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DISCRETIZATION AND STATE FEEDBACK CONTROL FOR UNCERTAIN LINEAR SYSTEM: new approaches considering linear multistep theory

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Natália Augusto Keles

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"Um passo a frente e você já não está no mesmo lugar." Chico Science

RESUMO

Esta tese apresenta três métodos para projetar controladores de tempo discreto que podem estabilizar sistemas incertos em tempo contínuo. Todas as condições de síntese propostas são baseadas na teoria de Lyapunov, usando uma função de Lyapunov constante ou dependente de parâmetros. As condições baseadas em LMI são desenvolvidas a partir de sistemas lineares de tempo discreto, que pertencem a domínios politópicos que são obtidos através da discretização de sistemas lineares contínuos incertos. O primeiro método de discretização proposto envolve um procedimento de otimização projetado para encontrar os coeficientes que minimizem o erro residual de discretização. Esses coeficientes otimizados são então usados para construir uma estrutura aumentada para o sistema de tempo discreto. O erro de discretização é calculado através de uma busca em grade e é incorporado na formulação do sistema de tempo discreto. Adicionalmente, duas outras metodologias de discretização são desenvolvidas utilizando métodos numéricos multipassos, especificamente, os métodos de Adams-Bashforth e Adams-Moulton, ou seja, uma estrutura de tempo discreto aumentada é obtida utilizando os coeficientes dos métodos abordados e instantes de tempo atrasados da aproximação do sistema de tempo contínuo em questão. Para cada sistema aumentado de tempo discreto resultante existe uma condição de síntese específica para fornecer uma lei de controle de tempo discreto que estabilize o sistema de tempo contínuo em malha fechada. Experimentos numéricos demonstram a validade dos métodos propostos em comparação com outros exemplos da literatura.

Palavras-chave: teoria de métodos de passo múltiplo; incerteza politópica; discretização de sistemas incertos. síntese de dontroladores.

ABSTRACT

This thesis presents three methods for designing discrete-time controllers that can stabilize continuous-time uncertain systems. All proposed synthesis conditions are based on the Lyapunov theory, using either constant or parameter-dependent Lyapunov function. The LMI-based conditions are based on discrete-time linear systems that belong to polytopic domains which are obtained through the discretization of uncertain continuous-time linear systems. The first proposed discretization method involves an optimization procedure designed to find the coefficients that minimize the residual discretization error. These optimized coefficients are then used to construct an augmented structure for the discrete-time system. The discretization error is computed via a grid search and it is incorporated in the discrete-time system formulation. Additionally, two other discretization methodologies are developed using multistep numerical methods, specifically, the Adams-Bashforth and the Adams-Moulton methods, in other words, an augmented discrete-time structure is obtained using the coefficients of the considered methods and delayed steps of the approximation of the continuous-time system that serves as a discrete representation of the original system. For each resulting discrete-time system, there is a specific synthesis condition to provide a discrete-time control law that stabilizes the continuous-time system in closed-loop. Numerical experiments demonstrate the validity of the proposed methods in comparison to other examples in the literature.

Keywords: multistep theory; polytopic uncertainty; discretization of uncertain systems; control synthesis.

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List of abbreviations and acronyms

qLPV	quasi-Linear Parameter Varying
LMI	Linear Matrix Inequality
BDF	Backward Differentiation Formula
ODE	Ordinary Differential Equation
LKF	Lyapunov-Krasoviskii functional
TS	Takagi-Sugeno
LTI	Linear Time-Invariant
HIGS	Hybrid Integrator-Gain System
BMI	Bilinear Matrix Inequality
LPV	Linear Parameter Varying

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Chapter 1

INTRODUCTION

Most processes in practice deal with continuous-time systems and although there are suitable synthesis conditions to be applied in the continuous-time domain (Gritli et al., 2021; da Ponte Caun et al., 2021; Aeinfar et al., 2020), the use of digital technologies (and discrete-time representations) makes implementation and maintenance less costly. In addition to that, computer-controlled systems provide better flexibility, scalability, and ease of implementation (Ogata, 1995; Chen and Francis, 1995; Hara et al., 1996), which promotes the development of new methodologies that employ discrete-time systems to yield a discrete-time control law that can control continuous-time systems (Pradhan and Das, 2022; Campos et al., 2022a; Lerch et al., 2021).

Despite the many benefits of working with discrete-time control, there are several challenges to obtaining a discrete-time system that accurately represents the dynamics of the continuous-time system and also to designing control laws that ensure the performance of the continuous-time counterpart.

Obtaining discrete-time controllers capable of stabilizing a continuous-time plant is not an easy task. In real applications, systems are subject to parameter variations, noise, external disturbances, inaccuracies in actuators or sensors, and discretization errors inherent to discretization processes (Ackermann et al., 1993), all of which can compromise the performance of the discrete-time control. To account for these real-world uncertainties, representations that incorporate such variations into their structure are used, and these are referred to as uncertain systems. The challenge increases when dealing with continuous-time uncertain systems, as their discretization is more complex than systems without uncertainties due to the need for approximating the exponential of an uncertain matrix.

There are three possible ways for synthesizing discrete-time controllers as illustrated in Figure 1:

- Continuous-based synthesis: a continuous-time controller based on continuous-time theory is designed and then it is converted to its digital implementation.

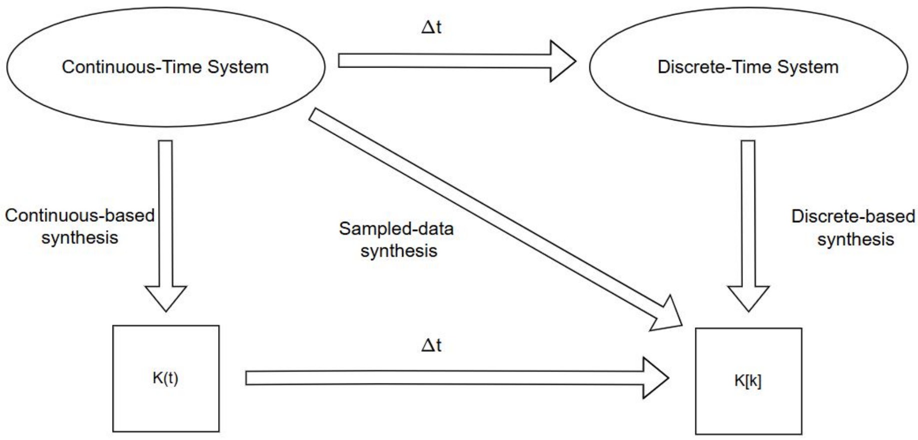

Figure 1 – Strategies for designing discrete-time controllers.

- Discrete-based synthesis: a discrete-time model is obtained from the continuous-time system at sampling instants, and then a digital controller is designed using the discrete-time control theory.
- Sampled-data synthesis: a digital controller is directly designed by taking into account the intersampling behavior.

The first approach can be divided into two categories: emulation and redesign. Emulation involves designing a controller in the continuous-time domain and then directly discretizing it for use in a digital system without further modifications (Astolfi et al., 2019; de Souza et al., 2021). On the other hand, redesign assumes the existence of a good continuous-time control law that stabilizes the system in closed-loop. The aim is then to design a discrete-time controller using some technique to discretize the continuous-time controller, achieving similar behavior, i.e., stabilizing the plant and optimizing a performance criterion related to the error between the closed-loop continuous-time and discrete-time trajectories (Kim et al., 2010; Koo et al., 2016; Maccari Jr. et al., 2013; Morais et al., 2017).

In the second approach, the control law is obtained in discrete-time directly based on an approximate discretized model of the plant, ignoring the intersampling behavior. This implies that the discrete-time model has to supply enough information from the original system so that the synthesis conditions can find a feasible digital controller (Kazantzis and Kravaris, 1999; Coutinho et al., 2020a; Campos et al., 2022a; Braga et al., 2013). The fact that the behavior between samples is ignored may allow the existence of undesired oscillatory phenomena. Therefore, techniques that reduce signal-to-noise ratio can be considered during the design of the digital control law, e.g. $\mathcal{H}_\infty$ norm (Kim and Hagiwara, 2022a; Xing et al., 2022).

Finally, the third approach involves designing the controller in the discrete-time domain while considering the inter-sampling behavior. This approach is often used when systems operate at different sampling rates. A simple solution to this problem is to reduce the overall sampling period to match the slowest rate. However, this can lead to other issues, such as reduced overall system performance due to missed high-frequency dynamics. To address this, mechanisms that manage inter-sampling behavior have been developed as presented in Coutinho et al. (2021); Ernesto et al. (2012).

The methods proposed in this work are based on the second approach, where a discrete-time system is obtained from the continuous-time one. In the literature, there are several ways to obtain a discretized uncertain system from a continuous-time uncertain one, such as the exact discretization approach (Kim et al., 2017; Campos et al., 2022b; Lee, 2013).

In some situations, over approximations are required to obtain a discrete-time uncertain system based on a continuous-time one, utilizing, for instance, the real Jordan Form (Heemels et al., 2010; Olaru and Niculescu, 2008) or Cayley-Hamilton theorem (Gielen et al., 2010, 2009). In both cases, the discretization method is only suitable for systems with few vertices. There are also some methods that employ Taylor Series Expansion applied to the exponential of uncertain matrices (Braga et al., 2013; Hetel et al., 2006). These approximated discrete-time models can suffer from performance degradation or even instability of the resulting sampled-data system when the approximation error is relatively large (Lee et al., 2014). Hence, to the best of our knowledge, stabilization of continuous-time uncertain linear systems via discrete-time controllers remains an open problem in the literature. In that regard, this thesis proposes a novel discretization approach that mixes discrete-time models obtained from different multiples of a fixed step-size, leading to a discretization approach similar to a multistep method, which are numerical methods that use delayed instants to construct an approximate solution of the system.

1.1 Motivation

Physical systems are usually defined in the continuous-time domain. However, considering the advent of technology, more and more digital devices connect to continuous-time systems and modify their dynamics when necessary, resulting in a type of hybrid system (continuous and discrete times coexist). Developing a stabilizing digital control law for a continuous-time uncertain system is a challenge, mainly due to the control law not being updated between sampling instants and due to the choice of a suitable sampling time. Shorter sampling times require more processing and, consequently, more powerful computers. On the other hand, longer sampling times may not capture the dynamics of the continuous-time system, affecting directly the performance of the closed-loop system.

Despite the growing efficiency and affordability of microcontrollers, many scenarios still involve limited computational resources, making it crucial to optimize digital control strategies. Applications such as unmanned aerial vehicles (UAVs) (Ahmed et al., 2021; Liu et al., 2018; Derrouaoui et al., 2021), oil extraction devices(Ma et al., 2024), autonomous underwater vehicles (AUVs)(Siddhartha and Mahapatra, 2022; Shet et al., 2024), satellitesLiu et al. (2024); Khosravi et al. (2024), medical implants (Lessard et al., 2004; Hadian, 2023), and industrial wireless sensor networks(Shi et al., 2021; Joelianto et al., 2008) often cannot rely on large computers due to constraints like size, weight, power consumption, or harsh environmental conditions.

For instance, UAVs and AUVs require lightweight and energy-efficient control systems to achieve optimal performance without incurring excessive computational overhead. Similarly, offshore oil extraction systems must function reliably in remote and extreme environments with minimal hardware. In medical implants like pacemakers, reducing processing power can extend battery life, ensuring long-term functionality.

To tackle these challenges, increasing the sampling period in digital control can help lower computational demands and energy consumption while maintaining stability and performance. This often necessitates the use of advanced discretization techniques and robust control strategies.

1.2 Objectives

The main objective of this thesis is to develop procedures for obtaining a discrete-time state feedback controller using a discretization-based approach. The goal is to develop discretization techniques that yield a discrete-time model accurately representing the continuous-time system. Additionally, conditions for synthesizing digital controllers are proposed, considering the discrete-time models obtained and their associated discretization errors. The developed approaches are based on Taylor Series Expansion and multistep theory to derive the discrete-time system, as well as robust control techniques for designing the discrete-time control law.

- Develop discrete-time systems using an augmented structure that accounts for delayed steps, based on Multistep theory.

- Incorporate discretization errors into the augmented discrete-time system or synthesis conditions to enhance robustness in the methodology.

- Develop synthesis conditions for discrete-time control laws that stabilize the continuous-time closed-loop system.

1.3 Main Contributions

The proposed methodologies resulted in the following main contributions:

- Three novel discretization approaches for time-invariant uncertain continuous-time systems are proposed based on multistep method theory. The first procedure mixes models obtained with different multiples of a fixed step-size (sampling time). An optimization problem is proposed in order to find the best way to mix these models, by enhancing the available information and minimizing the discretization error associated with the methodology. The second and third approaches are based on two multistep methods: Adams-Bashforth and Adams-Moulton. In which delayed steps are aggregates using the structure and coefficients based on their formulations.

- A new methodology based on LMIs to define an upper bound for the discretization error between the original system and the discretized one using the Multistep Methods: Adams-Bashforth and Adams-Moulton method.

- Two sufficient LMI-based conditions to design robust discrete-time controllers using the discretized models are proposed. Considering the Mixed Method discretization procedure there are two synthesis conditions: i) one based on a constant Lyapunov function and, ii) one based on a parameter-dependent Lyapunov function. In both cases, asymptotic stability is ensured for the continuous-time system in closed-loop with the discrete-time controller. On the other hand, utilizing the procedure based on the Adams-Family, the proposed synthesis conditions can be adapted to consider both constant Lyapunov functions and parameter-dependent Lyapunov functions and discrete-time control laws are feasible when the calculated convergence region is smaller region of the domain of validity.

1.4 Text Structure

The structure of this document consists of five chapters. This chapter introduced the topic of continuous-time uncertain system discretization, discussing relevant works in the literature that address the same problem. Additionally, the motivation for conducting this research is presented, along with the objectives guiding its execution.

Chapter 2 gives a brief review of the literature related to the main methodologies that provide synthesis conditions to obtain discrete-time controllers. Input-delay approach, hybrid system, lifting approach and other methods are depicted. On the other hand, chapter 3 presents fundamental concepts to understand the developments in the following sections. In this context are presented the concepts related to Taylor Series Expansion, Multistep Theory and Adams-Bashforth and Adams-Moulton formulation, uncertain linear systems, Finsler's Lemma, and S-procedure.

The first main result of this thesis is presented in Chapter 4. The discrete-time representation is obtained using an optimization process that provides the optimized coefficients to combine the discrete-time models obtained from different multiples of a fixed step size, aiming at the minimization of the discretization error. Two synthesis conditions are defined considering constant Lyapunov functions and parameter-dependent Lyapunov functions to design the discrete-time control law.

The second main development is described in Chapter 5. In this case, the augmented discrete-time system is obtained by combining models from the Adams-Bashforth or Adams-Moulton structure. Although there are two discrete-time representations, both use the same controller synthesis conditions. Furthermore, the LMI-based conditions can be adapted to either constant or parameter-dependent Lyapunov functions.

Chapter 6 provides three numerical examples to illustrate the effectiveness of the proposed methods. Additionally, tables are included to compare the results of these methodologies with each other and with examples from the literature.

Finally, conclusions and proposals for continuity are presented in Chapter 7.

Chapter 2

A BRIEF REVIEW OF THE LITERATURE - DISCRETE-TIME CONTROLLERS

In recent years there has been growing interest in developing approaches to stabilize continuous-time systems using discrete-time controllers. There are four main procedures to address closed-loop stability in this context: (1) Modeling a sampled-data system as a continuous-time one with delayed inputs, where Lyapunov-Krasoviskii functionals are applied to determine a control law to stabilize the closed-loop system; (2) Hybrid Systems representing the sampled-data system as an impulsive model where a time-varying periodic Lyapunov functional is used to certify stability; (3) the lifting approach consists on finding an equivalent discrete-time system that keeps the inter-sampling information; (4) Discretization-based approach consists of building a discrete-time system capable of adequately representing the dynamics of the original and from this system proposing synthesis conditions for controllers, observers, and filters. They will be discussed in-depth in the following sections.

2.1 Time delay Approach

In the input delay approach, the discrete-time control action is modeled as a delay for the control input of the continuous-time system (Fridman, 1992). This methodology became popular in the context of networked control systems.

The methodology developed by Sereni et al. (2023) proposes a control approach for the robust stabilization of Linear Time-Invariant (LTI) systems with sensor and actuator dynamics, subject to time-delayed signals. An augmented model, incorporating these dynamics and the time-delay effect via the Padé approximation, is used. Since plant state variables are unavailable, a static output-feedback law is designed based on sensor outputs.

Controller gains are computed through a two-stage LMI-based strategy, using homogeneous polynomial Lyapunov functions to reduce conservatism. A minimum decay rate criterion enhances transient response, while disturbance rejection is addressed via $\mathcal{H}_2$ guaranteed cost minimization.

A novel approach to robust sampled-data stabilization for continuous-time systems is proposed by Fridman et al. (2004), where the control input has a piecewise-continuous delay. Sufficient LMIs conditions for sampled-data state-feedback stabilization of such systems are derived via a descriptor approach to time-delay systems. The conditions obtained are robust with respect to different samplings with the only requirement that the maximum sampling interval is not greater than Δt. Moreover, the feasibility of the LMIs is guaranteed for small Δt if the corresponding continuous-time controller stabilizes the system.

The work of Yang et al. (2014) investigates the problem of sampled-data control for Takagi-Sugeno (TS) fuzzy systems with aperiodic sampling intervals based on an input delay approach. The authors proposed a stability criterion and a sampled-data controller design for TS fuzzy continuous-time systems based on delay-dependent discontinuous Lyapunov-Krasoviskii functionals (LKF). This methodology provides information on the time-varying input delays and all available information about the states. A bounding technique associated with reciprocally convex methods and linear convex combinations is presented for obtaining the time derivative of the functional, where Jensen's and Wirtinger's inequalities are applied. A feasible solution of the obtained criterion formulated as parameterized linear matrix inequalities is ultimately conceived.

Ramezanifar et al. (2012) also proposed a methodology based on input delay. The work develops a method for the design of sampled-data state feedback controllers for Linear Parameter Varying (LPV) systems. The goal is to ensure asymptotic stability and a specified level of performance on the closed-loop hybrid system. The approach involves using a parameter-dependent Lyapunov-Krasovskii functional to develop a delay-dependent synthesis method. This method is capable of handling fast-varying time delays, unlike existing approaches in the literature. To formulate the solution as a Linear Matrix Inequality (LMI) based optimization problem, the authors improved upon the conditions presented in Tuan et al. (2001) by introducing slack variables. This improvement helps relaxing the resulting condition in terms of an LMI problem. By using the derived formulation based on the slack variables, the synthesis condition for the state-feedback sampled-data control design becomes considerably simpler compared to existing methods.

Additionaly, Abolpour (2021) presents an algorithm for verifying the robust stability of an uncertain system. It employs affine parameter-dependent Lyapunov functions to assess stability over the model's uncertain space. These functions are defined for polygonal subspaces, each evaluating stability within its respective region. The algorithm iteratively refines these subspaces until robust stability is confirmed throughout. Notably, it guarantees stability for all positive delay values for the first time.

2.2 Hybrid System Approach

Hybrid systems combine continuous and discrete dynamics, serving as useful mathematical models for various physical and engineering systems, because they have these characteristics, discrete-time controllers are widely used in context. Examples of hybrid systems that benefit from digital control approaches are: event-triggered systems, sampled-data systems considering aperiodic or periodic time sampling. The control design for these systems is complex with different approaches presented in the literature for various classes of hybrid systems and control objectives, such as safety specification and verification. Stabilization is a key control objective in these cases.

Related to sample-data system, Naghshtabrizi et al. (2008) investigated the stability of nonlinear time-varying impulsive systems by using Lyapunov functions that include discontinuities at impulse times. The proposed stability conditions have the advantage that, when applied to linear impulsive systems, the stability conditions can be expressed as LMIs. The authors analyze LTI uncertain sampled-data systems facing two uncertainties: unknown process parameters that meet a polytopic condition, and variable sampling intervals. These are modeled as linear impulsive systems. Running this process, a positive constant determines an upper bound on the sampling intervals for which the stability of the closed loop is guaranteed, and discrete-time controller gains are found. Additionally, the authors consider systems with constant sampling intervals, resulting in less conservative outcomes than those with variable intervals.

In Zhang and Gharesifard (2021), the authors studied the event-triggering stabilization problem of time-delay systems from a hybrid control perspective. They used impulsive controls coupled with the event-triggered control mechanism to prevent Zeno behavior, which is an issue in event-triggered systems where an infinite number of events can occur in a finite time interval. The impulsive control method involves using sudden changes or jumps over a negligible time period to achieve the desired performance of the closed-loop system. The hybrid control algorithm operates as follows: initially, a minimal dwell time is set as the lower bound of each inter-execution time, which is the time interval between two successive control updates. If a measurement error reaches a certain threshold outside this dwell-time period, the control input is updated. Otherwise, the control signal remains unchanged until the end of the dwell-time period, at which point an impulsive control input is executed, and the feedback control signal is updated. This hybrid algorithm effectively prevents Zeno behavior. Using the Lyapunov function method and the Razumikhin technique, the authors establish conditions on the impulse inputs and impulse moments that ensure the asymptotic stability of the corresponding closed-loop system under this hybrid control algorithm.

In a study on obtaining digital control for aperiodic sampled-data systems, Lien et al. (2020) proposed a method for designing switching signal selections and aperiodic sampled-data state feedback controllers to achieve $\mathcal{H}_\infty$ performance of the switched system with time-delay and linear fractional perturbations. They developed a practical switching signal to address real-time switching challenges and provided additional results for better disturbance attenuation. The use of suitable free-weighting matrices can further improve the achieved results. The study utilized an LMI optimization approach to minimize the H_∞ performance of the considered system and observed better disturbance attenuation as well as larger upper bounds for time-delay and sampling intervals.

Lastly, in Fagundes and da Silva (2022), the authors addressed the design of stabilizing aperiodic sampled-data control laws for linear plants under control saturation. In particular, the method consists in the closed-loop system to be modeled by an impulsive system with a linear flow and nonlinear (due to saturation) jump dynamics. LMI-based optimization procedures are proposed, whose solutions provide the desired control law, which maximizes an estimate for the region of attraction of the sampled-data system.

2.3 Lifting Approach

The lifting approach plays a fundamental role in discretizing continuous-time systems and designing robust controllers, especially when dealing with dynamic systems that exhibit periodicity, time-dependent variations, or structured uncertainties. Furthermore, this method deals simultaneously with continuous and discrete-time signals (Megretski and Rantzer, 1997). Therefore, the lifting procedure offers a powerful strategy to reformulate the original control system into an equivalent representation in a higher dimension, where classical analysis and synthesis tools, such as LMIs, can be applied (Chen and Francis, 2012).

In certain scenarios, the lifting approach is applied in the discrete-time domain, in which an augmented discrete-time system is built incorporating a delay factor that can enhance data availability and optimize performance. These improvements are particularly evident under synthesis conditions that aim at stabilizing the system (Fridman, 2014; Richard, 2003; Frezzatto et al., 2018).

An example of a work utilizing the lifting approach to develop a simpler digital control law compared to other literature can be found in Bhattacharya and Balas (2003). This research presents an algorithm for the efficient implementation of digital control laws based on lifting techniques. The transformations applied to the discrete-time control law and the stability performance of the closed-loop systems were evaluated. The conclusion drawn was that the transformation did not significantly alter the behavior of the closed-loop system. However, the reduction in computational overhead was substantial in the analyzed example.

The lifting technique is also employed to address systems that experience failures in the feedback channel. This means that the control signal can occasionally be unavailable, causing the system to operate in an open-loop mode during those times. Seidel et al. (2024) proposed an LMI-based condition that utilizes a system built from multiple past inputs and outputs at the instants where control attempts were successful to obtain a discrete-time control law. This approach provides less conservative results when compared to methods that model failures as deterministic bounded losses and use Lyapunov stability to derive digital control laws.

2.4 Discretization-based approach

Discretization-based approach consists in proposing methodologies to build a discrete-time system and, based on this system, develop synthesis conditions to obtain a discrete-time controller that can stabilize the closed-loop system. Obtaining a discrete-time model that reproduces the dynamic of a continuous-time system and developing conditions that guarantee closed-loop stabilization can be a challenge due to the usage of an approximated discretization instead of an exact discretization, since the second might be impractical or impossible to obtain when uncertain systems are considered. Conditions considering intersample behavior are too conservative and a large computational effort can be employed when a short sampling time is necessary to provide a control signal to the real system.

Several authors devoted their attention to developing discretization methodologies to stabilize continuous-time systems. For example, Campos et al. (2022a) proposed an approach for discretizing nonlinear systems represented by TS fuzzy models. The procedure is based on the process of deriving an auxiliary system based on the original system using LMI techniques. If the proposed conditions are feasible, the matrices of the auxiliary system and upper bounds for the error between this auxiliar system and the original one over a sampling period are obtained. Then, this auxiliary system is discretized instead of the original one, and conditions for controller and observer designs can be applied to the discretized system. The results were applied to a real oscillating nonlinear Chuas's circuit (Torres and Aguirre, 2005). The state-feedback controller based on the procedure attained a maximum sampling period higher than several other available strategies in the literature.

To address the problem of robust stabilization, some researchers consider Taylor Series Expansion. For instance, Braga et al. (2014) uses the Taylor Series Expansion with an arbitrary degree to obtain a discrete-time model from a time-invariant continuous-time linear system in polytopic domains. The sampling period is assumed to be unknown but belongs to a given interval. The resulting discrete-time model consists of homogeneous polynomial matrices with parameters within a multi-simplex, along with an additive norm-bounded term representing the discretization residual error. The original continuous-time system is controlled through a communication network, introducing a time delay in the process. LMI relaxations, including a scalar parameter search, are proposed for designing a digital robust state feedback controller ensuring the closed-loop stability of the networked control system.

In Braga et al. (2019), the authors extend the results presented in Braga et al. (2014) and propose a discretization methodology that generates an artificial discrete-time descriptor system. This approach enables the tuning of stabilizing discrete-time state-feedback control law with longer sampling times. The resulting descriptor system has state-space matrices that are polynomially parameter-dependent and, additionally, these matrices are associated with norm-bounded uncertainties related to residuals from the discretization process, the chosen degree of the Taylor Series Expansion, the sampling time, and the uncertainties of the original continuous-time system. Leveraging the obtained discrete-time system, the authors propose synthesis conditions for controllers.

Also using Taylor Series Expansion to obtain discrete-time uncertain systems, Jungers et al. (2017) presented a methodology to explicitly define an upper limit for the discretization error by approximating the exponential of the matrix by Taylor Series Expansion of arbitrary degree. The proposed approach uses the generalization of the norm of a matrix and the calculation of the matrix measure applied to a convex polytope. One of the advantages presented for using an explicit methodology is the definition of the limit without the need to perform a fine grid procedure to find it since the computational cost increases dramatically with the increase in the number of vertices of the uncertain system addressed. According to the authors, the limit provided is the least conservative among the works in this context. Furthermore, LMI-based conditions were developed for the stability analysis and synthesis of controllers based on the developed discrete system. Numerical examples validate and reinforce the effectiveness of the results.

Chapter 3

THEORETICAL FOUNDATION

This section introduces key concepts of numerical methods and uncertain systems, essential for providing context on the topics used to develop the main results of this thesis. Additionally, a brief theoretical review is included to highlight the importance of obtaining discrete-time models for designing a discrete-time control law capable of stabilizing a continuous-time system.

3.1 Taylor Series Expansion and Lagrange's Form of the Remainder

The Taylor Series Expansion of a function is widely used to approximate trajectories of dynamic systems and obtain systems in a discrete-time domain (Fridman et al., 2004; Jungers et al., 2017; Coutinho et al., 2020b). It is defined as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n)!}(x-a)^n + R_n(x,a) \quad (3.1)$$

where $f(.)$ is the function to be approximated, n is the order of approximation and $f^{(n)}(a)$ denotes the n-th derivative of $f(.)$ evaluated at point a. Taylor polynomials are approximations of a function, which become generally more accurate as n increases. $R_n(x,a)$ represents the approximation error, that is, every derivative term with order higher than n that was not included in the Taylor Series approximation. $R_n(x,a)$ is called the Lagrange Remainder Form (Beesack, 1966), and is defined as

$$R_n(x,a) = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-a)^{n+1} \quad (3.2)$$

where $x^* \in [a,x]$. This formulation will be applied to construct the discrete-time systems developed in this thesis.

3.2 Linear Multistep Methods

Developing methods to approximate ordinary differential equations (ODEs) solutions is an important topic in numerical analysis. Generally, linear single-step methods are used for obtaining numerical solutions for ODE problems due to their simplicity of implementation and suitable accuracy in most scenarios. Nevertheless, for long simulations, improvements in accuracy and efficiency can be attained by employing linear multistep methods instead. Unlike single-step methods, which utilize only the current information for determining the solution, multistep methods rely on multiple previous points to compute the next value of interest.

The seminal work in multistep methods theory is attributed to Bashforth and Adams (1883). The main concept of this work was extending Euler's method by making the approximate solution at one point depend on the solution values and the derivative values at several previous steps, weighted by some constant factors. As a result, the obtained procedure is an explicit method. An extension to this approach was later developed by Moulton (1926), resulting in an implicit structure.

Years later, other authors introduced different multistep approaches, such as the Backward Differentiation Formula (BDF) (Curtiss and Hirschfelder, 1952) which has a special role in solving stiff problems. The development of modern theory in linear multistep theory is also credited to Curtiss and Hirschfelder (1952).

3.2.1 Adams Methods

In this thesis, two different Linear Multistep structures are used to obtain discrete-time uncertain systems: Adams-Bashforth and Adams-Moulton. These formulations are presented below.

The main characteristic of these methods is their use of several previous points and derivatives from previous steps. This approach provides more information and aims to enhance the efficiency of the approximation process, rather than discarding these data.

The Adams-Bashforth method is the most widely used family of explicit linear multistep methods for the solution of ordinary differential equations. The Adams-Moulton method, in contrast, is implicit. Both procedures have the goal of finding approximate solutions to initial value problems of the form

$$\dot{x} = f(t,x), \quad x(t_0) = x_0. \tag{3.3}$$

If a fixed step size, given by Δt, is considered, the approximate solution can be written as

$$x(t) = x(t-\Delta t) + \Delta t \left(\sum_{k=0 \text{ or } 1}^{N} \beta_k f(t-k\Delta t, x(t-k\Delta t)) \right) \tag{3.4}$$

where k starts either at 0 or 1 if (3.4) is related to Adams-Moulton or Adams-Bashforth method, respectively.

As demonstrated in Butcher (2008), the constants β_k are selected to ensure that the method reaches order N. To achieve this, the approximate solution (3.4) is reformulated as

$$x(t) = x(t-\Delta t) + \Delta t \left(\sum_{k=0 \text{ or } 1}^{N} \beta_k \dot{x}(t-k\Delta t) \right) \tag{3.5}$$

Considering the Adams-Bashforth method and expanding (3.5), one has

$$x(t) = x(t-\Delta t) + \Delta t\beta_1 \dot{x}(t-\Delta t) + \Delta t\beta_2 \dot{x}(t-2\Delta t) + \ldots + \Delta t\beta_N \dot{x}(t-N\Delta t) \tag{3.6}$$

For each term in (3.6) it is necessary to perform a Taylor Series Expansion around time t as defined in (3.1). The generalized Taylor series expansion of $x(t)$ is given by

$$\begin{aligned} x(t-\Delta t) = x(t) - \dot{x}(t)\Delta t + \ddot{x}(t)\frac{(\Delta t)^2}{2} - \dddot{x}(t)\frac{(\Delta t)^3}{6} + \ddddot{x}(t)\frac{(\Delta t)^4}{24} \\ + \ldots + x^{(N)}(t)\frac{(-\Delta t)^N}{N!} + \frac{x^{(N+1)}(c_0)}{(N+1)!}(-\Delta t)^{N+1} \end{aligned} \tag{3.7}$$

and the generalized Taylor series expansion of $\dot{x}(t)$ is described by

$$\begin{aligned} \dot{x}(t-k\Delta t) = \dot{x}(t) - \ddot{x}(t)k\Delta t + \dddot{x}(t)\frac{k^2(\Delta t)^2}{2} - \ddddot{x}(t)\frac{k^3(\Delta t)^3}{6} + \\ x^{(5)}(t)\frac{k^4(\Delta t)^4}{24} + \ldots + x^{(N)}(t)\frac{k^{N-1}(-\Delta t)^{N-1}}{(N-1)!} + \frac{x^{(N+1)}(c_k)}{N!}(-k\Delta t)^N \end{aligned} \tag{3.8}$$

where c_0 and c_k are real values such that $c_k \in [t_0, t]$, with $k = 0,1,\ldots,N$. Therefore, using (3.7) and (3.8), Adams-Bashforth expression (3.6) is rewritten as

$$\begin{aligned} x(t) = x(t) + \sum_{i=1}^{N} \frac{(-\Delta t)^i}{i!} x^{(i)}(t) + \frac{x^{(N+1)}(c_0)}{(N+1)!}(-\Delta t)^{N+1} \\ + \sum_{k=1}^{N} \Delta t\beta_k \left(\dot{x}(t) + \sum_{i=2}^{N} \frac{k^{i-1}(-\Delta t)^{i-1}}{(i-1)!} x^{(i)}(t) \right) + \sum_{k=1}^{N} \Delta t\beta_k \frac{x^{(N+1)}(c_k)}{N!}(-k\Delta t)^N, \end{aligned} \tag{3.9}$$

in which the β_k constants are chosen to ensure the equality holds when the higher orders are ignored. Solving the linear system for each order, it is possible to recover the constants presented in Table 1. Following the same steps and considering $k = 0$ in (3.5), a closed

Table 1 – Coefficients for Adams-Bashforth methods

k	β_1	β_2	β_3	β_4	β_5	β_6
1	1					
2	$\frac{3}{2}$	$-\frac{1}{2}$				
3	$\frac{23}{12}$	$-\frac{4}{3}$	$\frac{5}{12}$			
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{3}{8}$		
5	$\frac{1901}{720}$	$-\frac{1387}{360}$	$\frac{109}{30}$	$-\frac{637}{360}$	$\frac{251}{770}$	
6	$\frac{4277}{140}$	$-\frac{2641}{480}$	$\frac{4991}{720}$	$-\frac{3649}{720}$	$\frac{959}{480}$	$-\frac{95}{288}$

Table 2 – Coefficients for Adams-Moulton methods

k	β_0	β_1	β_2	β_3	β_4	β_5	β_6
0	1						
1	$\frac{1}{2}$	$\frac{1}{2}$					
2	$\frac{5}{12}$	$\frac{2}{3}$	$-\frac{1}{12}$				
3	$\frac{3}{8}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$			
4	$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{770}$		
5	$\frac{95}{288}$	$\frac{1427}{1140}$	$-\frac{133}{240}$	$\frac{241}{720}$	$-\frac{173}{1440}$	$\frac{3}{160}$	
6	$\frac{19087}{60480}$	$\frac{2713}{2520}$	$-\frac{15487}{20160}$	$\frac{586}{945}$	$-\frac{6737}{20160}$	$\frac{263}{2520}$	$-\frac{863}{60480}$

formulation to the Adams-Moulton Method can be obtained and the corresponding are shown in Table 2.

These coefficients are widely known in the literature and are usually presented until order $k = 6$ due to the multistep methods suffering with numerical stability when they are considered orders bigger than 6.

3.3 Uncertain Linear Systems

Mathematical models are widely used to describe the dynamics of real systems and processes. By definition, a model is an approximation, that is, it is not an exact representation of the system. The main causes for the difference between the mathematical models and the physical systems are (1) lack of knowledge about the physical structure of the system under study and (2) the use of too simple models to apply analysis and design techniques (Chesi et al., 2009).

This difference between the mathematical model and the original system may be called uncertainty. There are also other sources of uncertainty, for instance, parameter variations, noises, among others. However, when it comes to stability analysis and the design of robust controllers, it is interesting to incorporate them into the model.

In this section, two types of structured uncertainties widely disseminated in the literature in applications involving controller synthesis are introduced. The first is polytopic uncertainty, which is widely used due to its ease of representation and generality, as well as being widely used in systems that have different operating points. The second is the norm-bounded uncertainty, which allows for flexibility in modeling uncertainties, including parameter variations, unmodeled dynamics, and external disturbances, as long as they can be bounded by a norm (Caun et al., 2018). Both structured uncertainties are simple to deal with when the applied methodologies are based on Lyapunov Theory and LMIs.

The formulation of a continuous-time uncertain linear system with polytopic uncertainty is given by

$$\dot{x}(t) = A_\lambda x(t) + B_\lambda u(t) \tag{3.10}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input and matrices $A_\lambda \in \mathbb{R}^{n\times n}$ and $B_\lambda \in \mathbb{R}^{n\times n_u}$ are not precisely known, but belong to a convex bounded uncertain domain (polytopic domain $\mathcal{P}$) given by

$$\mathcal{P} = \left\{ A_\lambda = \sum_{r=1}^{R} \lambda_r A_r, \quad B_\lambda = \sum_{r=1}^{R} \lambda_r B_r \right\} \tag{3.11}$$

where A_r and B_r, $r = 1, \ldots, R$, are the vertices of the polytope, and $\lambda \in \Lambda_R$ with

$$\Lambda_R = \left\{ \lambda \in \mathbb{R}^R : \sum_{r=1}^{R} \lambda_r = 1,\ \lambda_r \ \geq 0,\ \ r = 1, \ldots, R \right\}.$$

An uncertain continuous-time linear system can be also represented with norm-bounded uncertainties:

$$\dot{x}(t) = (A + F_A\Delta_A)x(t) + (B + F_B\Delta_B)u(t). \tag{3.12}$$

The definitions of matrices and vectors in (3.12) are similar to the definitions presented for system (3.10) with the addition of F_A and F_B, which are known matrices that indicate the input directions of Δ_A and Δ_B. Matrices Δ_A and Δ_B, in turn, are unknown and satisfy $||\Delta_A||_2 < 1$ and $||\Delta_B||_2 < 1$ or equivalently $\Delta_A^T\Delta_A < I$ and $\Delta_B^T\Delta_B < I$.

Finally, it is possible to combine both representations and obtain an uncertain continuous-time representation considering polytopic and norm-bounded uncertainties:

$$\dot{x}(t) = (A_\lambda + F_A\Delta_A)x(t) + (B_\lambda + F_B\Delta_B)u(t). \tag{3.13}$$

In a similar way, there are also representations of uncertain discrete-time linear systems. For the polytopic system we have

$$x_{k+1} = A_{d_\lambda} x_k + B_{d_\lambda} u_k \tag{3.14}$$

where $x_k \in \mathbb{R}^n$ is a simplified way to write $x(k\Delta t)$ and, consequently $x_{k+1} \in \mathbb{R}^n$ is equivalent to $x((k+1)\Delta t)$. $A_{d_\lambda} \in \mathbb{R}^{n\times n}$ and $B_{d_\lambda} \in \mathbb{R}^{n\times n_u}$ are discrete-time uncertain matrices.

Norm-bounded discrete-time uncertain systems can be described as

$$x_{k+1} = (A_d + F_A\Delta_A)x_k + (B_d + F_B\Delta_B)u_k \tag{3.15}$$

and, lastly, a discrete-time representation that combines both types of uncertainty is given by

$$x_{k+1} = (A_{d_\lambda} + F_A\Delta_A)x_k + (B_{d_\lambda} + F_B\Delta_B)u_k. \tag{3.16}$$

The definitions presented to the discrete-time system (3.14) are also valid to the discrete-time representations (3.15) and(3.16).

3.4 S-Procedure

The S-procedure is a mathematical tool widely used in control and optimization theory to deal with problems involving quadratic inequalities and uncertainties in dynamic systems. It provides a way to relax nonlinear inequality conditions into conditions that can be expressed by LMIs. S-procedure sometimes provides conservative results, however it is often useful approximation. For this, the S-procedure presented in Lemma 1 is employed.

Lemma 1 (S-Procedure (Boyd et al., 1994))**.** *Let* $F_0 = F_0^T$ *and* $F_1 = F_1^T \ \in R^{n\times n}$

$$\forall\ z,\quad z^T F_1 z \geq 0 \ \longrightarrow\ z^T F_0 z \geq 0, \tag{3.17}$$

if and only if there is a non-negative scalar $\tau \in R$*, such that* $F_0 - \tau F_1 \geq 0$.

3.5 Finsler's Lemma

The Finsler Lemma is a fundamental result in control theory and optimization that provides a condition for the existence of a solution to a quadratic inequality. It's often used to derive LMIs for robust control and optimization.

Lemma 2. *Consider* $w \in \mathbb{R}^n$*,* $\mathcal{Q} \in \mathbb{R}^{n\times n}$ *and* $\mathcal{B} \in \mathbb{R}^{m\times n}$ *with rank*$(\mathcal{B}) < n$ *and* $\mathcal{B}^\perp$ *a base to the null space of* $\mathcal{B}$ *(that is,* $\mathcal{B}\mathcal{B}^\perp = 0$*). Then the following conditions are equivalent*

- $w^T\mathcal{Q}w \ < \ 0, \quad \forall w \neq 0 \ : \mathcal{B}w \ = \ 0$
- ${\mathcal{B}^\perp}^T\mathcal{Q}\mathcal{B}^\perp \ < \ 0$
- $\exists\mu \ \in \ \mathbb{R} \ : \ \mathcal{Q} - \mu\mathcal{B}^T\mathcal{B} \ < \ 0$
- $\mathcal{X} \ \in \ \mathbb{R}^{n\times m} \ : \ \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T\mathcal{X}^T \ < \ 0$

3.6 Schur Complement

Schur Complement provides a necessary and sufficient condition for the positive definiteness of block matrices.

Lemma 3. *Let*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

a symmetric matrix, where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$, *and* $C \in \mathbb{R}^{n\times n}$. *The following statements hold:*

1. *If* $C \succ 0$, *then* $M \succ 0$ *if and only if the Schur complement of* C *in* M, *given by*

$$S = A - BC^{-1}B^T, \tag{3.18}$$

   is positive definite, i.e., $S \succ 0$.

2. *If* $A \succ 0$, *then* $M \succ 0$ *if and only if the Schur complement of* A *in* M, *given by*

$$S' = C - B^T A^{-1} B, \tag{3.19}$$

   is positive definite, i.e., $S' \succ 0$.

Chapter 4

THE MIXED METHOD — A NEW APPROACH CONSIDERING DELAYED STEPS IN ITS FORMULATION

In this chapter, a new discretization method, henceforth called the Mixed Method, is proposed. It uses delayed values of the state to develop an augmented discrete-time system with more information (and less uncertainty) for the current state. Each subsection presented in the sequence details how the discrete-time uncertain system is constructed. Additionally, state-feedback controller synthesis conditions based on constant and parameter-dependent Lyapunov functions are proposed using the augmented discrete-time model.

4.1 The Mixed Method Approach

Consider the linear time-invariant continuous-time system presented in (3.10), with an explicit solution given by

$$x(t) = e^{A_\lambda t}x(0) + \int_0^t e^{A_\lambda(t-\tau)}B_\lambda u(\tau)d\tau. \tag{4.1}$$

Assuming $t \in [t_0,t_k]$, (4.1) can be rewritten as

$$x_k = e^{A_\lambda(t_k-t_0)}x_0 + \int_{t_0}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau, \tag{4.2}$$

where x_k is the short way to represent $x(t_k)$ and x_0 is the initial condition denoted by $x(t_0)$.

The Mixed Method discretizes the continuous-time system with polytopic uncertainty in (3.10) by *mixing* different solutions for the current state from several delayed steps. In order to propose and analyze the method, consider the discrete-time solution presented in (4.2) for the continuous-time system and time increments of Δt (time or delay step), $t_k = t_{k-1} + \Delta t$.

For one delay step, (4.2) can be rewritten as follows

$$x_k = e^{A_\lambda(t_k-(t_k-\Delta t))}x_{k-1} + \int_{t_k-\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau, \tag{4.3}$$

$$x_k = e^{A_\lambda \Delta t}x_{k-1} + \int_{t_k-\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau, \tag{4.4}$$

for two delay steps as

$$x_k = e^{A_\lambda(t_k-(t_k-2\Delta t))}x_{k-2} + \int_{t_k-2\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau, \tag{4.5}$$

$$x_k = e^{A_\lambda 2\Delta t}x_{k-2} + \int_{t_k-2\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau, \tag{4.6}$$

and so on. For an arbitrary N number of steps, the solution becomes

$$x_k = e^{A_\lambda(t_k-(t_k-N\Delta t))}x_{k-N} + \int_{t_k-N\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau, \tag{4.7}$$

$$x_k = e^{A_\lambda N\Delta t}x_{k-N} + \int_{t_k-N\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau. \tag{4.8}$$

The proposed Mixed Method joins all these different solutions, weighted by scalar values, to obtain a solution that uses different delays of the state. The general solution is then given by

$$x_k = \sum_{s=1}^{N} \alpha_s e^{A_\lambda s\Delta t}x_{k-s} + \sum_{s=1}^{N} \alpha_s \int_{t_k-s\Delta t}^{t_k} e^{A_\lambda(t-\tau)}B_\lambda u(\tau)d\tau, \tag{4.9}$$

where $\alpha_s \in \mathbb{R}$ with $s = 1, \ldots, N$, $\sum_{s=1}^{N} \alpha_s = 1$ are coefficients that represent the importance of each delayed discrete-time system and N refers to the highest delayed value considered in the Mixed Method.

Even though the input signal does not change between discrete time steps, it is still dependent on τ in (4.9). To indicate the dependence on the delayed input values, and not explicitly on τ, the integral term in (4.9) should undergo a transformation. By assuming that the control inputs are held constant in between time steps, it follows that

$$\begin{aligned}\int_{t_k-s\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}B_\lambda u(\tau)d\tau = \\ \int_{t_k-\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)}d\tau B_\lambda u_{k-1} + \int_{t_k-2\Delta t}^{t_k-\Delta t} e^{A_\lambda(t_k-\tau)}d\tau B_\lambda u_{k-2} + \\ \ldots + \int_{t_k-s\Delta t}^{t_k-(s-1)\Delta t} e^{A_\lambda(t_k-\tau)}d\tau B_\lambda u_{k-s}\end{aligned} \tag{4.10}$$

and, in a concise form, as

$$\int_{t_k-s\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)} B_\lambda u(\tau)d\tau = \sum_{\ell=1}^{s} \int_{t_k-\ell\Delta t}^{t_k-(\ell-1)\Delta t} e^{A_\lambda(t_k-\tau)} d\tau B_\lambda u_{k-\ell}. \tag{4.11}$$

Note that the limits of the integral in (4.11) are function of t_k, and Δt. To remove t_k a few transformations are required.

The first transformation uses $\tau = \bar{\tau} - (\ell-1)\Delta t$ and $d\tau = d\bar{\tau}$ to yield

$$\int_{t_k-s\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)} B_\lambda u(\tau)d\tau = \sum_{\ell=1}^{s} \int_{t_k-\Delta t}^{t_k} e^{A_\lambda(t_k-\bar{\tau}+(\ell-1)\Delta t)} d\bar{\tau} B_\lambda u_{k-\ell}. \tag{4.12}$$

A second transformation is performed considering $\bar{\bar{\tau}} = t_k - \bar{\tau} + (\ell-1)\Delta t$, $d\bar{\bar{\tau}} = -d\bar{\tau}$ and the new upper and lower limits are given by $\ell\Delta t$ and $(\ell-1)\Delta t$. The integral described in (4.12) is then written as

$$\int_{t_k-s\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)} B_\lambda u(\tau)d\tau = \sum_{\ell=1}^{s} \int_{\ell\Delta t}^{(\ell-1)\Delta t} -e^{A_\lambda\bar{\bar{\tau}}} d\bar{\bar{\tau}} B_\lambda u_{k-\ell} = \sum_{\ell=1}^{s} \int_{(\ell-1)\Delta t}^{\ell\Delta t} e^{A_\lambda\bar{\bar{\tau}}} d\bar{\bar{\tau}} B_\lambda u_{k-\ell}. \tag{4.13}$$

Finally, making $\tilde{\tau} = \bar{\bar{\tau}} - (\ell-1)\Delta t$ and $d\tilde{\tau} = d\bar{\bar{\tau}}$ in the last transformation yields

$$\int_{t_k-s\Delta t}^{t_k} e^{A_\lambda(t_k-\tau)} B_\lambda u(\tau)d\tau = \sum_{\ell=1}^{s} \int_{0}^{\Delta t} e^{A_\lambda(\tilde{\tau}+(\ell-1)\Delta t)} d\tilde{\tau} B_\lambda u_{k-\ell}. \tag{4.14}$$

Using (4.14), the condition presented in (4.9) can be rewritten as

$$x_k = \sum_{s=1}^{N} \alpha_s \left(e^{A_\lambda s\Delta t} x_{k-s} + \sum_{\ell=1}^{s} \int_{0}^{\Delta t} e^{A_\lambda(\tilde{\tau}+(\ell-1)\Delta t)} d\tilde{\tau} B_\lambda u_{k-\ell} \right). \tag{4.15}$$

The first exponential term in (4.15), that is multiplied by the discrete-time state x_{k-s}, is approximated by a first-order Taylor Series expansion as follows

$$e^{(A_\lambda s\Delta t)} \approx I + s\Delta t A_\lambda = A_{d_{s\lambda}}. \tag{4.16}$$

For the exponential in the integral term of (4.15), that is, the exponential that multiplies the matrix B_λ, a zero-order approximation of the Taylor series expansion is used, so that the degree of the homogeneous polynomial that describes the uncertainty is maintained

$$\int_{0}^{\Delta t} e^{A_\lambda(\tilde{\tau}+(\ell-1)\Delta t)} d\tilde{\tau} B_\lambda \approx \int_{0}^{\Delta t} I d\tilde{\tau} B_\lambda \approx \Delta t B_\lambda = B_{d_\lambda}. \tag{4.17}$$

Putting everything together, the resulting discrete-time model system is

$$x_k \approx \sum_{s=1}^{N} \alpha_s (I + A_\lambda s\Delta t) x_{k-s} + \sum_{s=1}^{N} \sum_{\ell=1}^{s} \alpha_s \Delta t B_\lambda u_{k-\ell} \tag{4.18}$$

or

$$x_k \approx \sum_{s=1}^{N} \alpha_s A_{d_{s\lambda}} x_{k-s} + \sum_{s=1}^{N} \sum_{\ell=1}^{s} \alpha_s B_{d_\lambda} u_{k-\ell} \tag{4.19}$$

To see how α_s affects B_d terms, (4.19) can be rewritten as

$$x_k \approx \sum_{s=1}^{N} \alpha_s A_{d_{s\lambda}} x_{k-s} + \sum_{\ell=1}^{N} \left(\sum_{s=\ell}^{N} \alpha_s \right) B_{d_\lambda} u_{k-\ell}. \tag{4.20}$$

The approximation model presented in (4.20) can be viewed as an explicit multistep method.

4.1.1 Norm-Bounded Uncertainty and coefficients definition

The approximation presented in (4.18) is a discrete model built using Taylor series expansion to approximate the exponential matrices from (4.15). Since there are differences between these models, they must be calculated to be incorporated as uncertainties related to the discretization process in the uncertain discrete-time model and then the error inherent in discretization schemes should be treated. Jungers et al. (2017) used the norm of the uncertain matrix and the measure of the square matrix to define an upper bound for the discretization error analytically; Braga et al. (2019) used Monte Carlo approach in a fine grid on the parameter vector λ to define the discretization error and use this result to represent the norm of bounded uncertainties in an uncertain discrete-time system. This method will be adopted below. For this, consider

$$\delta_{A_{si}} = || \underbrace{I + A_\lambda s\Delta t}_{A_{d_{s\lambda}}} - e^{A_\lambda s\Delta t} || \tag{4.21}$$

$$\delta_{B_{\ell i}} = || \underbrace{\Delta t B_\lambda}_{B_{d_\lambda}} - \int_0^{\Delta t} e^{A_\lambda (\tilde{\tau} + (\ell-1)\Delta t)} d\tilde{\tau} B_\lambda ||, \tag{4.22}$$

where $\delta_{A_{s_i}}$ and $\delta_{B_{\ell_i}}$, with $i = 1, \ldots, n$, are given by the Euclidean norm of the differences in the equations, considering s delay and i iteration. The more iterations are considered, the higher the computational cost, however, a better approximation is obtained. For each delay, there is a norm-bounded uncertainty defined by

$$||\Delta_{\bar{A}_s}|| \leq \max(\delta_{A_{si}}), \quad ||\Delta_{\bar{B}_\ell}|| \leq \max(\delta_{B_{\ell i}}). \tag{4.23}$$

Thus, an uncertain discrete-time system that incorporates the uncertainties can be written as

$$x_k = \sum_{s=1}^{N} \alpha_s (A_{d_{s\lambda}} + \Delta_{\bar{A}_s}) x_{k-s} + \sum_{\ell=1}^{N} \left(\sum_{s=\ell}^{N} \alpha_s \right) (B_{d_\lambda} + \Delta_{\bar{B}_\ell}) u_{k-\ell}. \tag{4.24}$$

The next step is to find the coefficients α_s. Note that, just as the individual delayed models are weighted by α_s, the norm-bounded uncertainties are also weighted by these coefficients. Then, an optimization method will be used. The objective function depends on the delayed discrete-time system. To minimize the squared norm of the uncertainties, a slack variable c that sets an upper bound for each uncertainty individually is used. So, from (4.24), it is possible to write the constrains

$$\alpha_s^2 \Delta_{\bar{A}_s}^2 \leq c \quad \text{and} \quad \left(\sum_{s=\ell}^{N} \alpha_s \right)^2 \Delta_{\bar{B}_\ell}^2 \leq c \tag{4.25}$$

with $s = 1, \ldots, N$ and $\ell = 1, \ldots, N$. The choice of which uncertainties are considered in the optimization is flexible, but it should cover cases for the uncertainties of matrices A_{d_s} or the uncertainties of matrices B_d or even both sets. The proposed optimization problem contains extra constraints to enforce the zero-stability of the numerical method used in the discretization and, in doing so, it guarantees good characteristics of the discretization procedure, such as consistency, zero-stability, and convergence. Consistency implies that the truncation error tends to zero whenever the sampling time is sufficiently small (it approaches zero). Zero stability means small changes in the initial conditions produce correspondingly small changes in the subsequent approximations. Finally, if the method is convergent, the solution of the discretized equations tends to the exact solution of the differential equation as the grid spacing tends to zero (Ferziger et al., 2002). The constraints of the Mixed Method can assure zero-stability and consistency, which is sufficient to ensure its convergence.

Lemma 4 (Consistency Butcher (2008))**.** *The multistep method is a consistent method due to how the methodology is composed, that is, if*

$$\sum_{s=1}^{N} \alpha_s = 1 \tag{4.26}$$

and

$$\alpha_1 + 2\alpha_2 + \ldots + k a_k = \beta_1 + \beta_2 + \ldots + \beta_k \tag{4.27}$$

Considering (4.18), our method could be interpreted as

$$x_k \approx \sum_{s=1}^{N} (\alpha_s x_{k-s} + s\alpha_s \Delta t \dot{x}_{k-s}) \approx \sum_{s=1}^{N} (\alpha_s x_{k-s} + \beta_s \Delta t \dot{x}_{k-s}) \tag{4.28}$$

with $\dot{x}_{k-s} = A_d x_{k-s} + B_d u_{k-s}$ and which is more along the usual lines of a multistep method. Since by definition $\beta_s = s\alpha_s$, our method is consistent by construction.

Lemma 5 (Root condition Butcher (2008)). *A linear multistep method is zero-stable if and only if every root r of the generating polynomial $\rho(z)$ satisfies $|r| \leq 1$, and any root r with $|r| = 1$ is simple.*

Lemma 6 (Convergence Butcher (2008)). *A stable consistent linear multistep method is convergent.*

Using Lemmas 4 and 5, a set of conditions that should guarantee a zero-stable numerical method is developed. To this end, the constraints applied in the optimization procedure use the approximation of x_k given by

$$x_k = \sum_{s=1}^{N} \alpha_s I x_{k-s}. \tag{4.29}$$

In the z-domain, (4.29) can be simplified to

$$1 = \sum_{s=1}^{N} \alpha_s z^{-s} \tag{4.30}$$

or, equivalently,

$$1 - \alpha_1 z^{-1} - \alpha_2 z^{-2} - \ldots - \alpha_N z^{-N} = 0. \tag{4.31}$$

The proposed numerical approximation scheme always has a generating polynomial with a root at $z = 1$. To avoid numerical instability in the optimization, the characteristic polynomial in (4.31) is divided by $z - 1$ and therefore the marginally stable root is removed. The resulting polynomial is shown in (4.32)

$$z^{-1} + (1 - \alpha_1)z^{-2} + (1 - \alpha_1 - \alpha_2)z^{-3} + \ldots + (1 - \alpha_1 - \alpha_2 - \ldots \alpha_{N-1})z^{-N} = 0. \tag{4.32}$$

By multiplying (4.32) by z^N with $z \neq 0$ on both sides and considering (4.26) yields

$$\alpha_N + (\alpha_N + \alpha_{N-1})z + \ldots + \left(\sum_{s=0}^{N-2} \alpha_{N-s}\right) z^{N-2} + z^{N-1} = 0. \tag{4.33}$$

The discrete characteristic polynomial in Equation (4.33) can be written in the controllable canonical form as

$$\Omega = \left[\begin{array}{c|c} 0_{(N-1)\times 1} & I_{(N-1)} \\ \hline -\alpha_N & \upsilon \end{array}\right]$$

, where

$$\upsilon = \begin{bmatrix} -\sum_{i=N-1}^{N} \alpha_i & -\sum_{i=N-2}^{N} \alpha_i & \dots & -\sum_{i=2}^{N} \alpha_i \end{bmatrix} \tag{4.34}$$

Since matrix Ω has (4.33) as its characteristic equation, the corresponding discretization method will be zero-stable, and convergent, as long as it is Schur stable. To ensure the Schur stability of matrix Ω, a set of Matrix Inequalities and a quadratic Lyapunov function will be employed. For this, consider

$$\phi_{k+1} = \Omega \phi_k \tag{4.35}$$

in which $\phi_k^T = [x_{k-N}^T \ \dots \ x_{k-2}^T \ x_{k-1}^T \ x_k^T]^T$, and Lyapunov function given by

$$V(\phi_k) = \phi_k^T P \phi_k \geq 0 \tag{4.36}$$

and

$$\Delta V(\phi_k) = \phi_{k+1}^T P \phi_{k+1} - \phi_k^T P \phi_k \leq 0 \tag{4.37}$$

replacing ϕ_{k+1}

$$\Delta V(\phi_k) = \phi_k^T \Omega^T P \Omega \phi_k - \phi_k^T P \phi_k \leq 0. \tag{4.38}$$

Then applying a Schur complement and the congruence transformation $\mathcal{T} = \text{diag}(P,I)$ one gets

$$\begin{bmatrix} -P & \Omega^T P \\ P\Omega & -P \end{bmatrix} \leq 0. \tag{4.39}$$

The coefficients of the Mixed Method are then determined by solving a non-convex as follow

$$\min_{c,\alpha_s,P} \quad c \tag{4.40}$$

$$\text{subject to:} \quad \sum_{s=1}^{N} \alpha_s = 1, \tag{4.41}$$

$$\alpha_s^2 \Delta_{\bar{A}_s}^2 \leq c, \quad s = 1, \dots, N, \tag{4.42}$$

$$\left(\sum_{s=\ell}^{N} \alpha_s \right)^2 \Delta_{B_\ell}^2 \leq c, \quad \ell = 1, \dots, N, \tag{4.43}$$

$$P \geq I, \tag{4.44}$$

$$\begin{bmatrix} -P & \Omega^T P \\ P\Omega & -P \end{bmatrix} \leq 0. \tag{4.45}$$

This set of constraints guarantees stability, consistency, and convergence for the methodology proposed in this chapter as defined in Lemma 6. It is worth noting that constraints (4.42), (4.43), and (4.45) is a Bilinear Matrix Inequality (BMI), involving a product between two decision variables, Ω and P. As a result, the optimization problem may lead to a local minimum instead of a global one. Thus, when utilizing constraint (4.45), there is no guarantee that the coefficients found represent the minimum uncertainty.

Although the best way to find the coefficients in the Mixed Method is still an open problem, the proposed procedure guarantees consistency, zero-stability, and convergence. Numerical examples will be given in the Chapter 6 to illustrate how these characteristics can result in a discretization method with interesting properties.

4.1.2 Augmented Representation

Using the discretization given by (4.18) and considering the uncertainties (represented by $\Delta_{\bar{A}_s}$ and $\Delta_{\bar{B}_\ell}$) have already been calculated, an augmented system is now proposed.

The augmented state vector is defined as

$$\xi_k = \begin{bmatrix} u^T_{k-(N-1)} & \cdots & u^T_{k-1} & x^T_{k-(N-1)} & \cdots & x^T_k \end{bmatrix}^T \tag{4.46}$$

With the new state vector just defined, the discrete-time system that represents the continuous-time system (3.10) has the following general form,

$$\begin{aligned} \xi_{k+1} = \bar{A}_\lambda \xi_k + \bar{B} u_k + F_A \Delta_{A_1} G_{A_1} \xi_k + \\ \ldots + F_A \Delta_{A_N} G_{A_N} \xi_k + F_A \Delta_{A_{N+1}} G_{A_{N+1}} \xi_k + \\ \ldots + F_A \Delta_{A_{2N-1}} G_{A_{2N-1}} \xi_k + F_B \Delta_B u_k, \end{aligned} \tag{4.47}$$

where $\Delta_{A_s} = |\alpha_s| \Delta_{\bar{A}_s}$ with $s = 1, \ldots, N$, $\Delta_{A_{N+\ell-1}} = \left(\sum_{s=\ell}^N \alpha_s\right) \Delta_{B_\ell}$, $\Delta_B = \sum_{i=2}^N \left(|\sum_{j=2}^i \alpha_j| \right) \Delta_{B_\ell}$ with $\ell = 2, \ldots, N$. $\bar{A}_\lambda$, $\bar{B}_\lambda$,F_A, G_{A_i} and F_B are defined in the sequel.

$$\bar{A}_\lambda = \begin{bmatrix} 0_{(N-2)n_u \times n_u} & I_{(N-2)n_u} & 0_{(N-2)n_u \times n} & 0_{(N-2)n_u \times (N-1)n} \\ 0_{n_u \times n_u} & 0_{n_u \times (N-2)n_u} & 0_{n_u \times n} & 0_{n_u \times (N-1)n} \\ 0_{(N-1)n \times n_u} & 0_{(N-1)n \times (N-2)n_u} & 0_{(N-1)n \times n} & I_{(N-1)n} \\ \alpha_N B_{d\lambda} & \Gamma_1 & \alpha_N A_{d_{N\lambda}} & \Gamma_2 \end{bmatrix} \tag{4.48}$$

with

$$\Gamma_1 = \begin{bmatrix} \sum_{s=N-1}^N \alpha_s B_{d\lambda} & \ldots & \sum_{s=2}^N \alpha_s B_{d\lambda} \end{bmatrix}, \tag{4.49}$$

$$\Gamma_2 = \begin{bmatrix} \alpha_{(N-1)} A_{d_{(N-1)\lambda}} & \ldots & \alpha_2 A_{d_{2\lambda}} & \alpha_1 A_{d_{1\lambda}} \end{bmatrix}, \tag{4.50}$$

$$\bar{B}_\lambda = \begin{bmatrix} 0_{(N-2)n_u \times n_u} \\ I_{n_u} \\ 0_{(N-1)n \times n_u} \\ B_{d\lambda} \end{bmatrix}, \tag{4.51}$$

$$F_A = \left[\begin{array}{c|c} 0_{2(N-1)n \times (N-1)(n+n_u)} & 0_{2(N-1)n \times n} \\ \hline 0_{n \times (N-1)(n+n_u)} & I_n \end{array}\right],$$

$$F_B = \left[\begin{array}{c} 0_{(N-1)(n+n_u) \times n} \\ \hline I_n \end{array}\right]$$

and

$$G_{A_i} = \left[\begin{array}{c|c} 0_{((N-1)n_u+Nn) \times (2(N-1)n)} & \Gamma_{3i} \end{array}\right]$$

where Γ_{3i} is a zero block matrix with the number of columns equal to n, and i denotes the block filled with an identity matrix, I_n. Finally, Δ_{A_i} e Δ_B are unknown matrices, satisfying $||\Delta_{A_i}|| \leq c$ and $||\Delta_B|| \leq c$. In the Appendix A there is an teaching example to demonstrate how to build these augmented matrices.

4.2 State-feedback Controllers Synthesis Conditions

Based on the augmented system (4.47), a digital state-feedback control law that can stabilize the closed-loop system is given by

$$u_k = \bar{K}\xi_k \tag{4.52}$$

two discrete-time control design procedures are proposed to obtain controllers that guarantee the closed-loop stability of the original continuous-time systems (3.10).

Theorem 1. *If there exist a matrix $Z \in \mathbb{R}^{n_u \times (Nn+(N-1)n_u)}$, scalar variables σ_{A_i} ,with $i = 1, \ldots, 2N-1$, σ_B and a symmetric positive definite matrix $P \in \mathbb{R}^{(Nn+(N-1)n_u) \times (Nn+(N-1)n_u)}$, with $N \in \mathbb{N}^*$ such that*

$$P \geq I, \tag{4.53}$$

and (4.55) in which

$$\Psi_1 = -P + \sigma_{A_1} F_A^T F_A + \ldots + \sigma_{A_N} F_A^T F_A + \sigma_{A_{N+1}} F_A^T F_A + \ldots + \sigma_{A_{2N-1}} F_A^T F_A + \sigma_B F_B^T F_B \tag{4.54}$$

then the digital state feedback control law (4.52) with $\bar{K} = ZP^{-1}$ can stabilize the continuous-time system (3.10).

$$\begin{bmatrix} -P & P\bar{A}_\lambda^T + Z^T\bar{B}_\lambda^T & \mu_{A_1}PG_{A_1}^T & \mu_{A_2}PG_{A_2}^T & \dots & \mu_{A_N}PG_{A_N}^T & \mu_{A_{N+1}}PG_{A_{N+1}}^T & \dots & \mu_{A_{2N-1}}PG_{A_{2N-1}}^T & \mu_B Z^T \\ \star & \Psi_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \star & \star & -\sigma_{A_1}I & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \star & \star & \star & -\sigma_{A_2}I & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \star & \star & \star & \vdots & \dots & -\sigma_{A_N}I & 0 & \dots & 0 & 0 \\ \star & \star & \star & \star & \dots & \star & -\sigma_{A_{N+1}}I & \dots & 0 & 0 \\ \star & \star & \star & \star & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \star & \star & \dots & \star & \star & \dots & -\sigma_{A_{2N-1}}I & 0 \\ \star & \star & \star & \star & \dots & \star & \star & \dots & \star & -\sigma_B I \end{bmatrix} \leq 0 \quad (4.55)$$

Proof. Considering the discrete-time system (4.47) and the Lyapunov stability theory, a Lyapunov function is given by

$$V(\xi_k) = \xi_k^T P^{-1}\xi_k \geq 0, \tag{4.56}$$

and

$$\Delta V(\xi_k) \leq 0 \tag{4.57}$$

Furthermore, it is possible to write

$$\Delta V(\xi_k) = \xi_k^T \bar{\mathcal{A}}^T P^{-1}\bar{\mathcal{A}}\xi_k - \xi_k^T P^{-1}\xi_k \leq 0 \tag{4.58}$$

where

$$\begin{aligned} \bar{\mathcal{A}} = \bar{A}_\lambda^T + K^T\bar{B}_\lambda^T + G_{A_1}^T\Delta_{A_1}F_A^T + G_{A_2}^T\Delta_{A_2}F_A^T + \ldots + G_{A_N}^T\Delta_{A_N}F_A^T \\ + G_{A_{N+1}}^T\Delta_B F_A^T + \ldots + G_{A_{2N-1}}^T\Delta_B F_A^T + K^T\Delta_B^T F_B^T, \end{aligned} \tag{4.59}$$

applying Schur complement in (4.58), one gets

$$\begin{bmatrix} -P^{-1} & \bar{\mathcal{A}}^T \\ \star & -P \end{bmatrix} \leq 0. \tag{4.60}$$

Condition (4.60) is split into $2N+2$ parts

$$\begin{aligned}\begin{bmatrix}-P^{-1} & \bar{A}_\lambda^T + K\bar{B}_\lambda^T \\ \bar{A}_\lambda + \bar{B}_\lambda K & -P\end{bmatrix} + \begin{bmatrix}0 & G_{A_1}^T\Delta_{A_1}^T F_A^T \\ F_A\Delta_{A_1}G_{A_1} & 0\end{bmatrix} + \dots \\ + \begin{bmatrix}0 & G_{A_N}^T\Delta_{A_N}^T F_A^T \\ F_A\Delta_{A_N}G_{A_N} & 0\end{bmatrix} + \begin{bmatrix}0 & G_{A_{N+1}}^T\Delta_{A_{N+1}}^T F_A^T \\ F_A\Delta_{A_{N+1}}G_{A_{N+1}} & 0\end{bmatrix} + \dots \\ + \begin{bmatrix}0 & G_{A_{2N-1}}^T\Delta_{A_{2N-1}}^T F_A^T \\ F_A\Delta_{A_{2N-1}}G_{A_{2N-1}} & 0\end{bmatrix} + \begin{bmatrix}0 & \bar{K}^T\Delta_B^T F_B^T \\ F_B\Delta_B\bar{K} & 0\end{bmatrix} \leq 0\end{aligned} \tag{4.61}$$

By considering a majoration given by

$$X^TY + Y^TX < \sigma X^TX + \frac{1}{\sigma}Y^TY \tag{4.62}$$

The result of the parts depending on F_A, Δ_{A_i} and G_{A_i}, with $i = 1, \dots, 2N-1$ can be written as

$$\begin{aligned}\begin{bmatrix}G_{A_i}^T\Delta_{A_i}^T \\ 0\end{bmatrix}\begin{bmatrix}0 & F_A\end{bmatrix} + \begin{bmatrix}0 \\ F_A^T\end{bmatrix}\begin{bmatrix}0 & \Delta_{A_i}G_{A_i}\end{bmatrix} \\ \leq \begin{bmatrix}\frac{1}{\sigma_{A_i}}G_{A_i}^T\Delta_{A_i}^T\Delta_{A_i}G_{A_i} & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & \sigma_{A_i}F_A^TF_A\end{bmatrix}\end{aligned} \tag{4.63}$$

and the result of the majoration applied to the term depending on F_B, Δ_B and $\bar{K}$ is given by

$$\begin{bmatrix}K^T\Delta_B^T \\ 0\end{bmatrix}\begin{bmatrix}0 & F_B\end{bmatrix} + \begin{bmatrix}0 \\ F_B^T\end{bmatrix}\begin{bmatrix}0 & \Delta_BK\end{bmatrix} \leq \begin{bmatrix}\frac{1}{\sigma_B}K^T\Delta_B^T\Delta_BK & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & \sigma_BF_B^TF_B\end{bmatrix} \tag{4.64}$$

Besides, the majorations described by

$$\Delta_{A_s}^T\Delta_{A_s} \leq ||\Delta_{A_s}||_2 I = \mu_{A_s}^2 I \tag{4.65}$$

$$\Delta_B^T\Delta_B \leq ||\Delta_B||_2 I = \mu_B^2 I \tag{4.66}$$

are applied to (4.61). It is noteworthy that the majoration applied to each Δ_{A_s} can be different, since the norm-bound is distinct for each delay state.

The following inequality is obtained by summing all transformed parts

$$\begin{bmatrix}-P^{-1} + \Psi_2 & \bar{A}_\lambda^T + \bar{K}^T\bar{B}_\lambda^T \\ \bar{A}_\lambda + \bar{B}_\lambda\bar{K} & -P + \Psi_3\end{bmatrix} \leq 0, \tag{4.67}$$

where Ψ_2 and Ψ_3 depend on how many delays are chosen to build the discrete-time system, and they are given by

$$\Psi_2 = \sum_{i=1}^{2N-1} \frac{1}{\sigma_{A_i}} \mu_{A_i}^2 G_{A_i}^T G_{A_i} + \frac{1}{\sigma_B} \mu_B^2 \bar{K}^T I \bar{K} \tag{4.68}$$

and

$$\Psi_3 = \sum_{i=1}^{2N-1} \sigma_{A_i} F_A^T F_A + \sigma_B \mu_B^2 F_B^T I F_B. \tag{4.69}$$

Applying the congruence transformation $\mathcal{T} = \mathrm{diag}(P,I)$ to the right and to left of (4.67) and then performing some Schur complements leads to inequality (4.55), and that concludes the proof. □

Remark 1. *Methodologies using the Lyapunov-Krasovskii functional are not applied because the discrete-time system obtained incorporates delayed steps. This increases complexity, as the functional grows with the number of delayed steps.*

The conditions proposed in Theorem 1 have some inherent conservativeness due to adopting a constant (quadratic) Lyapunov function. To reduce such conservativeness, a parameter-dependent Lyapunov function is considered and the attained conditions are reported in Theorem 2.

Theorem 2. *If there exist a matrix $\bar{Z} \in \mathbb{R}^{n_u \times (Nn+(N-1)n_u)}$, a slack matrix $M \in \mathbb{R}^{(Nn+(N-1)n_u)\times(Nn+(N-1)n_u)}$, scalar variables σ_i, with $i = 1,\ldots,2N-1$, σ_B and symmetric positive definite matrices $P_\lambda \in \mathbb{R}^{(Nn+(N-1)n_u)\times(Nn+(N-1)n_u)}$, $N \in \mathbb{N}^*$ such that*

$$P_\lambda \geq I, \tag{4.70}$$

and (4.71) with Ψ_1 given by (4.54), with P replaced by P_λ, then the digital state feedback control law (4.52) with $\bar{K} = ZM^{-1}$ can stabilize the continuous-time system (3.10).

$$
\begin{bmatrix}
P_\lambda - M^T - M & M\bar{A}_\lambda^T + \bar{Z}^T\bar{B}_\lambda^T & \mu_{A_1} M G_{A_1}^T & \mu_{A_2} M G_{A_2}^T & \dots & \mu_{A_N} M G_{A_N}^T & \mu_{A_{N+1}} M G_{A_{N+1}}^T & \dots & \mu_{A_{2N-1}} M G_{A_{2N-1}}^T & \mu_B \bar{Z}^T \\
\star & \Psi_1 & 0 & 0 & \dots & 0 & 0 & \dots & f0 & 0 \\
\star & \star & -\sigma_{A_1} I & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
\star & \star & \star & -\sigma_{A_2} I & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\
\star & \star & \star & \vdots & \dots & -\sigma_{A_N} I & 0 & \dots & 0 & 0 \\
\star & \star & \star & \star & \dots & \star & -\sigma_{A_{N+1}} I & \dots & 0 & 0 \\
\star & \star & \star & \star & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\star & \star & \star & \star & \dots & \star & \star & \dots & -\sigma_{A_{2N-1}} I & 0 \\
\star & \star & \star & \star & \dots & \star & \star & \star & \star & -\sigma_B I
\end{bmatrix} \leq 0 \quad (4.71)
$$

Proof. This proof follows the same steps presented in Theorem 1, but the congruence transformation is given by $\mathcal{T} = diag(M,I)$ and then the following upper bound

$$
-M^T P_\lambda^{-1} M \leq P_\lambda - M^T - M \tag{4.72}
$$

is used in the term at position (1,1) in the LMI with this condition (4.71) is recovered. □

Remark 2. *If $M = P$ in Theorem 2, Theorem 1 is recovered.*

Chapter 5

ADAMS-BASHFORTH AND ADAMS MOULTON BASED METHODS

In this chapter, another discretization method, that builds upon the concepts of Multistep theory, is developed. This method relies on the Adams-Bashforth and Adams-Moulton Methods. The first subsection explains how to calculate the upper bounded discretization error. The second subsection illustrates how to construct a discrete-time model using an augmented structure with delayed steps. Finally, state-feedback controller synthesis conditions that account for discretization error in their formulation are proposed, based on constant and parameter-dependent Lyapunov functions using the augmented discrete-time model.

5.1 Upper Bound for the Discretization Error

Determining upper-bounds for the discretization, as performed in Chapter 4, is an interesting procedure and can be applied to a variety of scenarios. However, the fine grid procedure employed in the previous approach is costly, and theoretical upper-bounds are usually very conservative. Hence, we propose a different approach in which an upper-bound for the discretization error between systems (3.10) and (3.14) is computed without the need of performing a fine grid. This result is presented in Theorem 3.

Theorem 3. *Consider matrices A_λ and B_λ from the continuous-time uncertain system (3.10), scalars $\eta \in (0,1)$, $\overline{x}$ and $\overline{u}$ such that $u^T u \leq \overline{u}^2$ and $x^T x \leq \overline{x}^2$, and a matrix with slack variables $\mathcal{V} \in \mathbb{R}^{(2N+2)n+n+n_u \times ((2N+2)n+n)}$. Defining $\mathcal{B} \in \mathbb{R}^{((2N+2)n+n)\times((2N+2)n+n+n_u)}$*

with

$$\mathcal{B}_{(i,j)} = \begin{cases} b_{11} = -B_\lambda \\ b_{ij} = -A_\lambda, & \text{if } i = j \text{ and } i \neq 1 \text{ and } i \neq N+2 \\ b_{ij} = I_n, & \text{if } j = i+1 \text{ and } i \neq N+2 \\ b_{ij} = 0_n, & \text{if } i = N+2 \text{ and } j \neq 1 \\ b_{ij} = 0_{n_u \times n}, & \text{if } i \neq 1 \text{and } j = 1 \\ b_{ij} = 0_{n \times n}, & \text{otherwise} \end{cases} \tag{5.1}$$

and $\mathcal{Q} \in \mathbb{R}^{((2N+2)n+n+n_u)\times((2N+2)n+n+n_u)}$ *described by*

$$\mathcal{Q}_{(i,j)} = \begin{cases} q_{(1,1)} = \frac{-\eta}{\overline{u}^2} I_{n_u}, \\ q_{(i,j)} = \frac{-(1-\eta)}{\overline{x}^2} I_n, & \text{if } (i = N+3,\ j = N+3), \\ q_{(i,j)} = \phi I_n, & \begin{cases} \text{if } (i = N+2,\ j = N+2), \\ \text{if } (i = N+2,\ j = 2N+4), \\ \text{if } (i = 2N+3,\ j = N+2), \\ \text{if } (i = 2N+4,\ j = 2N+4), \end{cases} \\ q_{(i,j)} = 0_{n \times n_u}, & \text{if } i \neq 1 \text{ and } j = 1, \\ q_{(i,j)} = 0_{n_u \times n}, & \text{if } i = 1 \text{ and } j \neq 1, \\ q_{(i,j)} = 0_{n \times n}, & \text{otherwise.} \end{cases} \tag{5.2}$$

with (i,j) respectively representing the row and column (position of the entry) in a matrix block and $\phi = \frac{1}{\zeta^2}$*. Then, if the condition*

$$\mathcal{Q} + \mathcal{V}\mathcal{B} + \mathcal{B}^T\mathcal{V}^T \leq 0 \tag{5.3}$$

is feasible, it is possible to obtain an upper bound ζ *for the higher order terms. And then, calculating the upper-bound discretization error,* χ*, defined by*

$$|e_k(t)| \leqslant \underbrace{\zeta \left(\left| \frac{(-\Delta t)^{N+1}}{(N+1)!} \right| + \sum_{k=1}^{N} |\beta_k| \left| \frac{(-k\Delta t)^N}{N!} \right| \right)}_{\chi}. \tag{5.4}$$

where Δt *is sampling time chosen and* β_k *are constants defined by the Adams-Bashorth method with* $k = 1,2,\ldots,N$ *or Adams-Moulton method with* $k = 0,1,\ldots,N$*.*

Proof. The discretization error, $e_k(t)$, is the difference between the expected value $x(t)$ and the approximation using a Linear Multistep method, in this case, Adams-Bashforth method presented in (3.9). Then, the discretization error, $e_k(t)$, is defined by

$$e_k(t) = x(t) - \left(\underbrace{x(t) + \sum_{i=1}^{N} \frac{(-\Delta t)^i}{i!} x^{(i)}(t)}_{I} + \frac{x^{(N+1)}(c_0)}{(N+1)!}(-\Delta t)^{N+1} \right.$$

$$\left. + \underbrace{\sum_{k=1}^{N} \beta_k \left(\dot{x}(t) + \sum_{i=2}^{N} \frac{k^{i-1}(-\Delta t)^{i-1}}{(i-1)!} x^{(i)}(t) \right)}_{II} + \sum_{k=1}^{N} \beta_k \frac{x^{(N+1)}(c_k)}{N!}(-k\Delta t)^N \right). \quad (5.5)$$

Note that, in obtaining the discrete-time model, the Taylor Series Expansion approximations of the time-derivatives of $x(t)$ were considered up to an arbitrary order N. In this way, all higher order terms are considered as residues associated with the truncation of the Taylor Series Expansion and are used to calculate the discretization error through the Lagrange Remainder Form (Beesack, 1966). By formulating the method, the values of β_k are chosen so that the system is consistent, that is, the terms I and II of (5.5) cancel each other.

Thus, the closed formula for the discretization error using the Adams-Bashforth Method is given by

$$e_k(t) = -\frac{x^{(N+1)}(c_0)}{(N+1)!}(-\Delta t)^{N+1} - \sum_{k=1}^{N} \beta_k \frac{x^{(N+1)}(c_k)}{N!}(-k\Delta t)^N. \quad (5.6)$$

Applying the Euclidean norm on both sides, we have

$$|e_k(t)| = |-1| \left| \frac{x^{(N+1)}(c_0)}{(N+1)!}(-\Delta t)^{N+1} + \sum_{k=1}^{N} \beta_k \frac{x^{(N+1)}(c_k)}{N!}(-k\Delta t)^N \right|, \quad (5.7)$$

that, by triangular inequality results in

$$|e_k(t)| \leq \left| \frac{(-\Delta t)^{(N+1)}}{(N+1)!} \right| \left| x^{(N+1)}(c_0) \right| + \sum_{k=1}^{N} \left| \beta_k \frac{(-k\Delta t)^N}{N!} \right| \left| x^{(N+1)}(c_k) \right| \quad (5.8)$$

and, finally, a transformation of variables is performed in which $|x^{(N+1)}(c_k)| \leq \zeta$, for $k = 0, 1, \ldots, N$, obtaining

$$|e_k(t)| \leqslant \underbrace{\zeta \left(\left| \frac{(-\Delta t)^{N+1}}{(N+1)!} \right| + \sum_{k=1}^{N} |\beta_k| \left| \frac{(-k\Delta t)^N}{N!} \right| \right)}_{\chi}, \quad (5.9)$$

where χ is the upper bound of the discretization error using the Adams-Bashforth method.

Furthermore, knowing that $|x^{(N+1)}(t)| \leq \zeta$, and

$$x^{(N+1)}(t) = A_\lambda^{(N+1)} x(t) + \sum_{i=0}^{N} A_\lambda^i B_\lambda u(t)^{(N-i)}. \quad (5.10)$$

If $u(t)$ is constant, every derivative of $u(t)$ is zero and (5.10) can be reduced to

$$x^{(N+1)}(t) = A_\lambda^{(N+1)}x(t) + A_\lambda^{(N)}B_\lambda u(t) \tag{5.11}$$

then, it is possible to write

$$\left\| A_\lambda^{(N+1)}x(t) + A_\lambda^{(N)}B_\lambda u(t) \right\| \leq \zeta. \tag{5.12}$$

In this case, limits are defined for the states and for the input

$$\frac{u^T(t)u(t)}{\overline{u}^2} \leq 1, \qquad \frac{x^T(t)x(t)}{\overline{x}^2} \leq 1. \tag{5.13}$$

Considering both constraints in (5.13) the following convex combination is considered

$$\eta\frac{u(t)^T u(t)}{\overline{u}(t)^2} + (1-\eta)\frac{x(t)^T x(t)}{\overline{x}(t)^2} \leq 1. \tag{5.14}$$

Squaring both sides of (5.12) and using (5.14), yields

$$\begin{aligned} &\frac{x^T(t)A_\lambda^{(N+1)^T}A_\lambda^{(N+1)}x(t)}{\zeta^2} + \frac{u^T(t)B_\lambda^T A_\lambda^{(N)^T}A_\lambda^{(N)}B_\lambda u(t)}{\zeta^2} \\ &\qquad + \frac{u^T(t)B_\lambda^T A_\lambda^{(N)}A_\lambda^{(N+1)}x(t)}{\zeta^2} + \frac{x^T(t)A_\lambda^{(N+1)^T}A_\lambda^{(N)}B_\lambda u(t)}{\zeta^2} \\ &\qquad\qquad - \eta\frac{u(t)^T(t)u(t)}{\overline{u}(t)^2} - (1-\eta)\frac{x^T(t)x(t)}{\overline{x}(t)^2} \leq 0 \end{aligned} \tag{5.15}$$

Using Finsler's Lemma (Finsler, 1937) and the variable transformations

$$\begin{aligned} Z_1 &= A_\lambda x(t) \\ Z_2 &= A_\lambda Z_1 \\ &\vdots \\ Z_N &= A_\lambda Z_{N-1} \end{aligned} \qquad \text{and} \qquad \begin{aligned} Y_1 &= B_\lambda u(t) \\ Y_2 &= A_\lambda Y_1 \\ &\vdots \\ Y_N &= A_\lambda Y_{N-1}, \end{aligned}$$

v can be defined as

$$v = \begin{bmatrix} u^T(t) & Y_1^T & \dots & Y_N^T & x^T(t) & Z_1^T & \dots & Z_N^T \end{bmatrix}^T. \tag{5.16}$$

Then, we have $v^T\mathcal{Q}v \leq 0$ with $\mathcal{B}v = 0$, which, by Finsler's Lemma, is met if

$$\mathcal{Q} + \mathcal{V}\mathcal{B} + \mathcal{B}^T\mathcal{V}^T \leq 0 \tag{5.17}$$

with $\mathcal{V}$ being a matrix of slack variables and dimensions given in the statement of the Theorem 3. The proof is complete. A teaching example is presented in Appendix B to ease the comprehension how to build this matrix structure. □

Remark 3. *The same procedure can be adapted to obtain an upper bound for the discretization error for Adams-Moulton method.*

5.2 Discretization Strategies

This section aims to present a methodology to obtain uncertain discrete-time systems, based on linear multistep methods presented in Section 3.2.1.

Remembering that a general discrete-time representation of the Adams methods is given by

$$x_{k+1} = x_k + \Delta t \left(\sum_{i=(0,1)}^{N} \beta_i \dot{x}_{k+1-i} \right), \tag{5.18}$$

in which coefficients β_i are defined as in (3.9) and i starts at 1 when the method chosen is Adams-Bashforth or 0 when it is Adams-Moulton.

Considering the definition of $\dot{x}$ from system (3.10), and replacing it in (5.18) yields

$$x_{k+1} = x_k + \sum_{i=(0,1)}^{N} \delta_i \left(A_\lambda x_{k+1-i} + B_\lambda u_{k+1-i} \right) \tag{5.19}$$

where $\delta_i = \Delta t \beta_i$. The proposal to use delayed time-instants of the system, in addition to the current instant, as seen in (5.19), is to verify whether the use of a multi-step method is capable of providing more information to the discretized system, thus enabling better approximations for the solution of system (3.10). From (5.19), an augmented structure for the system is proposed to absorb the previous steps in its formulation. As a result, the augmented state vector to Adams-Bashforth discretization method is defined as

$$\xi_k = \begin{bmatrix} u_{k-(N-1)}^T & \cdots & u_{k-1}^T & x_{k-(N-1)}^T & \cdots & x_k^T \end{bmatrix}^T \tag{5.20}$$

and to Adams-Moulton Methods is given by

$$\xi_k = \begin{bmatrix} u_{k-N}^T & \cdots & u_{k-1}^T & x_{k-(N-1)}^T & \cdots & x_k^T \end{bmatrix}^T. \tag{5.21}$$

To build the augmented system, we consider an approach with a zero-order holder (meaning that the control inputs are held constant between successive instants of time). At the time instant t, the control action (approaching the limit from the right) can be considered as u_k and will, therefore, be adjusted to this value. In this way, the discrete-time system that approximates system (3.10), while accounting for the associated discretization error, is given in the form

$$\overline{E}_\lambda \xi_{k+1} = \overline{A}_\lambda \xi_k + \overline{B}_\lambda u_k + \overline{B}_e e_k \tag{5.22}$$

with $\overline{A}_\lambda \in \mathbb{R}^{(Nn+(N-\kappa)n_u)\times(Nn+(N-\kappa)n_u)}$ the augmented dynamic matrix, $\overline{B}_\lambda \in \mathbb{R}^{(Nn+(N-\kappa)n_u)\times n_u}$ the augmented input matrix, $\overline{B}_e \in \mathbb{R}^{(Nn+(N-\kappa)n_u)\times n}$ a matrix referring to the discretization error and $\overline{E}_\lambda \in \mathbb{R}^{(Nn+(N-\kappa)n_u)\times(Nn+(N-\kappa)n_u)}$ the descriptor matrix of the system.

The system (5.22) can represent both the Adams-Bashforth and Adams-Moulton methods depending on the choices of the matrices $\overline{E}_\lambda$, $\overline{A}_\lambda$, $\overline{B}_\lambda$ and $\overline{B}_e$.

- Adams-Bashforth

  For this method, $\kappa = 2$ and the matrices are

  $\overline{E}_\lambda = I$,

$$\overline{A}_\lambda = \begin{bmatrix} 0_{(N-\kappa)n_u \times n_u} & I_{(N-\kappa)n_u} & 0_{(N-\kappa)n_u \times n} & 0_{(N-\kappa)n_u \times (N-1)n} \\ 0_{n_u \times n_u} & 0_{n_u \times (N-\kappa)n_u} & 0_{n_u \times n} & 0_{n_u \times (N-1)n} \\ 0_{(N-1)n \times n_u} & 0_{(N-1)n \times (N-\kappa)n_u} & 0_{(N-1)n \times n} & I_{(N-1)n} \\ \delta_N B_\lambda & \Gamma_1 & \delta_N A_\lambda & \Gamma_2 \end{bmatrix} \tag{5.23}$$

$$\overline{B}_\lambda = \begin{bmatrix} 0_{(N-\kappa)n_u \times n_u} \\ I_{n_u} \\ 0_{(N-1)n \times n_u} \\ \delta_1 B_c \end{bmatrix} \quad \text{and} \quad \overline{B}_e = \begin{bmatrix} 0_{(N-\kappa)n_u \times n_u} \\ 0_{n_u \times n_u} \\ 0_{(N-1)n \times n_u} \\ I_n \end{bmatrix} \tag{5.24}$$

  with

$$\Gamma_1 = \begin{bmatrix} \delta_{N-1} B_\lambda & \dots & \delta_3 B_\lambda & \delta_2 B_\lambda \end{bmatrix}, \tag{5.25}$$

$$\Gamma_2 = \begin{bmatrix} \delta_{N-1} A_\lambda & \dots & \delta_2 A_\lambda & (I + \delta_1 A_\lambda) \end{bmatrix}. \tag{5.26}$$

- Adams-Moulton

  For this method, $\kappa = 1$ and the matrices are

$$\overline{E}_\lambda = \begin{bmatrix} I_{Nn_u+(N-1)n} & 0_{Nn_u+(N-1)n \times n} \\ 0_{n \times Nn_u+(N-1)n} & I - \delta_0 A_\lambda \end{bmatrix} \tag{5.27}$$

$$\overline{A}_\lambda = \begin{bmatrix} 0_{(N-\kappa)n_u \times n_u} & I_{(N-\kappa)n_u} & 0_{(N-\kappa)n_u \times n} & 0_{(N-\kappa)n_u \times (N-1)n} \\ 0_{n_u \times n_u} & 0_{n_u \times (N-\kappa)n_u} & 0_{n_u \times n} & 0_{(n_u \times (N-1)n} \\ 0_{(N-1)n \times n_u} & 0_{(N-1)n \times (N-\kappa)n_u} & 0_{(N-1)n \times n} & I_{(N-1)n} \\ \delta_N B_\lambda & \Gamma_1 & \delta_N A_\lambda & \Gamma_2 \end{bmatrix} \tag{5.28}$$

$$\overline{B}_\lambda = \begin{bmatrix} 0_{(N-\kappa)n_u \times n_u} \\ I_{n_u} \\ 0_{(N-1)n \times n_u} \\ \delta_0 B_c \end{bmatrix}, \quad \text{and} \quad \overline{B}_e = \begin{bmatrix} 0_{(N-\kappa)n_u \times n_u} \\ 0_{n_u \times n_u} \\ 0_{(N-1)n \times n_u} \\ I_n \end{bmatrix} \tag{5.29}$$

  with

$$\Gamma_1 = \begin{bmatrix} \delta_{N-1} B_\lambda & \dots & \delta_2 B_\lambda & \delta_1 B_\lambda, \end{bmatrix}, \tag{5.30}$$

$$\Gamma_2 = \begin{bmatrix} \delta_{N-1} A_\lambda & \dots & \delta_2 A_\lambda & (I + \delta_1 A_\lambda) \end{bmatrix}. \tag{5.31}$$

5.3 State-feedback Controllers Synthesis Conditions

For the design of state-feedback controllers, we consider the discrete-time system presented in (5.22) and the control law given by

$$u_k = K\xi_k. \tag{5.32}$$

Thus, the discrete-time closed-loop system is

$$\overline{E}_\lambda \xi_{k+1} = (\overline{A}_\lambda + \overline{B}_\lambda K)\xi_k + \overline{B}_e e_k \tag{5.33}$$

In this way, Theorem 4 presents a sufficient set of LMIs for designing a discrete-time control law that is capable of stabilizing the continuous-time uncertain system in (3.10).

Theorem 4. *For a given $\kappa = \{1,2\}$, respectively, for Adams-Moulton or Adams-Bashforth method and given scalars $\bar{x}$, $\bar{u}$ and r, if there exist matrices $Z \in \mathbb{R}^{((N-\kappa)n_u+Nn)\times((N-\kappa)n_u+Nn)}$ and $G \in \mathbb{R}^{((N-1)n_u+Nn)\times((N-\kappa)n_u+Nn)}$, symmetric positive definite matrices $\overline{W} \in \mathbb{R}^{((N-\kappa)n_u+Nn)\times((N-\kappa)n_u+Nn)}$, positive scalar variables ϵ, υ, γ and τ_i, with $i = 1,\dots,\phi$, indicator matrices S_{u_i}, with $i = 1,\dots,\phi_1$, S_{x_i}, with $i = \phi_1,\dots,\phi$ composed of null values, in which the matrix block with identity is indicated by i, such that*

$$\begin{bmatrix} -\epsilon\left(G^T\overline{E}_\lambda^T + \overline{E}_\lambda G - \overline{W}\right) & 0 & G^T\overline{A}_\lambda^T + Z^T\overline{B}_\lambda^T \\ 0 & -\gamma I & \overline{B}_e^T \\ \overline{A}_\lambda G + \overline{B}_\lambda Z & \overline{B}_e & -\overline{W} \end{bmatrix} \leq 0 \tag{5.34}$$

$$\begin{bmatrix} -M & Z^T \\ Z & \frac{-\bar{u}^2}{r} I \end{bmatrix} \leq 0 \tag{5.35}$$

$$\begin{bmatrix} -\overline{W} + G^T\overline{E}_\lambda^T + \overline{E}_\lambda G & G^T \\ G & \upsilon I \end{bmatrix} \geq 0 \tag{5.36}$$

$$\begin{bmatrix} \sum_{i=1}^{\phi_1} \tau_i S_{u_i}^T S_{u_i} + \sum_{j=\phi_1+1}^{\phi} \tau_j S_{x_j}^T S_{x_j} & I \\ I & G + G^T - M \end{bmatrix} \geq 0 \tag{5.37}$$

$$r - \sum_{i=1}^{\phi_1} \tau_i \overline{u}^2 - \sum_{j=\phi_1}^{\phi} \tau_j \overline{x}^2 \geq 0 \tag{5.38}$$

is valid for a level set $M(\xi_k) \leq r$, then the control law given by $K = ZG^{-1}$ ensures the stability of system (5.22), and consequently (3.10), in closed-loop within a region limited by $\overline{x}^2$. Furthermore, the continuous-time system (3.10) is ultimately bounded by the region $\xi_k^T\xi_k \leq \frac{\gamma\upsilon\chi^2}{(1-\epsilon)}$ using χ calculated in Theorem 3.

Proof. Consider the discrete-time system (5.22) and a Lyapunov function given by

$$V(\xi_k) = \xi_k^T \overline{E}_\lambda^T \overline{P} \overline{E}_\lambda \xi_k \geq 0. \tag{5.39}$$

Assume that

$$V(\xi_{k+1}) - \epsilon V(\xi_k) - \gamma e_k^T e_k \leq 0, \tag{5.40}$$

with $V(\xi_{k+1}) = \xi_{k+1}^T \overline{E}_\lambda^T \overline{P} \overline{E}_\lambda \xi_{k+1}$. Replacing the definition of $\overline{E}_\lambda \xi_{k+1}$ presented in (5.22), yields

$$V(\xi_{k+1}) = \left[\xi_k^T \left(\overline{A}_\lambda^T + K^T \overline{B}_\lambda^T\right) + e_k^T \overline{B}_e^T\right] \overline{P} \left[\left(\overline{A}_\lambda + \overline{B}_\lambda K^T\right) \xi_k + \overline{B}_e e_k\right]. \tag{5.41}$$

Rewriting condition (5.40) using (5.41), we have

$$\begin{bmatrix} \xi_k^T & e_k^T \end{bmatrix} \begin{bmatrix} \overline{A}_\lambda^T + K^T \overline{B}_\lambda^T \\ \overline{B}_e^T \end{bmatrix} \overline{P} \begin{bmatrix} \overline{A}_\lambda + \overline{B}_\lambda K & \overline{B}_e \end{bmatrix} \begin{bmatrix} \xi_k \\ e_k \end{bmatrix} + \begin{bmatrix} \xi_k^T & e_k^T \end{bmatrix} \begin{bmatrix} -\epsilon \overline{E}_\lambda^T \overline{P} \overline{E}_\lambda & 0 \\ 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \xi_k \\ e_k \end{bmatrix} < 0. \tag{5.42}$$

Applying a Schur complement to (5.42) provides

$$\begin{bmatrix} -\epsilon \overline{E}_\lambda^T \overline{P} \overline{E}_\lambda & 0 & \overline{A}^T + K^T \overline{B} \\ 0 & -\gamma I & \overline{B}_e^T \\ \overline{A}_\lambda + \overline{B}_\lambda K & \overline{B}_e & -\overline{P}^{-1} \end{bmatrix} < 0. \tag{5.43}$$

Then, applying the congruence transformation $\mathcal{T} = diag(G, I, I)$ to the right and its transpose to the left of (5.43), we obtain

$$\begin{bmatrix} -\epsilon G^T \overline{E}_\lambda^T \overline{P} \overline{E}_\lambda G & 0 & G^T \overline{A}^T + G^T K^T \overline{B}^T \\ 0 & -\gamma I & \overline{B}_e^T \\ \overline{A}_\lambda G + \overline{B}_\lambda K G & \overline{B}_e & -\overline{P}^{-1} \end{bmatrix} < 0. \tag{5.44}$$

Considering the majoration

$$-\epsilon G^T \overline{E}_\lambda^T \overline{P} \overline{E}_\lambda G \leq -\epsilon(\overline{P}^{-1} - G^T \overline{E}_\lambda^T - \overline{E}_\lambda G) \tag{5.45}$$

and the variable transformations $\overline{P}^{-1} = \overline{W}$ and $Z = KG$, we recover the condition that guarantees asymptotic stability presented in (5.34). In Theorem 3 were specified regions to the states and control inputs in which the stability of the uncertain continuous-time system is desired. As the control inputs are bounded, that is, $u_k^T u_k \leq \overline{u}^2$, we have

$$\xi_k^T K^T K \xi_k \leq \overline{u}^2 \longleftrightarrow \frac{r}{\overline{u}^2} \xi_k^T K^T K \xi_k \leq r. \tag{5.46}$$

Furthermore, another proposed condition is the existence of a function $M(\xi_k) = \xi_k^T G^{-T} M G^{-1} \xi_k$ with a symmetric positive matrix $M \in \mathbb{R}^{(n_u(N-\kappa+1)+Nn)\times(n_u(N-\kappa+1)+Nn)}$

such that $\xi_k^T G^{-T} M G^{-1} \xi_k \leq r$. This function is a level curve that bounds the control inputs. Therefore, we can write

$$\frac{r}{\overline{u}^2}\xi_k^T K^T K \xi_k \leq \xi_k^T G^{-T} M G^{-1} \xi_k \longrightarrow \frac{r}{\overline{u}^2}\xi_k^T K^T K \xi_k - \xi_k^T G^{-T} T G^{-1} \xi_k \leq 0. \tag{5.47}$$

Multiplying the left side by G^T and the right side by G^{-1}, applying Schur Complements and performing the variable transformation $Z = KG$, we obtain (5.35).

In the sequel, defining σ as the smallest eigenvalue of $\overline{P}$, it is possible to write

$$\sigma I \leq \overline{E}_\lambda^T \overline{P E}_\lambda \tag{5.48}$$

multiplying on the left side by G^T and right side by G, we have

$$\sigma G^T G \leq G^T \overline{E}_\lambda^T \overline{P E}_\lambda G \tag{5.49}$$

and with a Schur Complement

$$\begin{bmatrix} G^T \overline{E}_\lambda^T \overline{P E}_\lambda G & G^T \\ G & \frac{1}{\sigma} I \end{bmatrix} \geq 0 \tag{5.50}$$

Using the variable transformation $\upsilon = \frac{1}{\sigma}$ and (5.45) with $\overline{P}^{-1} = \overline{W}$, yields (5.36). Moreover, as defined im (5.9), $e_k^T e_k \leq \chi^2$, therefore (5.40) can be written as

$$V(\xi_{k+1}) \leq \epsilon V(\xi_k) + \gamma\chi^2 \tag{5.51}$$

$$V_1 \leq \epsilon V_0 + \gamma\chi^2 \tag{5.52}$$

$$V_2 \leq \epsilon V_1 + \gamma\chi^2 \leq \epsilon(\epsilon V_0 + \gamma\chi^2) + \gamma\chi^2 \tag{5.53}$$

$$\vdots \tag{5.54}$$

$$V_k \leq \epsilon^k V_0 + \gamma\chi^2 \sum_{n=0}^{k-1} \epsilon^n \tag{5.55}$$

The sum can be approximated by a geometric progression and condition (5.55) can be rewritten as

$$V_k \leq \epsilon^k V_0 + \gamma \left(\frac{1-\epsilon^k}{1-\epsilon}\right) \chi^2. \tag{5.56}$$

Taking

$$\lim_{k\to\infty} V_k \leq \lim_{k\to\infty} \left[\epsilon^k V_o + \gamma \left(\frac{1-\epsilon^k}{1-\epsilon}\right) \chi^2\right] \tag{5.57}$$

and assuming $|\epsilon| < 1$, we have

$$\lim_{k\to\infty} V_k \leq \left(\frac{\gamma\chi^2}{1-\epsilon}\right) \tag{5.58}$$

Thus

$$\sigma\xi_k^T\xi_k \leq \xi_k^T\overline{E}_\lambda^T\overline{PE}_\lambda\xi_k \leq \frac{\gamma\chi^2}{1-\epsilon} \tag{5.59}$$

consequently,

$$\sigma\xi_k^T\xi_k \leq \frac{\gamma\chi^2}{1-\epsilon} \longleftrightarrow \xi_k^T\xi_k \leq \frac{\gamma\chi^2}{\sigma(1-\epsilon)}, \tag{5.60}$$

which delimits the region where the method is valid.

Finally, Lemma 1 is used to provide sufficient conditions to include the constraints related to the upper bounds of states, x, and control input u. That is

$$\begin{cases} u_{k-d}^T u_{k-d} & \leq \ \overline{u}^2, \quad d = 1, \ldots, N-1, \\ x_{k-d}^T x_{k-d} & \leq \ \overline{x}^2, \quad d = 0, \ldots, N-1. \end{cases} \tag{5.61}$$

in Adams-Bashforth case, or

$$\begin{cases} u_{k-d}^T u_{k-d} & \leq \ \overline{u}^2, \quad d = 1, \ldots, N, \\ x_{k-d}^T x_{k-d} & \leq \ \overline{x}^2, \quad d = 0, \ldots, N-1. \end{cases} \tag{5.62}$$

for the Adams-Moulton method. Conditions presented in (5.61) and (5.62) can be put in terms of ξ_k assuming that

$$u_i = S_{u_i}\xi_k, \quad x_j = S_{x_j}\xi_k \tag{5.63}$$

with $i = 1, \ldots, \phi_1$ and $j = \phi_1, \ldots, \phi$. In this formulation, we use N indicator matrices $S_{x_i} \in \mathbb{R}^{n\times(\phi_1 n_u + Nn)}$. Each matrix consists of N null matrix blocks of dimensions $\mathbb{R}^{n\times n}$ and $N-1$ null matrix blocks of dimensions $\mathbb{R}^{n\times n_u}$, arranged from right to left. Similarly, the indicator matrices $S_{u_i} \in \mathbb{R}^{n_u\times(\phi_1 n_u + Nn)}$ are composed of N null matrix blocks of dimensions $\mathbb{R}^{n_u\times n}$ and $N-1$ null matrix blocks of dimensions $\mathbb{R}^{n_u\times n_u}$, also arranged from right to left. In these matrices, the position i corresponds to where an identity matrix block is placed. It's important to note that the counter i resets when the dimension of the matrix block changes.

For example, when using an Adams-Bashforth discretization method with three delayed steps ($N = 3$), the indicator matrices would be described by

$$S_{x_1} = \begin{bmatrix} 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n} & 0_{n\times n} & I_n \end{bmatrix}, \tag{5.64}$$

$$S_{x_2} = \begin{bmatrix} 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n} & I_n & 0_{n\times n} \end{bmatrix}, \tag{5.65}$$

$$S_{x_3} = \begin{bmatrix} 0_{n\times n_u} & 0_{n\times n_u} & I_n & 0_{n\times n} & 0_{n\times n} \end{bmatrix}, \tag{5.66}$$

$$S_{u_1} = \begin{bmatrix} 0_{n_u\times n_u} & I_{n_u} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} \end{bmatrix}, \tag{5.67}$$

$$S_{u_2} = \begin{bmatrix} I_{n_u} & 0_{n_u\times n_u} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} \end{bmatrix}. \tag{5.68}$$

Thus, using (5.61), knowing $\xi_k G^{-T} T G^{-1} \xi_k \leq r$ and considering Lemma 1, it is possible to write

$$r - \xi_k^T G^{-T} T G^{-1} \xi_k - \sum_{i=1}^{N-1} \tau_i(\overline{u}^2 - \xi_k^T S_{u_i}^T S_{u_i} \xi_k) - \sum_{j=N-1}^{N-1} \tau_i(\overline{x}^2 - \xi_k^T S_{x_j}^T S_{x_j} \xi_k) \geq 0 \quad (5.69)$$

Inequality (5.69) can be separated in two parts, the constraint related to the scalar terms, to recover LMI (5.38), and the constraint using the terms composed by the matrices obtaining (5.37). The proof is concluded. □

Remark 4. *The conditions presented in Theorem 4 allow us to consider a constant Lyapunov function or a parameter-dependent Lyapunov function, that is,* $\overline{W} = \overline{P}^{-1}$ *can be either constant or parameter-dependent.*

Remark 5. *The dimensions of the augmented matrices and how many* τ_i *are considered in the conditions depend on the discretization chosen. Adams-Bashforth method considers* $\phi_1 = N - 1$ *and* $\phi = 2N - 1$ *due to the use of* N *delays in the states and* $N - 1$ *delays in the input control. On the other hand, using the Adams-Moulton procedure we have* $\phi_1 = N$ *and* $\phi = 2N$ *once it is used* N *delays in the states and* N *delays in the input control.*

Remark 6. *The proposed theorem addresses a multi-objective optimization problem, in which it is desired to minimize the values of* γ *and* υ*, thus guaranteeing the convergence of the discrete-time system to the smallest possible region. For this reason, the objective function is chosen using a convex sum of these two decision variables as*

$$\theta\upsilon + (1 - \theta)\gamma, \quad (5.70)$$

where $\theta \in [0, 1]$*. Then, a Pareto set is obtained changing the value of* θ *and the value that minimizes the region is chosen.*

Remark 7. *Consider that the region around the origin of the states is limited by* $\overline{x}$ *and the time evolution of the states converges to a region smaller than* $\bar{x}$*, that is*

$$x_{k+1} < x_k. \quad (5.71)$$

If the system is linear time-invariant and the ratio

$$\frac{x_{k+1}}{x_k} < 1 \quad (5.72)$$

is attended, then Theorem 4 can guarantee asymptotic stability. This condition to guarantee asymptotic convergence is similar to the idea adopted in the Small-Gain Theorem (Hill, 1991).

Chapter 6

ILLUSTRATIVE EXAMPLES

In this section, three numerical examples were chosen to illustrate the validity of the proposed procedures. The proposed methodologies are compared with each other, and finally, these results are compared with other results available in the literature.

6.1 Example 1

Consider an example taken from Fridman et al. (2004) with an open-loop unstable continuous-time polytopic system defined by

$$A_\lambda = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix}, \tag{6.1}$$

where $|g_1| \leq 0.1$, $|g_2| \leq 0.3$. The conditions proposed by Fridman et al. (2004) have been verified to enable the design of a discrete-time control law that stabilizes the continuous-time uncertain system using a sampling time smaller than $0.35s$. In this context, using the procedures developed in this work, we desire to provide a sampling time that overcomes the Fridman et al. (2004) so that the discrete-time control law obtained can be used in the process in which the energy-efficient and limited resources are required. Besides this, the sampling time smaller than $\Delta t = 0.35$, in this case, does not invalidate the result or is evaluated as a bad result, it is only a controller with less energetic efficiency.

Table 3 shows the Δt for each discretization order (delay) N and each methodology applied, where MM indicates the Mixed Method discretization approach, AB stands for the Adams-Bashforth method, and AM represents the Adams-Moulton discretization procedure. The Mixed Method utilizes two different theorems to derive a state-feedback control law. Theorem 1 employs a constant Lyapunov function and Theorem 2 adopts a parameter-dependent Lyapunov function. On the other hand, the Adams-Bashforth and Adams-Moulton methods rely only on Theorem 4, which allows the use of both constant and parameter-dependent Lyapunov functions, shown in Table 3 as W and W_λ, respectively.

Additionally, Table 3 displays the sampling times found by considering the discrete-time models built using the proposed discretization methods with a maximum discretization order of $N = \{1,2,\ldots,6\}$. Notably, choosing the Adams-Bashforth method and using $N = 1$ corresponds to the classic Forward Euler's procedure. Choosing the Adams-Moulton method and $N = 1$ recovers the Trapezoidal method. Both are classical numerical methods widely used in the discretization context. The sampling times that improve upon the results presented in Fridman et al. (2004) are highlighted in blue in the table. Note that the Adams-Moulton and the Mixed discretization methods obtained larger sampling times by means of constant and parameter-dependent Lyapunov functions. The best result was obtained by means of the Adams-Moulton discretization method and Theorem 4, using a parameter-dependent Lyapunov function with sampling time $\Delta t = 0.38s$. The results indicated by $'-'$ in Table 3 (and in all other tables) signify that the obtained result is not valid. This means that either the theorems are infeasible, or although the theorems can find a discrete-time control law, the calculated convergence region is larger than $\bar{x}^2$. As a result, the discrete-time control law becomes invalid.

For all methodologies, an exhaustive search is done to find the largest values of sampling time that can provide a discrete-time state-feedback control law that stabilizes the continuous-time system. The following subsections describe how each procedure was applied.

Table 3 – Values of sampling-time, Δt, considering a maximum delay given by N for each discretization method and convergence region (when Adams-Bashforth and Adams-Moulton Method are considered) for the system of Example 1.

Method	Theorem	N	1	2	3	4	5	6
MM	1 (P)	$\Delta t(s)$	0.1560	0.2557	0.3504	0.3127	0.2937	0.2266
	2 (P_λ)	$\Delta t(s)$	0.1560	0.2557	0.3759	0.3190	0.2943	0.2560
AB	4	$\Delta t(s)$	0.03	0.07	0.22	0.13	-	-
		Region	50.41	96.79	82.36	0.45	-	-
	4 ($\bar{W} = W_\lambda$)	$\Delta t(s)$	0.03	0.09	0.23	0.19	0.06	-
		Region	64.67	89.09	78.50	14.73	3.06×10^{-6}	-
’ ’ AM	4 ($\bar{W} = W$)	$\Delta t(s)$	0.27	0.09	0.22	0.33	0.31	0.15
		Region	70.97	78.77	98.75	81.12	1.87	6.93×10^{-6}
	4 ($\bar{W} = W_\lambda$)	$\Delta t(s)$	0.27	0.1	0.25	0.38	0.35	0.18
		Region	91.33	93.35	57.26	91.48	5.591	7.22×10^{-5}

Analyzing Table 3, we can observe that until a certain order of discretization methodology, the sampling time tends to grow and then decay. This happens because of the characteristic of multistep methods that decreases their numerical stability region as the order increases. Additionally, when we worked with Adams-Bashforth and Adams-Moulton discretization methods and Theorem 4, we calculated the stability region. The value of this region does not follow any pattern, once it is calculated based on upper bound discretization error, χ (calculated value), ϵ (chosen value), and γ and υ that are decision variables from the optimization problem.

6.1.1 Step-by-step execution of The Mixed Method for the system of Example 1

As shown in Table 3, the most significant values obtained by the Mixed Method are related to the discrete-time systems using the maximum delay of $N = 3$. Note that, when $N = 3$, the proposed theorems outperformed the results presented by Fridman et al. (2004). Specifically, Theorem 2 showed superior performance by utilizing a parameter-dependent Lyapunov function, which effectively reduced the conservativeness of the proposed conditions.

Given that the proposed methodology involves several steps to generate a discrete-time uncertain system, the best result, $\Delta t = 0.3759s$ shown in Table 3 will be revisited.

The Mixed Method is based on using delayed steps to enhance the approximation of x_k. To achieve this, it is essential to calculate the maximum discretization error for each delay, as presented in (4.21) to (4.23). As A_λ and B_λ are uncertain matrices, a fine grid search with 10^3 points is used to evaluate the parameter λ and determine the worst-case scenario for the discretization error. Figure 2 illustrates the behavior of the maximum discretization error calculated for each delay. As expected, the associated error increases with a larger number of time steps.

Results for the discretization error of B_λ in all performed experiments had little influence on the final result, hence they are not presented. For the system of Example 1, for any $N = \{1,2,\ldots,6\}$ and $\Delta t = 0.3759$, the discretization error for B_λ is $\Delta B = 0.089605$.

Note that the proposed approach may involve a significant computational effort, particularly when considering the fine grid used to assess the error of the matrix exponential. Furthermore, although using a fine grid brings the approximation closer to the exact bound, it requires a much higher computational effort, which can be prohibitive in some scenarios. Alternatively, this step could be replaced with an exact upper bound obtained from Jungers et al. (2017), albeit with more conservativeness, as is known in the literature. On the other hand, the other steps in the proposed methodology are considerably less computationally demanding in comparison. It is important to highlight that all of these steps are performed offline during the controller design process. The resulting control law is a usual state-feedback control law, although with some additional delayed states, and it

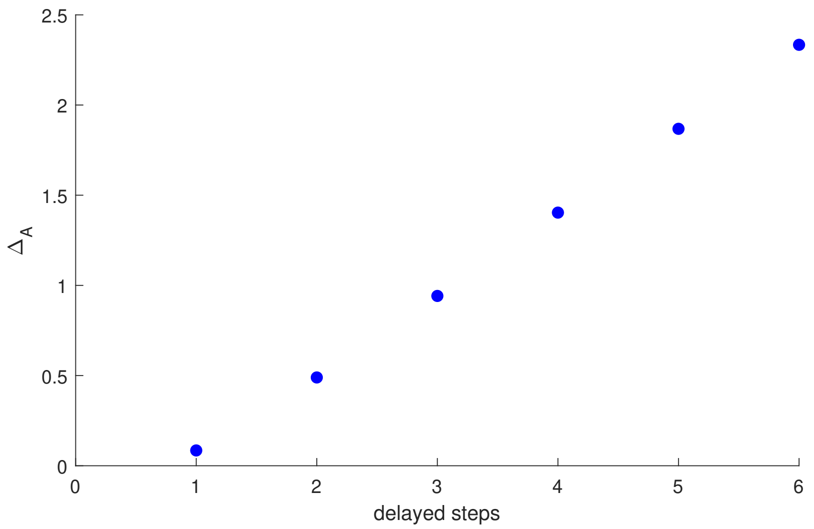

Figure 2 – Maximum discretization error found using $\Delta t = 0.3759s$ and a grid-search with 10^3 points considering different delays for the system of Example 1.

does not require a high computational effort during its implementation.

The second step comprises minimizing the optimization problem considering the discretization error obtained in the previous step. The main objective of this stage is to find the coefficients to combine the different delayed models, minimizing the discretization error and building the final formulation of the mixed method. Figure 3 shows that the values of the objective functions obtained for each delayed step are quite similar, leading to the possible conclusion that the chosen structure, used to define the discrete-time model, influences more in obtaining a less conservative model than in the uncertainty found.

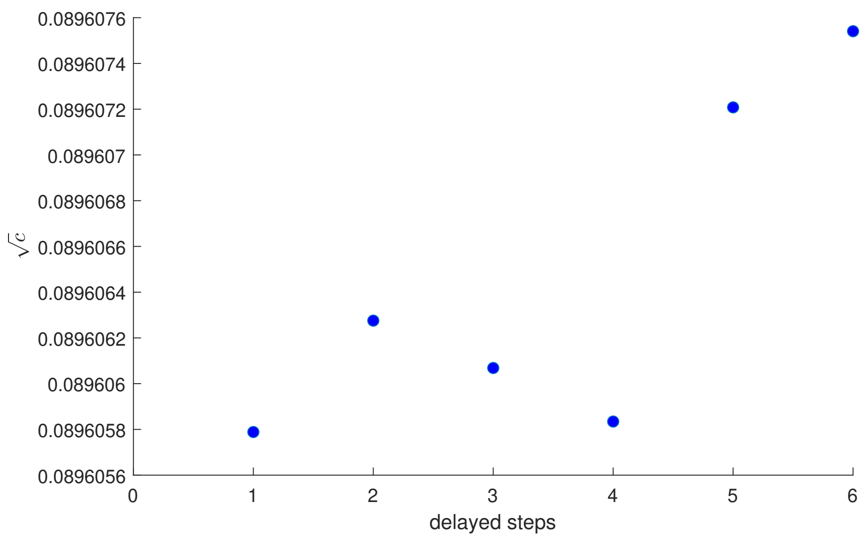

Figure 3 – Root squared values of the objective function in the optimization problem considering a sampling time $\Delta t = 0.3759s$ for the system of Example 1.

This example clearly shows that constructing a discrete-time system that includes delayed steps allows larger sampling times compared to Euler's methods ($N = 1$) and the procedure proposed by Fridman et al. (2004). This can be regarded as a measure of the conservativeness of the proposed approach, indicating that it can be less conservative than other available methods.

To further illustrate the results obtained with our approach, the closed-loop system was simulated for different values of $\lambda_r \in \Lambda_R$. All trajectories started from the initial condition $x(0) = [2,5]^T$ and the behavior for each state, x_1 and x_2, is shown in Figures 4 and 5, respectively, whereas the behavior of the control law is depicted in Figure 6. Finally, Figure 7 presents the phase plane illustrating that with 1000 different initial conditions ($x_0 \in [20,20]$) the closed-loop system is stabilized. These results are obtained using the Mixed Method with maximum delay $N = 3$, sampling time $\Delta t = 0.3759s$, and the control gain designed using Theorem 2, achieving a discrete-time control law given by

$$K = \begin{bmatrix} -0.6392 & -0.9308 & 0.0542 & -0.0005 & 0.0685 & 0.0769 & -3.2667 & -1.2350 \end{bmatrix}.$$

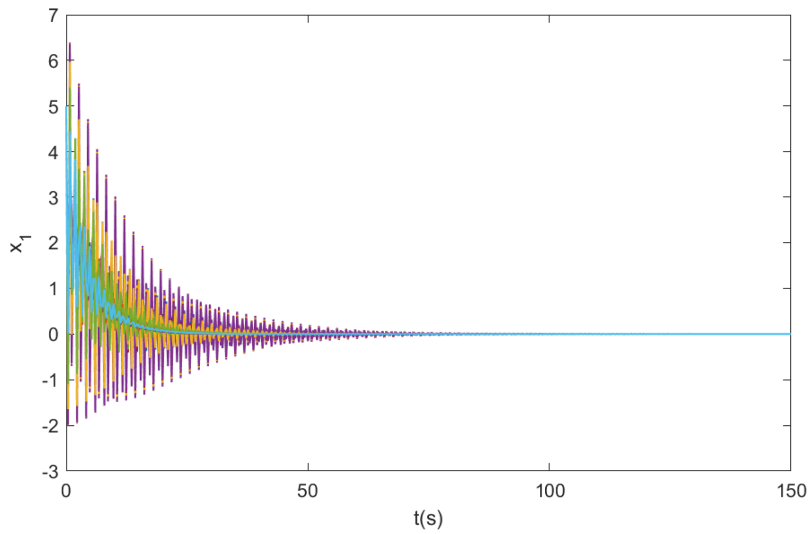

Figure 4 – Time evolution of states x_1 considering 1000 different combination to λ for the system of Example 1 using The Mixed Method and Theorem 2.

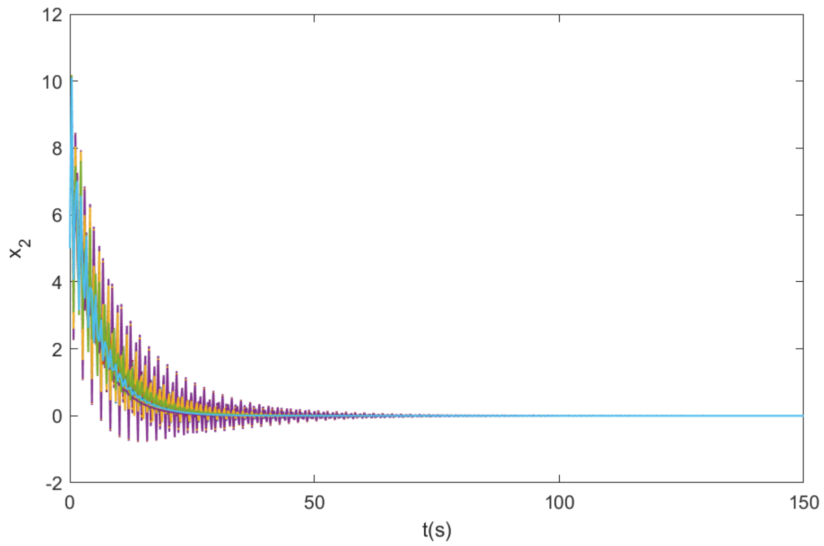

Figure 5 – Time evolution of states x_2 considering 1000 different combination to λ for the system of Example 1 using The Mixed Method and Theorem 2.

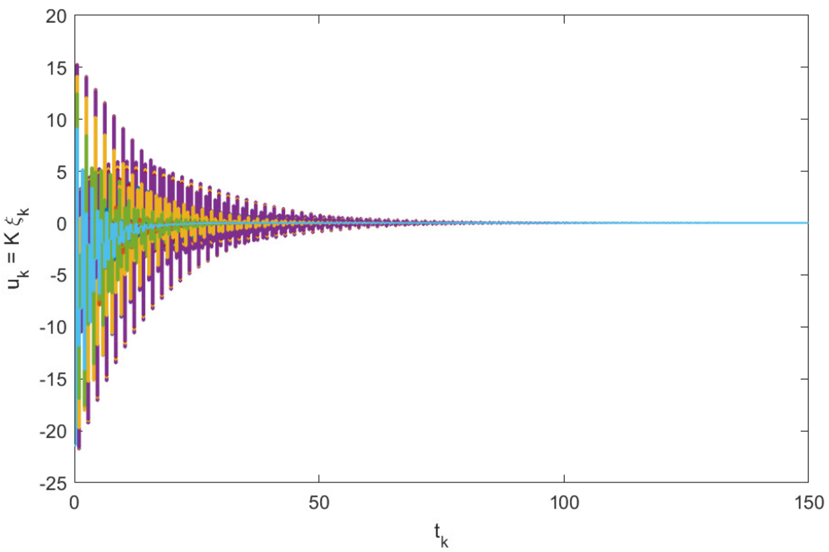

Figure 6 – Time evolution of control signal u_k for the system of Example 1 using the Mixed Method and Theorem 2.

6.1.2 Step-by-step execution of The Adams-Bashforth and Adams-Moulton Methods for the system of Example 1

For each discretization method, the conditions of both Theorem 3 and Theorem 4 were evaluated considering the order N, that is, how many delays were used to build the discrete-time uncertain system.

The first step to obtaining a control law was applying Theorem 3 to find the upper bound for the discretization error based on the matrices of the uncertain continuous-time system (3.10) and the bounds that limit the region of the evolution of the states and the

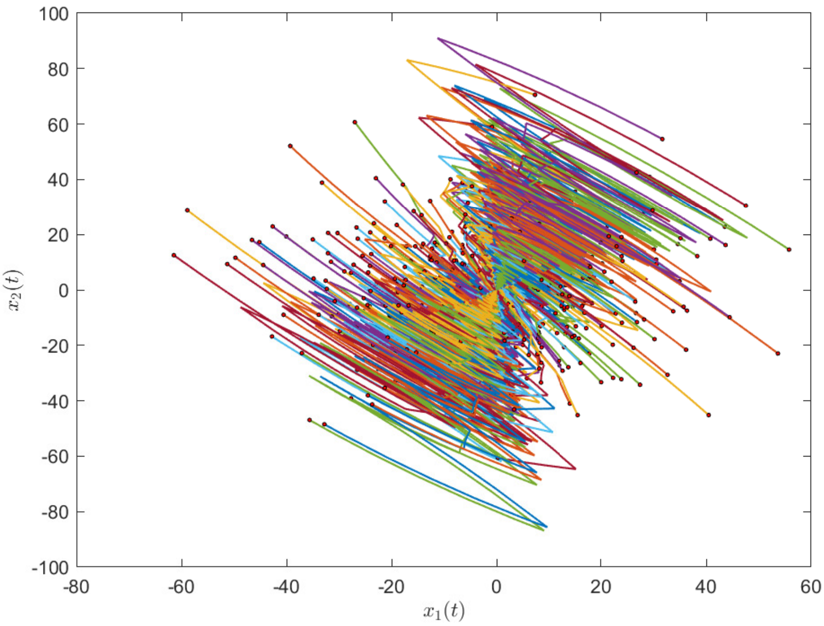

Figure 7 – Phase plane considering the time evolution of the states $x_1(t)$ and $x_2(t)$ in the closed loop system of the Example 1 from 1000 different initial conditions using The Mixed Method and Theorem 2.

control inputs, given by $\overline{x}^2 = 100$ and $\overline{u}^2 = 10000$. Then, Theorem 4 was applied to provide a valid discrete-time control law to stabilize the continuous-time system (3.10). This theorem was evaluated both in terms of a constant Lyapunov function, that is, $\overline{W} = W$, and a parameter-dependent Lyapunov function, $\overline{W} = W_\lambda$. In addition, it was important to analyze the convergence region, because the proposed sufficient conditions can only find a feasible control law if the convergence region, that is, $\frac{\gamma \upsilon \chi^2}{(1-\epsilon)}$ (γ and υ are decision variables, χ is calculated as in (5.9) and $\epsilon = 0.97$) is smaller than $\overline{x}^2$, as mentioned in Remark 7. Then, a grid search was applied in θ to define the best objective function, as discussed in Remark 6. In this case, $\theta = 0$ was used to compose the objective function considered in Theorem 4. The same steps and constants were applied considering the Adams-Bashforth and Adams-Moulton discretization methods. Table 4 summarizes all the constants used in Theorem 4, which are kept the same for all orders to ensure comparability between results. Several combinations of these scalars were tested and the best result was chosen. However, it is worth noting that the chosen constants are not optimal values and can be adjusted with different values for each order of the discretization methods applied. Table 3 summarizes the results from these methods.

Table 4 – Scalar considered in Theorem 4 for the system of Example 1.

$\bar{x}^2$	$\bar{u}^2$	r	θ	ϵ
100	10000	1	0	0.97

Note from Table 3 that the most significant result considering the Adams-Bashforth methodology was obtained using $N = 3$ and $\Delta t = 0.23s$, with parameter-dependent Lyapunov functions. The discrete-time system obtained through the Adams-Moulton method achieved improved results compared to the Adams-Bashforth method using both types of Lyapunov functions. In the Adams-Moulton case, the proposed approach overcame the sampling time found using the conditions of Chapter 4 (the Mixed Method), reaching a $\Delta t = 0.38s$ and a region of 91.48, using $N = 4$ and also using parameter-dependent Lyapunov function.

It becomes clear from this example that constructing a discrete-time system that includes delayed steps allows larger sampling times compared to Euler's methods ($N = 1$), and the Trapezoidal method, and also the methodologies proposed by Fridman et al. (2004). This can be regarded as a measure of the conservativeness of the proposed approach, indicating that the Adams-Moulton discretization method combined with Theorems 3 and 4 can be less conservative than all other methodologies presented. In addition, to illustrate Remark 5, several simulations were run in which the sampling time was maintained, and the starting region was chosen to be the same on the calculated region obtained in the last simulation, as demonstrated in Table 5. It is important to note that the calculated region is smaller than the region bounded by $(barx^2)$, which ensures the asymptotic stability of the closed-loop system when using the discrete-time control gain derived from Theorem 4. The main obstacle is that the initial condition should be chosen inside the region, that is $x_0 \leq \bar{x}$.

Table 5 – the behavior of convergence region according to the start region $\bar{x}^2$ decrease considering the closed loop system of Example 1 using discrete-time control gain from Theorem 4.

$\bar{x}^2$	Region
100	91.48
91.48	84.49
84.79	79.48
79.48	71.72

To further illustrate the results obtained with our approach, the closed-loop system was simulated for different values of $\lambda \in \Lambda_R$. All trajectories started from the initial condition $x^T(0) = [2{,}5]^T$, inside the method's feasibility region, and the behavior for each state, x_1 and x_2, is shown in Figures 8 and 9, respectively, whereas the behavior of the control law is depicted in Figure 10. Additionally, Figure 11 presents the phase plane illustrating that with 1000 different initial conditions ($x_0 \in [-10{,}10]$) the closed-loop system is stabilized These results were obtained using the Adams-Moulton discretization method with maximum delay $N = 4$, sampling time $\Delta t = 0.42s$, and the control gain, K, designed using Theorem 4 given by

$$K = \begin{bmatrix} 0.0170 & -0.0445 & 0.0920 & -0.1933 & 0.0211 & 0.0037 & -0.1197 \\ & & & -0.0291 & 0.2440 & 0.0578 & -3.0954 & -0.7833 \end{bmatrix}.$$

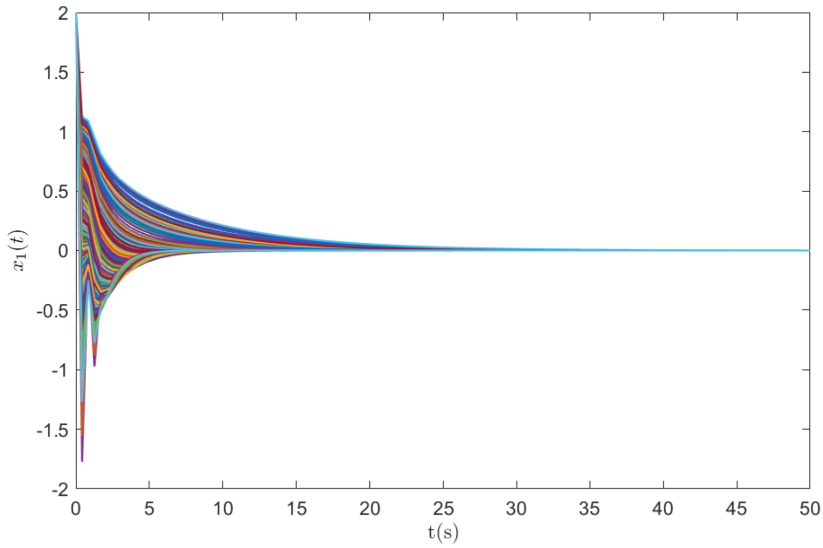

Figure 8 – Time evolution of states x_1 considering 1000 different combinations of λ for the system of Example 1 using The Adams-Moulton Method and Theorem 4.

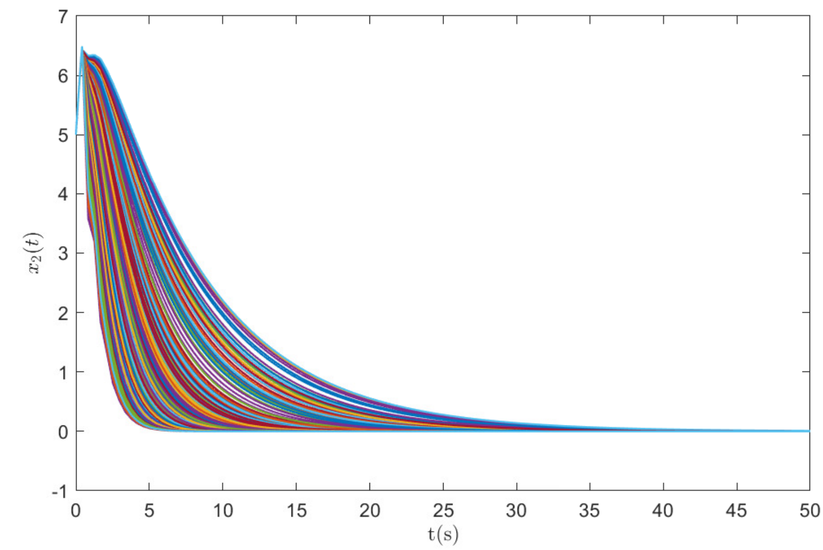

Figure 9 – Time evolution of states x_2 considering 1000 different combinations of λ for the system of Example 1 using The Adams-Moulton Method and Theorem 4.

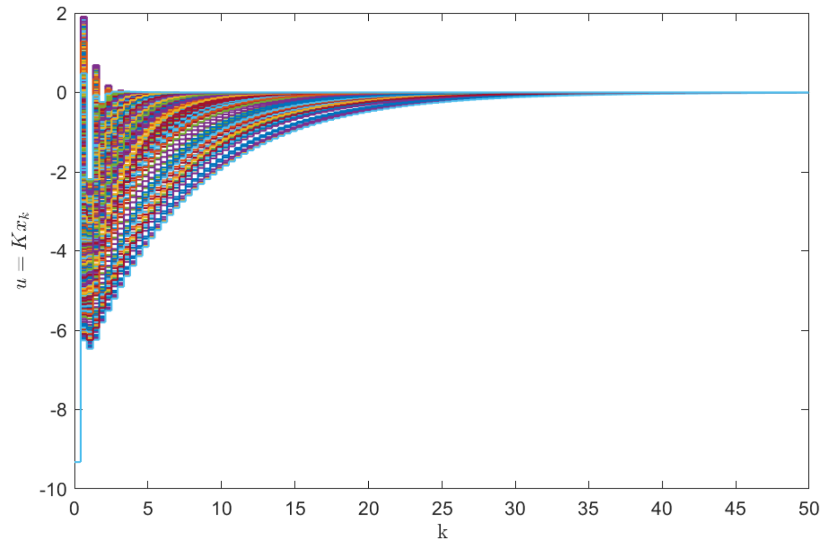

Figure 10 – Time evolution of control signal u_k for the system of Example 1 using The Adams-Moulton Method and Theorem 4.

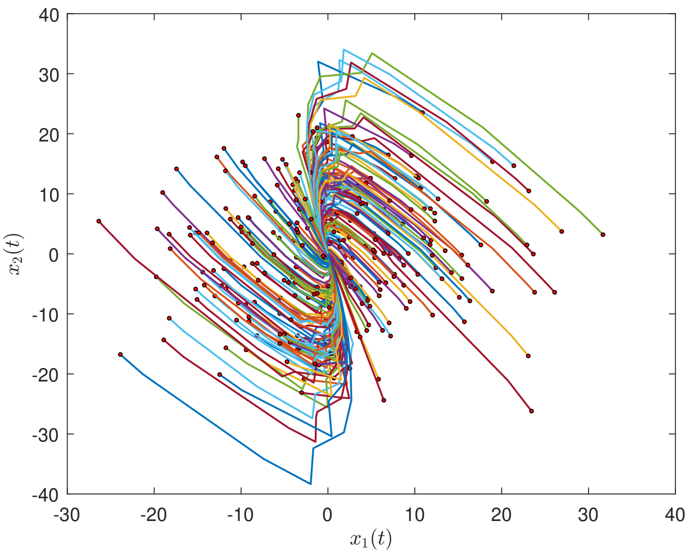

Figure 11 – Phase plane considering the time evolution of the states $x_1(t)$ and $x_2(t)$ in the closed loop system of the Example 1 from 1000 different initial conditions using The Adams-Moulton discretization method and Theorem 2.

6.2 Example 2

This second example consists of a continuous-time uncertain system adapted from Seuret and Briat (2015). The structure of the system follows 3.10 and their matrices are defined by

$$A_\lambda = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.2}$$

where $|c| \leq 0.1$. In Seuret and Briat (2015), c is time-dependent. However, for applying the methodologies developed in this work c is not considered time-dependent. Nevertheless, the sampling time $\Delta t = 0.7996$ can still be compared, as the time-dependent uncertainty provides a more general approximation. Consequently, this result establishes an upper bound for the sampling time from Seuret and Briat (2015).

The same steps used in Example 1 are followed in this example. Table 6 presents the results for the system in Example 2 considering each of the methodologies developed in this thesis. Note that The Mixed Method obtained a result that surpasses the procedure presented in Seuret and Briat (2015) when it is used with order $N = 1$ and Theorems 1 and 2. Adams-Bashforth and Adams-Moulton Methods also overcame the sampling-time $\Delta t = 0.7996s$ proposed by Seuret and Briat (2015). Considering these methodologies, the best result achieved was using Adams-Moulton discretization method and Theorem 4 applying a parameter-dependent Lyapunov function.

Table 6 – Values of sampling-time, Δt considering maximum delay given by N for each discretization method and convergence region (when Adams-Bashforth and Adams-Moulton Method are considered) for the system of Example 2.

Method	Theorem	N	1	2	3	4	5	6
MM	1 (P)	$\Delta t(s)$	0.93	0.6	0.45	0.3	0.24	-
	2 (P_λ)	$\Delta t(s)$	0.96	0.61	0.45	0.3	0.24	-
AB	4 $(\bar{W} = W)$	$\Delta t(s)$	0.16	0.3	0.53	0.63	0.33	0.14
		Region	973.54	921.88	963.25	87.54	0.71	2.3844×10^{-7}
	4 $(\bar{W} = W_\lambda)$	$\Delta t(s)$	0.16	0.3	0.8	0.4	0.39	0.19
		Region	981.9	996.33	930.62	790.9	0.931	2.17
AM	4 $\bar{W} = W$	$\Delta t(s)$	0.59	0.69	0.76	1.3	1.23	0.56
		Region	999.16	998.97	953.35	935	780.32	0.10
	4 $\bar{W} = W_\lambda$	$\Delta t(s)$	0.59	0.69	0.77	1.44	1.28	0.69
		Region	996.38	971.47	958.25	999.71	157.52	0.208

6.2.1 Step-by-step execution of The Mixed Method for the system of Example 2

Each value of Δt associated with the discrete-time system obtained by applying The Mixed Method and using Theorem 1 and Theorem 2 are shown in Table 6. Despite the procedure providing discrete-time controllers for each order ($N = 1,2,\ldots$) the best result is achieved using $N = 1$, that is, an augmented structure provides extra available information. However, the synthesis conditions could not take advantage of this new configuration considering higher orders, and a simple Euler approximation and Theorems 1 and 2 were better than the result presented by Seuret and Briat (2015).

As the best result does not use delayed steps the optimization procedure is unnecessary. Therefore elements related to this step are not presented. The time evolution of the states x_1 and x_2 and the time evolution of the discrete-time control law u_k for the best result ($N = 1$ using Theorem 2 and $\Delta t = 0.96s$) are presented in Figures 12, 13 and 14, respectively. The closed-loop system was simulated for 1000 different values of $\lambda_r \in \Lambda_R$, with all trajectories starting from initial condition $x^T(0) = [1, -3]^T$ and also Figure 11 presents the phase plane illustrating that with 1000 different initial conditions the closed-loop system is stabilized.

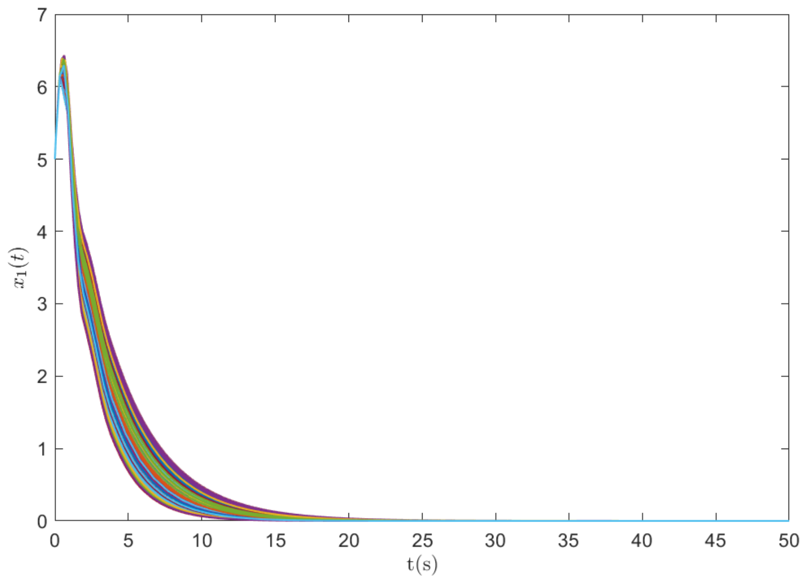

Figure 12 – Time evolution of states x_1 considering 1000 different combination to λ. Example 2 using The Mixed Method and Theorem 2

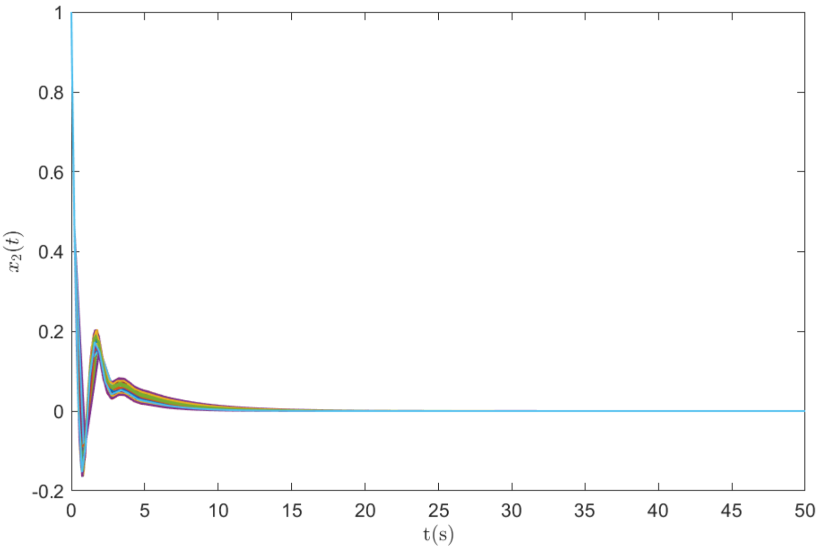

Figure 13 – Time evolution of states x_2 considering 1000 different combination to λ. Example 2 using The Mixed Method and Theorem (2)

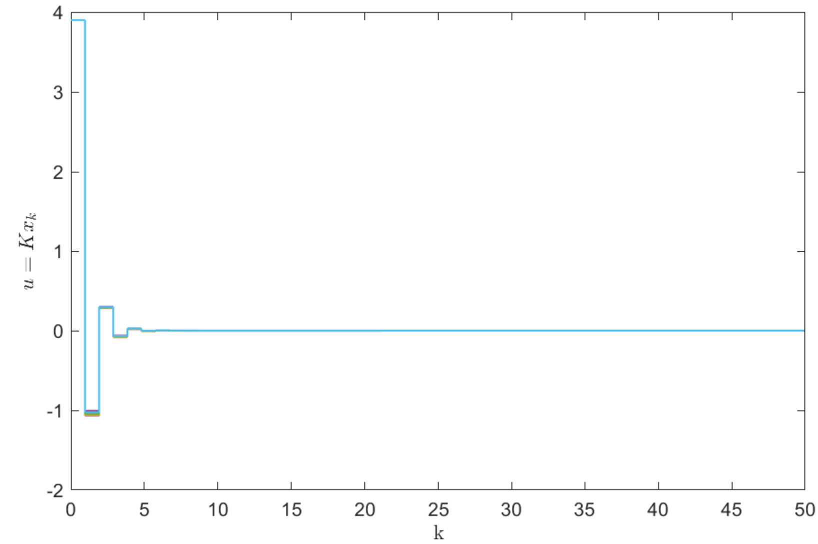

Figure 14 – Time evolution of control signal u_k. Example 2 using The Mixed Method and Theorem (2).

6.2.2 Step-by-step execution of The Adams-Bashforth and Adams-Moulton Methods for the system of Example 2

Following the same steps as in Subsection 6.2.1 for the system of Exemple 2, a stabilizing digital state-feedback control law will be determined by means of the conditions reported in Theorems 3 and 4. In the current scenario, the adopted constants are depicted in Table 7. Adjusting the constants could potentially improve the results.

In this example, both discretization procedures using Theorem 4 achieved equivalent or better results than those developed by Seuret and Briat (2015). Using Adams-

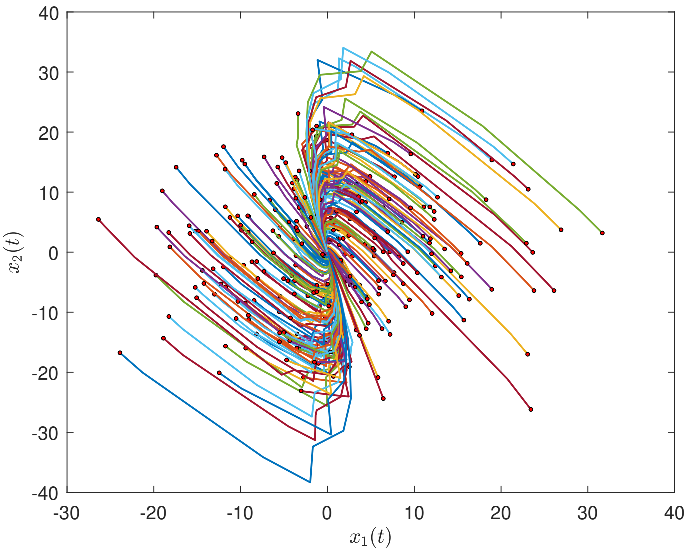

Figure 15 – Phase plane considering the time evolution of the states $x_1(t)$ and $x_2(t)$ in the closed loop system of the Example 1 from 1000 different initial conditions using The Mixed Method and Theorem 2.

Table 7 – Scalar considered in Theorem 4 for the system of Example 2.

$\bar{x}^2$	$\bar{u}^2$	r	θ	ϵ
1000	10000	1	0	0.99

Bashforth discretization method with $N = 4$, Theorems 3 and 4 with constants presented in Table 12 parameter-dependent Lyapunov functions, the best sampling time established was $\Delta = 0.8s$ with a convergence region smaller than $\bar{x}^2$ as reported in Table 6. Note that due to the chosen value of $\overline{x}$, the convergence regions have higher values when compared to Example 1. However, as described in Remark 7, the discrete-time control law guarantees the stability of the continuous-time system if the calculated value for the convergence region is lower than $\overline{x}$.

In turn, the Adams-Moulton discretization method and Theorems 3 and 4 provide even better results when used $N = 4$, that is, four delays are considered to build a discrete-time system. Using a constant Lyapunov function, the higher sampling time is $\Delta t = 1.3s$ with convergence region given by 935 and considering a parameter-dependent Lyapunov function $\Delta t = 1.44s$ and the convergence region is given by 999.71. For this case, the control gain is given by

$$K = \Big[0.0103 \quad -0.0708 \quad 0.1562 \quad -0.4008 \quad 0.0000 \quad 0.0050$$
$$-0.0020 \quad -0.0272 \quad 0.0100 \quad 0.0578 \quad -0.2068 \quad -0.5968\Big].$$

For this particular case and using the initial $x^T = [-1, 3]^T$, inside the method's feasibility region. A thousand different simulations were performed. The behavior and the convergence of the states for the closed-loop system are presented in Figures 16, and 17, whereas the time evolution of the digital control signal is shown in Figure 18. Beside of this, Figure 19 presents the phase plane illustrating that with 1000 different initial conditions ($x_0 \in [-31{,}31]$) the closed-loop system is stabilized. These results were obtained using the Adams-Moulton discretization method with maximum delay $N = 4$, sampling time $\Delta t = 0.42s$, and the control gain, K, designed using Theorem 4 given by

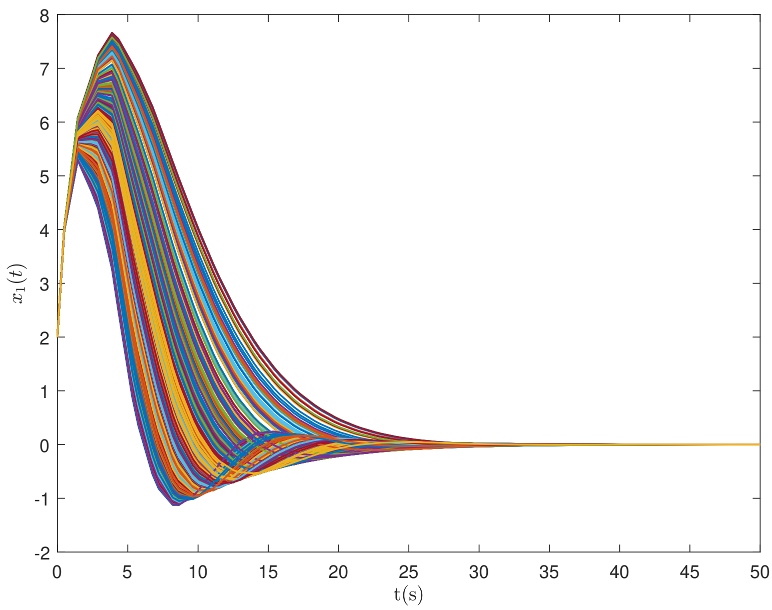

Figure 16 – Time evolution of states x_1 considering 1000 different combinations of λ in Example 2 using Adams-Moulton Method and Theorem (4).

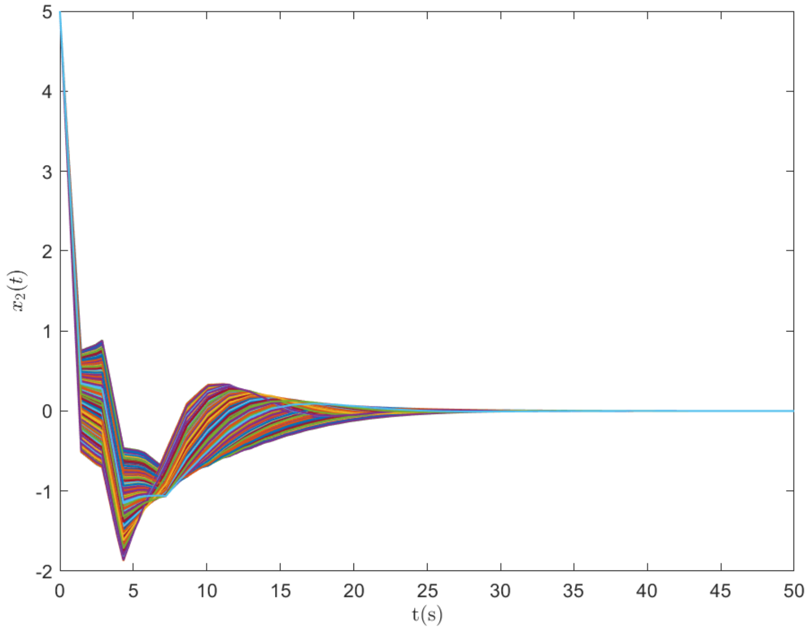

Figure 17 – Time evolution of states x_2 considering 1000 different combinations of λ in Example 2 using Adams-Moulton Method and Theorem (4).

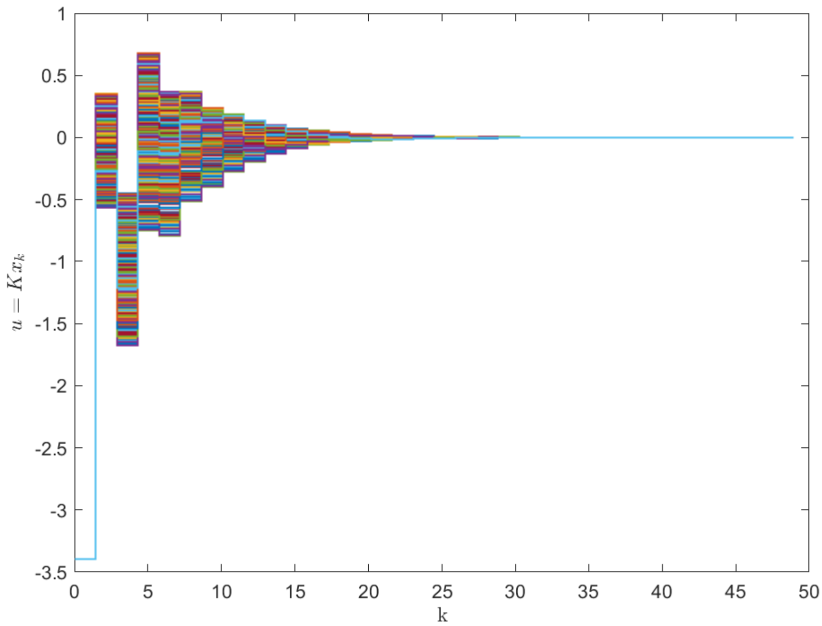

Figure 18 – Time evolution of control signal u_k in Example 2 using Adams-Moulton Method and Theorem (4).

6.3 Example 3

Consider the illustrative example adapted from Agulhari et al. (2010). This system has more states and, therefore, provides greater complexity for the controller synthesis conditions, making it more difficult to obtain discrete-time control laws that stabilize the

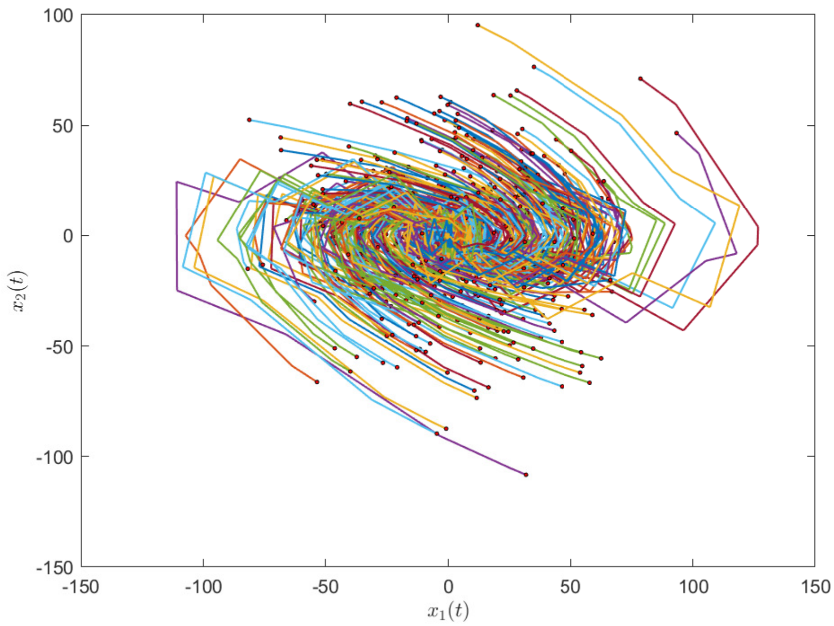

Figure 19 – Phase plane considering the time evolution of the states $x_1(t)$ and $x_2(t)$ in the closed loop system of the Example 1 from 1000 different initial conditions using The Adams-Moulton discretization method and Theorem 4.

continuous-time uncertain system.

In Agulhari et al. (2010), the discrete-time uncertain matrices are obtained from Euler first-order approximation with a sampling time of 0.1s and are depicted below

$$A_{d_i} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ \frac{-0.1(k_1+k_2)}{m_1} & \frac{0.1k_2}{m_1} & 1-\frac{0.1c_0}{m_1} & 0 \\ \frac{0.1k_2}{m_2} & \frac{-0.1k_2}{m_2} & 0 & 1-\frac{0.1c_0}{m_2} \end{bmatrix}, \tag{6.3}$$

where $m_1 = 2$, $m_2 = 1$, $1 \leq k_1 \leq 4$, $k_2 = 0.5$, $1 \leq c_0 \leq 4$ and

$$B_{d_i} = \begin{bmatrix} 0 & 0 & 0.5 & 0 \end{bmatrix}^T. \tag{6.4}$$

The continuous-time uncertain matrices were recovered by applying the inverse operation of the first-order Euler method to each vertex of the presented discrete-time system. Consequently, the discretization methodologies proposed in this work can be performed.

Although the system in Example 3 is adapted from Agulhari et al. (2010), this work does not focus on developing discretization strategies or determining the best sampling time. Therefore, there is no direct comparison between the methodologies depicted in this work with the paper developed by Agulhari et al. (2010).

Table 8 – Values of sampling-time, Δt, considering maximum delays given by N for each discretization method and convergence region (when Adams-Bashforth and Adams-Moulton Method are considered) for the system of Example 3.

<table>
<tr><th>Method</th><th>Theorem</th><th>N</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr>
<tr><td rowspan="2">MM</td><td>1 (P)</td><td>$\Delta t(s)$</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr>
<tr><td>2 (P_λ)</td><td>$\Delta t(s)$</td><td>0.01</td><td>-</td><td>0.02</td><td>0.02</td><td>0.01</td><td>0.01</td></tr>
<tr><td rowspan="4">AB</td><td rowspan="2">4 ($\bar{W} = W$)</td><td>$\Delta t(s)$</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr>
<tr><td>Region</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td><td>-</td></tr>
<tr><td rowspan="2">4 ($\bar{W} = W_\lambda$)</td><td>$\Delta t(s)$</td><td>-</td><td>-</td><td>0.13</td><td>0.09</td><td>0.04</td><td>-</td></tr>
<tr><td>Region</td><td>-</td><td>-</td><td>7.23</td><td>0.19</td><td>7.25^{-9}</td><td>-</td></tr>
<tr><td rowspan="4">AM</td><td rowspan="2">4 ($\bar{W} = W$)</td><td>$\Delta t(s)$</td><td>-</td><td>0.07</td><td>0.24</td><td>0.46</td><td>0.33</td><td>0.17</td></tr>
<tr><td>Region</td><td>-</td><td>7.7619</td><td>8.53</td><td>7.78</td><td>0.9</td><td>4.34×10^{-7}</td></tr>
<tr><td rowspan="2">4 ($\bar{W} = W_\lambda$)</td><td>$\Delta t(s)$</td><td>-</td><td>0.08</td><td>0.27</td><td>0.46</td><td>0.34</td><td>0.18</td></tr>
<tr><td>Region</td><td>-</td><td>8.44</td><td>9.34</td><td>2.81</td><td>0.9568</td><td>1.02×10^{-5}</td></tr>
</table>

6.3.1 Step-by-step execution of The Mixed Method for the system of Example 3

When Theorem 1 (which involves a constant Lyapunov function) is applied, it is not possible to find feasible discrete-time control laws using the chosen orders $N = 1, \ldots, 6$. However, when Theorem 2 is used, orders $N = 3$ and $N = 4$ obtain the best values of sampling time when The Mixed Method is applied, that is, $\Delta = 0.02s$.

Analyzing in detail the results obtained for this system, the optimization process often ends up being trapped in a local minimum. Increasing the sampling time causes the obtained coefficients to approach zero, consequently, using conditions from Theorem 4 cannot achieve a feasible control law.

Despite not obtaining significant results, to illustrate the validity of the methodology to find discrete-time control laws that stabilize the continuous-time system, the closed-loop system was simulated for different values of λ_r, considering an initial condition given by $x^T(0) = [5, \ -3, \ -1, \ 8]^T$. The evolution of the states are presented in Figures 20, 21, 22 and 23 for x_1, x_2, x_3 and x_4, respectively, and evolution of the control input is depicted in Figure 24.

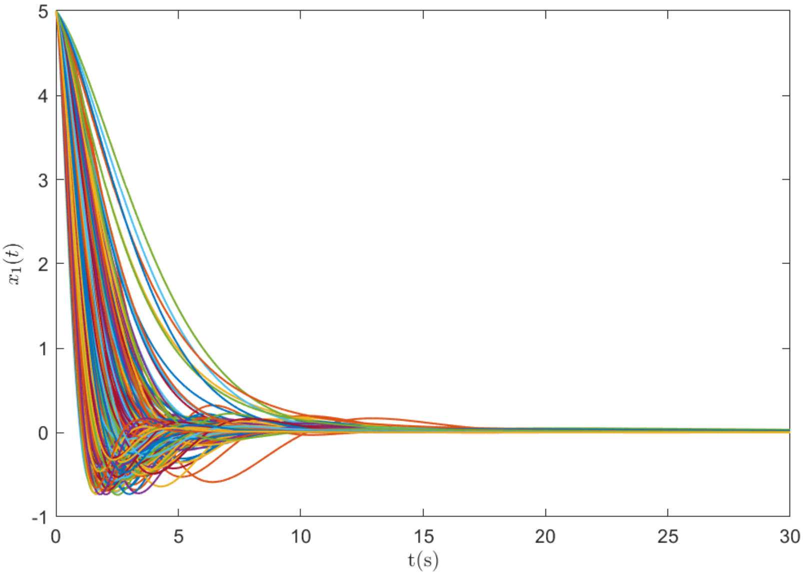

Figure 20 – Time evolution of states x_1 considering 1000 different combinations of λ in Example 3 using The Mixed Method and Theorem 2.

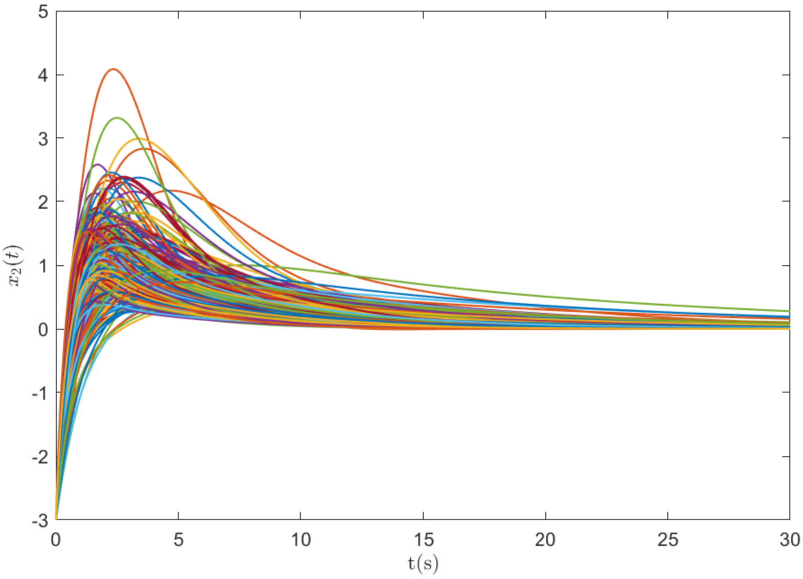

Figure 21 – Time evolution of states x_2 considering 1000 different combinations of λ in Example 3 using The Mixed Method and Theorem 2.

6.3.2 Step-by-step execution of The Adams-Bashforth and Adams-Moulton Methods for the system of Example 3

Following the same procedure depicted in Subsection 6.1.2 constants related to Theorems 3 and 4 are defined in Table 9 for this example. These constants are not optimal parameters and can be better adjusted for each discretization methodology. Nevertheless, they were conserved for all the tested procedures.

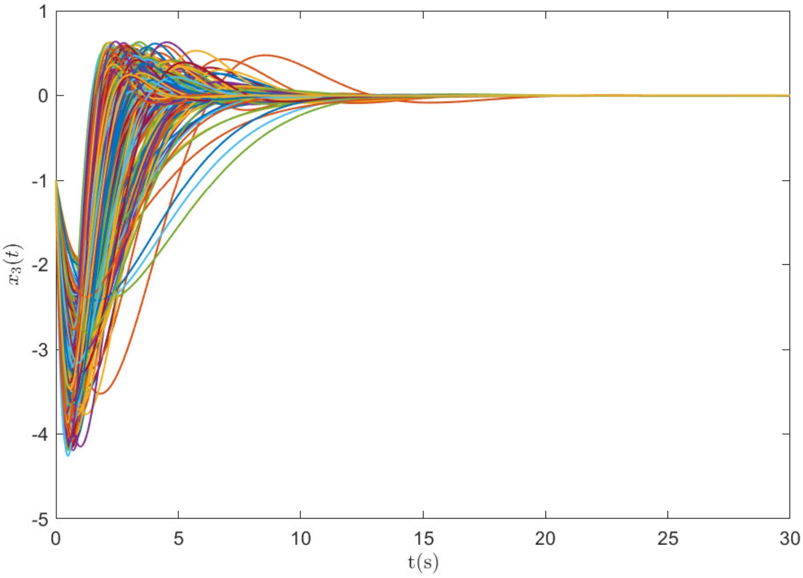

Figure 22 – Time evolution of states x_3 considering 1000 different combinations of λ in Example 3 using The Mixed Method and Theorem 2.

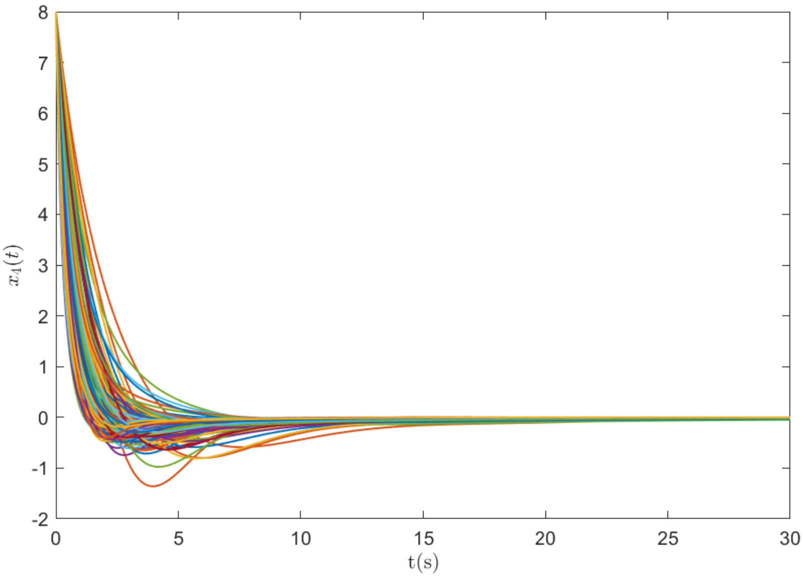

Figure 23 – Time evolution of states x_4 considering 1000 different combinations of λ in Example 3 using The Mixed Method and Theorem 2.

Table 9 – Scalar considered in Theorem 4 for the system of Example 3.

$\bar{x}^2$	$\bar{u}^2$	r	θ	ϵ
10	10000	100	0.1	0.99

Feasible results, with a smaller region of convergence than $\bar{x}^2$ were obtained by applying the Adams-Bashforth discretization only considering a parameter-dependent Lyapunov function, except for the case $N = 1$.

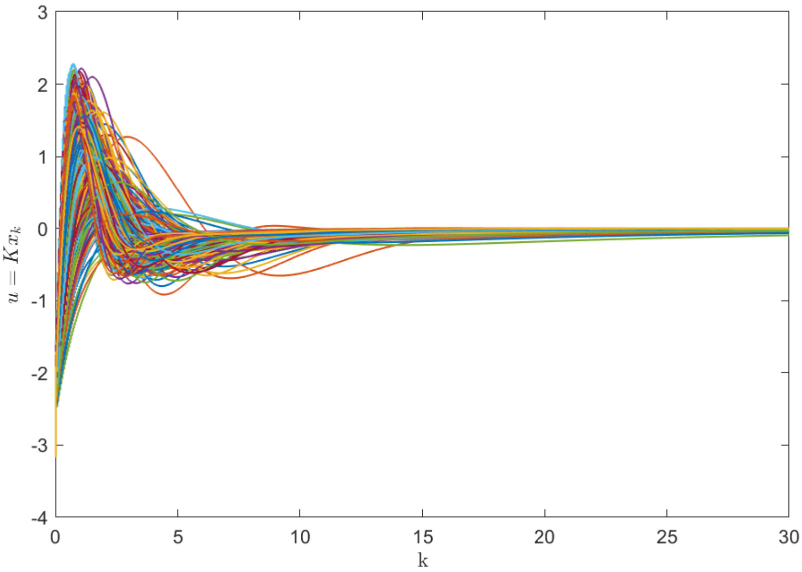

Figure 24 – Time evolution of control signal u_k in Example 3 using The Mixed Method and Theorem 2.

Using the Adams-Moulton discretization method, Theorems 3 and 4 provide improved results utilizing constant and parameter-dependent Lyapunov functions. The best sampling time was obtained in both cases considering a discrete-time system built with $N = 4$, achieving $\Delta t = 0.46s$. Note that, in this example, it is necessary to adjust the value of $\theta = 0.1$ as mentioned in Remark 6. For the case considering $N = 4$, constants presented in Table (9) and Theorem 4 using a parameter-dependent Lyapunov Function, the discrete-time control gain is given by

$$K = \begin{bmatrix} 0.0011 & -0.0085 & 0.0227 & -0.0096 & -0.0257 & 0.0024 \\ & -0.0155 & -0.0050 & 0.0296 & 0.0008 & 0.0105 & -0.0084 \\ & -0.0044 & 0.0304 & -0.0121 & 0.0330 & -0.0121 \\ & 0.0330 & -0.1187 & -0.1837 & -0.3733 & -0.1642 \end{bmatrix}$$

To illustrate the methodology, a simulation is performed for different values of λ_r, considering the initial condition defined by $x^T(0) = [-1,\, 2,\, -3,\, 1]^T$.

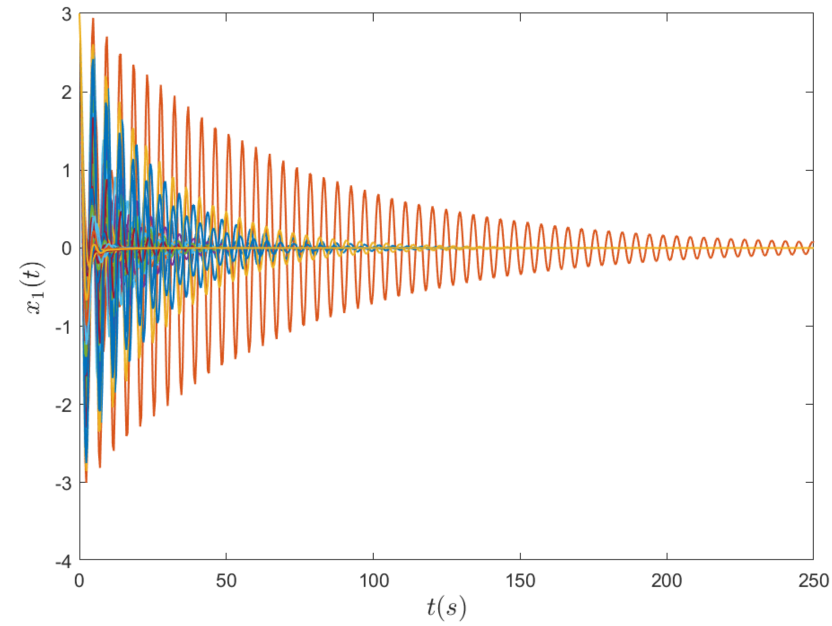

Figure 25 – Time evolution of states x_1 considering 1000 different combinations of λ in Example 3 using The Adams-Moulton Method and Theorem 4.

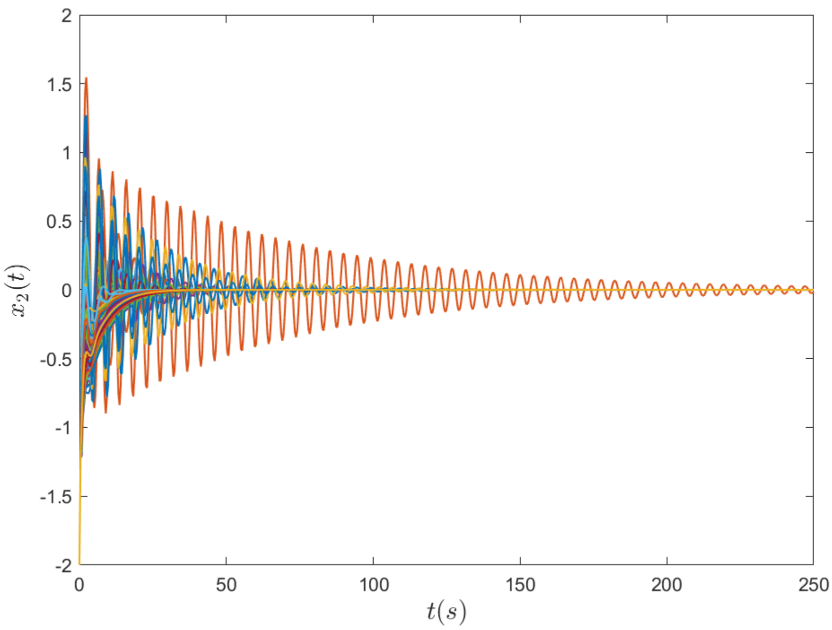

Figure 26 – Time evolution of states x_2 considering 1000 different combinations of λ in Example 3 using The Adams-Moulton Method and Theorem 4.

6.4 Comparative Results

In this subsection, the proposed methodologies developed are compared with the works of Fridman et al. (2004), Jungers et al. (2017), and Naghshtabrizi et al. (2008). Note that since the work of Agulhari et al. (2010) does not properly take into account the efforts of discretization of an uncertain system and the work developed by Seuret and

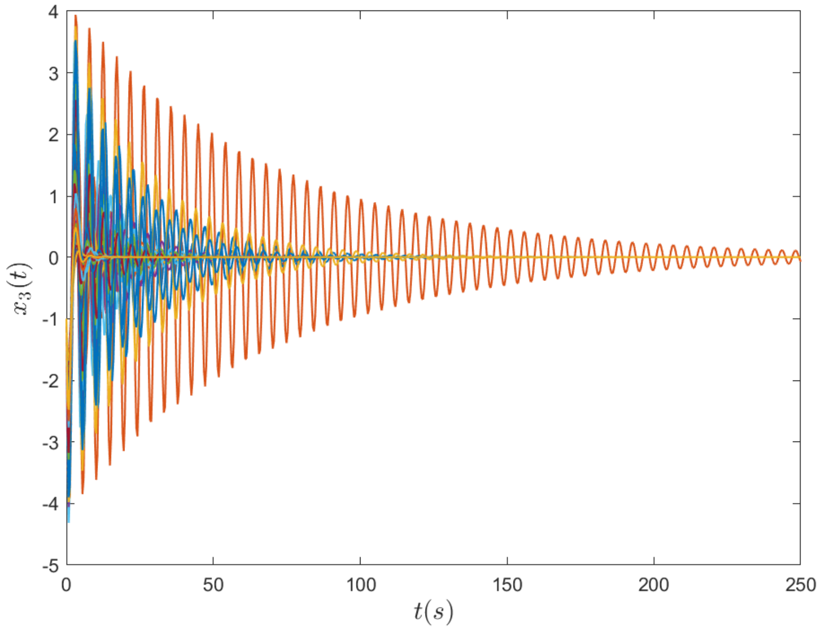

Figure 27 – Time evolution of states x_3 considering 1000 different combinations of λ in Example 3 using The Adams-Moulton Method and Theorem 4.

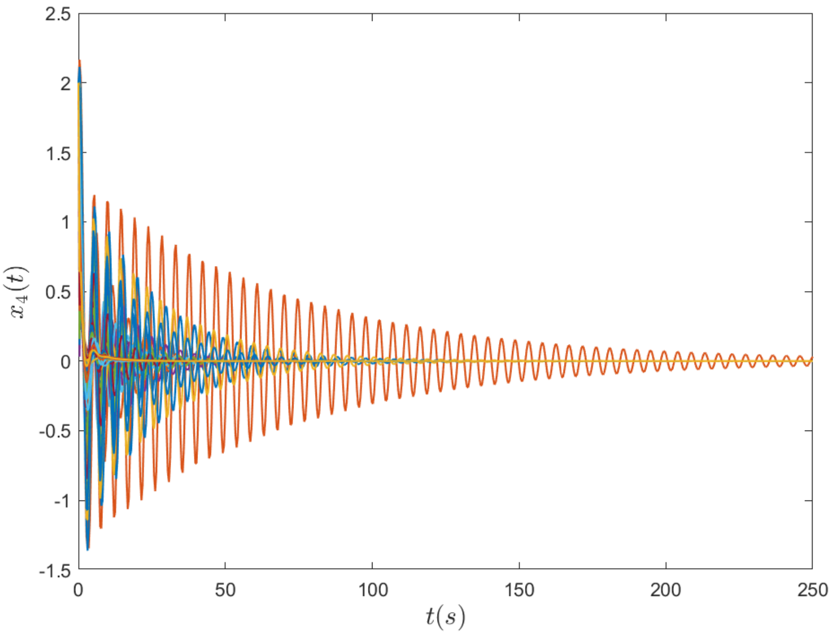

Figure 28 – Time evolution of states x_4 considering 1000 different combinations of λ in Example 3 using The Adams-Moulton Method and Theorem 4.

Briat (2015) does not have an approach for controller synthesis, they are not included in the comparison.

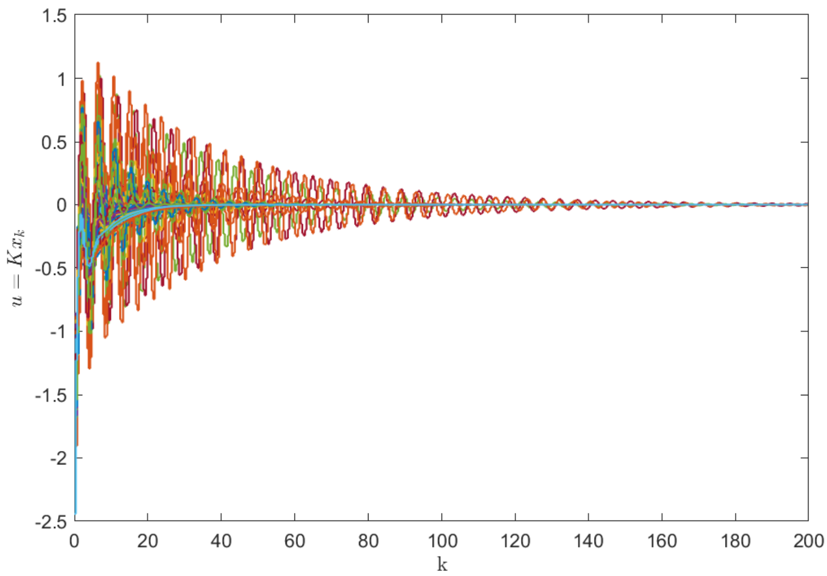

Figure 29 – Time evolution of control signal u_k in Example 3 using The Adams-Moulton Method and Theorem 4.

6.4.1 Comparison based on sampling time

Table 10 presents the best results related to the methodologies and results obtained from Examples 1, 2, and 3 considering the conditions defined in works of Fridman et al. (2004), Jungers et al. (2017) and Naghshtabrizi et al. (2008). The best results among the considered methodologies are highlighted in blue.

Table 10 – The best results obtained using The Mixed Method (Theorems 1 and 2), Adams-Bashforth and Adams-Moulton discretization methods (Theorems 3 and 4) and conditions from Fridman et al. (2004), Jungers et al. (2017), and Naghshtabrizi et al. (2008) considering Examples 1, 2, and 3.

Method		Example 1	Example 2	Example 3
MM	$\Delta t(s)$	0.3759	0.96	0.02
AB	$\Delta t(s)$	0.23	0.8	0.13
AM	$\Delta t(s)$	0.38	1.44	0.46
Fridman et al. (2004)	$\Delta t(s)$	0.35	1	0.36
Jungers et al. (2017)	$\Delta t(s)$	0.35	0.92	0.17
Naghshtabrizi et al. (2008)	$\Delta t(s)$	0.22	1.15	0.58

Note that, the Adams-Moulton discretization method provides better results in two of the three examples considered, achieving the best sampling times ($\Delta t = 0.38s$ for Example 1 and $\Delta t = 1.44s$ for Example 2) when compared with other methodologies developed in this work. This advantage arises due to Adams-Moulton being an implicit numerical method, which means this method uses the current value to correct the predicted value. This approach reduces truncation error and increases accuracy, avoiding the accumulation of errors in long simulations (Wang et al., 2023).

When Example 3 is addressed, the greatest result obtained is using the methodology depicted in Naghshtabrizi et al. (2008) achieving $\Delta t = 0.58s$. This demonstrates that different systems can adapt better to one or another discrete controller synthesis condition.

It is important to mention that among the literature examples chosen, the only one that has the flexibility to achieve higher sampling times is Jungers et al. (2017). This work provides the possibility of using larger degrees for homogeneous polynomials. This fact contributes to the representation having a greater capacity to capture non-linear variations and system uncertainties and consequently provides less conservative results and the best sampling times with the computational cost increasing as the order of the polynomial increases. For this reason, the results obtained from Jungers et al. (2017) used affine polynomials structure to compose polytopic representation for a fair comparison. Results considering larger degrees for homogeneous polynomials are depicted in the next subsection.

6.4.2 Comparison based on execution time

The methodology proposed by Jungers et al. (2017) can use high-order homogeneous polynomials to obtain discrete-time control laws with higher sampling times due to the obtained flexibility using this structure. However, this flexibility may incur significant computational costs. For this reason, a specific comparison with Jungers et al. (2017) noting the execution time is depicted in this section.

For analysis and comparison, the methodologies developed in this thesis consider a similar structure for the polytopic representation and $N = 1, 2, \ldots, 6$ delays for its composition. On the other hand, the methodology of Jungers et al. (2017) considers the order of the homogeneous polynomial with $N = 1, 2, \ldots, 6$. Tables 11, 12 and 13 present the results from these comparison.

Note that, when small orders are considered, for example, $N = 1$ or $N = 2$, the Mixed Method, the Adams-Bashforth and the Adams-Moulton discretization methods can provide better sampling times than the approach developed by Jungers et al. (2017). However, increasing the order of the Jungers et al. (2017) method corresponds to raising the degree of the homogeneous polynomial. By definition, this increase results in less conservatism but also leads to a higher computational cost. In the approaches developed in this work, raising the order involves incorporating more delayed steps, which yields a more conservative solution compared to results from Jungers et al. (2017). However, this is achieved with a lower computational cost, especially when considering methodologies

Table 11 – Results obtained using The Mixed Method (Theorems 1 and 2), Adams-Bashforth and Adams-Moulton discretization methods (Theorems 3 and 4) and conditions from Jungers et al. (2017) considering Example 1.

Method	N	1	2	3	4	5	6
Jungers et al. (2017)	$\Delta t(s)$	0.35	0.67	0.83	0.86	0.89	0.8962
	Exec. Time (s)	1.41	1.29	3.22	10.7	47.53	289.81
MM	$\Delta t(s)$	0.15	0.2557	0.3759	0.319	0.2943	0.256
	Exec. Time (s)	7.74	8.23	8.4104	11.4486	13.1955	15.6471
AB	$\Delta t(s)$	0.11	0.09	0.23	0.19	0.06	-
	Exec. Time (s)	0.052	0.043	0.042	0.047	0.052	-
AM	Δt	0.27	0.1	0.25	0.38	0.35	0.18
	Exec. Time (s)	0.051	0.42	0.042	0.047	0.053	0.075

Table 12 – Results obtained using The Mixed Method (Theorems 1 and 2), Adams-Bashforth and Adams-Moulton discretization methods (Theorems 3 and 4) and conditions from Jungers et al. (2017) considering Example 2.

Method	N	1	2	3	4	5	6
Jungers et al. (2017)	$\Delta t(s)$	0.92	1.49	1.88	2.22	2.54	2.84
	Exec. Time (s)	0.5849	0.69	0.7058	1.058	1.54	2.6
MM	$\Delta t(s)$	0.96	0.61	0.45	0.3	0.24	-
	Exec. Time (s)	6.88	7.09	8.49	11.45	13.67	-
AB	$\Delta t(s)$	0.16	0.3	0.55	0.8	0.4	0.19
	Exec. Time (s)	0.061	0.043	0.04	0.047	0.051	0.056
AM	$\Delta t(s)$	0.59	0.69	0.77	1.44	1.28	0.69
	Exec. Time (s)	0.065	0.042	0.041	0.046	0.0514	0.0567

Table 13 – Results obtained using The Mixed Method (Theorems 1 and 2), Adams-Bashforth and Adams-Moulton discretization methods (Theorems 3 and 4) and conditions from Jungers et al. (2017) considering Example 3.

Method	N	1	2	3	4	5	6
Jungers et al. (2017)	$\Delta t(s)$	0.17	0.247	0.387	0.505	0.621	0.7212
	Exec. Time (s)	0.7219	1.22	3.47	11.19	49.5	289.66
MM	$\Delta t(s)$	0.01	-	0.02	0.02	0.01	0.01
	Exec. Time (s)	7.11	-	9.83	18.33	20.65	31.56
AB	$\Delta t(s)$	-	-	0.13	0.09	0.04	-
	Exec. Time (s)	-	-	1.08	1.20	1.84	-
AM	$\Delta t(s)$	-	0.08	0.27	0.46	0.34	-
	Exec. Time (s)	-	0.058	0.052	0.062	0.104	-

Table 14 – Execution time for Examples 1, 2 and 3 considering methodologies from Fridman et al. (2004) and Naghshtabrizi et al. (2008).

		Example 1	Example 2	Example 2
Fridman et al. (2004)	$\Delta t(s)$	0.35	1	0.36
	Exec. Time (s)	0.82	0.61	0.67
Naghshtabrizi et al. (2008)	$\Delta t(s)$	0.22	1.15	0.58
	Exec. Time (s)	4.78	2.64	7.64

based on Adams-Bashforth and Adams-Moulton. In addition, the methods based on Adams-Bashforth and Adams-Moulton cannot increase the orders indefinitely, once the region of stability of these methods tends to decrease as the order increases.

Beside of this, Table 14 provides the execution time to other works considering in this comparison. Note that, in most cases, these methodologies do not outperform the results of the Adams-Moulton method and Theorem 4 (except in Example 3 with Naghshtabrizi et al. (2008)). For all examples, these works reported the execution times higher than the Adams-Moulton method.

By examining the numerical results, it is possible to observe that the methodologies proposed in this work achieve good results by including delayed time instances in the formulation of the discrete system. From this system, synthesis conditions for controllers are developed, allowing the recovery of a discrete control law capable of stabilizing the continuous system.

The mixed method, despite yielding less impressive results regarding the achievement of the largest sampling time and having higher computational cost, is the methodology that ensures a global result—that is, the obtained control law does not depend on the initial condition. On the other hand, the Adams-Bashforth and Adams-Moulton methods require less computational effort and yield the best results in terms of sampling time, especially the Adams-Moulton discretization method, which is based on an implicit numerical method and, therefore, is less affected by truncation issues. However, these methodologies demand more from the designer, as there are parameters to be defined before executing the synthesis conditions. Furthermore, this methodology provides local results, meaning the initial condition must be within the starting region, i.e., $x_0 \leq \bar{x}$.

Chapter 7

Final Remarks

This thesis proposed three methodologies for designing discrete-time controllers to stabilize time-invariant continuous-time systems within polytopic domains. The LMI-based conditions were derived using Lyapunov theory, initially considering constant Lyapunov functions and later extending to parameter-dependent Lyapunov functions to reduce conservatism. The LMI-based conditions were constructed from discrete-time uncertain linear systems obtained through a multistep discretization approach that incorporated delayed steps, resulting in augmented discrete-time models.

Among the proposed methods, the Mixed Method combined delayed steps through an optimization process to determine the best coefficients that minimize discretization error. This methodology ensures a global solution—meaning the resulting control law is independent of the initial condition—but is computationally expensive due to the fine-grid approach used to define uncertainties. Additionally, as the sampling time increases, the optimization may fail to find feasible solutions.

In contrast, the Adams-Bashforth and Adams-Moulton discretization methods employed pre-defined coefficients based on the order of the numerical method. These approaches demand less computational effort and yielded superior results in terms of achievable sampling time when compared to those found in the literature. The Adams-Moulton method, in particular, provided the best performance due to its implicit structure, which mitigates truncation errors. However, these methods produce local solutions, requiring the initial condition to lie within the defined starting region, i. e., $x_0 \leq \bar{x}$.

Finally, the numerical examples illustrate the validity of each discretization methodology and state-feedback synthesis conditions developed.

7.1 Future Research

Some suggestions of possible next steps for this research are discussed in this section. They are based on further developing the main objective of this thesis: to use lifting techniques to propose state-feedback discrete-time control strategies for the stabilization of continuous-time systems in polytopic domains.

- Extending the proposed methodologies to compute $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance

  $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are important performance robustness criteria for systems when they are affected by uncertainties. While $\mathcal{H}_2$ measures the energy of the system's response to noise inputs, $\mathcal{H}_\infty$ provides the worst response of the system to the worst disturbance (Zhou and Doyle, 1998). Hence, many authors devote their effort to obtaining these performance criteria when discrete-time uncertain systems and state feedback synthesis are developed (Ebihara et al., 2011; Kim and Hagiwara, 2022b; Yue et al., 2024) showing the importance of providing a measure of robustness for the proposed methodologies.

- Extending the proposed methodologies to consider linear parameter-varying (LPV) systems

  LPV system theory has been widely studied in recent years due to extending the concept of linear time-invariant (LTI) systems by allowing system dynamics to vary with time-dependent parameters. These systems provide a structured way to model nonlinear or time-varying dynamics as a linear system whose coefficients depend on measurable, time-varying parameters. Rugh and Shamma (2000). The combination of LPV system formulation and lifting techniques (that improve the precision of the discrete-time system) based on multistep theory can be a strong tool to obtain efficient and robust state-feedback synthesis conditions to stabilize the continuous-time system.

- Using the multistep theory to develop redesign control techniques

  The methodology that converts an existing state-feedback control in the continuous-time domain into an equivalent digital one is called the digital redesign method. This methodology is widely used due to the easy implementation and maintenance of the proprieties of the continuous-time system (Hu, 2023; Jang et al., 2022; Fang et al., 2022). One of the problems of the digital redesign is the dependence on sampling time. Therefore, the usage of multistep theory to build conditions that increase the available information can be considered in future works.

7.2 Developed Papers

The methodology reported in Chapter 4 has been published in the following paper:

[1] N. A. Keles, L Frezzato, E. M. A. M. Mendes, and V. C. S. Campos, "Discretization and state feedback control for uncertain linear systems - a new approach considering linear multistep method theory," *Journal of the Franklin Institute, 2023*.

Furthermore, the results reported in Chapter 5 are also submitted to a journal and the paper following paper is currently under review. The information associated with this paper is presented in the sequel.

Title: *Discretization and State Feedback Control for Uncertain Linear Systems - A new approach considering Adams-Bashforth and Adams-Moulton Methods*;
Authors: *N. A. Keles, L. Frezzato, E. M. A. M. Mendes, V. C. S. Campos*;

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Appendix A

Teaching example to build the augmented discrete-time system based on the Mixed Method

Consider the Mixed method presented in (4.24) using $N = 4$ delays, $n_u = 1$ and $n = 2$. The difference equation, in this case, is given by

$$
\begin{aligned}
x_k = {} & \alpha_1(A_{d_{1\lambda}} + \Delta_{\bar{A}_1})x_{k-1} && \text{(A.1)}\\
& +\alpha_2(A_{d_{2\lambda}} + \Delta_{\bar{A}_2})x_{k-2} && \text{(A.2)}\\
& +\alpha_3(A_{d_{3\lambda}} + \Delta_{\bar{A}_3})x_{k-3} && \text{(A.3)}\\
& +\alpha_4(A_{d_{4\lambda}} + \Delta_{\bar{A}_4})x_{k-4} && \text{(A.4)}\\
& +\sum_{\ell=1}^{4}(B_{d_\lambda} + \Delta_{\bar{B}_1})u_{k-1} && \text{(A.5)}\\
& +\sum_{\ell=2}^{4}(B_{d_\lambda} + \Delta_{\bar{B}_2})u_{k-2} && \text{(A.6)}\\
& +\sum_{\ell=3}^{4}(B_{d_\lambda} + \Delta_{\bar{B}_3})u_{k-3} && \text{(A.7)}\\
& +\sum_{\ell=4}^{4}(B_{d_\lambda} + \Delta_{\bar{B}_4})u_{k-4} && \text{(A.8)}\\
& && \text{(A.9)}
\end{aligned}
$$

From (A.9) and shifting the delay in one unit is possible to write the augmented state vector as

$$\xi_k = \begin{bmatrix} u_{k-3}^T & u_{k-2}^T & u_{k-1}^T & x_{k-3}^T & x_{k-2} & x_{k-1}^T & x_k \end{bmatrix}^T \tag{A.10}$$

Using the definition of $\bar{A}_\lambda$ in the Mixed Method, we have

$$\Gamma_1 = \begin{bmatrix} \sum_{\ell=3}^{4} \alpha_\ell B_{d_\lambda} & \sum_{\ell=2}^{4} \alpha_\ell B_{d_\lambda} \end{bmatrix}, \tag{A.11}$$

that is, $\Gamma_1 \in \mathbb{R}^{n\times(N-2)n_u}$ and

$$\Gamma_2 = \begin{bmatrix}\alpha_3 A_{d_{3\lambda}} & \alpha_2 A_{d_{2\lambda}} & \alpha_1 A_{d_{1\lambda}}\end{bmatrix}, \tag{A.12}$$

then $\Gamma_2 \in \mathbb{R}^{n\times(N-1)n.}$

$$\bar{A}_\lambda = \left[\begin{array}{c|cc|c|ccc}
0_{n_u\times n_u} & I_{n_u} & 0_{n_u\times n_u} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} \\
0_{n_u\times n_u} & 0_{n_u\times n_u} & I_{n_u} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} \\
\hline
0_{n_u\times n_u} & 0_{n_u\times n_u} & 0_{n_u\times n_u} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} \\
\hline
0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n} & I_n & 0_{n\times n} & 0_{n\times n} \\
0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n} & 0_{n\times n} & I_n & 0_{n\times n} \\
0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & I_n \\
\hline
\alpha_4 B_{d\lambda} & \sum_{\ell=3}^{4}\alpha_\ell B_{d\lambda} & \sum_{\ell=2}^{4}\alpha_\ell B_{d\lambda} & \alpha_4 A_{d_{4\lambda}} & \alpha_3 A_{d_{3\lambda}} & \alpha_2 A_{d_{2\lambda}} & \alpha_1 A_{d_{1\lambda}}
\end{array}\right]$$

This matrix has the same partitions presented in (4.48). Following the same, idea $\bar{B}_{d\lambda}$ using $N = 4$ is given by

$$\bar{B}_\lambda = \left[\begin{array}{c}
0_{n_u\times n_u} \\
0_{n_u\times n_u} \\
\hline
0_{n_u\times n_u} \\
\hline
0_{n\times n_u} \\
0_{n\times n_u} \\
0_{n\times n_u} \\
\hline
B_{d\lambda}
\end{array}\right].$$

Appendix B

Teaching example to build matrices of Theorem 3

Following the steps of Theorem 3, and from the step defined in (5.15) we consider Adams-Bashforth or Adams-Moulton discretization method using $N = 3$ to develop a teaching example of this Theorem.

This way, (5.15) using $N = 3$ can be write as

$$\begin{aligned} &\frac{x^T(t)A_\lambda^{(4)^T}A_\lambda^{(4)}x(t)}{\zeta^2} + \frac{u^T(t)B_\lambda^T A_\lambda^{(3)^T}A_\lambda^{(3)}B_\lambda u(t)}{\zeta^2} \\ &\qquad + \frac{u^T(t)B_\lambda^T A_\lambda^{(3)}A_\lambda^{(4)}x(t)}{\zeta^2} + \frac{x^T(t)A_\lambda^{(4)^T}A_\lambda^{(3)}B_\lambda u(t)}{\zeta^2} \\ &\qquad\qquad - \eta\frac{u(t)^T(t)u(t)}{\overline{u}(t)^2} - (1-\eta)\frac{x^T(t)x(t)}{\overline{x}(t)^2} \leq 0 \end{aligned} \tag{B.1}$$

Considering the variable transformations

$$\begin{aligned} Z_1 &= A_\lambda x(t) \\ Z_2 &= A_\lambda \underbrace{A_\lambda x(t)}_{Z_1} \\ Z_3 &= A_\lambda \underbrace{A_\lambda A_\lambda x(t)}_{Z_2} \\ Z_4 &= A_\lambda \underbrace{A_\lambda A_\lambda A_\lambda x(t)}_{Z_3} \end{aligned} \quad \text{and} \quad \begin{aligned} Y_1 &= B_\lambda u(t) \\ Y_2 &= A_\lambda \underbrace{B_\lambda u(t)}_{Y_1} \\ Y_3 &= A_\lambda \underbrace{A_\lambda B_\lambda u(t)}_{Y_2} \\ Y_4 &= A_\lambda \underbrace{A_\lambda B_\lambda u(t)}_{Y_3} \end{aligned} \tag{B.2}$$

By the Finsler's Lemma, in which $w^T \mathcal{Q} w \leq 0$ and $\mathcal{B}w = 0$. Then, w, $\mathcal{B}$ and $\mathcal{Q}$ are given by, respectively

$$w^T = \begin{bmatrix} u(t)^T & Y_1^T & Y_2^T & Y_3^T & Y_4^T & x(t)^T & Z_1^T & Z_2^T & Z_3^T & Z_4^T \end{bmatrix} \tag{B.3}$$

$$\mathcal{B} = \begin{bmatrix} B_\lambda & -I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & A_\lambda & -I_n & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & A_\lambda & -I_n & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & A_\lambda & -I_n & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & A_\lambda & -I_n & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & A_\lambda & I_n & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & A_\lambda & -I_n & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & 0_{n\times n_u} & A_\lambda & -I_n \end{bmatrix} \tag{B.4}$$

$$\mathcal{Q} = \begin{bmatrix} \frac{-\eta}{\bar{u}^2} I_{n_u} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n_u\times n} & 0_{n\times n_u} & 0_{n_u\times n} & 0_{n\times n_u} & 0_{n\times n_u} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & \Phi I_n & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & \Phi I_n \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & \frac{-(1-\eta)}{\bar{x}^2} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n_u} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & \Phi I_n & 0_{n\times n} & 0_{n\times n} & 0_{n_u\times n} & 0_{n\times n} & \Phi I_n \end{bmatrix} \tag{B.5}$$

This is the explicit way to build the matrices used in Theorem 3.