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Belo Horizonte, Minas Gerais, Brasil

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"-Why do you want to fight?"

"-Because I can't sing or dance."

Rocky

Resumo

Nesta tese estudamos germes de hipersuperfícies real-analíticas Levi-flat singulares com dois propósitos distintos. Mostramos a existência de formas normais para germes de hipersuperfícies real-analítica Levi-flat singulares que são definidas pelo anulamento da parte real de polinômios complexos quase homogêneos com singularidade isolada. Este resultado generaliza resultados prévios de Burns-Gong [7] e Fernández-Pérez [15]. Ademais, mostramos a existência de duas novas formas normais rígidas para germes de hipersuperfícies real-analítica Levi-flat singulares que são preservadas por uma mudança de coordenadas isócoras, isto é, uma mudança de coordenadas que preserva volume. Além disso, tratamos do problema de encontrar condições suficientes para garantir a coincidência das curvas de nível de uma função holomorfa com as folhas da folheação de Levi em um germe de hipersuperfície real-analítica Levi-flat com singularidade isolada. Para um germe de hipersuperfície real-analítica irredutível em $0 \in \mathbb{C}^2$ com singularidade isolada não-dicrítica, mostramos que as folhas da folheação de Levi sempre coincidem com as curvas de nível de valores reais de uma função holomorfa. No caso dicrítico, um contra-exemplo deste resultado é dado.

Palavras-chave: Hipersuperfícies Levi-flat. Folheações holomorfas. Coordenadas isócoras.

Abstract

In this thesis we study germs of singular real-analytic Levi-flat hypersurfaces with two distinct purposes. We show the existence of normal forms for germs of singular Levi-flat hypersurfaces which are defined by the vanishing of the real part of complex quasihomogeneous polynomials with isolated singularity. This result generalizes previous results of Burns-Gong [7] and Fernández-Pérez [15]. Furthermore, we show the existence of two new rigid normal forms for germs of singular real-analytic Levi-flat hypersurfaces which are preserved by a change of isochore coordinates, that is, a change of coordinates that preserves volume. Moreover, we address the problem of finding sufficient conditions to guarantee the coincidence of the level sets of a holomorphic function with the leaves of the Levi foliation on a germ of a real-analytic Levi-flat hypersurface with isolated singularity. For a germ of irreducible real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$ with a nondicritical isolated singularity, we show that the leaves of the Levi foliation coincide with the level sets of real values of a holomorphic function. In the dicritical case, a counter-example of this result is given.

Keywords: Levi-flat Hypersurfaces. Holomorphic Foliations. Isochoric coordinates.

Contents

Introduction

Let M be a germ of a real codimension one irreducible real-analytic subvariety at $0 \in \mathbb{C}^n$, $n \geq 2$. Without loss of generality we may assume that $M = \{F = 0\}$, where F is a germ of irreducible real-analytic function at $0 \in \mathbb{C}^n$. We define the *singular set* of M as

$$\mathsf{Sing}(M) = \{F = 0\} \cap \{dF = 0\}$$

and its *regular part* is defined as $M^* = M \setminus \mathsf{Sing}(M)$. Consider the distribution of complex hyperplanes L on M^* given by

$$L_p := ker(\partial F(p)) \subset T_p M^* = ker(dF(p)),\ p \in M^*.$$

This distribution is called *Levi distribution*. When L is integrable, in the sense of Frobenius, then we say that M is *Levi-flat*. Since M^* admits an integrable complex distribution, it is locally foliated by a real-analytic codimension-one foliation $\mathcal{L}$ on M^*, the *Levi foliation*. Each leaf of $\mathcal{L}$ is a codimension-one holomorphic submanifold immersed in M^*.

In 1932, the French mathematician Élie Cartan established an important result in the subject (see [9]): given a smooth real-analytic Levi-flat hypersurface M and $p \in M$, there exists a local coordinate system $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ of p such that M is locally given by $M = \{\mathcal{R}e(w) = 0\}$. This is the *local normal form of* M *at* p. In this case, the leaves of the Levi foliation are given by $\{w = ic\}$, $c \in \mathbb{R}$.

The problem of understanding the local structure of a real-analytic Levi-flat hypersurface is solved in the smooth case. The natural question that follows is the study of normal forms for germs of *singular* real-analytic Levi-flat hypersurfaces. This question has been answered only in a few specific cases.

We refer to Bedford [4], D. Burns and X. Gong [7], M. Brunella [6] and J. Lebl [23] for previous studies on germs of singular real-analytic Levi-flat hypersurfaces. Normal forms of real-analytic hypersurfaces with non-degenerate Levi form have been studied by Tanaka [31] and Chern and Moser [11]. Of particular interest is the following result from [7]: Let $M = \{F = 0\}$ be a germ of real-analytic Levi-flat hypersurface, where

$$F(z) = \mathcal{R}e(z_1^2 + ... + z_n^2) + H(z, \overline{z}),$$

and $H(z, \overline{z}) = O(|z|^3)$, $H(z, \overline{z}) = \overline{H(\overline{z}, z)}$. Then there exists a germ of biholomorphism $\phi \in \mathsf{Diff}(\mathbb{C}^n, 0)$ such that

$$\phi(M) = \{\mathcal{R}e(z_1^2 + ... + z_n^2) = 0\}.$$

Moreover, Burns and Gong classified all quadratic Levi-flat hypersurfaces as follows:

Type	Normal Form	Singular set
$Q_{0,2k}$	$\mathcal{R}e(z_1^2 + z_2^2 + ... + z_k^2) = 0$	$\mathbb{C}^{n-k}$
$Q_{1,1}$	$z_1^2 + 2z_1\overline{z_1} + \overline{z}_1^2 = 0$	empty
$Q_{1,2}^{\lambda}$ ($\lambda \in (0,1)$)	$z_1^2 + 2\lambda z_1\overline{z_1} + \overline{z}_1^2 = 0$	$\mathbb{C}^{n-1}$
$Q_{2,2}$	$(z_1 + \overline{z_1})(z_2 + \overline{z_2}) = 0$	$\mathbb{R}^2 \times \mathbb{C}^{n-2}$
$Q_{2,4}$	$z_1\overline{z_2} - \overline{z_1}z_2 = 0$	$\mathbb{C}^{n-2}$

Table 1: Quadratic Levi-flat hypersurfaces [7]

In the same work, D. Burns and X. Gong show that not every singular real-analytic Levi-flat hypersurface can be defined as the vanishing of the real part of a holomorphic or a meromorphic function. This fact leads us to the investigation of conditions for the existence of normal forms for these hypersurfaces.

The problem of extending the Levi foliation of a germ of a real-analytic Levi-flat hypersurface to a neighbourhood of the origin has been studied by D.

Cerveau and A. L. Neto in [10], using techniques of the theory of holomorphic foliations. In short, it is shown that if a germ of real-analytic Levi-flat hypersurface M has sufficiently low dimension singular set, then it is given by the vanishing of the real part of a non-constant holomorphic function. This also shows that the Levi foliation $\mathcal{L}$ is a restriction to M^* of a holomorphic foliation in the ambient space.

Those techniques were further used by Fernández-Pérez to describe normal forms in several cases (see [14], [15], [16]). In particular, the following normal forms for real analytic Levi-flat hypersurfaces M are described, where M is given by $M = \{F = 0\}$, F defined as the vanishing of real part of germs of functions of type A_k, D_k, E_k (as Arnold's classification, see [1]) plus real-analytic higher order terms:

Type	Normal Form	Conditions
A_k	$\mathcal{R}e(z_1^2 + z_2^{k+1} + ... + z_n^2) = 0$	$k \geq 1$
D_k	$\mathcal{R}e(z_1^2 z_2 + z_2^{k-1} + z_3^2 + ... + z_n^2) = 0$	$k \geq 4$
E_6	$\mathcal{R}e(z_1^4 + z_2^3 + z_3^2 + ... + z_n^2) = 0$	
E_7	$\mathcal{R}e(z_1^3 z_2 + z_2^3 + z_3^2 + ... + z_n^2) = 0$	
E_8	$\mathcal{R}e(z_1^5 + z_2^3 + z_3^2 + ... + z_n^2) = 0$	

Table 2: Normal forms for Levi-flat hypersurfaces with A_k, D_k, E_k singularities [15]

Also, in the case when M is given as the vanishing of the real part of a germ of function having an isolated line singularity, as in D. Siersma's work (see [29]), the normal forms in Table 3 are obtained.

Type	Normal Form	Conditions
A_∞	$\mathcal{R}e(y_1^2 + y_2^2 + ... + y_n^2) = 0$	
D_∞	$\mathcal{R}e(xy_1^2 + y_2^2 + y_3^2 + ... + y_n^2) = 0$	
$J_{k,\infty}$	$\mathcal{R}e(x^k y_1^2 + y_1^3 + y_2^2 + ... + y_n^2) = 0$	$k \geq 2$
$T_{\infty,k,2}$	$\mathcal{R}e(x^2 y_1^2 + y_1^k + y_2^2 + ... + y_n^2) = 0$	$k \geq 4$
$Z_{k,\infty}$	$\mathcal{R}e(xy_1^3 + x^{k+2} y_1^2 + y_2^2 + ... + y_n^2) = 0$	$k \geq 1$
$W_{1,\infty}$	$\mathcal{R}e(x^3 y_1^2 + y_1^4 + y_2^2 + ... + y_n^2) = 0$	
$T_{\infty,q,r}$	$\mathcal{R}e(xy_1 y_2 + y_1^q + y_2^r + y_3^2 + ... + y_n^2) = 0$	$q \geq r \geq 3$
$Q_{k,\infty}$	$\mathcal{R}e(x^k y_1^2 + y_1^3 + xy_2^2 + y_3^2 + ... + y_n^2) = 0$	$k \geq 2$
$S_{1,\infty}$	$\mathcal{R}e(x^2 y_1^2 + y_1^2 y_2 + y_3^2 + ... + y_n^2) = 0$	

Table 3: Normal forms for Levi-flat hypersurfaces with isolated line singularities [16]

In this thesis we study germs of singular real-analytic Levi-flat hypersurfaces with two distinct purposes. The first part of this work is dedicated to normal forms for germs of real-analytic Levi-flat hypersurfaces. Its second part studies the connectedness of the closure of a leaf on a germ of a real-analytic Levi-flat hypersurface with isolated singularity. It is divided in four chapters.

In Chapter 1 we introduce some definitions and examples.

In Chapter 2, we state some results about normal forms for complex quasihomogeneous polynomials. In particular, we prove the following theorem:

Theorem 1. *Let $M = \{F = 0\}$ be a germ at $0 \in \mathbb{C}^2$ of irreducible real-analytic Levi-flat hypersurface such that*

(a) $F(z) = \mathcal{R}e(Q(z)) + H(z, \overline{z})$,

(b) Q is a complex quasihomogeneous polynomial of quasihomogeneous degree d with an isolated singularity at $0 \in \mathbb{C}^2$,

(c) $H(z,\overline{z}) = O(|z|^{deg(Q)+1})$ *and* $H(z,\overline{z}) = \overline{H(\overline{z},z)}$.

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *such that*

$$\phi(M) = \left\{ \mathcal{R}e \left(Q(z) + \sum_{j=1}^{s} c_j e_j(z) \right) = 0 \right\},$$

where $e_1, ..., e_s$ *are elements of the monomial basis of the local algebra of* Q *such that* $deg(e_j) > d$ *and* $c_j \in \mathbb{C}$.

A similar result has already been proved in dimension 3 or higher in [15] and, therefore, this theorem completes the study of germs of singular real-analytic Levi-flat hypersurfaces which are defined by the vanishing of the real part of a complex quasihomogeneous polynomial with isolated singularity. It is particularly interesting for several reasons. First, Theorem 1 generalizes the particular cases proved in [15] for Levi-flat real-analytic hypersurfaces defined by the vanishing of the real part of the Arnold singularities A_k, D_k, E_6, E_7 and E_8. Second, as a consequence of it, we have that for every Levi-flat real-analytic hypersurface given by the vanishing of the real part of complex quasihomogeneous polynomial plus real-analytic higher order terms, there exists a coordinate system in which this hypersurface is the vanishing real part of a complex polynomial (without added real-analytic terms), since each e_j is a polynomial itself.

In Chapter 3 we prove the existence of normal forms for germs of real-analytic Levi-flat hypersurfaces which are preserved under a change of *isochore* coordinates, that is, a change of coordinates that *preserves volume*. Our main motivations are the Morse-type results for singularities of holomorphic functions given by J. Vey [32] and J-P Françoise [18]. More precisely, Vey proved an isochore version of the Morse Lemma for germs of holomorphic functions at $0 \in \mathbb{C}^n$, $n \geq 2$, and Françoise gave a new proof of the same result. A much more general statement was given by Garay [20]. Motivated by these results, we propose an analogous version of Vey's theorem for real-analytic Levi-flat hypersurfaces defined by the vanishing of the real part of a generic Morse singularity. We state the following result.

Theorem 2. *Let* $M = \{F(z) = 0\}$ *be a germ at* $0 \in \mathbb{C}^n$*,* $n \geq 2$*, of irreducible real-analytic Levi-flat hypersurface such that*

$$F(z) = \mathcal{R}e(z_1^2 + ... + z_n^2) + H(z, \overline{z}),$$

where $H(z,\overline{z}) = O(|z|^3)$*,* $H(z,\overline{z}) = \overline{H(\overline{z}, z)}$*. Then, there exists a volume-preserving germ of biholomorphism* $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *and an automorphism* $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *such that*

$$\phi(M) = \{\mathcal{R}e(\psi(z_1^2 + ... + z_n^2)) = 0\}.$$

The above theorem can be viewed as an isochore version of Burns-Gong's theorem [7]. In order to establish our next result we need some definitions and notation: for a germ of a singular real-analytic Levi-flat hypersurface M in $\mathbb{C}^n$ with Levi foliation $\mathcal{L}$ and singular set $\mathsf{Sing}(M)$, define its complexification $M_{\mathbb{C}}$ of M, which will be a germ of complex analytic subvariety in $\mathbb{C}^{2n}$. The singular set of $M_{\mathbb{C}}$ will be denoted by $\mathsf{Sing}(M_{\mathbb{C}})$. We will see that $M_{\mathbb{C}}$ is equipped with a germ of a singular codimension-one holomorphic foliation $\mathcal{L}_{\mathbb{C}}$, which is the complexification of the Levi foliation $\mathcal{L}$. The singular set of $\mathcal{L}_{\mathbb{C}}$ will be denoted by $\mathsf{Sing}(\mathcal{L}_{\mathbb{C}})$.

Recently, A. Szawlowski [30] presented a volume-preserving normal form for holomorphic germs of functions that are right-equivalent to the product of all coordinates. Two germs of holomorphic functions f and g are right-equivalent $f \sim_R g$, if there exists a germ of biholomorphism ϕ around the origin such that $f \circ \phi^{-1} = g$. Motivated by this, we prove an analogous version for Levi-flat real-analytic hypersurfaces.

Theorem 3. *Let* $M = \{F = 0\}$ *be a germ of an irreducible singular real-analytic Levi-flat hypersurface at* $0 \in \mathbb{C}^n$*,* $n \geq 2$*, such that* $F(z) = \mathcal{R}e(z_1 \cdots z_n) + H(z, \overline{z})$*,* *where* $H(z,\overline{z}) = O(|z|^{n+1})$ *and* $H(z,\overline{z}) = \overline{H(\overline{z},z)}$. *Suppose that*

$$\textsf{Sing}(M_{\mathbb{C}}) = \bigcup_{\substack{1\leq i<j\leq n \\ 1\leq k<\ell\leq n}} V_{ijk\ell},$$

where

$$V_{ijk\ell} = \{(z,w)\in\mathbb{C}^n\times\mathbb{C}^n : z_i = z_j = w_k = w_\ell = 0\}$$

and $\textsf{Sing}(M_{\mathbb{C}}) \subset \textsf{Sing}(\mathcal{L}_{\mathbb{C}})$. *Then, there exists a germ of codimension-one holomorphic foliation* $\mathcal{F}_M$ *tangent to* M, *with a non-constant holomorphic first integral* $f(z) = z_1\cdots z_n + O(|z^{n+1}|)$ *such that*

$$M = \{\mathcal{R}e(f(z)) = 0\}.$$

As a consequence of the above theorem and the main result of Szawlowski [30], we have the following corollary.

Corollary 1. *Let* M *be a germ of irreducible singular real-analytic Levi-flat hypersurface as in Theorem 3. If* f *is right-equivalent to the product of all coordinates,* $f\sim_R z_1\cdots z_n$, *then there exists a germ of a volume-preserving biholomorphism* $\phi:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ *and a germ of an automorphism* $\psi:(\mathbb{C},0)\to(\mathbb{C},0)$ *such that*

$$\phi(M) = \{\mathcal{R}e(\psi(z_1\cdots z_n)) = 0\},$$

where ψ *is uniquely determined by* f *up to a sign.*

We remark that the normal form of Theorem 3 is a germ of real-analytic Levi-flat hypersurface whose singular set is of positive dimension. In general, the problem of finding normal forms for real-analytic Levi-flat hypersurfaces with non-isolated singularities is very difficult, and has been solved only in a few specific cases, see, for instance, [16].

In order to prove Theorems 1, 2 and 3 we use techniques from the theory of holomorphic foliations developed by D. Cerveau and A. Lins Neto in [10]. These techniques are fundamental in order to find normal forms of real-analytic Levi-flat hypersurfaces. Specifically, we apply a result of Cerveau-Lins Neto that gives sufficient conditions for a real-analytic Levi-flat hypersurface to be defined by the zeroes of the real part of a holomorphic function.

In Chapter 4, we deal with the following problem: let M be a germ of real-analytic Levi-flat hypersurface in $\mathbb{C}^n$, $n \geq 2$, and $\mathcal{L}$ the Levi foliation in M^* with an isolated singularity at the origin. Under what conditions the leaves of the Levi foliation coincide with the level sets of a holomorphic function f?

One such condition is that the level sets of f are connected near the singularity of $\mathcal{L}$. The problem of describing the topology of level sets of functions has several applications in the theory of singularities, ordinary differential equations and foliations. Conditions for the connectedness of level sets of holomorphic functions have been studied previously in [5] and [25] in dimension 2 and then generalized in [27] for higher dimensions and in some meromorphic cases.

In [28], it is assumed that the leaves of a Levi foliation coincide with connected level sets of a holomorphic function near a non-dicritical singularity in order to study uniformly laminar currents. Our goal is to establish conditions for this situation. The following result gives an answer to the question in dimension two, for leaves near the origin, which is assumed to be a nondicritical singularity of the Levi foliation.

Theorem 4. *Let M be a germ of real-analytic Levi-flat hypersurface in $\mathbb{C}^2$ and $\mathcal{L}$ be the Levi foliation on M^*. Assume that the origin is a nondicritical singularity. Then there exists a germ of holomorphic function $f \in \mathcal{O}_2$ whose connected real level sets $f^{-1}(c) \cap M$, $c \in \mathbb{R}$ coincide with the leaves of $\mathcal{L}$ near the origin.*

It is also discussed the more restrictive case when $0 \in \mathbb{C}^2$ is a dicritical singularity. In this case several assumptions must be made to achieve a similar result.

In particular, one must assume that $\mathcal{L}$ is a restriction of a germ of holomorphic foliation of the ambient space to M, in which case a meromorphic first integral g is obtained. Over this g, non-ressonance assumptions must be made to ensure that its level sets are connected. A counterexample is presented for a case in which the isolated singularity $0 \in \mathbb{C}^n$ is dicritical and the leaves passing through the origin are disconnected. In higher dimensions we relate the difficulties of giving an answer to this problem to the problem of finding local holomorphic extensions to foliations.

1 Preliminaries

1.1 Basic definitions

Let $\mathbb{C}^n$ be the n-dimensional complex euclidean space with coordinates $z = (z_1, ..., z_n)$, in which each z_j can be written as $z_j = x_j + iy_j$. Denote by $x_j = \mathcal{R}e(z_j)$ the *real part* of z_j, and by $y_j = \mathcal{I}m(z_j)$, the *imaginary part* of y_j, for each $j = 1, ..., n$. We also denote $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = x + iy$. This gives us an obvious identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.

We denote the complex conjugation $\overline{z} = (\overline{z_1}, ..., \overline{z_n})$, in which $\overline{z_j} = x_j - iy_j$ for each j and therefore $\overline{z} = x - iy$. Since

$$x = \frac{1}{2}(z + \overline{z}), y = \frac{1}{2i}(z - \overline{z}),$$

a function f defined in an open set U of $\mathbb{C}^n$ may be written either as $f(x, y)$ or as $f(z, \overline{z})$.

For simplicity, we refer to germs and their representants by the same letter. Let us first introduce a few notation.

- $\mathcal{O}_n$ is the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$.
- $\mathcal{O}_n^* = \{f \in \mathcal{O}_n | f(0) \neq 0\}$.
- $\mathcal{M}_n = \{f \in \mathcal{O}_n | f(0) = 0\}$, the maximal ideal of $\mathcal{O}_n$.
- $\mathcal{A}_n$ is the ring of germs at $0 \in \mathbb{C}^n$ of **complex** valued real-analytic functions.
- $\mathcal{A}_{n\mathbb{R}}$ is the ring of germs at $0 \in \mathbb{C}^n$ of **real** valued real-analytic functions. Note that $F \in \mathcal{A}_n$ is also in $\mathcal{A}_{n\mathbb{R}}$ if and only if $F = \overline{F}$.

- $\mathsf{Diff}(\mathbb{C}^n, 0)$ is the group of germs at $0 \in \mathbb{C}^n$ of biholomorphisms $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ under composition.

- $j_0^k(f)$ is the k-jet at $0 \in \mathbb{C}^n$ of $f \in \mathcal{O}_n$.

- If f is a germ of function in $(\mathbb{C}^n, 0)$, then $\{f = 0\} = \{z \in \mathbb{C}^n : f(z) = 0\}$.

The notation $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ means that ϕ is a (germ of a) mapping from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $\phi(0) = 0$. A germ of function $f \in \mathcal{O}_n$ is *irreducible* if it cannot be written as a product of non-units in $\mathcal{O}_n$.

Definition 1.1. Two germs $f, g \in \mathcal{O}_n$ are *right-equivalent* if there is a biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f \circ \phi^{-1} = g$. We denote this as $f \sim_R g$.

Definition 1.2. The local algebra of $f \in \mathcal{O}_n$ is

$$A_f = \mathcal{O}_n \Big/ \left(\frac{\partial f}{\partial z_1}, ..., \frac{\partial f}{\partial z_n} \right).$$

The *Milnor number* of f at $0 \in \mathbb{C}^n$ is $\mu(f, 0) = \dim_{\mathbb{C}} A_f$.

Morse's lemma can now be stated by saying that if $0 \in \mathbb{C}^n$ is an isolated singularity of $f \in \mathcal{O}_n$ with Milnor number $\mu(f, 0) = 1$, then f is right-equivalent to its second jet. The following lemma is a generalization of Morse's Lemma from [2].

Lemma 1.1 (Tougeron's Lemma)**.** *Suppose that $f \in \mathcal{M}_n$ has an isolated singularity at $0 \in \mathbb{C}^n$. If its Milnor number is μ, then f is right-equivalent to $j_0^{\mu+1}(f)$.*

Definition 1.3. A smooth *real-analytic submanifold of codimension m* is a subset M of $\mathbb{C}^n$ such that, for all $p \in M$ there exists an open set $U \subset \mathbb{C}^n$ and m real, analytic functions ϕ_1, ..., $\phi_m : U \to \mathbb{R}$ such that

$$M \cap U = \{z \in U : \phi_1(z) = ... = \phi_m(z) = 0\}$$

and the exterior differentials $d\phi_1$, ..., $d\phi_m$ are linearly independent in M. In the case $m = 1$, M is called a *real-analytic hypersurface.*

We refer to [3] for a more complete text on these first elements. We present the following lemma from [10], which will be needed in some of our proofs.

Lemma 1.2. *Let $f \in \mathcal{O}_n$, not identically zero and $f(0) = 0$. Suppose that f is not a power in $\mathcal{O}_n$. Then $\mathcal{I}m(f)$ and $\mathcal{R}e(f)$ are irreducible in $\mathcal{A}_{n\mathbb{R}}$.*

1.2 Holomorphic Foliations

Let us give an overview on germs of singular holomorphic foliations. The main references here are [8] and [26].

Definition 1.4. Let M be a complex manifold of dimension $n \geq 2$. A codimension one holomorphic foliation $\mathcal{F}$ in M is given by collections $\{\omega_\alpha\}_{\alpha \in A}$, $\{U_\alpha\}_{\alpha \in A}$ and $\{g_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$ such that

(i) $\{U_\alpha\}_{\alpha \in A}$ is an open covering of M.

(ii) Each ω_α is a holomorphic, not identically zero, 1-form in U_α that verifies $\omega_\alpha \wedge d\omega_\alpha = 0$ (*Frobenius integrability*).

(iii) $g_{\alpha\beta}$ is holomorphic and non vanishing in $U_\alpha \cap U_\beta$.

(iv) If $U_\alpha \cap U_\beta \neq \emptyset$ then $\omega_\alpha = g_{\alpha\beta}\omega_\beta$ in $U_\alpha \cap U_\beta$.

For each 1-form ω_α, its singular set is defined by

$$S_\alpha := \mathsf{Sing}(\omega_\alpha) = \{p \in U_\alpha | \omega_\alpha(p) = 0\}.$$

Each S_α is an analytic subvariety of U_α. By definition we have $S_\alpha \cap U_\alpha \cap U_\beta = S_\beta \cap U_\alpha \cap U_\beta$, and therefore we may define the *singular set of* $\mathcal{F}$ as

$$\mathsf{Sing}(\mathcal{F}) = \bigcup_{\alpha \in A} S_\alpha.$$

The foliation $\mathcal{F}$ is *regular* in the open set $U = M \setminus S$, and the leaves of $\mathcal{F}$ are the leaves of the restriction of $\mathcal{F}$ to U, denoted $\mathcal{F}|_U$. If $\mathsf{Sing}(\mathcal{F}) = \emptyset$, then we say that $\mathcal{F}$ is *regular*.

A *germ* of codimension one holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^n$ is given by a germ of holomorphic 1-form $\omega = \sum_{i=1}^{n} a_i dx_i$, $a_i \in \mathcal{O}_n$, which verifies

(i) $\omega \wedge d\omega = 0$ (*Frobenius integrability*),

(ii) $\mathsf{Sing}(\omega) = \{a_1 = ... = a_n = 0\}$.

It'is usual to write $\omega = 0$ in order to denote the foliation $\mathcal{F}$ induced by ω in this case.

We remark that if ω and η are two germs of holomorphic 1-forms such that and $\eta = f\omega$, where f is a non-vanishing germ of holomorphic function around the origin, then the germs of holomorphic foliations induced by ω and η coincide. In particular, if ω is integrable, so is η and the foliations defined by $\omega = 0$ and $\eta = 0$ are the same.

Definition 1.5. Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$ of holomorphic foliation in M. A holomorphic (respectively, meromorphic) first integral for $\mathcal{F}$ is a non-constant holomorphic (respectively, meromorphic) germ of function f on M, such that f is constant along the leaves of $\mathcal{F}$.

When $\mathcal{F}$ has codimension one in M and is given by a integrable holomorphic 1-form ω, then a holomorphic (or meromorphic) function f is a first integral of $\mathcal{F}$ if and only if $\omega \wedge df \equiv 0$.

Definition 1.6. Let $\mathcal{F}$ be a holomorphic foliation in a connected manifold M given by an integrable holomorphic 1-form ω and f be a non-constant holomorphic function in M. We say that the set $\{f = 0\}$ is *invariant* by $\mathcal{F}$ if its irreducible components are the closures of leaves of $\mathcal{F}$.

Proposition 1.3. *In the context of the above definition, the set* $\{f = 0\}$ *is invariant by* $\mathcal{F}$ *if and only if there exists a holomorphic 2-form* Θ *in* M *such that*

$$\omega \wedge df = f\Theta.$$

See [26] for a proof of this fact.

Definition 1.7. A germ of foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$ is *dicritical* if there are infinitely many leaves that passes through the origin $0 \in \mathbb{C}^2$. These leaves are called the *separatrices* of the foliation $\mathcal{F}$. Otherwise, the foliation is called *non-dicritical.*

1.3 Levi-flat hypersurfaces

We present some definitions and examples on Levi-flat hypersurfaces.

Definition 1.8. Let M be a germ at $0 \in \mathbb{C}^n$ of an irreducible, codimension one, real-analytic hypersurface given as the set of zeroes of $F \in \mathcal{A}_{n\mathbb{R}}$. The *singular set* of M is given by

$$\mathsf{Sing}(M) = \{F = 0\} \cap \{dF = 0\}$$

and its regular part is $M^* = \{F = 0\} \setminus \{dF = 0\}$.

We remark that $\mathsf{Sing}(M)$ contains all points $p \in M$ such that the codimension of M at p is at least two. Consider the *Levi distribution* L on M^*, defined by

$$L_p := Ker(\partial F(p)) \subset T_pM^* = Ker(dF(p)),$$

for each $p \in M^*$.

Definition 1.9. M is Levi-flat if the Levi distribution L is integrable (in the sense of Frobenius).

When L is integrable, the hypersurface M is foliated by complex submanifolds immersed in M having complex codimension one. This foliation is called *Levi foliation* and is denoted by $\mathcal{L}$.

This distribution L in M^* can also be defined by the real-analytic 1-form $\eta = i(\partial F - \overline{\partial} F)$, which is called *Levi form* of F. The integrability condition over η is equivalent to

$$(\partial F - \overline{\partial} F) \wedge \partial\overline{\partial} F|_{M^*} = 0$$

or, using the fact that $dF = \partial F + \overline{\partial} F$,

$$\partial F(p) \wedge \overline{\partial} F(p) \wedge \partial\overline{\partial} F(p)|_{M^*} = 0, \forall p \in M^*.$$

Example 1.1. Consider the real hypersurface M in $\mathbb{C}^n$ given by $\{F = 0\}$ in which

$$F(z_1, ..., z_n) = \mathcal{R}e(z_1^2 + ... + z_n^2).$$

This hypersurface is called the *real cone*, which is Levi-flat and its singular set is only the origin of $\mathbb{C}^n$.

Example 1.2. In $\mathbb{C}^n$, $n \geq 2$, let M be given as the set of zeroes of

$$F(z_1, ..., z_n) = z_1\overline{z_1} - z_2\overline{z_2}.$$

The function F is real-analytic and M is Levi-flat. Its singular set is biholomorphic to $\mathbb{C}^{n-2}$. This hypersurface is the *complex cone*.

Example 1.3. Let $f \in \mathcal{O}_n$ be a germ of non-constant holomorphic function fith $f(0) = 0$. Then the analytic set $M = \{\mathcal{R}e(f) = 0\}$ is Levi-flat and its singular set is given by $\mathsf{crit}(f) \cap M$, where $\mathsf{crit}(f)$ is the set of critical points of f. The leaves of the Levi foliation $\mathcal{L}$ on M are the imaginary levels of f.

1.4 Complexification of real-analytic hypersurfaces

Let $F \in \mathcal{A}_n$. Its Taylor series around $0 \in \mathbb{C}^n$ is written as

$$F(z, \overline{z}) = \sum_{\mu,\nu} G_{\mu,\nu} z^{\mu} \overline{z}^{\nu},$$

where $G_{\mu,\nu} \in \mathbb{C}$, $\mu = (\mu_1, ..., \mu_n)$, $\nu = (\nu_1, ..., \nu_n)$, $z^{\mu} = z_1^{\mu_1} \cdots z_n^{\mu_n}$ and $\overline{z}^{\nu} = \overline{z_1}^{\nu_1} \cdots \overline{z_n}^{\nu_n}$.

When $F \in \mathcal{A}_{n\mathbb{R}}$, then each coefficient verifies

$$\overline{F}_{\mu\nu} = F_{\nu\mu}.$$

Definition 1.10. The *complexification* $F_{\mathbb{C}} \in \mathcal{O}_{2n}$ of F is defined as the power series

$$F_{\mathbb{C}}(z, w) = \sum_{\mu,\nu} F_{\mu,\nu} z^{\mu} w^{\nu}.$$

If the defining power series for F is convergent in a polydisk $D_r^n = \{z \in \mathbb{C}^n | |z_j| \leq r\}$ then the power series of the complexification $F_{\mathbb{C}}$ of F is convergent in the polydisk D_r^{2n} and therefore is holomorphic around the origin of $\mathbb{C}^{2n}$. Moreover, $F(z, \overline{z}) = F_{\mathbb{C}}(z, \overline{z})$, $\forall z \in D_r^n$.

We remark that the complexification does not depend on the choice of the coordinate system, that is, by taking a coordinate system induced by a biholomorphism $\phi \in \mathsf{Diff}(\mathbb{C}^n, 0)$, then there exists a single element $\phi_{\mathbb{C}} \in \mathsf{Diff}(\mathbb{C}^{2n}, 0)$ such that

$$(F \circ \phi)_{\mathbb{C}} = F_{\mathbb{C}} \circ \phi_{\mathbb{C}}.$$

Indeed, by writing the Taylor series of ϕ as $\phi(x) = \sum_{\mu} \phi_{\mu} x^{\mu}$, and letting $\overline{\phi}(y) = \sum_{\mu} \overline{\phi}_{\mu} y^{\mu}$, then we see that

$$\phi_{\mathbb{C}}(u, v) = (\phi(u), \overline{\phi}(v)),$$

and thus $(F \circ \phi)_{\mathbb{C}} = F_{\mathbb{C}} \circ \phi_{\mathbb{C}}$.

Suppose that $F(0) = 0$ and that $M = \{F = 0\}$ is Levi-flat. As seen before, we have the Levi 1-form $\eta = i(\partial F - \overline{\partial} F)$. The *complexification* of the Levi form is the holomorphic 1-form

$$\begin{aligned}\eta_{\mathbb{C}} &= i\sum_{j=1}^{n}\left(\frac{\partial F_{\mathbb{C}}}{\partial z_j}dz_j - \frac{\partial F_{\mathbb{C}}}{\partial w_j}dw_j\right)\\ &= i\sum_{\mu,\nu}(F_{\mu\nu}w^{\nu}d(z^{\mu}) - F_{\mu\nu}z^{\mu}d(w^{\nu})).\end{aligned}$$

Finally, the complexification of M is the complex analytic variety defined as $M_{\mathbb{C}} = \{F_{\mathbb{C}} = 0\}$. As before, $M_{\mathbb{C}}$ does not depend on the choice of a coordinate system. The *smooth part* of $M_{\mathbb{C}}$ is $M_{\mathbb{C}}^{*} = \{F_{\mathbb{C}} = 0\} \setminus \{dF_{\mathbb{C}} = 0\}$ and its singular part is $\mathsf{Sing}(M_{\mathbb{C}}) = \{F_{\mathbb{C}} = 0\} \cap \{dF_{\mathbb{C}} = 0\}$. Since η is integrable, then $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^{*}}$ is integrable and defines a codimension one holomorphic foliation in $M_{\mathbb{C}}^{*}$, which is denoted by $\mathcal{L}_{\mathbb{C}}$ and called the *complexification* of $\mathcal{L}$.

Remark 1.1. We can write $\eta_{\mathbb{C}} = i(\alpha - \beta)$, where

$$\alpha = \sum_{j=1}^{n}\frac{\partial F_{\mathbb{C}}}{\partial z_j}dz_j \quad \text{and} \quad \beta = \sum_{j=1}^{n}\frac{\partial F_{\mathbb{C}}}{\partial w_j}dw_j.$$

Note that $dF_{\mathbb{C}} = \alpha + \beta$, then

$$\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^{*}} = (\eta_{\mathbb{C}} + idF_{\mathbb{C}})|_{M_{\mathbb{C}}^{*}} = 2i\alpha|_{M_{\mathbb{C}}^{*}}.$$

Analogously

$$\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^{*}} = (\eta_{\mathbb{C}} - idF_{\mathbb{C}})|_{M_{\mathbb{C}}^{*}} = -2i\beta|_{M_{\mathbb{C}}^{*}}.$$

In particular, $\alpha|_{M_{\mathbb{C}}^{*}}$ and $\beta|_{M_{\mathbb{C}}^{*}}$ define $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^{*}$ and $\mathsf{Sing}(\mathcal{L}_{\mathbb{C}}) = \mathsf{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^{*}})$.

Definition 1.11. Let M be a germ at $0 \in \mathbb{C}^n$ of real-analytic Levi-flat hypersurface and $M_{\mathbb{C}}$ be its complexification. We define the *algebraic dimension* of $\mathsf{Sing}(M)$ as the complex dimension of the singular set of $M_{\mathbb{C}}$.

Assume that the Taylor series of F converges in a polydisk D_r^n. Let $W = M_{\mathbb{C}}^{*} \setminus \mathsf{Sing}(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^{*}})$ and let L_p be the leaf of $\mathcal{L}_{\mathbb{C}}$ through $p \in W$. We have the following lemma from [10]:

Lemma 1.4. *For any $p \in W$, the leaf L_p is closed (with respect to the induced topology) in $M^*_{\mathbb{C}}$.*

The following theorem, due to D. Cerveau and A. Lins Neto (see [10]), is central in the proofs of our main results. In general terms, it asserts that if the singularities of M have a sufficiently small algebraic dimension, then M is given as the vanishing of the real part of a holomorphic function.

Theorem 1.5 (Cerveau, Lins Neto [10]). *Let $M = \{F = 0\}$ be a germ of irreducible real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, with Levi form η. Assume that the algebraic dimension of $\mathsf{Sing}(M)$ is less than or equal to $2n-4$. Then there exists a unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, as long as one of the following conditions is fulfilled:*

(a) $n \geq 3$ and $\mathsf{cod}_{M^*_{\mathbb{C}}}(\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) \geq 3$.

(b) $n \geq 2$, $\mathsf{cod}_{M^*_{\mathbb{C}}}(\mathsf{Sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) \geq 2$ and $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

Moreover, in both cases the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral f such that $M = \{\mathcal{R}e(f) = 0\}$.

We remark that the foliation $\mathcal{F}_M$ obtained in the previous theorem is unique. In fact, if M is Levi-flat and $\mathcal{F}$ is a holomorphic foliation tangent to it, then $\mathcal{F}$ is unique. This is due to the following facts:

(a) If two holomorphic foliations, defined in a connected open set, coincide in a non-empty open subset then they are equal (see [26]).

(b) At each $p \in M^*$, by Cartan's theorem there exist holomorphic coordinates $z = (z_1, ..., z_n)$ such that M is locally given as $M = \{\mathcal{R}e(z_n) = 0\}$. In this case, the unique holomorphic foliation extension of the Levi foliation to a neighborhood V of p is a foliation $\mathcal{G}$ whose leaves are the levels $z_n = constant$. In particular, $\mathcal{F}_M|_V = \mathcal{G}$, and the uniqueness of the foliation tangent to M follows.

In the last part of this work, we will need the following theorem.

Theorem 1.6 (Cerveau, Lins Neto [10])**.** *Let $\mathcal{F}$ be a germ at the origin of $\mathbb{C}^n$, $n \geq 2$, of holomorphic codimension one foliation tangent to a germ of codimension one irreducible real-analytic variety M. Then $\mathcal{F}$ has a non-constant meromorphic first integral.*

Moreover, in the case of dimension two, we have the following:

(a) if $\mathcal{F}$ is dicritical, it has a non-constant meromorphic first integral f/g, where $f, g \in \mathcal{O}_2$ and $f(0) = g(0) = 0$.

(b) If $\mathcal{F}$ is non-dicritical, it has a non-constant holomorphic first integral.

1.5 Weighted projective varieties and weighted blow ups

In this section we present an overview of weighted projective spaces and weighted blow-ups. We refer to [12] and [21] for a more extensive presentation of the subject.

Let $\sigma = (a_0, ..., a_n)$ a $n-$tuple of positive integers. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n-1} \setminus \{0\}$ given by

$$\lambda \cdot (x_0, ..., x_n) = (\lambda^{a_0} x_0, ..., \lambda^{a_n} x_n).$$

We may take the quotient space by this action, which is called *weighted projective space of type σ*, denoted $\mathbb{P}(a_0, ..., a_n)$ or simply $\mathbb{P}_\sigma$. Note that, if all $a_i = 1$, then the quotient space is exactly the projective space $\mathbb{P}^n$. In the case that $a_i > 1$ for some i, we have that $\mathbb{P}_\sigma$ is a compact algebraic variety with cyclic quotient singularities.

Let $[x_0 : ... : x_n]$ be the homogeneous coordinates on $\mathbb{P}_\sigma$. Assuming that $x_i \neq 0$ for some i, the affine piece obtained is isomorphic to $\mathbb{C}^n/\mathbb{Z}_{a_i}$ ($\mathbb{Z}_{a_i}$ is the quotient group modulo a_i). If ϵ is a a_i^{th}-primitive root of unity, $\mathbb{Z}_{a_i}$ acts on $\mathbb{C}^n$ by

$$z_j \mapsto \epsilon^{a_j} z_j$$

for all $j \neq i$ on the coordinates $(z_0, ..., \widehat{z_j}, ..., z_n)$ of $\mathbb{C}^n$ (the term $\widehat{z_j}$ is removed from the entries of this element). We might see z_j as the quotient $x_j/x_i^{1/a_i}$. When $a_i = 1$, we are in $\mathbb{P}^n$ and this is the exact representation of affine coordinates $z_j = x_j/x_i$.

Definition 1.12. The space $\mathbb{P}_\sigma$ is *well formed* if, for each i,

$$gcd(a_0, ..., \widehat{a_i}, ..., a_n) = 1.$$

Definition 1.13. Let X be a closed subvariety of a weighted projective space $\mathbb{P}_\sigma$ and let $\rho : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_\sigma$ be the canonical projection. The *punctured affine cone* $\mathcal{C}_X^*$ over X is the set $\mathcal{C}_X^* = \rho^{-1}(X)$. The *affine cone* $\mathcal{C}_X$ over X is the completion of $\mathcal{C}_X^*$ in $\mathbb{C}^{n+1}$.

Lemma 1.7. *The punctured affine cone $\mathcal{C}_X^*$ has no isolated singularities.*

If the affine cone $\mathcal{C}_X$ of an m-dimensional subvariety $X \subset \mathbb{P}_\sigma$ is smooth of dimension $m+1$, except at the origin $0 \in \mathbb{C}^{n+1}$ (the origin is called the *vertex* of the cone), then X is called *quasi-smooth*.

Quasi-smooth varieties have an important property: their singularities are given by the $\mathbb{C}^*$-action and therefore are cyclic quotient singularities. They are also complex spaces locally isomorphic to the quotient of a complex manifold by a finite group of holomorphic automorphisms.

Let $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, ..., a_n)$ be a cyclic quotient singularity, that is, X is the quotient variety $\mathbb{C}^n/\tau$, where τ is the action given by

$$x_i \mapsto \epsilon^{a_i} x_i$$

for all i, and ϵ is an m^{th}-primitive root of unity. Let us overview weighted blow ups. For that, we need toric varieties (see [19]). Let

$$e_1 = (1, 0, ..., 0), ..., e_n = (0, ..., 0, 1)$$

and

$$e = \frac{1}{m}(a_1, ..., a_n).$$

Then $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, ..., a_n)$ is the toric variety corresponding to the *lattice* $N = \mathbb{Z}e_1 + ... + \mathbb{Z}e_n + \mathbb{Z}e$ and the cone $C = \mathbb{R}_+ e_1 + ... + \mathbb{R}_+ e_n$. Let $\triangle$ the *fan* associated to X, which consists of all the faces of C.

Take $\nu = \frac{1}{m}(a_1, ..., a_n) \in N$ and assume that the lattice N is generated by $e_1, ..., e_n$ and ν. This $\nu \in N$ is called a *weight.*

The *weighted blow up with weight* ν, denoted $E : \tilde{X} \to X$, is given as follows: for each $i = 1, ..., n$ construct the new cone C_i defined as

$$C_i = \mathbb{R}_+ e_1 + ...+ \overset{i^{th}}{\mathbb{R}_+ \nu} +... + \mathbb{R}_+ e_n.$$

(the i-th term from the expression of C is substituted by $\mathbb{R}_+\nu$ to obtain C_i). We remark that the usual notion of a blow up on a point may be obtained from the weighted blow up by making all the weights a_i equal to one.

Let $\triangle'$ be the new fan consisting of all the faces C_i, $i = 1, ..., n$. Then $\tilde{X}$ is the toric variety corresponding to N and $\triangle'$ and E is the morphism induced from the natural map of fans $(N, \triangle') \to (N, \triangle)$.

This new variety $\tilde{X}$ is covered by n affine open sets $\tilde{U}_1$, ..., $\tilde{U}_n$ which correspond respectively to the cones C_1, ..., C_n. They are described as follows:

$$\tilde{U}_i = \mathbb{C}^n/\mathbb{Z}_{a_i}\left(-a_1, ..., \overbrace{m}^{i^{th}}, ..., -a_n\right),$$

and the mapping E is written in coordinates as

$$E|_{\tilde{U}_i} : \tilde{U}_i \ni (y_1, ..., y_n) \mapsto \left(y_1 y_i^{\frac{a_1}{m}}, ..., \overbrace{y_i^{\frac{a_i}{m}}}^{i^{th}}, ..., y_n y_i^{\frac{a_n}{m}}\right) \in X.$$

The exceptional divisor D of E is isomorphic to the weighted projective space $\mathbb{P}(a_1, ..., a_n)$ and $D \cap \tilde{U}_i = \{y_i = 0\}/\mathbb{Z}_{a_i}$.

1.6 Holonomy and First Integrals

We refer to [8], [13] and [26] for a more extensive treatment on the holonomy of a leaf of a foliation.

Consider the following situation: let M be an irreducible germ of real-analytic Levi-flat hypersurface at the origin of $\mathbb{C}^n$, let $\mathcal{L}$ be the Levi foliation on M^*, and $M_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{C}}$ be their respective complexifications. Let D be the exceptional divisor of a blow up E at the origin of $\mathbb{C}^{2n}$. Denote by $\tilde{M}_{\mathbb{C}}$ the strict transform of $M_{\mathbb{C}}$ under E and by $\tilde{\mathcal{F}} = E^*(\mathcal{F}_{\mathbb{C}})$ the foliation on $\tilde{M}_{\mathbb{C}}$.

Suppose that $\tilde{M}_{\mathbb{C}}$ is smooth and that $\tilde{C} = \tilde{M}_{\mathbb{C}} \cap D$. Assume that $\tilde{C}$ is invariant by $\tilde{\mathcal{F}}$ and that one of the following cases is true:

(i) $\mathsf{Sing}(\tilde{\mathcal{F}}) \cap \mathsf{Sing}(D) = \emptyset$;

(ii) $\mathsf{Sing}(D) \subsetneq \mathsf{Sing}(\tilde{\mathcal{F}})$

Let $S = \tilde{C} \setminus \mathsf{Sing}(\tilde{\mathcal{F}})$. Then S is a smooth leaf of $\tilde{\mathcal{F}}$. Take a point p_0 in $S \setminus \mathsf{Sing}(D)$ and a transverse section Σ passing through p_0. Let $G \subset \mathsf{Diff}(\Sigma, p_0)$ be the holonomy group of the leaf S of $\tilde{\mathcal{F}}$. Since $\dim(\Sigma) = 1$, we may assume that $G \subset \mathsf{Diff}(\Sigma, 0)$. In this context we have the following result, which was proved in [15] and we prove it here for the sake of completeness.

Lemma 1.8 ([15]). *Assume the following:*

(a) For any $p \in M^* \setminus \mathsf{Sing}(\mathcal{F})$ the leaf L_p of $\mathcal{F}$ through p is closed in M^* (as in Lemma 1.4).

(b) $g'(0)$ is a primitive root of unity, for all $g \in G$, with $g \neq \mathsf{id}$.

Then $\mathcal{F}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

Proof. Consider the multiplicative group

$$G' = \{g'(0) : g \in G\}$$

and the homomorphism

$$\phi : G \to G', \ \phi(g) = g'(0).$$

We claim that ϕ is injective. If we assume that $\phi(g) = 1$ but $g \neq \mathsf{id}$, then we may write g as

$$g(z) = z + az^{r+1} + ..., \ a \neq 0.$$

According to [24], the pseudo-orbits of g accumulate at $0 \in (\Sigma, 0)$. This leads to a contradiction with the assumption that the leaves L_p of $\tilde{\mathcal{F}}$ through p are closed in S, and therefore, ϕ is injective.

Note that every element $g \in G$ has finite order (as in [25]). This comes from the fact that $\phi(g) = g'(0)$ is a root of unity, and, since ϕ is injective, g has finite order. Therefore, all elements of G have finite order and G is linearizable.

This means that there is a coordinate system w in $(\Sigma, 0)$ such that

$$G = \langle w \to \lambda w \rangle,$$

where λ is a d^{th}-primitive root of unity. In particular, $\psi(w) = w^d$ is a first integral of the group G, meaning that $\psi \circ g = \psi$, for any $g \in G$.

From [24], there are finitely many leaves of $\mathcal{F}$ passing through the origin 0. Let Γ be the union of these leaves and $\tilde{\Gamma}$ be its strict transform under E. The first integral ψ of G can be extended to a first integral $\phi : \tilde{Y} \setminus \tilde{\Gamma} \to \mathbb{C}$ by setting

$$\phi(q) = \psi\left(\tilde{L}_q \cap \Sigma\right),$$

where $\tilde{L}_p$ denotes the leaf through $\tilde{\mathcal{F}}$ through q.

Since ψ is bounded in a small compact neighborhood around $0 \in \Sigma$, then so is ϕ. It follows from Riemann's Extension Theorem that ϕ extends itself holomorphically to $\tilde{\Gamma}$, with $\phi(\tilde{\Gamma}) = 0$. This provides the first integral of $\mathcal{F}$. □

Example 1.4. We present an example on how we compute the holonomy of a foliation. Let $\mathcal{F}$ be a foliation around the origin of $\mathbb{C}^2$ given by the holomorphic 1-form

$$\omega = xdy - \lambda y(1 + A(x, y))dx,$$

where A is a complex polynomial of degree 1 or higher. Let L be the hypersurface given by $\{y = 0\}$. According to Proposition 1.3, L is invariant by $\mathcal{F}$ and therefore $L \setminus \{0\}$ is a leaf of this foliation.

This leaf can be taken as $\mathbb{D} \setminus \{0\}$, where $\mathbb{D}$ is a small polydisk around the origin (containing, say, the subsets $|x| \leq 1$, $|y| \leq 1$).The fundamental group $\pi_1(L, q)$ of L is, therefore, $\mathbb{Z}$, generated by (a class of equivalence of) a single loop

$$\gamma(t) = (e^{2\pi it}, 0), t \in \mathbb{D}.$$

Let $q = (1,0)$ and $\Sigma = \{(1,y) : y \in \mathbb{C}\}$ a transversal section of $L/\{0\}$ passing through q. This loop γ lifts to a loop

$$\tilde{\Gamma}(y,t) = (e^{2\pi it}, \Gamma(y,t)),$$

where Γ is a funcion that verifies $\Gamma(0,t) = 0$ and $\Gamma(y,0) = y$. Assuming that this lifting is contained in a leaf of $\mathcal{F}$, then, using the expression for ω, we get

$$e^{2\pi it}\frac{\partial \Gamma}{\partial t}(y,t) - 2\pi i e^{2\pi it}\lambda\Gamma(y,t)(1 + A(e^{2\pi it}, \Gamma(y,t))) = 0.$$

Let the Taylor series of Γ be $\Gamma(y,t) = \sum_{k=1}^{\infty} \gamma_k(t)y^k$. From the previous expression we have

$$\gamma_1'(t) - 2\pi i\lambda\gamma_1(t) = 0,$$

which, along with the condition $\gamma_1(0) = 1$, gives us $\gamma_1(t) = e^{2\pi it}$. The holonomy group is generated by only one germ of function, h_γ, corresponding to the equivalence class of γ in $\pi_1(L,q)$. The holonomy mapping is, by definition, $h_\gamma(y) = \Gamma(y,1)$, which gives us

$$h_\gamma(y) = e^{2\pi i\lambda}y + ...$$

allowing us to conclude that

$$h_\gamma'(0) = e^{2\pi i\lambda}.$$

2 Normal Forms in the quasihomogeneous case

A germ of polynomial function $f \in \mathcal{O}_2$ is *quasihomogeneous* with weights $a_1, ..., a_n \in \mathbb{Z}_+^*$ if, for each $\lambda \in \mathbb{C}^*$,

$$f(\lambda^{a_1} z_1, ..., \lambda^{a_n} z_n) = \lambda^d f(z_1, ..., z_n).$$

The number d is called the *quasihomogeneous degree* of f. We will denote the usual polynomial degree of f as $deg(f)$, which may be different from d.

This definition is equivalent to the following: a polynomial $f(z)$ is quasi-homogeneous of *type* $(w_1, ..., w_n)$ if it can be expressed as a linear combination of monomials $z_1^{i_1} z_2^{i_2} \cdots z_n^{i_m}$ such that the equality

$$i_1 w_1 + ... + i_n w_n = d$$

holds for all these monomials. The number d is the same as the quasihomogeneous degree defined above.

Definition 2.1. Let $f = \sum a_{ij} x^i y^j$ be a polynomial in two variables. The *Newton support* of f is

$$supp(f) = \{(i, j) : a_{ij} \neq 0\}.$$

For a germ of quasihomogeneous holomorphic function at the origin of $\mathbb{C}^n$ with weights $(\alpha_1, ..., \alpha_n)$ that is written as a power series

$$f(z) = \sum a_k x^k, \; k = (k_1, ..., k_n), \; x^k = x_1^{k_1} ... x_n^{k_n}$$

we have

$$supp(f) = \{k : \alpha_1 k_1 + ... + \alpha_n k_n = d\}.$$

The *quasihomogeneous filtration* in the ring $\mathcal{O}_n$ with weights $(\alpha_1, ..., \alpha_n)$ is defined as follows: let

$\mathcal{A}_d = \{$Q: quasihomogeneous degrees of monomials from $supp(Q)$ are greater than $d\}$.

Each $\mathcal{A}_d$ is an ideal in $\mathcal{O}_n$ such that $\mathcal{A}_{d'} \subset \mathcal{A}_d$ whenever $d < d'$. The quasihomogeneous filtration of $\mathcal{O}_n$ consists of this decreasing family of ideals.

Definition 2.2. A function f is *semiquasihomogeneous* if $f = Q + F'$, where Q is quasihomogeneous of quasihomogeneous degree d and $\mu(Q, 0) < \infty$, and $F' \in \mathcal{A}_{d'}$, $d' > d$.

From [1] we have the following lemma:

Lemma 2.1. *Let $f = Q + F'$ be a semiquasihomogeneous function. Then f is right-equivalent to a function $Q(z) + \sum_j c_j e_j(z)$ where $e_1, ..., e_j$ are elements of the monomial basis of the local algebra A_Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.*

The following factorization for quasihomogeneous polynomials in two variables plays a significant role in our analysis.

Lemma 2.2. *If $f \in \mathcal{O}_2$ is a quasihomogeneous polynomial of quasihomogeneous degree d, then f factors itself uniquely as*

$$f(z_1, z_2) = \mu z_1^m z_2^n \prod_{\ell=1}^{k} (z_2^p - \lambda_\ell z_1^q),$$

where $m, n, p, q \in \mathbb{Z}_+^*, \mu, \lambda_\ell \in \mathbb{C}^*$ for each $\ell = 1, ..., k$, and $gcd(p, q) = 1$.

Proof. Let (a, b) be the weights of f and d be its quasihomogeneous degree. Let us first show that f may be written as

$$f(z_1, z_2) = \mu z_1^m z_2^n Q(z_1, z_2),$$

where m, $n \in \mathbb{Z}_+^*$, $\mu \in \mathbb{C}^*$, for some polynomial Q.

Let A be the subset of $\mathbb{Z} \times \mathbb{Z}$ of all pairs of positive integer solutions (α, β) of the equation

$$\alpha a + \beta b = d. \tag{2.1}$$

Since we consider only positive solutions, the set A is finite. We may write f in the form

$$f(z_1, z_2) = \sum_{(\alpha,\beta)\in A} c_{\alpha,\beta} z_1^{\alpha} z_2^{\beta},$$

for adequate complex numbers $c_{(\alpha,\beta)}$.

If f has only one term, we are done since $Q \equiv 1$. If not, any two distincts solutions (α_1, β_1) and (α_2, β_2) of (2.1) with non-zero coefficients $c_{(\alpha,\beta)}$ allow us to find two positive integers p, q such that

$$\alpha q + \beta p = d$$

and therefore $gcd(p, q) = 1$. By choosing $(\alpha_0, \beta_0) \in A$ with a maximal α_0, any element (α, β) is in A if and only if $(\alpha - \alpha_0)p + (\beta - \beta_0)q = 0$. This means that there exists $s \in \mathbb{Z}$ such that

$$\alpha - \alpha_0 = -sq \text{ and } \beta - \beta_0 = sp.$$

Since α_0 is maximal, $\alpha = \alpha_0 - sq$ implies that $s \geq 0$. By choosing the maximal $s_1 \in \mathbb{Z}$, with $\alpha_0 = \alpha_1 - s_1 q \geq 0$, then for any $(\alpha, \beta) \in A$ we have

$$\alpha = \alpha_1 + (s - s_1)q, \;\; \beta = \beta_0 + sp.$$

Note that $s_1 - s$ is a positive integer. In this way, every monomial $z_1^{\alpha} z_2^{\beta}$ may be written as

$$z_1^{\alpha} z_2^{\beta} = z_1^{\alpha_1} z_2^{\beta_0} (z_1)^{q(s_1 - s)} (z_2)^{ps}.$$

and therefore

$$f(z_1, z_2) = \mu z_1^{\alpha_1} z_2^{\beta_0} Q(z_1, z_2), \tag{2.2}$$

as intended.

From the equation (2.2), we may substitute $x_1 = z_1^q$ and $x_2 = z_2^p$, obtaining the polynomial $Q(x_1, x_2)$. Assume that the degree of $Q(x_1, x_2)$ as a polynomial in two variables is k. Then we may write

$$Q(x_1, x_2) = c_0 x_2^k + c_1 x_2^{k-1} x_1 + ... + c_{k+1} x_1^k,$$

for some $c_0, c_1, ..., c_{k+1} \in \mathbb{C}$, not all zero. Without any loss of generality, assume that $c_{k+1} = 1$. Then we may write

$$Q(x_1, x_2) = x_1^k Q\left(1, \frac{x_2}{x_1}\right).$$

Let $w = \frac{x_2}{x_1}$. Then $Q(1, w)$ is a complex polynomial in one variable, which factors itself in irreducible factors as

$$Q(1, w) = \prod_{\ell=1}^{k} (w - \lambda_\ell),$$

where λ_1, ..., λ_k are the roots of $Q(1, w)$. Using this and the previous expression for Q, we have

$$Q(x_1, x_2) = x_1^k \prod_{\ell=1}^{k} \left(\frac{x_2}{x_1} - \lambda_\ell\right) = \prod_{\ell=1}^{k} (x_2 - \lambda_\ell x_1),$$

therefore

$$f(z_1, z_2) = \mu z_1^m z_2^n \prod_{\ell=1}^{k} (z_2^p - \lambda_\ell z_1^q),$$

which finishes the proof. □

We are ready to prove Theorem 1. Let us first show the following result.

Proposition 2.3. *Let M be a germ of an irreducible singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$ satisfying the hypotheses of Theorem 1. Then we have the following:*

(a) the algebraic dimension of $\mathsf{Sing}(M)$ *is* 0*;*

(b) $\mathsf{cod}_{M^*_{\mathbb{C}}}(\mathsf{Sing}(\mathcal{L}_{\mathbb{C}})) = 2$*;*

(c) $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

Proof. Let M be as in Theorem 1. Then M is given by $M = \{F = 0\}$, where

$$F(z) = \mathcal{R}e(Q(z)) + H(z, \overline{z}),$$

Q is a complex quasihomogeneous polynomial of quasihomogeneous degree d with an isolated singularity at $0 \in \mathbb{C}^2$ and $H(z, \overline{z}) = O(|z|^{deg(Q)+1})$. It follows from Lemma 2.2 that Q can be written as

$$Q(x, y) = \mu x^m y^n \prod_{\ell=1}^{k} (y^p - \lambda_\ell x^q), \tag{2.3}$$

where $m, n, p, q \in \mathbb{Z}_+^*$, $\mu, \lambda_\ell \in \mathbb{C}^*$ for each $\ell = 1, ..., k$, and $\gcd(p, q) = 1$. Since Q has an isolated singularity at $0 \in \mathbb{C}^2$, then we necessarily have that both m and n are either 0 or 1.

On the other hand, we can assume that Q has weights (a, b) with $\gcd(a, b) = 1$. Using the factorization (2.3) and the fact that each polynomial $(y^p - \lambda_\ell x^q)$ has also weights (a, b), we can conclude that $aq = bp$ and, since p, q are relatively prime, we get $a = p$ and $b = q$.

For simplicity, using (2.3), we write

$$Q(x, y) = \mu x^m y^n \prod_{\ell=1}^{k} Q_\ell(x, y),$$

where $Q_\ell(x, y) = (y^p - \lambda_\ell x^q)$. Without loss of generality, we can assume that Q has real coefficients. Then the complexification $F_{\mathbb{C}}$ of F is given by

$$F_{\mathbb{C}}(x, y, z, w) = \frac{1}{2} Q(x, y) + \frac{1}{2} Q(z, w) + H_{\mathbb{C}}(x, y, z, w).$$

Since Q has an isolated singularity at $0 \in \mathbb{C}^2$, we get that $M_{\mathbb{C}} = \{F_{\mathbb{C}} = 0\} \subset (\mathbb{C}^4, 0)$ has an isolated singularity at $0 \in \mathbb{C}^4$ and so the algebraic dimension of $\mathsf{Sing}(M)$ is zero. Hence item (a) is proved.

Consider the algebraic subvariety contained in $\mathbb{P}(a,b,a,b)$

$$V_{M_{\mathbb{C}}} = \{Q(Z_0, Z_1) + Q(Z_2, Z_3) = 0\},$$

where $[Z_0 : Z_1 : Z_2 : Z_3] \in \mathbb{P}(a,b,a,b)$. It is not difficult to see that $\mathsf{Sing}(M_{\mathbb{C}}) \subset \mathsf{Sing}(V_{M_{\mathbb{C}}})$. Note that $V_{M_{\mathbb{C}}}$ can be considered in the inclusion

$$V_{M_{\mathbb{C}}} \subset Z \simeq \mathbb{C}^4/\mathbb{Z}(a,b,a,b).$$

Now we consider the weighted blow-up $E : \tilde{Z} \to Z$, with weight $\delta = (a,b,a,b)$. Let $\tilde{M}_{\mathbb{C}}$ be the strict transform of $M_{\mathbb{C}}$ by E and $D \simeq \mathbb{P}_\delta$ be the exceptional divisor, with coordinates $(Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 \setminus \{0\}$. The intersection of $\tilde{M}_{\mathbb{C}}$ with $\mathbb{P}_\delta$ is

$$\tilde{C} := \tilde{M}_{\mathbb{C}} \cap \mathbb{P}_\delta = \{Q(Z_0, Z_1) + Q(Z_2, Z_3) = 0\}.$$

It follows from Remark 1.1 that $\mathcal{L}_{\mathbb{C}}$ is given by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\begin{aligned} \alpha &= \frac{\partial F_{\mathbb{C}}}{\partial x}dx + \frac{\partial F_{\mathbb{C}}}{\partial y}dy \\ &= Q(x,y)\left[\left(\frac{m}{x} - qx^{q-1}\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x,y)}\right)dx + \left(\frac{n}{y} + py^{p-1}\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x,y)}\right)dy\right] + \theta \end{aligned}$$

where θ is a 1-form composed of higher order terms coming from $H_{\mathbb{C}}(z,\overline{z})$. Due to Q having an isolated singularity at the origin, this 1-form has an isolated singularity at $(0,0)$ which implies that $\mathsf{Sing}(\mathcal{L}_{\mathbb{C}})$ has codimension two, proving item (b).

The rest of the proof is devoted to the proof of item (c). The idea is to use Lemma 1.8. For this we need to show that the linear parts of the elements of the holonomy group G are primitive roots of the unit. We will describe an expression for the pullback of the 1-form α under the weighted pull-back and examine the singular set of $\tilde{\mathcal{L}}_{\mathbb{C}}$ over the exceptional divisor. We finish by finding a decomposition for this singular set in connected components, which allow us to describe the fundamental group of the complement of a regular leaf and using this to compute the holonomy.

For each $i = 1, 2, 3, 4$, we have the affine open sets

$$\tilde{U}_i = \mathbb{C}^4/\mathbb{Z}_{a_i}(-a, ..., \underbrace{1}_{i-th}, ..., -b).$$

We work in $\tilde{U}_3$ with coordinates (x_1, y_1, z_1, w_1). In this open subset, the blow-up E has the following expression

$$E(x_1, y_1, z_1, w_1) = (x_1 z_1^a, y_1 z_1^b, z_1^a, w_1 z_1^b),$$

with $D \cap \tilde{U}_3 = \{z_1 = 0\} / \mathbb{Z}_a$. In this chart, the pull-back of α by E is given by

$$E^*\alpha = z_1^{pm+qn+kpq-1}\alpha_1,$$

where

$$\begin{aligned}
\alpha_1 &= Q(x_1, y_1)\left[\left(\frac{mz_1}{x_1} - qx_1^{q-1}z_1\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_1, y_1)}\right)dx_1 + \right. \qquad (2.4)\\
&\left(\frac{nz_1}{y_1} + py_1^{p-1}z_1\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_1, y_1)}\right)dy_1 + \\
&\left.\left(pm + qn - pqx_1^q\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_1, y_1)} + pqy_1^p\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_1, y_1)}\right)dz_1\right] + z_1\theta_1 \\
&= Q(x_1, y_1)\left[\left(\frac{mz_1}{x_1} - qx_1^{q-1}z_1\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_1, y_1)}\right)dx_1 + \right. \qquad (2.5)\\
&\left.\left(\frac{nz_1}{y_1} + py_1^{p-1}z_1\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_1, y_1)}\right)dy_1 + (pm + qn + pqk)\,dz_1\right] + z_1\theta_1
\end{aligned}$$

and $\theta_1 = E^*\theta / z_1^{pm+qn+kpq}$. The pull-back foliation $\tilde{\mathcal{L}}_{\mathbb{C}}$ is defined by $\alpha_1|_{\tilde{M}^*_{\mathbb{C}}} = 0$. The intersection of $\tilde{C}$ with the open subset $\tilde{U}_3$ is

$$\tilde{C} \cap \tilde{U}_3 = \{z_1 = Q(x_1, y_1) + Q(1, w_1) = 0\}/\mathbb{Z}_a,$$

which implies that $\tilde{C}$ is invariant by $\tilde{\mathcal{L}}_{\mathbb{C}}$, and

$$\mathsf{Sing}(\tilde{\mathcal{L}_{\mathbb{C}}} \cap \tilde{U}_3) = \{z_1 = Q(x_1, y_1) = Q(1, w_1) = 0\}/\mathbb{Z}_a.$$

In the chart $\tilde{U}_4$, with coordinates (x_2, y_2, z_2, w_2), the blow-up is

$$E(x_2, y_2, z_2, w_2) = (x_2 w_2^a, y_2 w_2^b, z_2 w_2^a, w_2^b)$$

and $D \cap \tilde{U}_4 = \{w_2 = 0\}/\mathbb{Z}_b$. In this chart, the pull-back of α is

$$E^*\alpha = w_2^{pm+qn+kpq-1}\alpha_2,$$

where

$$\begin{aligned}
\alpha_2 &= Q(x_2,y_2)\left[\left(\frac{mw_2}{x_2} - qx_2^{q-1}w_2\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_2,y_2)}\right)dx_2 + \right.\\
&\left(\frac{nw_2}{y_2} + py_2^{p-1}w_2\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_2,y_2)}\right)dy_2 + \\
&\left.\left(pm + qn - pqx_2^q\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_2,y_2)} + pqy_2^p\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_2,y_2)}\right)dw_2\right] + w_2\theta_2 \\
&= Q(x_2,y_2)\left[\left(\frac{mw_2}{x_2} - qx_2^{q-1}w_2\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_2,y_2)}\right)dx_2 + \right.\\
&\left.\left(\frac{nw_2}{y_2} + py_2^{p-1}w_2\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_2,y_2)}\right)dy_2 + (pm+qn+pqk)\,dw_2\right] + w_2\theta_2
\end{aligned}$$

and $\theta_2 = E^*\theta/w_2^{pm+qn+kpq}$. The pull-back foliation $\tilde{\mathcal{L}}_{\mathbb{C}}$ is given by $\alpha_2|_{\tilde{M}^*_{\mathbb{C}}} = 0$. Similarly as before, the intersection of $\tilde{C}$ with the open subset $\tilde{U}_4$ is

$$\tilde{C} \cap \tilde{U}_4 = \{w_2 = Q(x_2,y_2) + Q(z_2,1) = 0\}/\mathbb{Z}_b,$$

which is invariant by $\tilde{\mathcal{L}}_{\mathbb{C}}$, and

$$\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}} \cap \tilde{U}_4) = \{w_2 = Q(x_2,y_2) = Q(z_2,1) = 0\}/\mathbb{Z}_b.$$

Now, we focus in the chart $\tilde{U}_3$. In this open subset, the action of the group is given by

$$\begin{aligned}
x_1 &\mapsto x_1,\\
y_1 &\mapsto e^{\frac{2bi\pi}{a}}y_1,\\
w_1 &\mapsto e^{\frac{2bi\pi}{a}}w_1.
\end{aligned}$$

The singularities of the exceptional divisor in this chart are given by

$$\mathsf{Sing}(D \cap \tilde{U}_3) = \{y_1 = z_1 = w_1 = 0\}/\mathbb{Z}_a$$

and therefore the intersection of the singular set of $\tilde{\mathcal{L}}_{\mathbb{C}}$ with the singular set of the exceptional divisor is

$$\mathsf{Sing}(D) \cap \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3 = \{y_1 = z_1 = w_1 = Q(1,0) = 0\}/\mathbb{Z}_a.$$

Due to the factorization of Q given in (2.3), we investigate four cases.

- $m = n = 1$. In this case, $Q(1,0) = 0$ and, since $Q(1,w_1)$ is a complex polynomial in w_1, there exists another complex polynomial $\tilde{Q}$ such that $Q(1,w_1) = w_1\tilde{Q}(w_1)$ and $\tilde{Q}(0) \neq 0$. Note that the power for w_1 may not be higher than one, because this would conflict with the fact that $n = 1$ in the factorization of Q. Now, if r is a root of $\tilde{Q}$, then $r \neq 0$ and therefore $(0,0,0,r) \in \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ and $(0,0,0,r) \notin \mathsf{Sing}(D) \cap \tilde{U}_3$. Therefore, we have that $\mathsf{Sing}(D) \cap \tilde{U}_3 \subsetneq \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$.
- $m = 0$, $n = 1$. The same argument as the previous one holds in this case and therefore we have $\mathsf{Sing}(D) \cap \tilde{U}_3 \subsetneq \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$.
- $m = 1, n = 0$. In this case, $Q(1,0) \neq 0$ and therefore $\mathsf{Sing}(D) \cap \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3 = \emptyset$.
- $m = 0$. Same as before, we conclude that $\mathsf{Sing}(D) \cap \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3 = \emptyset$.

We arrive to the same conclusions working in the chart $\tilde{U}_4$. In both cases we have shown that, either $\mathsf{Sing}(D) \cap \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) = \emptyset$ or that $\mathsf{Sing}(D) \subsetneq \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}})$.

Consider the set $S := \tilde{C} \setminus \mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}})$. This set is a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Let q_0 be a point in $S \setminus \mathsf{Sing}(D)$ and a section Σ transverse to S passing through q_0. Working on the chart $\tilde{U}_3$, we may assume without loss of generality that $q_0 = (1,0,0,0)$ and $\Sigma = \{(1,0,t,0) | t \in \mathbb{C}\}$. Let G be the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in Σ. Recall that

$$\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3 = \{z_1 = Q(x_1,y_1) = Q(1,w_1)\}/\mathbb{Z}_a.$$

This set splits into several connected components, separated in the following cases:

- $m = 1$, $n = 1$. In this case,

$$Q(x,y) = xy \prod_{\ell=1}^{k} Q_\ell(x,y),$$

where $Q_\ell(x,y) = (y^p - \lambda_\ell x^q)$, $\gcd(p,q) = 1$.

The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ splits as the union of the following connected components:

$$C^{\ell}_{rs} = \{z_1 = Q_\ell(x_1, y_1) = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0\}/\mathbb{Z}_a,$$

$$C^{x_1}_{rs} = \{z_1 = x_1 = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0\}/\mathbb{Z}_a,$$

$$C^{y_1}_{rs} = \{z_1 = y_1 = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1, ..., k\}$ and $r \in \{1, ..., p\}$ and for each r, $\varepsilon^{(r)}_p(\lambda_s)$ is an *p-th* root of λ_s. According to [33], the fundamental group $\pi_1(S, q_0)$ may be written in terms of generators and its relations as

$$\pi_1(S, q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs}, \xi_{rs}, \tau_{rs} : \gamma^p_{\ell rs} = \delta^q_{\ell rs} \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where, for each ℓ, r, s, the elements $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around the connected component C^{ℓ}_{rs} of $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$, ξ_{rs} are loops around $C^{x_1}_{rs}$ and τ_{rs} a loop around $C^{y_1}_{rs}$. If G is the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ, then

$$G = \langle f_{\ell rs}, g_{\ell rs}, h_{rs}, k_{rs} \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$, h_{rs} and k_{rs} correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$, ξ_{rs}, τ_{rs} in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, $\Xi_{rs}(t)$, $\Upsilon_{rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M^*_{\mathbb{C}}} = 0$ (see (2.4)). The coefficients of the linear terms of the holonomy maps are

$$\begin{aligned}
f'_{\ell rs}(0) &= e^{-\frac{2(1+qk)}{p+q+pqk}\pi i},\\
g'_{\ell rs}(0) &= e^{-\frac{2}{q}\left(\frac{p+pqk}{p+q+pqk}\right)\pi i},\\
h'_{rs}(0) &= 1,\\
k'_{rs}(0) &= e^{-2\left(\frac{1+pk}{p+q+pqk}\right)\pi i}.
\end{aligned}$$

These computations are shown in Appendix A. This shows that each element of $G/\{\mathsf{id}\}$ has finite order. According to Lemma 1.8, the foliation $\tilde{\mathcal{L}}_{\mathbb{C}}$ has a holomorphic non-constant first integral and the proof in this case is finished.

- $m = 0, n = 1$. In this case,

$$Q(x,y) = y\prod_{\ell=1}^{k} Q_\ell(x,y),$$

where $Q_\ell = (y^p - \lambda_\ell x^q)$, $\gcd(p,q) = 1$. The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ splits as the union of the following connected components:

$$C^\ell_{rs} = \{z_1 = Q_\ell(x_1,y_1) = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

$$C^{y_1}_{rs} = \{z_1 = y_1 = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1, ..., k\}$, $r \in \{1, ..., p\}$ and, for each r, $\varepsilon_p^{(r)}(\lambda_s)$ is a p-th root of λ_s. The group $\pi_1(S, q_0)$ is written in terms of generators and its relations as

$$\pi_1(S,q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs}, \tau_{rs} : \gamma^p_{\ell rs} = \delta^q_{\ell rs}\rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where, for each ℓ, r, s, $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around C^ℓ_{rs} and τ_{rs} a loop around $C^{y_1}_{rs}$. If G is the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ then

$$G = \langle f_{\ell rs}, g_{\ell rs}, k_{rs}\rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$ and k_{rs} correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$, τ_{rs} in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, $\Upsilon_{rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M^*_{\mathbb{C}}} = 0$ (see for instance (2.4)). The coefficients of the linear terms of the holonomy maps are

$$\begin{aligned} f'_{\ell rs}(0) &= e^{-\frac{2\pi i}{p}}, \\ g'_{\ell rs}(0) &= e^{-\frac{2\pi i}{q}}, \\ k'_{rs}(0) &= 1. \end{aligned}$$

These computations are shown in Appendix A. This shows that each element of $G/\{\mathsf{id}\}$ has finite order. Using Lemma 1.8, the proof in this case is finished.

- $m = 1, n = 0$. In this case

$$Q(x,y) = x\prod_{\ell=1}^{k} Q_\ell(x,y),$$

where $Q_\ell = (y^p - \lambda_\ell x^q)$, $\gcd(p,q) = 1$.. The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ splits as the union of the following connected components:

$$C^\ell_{rs} = \{z_1 = Q_\ell(x_1,y_1) = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0\}/\mathbb{Z}_a,$$

$$C^{y_1}_{rs} = \{z_1 = x_1 = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1, ..., k\}$, $r \in \{1, ..., p\}$ and, for each r, $\varepsilon^{(r)}_p(\lambda_s)$ is a p-th root of λ_s. The group $\pi_1(S, q_0)$ is written in terms of generators and its relations as

$$\pi_1(S,q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs}, \xi_{rs} : \gamma^p_{\ell rs} = \delta^q_{\ell rs}\rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where, for each ℓ, r, s, $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around C^ℓ_{rs} and ξ_{rs} a loop around $C^{x_1}_{rs}$. If G is the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ then

$$G = \langle f_{\ell rs}, g_{\ell rs}, h_{rs}\rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$ and h_{rs} correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$, ξ_{rs} in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, $\Xi_{rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M^*_{\mathbb{C}}} = 0$ (see (2.4)). The coefficients of the linear terms of the holonomy maps are

$$\begin{aligned} f'_{\ell rs}(0) &= e^{-\frac{2\pi i}{q}}, \\ g'_{\ell rs}(0) &= e^{-\frac{2\pi i}{p}}, \\ k'_{rs}(0) &= 1. \end{aligned}$$

These computations are shown in Appendix A. This shows that each element of $G/\{\mathsf{id}\}$ has finite order. Using Lemma 1.8, the proof in this case is finished.

- $m = 0, n = 0$. In this case, $Q(x,y) = \prod_{\ell=1}^{k} Q_\ell(x,y)$, where $Q_\ell = (y^p - \lambda_\ell x^q)$, $\gcd(p,q) = 1$. The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ splits as the union of the following connected components:

$$C^\ell_{rs} = \{z_1 = Q_\ell(x_1,y_1) = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1,...,k\}$, $r \in \{1,...,p\}$ and, for each r, $\varepsilon_p^{(r)}(\lambda_s)$ is a p-th root of λ_s. The group $\pi_1(S,q_0)$ is written in terms of generators and its relations as

$$\pi_1(S,q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^q \rangle_{\substack{\ell, s = 1,...,k \\ r = 1,...,p}}$$

where, for each ℓ, r, s, $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around C^ℓ_{rs}. If G is the holonomy of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ then

$$G = \langle f_{\ell rs}, g_{\ell rs} \rangle_{\substack{\ell, s = 1,...,k \\ r = 1,...,p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$ correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$ in $\pi_1(S,q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M^*_{\mathbb{C}}} = 0$ (see (2.4)). The coefficients of the linear terms of the holonomy maps are

$$\begin{aligned} f'_{\ell rs}(0) &= e^{-\frac{2\pi i}{q}}, \\ g'_{\ell rs}(0) &= e^{-\frac{2\pi i}{p}}. \end{aligned}$$

These computations are shown in Appendix A. This shows that each element of $G/\{\mathsf{id}\}$ has finite order. Using Lemma 1.8, this case is finished and Proposition 2.3 is proven.

□

Now we can prove Theorem 1.

Theorem 1. *Let $M = \{F = 0\}$ be a germ at $0 \in \mathbb{C}^2$ of irreducible real-analytic Levi-flat hypersurface such that*

(a) $F(z) = \mathcal{R}e(Q(z)) + H(z, \overline{z})$,

(b) *Q is a complex quasihomogeneous polynomial of quasihomogeneous degree d with an isolated singularity at $0 \in \mathbb{C}^2$,*

(c) $H(z, \overline{z}) = O(|z|^{deg(Q)+1})$ *and* $H(z, \overline{z}) = \overline{H(\overline{z}, z)}$.

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$\phi(M) = \left\{ \mathcal{R}e \left(Q(z) + \sum_{j=1}^{s} c_j e_j(z) \right) = 0 \right\},$$

where $e_1, ..., e_s$ are elements of the monomial basis of the local algebra of Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.

Proof. Using Proposition 2.3, we have that the hypotheses of Theorem 1.5, part (b) are verified. Then there exists a foliation $\mathcal{F}_M$ with a non-constant holomorphic first integral $f \in \mathcal{O}_2$ such that $M = \{\mathcal{R}e(f) = 0\}$. Without loss of generality, we can assume that f is not a power in $\mathcal{O}_2$ and therefore so $\mathcal{R}e(f)$ is irreducible by Lemma 1.2. This implies that

$$\mathcal{R}e(f) = U \cdot F,$$

where $U \in \mathcal{A}_{n\mathbb{R}}$ and $U(0) \neq 0$.

Note that, since Q is quasihomogeneous with quasihomogeneous degree d and weights (a, b), then its real and imaginary parts are also quasihomogeneous polynomials with the same weights and quasihomogeneous degree d. Write f as the decomposition

$$f = \sum_{\ell \geq d} f_\ell,$$

where each f_ℓ is quasihomogeneous of quasihomogeneous degree ℓ with weights (a, b) (see [1]). Let the power series of U around $0 \in \mathbb{C}^2$ be

$$U(z) = U(0) + \sum_{\substack{\mu_1+\mu_2 \geq 1 \\ \nu_1+\nu_2 \geq 1}} c_{\mu_1\mu_2\nu_1\nu_2} z_1^{\mu_1} z_2^{\mu_2} \overline{z_1}^{\nu_1} \overline{z_2}^{\nu_2} = U(0) + \tilde{U}(z).$$

Then

$$\begin{aligned} \mathcal{R}e(f) &= (U(0) + \tilde{U})(\mathcal{R}e(Q) + H) \\ &= U(0)\mathcal{R}e(Q) + \tilde{U}\mathcal{R}e(Q) + U(0)H + \tilde{U}H. \end{aligned}$$

We need to investigate what terms on the previous equality have quasihomogeneous degree d with weights (a, b), which will be equal to $\mathcal{R}e(f_d)$. Let us denote the term $U(0)H + \tilde{U}H$ simply by $\tilde{H}$ since the quasihomogeneous degrees of its decomposition are all greater than d. Since

$$\mathcal{R}e(f) = \mathcal{R}e(f_d) + \sum_{\ell > d} \mathcal{R}e(f_\ell),$$

we have

$$\begin{aligned} \mathcal{R}e(f(\lambda^a z_1, \lambda^b z_2)) &= U(0)\mathcal{R}e(Q(\lambda^a z_1, \lambda^b z_2)) \\ &\quad + \tilde{U}(\lambda^a z_1, \lambda^b z_2)\mathcal{R}e(Q(\lambda^a z_1, \lambda^b z_2) + \tilde{H} \\ &= U(0)\mathcal{R}e(\lambda^d Q(z_1, z_2)) \\ &\quad + \tilde{U}(\lambda^a z_1, \lambda^b z_2)\mathcal{R}e(\lambda^d Q(z_1, z_2)) + \tilde{H} \\ &= U(0)\left(\frac{\lambda^d Q(z) + \overline{\lambda}^d \overline{Q(z)}}{2}\right) \\ &\quad + \Big(c_{1000}\lambda^a z_1 + c_{0100}\lambda^b z_2 + c_{0010}\overline{\lambda}^a \overline{z_1} \\ &\quad + c_{0001}\overline{\lambda}^b \overline{z_2} + ...\Big)\, \mathcal{R}e(\lambda^d Q(z)) + \tilde{H} \\ &= \underbrace{U(0)\left(\frac{\lambda^d Q(z) + \overline{\lambda}^d \overline{Q(z)}}{2}\right)}_{\text{quasihomogeneous degree is } d} \\ &\quad + \underbrace{c_{1000}\lambda^a \lambda^d z_1 \mathcal{R}e(Q(z)) + ... + \tilde{H}}_{\text{quasihomogeneous degree is greater than } d} \end{aligned}$$

which means that $f_d(z) = U(0)Q(z)$, so $f(z) = U(0)Q(z) + H(z, \overline{z})$. Without any loss of generality we may assume that $U(0) = 1$.

In particular, $\mu(f, 0) = \mu(Q, 0)$ since Q has an isolated singularity at the origin. From Lemma 2.1, there exists a germ of biholomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$f \circ \phi^{-1}(z) = Q(z) + \sum_j c_j e_j(z),$$

where $c_j \in \mathbb{C}$ and e_j are elements of the monomial basis of A_Q with $\deg(e_j) > d$. Hence

$$\phi(M) = \left\{ \mathcal{R}e \left(Q(z) + \sum_j c_j e_j(z) \right) = 0 \right\}$$

and this finishes the proof of Theorem 1.

□

3 Volume-preserving Normal Forms

Let $f \in \mathcal{O}_n$ be a germ of holomorphic function such that $f(0) = 0$ for which the Hessian form

$$h = \sum_{i=1,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial z_j}(0) z_i z_j$$

is non-degenerate. The classical Morse's lemma asserts that f is right-equivalent to h.

Let $\omega = a(z) dz_1 \wedge ... \wedge dz_n$, $a(0) \neq 0$ be a holomorphic volume form on a coordinate system $(z_1, ..., z_n)$ on an open set around $0 \in \mathbb{C}^n$. A coordinate system is *isochore* or *volume-preserving* if ω may be written as $dz_1 \wedge ... \wedge dz_n$ in it. We say that a biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is isochore or volume-preserving if the coordinate system induced by it is volume-preserving.

In 1977, J. Vey [32] has posed the following question:

Is it possible to find a volume-preserving coordinate system in which a germ of function $f \in \mathcal{O}_n$ is right-equivalent to its hessian h?

The answer is, in general, negative. However, Vey proved that right-equivalence with the composition of h with a holomorphic function in one variable may be obtained:

Lemma 3.1 (Vey, [32])**.** *Let $f \in \mathcal{O}_n$, $n \geq 2$, with isolated singularity at the origin of $\mathbb{C}^n$ and non-degenerate Hessian form h. Then there exists a volume-preserving germ of biholomorphism $\phi \in \mathit{Diff}(\mathbb{C}^n, 0)$ and $\psi \in \mathcal{O}_1$, with $\psi(0) = 0$ and $\psi'(0) = 1$, such that*

$$f \circ \phi^{-1} = \psi(h).$$

The function ψ is uniquely determined by f up to a sign.

This result was also demonstrated by J-P Françoise [18]. The approach used by Françoise was later generalised by A. Szawlowski [30] to quasihomogeneous polynomials and to the germs of holomorphic functions which are right-equivalent to the product of coordinates $z_1 \cdots z_n$, as stated in the following theorem:

Lemma 3.2 (Szawlowski, [30])**.** *Let $f \in \mathcal{O}_n$, $n \geq 2$ be a germ of holomorphic function that is right-equivalent to the product of all coordinates: $f \sim_R z_1 \cdots z_n$. Then there exists a germ of volume-preserving biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\psi \in \mathcal{O}_1$, with $\psi(0) = 0$, such that*

$$f \circ \phi(x) = \psi(z_1 \cdots z_n).$$

The function ψ is uniquely determined by f up to a sign.

Note that the above normal form for f is a germ of holomorphic function whose singular set is of positive dimension (non-isolated). In general, normal forms of germs of functions with non-isolated singularities are very difficult to find, even assuming isochore coordinates.

Remark 3.1. Not all germs at the origin of $\mathbb{C}^n$ having the form $z_1 \cdots z_n + O(|z|^{n+1})$ are right-equivalent to the product of coordinates. Since the origin of $\mathbb{C}^n$ is not an isolated singularity for the germ $z_1 \cdots z_n$, Morse's lemma may not be applied.

Example 3.1. Let f be the holomorphic function in $\mathbb{C}^3$ defined as

$$f(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^4 + z_2^4 + z_3^4.$$

The singular set of this function in $\mathbb{C}^n$ is composed of 17 distinct, isolated points, one of them being the origin, and therefore f is not right-equivalent to the product of coordinates $z_1 z_2 z_3$ whose singular set is composed of the union of the lines

$$\{z_i = z_j = 0\}, 1 \leq i, j \leq 3, i \neq j.$$

To prove Theorem 2 we use the following result from [14], although it is not stated there as a separate theorem. We restate it here for the sake of completeness.

Theorem 3.3 (Fernández-Pérez [14]). *Let M be a germ of irreducible real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$ such that $M = \{F = 0\}$, and*

1. $F(z) = \mathcal{R}e(P(z)) + H(z, \overline{z})$,
2. *P is a homogeneous polynomial of degree k with an isolated singularity at $0 \in \mathbb{C}^n$,*
3. *$H(z, \overline{z}) = O(|z|^{k+1})$ and $H(z, \overline{z}) = \overline{H(\overline{z}, z)}$.*

Then there exists a germ at $0 \in \mathbb{C}^n$ of holomorphic codimension-one foliation $\mathcal{F}_M$ tangent to M. Moreover, the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral $f(z) = P(z) + O(|z|^{k+1})$, and $M = \{\mathcal{R}e(f) = 0\}$.

We can now prove Theorem 2.

Theorem 2. *Let $M = \{F(z) = 0\}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of irreducible real-analytic Levi-flat hypersurface such that*

$$F(z) = \mathcal{R}e(z_1^2 + ... + z_n^2) + H(z, \overline{z}),$$

where $H(z, \overline{z}) = O(|z|^3)$, $H(z, \overline{z}) = \overline{H(\overline{z}, z)}$. Then, there exists a volume-preserving germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and an automorphism $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$\phi(M) = \{\mathcal{R}e(\psi(z_1^2 + ... + z_n^2)) = 0\}.$$

Proof. Let $M = \{F = 0\}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$ of irreducible real-analytic Levi-flat hypersurface such that

$$F(z) = \mathcal{R}e(z_1^2 + ... + z_n^2) + H(z, \overline{z}),$$

where $H(z,\overline{z}) = O(|z|^3)$, $H(z,\overline{z}) = \overline{H(\overline{z},z)}$. Since $P(z_1,\ldots,z_n) = z_1^2+\ldots+z_n^2$ is a complex homogeneous polynomial of degree 2, we can apply Theorem 3.3, so that there exists $f \in \mathcal{O}_n$ such that $f(z) = z_1^2+\ldots+z_n^2+O(|z|^3)$ and $M = \{\mathcal{R}e(f) = 0\}$. On the other hand, applying Lemma 3.1 to f, there exists a volume-reserving $\phi \in \mathsf{Diff}(\mathbb{C}^n, 0)$ and an automorphism $\psi_1 \in \mathcal{O}_1$, with $\psi_1(0) = 0$, such that

$$f \circ \phi^{-1} = \psi_1(2P), \quad \psi_1'(0) = 1.$$

Taking $\psi := \psi_1(t/2) \in \mathcal{O}_1$, we have $f \circ \phi^{-1} = \psi \circ P$. Finally,

$$\phi(M) = \{\mathcal{R}e(\psi(P)) = 0\}$$

and the proof of Theorem 2 ends. □

Let us now prove Theorem 3 and Corollary 1. Here we will use the same idea of the proof of Theorem 1. First of all, note that, in dimension two, under the change of variables $z_1 = y + ix$, $z_2 = y - ix$, and we have $z_1 z_2 = x^2 + y^2$, from what Theorem 3 follows from Theorem 2. In this case, the singular set of $M_{\mathbb{C}}$ is composed only of the origin of $\mathbb{C}^4$. Therefore, we only consider the case $n \geq 3$.

Proposition 3.4. *Let M be a germ of a singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 3$, satisfying the hypotheses of Theorem 3. Then $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.*

Proof. Let M be as in Theorem 3. Then, M is given by the $\{F = 0\}$ where

$$F(z) = \mathcal{R}e(z_1 \cdots z_n) + H(z_1, ..., z_n),$$

and $H(z,\overline{z}) = O(|z|^{n+1})$. Its complexification is

$$F_{\mathbb{C}}(z,w) = \frac{1}{2}(z_1 \cdots z_n) + \frac{1}{2}(w_1 \cdots w_n) + H_{\mathbb{C}}(z,w), \tag{3.1}$$

and therefore $M_{\mathbb{C}} = \{F_{\mathbb{C}} = 0\} \subset (\mathbb{C}^{2n}, 0)$. By hypotheses, $\mathsf{Sing}(M_{\mathbb{C}})$ is the union of the sets

$$V_{ijk\ell} = \{z_i = z_j = w_k = w_\ell = 0\}, \quad 1 \leq i \neq j \leq n, \quad 1 \leq k \neq \ell \leq n.$$

Since $V_{ijk\ell}$ has complex dimension $2n-4$, then the algebraic dimension of $\mathsf{Sing}(M)$ is $2n-4$.

On the other hand, it follows from Remark 1.1 that $\mathcal{L}_{\mathbb{C}}$ is given by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = \sum_{i=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial z_i} dz_i.$$

Using (3.1) we can write α in coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ as

$$\alpha = \frac{1}{2}\sum_{i=1}^{n} \left(z_1 \cdots \widehat{z_i} \cdots z_n + \frac{\partial R}{\partial z_i} \right) dz_i,$$

where $\dfrac{\partial R}{\partial z_i} = 2\dfrac{\partial H_{\mathbb{C}}}{\partial z_i}$ for all $i = 1 \ldots, n$. Then we can consider that the $\mathcal{L}_{\mathbb{C}}$ is defined by $\tilde{\alpha}|_{M_{\mathbb{C}}} = 0$, where

$$\tilde{\alpha} = \sum_{i=1}^{n} \left(z_1 \cdots \widehat{z_i} \cdots z_n + \frac{\partial R}{\partial z_i} \right) dz_i.$$

Let us prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral. We start with the blow-up π_1 at $0 \in \mathbb{C}^{2n}$ with exceptional divisor $D_1 = \mathbb{P}^{2n-1}$. Let $[r : \ell] = [r_1 : ... : r_n : \ell_1 : ... : \ell_n]$ be the homogeneous coordinates of D_1. The intersection of $\tilde{M}_{\mathbb{C}} = \pi_1^*(M_{\mathbb{C}})$ with the divisor D_1 is the algebraic hypersurface

$$Q_1 := \tilde{M}_{\mathbb{C}} \cap D_1 = \{[r_1 : ... : r_n : \ell_1 : ... : \ell_n] \in \mathbb{P}^{2n-1} | r_1 \cdots r_n + \ell_1 \cdots \ell_n = 0\}.$$

In the chart $(W, (r, \ell) = (r_1, ..., r_n, \ell_1, ..., \ell_n))$ of $\tilde{\mathbb{C}}^{2n}$ where

$$\pi_1(r, \ell) = (\ell_1 r_1, ..., \ell_1 r_2, ..., \ell_1 r_n, \ell_1, \ell_1\ell_2, ..., \ell_1\ell_n).$$

Then

$$\begin{aligned} \tilde{F}_{\mathbb{C}}(r, \ell) &= F_{\mathbb{C}} \circ \pi_1(r, \ell) = \ell_1^n r_1 \cdots r_n + \ell_1^n \ell_2 \cdots \ell_n + R(\pi_1(r, \ell)) \\ &= \ell_1^n (r_1 \cdots r_n + \ell_2 \cdots \ell_n + \ell_1 R_1(r, \ell)), \end{aligned}$$

where $R_1(r, \ell) = \dfrac{R(\pi_1(r, \ell))}{\ell_1^{n+1}}$. Therefore

$$\tilde{M}_{\mathbb{C}} \cap W = \{r_1 \cdots r_n + \ell_2 \cdots \ell_n + R_1(r, \ell) = 0\},$$

and

$$Q_1 \cap W = \{\ell_1 = r_1 \cdots r_n + \ell_2 \cdots \ell_n = 0\}.$$

On the other hand, the pull-back of $\tilde{\alpha}$ by the blow-up π_1 is

$$\begin{aligned} \pi_1^*(\tilde{\alpha}) &= \sum_{i=1}^{n} \ell_1^{n-1} \left(r_1 \ldots \widehat{r}_i \cdots r_n\right) d(\ell_1 r_i) + \theta \\ &= \ell_1^{n-1} \left(\sum_{i=1}^{n} \ell_1 r_1 \cdots \widehat{r}_i \cdots r_n dr_i + n r_1 \cdots r_n d\ell_1 + \ell_1 \theta_1 \right), \end{aligned}$$

where $\theta_1 = \theta/\ell_1^n$. In the chart W, the exceptional divisor is written as $D_1 = \{\ell_1 = 0\}$ and $\tilde{\mathcal{L}}_{\mathbb{C}}$ is given by $\alpha_1|_{\tilde{M}^*_{\mathbb{C}}} = 0$, where

$$\alpha_1 = \sum_{i=1}^{n} \ell_1 r_1 \cdots \widehat{r}_i \cdots r_n dr_i + n r_1 \cdots r_n d\ell_1 + \ell_1 \theta_1.$$

Note that $D_1 \cap \tilde{M}_{\mathbb{C}}$ is invariant by $\tilde{\mathcal{L}}_{\mathbb{C}}$ and moreover

$$\mathsf{Sing}(\tilde{M}_{\mathbb{C}}) \cap W = \bigcup_{i,j,k,s} W_{i,j,k,s},$$

where

$$W_{i,j,k,s} := \{r_i = r_j = \ell_k = \ell_s = 0\}_{\,1 \le i,j,k,s \le n}$$

where $i \neq j$, $k \neq s$ and $k \neq 1$, $s \neq 1$.

Consider the irreducible component $W_{1,2,2,3}$ of $\mathsf{Sing}(\tilde{M}_{\mathbb{C}}) \cap W$. We make a blow-up along this component; the process of desingularization around the other components of $\mathsf{Sing}(\tilde{M}_{\mathbb{C}}) \cap W$ are similarly obtained by exchanging coordinates. Let E be the exceptional divisor of $\pi_\ell : \tilde{\mathbb{C}}^{2n} \to \mathbb{C}^{2n}$. Let $\tilde{\tilde{M}}_{\mathbb{C}}$ be the strict transform of $\tilde{M}_{\mathbb{C}}$ and $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$ be the pull-back of $\tilde{\mathcal{L}}_{\mathbb{C}}$ by π_ℓ respectively. Let U be a open set with coordinates $(x_1, ..., x_{2n})$ where the blow-up is

$$\pi_\ell(x_1, ..., x_{2n}) = (x_1 x_{n+3}, x_2 x_{n+3}, x_3, ..., x_n, x_{n+1}, x_{n+2} x_{n+3}, x_{n+3}, x_{n+4}, ..., x_{2n}).$$

We have

$$\tilde{\tilde{F}}_{\mathbb{C}} = \tilde{F}_{\mathbb{C}} \circ \pi_\ell = x_{n+1}^n x_{n+3}^2 (x_1 \cdots x_n + x_{n+1} x_{n+2} x_{n+4} \cdots x_{2n} + x_{n+1} x_{n+3} R_2),$$

where $R_2 = \dfrac{R_1(\pi_\ell(x_1, ..., x_{2n}))}{x_{n+3}^3}$. Therefore

$$\tilde{\tilde{M}}_{\mathbb{C}} \cap U = \{x_1 \cdots x_n + x_{n+2}x_{n+4} \cdots x_{2n} + x_{n+1}x_{n+2}R_2 = 0\}$$

hence

$$\tilde{\tilde{M}}_{\mathbb{C}} \cap U \cap E = \{x_{n+1} = x_{n+3} = x_1 \cdots x_n + x_{n+2}x_{n+4} \cdots x_{2n} = 0\}.$$

The pull-back of α_1 by π_ℓ is

$$\begin{aligned} \pi_\ell^*(\alpha_1) \;=\; & x_{n+3}\left(x_2 \cdots x_n x_{n+1} x_{n+3} dx_1 + x_1 x_3 \cdots x_n x_{n+1} x_{n+3} dx_2 + \right. \\ & \sum_{i=3}^{n} \frac{x_1 \cdots x_n x_{n+1} x_{n+3}}{x_i} dx_i + \\ & \left. n x_1 \cdots x_n x_{n+3} dx_{n+1} + 2x_1 x_2 \cdots x_n x_{n+1} dx_{n+3} + x_{n+1}x_{n+3}\theta_2\right), \end{aligned}$$

where $\theta_2 = \theta_1 / x_{n+3}^2$. In the chart U, the exceptional divisor is written as

$$D = D_1 \cup D_2 = \{x_{n+1} = 0\} \cup \{x_{n+3} = 0\}$$

and $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$ is given by $\alpha_2|_{\tilde{\tilde{M}}_{\mathbb{C}}^*} = 0$, where

$$\begin{aligned} \alpha_2 \;=\; & x_2 \cdots x_n x_{n+1} x_{n+3} dx_1 + x_1 x_3 \cdots x_n x_{n+1} x_{n+3} dx_2 + \\ & \sum_{i=3}^{n} \frac{x_1 \cdots x_n x_{n+1} x_{n+3}}{x_i} dx_i + \\ & n x_1 \cdots x_n x_{n+3} dx_{n+1} + 2x_1 x_2 \cdots x_n x_{n+1} dx_{n+3} + x_{n+1}x_{n+3}\theta_2, \end{aligned} \tag{3.2}$$

which allows us to conclude that $D \cap \tilde{\tilde{M}}_{\mathbb{C}}$ is invariant by $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$. The singularities of the foliation $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$ on the exceptional divisor in this chart are given by

$$\mathsf{Sing}\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}} \cap U \cap D = \{x_{n+1} = x_{n+3} = x_1 \cdots x_n = x_{n+2}x_{n+4} \cdots x_{2n} = 0\}.$$

If we define $\mathcal{C}_{i,n+j} = \{x_{n+1} = x_{n+3} = x_i = x_{n+j} = 0\} \simeq \mathbb{C}^{2(n-2)}$, then we can write

$$\mathsf{Sing}\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}} \cap U \cap D = \bigcup_{\substack{1 \leq i, j \leq n \\ j \neq 1, 3}} \mathcal{C}_{i,n+j}.$$

Since $D \cap \tilde{\tilde{M}}_{\mathbb{C}}$ is invariant by $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$, then

$$S := (D \cap \tilde{\tilde{M}}_{\mathbb{C}}) \setminus \mathsf{Sing}\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$$

is a leaf of $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$. Let G be its holonomy group, and $p_0 \in S$ given by

$$p_0 = (x_1, ..., x_n, x_{n+1}, x_{n+1}, x_{n+3}, x_{n+4}, ..., x_{2n}) = (1, ..., 1, 0, -1, 0, 1, ..., 1).$$

Take Σ the transversal section through p_0 given by

$$\Sigma = \{(1, ..., 1, \lambda, -1, \lambda, 1, ..., 1) | \lambda \in \mathbb{C}\}.$$

Let $\delta_{i,j}(\theta)$ be a loop around $\mathcal{C}_{i,n+j}$, for $1 \leq i \leq n$ and $4 \leq j \leq n$, and $\delta_{i,2}(\theta)$ a loop around $C_{i,n+2}$, $1 \leq i \leq n$ with $\theta \in [0, 1]$. Each one of these loops lifts up to $\Gamma_{i,j}(\lambda, \theta)$ and $\Gamma_{i,2}(\lambda, \theta)$, respectively, such that

$$\Gamma_{i,j}(0, \theta) = 0, \Gamma_{i,j}(\lambda, 0) = \lambda \text{ and } \Gamma_{i,j}(\lambda, \theta) = \sum_{k=1}^{\infty} \delta_k^{i,j}(\theta)\lambda^k,$$

where $i = 1, ..., n$ and $j = 2, 4, 5, ..., n$.

The holonomy map with respect to these loops are

$$h_{\delta_{i,j}}(\lambda) = \Gamma_{i,j}(\lambda, 1).$$

Using the expression of α_2 given in (3.2), we get

$$h'_{\delta_{i,j}}(0) = e^{-\frac{2\pi i}{n+2}}, \text{ for } i = 1, ..., n \text{ and } j = 2, 4, 5, ..., n.$$

It follows from Lemma 1.8 that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral on $M_{\mathbb{C}}$. □

We are ready to prove Theorem 3.

Theorem 3. *Let $M = \{F = 0\}$ be a germ of an irreducible singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, such that $F(z) = \mathcal{R}e(z_1 \cdots z_n) + H(z, \overline{z})$,* *where* $H(z,\overline{z}) = O(|z|^{n+1})$ *and* $H(z,\overline{z}) = \overline{H(\overline{z},z)}$. *Suppose that*

$$\mathsf{Sing}(M_{\mathbb{C}}) = \bigcup_{\substack{1\le i<j\le n\\ 1\le k<\ell\le n}} V_{ijk\ell},$$

where

$$V_{ijk\ell} = \{(z,w)\in\mathbb{C}^n\times\mathbb{C}^n : z_i = z_j = w_k = w_\ell = 0\}$$

and $\mathsf{Sing}(M_{\mathbb{C}}) \subset \mathsf{Sing}(\mathcal{L}_{\mathbb{C}})$. *Then, there exists a germ of codimension-one holomorphic foliation* $\mathcal{F}_M$ *tangent to* M*, with a non-constant holomorphic first integral* $f(z) = z_1\cdots z_n + O(|z|^{n+1})$ *such that*

$$M = \{\mathcal{R}e(f(z)) = 0\}.$$

Proof. Note that Proposition 3.4 implies that the hypotheses of Theorem 1.5, part (b) are verified. Then we get $f\in\mathcal{O}_n$ such that the foliation $\mathcal{F}$ given by $df = 0$ is tangent to M and $M = \{\mathcal{R}e(f) = 0\}$. Without loss of generality we may assume that f is not a power in $\mathcal{O}_n$ and therefore $\mathcal{R}e(f)$ is irreducible in $\mathcal{A}_{n\mathbb{R}}$. We must have $Re(f) = U\cdot F$ where $U\in\mathcal{A}_{n\mathbb{R}}$, $U(0)\neq 0$. If the Taylor expansion of f at $0\in\mathbb{C}^n$ is

$$f = \sum_{j\ge n} f_j,$$

where each f_j is a homogeneous polynomial of degree j, then

$$\mathcal{R}e(f_n) = j_0^n(\mathcal{R}e(f)) = j_0^n(U\cdot F) = U(0)\mathcal{R}e(z_1\cdots z_n),$$

which means $f_n(z) = U(0)z_1\cdots z_n$. We can assume that $U(0) = 1$ and therefore

$$f(z) = z_1\cdots z_n + O(|z|^{n+1}).$$

This proves Theorem 3. □

Finally we prove Corollary 1.

Corollary 1. *Let M be a germ of irreducible singular real-analytic Levi-flat hypersurface as in Theorem 3. If f is right-equivalent to the product of all coordinates, $f \sim_R z_1 \cdots z_n$, then there exists a germ of a volume-preserving biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a germ of an automorphism $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that*

$$\phi(M) = \{\mathcal{R}e(\psi(z_1 \cdots z_n)) = 0\},$$

where ψ is uniquely determined by f up to a sign.

Proof. Assume that $f(z) \sim_R z_1 \cdots z_n$. It follows from Theorem 3.2 that there exists a germ of a volume-preserving biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a germ of an automorphism $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that

$$f \circ \phi^{-1}(z) = \psi(z_1 \cdots z_n).$$

Hence

$$\phi(M) = \{\mathcal{R}e(\psi(z_1 \cdots z_n)) = 0\}.$$

This finishes the proof of Corollary 1. ☐

4 Connected Level sets and separatrices

Let M be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of real-analytic Levi-flat hypersurface with Levi foliation $\mathcal{L}$. Suppose that M has an isolated singularity at the origin $0 \in \mathbb{C}^n$. In this chapter, we focus on the problem of finding conditions for the existence of $f \in \mathcal{O}_2$ such that the leaves of $\mathcal{L}$ near the origin coincide with the *real* level sets of f, that is, level sets $f^{-1}(c)$, $c \in \mathbb{R}$.

This question is motivated by a recent work with uniformly laminar currents near non-dicritical singularities of irreducible real-analytic Levi-flat hypersurfaces from S. Pinchuk, R. Shafikov and A. Sukhov (see [28]), where it is assumed that each leaf of the Levi foliation is a level set of a real value of a holomorphic function. We check below that this is always true in the particular case of $n = 2$ under the hypothesis that the Levi foliation is non-dicritical. We examine the same problem in the dicritical case and when $n \geq 3$.

A singular point p of a Levi-flat hypersurface is *dicritical* if infinitely many leaves of the Levi foliation have p in their closure.

4.1 The bidimensional case with a non-dicritical singularity

Let $M \subset \mathbb{C}^n$ be a real-analytic Levi-flat hypersurface of codimension 1. Let M_{reg} be the of *regular* points of M, that is, points near which M is a real-analytic submanifold of any dimension, and let M_s be its singular locus, defined by $M \setminus M_{reg}$.

We remark the slight difference between the definitions of *smooth part* M^* and *singular set* $\mathsf{Sing}(M)$, as defined previously in Chapter 1, and *regular part* M_{reg} and *singular locus* M_s used in the last theorem. The set M_s is not necessarily equal to $\mathsf{Sing}(M)$, even if M is irreducible. See [6] and [22] for examples on this subject.

In this context, we have the following result, which is due to J. Lebl (see [17] and [23]) and is analogous to Theorem 1.5.

Theorem 4.1 (Lebl). *Let $U \subset \mathbb{C}^n$ be an open subset and $M \subset U$ be a Levi-flat real-analytic subvariety that is irreducible as a germ at $p \in \overline{M_{reg}} \cap U$. Suppose that one of the two below happens:*

(a) $dim(M_s) = 2n - 4$ and p is non-dicritical.

(b) $dim(M_s) < 2n - 4$.

Then the Levi foliation $\mathcal{L}$ extends to a singular holomorphic foliation on a neighborhood of p.

Suppose that M is given locally as $\{F(z, \overline{z}) = 0\}$. Let $M_{\mathbb{C}}$ be the complexification of M and $\mathcal{L}_{\mathbb{C}}$; then $M_{\mathbb{C}}$ is given locally around the origin of $\mathbb{C}^4$ as $\{F(z, w) = 0\}$. For a representative of $M_{\mathbb{C}} \subset (\mathbb{C}^4, 0)$ in a small enough open set $U \times U$, $U \subset \mathbb{C}^2$ around the origin, we define the set

$$Q_w = \{z \in U : F(z, \overline{w}) = 0\}.$$

This set is the *Segre variety* associated to M at w. We say that $q \in U$ is *Segre-degenerate* if the complex dimension of Q_q is n (in our context, $n = 2$). From [28] we have the following result:

Theorem 4.2. *Let $M = \{F = 0\}$ be a germ of irreducible real-analytic Levi-flat hypersurface in $\mathbb{C}^n$ and $0 \in \overline{M^*}$. Then 0 is a dicritical singularity of $\mathcal{L}$ if and only if it is Segre-degenerate.*

The following lemma is due to J. F. Mattei and R. Moussu (see [25]):

Lemma 4.3. *Let $f \in \mathcal{O}_2$ be a germ of holomorphic funtion which is not a power. Then there exists an open set U around $0 \in \mathbb{C}^2$ such that the level sets of $f|_U$ are connected.*

The following result, due to E. Paul [27], is a generalization of the previous one:

Theorem 4.4. *Let $f = f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ be a germ of Liouvillian function which is not a power in $\mathcal{O}_n$. Suppose that the complex numbers λ_i do not satisfy linear relations with positive integer coefficients, and that the ratios of elements of the form $\sum n_i \lambda_i$, $n_i \in \mathbb{Z}_+$, are not real negative numbers. Then, there exists a fundamental system of neighborhoods of the origin in which the level sets of a representative of this function f are connected.*

Intuitively, complex Liouvillian functions are the ones obtained from complex rational functions after a finite sequence of integrations, exponentiations and algebraic operations. It is clear from this result that if f is a germ of holomorphic function that is not a power, then its level sets are connected in a neighborhood of the origin.

Theorem 4. *Let M be a germ of real-analytic Levi-flat hypersurface in $\mathbb{C}^2$, and $\mathcal{L}$ be the Levi foliation on M^*. Assume that the origin is a non-dicritical singularity. Then there exists a germ of holomorphic function $f \in \mathcal{O}_2$ whose connected real level sets $f^{-1}(c) \cap M$, $c \in \mathbb{R}$ coincide with the leaves of $\mathcal{L}$ near the origin.*

Proof. Let $M = \{F = 0\} \subset (\mathbb{C}^2, 0)$ be a germ of irreducible real-analytic Levi-flat hypersurface. Let $\mathcal{L}$ be its Levi foliation with a non-dicritical singularity at the origin $0 \in \mathbb{C}^2$.

According to Theorem 4.1, the foliation $\mathcal{L}$ is the restriction to M of a germ of a codimension one holomorphic foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$. In particular, M is invariant by $\mathcal{F}$ and $0 \in \mathbb{C}^2$ is a non-dicritical singularity of $\mathcal{F}$.

Due to Theorem 1.6, $\mathcal{F}$ has a non-constant holomorphic first integral $f \in \mathcal{O}_2$. Without any loss of generality, we may assume that f is not a power in $\mathcal{O}_2$. According to [10], Lemma 4.2, f may be taken in such a way that $f(M) \subset (\mathbb{R}, 0)$, and each leaf of $\mathcal{L}$ is written as $L_c = f^{-1}(c) \cap M$, $c \in \mathbb{R}$.

Now we can use Lemma 4.3, allowing us to concluce that there exists an open set U around the origin of $\mathbb{C}^2$ in which the level sets $f^{-1}(z)$ of $f|_U$ are connected. In particular,the real level sets $f^{-1}(c)$, $c \in \mathbb{R}$ are also connected. Therefore, the leaves of $\mathcal{L}$ coincide with the real level sets of f, which finishes the proof. □

4.2 Dicritical singularities and higher dimensions

In this section we deal with the case of a dicritical singularity at the origin in dimension two. We finish with a discussion of the problem when $n \geq 3$.

Let us consider the dicritical case in dimension two. Let $M = \{F = 0\}$ be a germ of real-analytic Levi-flat hypersurface at the origin. Let $\mathcal{L}$ be the Levi foliation on M^* with a dicritical singularity at $0 \in \mathbb{C}^2$. We assume that the foliation $\mathcal{L}$ extends to a holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$. In this situation, Theorem 1.6 implies that there exists a non-constant meromorphic first integral f for $\mathcal{F}$. Without any loss of generality, we may assume that f is not a power.

Write $f = f_1^{\lambda_1} \cdots f_n^{\lambda_n}$, where $\lambda_j \in \mathbb{C}$ and $f_j \in \mathcal{O}_n$, for $1 \leq j \leq p$. If we assume that the complex numbers λ_i do not satisfy linear relations with positive integer coefficients, and that the ratios of elements of the form $\sum n_i \lambda_i$, $n_i \in \mathbb{Z}_+$, are not real negative numbers, then we can apply Theorem 4.4 to conclude that the leaves of $\mathcal{F}$ coincide with the real level sets of f near the origin, which, due to our assumptions, are connected.

We remark, however, that there are no known results regarding the extension of the Levi foliation $\mathcal{L}$ on M to a holomorphic foliation $\mathcal{F}$ in the dicritical case. Also, in the situation where a meromorphic first integral f is obtained, in general there is no control over the elements λ_j from its factorization. Moreover, Theorem 4.4 may not be assumed as valid in all meromorphic cases, such as in the example below.

Example 4.1. Let us provide an example of a real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$ with non-connected separatrices. Consider the meromorphic function

$$f(x, y) = \frac{(x^2 - y^3)^3}{(x^3 - y^2)^4}.$$

This germ of meromorphic function does not have connected level sets around the origin (see [27]). Consider the germ of real-analytic hypersurface in $\mathbb{C}^2$:

$$\begin{aligned} M &= \left\{ \frac{(x^2 - y^3)^3}{(x^3 - y^2)^4} = \frac{(\overline{x}^2 - \overline{y}^3)^3}{(\overline{x}^3 - \overline{y}^2)^4} \right\} \\ &= \{(x^2 - y^3)^3(\overline{x}^3 - \overline{y}^2)^4 - (x^3 - y^2)^4(\overline{x}^2 - \overline{y}^3)^3 = 0\} \end{aligned}$$

This hypersurface is Levi-flat with an isolated singularity at the origin. If $\mathcal{L}$ is the Levi foliation on M^*, then its leaves contain the real level sets of f,

$$f^{-1}(c) = \{(x^2 - y^3)^3 - c(x^3 - y^2)^4 = 0\}, \ c \in \mathbb{R},$$

Note that M has a dicritical singularity at $0 \in C^2$. Moreover the function f is a meromorphic first integral to $\mathcal{L}$ on M^*. By definition, the leaves of $\mathcal{L}$ are connected, but the level set $f^{-1}(0)$ is not, and therefore they do not coincide.

We remark that the foliation $\mathcal{L}$ extends to a germ of singular foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ given locally by the 1-form $\alpha = \dfrac{df}{f}$.

Remark 4.1. Let M be a germ of real-analytic Levi-flat hypersurface at $(\mathbb{C}^n, 0)$, $n \geq 3$, $\mathcal{L}$ its Levi foliation on M^*. Assume that one of the following situation happens:

(a) $dim(M_s) = 2n - 4$ and the origin is non-dicritical.

(b) $dim(M_s) < 2n - 4$.

Then we may apply Theorem 4.1. The foliation $\mathcal{L}$ extends to a holomorphic foliation $\mathcal{F}$ in a neighborhood around the origin. Theorem 1.6 gives us a germ of non-constant meromorphic first integral

$$h = h_1^{\lambda_1} \cdots h_p^{\lambda_p}$$

for $\mathcal{F}$. However, we may not guarantee that this meromorphic function h has connected level sets, due to the lack of control over its powers λ_i, so that we cannot apply Theorem 4.4.

The problem below is related to the investigation of coincidence the of the leaves of the Levi foliation of a real-analytic Levi-flat hypersurface with real level sets of functions in the dicritical case. So far, no general answer for it has been obtained, even in low dimensions.

Open question: Let M be a germ of irreducible real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$. Suppose that 0 is a dicritical singularity of M. Under what conditions does the Levi foliation $\mathcal{L}$ of M^* extend to a neighborhood of $0 \in \mathbb{C}^2$?

APPENDIX A – Holonomy calculations

A.1 Holonomy in the quasihomogeneous case

Recall that the Levi foliation is given by (2.4):

$$\alpha_1 = Q(x_1,y_1)\left[\left(\frac{mz_1}{x_1} - qx_1^{q-1}z_1\sum_{\ell=1}^{k}\frac{\lambda_\ell}{Q_\ell(x_1,y_1)}\right)dx_1 + \left(\frac{nz_1}{y_1} + py_1^{p-1}z_1\sum_{\ell=1}^{k}\frac{1}{Q_\ell(x_1,y_1)}\right)dy_1 + (pm+qn+pqk)\,dz_1\right] + z_1\theta_1$$

where

$$Q(x_1,y_1) = \mu x_1^m y_1^n \prod_{\ell=1}^{k} Q_\ell(x_1,y_1)$$

and, for each ℓ,

$$Q(x_1,y_1) = y_1^p - \lambda_\ell x_1^q,$$

where $p,q \in \mathbb{Z}_+^*$, $m,n \in \{0,1\}$, $\lambda_\ell \in \mathbb{C}^*$ for $\ell = 1,...,k$ and $\gcd(p,q) = 1$. Also recall that $a = p$, $b = q$.

We have four distinct cases to consider:

- $m = 1$, $n = 1$. In this case,

$$Q(x,y) = xy\prod_{\ell=1}^{k} Q_\ell(x,y),$$

where $Q_\ell(x,y) = (y^p - \lambda_\ell x^q)$, $\gcd(p,q) = 1$.

The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ splits as the union of the following connected components:

$$C_{rs}^{\ell} = \{z_1 = Q_\ell(x_1,y_1) = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

$$C_{rs}^{x_1} = \{z_1 = x_1 = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

$$C_{rs}^{y_1} = \{z_1 = y_1 = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1, ..., k\}$ and $r \in \{1, ..., p\}$ and for each r, $\varepsilon_p^{(r)}(\lambda_s)$ is an p-th root of λ_s. According to [33], the fundamental group $\pi_1(S, q_0)$ may be written in terms of generators and its relations as

$$\pi_1(S, q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs}, \xi_{rs}, \tau_{rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^q \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where, for each ℓ, r, s, the elements $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around the connected component C_{rs}^{ℓ} of $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$, ξ_{rs} are loops around $C_{rs}^{x_1}$ and τ_{rs} a loop around $C_{rs}^{y_1}$.

Take the following parametrizations for these loops:

$$\begin{aligned}
\gamma_{\ell rs} &= (e^{2\pi it}, 0, 0, \varepsilon_p^{(r)}(\lambda_s)),\\
\delta_{\ell rs} &= (e^{\frac{2p\pi it}{q}}, 0, 0, \varepsilon_p^{(r)}(\lambda_s)),\\
\xi_{rs} &= (e^{2p\pi it}, e^{2q\pi it}, 0, \varepsilon_p^{(r)}(\lambda_s)),\\
\tau_{rs} &= (0, e^{2\pi it}, 0, \varepsilon_p^{(r)}(\lambda_s)).
\end{aligned}.$$

If G is the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ, then

$$G = \langle f_{\ell rs}, g_{\ell rs}, h_{rs}, k_{rs} \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$, h_{rs} and k_{rs} correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$, ξ_{rs}, τ_{rs} in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, $\Xi_{rs}(t)$, $\Upsilon_{rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M_{\mathbb{C}}^*} = 0$ (see (2.4)). These conditions guarantee the existence of functions $\Gamma(z_1, t)$, $\Delta(z_1, t)$, $\Xi(z_1, t)$, $\Upsilon(z_1, t)$ such that the parametrizations of those liftings may be written as

$$\begin{aligned}
\Gamma_{\ell rs}(z_1, t) &= (e^{2\pi it}, 0, \Gamma(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),\\
\Delta_{\ell rs}(z_1, t) &= (e^{\frac{2p\pi it}{q}}, 0, \Delta(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),\\
\Xi_{rs}(z_1, t) &= (e^{2p\pi it}, e^{2q\pi it}, \Xi(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),\\
\Upsilon_{rs}(z_1, t) &= (0, e^{2\pi it}, \Upsilon(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),
\end{aligned}$$

where

$$\Gamma(0,t)=0,\Gamma(z_1,0)=z_1,$$

$$\Delta(0,t)=0,\Delta(z_1,0)=z_1,$$

$$\Xi(0,t)=0,\Xi(z_1,0)=z_1,$$

$$\Upsilon(0,t)=0,\Upsilon(z_1,0)=z_1.$$

Let the Taylor series of Γ be $\Gamma(z_1,t)=\sum_{k=1}^{\infty}\gamma_k(t)z_1^k$. Since $\Gamma_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Gamma}{e^{2\pi it}}+qe^{2(q-1)\pi it}\Gamma\frac{k}{e^{2q\pi it}}\right)2\pi ie^{2\pi it}+\left(p+q+pqe^{2q\pi it}\frac{k}{e^{2q\pi it}}\right)\Gamma'=0,$$

which yields

$$\Gamma'=-2\left(\frac{1+qk}{p+q+pqk}\pi i\right)\Gamma.$$

Using the Taylor series for Γ, along with the condition $\gamma_1(0)=1$, we have

$$\gamma_1(t)=e^{-\frac{2(1+qk)}{p+q+pqk}\pi it}.$$

The holonomy mapping is given by $f_{\ell rs}(z_1)=\Gamma_{\ell rs}(z_1,1)$, which gives us

$$f_{\ell rs}(z_1)=e^{-\frac{2(1+qk)}{p+q+pqk}\pi i}z_1+...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$f'_{\ell rs}(0)=e^{-\frac{2(1+qk)}{p+q+pqk}\pi i}.$$

Let the Taylor series of Δ be $\Delta(z_1,t)=\sum_{k=1}^{\infty}\delta_k(t)z_1^k$. Since $\Delta_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Delta}{e^{\frac{2p\pi it}{q}}}+qe^{\frac{2p(q-1)\pi it}{q}}\Delta\frac{k}{e^{2p\pi it}}\right)2\pi ie^{2\pi it}+\left(p+q+pqe^{2\pi it}\frac{k}{e^{2p\pi it}}\right)\Delta'=0.$$

which yields

$$\Delta'=-\frac{2}{q}\left(\frac{p+pqk}{p+q+pqk}\right)\pi i\Delta.$$

Using the Taylor series for Δ, along with the condition $\delta_1(0) = 1$, we have

$$\delta_1(t) = e^{-\frac{2}{q}\left(\frac{p+pqk}{p+q+pqk}\right)\pi it}.$$

The holonomy mapping is given by $g_{\ell rs}(z_1) = \Delta_{\ell rs}(z_1, 1)$, which gives us

$$g_{\ell rs}(z_1) = e^{-\frac{2}{q}\left(\frac{p+pqk}{p+q+pqk}\right)\pi it} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$g'_{\ell rs}(0) = e^{-\frac{2}{q}\left(\frac{p+pqk}{p+q+pqk}\right)\pi i}.$$

Let the Taylor series of Ξ be $\Xi(z_1, t) = \sum_{k=1}^{\infty} \xi_k(t) z_1^k$. Since $\Xi_{rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\begin{aligned}
&\left(\frac{\Xi}{e^{2p\pi it}} - qe^{2p(q-1)\pi it}\Xi \sum_{\ell=1}^{k} \frac{\lambda_\ell}{e^{2pq\pi it}(1-\lambda_\ell)}\right) 2p\pi i e^{2p\pi it} \\
&+ \left(\frac{\Xi}{e^{2q\pi it}} + pe^{2q(p-1)\pi it}\Xi \sum_{\ell=1}^{k} \frac{1}{e^{2qp\pi it}(1-\lambda_\ell}\right) 2q\pi i e^{2q\pi it} \\
&+ \left(p + q - pqe^{2qp\pi it} \sum_{\ell=1}^{k} \frac{\lambda_\ell}{e^{2pq\pi it}(1-\lambda_\ell)} + pqe^{2pq\pi it} \sum_{\ell=1}^{k} \frac{1}{e^{2pq\pi it}(1-\lambda_\ell)}\right) \Xi' = 0
\end{aligned}$$

which yields

$$\Xi' = -2\pi i \Xi.$$

Using the Taylor series for Ξ, along with the condition $\xi_1(0) = 1$, we have

$$\xi_1(t) = e^{-2\pi it}.$$

The holonomy mapping is given by $h_{rs}(z_1) = \Xi_{rs}(z_1, 1)$, which gives us

$$h_{rs}(z_1) = e^{-2\pi it} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$h'_{rs}(0) = e^{-2\pi i} = 1.$$

Let the Taylor series of Υ be $\Upsilon(z_1,t)=\sum_{k=1}^{\infty}\tau_k(t)z_1^k$. Since $\Upsilon_{rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Upsilon}{e^{2\pi it}}+pe^{2(p-1)\pi it}\Upsilon\frac{k}{e^{2p\pi it}}\right)2\pi ie^{2\pi it}+\left(p+q+pqe^{2p\pi it}\frac{k}{e^{2p\pi it}}\right)\Upsilon'=0$$

which yields

$$\Upsilon'=-2\left(\frac{1+pk}{p+q+pqk}\right)\pi i\Upsilon.$$

Using the Taylor series for Υ, along with the condition $\tau_1(0)=1$, we have

$$\tau_1(t)=e^{-2\left(\frac{1+pk}{p+q+pqk}\right)\pi it}.$$

The holonomy mapping is given by $k_{rs}(z_1)=\Upsilon_{rs}(z_1,1)$, which gives us

$$k_{rs}(z_1)=e^{-2\left(\frac{1+pk}{p+q+pqk}\right)\pi it}z_1+...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$k'_{rs}(0)=e^{-2\left(\frac{1+pk}{p+q+pqk}\right)\pi i}.$$

- $m=0,n=1$. In this case,

$$Q(x,y)=y\prod_{\ell=1}^{k}Q_\ell(x,y),$$

where $Q_\ell(x,y)=(y^p-\lambda_\ell x^q)$, $\gcd(p,q)=1$.

The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}})\cap\tilde{U}_3$ splits as the union of the following connected components:

$$C_{rs}^{\ell}=\{z_1=Q_\ell(x_1,y_1)=w_1-\varepsilon_p^{(r)}(\lambda_s)=0\}/\mathbb{Z}_a,$$

$$C_{rs}^{y_1}=\{z_1=y_1=w_1-\varepsilon_p^{(r)}(\lambda_s)=0\}/\mathbb{Z}_a,$$

where $s,\ell\in\{1,...,k\}$, $r\in\{1,...,p\}$ and, for each r, $\varepsilon_p^{(r)}(\lambda_s)$ is a p-th root of λ_s. The group $\pi_1(S,q_0)$ is written in terms of generators and its relations as

$$\pi_1(S,q_0)=\langle\gamma_{\ell rs},\delta_{\ell rs},\tau_{rs}:\gamma_{\ell rs}^p=\delta_{\ell rs}^q\rangle_{\substack{\ell,s=1,...,k\\ r=1,...,p}}$$

where, for each ℓ, r, s, $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around C^{ℓ}_{rs} and τ_{rs} a loop around $C^{y_1}_{rs}$.

Take the following parametrizations for these loops:

$$\begin{aligned}
\gamma_{\ell rs} &= (0, e^{2\frac{q}{p}\pi it}, 0, \varepsilon_p^{(r)}(\lambda_s)),\\
\delta_{\ell rs} &= (0, e^{2\pi it}, 0, \varepsilon_p^{(r)}(\lambda_s)),\\
\tau_{rs} &= (e^{2p\pi it}, e^{2q\pi it}, 0, \varepsilon_p^{(r)}(\lambda_s)).
\end{aligned}$$

If G is the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ then

$$G = \langle f_{\ell rs}, g_{\ell rs}, k_{rs}\rangle_{\substack{\ell, s = 1, \ldots, k\\ r = 1, \ldots, p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$ and k_{rs} correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$, τ_{rs} in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, $\Upsilon_{rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M^*_{\mathbb{C}}} = 0$ (see for instance (2.4)). These conditions guarantee the existence of functions $\Gamma(z_1, t)$, $\Delta(z_1, t)$, $\Upsilon(z_1, t)$ such that the parametrizations of those liftings may be written as

$$\begin{aligned}
\Gamma_{\ell rs}(z_1, t) &= (0, e^{2\frac{q}{p}\pi it}, \Gamma(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),\\
\Delta_{\ell rs}(z_1, t) &= (0, e^{2\pi it}, \Delta(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),\\
\Upsilon_{rs}(z_1, t) &= (e^{2p\pi it}, e^{2q\pi it}, \Upsilon(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma(0, t) &= 0, \Gamma(z_1, 0) = z_1,\\
\Delta(0, t) &= 0, \Delta(z_1, 0) = z_1,\\
\Upsilon(0, t) &= 0, \Upsilon(z_1, 0) = z_1.
\end{aligned}$$

Let the Taylor series of Γ be $\Gamma(z_1, t) = \sum_{k=1}^{\infty} \gamma_k(t) z_1^k$. Since $\Gamma_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Gamma}{e^{\frac{2q\pi it}{p}}} + pe^{\frac{2q(p-1)\pi it}{p}}\Gamma\frac{k}{e^{2q\pi it}}\right)\frac{2q\pi i}{p}e^{\frac{2q\pi it}{p}} + \left(p + pqe^{2q\pi it}\frac{k}{e^{2q\pi it}}\right)\Gamma' = 0,$$

which yields

$$\Gamma' = -\frac{2\pi i}{p}\Gamma.$$

Using the Taylor series for Γ, along with the condition $\gamma_1(0) = 1$, we have

$$\gamma_1(t) = e^{-\frac{2\pi i}{p}t}.$$

The holonomy mapping is given by $f_{\ell rs}(z_1) = \Gamma_{\ell rs}(z_1, 1)$, which gives us

$$f_{\ell rs}(z_1) = e^{-\frac{2\pi i}{p}} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$f'_{\ell rs}(0) = e^{-\frac{2\pi i}{p}}.$$

Let the Taylor series of Δ be $\Delta(z_1, t) = \sum_{k=1}^{\infty} \delta_k(t) z_1^k$. Since $\Delta_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Delta}{e^{2\pi it}} + qe^{2(p-1)\pi it}\Delta\frac{k}{e^{2p\pi it}}\right)2\pi ie^{2\pi it} + \left(q + pqe^{2p\pi it}\frac{k}{e^{2p\pi it}}\right)\Delta' = 0,$$

which yields

$$\Delta' = -\frac{2\pi i}{q}\Delta.$$

Using the Taylor series for Δ, along with the condition $\delta_1(0) = 1$, we have

$$\delta_1(t) = e^{-\frac{2\pi i}{q}t}.$$

The holonomy mapping is given by $g_{\ell rs}(z_1) = \Delta_{\ell rs}(z_1, 1)$, which gives us

$$g_{\ell rs}(z_1) = e^{-\frac{2\pi i}{q}t} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$g'_{\ell rs}(0) = e^{-\frac{2\pi i}{q}}.$$

Let the Taylor series of Υ be $\Upsilon(z_1,t)=\sum_{k=1}^{\infty}\tau_k(t)z_1^k$. Since $\Upsilon_{rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\begin{aligned}&\left(-qe^{2p(q-1)\pi it}\Upsilon\sum_{\ell=1}^{k}\frac{\lambda_\ell}{e^{2pq\pi it}(1-\lambda_\ell)}\right)2p\pi ie^{2p\pi it}\\&+\left(\frac{\Upsilon}{e^{2q\pi it}}+pe^{2q(p-1)\pi it}\Upsilon\sum_{\ell=1}^{k}\frac{1}{e^{2pq\pi it}(1-\lambda_\ell)}\right)2q\pi ie^{2q\pi it}\\&+\left(q-pqe^{2pq\pi it}\sum_{\ell=1}^{k}\frac{\lambda_\ell}{e^{2pq\pi it}(1-\lambda_\ell)}+pqe^{2pq\pi it}\sum_{\ell=1}^{k}\frac{1}{e^{2pq\pi it}(1-\lambda_\ell)}\right)\Upsilon'=0\end{aligned}$$

which yields

$$\Upsilon'=-2\pi i\Upsilon.$$

Using the Taylor series for Υ, along with the condition $\tau_1(0)=1$, we have

$$\tau_1(t)=e^{-2\pi it}.$$

The holonomy mapping is given by $k_{rs}(z_1)=\Upsilon_{rs}(z_1,1)$, which gives us

$$k_{rs}(z_1)=e^{-2\pi it}z_1+...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$k'_{rs}(0)=e^{-2\pi i}=1.$$

- $m=1,n=0$. In this case,

$$Q(x,y)=x\prod_{\ell=1}^{k}Q_\ell(x,y),$$

where $Q_\ell(x,y)=(y^p-\lambda_\ell x^q)$, $\gcd(p,q)=1$.

The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}})\cap\tilde{U}_3$ splits as the union of the following connected components:

$$C_{rs}^{\ell}=\{z_1=Q_\ell(x_1,y_1)=w_1-\varepsilon_p^{(r)}(\lambda_s)=0\}/\mathbb{Z}_a,$$

$$C_{rs}^{y_1}=\{z_1=x_1=w_1-\varepsilon_p^{(r)}(\lambda_s)=0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1, ..., k\}$, $r \in \{1, ..., p\}$ and, for each r, $\varepsilon_p^{(r)}(\lambda_s)$ is a p-th root of λ_s. The group $\pi_1(S, q_0)$ is written in terms of generators and its relations as

$$\pi_1(S, q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs}, \xi_{rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^q \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where, for each ℓ, r, s, $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around C_{rs}^{ℓ} and ξ_{rs} a loop around $C_{rs}^{x_1}$.

Take the following parametrizations for these loops:

$$\begin{aligned} \gamma_{\ell rs} &= (e^{\frac{2p\pi it}{q}}, 0, 0, \varepsilon_p^{(r)}(\lambda_s)), \\ \delta_{\ell rs} &= (e^{2\pi it}, 0, 0, \varepsilon_p^{(r)}(\lambda_s)), \\ \xi_{rs} &= (e^{2p\pi it}, e^{2q\pi it}, 0, \varepsilon_p^{(r)}(\lambda_s)). \end{aligned}$$

If G is the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ then

$$G = \langle f_{\ell rs}, g_{\ell rs}, h_{rs} \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$ and h_{rs} correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$, ξ_{rs} in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, $\Xi_{rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M_{\mathbb{C}}^*} = 0$ (see (2.4)). These conditions guarantee the existence of functions $\Gamma(z_1, t)$, $\Delta(z_1, t)$, $\Xi(z_1, t)$ such that the parametrizations of those liftings may be written as

$$\begin{aligned} \Gamma_{\ell rs}(z_1, t) &= (e^{\frac{2p\pi it}{q}}, 0, \Gamma(z_1, t), \varepsilon_p^{(r)}(\lambda_s)), \\ \Delta_{\ell rs}(z_1, t) &= (e^{2\pi it}, 0, \Delta(z_1, t), \varepsilon_p^{(r)}(\lambda_s)), \\ \Xi_{rs}(z_1, t) &= (e^{2p\pi it}, e^{2q\pi it}, \Xi(z_1, t), \varepsilon_p^{(r)}(\lambda_s)), \end{aligned}$$

where

$$\begin{aligned} &\Gamma(0, t) = 0, \Gamma(z_1, 0) = z_1, \\ &\Delta(0, t) = 0, \Delta(z_1, 0) = z_1, \\ &\Xi(0, t) = 0, \Xi(z_1, 0) = z_1, \end{aligned}$$

Let the Taylor series of Γ be $\Gamma(z_1,t) = \sum_{k=1}^{\infty} \gamma_k(t) z_1^k$. Since $\Gamma_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Gamma}{e^{\frac{2p\pi it}{q}}} + qe^{\frac{2p(q-1)\pi it}{q}} \Gamma \frac{k}{e^{2p\pi it}} \right) \frac{2p\pi i}{q} e^{\frac{2p\pi it}{q}} + \left(p + pqe^{2p\pi it} \frac{k}{e^{2p\pi it}} \right) \Gamma' = 0,$$

which yields

$$\Gamma' = -\frac{2\pi i}{q}\Gamma.$$

Using the Taylor series for Γ, along with the condition $\gamma_1(0) = 1$, we have

$$\gamma_1(t) = e^{-\frac{2\pi i}{q}t}.$$

The holonomy mapping is given by $f_{\ell rs}(z_1) = \Gamma_{\ell rs}(z_1, 1)$, which gives us

$$f_{\ell rs}(z_1) = e^{-\frac{2\pi i}{q}} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$f'_{\ell rs}(0) = e^{-\frac{2\pi i}{q}}.$$

Let the Taylor series of Δ be $\Delta(z_1,t) = \sum_{k=1}^{\infty} \delta_k(t) z_1^k$. Since $\Delta_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Delta}{e^{2\pi it}} + qe^{2(q-1)\pi it} \Delta \frac{k}{e^{2q\pi it}} \right) 2\pi i e^{2\pi it} + \left(p + pqe^{2q\pi it} \frac{k}{e^{2q\pi it}} \right) \Delta' = 0,$$

which yields

$$\Delta' = -\frac{2\pi i}{p}\Delta.$$

Using the Taylor series for Δ, along with the condition $\delta_1(0) = 1$, we have

$$\delta_1(t) = e^{-\frac{2\pi i}{p}t}.$$

The holonomy mapping is given by $g_{\ell rs}(z_1) = \Delta_{\ell rs}(z_1, 1)$, which gives us

$$g_{\ell rs}(z_1) = e^{-\frac{2\pi i}{p}t} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$g'_{\ell rs}(0) = e^{-\frac{2\pi i}{p}}.$$

Let the Taylor series of Ξ be $\Xi(z_1, t) = \sum_{k=1}^{\infty} \xi_k(t) z_1^k$. Since $\Upsilon_{rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(\frac{\Xi}{e^{2p\pi it}} - qe^{2p(q-1)\pi it}\Xi\sum_{\ell=1}^{k}\frac{\lambda_\ell}{e^{2pq\pi it}(1-\lambda_\ell)}\right)2p\pi ie^{2p\pi it}$$
$$+\left(pe^{2q(p-1)\pi it}\Xi\sum_{\ell=1}^{k}\frac{1}{e^{2pq\pi it}(1-\lambda_\ell)}\right)2q\pi ie^{2q\pi it}$$
$$+\left(p - pqe^{2pq\pi it}\sum_{\ell=1}^{k}\frac{\lambda_\ell}{e^{2pq\pi it}(1-\lambda_\ell)} + pqe^{2pq\pi it}\sum_{\ell=1}^{k}\frac{1}{e^{2pq\pi it}}\right)\Xi' = 0$$

which yields

$$\Xi' = -2\pi i\Xi.$$

Using the Taylor series for Ξ, along with the condition $\xi_1(0) = 1$, we have

$$\xi_1(t) = e^{-2\pi it}.$$

The holonomy mapping is given by $h_{rs}(z_1) = \Xi_{rs}(z_1, 1)$, which gives us

$$h_{rs}(z_1) = e^{-2\pi it}z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$h'_{rs}(0) = e^{-2\pi i} = 1.$$

- $m = 0, n = 0$. In this case, $Q(x, y) = \prod_{\ell=1}^{k} Q_\ell(x, y)$, where $Q_\ell = (y^p - \lambda_\ell x^q)$, $\gcd(p, q) = 1$. The set $\mathsf{Sing}(\tilde{\mathcal{L}}_{\mathbb{C}}) \cap \tilde{U}_3$ splits as the union of the following connected components:

$$C^\ell_{rs} = \{z_1 = Q_\ell(x_1, y_1) = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0\}/\mathbb{Z}_a,$$

where $s, \ell \in \{1, ..., k\}$, $r \in \{1, ..., p\}$ and, for each r, $\varepsilon_p^{(r)}(\lambda_s)$ is a p-th root of λ_s. The group $\pi_1(S, q_0)$ is written in terms of generators and its relations as

$$\pi_1(S, q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^q \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where, for each ℓ, r, s, $\gamma_{\ell rs}$ and $\delta_{\ell rs}$ are loops around C_{rs}^ℓ.

Take the following parametrizations for these loops:

$$\gamma_{\ell rs} = (e^{\frac{2p\pi it}{q}}, 0, 0, \varepsilon_p^{(r)}(\lambda_s)),$$
$$\delta_{\ell rs} = (e^{2\pi it}, 0, 0, \varepsilon_p^{(r)}(\lambda_s)).$$

If G is the holonomy of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$ in the section Σ then

$$G = \langle f_{\ell rs}, g_{\ell rs} \rangle_{\substack{\ell, s = 1, ..., k \\ r = 1, ..., p}}$$

where $f_{\ell rs}$, $g_{\ell rs}$ correspond to the equivalence classes of the loops $\gamma_{\ell rs}$, $\delta_{\ell rs}$ in $\pi_1(S, q_0)$, respectively. Each one of these loops lifts up to $\Gamma_{\ell rs}(t)$, $\Delta_{\ell rs}(t)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_1|_{M_{\mathbb{C}}^*} = 0$ (see (2.4)). These conditions guarantee the existence of functions $\Gamma(z_1, t)$, $\Delta(z_1, t)$ such that the parametrizations of those liftings may be written as

$$\Gamma_{\ell rs}(z_1, t) = (e^{\frac{2p\pi it}{q}}, 0, \Gamma(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),$$
$$\Delta_{\ell rs}(z_1, t) = (e^{2\pi it}, 0, \Delta(z_1, t), \varepsilon_p^{(r)}(\lambda_s)),$$

where

$$\Gamma(0, t) = 0, \Gamma(z_1, 0) = z_1,$$
$$\Delta(0, t) = 0, \Delta(z_1, 0) = z_1.$$

Let the Taylor series of Γ be $\Gamma(z_1, t) = \sum_{k=1}^\infty \gamma_k(t) z_1^k$. Since $\Gamma_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(q + e^{\frac{2p\pi it}{q}(q-1)} \Gamma \frac{k}{e^{2p\pi it}}\right) \frac{2p\pi i}{q} e^{\frac{2p\pi it}{q}} + \left(pq e^{2p\pi it} \frac{k}{e^{2p\pi it}}\right) \Gamma' = 0,$$

which yields

$$\Gamma' = -\frac{2\pi i}{q}\Gamma.$$

Using the Taylor series for Γ, along with the condition $\gamma_1(0) = 1$, we have

$$\gamma_1(t) = e^{-\frac{2\pi i}{q}t}.$$

The holonomy mapping is given by $f_{\ell rs}(z_1) = \Gamma_{\ell rs}(z_1, 1)$, which gives us

$$f_{\ell rs}(z_1) = e^{-\frac{2\pi i}{q}} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$f'_{\ell rs}(0) = e^{-\frac{2\pi i}{q}}.$$

Let the Taylor series of Δ be $\Delta(z_1, t) = \sum_{k=1}^{\infty} \delta_k(t) z_1^k$. Since $\Delta_{\ell rs}(t)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 2.4, obtaining:

$$\left(qe^{2(q-1)\pi it}\Delta\frac{k}{e^{2q\pi it}}\right)2\pi ie^{2\pi it} + \left(pqe^{2q\pi it}\frac{k}{e^{2q\pi it}}\right)\Delta' = 0,$$

which yields

$$\Delta' = -\frac{2\pi i}{p}\Delta.$$

Using the Taylor series for Δ, along with the condition $\delta_1(0) = 1$, we have

$$\delta_1(t) = e^{-\frac{2\pi i}{p}t}.$$

The holonomy mapping is given by $g_{\ell rs}(z_1) = \Delta_{\ell rs}(z_1, 1)$, which gives us

$$g_{\ell rs}(z_1) = e^{-\frac{2\pi i}{p}t} z_1 + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$g'_{\ell rs}(0) = e^{-\frac{2\pi i}{p}}.$$

A.2 Holonomy in the volume-preserving case

We take the irreducible component $W_{1,2,2,3}$ of $\mathsf{Sing}(\tilde{M}_{\mathbb{C}}) \cap W$, as in the proof of Proposition 3.4. A similar situation occurs in other charts, due to the symmetry of variables. In the chart U, the exceptional divisor is written as

$$D = D_1 \cup D_2 = \{x_{n+1} = 0\} \cup \{x_{n+3} = 0\}$$

and $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$ is given by $\alpha_2|_{\tilde{\tilde{M}}^*_{\mathbb{C}}} = 0$, where

$$\begin{aligned}\alpha_2 \quad = \quad & x_2 \cdots x_n x_{n+1} x_{n+3} dx_1 + x_1 x_3 \cdots x_n x_{n+1} x_{n+3} dx_2 + \\ & \textstyle\sum_{i=3}^n \dfrac{x_1 \cdots x_n x_{n+1} x_{n+3}}{x_i} dx_i + \\ & n x_1 \cdots x_n x_{n+3} dx_{n+1} + 2 x_1 x_2 \cdots x_n x_{n+1} dx_{n+3} + x_{n+1} x_{n+3} \theta_2,\end{aligned}$$

which allows us to conclude that $D \cap \tilde{\tilde{M}}_{\mathbb{C}}$ is invariant by $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$. The singularities of the foliation $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$ on the exceptional divisor in this chart are given by

$$\mathsf{Sing}\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}} \cap U \cap D = \{x_{n+1} = x_{n+3} = x_1 \cdots x_n = x_{n+2} x_{n+4} \cdots x_{2n} = 0\}.$$

If we define $\mathcal{C}_{i,n+j} = \{x_{n+1} = x_{n+3} = x_i = x_{n+j} = 0\} \simeq \mathbb{C}^{2(n-2)}$, then we can write

$$\mathsf{Sing}\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}} \cap U \cap D = \bigcup_{\substack{1 \le i, j \le n \\ j \neq 1,3}} \mathcal{C}_{i,n+j}.$$

Since $D \cap \tilde{\tilde{M}}_{\mathbb{C}}$ is invariant by $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$, then

$$S := (D \cap \tilde{\tilde{M}}_{\mathbb{C}}) \setminus \mathsf{Sing}\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$$

is a leaf of $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$.

Let G be its holonomy group, and $p_0 \in S$ given by

$$p_0 = (x_1, ..., x_n, x_{n+1}, x_{n+1}, x_{n+3}, x_{n+4}, ..., x_{2n}) = (1, ..., 1, 0, -1, 0, 1, ..., 1).$$

Then

$$G = \langle f_{i,j} \rangle_{\substack{1 \le i \le n \\ j = 2,4,5,...,n}}$$

where $f_{i,j}$ correspond to the equivalence classes of the loop $\delta_{i,j}$. Take Σ the transversal section through p_0 given by

$$\Sigma = \{(1, ..., 1, \lambda, -1, \lambda, 1, ..., 1) | \lambda \in \mathbb{C}\}.$$

Let $\delta_{i,j}(\theta)$ be a loop around $\mathcal{C}_{i,n+j}$, for $1 \leq i \leq n$ and $j = 2, 4, 5, ..., n$, with $\theta \in [0, 1]$. Take the following parametrizations for these loops:

$$\begin{aligned}
\delta_{i,j}(\theta) &= (1, ..., 1, \underbrace{e^{2\pi i\theta}}_{i-th}, 1, ..., \underbrace{1}_{n-th}, 0, -1, 0, 1, ..., 1, \underbrace{e^{2\pi i\theta}}_{(n+j)-th}, 1, ..., 1), \text{ if } j \neq 2 \\
\delta_{i,2}(\theta) &= (1, ..., 1, \underbrace{e^{2\pi i\theta}}_{i-th}, 1, ..., \underbrace{1}_{n-th}, 0, -e^{2\pi i\theta}, 0, 1, ..., 1)
\end{aligned} .$$

Each one of these loops lifts up to $\tilde{\Gamma}_{i,j}(\lambda, \theta)$, respectively, under the condition that each one of these belong on the leaves of $\tilde{\tilde{\mathcal{L}}}_{\mathbb{C}}$ and that this foliation is defined by $\alpha_2|_{\tilde{M}^*_{\mathbb{C}}} = 0$. These conditions guarantee the existence of functions $\Gamma_{i,j}(\lambda, t)$, such that the parametrizations of those liftings may be written as

$$\begin{aligned}
\Gamma_{i,j}(\lambda, t) &= (1, ..., 1, e^{2\pi i\theta}, 1, ..., 1, \Gamma_{i,j}(\lambda, \theta), -1, \Gamma_{i,j}(\lambda, \theta), 1, ..., 1, e^{2\pi i\theta}, 1, ..., 1), \text{if } j \neq 2, \\
\Gamma_{i,2}(\lambda, t) &= (1, ..., 1, e^{2\pi i\theta}, 1, ..., 1, \Gamma_{i,2}(\lambda, \theta), -e^{2\pi i\theta}, \Gamma_{i,2}(\lambda, \theta), 1, ..., 1),
\end{aligned}$$

where

$$\Gamma_{i,j}(0, \theta) = 0, \Gamma_{i,j}(\lambda, 0) = \lambda$$

and its Taylor series is

$$\Gamma_{i,j}(\lambda, \theta) = \sum_{k=1}^{\infty} \delta_k^{i,j}(\theta)\lambda^k$$

for $i = 1, ..., n$ and $j = 2, 4, 5, ..., n$. The holonomy mapping is $h_{\delta_{i,j}}(\lambda) = \Gamma_{i,j}(\lambda, 1)$.

Since $\Gamma_{i,j}(\theta)$ belong on a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$, we may substitute its expression in 3.2, for each $i = 1, ..., n$ and $j = 2, 4, 5, ..., n$, obtaining:

$$\Gamma_{i,j}(\lambda, \theta) 2\pi i e^{2\pi i\theta} d\theta + n e^{2\pi i\theta} \Gamma_{i,j} \frac{\partial \Gamma_{i,j}}{\partial \theta} d\theta + 2e^{2\pi i\theta} \Gamma_{i,j}(\lambda, \theta) \frac{\partial \Gamma_{i,j}}{\partial \theta} d\theta = 0$$

which yields

$$\frac{\partial \Gamma_{i,j}}{\partial \theta} = \frac{-2\pi i}{n+2} \Gamma_{i,j}.$$

Using the Taylor series for $\Gamma_{i,j}$, along with the condition $\delta_1^{i,j}(0) = 1$, we have

$$\delta_1^{i,j}(\theta) = e^{\frac{-2\pi i\theta}{n+2}}.$$

The holonomy mapping is given by $f_{i,j}(\lambda) = \Gamma_{i,j}(\lambda, 1)$, which gives us

$$f_{i,j}(\lambda) = e^{\frac{-2\pi i}{n+2}}\lambda + ...$$

which allow us to conclude that the coefficient of the linear term of this holonomy mapping is

$$f'_{i,j}(0) = e^{\frac{-2\pi i}{n+2}}.$$

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