0)^{-1} \cdot \varphi.$$

In general, if an automorphism fixes a point then it corresponds to an automorphism of the blow-up at that point. For linear germs in $(\mathbb{C}^2, 0)$, we will exhibit this property in the following lemma.

Lemma 2.2. *Let G be a finite subgroup of* $\mathrm{Diff}(\mathbb{C}^2, 0)$ *and let* $\pi : (X, E) \to (\mathbb{C}^2, 0)$ *be the blow-up at the origin, where E denotes the exceptional divisor. Then G lifts to a group of automorphisms of X that leaves E invariant.*

Proof. Fix an element of G, say $\varphi(x, y) = (ax+by, cx+dy)$, $ad-bc \neq 0$. Choosing coordinates $(x, y; u : t)$ on (X, E) such that $\pi(x, y; u : t) = (x, y)$, we define an automorphism of (X, E) by

$$\psi(x, y; u : t) = (ax + by, cx + dy; au + bt : cu + dt).$$

It follows that $\varphi \circ \pi = \pi \circ \psi$. $\square$

The automorphisms that we will work with are the ones that preserve some foliation. It means that a leaf of the foliation is sent into another leaf by such automorphisms. The precise definition is the following.

Definition 2.3. Let $\mathcal{F}$ be a germ of foliation on $(\mathbb{C}^2, 0)$ given by a 1–form ω. We say that $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ is an automorphism of $\mathcal{F}$ if $\varphi^*\omega$ also defines $\mathcal{F}$. This occurs if and only if $\varphi^*\omega = f\omega$ where f is a germ of holomorphic function that does not vanish at $0 \in \mathbb{C}^2$. Equivalently, if $\mathcal{F}$ is given by a vector field v, the map φ preserves $\mathcal{F}$ if $\varphi_* v = fv$ for some invertible holomorphic germ of function f.

2.2 Automorphisms of Regular Foliations

Let $\mathcal{F}$ be a germ of regular foliation on $(\mathbb{C}^2, 0)$. If $\mathcal{F}$ is given by a vector field v, then $v(0)$ defines a nontrivial eigenspace of $D\varphi_0$ for any φ that preserves $\mathcal{F}$. Hence, this yields serious restrictions on automorphism groups that stabilize a regular point of a foliation. As we will see, it turns out that finite groups of such automorphisms are abelian, or even cyclic if we impose mild restrictions on $\mathcal{F}$.

Proposition 2.4. *Let $\mathcal{F}$ be a germ of regular foliation on* $(\mathbb{C}^2, 0)$. *If G is a finite group of automorphisms of $\mathcal{F}$, then G is abelian. Moreover, there are coordinates (x, y) where G is linear diagonal and $\mathcal{F}$ is given by a vector field of the form*

$$v = P(x, y)\partial_x + Q(x, y)\partial_y$$

where either $P(0) = 0$ or $Q(0) = 0$.

Proof. By Lemma 2.1, we can choose coordinates (x, y) where G is linearized. Let v be a vector field defining $\mathcal{F}$. Without loss of generality we can suppose that v is written as

$$v = P(x, y)\partial_x + \partial_y.$$

Up to a linear change o coordinates we may suppose that $v(0) = \partial_y$, equivalently $P(0) = 0$. Let $\varphi(x, y) = (ax + by, cx + dy)$ be an element of G of order n. Then,

$$\lambda\partial_y = \lambda v(0) = (\varphi_* v)(0) = D\varphi_0 \cdot v(0) = b\partial_x + d\partial_y.$$

Hence, the invariance implies that $b = 0$. If $a = d$, for any $k > 0$ we have that

$$\varphi^k(x, y) = (a^k x, kca^{k-1}x + a^k y).$$

It follows that $c = 0$ since φ has finite order.

Suppose now that $c \neq 0$, hence $a \neq d$. Then

$$\varphi^k(x, y) = \left(a^k x, \frac{a^k - d^k}{a - d}cx + d^k y\right)$$

for any $k \geq 0$. Since φ has order n, we have that $a^n = d^n = 1$ and c is any complex number. After a linear change of coordinates T, we may assume that $\varphi(x, y) = (ax, by)$, namely

$$T(x, y) = \left(x, x + \frac{d - a}{c}y\right).$$

Observe that in these new coordinates $\mathcal{F}$ is given by a vector field

$$w = Q(x, y)\partial_x + \partial_y$$

such that $Q(0) = 0$. In fact,

$$T_*v = \left(P \circ T^{-1}\right)(x, y)\partial_x + \frac{d - a}{c}\left[1 - \left(P \circ T^{-1}\right)(x, y)\right]\partial_y$$

and the coefficient of ∂_y is invertible, since $P(0, 0) = 0$. If $\psi \in G$ commutes with φ, then T transforms ψ into a diagonal form. If they don't, ψ is written as above: $\psi(x, y) = (ex, gx + hy)$, $e^m = h^m = 1$ and $g \neq 0$. We will now show that such ψ cannot exist. The commutator $\rho = \varphi\psi\varphi^{-1}\psi^{-1} \in G$ is written

$$\rho(x, y) = \left(x, \frac{g}{e}\left(\frac{d}{a} - 1\right)x + y\right).$$

Since $a \neq d$, ρ has infinite order, which contradicts the finiteness of G. Therefore, G is abelian. □

We have seen that there are coordinates where $\mathcal{F}$ is tangent to one of the axes. For an abelian diagonal group, the level sets of each coordinate function gives other germs of invariant regular foliations. In the chosen coordinates, one of them is tangent to $\mathcal{F}$ at the origin. If this foliation does not share a leaf with $\mathcal{F}$, then the group is cyclic. In other words, if we assume that the axes are not $\mathcal{F}$–invariant, the only finite groups that can preserve $\mathcal{F}$ are cyclic.

Proposition 2.5. *Let $\mathcal{F}$ be a regular germ of foliation on $(\mathbb{C}^2, 0)$ such that the line L_0 tangent to the leaf through 0 is not $\mathcal{F}$–invariant. Let G be a finite group of* *automorphisms that preserve $\mathcal{F}$. Then G is cyclic and there are coordinates (x, y) where $L_0 = \{x = 0\}$ and G is generated by an element of the form*

$$\varphi(x, y) = (l^{k+1}x, ly)$$

where l is a root of the unity of order $|G|$ and $k = tang(\mathcal{F}, L_0, 0) \geq 1$. Moreover, G has a subgroup H of order $\gcd(k + 1, |G|)$ that sends each line $\{x = c\}$, $c \in \mathbb{C}$, onto itself.

Proof. Since $\mathcal{F}$ is regular at the origin, Proposition 2.4 implies that there are coordinates (x, y) such that G is abelian diagonal and $\mathcal{F}$ is given by a vector field

$$v = P(x, y)\partial_x + \partial_y.$$

It remains to prove that G is cyclic. First, let us prove that every element in G can be written in the form described in the proposition.

Let $\varphi(x, y) = (l^a x, l^b y) \in G$, $l^n = 1$ and $\gcd(a, b, n) = 1$. The 1–form dual to v is

$$\omega = dx - P(x, y)dy$$

and we have that $L_0 = \{x = 0\}$. The line L_0 is not $\mathcal{F}$–invariant, hence x does not divide P. Then, by Weierstrass Preparation Theorem, we may write

$$P(x, y) = h(x, y)(y^k - xg(x, y))$$

for some $k > 0$ and holomorphic functions g and h, $h(0) \neq 0$. Then, we have that

$$\varphi^*\omega = l^a dx - (P \circ \varphi)l^b dy = l^a dx - (h \circ \varphi)(l^{bk}y^k + (xg) \circ \varphi)l^b dy$$

hence $l^a = l^{b(k+1)}$, since $h(0, 0) \neq 0$, which is equivalent to $a \equiv b(k + 1) \pmod{n}$. This congruence holds whenever $\gcd(b, n)$ divides a. Then $\gcd(a, b, n) = 1$ implies that $\gcd(b, n) = 1$. Let $t \in \mathbb{Z}$ such that $bt \equiv 1 \pmod{n}$. Then

$$\varphi^t(x, y) = (l^{k+1}x, ly),$$

which generates the same group as φ.

Now we proceed to prove that G is cyclic. Suppose φ is an element with maximal order in G and let ψ be another element in G. Then we can write

$$\psi(x, y) = (\zeta^{k+1}x, \zeta y),$$

$\zeta^m = 1$ and $m \leq n$. The group G must have an element of order $\operatorname{lcm}(m, n) \geq n$. Hence, by the maximality of n, we have that $n = md$ for some $d \geq 1$. Therefore, $\zeta = l^{sd}$ for some s invertible modulo n, which implies $\psi \in <\varphi>$. Hence G is cyclic.

Let $r = \gcd(k+1, n)$, then $n = pr$ for some p and $k + 1 \equiv 0 \pmod{r}$. This implies that

$$\varphi^p(x, y) = (x, l^p y)$$

sends each line $\{x = c\}$, $c \in \mathbb{C}$ onto itself. Conversely, any element of order m in G with this property must be of this form. Then m divides both n and $k+1$ and hence r. This show that G has a subgroup H of order $r = \gcd(k+1, n)$ that sends each line $\{x = c\}$, $c \in \mathbb{C}$ onto itself.

To finish we must show that $k = \operatorname{tang}(\mathcal{F}, \{x = 0\}, 0)$. By definition,

$$\operatorname{tang}(\mathcal{F}, \{x = 0\}, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{\langle x, v(x) \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{\langle x, P \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,0}}{\langle x, y^k \rangle} = k$$

□

2.3 Automorphisms of Singular Foliations

Naturally, the behavior of a foliation near its singular points also imposes many restrictions on the automorphisms that leave it invariant. Specifically for foliation with non-vanishing first jet we have the following:

Proposition 2.6. *Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^2, 0)$ given by a germ of vector field v such that $v(0) = 0$ and $Dv_0 \neq 0$. If $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ is a linearizable automorphism of $\mathcal{F}$, then:*

1. *If $Dv_0 = I$, then φ can be any element of $GL(2, \mathbb{C})$;*
2. *If $Dv_0 \neq I$ is diagonalizable with eigenvalues 1, λ then φ is diagonal for any λ or $\varphi(x, y) = (by, cx)$ for $\lambda = -1$;*
3. *If Dv_0 is invertible but not diagonalizable, then $\varphi(x, y) = (ax + by, ay)$;*
4. *If Dv_0 is nilpotent, then $\varphi(x, y) = (ax + by, dy)$.*

Proof. We may choose coordinates (x, y) such that φ is linearized, it is written as

$$\varphi(x, y) = (ax + by, cx + dy)$$

where $ad - bc \neq 0$.

Let v be a germ of vector field defining $\mathcal{F}$. By hypothesis, Dv_0 is not trivial and the relation $\varphi_* v = f v$, $f(0) \neq 0$, implies that

$$f(0) Dv_0 = D\varphi_0 \cdot Dv_0 \cdot (D\varphi_0)^{-1}$$

since $v(0) = 0$. Up to a linear change of coordinates, we may assume that Dv_0 is in one of the following canonical forms:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

where $\lambda, \zeta \in \mathbb{C}$, $\zeta \neq 0$. Therefore, by imposing the relation above and straightforward calculations lead to the classification. Fix

$$\varphi(x, y) = (ax + by, cx + dy).$$

First case:

$$f(0)(ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} ad - \lambda bc & ab(\lambda - 1) \\ cd(1 - \lambda) & ad\lambda - bc \end{pmatrix}.$$

If $\lambda = 1$, then $Dv_0 = I$ and φ can be any element of $\mathrm{GL}(2, \mathbb{C})$. Suppose that $\lambda \neq 1$. Then $ab = cd = 0$ and, since $ad - bc \neq 0$, we have that either $b = c = 0$ or $a = d = 0$. If $b = c = 0$, then

$$\begin{cases} f(0)ad = ad \\ f(0)ad\lambda = ad\lambda \end{cases}$$

Hence $f(0) = 1$ and $a, d \in \mathbb{C}^*$ can be any numbers. If $a = d = 0$, then

$$\begin{cases} -f(0)bc = -\lambda bc \\ -f(0)bc\lambda = -bc \end{cases}.$$

which implies that $f(0) = \lambda = -1$ and $b, c \in \mathbb{C}^*$ can be any numbers.

Second case:

$$f(0)(ad - bc)\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ad - bc - ac\zeta & a^2\zeta \\ -c^2\zeta & ad - bc + ac\zeta \end{pmatrix}.$$

Then, $c = 0$ and

$$\begin{cases} f(0)ad = ad \\ f(0)ad\zeta = a^2\zeta \end{cases}.$$

Therefore, $f(0) = 1$, $a = d \in \mathbb{C}^*$ and $b \in \mathbb{C}$ can be any numbers.

Third case:

$$f(0)(ad - bc)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix}.$$

Hence $c = 0$ and $f(0)d = a$. □

Reduced singularities are those in case 2 of Proposition 2.6 such that $\lambda \notin \mathbb{Q}_{>0}$. The finite groups of automorphism that preserve these types of singularities can be described in a more geometric way as we state in the following proposition.

Proposition 2.7. *Let $\mathcal{F}$ be a germ of singular foliation on $(\mathbb{C}^2, 0)$ with reduced singularity whose eigenvalue is λ. If G is a finite group of automorphisms of $\mathcal{F}$, then G has an abelian subgroup H whose index is at most 2. In appropriate coordinates, H has generators with the form*

$$\varphi(x, y) = (l^a x, l^b y),$$

where l is a nth root of the unity, $\gcd(a, b, n) = 1$. Moreover, if $\lambda \neq -1$, $G = H$.

Proof. By Lemma 2.1, we may take coordinates (x, y) such that G is linear. Up to a linear change of coordinates we may also suppose that $\mathcal{F}$ is given by the 1–form

$$\omega = (\lambda y + \dots)dx - (x + \dots)dy.$$

Fix an element $\varphi \in G$. Let $\pi : (X, E) \to (\mathbb{C}^2, 0)$ denote the blow-up of $(\mathbb{C}^2, 0)$ with exceptional divisor E, and let $\widetilde{\mathcal{F}}$ be the foliation pulled back to X. Since the singularity is reduced, $\widetilde{\mathcal{F}}$ has two reduced singularities on E (cf. [5] p.14), say 0 and ∞.

The germ of surface (X, E) is given by

$$X = \left\{(x, y; u : t) \in (\mathbb{C}^2, 0) \times \mathbb{P}^1;\ xt = yu\right\},$$

that is covered by two affine charts: $U_0 = \{u \neq 0\}$ and $U_\infty = \{t \neq 0\}$. The projections are $\pi_0(x, t) = (x, xt)$ and $\pi_\infty(u, y) = (uy, y)$, respectively.

Denote by ψ the automorphism of (X, E) such that $\varphi \circ \pi = \pi \circ \psi$ and let η_0 and η_∞ be local 1-forms generating $\widetilde{\mathcal{F}}$ at open subsets U_0 and U_∞, respectively. In the coordinates (x, t) of U_0,

$$\eta_0 = [(\lambda - 1)t + \cdots]dx - (x + \cdots)dt.$$

In coordinates (u, y) of U_∞,

$$\eta_\infty = (\lambda y + \cdots)du - [(1 - \lambda)u + \cdots]dy.$$

The map π is a biholomorphism outside the exceptional divisor, hence ψ preserves $\widetilde{\mathcal{F}}$ outside E. In particular, ψ permutes the singularities.

If ψ exchanges the singularities on E, that is,

$$\psi(0, 0; 1 : 0) = (0, 0; 0 : 1) \,\text{and}\, \psi(0, 0; 0 : 1) = (0, 0; 1 : 0),$$

then $\psi(U_0) = U_\infty$ and $\psi(x, y; u : t) = (by, cx; bt : cu)$. In coordinates, $(u, y) = \psi(x, t) = (bt/c, cx)$ and we have the relation $\psi^*\eta_\infty = f\eta_0$.

$$\begin{aligned}
\psi^*\eta_\infty &= (\lambda cx + \cdots)\frac{b}{c}dt - \left[(1 - \lambda)\frac{b}{c}t + \cdots\right]cdx \\
&= -b[(1 - \lambda)t + \cdots]dx + b(\lambda x + \cdots)dt = f\eta_0.
\end{aligned}$$

Comparing the linear parts, it follows that $\lambda^2 = 1$. And the only solution with $\lambda \notin \mathbb{Q}_{>0}$ is $\lambda = -1$. Therefore, if $\lambda \neq -1$, ψ fixes both singularities.

The group G is finite and preserves $\mathcal{F}$, then G lifts to a group of automorphisms that preserve $\widetilde{\mathcal{F}}$ and projects to a group $\bar{G}$ of automorphisms of the exceptional divisor.

The exact sequence

$$1 \to \mathbb{C}^* \to \mathrm{GL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C}) \to 1$$

restricts to G as

$$1 \to G' \to G \to \bar{G} \to 1$$

where $G' = G \cap \mathbb{C}^*$ and $\mathbb{C}^*$ is regarded as the center of $\mathrm{GL}(2, \mathbb{C})$. This says that G is a central extension of $\bar{G}$ by G'.

If $\lambda \neq -1$, $\bar{G}$ fixes two points, hence it is cyclic. Consequently, G is abelian. If $\lambda = -1$, $\bar{G}$ can permute the two singularities, hence it is at most a dihedral group. Therefore, G has an abelian (normal) H subgroup of index at most two. Moreover, H is conjugated to a finite diagonal subgroup of $\mathrm{GL}(2, \mathbb{C})$, hence has generators of the form

$$\varphi(x, y) = (l^a x, l^b y)$$

where l is a nth root of the unity, $\gcd(a, b, n) = 1$. $\square$

Furthermore, we point out an interesting fact that will help us throughout the following chapters. We can describe the automorphisms that preserve two foliations, one having a reduced singularity and the other being regular. It turns out that the tangency between them plays an important role.

Proposition 2.8. *Let $\mathcal{F}$ be a germ of reduced foliated singularity on $(\mathbb{C}^2, 0)$ with eigenvalue λ. Let G be a finite abelian group of automorphisms of $\mathcal{F}$. Let*

$$g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$$

be a germ of holomorphic function such that the fibration given by its level sets is smooth and G–invariant but not $\mathcal{F}$–invariant. Then, for some choice of coordinates, G is cyclic generated by an automorphism of the form

$$\varphi(x, y) = (l^k x, ly),$$

where l is an nth root of the unity and $k = \mathit{tang}(\mathcal{F}, \{g = 0\}, 0) \geq 1$. Moreover, G may have a subgroup of order $\gcd(k, |G|)$ that sends each fiber of g onto itself.

Proof. By Proposition 2.7, we can choose coordinates (x, y) such that $\mathcal{F}$ is given by a vector field v written as

$$v = (x + P(x, y))\partial_x + (\lambda y + Q(x, y))\partial_y$$

where P and Q are germs of holomorphic functions that vanish at the origin with order at least two and G is diagonal. Let $\varphi \in G$. It can be written as

$$\varphi(x, y) = (l^a x, l^b y)$$

where l is a nth root of the unity, $\gcd(a, b, n) = 1$.

The curve $\{g=0\}$ is smooth and G–invariant. Then, either $\partial g/\partial x(0)\neq 0$ or $\partial g/\partial y(0)\neq 0$ and we have that $g\circ\varphi = ug$ for some invertible function u. Hence

$$\begin{gathered} d(g\circ\varphi)_0 = u(0)dg_0 \\ \frac{\partial g}{\partial x}(0)l^a dx + \frac{\partial g}{\partial y}(0)l^b dy = u(0)\frac{\partial g}{\partial x}(0)dx + u(0)\frac{\partial g}{\partial y}(0)dy. \end{gathered}$$

If $\{g=0\}$ is transverse to both axes then both derivatives do not vanish at the origin. We have that $a\equiv b \pmod n$. Then G is cyclic generated by

$$\varphi(x,y)=(lx,ly).$$

In this case, $\mathrm{tang}(\mathcal{F},\{g=0\},0)=1$.

Suppose now that $\{g=0\}$ is tangent to one of the axes. We separate in two cases: either $\lambda=0$ or not.

First case : $\lambda\neq 0$.

In this case, we can assume that $\{g=0\}$ is tangent $\{x=0\}$, hence we have $\partial g/\partial y(0)=0$. If x divides g, we write

$$g(x,y)=u(x,y)x$$

with $u(0)\neq 0$ since $\{g=0\}$ is smooth. By hypothesis, $\{g=0\}$ is not $\mathcal{F}$–invariant, hence x may not divide $x+P(x,y)$. By the Weierstrass Preparation Theorem,

$$x+P=c(x,y)(x-y^k s(y))$$

where c and s are germs of invertible functions and $k>0$. Dividing by c does not affect $\mathcal{F}$, then we may assume that

$$v=(x-y^k s(y))\partial_x + (\lambda y + Q(x,y))\partial_y$$

and, since $\varphi_* v = fv$ for some invertible f, we have that

$$f[(x-y^k s(y))\partial_x+(\lambda y+Q(x,y))\partial_y] = (l^{-a}x-l^{-kb}y^k s(y))l^a\partial_x+(\lambda l^{-b}y+Q\circ\varphi^{-1})l^b\partial_y.$$

Hence $a\equiv kb \pmod n$ and this congruence holds whenever $\gcd(b,n)$ divides a. Then $\gcd(a,b,n)=1$ implies that $\gcd(b,n)=1$. Let $t\in\mathbb{Z}$ such that $bt\equiv 1 \pmod n$, then

$$\varphi^t(x,y)=(l^k x, ly),$$

which generates the same group as φ. Furthermore, as in the proof of Proposition 2.5, we see that this implies that G is cyclic and may have a subgroup of order $\gcd(k,n)$ that preserves the fibers $\{g=c\}$, $c\in\mathbb{C}$. In this case we have that

$$\mathrm{tang}(\mathcal{F},\{g=0\},0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathbb{C}^2,0}}{\langle g, v(g)\rangle} = \dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathbb{C}^2,0}}{\langle x, v(x)\rangle} = \dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathbb{C}^2,0}}{\langle x, y^k\rangle} = k.$$

If x does not divide g then, by the Weierstrass Preparation Theorem,

$$g(x,y) = u(x,y)(x - y^r h(y))$$

where u and h are germs of invertible functions, and $r > 0$. Then we may change the coordinates by

$$T(x,y) = (x + y^r h(y), y).$$

It is clear that x divides $g \circ T$ and, since the level sets of g are G–invariant, T and φ commute. Moreover, T is tangent to the identity, since $r > 0$, then $T_* v$ and v have the same linear part. Therefore, the proof follows as in the previous case.

Second case: $\lambda = 0$.

If $\{g = 0\}$ is tangent to $\{x = 0\}$ the proof works as above. It remains to prove the case where $\{g = 0\}$ is tangent to $\{y = 0\}$. Moreover, by the previous observation, we may assume that $g(x,y) = u(x,y)y$ with u invertible. By hypothesis $\{y = 0\}$ is not $\mathcal{F}$–invariant, hence y does not divide Q and, by the Weierstrass Preparation Theorem,

$$Q = a(x,y)(x^m - b(x,y)),$$

where a and b are germs or holomorphic functions, $m > 0$, a is invertible and b is a polynomial in x that vanishes at the origin. The relation $\varphi_* v = fv$ implies that

$$f[(x+P)\partial_x + Q(x,y)\partial_y] = (l^{-a}x + P \circ \varphi^{-1})l^a \partial_x + a \circ \varphi^{-1}(l^{-ma}x^m - Q \circ \varphi^{-1})l^b \partial_y.$$

Hence $ma \equiv b \pmod{n}$. As we proved above, this implies that $\gcd(a,n) = 1$ and G is cyclic generated by

$$\varphi^s(x,y) = (lx, l^m y),$$

where $as \equiv 1 \pmod{n}$.

Finally, let us calculate $\mathrm{tang}(\mathcal{F}, \{g = 0\}, 0)$.

$$v(g)|_{\{g=0\}} = u\, v(y)|_{\{y=0\}} = Q(x,0) = x^m a(x,0)u(x,0)$$

Therefore, $\mathrm{tang}(\mathcal{F}, \{g = 0\}, 0) = m$. □

Remark 2.9. Throughout this chapter we have only dealt with reduced singularities and this will be our focus in the next chapters. In the analysis of bimeromorphism groups, it is natural to work with foliations such that all the singularities are reduced. However, one can use Lemma 2.2 and the propositions above to deal with non-reduced singularities. In general, this analysis will depend on the divisor that appears in the process of reduction of the singularity, hence it is very tricky to deal with.

To end this chapter we present a simple degenerated example.

Example 2.10. Let $\mathcal{F}$ be the foliation on $\mathbb{C}^2$ defined by

$$\omega = (x^4 - x^2y^5)dx - (y^2 - x^3y^4)dy$$

This is one of the Żołądek's examples studied in [43]. In his notation, $a = b = 2$ and $c = d = 4$. It has a cyclic group of automorphisms fixing the origin of order

$$N = (b+1)(c+1) - (a-c)(d-b) = 19$$

It has a singularity at the origin with zero linear part. Its minimal resolution is achieved by blowing up the origin and other three infinitely near points. Let E_i be the exceptional divisor of the i-th blow-up. They are arranged as in the figure:

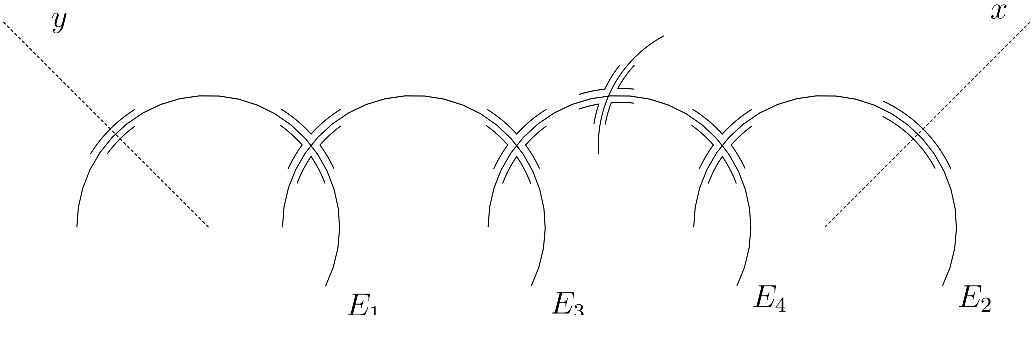

Figure 2.1: Reduction of the singularity of ω at 0.

All singularities are nondegenerate and only one lies away from the corners in E_4. The automorphisms of $\mathcal{F}$ lift to the blown-up surface and preserves the divisor, in particular they preserve E_4 and the singularities that lie there. Hence, any such automorphism fixes E_4 pointwise.

Around the corner $E_4 \cap E_3$ the foliation is given, in coordinates (z, w), by

$$\eta = w(-3 + 2z + z^{12}w^{19})dz - z(5 - 3z - 2z^{12}w^{19})dw,$$

where E_4 has equation $w = 0$ in this chart. Hence, the group of automorphisms is cyclic generated by

$$\widetilde{\varphi}(z, w) = (z, lw)\,,\, l^{19} = 1$$

Blowing all the way down to $\mathbb{C}^2$ we see that this generator is transformed into

$$\varphi(x, y) = \left(l^3 x, l^5 y\right).$$

Chapter 3

Bounds for Foliations on $\mathbb{P}^2$

This chapter is dedicated to study the automorphism groups of foliations on the projective plane. Initially we have to state some useful known facts about $\mathbb{P}^2$ and its automorphism group, specially the finite subgroups. For the proofs we refer to Hartshorne [12] or Huybrechts [14].

3.1 General Facts about Foliations on $\mathbb{P}^2$

Recall that the complex projective plane $\mathbb{P}^2$ is the set of lines in $\mathbb{C}^3$ that contain the origin, equivalently

$$\mathbb{P}^2 = \frac{(\mathbb{C}^3 \backslash \{0\})}{\mathbb{C}^*},$$

where $\mathbb{C}^*$ acts by scalar multiplication. The points in $\mathbb{P}^2$ are written as $(X : Y : Z)$, called homogeneous coordinates. In this notation $(\lambda X : \lambda Y : \lambda Z)$ and $(X : Y : Z)$ represent the same point. The Picard group of $\mathbb{P}^2$ is cyclic:

$$\mathrm{Pic}(\mathbb{P}^2) \simeq \mathbb{Z},$$

generated by the class of (the line bundle associated to) a line H. We denote the line bundles associated to a divisor linearly equivalent to dH by $\mathcal{O}_{\mathbb{P}^2}(d)$ as usual.

The most useful tool to describe the foliations on $\mathbb{P}^2$ is the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \longrightarrow T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow 0,$$

induced by the natural inclusion of the tautological line bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$ in the trivial bundle of rank 3. Dualizing this sequence we get

$$0 \longrightarrow \Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,$$

where the last map is the contraction by the radial vector field

$$R := X\partial_X + Y\partial_Y + Z\partial_Z.$$

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ defined by a global section of $T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(k)$. for some $k \in \mathbb{Z}$, then $K_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2}(k)$. The degree d of $\mathcal{F}$ is defined as the tangency number between $\mathcal{F}$ and a general line L, that is,

$$d = \operatorname{tang}(\mathcal{F}, L) = K_{\mathcal{F}} \cdot L + L^2 = k + 1.$$

Hence, a degree d foliation $\mathcal{F}$ has canonical bundle isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d-1)$. The normal bundle is determined by adjunction:

$$N_{\mathcal{F}} = K_{\mathcal{F}} \otimes K_{\mathbb{P}^2}^{\vee} = \mathcal{O}_{\mathbb{P}^2}(d+2).$$

Tensorizing the Euler sequence with $\mathcal{O}_{\mathbb{P}^2}(d)$ and taking the long exact sequence of cohomology,

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \longrightarrow \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))^{\oplus 3} \longrightarrow \mathrm{H}^0(\mathbb{P}^2, T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(d-1)) \longrightarrow 0,$$

since $\mathrm{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) = 0$. This implies that a holomorphic global section of $T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(d-1)$ is identified with a polynomial vector field in $\mathbb{C}^3$

$$v = A\partial_X + B\partial_Y + C\partial_Z,$$

where A, B and C are homogeneous of degree d, and $v + GR$ defines the same section for any homogeneous polynomial G of degree $d-1$.

Recall that the divergence of a vector field v is defined in this case by

$$\operatorname{div}(v)\mu = \left(\frac{\partial A}{\partial X} + \frac{\partial B}{\partial Y} + \frac{\partial C}{\partial Z}\right)\mu = d(\iota_v \mu)$$

where $\mu = dX \wedge dY \wedge dZ$, ι_v is the contraction by v and d is the differential operator. For $v + GR$, a direct calculation shows that $\operatorname{div}(v + GR) = \operatorname{div}(v) + (d+2)G$. Hence, any foliation can be represented by a vector field with zero divergence, just put $G = -\operatorname{div}(v)/(d+2)$. The interaction of automorphisms and the divergence of vector fields in $\mathbb{C}^3 \backslash \{0\}$ leads to an interesting fact.

Lemma 3.1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ given by a vector field v. If F is an element of $Aut(\mathcal{F}) < Aut(\mathbb{P}^2)$, then*

$$\operatorname{div}(F_*^{-1} v) = F^* \operatorname{div}(v).$$

Moreover, if $\operatorname{div}(v) = 0$ then $F_* v = \lambda v$ for some $\lambda \in \mathbb{C}^*$.

Proof. Any such automorphism F is the class of an element of $\mathrm{GL}(3, \mathbb{C})$ that we also call F. We have that $F^*\mu = \det(F)\mu$. Then

$$\begin{aligned}\operatorname{div}(F_*^{-1} v)\mu = d(\iota_{F_*^{-1} v}\mu) &= \frac{1}{\det(F)} d(\iota_{F_*^{-1} v} F^*\mu) = \frac{1}{\det(F)} d(F^*(\iota_v \mu)) \\ &= \frac{1}{\det(F)} F^*(\operatorname{div}(v)\mu) = (F^*\operatorname{div}(v))\mu.\end{aligned}$$

We have that $F \in \mathrm{Aut}(\mathcal{F})$ means that F_*v defines $\mathcal{F}$. Then $F_*v = \lambda v + GR$ with $\lambda \in \mathbb{C}^*$ and G a homogeneous polynomial. If $\mathrm{div}(v) = 0$, then

$$0 = \mathrm{div}(F_*v) = \mathrm{div}(\lambda v + GR) = (d+2)G$$

hence, $G = 0$. □

The foliation $\mathcal{F}$ can also be defined by a global section of $\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$. By taking global sections in the dual of the Euler sequence and observing that $\mathrm{H}^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)) = 0$, since $d+2 \neq 0$, we achieve the following exact sequence:

$$0 \to \mathrm{H}^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)) \to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+1))^{\oplus 3} \to \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+2)) \to 0$$

Then a global section of $\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$ is identified with a polynomial 1-form

$$\omega = AdX + BdY + CdZ$$

in $\mathbb{C}^3$ where A, B and C are homogeneous of degree $d+1$, such that the contraction by R is zero, that is $XA + YB + ZC = 0$.

3.2 Finite Subgroups of Aut($\mathbb{P}^2$)

We are interested in foliations with finite automorphism groups. For the sake of convenience we recall the classification of finite subgroups of $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3, \mathbb{C})$, established by Blichfeldt in [29].

Theorem 3.2 (Blichfeldt). *Any finite subgroup of $PGL(3, \mathbb{C})$ is conjugated to one of the following.*
Intransitive groups:
(A) An abelian group generated by diagonal matrices;
(B) A finite subgroup of $\mathrm{GL}(2, \mathbb{C})$.
Imprimitive groups:
(C) A group generated by an abelian group and

$$T(X : Y : Z) = (Y : Z : X);$$

(D) A group generated by a group of type (C) and a transformation

$$R(X : Y : Z) = (aX : bZ : cY).$$

Primitive groups having a non-trivial normal subgroup:
(E) The group of order 36 *generated by* $S(X : Y : Z) = (X : \lambda Y : \lambda^2 Z)$, T *and*

$$V(X : Y : Z) = (X + Y + Z : X + \lambda Y + \lambda^2 Z : X + \lambda^2 Y + \lambda Z),$$

where λ is a primitive cubic root of 1;

(F) *The group of order* 72 *generated by* S, T, V *and* UVU^{-1} *where*

$$U(X:Y:Z) = (X:Y:\lambda Z);$$

(G) *The Hessian group of order* 216 *generated by* S, T, V, *and* U.

Simple groups:

(H) *The icosahedral group isomorphic to* $\mathfrak{A}_5$ *of order* 60 *generated by* T *and the transformations* $R_1(X:Y:Z) = (X:-Y:-Z)$ *and*

$$R_2(X:Y:Z) = (X+\varphi Y-\varphi^{-1}Z : \varphi X-\varphi^{-1}Y+Z : -\varphi^{-1}X+Y+\varphi Z)$$

where $\varphi = \frac{1}{2}(1+\sqrt{5})$.

(I) *The Valentiner group isomorphic to* $\mathfrak{A}_6$ *of order* 360 *generated by* T, R_1, R_2 *and* S.

(J) *The Klein group isomorphic to* $\mathrm{PSL}(2,\mathbb{F}_7)$ *of order* 168 *generated by* T *and the transformations* $R'(X:Y:Z) = (X:\beta Y:\beta^3 Z)$ *and*

$$S'(X:Y:Z) = (aX+bY+cZ : bX+cY+aZ : cX+aY+bZ)$$

where $\beta^7 = 1$, $a = \beta^4-\beta^3$, $b = \beta^2-\beta^5$ *and* $c = \beta-\beta^6$.

Later Dolgachev and Iskovskikh classified the transitive imprimitive groups, the groups of type (C) and (D). If G is a group of one of these types, it has an abelian normal subgroup H such that G/H permutes transitively the three H–invariant lines. It turns out that G/H is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or the symmetric group $\mathcal{S}_3$ on three elements. They have proved the following:

Theorem 3.3 (Dolgachev–Iskovskikh)**.** *Let* G *be a transitive imprimitive subgroup of* $PGL(3,\mathbb{C})$. *Then* G *is conjugate to one one of the following groups:*

(C1) $G \simeq (\mathbb{Z}/n\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$ *generated by*

$$\begin{aligned} S_1(X:Y:Z) &= (lX:Y:Z),\\ S_2(X:Y:Z) &= (X:lY:Z),\\ T(X:Y:Z) &= (Y:Z:X) \end{aligned}$$

where l *is a primitive* n*–th root of the unity.*

(D1) $G \simeq (\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mathcal{S}_3$ *generated by* S_1, S_2, T *and*

$$R(X:Y:Z) = (Y:X:Z)$$

(C2) $G \simeq (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/3\mathbb{Z})$ *generated by*

$$\begin{aligned} S_3(X:Y:Z) &= (l^kX:Y:Z),\\ S_4(X:Y:Z) &= (l^sX:lY:Z),\\ T(X:Y:Z) &= (Y:Z:X) \end{aligned}$$

where $k > 1$, $n = mk$ *and* $s^2-s+1 \equiv 0 \pmod{k}$.

(D2) $G \simeq (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes \mathcal{S}_3$ *generated by* S_3, S_4, T *and* R, *where* $k = 3$ *and* $s = 2$.

Observe that groups of type (A) or (B) fix the point $(1 : 0 : 0)$ and all the others do not fix any point but have a subgroup of type (C). In particular, they contain T as an element. Then we have restrictions on foliations with automorphism groups without fixed points, as we state in the following lemma:

Lemma 3.4. *Let* $\mathcal{F}$ *be a foliation on* $\mathbb{P}^2$ *of degree* $d \geq 0$ *and let* $G < Aut(\mathcal{F})$ *be a finite subgroup. Then, either*

1. *G has a fixed point, or*
2. *up to linear change of coordinates, $Aut(\mathcal{F})$ contains the automorphism $T(X : Y : Z) = (Y : Z : X)$ and $\mathcal{F}$ is defined by a vector field of te form*

$$v = A(X, Y, Z)\partial_X + \lambda^2 A(Y, Z, X)\partial_Y + \lambda A(Z, X, Y)\partial_Z$$

where $\lambda^3 = 1$ *and* A *is a homogeneous polynomial of degree* d.

Moreover, if $Aut(\mathcal{F})$ *contains both* T *and* $R(X : Y : Z) = (Y : X : Z)$, *then* $\lambda = 1$ *and* $A \circ T \circ R = \pm A$.

Proof. If G does not fix any point in $\mathbb{P}^2$ then, up to conjugation, we may suppose that $T \in G$. Let $v = A\partial_X + B\partial_Y + C\partial_Z$ be a vector field inducing $\mathcal{F}$ such that $\operatorname{div}(v) = 0$. The vector field $T_*^{-1}v$ also induces $\mathcal{F}$ and, from Lemma 3.1, we have a relation $T_*^{-1}v = \lambda v$ for some $\lambda \in \mathbb{C}^*$. This means that

$$(A \circ T)\partial_X + (B \circ T)\partial_Y + (C \circ T)\partial_Z = \lambda(B\partial_X + C\partial_Y + A\partial_Z),$$

which implies that $A \circ T = \lambda B$, $B \circ T = \lambda C$, $C \circ T = \lambda A$, $\lambda^3 = 1$ and

$$v = A(X, Y, Z)\partial_X + \lambda^2 A(Y, Z, X)\partial_Y + \lambda A(Z, X, Y)\partial_Z$$

Suppose now that $\operatorname{Aut}(\mathcal{F})$ has a subgroup G of type (D). By Theorem 3.3, G is conjugate to a group of type (D1) or (D2). Then, up to a change of homogeneous coordinates, G contains T and

$$R(X : Y : Z) = (Y : X : Z)$$

It follows that $R_*v = \alpha v$ and $(T \circ R)_*v = \beta v$, $\alpha^2 = \beta^2 = 1$, since these automorphisms have order two. However,

$$\beta v = (T \circ R)_*v = T_*(R_*v) = \lambda^2 \alpha v,$$

hence $\lambda = 1$ and $\alpha = \beta$. In particular,

$$v = A(X, Y, Z)\partial_X + A(Y, Z, X)\partial_Y + A(Z, X, Y)\partial_Z$$

Calculating the pushforward by R, we have

$$R_*v = A(X, Z, Y)\partial_X + A(Y, X, Z)\partial_Y + A(Z, Y, X)\partial_Z$$

then $A(X, Z, Y) = \alpha A(X, Y, Z)$, that is, $A \circ T \circ R = \pm A$. □

Any automorphism of a foliation leaves invariant its singular set. Then it is useful to know how the action on a finite set can be. We first recall the orbit-decomposition formula that will play an important role. For a more complete discussion, see [18, page 25].

Let G be a finite group acting on a finite set $X = \{x_1, \dots x_r\}$. We can define an equivalence relation in X by $x_i \sim x_j$ if these two points lie in the same G-orbit, that is, $x_i = g(x_j)$ for some $g \in G$. Hence, X splits into disjoint orbits. Suppose that there are s disjoint orbits generated by $x_1, \dots, x_s$. Then

$$r = \#X = \sum \#\mathrm{Orb}(x_i) = \sum \frac{|G|}{|H_i|}$$

where H_i is the stabilizer of x_i in G.

A group of automorphisms that preserves a foliation $\mathcal{F}$ acts on the singular set $\mathrm{Sing}(\mathcal{F})$ and, in particular, preserves the analytical invariants of the singularities. Concerning Milnor numbers, we have the following:

Proposition 3.5. *Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ of degree d and let $G < Aut(\mathcal{F})$ be a finite subgroup. Then*

$$d^2 + d + 1 = \sum_{i=1}^{s} \frac{|G|}{|H_i|} \mu(\mathcal{F}, x_i)$$

where $x_1, \dots, x_s \in Sing(\mathcal{F})$ lie on disjoint orbits and H_i is the stabilizer of x_i in the group G.

Proof. Let e be the number of singularities without multiplicity, then

$$e = \sum_{i=1}^{s} \#\mathrm{Orb}(x_i) = \sum_{i=1}^{s} \frac{|G|}{|H_i|}$$

where the $x_1, \dots, x_s$ generate the disjoint orbits and H_i is the stabilizer of x_i. Let v be a germ of vector field that defines $\mathcal{F}$ at a singularity p and let $g \in G$, then $g_* v$ defines $\mathcal{F}$ at $g(p)$ and

$$\mu(\mathcal{F}, p) = \mu(\mathcal{F}, g(p)),$$

since the Milnor number is invariant by biholomorphisms. This implies that all singularities in the same orbit have the same Milnor number, hence

$$d^2 + d + 1 = \sum_{p \in \mathrm{Sing}(\mathcal{F})} \mu(\mathcal{F}, p) = \sum_{i=1}^{s} \#\mathrm{Orb}(x_i) \mu(\mathcal{F}, x_i) = \sum_{i=1}^{s} \frac{|G|}{|H_i|} \mu(\mathcal{F}, x_i)$$

□

Corollary 3.6. *Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ of degree d and let $G < Aut(\mathcal{F})$ be a finite p–group, for some prime p. Then*

$$d^2 + d + 1 \equiv \sum \mu(\mathcal{F}, x) \pmod{p}$$

where the sum is taken over the singularities fixed by G. In particular, if G does not fix any singularity, then p divides d^2+d+1.

Proof. By hypothesis, G has order p^n for some n and all its subgroups have a power of p as its order. Then

$$d^2+d+1=\sum_{i=1}^{s} p^{n_i}\mu(\mathcal{F},x_i)=p\sum_{i=1}^{u} p^{n_i-1}\mu(\mathcal{F},x_i)+\sum_{i=u+1}^{s}\mu(\mathcal{F},x_i)$$

where p^{n_i} is the index of H_i in G and $H_j=G$ for $j=u+1,\ldots,s$. If G does not fix any singularity,

$$d^2+d+1=\sum_{i=1}^{s} p^{n_i}\mu(\mathcal{F},x_i)=p\sum_{i=1}^{u} p^{n_i-1}\mu(\mathcal{F},x_i)$$

hence p divides d^2+d+1. □

3.3 Bounds for Foliations on $\mathbb{P}^2$

Every element $\varphi\in \mathrm{PGL}(3,\mathbb{C})$ of finite order is diagonalizable and either has three isolated fixed points or fixes a line pointwise and fixes a point outside that line. An element in this last case is called a pseudo-reflection, and that occurs when φ has only two distinct eigenvalues. We begin our analysis by these simple automorphisms.

Proposition 3.7. *Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{P}^2$ invariant by an automorphism of the form $\varphi(X:Y:Z)=(X:Y:lZ)$, $l^n=1$. Then, $n\le d+1$ and $(0:0:1)\notin Sing(\mathcal{F})$ if and only if*

1. *$n|d$ and $\{Z=0\}$ is $\mathcal{F}$–invariant, or*
2. *$n|d+1$ and $\{Z=0\}$ is not $\mathcal{F}$–invariant.*

Proof. Let $v=A\partial_X+B\partial_Y+C\partial_Z$ be a vector field inducing $\mathcal{F}$ such that $\mathrm{div}(v)=0$. Then, for some $\lambda\in\mathbb{C}^*$,

$$(A\circ\varphi)\partial_X+(B\circ\varphi)\partial_Y+(C\circ\varphi)\partial_Z=\lambda(A\partial_X+B\partial_Y+lC\partial_Z)$$

and this implies that $A\circ\varphi=\lambda A$, $B\circ\varphi=\lambda B$, $C\circ\varphi=\lambda lC$ and $\lambda=l^r$ for some $r>0$. We may expand these polynomials in terms of Z:

$$A(X,Y,Z)=\sum_k a_{d-k}(X,Y)Z^k$$
$$B(X,Y,Z)=\sum_k b_{d-k}(X,Y)Z^k$$
$$C(X,Y,Z)=\sum_k c_{d-k}(X,Y)Z^k,$$

where a_{d-k}, b_{d-k} and $c_{d-k}$are homogeneous of degree $d-k$. Therefore, the relation above implies that a_{d-k} and b_{d-k} are trivial unless $k \equiv r \pmod{n}$, and c_{d-k} is trivial unless $k \equiv r+1 \pmod{n}$.

By definition, $\mathcal{F}$ has isolated singularities, hence A, B and C do not share a nontrivial common factor. In particular, Z may not divide them all. Suppose that Z does not divide C, hence $\{Z=0\}$ is not $\mathcal{F}$–invariant, that is, $c_d \neq 0$. This implies that $0 \equiv r+1 \pmod{n}$, hence $a_{d-k} = b_{d-k} = 0$ if $k \not\equiv -1 \pmod{n}$. Therefore Z^{n-1} divides both A and B and, moreover, $n-1 \leq d$. In this case

$$v = Z^{n-1}\widetilde{A}(X,Y,Z^n)\partial_X + Z^{n-1}\widetilde{B}(X,Y,Z^n)\partial_Y + \widetilde{C}(X,Y,Z^n)\partial_Z$$

If $\{Z=0\}$ is $\mathcal{F}$–invariant, Z divides C, then Z does not divide A or B. We have that either $a_d \neq 0$ or $b_d \neq 0$. In both cases $0 \equiv r \pmod{n}$ hence A and B are φ–invariant, that is, polynomials in X, Y and Z^n. In particular, $n \leq d$. In this case

$$v = \widetilde{A}(X,Y,Z^n)\partial_X + \widetilde{B}(X,Y,Z^n)\partial_Y + Z\widetilde{C}(X,Y,Z^n)\partial_Z$$

In the chart $\{Z \neq 0\}$, let $(x,y) = (X/Z, Y/Z)$ be the coordinates. Then $(0:0:1)$ corresponds to the origin, v is written as

$$v(x,y) = [A(x,y,1) - xC(x,y,1)]\partial_x + [B(x,y,1) - yC(x,y,1)]\partial_y$$

and v has a singularity at the origin if and only if $A(0,0,1) = B(0,0,1) = 0$. Therefore, $(0:0:1) \notin \mathrm{Sing}(\mathcal{F})$ if and only if either $a_0 \neq 0$ or $b_0 \neq 0$. In both cases we have:

1. $d = k \equiv r \equiv -1 \pmod{n}$, if $\{Z=0\}$ is not $\mathcal{F}$–invariant, or
2. $d = k \equiv r \equiv 0 \pmod{n}$, if $\{Z=0\}$ is $\mathcal{F}$–invariant.

□

Remark 3.8. We recall that the quotient of $\mathbb{P}^2$ by the action of the cyclic group generated by a pseudo-reflection $\varphi(X:Y:Z) = (X:Y:lZ)$, $l^n = 1$, is the weighted projective plane $\mathbb{P}(1,1,n)$ and the quotient projection gives a bijection between foliations in $\mathbb{P}(1,1,n)$ and foliations on $\mathbb{P}^2$ that are invariant by φ. This fact has been pointed out by Lizarbe in [21].

The projection is totally ramified on the line $\{Z=0\}$ and this relates the degree of $\mathcal{F}$ to the degree of its projection on the weighted projective plane. In particular the above proposition can recover a result proved by Rodríguez in [33] on whether the singularity of $\mathbb{P}(1,1,n)$ is a singularity for a foliation of degree d.

However, it is not our aim to deal with foliations on weighted projective planes. We leave this for the reader to check.

Our next step is to analyze the automorphisms that fix only regular points. Proposition 2.4 gives a glimpse on what we might find. Furthermore, the global situation imposes more restrictions.

Theorem 3.9. *Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ of degree d. Let $G < Aut(\mathcal{F})$ be a finite subgroup that has fixed points but does not fix any singularity of $\mathcal{F}$. Then, G is cyclic generated by*

$$\varphi(X : Y : Z) = (X : lY : l^{k+1}Z)$$

where $k = tang(\mathcal{F}, L_p, p) > 0$ and L_p is the line tangent to $\mathcal{F}$ at a fixed point p. Moreover, either

- *$|G|$ divides $d^2 + d + 1$ if all singularities have trivial stabilizers in G; or*
- *$|G|$ divides $d(d + 1)$ if a singularity has nontrivial stabilizer.*

Proof. Let $p \notin \mathrm{Sing}(\mathcal{F})$ be a fixed point. Then, G is abelian by Proposition 2.4. We can choose coordinates such that G is formed by diagonal automorphisms, G fixes $p = (1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$, and $\mathcal{F}$ is tangent to $\{Z = 0\}$ at p.

We can suppose, also, that $\{Z = 0\}$ is not $\mathcal{F}$–invariant. In fact, $\mathcal{F}$ is regular at these three points, hence if $\{Z = 0\}$ is invariant, neither $\{X = 0\}$, nor $\{Y = 0\}$ can be $\mathcal{F}$–invariant and we may choose p to be $(0 : 0 : 1)$ and permute the coordinates.

Therefore, Proposition 2.5 implies that G is cyclic generated in these coordinates by

$$\varphi(X : Y : Z) = (X : lY : l^{k+1}Z)$$

where $l^n = 1$ and $k = \mathrm{tang}(\mathcal{F}, \{Z = 0\}, p) > 0$.

We can decompose $\mathrm{Sing}(\mathcal{F})$ into disjoint G–orbits. By Lemma 3.5,

$$d^2 + d + 1 = \sum \frac{|G|}{|H_i|}\mu(\mathcal{F}, t_i)$$

where $t_i \in \mathrm{Sing}(\mathcal{F})$ lie on disjoint orbits and H_i is the stabilizer of t_i. If all singularities have trivial stabilizers, then $n = |G|$ divides $d^2 + d + 1$. This occurs, in particular, when φ is a pseudo-reflection: either $k \equiv 0 \pmod{n}$ or we have that $k \equiv -1 \pmod{n}$. However, this is not possible. By Proposition 3.7, n would divide d or $d + 1$ and

$$\gcd(d, d^2 + d + 1) = \gcd(d + 1, d^2 + d + 1) = 1$$

hence φ would be the identity.

Suppose now that there is a singularity with nontrivial stabilizer, say t_1. Any element of H_1 fixes more than three points, hence H_1 is cyclic generated by a pseudo-reflection $\psi = \varphi^a$ for some $a > 0$. We have two distinct cases:

1. $u = \gcd(k + 1, n) > 1$, $n = ua$ and

$$\psi(X : Y : Z) = (X : l^aY : Z).$$

2. $u = \gcd(k, n) > 1$, $n = ua$ and

$$\psi(X : Y : Z) = (l^{-a}X : Y : Z).$$

In particular, $|H_1| = u > 1$ and $a > 0$ since H_1 is a nontrivial subgroup of G.

For the first case, the line $\{Y = 0\}$ is not $\mathcal{F}-$ invariant since $\mathcal{F}$ is tangent to $\{Z = 0\}$ at p. By Proposition 3.7, u divides $d+1$ and $\mathcal{F}$ is given by a vector field of the form

$$v = Y^{u-1}A(X, Y^u, Z)\partial_X + B(X, Y^u, Z)\partial_Y + Y^{u-1}C(X, Y^u, Z)\partial_Z$$

It is straightforward to verify that $\mathcal{F}$ is tangent to $\{X = 0\}$ at $(0 : 0 : 1)$. Hence there are d points (counted with multiplicities) where $\mathcal{F}$ is tangent to $\{Y = 0\}$ and none is fixed by G.

Therefore, $K = G/H_1$ has order a and acts on $\{Y = 0\}$ permuting the points of tangency and the stabilizer of any point is trivial. By the same argument as in Lemma 3.5,

$$d = a \sum_t \operatorname{tang}(\mathcal{F}, \{Y = 0\}, t)$$

hence

$$n = au \mid d(d+1)$$

In the second case, either $\{X = 0\}$ is $\mathcal{F}$–invariant and u divides d or $\{X = 0\}$ is not $\mathcal{F}$–invariant and u divides $d+1$. If $\{X = 0\}$ is $\mathcal{F}$–invariant, $K = G/H_1$ permutes the singularities that lie on $\{X = 0\}$ and acts with trivial stabilizers. Then, as above we have that

$$d + 1 = a \sum_t \mathrm{Z}(\mathcal{F}, \{X = 0\}, t).$$

hence

$$n = ua \mid d(d+1)$$

If $\{X = 0\}$ is not $\mathcal{F}$–invariant, $\mathcal{F}$ is given by a vector field of the form

$$v = A(X^b, Y, Z)\partial_X + X^{b-1}B(X^b, Y, Z)\partial_Y + X^{b-1}C(X^b, Y, Z)\partial_Z$$

hence $\mathcal{F}$ is tangent to $\{Y = 0\}$ at $(0 : 0 : 1)$ and to $\{Z = 0\}$ at $(0 : 1 : 0)$. As above, K acts on $\{X = 0\}$ permuting the points of tangency and the stabilizer of any point is trivial. Then

$$n = ua \mid d(d+1).$$

□

Now, we turn our attention to the groups that fix some singularity. We restrict ourselves to the reduced ones.

Proposition 3.10. *Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ of degree $d \geq 2$. Let $G < Aut(\mathcal{F})$ be a finite subgroup that fixes a reduced singularity p with eigenvalue λ, such that p does not have a straight line as a separatrix. Then*

$$|G| \leq 2(d^2 - 1).$$

Moreover,

$$|G| \leq (d^2 - 1)$$

if G is abelian, which is always the case when $\lambda \neq -1$.

Proof. By Theorem 3.2, G is of type (A) or (B). We may choose coordinates such that G fixes $p = (0 : 0 : 1)$ then, by Proposition 2.7, the case (B) can only occur when $\lambda = -1$. In general, G has a diagonal subgroup H of index at most two. Hence, we may suppose that G is abelian diagonal and later double the bound.

By hypothesis, the lines $\{X = 0\}$ and $\{Y = 0\}$ are not $\mathcal{F}$–invariant, hence Proposition 2.8 implies that G is cyclic generated either by

$$\varphi(X : Y : Z) = (l^k X : lY : Z)$$

where $l^n = 1$ and $k = \operatorname{tang}(\mathcal{F}, \{X = 0\}, p)$ or by

$$\psi(X : Y : Z) = (\zeta X : \zeta^q Y : Z)$$

where $\zeta^n = 1$ and $q = \operatorname{tang}(\mathcal{F}, \{Y = 0\}, p)$. Therefore, $\psi = \varphi^r$ for some r, $\gcd(r, n) = 1$ and this implies that $\zeta = l^r$ and $kq \equiv 1 \pmod{n}$. Hence,

$$n \,|\, kq - 1 \leq d^2 - 1$$

if $k \not\equiv 1 \pmod{n}$. Otherwise, if $k \equiv 1 \pmod{n}$, φ is a pseudo-reflection. Then, by Proposition 3.7,

$$n \leq d + 1$$

□

The groups that do not fix any point contain the automorphism T described in Theorem 3.2. For the simplest case, type (C1), we have the following:

Proposition 3.11. *Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ of degree d. Suppose that for some choice of coordinates, $Aut(\mathcal{F})$ contains T and a diagonal pseudo-reflection φ of order n, that is, $Aut(\mathcal{F})$ contains a subgroup of type (C1). Then*

- *$n \,|\, d - 1$ if the curve $\{XYZ = 0\}$ is $\mathcal{F}$–invariant, or*
- *$n \,|\, d + 2$ if the curve $\{XYZ = 0\}$ is not $\mathcal{F}$–invariant.*

Moreover, $\mathcal{F}$ always has singularities at the points $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $(1 : 0 : 0)$. In the second case these singularities are reduced only if $n = 2$.

Proof. By Lemma 3.4, we have that $\mathcal{F}$ is generated by a homogeneous vector field

$$v = A(X, Y, Z)\partial_X + \lambda^2 A(Y, Z, X)\partial_Y + \lambda A(Z, X, Y)\partial_Z$$

where $\lambda^3 = 1$, since $T \in \operatorname{Aut}(\mathcal{F})$. If we write $\varphi(X : Y : Z) = (X : Y : \zeta Z)$, $\zeta^n = 1$, then

$$\begin{aligned} T \circ \varphi \circ T^2(X : Y : Z) &= (X : \zeta Y : Z) \text{ and} \\ T^2 \circ \varphi \circ T(X : Y : Z) &= (\zeta X : Y : Z) \end{aligned}$$

also belong to $\mathrm{Aut}(\mathcal{F})$.

One of the lines $\{X = 0\}$, $\{Y = 0\}$ or $\{Z = 0\}$ is $\mathcal{F}$–invariant if and only if the others are invariant, since they are permuted by T. Suppose that they are invariant. By the proof of Proposition 3.7,

$$v = X\widetilde{A}(X^n, Y^n, Z^n)\partial_X + \lambda^2 Y\widetilde{A}(Y^n, Z^n, X^n)\partial_Y + \lambda Z\widetilde{A}(Z^n, X^n, Y^n)\partial_Z.$$

In particular, $\widetilde{A}(X, Y, Z)$ is a homogeneous polynomial of degree k, that satisfies $d = nk + 1$. Hence n divides $d - 1$. Suppose that the curve $\{XYZ = 0\}$ is not $\mathcal{F}$–invariant. By the proof of Proposition 3.7, $A(X, Y, Z) = Y^{n-1}Z^{n-1}\widetilde{A}(X^n, Y^n, Z^n)$ and

$$\begin{aligned} v =& Y^{n-1}Z^{n-1}\widetilde{A}(X^n, Y^n, Z^n)\partial_X + \lambda^2 X^{n-1}Z^{n-1}\widetilde{A}(Y^n, Z^n, X^n)\partial_Y \\ &+ \lambda X^{n-1}Y^{n-1}\widetilde{A}(Z^n, X^n, Y^n)\partial_Z. \end{aligned}$$

In particular, $\widetilde{A}(X, Y, Z)$ is a homogeneous polynomial of degree k, that satisfies $d + 2 = n(k + 2)$.

On the affine chart $\{Z \neq 0\}$ with the standard coordinates $(x, y) = (X/Z, Y/Z)$, v is written as

$$\begin{aligned} v =& y^{n-1}\left(\widetilde{A}(x^n, y^n, 1) - x^n\lambda^2\widetilde{A}(1, x^n, y^n)\right)\partial_x \\ &+ \lambda x^{n-1}\left(\lambda\widetilde{A}(y^n, 1, x^n) - y^n\widetilde{A}(1, x^n, y^n)\right)\partial_y \end{aligned}$$

The singularity at $(0 : 0 : 1)$ is reduced only if $n = 2$ and

$$\widetilde{A}(0, 1, 0)\widetilde{A}(0, 0, 1) \neq 0.$$

$\square$

To conclude this chapter we state a theorem combining the results that we have proved so far.

Theorem 3.12. *Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ of degree $d \geq 2$ such that Aut($\mathcal{F}$) is finite and imprimitive, that is, Aut($\mathcal{F}$) leaves invariant the union of three lines, L_1, L_2 and L_3, in general position (meeting in three distinct points). If these lines are not $\mathcal{F}$–invariant and support at most reduced singularities, then*

$$|Aut(\mathcal{F})| \leq 3(d^2 + d + 1)$$

except for $d = 2$ and $\mathcal{F}$ given by

$$v = YZ\partial_X + XZ\partial_Y + XY\partial_Z$$

Proof. Up to a linear change of coordinates, we may suppose that L_1, L_2 and L_3 are $\{X=0\}$, $\{Y=0\}$ and $\{Z=0\}$ and $\mathrm{Aut}(\mathcal{F})$ is in one of the forms in Theorem 3.2 from (A) to (D).

First case: $\mathrm{Aut}(\mathcal{F})$ is of type (A). The points $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$ are regular or at least one of them is a singularity. If they are regular, by Theorem 3.9,

$$|\mathrm{Aut}(\mathcal{F})| \leq d^2+d+1$$

If at least one of these points belongs to $\mathrm{Sing}(\mathcal{F})$ and the lines that contain this point are not $\mathcal{F}$–invariant, by Proposition 3.10,

$$|\mathrm{Aut}(\mathcal{F})| \leq d^2-1$$

Second case: $\mathrm{Aut}(\mathcal{F})$ is of type (B). The group $\mathrm{Aut}(\mathcal{F})$ has a subgroup of type (A) whose index is two and has a singularity in one of the three points. Hence we double the previous bound:

$$|\mathrm{Aut}(\mathcal{F})| \leq 2(d^2-1)$$

Third case: $\mathrm{Aut}(\mathcal{F})$ is of type (C). By Theorem 3.3, we fall in two cases: (C1) or (C2). For groups of type (C1), Proposition 3.11 implies that

$$\mathrm{Aut}(\mathcal{F}) = (\mathbb{Z}/n\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$$

with $n=2$, since $\{XYZ=0\}$ is not $\mathcal{F}$–invariant and the singularities are reduced. Hence $|\mathrm{Aut}(\mathcal{F})| = 12$. For groups of type (C2), we have that

$$\mathrm{Aut}(\mathcal{F}) \simeq (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/3\mathbb{Z})$$

generated by

$$\begin{aligned}\varphi(X:Y:Z) &= (l^k X : Y : Z),\\ \psi(X:Y:Z) &= (l^s X : lY : Z),\\ T(X:Y:Z) &= (Y:Z:X)\end{aligned}$$

where $l^n=1$, $k>1$, $n=mk$ and $s^2-s+1 \equiv 0 \pmod{k}$. If $k<n$, by the proof of Proposition 3.11, $n=2k$ and $\mathcal{F}$ is given by a vector field of the form

$$\begin{aligned}v =\, & YZA(X^2,Y^2,Z^2)\partial_X + \lambda^2 XZA(Y^2,Z^2,X^2)\partial_Y\\ & + \lambda XYA(Z^2,X^2,Y^2)\partial_Z\end{aligned}$$

Moreover $A(0,1,0)A(0,0,1) \neq 0$ and this implies that either $A(X^2,Y^2,Z^2)$ is constant, hence $d=2$, or $A(X^2,Y^2,Z^2)$ has monomials gY^{d-2} and hZ^{d-2} for some $g,h \neq 0$, if $d>2$. If A is constant, $\mathcal{F}$ is given by

$$v = YZ\partial_X + \lambda^2 XZ\partial_Y + \lambda XY\partial_Z$$

which has automorphism group of type (C1) (or (D1) if $\lambda = 1$). In fact, by taking the pushforward of v by φ^{-1} we see that

$$1 - s \equiv s - 1 \equiv s + 1 \pmod{n}$$

hence we would have $n = 2$ and $k = 1$, which is an absurd. If A is not constant, the vector field

$$w = (gY^{d-1}Z + hYZ^{d-1})\partial_X + \lambda^2(gXZ^{d-1} + hX^{d-1}Z)\partial_Y + \lambda(gX^{d-1}Y + hXY^{d-1})\partial_Z$$

is also invariant by $\mathrm{Aut}(\mathcal{F})$. Taking the pushforward of w by φ^{-1} we see that n divides $d - 2$. Therefore,

$$|\mathrm{Aut}(\mathcal{F})| = \frac{3n^2}{k} = 6n \leq 6(d-2).$$

Now suppose that $k = n$. We have that

$$\mathrm{Aut}(\mathcal{F}) \simeq (\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/3\mathbb{Z})$$

and is generated by

$$\begin{aligned} \psi(X : Y : Z) &= (l^s X : lY : Z), \\ T(X : Y : Z) &= (Y : Z : X) \end{aligned}$$

where $l^n = 1$ and $s^2 - s + 1 \equiv 0 \pmod{n}$. Hence, $\mathrm{Aut}(\mathcal{F})$ has a (normal) cyclic subgroup of index 3. By the bounds for the type (A) we have that

$$|\mathrm{Aut}(\mathcal{F})| \leq 3(d^2 + d + 1).$$

Fourth case: $\mathrm{Aut}(\mathcal{F})$ is of type (D). Again we fall in two cases described by Theorem 3.3, (D1) and (D2). The case (D1) is similar to (C1). It follows that

$$|\mathrm{Aut}(\mathcal{F})| = 24.$$

If $\mathrm{Aut}(\mathcal{F})$ is of type (D2), then it has a subgroup of type (C2) where $k = 3$ and $s = 2$. We have seen above that either that $n = k$ or $n = 2k$. Therefore

$$|\mathrm{Aut}(\mathcal{F})| = \frac{6n^2}{k} = 18 \,\text{or}\, 72.$$

Comparing these bounds we have that under the hypotheses,

$$|\mathrm{Aut}(\mathcal{F})| \leq 3(d^2 + d + 1)$$

except for the foliation defined by

$$v = YZ\partial_X + XZ\partial_Y + XY\partial_Z$$

which has automorphism group of type (D1) and order 24.

To see this we only have to note that, in the case (D2) for the group of order 72, we have that $d > 2$ and $d - 2$ is divisible by $n = 6$. Hence $d \geq 8$ and $72 < 3(d^2 + d + 1)$.

□

We will see that this bound is sharp. It turns out that it is achieved by the family of examples given by Jouanoulou in [15].

Example 3.13. Let $\mathcal{J}_d$ be the Jouanolou's example in $\mathbb{P}^2$, of degree $d \geq 2$, which is given by

$$v = Z^d \partial_X + X^d \partial_Y + Y^d \partial_Z.$$

The automorphism group of $\mathcal{J}_d$ is

$$\mathrm{Aut}(\mathcal{J}_d) \simeq \mathbb{Z}/(d^2 + d + 1)\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$$

generated by T and

$$\varphi(X : Y : Z) = (lX : l^d Y : l^{d^2} Z),$$

$l^{d^2+d+1} = 1$, with l primitive. It is a group of type (C2) with $n = k = d^2 + d + 1$ and $s = -d$.

Chapter 4

Bounds for Foliations on Ruled Surfaces

In this chapter we will analyze foliations on geometrically ruled surfaces over irrational projective curves. First we recall some general properties of these surfaces, the proofs can be found in [12], for example. We start by recalling some properties of $\mathbb{P}^1$–bundles and of foliations on such surfaces.

4.1 Foliations on $\mathbb{P}^1$–bundles

A geometrically ruled surface is a surface X together with a surjective holomorphic map $\pi : X \longrightarrow C$ onto a smooth compact curve, C, that admits a section and, for every $y \in C$, the fiber $\pi^{-1}(y)$ is isomorphic to $\mathbb{P}^1$.

For every geometrically ruled surface $\pi : X \longrightarrow C$, there exists a rank two vector bundle E over C such that $X \simeq \mathbb{P}(E)$. E is not uniquely determined but, if E' is another vector bundle over C such that $X \simeq \mathbb{P}(E')$, then $E \simeq E' \otimes L$ for some line bundle L over C.

Let $C_0 \subset X$ be (the image of) a section. Then

$$\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot C_0 \oplus \pi^* \mathrm{Pic}(C)$$

and

$$\mathrm{Num}(X) \simeq \mathbb{Z} \cdot C_0 \oplus \mathbb{Z} \cdot f$$

where f is the class of a fiber. The self-intersection numbers of the sections are bounded from below and its minimum defines an invariant e of X. Precisely,

$$e = -\min\{D^2;\ D \text{ is (the image of) a section of } \pi\}.$$

A section D is called minimal if $D^2 = -e$. We will assume that the generator C_0 of $\mathrm{Num}(X)$ is the class of a minimal section.

Equivalently, we can define e by the line sub-bundles of E. In fact, there is a bijection between line subbundles of E and sections of π, see [25, Lemma 1.14]. The degree of these subbundles are bounded from above and the maximum is exactly the number e.

Definition 4.1. If L is a divisor on X, the class of L is also denoted by L and is expressed by $L \equiv aC_0 + bf$, where $\equiv$ stands for numerical equivalence. We call (a, b) the bidegree of L. A foliation $\mathcal{F}$ on X has bidegree (a, b) if $K_{\mathcal{F}} \equiv aC_0 + bf$.

By applying adjunction formula it follows that the canonical class of X is $K_X \equiv -2C_0 + (2g - 2 - e)f$. The tangent bundle of X has a sub-line bundle τ defined by the kernel of the jacobian of π,

$$0 \longrightarrow \tau \longrightarrow T_X \longrightarrow \pi^* T_C = N \longrightarrow 0,$$

where N is the normal bundle of the ruling. Therefore, $K_X \simeq \tau^* \otimes N^*$. From $\pi^* T_C = N$, we know that $N \equiv (2 - 2g)f$, hence $\tau \equiv 2C_0 + ef$. Next we present an useful result concerning the numerical properties of divisors on rules surfaces. For the proof, see [12, V, Propositions 2.20 and 2.21]. Recall that, on a surface, a Cartier divisor D is nef if $D \cdot Y \geq 0$ for every curve Y on X.

Proposition 4.2. *Let X be a ruled surface over a curve C with invariant e and let $Y \equiv aC_0 + bf$ be an irreducible curve different from C_0 or a fiber. Then*

a) if $e \geq 0$, then $a > 0$ and $b \geq ae$;

b) if $e < 0$, then either $a = 1$, $b \geq 0$ or $a \geq 2$, $2b \geq ae$.

A divisor $D \equiv aC_0 + bf$ is nef if and only if

a) $e \geq 0$, $a \geq 0$ and $b \geq ae$;

b) $e < 0$, $a \geq 0$ and $2b \geq ae$.

D is ample when the inequalities are strict.

If $e > 0$ and D is an effective divisor such that C_0 is not in its support, then D is nef. In general, $D \equiv nC_0 + E$ for some E nef divisor and $n \geq 0$. If $e \leq 0$ then every effective divisor is nef.

Now we will follow [9] to recall some properties of foliations on geometrically ruled surfaces. However, our notation is different and we have refined some results. Let $\mathcal{F}$ be a foliation of bidegree (a, b) on $X \longrightarrow C$ with invariant e and $g(C) = g$. Then,

1. $\#\mathrm{Sing}(\mathcal{F}) = c_2(T_X \otimes K_{\mathcal{F}}) = (a + 1)(2b - ae + 2 - 2g) + 2 - 2g$;
2. $N_{\mathcal{F}}^2 = (a + 2)(2b - ae + 4 - 4g)$;
3. $\mathrm{tang}(\mathcal{F}, F) = K_{\mathcal{F}} \cdot f + f^2 = a$ for F a fiber not $\mathcal{F}$-invariant;
4. $\mathrm{Z}(\mathcal{F}, F) = \chi(F) + K_{\mathcal{F}} \cdot f = 2 + a$ and $\mathrm{CS}(\mathcal{F}, F) = f^2 = 0$ for F an $\mathcal{F}$-invariant fiber.

Since $c_2(T_X \otimes K_{\mathcal{F}})$ is the number of singularities of $\mathcal{F}$, see Proposition 1.3, then it cannot be negative. Hence, $a \geq 0$ and $2b - ae \geq 2g - 2$, if $g \geq 1$. The later equality only holds if $g = 1$ and $2b = ae$.

Suppose that $\mathcal{F}$ is not tangent to the fibers of the ruling. Let $\{U_i\}$ be a fine open cover of X such that there exists vector fields v_i generating $\mathcal{F}$ and 1-forms ω_i generating the ruling (regarded as a foliation). The holomorphic functions $g_i = \omega_i(v_i)$ define a nontrivial divisor on X. This is the tangency divisor between $\mathcal{F}$ and the ruling, denoted by $\operatorname{tang}(\mathcal{F}, \tau)$. By construction, we have that

$$\operatorname{tang}(\mathcal{F}, \tau) \sim K_F + N \equiv aC_0 + (b + 2 - 2g)f$$

and it is effective. In order to analyze $\operatorname{tang}(\mathcal{F}, \tau)$ we need a technical lemma.

Lemma 4.3. *Let $\mathcal{F}$ be a foliation on a smooth compact complex surface and $C_1, \ldots, C_k$ be disjoint smooth $\mathcal{F}$-invariant curves. If $D = C_1 + \ldots C_k$ as a divisor, then*

$$c_2(T_X(-\log D) \otimes K_{\mathcal{F}}) = \#\mathit{Sing}(\mathcal{F}) - \sum_{i=1}^{k} \mathrm{Z}(\mathcal{F}, C_i)$$

Proof. The curves C_i being smooth and disjoint implies that $T_X(-\log D)$ is locally free by Saito's criterion and fits into an exact sequence:

$$o \longrightarrow T_X(-\log D) \longrightarrow T_X \longrightarrow \mathcal{O}_D(D) \longrightarrow 0,$$

see [20] for details.

Taking the total Chern class we have that $c_1(T_X(-\log D)) = -K_X - D$ and $c_2(T_X(-\log D)) = c_2(X) + K_X \cdot D + D^2$. Hence, by direct calculation we can show that

$$\begin{aligned} c_2(T_X(-\log D) \otimes K_{\mathcal{F}}) &= c_2(T_X(-\log D)) + c_1(T_X(-\log D)) \cdot K_{\mathcal{F}} + K_{\mathcal{F}}^2 \\ &= c_2(X) + K_X \cdot D + D^2 + -(K_X + D) \cdot K_{\mathcal{F}} + K_{\mathcal{F}}^2 \\ &= c_2(T_X \otimes K_{\mathcal{F}}) - \chi(D) - K_{\mathcal{F}} \cdot D. \end{aligned}$$

The lemma follows by Propositions 1.3 and 1.5. □

In his thesis [24], Machado proved that, for a one dimensional foliation $\mathcal{F}$ on a compact complex manifold X of dimension n with isolated singularities and an $\mathcal{F}$–invariant smooth hypersurface S, the class $c_n(T_X(-\log S) - T_{\mathcal{F}})$ can be computed as the sum of the Milnor numbers of the singularities away from S, under mild hypotheses on the singularities that lie on S:

$$\int_X c_n(T_X(-\log S) - T_{\mathcal{F}}) = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cap \{X \setminus S\}} \mu(\mathcal{F}, p).$$

The restriction that is imposed is the vanishing of a logarithmic index $\mathrm{Ind}_{logS,p}$ that in our case is

$$\mathrm{Ind}_{logS,p} = \mu(\mathcal{F}, p) - \mathrm{Z}(\mathcal{F}, S, p).$$

For reduced singularities, direct calculation shows that the vanishing of this index holds for any separatrix of a nondegenerate singularity and for the weak separatrix of a saddle-node (when it converges). However, it fails for the strong separatrix.

We state a particular case of this result that will serve our purposes:

Lemma 4.4. *Let X be a compact complex surface and let $\mathcal{F}$ be foliation on X. If S is a smooth $\mathcal{F}$–invariant curve such that Sing$(\mathcal{F}) \subset S$ and $\mu(\mathcal{F}, p) = \mathrm{Z}(\mathcal{F}, S, p)$ for all $p \in$ Sing$(\mathcal{F})$, then*

$$c_2(T_X(-\log S) \otimes K_{\mathcal{F}}) = 0.$$

This result applied to the analysis of the invariant fibers for a foliation yields the following classification:

Theorem 4.5. *Let $X \longrightarrow C$ be a $\mathbb{P}^1$-bundle over a smooth curve whose genus is g and let e denote the invariant of X. Let $\mathcal{F}$ be a foliation on X of bidegree (a, b) such that $\mu(\mathcal{F}, p) = \mathrm{Z}(\mathcal{F}, F, p)$ for every p in an $\mathcal{F}$–invariant fiber F. Then one of the following is true:*

1. *$\mathcal{F}$ is tangent to the ruling, or*
2. *$a = 0$ and $\mathcal{F}$ is Riccati, or*
3. *$b = e = 0$, $a > 0$, $g = 1$ and $\mathcal{F}$ is, up to an unramified cover, a regular foliation on $C \times \mathbb{P}^1$, or*
4. *$\mathcal{F}$ is a foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ with $a > 0$, $b = 0$ and all singularities lie on the two $\mathcal{F}$-invariant fibers. In particular, $\mathcal{F}$ is Riccati with respect to the other projection. Or*
5. *$e > 0$, $b = (a + 1)e$ and all singularities lie on $\mathcal{F}$-invariant fibers, or*
6. *$a > 0$, and $\mathcal{F}$ has a singularity that lies on a fiber which is not $\mathcal{F}$-invariant.*

Proof. First we suppose that $\mathcal{F}$ is not tangent to the ruling, then a general fiber F is transverse to $\mathcal{F}$ and $a = \mathrm{tang}(\mathcal{F}, F) \geq 0$. Equality holds if and only if $\mathcal{F}$ is a Riccati foliation. These are our first two cases.

Now suppose that $a > 0$. The transversality between $\mathcal{F}$ and a general fiber implies that $\mathrm{tang}(\mathcal{F}, \tau)$ is a nontrivial effective divisor. A fiber F in the support of $\mathrm{tang}(\mathcal{F}, \tau)$ is $\mathcal{F}$-invariant. Let $F_1, \cdots, F_s$ be such fibers, and define

$$\begin{aligned} D &= F_1 + \cdots + F_s \equiv sf, \\ \Delta &= \mathrm{tang}(\mathcal{F}, \tau) - D \equiv aC_0 + (b + 2 - 2g - s)f. \end{aligned}$$

Both are effective divisors, by construction. It follows that Δ is trivial if and only if $\mathcal{F}$ is Riccati or tangent to the ruling.

Suppose every singularity of $\mathcal{F}$ lies on a invariant fiber. Then, by Lemma 4.4,

$$\begin{aligned} 0 &= c_2(T_X(-\log(D)) \otimes K_{\mathcal{F}}) \\ &= (a+1)(2b - ae + 2 - 2g) + 2 - 2g - s(a+2) \\ &= (a+2)(b + 2 - 2g - s) + a(b - ae - e). \end{aligned}$$

Hence, $(a+2)(b+2-2g-s) = -a(b-ae-e)$ and

$$\Delta = K_{\mathcal{F}} + N - D \equiv aC_0 + \frac{-a(b-ae-e)}{a+2} f.$$

If $e \leq 0$, every effective divisor is nef. It follows that $-2a(b-ae-e) \geq (a+2)ae$, and this implies that $0 \geq e \geq 2b - ae \geq 2g-2 \geq 0$, when $g \geq 1$. Hence, $2b = e = 0$, $g = 1$ and $\mathcal{F}$ is a regular foliation on an elliptic ruled surface $X \longrightarrow C$. By [9, Theorem 3.3], there exists an unramified cover $C \times \mathbb{P}^1 \longrightarrow X$ and $\mathcal{F}$ lifts to a foliation described in [9, Proposition 3.2]. That is case 3.

If $g = 0$, $e \geq 0$ always. In fact, a theorem due to Nagata says that $e \geq -g$, see [12, page 384] or [25]. Hence, $e = 0$ and $0 \geq 2b \geq -2$, which implies that $b = 0, -1$ and $X = \mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, $(a+2)$ divides ab, since Δ is an effective integral divisor, then $b = 0$ and, consequently, $s = 2$. In particular, $\mathcal{F}$ is Riccati with respect to the other projection. This gives us case 4.

If $e > 0$ and C_0 is not contained in the support of Δ, again Δ is nef and this implies that $-a(b-ae-e) \geq (a+2)ae$. Then

$$0 > -2e \geq 2b \geq ae > 0,$$

which is an absurd. Therefore, C_0 is in the support of Δ and this means that for a general point $p \in C_0$, $\mathcal{F}$ is regular at p and the leaf that passes through p is transverse to C_0. Hence

$$0 \leq \operatorname{tang}(\mathcal{F}, C_0) = K_{\mathcal{F}} \cdot C_0 + C_0^2 = b - ae - e.$$

On the other hand, $\Delta = mC_0 + E$, E a nef divisor. Then

$$E \equiv (a-m)C_0 + \frac{-a(b-ae-e)}{a+2} f$$

and we have that $(a-m) \geq 0$ and $-a(b-ae-e) \geq (a-m)(a+2)e$. Hence,

$$0 \geq -a(b-ae-e) \geq (a-m)(a+2)e \geq 0.$$

Consequently, $a = m$, $b = (a+1)e$, $\Delta = aC_0$ and there are $s = b+2-2g$ invariant fibers, which is case 5.

If $\mathcal{F}$ does not fit in any of the previous cases, it has a singularity on a generically transverse fiber. □

4.2 Finite Automorphism Groups of Ruled Surfaces

The automorphism groups of geometrically ruled surfaces have been classified by Maruyama in [26]. As one may expect, it is closely related to automorphisms of rank two vector bundles over a curve. In fact, for a ruled surface $\pi : X \longrightarrow C$, Maruyama proved that if either C is irrational or C is rational and $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathrm{Aut}(X)$ fits in an exact sequence

$$1 \longrightarrow \mathrm{Aut}_C(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(C)$$

where $\mathrm{Aut}_C(X)$ is the subgroup that sends each fiber of π onto itself. Alternatively, it is the automorphism group of X seen as a scheme over C. A lemma due to Grothendieck relates this subgroup to the automorphism group of a vector bundle E such that $X \simeq \mathbb{P}(E)$ by the following exact sequence:

$$1 \longrightarrow \mathrm{Aut}(E)/\mathrm{H}^0(C, \mathcal{O}_C^*) \longrightarrow \mathrm{Aut}_C(X) \longrightarrow \Delta \longrightarrow 1$$

where $\Delta = \{N \in \mathrm{Pic}(C)\,;\, E \simeq E \otimes N\}$ and, in our case, $\mathrm{H}^0(C, \mathcal{O}_C^*) = \mathbb{C}^*$ acts on $\mathrm{Aut}(E)$ by rescaling. See [11] for the proof. Although Maruyama works in a arbitrary algebraically closed field, we restrict ourselves to $\mathbb{C}$. For the proof of the next result, see [26, Theorem 2].

Theorem 4.6. *Let $\pi : X \longrightarrow C$ be a ruled surface with invariant e.*

1. *If $e < 0$, then $Aut_C(X) \simeq \Delta$.*
2. *If $e \geq 0$, X is indecomposable and if C_0 is the unique minimal section, then*

$$Aut_C(X) \simeq \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & t_r \\ 0 & 1 \end{pmatrix} \right) \mid t_i \in \mathbb{C} \right\}$$

   where $r = h^0(C, \mathcal{O}_C(-\pi(C_0^2))) = h^0(C, \det(E)^{-1} \otimes L_{C_0}^{\otimes 2})$, $X \simeq \mathbb{P}(E)$.
3. *If X is decomposable and if X does not carry two minimal sections, C_0 and C_1, such that $\pi(C_0^2) = \pi(C_1^2)$ (as divisor classes), then*

$$Aut_C(X) \simeq \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha & t_r \\ 0 & 1 \end{pmatrix} \right) \;\middle|\; \begin{array}{c} t_i \in \mathbb{C} \\ \alpha \in \mathbb{C}^* \end{array} \right\}$$

   where $r = h^0(C, \mathcal{O}_C(-\pi(C_0^2)))$.
4. *If X is decomposable, $X \not\simeq \mathbb{P}^1 \times C$ and if X has two distinct minimal sections, C_0 and C_1, such that $\pi(C_0^2) = \pi(C_1^2)$ (accordingly $e = 0$), then*

$$Aut_C(X) \simeq \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{C}^* \right\} \bigcup \left\{ \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \mid \beta \in \mathbb{C}^* \right\}$$

5. *If $X \simeq \mathbb{P}^1 \times C$, then*
$$Aut_C(X) \simeq PGL(2, \mathbb{C})$$

From this classification we can easily extract the finite subgroups in each case.

Corollary 4.7. *Let $\pi : X \longrightarrow C$ be a ruled surface with invariant e and let $G < Aut_C(X)$ be a finite subgroup.*

1. *If $e < 0$, then*
$$G \simeq (\mathbb{Z}/2\mathbb{Z})^r$$
for some $r \geq 0$.
2. *If $e \geq 0$ and X is indecomposable, then G is trivial.*
3. *If X is decomposable and if X does not carry two minimal sections, C_0 and C_1, such that $\pi(C_0^2) = \pi(C_1^2)$ (as divisor classes), then*
$$G \simeq \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha & t_r \\ 0 & 1 \end{pmatrix} \right) \;\middle|\; \begin{matrix} \alpha \in \mathbb{C}^* \\ \alpha^n = 1 \end{matrix} \right\}$$
is cyclic, where $r = h^0(C, \mathcal{O}_C(-\pi(C_0^2)))$ and $t_i \in \mathbb{C}$ can be any (fixed) numbers.
4. *If X is decomposable, $X \not\simeq \mathbb{P}^1 \times C$ and if X has two distinct minimal sections, C_0 and C_1 such that $\pi(C_0^2) = \pi(C_1^2)$ (accordingly $e = 0$), then*
$$G \simeq \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} \alpha \in \mathbb{C}^* \\ \alpha^n = 1 \end{matrix} \right\} \bigcup \left\{ \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \;\middle|\; \begin{matrix} \beta \in \mathbb{C}^* \\ \beta^m = 1 \end{matrix} \right\}$$
for some integers n and m.
5. *If $X \simeq \mathbb{P}^1 \times C$, then G is a finite subgroup of $PGL(2, \mathbb{C})$, namely:*
   - *cyclic groups;*
   - *dihedral groups;*
   - *the tetrahedral group isomorphic to the alternating group $\mathcal{A}_4$;*
   - *the octahedral group isomorphic to the symmetric group $\mathcal{S}_4$;*
   - *the icosahedral group isomorphic to the alternating group $\mathcal{A}_5$.*

Proof. The first case comes from the fact that Δ is a subgroup of the 2-torsion part of $\mathrm{Pic}(C)$: if $E \simeq E \otimes N$, then $\det(E) \simeq \det(E) \otimes N^{\otimes 2}$, hence $N^{\otimes 2}$ is trivial. For the second case, observe that any element of the form
$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$
has infinite order, unless $t = 0$. For the third case, the action of any such automorphism is given by the following steps.

Fix a basis $\{s_1, \dots s_r\}$ of $\mathrm{H}^0(C, \mathcal{O}_C(-\pi(C_0^2)))$ and let $t_1, \dots t_n \in \mathbb{C}$. Define $\gamma \in \mathrm{H}^0(C, \mathcal{O}_C(-\pi(C_0^2)))$ by
$$\gamma = \sum_{i=1}^{n} t_i s_i \, .$$

Fix homogeneous coordinates $(z : w)$ on the fiber over a point $x \in C$, then an element T of $\mathrm{Aut}_C(X)$ defined by the t_i's acts by

$$T_x(z, w) = (\alpha z + \gamma(x)w : w).$$

Then the only finite subgroups are cyclic, γ can be any section but α is a root of the unity.

The fourth case is straightforward, note that n or m can be zero. The fifth case is the well-known classification of finite subgroups of $\mathrm{PGL}(2, \mathbb{C})$, see [29] for example.

□

4.3 Bounds for Foliations on Ruled Surfaces

The classification finite subgroups of automorphisms and the properties of foliations on ruled surfaces presented in the previous sections are our main tools to provide the bounds. We will use Theorem 4.5 in order to split in two cases, if $\mathcal{F}$ has a singularity lying on a non-invariant fiber or not. Let us begin by the first case.

Theorem 4.8. *Let $X \longrightarrow C$ be a $\mathbb{P}^1$-bundle over a smooth curve whose genus is $g \geq 1$ and let e denote the invariant of X. Let $\mathcal{F}$ be a foliation on X of bidegree (a, b) such that*

1. *$Aut(\mathcal{F})$ is finite;*
2. *$\mathcal{F}$ has a singularity at $p \in X$ which lies on a fiber F not $\mathcal{F}$–invariant.*
3. *If $\mathcal{F}$ has a singularity at $q \in X$ that lies in a $\mathcal{F}$–invariant fiber F, then $\mu(\mathcal{F}, q) = \mathrm{Z}(\mathcal{F}, F, q)$.*

Then

- *either $|Aut(\mathcal{F}) \leq (8g + 4)[(a + 1)(2b - ae + 2 - 2g) + 2 - 2g]$ if $e < 0$*
- *or $|Aut(\mathcal{F})| \leq a(4g + 2)[(a + 1)(2b - ae + 2 - 2g) + 2 - 2g]$ if $e \geq 0$.*

Proof. Let G be the stabilizer of p in $\mathrm{Aut}(\mathcal{F})$. Then G has index at most

$$c_2(TX \otimes K_{\mathcal{F}}) = (a + 1)(2b - ae + 2 - 2g) + 2 - 2g.$$

The projection π induces a group homomorphism, hence an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

where $K < \mathrm{Aut}_C(X)$ and $H < \mathrm{Aut}(C)$. Since G fixes p, K also fixes p and H fixes $\pi(p)$. In particular, H is cyclic. We have that K is also cyclic. In fact, in a small neighborhood of p with coordinates (x, y) with $p = (0, 0)$ and the fibers are $\{x = c\}, c \in \mathbb{C}$, any element φ of K is given by

$$\varphi(x, y) = (x, ly),\ l \in \mathbb{C}^*$$

hence K injects into $\mathbb{C}^*$, which implies that it is cyclic.

Suppose that $e < 0$. Then, by Corollary 4.7, $K < \mathbb{Z}/2\mathbb{Z}$. By Theorem 0.1, $|H| \leq 4g + 2$. Therefore,

$$|G| = |H||G| \leq 8g + 4.$$

If $e \geq 0$, we also have that $|H| \leq 4g + 2$. The fiber F that contains p is not $\mathcal{F}$–invariant, then, by Proposition 2.8,

$$|K| \leq \operatorname{tang}(\mathcal{F}, F, p) \leq a.$$

In fact, if p is degenerated, the last part of the proof of Proposition 2.8 still works here since the generator of K has the form described above in some neighborhood of p. Therefore,

$$|G| = |H||G| \leq a(4g + 2).$$

□

When all singularities lie on invariant fibers, Theorem 4.5 imposes restrictions the foliations that may occur. We give them the following bound:

Theorem 4.9. *Let $X \longrightarrow C$ be a $\mathbb{P}^1$-bundle over a smooth curve whose genus is $g \geq 1$ and let e denote the invariant of X. Let $\mathcal{F}$ be a foliation on X of bidegree (a, b), $ab \neq 0$, such that*

1. *$Aut(\mathcal{F})$ is finite;*
2. *All singularities lie on $\mathcal{F}$–invariant fibers and satisfy $\mu(\mathcal{F}, q) = \mathrm{Z}(\mathcal{F}, F, q)$;*
3. *$\mathcal{F}$ has at least one singularity p with $\mathrm{Z}(\mathcal{F}, F, p) < a + 2$, where F is the fiber that contains p.*

Then

- *either $|Aut(\mathcal{F})| \leq 84(g - 1)(a + 2)$ if $g \geq 2$*
- *or $|Aut(\mathcal{F})| \leq 6(a + 2)(a + 1)e = 6c_2(TX \otimes K_{\mathcal{F}})$ if $g = 1$*

Proof. By Theorem 4.5, we have that $e > 0$, $b = (a + 1)e$ and the tangency divisor between $\mathcal{F}$ and the ruling is composed of $b + 2 - 2g$ fibers and the negative section C_0. Moreover, for any fiber F which is not $\mathcal{F}$–invariant, F and $\mathcal{F}$ have a single tangency point of multiplicity a that lies on the intersection with C_0. Since $\operatorname{tang}(\mathcal{F}, C_0) = b - (a + 1)e = 0$, C_0 does not support any singularity of $\mathcal{F}$.

As we have mentioned in the previous theorem, $\mathrm{Aut}(\mathcal{F})$ fits in an exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Aut}(\mathcal{F}) \longrightarrow H \longrightarrow 1$$

where $K < \mathrm{Aut}_C(X)$ and $H < \mathrm{Aut}(C)$. By Corollary 4.7, K is cyclic and fixes C_0 pointwise. For the fiber F through p, $F \cap C_0$ is a regular point of $\mathcal{F}$ fixed by K. Since $\mathrm{Z}(\mathcal{F}, F, p) < a + 2$, then the fiber F through p has other singularities. The group K must permute these singularities and we have that

$$|K| \leq a + 2.$$

It remains to bound the order of H which is a finite subgroup of $\mathrm{Aut}(C)$. If $g \geq 2$, then

$$|H| \leq 84(g-1)$$

by Hurwitz' theorem. If $g = 1$, then $H < \mathbb{Z}/n\mathbb{Z} \ltimes \mathbb{Z}/m\mathbb{Z}$ where $n \leq 6$, $\mathbb{Z}/n\mathbb{Z}$ has a fixed point and $\mathbb{Z}/m\mathbb{Z}$ is a group generated by translations. we have that H permute the images of the $b + 2 - 2g = b$ invariant fibers, then

$$|H| \leq 6b = 6(a+1)e.$$

Combining these bounds we have

1. $|\mathrm{Aut}(\mathcal{F})| \leq 84(g-1)(a+2)$ if $g \geq 2$ or
2. $|\mathrm{Aut}(\mathcal{F})| \leq 6(a+2)(a+1)e = 6c_2(TX \otimes K_{\mathcal{F}})$ if $g = 1$

□

Chapter 5

Bounds for Foliations on Non-ruled Surfaces

Along this chapter we will analyze the automorphism groups of foliations on surfaces that are not birationally ruled, that is, they have nonnegative Kodaira dimension. In particular, such surfaces do not support pencils of rational curves and this, among other features, will play a important role on our approach to bound the order of the groups we are interested in. We may begin with the simplest case: regular foliations. It turns out that they live naturally on general type surfaces.

5.1 Regular Foliations of General Type

Let X be a smooth projective surface and $\mathcal{F}$ be a foliation on X of general type and suppose that the minimal model $(Y, \mathcal{G})$ is regular. In this situation, the Baum-Bott formulae express the Chern numbers of Y in terms of the foliation.

Lemma 5.1. *Let $\mathcal{G}$ be a regular foliation of general type on a smooth compact surface Y, then*

$$c_1(Y)^2 > 2c_2(Y).$$

Proof. From the Baum-Bott formulae in Proposition 1.3, we have

$$\begin{aligned} c_2(Y) &= K_X \cdot K_{\mathcal{G}} - K_{\mathcal{G}}^2, \\ c_1(Y)^2 &= 2K_Y \cdot K_{\mathcal{G}} - K_{\mathcal{G}}^2. \end{aligned}$$

In particular, $c_1(Y)^2 - 2c_2(Y) = K_{\mathcal{G}}^2$. And we are reduced to prove that $K_{\mathcal{G}}^2 > 0$.

Since $\mathcal{G}$ is regular of general type, we have that $K_{\mathcal{G}}$ is big. Then, we can write $mK_{\mathcal{G}} = A + E$, for some $m \gg 0$, with A ample and $E = \sum a_i C_i$ effective (see [19, Corollary 2.2.6]). If $C_i^2 \geq 0$, we have:

$$mK_{\mathcal{G}} \cdot C_i = A \cdot C_i + E \cdot C_i > 0$$

If $C_i^2 < 0$, the Camacho-Sad formula says that C_i cannot be invariant (it would have zero self-intersection). Then,

$$K_{\mathcal{G}} \cdot C_i = -C_i^2 + \operatorname{tang}(\mathcal{G}, C_i) > 0.$$

Therefore,

$$m^2 K_{\mathcal{G}}^2 = mK_{\mathcal{G}}(A+E) = (A+E)A + m\sum a_i K_{\mathcal{G}} \cdot C_i > m\sum a_i K_{\mathcal{G}} \cdot C_i > 0$$

□

Theorem 5.2. *Let $\mathcal{F}$ be a foliation of general type on a smooth surface X. If $\mathcal{F}$ is birationally regular, then*

$$|Bim(\mathcal{F})| \leq (42K_Y)^2$$

where $(Y, \mathcal{G})$ is its (regular) minimal model.

Proof. Let $(Y, \mathcal{G})$ be the minimal model of $(X, \mathcal{F})$, which is regular by hypothesis. Brunella's classification of regular foliations on surfaces [3, Théorème 2] implies that all surfaces Y with Kodaira dimension at most one that support regular foliations satisfy $c_1(Y)^2 = 2c_2(Y)$. By Lemma 5.1, we conclude that Y is a surface of general type. Brunella also proves (cf. [3, Corollaire 1]) that Y is minimal.

We also have that $\mathrm{Bim}(\mathcal{F}) \simeq \mathrm{Aut}(\mathcal{G})$. Hence, Xiao's bound (Theorem 0.3) implies that

$$|\mathrm{Bim}(\mathcal{F})| = |\mathrm{Aut}(\mathcal{G})| \leq |\mathrm{Aut}(Y)| \leq (42K_Y)^2$$

□

Therefore, we may assume that the foliations are singular in the rest of this chapter. Let $(X, \mathcal{F})$ be a foliated surface such that $\mathrm{Kod}(X) \geq 0$ and let G be a finite subgroup of $\mathrm{Aut}(F)$. The principal part of our strategy is to construct a G–equivariant fibration on a surface given by X blown-up at several points and infer the order of G using its action on the base and on a general fiber. Hence, we may state some technical results about fibrations and linear systems that will be useful.

5.2 Fibrations

The first fact to observe is that a fibration $X \longrightarrow B$ gives rise to a morphism of automorphism groups. The following lemma is well known, but we state for the sake of convenience.

Lemma 5.3. *Let X be a compact complex surface, B be a smooth compact curve and $G < Aut(X)$. Let $f : X \longrightarrow B$ be a fibration. Then there are groups $K < G$ and $H < Aut(B)$ such that the sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

is exact.

Proof. It follows from the fact that f induces a group homomorphism $G \longrightarrow \operatorname{Aut}(B)$. For each $g \in G$ and $x \in B$, $g(x) := f(g(y))$ for any $y \in f^{-1}(x)$. H is defined as the image of G in $\operatorname{Aut}(B)$ and K is the kernel of this homomorphism. □

When the basis B of the fibration is not rational, bounding the order of H is a simple task. We may remark some properties of fibrations with rational basis, $B \simeq \mathbb{P}^1$. In order to understand the behavior of the action of a group on a fibration, it is important to know whether special fibers can appear and how many of them can exist.

Consider $f : X \to \mathbb{P}^1$ relatively minimal, that is, the fibers do not contain exceptional curves of the first kind. If f is not trivial and $g \geq 1$, it is known that there are at least 2 singular fibers, 3 if it is not isotrivial (cf. [22]). The isotrivial elliptic fibrations with two singular fibers have been classified by Schmickler-Hirzebruch in [34], a complete description is given (cf [38]).

For $g \geq 2$, Gong, Lu and Tan prove in [10] that X is a ruled surface, the geometric genus of the singular fibers coincide with the irregularity of X and $h^{1,1}(Y)$ is the number of irreducible components of the singular fibers together.

Therefore, if we suppose that X is not ruled, $f : X \to \mathbb{P}^1$ has at least 3 singular fibers, maybe more if it is not relatively minimal. In a more general setting, Viehweg and Zuo have shown that this holds for an arbitrary surjective morphism $Y \to \mathbb{P}^1$ provided that Y is a complex projective manifold of non-negative Kodaira dimension:

Theorem 5.4 ([39], Theorem 0.2)**.** *Let X be a complex projective manifold of non-negative Kodaira dimension. Then a surjective morphism $X \to \mathbb{P}^1$ has at least 3 singular fibers*

The idea in the case of surfaces is given in the following remark:

Remark 5.5. Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration. X does not have a moving family of rational curves, then the genus of a general fiber is $g \geq 1$. The number of singular fibers is $r \geq 2$ by Corollary 5.8 of [22]. Suppose that $r = 2$.

If $g = 1$, the classification given by Schmickler-Hirzebruch in [34] implies that $\operatorname{Kod}(X) = -\infty$, which contradicts our hypothesis. Then we may assume that $g \geq 2$.

The semi-stable model $\tilde{f} : \widetilde{X} \longrightarrow B$ of f is built taking a n-cyclic base change, $n \gg 0$, totally ramified on the images of the two singular fibers. In particular $B \simeq \mathbb{P}^1$ and $\tilde{f}$ has at most two singular fibers.

If $\widetilde{f}$ is not trivial, the Szpiro inequality for higher genus fibrations (cf. [2]) implies that

$$N < (4g+2)(r-2) = 0$$

where N is the total number of singular points of the fibers, hence $N \geq 0$ and we have a contradiction. Hence $\tilde{f}$ is trivial, $\widetilde{X} \simeq F \times \mathbb{P}^1$, F a general fiber of $\tilde{f}$. In particular, X is ruled, which is no our case.

Another well-known result states an upper bound for the number of singular fibers by calculating Euler characteristics.

Lemma 5.6. *Let $f : X \to B$ be a genus g fibration over a projective curve B whose genus is b. Let $F_1, \dots, F_r$ be the singular fibers and suppose that $\chi(F_i) > 2 - 2g$ for $1 \leq i \leq s$, where $s \leq r$ and $\chi(Y)$ denotes the topological Euler characteristic of Y. Then,*

$$s \leq c_2(X) - 4(g-1)(b-1)$$

Proof. Let F denote a general fiber of f, then

$$\begin{aligned} c_2(X) = \chi(X) &= \chi(X \backslash (F_1 \cup \cdots \cup F_r)) + \chi(F_1 \cup \cdots \cup F_r) \\ &= (2-2g)(2-2b-r) + \sum_{i=1}^{r} \chi(F_i) \\ &= (2-2g)(2-2b) + \sum_{i=1}^{s} (2g-2+\chi(F_i)) \\ &\geq 4(1-g)(1-b) + s \end{aligned}$$

□

We have that $e(F) = 2-2g$ for a singular fiber if and only if $g = 1$ and $F = nE$, where E is a smooth elliptic curve. Then, for $g \geq 2$, $r = s$. Consider $f : X \to \mathbb{P}^1$ a relatively minimal elliptic fibration with $s = 2$ and X non-ruled. Then $r \geq 3$ and there is at least one multiple fiber.

Let $p_g(X) = h^2(\mathcal{O}_X)$, $q(X) = h^1(\mathcal{O}_X)$ and $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$ denote, respectively, the geometric genus of X, the irregularity of X and the Euler characteristic of $\mathcal{O}_X$. It is well-known that, for elliptic fibrations we have the inequality, $\chi(\mathcal{O}_X) = \deg\left(f_* K_{X/\mathbb{P}^1}\right) \geq 0$ and equality holds only if f is isotrivial and the singular fibers are multiples of smooth curves. For the proof see Chapter V of [1], for example. Combining this with [23, Corollary 1.11] we have:

$$0 < \chi(\mathcal{O}_X) = \deg\left(f_* K_{X/\mathbb{P}^1}\right) \leq (1 - q(X))(s-1) = 1 - q(X)$$

which implies that $p_g(X) = q(X) = 0$. Surfaces with these invariants have been studied extensively in the literature. The first known example is due to Enriques. We refer to Dolgachev's work in [8] for further discussion.

Due to technical difficulties we will avoid this class of surfaces, which will be called *Dolgachev–Enriques surfaces.*

Definition 5.7. A surface X is called a *Dolgachev–Enriques surface* if it is not of general type and $p_g(X) = q(X) = 0$.

The fibrations that we will construct arise from the action of the automorphism group of a foliation on some very ample linear systems associated to the canonical bundle. The existence of these linear systems can be proved using the following theorem due to Reider in [32].

Theorem 5.8 ([32], Theorem 1). *Let S be a complex algebraic surface and L a nef divisor on S.*

1. *If $L^2 \geq 5$ and p is a base point of $|K_S + L|$, then there exists an effective divisor E passing through p such that*
$$\begin{aligned} \text{either} \quad & L \cdot E = 0, E^2 = -1 \\ \text{or} \quad & L \cdot E = 1, E^2 = 0 \end{aligned}$$
2. *If $L^2 \geq 10$ and points p, q are not separated by $|K_S + L|$ (p, q can be infinitely near), then there exists an effective divisor E passing through p and q such that*
$$\begin{aligned} \text{either} \quad & L \cdot E = 0, E^2 = -1, -2 \\ \text{or} \quad & L \cdot E = 1, E^2 = -1, 0 \\ \text{or} \quad & L \cdot E = 2, E^2 = 0 \end{aligned}$$

Corollary 5.9. *Let X be a complex algebraic surface and L an ample divisor on X. Then $|K_X + 4L|$ is very ample.*

Proof. For any effective divisor E in X, it holds that $L \cdot E \geq 1$. Then $nL \cdot E \geq n$ and $(nL)^2 \geq n^2$ and this implies that the exceptional cases of the theorem do not occur for $n \geq 4$. □

5.3 Bounds for Foliations on Non-ruled Surfaces

In [7], Corrêa and Fassarella show that, in general, foliations with ample canonical bundle have finite automorphism groups. There is only one kind of exception. They summarize this in the following proposition:

Proposition 5.10 ([7], Proposition 2.2). *If $\mathcal{F}$ is a foliation on a smooth projective surface X with $K_{\mathcal{F}}$ ample and* $\mathrm{Aut}(\mathcal{F})$ *infinite then, up to a birational map, $\mathcal{F}$ is preserved by the flow of a vector field $v = v_1 \oplus v_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, if v is not tangent to a foliation by rational curves then $\mathcal{F}$ is given by a global vector field on $\mathbb{P}^1 \times \mathbb{P}^1$.*

Consequently, non-rational surfaces only support foliations with ample canonical bundles that have finite automorphism groups.

Corollary 5.11. *If $(X, \mathcal{F})$ is a foliated surface such that X is not rational and $K_{\mathcal{F}}$ is ample, then $Aut(\mathcal{F})$ is finite.*

Moreover, if we suppose that $\mathcal{F}$ is reduced, $(X, \mathcal{F})$ is minimal. In fact, if $(X, \mathcal{F})$ is not minimal then there is an $\mathcal{F}$–exceptional curve E. We have that E is $\mathcal{F}$–invariant and

$$2 \geq \mathrm{Z}(\mathcal{F}, E) = \chi(E) + K_{\mathcal{F}} \cdot E = 2 + K_{\mathcal{F}} \cdot E$$

hence $K_{\mathcal{F}} \cdot E \leq 0$, which would imply that $K_{\mathcal{F}}$ is not ample.

Then, for reduced non-ruled foliated surfaces with ample canonical bundle, $\mathrm{Bim}(\mathcal{F}) = \mathrm{Aut}(\mathcal{F})$ and we have the following bound:

Theorem 5.12. *Let (X, F) be a reduced foliated surface such that $K_{\mathcal{F}}$ is ample and X is non-ruled. Suppose that $\mathcal{F}$ has a singularity at $p \in X$ which does not lie on an $\mathcal{F}$–invariant algebraic curve, then*

$$\begin{aligned} |Aut(\mathcal{F})| \leq\, & 2(c_2(X) + (K_X + 4K_{\mathcal{F}})(5K_X + 12K_{\mathcal{F}})) \\ & \cdot [4(K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) + 6]c_2(T_X \otimes K_{\mathcal{F}}) \end{aligned}$$

unless X is a Dolgachev–Enriques surface.

Proof. By Corollary 5.11, $\mathrm{Aut}(\mathcal{F})$ is finite. We may take the stabilizer of p in $\mathrm{Aut}(\mathcal{F})$ whose index is bounded by $c_2(T_X \otimes K_{\mathcal{F}})$, which is the number of singularities. By Proposition 2.7, this group has an abelian subgroup G of index at most two, then the index of G in $\mathrm{Aut}(\mathcal{F})$ is

$$(\mathrm{Aut}(\mathcal{F}) : G) \leq 2c_2(T_X \otimes K_{\mathcal{F}}).$$

The line bundle $K_X \otimes K_{\mathcal{F}}^{\otimes 4}$ is G-invariant, since K_X and $K_{\mathcal{F}}$ are G–invariant, and, by Corollary 5.9, it is very ample. The action of G lifts to the global sections and is diagonalizable, hence we can choose a G–invariant pencil, Λ. We can even choose a pencil that is smooth at p. Let us prove this briefly.

Let $s_0, \ldots, s_N$ be a basis of $\mathrm{H}^0(X, K_X \otimes K_{\mathcal{F}}^{\otimes 4})$ composed by G–semi-invariants: $s_i \circ g = \lambda_i(g)s_i$, where $g \in G$ and λ_i is the corresponding character. Let Λ_{ij} be the pencil generated by s_i and s_j and let

$$\Phi : X \longrightarrow \mathbb{P}^N$$

be the map defined by $\Phi(x) = (s_0(x), \ldots, s_N(x))$. Since $K_X \otimes K_{\mathcal{F}}^{\otimes 4}$ is very ample, Φ is an embedding. This implies, in particular, that for some i, $s_i(p) \neq 0$.

Suppose, without loss of generality, that $s_0(p) = 1$. If $\Phi(p) = (1, a_1, \ldots, a_N)$, the element of Λ_{0j} that vanishes at p is $f_j = a_j s_0 - s_j$. On the affine open set $U = \{s_0 \neq 0\}$, we define $t_i = s_j/s_0$ and we have that the germs of dt_i at p span $\Omega^1_{X,p}$. In particular there is a j such that $(dt_i)(p) \neq 0$, hence

$$d\left(\frac{f_j}{s_0}\right)(p) = (dt_j)(p) \neq 0$$

and $\{f_j = 0\}$ is smooth at p. Therefore, we may choose $\Lambda = \Lambda_{0j}$.

Next we construct a G-equivariant fibration. After removing the (maybe trivial) fixed divisor of Λ, it defines a rational map $\Lambda : X \dashrightarrow \mathbb{P}^1$ which is undefined at most at isolated base points. Take $\pi : Y \longrightarrow X$ the blow-up of the base points (including the infinitely near ones). After a Stein's factorization we have a fibration

$$f : Y \longrightarrow B$$

where B is a smooth curve and $c_2(Y) \leq c_2(X) + (K_X + 4K_{\mathcal{F}})^2$ since each blow-up adds one to the Chern class c_2. The general fiber of f has genus, say g, at most the genus of a smooth curve in the complete linear system $|K_X + 4K_{\mathcal{F}}|$, that is

$$g \leq (K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) + 1$$

by adjunction formula. We also have that $g \geq 1$ since $\mathrm{Kod}(X) \geq 0$.

The action of G lifts to Y since for each point which we have blown-up, we also have blown up its whole orbit, see Lemma 2.2. The map f is made to be equivariant just letting G act on B via f. By Lemma 5.3, we construct groups K and H whose orders we can bound.

The subgroup $K < G$ sends each fiber onto itself, hence K is isomorphic to a subgroup of the automorphism group of a general fiber. We have also that K fixes $q = \pi^{-1}(p)$ and the component of the fiber where it lies, which is smooth. Then K is a cyclic group. By Wiman's bound (Theorem 0.1),

$$|K| \leq 4g + 2 \leq 4(K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) + 6.$$

Of course, the bound is for $g \geq 2$. However, for $g = 1$, K has a fixed point on a nearby smooth fiber, hence its order is at most $6 = 4g + 2$. This is true by the proof of Proposition 2.8.

Let $b = g(B)$. We have that $b > 0$ only if Λ does not have base points. In fact, if there is a base point, then there exists some rational curve E in Y not contained on a fiber of f. The restriction of f to E defines a holomorphic map to B. By Hurwitz Formula, B must be rational.

The group H is a subgroup of $\mathrm{Aut}(B)$ that fixes $f(q)$, in particular it is cyclic. If $b \geq 2$, then

$$|H| \leq 4b + 2.$$

In this case, for $g \geq 2$, by Lemma 5.6

$$|G| \leq 4(2g + 1)(2b + 1) \leq 100(g - 1)(b - 1) \leq 25c_2(X).$$

If $b = 1$ and $g \geq 2$, then

$$|G| \leq 6|K| \leq 24(K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) + 36.$$

If $b = g = 1$, then

$$|G| \leq 36.$$

Now we turn to the case $b = 0$. H permutes the images, via f, of the singular fibers and, by Theorem 5.4, there are $r \geq 3$ such fibers. Since $H < \mathrm{PGL}(2, \mathbb{C})$ is cyclic, it has another fixed point. If $g \geq 2$, then

$$\begin{aligned}|H| \leq r &\leq c_2(Y) + 4(g-1) \\ &\leq c_2(X) + (K_X + 4K_{\mathcal{F}})^2 + 4(K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) \\ &= c_2(X) + (K_X + 4K_{\mathcal{F}})(5K_X + 12K_{\mathcal{F}})\end{aligned}$$

Then we have

$$\begin{aligned}|G| = |H|\,|K| \leq &(c_2(X) + (K_X + 4K_{\mathcal{F}})(5K_X + 12K_{\mathcal{F}})) \\ &\cdot [4(K_X + 4K_{\mathcal{F}})(K_X + 2K_{\mathcal{F}}) + 6].\end{aligned}$$

Now suppose that $g = 1$. Then there are at least 3 fibers that are not multiples of a smooth curve, unless X is a *Dolgachev–Enriques surface*, which is ruled out by hypothesis. Then

$$|H| \leq r \leq c_2(Y) \leq c_2(X) + (K_X + 4K_{\mathcal{F}})^2$$

and, consequently:

$$|G| = |H|\,|K| \leq 6[c_2(X) + (K_X + 4K_{\mathcal{F}})^2].$$

Therefore, we have $|\mathrm{Aut}(\mathcal{F})| \leq 2c_2(T_X \otimes K_{\mathcal{F}})|G|$ and this concludes the proof. □

Next we consider a minimal foliated surface $(X, \mathcal{F})$ of general type such that $\mathcal{F}$ has holomorphic first integral.

Let $f : X \longrightarrow C$ be a first integral for $\mathcal{F}$, that is, the leaves of $\mathcal{F}$ are contained on the level sets $\{f = a\}$, $a \in C$. Up to a Stein's factorization, we may suppose that f has connected fibers. By the classification of foliated surfaces of lower Kodaira dimension, $\mathcal{F}$ is a nonisotrivial fibration of genus $g \geq 2$. See also [37, Theorem 2.1].

For these foliations we have the following bound:

Theorem 5.13. *Let (X, F) be a minimal foliated surface of general type. Suppose that $\mathcal{F}$ has a holomorphic first integral $f : X \longrightarrow C$ with connected fibers. If a general leaf of $\mathcal{F}$ has genus g and $g(C) = b$ then*

- $|Aut(\mathcal{F})| \leq 60c_2(X)c_2(T_X \otimes K_{\mathcal{F}})$ *if* $b \geq 2$,
- $|Aut(\mathcal{F})| \leq 48(g+1)c_2(T_X \otimes K_{\mathcal{F}})$ *if* $b = 1$,
- $|Aut(\mathcal{F})| \leq 8(g+1)(c_2(X) + 4(g-1))c_2(T_X \otimes K_{\mathcal{F}})$ *if* $b = 0$.

Proof. Let $p \in \mathrm{Sing}(\mathcal{F})$. Then there is an abelian subgroup G of $\mathrm{Aut}(\mathcal{F})$ that fixes p and has index bounded by $2c_2(T_X \otimes K_{\mathcal{F}})$. Then, by Lemma 5.3, f induces an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

K fits into the automorphism group of a general fiber, hence

$$|K| \leq 4g + 4$$

by Nakajima's bound (Theorem 0.2).

If $b = g(C) \geq 2$, by Wiman's bound (Theorem 0.1), $|H| \leq 4b + 2$. Hence,

$$|G| \leq 8(g+1)(2b+1) \leq 120(g-1)(b-1) \leq 30c_2(X)$$

by Lemma 5.6.

If $b = 1$, then $|H| \leq 6$ then

$$|G| \leq 24(g+1).$$

If $b = 0$, Theorem 5.4 implies that there are $r \geq 3$ singular fibers. Then

$$|H| \leq r \leq c_2(X) + 4(g-1)$$

by Lemma 5.6. Hence

$$|G| \leq 4(g+1)(c_2(X) + 4(g-1)).$$

□

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