Categorical and Geometrical Methods in Physics

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Categorical and Geometrical Methods in Physics

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Abstract

In this work we develop the higher categorical language aiming to apply it in the foundations of physics, following an approach based in works of Urs Schreiber, John Baez, Jacob Lurie, Daniel Freed and many other, whose fundamental references are [182, 20, 124, 127, 125]. The text has three parts. In Part I we introduce categorical language with special focus in algebraic topological aspects, and we discuss that it is not abstract enough to give a full description for the foundations of physics. In Part II we introduce the *categorical process*, which produce an abstract language from a concrete language. Examples are given, again focused on Algebraic Topology. In Part III we use the categorification process in order to construct arbitrarily abstract languages, the higher categorical ones, including the cohesive ∞-topos. An emphasis on the formalization of abstract stable homotopy theory is given. We discuss the reason why we should believe that cohesive ∞-topos are natural languages to use in order to attack Hilbert's sixth problem.

Remark. The core of this text was written as lecture notes for minicourses, courses and a large number of seminars given at UFLA and UFMG between the years 2016 and 2018.

Resumo

Neste trabalho, desenvolvemos a linguagem categórica em altas dimensões visando aplicá-la nos fundamentos da física, seguindo uma abordagem baseada em obras de Urs Schreiber, John Baez, Jacob Lurie, Daniel Freed, e muitos outros, cujas referências fundamentais são [182, 20, 124, 127, 125]. O texto possui três partes. Na Parte I, introduzimos a linguagem categórica, com foco especial em aspectos algebro-topológicos, e discutimos que esta linguagem não é abstrata o bastante para fornecer uma descrição completa dos fundamentos da física. Na Parte II, introduzimos o *processo de categorificação*, o qual produz linguagens abstratas a partir de linguagens concretas. Exemplos são dados, novamente focando na Topologia Algébrica. Na Parte III, usamos o processo de categorificação para construir linguagens arbitrariamente abstratas (as linguagens categóricas em altas dimensões), incluindo os ∞-topos coesivos. Um enfoque na formalização da teoria da homotopia estável abstrata é dado. Discutimos a razão pela qual se deveria acreditar que os ∞-topos coesivos são linguagens naturais a serem usadas para atacar o sexto problema de Hilbert.

Remark. O núcleo deste texto foi escrito como notas de aula para minicursos, cursos e vários seminários apresentados entre os anos 2016 e 2018 na UFLA e na UFMG.

Contents

Introduction

There are two kinds of mathematics: *naive* (or *intuitive*) *mathematics* and *axiomatic* (or *rigorous*) *mathematics*. In naive mathematics the fundamental objects are *primitives*, while in axiomatic mathematics they are defined in terms of more elementary structures. For instance, we have *naive set theory* and *axiomatic set theory*. In both cases (naive or axiomatic) we need a *background language* (also called *logic*) in order to develop the theory, as in the diagram below. In the context of set theory this background language is just *classical logic*.

$$\begin{tikzcd} & \textbf{naive math} \\ \textbf{background language} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & \textbf{axiomatic math} \end{tikzcd}$$

Notice that, when a mathematician is working (for instance, when he is trying to prove some new result in his area of research), at first he does not make use of completely rigorous arguments. In fact, he first uses of his intuition, doing some scribbles in pieces of paper or in a blackboard and usually considers many wrong strategies before finally discovering a good sequence of arguments which can be used to prove (or disprove) the desired result. It is only in this later moment that he tries to introduce rigor in his ideas, in order for his result to be communicated and accepted by the other members of the academy.

Thus we can say that in the process of producing new mathematics, *naive arguments come before rigorous ones*. More precisely, we can say that naive mathematics produces conjectures and rigorous mathematics turns these conjectures into theorems. It happens that the same conjecture generally can be proved (or disproved) in different ways, say by using different tools or by considering different models. For example, as will be discussed in some moments in the text, certain results can be proven using algebraic tools as well as geometric tools, reflecting the existence of a duality between algebra and geometry.

$$
\begin{tikzcd}
 & \textbf{naive math} \arrow[r, Rightarrow, "\text{produces}"] \arrow[d, Rightarrow, "\text{model}"] & \textbf{conjectures} \arrow[d, Rightarrow, shift right, "\text{algebra}"'] \arrow[d, Rightarrow, shift left, "\text{geometry}"] \\
\textbf{background language} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & \textbf{axiomatic math} \arrow[r, Rightarrow, "\text{produces}"'] & \textbf{theorems}
\end{tikzcd}
\tag{0.0.1}
$$

When compared with mathematics, *physics is a totally different discipline*. Indeed, in physics we have a restriction on the *existence*, meaning that we have a connection with *ontology*, which is given by *empiricism*. More precisely, *while math is a strictly logical discipline, physics is logical and ontological*. This restriction on existence produces many difficulties. For instance, logical consistence is no longer sufficient in order to establish a given sentence as physically true: ontological consistence is also needed. So, *even though a sentence is logical consistent, in order to be considered physically true it must be consistent with all possible experiments!* This fact can be expressed in terms of a commutative diagram:

$$
\begin{tikzcd}
\textbf{phylosophy} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[d, Rightarrow, "\text{mathematical modelling}"] \\
\textbf{ontology} \arrow[r, Rightarrow, "\text{empiricism}"'] & \textbf{physics}
\end{tikzcd}
$$

But now, recall that logic determines the validity of mathematical arguments, which was also translated in terms of the commutative diagram (0.0.1). So, gluing the diagram of physics with the diagram of math, we have a new commutative diagram:

$$
\begin{tikzcd}
 & \textbf{conjectures} \arrow[r, Rightarrow] & \textbf{theorems} \\
 & \textbf{naive math} \arrow[u, Rightarrow] & \\
\textbf{phylosophy} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & \textbf{axiomatic math} \arrow[uu, Rightarrow] \\
\textbf{ontology} \arrow[r, Rightarrow, "\text{empiricism}"'] & \textbf{physics} &
\end{tikzcd}
$$

The most important fact concerning the relation between physics and mathematics is that, in the above diagram, the arrow

$$
\textbf{logic} \Longrightarrow \textbf{physics} \qquad \text{has an inverse} \qquad \textbf{physics} \Longrightarrow \textbf{logic}\,.
$$

This last is called *physical insight*. The composition of *physical insight* with axiomatic math produces a new arrow, called *mathematical physics*.

$$
\begin{tikzcd}
\textbf{logic} \arrow[r, Rightarrow] & \textbf{axiomatic math} \arrow[dl, Rightarrow, "\text{math. phys.}"] \\
\textbf{physics} \arrow[u, Rightarrow, "\text{insight}"] &
\end{tikzcd}
$$

Therefore, the full relation between physics and mathematics is given by the following diagram

$$
\begin{tikzcd}
 & \textbf{conjectures} \arrow[r, Rightarrow] & \textbf{theorems} \\
 & \textbf{naive math} \arrow[u, Rightarrow] & \\
\textbf{phylosophy} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{logic} \arrow[r, Rightarrow] \arrow[u, Rightarrow] \arrow[d, Leftrightarrow] & \textbf{axiomatic math} \arrow[uu, Rightarrow] \arrow[dl, Rightarrow] \\
\textbf{ontology} \arrow[r, Rightarrow, "\text{empiricism}"'] & \textbf{physics} &
\end{tikzcd}
$$

leading us to the following conclusion:

Conclusion: *we can use physical insight in order to do naive mathematics and, therefore, in order to produce conjectures. These conjectures can eventually be proven, producing theorems! Furthermore, with such theorems on hand we can create mathematical models for physics* (*mathematical physics*).

Now, we can say in few words which is the fundamental objective of this text: *to study some examples of the following sequence*:

$$
\textbf{physics} \Longleftrightarrow \textbf{logic} \Longrightarrow \begin{array}{c}\textbf{naive}\\ \textbf{math}\end{array} \Longrightarrow \textbf{conjectures} \Longrightarrow \textbf{theorems}
$$

Hilbert's Sixth Problem

More honestly, our focus will be on the *axiomatization problem of physics*[1]. In the year of 1900, David Hilbert published [95] a list containing 23 problems (usually known as the *Hilbert's problems*) which in his opinion would strongly influence the 20th century mathematics. The sixth of these problems was about the axiomatization of the whole physics and, presently, it remains partially unsolved. Our aim is to present an approach to this problem following works of Schreiber, Freed, Baez, Lurie and many others.

[1] As will be discussed, there exists another different (but strictly related) problem: the *unification of physics*.

The idea is the following. Recall that the existence of physical insight gives the "mathematical physics" arrow, as below. An approach to Hilbert's sixth problem can then be viewed as a way to present mathematical physics as a **surjective** arrow.

$$\begin{tikzcd}
\textbf{axiomatic math} \arrow[r, Rightarrow, "\text{mathematical}", "\text{physics}"'] & \textbf{physics}
\end{tikzcd}$$

Because axiomatic math is described by some logic, the starting point is to select a proper background language. The selected background language determines directly the **naive** math, so the next step is to analyze the following loops:

$$\begin{tikzcd}
& \textbf{naive math} \arrow[dr, Rightarrow] & \\
\textbf{background language} \arrow[ur, Rightarrow] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}$$

Then, selecting some model, we lift to axiomatic math, as in the first diagram below. The final step is to verify if the corresponding "mathematical physics" arrow really is surjective. In other words, *we have to verify if the axiomatic concepts produced by the selected logic are general enough to model all physical phenomena*.

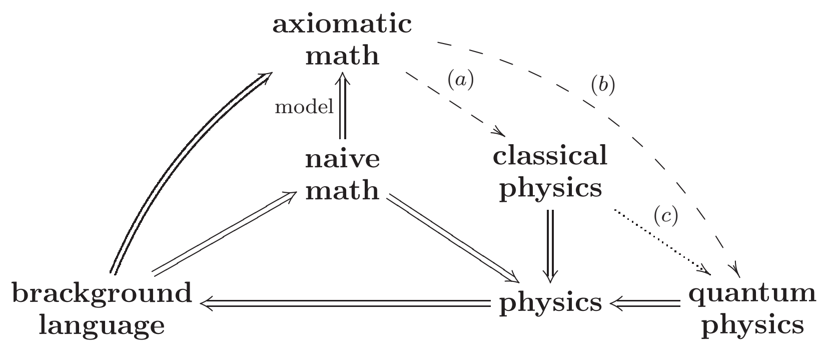
$$\begin{tikzcd}
& \textbf{axiomatic math} & \\
& \textbf{naive math} \arrow[u, Rightarrow, "\text{model}"] \arrow[dr, Rightarrow] & \\
\textbf{background language} \arrow[uur, Rightarrow, bend left] \arrow[ur, Rightarrow] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \textbf{axiomatic math} \arrow[ddr, dashed, bend left] & \\
& \textbf{naive math} \arrow[u, Rightarrow, "\text{model}"] \arrow[dr, Rightarrow] & \\
\textbf{background language} \arrow[uur, Rightarrow, bend left] \arrow[ur, Rightarrow] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}$$

Presently, physical theories can be divided into two classes: the *classical theories* and the *quantum theories*, as below. They are empirically classified by their domain of validity: classical theories describe those phenomena which involve the same scales of our everyday life, while quantum theories are used to describe phenomena appearing in extremal scales.

$$\begin{tikzcd}
\textbf{classical physics} \arrow[d, Rightarrow] & \\
\textbf{physics} & \textbf{quantum physics} \arrow[l, Rightarrow]
\end{tikzcd}$$

Thus, there are essentially two ways to build a surjective "mathematical physics" arrow. Either we build the arrow directly (as in the second diagram above), or we first build surjective arrows

(a) and (b), which respectively axiomatize classical and quantum physics, and then another arrow (c) linking these two axiomatizations, as in the diagram below.



We emphasize the difference between the two approaches: in the first, all physical theories are described by the **same** set of axioms, while in the second classical and quantum theories are described by **different** axioms, but that are related by some process. This means that if we choose the first approach we need to *unify all physical theories*.

There are some models to this unification arrow. One of them is given by *string theory*. It is based on the assumption that the "building blocks" of nature are not particles, but indeed one dimensional entities called *strings*. More precisely, what we call "string" is just a connected one-dimensional manifold, which is diffeomorphic to some interval (when it has boundary or when it is not compact) or to the circle (if it is boundaryless and compact). In the first case we say that we have *open strings*, while in the second we say that we have *closed strings*.

From the mathematical viewpoint, string theory is a very fruitful idea. But we need to recall that physics is not completely determined by the arrow **logic** $\Rightarrow$ **physics**. Indeed, we also have the ontological (i.e, empirical) branch. It happens that presently there is no empirical indicative that strings (instead of particles) really are the most fundamental objects of nature[2].

In this text, we will follow the second approach. Indeed, as we will see, if we start with a sufficiently abstract (or powerful) background language, then we can effectively axiomatize **separately** all interesting classical and quantum theories. In this context, the construction of the quantization process linking classical and quantum theories is presently incomplete. In part this comes from the fact that the underlying background language is itself under construction. Even so, there is a very promising idea, which behaves very well for many interesting cases, known as *motivic/cohomological/pull-push quantization* [182, 159].

Towards The Correct Language

As commented in the last subsection, independently of the approach used in order to attack Hilbert's sixth problem, the starting point is to select a proper background language. The most obvious choice is the *classical logic* used to describe set theory. This logic produces, via the arrow **logic** $\Rightarrow$ **axiomatic math**, *not only set theory, but indeed all classical areas of math, such as group theory, topology, differential geometry, etc!* Therefore, for this choice of background

[2]Need reference.

language, the different known areas of mathematics should be used in order to describe all classical and all quantum physical phenomena.

Since 1900, when Hilbert published his list of problems, many classical and quantum theories were formalized by means of these bye now well known areas of mathematics. Indeed, quantum theories were observed to have more algebraic and probabilistic nature, while classical theories were presumably more geometric in character.

For instance, a system in Quantum Mechanics (which is about quantum **particles**) was formalized[3] as a pair $(\mathcal{H},\hat{H})$, where $\mathcal{H}$ is a complex (separable) Hilbert space and $\hat{H}:D(\hat{H})\rightarrow\mathcal{H}$ is a self-adjoint operator defined in a (dense) subespace $D(\hat{H})\subseteq\mathcal{H}$. We say that $\hat{H}$ is the *Hamiltonian* of the system and the fundamental problem is to the determine its *spectrum*, which is the set of all information about $\hat{H}$ that can be accessed experimentally. The dynamics of the system from a instant t_0 to a instant t_1 is guided by the unitary operator $U(t_1;t_0)=e^{\frac{i}{\hbar}(t_1-t_0)\hat{H}}$ operator associated to $\hat{H}$ or, equivalently, by the (time independent) Schrödinger equation $i\hbar\frac{d\psi}{dt}=\hat{H}\psi$. The probabilistic nature of Quantum Mechanics comes from the fact that the *dynamics does not determine the states of the particles, but only the time evolution of the probabilities associated to them*.

On the other hand, classical theories for **particles** are given by some *action functional* $S:X\rightarrow\mathbb{R}$, defined in some "space of fields". These fields generally involve paths $\gamma:I\rightarrow M$ in a four dimensional M, which are interpreted as the trajectories of particles moving into some "spacetime". It happens that in the typical situations, the particle is generally subjected to interactions, whose effects on the movement of the particle are measured by a corresponding quantity called *force*.

The interactions may (or not) be intrinsic to the spacetime. For instance, since the developments of General Relativity in 1916, gravity is supposed to be a intrinsic force, meaning that it will act on **any** particle. The presence of a intrinsic force is formalized by the assumption of a certain additional geometric structure on the spacetime M. For example, gravity is modeled by a Lorentzian metric g on M. Other intrinsic interactions are modeled by other types of geometric structure. It happens that **not all** manifolds may carry a given geometric structure. This means that **not all** manifolds can be used to model the spacetime. Indeed, each geometric structure exists on a given manifold only if certain quantities, called "obstruction characteristic classes" vanishe. For example, a compact manifold admits a Lorentzian metric iff its Euler characteristic $\chi(M)$ vanishes, which implies that $\mathbb{S}^4$ cannot be used to model the universe[4].

An important class of non-intrinsic interactions are given by the Yang-Mills fields. These depend on a Lie group G, called the *gauge group*, and on a structure called a *G-principal bundle* $P\rightarrow M$ over the spacetime M. The interaction is then modelled by a connection of P, which is just an equivariant $\mathfrak{g}$-valued 1-form $A:TP\rightarrow\mathfrak{g}$. We can think of A as the "potential" of the interaction, so that the "force" is just the (exterior covariant) derivative dA of A. Here, the standard example is electromagnetism, for which G is the abelian group $U(1)$ and A is the eletromagnetic vector potential. We can think of an arbitrary Yang-Mills interaction as some kind of "nonabelian" version of electromanetism.

Remark. There is a very important difference between classical and quantum theories. Be-

[3] Need reference.

[4] For other examples of restrictions to the possible topologies of the universe, see [50, 139]

cause quantum theories are probabilistic, we can only talk about the probability of a certain event occurring in nature. Consequently, all possible configurations of a system can, in principle, be accessed. At the energy level where the classical theories lives, on the other hand, a surprising fact occurs: *only the configurations which minimize the action functional S are observed experimentally!*

Now, let us return to focus on Hilbert's sixth problem. It requires answering for questions like these:

1. *what is a classical theory?*
2. *what is a quantum theory?*
3. *what is quantization?*

Notice that the previous discussion **does not** answer these questions. Indeed, it only reveals some examples/aspects of the classical and quantum theories; it do notes say, axiomatically, what they really are. This leads us to the following conclusion:

Conclusion: *classical logic, regarded as a background language, is very nice in order to formulate and study properties of a classical or quantum theory isolately. On the other hand, it does not give tools to study more deeper questions as those required by Hilbert's sixth problem.*

The Role of Category Theory

The above conclusion shows that, in order to attack Hilbert's sixth problem, we need to replace classical logic by a more abstract background language. *What kind of properties this new language should have?*

Recall that by making use of classical logic we learn that classical theories are generally described by *geometric notions*, while quantum theories are described by *algebraic and probabilistic tools*. Furthermore, the quantization process should be some kind of process linking classical theories to quantum theories and, therefore, geometric areas to algebraic/probabilistic areas. So, the idea is to search for a language which formalizes the notion of "area of mathematics" and the notion of "map between two areas".

This language actually exists: it is *categorical language*. In categorical language, an area of mathematics is determined by specifying which are the objects of interest, which are the mappings between these objects and which are the possible ways to compose two given mappings. We say that this data defines a *category*. The link between two areas of mathematics, say described by categories $\mathbf{C}$ and $\mathbf{D}$, is formalized by the notion of *functor*. This is given by a rule $F : \mathbf{C} \to \mathbf{D}$ assigning objects into objects and mappings into mappings in such a way that compositions are preserved.

Notice that we have a category **Set**, describing set theory, whose fundamental objects are sets, whose mappings are just functions between sets and whose composition laws are the usual compositions between functions. That all classical areas of mathematics can be described by categorical language comes from the the fact that in each of them the fundamental objects are just *sets endowed with some further structure*, while the mappings are precisely the f*unctions between the underlying sets which preserve this additional structure*. For instance, Linear Algebra is the area of mathematics which study vector spaces and linear transformations. But a vector space is just a set endowed with a linear structure, while a linear transformation is just a function preserving the linear structure.

Therefore, each classical area of math defines a category $\mathbf{C}$ equipped with a "inclusion functor" $\imath : \mathbf{C} \hookrightarrow \mathbf{Set}$ which only forgets all additional structures (in the context of Linear Algebra, this functor forgets the linear structure). The categories which can be included into $\mathbf{Set}$ are called *concretes*. So, *in order to recover classical logic from categorical language it is enough to restrict to the class of concrete categories.*

The fact that categorical language is really more abstract that classical logic comes from the existence of *non-concrete* (also called *abstract*) categories. There are many of them. For instance, given a natural number p, we can build a category $\mathbf{Cob}_{p+1}$ whose objects are p-dimensional smooth manifolds and whose mappings $\Sigma : M \to N$ are *cobordisms* between them, i.e, $(p+1)$-manifolds Σ such that $\partial\Sigma = M \sqcup N$. The abstractness of this category is due to the fact that the mappings are not functions satisfying some condition, but actually manifolds.

Example. If $p = 0$, the objects of $\mathbf{Cob}_1$ are just 0-manifolds: finite collection of points. The cobordisms between them are 1-manifolds having these 0-manifolds as boundaries. In other words, the morphisms are just disjoint unions of intervals, while the composition between intervals $[t_0; t_1]$ and $[t_1; t_2]$ is the interval $[t_0; t_1]$.

With categorical language on hand, let us try to attack Hilbert's sixth problem. We start by recalling that the dynamics of a system in Quantum Mechanics is guided by the time evolution operators $U(t_1; t_0) = e^{\frac{i}{\hbar}(t_1 - t_0)\hat{H}}$. Notice that when varying t_0 and t_1, all corresponding operators can be regarded as a unique functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$, where $\mathbf{Vec}_{\mathbb{C}}$ denotes the category delimiting complex linear algebra (i.e, it is the category of complex vector spaces and linear transformations). Such a functor assigns to any instant t_0 a complex vector space $U(t) = \mathcal{H}_t$ and to any interval $[t_0; t_1]$ an operator $U(t_1; t_0) : \mathcal{H}_0 \to \mathcal{H}_1$.

At this point, the careful reader could do some remarks:

1. as commented previously, a system in Quantum Mechanics is defined by a pair $(\mathcal{H}, \hat{H})$, where we have **only one** space $\mathcal{H}$ which does not depend on time, so that for any interval $[t_0; t_1]$ the time evolution operator $U(t_1; t_0)$ is defined in the **same** space. On the other hand, for a functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$ we have a space $\mathcal{H}_t$ for each instant of time and, therefore, for any interval the corresponding operators are defined on **different** spaces;

2. in Quantum Mechanics, the evolution is guided not by an arbitrary operator, but by a **unitary** operator. Furthermore, in Quantum Mechanics the space $\mathcal{H}$ for systems describing more than one (say k) particles decomposes as a tensor product $\mathcal{H}^1 \otimes ... \otimes \mathcal{H}^k$. It happens that both conditions **are not** contained in the data defining a functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$.

About the first remark, notice that if there is a interval $[t_0; t_1]$ connecting two different time instants t_0 and t_1, then there is a inverse interval $[t_0; t_1]^{-1}$ such that, when composed (in $\mathbf{Cob}_1$) with the original interval, we get precisely the trivial interval. In fact, the inverse is obtained simply by flowing the time in the inverse direction. In more brief terms, any morphism in the category $\mathbf{Cob}_1$ is indeed an isomorphism. But functors preserve isomorphisms, so that for any internal $[t_0; t_1]$ the corresponding spaces $\mathcal{H}_{t_0} = U(t_0)$ and $\mathcal{H}_{t_1} = U(t_1)$ are isomorphic. This means that the spaces $\mathcal{H}_t$ **do not** dependent on the time: up to isomorphisms they are all the same.

On the second remark, let us say that there are good reasons to believe that the unitarity of the time evolution does not should be **required** as a fundamental axiom of the true fundamental physics, but instead it indeed should emerge as a consequence (or as an additional assumption) of the correct axioms. For instance, in Quantum Mechanics itself the unitarity of $U(t_0;t_1)$ can be viewed as a consequence of the Schrödinger equation, because its solutions involve the exponential of a hermitean operators which is automatically unitary[5].

On the other hand, it is really true that an arbitrary functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$ does not take into account the fact that in a system with more that one particle the total space of states decomposes as a tensor product of the state of spaces of each particle. In order to incorporate this condition, notice that when we say that we have a system of two independent particles, we are saying that time intervals corresponding to their time evolution are **disjoint**. So, we can interpret the time evolution of a system with many particles as a disconnected one dimension manifold, i.e *as a disconnected morphism of* $\mathbf{Cob}_1$. Consequently, the required condition on the space of states can be obtained by imposing the properties

$$U(t \sqcup t') \simeq U(t) \otimes U(t') \quad \text{and} \quad U(\varnothing) \simeq \mathbb{C},$$

where the second condition only means that a system with zero particles must have a trivial space of states.

Now, notice that both categories $\mathbf{Cob}_1$ and $\mathbf{Vec}_{\mathbb{C}}$ are equipped with operations (respectively given by $\sqcup$ and $\otimes$), which are associative and commutative up do isomorphisms, together with distinguished objects (given by $\varnothing$ and $\mathbb{C}$), which behaves as "neutral elements" for these operations. A category with this kind of structure is called a *symmetric monoidal category*. A functor between monoidal categories which preserves the operations and the distinguished object is called a *monoidal functor*. Therefore, this discussion leads us to the following conclusion:

Conclusion. *A system in Quantum Mechanics is a special flavor of a monoidal functor from the category* $\mathbf{Cob}_1$ *of 1-dimensional cobordisms to the category* $\mathbf{Vec}_{\mathbb{C}}$ *of complex vector spaces.*

Consequently, with our eyes on Hilbert's sixth problem we can use the above characterization in order to **axiomatize** a quantum theory of **particles** as being an *arbitrary monoidal functor*

$$U : (\mathbf{Cob}_1, \sqcup, \varnothing) \to (\mathbf{C}, \otimes, 1)$$

taking values in some symmetric monoidal category. A natural question is on the viability of using the same kind of argument in order to get an axiomatization of the classical theories of **particles**. This really can be done, as we will outline.

We start by recalling that a classical theory of **particles** is given by an action functional $S : X \to \mathbb{R}$, defined into some "space of fields". Therefore, the first step is to axiomatize the notion of "space of fields". In order to do this, recall that it is generally composed by smooth paths $\gamma : I \to M$, representing the trajectories of the particle into some spacetime M, and by the "interacting fields", corresponding to configurations of some kind of geometric structure put in M.

[5] There are more fundamental reasons involving the possibility of topology change in quantum gravity. See [18] for an interesting discussion.

The canonical examples of "interacting fields" are metrics (describing gravitational interaction) and connections over bundles (describing gauge interactions).

Now we ask: *which properties smooth functions, metrics and connections have in common?* The answer is: *locality*. In fact, in order to conclude that a map $\gamma : I \to M$ is smooth, it is enough to analyze it relative to a open cover $U_i \hookrightarrow M$ given by coordinate charts $\varphi_i : U_i \to \mathbb{R}^n$. Similarly, a metric g on M is totally determined by its local components g_{ij}. Finally, it can also be shown that to give a connection $A : TP \to \mathfrak{g}$ is the same as giving a family of 1-forms $A_i : U_i \to \mathfrak{g}$ fulfilling compatibility conditions at the intersections $U_i \cap U_j$.

Therefore, we can say that a "space of fields" over a fixed spacetime M is some kind of set $\mathrm{Fields}(M)$ of structures which are "local" in the sense that, for any cover $U_i \hookrightarrow M$ by coordinate systems, we can reconstruct the total space $\mathrm{Fields}(M)$ from the subset of all $s_i \in \mathrm{Fields}(U_i)$ that are compatible in the intersections $U_i \cap U_j$.

Notice that we are searching for a notion of *space of fields* (without mention of any spacetime), but up to this point we got a notion of *space of fields over a* ***fixed*** *spacetime*. So, the main idea is to consider the rule $M \mapsto \mathrm{Fields}(M)$ and the immediate hypothesis is to suppose that it is functorial. Therefore, we can axiomatize a space of fields as a functor

$$\mathrm{Fields} : \mathbf{Diff}^{op} \to \mathbf{Set}$$

assigning to any manifold a set of local structures. This kind of functor is called a *sheaf on the site of manifolds and coordinate coverings* (or, more succinctly, a *smooth sheaf*).

This is still not the correct notion of "space of fields". Indeed, recall that in typical situations the set $\mathrm{Fields}(M)$ contains geometric structures. It happens that when doing geometry we only consider the entities up to their natural equivalences (their "congruences"). This means that for a fixed space M we should take the quotient space

$$\mathrm{Fields}(M)/\text{congruences}. \tag{0.0.2}$$

In order to do calculations with a quotient space we have to **select an element** into each equivalence class and then prove that the calculus does not depend of this choice. The problem is that when we select a element we are automatically privileging it, but *there is no physically privileged element*. So, in order to be physically correct we have to work with all elements of the equivalent class **simultaneously**. This can be done by replacing the set (0.0.2) with the category whose objects are elements of $\mathrm{Fields}(M)$ and such that there is a mapping between s, s' iff they are equivalent. In this category, all mappings are obviously isomorphisms, so that it is indeed a *groupoid*. The sheaves on the site of manifolds which take values in **Gpd** are called *smooth stacks* and they finally give the correct way of thinking about the "space of fields" for the case of **particles**.

Therefore, in order to end the axiomatization of the classical theories for **particles** we need to define what is the action functional. Given a spacetime M this should be some kind of map between the space of fields $\mathrm{Fields}(M)$ to $\mathbb{R}$. Notice that $\mathbb{R}$ is a **set**, but we had promoted the space of fields to a **groupoid**. So, in order to define a map between these two entities we also need to promote $\mathbb{R}$ to a groupoid. This is done defining a groupoid whose objects are real numbers and having only trivial morphisms.

We then finally say that *a classical theory of* ***particles*** *is given by a smooth stack* Fields (*describing the space of fields*) *and by a rule* S (*describing the action functional*) *which assign to* *any manifold M a functor*

$$S_M : \mathrm{Fields}(M) \to \mathbb{R}.$$

The Need of Higher Category Theory

Up to this point we have seen that *starting with categorical logic as background language we can effectively axiomatize the notions of classical and quantum theories for* ***particles***. Indeed, a quantum theory is just a monoidal functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$, while a classical theory is given by a smooth stack Fields and by an action functional $S_{(-)} : \mathrm{Fields}(-) \to \mathbb{R}$.

It happens that *we do not know if particles are really the correct building blocks of nature*, so that we need a background language with allows us to axiomatize classical and quantum theories not only for particles, but for objects of any dimension. In this regard, categorical logic fails as background language. Indeed, we have at least the following problems:

- *in quantum theories*. Recall that for any p we have the category $\mathbf{Cob}_{p+1}$, so that we could immediately extend the notion of quantum theory for **particles** by defining a quantum theory for p**-branes** as a monoidal functor $U : \mathbf{Cob}_{p+1} \to \mathbf{Vec}_{\mathbb{C}}$. The problem is that in this case we are only replacing the assumption that "particles are the correct building blocks of nature" by the assumption that "p-branes are the correct building blocks of nature". Indeed, in both cases we can only talk about quantum theories *for a specific kind of object*, while Hilbert's sixth problem requires an *absolute notion of quantum theory*;

- *in classical theories*. The motion of particles (which are 0**-dimensional** objects) on a spacetime M was described by smooth paths $\varphi : I \to M$, which are smooth maps defined on a 1**-dimensional** manifold I. Therefore we can easily define the motion of p-branes (which are p**-dimensional** objects) on M as smooth maps $\varphi : \Sigma \to M$ defined on a $(p+1)$**-dimensional** manifold Σ. The problem is that in the "space of fields" of a classical theory we have to consider not only the motions but indeed the interactions which act on the object. For particles we can consider these interactions as modeled by connections because the *notion of connection is equivalent to the notion of parallel transport along paths*. Unhappily, there is no global notion of parallel transport along higher dimensional manifolds, so that *in principle the notion of connection (and, therefore, the description of the interactions) cannot be extended to the context of strings and branes.*

Summarizing,

Conclusion. *in order to axiomatize the notions of classical and quantum theories for higher dimensional objects we need to start with a language which is more abstract than categorical language.*

Therefore, the immediate idea is to try to build some kind of process which takes a language and returns a more abstract language. Before discussing how this can be done, recall that for a selected background language, the first step in solving Hilbert's sixth problem is to consider a loop involving naive math, the language itself and physics. So, having constructed a more abstract language from a given one, we would like to consider loops for the new language as arising by extensions of the loop for the starting language, as in the diagram below.

$$\begin{tikzcd}
 & & \textbf{naive math} \arrow[dr] & \\
\textbf{language} \arrow[r, dashed, Rightarrow] \arrow[urr, bend left, Rightarrow] \arrow[rrr, bend right, Rightarrow] & \textbf{abstract language} \arrow[ur] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}$$

This condition is very important, because it ensures that the physics axiomatized by the initial language is contained in the physics axiomatized by the new language. So, this means that when applying the process of "abstractification" we are getting languages that axiomatize more and more physics. Consequently, this seems that the iterating the process and taking the limit one hopes get a background language which is abstract enough in order to axiomatize the whole physics.

$$\begin{tikzcd}
 & & & & \textbf{naive math} \arrow[dr] & \\
\textbf{language} \arrow[r, dashed, Rightarrow] \arrow[urrrr, bend left, Rightarrow] \arrow[rrrrr, bend right, Rightarrow] & \textbf{abstract language} \arrow[r, dashed, Rightarrow] \arrow[urrr, bend left, Rightarrow] \arrow[rrrr, bend right, Rightarrow] & \textbf{more abstract language} \arrow[r, dashed, Rightarrow] \arrow[urr, bend left, Rightarrow] \arrow[rrr, bend right, Rightarrow] & \cdots \arrow[ur] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}$$

Highering

We saw that category theory is useful to axiomatize particle physics, but not string physics, so that we need to build some "abstractification process" which will be used to replace category theory by other more abstract theory. Notice that categorical language is more abstract than classical language, so that learning how to characterize the passage from classical logic to categorical logic would help to know how to iterate the construction, getting languages more abstract than categorical language. In other words, the main approach to the "abstractification process" is as some kind of "categorification process".

In order to get some feeling on this categorification process, notice that a *set* contains less information than a *category*. Indeed, sets are composed of a single type of information: *their elements*. On the other hand, categories have three kinds of information: *objects*, *morphisms* and *compositions*. Thus, we can understood the passage from set theory to category theory (and, therefore, from classical logic to categorical logic) as the "*addition of information layers*" (see [19] for an interesting discussion).

So, when iterating this process we expect to get a language describing entities containing more information than usual categories. Indeed, we expect to have not only objects, morphisms between objects and compositions between morphisms, but also *morphisms between morphisms* (called 2-morphisms) and compositions of 2-morphisms. Thus, if we call such entities 2-*categories*, adding another information layer we get 3-*categories*, and so on. Taking the limit we then get ∞-*categories*, leading us to the following conclusion:

Conclusion. *A natural candidate for a background language sufficient to solve Hilbert's sixth problem is ∞-categorical language.*

With this conclusion in mind we need to build a math-language-physics loop for ∞-categorical language by extending the loop for categorical language. This can be done as follows:

- *in quantum theories*. Recall that the problem with the definition of quantum theories as monoidal functors $U : \mathbf{Cob}_{p+1} \rightarrow \mathbf{Vec}_{\mathbb{C}}$ involves the fact that such functors take into account only p-branes for a **fixed** p, implying that we need to know previously which is the correct building blocks of nature. Let us see that this problem can be avoided in the ∞-categorical context. Indeed, we can define a ∞-category $\mathbf{Cob}(\infty)$ having 0-manifolds as objects, 1-cobordisms (i.e cobordisms between 0-manifolds) as morphisms, 2-cobordisms (i.e, cobordisms between 1-cobordisms) as 2-morphisms and so on. Notice that differently from $\mathbf{Cob}_{p+1}$ (which contains only p-manifolds and cobordisms between them), the defined ∞-category $\mathbf{Cob}(\infty)$ contains cobordisms of all orders and, therefore, describe p-branes for every p **simultaneously**. So, we can define an *absolute (or extended) quantum theory* (as required by Hilbert's sixth problem) as some kind of ∞-functor $U : \mathbf{Cob}(\infty) \rightarrow \infty\mathbf{Vec}_{\mathbb{C}}$, where $\infty\mathbf{Vec}_{\mathbb{C}}$ is a ∞-categorical version of $\mathbf{Vec}_{\mathbb{C}}$ (i.e, is some ∞-category of ∞-vector spaces). See [20, 124];

- *in classical theories*. The problem with the axiomatization of classical theories via categorical language is that in the space of fields we have to consider interacting fields. For particles, these fields are modeled by connections on bundles, which are equivalent to parallel transport along paths. But, as commented, there is no canonical notion of transportation along higher dimensional manifolds. Another way to see the problem is the following: in order to define a connection A locally (i.e, in terms of data over an open covering $U_i \hookrightarrow M$) we need a family of 1-forms on A_i in U_i subjected to compatibility conditions at $U_i \cap U_j$. The data U_i and $U_i \cap U_j$ belong to an usual category $\check{C}(U_i)$, whose objects are elements $x_i \in U_i$ and there is a morphism $x_i \rightarrow x_j$ iff $x_i, x_j \in U_i \cap U_j$ (this is the *Čech groupoid* of the covering). On the other hand, if we try to define higher transportation locally we need to take into account data on U_i which is compatible not only at $U_i \cap U_j$, but also at $U_i \cap U_j \cap U_k$, at $U_i \cap U_j \cap U_k \cap U_l$, and so on. Certainly, all this information cannot be put inside an usual category, justifying the nonexistence of connections along higher dimensional manifolds. It happens that it can be put inside a ∞-category $\check{C}_{\infty}(U_i)$, the *Čech ∞-groupoid*) meaning that we actually have a notion of ∞-connection when we consider ∞-categorical language (see [24, 23, 179, 184]). More precisely, the initial problem is avoided if we define the space of fields not as a smooth stack, but as a *smooth ∞-stack*: this is a ∞-functor $\mathrm{Fields} : \mathbf{Diff}^{op} \rightarrow \infty\mathbf{Gpd}$ such that for any M the quantity $\mathrm{Fields}(M)$ is determined not only by $\mathrm{Fields}(U_i)$ and $\mathrm{Fields}(U_i \cap U_j)$, but also by[6]

$$\mathrm{Fields}(U_i \cap U_j \cap U_k), \quad \mathrm{Fields}(U_i \cap U_j \cap U_k \cap U_l) \quad \text{and so on.}$$

[6]The suggestion that not only ∞-connections but indeed all interesting physical fields fits into a (super)smooth ∞-stack appears is [182]

The Need of Synthetic Languages

All that we discussed up to this point is wrong! More precisely, all that was discussed is *incomplete*. Indeed, notice that at all moment we assumed that the spacetime is modeled by a **manifold** M, so that when fixed a local chart $\varphi : U \rightarrow \mathbb{R}^n$ its structure becomes totally described by the coordinate functions $x_i : U \rightarrow \mathbb{R}$. They belong to the algebra $C^{\infty}(U;\mathbb{R})$ of smooth real functions, which is a commutative algebra, meaning that the variables x_i commutes, i.e, $x_i \cdot x_j = x_j \cdot x_i$ and, therefore, $[x_i, x_j] = 0$. Consequently, if a smooth function $\varphi : \Sigma \rightarrow M$ describes the motion of a p-brane into M, then locally such a motion is also modeled by a totally commutative set of variables φ_i.

It happens that, thanks to Spin-Statistics theorem, commutative variables only describe bosons. In order to describe fermions, we should have anticommutative variables too. But this would imply that that the object modelling the spacetime M is not a manifold, but indeed some other kind of object that is locally described not only by commuting coordinates x_i, but also by *anticommuting commuting* ξ_a, in the sense they belong to some anticommutative, i.e, $\xi_a \wedge \xi_b = -\xi_b \wedge \xi_a$. These objects are called *supermanifolds*.

This problem could appear only a mathematical whim. However, the Stern-Gerlach experiment proves that fermionic objects actually exist in nature, being the electron an example. So, *by the ontological branch of physics we really have to replace our model for the spacetime of manifolds by supermanifolds.*

This conclusion has several consequences. For instance, recall that we described the interaction acting on a particle as a connection on a **manifold**. Therefore, if the spacetime is not a manifold, but indeed a supermanifold, then we need to modify our understanding about the notion of connection, lifting it from manifolds to supermanifolds. Fortunately, this can be done, giving the concept of *superconnection on a supermanifold.*

Therefore, the problem above was only a scare. But it also gives an important learning: the relation between physics and ontology makes Hilbert's sixth problem very unstable. Indeed, even if we develop a very powerful language be able to axiomatize every current physics, a future experiment discovering new properties of the matter/energy/light will imply that all previous work need to be reformulated. This means that:

Conclusion. *if we are taking Hilbert's sixth problem seriously we need to consider only languages which does not depend of any explicit object (as a manifold), but which are totally build in terms of abstract/axiomatic properties.*

A language built only in terms of axiomatic properties is called *synthetic*. Therefore, we can summarize the above discussion by saying that *Hilbert's sixth problem requires synthetic languages because this class of languages give a "safety margin" respectively to new empirical discoverings.*

Notice that working with synthetic languages imply to reformulate synthetically concrete notions, such as the notion of "connection of a G-bundle over a manifold", without any mention of the manifold structure! It may seems very strange the existence of this kind of abstract formulation, but it indeed exists. In the following we will try to convince the reader of this fact.

In order to reformulate the notion of "connection of a G-bundle over a manifold" let us first see that the notion of "G-bundle" can be synthetically described. The concept of "G-bundle" depends of a group structure G, so that we need to give a synthetic formulation for this notion too. But this is easy. Indeed, a group is just a set endowed with an operation $* : G \times G \rightarrow G$, a distinguished element e (playing the role of a neutral element, which can be identified with a map $1 \to G$, where 1 is an unit set) and a rule $\mathrm{inv} : G \to G$ assigning to any element its inverse. This data fits into the commutative diagrams below, where $\Delta(g) = (g, g)$ is the diagonal map.

$$\begin{tikzcd} G \times G \times G \arrow[r, "* \times id"] \arrow[d, "id \times *"'] & G \times G \arrow[d, "*"] \\ G \times G \arrow[r, "*"'] & G \end{tikzcd} \qquad \begin{tikzcd} 1 \times G \arrow[r] \arrow[dr, "\pi_1"'] & G \times G \arrow[d, "*"] & G \times 1 \arrow[l] \arrow[dl, "\pi_2"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[r, "e"] & G \\ G \times G \arrow[rr, "id \times \mathrm{inv}"'] & & G \times G \arrow[u, "*"'] \end{tikzcd}$$

Therefore, we can talk about "group-like objects" in any category in which these diagrams makes sense, meaning that the notion of group can be synthetically presented by these diagram. For instance, in **Diff** these diagrams reproduces precisely the concept of Lie group. We could also use this diagrams to internalize the notion of group in the category of supermanifolds, getting *super Lie groups*.

With a synthetic formulation of "group" we can try to get an abstract version of "G-bundles". Recall that, as discussed previously, a G-bundle over a space M is a "local entity" in the sense that it becomes totally determined when given an open covering $U_i \hookrightarrow M$ fulfilling conditions at the intersections $U_i \cap U_j$. In other words, it is an example of *smooth stack*. As a smooth stack, it can be proven that it is globally classified by a map $f : M \to \mathbf{B}G$, where $\mathbf{B}G$ is the *delooping groupoid of* G, whose set of objects is a unit set and whose set of morphisms is G. Locally, in turn, it is classified by a family of maps $g_i : U_i \to G$ which, when restricted to the intersections (where they pass to be denoted by g_{ij}), are required to satisfy the *cocycle conditions*

$$g_{ij} \cdot g_{jk} = g_{ik} \quad \text{and} \quad g_{ii} = e.$$

Notice that the delooping groupoid can be defined in any context in which the notion of group makes sense. Therefore, if $\mathbf{H}$ is a category in which not only the notion of group, but also the notion of "local entity" makes sense, then for any group-like object G and any object M we will be able to define synthetically a "G-bundle over M" as the "local entity" classified by a morphism $f : M \to \mathbf{B}G$. In the usual sense, a "local entity" is one that becomes totally determined by data over any covering $U_i \hookrightarrow M$ fulfilling compatibility conditions at the intersections $U_i \cap U_j$, Therefore, in order to axiomatize "local entity" we only need to axiomatize "coverings" and "intersections".

The fundamental property of the open coverings $U_i \hookrightarrow M$ is that if $f : N \to M$ is any continuous map, the preimages $f^{-1}(U_i) \hookrightarrow N$ give an open covering for N. These preimages can be characterized as *pullbacks*, so that this last property only says that the open coverings are stable under pullbacks. Indeed, a pullback is a categorical construction which take two maps $f : A \to X$ and $g : B \to X$ and return a space $A \times_X B$; the pre-image $f^{-1}(U_i)$ is just $U_i \times_M N$. It happens that the intersections $U_i \cap U_j$ can also be viewed as pullbacks $U_i \times_X U_j$.

Consequently, we can talk of "local entities" in any category with pullbacks! A category $\mathbf{H}$ with pullbacks in which a class J of coverings was fixed is called a *site*. An entity which is local with respect to the data $(\mathbf{H}, J)$ is called a *stack in the site* $(\mathbf{H}, J)$, while the category of sites is called a *Grothendieck topos*. So, we concluded that *the notion of "G-bundles" can be synthetically defined in any Grothendieck topos.*

Differential Cohesion

We are trying to show that the notion of "connection of a G-bundle over a manifold" can be defined synthetically. Up to this moment we concluded that the concept of G-bundle makes sense synthetically internal to any Grothendieck topos $\mathbf{H}$. Recall that a connection on a smooth bundle is a local object (i.e, a smooth stack), so that the idea is to verify if this abstract characterization of the notion of connection also makes sense in an arbitrary Grothendieck topos $\mathbf{H}$.

A connection on a G-bundle $P \to M$ was globally defined as an 1-form $A : TP \to \mathfrak{g}$, but locally (i.e, when given an open covering $U_i \hookrightarrow M$) it is determined by a family of 1-forms $A_i : TU_i \to \mathfrak{g}$ subjected to conditions at the intersections $U_i \cap U_j$. Up to this point we have not said which are these conditions. So, let us say that they are given by the *gauge compatibility condition*:

$$g_{ij} \cdot (A_j - A_i) \cdot g_{ij} = dg_{ij},$$

where here $g_{ij} : U_i \cap U_j \to G$ are part of the local data that classifies the bundle $P \to M$. In the last expression we say that the local 1-forms A_i and A_j are related by a *gauge transformation with parameter* g_{ij}.

In order to give an abstract (i.e, functorial) description of this data, let us start by considering the functor $\mathbf{B}G_{\mathrm{conn}}$, which to any manifold M assigns the groupoid $\mathbf{B}G_{\mathrm{conn}}(M)$, whose objects are 1-forms $\omega : TM \to \mathfrak{g}$ and whose morphisms between ω and ω' are gauge transformations, i.e, smooth functions $g : M \to G$ such that

$$g \cdot (\omega' - \omega) \cdot g = dg.$$

With this functor on hand, we can see that a connection on a G-bundle classified by a morphism $f : M \to \mathbf{B}G$ is nothing but a lifting of f from $\mathbf{B}G$ to $\mathbf{B}G_{\mathrm{conn}}$, as in the diagram below. There, conn is the canonical projection.

$$\begin{tikzcd} & \mathbf{B}G_{\mathrm{conn}} \arrow[d, "\mathrm{conn}"] \\ M \arrow[ur, dashed, "A"] \arrow[r, "f"'] & \mathbf{B}G \end{tikzcd}$$

So, we conclude that *we can define connection on a bundle synthetically internal to any Grothendieck topos in which the object* $\mathbf{B}G_{\mathrm{conn}}$ *can be* constructed. Notice that the definition of $\mathbf{B}G_{\mathrm{conn}}$ involves much more than only internal groups and intersections, as was required to define bundles synthetically. Indeed, it involves the notion of "Lie algebra of a Lie group" and the notion of "exterior differential of a 0-form". Consequently, $\mathbf{B}G_{\mathrm{conn}}$ (and, therefore, the notion of connection) can be internalized only in the Grothendieck topos in which we have well defined Lie theory and de Rham theory.

Very surprisingly, as remarked by Schreiber, the Grothendieck topos satisfying an additional property, called *cohesion* (firstly introduced by Lawvere), accomplish these desired conditions! Indeed, in order to have a well defined Lie theory we need a special flavor of cohesion called *differential cohesion*. In few words we can explain this fact in the following way. Notice that "connection" is a notion of **differential** geometry. Cohesion is exactly the property of a Grothendieck topos $\mathbf{H}$ that allows us to internalized "geometry" in it. In order to internalize **differential** geometry we need **differential** cohesion.

Conclusion. *The notion of connection on a bundle can be synthetically defined in any (differential) cohesive Grothendieck topos.*

There is a further detail to be considered here. Recall that "connection on a bundle" is the notion which we used to axiomatize the interaction acting on **particles**. In order to axiomatize interaction between **strings** or other objects were necessary to consider "higher connections", which are ∞-smooth stacks.

We recall that these ∞-smooth stacks are entities which are "higher local" in the sense that when given a covering $U_i \hookrightarrow M$ they are determined by local data on U_i fulfilling compatibility conditions not only at the intersections $U_i \cap U_j$, but also at $U_i \cap U_j \cap U_k$, and so on. The notion of "local object" (which depends only of conditions at $U_i \cap U_j$) could be defined in any site, because the intersections are just pullbacks. The iterated intersections can be understood as "higher pullbacks", so that we can define a "higher local object" (i.e, a ∞-stack) not in a site, but indeed in a ∞-*site* $(\mathbf{C}, J)$, which is a ∞-category $\mathbf{C}$ endowed with a rule J assigning to any object X a collection $J(x)$ of "higher coverings", which are stable under "higher pullbacks".

If a category of "local objects" in a site is a Grothendieck topos, a category of "higher local objects" in a ∞-site is a Grothendieck ∞-topos. We can extend the notion of "cohesion" to the higher case, so that *we talk of differential cohesive Grothendieck ∞-topos, which is precisely the language in which the notion of "∞-connection" can be synthetically defined.*

Cohomological Interpretation

For any object X in any category $\mathbf{C}$ we can define a very simple and canonical functor

$$[-, X] : \mathbf{C} \to \mathbf{Set},$$

which take any other Y and assigns to it the set of morphisms $Y \to X$. This functor is usually called the *hom-functor* (or the *representable functor*) defined by X. We also say that X is its *representing object*. If instead of a category we now have a ∞-category, then we can define analogue functors, but now they will take values not in **Set**, but indeed in the category of ∞-categories.

On the other hand, in Algebraic Topology and Homological Algebra, many powerful invariants, called *cohomology theories*, are build. It is a very remarkable fact that *all these cohomology theories are, indeed, representable functors in* (*the homotopy category of*) *some ∞-topos!* This motivate us to think of the functor $[-, X]$ in a ∞-topos as some kind of "abstract cohomology" with coefficients in X.

Example (*nonabelian cohomology*)**.** We have seen that, if G is a Lie group, then G-bundles over a manifold M are totally classified by (homotopy classes of) maps $M \to \mathbf{B}G$. This means that the space of bundles is equivalent to $[M, \mathbf{B}G]$, i.e, *G-bundles are classified by abstract cohomology with coefficients in* $\mathbf{B}G$. We say that this is the *nonabelian cohomology of the group G.*

Example (*differential nonabelian cohomology*)**.** Similarly, we have seen that the connections of G-bundles are classified by (homotopy) liftings from $\mathbf{B}G$ to $\mathbf{B}G_{\mathrm{conn}}$. So, in other words, *$G$-connections are classified by abstract cohomology with coefficients in the refined object* $\mathbf{B}G_{\mathrm{conn}}$. It is usually known as the *differential* (or the *differential refinement of*) *nonabelian cohomology*.

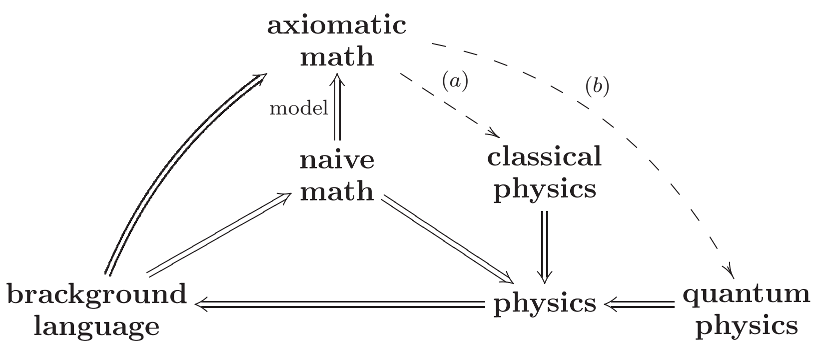
Now, notice that the "space of fields" of a classical theory is generally given by maps $\Sigma \to M$, representing motions of branes into M, and ∞-connections over M, representing the interaction acting on the branes. Therefore, in typical cases we have

$$\mathrm{Fields}(\Sigma) = [\Sigma, M] \times [M; \mathbf{B}G_{\mathrm{conn}}],$$

so that the *space of fields can be generally regarded as* (the product of) *some abstract cohomology in a cohesive ∞-topos*. This is a very important result, as will be explained in the next subsection.

Quantization

We have discussed that the notions of "classical theory" and of "quantum theory" can be synthetically defined in any cohesive ∞-topos as, for instance, the ∞-topos of smooth ∞-stacks. In other words, we have seen that the language of cohesive ∞-topos can be used in order to produce the following diagram:



We notice, however, that this is **not** sufficient to get a complete solution of Hilbert's sixth problem. Indeed, recall that the problem requires not only an axiomatization for "classical theories" and "quantum theories", but also the existence of a "quantization process" assigning to **any** classical theory a corresponding quantum version of it.

In order to get some feeling on what the quantization should be, the immediate idea is to select a concrete ∞-topos and look at particular/concrete constructions in them. In this level, there are many approaches to the quantization process. The most known in the quantum field theory literature is the "path integral quantization". It relies on the existence of a hypothetical notion of "integration" in the space of fields of any classical theory. With this "integration" on hand, if $S_\Sigma : \mathrm{Fields}(\Sigma) \to \mathbb{R}$ is an action funcional representing a classical theory, the quantum theory assigned to it is given by[7]:

$$U(\Sigma) = \int_{\mathrm{Fields}(\Sigma)} e^{\frac{i}{\hbar} S_\Sigma[\varphi, A]} D\varphi DA.$$

The problem is that in general the space of fields has no "finite-dimensional structure", meaning that there is no canonical way to introduce the measure $\mu = D\varphi DA$ (respectively to which the "path integral" would be defined as a Lebesgue integral). In view of this problem, the quantum

[7]Here, $\hbar$ is the (reduced) *Plank's constant*, the fundamental constant of the quantum world

field theorists of particles try to define the path integral perturbatively: they decompose the action as a sum

$$S[\varphi, A] = S_{\text{free}}[\varphi] + S_{\text{back}}[A] + S_{\text{int}}[\varphi, A]$$

and expand the exponential $e^{\frac{i}{\hbar}S_{\Sigma}[\varphi,A]}$ in Taylor's series in order to get a perturbative series, which is diagrammatically represented by the so-called *Feynman diagrams*. The mixture of fields φ and A appearing in $S_{\text{int}}[\varphi, A]$ are called *vertexes* and it is over the number of these vertex that the expansion is made. Each diagram in the series give a contribution to what would be the result of the integration, so that summing over them and taking the limit $n \to \infty$ over the number of vertexes we could, in principle, "calculate" the path integral without defining it!

It happens that there are theories for which the contributions of a "dense amount" of Feynman diagrams is infinite, so that in this case the series is not well defined and, therefore, the perturbative approach fails. These are called *nonrenormalizable theories*, of which General Relativity is an example. We notice that this does **not** means that gravity cannot be quantized; this only means that the perturbative method does not hold here.

Because of this "renormalization problem" it is natural to search for nonpertubative methods of quantization. There is a traditional one, called *geometric quantization* (developed by Kirillov, Kostant and Souriau), which can be applied when the space of fields has the structure of a (possibly infinite-dimensional) symplectic manifold. Unfortunately, this is a very strong restriction, so that this approach only works for a small class of theories. On the other hand, more recently Guillemin and others have been observed that geometric quantization can be understood as some kind of operation in a cohomology theory called *complex K-theory*.

Now, recall that, as commented in the last subsection, in the language of cohesive ∞-topos the space of fields of any classical theory is given by some abstract cohomology theory. Therefore, following the approach of Guillemin and company, we can try to define the quantum theory assigned to a classical theory as some operation in the abstract cohomology that classifies the given classical theory. This is the fundamental idea behind the *cohomological/motivic quantization* recently developed by Freed, Schreiber and many others.

We will outline the construction. For $S : \text{Fields} \Rightarrow \mathbb{R}$ an action functional describing a classical theory in some cohesive ∞-topos $\mathbf{H}$, we need to define a corresponding quantum theory U, which assign to any manifold Σ certain linear structure $U(\Sigma)$ (which in general is some kind of "higher vector space") and to any cobordism $\Sigma : \Sigma_0 \to \Sigma_1$ a corresponding "higher linear map"

$$U(\Sigma) : U(\Sigma_0) \to U(\Sigma_1).$$

So, given maps as in the first diagram below (representing a cobordism), applying Fields we get the upper part of the second diagram, while the exponentiated action functional (which is the object appearing in the path integral) defines the lozenge. The data assigned by the quantum theory should be some kind of "higher module". A structure of linear module depends of the choice of a ring of coefficients, so that our starting point is the choice of a "higher ring" R internal to $\mathbf{H}$. From any usual ring R we can extract its group of units $GL_1(R)$: this is the group of all elements $r \in R$ which has a multiplicative inverse. Therefore, it is expected that, similarly, from any "higher ring R" we can extract some "higher group" $GL_1(R)$ internal to $\mathbf{H}$. The next step is to select some morphism of higher groups ρ, as in the second diagram below. There, the dotted arrows χ_i are the composition of the unidentified arrows with ρ.

$$
\begin{tikzcd}
& \Sigma & \\
\Sigma_0 \arrow[ur, "\imath_0"] & & \Sigma_1 \arrow[ul, "\imath_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{Fields}(\Sigma) \arrow[dl, "\mathrm{Fields}(\imath_0)"'] \arrow[dr, "\mathrm{Fields}(\imath_1)"] & \\
\mathrm{Fields}(\Sigma_0) \arrow[rr, Rightarrow, "e^{\frac{i}{\hbar}S_\Sigma}"] \arrow[dr] \arrow[ddr, dotted, "\chi_0"'] & & \mathrm{Fields}(\Sigma_1) \arrow[dl] \arrow[ddl, dotted, "\chi_1"] \\
& \mathbf{B}U(1) \arrow[d, dashed, "\rho"] & \\
& \mathbf{B}GL_1(R) &
\end{tikzcd}
$$

Now, recall that in any ∞-topos we can talk of "G-bundles over an object X" and these are classified by morphisms $X \to \mathbf{B}G$. Therefore, the choice of ρ give to us four $GL_1(R)$-bundles: one over $\mathrm{Fields}(\Sigma_0)$, two over $\mathrm{Fields}(\Sigma)$ and one over $\mathrm{Fields}(\Sigma_1)$, which will be respectively denoted by E_0, E, E' and E_1. Notice that the exponentiated action functional $e^{\frac{i}{\hbar}S_\Sigma}$ induces a morphism between the maps classifying E and E', so that $E \simeq E'$. Consequently, taking the global section functor Γ we then get the following diagrams. Because the bundles are structured over $GL_1(R)$, it is expected that each space of sections has, indeed, a structure of "higher R-module", motivating us to define $U(\Sigma_0) := \Gamma(E_0)$.

$$
\begin{tikzcd}
& \Gamma(E) \arrow[r, "\simeq"] & \Gamma(E) & \\
\Gamma(E_0) \arrow[ur, "\Gamma(\chi_0)"] & & & \Gamma(E_1) \arrow[ul, "\Gamma(\chi_1)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Gamma(E) \arrow[r, "\simeq"] & \Gamma(E) \arrow[dr, "\Gamma(\chi_1)!"] & \\
\Gamma(E_0) \arrow[ur, "\Gamma(\chi_0)"] \arrow[rrr, dashed, "U(\Sigma)"'] & & & \Gamma(E_1)
\end{tikzcd}
$$

In order to end the construction we need to define the "higher linear map" between the "higher modules" $\Gamma(E_0)$ and $\Gamma(E_1)$. If in the first diagram above the right-hand arrow has an adjoint $\Gamma(\chi_1)!$, then we can build the desired map as the composition in the second diagram above. This happens, for instance, if the bundle classified by χ_1 is *oriented* in some sense. So, "orientation" is a condition involved with the quantization problem.

Indeed, as discussed previously, a classical system may have symmetries. A priori, in the process of quantizing the theory, the symmetries could not be preserved, which is an undesired situation. In these cases we say that the theory has *quantum anomalies*. Thanks to works of Freed, Witten, Kapustin and others, the orientation of the bundles appearing in the quantization could be understood as a condition implying the cancelation of quantum anomalies. Therefore, under the orientatibility hypothesis the symmetries survive to the (cohomological) quantization.

Summarizing

If questioned by an arbitrary people about the subject of this text we would say: have you some time (maybe some days)? For a positive answer we would explain all that was discussed in this Introduction. On the other hand, for a people with no time, we would only say that we wrote a text with the following objectives:

- to prepare the reader to more advanced books/texts/articles on algebraic topology, physics and higher category theory;
- to convince the reader that physics and mathematics constitute a mutualistic symbiosis system[8].
- to convince the reader that by making use of more abstract languages we can axiomatize more and more physical theories, but in order to axiomatize **every** theories/laws/concepts/ideas of physics the correct approach seems to work **synthetically**;
- to convince the reader that a nice class of abstract background languages to axiomatize physics are the **differential cohesive ∞-toposes**. A model is given by the $(\infty, 1)$-category of ∞-stacks on the site of super-formal-smooth manifolds.
- to convince the reader that, even after more than 100 years, Hilbert's sixth problem remains partially unsolved, being a source of many important mathematical works.

Finally, for someone without any patience we could say simply that this text is about the development of abstract stable homotopy theory aiming applications in the foundations of physics.

[8] In Biology, a mutualistic symbiosis system is composed by two or more individuals, in which each of them does not survive without the others, at the same time that this interdependence is very fruitful to the whole system.

Part I

Categorical Language

Prologue

Recall that, as discussed in Introduction, in order to attack Hilbert's sixth problem the first step is to select a suitable background language and then analyze how the naive mathematics described by it interact with the foundations of physics. More precisely, fixed some logic, first of all we need to study the arrow **logic** $\Rightarrow$ **physics**, given by the following sequence:

$$
\begin{tikzcd}
\begin{array}{c}\textbf{abstract}\\ \textbf{language}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\textbf{naive}\\ \textbf{math}\end{array} \arrow[r, Rightarrow] & \textbf{physics}
\end{tikzcd}
$$

Then, we have to make use of physical insight in order to get new mathematical felling, in the sense of the arrow **physics** $\Rightarrow$ **logic**. So, in few words, the first step to Hilbert's sixth problem is to build loops as below:

$$
\begin{tikzcd}
 & \begin{array}{c}\textbf{naive}\\ \textbf{math}\end{array} \arrow[dr, Rightarrow] & \\
\begin{array}{c}\textbf{abstract}\\ \textbf{language}\end{array} \arrow[ur, Rightarrow] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

In this part of the text we will build concretely each arrow in the diagram above for the background language given by (higher) *categorical language*. More precisely, we begin in the Chapter 1, where we present the fundamental aspects of category theory, meaning the arrow (here our arrow take values in **naive** math because the definition of category that will be given depends of the notion of "collection" which we assume primitive):

$$
\begin{tikzcd}
\begin{array}{c}\textbf{categorical}\\ \textbf{language}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\textbf{naive}\\ \textbf{math}\end{array}
\end{tikzcd}
$$

Then, in the Chapter 2 we argue that category theory really is a abstract theory, but not abstract enough to build a nice arrow from naive math to physics. In order to solve the problem, we **conjecture** the existence of a categorification process, which take a classical concept and give a categorical version of it in such a way that by iteration we get a sequence

$$
\begin{tikzcd}
\begin{array}{c}\textbf{set theory}\\ \text{(classical logic)}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\textbf{category theory}\\ \text{(categorical logic)}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\textbf{2-category theory}\\ \text{(2-categorical logic)}\end{array} \arrow[r, Rightarrow] & \cdots
\end{tikzcd}
$$

We then discuss that when we categorify the language that describes p-branes we get a language describing not only p-branes, but also $(p+1)$-branes, meaning that the diagram below should be commutative (as required by the "abstractification process" discussed in the Introducion). Specially, this will suggest that p-category theory is the natural language to describe physical systems containing k-branes, for $0 \leq k \leq p$.

$$
\begin{tikzcd}
 & & & & \textbf{naive math} \arrow[dr, Rightarrow] & \\
\begin{array}{c}\textbf{classical}\\ \textbf{language}\end{array} \arrow[r, Rightarrow, dashed] \arrow[urrrr, Rightarrow, bend left] \arrow[rrrrr, Rightarrow, bend right] & \begin{array}{c}\textbf{categorical}\\ \textbf{language}\end{array} \arrow[r, Rightarrow, dashed] \arrow[urrr, Rightarrow, bend left] \arrow[rrrr, Rightarrow, bend right] & \begin{array}{c}\textbf{2-categorical}\\ \textbf{language}\end{array} \arrow[r, Rightarrow, dashed] \arrow[urr, Rightarrow, bend left] \arrow[rrr, Rightarrow, bend right] & \cdots \arrow[ur, Rightarrow] & & \textbf{physics} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

The part ends in the Chapter 3, where we discuss the implications of the existence of these hypothetical higher categorical languages to the foundations of physics. In other words, we end by realizing the following arrows

$$
\begin{array}{c}p\textbf{-categorical}\\ \textbf{language}\end{array} \Longleftarrow \begin{array}{c}p\textbf{-brane}\\ \textbf{physics}\end{array}
$$

Chapter 1

Categories

In this chapter we will discuss our first example of abstract background language: the *categorical language*. We start at the first section presenting the definition of category and giving many examples. A category is essentially an area of mathematics, so that these examples reflect the existence of different areas of math. They can be divided in two classes: the *concrete areas* and the *abstract areas*.

The first class corresponds to the cases in which the fundamental objects of study are simply sets endowed with some additional structure; we are as well interested on functions between the underlying sets preserving the additional structures. The other examples are described by entities which are more distant from intuition. The fundamental example is the area of mathematics known as *cobordism theory*. There we are interested on manifolds that are related not by usual smooth functions, but by other manifolds!

Still in the first section we show that there is a notion of mapping between categories, producing a notion of mapping between different areas of mathematics. These are the *functors*. Generally, in any area of mathematics we have a canonical way to identify two objects and we are interested not on the object, but on its equivalence class. The fundamental property of the functors is that they always maps equivalent objects into equivalent objects, allowing us to build many powerful invariants. The second section is totally devoted to giving examples of invariants that comes from functors.

The chapter ends in the third section, where we discuss three important principles that permeates the categorical language. These are the *duality principle*, the *relativity principle* and the *weakening principle*. The first principle allow us to produce many dual theorems. The second one gives a new viewpoint to mathematics and the third one teach us that any concept can be abstracted.

1.1 Structure

Giving a *category* is just the same thing as specifying an area of mathematics. More precisely, a category is an entity composed of three kinds of information: *objects*, *mappings* (also called *morphisms*) between objects and an associative *composition* of mappings. We also require the existence of *identity morphisms*. This means that a category theory is just the natural ambient in which we can talk about *commutative diagrams*.

Formally, a category $\mathbf{C}$ is defined giving the following data:

1. a collection of *objects* $\mathrm{Ob}(\mathbf{C})$;
2. for any two objects X and Y a correspondent collection $\mathrm{Mor}_{\mathbf{C}}(X, Y)$, whose elements are called *morphisms between* X *and* Y, being represented by arrows $f : X \to Y$;
3. for any object a distinguished morphism $id_X : X \to X$ called the *identity of* X;
4. for any three objects a *composition law of morphisms*, which is represented by a function

$$\circ : \mathrm{Mor}_{\mathbf{C}}(X, Y) \times \mathrm{Mor}_{\mathbf{C}}(Y, Z) \to \mathrm{Mor}_{\mathbf{C}}(X, Z)$$

and that are required to be associative and unital in the sense that

$$id_X \circ f = f = f \circ id_X \quad \text{and} \quad (h \circ g) \circ f = h \circ (g \circ f)$$

wherever these compositions are defined.

Even though we have the previous concrete definition, we will try to think about a category as being, effectively, an area of math. This will provide more intuition to our arguments. Following this philosophy, let us try to give some examples of categories.

Example 1.1 (*sets*)**.** Classical mathematics is developed using linear logic as background logic. The most elementary area of mathematics that can be built over this logic is *set theory.* So, it is expected the existence of a category **Set** describing set theory. Indeed, the fundamental objects in set theory are just the sets and the canonical maps between them are precisely functions. The composition of functions is naturally associative and we have the identity functions. Together, this data defines the required category **Set**.

Example 1.2 (*algebraic categories*)**.** We have many areas of math lying inside Algebra. For instance, we have *group theory*, *ring theory*, *R-module theory*, *linear algebra*, and so on. Therefore we can try to define *algebraic categories* describing each of these areas of Algebra. For every case, the fundamental objects are sets with additional operations on the elements and the canonical mappings between them are *homomorphisms* (i.e, functions between the underlying sets which preserves the additional structures). The composition of such mappings is well defined and the identity function is a homormorphism. Consequently, we have the category **Grp** of groups, the category **Rng** of rings, the category $\mathbf{Mod}_R$ of R-modules, the category $\mathbf{Vec}_{\mathbb{K}}$ of vectors spaces, and son on.

Example 1.3 (*topological categories*)**.** Topology is another area of math. Indeed, it is precisely the area of math in which we can differentiate local properties from global properties. There, the fundamental objects are the topological spaces and the fundamental mappings are continuous functions. Composition is well defined and the identity function is clearly continuous. Therefore, there is a category **Top** describing topology. There are other flavors of topological categories. Indeed, in some cases it is more natural to work with topological spaces endowed with a distinguished point. The mappings are then continuous functions preserving this distinguished point. More precisely, we can define a category $\mathbf{Top}_*$ whose objects are pairs (X, x), with $x \in X$ and $X \in \mathbf{Top}$, and whose morphisms are maps $f : X \to Y$ such that $f(x) = y$.

Example 1.4 (*analytic categories*)**.** We also expect to have categories describing areas of Analysis. For instance, the canonical ambient in which we can do analysis are the finite dimensional euclidean spaces $\mathbb{R}^n$ or hermitian spaces $\mathbb{C}^n$. There are many natural maps that can be considered and for each of them we will have a different category. Indeed, we can consider the morphisms as being linear maps, adjoint maps, unitary maps, smooth maps, and so on. But analysis on $\mathbb{R}^n$ or $\mathbb{C}^n$ is very limited and in several situations we need to consider more general theories. We list some approaches:

1. *finite dimensional manifolds*. we can abstract the analysis on $\mathbb{R}^n$ or $\mathbb{C}^n$ by considering spaces which are "modeled over $\mathbb{R}^n$ or $\mathbb{C}^n$" in the sense that they are only locally equivalent to $\mathbb{R}^n$ or $\mathbb{C}^n$. This approach will produce the categories $\mathbf{Diff}$ of smooth real manifolds with smooth maps and $\mathbf{Diff}_{\mathbb{C}}$ of complex manifolds with holomorphic maps. We can also fix some natural n and build the categories $\mathbf{Diff}^n$ and $\mathbf{Diff}^n_{\mathbb{C}}$ which are obtained from the previous ones by considering only manifolds with real or complex dimension n;

2. *functional analysis.* on the other hand, we can abstract analysis on $\mathbb{R}^n$ by considering spaces which are not necessarily finite dimensional. This is the domain of *functional analysis.* Here there two classes of canonical objects to be considered:

    (a) *Banach spaces*. These are simply normed linear spaces which are topologically complete in the metric induced by the norm. We have many types of morphisms, each of them producing a different category. For instance, we can consider the category $\mathbf{Ban}_{\mathbb{C}}$ of complex Banach spaces and linear maps. But Banach spaces are not only vector spaces; they have an additional structure, the *norm.* It happens that linear maps only preserve the linear structure, so that it is more natural to consider the *continuous* (or, equivalently, the *bounded*) linear maps, which preserve not only the linear structure, but also the norm. This defines a category $B\mathbf{Ban}_{\mathbb{C}}$;

    (b) *Hilbert spaces.* They are vector spaces which have a complete inner product instead of only a complete norm. We have the full subcategory $\mathbf{Vec}_{\mathbb{C}}$ having Hilbert spaces as objects and linear maps as morphisms. But, as for Banach spaces, this is not the most natural category to be considered, because linear maps do not preserve the inner product. So, the main idea is to consider maps which also preserve the inner product. There are some candidates. Indeed, we can consider *unitary embeddings* or *self-adjoint* maps. These are respectively linear maps $T : \mathcal{H} \to \mathcal{H}'$ and $A : \mathcal{H} \to \mathcal{H}$ such that

    $$\langle v, w\rangle = \langle T(v), T(w)\rangle' \quad \text{or} \quad \langle A(v), w\rangle = \langle v, A(w)\rangle$$

    for each $v, w \in \mathcal{H}$. Here we observe that the nondegeneracy of the inner product requires the injectivity of the unitary embeddings. Furthermore, a self-adjoint operator is only defined from some space to itself. The correspondent categories of Hilbert spaces and each of those classes of maps will be denoted by $\mathbf{Hilb}_{\mathbb{C}}$ and $\mathbf{Adj}_{\mathbb{C}}$.

3. *infinite dimensional manifolds*. in a third approach we can mix both previous cases by considering entities which are locally modeled over Banach spaces or over Hilbert spaces in the same way that a manifold is locally modeled over $\mathbb{R}^n$ or $\mathbb{C}^n$. These are the *Banach manifolds* and the *Hilbert manifolds.* There are many types of morphisms between them.

For instance, we can consider smooth maps with bounded derivative or smooth maps whose derivative is self-adjoint, etc. The correspondent categories will not be used directly in this text, so that we will not give a special name to them. Even so, they are important in physics appearing, for example, in hydrodynamics and in geometric quantization (see [??]).

Subcategories

Similarly to what happens in Algebra, Topology or Analysis, where we have the notion of *substructure*, we can also talk about *subcategories*. We say that $\mathbf{D}$ is a *subcategory* of $\mathbf{C}$ (writing $\mathbf{D} \subset \mathbf{C}$ to indicate this fact) when its objects and morphisms are also objects and morphisms of $\mathbf{C}$ and its compositions and identities coincide which those of $\mathbf{C}$. Thus, if a category defines an area of math, then any of its subcategories determines a correspondent subarea.

For instance, a subarea of Algebra is Commutative Algebra, so that we have categories describing *abelian group theory*, *commutative ring theory*, and so on. Similarly, in some cases we need to work with topological spaces with some extra condition (say compactness, separability, etc). The study of each of these classes of topological spaces determines an subarea of Topology and, therefore, a subcategory of $\mathbf{Top}$.

Both examples of subareas of Algebra and Topology have a common characteristic: *the mappings between objects on the subarea are just the morphisms between the objects viewed in the global area*. For instance, a mapping between abelian groups is just a homomorphism between the underlying groups. Similarly, a map between compact topological spaces is just a continuous function. In terms of subcategories $\mathbf{D} \subset \mathbf{C}$ this means that for any objects $X, Y \in \mathbf{D}$ we have

$$\mathrm{Mor}_{\mathbf{D}}(X;Y) = \mathrm{Mor}_{\mathbf{C}}(X;Y).$$

A subcategory satisfying such condition is called *full*. For instance, the previous examples of subcategories are full and $\mathbf{Ban}_{\mathbb{C}}$ is also a full subcategory of $\mathbf{Vec}_{\mathbb{C}}$. On the other hand, in the analytical context, the important categories $B\mathbf{Ban}_{\mathbb{C}}$, $\mathbf{Hilb}_{\mathbb{C}}$ and $\mathbf{Adj}_{\mathbb{C}}$ are subcategories of $\mathbf{Vec}_{\mathbb{C}}$ which are *not* full. Indeed, its morphisms are respectively continuous linear maps, unitary embeddings and self-adjoint maps, which are linear maps satisfying additional conditions. But we usually consider full subcategories of each of these non-full subcategories. Indeed, we generally have interest in the special class of the so called *separable* Banach and Hilbert spaces. They are the natural ambient in which quantum mechanical systems (and quantum field theories in its algebraic approach) are described.

In the algebraic context we also have important examples of non-full subcategories. For instance, given a commutative ring R we can consider the subarea of R-module theory which describes *G-graded R-modules* for some abelian group G. There, the fundamental objects are R-modules M which admit a direct sum decomposition parametrized by the elements of the given group G. More precisely, we say that M is *G-graded* when for any $g \in G$ there is another R-module M_g such that $M = \bigoplus_G M_g$. A mapping between two graded modules M and N is **not** only a homomorphism of modules, but indeed a homomorphism $f : M \to N$ which *preserves the grading* in the sense that if $m \in M_g$ then $f(m) \in N_g$. Therefore, we have a **non-full** subcategory $G\mathbf{Grad}_R \subset \mathbf{Mod}_R$ describing the theory of G-graded R-modules.

On the other hand, we could considered the **full** subcategory of $\mathbf{Mod}_R$ given by the G-graded R-modules. The difference is that here we are **not** supposing that a morphism $f : M \to N$ between two graded modules preserves the grading. We will use $G\mathbf{Mod}_R$ to denote the corresponding category. The interesting fact is that *this full subcategory is completely described by certain non-full pieces*. Indeed, we observe that there are some special classes of morphisms $f : M \to N$: those for which we have an element $\deg(f) \in G$ such that f maps M_g into $N_{g+\deg(f)}$ for any g. We call $\deg(f)$ the *degree* of f. For instance, the morphisms with degree equal to zero are just the morphisms preserving the grading (i.e, the morphisms of the non-full category $G\mathbf{Grad}_R$). Clearly, not every morphism have a degree. On the other hand, it can be verified that any morphism can be written as a direct sum of morphisms which have degree. This means that *the fundamental category of graded modules is those whose morphisms are required to preserve the grading.*

In some useful situations we are more interested in the graded modules M which come equipped with a distinguished endomorphism $\partial : M \to M$, called *differential operator*. These modules are the *differential G-graded R-modules* (if ∂ has degree we say it is also the degree of M). Now we have an additional structure (the differential operator), so that we generally suppose that the morphisms between this new objects preserve not only the grading, but also the differential structure. This means that a morphism f between differential graded modules (M, ∂) and (N, d), say with $\deg(\partial) = \alpha = \deg(d)$, is a family of R-module homorphisms $f_g : M_g \to N_g$ such that the diagram below is commutative.

$$\begin{tikzcd} \cdots \arrow[r] & M_{g-\alpha} \arrow[r, "\partial"] \arrow[d, "f_{g-\alpha}"'] & M_g \arrow[r, "\partial"] \arrow[d, "f_g"] & M_{g+\alpha} \arrow[r, "\partial"] \arrow[d, "f_{g+\alpha}"] & \cdots \\ \cdots \arrow[r] & N_{g-\alpha} \arrow[r, "d"'] & N_g \arrow[r, "d"'] & N_{g+\alpha} \arrow[r] & \cdots \end{tikzcd}$$

Therefore, *for any given $\alpha \in G$ we have another non-full subcategory $G\mathbf{DGrad}_R^\alpha \subset G\mathbf{Grad}_R$ of differential G-graded R-modules whose differential operator has degree α.* Summarizing, for any ring R we have a chain of **non-full** subcategories

$$G\mathbf{DGrad}_R^\alpha \subset G\mathbf{Grad}_R \subset \mathbf{Mod}_R.$$

Remark. As will become clear in the next section, there is further interest in the case $G = \mathbb{Z}$ with $\deg(\partial) = \pm 1$ and in which the differential operator ∂ satisfy $\partial \circ \partial = 0$. In this situation, a $\mathbb{Z}$-graded module M is identified with a collection (M_n) of modules, while a differential $\mathbb{Z}$-graded with degree $\deg(\partial) = 1$ or $\deg(\partial) = -1$ is identified with a sequence of homomorphisms

$$\partial_n : M_n \to M_{n+1} \quad \text{or} \quad \partial^n : M_n \to M_{n-1}$$

and the condition $\partial \circ \partial = 0$ writes

$$\partial_{n+1} \circ \partial_n = 0 \quad \text{or} \quad \partial^n \circ \partial^{n+1} = 0.$$

In the topological context we also have very important examples of non-full subcategories of **Top**. Maybe the most important is given by the subcategory of *cell complexes*. These are the topological version of $\mathbb{N}$-graded modules with degree $\alpha = 1$. Indeed, as remarked above, anyone of these modules can be identified with a sequence M_n of modules linked by homomorphisms $M_n \to M_{n+1}$. This identification is obtained by considering the limiting space $\lim_{n\to\infty} A_n$, where

$$A_n = M_0 \oplus M_1 \oplus ... \oplus M_n.$$

Similarly, a *cell complex* is a topological space obtained as a limiting process. We start with a sequence of topological spaces X_n, called the *n-skeletons* of the construction. Then, fixing continuous maps $X_n \to X_{n+1}$, called *linking maps*, we take the limiting space which will be our cell complex. Recall that the morphisms between graded modules are not only module homomorphisms, but indeed the homomorphisms which preserve the grading. Similarly, in the present topological context, the mappings between cell complex are not only continuous maps $X \to Y$, but indeed continuous functions which preserve the cell decomposition. This means that they can be identified with sequences $f_n : X_n \to Y_n$ commuting with the linking maps. Thus we really have a non full subcategory $\mathbf{Cell} \subset \mathbf{Top}$.

Now, in the same way as there is special interest in the $\mathbb{Z}$-graded modules for which $\partial \circ \partial = 0$, we also have a special class of cell complexes which defines a full subcategory $\mathbf{CW} \subset \mathbf{Cell}$. Indeed, it is the class of the so called *CW-complexes*. These are cell complexes for which X_n is obtained from X_{n-1} by attaching n-cells. More precisely, starting from the $(n-1)$-skeleton X_{n-1} we consider a family of attaching maps $f_1^n, ..., f_k^n : \mathbb{S}^{n-1} \to X_{n-1}$ and then we glue disks $\mathbb{D}^n$ into X_{n-1} along their boundaries $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$ using each of these maps. In a CW-complex, the resulting space is precisely the n-skeleton X_n. So, a CW-complex is totally determined by its 0-skeleton and by the attaching maps. Notice that this condition is just the topological analogue of the condition $A_n = M_0 \oplus M_1 \oplus ... \oplus M_n$ valid for $\mathbb{N}$-graded modules. The interest in CW-complexes will become clear later.

Concreteness

We observe that all the previous examples of categories were built following a certain recipe:

1. considering as objects sets endowed with additional structure;
2. taking morphisms between objects as being simply functions between the underlying sets which preserve the additional structure;
3. fixing compositions as the composition of the underlying functions and the identities as being the identity functions.

A category built in this way is usually called *concrete*. So, in other words, *a concrete category is simply a subcategory of* **Set**. We observe, on the other hand, that there are very important categories which are not concrete. In the following, we will present some of them.

Example 1.5 (*cobordism category*)**.** Generally the mappings between two n-manifolds are the smooth maps. But given two n-manifolds M_0 and M_1 we can think of them as being the initial and the final configuration of a system evolving in time. In this perspective, a morphism between M_0 and M_1 is an $(n+1)$-dimensional manifold M such that $\partial M = M_0 \sqcup M_1$. For instance, the trivial dynamics is that given by considering any intermediate state (including the final state) as equal to the initial state. The morphism is this case is simply the cylinder $M_0 \times [0, 1]$. Now we can try to define a non-concrete category $\mathbf{Cob}_n$ whose objects are n-manifolds, as in $\mathbf{Diff}_n$, but whose morphisms are time evolutions, also called *cobordisms*, instead of smooth maps. The identity id_M will be the cylinder, of course. In order to define a genuine categorical structure we need an associative and unital composition. A composition between cobordisms $M : M_0 \to M_1$ and $M' : M_1 \to M_0'$ needs to be a cobordism $M' \circ M : M_0 \to M_0'$. So, starting from a manifold with boundary $M_0 \sqcup M_1$ and a manifold with boundary $M_1 \sqcup M_0'$ we need to build another manifold with boundary $M_0 \sqcup M_0'$. So, the main idea is to define $M' \circ M$ as being the entity obtained by identifying (i.e, gluing) the components M_1 of the boundaries of M and M'. However, this construction **is not** well defined because in principle there is no canonical way to select the attaching maps. But even if we obtain canonical attaching maps the gluing will be defined only up to diffeomorphisms. This shows that we need to redefine the notion of cobordism. Indeed, a *cobordism* between M_0 and M_1 is correctly defined as manifold M with two boundary components, endowed with a labelling map $p : \partial M \to \{0, 1\}$ that labels each component of the boundary, as well as with diffeomorphisms θ_i from each labeled component to M_i. Two cobordisms M and M' are called *diffeomorphic* when there is a diffeomorphism $f : M \to M'$ which commutes with the labelling maps and with the boundary diffeomorphisms θ_i. So finally we can define a genuine category $\mathbf{Cob}_n$ by considering cobordisms as objects, diffeomorphism classes of cobordisms as morphisms, gluings as compositions and cylinders as identities.

Example 1.6 (*homotopy categories*)**.** In Topology we have the notion of *homotopy* between continuous maps $f, g : X \to Y$. This is simply another continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. For a suitable class of spaces this is equivalently a continuous path $H : I \to \mathrm{Map}(X; Y)$ linking f and g. So we can think of a homotopy as being a way to deform f continuously into g. The homotopy relation is, indeed, an equivalence relation and some results in Topology hold equally well replacing a map by their homotopy class. The study of topological spaces linked by homotopy classes of maps determines, therefore, an area of math: *homotopy theory*. The corresponding category, denoted by $\mathrm{Ho}(\mathbf{Top})$ is clearly non-concrete. We also have the notion of homotopy between based functions. Indeed, if $f, g : X \to Y$ are morphisms preserving some base point $x \in X$, then a homotopy between then is simply a classical homotopy H such that $H_t : X \to Y$ also preserves x for any $t \in I$. Consequently we also have a non-concrete category $\mathrm{Ho}(\mathbf{Top}_*)$.

The previous example is a particular case of a more general construction which always produces a non-concrete category. Indeed, starting with an arbitrary category $\mathbf{C}$, let us suppose that we have a function $\simeq$ establishing an equivalence relation $\simeq_{XY}$ in any set of morphisms $\mathrm{Mor}_{\mathbf{C}}(X, Y)$, which preserve the compositions. Then we can define a new category $\mathbf{C}/\simeq$, called *quotient category*, whose objects are just the objects of $\mathbf{C}$, but whose morphisms are the equivalence classes of morphisms of $\mathbf{C}$. More precisely,

$$\mathrm{Ob}(\mathbf{C}/\simeq) := \mathrm{Ob}(\mathbf{C}) \quad \text{and} \quad \mathrm{Mor}_{\mathbf{C}/\simeq}(X, Y) := \mathrm{Mor}_{\mathbf{C}}(X, Y)/\simeq_{XY} .$$

The compositions are the compositions of $\mathbf{C}$ after passing to the quotient (which is well defined by hypothesis). The identities are evidently the equivalence classes of the identities of $\mathbf{C}$. In the previous example, $\simeq$ is simply the rule assigning to any pair of topological spaces the correspondent homotopy relation on the maps between them.

Example 1.7 (*categories of bundles*)**.** In topology we frequently work with spaces X such that in any point $x \in X$ we have a certain additional structure E_x. We then can joint all these structures into a unique object $E = \sqcup_X E_x$ for which we have a canonical projection $\pi : E \to X$ mapping any E_x into x. The continuity of this map says that the structures E_x change continuously when the points x vary. We usually say that this process defines a *bundle* over the space X. The structure E_x is called the *fiber* on x. We can define a notion of mapping between bundles $\pi : E \to X$ and

$\pi' : E' \to X$. These are given by continuous maps $g : E \to E'$ which preserve the fiber structure i.e, such that if $e_x \in E_x$, then $g(e_x) \in E'_x$. Equivalently, this means that $\pi' \circ g = \pi$. This defines a non-concrete category $\mathbf{Bun}_X$ describing the part of topology which deals with bundles over X. But, this category is too huge to be analyzed (corresponding to the fact that in this general case the structures E_x are not subjected to any relation). So, generally we are interested in certain *subcategories* of $\mathbf{Bun}_X$. Important examples are the following:

- *bundes with typical fiber*. The most simple way to relate the fibers of a bundle $\pi : E \to X$ is by requiring that they have a model. This means that for any $x \in X$ the structure E_x is equivalent to some fixed structure F. In this case, we say that the bundle has a *typical fiber* given by F. Now, notice that the requirement of a typical fiber does not correspond to the introduction of some additional structure (we are only adding a *canonical model* to the fibers). Therefore, it is natural to consider a morphism between two bundles with a same typical fiber F as being simply a morphism between the underlying bundles, defining a **full** subcategory of $\mathbf{Bun}_X$.

- *G-structured bundles*. In many situations there is a group G acting on each fiber of a bundle. So, we can produce a notion of compatibility between the fibers by requiring some compatibility between the corresponding actions. For instance, we can require that the actions $G \times B_p \to B_p$ *vary continuously when* p *vary*. One way to make precise this idea is assigning to any point p a neighbourhood U_i in which we have a continuous map $t_i : U_i \to G$, called *coordinates* of the bundle (in analogy to the coordinates of a manifold), such that t_i and t_j are *compatible* in the intersection $U_{ij} = U_i \cap U_j$. Generally we require the *cocycle conditions* $t_{ij} = t_{ik} \cdot t_{kj}$ and $t_{ii} = 1$. A bundle with such structure is called *structured by the group* G. Differently from the case of bundles with typical fibers, a G-structured bundle is a bundle endowed with an additional structure: the action on the fibers. Therefore, the natural morphisms between them are the morphisms between the underlying bundles which preserve these actions (i.e that are equivariant when restricted to each fiber). This defines a **non full** subcategory of $\mathbf{Bun}_X$.

- *principal bundles*. The most important bundles having typical fiber and structural group are, surprisingly, the most simple of them. Indeed, it is the class of the *G-principal bundles*, which have G as typical fiber and whose action $G \times G \to G$ is given by left translation (the category of G-principal bundles will be denoted by $G\mathrm{Princ}_X$). They constitute the most important class because *they determines any other class of structured bundles*. Indeed, notice that if $E \to X$ is a G-bundle, say with typical fiber F, then the action $G \times F \to F$ can be equivalently viewed as a representation of G into the automorphism group $\mathrm{Aut}(F)$ of F. Therefore, the coordinates $t_i : U_i \to G$ induce, by composition with the representation, coordinates $\overline{t}_i : U_i \to \mathrm{Aut}(F)$, showing that we can enlarge the structure group by considering it as the automorphism group of F. Now, we can build an $\mathrm{Aut}(F)$-principal bundle, called the *associated frame bundle* of E. This is the bundle $\mathrm{Fr}(E)$ whose fiber $\mathrm{Fr}(E)_x$ on $x \in X$ is simply the set of isomorphisms $\mathrm{Iso}(E_x; F)$. The action
$$\mathrm{Aut}(F) \times \mathrm{Fr}(E)_x \to \mathrm{Fr}(E)_x \tag{1.1.1}$$
is by composition. The coordinates are the same coordinates $\overline{t}_i$. This bundle is principal because we have $\mathrm{Iso}(E_x; F) \simeq \mathrm{Aut}(F)$ and with this identification the action (1.1.1) is

simply the action by left translation. Therefore, *any G-bundle with fiber F induces a principal bundle.* Reciprocally, if $P \to X$ is a G-principal bundle, then for any action $G \times F \to F$ we have a representation and this representation induces new coordinates $\overline{t}_i$ as above, producing a bundle with fiber F.

- *vector bundles*. A particular interesting class of bundles are *rank n vector bundles*. These are bundles with typical fiber $\mathbb{R}^n$ that are structured over $GL(n) = \operatorname{Aut}(\mathbb{R}^n)$ with respect to the canonical action. A classical example is the tangent bundle of a manifold M, whose fiber on $p \in M$ is the tangent space TM_p. From the mathematical viewpoint, vector bundles are interesting because we can do linear algebra with them. More precisely, *any continuous operation that can be done on the fibers extends to the whole bundle*. In technical terms this means that any functor[1] on the category $\mathbf{Vec}_{\mathbb{R}}$ of vector spaces have an analogue on the category $\mathbf{Vec}_X$ of vector bundles over X (for a proof see e.g [50]). So, for instance, we can talk about the direct sum and the tensor product between two vector bundles E and E' over X. These will be bundles $E \oplus E$ and $E \otimes E'$ over X whose fibers at each x are exactly the direct sum $E_x \oplus E'_x$ and the tensor product $E_x \otimes E'_x$. From the physical viewpoint, the class of vector bundles are interesting because the configurations of many physical systems are given by sections of this type of bundles, as will be discussed in the chapters 11-13.

Functors

Now, in order to see that categorical language really is the natural language to talk about connections between different areas of mathematics, we notice that the action "to specify an area of math" defines itself an area of mathematics! This means that we can talk about mappings between different categories (and therefore, between different areas of mathematics). These mappings are called *functors*. In other words, there is a category **Cat**, whose objects are categories and whose morphisms are functors. The area of math delimited by **Cat** is just *category theory*.

But, what is a functor? To define it, remember that in general a mapping is a kind of rule that preserve all structures. A category is an entity with objects, morphisms, compositions and identities. So, a functor between $\mathbf{C}$ and $\mathbf{D}$ is just a rule $F : \mathbf{C} \to \mathbf{D}$ that take objects and morphisms of $\mathbf{C}$ and return objects and morphisms of $\mathbf{D}$. Furthermore, F must satisfy

$$F(g \circ f) = F(g) \circ F(f) \quad \text{and} \quad F(id_X) = id_{F(X)}. \tag{1.1.2}$$

Example 1.8 (*canonical functors*)**.** There are functors which can be build in any category, meaning that they are not special features of certain theories nor nontrivial ways to connect distinct areas of mathematics, but simply part of the general categorical language. One example is the *inclusion functor* $\imath : \mathbf{D} \to \mathbf{C}$ of a subcategory into the larger category. It acts on objects and on morphisms as inclusion functions. We also have the *projection functor* $\jmath : \mathbf{C} \to \mathbf{C}/\simeq$ of a category into some of its quotients. At objects it act as the identity function and at morphisms as quotient maps. For any given object $X \in \mathbf{C}$ we can build two functors h_X and h^X from $\mathbf{C}$ to **Set**, called the *hom-functors of X*, defined as follows: at objects they assign to any Y the set of morphisms $Y \to X$ and the set of morphisms $X \to Y$, respectively. In other words,

$$h_X(Y) := \operatorname{Mor}_{\mathbf{C}}(Y; X) \quad \text{and} \quad h^X(Y) := \operatorname{Mor}_{\mathbf{C}}(X; Y).$$

[1] Indeed, any *continuous functor*.

At morphisms, on the other hand, they act by composition on the left and on the right, respectively.

To explain the relevance of functors, recall that in the most diverse areas of math we consider entities only up to certain notion of equivalence. For example, in set theory we consider only sets up to bijections and in topology we are interested in topological spaces only up to homeomorphisms. In both cases, an *isomorphism* between two of these objects is a morphism $f : X \to Y$ which admits an inverse, in the sense that there is another morphism $g : Y \to X$ satisfying $g \circ f = id_X$ and $f \circ g = id_Y$. We observe that this notion makes sense not only in **Set** and **Top**, but indeed in *any category*, so that in *any area of math* we have a natural notion of isomorphism and there are no interest in the objects itself, but only in their classs of isomorphisms.

Example 1.9 (*isomorphisms*)**.** In algebraic categories the previous notion of isomorphism coincides with the classical notion of linear isomorphism between groups, rings, modules, vector spaces, and so on. Clearly, in topological categories, two topological spaces are isomorphic iff they are homeomorphic. In **Cell** two cell-complexes are isomorphic precisely when there are homeomorphisms which preserve the cellular structure. In $\mathbf{Hilb}_{\mathbb{C}}$ two Hilbert spaces are isomorphic when they are unitarily equivalent in the sense that there are linear isomorphisms between the underlying spaces which (and its inverses) are unitary operators. In **Diff** two manifolds are isomorphic iff they are diffeomorphic.

The fundamental fact concerning functors is that the properties (1.1.2) automatically imply that any functor maps isomorphic objects of a certain category into isomorphic objects in another category. Therefore, *in order to show that two objects $X, Y \in \mathbf{C}$ are not isomorphic, it is enough to give a functor $F : \mathbf{C} \to \mathbf{D}$ such that $F(X)$ and $F(Y)$ are not isomorphic*. In other words, *functors are very natural sources of invariants*!

Example 1.10 (*automorphism functor*)**.** We define the set of *automorphisms* of an object $X \in \mathbf{C}$ as the collection of all isomorphisms $X \to X$. This set is clearly a group with the operation of composition of morphisms (the neutral element is just the identity id_X). As can be easily verified this construction extends to a functor $\mathrm{Aut} : \mathbf{C} \to \mathbf{Grp}$. Therefore, each category has a canonical algebraic invariant. It happens that *this invariant in general is very difficult to compute!* For instance, when $\mathbf{C} = \mathbf{Top}$ the correspondent invariant is the homeomorphism group of a topological space and presently it is an open problem to determine even the complete structure of $\mathrm{Homeo}(\mathbb{R}^n)$! This shows that simply building functors is not sufficient. Indeed, *we need to search for functors whose correspondent invariant can be easily calculated.*

1.2 Invariants

In this subsection we will analyze some useful examples of invariants which are defined by functors and for which we have good strategies of computation. They will be important for future discussion, specially as motivation to very abstract constructions (we are of the opinion that categorical language becomes much more clear when we have some examples in mind).

The immediate question is: where we can find this class of functors? We notice that the classification problem for algebraic structures is in general more simpler than the classification problem for topological structures. Indeed, a homormorphism between algebraic entities in a isomorphism precisely when it is bijective, what is clearly not valid in the topological context. Really, there exists functions which are continuous and bijective but whose inverse is not continuous. The classical example is the complex exponentiation map $e^i : \mathbb{R} \to \mathbb{S}^1$ restricted to some interval as $[0, 2\pi)$.

Therefore, with the classification problem in mind, the functors $F : \mathbf{Top} \to \mathbf{Alg}$ are very natural. The area of mathematics that attempt to build and study these functors is the *Algebraic Topology*. The most known (and also the most important) examples of these functors are described by some hom-functors and the invariants associated to them are different flavors of homotopy and cohomology theories, for which we have nice computational strategies. It is exactly these invariant that will be discussed here.

Remark. The present section can be understood as a crash course on Algebraic Topology (for complete references on the subject see [141, 199, 200, 57] and the references therein. A more analytic approach is presented in [32]).

Homotopy Groups

We start with the *homotopy groups*. They are invariants of pointed topological spaces defined as follows: fix in the sphere $\mathbb{S}^n$ the canonical base point. Then, for any $X \in \mathbf{Top}_*$ with base point $x \in X$, define its *nth homotopy group based on x* as being the set

$$\pi_n(X, x) := \mathrm{Mor}_{\mathrm{Ho}(\mathbf{Top}_*)}(\mathbb{S}^n, X) \equiv [\mathbb{S}^n, X].$$

The rules $\pi_n : \mathrm{Ho}(\mathbf{Top}_*) \to \mathbf{Set}$ are clearly functorial and, for $n \geq 2$, they indeed takes values in the category **AbGrp** of abelian groups, as will become clear later.

Let us expend some time analyzing this example in more detail. With the homotopy groups in hand, we can ask: what can be done with them? We can, for instance, build a *more softer homotopy theory*. This is done analyzing the similarities between two spaces by comparing their homotopy groups. More precisely, we define a new category $\mathbf{Top}_*[W^{-1}]$, usually called the *topological derived category*, whose objects are just topological spaces but whose isomorphisms are the *weak homotopy equivalences*: continuous functions $f : X \to Y$ which induces isomorphism between $\pi_n(X, x)$ and $\pi_n(Y, y)$ for any n.

We have the following analogy: if we think of $\mathbf{Top}_*$ as being the set of smooth real functions, then the nth homotopy group based on x is the analogue of the nth derivative at x. Given a smooth function we can consider its Taylor expansion around a certain point. Similarly, given a space we can consider its sequence of homotopy groups based on a point. In this analogy, weak homotopic spaces corresponds to smooth functions having the same Taylor expansion.

In principle, a smooth function can or cannot be represented by its Taylor series. The *analytic functions* are those that can be represented. So, we can ask: is there a weak homotopical notion analogue to that analytic function? More precisely, is there some class of topological spaces in which two spaces are homotopically equivalent iff they have the same homotopy groups? A classical result in homotopy theory, called *Whitehead's Theorem*, says that such class of spaces really exist: it is the class of CW-complexes introduced in the last section.

Now, we notice that there are few analytic functions in comparison to smooth functions. More precisely, the space of smooth functions that fail to be analytic in each point is dense (when considered in the space of continuous functions, with the canonical C^0-topology). Up to this point we considered analytical facts and searched for analogues in homotopy theory. So, following such philosophy, it is expected that there are few CW-complexes in comparison to arbitrary topological spaces. But, surprisingly, this is not the case: it can be shown that *any topological space is weakly homotopic to a CW-complex*! This shows that the weak homotopy theory produced by the homotopy groups is very well behaved.

weak homotopy theory	real analysis
topological space X	C^{∞} function $X : \mathbb{R} \to \mathbb{R}$
nth homotopy group $\pi_n(X, x)$	nth derivative $D^n X_x$
topological derived category	smooth functions and Taylor series
weak homotopic (w.h.) spaces	smooth functions having the same Taylor expansion
CW-complexes	analytic functions
every space is w.h. to a CW-complex	???

Table 1.1: weak homotopy theory *vs* real analysis

The lesson from the previous discussion is the following:

Conclusion. Generally, an arbitrary category $\mathbf{C}$ is too complex. Functors $F : \mathbf{C} \to \mathbf{D}$ (or, more generally, sequences of functors $F_n : \mathbf{C} \to \mathbf{D}$) can then be used in order to produce a mathematical theory which is softer than the theory described by $\mathbf{C}$. In this new theory, two objects $X, Y \in \mathbf{C}$ are distinguished by comparison of the invariants $F_n(X)$ and $F_n(Y)$. More precisely, the functors F_n select a distinguished class W of morphisms in $\mathbf{C}$: the maps $f : X \to Y$ such that $F_n(f) : F(X) \to F(Y)$ is an isomorphism for each n. We then build the *derived category* $\mathbf{C}[W^{-1}]$ with respect to this distinguished class, in which any $f \in W$ turns to be an isomorphism. We can make an analogy between this new theory and smooth maps as was done for the homotopy groups. This new more softer version of $\mathbf{C}$ may or not be well behaved in the sense that the two final lines of the previous table may or not holds for F_n.

Now, we can return to giving examples of invariants which are defined by functors. We will discuss many flavors of cohomology theories, starting by the *nonabelian cohomology*.

Nonabelian Cohomology

For a given topological group G we can build a space BG, called the *universal space* for G. There are many ways to do this. In the following construction, the obtained space will be a CW-complex but in general it is only weakly homotopic to one (recall the previous discussion).

We start by considering products $G^n = G \times ... \times G$ and thinking of them as being a space of n-cells. In order to glue the cells, we need attaching maps $G^{n-1} \to G^n$. For each n we have $n-1$ of them: those that for each sequence of $n-1$ elements assign a sequence with these elements in the same order plus the neutral element $e \in G$ in the ith position:

$$(g_1, ..., g_{n-1}) \mapsto (g_1, ..., g_{i-1}, e, g_i, ..., g_{n-1}).$$

Gluing the cells with such attaching maps and taking the limit as $n \to \infty$ we get the sapce BG (this is the *geometric realization process*, which will be discussed in a more general context later). Now we can consider the hom-functor $[-, BG]$ on $\mathrm{Ho}(\mathbf{Top})$ represented by BG.

The invariant associated to it is called the *nonabelian cohomology* with coefficients on G. The classification theorem of bundles states precisely that such invariant computes the isomorphic classes of G-principal bundles. More precisely, for a suitable space X (paracompactness is enough), let $\mathrm{Iso}_G(X)$ be the space of isomorphism classes of G-bundles over X. Then we have natural bijections

$$\mathrm{Iso}_G(X) \simeq [X, BG], \tag{1.2.1}$$

which means that (up to equivalences) to give a G-bundle $P \to X$ is just the same as giving a continuous function $X \to BG$!

The equivalence (1.2.1) is given by the *pullback construction*. More precisely, there is a canonical bundle $EG \to BG$ (with EG weakly equivalent to the trivial space) and for any continuous $f : X \to BG$ we can build a bundle $f^*EG \to X$, called the *pullback of EG by f*, together with maps π_1 and π_2, which are characterized as being the universal data that turns commutative the following diagram:

$$\begin{tikzcd} f^*EG \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & EG \arrow[d] \\ X \arrow[r, "f"'] & BG \end{tikzcd}$$

This construction is such that if $g \simeq f$, then the bundles f^*EG and g^*EG are isomorphic, which theefore produces a map $[X, BG] \to \mathrm{Iso}_G(X)$. The classification theorem says precisely that this map is bijective.

Generalized Cohomology

Other invariants obtained in Algebraic Topology are the *generalized cohomology theories*. They can be understood as being the complete dualization of the homotopy groups. In order to make this precise, notice that we can obtain $\mathbb{S}^n$ recursively as being $\Sigma\mathbb{S}^{n-1} \simeq \mathbb{S}^n$, where here Σ is the *reduced suspension*[2]. A sequence of pointed spaces $\mathbb{E} = (E_n)$ together equivalences $\Sigma E_n \simeq E_{n+1}$ is called a *suspension spectrum*. So, the homotopy groups of a space X are defined precisely by the suspension spectrum $E_n = \mathbb{S}^n$:

$$\pi_n(X, x_o) = [E_n, X].$$

It happens that Σ has a dual operation Ω (the *loop space operation*[3]) in the sense that for any X and Y we have bijections $[\Sigma X, Y] \simeq [X, \Omega Y]$. Therefore, we can talk about *spectrum* (or Ω-*spectrum*). These are sequences $\mathbb{A} = (A^n)$ of spaces such that $A^n \simeq \Omega A^{n+1}$. The total dualization of homotopy groups are then the sets defined by

$$H^n(X, \mathbb{A}) := [X, A^n],$$

[2] We recall that ΣX is the pointed space obtained by collapsing the top and the base of the cilinder $X \times I$ and then considering the base point given by the equivalence class of the base point of X.

[3] As will be proved later, for large class of spaces, the loop space ΩX is precisely the set of all continuous maps $f : \mathbb{S}^1 \to X$ endowed with the compact-open topology, justifying the name "loop space". In other words, we will see that the *loop space of X is just the space of loops into X*.

each called the nth *reduced generalized cohomology group of X with coefficients in* $\mathbb{A}$. Important examples to have in mind includes the following.

Example 1.11 (*singular cohomology*)**.** Let G be an abelian topological group. The *Eilenberg-Mac Lane* spaces for G are some sequence of spaces $K(G, n)$ satisfying

$$\pi_i(K(G,n)) \simeq \begin{cases} G, & i = n \\ 0, & i \neq n. \end{cases}$$

Note that such sequence of spaces is unique only up to weak homotopy equivalences. Indeed, by definition we are constraining all homotopy groups. For instance, when the group G is discrete (as the additive groups $\mathbb{Z}$ or $\mathbb{Q}$), we have $K(G, n) \simeq B^n G$, where $B^n G = B...(B(BG))$ and BG is the classifying space of G. We notice that in the general case the sequence $K(G, n)$ is a Ω-spectrum: the *Eilenberg-Mac Lane spectrum*. Indeed,

$$\pi_i(\Omega K(G, n+1)) \simeq \pi_{i+1}(K(G, n+1))$$

which is nontrivial only for $i = n$. Therefore, the Eilenberg-Mac Lane spaces being determined up to weak equivalences, we will have

$$\Omega K(G, n+1) \simeq K(G, n),$$

showing that they really define a spectrum. The correspondent cohomology is the *reduced singular cohomology with coefficients in G*. For $G = \mathbb{Z}$ it is also called the *standard cohomology* or the *ordinary cohomology* and the cohomology groups are generally denoted by $H^n_{\text{sing}}(X)$.

Example 1.12 (*complex K-theory*)**.** Let BU be the space obtained taking the limit $n \to \infty$ over the classifying space $BU(n)$ of the unitary group $U(n)$. Similarly, let U be $\lim U(n)$. We assert that the sequence $(\mathbb{KU})_n = \Omega^n BU$ is a spectrum (usually called the *Bott spectrum*). First of all, as will be proven later, for any group G we have[4] $\Omega BG \simeq G$. Therefore,

$$\begin{aligned} \Omega BU &= [\mathbb{S}^1; BU] \\ &= [\mathbb{S}^1; \lim BU(n)] \\ &\simeq \lim [\mathbb{S}^1; BU(n)] \\ &= \lim \Omega BU(n) \\ &\simeq \lim U(n) = U, \end{aligned}$$

where we used that hom-functors preserve limits, which will become clear later. On the other hand, the *Bott periodicity theorem* states that $\Omega^2 BU \simeq BU \times \mathbb{Z}$. Therefore,

$$\begin{aligned} \Omega^3 BU &\simeq \Omega(BU \times \mathbb{Z}) \\ &\simeq \Omega BU \times \Omega \mathbb{Z} \\ &\simeq U \times [\mathbb{S}^1, K(\mathbb{Z}, 0)]. \\ &\simeq U \times H^0(\mathbb{S}^1; \mathbb{Z}) \\ &\simeq U \times pt \simeq U, \end{aligned}$$

[4] Thanks to this fact we also say that the classifying space of a group is its *delooping space*.

where we used that hom-functors preserve products. Thus, we have $\Omega^3 BU \simeq \Omega BU$ showing that the sequence $(\mathbb{KU})_n$ is periodic and, therefore, that $\mathbb{KU}$ really is a spectrum. The corresponding cohomology is the *complex topological K-theory* and the only nonequivalent cohomology groups are denoted by $KU(X)$ and $KU^1(X)$. Textbooks on (complex) K-theory include [11, 109, 164].

Some remarks on K-theory.

1. *comparison to nonabelian cohomology.* Notice that, for G a Lie group, the nonabelian cohomology with values in G classify G-bundles. This flavor of cohomology is given by the hom-functor represented by the classified space BG. So, for a **fixed** n, the space $BU(n)$ classify all $U(n)$-bundles (i.e, all n-dimensional complex vector bundles). Thanks to the periodicity, the spectrum of K-theory is composed only by two distinguished spaces: $KU_0 = BU$ and $KU_1 = \Omega BU$. Notice that $BU = \lim BU(n)$, which take into account $U(n)$ for **all** n. Therefore, *while nonabelian cohomology classify bundles with **fixed** dimension, the 0th cohomology group $[-, BU]$ of K-theory classify complex bundles of **arbitrary dimension**.* This suggest that K-theory can be obtained as some "completion" of nonabelian cohomology, which is really the case.

2. *comparison to ordinary cohomology*. Above we compared K-theory (a generalized cohomology theory) with nonabelian cohomology (which is not a generalized cohomology). In order to compare K-theory with another ordinary cohomology (which is also a generalized cohomology theory), notice that
$$
\begin{aligned}
H^n_{\mathrm{sing}}(\mathbb{S}^2) &= [\mathbb{S}^2, K(n;\mathbb{Z})] \\
&\simeq [\Sigma^2\mathbb{S}^0, K(n;\mathbb{Z})] \\
&\simeq [\mathbb{S}^0, \Omega^2 K(n;\mathbb{Z})] \\
&\simeq [\mathbb{S}^0, K(n-2;\mathbb{Z})] \\
&\simeq \pi_0(K(n-2;\mathbb{Z})),
\end{aligned}
$$
which is nontrivial (and equal to $\mathbb{Z}$) only for $n = 2$. Particularly, $H^0_{\mathrm{sing}}(\mathbb{S}^2)$ is trivial, but
$$
\begin{aligned}
KU(\mathbb{S}^2) &= [\mathbb{S}^2, BU] \\
&= [\Sigma^2\mathbb{S}^0, BU] \\
&\simeq [\mathbb{S}^0, \Omega^2 BU] \\
&\simeq [\mathbb{S}^0, BU] \\
&\simeq \pi_0(BU) \simeq \mathbb{Z},
\end{aligned}
$$
where the first isomorphism is given by the Bott periodicity. This is a crucial difference. Indeed, thanks to a result of John Milnor [153], *all generalized cohomology satisfying $H^0(X, \mathbb{A}) \simeq 0$ over path connected CW-complexes are isomorphic, so that they describe the same invariant* (see also [115]). The fact $KU(\mathbb{S}^2) \simeq \mathbb{Z}$ then reveals that K-theory really give a new invariant.

3. *real K-theory.* Up to this point we introduced only **complex** K-theory. On the other hand, we could also considered **real** K-theory. The construction would be very similar,

only replacing the unitary group $U(n)$ by the orthogonal group $O(n)$. Indeed, in this case we would define

$$O = \lim O(n) \quad \text{and} \quad BO = \lim BO(n),$$

but, instead of periodicity of degree two we would have periodicity of degree eight, i.e, $\Omega^8 BO \simeq BO$. This would imply that $(\mathbb{KO})_n = \Omega^n BO$ really is a spectrum whose nonequivalent cohomology groups are

$$KO(X),\ KO^1(X),\ ...,\ KO^8(X).$$

4. *Bott periodicity and Clifford algebras*. Notice that the procedure applied above in order to get flavors of K-theory relies in the following steps:

   (a) selecting a subgroup $G(n) \subset GL(n)$ for each n;

   (b) considering their stabilization $G = \lim O(n)$;

   (c) showing that the sequence $\Omega^k BG$ is periodic.

   The fact that this strategy works for $O(n)$ and $U(n)$ is intimately related with periodic properties of algebras over $\mathbb{R}$ and $\mathbb{C}$. Indeed, each of these groups is defined by looking at the matrices which preserve certain nongenerated bilinear form on $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively (they are just the canonical inner product and the canonical hermitean inner product). It happens that any $\mathbb{K}$-vector space V endowed with a quadratic form q defines a corresponding algebra $\mathcal{C}\ell(V,q)$, called the *Clifford algebra* of the given pair (V,q). If we write $\mathcal{C}\ell_n(\mathbb{R})$ and $\mathcal{C}\ell_n(\mathbb{C})$ in order to denote the Clifford algebras of $\mathbb{R}^n$ and $\mathbb{C}^n$ with the quadratic forms induced by the canonical inner/hermitean product, then the classification theorem of real and complex Clifford algebras [119] gives[5]

   $$\mathcal{C}\ell_{n+8}(\mathbb{R}) \simeq \mathcal{C}\ell_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{C}\ell_8(\mathbb{R}) \quad \text{and} \quad \mathcal{C}\ell_{n+2}(\mathbb{C}) \simeq \mathcal{C}\ell_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{C}\ell_2(\mathbb{C}).$$

   We say that two $\mathbb{K}$-algebras A and A' are *Morita equivalent* (writing $A \simeq_M A'$) when they are isomorphic up to the tensor product with another $\mathbb{K}$-algebra. So, the classification of Clifford algebras states precisely that for any n we have

   $$\mathcal{C}\ell_{n+8}(\mathbb{R}) \simeq_M \mathcal{C}\ell_n(\mathbb{R}) \quad \text{and} \quad \mathcal{C}\ell_{n+2}(\mathbb{C}) \simeq_M \mathcal{C}\ell_n(\mathbb{C}),$$

   which is an algebraic version of the Bott periodicity for $O(n)$ and $U(n)$. Indeed, we can effectively prove the Bott periodicity from this algebraic version. This was first done by Atiyah, Bott and Shapiro in [13] (see also [119]) for the case of $X = *$. More precisely, there they proved that $KU^{n+2}(*) \simeq KU^n(*)$ and $KO^{n+8}(*) \simeq KO^8(*)$ as a consequence of the above Morita equivalences. A completely algebraic prove of the Bott periodicity in the general case can be founded in [12, 108, 94, 109].

[5] The Clifford algebras are very important in physics. They appear, for instance, when we need to describe physical objects with internal degrees of freedom as the spin. This will be discussed in Chapter 11.

Twisted Cohomology

Another class of invariants arising from functors are the *twisted generalized cohomology theories*. When compared with generalized cohomology theories, they are "twisted" in the same sense in which a nontrivial bundle is "twisted" when compared with a trivial bundle.

More precisely, recall that generalized cohomology are sequences of functors $H^n = [-, A^n]$ for a certain spectrum $\mathbb{A}$. On the other hand, let $\pi : P \to X$ be a bundle. Its *space of sections* is the collection $\Gamma(P)$ of all left inverses of π (i.e, of all maps $s : X \to P$ such that $\pi \circ s = id_X$). It can be considered as a topological space of maps and, therefore, we can take its homotopy class, which will be represented by the same notation.

Now, for a trivial bundle $X \times F \to X$ the space of sections is just the space of maps $X \to F$. In other words, $\Gamma(X \times F) \simeq [X; F]$, which means that *generalized cohomology functors H^n with coefficients on $\mathbb{A}$ are just the global sections functor Γ of trivial bundles with typical fiber A^n*. This motivates us to consider global sections functors for not necessarily trivial bundles whose typical fiber is A^n. These functors are called the *twisted generalized cohomology with coefficients on $\mathbb{A}$*.

When the spectrum is a *ring spectrum* (in a sense that will be introduced in Section 5.3), there is a canonical way to build twisted cohomology theories from ordinary cohomology theories. For instance, each of the three flavors of generalized presented in the last subsection are, indeed, induced by ring spectra. The notion of "twisted cohomology" is important in the quantization process, as will be discussed in Section 9.4. At this moment we will give only one example (see [14, 104, 145]):

Example 1.13 (*twisted K-theory*)**.** Let $\mathcal{H}$ be a separable complex Hilbert space. We say that a linear operator $T : \mathcal{H} \to \mathcal{H}$ is *Fredholm* when it is continuous and has finite dimensional kernel and cokernel[6]. The collection $\mathrm{Fred}(\mathcal{H})$ of such operators is a subset of the space $B(\mathcal{H})$ of all bounded linear operators and, therefore, has an induced topology. Surprisingly, the obtained topological space is weakly equivalent to $\mathbb{KU}_0 = BU$! This reveals an important relation between complex K-theory and functional analysis. Such relation culminates in the *Atiyah-Singer index theorem*, which generalizes many classical results as, for instance, the Gauss-Bonnet theorem of differential geometry and the Riemann-Roch theorem of complex analysis. Now, let $\mathrm{Aut}_{\mathbf{Hilb}_{\mathbb{C}}}(\mathcal{H})$ be the automorphism group of $\mathcal{H}$, which coincides with the group $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$. Such group acts on the space $\mathrm{Fred}(\mathcal{H})$ by conjugation. But observe that if we multiply an unitary operator by a phase (i.e,by an element of the form $e^{i\theta}$) then the result is also unitary. Therefore, we can consider the projective group $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ which also acts by conjugation on $\mathrm{Fred}(\mathcal{H})$. By the previous relation between K-theory and functional analysis we then have an action

$$PU(\mathcal{H}) \times KU_0 \to KU_0.$$

Consequently, any $PU(\mathcal{H})$-principal bundle $\pi : P \to X$ induces an $PU(\mathcal{H})$-bundle over X with typical fiber $\mathbb{KU}_0$. Its space of sections is the so called *twisted K-theory of X with twisting π*.

In the last example we concluded that any $PU(\mathcal{H})$-principal bundle produces a twisted version of complex K-theory. A natural question is then about the existence of nontrivial bundles. We

[6]Recall that the *kernel* of a linear operator T is the subespace $\ker T$ of all vectors $\psi \in \mathcal{H}$ such that $T(\psi) = 0$, while their *cokernel* is the quotient space $\mathrm{coker}(T)$ of $\mathcal{H}$ by the image of T.

will would like to do a brief remark concerning this question (see [14]).

Remark. It can be show that the topological group $U(\mathcal{H})$ is contractible for any *infinite-dimensional*[7] $\mathcal{H}$. Furthermore, the action of $U(1)\times U(\mathcal{H})\rightarrow U(\mathcal{H})$ is free. Consequently, the unitary group is a model to $EU(1)$, so that the quotient $U(\mathcal{H})/U(1)$ is a model to $BU(1)$. But this quotient is just the projective group. Therefore, *bundles structured by $PU(\mathcal{H})$ are the same as bundles structured by $BU(1)$*. It happens that this classifying space models the Eilenberg-MacLane space $K(\mathbb{Z},2)$. Indeed,

$$\begin{aligned}
\pi_n(BU(1)) &= [\mathbb{S}^n, BU(1)]\\
&\simeq [\Sigma\mathbb{S}^{n-1}, BU(1)]\\
&\simeq [\mathbb{S}^{n-1}, \Omega BU(1)]\\
&\simeq [\mathbb{S}^{n-1}, U(1)]\\
&\simeq [\mathbb{S}^{n-1}, \mathbb{S}^1]\\
&= \pi_{n-1}(\mathbb{S}^1)
\end{aligned}$$

which is nontrivial and equal to $\mathbb{Z}$ only when $n=2$. Therefore, by the classification theorem of bundles and by the previous identifications we have

$$\begin{aligned}
\mathrm{Iso}_{PU(\mathcal{H})}(X) &\simeq [X, BPU(\mathcal{H})]\\
&\simeq [X, BK(\mathbb{Z},2)]\\
&\simeq [X, K(\mathbb{Z},3)]\\
&\simeq H^3(X;\mathbb{Z}),
\end{aligned}$$

showing that *any singular cohomology class of degree 3 on X defines a different twisted K-theory over X*. Because of this, we usually say that *complex K-theory receives twisting from the singular cohomology.*

Algebraic Cohomology

All the previous examples of invariants were obtained directly from methods of Algebraic Topology meaning that the underlying functors were always defined in some topological category. Now will give an example of an invariant that can be produced by different (but similar) methods. It is the *algebraic cohomology*.

Recall that a $\mathbb{Z}$-graded R-module M with degree $z\geq 0$ can be viewed as a sequence of R-modules M_n together with connecting maps $\partial_n : M_n\rightarrow M_{n+z}$. Our problem here is to find the relation between the structures of M_n and its neighbors. For instance, the kernel of ∂_n is on M_n and its image is on M_{n+z}, so that we can use these submodules to compare the structure of the whole module. Particularly, for any n we have that both $\ker(\partial_{n+z})$ and $\mathrm{img}(\partial_n)$ are contained in the same module, so that the quotient

$$H^n(M;R) := \ker(\partial_{n+z})/\mathrm{img}(\partial_n) \tag{1.2.2}$$

[7] We observe that the result is **false** in the finite dimensional case. For instance, if we consider $\mathcal{H}=\mathbb{C}$ (an unidimensional space), then $U(\mathcal{H})\simeq U(1)\simeq\mathbb{S}^1$ which clearly is not contractible.

says, in a certain sense, which elements of M_{n+z} are not totally determined by elements of M_n, which is a very nice information in the context of our problem! But observe that such quotient makes sense only if $\mathrm{img}(\partial_n) \subseteq \ker(\partial_{n+z})$ which is true iff $\partial_{n+z} \circ \partial_z = 0$. In this case, the quotient module (1.2.2) is called the *nth cohomology group* of the graded module M.

The previous discussion clarifies the interest in those $\mathbb{Z}$-graded modules of degree $z \geq 0$ where we always have $\partial \circ \partial = 0$. These are called the *cochain complexes of degree* z and for them the algebraic cohomology groups can always be constructed. We could similarly talk about *chain complexes of degree* z, with $z \leq 0$. In this case would have a dual construction, corresponding to the *homology groups*, which would be defined by

$$H_n(M; R) := \mathrm{img}(\partial_n)/\ker(\partial_{n+z}).$$

We notice that the cohomology and homology group constructions extend to functors H^n and H_n, respectively defined on the full subcategories $\mathbf{CCh}_R^z$ and $\mathbf{Ch}_R^z$ of cochain complex and chain complex of degree z. We usually consider $z = \pm 1$, writing $\mathbf{CCh}_R$ and $\mathbf{Ch}_R$ to denote the correspondent categories. The area of mathematics determined by such categories is *homological algebra*.

Remark. In some cases it is more interesting to work with cochain and chain complexes that are bounded, meaning that they are graded over $\mathbb{N}$ (instead of $\mathbb{Z}$) or, equivalently, that they are trivial in negative n. The correspondent categories will be denoted by $\mathbf{CCh}_R^+$ and $\mathbf{Ch}_R^+$.

As discussed in the context of homotopy groups, the sequence of functors H^n determines a distinguished class of morphisms: the so called *quasi-isomorphisms* $f : M \to N$, which induce an isomorphism $H^n(f) : H^n(M; R) \to H^n(N; R)$ between the nth cohomology group, for any n. With them in hand we define the *algebraic derived category* $\mathbf{CCh}_R[W^{-1}]$ as a first approximation to the category of cochain complexes. We then ask if this approximation is really good in the sense that we can build a table for the H^n analogous to the Table 1.1. Surprisingly, the answer is *yes*! Indeed, *restricting ourselves to the subcategory of bounded complexes over* $\mathbb{N}$ we can build at least *two* dual copies of this table in the algebraic context. This means that there are *two* good models to the homotopy theory defined by the algebraic cohomology groups.

In the first, the analogue of the CW-complexes are the cochain complexes of *projective modules* and in the second model the analogue are the cochain complexes of *injective modules*. Because of this we say that we have the *projective model* and the *injective model* to homotopy theory described by the algebraic one. For instance, the last line of Table 1.1 applied to each of these models says that any cochain complex is quasi-isomorphic to a projective complex and to an injective complex.

The similarity between homological algebra and classical homotopy theory is even more interesting. Indeed, recall that the homotopy groups π_n are defined on the *homotopy category* $\mathrm{Ho}(\mathbf{Top}_*)$. Such category is obtained from $\mathbf{Top}_*$ by defining an equivalence relation on each set of morphisms. Furthermore, by the Whitehead's theorem the CW-complexes are precisely the category of topological spaces for which the isomorphic classes of $\mathrm{Ho}(\mathbf{Top}_*)$ is equivalent to the isomorphic classes of the derived category $\mathbf{Top}_*[W^{-1}]$.

It happens that in homological algebra we have totally analogous facts! More precisely, we can also define the notion of *algebraic homotopy* between cochain maps which induces equivalence relations in each set of morphisms of $\mathbf{CCh}_R$, allowing us to define the quotient category $\mathrm{Ho}(\mathbf{CCh}_R)$.

In the context of bounded cochain complexes an algebraic version of Whitehead's theorem also holds: *for projective or injective complexes, the class of quasi-isomorphisms is equivalent to the class of algebraic homotopy equivalences, meaning that two projective or injective cochain complexes are quasi-isomorphic iff they are algebraically homotopic* (for details, see any nice text of Homological Algebra, e.g [78], or any text on model category theory, e.g [100, 96, 146]).

This similarity between homological algebra and classical homotopy will be explored later.

Quantum Field Theories

Other important invariants are those coming from functors defined on the category $\mathbf{Cob}_n$ of cobordisms and taking values in some algebraic category. From the mathematical viewpoint, they are relevant because they help us identify when two given manifolds are not cobordant. From the physical viewpoint, such functors are essentially the structure that describes *quantum field theories*, as will be discussed in the chapter 14.

We observe that although they have not been defined by the methods of Algebraic Topology, it is expected that quantum field theories can somehow be classified by a spectrum and, therefore, by a genuine cohomology theory! We will try to motivate this fact here. We start by noticing that any functor $U:\mathbf{Cob}_n\rightarrow\mathbf{Alg}$ maps cobordism classes into isomorphism classes of algebraic structures. So, in classifying these cobordism classes we are essentially classifying the functors. The fundamental fact (due to René Thom) is that the cobordism classes are determined by a suspension spectrum.

To be more precise, let $V\rightarrow X$ be a vector bundle associated to some $O(n)$-principal bundle over X. So we have an inner product g_x on each fiber V_x which varies continuously with the point x (this will be discussed in more details in the sections 5.3 and 11.3). With such a metric in hand we can define two new bundles: the *disk bundle* $D(V)\rightarrow X$ and the *sphere bundle* $S(V)\rightarrow X$. Its fiber at each $x\in X$ is respectively given by the vectors $v\in V_x$ such that $\Vert v\Vert\leq 1$ and $\Vert v\Vert=1$, where the norm is that defined by g_x. The quotient $D(V)/S(V)$ is the *Thom space* of the bundle V, denoted by $T(V)$.

Now, recall that we have an universal $O(n)$-principal bundle $EO(n)\rightarrow BO(n)$ as well as a canonical action $O(n)\times\mathbb{R}^n\rightarrow\mathbb{R}^n$, so that we also have a canonical vector bundle $V(n)\rightarrow BO(n)$. Applying the previous construction to it we get the Thom space $\mathbb{MO}_n:=T(V(n))$. We assert that the sequence of these spaces is a suspension spectrum: the *Thom spectrum* $\mathbb{MO}$. Indeed,

$$\begin{aligned}\Sigma\mathbb{MO}_n &= \Sigma T(V(n))\\ &\simeq T(V(n)\oplus\mathbb{R})\\ &\simeq T(V(n+1))=\mathbb{MO}_{n+1},\end{aligned}$$

where the first equivalence is a particular case of the general homeomorphism $T(V\oplus\mathbb{R}^n)\simeq\Sigma^nT(V)$ as well and the second equivalence comes from fundamental relations between the classifying spaces $BO(n)$ and $BO(n+1)$. More precisely,

The *Thom theorem* states that the spectrum MO_n determines the set of isomorphism classes of the cobordism categories, in the sense that for each n we have a natural bijection

$$\operatorname{Iso}(\mathbf{Cob}_n)\simeq\lim_k\pi_{n+k}(\mathbb{MO}_k), \tag{1.2.3}$$

where the left-hand side is the set of isomorphic classes of cobordism between n-manifolds. We notice that $\mathrm{Iso}(\mathbf{Cob}_n)$ has the structure of an abelian group induced by the disjoint union of manifolds and, indeed, a structure of $\mathbb{N}$-graded ring given by the cartesian product of manifolds. On the other hand, the Thom spectrum is a special spectrum, called *ring spectrum*, so that summing over n the right-hand side is also a $\mathbb{N}$-graded ring. So, being more precise, the Thom theorem state that the previous bijection is an isomorphism of graded rings.

Remark. The Thom theorem can be generalized in many directions. For instance, it can be generalized by replacing $O(n)$ by some other group $O^k(n)$, obtained from $O(n)$ by killing their first s homotopy groups and maintaining the others. In other words, by a group weakly homotopically characterized by[8]

$$\pi_i(O^s(n)) \simeq \begin{cases} 0, & i < s \\ \pi_i(O(n)), & i \geq s, \end{cases}$$

As an example, $O^0(n) \simeq O(n)$ and $O^1(n) \simeq SO(n)$. The case $O^2(n)$ corresponds to the so-called *spin group* $\mathrm{Spin}(n)$, which will be discussed in Section **??**. The next nonequivalent $O^s(n)$ is for $s = 8$. It can be show that the sequence of the Thom spaces of the universal vector bundles with structural group $O^s(n)$ define a suspension spectrum $\mathbb{MO}^s$. We say that a manifold M has $O^s(n)$-*structure* when their frame bundle $F(M)$ is structured by $O^s(n)$. We can define the category $\mathbf{Cob}_n^s$ of $O^s(n)$-cobordisms. In this case (see [114, 175]), Thom's theorem generalizes as isomorphism of graded rings

$$\mathrm{Iso}(\mathbf{Cob}_n^s) \simeq \lim_k \pi_{n+k}(\mathbb{MO}_k^s).$$

If we take $s = \infty$, then $O^\infty(k) \simeq *$, so that we are working with manifolds whose frame bundle is trivial: these are the *framed manifolds*. The Thom spaces are just spheres $\mathbb{MO}_k^\infty \simeq \mathbb{S}^k$, meaning that the Thom spectrum in the framed case is precisely the sphere spectrum (which define the homotopy groups). So, in this situation that the Thom theorem reduces to the so called *Thom-Pontryagin theorem*:

$$\mathrm{Iso}(\mathbf{Cob}_n^{\mathrm{fram}}) \simeq \lim_k \pi_{n+k}(\mathbb{S}^k).$$

In a totally similar way we can define *complex cobordisms* for complex manifolds, giving a complex Thom spectrum $\mathbb{MU}$, for which we have a corresponding Thom theorem.

Remark. For each of these cobordism theories (say for a group G, which can be $O(n)^s$, $U(n)$ or some other thing) the underlying spectrum is a ring spectrum, so that taking the isomorphisms classes we get a graded ring ΩG_*.

1.3 Principles

Ending our brief discussion on categories and categorical invariants, we will see that category theory relies at three very important principles. The first of them can be used to get a "dual version" of many results. This principle is usually known as the *duality principle*. The others are the *relativity principle* and the *weakening principle*.

[8] This is precisely the definition of the sth step in the Whitead tower of $O(n)$, as will be discussed in the section 8.3.

The relativity principle is the strategy of analysing an object X in a category $\mathbf{C}$ by looking to its interaction with the other objects of $\mathbf{C}$. It is Grothendieck's relative viewpoint used in his formulation of the categorical foundations of algebraic geometry (see [??]). Manifestations of this principle are the use of hom-functors to define invariants and the study of a space by the properties of the bundles over it.

The weakening principle says that every categorical definition/result defined/obtained using only commutative diagrams of functors can be extended to a more general and powerful version.

Duality

The duality principle relies on the following fact: *a category is essentially an ambient in which we can talk about commutative diagrams*. So, by reverting the orientations of each of the arrows of a commutative diagram we get a different diagram (called *dual* to the first) which is also commutative. Therefore, we have the

Duality Principle: *if a result can be proved using only commutative diagrams, then a totally dual result is also valid, whose proof is obtained simply considering the dual diagrams.*

The formalization of this idea comes from the existence of a functor $(-)^{op} : \mathbf{Cat} \to \mathbf{Cat}$, which takes any category $\mathbf{C}$ and gives a new category $\mathbf{C}^{op}$ (called the *opposite category*) having the same objects as $\mathbf{C}$, but whose morphisms $f^{op} : Y \to X$ are precisely the morphisms $f : X \to Y$. Furthermore, to any $F : \mathbf{C} \to \mathbf{D}$ we get a new functor $F^{op} : \mathbf{C}^{op} \to \mathbf{D}^{op}$ such that

$$F^{op}(X) = F(X) \quad \text{and} \quad F^{op}(f^{op}) = F(f)^{op}.$$

A functor defined in the opposite category is usually called a *contravariant functor*. For instance, the functors which define cohomology theory are all *contravariant*. On the other hand, the functors defining the homotopy groups are *covariant*. This emphasizes the duality between homotopy groups and cohomology groups.

Relativity

In set theory (i.e in classical logic) we analyze an object by looking at its elements. With the development of categorical logic we can try a different approach. Indeed, when we define a category we are saying which are the important objects and which are the important mappings between them. We can think about these mappings as ways to introduce some relation between two objects. So we can try to study an object not by looking at it, but by looking at the relations (i.e at the mappings) between it and other objects. This is precisely what the relativity principle tells us.

Relativity Principle: *in order to study a given object we can look to the space of morphisms between this object and other objects.*

The formalization of this fact comes from the existence of a functor $\mathrm{Arr} : \mathbf{Cat} \to \mathbf{Cat}$ that assigns to any category $\mathbf{C}$ another category $\mathrm{Arr}(\mathbf{C})$, called the *arrow categoy*, whose objects are the morphisms of $\mathbf{C}$. So, this functor *changes the focus on objects replacing it with a focus on the morphisms*. A morphism $h : f \Rightarrow g$ in the arrow category (say between maps $f : X \to X'$

and $g : Y \to Y'$) is a *commutative square* between f and g, as in the following diagram. Such commutative squares can be identified with pairs (h, h') such that $h' \circ f = g \circ h$. The composition is given by pasting diagrams.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "h'"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

Some particular subcategories of $\mathrm{Arr}(\mathbf{C})$ are specially interesting. For example, given an object $A \in \mathbf{C}$ we can focus our attention only on the arrows having A as the target space. This define a subcategory $\mathbf{C}/A$, called the *over category*. Its morphisms are commutative squares (h, id_A) and, therefore, can be identified with *commutative triangles* having A at the vertex, as above. By the duality principle we have a dual category $A/\mathbf{C}$, the *under category*. The functoriality of Arr induces functors $\mathbf{C}/$ and $/\mathbf{C}$ from $\mathbf{C}$ to $\mathbf{Cat}$.

Natural Transformations

The next step is to discuss the *weakening principle* in category theory. It relies on the existence of a notion of "mappings between two functors", called *natural transformations*, such that for any two given categories $\mathbf{C}$ and $\mathbf{D}$ we can form a new category $\mathrm{Func}(\mathbf{C}; \mathbf{D})$ whose objects are functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are these natural transformations.

Remark. Recall that any functor define a invariant, so that the existence of a natural transformation between two given functors imply the existence of a connection between the underlying invariants. In other words, *functors that are connected by a transformation will produce non-independent invariants.*

In formal terms, a *natural transformation* between two functors $F, G : \mathbf{C} \to \mathbf{D}$ is simply a rule $\xi : F \Rightarrow G$ which assigns to any object $X \in \mathbf{C}$ a morphism $\xi_X : F(X) \to G(X)$ such that for any mapping $f : X \to Y$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\xi_X"'] & F(Y) \arrow[d, "\xi_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

There is a result, called *Yoneda lemma*, which identify the collection of all natural transformations between two functors when at least one of then is a hom-functor. Indeed, if $F : \mathbf{C}^{op} \to \mathbf{Set}$ is any contravariant or covariant $F : \mathbf{C} \to \mathbf{Set}$ functor, then for a fixed $X \in \mathbf{C}$ the Yoneda lemma gives

$$\mathrm{Nat}(h_X, F) \simeq F(X) \quad \text{or} \quad \mathrm{Nat}(h^X, F) \simeq F(X).$$

Remark. Maybe this seems only a technical result, but it is one of the more important results in categorical language and in the approach to Hilbert's sixth problem which we are developing here. A proof and the intuitive meaning of Yoneda lemma will be discussed in the next chapter by making use of some results developed in Appendix A. Its fundamental role will become clear in the development of the text.

Before presenting the universality principle, let us try to understand the concept of natural transformation by studying the examples below. All of them arise naturally in the context of algebraic topology and play an important role in the axiomatization of physics. So, if the last section could be understood as a crash course on algebraic topology, the next examples can be understood as a second part of this crash course.

Example 1.14 (*characteristic classes and cohomology operations*)**.** Recall that the different flavors of cohomology theories are hom-functors, so that the natural transformations betweem them and any other functor can be characterized via Yoneda lemma. Indeed, if A_k is an object representing some cohomology group and $F : \mathrm{Ho}(\mathbf{Top}_*) \to \mathbf{Set}$ is any contravariant functor, then the transformations $\xi : F \Rightarrow H(-; A_k)$ are in bijection with the elements of $F(A_k)$. The most interesting situations are when F is another cohomology theory (say represented by B_l), because in them the transformations $\xi : H(-, B_l) \Rightarrow H(-; A_k)$ are totally characterized by morphisms $B_l \to A_k$ between the underlying representing objects. Two cases are further special.

- *characteristic classes.* These happen when F is nonabelian G-cohomology and A_k is part of a spetrum $\mathbb{A}$ representing a generalized cohomology theory. Thus, F is a hom-functor represented by BG and the natural transformations $\xi : [-, BG] \Rightarrow H^n_{\mathbb{A}}$ are in bijection with maps $BG \to A_n$. These transformations are called *characteristic classes*. In other words, *characteristic classes are maps from nonabelian cohomology to generalized cohomology.* We notice that, by the classification theorem of bundles we have $[-, BG] \simeq \mathrm{Iso}_G$, so that characteristic classes can also be understood as transformations $\mathrm{Iso}_G \Rightarrow H^n_{\mathbb{A}}$. In this perspective, they are rules assigning to any topological space X a morphism $\xi_X : \mathrm{Iso}_G(X) \to H^n(X, \mathbb{A})$ that associate cohomology classes of X to bundles over X. Therefore, *characteristic class are natural source of bundle invariants.*

- *cohomology operations*. These correspond to the case in which F is a cohomology theory of the same flavor that those classified by A_k. More precisely, if A_k is part of a spectrum $\mathbb{A}$, we then consider F as the cohomology group $H^n_{\mathbb{E}}$ represented by another spectrum $\mathbb{E}$. In this situation, we have transformations $\xi : H^n_{\mathbb{E}} \Rightarrow H^k_{\mathbb{A}}$ which correspond bijectively with maps $E_n \to A_k$. On the other hand, if $A_k = BG$ is the classifying space of some group, then we take F as the nonabelian cohomology for some group H, so that we have transformations $\xi : [-, BH] \Rightarrow [-, BG]$ which are classified by morphisms $BH \to BG$. In any situation we say that the natural transformations in question are *cohomology operations*.

Concrete examples of characteristic classes and cohomology operations are the following:

1. *Stiefel-Whitney and Chern classes*. There are fundamental morphisms

$$c_i : BU(n) \to K(i, \mathbb{Z}) \quad \text{and} \quad w_i : BO(n) \to K(i, \mathbb{Z}_2),$$

   respectively called *Chern morphisms* and *Stiefel-Whitney morphisms*. They define characteristic classes for rank n complex/real bundles. They are fundamental because any other class $\alpha : BU(n) \to K(k, \mathbb{Z})$ can be written in terms of $c_1, ..., c_k$ (similar condition holds for the Stiefel-Whitney classes). In order to be more precise, recall that, as commented in the last section, the Thom spectrum is special because it is an *ring spectrum*, which will mean that its cohomology groups acquires a structure of graded ring. Here, similarly, if R is a ring, then the Eilenberg-Mac Lane spectrum $K(n, R)$ becomes a ring spectrum and, therefore, the sum $H(X; R)$ of the cohomology groups $H^n(X; R)$ is a ring, meaning that we know how to multiply cohomology classes. It can be show that we have ring isomorphisms

$$H(BU(n); \mathbb{Z}) \simeq \mathbb{Z}[c_1, ..., c_n] \quad \text{and} \quad H(BO(n); \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, ..., w_n].$$

2. <u>*Chern character*</u>. As commented, the fundamental characteristic classes for complex bundles are the Chern classes, meaning that any other class can always be built in terms of them. An important example is the *Chern character*. In order to build it we will make use of the presentation of complex K-theory by vector bundles. So, given a space, let $G(\mathrm{Vect}_X) \simeq [X, BU \times \mathbb{Z}]$ be the Grothendieck ring completion of isomorphism classes of vector bundles over X. A fundamental result in K-theory is the *splitting principle*, which says that any bundle can be pulled back to a decomposition into line bundles in such a way that both K-theory and singular cohomology of the initial bundle are embedded into the K-theory and singular cohomology of the decomposed bundle (see [11, 109]). More precisely, for any bundle $E \to X$ over X there is a map $p : F(E) \to X$ such that p^*E decomposes as a direct sum $L_1 \oplus ... \oplus L_n$ of line bundles $L_i \to F(E)$ at the same time as the induced maps

$$p_* : KU(X) \to KU(F(E)) \quad \text{and} \quad p_* : H_{\mathrm{sing}}(X; \mathbb{Z}) \to H_{\mathrm{sing}}(F(E); \mathbb{Z})$$

are embeddings. Now, following [141, 119], let us make use of the splitting principle in order to build a morphism[9] $\hat{f} : KU(X) \to H_{\mathrm{sing}}(X; R)$ starting from any formal sum $f(t) = \sum a_i t^i$ defined in an arbitrary ring extension R of $\mathbb{Z}$. The Chern character will then be obtained as a particular case for $R = \mathbb{Q}$ and $f(t) = \sum t^i/i!$. Let $E \to X$ be a complex bundle representing a class in the K-theory $KU(X)$ of X. By the splitting principle this class has a representative that decomposes as a sum of line bundles $L_1 \oplus ... \oplus L_n$, so that it is enough to define $\hat{f}(E) \in H_{\mathrm{sing}}(X; R)$ in this situation. Indeed, we put

$$\hat{f}(E) = \sum_{i=1}^{n} \sum a_j f(c_1(L_i))^j.$$

3. <u>*Steenrod's operations*</u>. The examples above were about characteristic classes. Now, let us give examples of cohomology operations. There are operations $Sq^i : K(n, \mathbb{Z}_2) \to K(n + i, \mathbb{Z}_2)$, which are totally characterized by some properties, being called the *Steenrod "square" operations*. The name comes from one of these characterizing properties. Indeed, for any space X the morphisms

$$Sq^i_X : H^n(X; \mathbb{Z}_2) \to H^{n+i}(X; \mathbb{Z}_2) \quad \text{are such that} \quad Sq^i_X(x) = \begin{cases} x^2, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}$$

[9] Notice that this morphism is between the **whole** K-theory ring and the **whole** singular cohomology ring. Therefore, this should not be a characteristic class as the Chern and the Stiefel-Whitney class, but indeed a genuine "morphism of spectra" between $\mathbb{KU}$ and the Eilenberg-Mac Lane spectrum of R.

The other characterizing properties are $Sq^0_X(x) = x$ and the usually called *Cartan product formula*:

$$Sq^k_X(x \cdot y) = \sum_{i+j=k} Sq^i_X(x) \cdot Sq^j_X(y).$$

These square operations are "the fundamental operations" in the same sense as the Chern classes are the fundamental classes. More precisely, we have an embedding

$$H(K(2, \mathbb{Z}_2); \mathbb{Z}_2) \hookrightarrow \mathbb{Z}_2[Sq^0, Sq^1, ...],$$

so that any cohomology operation of degree 2 in $\mathbb{Z}_2$-cohomology can be written as a polynomial into the Steenrod operations. For the construction and applications of Steenrod's operations, see [151].

4. Adam's operations. A fundamental problem in topology is to determined if the tangent bundle TM of a given n-manifold is trivial or not (in the affirmative case we say that the manifold is *paralellizable*). This is equivalent to asking if M admits precisely n linearly independent vector fields. It is easy to verify that any Lie group is paralellizable, so that the spheres $\mathbb{S}^1 \simeq U(1)$ and $\mathbb{S}^3 \simeq SU(2)$ are parallelizable. A natural problem is to determine if the other spheres $\mathbb{S}^n$, for $n \neq 1, 3$ also are parallelizable. When we have a counting problem like this, the main idea is to make use of axiomatic properties of operations in some cohomology theory. We could try to use Steenrod's operations, but exactly because they are "squared" they only give information on the **even dimensional** spheres. A manifestation of this fact is the Poincaré-Hopf theorem, which can be stated by making use of the singular cohomology Betti numbers (the cohomology in which Steenrod's operations live). Indeed, by Poincaré-Hopf any vector field in an even dimensional sphere has a singularity, so that $\mathbb{S}^n$, with n **even**, is not parallelizable. Thanks to Milnor's uniqueness theorem, all ordinary cohomology has the same Betti-numbers and, therefore, will led us to the same conclusion. So, in order to study the above problem for odd-dimensional spheres we have to consider cohomology operations in non-ordinary cohomology. The natural candidate are operations in K-theory, of which there are *Adam's operations*. It can be shown that these operations are sufficient to solve the problem: only for $n = 1, 3, 7$ the sphere $\mathbb{S}^7$ is parallelizable. This is a classical result, usually known as the *Adam-Athiyah theorem*. The proof will be sketched in Section 4.3.

5. J-homomorphism. Up to this point we given examples of characteristic classes (i.e, maps from nonabelian cohomology to generalized cohomology) and of cohomology operations in generalized cohomology. Now we will give an important example of cohomology operation in nonabelian cohomology. The idea is the following. A fundamental step in the proof of the Adam-Atiyah theorem is the *Hopf construction*. This construction take any map $f : X \times Y \to Z$ and return another map $h_f : X * Y \to \Sigma Z$, where there $X * Y$ is the *join* between X and Y (this is certain "homotopical version" of the product $X \times Y$). This join has the property that $\mathbb{S}^n * \mathbb{S}^m \simeq \mathbb{S}^{n+m-1}$. Now, notice that any element of the orthogonal group $O(n)$ induces a continuous map $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. So, for any fixed k a continuous function $f : \mathbb{S}^k \to O(n)$ can be regarded as a map $\hat{f} : \mathbb{S}^k \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. Applying the Hopf construction we then get a map $h_{\hat{f}} : \mathbb{S}^{k+n} \to \mathbb{S}^n$, where we used $\mathbb{S}^k * \mathbb{S}^{n-1} \simeq \mathbb{S}^{k+n}$ and $\Sigma\mathbb{S}^{n-1} \simeq \mathbb{S}^n$. The starting map f represents a class in the homotopy group $\pi_k(O(n))$, while

the final map $h_{\hat{f}}$ represents a class in $\pi_{n+k}(\mathbb{S}^n)$. It can be proven that the Hopf construction preserve the composition and the identities, so that we just build a homomorphism

$$J_k : \pi_k(O(n)) \to \pi_{n+k}(\mathbb{S}^n),$$

called *kth J-homomorphism*. Taking the limit $n \to \infty$ we then get homomorphisms

$$J_k : \pi_k(O) \to \pi_k(\Omega^\infty \mathbb{S}^\infty).$$

Recall that the homotopy groups of a space determines the whole structure of this space in the topological derived category. Therefore, all the above homomorphisms refines to a map $J : O \to \Omega^\infty \mathbb{S}^\infty$ in the derived category. But both spaces are, indeed, groups in the derived category, so that applying B we get a map $BJ : BO \to B\Omega^\infty \mathbb{S}^\infty$ between the corresponding classifying spaces and, therefore, a cohomology operation in the underlying nonabelian cohomologies.

Universality

With the notion of natural transformation clear in our mind, we can finally introduce the universality principle. It is about the possibility of *weakening* any categorical concept defined using only commutative diagrams. Indeed, recall that the commutativity of a diagram internal to some category $\mathbf{C}$ corresponds, in the end, to the equality between two morphisms of $\mathbf{C}$. Particularly, commutative diagrams in **Cat** correspond to equality between functors (say $F = G$). But we have the notion of *mappings between functors*. So, instead of requiring such equality, we could require only the existence of a mapping ξ between F and G. In other words, we could require that the given diagram of functors commutes only up to natural transformations.

Let us give a more playful interpretation. In order to do this, suppose that the equality $F = G$ describes the validity of some property. So, the existence of a natural transformation ξ between F and G can be understood as an *approximation* to the equality and, therefore, to the validity of the desired property. But we may have morphisms $\xi : F \Rightarrow G$ from F to G as well as morphisms $\varphi : G \Rightarrow F$ from G to F, so that we can take approximations in two different directions. The first will correspond to a *left approximation* and the second to a *right approximation*.

On the other hand, there may exist many such approximations. So, in order to add more rigidity we can look only to certain "good" approximations. Here, being good means that the approximations satisfy some additional condition, which can be described in terms of a commutative diagram involving ξ and φ. Therefore, in the end, we are replacing the requirement of equality between functors by the equality between natural transformations. This is the *weakening principle*.

Weakening Principle: *any concept defined using only commutative diagrams of functors can be weakened by requiring that such diagrams commutes only up to certain well behaved natural transformations.*

Remark. We notice some similarity with homotopy theory: there (as introduced in the last section) we have the notion of homotopy between continuous functions, so that we can replace equality between two maps by the existence of a homotopy between them. Consequently, we can talk about diagrams which are "commutative up to homotopy", allowing us to define "homotopical versions" of many classical concepts. We will return to explore such similarities in future chapters.

There are many conditions that can be required on the natural transformations. For instance, we could require that $\xi : F \Rightarrow G$ and $\varphi : G \Rightarrow F$ satisfy the commutative conditions $\varphi \circ \xi = id_F$ and $\varphi \circ \xi = id_G$, which means simply that ξ is an isomorphism in $\mathrm{Func}(\mathbf{C};\mathbf{D})$ between F and G, whose inverse is φ. In this case, ξ is called a *natural isomorphism* between such functors.

On the other hand, we could require that ξ and φ be the "best approximations possible" in the sense that any other approximation factors uniquely into ξ and φ. More precisely, if $\xi' : F \Rightarrow G$ and $\varphi' : G \Rightarrow F$ are other left and right approximations, then there is a unique u such that $\xi' = u \circ \xi$ and a unique v such that $\varphi' = \varphi \circ v$). In this case, we say that ξ and φ are *universal* or that they satisfy *universality conditions*.

Example 1.15 (*equivalences and adjunctions*)**.** Being a category, **Cat** has an internal notion of isomorphism: two categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* when there are functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F = id_{\mathbf{C}}$ and $F \circ G = id_{\mathbf{D}}$. By the Weakening Principle, we can define new concepts by weakening the previous equality by supposing only the existence of transformations $\xi : G \circ F \Rightarrow id_{\mathbf{C}}$ and $\varphi : F \circ G \Rightarrow id_{\mathbf{D}}$ satisfying additional conditions. Each new concept will give a different way to say that the categories $\mathbf{C}$ and $\mathbf{D}$ are indiscernible. For instance, if ξ and φ are natural isomorphisms, then $\mathbf{C}$ and $\mathbf{D}$ are called *equivalent*. If, on the other hand, they satisfy the commutative condition present in the diagram below, we say that the categories are *adjoint*. We also say that the functors F and G are *adjoints*, writing $F \leftrightharpoons G$.

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\circ\varphi"] \arrow[dr, Rightarrow, "id_F"'] & F \circ G \circ F \arrow[d, Rightarrow, "\varphi\circ F"] \\
& F
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\xi\circ G"] \arrow[dr, Rightarrow, "id_G"'] & G \circ F \circ G \arrow[d, Rightarrow, "G\circ\varphi"] \\
& G
\end{tikzcd}
$$

It can be verified that two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are adjoints iff for any X, Y we have the following bijections, which are supposed to extend to natural isomorphisms:

$$\mathrm{Mor}_{\mathbf{D}}(F(X);Y) \simeq \mathrm{Mor}_{\mathbf{C}}(X;G(Y)).$$

Example 1.16 (*suspension, loop spaces and bundles*)**.** Recall that, as discussed in the last section, in the category of base topological spaces we have two functors Σ and Ω which play a *dual* role in homotopy theory. We introduced this duality as the existence of bijections $[\Sigma X, Y] \simeq [X, \Omega Y]$. Now we understand that they actually means that Σ and Ω are adjoint functors. Similarly, in the example ?? was presented a dual relation between principal bundles and vector bundles. More precisely, recall that we have a rule Fr_n assigning to any rank n vector bundle its frame bundle: a $\mathrm{Aut}(\mathbb{R}^n)$-principal bundle. Reciprocally, given any $\mathrm{Aut}(\mathbb{R}^n)$-principal bundle we can build a vector bundle of rank n: those associated to the canonical action $\mathrm{Aut}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^n)$. This dual relation is the manifestation of the existence of an adjunction between the category of principal bundles and the category of vector bundles.

Example 1.17 (*free objects*)**.** Another very occurring example is a left adjoint $L : \mathbf{D} \to \mathbf{C}$ to the inclusion functor $\imath : \mathbf{D} \hookrightarrow \mathbf{C}$ of a subcategory $\mathbf{D} \subset \mathbf{C}$. If this adjoint exists we say that the category $\mathbf{D}$ *is freely generated by* $\mathbf{C}$. Particularly, if $X \in \mathbf{D}$ is such that $X = L(B)$ for some

$B \in \mathbf{C}$ we say that X *is a free object with basis* B. This generally occurs when $\mathbf{C} = \mathbf{Set}$ and $\mathbf{D}$ is some algebraic category. Indeed, in this case the existence of the adjunction corresponds to the existence of natural bijections

$$\mathrm{Mor}_{\mathbf{Set}}(B; \imath(Y)) \simeq \mathrm{Mor}_{\mathbf{D}}(L(B); Y).$$

It means that an object X is free with basis B iff any function $B \to Y$ extend uniquely to a linear homomorphism $X \to Y$. In other words, X has basis B iff any morphism $X \to Y$ becomes totally determined in B. But this is exactly the condition that defines basis of vector spaces, free modules, free algebras, free groups, and so on.

Remark

We started the chapter giving a "definition" of category. Recall that a category was formally defined as being composed of a **collection** of objects and for any two objects a **collection** of morphisms which are linked by associative operations having neutral elements. We then presented many examples of entities which we expect to satisfy this definition. We would like to end the present chapter saying, on the other hand, that the given definition of category actually is *not* good in order to incorporate these examples.

Let us be more precise. Clearly, the given definition of category depends on the notion of collection. When doing **naive** mathematics we immediately think of a collection as being synonymous to a set, which itself is considered as being a primitive concept. Therefore, in this **naive** approach, there is no problem. On the other hand, if we are trying to do **axiomatic** mathematics, then we need to fix some definition of set. We then say that we are fixing a *presentation* of the notion of set and, therefore, of the notion of category, as below.

$$\begin{array}{c}\textbf{definition of}\\ \textbf{collection}\end{array} \Longrightarrow \begin{array}{c}\textbf{definition of}\\ \textbf{category}\end{array}$$

The canonical presentation is that given by the Zermelo-Fraenkel formulation. Assuming such formulation, we could try to do axiomatic category theory by defining collection as before: *as being a synonymous of set*. But in this case we would meet with "Russell's-like" paradox. For instance, the category **Set** would be composed of the **set of all sets** and of the **set of all functions**, which does not makes sense in the Zermelo-Fraenkel formulation. This means that *this formulation produces a definition of category that is rigorous but not useful*, so that in order to incorporate axiomatically many intuitive examples we need to work in another formulation. Particularly, we need a formulation in which we can define a collection as being something for which the "collection of all sets" really makes sense.

Generally it is fixed the von Neumann–Bernays–Gödel formulation, in which we have the notion of *class of all sets*. Then, setting collection as synonymous of class, our definition of category becomes rigorous and useful. This shows that *the concept of category admits at least one good presentation*. However, almost all the time we will prefer to work in the naive approach instead of in some concrete presentation. By the discussion at the introduction, this means that we will produce *conjectures*, which can be turned into *theorems* when some presentation is fixed.

Chapter 2

Unification

In the previous chapter we constructed our first example of abstract background language: the categorical language. Now we can analyze the relation between the logic underlying this language and the description of the fundamental laws of Physics. As discussed at the Introduction, this relation is a double lane. This means that the logic has a direct influence in the description of the physical laws as well as the physical insight can be used to produce new logic, what can be represented in a diagram:

$$\begin{array}{c}\textbf{categorical}\\ \textbf{logic}\end{array} \underset{\text{insight}}{\overset{\text{modelling}}{\rightleftarrows}} \begin{array}{c}\textbf{physical}\\ \textbf{laws}\end{array}$$

In this chapter we are interested in the possible influences of **categorical language** in the **modelling of physical laws**. In other words, this chapter is primarily about the arrow **logic** $\Rightarrow$ **physics**. We are particularly interested in the unification problem of Physics, so that in the first section we show that the categorical logic really is an abstract logic in the sense that many different mathematical concepts can be unified into a unique universal categorical concept: the *Kan extensions*. We also show that this concept is **coherent**: arbitrary Kan extensions can be totally described knowing few of them.

Having showed that category theory is an abstract language it is natural to ask what categories realize such abstraction. In other words, it is natural to ask about the existence of Kan extensions in arbitrary categories. In the second section we discuss that there are many categories which have fell properties and, therefore, that are not good models to describe physics. But we show that any poor category can be embedded into a category having all Kan extensions. This reveals that *category theory is a language that allows us to abstract any poor area of mathematics in order to produce a new very rich area.*

In the third section, on the other hand, we will see that, despite being very abstract, the categorical language is **not** sufficiently abstract to produce a complete axiomatization of all laws of Physics. We give examples suggesting that more general languages really must exist. We then conjecture the existence of these more abstract languages by giving an idea of how they can be constructed: following a naive process called *categorification*.

Finally, in the fourth section we discuss some direct consequences of the hypothetical categorification process in the description of the foundations of Physics. Particularly, we will see that categorifying the concept of particle physics we get the concept of string physics! We then study corollaries of this fact on the classical and quantum approaches to physics.

2.1 Unifying

In the last chapter we developed a good logic: the *categorical logic*. Let us now see that it is really very abstract in the sense that it can be used to unify many apparently different mathematical concepts. More precisely we will see that many concepts are indeed particular cases of a unique idea: the *Kan extension*. In typical cases, Kan extensions are simply ways to "weakly enlarge" the domain of definition of certain functors.

Let $\mathbf{C}$ be a category and let $F : \mathbf{A} \to \mathbf{D}$ be some functor defined on a subcategory $\mathbf{A} \subset \mathbf{C}$. An *extension* of F from $\mathbf{A}$ to $\mathbf{C}$ is another functor $\overline{F} : \mathbf{C} \to \mathbf{D}$ which coincides with F when restricted to $\mathbf{A}$. This means precisely that $\overline{F} \circ \imath = F$, where $\imath : \mathbf{A} \hookrightarrow \mathbf{C}$ is the inclusion. But this is an *equality between functors*. Therefore, by the Weakening Principle, we can get different notions of "weak extensions" by replacing such equality by the existence of natural transformations satisfying additional conditions.

With this in mind, we define the *left Kan extension* of a functor $F : \mathbf{A} \to \mathbf{D}$ with respect to $\imath : \mathbf{A} \to \mathbf{C}$ as being the left universal approximation to some extension of F from $\mathbf{A}$ to $\mathbf{C}$. In explicit terms this means that it is a functor $L : \mathbf{C} \to \mathbf{D}$ together with a natural transformation $\xi : F \Rightarrow L \circ \imath$ which is universal in the sense that, if $\xi' : F \Rightarrow L' \circ \imath$ is any other transformation, then there is a unique $u : L \Rightarrow L'$ such that the first diagram below is commutative. Similarly, a *right Kan extension* of F with respect to $\imath$ is a functor R together with an universal natural transformation $\varphi : R \circ \imath \Rightarrow F$. That is, such that if $\varphi' : R' \circ \imath \Rightarrow F$ is any other transformation, then there is a unique $u : R' \Rightarrow R$ making commutative the second diagram below (notice that left and right Kan extensions are related precisely by second order duality).

$$
\begin{tikzcd}
 & L' \circ \imath \\
F \arrow[ur, Rightarrow, "\xi'"] \arrow[r, Rightarrow, "\xi"'] & L \circ \imath \arrow[u, Rightarrow, dashed, "u \circ \imath"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & R \circ \imath \arrow[d, Rightarrow, "\varphi"] \\
R' \circ \imath \arrow[ur, Rightarrow, dashed, "u \circ i"] \arrow[r, Rightarrow, "\varphi'"'] & F
\end{tikzcd}
$$

Now, in order to see that Kan extensions really generalize many mathematical concepts, let us consider the most simple case: extensions to the trivial category $\mathbf{C} = \mathbf{1}$ with a unique object $*$ and a unique morphism. In this case, right and left Kan extensions of $F : \mathbf{A} \to \mathbf{D}$ are respectively called *limit of* F and *colimit of* F. Note that in this situation, the right and left Kan extensions are functors $R, L : \mathbf{1} \to \mathbf{D}$ and, therefore, they can be identified with their respective images $R(*) \in \mathbf{D}$ and $L(*) \in \mathbf{D}$, which are usually denoted by $\lim F$ and $\operatorname{colim} F$. Furthermore, the natural transformations $\varphi : R \circ \imath \Rightarrow F$ and $\xi : F \Rightarrow L \circ \imath$ in this case are simply rules that assign to any object $X \in \mathbf{A}$ a morphism $\varphi_X : \lim F \to F(X)$ or $\xi_X : F(X) \to \operatorname{colim} F$ such that the respective diagrams below are commutative.

$$
\begin{tikzcd}
 & F(X) \arrow[dd, "F(f)"] \\
\lim F \arrow[ur, bend left, "\varphi_X"] \arrow[dr, bend right, "\varphi_Y"'] & \\
 & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[dd, "F(f)"'] \arrow[dr, bend left, "\xi_X"] & \\
 & \operatorname{colim} F \\
F(Y) \arrow[ur, bend right, "\xi_Y"'] &
\end{tikzcd}
\tag{2.1.1}
$$

The previous diagrams are usually called the *cone with vertex* $\lim F$ and the *cocone with vertex* $\operatorname{colim} F$. The universality required in the definition of right Kan extensions means precisely that any other cone (with vertex in any other object A) *collapses* in the cone with vertex $\lim F$, in the sense that there exists a unique $u : A \to \lim F$ making commutative the first diagram below. Dual analysis holds, of course, for the universality of left Kan extensions, producing the second diagram below.

$$
\begin{tikzcd}
 & & F(X) \arrow[dd, "F(f)"] \\
A \arrow[r, dashed, "u"] \arrow[urr, bend left] \arrow[drr, bend right] & \lim F \arrow[ur, "\varphi_X"] \arrow[dr, "\varphi_Y"'] & \\
 & & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[dd, "F(f)"'] \arrow[dr, "\xi_X"] \arrow[drr, bend left] & & \\
 & \operatorname{colim} F \arrow[r, dashed, "u"] & A \\
F(Y) \arrow[ur, "\xi_Y"'] \arrow[urr, bend right] & &
\end{tikzcd}
$$

Analyzing the previous conditions for functors F defined on different categories $\mathbf{A}$ we will get different mathematical concepts, which are particular examples of limits/colimits and, therefore, of Kan extensions. In the next subsection we will specialize this discussion to some concrete functors F, defined on some special categories $\mathbf{A}$. Varying these categories we will get different *types/shapes/flavors* of limits/colimits.

Examples

In order to pass from abstract Kan extensions to limits/colimits we considered previously the most simple situation: those in which $\mathbf{C} = \mathbf{1}$. Here we will consider another very simple situation: those in which $\mathbf{A}$ *is a category generated by some quiver*.

A quiver is simply an oriented graph. We have the category **Quiv** of quivers and an obvious inclusion functor $\imath : \mathbf{Cat} \hookrightarrow \mathbf{Quiv}$. Indeed, we can see any category as being a quiver whose vertices are given by the objects and whose arrows linking the vertices are the morphisms between the objects. It happens that this functor has an adjoint P, so that we can talk about *the category freely generated by a quiver*.

In the following we will look to functors $F : P(Q) \to \mathbf{D}$ defined on categories $P(Q)$ generated by very simple quivers Q (these functors are in bijection with copies of the quiver Q internal to $\mathbf{D}$). Even so, the correspondent notions of limits/colimit will be very abstract. Indeed, they will incorporate many important concepts as, for instance, kernels and cokernels of linear maps, gluing of topological spaces, cartesian product of manifolds, direct sum of modules, quotient of groups, orbit spaces of actions, etc.

Example 2.1 (*products and coproducts*)**.** The trivial quiver is composed by a unique vertex and no arrows. Therefore, the most simple nontrivial quiver is one having two vertices (say 1 and 2) and no arrow linking them. Let $P(Q)$ be the category generated by them. So, a functor $F : P(Q) \to \mathbf{D}$ is represented simply by two objects of $\mathbf{D}$ labelled by the vertex of Q. Consequently, a cone for F is simply an object A with maps $\pi_i : A \to X_i$. A limit for F is an universal cone. Universality means that for any other cone $\pi'_i : A' \to X_i$ there is a unique $u : A' \to A$ such that $\pi_i \circ u = \pi'_i$ (see the first diagram below). The universal vertex is called the *binary product* between the objects X_1 and X_2, denoted by $X_1 \times X_2$ instead of $\lim F$. The maps

π_i are called *the canonical projections.*

$$
\begin{tikzcd}
 & & X_1 \\
A' \arrow[r, dashed, "u"] \arrow[urr, bend left, "\pi_1'"] \arrow[drr, bend right, "\pi_2'"'] & X_1 \times X_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[dr, "\imath_1"] \arrow[drr, bend left, "\imath_1'"] & & \\
 & X_1 \oplus X_2 \arrow[r, dashed, "u"] & A' \\
X_2 \arrow[ur, "\xi_Y"] \arrow[urr, bend right] & &
\end{tikzcd}
$$

A similar discussion shows that the colimit of any functor $F : P(Q) \to \mathbf{D}$ is totally characterized by the second diagram above. The vertex is called the *binary coproduct*, as well as the maps $\imath_i$ are the *canonical inclusions.* Furthermore, the given characterization is analogously obtained when P is a quiver with an arbitrary set of vertex and, again, no arrows linking them. Concrete examples to have in mind are the following:

- *products.* When $\mathbf{D}$ is the category **Set** we can effectively build an universal binary product cone for any two given sets X_1 and X_2: the vertex $X_1 \times X_2$ is simply the cartesian product and the canonical projections π_i are the projections pr_i in each variable. More generally, we can build the product between an arbitrary family of sets. A similar construction holds in concrete categories $\mathbf{D} \subset \mathbf{Set}$. Indeed, in this case the idea is to define the binary product as being the cartesian product endowed with the natural structure that turns the projections pr_i into morphisms of $\mathbf{D}$. For instance, in **Top** the most natural topology in $X_1 \times X_2$ making the projections continuous is the product topology: those generated by the set of all products $U \times V$, where $U \subset X_1$ and $V \subset X_2$ are open sets. In algebraic categories, on the other hand, the most natural linear structure that turns the projections into homomorphisms are those defined componentwise. This corresponds, for instance, to the concept of direct product of groups, direct product of rings, product of modules, etc. Similar analysis shows that **Diff** and the other categories of analysis also have products.

- *coproducts.* Dually, the category of sets and any concrete category $\mathbf{D}$ freely generated by sets admits binary coproducts. Indeed, the binary coproduct of two sets is simply their disjoint union $X_1 \sqcup X_2$ and the canonical inclusions are just inclusion maps. In any concrete category $\mathbf{D}$, the coproduct $X_1 \oplus X_2$ is the object freely generated by $X_1 \sqcup X_2$. For instance, in **Top** this corresponds to the topological sum. In groups/rings this is the free product of groups/rings, which in the abelian case is simply the direct sum. We observe that **Diff** is **not** generated by **Set** but it also has coproducts, given by the disjoint union of manifolds.

Example 2.2 (*equalizers and coequalizers*)**.** Now, let us consider the case of a quiver with two vertices and two arrows linking them. A cone for a functor is then an object A together with maps a and b turning commutative the first diagram below. This commutativity means simply that $f \circ a = a \circ g$. So, we can equivalently express a cone by the second diagram supplemented with this condition.

$$
\begin{tikzcd}
 & X_1 \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\
A \arrow[ur, bend left, "a"] \arrow[dr, bend right, "b"'] & \\
 & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "a"] & X_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & X_2
\end{tikzcd}
$$

Therefore, a cone is universal when, for every other cone $a' : A' \to X_1$ satisfying $f \circ a' = a' \circ g$ there is a unique $u : A' \to X_o$ such that $u \circ a = a'$. The universal vertex is called *the equalizer between f and g,* being denoted by $eq(f, g)$.

$$
\begin{tikzcd}
A' \arrow[r, dashed, "u"] \arrow[rr, bend right, "a'"'] & A \arrow[r, "a"] & X_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & X_2
\end{tikzcd}
$$

Simply reverting the arrows we define the colimit of a functor defined on such quiver. The universal vertex is then called the *coequalizer between f and g*, and it is represented by $coeq(f, q)$. Now, let us present some concrete examples:

- *equalisers*. We start by observing that in **Set** we can always build the equalizer between two given functions $f, g : X \to Y$: it is the smallest subset (possibly empty) in which such functions coincide. The map $a : eq(f, g) \to X$ is simply the inclusion map. In **Top** we have a similar construction: given two continuous functions we consider the equalizer between then in **Set** and give to it the subspace topology. This strategy also works for algebraic categories. Indeed, in general the set $\mathbf{Mor}_{\mathbf{Alg}}(X; Y)$ has a group structure, so that we can take the difference $f - g$. Therefore, the smallest subset in which f, g coincides is actually the subset in which the difference $f - g$ equals to the zero map. But this is just the kernel of $f - g$, which has a canonical induced linear structure. This shows that: *in a category whose sets of morphisms are groups and in which we have the notion of kernel we can take the equalizer between any two morphisms f, g by defining $eq(f, g) := \ker(f - g)$.* This clarifies why in the literature (specially in the literature about homological algebra) the name *difference kernel* is taken as a synonymous of equalizer. Particularly, notice that the equalizer between f and 0 is simply the $\ker(f)$.

- *coequaliser*. Similarly, in **Set** we can always take the coequalizer between $f, g : X \to Y$. It is the quotient of Y by the relation $f(x) \sim g(x)$ and the map $a : Y \to coeq(f, g)$ is the projection map. Using this, in **Top** we can build the coequalizer between two maps by considering the coequalizer in **Set** and taking the universal topology which makes the projection map continuous: this is the quotient topology. For algebraic categories the condition $f(x) \sim g(x)$ is valid iff $f - g \sim 0$, so that the coequalizer can be identified with the cokernel of the difference $f - g$. Therefore, *in a category whose sets of morphisms are groups and in which we have the notion of cokernel we can take the coequalizer between any two morphisms f, g by defining $coeq(f, g) :=$* $\mathrm{coker}(f - g)$. In particular, the cokernel of a linear map is simply the coequalizer between it and the trivial map. As a special case, the cokernel of the inclusion $A \to X$ of a subespace is the quotient X/A.

Completeness

Having shown some examples of limits and colimits we can ask if they are totally independent or if there are some of them that can be built in terms of the others. For instance, in the previous subsection we discussed **separately** products and equalisers (as well as its dual versions) in some categories. We can ask if the **simultaneous** existence of such limits/colimits implies the existence of other limits/colimits. The following examples give an affirmative answer.

Example 2.3 (*pullbacks and pushouts*)**.** Let $\mathbf{D}$ be a category in which we have products $X \times Y$ as well as equalisers between any two morphisms. So, given any two morphisms $f : X \to Z$ and $g : Y \to Z$ we can consider the equalizer between $f \circ \pi_1 \equiv f_1$ and $g \circ \pi_2 \equiv g_2$. Looking to the first diagram bellow we see that this equalizer is a way to forcing the commutativity of the square (see the second and the third diagrams). Specially, the universality of products and equalizers imply that the obtained square is also universal in the sense that for any other morphisms $a_1 : A \to X$ and $a_2 : A \to Y$ turning commutative the fourth diagram, there exists a unique dotted arrow u for which the whole diagram remains commutative.

$$
\begin{tikzcd}
X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
eq(f_1, g_2) \arrow[dr, "a"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
eq(f_1, g_2) \arrow[r, "\pi_2 \circ a"] \arrow[d, "\pi_1 \circ a"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, dashed, "u"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& eq(f_1, g_2) \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

Now, notice that a pair of arrows $f : X \to Z$ and $g : Y \to Z$ determines a quiver inside $\mathbf{D}$ and, therefore, a functor $F : P(Q) \to \mathbf{D}$ defined on the category generated by the abstract quiver Q with three vertices and two arrows $1 \to 3$ and $2 \to 3$. A moment of reflection reveals that the third diagram is, indeed, a cone for this functor, so that the fourth diagram is a manifestation of the universality. In other words, $\operatorname{eq}(f_1, g_2) = \lim F$. The limit of F is usually called the *pullback between f and g*, being denoted by $\operatorname{pb}(f, g)$. So, our discussion shows that: *if a category have binary products and equalizers, then it also have pullbacks.* The colimit of a functor as F is called the *pushout* and denoted by $\operatorname{ps}(f, g)$. By second order duality we conclude a dual statement: *if a category have binary coproducts and coequalizers, then it also have pushouts.* Let us discuss some specific examples of this construction.

- *pullbacks*. Let $A \subset X$ be a subset and $f : Y \to X$ be any function. Then the pullback between f and the inclusion map $\imath : A \to X$ is the collection of all pairs $(a, y) \in A \times Y$ such that $\imath(a) = f(y)$. For a fixed a there exists one such y iff $y \in f^{-1}(a)$. Therefore, the pullback is nothing more than the preimage $f^{-1}(A)$. Similar construction holds for **Top** and algebraic categories. Particularly we observe that if $\pi : E \to X$ is a bundle and $f : Y \to X$ is any continuous function, then we can take their pullback in **Top**. The result will be a space $\operatorname{pb}(f, g)$ together with maps $\pi_1 : \operatorname{pb}(f, g) \to Y$ and $\pi_2 : \operatorname{pb}(f, g) \to E$. The first of them is also a bundle and coincides with the canonical construction f^*E of the pullback bundle. The second map is the identity when restricted to each fibers, which characterizes the fact that the fibers of the pullback bundle are homeomorphic to the bundle which was pulled-back.

- <u>*pushouts*</u>. Let $A \subset X$ be a subset and $f : A \to Y$ be any function. The pushout between f and the inclusion map is given by the quotient space of the disjoint union $X \sqcup Y$ under the relation $\imath(a) \simeq f(a)$. This means that we are gluing X into Y by identifying X with $f(A)$. Therefore, the identical construction in **Top** reproduces the canonical notion of gluing between topological spaces. On the other hand, for any action $* : G \times X \to X$ (say of a group on a set or of a topological group on a topological space) we can consider the pushout between them and the projection $\pi_2 : G \times X \to X$. The result is identified with the orbit space X/G. A concrete example of pushout in **Top** is the construction of the reduced suspension ΣX of given based space (which were used in order to discuss suspension spectra in the section 1.2). Indeed, there are at least two ways to build:

  1. or we take the cylinder $X \times [0, 1]$ and collapses $X \times 0$ and $X \times 1$ simultaneously;
  2. or we consider two copies of the cylinder, collapses $X \times 0$ in one and $X \times 1$ in the other (getting two cones) and them glue these cones in them at their boundaries.

  Each of these constructions is presented as consecutive pushouts as in the diagrams below.

$$
\begin{tikzcd}
\Sigma X & \mathrm{ps} \arrow[l, dotted] & * \arrow[l, dashed] \\
 & X \times I \arrow[u, dashed] & X \arrow[l] \arrow[u] \\
* \arrow[uu, dotted] & X \arrow[l] \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma X & \mathrm{ps} \arrow[l, dotted] & * \arrow[l, dashed] \\
\mathrm{ps} \arrow[u, dotted] & X \times I \arrow[l, dashed] \arrow[u, dashed] & X \arrow[l] \arrow[u] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u] &
\end{tikzcd}
$$

The last example reveals that pullbacks can be built from products and equalizers. The surprising fact is that we could extend this list of examples in order to include *all limits*! More precisely, it can be proved that *if a category* $\mathbf{D}$ *has arbitrary products and equalizers, then any functor* $F : \mathbf{A} \to \mathbf{D}$*, defined on any category* $\mathbf{A}$*, has limit.* This is really surprising because in principle the category $\mathbf{A}$ can be *very huge*! The situation is even more interesting. Indeed, it can be proved that products and equalizers determine not only all limits, but all Kan extensions!

These results are not difficult to prove, but in order to maintain the focus on our primary objective we prefer to discuss them in the Appendix A. Here we will only say that if a category $\mathbf{D}$ has products and equalizers, then for any functor $F : \mathbf{A} \to \mathbf{D}$, its right Kan extension with respect to an arbitrary $\imath : \mathbf{A} \to \mathbf{C}$ is given by the equalizer between the arrows a, b on the following diagram:

$$
\begin{tikzcd}
H_y(s(f), s(f)) \arrow[r, "H_y(id,f)"] & H_y(s(f), t(f)) \\
\Pi_X H_y(X, X) \arrow[u, "\pi_{s(f)}"] \arrow[d, "\pi_{t(f)}"'] \arrow[ur] \arrow[dr] \arrow[r, dashed, shift left, "a"] \arrow[r, dashed, shift right, "b"'] & \Pi_f H_y(s(f), t(f)) \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\
H_y(s(f), s(f)) \arrow[r, "H_y(f,id)"'] & H_y(t(f), s(f))
\end{tikzcd}
\qquad \text{with} \quad H_y(X, Z) = \Pi_{\mathrm{Mor}_{\mathbf{D}}(Y;\imath(X))} F(Z)
$$

By the duality principle, a totally dual construction produces any left Kan extension of F when we have coproducts and coequalizers. These right and left Kan extensions are usually represented by the following "integral operator notation"

$$R(Y) = \int_X H_y(X,X) \quad \text{and} \quad L(Y) = \int^X H^y(X,X),$$

so that we can take double integrals and argue about the existence of some "Fubini theorem" for these entities. Such result really exists and together with the "integral operator notation" it will also be discussed in Appendix A.

Now, let us make two remarks about the previous result (each of them reflecting a different way of understanding it):

1. completeness. A category $\mathbf{D}$ is called *complete* (resp. *cocomplete*) when any $F : \mathbf{A} \to \mathbf{D}$ has a right (resp. left) Kan extension with respect to each $\imath : \mathbf{A} \to \mathbf{C}$. Therefore, the previous can be stated in terms of **completeness**: it says that a category is complete (resp. cocomplete) iff it has all products and equalizers (resp. all coproducts and coequalizers).

2. coherence. Given a category $\mathbf{D}$, consider the following problem: to determine if $F : \mathbf{A} \to \mathbf{D}$ has right and left Kan extensions with respect to each $\imath : \mathbf{A} \to \mathbf{C}$. In principle, solving this problem could be **impracticable**, because for any fixed F (which actually depends of an *arbitrary* category $\mathbf{A}$) we would need to know if it has or not Kan extensions with respect to each $\imath$ (which is itself *arbitrary* and depends of another *arbitrary* category $\mathbf{C}$). Therefore, the number of sentences to be considered is, in principle, parametrized over the collection of all categories and all functors. On the other hand, the previous result says that the collection of *independent* sentences is very small, allowing us to effectively study the problem. Therefore, the previous result is not only about completeness. In fact, it is much more fundamental: it is also about the **coherence** of the Kan extension concept. Coherence conditions will also be very important in future chapters (at the definition of weak higher categories) as well as in Appendix B (in the proof of the Cobordism Hypothesis).

Gluing

We end this section by noticing that pullbacks and pushouts satisfy an additional properties which is very useful in order to do calculations. Indeed, recall that if we have two commutative diagrams with a common arrow, then gluing then at the common arrow we get another commutative diagram. The fundamental fact is that when gluing pullback/pushout squares, the resulting diagram is also a pullback/pushout square! More precisely, if the first two diagrams below are pullback squares, then the last two (which were obtained by gluing) are pullbacks too. The condition for pushouts is analogue.

$$\begin{tikzcd} \mathrm{pb} \arrow[r, dashed] \arrow[d, dashed] & X' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb} \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb} \arrow[rr, dashed] \arrow[d, dashed] & & Y \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb} \arrow[r, dashed] \arrow[dd, dashed] & X' \arrow[d] \\ & Y \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

In the last subsection we have seen that the reduced suspension ΣX can be built at least in two different ways. As an application of the gluing law for pushouts, let us see that in general ΣX can be built indeed in infinitely many ways. The idea is the following: above we have seen that ΣX can be understood as the gluing of two cones. Here we will see that any map $f : X \to Y$ induces a notion of "cone" C_f and that the collapsing of Y into this cone is model of ΣX.

We start by noticing that the in the construction of ΣX by cones, each cone corresponds precisely to the pushout below, where $\imath_j : X \to X \times I$ is the inclusion of X as $X \times j$ with $j = 0, 1$. We then define the *cone* C_f of $f : X \to Y$ as the result of the consecutive pushout below. Notice that it is just like the construction of ΣX by gluing cones, but with one of the collapsing maps $X \to *$ replaced by f.

$$\begin{tikzcd}
\mathrm{ps} & X \times I \arrow[l, dashed] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u, "\imath_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C_f & \mathrm{ps} \arrow[l, dotted] & Y \arrow[l, dashed] \\
\mathrm{ps} \arrow[u, dotted] & X \times I \arrow[l, dashed] \arrow[u, dashed] & X \arrow[l, "\imath_1"] \arrow[u, "f"'] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u, "\imath_0"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{ps}' & * \arrow[l, squiggly] \\
\mathrm{ps} \arrow[u, squiggly] & Y \arrow[l, dashed] \arrow[u, dashed]
\end{tikzcd}$$

Notice that the cone of $X \to *$ is precisely ΣX, so that in order to build ΣX from C_f the idea is to replace f by this terminal map. This can be done as follows. Despite the pushouts defining C_f we also have the last pushout above (in it ps is the pushout between f and $\imath_1$). Gluing it with the diagram of C_f we get another commutative diagram which (by the gluing property of pushouts) is also a pushout: just the quotient C_f/Y. It happens that, as explained below, this resulting diagram is exactly the diagram defining ΣX, so that $C_f/Y \simeq \Sigma X$ by uniqueness.

$$\begin{tikzcd}
C_f/Y & & * \arrow[ll] \\
C_f \arrow[u] & \mathrm{ps} \arrow[l, dotted] & Y \arrow[l, dashed] \arrow[u, dashed]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C_f/Y & & * \arrow[ll] \\
C_f \arrow[u] & \mathrm{ps} \arrow[l, dotted] & Y \arrow[l, dashed] \arrow[u, dashed] \\
\mathrm{ps} \arrow[u, dotted] & X \times I \arrow[l, dashed] \arrow[u, dashed] & X \arrow[l, "\imath_1"] \arrow[u, "f"'] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u, "\imath_0"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\Sigma X & \mathrm{ps} \arrow[l, dotted] & * \arrow[l] \\
\mathrm{ps} \arrow[u, dotted] & X \times I \arrow[l, dashed] \arrow[u, dashed] & X \arrow[l, "\imath_1"] \arrow[u] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u, "\imath_0"']
\end{tikzcd}$$

Remark. Recall that, as discussed in the last chapter, it is not enough to build invariants: we need to have tools to do calculations with them. When the invariants are given by abelian groups the fundamental tools are *exact sequences*. They allow us to reconstruct the invariant of a term in the sequence when we known the invariants associated to the other terms. Here we would like to explain that *generalized cohomology theories are very useful invariants because for them we have many of exact sequences*. This is a direct consequence of the presentation of ΣX by the cone of any function (and, therefore, a direct consequence of the gluing property of pushouts). Indeed, from the previous construction we see that any map f induces a canonical sequence as below. We usually say that it is the *fiber sequence* of f.

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X$$

Iterating the fiber sequence by making use of the functor Σ we get the following long sequence, called *fibration sequence* (or *Barratt-Puppe sequence*) *of* f:

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \longrightarrow \Sigma C_f \longrightarrow \Sigma^2 X \longrightarrow \cdots \tag{2.1.2}$$

Now, If $\mathbb{E}$ is some spectrum representing a cohomology theory, then applying the 0th-cohomology functor $H^0_{\mathbb{E}} = [-, E_0]$ to the fibration sequence of $f : X \to Y$, making use that Σ and Ω are adjoints and that $\mathbb{E}$ is a Ω-spectrum (i.e, $E_n \simeq \Omega E_{n+1}$) we get a long sequence of abelian groups containing the $\mathbb{E}$-cohomology groups of X. But, why this sequence is **exact**? Well, as will be brief discussed in the next section, this come from a very special property of the objects appearing in the Barratt-Puppe sequence: they are all invariant by homotopy equivalences.

$$\cdots \longrightarrow H^n(X,\mathbb{E}) \longrightarrow H^n(Y,\mathbb{E}) \longrightarrow H^n(C_f,\mathbb{E}) \longrightarrow H^{n-1}(X,\mathbb{E}) \longrightarrow \cdots$$

2.2 Embedding

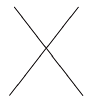
We proved that **Set**, **Top** and the standard algebraic categories have products and equalizers, as well as coproducts and coequalizers. Thanks to the result discussed in the last subsections we can now affirm that such categories have all limits/colimits and, more generally, all Kan extensions with respect any to other functor. Now, we can ask if this property of being complete/cocomplete is *generic* in the collection of all categories.

For instance, recall that the construction of products and equalizers for **Top** (and, similarly, for the algebraic categories) was obtained using the concreteness of **Top** and the existence of such limits in **Set**. More precisely, to build the product between two topological spaces (X, τ) and (X', τ') we first consider the product $X \times X'$ between the underlying sets and then we introduce the canonical topology in which the projections become continuous. Similarly, the equalizer between two continuous functions $f, g : (X, \tau) \to (X', \tau')$ is constructed from the equalizer $\mathrm{eq}(f, g)$ between the underlying functions $f, g : X \to X'$ by introducing the canonical topology in which the inclusion map $\imath : \mathrm{eq}(f, g) \to X$ becomes continuous.

Now, we are tempted to do a similar analysis in **any concrete category** concluding that each of them is complete/cocomplete. Indeed, if $\mathbf{D} \subset \mathbf{Set}$ is concrete, then it is natural to consider the product $X \times Y$ (resp. the equalizer between f and g) as being simply the product (resp. the equalizer) between the underlying sets (resp. functions) endowed with the **canonical structure** that turns the projections (resp. inclusions) into morphisms of $\mathbf{D}$. But in order to apply such a strategy we need first to show that this canonical structure **exists** in any situation. The following example clarifies that there are concrete categories in which this canonical structure does not exists, so that *being complete/cocomplete is not a generic property of the concrete categories.*

Example 2.4 (**Diff** *is incomplete*)**.** Consider the smooth function $x \mapsto x^2$ between $\mathbb{R}$ and $\mathbb{R}$, viewed as smooth manifolds with the trivial atlas. Now, considering this map in **Set** we can compute the pullback presented in the first diagram below. The result is the set of all (x, y) such that $x^2 = y^2$ that looks like the letter “X”, which **cannot** admit any smooth structure[1]. Similarly, viewing the point as a trivial zero dimensional manifold we can consider the pushout presented in the third diagram in **Set**. The result is obtaining by attaching two copies of $\mathbb{R}$ along a common point (say the origin), so that it is also equivalent to an “X” and, therefore, cannot admit any smooth structure.

$$\begin{tikzcd} \mathrm{pb} \arrow[r, dashed] \arrow[d, dashed] & \mathbb{R} \arrow[d, "x^2"] \\ \mathbb{R} \arrow[r, "y^2"'] & \mathbb{R} \end{tikzcd} \quad \simeq$$



$$\simeq \quad \begin{tikzcd} \mathrm{ps} & \mathbb{R} \arrow[l, dashed] \\ \mathbb{R} \arrow[u, dashed] & * \arrow[l] \arrow[u] \end{tikzcd}$$

Now we can ask if the problem appears exactly because we are working with concrete categories. More precisely, we can ask if completeness is a generic property of non-concrete categories. The following example shows that the answer remains negative.

Example 2.5 (Ho(**Top**) *is incomplete*)**.** Suppose, for a moment, that pullbacks exist in the homotopy category of topological spaces. Then for any two morphisms $a : X \to Z$ and $b : Y \to Z$ we can take the pullback between them. But recall that Ho(**Top**) is the quotient category of **Top** by the homotopy equivalence, so that a and b are indeed homotopy classes of continuous maps. So, fixing a representative for a and b the pullback between them can be computed as in the category of topological spaces, but it must be independent of the choice of representatives, in the sense that the pullback of two different representatives must produce homotopic spaces. Now, consider the pullback in **Top** presented in the first diagram, whose result is simply the product $\mathbb{S}^{n-1} \times \mathbb{D}^n$. We ask if it is also a pullback in the quotient category Ho(**Top**). We affirm that this is **not** the case. Indeed, being $\mathbb{D}^n$ contractible, $id : \mathbb{D}^n \to \mathbb{D}^n$ is homotopic to a constant c map, say taking values in $\mathbb{S}^{n-1}$. But when we compute the second pullback below we get a space homeomorphic to a point, therefore not homotopic to $\mathbb{S}^{n-1} \times \mathbb{D}^n$. Similarly, the third and the fourth diagrams below reveals that arbitrary pushouts cannot exist in Ho(**Top**). See Section 20.1 of [199].

$$\begin{tikzcd} \mathrm{pb} \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d, "id"] \\ \mathbb{S}^{n-1} \arrow[r, "\imath"'] & \mathbb{D}^n \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb}' \arrow[r] \arrow[d] & \mathbb{D}^n \arrow[d, "c"] \\ \mathbb{S}^{n-1} \arrow[r, "\imath"'] & \mathbb{D}^n \end{tikzcd} \qquad \begin{tikzcd} \mathrm{ps} & \mathbb{D}^n \arrow[l] \\ * \arrow[u] & \mathbb{S}^{n-1} \arrow[l] \arrow[u, "\imath"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{ps} & * \arrow[l] \\ * \arrow[u] & \mathbb{S}^{n-1} \arrow[l] \arrow[u] \end{tikzcd}$$

We end this subsection with a remark.

Remark. In the last example, in order to prove that Ho(**Top**) is not complete/cocomplete we used the argument that the usual limits/colimits of **Top** are not homotopy invariants. It happens that it is always possible to modify a limit in order to turn it homotopy invariant! Indeed, this is done by replacing any continuous map and any object in the usual limit by entities which are more well behaved with respect to homotopies. More precisely, we need to replace any map by another which satisfy some homotopy liftings/extension property. These are the so called *fibrations/cofibrations*. So, in other words, we can move from a “bad behaved limit/colimit” to a “homotopy limit/colimit” by replacing any by a fibration/cofibration. Concretely this is done by

[1]Indeed, as a consequence of the invariance of domain theorem, this set cannot admit any topological manifold structure.

replacing X by the first diagram below (if are trying to get a homotopy limit) or by the second diagram below (if we are trying to get a homotopy colimit). Here $\imath_j$ are the canonical inclusions of X into $X \times I$ as $X \times j$, while π_j are the canonical projections of the space of paths $[I, X]$ in X by $\pi_j(\gamma) = \gamma(j)$.

$$
\begin{tikzcd}
 & X \\
X & {[I,X]} \arrow[l, "\pi_0"'] \arrow[u, "\pi_1"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \times I & X \arrow[l, "\imath_1"'] \\
X \arrow[u, "\imath_0"] &
\end{tikzcd}
$$

So, for instance, applying the rule above we see that ΣX is just the homotopy version of the cokernel of $X \to *$, as below. In other words, the reduced suspension is a natural homotopical object. The same hold with the cone C_f, so that the Barratt-Puppe sequence is homotopically well behaved, because it was constructed by making use only of homotopy colimits.

$$
\begin{tikzcd}
\mathrm{ps} & * \arrow[l, dashed] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u]
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\Sigma X & \mathrm{ps} \arrow[l, dotted] & * \arrow[l, dashed] \\
 & X \times I \arrow[u, dashed] & X \arrow[l] \arrow[u] \\
* \arrow[uu, dotted] & X \arrow[l] \arrow[u] &
\end{tikzcd}
=
\begin{tikzcd}
\Sigma X & \mathrm{ps} \arrow[l, dotted] & * \arrow[l, dashed] \\
\mathrm{ps} \arrow[u, dotted] & X \times I \arrow[l, dashed] \arrow[u, dashed] & X \arrow[l] \arrow[u] \\
* \arrow[u, dashed] & X \arrow[l] \arrow[u] &
\end{tikzcd}
$$

Now, recall that representable contravariants functors map usual colimits into limits. The cohomology functors are contravariant, so that they map colimits into limits. The problem is that the cohomology functors are generally defined in Ho(**Top**) which is incomplete and, therefore, has few colimits. Happily, the representable functors in a homotopy category preserve not only the usual functors, but also the homotopy colimits! It is exactly by this reason that when applying $H^0_{\mathbb{E}}$ in the Barratt-Puppe sequence we get an **exact** long sequence on cohomology. For details, see [141, 199, 100].

Yoneda Embedding

In the last subsection we concluded that incomplete categories are common. But the discussion of the previous sections shows that the most interesting constructions are given by limits and colimits. Consequently, incomplete categories are lack of good properties. Given an incomplete category **D** we can remedy its shortage of properties following one of the strategies below:

1. *specifying*. The first idea is to search for subcategories $\mathbf{C} \subset \mathbf{D}$ with more limits/colimits and, therefore, with more properties. This is the canonical approach in differential geometry: the category **Diff** (over which differential geometry is generally developed) is incomplete. This means, for instance, that we cannot take arbitrary pullbacks and pushouts between smooth maps. On the other hand, if we restrict to the subcategory $\mathbf{C} \subset \mathbf{Diff}$ of smooth manifolds and submersions (instead of only smooth maps), then this category has pullbacks[2]. Similarly, if we restrict our attention to Lie groups acting properly discontinuously

[2]Indeed, it is easy to see that the pullback between two smooth maps intersecting transversally is a genuine smooth manifold. Particularly, the pullback between submersions always exists in **Diff**.

on manifolds, then the orbit space (which is a pushout) is also a manifold[3].

2. generalizing. We can also try to do the opposite. Thinking about the non existence of limits as a consequence of the rigidity of $\mathbf{D}$, we expect that "relaxing" the category $\mathbf{D}$ we can get more limits. More precisely, this strategy consists in the search for embeddings $\imath : \mathbf{D} \hookrightarrow \mathbf{D}'$ into some other category with more limits.

3. weakening. Instead of modifying the category $\mathbf{D}$ we can try to weak the notion of limit: the limits are given by universal commutative diagrams. We can, for instance, define "weak limits" by requiring "weak commutativity" and/or "weak universality".

Now we need to analyze the applicability of such strategies. The first is the most obvious but at the same time **impracticable**. Indeed, following this strategy, every time that we need to incorporate a limit in some theory (say described by $\mathbf{D}$) we must search for special classes of objects/morphisms (characterizing a subcategory of $\mathbf{D}$) in which this limit exists. But to get these conditions may be very difficult and, in some cases, impossible. More precisely, *there is no way to secure the existence of the subcategory with arbitrary limits/colimits*.

The third strategy is very promising. We notice, on the other hand, that it cannot be applied in arbitrary categories. Indeed, recall that the notion of limit/colimit is given by commutative diagrams in $\mathbf{D}$. So, the main idea is to apply the Weakening Principle, which says that any concept defined using only commutative diagrams can be weakened, right? But the *Weakening Principle holds only for diagrams in* $\mathbf{Cat}$! More precisely, recall that the Weakening Principle is based on the fact that we have a notion of mapping between functors (given by the concept of natural transformation). A commutative diagram of functors corresponds to equality between functors and we can replace this equality by the existence of a natural transformation between them. On the other hand, a commutative diagram in $\mathbf{D}$ involves equality of morphisms in $\mathbf{D}$ and a priori *there is no notion of mapping between such morphisms*.

Even so, in some categories this strategy can be effectively applied. For instance, the notion of homotopy between continuous maps can be interpreted as certain "mapping between continuous functions", so that *any commutative diagram in* $\mathbf{Top}$ *can be weakened by replacing the equality of functions with the existence of a homotopy between them*. Particularly, the notion of "weak limits" obtained in this way corresponds precisely to the notion of *homotopy limits/colimits* presented in the last subsection (see [199, 96, 146, 140]). This is a more interesting similarity between homotopy theory and category theory which will be very important later.

After the previous discussion, it seems that the only remaining strategy (the second one) is the most canonical. This really is the case. In fact, as we will explain now, *any category can be embedded into a complete and cocomplete category*. This means that *any area of math can be enlarged in order to be very well behaved, in the sense that any categorical construction can be done internal to it*. More precisely, recall that any object X in a category $\mathbf{C}$ determines a canonical functor $h_X : \mathbf{C}^{op} \to \mathbf{Set}$ which assign to any Y the set of morphisms from Y to X. It happens that this assignment is itself functorial, so that we have a functor

$$h_{-} : \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{op}; \mathbf{Set}). \tag{2.2.1}$$

[3] Indeed, it is very clear that if the total space E of a covering space (i.e a locally trivial bundle whose fibers are discrete spaces) $\pi : E \to X$ is a manifold, then there exists a unique smooth structure in X turning the projection π a local diffeomorphism. It happens that, for a given smooth action of a Lie group G on a manifold M, the quotient map $M \to M/G$ is a covering space iff the action is properly discontinuous.

Now, there are two fundamental facts which will provide a formalization of the previous assertion that any mathematical theory can be enlarged to be well behaved:

1. *the category at the right-hand side of (2.2.1) is complete and cocomplete*. This follows from the fact that **Set** is complete and cocomplete. Indeed, it can be shown that any category of functors that takes values in a complete/cocomplete category is, itself, complete/cocomplete.

2. *the functor (2.2.1) is, indeed, an embedding*. In other words, the category $\mathbf{C}$ is equivalent to its image $h_{\mathbf{C}}$ by h_{-}. Equivalently, h_{-} is injective at morphisms and injective up to isomorphisms at objects (which happens, particularly, when the functor is bijective at morphisms). In fact, note that the diagram computing Kan extensions in terms of products and equalisers makes sense for any bifunctor $H : \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{Set}$, not only for the H_y used there (see Appendix A for details). Particularly, given two functors $F, G : \mathbf{D} \to \mathbf{Set}$ we can compute this diagram for the bifunctor defined as
$$H(X, Y) = \mathrm{Mor}_{\mathbf{Set}}(F(X); G(Y)),$$
producing the set of natural transformations $\xi : F \Rightarrow G$ as a result. On the other hand, repeating the process for $F = h_Z$ we see that the diagram computes $F(Z)$. Consequently, we have
$$\mathrm{Mor}_{\mathrm{Func}(\mathbf{D}^{op};\mathbf{Set})}(h_Z, G) \simeq G(Z),$$
which is known as the *Yoneda lemma*[4]. Now, taking $F = h_X$ we see that the set of morphisms between the objects X and Z is in bijection with the set of morphisms between the functors h_X and h_Y, meaning that h_{-} is bijective on morphisms and, therefore, that it is a full embedding.

Locality

We have seen that, thanks to Yoneda embedding, any category $\mathbf{C}$ can be embedded into a complete and cocomplete category. Under this embedding, the objects $X \in \mathbf{C}$ are viewed as functors $F : \mathbf{C}^{op} \to \mathbf{Set}$ *fulfilling* ***additional conditions***. We notice that these "additional conditions" are very important in order to describe all properties of X. For instance, if X is a smooth manifold, then their structure is totally determined when we give coordinate systems $\varphi_i : U_i \to \mathbb{R}^n$, but this local property is not described by any functor $F : \mathbf{Diff}^{op} \to \mathbf{Set}$. In other words, under the Yoneda embedding, a manifold should be described by some kind of functor which also satisfy local conditions.

But what we mean with "local functor"? Observe that a manifold X is local object precisely because in order to describe the whole entity it is necessary and enough to study certain data (the coordinate systems φ_i) defined on some open covering $U_i \hookrightarrow X$ which are compatible at the intersections $U_i \cap U_j$ (here meaning that the transition functions $\varphi_j \circ \varphi_i^{-1}$ are smooth). So, if we think of functor $F : \mathbf{Diff}^{op} \to \mathbf{Set}$ as a rule that assign to any manifold X some data $F(X)$, this functor will be "local" precisely if there is some class of open coverings $U_i \hookrightarrow X$ such that the global data $F(X)$ can be recovered from the local data $F(U_i)$ subjected to compatibility

[4]Recall that the Yoneda lemma was used in the last chapter at the section 1.2 in order to classify characteristic classes and cohomology operations.

conditions at the intersections $F(U_i \cap U_j)$. Formally, F *is local if for any* X *we have a class of coverings* $U_i \hookrightarrow X$ *such that the canonical diagram below is an equalizer.*

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

In the literature a functor $F : \mathbf{Diff}^{op} \to \mathbf{Set}$ which is "local" in the above sense is usually called a *smooth sheaf* . So, what we are saying is that under the Yoneda embedding a smooth manifold is not an arbitrary functor, but indeed a *smooth sheaf*. We notice that the notion of "smooth sheaf" is totally characterized by the last diagram. It happens that in order to build this diagram we used only two properties of the category **Diff**:

1. for each object X we have a notion of covering given by certain smooth maps $U_i \hookrightarrow X$;

2. for any of these coverings the intersection $U_i \cap U_j$ is well defined.

Therefore we can talk of sheaves (or local functors) not only in **Diff**, but indeed in any category in which the two properties above makes sense. For instance, we can talk of sheaves in **Top** if we consider the notion of "covering" as the usual notion of "open covering", because for them the intersection is obviously defined. Note that the intersections $U_i \cap U_j$ are just pullback betweem $U_i \hookrightarrow X$ and $U_j \hookrightarrow X$, so that the concept of "intersection" can be axiomatized in any category with pullbacks. This means that if we start with category $\mathbf{H}$ with pullbacks, then any notion of "coverings" in it will induce a corresponding notion of "sheaves in $\mathbf{H}$".

The fundamental property of the open coverings is that they are *stable under pullbacks*. More precisely, if $\pi_i : U_i \to X$ is an open covering of a topological space X, then for any continuous map $f : Y \to X$ the preimages $f^{-1}(U_i)$ fits into an open covering for Y. Notice that these preimages are simply pullbacks between f and each π_i, so that the notion of covering can also be formalized in any category with pullbacks. Indeed, if $\mathbf{H}$ has pullbacks we define a *Grothendieck topology* in it as a rule J assigning to any object X a family $J(X)$ of morphisms $\pi : U \to X$ which contains any isomorphism of X and which is stable under pullbacks, i.e, if $f : Y \to X$ is any morphism and $\pi : U \to X$ belongs to $J(X)$, then $\mathrm{pb}(f, \pi)$ belongs to $J(Y)$.

A pair $(\mathbf{H}, J)$, where $\mathbf{H}$ has pullbacks and J is a Grothendieck topology is called a *site*. So, summarizing, *we can talk of sheaves in any site*. This is just a functor $F : \mathbf{H}^{op} \to \mathbf{Set}$ such that for any X and any covering $\pi_i : U_i \to X$ in $J(X)$ the diagram below is an equalizer.

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

The notion of "local functor" can be generalized even more. Indeed, notice that the above diagram (as an equalizer) makes sense if we replace **Set** by any other category with binary products and equalizer. Particularly, it makes sense in any complete category $\mathbf{D}$. This allows us to define $\mathbf{D}$*-valued sheaves in any site* $(\mathbf{H}, J)$. These are functors $F : \mathbf{H}^{op} \to \mathbf{D}$ such that for any X and any $\pi : U \to X$ in $J(X)$ the above diagram (as a diagram in $\mathbf{D}$) is an equalizer. Because sheaves are special flavors of functors, we have a full subcategory

$$\mathrm{Shv}_{\mathbf{D}}(\mathbf{H}, J) \subset \mathrm{Func}(\mathbf{H}^{op}; \mathbf{D}).$$

This category of sheaves is called a *Grothendieck topos*. It can be proven that the inclusion functor has a left-adjoint $\mathscr{L}$, which assign to any functor F their *sheafication* $\mathscr{L}F$. In other words, *the category of functors from* $\mathbf{H}^{op}$ *to* $\mathbf{D}$ *is freely generated by the* $\mathbf{D}$*-valued sheaves in* $(\mathbf{H}, J)$ *for any Grothendieck topology* J (i.e, the category of functors is freely generatered by any Grothedieck topos). Because $\mathscr{L}$ is a left-adjoint it preserve colimits. It can be shown, on the other hand, that it also preserve finite limits. Indeed, the functor $\mathscr{L}$ totally characterize the category of sheaves in the sense that any subcategory $\mathbf{C}$ of $\mathrm{Func}(\mathbf{H}^{op}; \mathbf{D})$ for which the inclusion has a left adjoint preserving finite-limits is a Grothendieck topos. Thanks to this equivalence we can also show that a category $\mathbf{C}$ is Grothendieck topos (i.e, it can be geometrically embedded into the category of functors) iff it satisfy the following *Giraud's axioms*:

1. there is a set S of morphisms (called *generating set*) such that two parallel arrows $f, g : X \to Y$ are equal iff they satisfy $f \circ s = g \circ s$ for any $s \in S$ for which the composition makes sense;
2. it is finitely complete (i.e, it has finite limits);
3. it is cocomplete;
4. colimits commute with pullbacks;
5. given two objects X and Y, the pullback between the canonical inclusion of them into $X \oplus Y$ is the terminal object $*$.

The axioms 2-5 imply that any Grothendieck topos has many of the fundamental properties of the category of sets like as the existence of object classifiers (which are essentially the objects which allows us to decide if a given sentence is true or not), the cartesian-closed property (which corresponds to the notion of power of a set by another arbitrary set) and analogue versions of the axiom of choice and the construction (by Peano's axioms) of the natural numbers (for details on this discussion, see [107, 132, 81, 147]). So, we have the following

Conclusion. *Starting from any category* $\mathbf{H}$ *(possibly with few limits/colimits) we can build a large number of very well categories (toposes) which behave much like as the category* **Set** *of sets and which can be used to replace it in the foundations of mathematics.*

The conclusion above is just one more manifestation of the abstractness of the categorical language. Indeed, it says that categorical language is a factory of set-type background languages.

2.3 Abstracting

In the last section we concluded that categorical logical is a very interesting logic. More precisely, we have seen that it is sufficiently abstract to unify many distinct mathematical concepts into the very natural notion of Kan extensions. We have also seen that this natural notion is coherent in the sense that the existence of few many Kan extensions are sufficient to ensure the existence of all the others. Finally we proved that categories with few properties can always be embedded into more well behaved categories, meaning that *categorical language allows us to enlarge poor mathematical theories, turning them richer in properties.*

Therefore, it is natural to expect that categorical logic can be used to unify different laws of Physics. Indeed, as will be discussed later, classical mechanics, quantum mechanics and many quantum field theories can be described by making use of this language. For instance, recall that (as commented in the first chapter and as will be discussed in detail in Chapter 14) a quantum field theory can be defined as a certain kind of functor from the category of cobordism into some algebraic category.

It happens that in many physical theories (specially for gauge aspects of string theory, as will be discussed still in this chapter in more details in chapters 11-13), *the categorical logic is not sufficiently abstract in order to describes all phenomena*. So, we need to look for more abstract languages. The first question is, naturally, about existence: *can there exist languages which are more abstract than categorical language?* The following example suggests an affirmative answer.

Example 2.6 (*homological algebra and classical homotopy theory*)**.** Homological algebra and homotopy theory are different areas of math, described (using categorical language) by different categories: the category $\mathbf{Ch}_R^+$ of bounded cochain complex describes homological algebra and the category $\mathbf{CW}_*$ of CW-complexes with distinguished base points describes homotopy theory. On the other hand, such theories are **very similar**! Some of those similarities are the following (see also the Table 2.1):

1. *structure of the objects*. The objects in $\mathbf{Ch}_R^+$ can be identified with increasing sequences (X_n) of R-modules entities linked by certain coboundary operators. Equivalently, we can see such objects as being $\mathbb{N}$-graded R-modules entities $X \in \mathbf{Ab}$ which are the limit of a process: $X = \lim \oplus_n X_n$. Similarly, the objects of $\mathbf{CW}_*$ are also limiting spaces of increasing sequences (the skeleton sequence) in which X_n was obtained from X_{n-1} by attaching n-cells.

2. *structure of the morphisms*. In both categories, the morphisms $f : X \to Y$ are families of morphisms $f_n : X_n \to Y_n$ preserving the decomposition: for cochain complexes, this means that f_n commutes with the coboundary operators, as well as for CW-complexes each f_n must commute with the attaching maps.

3. *existence of homotopy category*. In both categories we also have a canonical notion of homotopy, so that we can build the homotopy categories $\mathrm{Ho}(\mathbf{CW}_*)$ and $\mathrm{Ho}(\mathbf{Ch}_R^+)$.

4. *canonical homotopy invariants*. Additionally, there are canonical functors defined in both categories: they are the homotopy groups π_n and the cohomology functors H^n. In each case such functors respect homotopies in the sense that it passes from the original category to the corresponding homotopy category.

5. *well behaved derived category*. In each theory we have a distinguished class of morphisms, both given by the morphisms that are mapped into isomorphisms by each canonical functor. Indeed, on one hand we have the *weak homotopy equivalences* and on the other we have the *quasi-isomorphisms*. Localizing with respect to each of these classes we get the corresponding derived categories $\mathbf{CW}_*[W^{-1}]$ and $\mathbf{Ch}_R^+[W^{-1}]$. The weak homotopy theory described by them are both well behaved in the sense that we have versions of the Whitehead theorem for each of them.

classical homotopy theory	homological algebra
CW-complex	cochain complex
cellular map	cochain map
classical homotopy category	algebraic homotopy category
homotopy groups	cohomology groups
weak homotopy equivalences	quasi-isomorphisms
topological derived category	algebraic derived category
Whitehead theorem	algebraic Whitehead theorem

Table 2.1: Homotopy theory of cell complex *vs* homotopy theory of cochain complex

The previous example clearly suggests the existence of a *more abstract language, say describing* **abstract homotopy theory***, of which classical homotopy theory and homological algebra are only particular examples.* But now, the careful reader could say that despite the previous surprising similarities between homological algebra and homotopy theory, there are some differences between them:

1. cochain complexes can be defined more generally as being sequences parametrized by the *integers* $\mathbb{Z}$, but CW-complex are sequences parametrized only over the *naturals* $\mathbb{N}$;

2. cochain complexes are *stable* entities, but CW-complex generally are not. More precisely, we have functors $\Sigma, \Omega : d\mathbf{Ab} \to d\mathbf{Ab}$ which are the homological analogues of suspension and loop functors. We can see that they are simply shifts: for any complex X, we have $(\Sigma X)_n = X_{n+1}$ and $(\Omega X)_n = X_{n-1}$. Consequently, Σ and Ω are inverses one of the other and there are homotopy equivalences $X \simeq \Sigma X$. These facts are not valid for CW-complexes;

3. the category of cochain complexes is *additive*. This means that the set of morphisms between two chain complexes (i.e, the set of cochain maps) is not only a set, but it have a natural structure of abelian group. Furthermore, the composition of chain maps is bilinear with respect to such group structure. Additionally, the algebraic cohomology functors H^n are also additive in the sense that they preserve the sum of cochain maps. Consequently, this induce a natural additive structure in the derived category $\mathbf{Ch}_R[W^{-1}]$. On the other hand, there is no natural operation between weak homotopy equivalences to turn $\mathbf{CW}_*[W^{-1}]$ into an additive category.

There are even more differences. For instance, the algebraic derived category satisfies certain conditions that turn it into a *triangulated category*, which not happens in the topological case. So, we can ask if there exists some other class of topological entities that satisfy the conditions in Table 2.1, but for which the previous differences disappear.

The required class of spaces must be given by a family of based topological spaces $\mathbb{X} = (X_n)$, indexed over $\mathbb{Z}$, together with homotopy equivalences $\sigma_n : \Sigma X_n \to X_{n+1}$ and, therefore, together with equivalences $\Omega X_{n+1} \simeq X_n$. This looks very similar to the notion of Ω-spectrum used to define a generalized cohomology theory in the Section 1.2. The only difference is that the Ω-spectra used there are indexed over $\mathbb{N}$, but a priori there is no reason to restrict to such cases. Because of this, the extended sequences also will be called *spectra*. Therefore, the main idea is to repeat the construction done with CW-complexes replacing them with spectra.

For instance, in order to maintain the similarity with the cochain maps we define a morphism between two spectra $\mathbb{X}$ and $\mathbb{Y}$ as being a *spectrum map*, i.e a sequence of continuous maps $f_n : X_n \to Y_n$ commuting with the equivalences $\sigma_n : \Sigma X_n \to X_{n+1}$ and $\delta_n : \Sigma Y_n \to Y_{n+1}$, in the sense that $f_{n+1} \circ \sigma_n = \delta_n \circ \Sigma f_n$. The corresponding category will be denoted by **Spec**. Now, we can define functors

$$\pi_n^S : \mathbf{Spec} \to \mathbf{Set} \quad \text{by} \quad \pi_n^S(\mathbb{X}) = \lim_k \pi_{n+k}(X_k),$$

which assign to each spectrum its nth *stable homotopy group*. Therefore, here we also have distinguished morphisms: the *stable weak equivalences*, given by the morphisms between spectra that induce a isomorphism in each stable homotopy groups. Localizing with respect to this distinguished class we have the *stable derived category* $\mathbf{Spec}[W^{-1}]$ and the corresponding weak homotopy theory is also well behaved, evidently stable and, indeed, additive. Therefore, *it is expected the existence of an abstract language describing* **stable abstract homotopy theory** *which unifies unbounded homological algebra and stable homotopy theory.*

Highering

In the last section we concluded that there are good reasons to believe that there are background languages more abstract than categorical language. Now, the central question is: *how to build them?* Recall that set theory is simpler than category theory, which means that classical logic is less abstract than categorical logic. Thus, formalizing the passage from set theory to category theory we can iterate such *categorification process* to get languages which are more and more abstract, as presented in the following diagram:

$$\underset{\text{(classical logic)}}{\textbf{set theory}} \Longrightarrow \underset{\text{(categorical logic)}}{\textbf{category theory}} \Longrightarrow \underset{\text{(more abstract logic)}}{\textbf{more abstract theory}} \Longrightarrow \cdots .$$

In order to get some feeling about this categorification process, notice that a *set* contains less information than a *categor*y. Sets are composed of a unique type of information: its elements. Categories, on the other hand, have three kinds of information: *objects*, *morphisms* and *compositions*. Thus, the passage from set theory to category theory can be characterized by the "*addition of information layers*". Therefore, it is expect that the passage from category theory to another more general theory can also be characterized by the addition of information layers.

In other words, it is expected that the entities described by this more abstract theory will be composed not only by objects, morphisms and one composition, but also by *morphisms between morphisms* which can be *composed in two different ways.* Thus, if we call such entities of 2-*categories*, adding another layer of information we get 3-*categories*, and so on. So, the previous diagram can be translated as

$$\underset{\text{(classical logic)}}{\textbf{set theory}} \Longrightarrow \underset{\text{(categorical logic)}}{\textbf{category theory}} \Longrightarrow \underset{\text{(2-categorical logic)}}{\textbf{2-category theory}} \Longrightarrow \cdots$$

We will try to formalize the categorification process in the next chapters. After we do this, we can ask if the theory obtained by categorifying category theory really is more abstract. More precisely, we can ask if such "higher category theory" admits the very abstract notion of "higher Kan extensions", generalizing the usual Kan extensions and, therefore, generalizing the usual notions of limits and colimits, which includes pullbacks and pushouts, products and coproducts, equalizers and coequalizers, and so on.

Recall that the notion of Kan extensions was obtained by applying the Weakening Principle. Therefore, in order to define higher Kan extensions we need first of all to develop some "Abstract Weakening Principle". But the classical Weakening Principle relies precisely on the fact that we have a notion of "mapping between functors". In the higher categorical language this means precisely that **Cat** is indeed a 2-category! But in an 2-category we have a notion of mapping between morphisms, so there is expected to exist a similar principle. More generally, we expect to get the following

Abstract Weakening Principle: *any concept internal to an n-category defined using only commutative diagrams of k-morphisms can be weakened by requiring that such diagrams commutes only up to $(k+1)$-morphisms.*

Such principle allows us to define, for instance, the notion of "weak limit" in any 2-category. Recall that limits in $\mathbf{C}$ corresponds to commutative diagrams of morphisms, so that if we have the notion of 2-morphisms we can replace such commutativity by the existence of a 2-morphisms between the underlying morphisms. But what we are trying to get is not only a notion of "weak limit", but a notion of "higher Kan extension". For the classical Kan extensions we need a 2-categorical structure in **Cat**. We have an intuitive notion of n-functor between n-categories: it is a rule preserving not only objects and morphisms, but also each k-morphism, with $1 \leq k \leq n$. So, we can build a **usual** category $n\mathbf{Cat}$ of all n-categories. These "higher Kan extensions" should be diagrams of n-functors commuting up to some higher morphism. Therefore, *showing that $n\mathbf{Cat}$ is indeed a $(n+1)$-category we can then apply the Abstract Weakening Principle to commutative diagrams of n-functors, producing the required notions of higher Kan extension.*

2.4 Modelling

Many times in this text we said that more abstract background languages tend to help us in the axiomatization problem of Physics. In the last section, on the other hand, we presented an intuitive process that transforms any mathematics in a more abstract version of it: the *categorification process*. So we can ask if such process has a real effect in the axiomatization problem. In the present subsection we will try to convince the reader that such an effect does indeed exists. In fact we will show that in *categorifying a theory of particles we get a theory describing not only particles, but also strings.*

Warning. Up to this moment we do not have a full formalization of the categorification process, so that what will be done here is only to give motivations to the introduction of higher categorical methods in Physics.

We start by observing that in the modern abstract viewpoint, there are two kind of physical theories: *classical theories* and *quantum theories*. Both are based on the notion of *abstract motion* of a certain collection of p-dimensional objects, generically called *p-branes*. Such motion is given by a manifold of dimension $p+1$ which is interpreted as the time evolution of the system of p-branes. The number of connected components of this manifold describes the number of branes existing in the system.

In the most general view, a *classical theory* deals with possible ways to materialize the abstract motion Σ in some ambient M. This materialization is given by distinguished smooth maps $\varphi : \Sigma \rightarrow M$, called *configurations* of the system. The choice of these distinguished maps is generally made by giving a functional $S : \mathrm{Map}(\Sigma; M) \rightarrow \mathbb{R}$, called the *action functional*, and looking for the configurations that minimizes it. On the other hand, *quantum theories* can be understood as ways to represent the abstract motions in terms of linear data (as vector spaces and linear transformations). In other words, they are ways to do *representation theory of manifolds*.

We will discuss classical and quantum theories in more details later. At this moment, we will focus on the abstract motions. So, let us examine some cases in more detail.

Example 2.7 (*motion of particles*)**.** When $p = 0$, p-branes are *particles*. In such case, the abstract motion Σ of a particle system is a one dimensional manifold usually called *worldline*. Such manifolds may or may not have boundary. In both cases, each connected component of Σ (say Σ_i) is determined by the classification of one dimensional manifolds. Indeed, if a component is compact, then it must be $\mathbb{S}^1$ or $[0, 1]$, while if it is not compact then it must be $\mathbb{R}$ or some interval as $(0, 1]$ or $[0, 1)$. The interpretation in each situation is:

- a component diffeomorphic to $\mathbb{S}^1$ corresponds to an abstract *periodic motion*. The corresponding configurations (which are maps $\varphi : \mathbb{S}^1 \rightarrow M$) are just *loops* on M;
- a component diffeomorphic to $\mathbb{R}$ describes an abstract *eternal motion*. The corresponding configurations are *paths* on M;
- for a component diffeomorphic to some interval we have particles moving for s certain finite time inerval. The corresponding configurations are also paths.

Now we observe that, for $p > 1$, there is no complete classification of $(p + 1)$-manifolds up to diffeomorphism, so that we **cannot** do a similar analysis. Nevertheless, *there is a natural class of abstract motions* Σ *for each* p: the $(p + 1)$-manifolds Σ which are *cobordisms* between p-manifolds. There are many reasons to consider this class of abstract motions. We list some of them:

1. *every worldline is a cobordism*. By the discussion in the previous example, each worldline is a collection of circle and intervals (finite or not) which, in turn, can be seen as cobordisms: $[0, 1]$ is a cobordism between 0 and 1, as well as $(0, 1]$ is a cobordism from $\varnothing$ to 1 and, finally, the circle $\mathbb{S}^1$ can be seen as a cobordism between empty set manifolds. This indicates that looking at motions of p-branes given by cobordisms is the natural extension of motions of particles;

2. *boundary conditions*. As briefly commented, classical theories are generally given by the action functional S, which assign to any configuration a real number. From the classical viewpoint, not all of these configurations can be observed in nature, only those which minimize S. Minimization problems can be usually considered by some variational calculus approach, so that the extremization condition is described by a system of partial differential equations. In order to solve these equations it is natural to require some boundary conditions, which implies that the whole abstract motion has a boundary and, therefore, can be naturally regarded as a cobordism. Such boundary conditions are often of Dirichlet type,

so that when working with p-branes, the boundaries (or eventually its image under some configuration $\varphi : \Sigma \rightarrow M$, which are exactly where the boundary conditions are satisfied) are usually called D-branes ("D" of Dirichlet);

3. *holography*. There is another reason, more speculative, for using cobordisms in physical theories. Presently there are examples (such as AdS/CFT correspondence) suggesting the existence of a duality, called the *holographic principle*, which asserts that a physical theory in some domain is determined on the boundary of this domain by another theory. We will discuss this in more details in the next chapter and in other parts of the text.

There is also a purely mathematical motivation to the use of cobordisms instead of arbitrary manifolds: it is the *classification problem of manifolds*. As commented above there is no complete classification of smooth p-manifolds up to diffeomorphism for $p > 1$. On the other hand, as discussed in the first chapter, the Thom theorem give us a complete classification of manifolds up to cobordisms. Indeed, recall that there is a spectrum $\mathbb{MO}$ (the Thom spectrum) for which we have an isomorphism

$$\pi_n^S(\mathbb{MO}) \simeq \mathrm{Iso}(\mathbf{Cob}_n). \tag{2.4.1}$$

The left-hand side is simply the nth stable homotopy group of the Thom spectrum, while the right-hand side is the set of cobordism classes between n-manifolds. But the Thom theorem is a depper result. Recall that both sides of the expression (1.2.3) have an $\mathbb{N}$-graded abelian group and there are graded products that turn them into graded rings. Let $\mathrm{Iso}(\mathbf{Cob})$ be the graded ring obtained from the cobordism categories. It is natural to ask if there is some more general category $\mathbf{Cob}$ whose set of isomorphisms classes are precisely the graded ring $\mathrm{Iso}(\mathbf{Cob})$. Because $\mathrm{Iso}(\mathbf{Cob}_{p+1}) \subset \mathrm{Iso}(\mathbf{Cob})$, certainly this category must contain cobordisms between manifolds of arbitrary dimension.

For example, the most obvious approach to define $\mathbf{Cob}$ is by considering it as being the category whose objects are arbitrary manifolds, whose morphisms are cobordisms and whose compositions are gluing. Recall that cobordisms can be defined only between manifolds of the same dimension. Therefore, in this attempt in defining $\mathbf{Cob}$ there will be morphisms between two manifolds M and N only when they have equal dimension. In particular, we can compose only cobordism of the **same dimension**. On the other hand, the empty set $\varnothing$ can be regarded as a manifold of any dimension, so that any manifold can be considered as a cobordism $\varnothing \rightarrow \varnothing$. This implies that for arbitrary manifolds M and N (even having **different dimensions**) we can take the composition between $M : \varnothing \rightarrow \varnothing$ and $N : \varnothing \rightarrow \varnothing$, which is a contradiction.

The problem with this attempt in defining $\mathbf{Cob}$ is that the different manifold structures that can be introduced on $\varnothing$ are set on the same categorical level. Now, recall the intuitive notion of n-category presented in the last section. They are composed of objects, 1-morphisms, 2-morphisms, and so on, up to n-morphisms. This seems to be the natural language to place manifolds of different dimensions on different levels. Indeed, let M be a cobordism between $(n-1)$-manifolds M_0 and M_1. The manifolds M_0 and M_1 can also be viewed as cobordisms between $(n-2)$-manifolds, which in turn can be viewed as cobordisms between $(n-3)$-manifolds, etc. This suggests that for each n we can build a n-category $\mathbf{Cob}(n)$ whose objects are just 0-manifolds, whose 1-morphisms are 1-manifolds that are cobordisms between 0-manifolds, and so on, up to the n-morphisms which are cobordisms between $(n-1)$-manifolds.

The table below makes explicit the structure of the $(n-1)$-category of cobordisms and of the n-category of cobordisms. Looking at it we see that *we can move from* $\mathbf{Cob}(n-1)$ *to* $\mathbf{Cob}(n)$ *by adding one more layer of information*. Therefore, this is a manifestation of the categorification process discussed in the last section. But recall that a cobordism between p-manifolds is also interpreted as a fundamental abstract motion of p-branes. This means that $\mathbf{Cob}(p)$ is a category describing the abstract movements of each n-brane for $n \leq p$. So, we conclude that in *categorifying a theory which describes n-branes, for* $n \leq p$*, we get a theory describing n-branes, for* $n \leq p+1$*. Particularly, categorifying a theory of particles we get a theory describing not only particles, but also strings*!

$\mathbf{Cob}(n-1)$	$\mathbf{Cob}(n)$
0-manifolds	0-manifolds
1-cobordisms	1-cobordisms
$\vdots$	$\vdots$
$(n-2)$-cobordisms	$(n-2)$-cobordisms
$(n-1)$-cobordisms	$(n-1)$-cobordisms
????	n-cobordisms

Table 2.2: Categorifying cobordism category

Classical

The discussion above shows that string theories can be viewed as categorifications of particles theories. Consequently, it is expected that the fundamental aspects of particle physics can be categorified in order to produce fundamental aspects of string physics and, more generally, of p-brane physics. But recall that there are two flavors of physics: classical and quantum. Categorification should preserve these flavors in the sense that fundamental aspects of classical or quantum physics of particles are respectively categorified to classical or quantum aspects of string physics.

Le us analyze first the implications of categorification in **classical** physics. The consequences in **quantum** physics will be discussed in the next subsection. In the classical context, we have the following dichotomies in particle physics which we then could try to categorify:

1. *charge-force dichotomy*. Recall that in the study of classical electromagnetism, two **particles** can **interact** electromagnetically only if they have electric **charge**. This is a manifestation of a general fact: *to any fundamental force corresponds a fundamental charge and two particles can interact iff the corresponding charge in non trivial.*

2. *force-field dichotomy*. The electromagnetic force is mediated by a vector field: the electromagnetic field. More precisely, the electrically charged particles create an abstract entity (the **electromagnetic field**) which permeates the whole space. This entity is sensitive only to other electrically charged particles and, in the presence of one of them, both particles make a perturbation (a wave) in the field, which travels from one particle to the other at the speed of light. Such a perturbation carries information from one particle (say energy and linear momentum) and when it arrives at the other the information carried by the

wave is transferred to it. The **electric force** is precisely a measurement of the difference between the initial and the final energy/momentum. This is indeed the manifestation of another general fact: *any charged particle is the source of a vector field and the force is just a measurement of infinitesimal changes in the state of a test particle in the presence of such field*.

The first of those dichotomies clarifies when two particles can interact and the second says how this interaction happens. They are the basis of particle physics and, together, they justify why the Standard Model for particle physics (which is a theory describing three of the four interactions that presently are supposed to be fundamental) is a *gauge theory*. This type of theory will be discussed in detail in the chapters 11-13. However, let us say a few more words about them.

Electromagnetism is a classical theory and, therefore, as commented later, the configurations of this system corresponds to embeddings $\Sigma \to M$ of a 1-dimensional manifold (the worldline of a charged particle) into an ambient space. The second dichotomy above says that the interaction between charged particles is mediated by a **vector field** (i.e, the section of some vector bundle) defined on the whole **space**. This space is just the ambient space M in which the motion of the charged particle is realized. Therefore, the electromagnetic interaction is described by a **vector bundle** E over M. The **force**, on the other hand, can be measured by just embedding a charged particle in M and by analysing the **infinitesimal effect** produced by the sections of E. Mathematically, this infinitesimal effect can be described by the curvature of some **connection** on the bundle E. A gauge theory is just a theory about connections on general bundles, so that electromagnetism is an example of a gauge theory.

Having clarified this fundamental aspect of particle physics we can return to our discussion about the role of categorification in physics. If gauge theories are the basis of particle physics, it is expected that *higher gauge theory* (i.e categorified gauge theory) lies on the foundations of p-brane physics. But gauge theory is about connections on bundles, so that higher gauge theory must be about *higher connections* on *higher bundles*, as illustrated in the following table.

particle physics	$\Longrightarrow$	p-brane physics
particle	$\Longrightarrow$	p-brane
gauge theory	$\Longrightarrow$	higher gauge theory
bundle	$\Longrightarrow$	higher bundle
connection	$\Longrightarrow$	higher connection
geometry	$\Longrightarrow$	higher geometry

Table 2.3: Categorifying gauge theories of particles

Being "higher" versions, these concepts are supposed to be described not only by category theory and geometry, but by higher category theory. So, particularly, *it is expected that problems will occur when trying to describe gauge aspects of p-branes using only categorical language and classical geometry. Furthermore, it is also expected that these problems can be solved by considering categorified versions of the classical geometrical concepts.*

<u>Quantum</u>

Here we will analyze the effects of categorification in quantum physics. We recall that a quantum field theory of p-branes is a functor $U : \mathbf{Cob}_{p+1} \rightarrow \mathbf{Vec}_{\mathbb{C}}$. In particular, a quantum theory of particles is a functor $U : \mathbf{Cob}_1 \rightarrow \mathbf{Vec}_{\mathbb{C}}$. In the categorification process, categories lift to 2-categories and functors lift to 2-functors. On the other hand, as was discussed later, categorifying $\mathbf{Cob}_1$ we get $\mathbf{Cob}(2)$, so that it is expected that the categorification of a quantum theory of particles produces a 2-functor $U : \mathbf{Cob}(2) \rightarrow 2\mathbf{Vec}_{\mathbb{C}}$ taking values in some 2-category of vector spaces.

We observe that the resulting functor is **not** a quantum field theory of strings in the usual sense. Indeed, such quantum theory is defined on $\mathbf{Cob}(2)$ instead of on $\mathbf{Cob}_2$. Therefore, it is not only about strings, but also about particles (which is an usual effect of categorification). Therefore, *when we categorify a quantum field theory for particles we get an **extended** quantum field theory for strings.*

In the last subsection we concluded that we **must** use higher category theory in order to describe the **classical** aspects of interacting string theory. On the other hand, it should be observed that we can talk about **quantum** theories (a 1-categorical concept) of strings **without problem**. These are simply the usual functors $U : \mathbf{Cob}_2 \rightarrow \mathbf{Vec}_{\mathbb{C}}$. However, the categorification process reveals that **extended** quantum field theories (a higher categorical concept) tends to be a more natural object. So, we can ask: *what is the correct notion of quantum theory?* We present some points in favor of the extended quantum theories:

1. <u>*quantization*</u>. As will be commented in the next chapter (specially in Section 3.2), it is natural to expect the existence of a process, called *quantization*, which assigns to any classical theory of p-branes a quantum theory (in the usual sense) of p-branes. The existence of such a process will be discussed in chapters 15-16. There we will see that some examples of quantized theories are, hiddenly, extended quantum theories;

2. <u>*coherence*</u>. Another fundamental question in physics is: *what is the shape of the fundamental objects of nature?* this question is really difficult to answer, because in physics we cannot use inductive arguments, as will commented in Section 3.1. Now, suppose that we are studying the foundations of physics. When working in this problem with the canonical 1-categorical notion of quantum theory, certainly we need to know the answer to the above question, because the usual quantum field theories are about p-branes for a **fixed** value of p. On the other hand, an extended theory of p-branes is defined on $\mathbf{Cob}(p+1)$, which takes into account **all** k-branes with $k \leq p$. Therefore, if we know an upper value of p for which any l-brane, with $l \geq p$, is not a fundamental object, then the theory defined on $\mathbf{Cob}(p+1)$ will describe the foundations of physics without necessity of answering the above question. Concretely, there are many reasons to believe that p-branes, with $p \geq 2$, are not fundamental entities. Therefore, particles or strings are the only candidates for fundamental objects of nature and the winner is determined answering the above question. In the canonical approach to quantum theories we will then need to decide between those defined on $\mathbf{Cob}_1$ or $\mathbf{Cob}_2$, but in the extended approach both are unified in a unique category $\mathbf{Cob}(2)$, so that we do not need to decide between particles or strings.

3. <u>*invariants*</u>. In Section 1.2 we saw that functors are natural sources of invariants. Therefore, being functors, the canonical quantum field theories of p-branes can be used to get invariants

of p-manifolds, for a **fixed** p. On the other hand, an extended quantum field theory is defined on $\mathbf{Cob}(p+1)$ and, therefore, it can be used to get invariants for **all** k-manifolds, with $k \leq p$.

4. *classification.* Mathematically, the above point reveals that the extended theories are more **powerful**. On the other hand, being more **complex** objects, it is natural to expect that the extended field theories will be more difficult to build than canonical quantum field theories. But surprisingly, such extended theories are generally much simpler to describe than the usual theories! In fact, for any value of p, every extended quantum field theory $U : \mathbf{Cob}(p+1) \to \mathbf{C}$ is completely determined by its value on the trivial 0-manifold with a unique point! This very interesting fact is usually known as the Cobordism Hypothesis and was first conjectured by Baez and Dolan in [20]. A sketch of the prove in the general case was given by Lurie in [124] and some clarifications was presented in [216]. A more concrete proof of the $p = 1$ case was given by Schommer-Pries in [180]. Interesting reviews/expositions are [74, 29, 191, 201, 76]. A rough idea of the Lurie's proof is presented in Appendix B.

These points will become much more clear along the text. Indeed, in Chapter 14 we will insist on working with the canonical 1-categorial concept of quantum theory, but the constructions and the results obtained will admit a natural extension to the extended higher categorical context.

Chapter 3

Insight

In the previous chapters (specially in Section 2.3) we considered the consequences produced in physics by changes in logic. More precisely, we analysed in some detail the arrow **logic** $\Rightarrow$ **physics**. But recall that the relation between logic and physics is doubly directed, meaning that we also have an arrow **physics** $\Rightarrow$ **logic**, given by the *physical insight*. In this chapter we will present some (almost) concrete examples of this influence of physics on the foundations of mathematics. More precisely, we will look for situations in which **physical insight** can be effectively used to do **naive math** and, therefore, to produce **conjectures**. Succinctly, we will give some realizations of the following sequence:

$$\textbf{physics} \xRightarrow[\text{insight}]{\text{physical}} \textbf{logic} \Longrightarrow \begin{matrix}\textbf{naive}\\ \textbf{math}\end{matrix} \Longrightarrow \textbf{conjectures}\,. \tag{3.0.1}$$

The present chapter is divided in three sections. This means that we will give three kinds of realizations of the above sequence, each of them based on some physical principle coming from the connection between physics and ontology. The first kind concerns certain dualities in string theory and the physical insight corresponding to them will produce conjectures about the existence of high nontrivial relations between different areas of mathematics.

The second kind of examples is based on the the fact that inductive arguments cannot be used in order to do physics (i.e, in physics any inductive argument is falsifiable). If on one hand this fact clarifies the complexity of physics, on the other it gives a strategy to build conjectures. Indeed, *because inductive arguments are always falsifiable we can only make use of incomplete induction, which produces conjectures about the extension of a partially true sentence.*

Recall that in the previous chapter we discussed that more abstract background languages give more powerful tools to unify different physical laws. But we did not give any real reasons to believe in the existence of some unifying theory. Here, as an application of incomplete induction we will see that these reasons really exist. Especially, we will conjecture the existence of a theory (called *theory of everything*) describing the whole physics and being connected to the other known physical theories by only three mechanisms: as *effective theories*, by *compactification* and by *effective compactification*. In this section, also by making use of incomplete induction we will conjecture the existence of a *quantization process*, assigning to any classical theory a corresponding quantum theory.

In the third section we give another example of conjecture, also coming from physical insight. But, differently of the conjectures presented in the other two sections, immediately after stating it we prove it to be false! This conjecture is about the existence of a mathematical equivalence between two different presentations of the notion of interaction between particles: the *internal interactions* and the *external interactions*. Its failure have very nice consequences in mathematics, as the existence of modern invariants computing exotic smooth structures. We end the chapter extending these analyses to the context of p-branes and giving good reasons to believe that p-branes with $p \geq 2$ are not the correct building blocks of nature.

In a sense if Section 1.2 could be understood as a crash course on Algebraic Topology, the present chapter can be understood as a crash course on String Theory (especially, in its consequences for mathematics). Therefore, *string theory can be seen as a machine producing interesting mathematical conjectures.*

Remark. The implications of String Theory in mathematics that will be presented here are very abstract (or philosophical). There are, however, more concrete implications. Some of them include: an easy proof of the Atiyah-Singer index theorem for the case of Dirac operators [5, 79, 80], construction of new quantum invariants for knots [213, 160] and the construction of higher nontrivial examples of exotic manifolds [212, 189, 190] generalizing those given by Donaldson [58, 75]. Notice that each of these concrete implications are about geometry. Therefore, they appear at the geometric realization of sequence (3.1), as presented below. This is why they are not discussed in the present chapter.

$$\textbf{physics} \xRightarrow[\text{insight}]{\text{physical}} \textbf{logic} \Longrightarrow \begin{array}{c}\textbf{naive}\\ \textbf{math}\end{array} \Longrightarrow \textbf{conjectures} \xRightarrow{\text{realization}} \textbf{theorems}\,.$$

3.1 Dualities

Recall that in the last chapter we presented a naive idea concerning a process (the *cotegorification process*) which transforms any **mathematical concept** into a more abstract version of it. We would like to reinforce it: *the categorification process constructs new **concepts***. These new concepts are only abstract definitions which must be realized by examples. Evidently, a given concept can be realized by two different examples. On the other hand, *there is no reason why there should be some relation between two examples realizing the same categorified concept.* In fact, the only a priori relation between them is the fact that they realize the same definition.

For instance, let us consider the classical **concept** of monoid. This is just a set X endowed with an associative operation $* : X \times X \to X$ and with a distinguished neutral element $e \in X$. The categorification of this concept is obtained by adding one layer of information, which corresponds to promoting sets to categories, functions to functors and distinguished elements to distinguished objects.

So, the categorification of the concept of monoid produces a new **concept** (i.e, a new definition): the concept of *strict monoidal category*. This is a category $\mathbf{C}$ endowed with an **associative** bifunctor $* : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ together with a distinguished object $1 \in \mathbf{C}$ behaving as a “neutral element” for $*$. But notice that the associativity of an operation is described by a commutative diagram, so that the associativity of the bifunctor $*$ in a strict monoidal category will correspond to a commutative diagram in **Cat**. Now, recall that **Cat** is a 2-category, so that the Weakening Principle applies. This means that we can weaken the **concept** of strict monoidal category by requiring associativity only up to natural isomorphisms. The result is the **concept** of *weak* *monoidal category.*

Let us search for **examples** realizing this new categorified **concept** of weak monoidal category. The category **Set** of sets endowed with the *cartesian product bifunctor* $\times$, which assigns to any pair of sets its cartesian product, and to any pair of functions $f : X \to Y$ and $g : X' \to Y'$ the new function $f \times g : X \times X' \to Y \times Y'$, given by $f \times g(x, x') = (f(x), f(x'))$, is an example of weak monoidal category (any unit set can be used as a distinguished object). Similarly, the category $\mathbf{Vec}_{\mathbb{R}}$ of real vector spaces is a weak monoidal category when endowed with the tensor product $\otimes$. Notice that these realizations of the weak monoidal category concept are **not** equivalent. Indeed, the equivalence between **Set** and $\mathbf{Vec}_{\mathbb{R}}$ would imply, for instance, that the tensor product and the cartesian product of two spaces are isomorphic, which actually is not the case: if V and W are spaces of respective dimensions n and m, then $V \times W$ has dimension $n + m$ while $V \otimes W$ has dimension $n \cdot m$.

Recall that when categorifying a theory of p-branes we get a theory of $(p+1)$-branes. In particular, when we categorify a theory of particles we get a theory of strings. More precisely, categorifying the **concept** of particle physics we recover the **concept** of string physics. Therefore, as was the case with the categorified concept of weak monoidal category, it is natural to expect that the categorified concept of string theory could be realized by many **examples**, each of them possibly lying in a different area of mathematics, *which a priori need not be related*. On the other hand, notice that each of these realizations for string theory will describe the *same kind of physics*. But physics is connected with ontology, so that two theories cannot be physically distinguished when both reproduce the same correct empirical results. In our context, this means that if there are two realizations of the string theory concept reproducing the same correct empirical results, then they must be equivalent in some sense! This allow us to enunciate the following:

Conjecture: *different physically coherent realizations of the abstract concept of string theory* (*and, therefore*, *the different areas of math described by them*) *should be related in a nontrivial way.*

Now, natural questions concern the existence of such physically coherent string theories and the validity of the above conjecture for them. As will be discussed with more detail in chapters 11-13, there are at least **five** such nice string theories, called Type I, Type II A, Type II B, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic string theory. Surprisingly, such theories are indeed related by nontrivial isomorphisms called T*-duality* and S*-duality*, as presented in the diagram below (the dotted arrows correspond to indirect relations, obtained composing the other arrows). Furthermore, there are good reasons to believe that these are the unique coherent string theories. For instance, they are the **only ones** for which we have quantum anomalies cancellations, as will be discussed in chapters 11-13 and 14-16. For details, see also [??,??].

$$
\begin{tikzcd}
 & E_8 \times E_8-\text{Heterotic} \arrow[r, Rightarrow, dashed, "\text{indirect}"] \arrow[dd, Leftrightarrow, "T-\text{dualiy}"] & \text{Type II A} \arrow[dd, Leftrightarrow, "T-\text{dualiy}"] \\
\text{Type I} \arrow[ur, Rightarrow, dashed, "\text{indirect}"] \arrow[dr, Rightarrow, "S-\text{duality}"'] & & \\
 & SO(32)-\text{Heterotic} \arrow[r, Rightarrow, "S-\text{dualiy}"'] & \text{Type II B}
\end{tikzcd}
\tag{3.1.1}
$$

Holography

Because of the connection between physics and ontology, we cannot use inductive arguments to conclude that some sentence is true. Indeed, There is no a **enumerable** set of "fundamental experiments" under with we can apply induction, getting information on **all** experiments, because there exists **nonenumerable** independent physical variables which could be analysed. In other words, in physics *every inductive argument is falsifiable.* This reveals the complexity underlying the empiric nature of physics. For instance, we cannot use the veracity of particular cases to conclude the validity of the general case: even if a certain sentence agree with one million of experiments, this does not means that it will agree with **all** existing experiments. But in order to discard a sentence it is enough to build one experiment for which the prediction of the sentence is not correct.

We notice, on the other hand, that if a sentence agrees with many reliable experiments, then we can say that it is effectively true in a certain *domain*: the domain contemplated by the experiments correctly described by the sentence. If this domain is sufficiently dense, then it is natural to *conjecture* the possibility of extending it, as presented in the following diagram:

$$\begin{tikzcd} & \textbf{logic} \\ \begin{matrix}\textbf{certain}\\ \textbf{domain}\end{matrix} \arrow[ur, Rightarrow, "\text{true assertion}"] \arrow[r, Rightarrow, "\text{inclusion}"'] & \begin{matrix}\textbf{whole}\\ \textbf{physics}\end{matrix} \arrow[u, Rightarrow, dashed, "\begin{matrix}\text{incomplete}\\ \text{induction}\end{matrix}"'] \end{tikzcd}$$

In other words, in physics we cannot get **theorems** using induction. Therefore, using (incomplete) induction we will get **conjectures**. So, incomplete induction is a source of concrete realizations for the sequence (3.1). Consequently, *in order to get mathematical conjectures we must search for partially true physical facts.*

Example 3.1 (*source at the boundary*)**.** As commented later, the physical systems are generally described by *fields* defined on a $(p+1)$-manifold Σ with boundary (a cobordism). The manifold Σ is the abstract motion of a p-brane and the fields correspond to the existence of some interaction. Many of these systems have a peculiar behavior: when restricted to the boundary $\partial\Sigma$, they are totally determined by certain *source*. Some examples to have in mind are the following:

1. *gauge theory*. When analysed together, the charge-force and the force-field dichotomies imply that any gauge theory (for particles) has the required behavior. Indeed, charge-force dichotomy says that a particle can interact iff it is charged. On the other hand, the force-field dichotomy says that such interaction is determined by the background field, which itself is generated by the charged particle. In other words, the interaction have a source: the charged particles (which lie at the boundary of the abstract motion Σ);

2. *black holes*. As will be discussed in more details in chapters 11-13 gravitational theories (as, for instance, General Relativity) are generally described by considering a Lorentzian 4-manifold (Σ, g), called *spacetime*, which is supposed to satisfy certain conditions on the curvature (for instance, the Einstein's equation). In any Lorentzian manifold we have three kinds of geodesics γ: those for which $g(\gamma', \gamma')$ is positive, zero or negative, being respectively called *time-like*, *null-like* and *space-like* geodesics. Black holes are special gravitational

systems characterized by having a region $H \subset \Sigma$ such that any null-like geodesic crossing the boundary ∂H will remain inside H forever[1]. The boundary ∂H is called the *event horizon* of the black hole. But many fundamental aspects of the black hole, say its thermodynamic aspects, are determined simply by the area of its event horizon. This means that the black hole system is partially described by another system at the boundary: its event horizon.

Thanks to the previous example, the sentence "any physical theory defined on Σ is determined on the boundary $\partial\Sigma$ by another theory" is true in some domain. We usually say that the theory on Σ is the *bulk theory* and that the theory in $\partial\Sigma$ is the *source theory*. We also say that they are *holographically dual*. Therefore, applying incomplete induction, we get the following conjecture, called *holographic duality principle*:

Conjecture: *any physical theory of* p*-branes* (*and, therefore, any area of mathematics describing it*) *developed over a manifold* Σ *has some holographically dual theory of* $(p-1)$*-branes lying at the boundary* $\partial\Sigma$.

Many important realizations of this conjecture are given by examples of AdS-CFT correspondence. Such examples involve certain "compactification process" and "effective limits", as will be briefly explained in the next section.

3.2 Limiting

In the last section we concluded that to any partially true physical sentence we can associate (via incomplete induction) a mathematical conjecture. But this correspondence is **not stable** in the sense that if we make an experiment (outside of the initial domain of the sentence, of course) which does not agree with the predictions of the sentence, then incomplete induction immediately fails and the corresponding conjecture tends to fail too.

We can consider, for instance, a **partially defined** physical theory (i.e, one that is defined only for certain abstract motions Σ) as presenting a **partially true** physical sentence. In this case, saying that the domain of veracity of the sentence cannot be enlarged corresponds to saying that the domain of definition of the theory cannot be enlarged. There are several examples of physical theories in this situation. For instance, newtonian mechanics describes very nicely the phenomena in the everyday. On the other hand, the classical experiments in the beginning of the 20th century reveals that newtonian mechanics does not hold when considering particles whose mass is very small. One way to get an insight into this fact is by recalling Newton's second law, which says that the total force acting on a particle is given by $F = m \cdot a$, where m is the mass. Then, in the limit $m \to 0$ there is no effective contribution of any force and the Newton's law breakdowns. Therefore, incomplete induction fail for newtonian mechanics and, consequently, to the corresponding mathematical conjecture associated to it.

But if a physical theory is **correct** in a certain domain D, then it is natural to expect that any other theory replacing it must reproduce the initial theory in some limit converging to D. Indeed, in that limit both theories will describe the same physical phenomena and, therefore, following an argument used later, it is expected that they must be equivalent. More precisely,

[1] Null geodesics are generally interpreted as being light rays. Therefore, a black hole solution is characterized by having a region where light never scapes, meaning that they really are "black".

if the new theory has domain D', we expect that there is a one parameter family $\Sigma_\mu \in D'$, say with $\mu \in \mathbb{R}$, such that $D' \to D$ when $\mu \to 0$.

As will be discussed now, there are at least three ways to realize these limits: as effective theories, by effective compatifications and by compatictifications of the ambient space.

Effective Theories

If the initial theory is about p-branes, let is suppose that the enlarged theory is also about p-branes. The parameter μ then can be consider as being a certain "fundamental scale" of the abstract world of p-branes. It can be, for instance, the *energy scale*. We say that the oldest theory is an *effective theory* of the newest. Concrete examples to have in mind are the following:

Example 3.2 (*quantum mechanics*)**.** As discussed above, incomplete induction for newtonian mechanics fails in the limit $m \to 0$. Quantum mechanics, on the other hand, is a new theory which behaves very well in this limit. Both theories are about particles, so it is expected that newtonian mechanics be an effective theory of quantum mechanics. Indeed, the fundamental parameter in quantum physics is the *reduced Planck constant* $\hbar$ and in the limit $\hbar \to 0$ we recover newtonian mechanics.

Example 3.3 (*relativistic theories*)**.** In classical electrodynamics, any massive charged particle moving in some electromagnetic ambient space cannot reach the speed of light. This is a theoretical fact that agrees very well with the experiments. On the other hand, newtonian mechanics does not predict such phenomena, so that this theory also fail in another limit: when the particles are moving very rapidly, i.e when $v \to c$, where c is the speed of light predicted by electrodynamics. This reveals that there should exist another theory describing classical electrodynamics and having newtonian mechanics as an effective theory. This is the *particle relativistic field theory*, which has the velocity of light c as an internal parameter. In the limit $c \to \infty$ we recover newtonian physics.

Example 3.4 (*quantum field theories*)**.** Both quantum mechanics and relativistic field theory have newtonian mechanics as an effective theory. But these theories describe different types of phenomena (quantum mechanics applies to particles with very small mass while relativistic theories to very fast particles). Therefore, it is expected that both theories fail in the description of particles which are simultaneously small and fast. Indeed, presently there are particle accelerator experiments (as those given by the Large Hadron Collider, also known as LHC) that do not agree with the quantum mechanics nor with the relativistic field theory predictions. Therefore, there should exist another theory having both quantum mechanics and relativistic field theory as effective theories. Particularly, it is expected that the theory should be some kind of "quantum relativistic field theory" and simultaneously a "relativistic quantum theory", meaning that the diagram below is commutative. This type of theory is what is canonically called a *quantum field theory*. One important example is the Standard Model for particle physics which agrees very well[2] with the LHC experiments.

$$\begin{tikzcd} \textbf{particle quantum field theory} \arrow[r, Rightarrow, "\hbar\to 0"] \arrow[d, Rightarrow, "c\to\infty"'] & \textbf{particle relativistic field theory} \arrow[d, Rightarrow, "c\to\infty"] \\ \textbf{quantum mechanics} \arrow[r, Rightarrow, "\hbar\to 0"'] & \textbf{newtonian mechanics} \end{tikzcd} \tag{3.2.1}$$

Example 3.5 (*beyond Standard Model*)**.** The standard model, on the other hand, predicts that certain particles (called neutrinos) are massless, which do not coincide with actual experiments. Therefore, it is expected that there exist extended theories having the Standard Model as an effective theory. They describe the so-called "particle physics beyond the Standard Model". There are several models for such theories as, for instance, the Standard-Model Extension theory of Kostelecky and Samuel (see, for instance, [??]);

Example 3.6 (*gravity*)**.** In the above examples, gravity has not appeared. This is why gravity is considered as being a fundamental force which is not described by the Standard Model. Indeed, presently there is no a complete approach to "quantum particle theory of gravity" for a reason that will become more clear in Section 3.3. But from viewpoint of classical theories we can study gravity perfectly. Particularly, the most tested and accepted model is given by General Relativity (or some of its gauge extensions), which is a relativistic field theory having an additional internal scale G (the Newton's gravitational constant). In the limit $G \to 0$ the gravitational interaction is decoupled from the other forces, meaning that we have the additional arrow below. Its composition with the limit $c \to \infty$ is the so called *newtonian limit* and it is the canonical way to recover Newton's universal law of gravitation from General Relativity.

$$\begin{tikzcd} \textbf{particle gravity} \arrow[r, Rightarrow, "G\to 0"] \arrow[rr, Rightarrow, bend right, "\text{newtonian limit}"'] & \textbf{particle relativistic field theory} \arrow[r, Rightarrow, "c\to\infty"] & \textbf{newtonian mechanics} \end{tikzcd}$$

Example 3.7 (*string theory*)**.** The abstract motions of string theory are generally cobordisms between one dimensional manifolds and, therefore, are cobordisms between intervals or circles. When looking only at the first case (i.e, when the shape of the fundamental strings are supposed to be intervals) we say that we are doing *open string theory*; in the second case we say that we are doing *closed string theory*. When both cases are considered simultaneously we say that we have an *open-closed string theory*. It is expected that any string theory becomes equipped with an internal parameter α', called *Regge slope parameter*, and that any model including closed strings describes some gravity theory in the limit $\alpha' \to 0$. In other words, it is supposed that any closed and open/closed string theory has a theory of gravity as an effective theory. Indeed, each of the five models of string theory commented later actually describes closed strings, so that it is expected to have at least five good models of string gravity. We observe that, in principle, such models are **different** of those discussed in the last example: the effective theory obtained taking $\alpha' \to 0$ is about **strings** while the theory of the last example is about **particles**. On the other

[2]for certain experiments this agreement is of the order of nine decimal cases! See [??].

hand, as will be discussed in the next section, they have "under the surface" the **same** flavor.

$$\textbf{closed string theory} \xRightarrow{\alpha'\to 0} \textbf{string gravity}$$

Effective Compactification

In this approach, if the initial theory is about p-branes, then the extended theory must be about $(p+1)$-branes. A theory of $(p+1)$-branes has $(p+2)$-cobordisms between $(p+1)$-manifolds as abstract motions. Some of these cobordisms $\Sigma : \Sigma_0 \to \Sigma_1$ are between manifolds of the form $\Sigma_i \simeq \Theta_i \times X_i$, where Θ_i is a 1-manifold and X_i is a p-manifold. Therefore, Θ_i is either an interval or a circle, so that we can assign to it a parameter ℓ_{p+1} (taking the role of μ), called the $(p+1)$-*brane scale*, which captures the "length" of Θ_i: if it is an interval, then ℓ_{p+1} can be regardered as its measure, while if it is a circle, then ℓ_{p+1} is its radius.

Observe that in the limit $\ell_{p+1} \to 0$ the manifold Θ_i collapses to a point, so that Σ_i collapses to the manifold X_i and the cobordism Σ collapses to a cobordism between p-manifolds. This reveals that in the limit we recover the initial theory of p-branes having as abstract motions some of the collapsed cobordisms. We then say that the p-brane theory was obtained from *effective compactification* (or *compactification of the worldvolume*) of the $(p+1)$-brane theory. In our context, the most important example is the following:

Example 3.8 (*from strings to particles*)**.** Applying the above procedure to a theory of strings we get a theory of particles. But by the examples discussed in the previous section, string theories also have gravity theories as effective theories. Therefore, it is natural to expect that the diagram (3.2) can be enlarged to the following diagram (the segmented arrows corresponds to effective compactification limits):

$$\begin{tikzcd}
\textbf{string theory} \arrow[r, Rightarrow, "\alpha'_s\to 0"] \arrow[dr, Rightarrow, dashed, "\ell_s\to 0"'] & \textbf{string gravity} \arrow[r, Rightarrow, dashed, "\ell_s\to 0"] & \textbf{particle gravity} \arrow[d, Rightarrow, "G\to 0"] \\
& \textbf{quantum field theory} \arrow[r, Rightarrow, "\hbar\to 0"] \arrow[d, Rightarrow, "c\to\infty"'] & \textbf{relativistic field theory} \arrow[d, Rightarrow, "c\to\infty"] \\
& \textbf{quantum mechanics} \arrow[r, Rightarrow, "\hbar\to 0"'] & \textbf{newtonian mechanics}
\end{tikzcd}
\qquad
\begin{tikzcd}
\textbf{string theory} \arrow[r, Rightarrow, dashed, "\ell_s\to 0"] \arrow[d, Rightarrow, dashed, "\ell_s\to 0"'] & \textbf{particle gravity} \arrow[d, Rightarrow, "G\to 0"] \\
\textbf{quantum field theory} \arrow[r, Rightarrow, "\hbar\to 0"] \arrow[d, Rightarrow, "c\to\infty"'] & \textbf{relativistic field theory} \arrow[d, Rightarrow, "c\to\infty"] \\
\textbf{quantum mechanics} \arrow[r, Rightarrow, "\hbar\to 0"'] & \textbf{newtonian mechanics}
\end{tikzcd} \tag{3.2.2}$$

But now, an important fact is that the fundamental string length ℓ_s is related to the Regge slope parameter α' by $\ell_s^2 = \alpha'$, so that taking the effective limit $\alpha' \to 0$ we are automatically taking the effective compactification limit $\ell_s \to 0$, clarifying that the first diagram above collapses to the second. This fact has a very intuitive meaning: in the limit $\alpha' \to 0$ when the energy of the strings come down, the strings themselves behaves like particles and, therefore, the classical theories of gravity for particles (like General Relativity) appear naturally. In other words, *differently from what happens with the Standard Model, the models for string theory describe not only quantum theory of particles but also gravity!*

Compactification

There is a third approach that can be used in order to connect an enlarged theory with the smaller theory in a limiting way. It differs from the other approaches in the following point: *while the others are constructed independently of the ambient space, here this dependence is explicit.* The fundamental aspects of string theory are obtained by making use of this new approach, so this is a good reason to say that *string theory is a background dependent theory.*

Our starting point is a theory of p-branes moving inside some compact ambient space M, say of dimension n. This means that we are considering certain configurations $\varphi : \Sigma \to M$. Each such configuration induces a theory on $\varphi(\Sigma)$. Now, suppose we are given another theory on the product $M \times \varphi(\Sigma)$. If μ is the volume of M (which is well defined because the manifold is assumed compact), in the limit $\mu \to 0$ we have that M becomes trivial and the theory on $M \times \varphi(\Sigma)$ reproduces the theory induced on $\varphi(\Sigma)$ from the initial theory of p-branes. We then say that *the p-brane theory arises from the theory defined on* $M \times \varphi(\Sigma)$ as *a compactification on* M.

Generally this compactification is modeled by the so called *Kaluza-Klein mechanism.* It applies when the theory on the product $M \times \varphi(\Sigma)$ is some flavor of gravity. The resultant theory on $\varphi(\Sigma)$ is gravity coupled with a gauge theory and with a scalar field called *dilaton.* This will be discussed in more details in chapters 11-13. At this moment, let us see how such compactification process, together with effective limits and effective compactifications, allow us to produce a concrete version (called $\mathrm{AdS}_5 - \mathrm{CFT}_4$ *duality*) of the holographic principle conjectured in the section 3.1.

Example 3.9 ($\mathrm{AdS}_5 - \mathrm{CFT}_4$)**.** We start by considering a gauge theory of open-closed strings with configurations $\varphi : \Sigma \to M$, defined on some cobordism Σ and taking values in a ten-dimensional ambient space M. Then we have the image $\varphi(\partial\Sigma)$ which we we assume ends in some $D3$-brane S (which actually is a 4-brane). The gauge theory on the ambient spaces induces a theory on the $D3$-brane. The string theory has the Rugge slope parameter α' and in the low energy limit $\alpha' \to 0$ it becomes a gravity theory in ten dimensions. So, in this limit we have the gauge theory of the 4-brane and the ten dimension gravity theory on the ambient space. Such gravity theory is generally a black brane theory, which in the near-horizon limit has geometry $\mathrm{AdS}_5 \times \mathbb{S}^5$. In this limit, the $D3$-brane stays at the asymptotic boundary of AdS_5. Kaluza-Klein compactification on the five sphere $\mathbb{S}^5$ then gives a gravitational theory on AdS_5 which must reproduce the initial gauge theory on the $D3$-brane. This means that *the gravitational system* AdS_5 *is determined by a four dimensional gauge theory at its boundary, which is a manifestation of the holographic principle.*

As an additional example of the role of compactification in string theory, recall that in Section 3.1 it was predicted the existence of dualities between different realizations of the abstract notion of string theory. We commented that there are at least five such realizations which actually are related by the so-called S-duality and T-duality. Now we can say with little more detail what T-duality is about:

Example 3.10 (*T-duality*)**.** We start by observing that, as presented in diagram (3.2), T-duality appears only between string theories of the "same flavor": we have T-duality only between Type II A and Type II B string theories, as well as between $SO(32)$-Heterotic and $E_8 \times E_8$-Heterotic string theories. The reason is the following: Type II and the Heterotic flavors are about *oriented strings*, meaning that the worldsheet covered by these strings are *oriented 2-manifolds*. The two **different** orientations of produces theories about the **same** string with **different** symmetries, justifying the existence of two kinds of Type II theories and two kinds of Heterotic theories. T-duality is precisely about the equivalence of the theories of the **same** string with **different** orientations when compactified on some torus (thus, **T**-duality can also be called **Toroidal** Duality). More precisely, starting with Type II A string theory on an ambient space $M \times \mathbb{T}^k_R$, where $\mathbb{T}^k_R = \mathbb{S}^1_{R_1} \times ... \times \mathbb{S}^1_{R_k}$ is the k-torus with radius $R_1, ..., R_k$, taking the low energy limit $\alpha' \to 0$ and finally compactifying the resulting effective theory on the k-torus we get a certain theory of gravity on M coupled to a gauge theory. On the other hand, starting with Type II B string theory on the ambient space $M \times \mathbb{T}^k_{1/R}$ and doing the same process, we get another gravity theory on M. Now, *T-duality states that both theories are the same*.

Theory of Everything

From the beginning of this text we have been talking about the *unification/axiomatization problems of physics*. We discussed, for instance, that the development of more and more abstract background languages gives more and more powerful tools to work in these problems. But up to this point we have not looked at the following very natural questions:

1. Are there reasons to believe that the problems have some solution?
2. If this solution really exists, in which sense the unified theory describes the actual physical laws which agree very well with many classes of experiments?

Now,notice that the examples discussed in the previous subsections reveals that the sentence "any physical theory has a limiting extension" is partially true. Consequently, by the arguments of the last section, incomplete induction produces a corresponding mathematical conjecture which gives an answer to both questions above:

Conjecture. *There exists a maximal theory* (*described by a very abstract mathematical language*) *of which any other known physical theory can be obtained by compactification methods or as an effective theory.*

Such a hypothetical physical theory is called *theory of everything*. The example 3.8 reveals that string theory unifies gravity and the Standard Model, so that it unifies all forces that presently are considered as being really fundamental. Furthermore, by the discussion at Section 2.3, string theory must be described by a Higher Category Theory, meaning that (as suggested by the above conjecture) the background language used to describe it is very abstract. Therefore, *string theory seems a very nice candidate to a theory of everything*.

On the other hand, there is a strong reason to believe that string theory is **not** the correct model to the everything theory. Indeed, there are at least **five** coherent models of string theory and we would like to have a **unique** coherent model to the theory of everything. Evidently, if such a model really exists, then starting from it and doing effective limits and compactifications we need to recover each string theory model.

The most immediate strategy is to try to consider it as being some **coherent** theory having 2-branes (instead of strings, which are 1-branes) as fundamental objects. Such theories arise from categorification of the string theories and, therefore, are described by abstract languages, making it clear that they really are canonical candidates. On the other hand, as will be discussed in the next section, there are reasons to believe that a such **coherent** theory of 2-branes does not exist.

If 2-brane theories are not the solution, how to proceed? Recall that the coherent models to string theory are connected by dualities. Some of these dualities are *indirect* in the sense that they are obtained by composition of other dualities. So, we can try to search for a theory which turns such indirect dualities into direct dualities. One of such a hypothetical model to the theory of everything is called *M-theory*.

Quantization

Recall the concept of effective theories: we say that a theory with domain D is an *effective theory* of some other theory with domain D' when there is a parameterized family of abstract motions $\Sigma_\mu \in D'$ such that $D' \to D$ when $\mu \to 0$. So, the effective theories are produced starting with a theory and looking for a new theory which reproduces the initial theory in some limit. Previously were given many realizations of this idea. In some of them, the parameter μ were precisely the reduced Planck's constant $\hbar$. They include the newtonian mechanics as an effective theory of **quantum** mechanics and particle relativistic field theory asan effective theory of particle **quantum** field theory.

Newtonian mechanics and relativistic field theory can be seen as classical theories in the sense that they are determined by selecting certain embeddings $\varphi : \Sigma \to M$. On the other hand, quantum mechanics and particle quantum field theory really are quantum theories, meaning that they are defined by some functor $\mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$. Therefore, these examples reveals that the sentence "for any classical physical theory corresponds a quantum physical theory which have the initial theory as an effective theory on the Planck's constant" is partially true. Incomplete induction then gives the following conjecture:

Conjecture. *There exists a process assigning to each classical theory a quantum theory parameterized over $\hbar$, in such a way that the classical theory can be recovered in the limit $\hbar \to 0$.*

This hypothetical process is called a *quantization process.* Its existence/uniqueness will be analysed in chapters 15-16. At this moment we will only comment on the mathematical interest in int. Let us start by recalling that, being functors defined on cobordism categories, quantum theories are natural sources of invariants. On the other hand, classical theories have a very geometrical nature and, consequently, are very easy to build. Therefore, this quantization process will allow us to produce powerful invariants from easily constructed classical theories. In other words, from a mathematical viewpoint, *quantization is a "machine" producing invariants for smooth manifolds.*

3.3 Interactions

In Section 2.4 we discussed some important aspects of particle physics. We saw, for instance, that to any fundamental interaction corresponds a certain fundamental charge and two particles can interact iff they have non-zero charge. But now we observe that given a charged particle, there are **two** kinds of interactions that can be considered: *internal* interactions and *external* interactions. Indeed, assuming that the given particle is interacting with another particle, we can build two types of systems: one in which both particles are included and other in which the particles are analysed separately. In the first case we say that the interaction between the particles is *internal to the system*. In the second we say that each particle is subjected to an *external interaction* determined by the other particle.

In other words, recalling that charged particles can be viewed as the sources of the interactions, a system has internal interactions when it contains the sources of the interaction, while it has external interactions when it does not contain the sources of the interaction.

We commented that the canonical way to model physical systems is by considering abstract motions: these are $(p+1)$-manifolds Σ (generally cobordisms) representing a possible time evolution of a p-brane. Certainly, this model contemplates the *external interaction*. Indeed, by the force-field dichotomy, having realized Σ into some ambient space M by a map $\varphi : \Sigma \to M$, we can model the interaction of the particle with worldvolume $\varphi(\Sigma)$ as being mediated by a section of a certain bundle $E \to M$. The interaction is **external** because *it is not a property of the abstract motion Σ, but of the environment M.*

Now we would like to observe that at least for **particles** the previous model for physical systems does **not** contemplate **internal** interactions. In fact, being internal, such interactions should be *totally described by the abstract motion* Σ (which up to this moment was assumed to be a 1-manifold). Typical examples of internal interactions between particles are the *scatterings*, which include, for example, *collisions*. But the worldline underlying this kind of interaction has a *vertex* (the point at which the particles scatter). Closely to the vertex, the worldline is like an "X" and, therefore, it cannot be a manifold (recall that **Diff** is poor of limits and colimits) contradicting the supposed smooth structure of the abstract motions.

The paragraph above clarifies that in order to model systems of **particles** with **internal** interactions we need to work with a different approach, i.e., we must allow that the abstract motions of a system of interacting particles can be objects that are more general than smooth manifolds. But, what kind of objects? This will depend on the model chosen to describe the internal interactions. One useful model is the following:[3]

1. first we consider two systems describing the particles separately. In them, each particle follows an abstract motion (say Σ_1 and Σ_2) which is supposed to be a genuine 1-manifold, as before. This means that Σ_i can be circles, lines or intervals. In our model, such abstract motions are called the *propagators* of the particles;

2. we then select a distinguished point in each Σ_i, corresponding to the point at which the interaction between the particles will occur. More precisely, instead of working with smooth 1-manifolds, we assume that Σ_1 and Σ_2 are *based 1-manifolds*. A fundamental interaction between a particle with abstract motion Σ_1 and other particle with abstract motion Σ_2 can be described by gluing such manifolds at the distinguished point. The result generally is not a manifold, but a pointed space with a smooth singularity, here called the *fundamental vertex* (or first order interaction). Notice that in principle we can glue Σ_1 into itself, which characterizes a theory describing particles with *self interactions*;

3. finally, we build a new system, now including both particles, whose abstract motion Σ is given by fundamental vertexes, fundamental vertexes glued onto fundamental vertexes

[3]We will assume that we are trying to describe a system in which two particles interact, because the abstraction to the case of systems containing more than two particles will be immediate.

(called *second order interactions*), and so on. Examples of possible worldlines are presented below. The first picture corresponds to a first order interaction, the second and the third to a second order interaction and the fourth to a third order interaction.

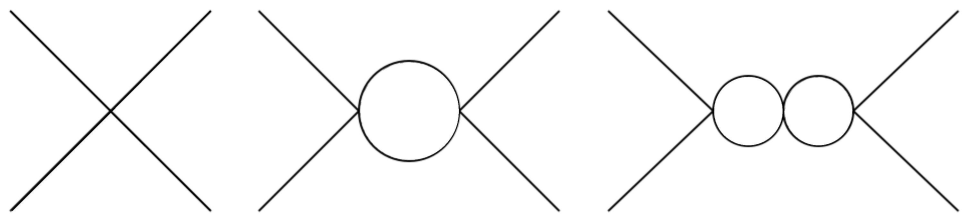


Figure 3.3.1: Some internal interactions

Note that in each of the above pictures we have a unique type of line (a continuous line). This is because we pictured interactions between two particles with the **same properties**. When considering interaction between particles with **different properties**, the propagator of each of them is usually represented by different lines (say one continuous and the other dotted, as in the diagram below), emphasizing that the particles interacting have different properties.

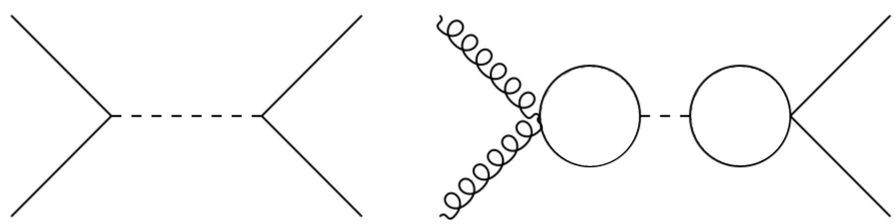


Figure 3.3.2: Particles with different properties interacting

Now, we ask: in this model of internal interactions, what kind of structure the abstract motions have? They cannot be manifolds, of course, but at first glance we can try to consider them as *one dimensional orbifolds* (i.e, as objects Σ modeled over some quotient space $\mathbb{R}/G$ by a *finite* group G), which are the natural generalizations of manifolds to incorporate smooth singularities (the singularities live precisely at the fixed points of the G-action on $\mathbb{R}$). However, this does not work, because in principle we may have interactions of *arbitrary* orders (instead of up to a fixed *finite* order), as below.

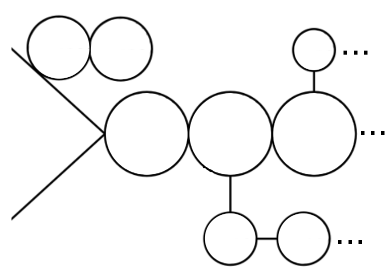


Figure 3.3.3: Particles with different properties interacting

Therefore the next strategy is trying to characterize the worldlines as being entities modeled over quotients $\mathbb{R}/G$ by groups that are not necessarily finite. In some situations this can be effectively done, as in the case of gauge theories. Indeed, in them we have a distinguished group: the gauge group. The propagators are then labelled by representations of such groups over a finite complex space and the fundamental vertex are intertwiners between representations (see, for instance, [27, 17]).

On the other hand, in the general case there is no way to do this. But there are no obstructions to consider each abstract motion as being a graph, called the *Feynman graph* of the interaction. If the initial abstract motions Σ_1 and Σ_2 are assumed to be **oriented** 1-manifolds, then the resultant graph is indeed an **oriented** graph (a diagram) and in such cases we talk about *Feynman diagrams*.

A theory of interacting particles that can be described by the external perspective is usually called *full* or (*non perturbative*) and a theory described by the internal perspective is called *perturbative*. So, we conclude that *Feynman diagrams model perturbative theories of interacting particles.*

But now, notice that **independently of the model** that is fixed, in principle we can move from a system with internal interaction to a system with external interaction by simply isolating the particles. Reciprocally, if we are in a system with external interaction we can turn it into a system with internal interaction by enlarging the system to include the sources of the interactions. This suggests that both perspectives are *physically equivalent* and, therefore, this also suggests a underlying *equivalence between the mathematical models* describing external and internal interactions, motivating us to enunciate the following conjecture:

Conjecture. *In any model, full and perturbative theories of* ***particles*** *are mathematically equivalent.*

Topological Defects

Let us analyse the last conjecture. In order to do this, recall that our model of full theories is about theories with external interaction. This means that the theory does not contain the source of the interaction, so that it is a property of the **ambient space** in which the abstract motions will be realized. Furthermore, the possible realizations $\varphi : \Sigma \to M$ (which in the last instance will give the possible interactions) are those that minimize a certain action functional $S : \mathrm{Map}(\Sigma; M) \to \mathbb{R}$. More precisely, this minimization process produces a partial differential equation whose solutions are the physically interesting configurations (i.e, the physically interesting ways to realize the external interaction).

On the other hand, the selected model of perturbative theory involve internal interactions. This means that the theory contains the sources of the interaction, so that they are totally described by the **abstract motions** (i.e, before the realization in some ambient space). In special, the discussion in the last subsection reveals that in this approach the abstract motions cannot be assumed to be 1-manifolds: generally they are only graphs. These graphs have fundamental pieces: the propagators and the vertex. The graphs describing the possible abstract motions are exactly those obtained by gluing the fundamental pieces, which allows us to organize them into an increasing sequence parametrized by the order of the interaction. This sequence is called the *perturbation sequence*.

We notice that (in the above models) any full theory really induces a perturbative theory. More precisely, any full theory (represented by some action functional) induces a canonical perturbation sequence. Indeed, let S be an action representing a full theory. Generally it can be written in the form $S = S_0 + S_{\text{int}}$, where the first term is the theory of the particle without the external interaction (called the *free theory*) while the second part is the external interaction. We usually introduce a parameter λ in S_{int} (called *coupling parameter*) quantifying the power of the interaction. Therefore, we can define a family of full theories $S_\lambda = S_{\text{free}} + \lambda S_{\text{int}}$, so that in the limit $\lambda \to 0$ we recover the theory without interaction. This means that *the free theory can be seen as an effective theory of the interacting theory.*

Now, consider e^{S_λ} as a function of λ. Expanding it on Taylor's series around $\lambda = 0$ we get an expression like

$$S_{\text{free}} + \lambda S_{\text{int}} + \frac{\lambda^2}{2!} S_{\text{int}}^2 + \mathcal{O}(\lambda^3), \tag{3.3.1}$$

called the *perturbative expansion of S*, suggesting that the construction of a perturbation sequence is one in which the nth order interactions should be built from the terms involving nth power of the coupling parameter λ. Particularly, the propagators and the fundamental vertexes of the corresponding perturbative theory should be respectively determined by S_{free} and by S_{int}. Such a correspondence really exists and is given by the *Feynman rules* [207, 148, 215], which can be summarized in the following diagram:

$$\begin{array}{c}\textbf{full theory}\\ \textbf{(external interaction)}\end{array} \xRightarrow{\text{Feynman rules}} \begin{array}{c}\textbf{perturbative theory}\\ \textbf{(internal interaction)}\end{array}$$

In order to test the conjecture described in the last subsection we must ask if the above arrow may has an inverse. More precisely, we must ask if the perturbative expansion of a given action S by (3.3.1) can describe **all** configurations $\varphi : \Sigma \to M$ minimizing S (corresponding to the surjectivity of the above arrow) and if the action is uniquely determined by its perturbative expansion (meaning that the arrow is injective). However, observe that if instead of S we consider another perturbed action

$$\overline{S} := S + \frac{1}{\lambda} S_{\text{def}} = S_{\text{free}} + \lambda S_{\text{int}} + \frac{1}{\lambda} S_{\text{def}},$$

then both S and $\overline{S}$ will have the same perturbative expansion and, therefore, they will define the same pertubation theory. Since in the limit $\lambda \to 0$ the function $e^{\frac{1}{\lambda}}$ and each of its derivatives goes to zero, we conclude that the arrow in the previous diagram **cannot be injective**.

We assert that this arrow also **cannot be surjective**. Indeed, in order to prove this, we first observe that the term S_{def} is generally constructed in the following way. Let $\mathcal{C}$ be the space of all configurations minimizing S. It can be endowed with a canonical topology, so that we have a notion of proximity between two configurations. There are some minimizing configurations that are special: those which also minimize the interacting part S_{int} (these are the so called *vacuum configurations*). Let $\mathcal{C}_{\text{vac}} \subset \mathcal{C}$ be the subspace of these vacuum configurations. In many cases $\mathcal{C}_{\text{vac}}$ is not contractible, so that there are "topologically disconnected vacuums", meaning that $\pi_n(\mathcal{C}_{\text{vac}}) \neq 0$ for some n. Generally we also have a distinguished trivial vacuum solution φ_o. The other vacuum solutions topologically disconnected to φ_o are the *topological defects*. With this in mind we can consider S_{def} as being simply S_{int} restricted to these defects. Usual situations are those given by gauge theories with *spontaneous symmetry breaking* in which important examples of topological defects are given by the *magnetic monopoles* in classical electromagnetism [??] and *instantons* in nonabelian Yang-Mills theory [??]. Now, it is clear that not all configurations minimizing S have to minimize $\overline{S}$. Indeed, the vacuum solutions connected to φ_o minimize S_{int} but not S_{def}. But both actions have the same perturbation expansion, so the arrow in the previous diagram really cannot be surjective.

Conclusion. The discussion above reveals that, at least for the mathematical models describing full and perturbative theories developed here, *the conjecture presented in the last subsection is wrong!*

In this chapter we discussed many examples of mathematical conjectures arising from physical insight. Every one of them (except the last) is believed to be true. So, we can ask: which is the impact in mathematics given by the failure of the last conjecture? Surprisingly, the impact is very nice! Indeed, in some interesting cases the moduli space $\mathcal{C}_{\mathrm{def}}$ of topological defects is smaller enough in order to accommodate a finite-dimensional smooth structure, whose homology/cohomology classes provides powerful invariants for smooth manifolds, which can be used, for instance, to produce nontrivial examples of exotic manifolds[4]. In the context of nonabelian Yang-Mills theory, we have the instantons and its moduli space originates the Donaldson's invariants, for which Donaldson got his Fields Medal [58, 75]. On the other hand, in the context of super Yang-Mills theory we have even more powerful invariants: the Seiberg-Witten invariants (the original articles of Seiberg and Witten are [212, 189, 190]. For nice expositions see [136, 149, 150, 177, 157] and for a relation with the called Gromov-Witten invariants (which arise in symplectic topology) see [204]).

Renormalizability

In the last subsection we learned that *full theories of* ***particles*** *really contain more mathematical information than perturbative theories*. We would like to generalize this conclusion for theories of p-**branes**. This is not immediate, because in principle we only developed a model for internal interactions between **particles**. So, our starting point is to define what is a perturbative theory of p-branes.

Recalling that string physics can be seen as being a categorification of particle physics, it is natural to define the perturbative theories of strings as being the categorification of the model for perturbative theories of particles. Iterating this process we will them get a model for the perturbative theories of arbitrary p-branes. As discussed, in a system of particles with internal interactions, the abstract motions generally do not have the structure of manifolds, but are generally graphs obtained following a certain recipe. So, we can get a model to internal interactions between strings by categorifying this recipe, as presented below:

1. we start by considering two systems describing the strings separately. Each string follows an abstract motion (say Σ_1 and Σ_2) which is assumed to be cobordisms between genuine 1-manifolds, here called the *propagators* of the strings;

2. we then select arbitrary boundary components in each Σ_i, corresponding to regions in which

[4] A general perspective in the construction of these invariants can be seen in [205] around the pages 13-23.

the interaction will occur. The first order interactions are obtained by gluing the cobordisms Σ_i at the distinguished boundaries. We may have self interactions, as happened previously in the case of particles;

3. finally, we build a new system whose abstract motions Σ are those given by first order interactions, second order interactions, and so on.

Therefore, the situation for strings seems very similar to the situation for particles. For instance, we can also organize these "higher-dimensional Feynman diagrams" in a perturbation sequence, where the number of loops is replaced by the number of holes (the genus) of the abstract motion Σ, as pictured below. The first is an example of diagram for closed string theory. The second is for open string theory and the third is about open/closed string theory.

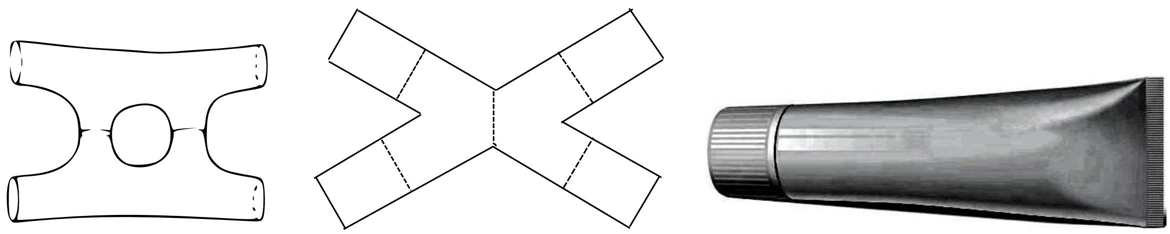


Figure 3.3.4: Feynman diagrams for strings

Now, recall that our model of full theories applies not only to particles, but equally well to strings. This means that a full theory of strings is represented by an action functional S defined in some configurations $\varphi : \Sigma \to M$, where Σ is a 2-cobordism. The physically interesting configurations are the critical points of S, meaning that they are **solutions** of a system of **partial differential equations** obtained from variational calculus applied to S. Such equations also admit topological defects, so that the conjecture present in the last subsection also fail in the string context. Therefore, *full theories of **strings** contain more mathematical information than perturbative theories.*

We notice, on the other hand, that in order to solve a system of partial differential equations we generally require some boundary conditions, as Dirichlet/Neumann conditions. These conditions are constraints to the possible submanifolds $S \subset M$ containing the boundaries $\varphi(\partial\Sigma)$. But these submanifolds give information on the **ambient space** and, therefore, are special features of the **external interaction**. Thus, it as happened for the topological defects, such data is not captured by the perturbation sequence. But for Dirichlet boundary conditions this data describes precisely D-branes! This reveals that *D-branes* (and, therefore, *black branes*) *also are non-perturbative effects.*

Despite the similarities between full/perturbative aspects of particle and string physics, we observe that there is a crucial difference between them. Indeed, while particles interact at a *vertex* (which is a smooth singularity of the abstract motion), strings interact by gluing at boundaries, which is a well defined smooth process.

On the other hand, when working with open and open/closed strings, the Feynman diagrams have a kind of singularity not appearing in the diagrams for particles. Indeed, the abstract motions of strings are cobordisms between 1-manifolds and when we have open strings or open/closed strings we need to consider cobordisms between intervals or between intervals and circles. These have smooth singularities as presented, for instance, in the last picture. So, this fundamental difference between perturbative theory of particles and strings can be summarized in the following table:

singularities type $\Rightarrow$ fundamental objects $\Downarrow$	**cobordism**	**interactions**
particle	–	$\times$
closed strings	–	–
open strings	$\times$	–
open/closed strings	$\times$	–

Table 3.1: Smooth singularties of particles *vs* strings

We concluded that both perturbative theories for particles and for strings have smooth singularities, but of different origins. We can then ask: *what is the effect of such singularities in the physical theories?* Recall that in the last section we conjectured the existence of a quantization process assigning to any full classical theory a corresponding quantum theory. As will be discussed in Chapter 15, we can try to realize this quantization process by first passing to the perturbative theory as in the following diagram:

$$\begin{matrix}\textbf{full} \\ \textbf{theory}\end{matrix} \xRightarrow[\text{expansion}]{\text{perturbative}} \begin{matrix}\textbf{perturbative} \\ \textbf{theory}\end{matrix} \xRightarrow{\text{quantization}} \begin{matrix}\textbf{quantum} \\ \textbf{theory}\end{matrix}$$

The idea is the following: given an action S we build a perturbative sequence of Feynman diagrams. We then define the corresponding quantum theory by "summing" over all such diagrams. The quantum theory is well defined when **each term** of the sum is **finite** (the series itself **need not converge**. Actually, the presence of topological defects and other nonperturbative effects, as D-branes, implies that such series must **diverge** [60]). It happens that generally the smooth singularities in the diagrams (specially those arising from loops) give infinite contributions, meaning that the term at the sum corresponding to a diagram with smooth singularities is infinity and, therefore, quantization cannot be applied to it.

On the other hand, many of these infinities coming from smooth singularities can be absorbed by redefining the parameters of the theory. A theory in which all infinities coming from singularities can be absorbed is called *fully renormalizable*. In every theory we have only a finite quantity of parameters, so that a given theory is fully renormalizable only if each of its Feynman diagrams contains a finite amount of smooth singularities. As shown in the last table, these singularities appear in two types: on the vertex of interactions between particles and on cobordisms between open or open/closed strings. So, when working with **particles**, we need to analyse only the first kind of singularities. On the other hand, when working with **strings** we need to analyse only the second kind.

As discussed in previous subsections, a general Feynman diagram for **particles** may have arbitrary smooth singularities when $\lambda \to \infty$. This does **not** mean that each particle theory is nonrenormalizable, but it is only a clue about the existence of this kind of theories. A similar situation happens with a general Feynman diagram between open **strings**: in each diagram we have a finite number of cobordism singularities (see the last pictures), but we are gluing singular cobordism into singular cobordisms, so that such singularities propagate and when $\lambda \to$

∞ they can grow arbitrarily. Therefore, we **cannot** say that every theory of open strings is nonrenormalizable; we can only believe in the existence of a theory with this property.

In closed string theories, on the other hand, the propagators are well defined cobordisms and the interactions are given by gluing these cobordisms, so that there are no smooth singularities. This suggests that this kind of theory is fully renormalizable. The classical theories of gravity, considered as particle theories, generally are nonrenormalizable. This means that particle theory does not give a nice way to talk about "quantization of gravity". But, as commented later, closed strings generally describe theories of gravity and, by the above discussion, they tends to be fully renormalizable. Therefore, *string theory is a nice approach to quantization of gravity.*

Now, let us analyse what happens in open/closed string theories. As in the case of open strings, the Feynman diagrams for such theories only have cobordism-type singularities. But there is a difference: an open/closed string theory admits cobordisms between open and closed strings. This means that a general propagator in such a theory is given by a cobordism between an interval and a circle. These cobordisms always have an **uncountable** amount of smooth singularities, so that the Feynman diagrams for such theories also have uncountable singularities (see the last picture). Therefore, *a general open/closed string theory is nonrenormalizable.* Notice that this not mean that **every** open/closed string theory is nonrenormalizable. Indeed, the *uncountable smooth singularities appear exactly when open strings are interacting with closed strings.* So, if we consider open/closed string theories in which we only have interactions open-open and closed-closed strings, the uncountable singularities disappear. In other words, *the well-behaved open/closed string theories are those that can be divided into two sectors: the open string sector and the closed string sector.*

Finally, similar analyses can be done for theories of p-branes. In them, the propagators are p-cobordisms with a distinguished boundary, the interactions are given by gluing p-cobordisms at the distinguished boundary, and so on. Therefore, as for strings, we only have smooth singularities of cobordism type. But here we do not have a complete classification of the possible boundaries, so that there is no satisfactory way to divide general theories into open p-brane theories, closed p-brane theories and open/closed p-brane theories. Even so, we still have a good notion of **open** p-branes: they are those diffemorphic to the cartesian product $I_1 \times ... \times I_p$ between p intervals, being a direct analogy to the string (i.e, $p = 1$) case. Therefore we can talk about **open** p-brane theories: those in which the propagators are cobordisms between products $I_1 \times ... \times I_p$ with a distinguished boundary, the interactions are given by gluing, and so on.

For $p = 1$, the open brane I is perfectly smooth. For $p = 2$, the open brane $I_1 \times I_2$ looks like a rectangle and, therefore, have finite (four, indeed) singularities lying at its vertexes. On the other hand, for $p = 3$ (and similarly for $p > 3$) the open branes are cubes and not only its vertexes, but each point in its edges is a smooth singularity. This reveals that *any theory of open p-branes, with $p \geq 3$, is nonrenormalizable.* Now recall that the propagators in a theory of branes are cobordisms. So, for a given open p-brane $I_1 \times ... \times I_p$ we have the trivial cobordism $I_1 \times ... \times I_p \times [0,1]$. But this trivial cobordism is an open $(p+1)$-brane. In particular, the trivial cobordism associated to an open 2-brane is an open 3-brane. Consequently, *any theory of open 2-branes containing trivial propagators is nonrenormalizable.* This discussion can be summarized in the following table:

singularity type	singularities behavior	nonrenormalizable?
interactions	infinite when $\lambda \to \infty$	???
closed 1-cobordisms	—	**no**
open p-cobordisms	infinite for $p \leq 1$and $\lambda \to \infty$ uncountable for $p > 1$	??? for $p{\leq}1$ **yes** for $p{>}1$
open/closed 1-cobordisms	uncountable	**yes**

Table 3.2: Nonrenormalizability of $(p > 1)$-branes

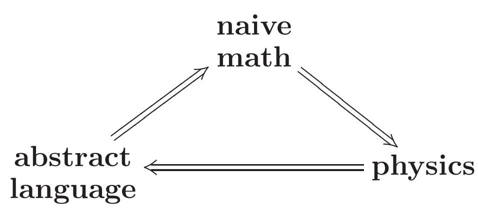
In the last section (at the discussion about everything theories) we commented that there are good reasons to believe that p-branes, with $p \geq 2$, are not the correct "building blocks" of nature. One of the reasons is the nonrenormalizability criterion discussed here...

Part II

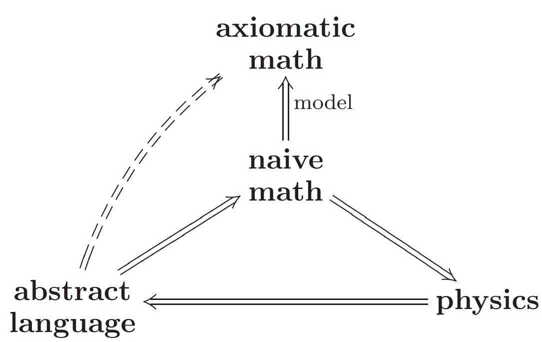
The Categorification Process

Prologue

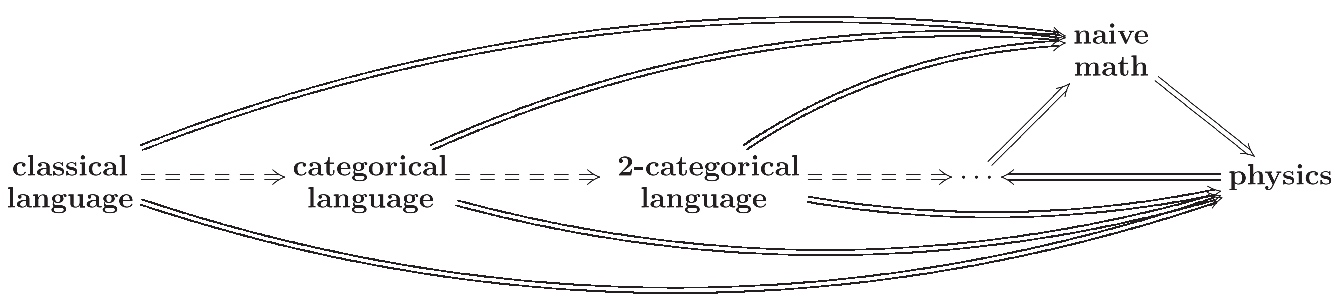
Recall that our approach to Hilbert's sixth problem is based in five steps. After selected a background language, the first of them is to study the relation between the naive mathematics produced by the language and the foundations of physics, as schematized in the following diagram.



The next step is to lift from naive mathematics to axiomatic mathematics by making use of some model, as below.



In the first part of the text we discussed that categorical language is a very interesting language, but it is **not** abstract enough in order to axiomatize **every** physical law. On the other hand, we saw that under the presence of a hypothetical "categorification process", categorical language can be replaced by another language as abstract as we like, suggesting the existence of the required unifying language.



This part of the text is devoted to formalize this “categorification process”. We start in Chapter 4 by categorifying the very simple concept of “monoid” (a set endowed with an associative operation and with a distinguished “neutral element”). In Chapter 5 we categorify a little more complex concept: “monoids fulfilling additional conditions”, as commutativity. Finally, in Chapter 6 we test the full power of the categofication process by effectively categorifying the categorical language.

One time formalized the “categorification process”, in the next part we will be able to build a lifting from “naive math” to “axiomatic math” in each step of the diagram above.

Chapter 4

Monoids

In very few words, *this chapter is about categorification of the classical concept monoid*. We start at the first section by discussing strategies to formalize what is a "categorification process". We give two approaches based in other two process: the *internalization process* and the *enrichment process*. In each of them, categorification is something as "internalization in **Cat**" or "enrichment over **Cat**".

Categorifying the concept of monoid by internalization in **Cat** we get the notion of *monoidal category* which was used at Section 3.1 in order to conjecture the existence of nontrivial isomorphism between the mathematical areas describing different models to string theory. On the other hand, categorifying monoid by enrichement we get an abstract notion of "monoid object".

The second section is purely devoted to examples. Our main objective there is to clarify that the categorification process is really very powerful, in the sense that the categorification of a very simple concept (as the monoid concept) produces a new notion that can be used to unify many different ideas/results. For instance, there we see that rings, algebras, superalgebras, Lie groups, and son on, are only different flavors of monoid objects. We also see that the Künneth theorem in homological algebra can be understood as the assertion that algebraic cohomology is an example of categorified morphism between monoids.

This chapter ends in the third section, where we outline the proof of the Adam's theorem, which says that the unique spheres admitting monoid object structure up to homotopy are $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$, realized as the space of normalized vectors of the normed algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively. A direct consequence is the classical result that these are the only parallelizable spheres. The result also imply that only $\mathbb{S}^0 \simeq \mathbb{Z}_2$, $\mathbb{S}^1 \simeq U(1)$ and $\mathbb{S}^3 \simeq SU(2)$ can be regarded as a Lie group.

4.1 Categorifying

In the last chapters we concluded that in order to attack Hilbert's sixth problem we need to get more and more abstract languages. Furthermore, we commented that in order to get very abstract languages it is enough to develop a *categorification process*, allowing us to pass from *classical concepts* to *categorical concepts*. Ideologically, this process consists in the "addition of layers of information". In the present section we will try to formalize this idea. There are at least two ways to do this, as will be briefly discussed here. They are the following:

1. categorification by *internalization*;

2. categorification by *enrichment*.

Internalization and enrichment are themselves processes that attempt to abstract a given concept. Not all concepts can be abstracted by making use of internalization or enrichment: *they apply only to concepts which can be totally characterized by a collection of commutative diagrams.* The difference between them is what they do with the commutative diagrams in order to return a new concept.

In more precise terms, the *internalization process* consists of two steps:

1. characterization of a given concept $\mathcal{P}$ in terms of purely categorical structures (objects, morphisms, limits, colimits, etc), which satisfy certain commutativity conditions;

2. definition of an analogue in any category $\mathbf{H}$ that **has such structures**. We say that the result is a version of the concept $\mathcal{P}$ *internalized in* $\mathbf{H}$ (called the *ambient* of internalization).

Enrichment is slightly different. Indeed, we can say that *enrichment is a certain kind of partial internalization* (or a certain kind of *internalization of diagrams*). More precisely, enrichment is also composed of two steps:

1. categorical characterization of the concept $\mathcal{P}$ to be enriched (exactly as in the internalization process);

2. definition of an analogue of $\mathcal{P}$ in any category $\mathbf{H}$, here called the *enrichment ambient*, in which the commutative conditions **make sense**.

Remark. Notice that we can internalize a concept *only* in a category which has the same categorical properties characterizing the given concept. On the other hand, in principle we can enrich a concept over a category which does not have these properties.

The process of internalization in **Cat** and enrichment over **Cat** corresponds to what were called above, in the beginning of the section, *categorification by internalization* and *categorification by enrichment*.

Monoids

The process of internalization and enrichment (and the difference between them) will be clear after working through an example. So, for instance, let us consider $\mathcal{P}$ as being the classical concept of monoid: a set X endowed with a binary operation $* : X \times X \to X$ and a distinguished element $e \in X$ such that

$$(x * y) * z = x * (y * z) \quad \text{and} \quad x * e = x = e * x \tag{4.1.1}$$

for any elements $x, y, z \in X$. Recall that, in order to internalize or enrich some concept, the first step is always to characterize it in terms of purely categorical information. The binary operation is a morphism defined in a binary product of **Set**, so that $*$ admits a categorical characterization. The distinguished element is the same as a function $1 \to X$, where 1 is any set with a unique element. But 1 is a terminal object in **Set**, so that the $e \in X$ also admits a categorical characterization. Finally, the conditions (4.1.1) can be translated into the commutativity of the following diagrams:

$$\begin{tikzcd} X \times (X \times X) \arrow[r, "\simeq"] \arrow[d, "id\times *"'] & (X \times X) \times X \arrow[r, "*\times id"] & X \times X \arrow[d, "*"] \\ X \times X \arrow[rr, "*"'] & & X \end{tikzcd} \qquad \begin{tikzcd} 1 \times X \arrow[r] \arrow[dr, "\simeq"'] & X \times X \arrow[d, "*"] & X \times 1 \arrow[l] \arrow[dl, "\simeq"] \\ & X & \end{tikzcd} . \tag{4.1.2}$$

Therefore, the concept of monoid can be seen as a collection of categorical information (objects, morphisms, binary product and terminal object) subjected to certain commutativity conditions, so that the internalization and the enrichment processes applies. Let us analyze each of them separately.

- Internalization. We only can internalize "monoid" in a category with binary products and terminal objects (because these are the categorical data used to define the classical concept of monoid). So, let **H** be a category with these properties. A *monoid internal to* **H** is an object $X \in \mathbf{H}$ endowed with a morphism $* : X \times X \to X$ from the *binary product* $X \times X$ to X and with a morphism $1 \to X$ defined on a *terminal object* $1 \in \mathbf{H}$, such that the commutativity conditions (4.2) are satisfied.

- Enrichment. In order to enrich a monoid in **H**, the category *does not need* to have binary products and a terminal object. Indeed, we only need that diagrams (4.2) make sense in **H**. This happens when **H** has **some** *notion of associative product* $X \otimes Y$ (not necessarily given by the binary product) and **some** *distinguished object* $\mathbb{I}$ satisfying $\mathbb{I} \otimes X \simeq \mathbb{I} \simeq X \otimes \mathbb{I}$ (not necessarily a terminal object). Let **H** be some category with these structures (say with product $\otimes$ and distinguished object $\mathbb{I}$). A *monoid enriched over* **H** (also called a *monoid object on* **H**) is an object $X \in \mathbf{H}$ endowed with morphisms $* : X \otimes X \to X$ and $\mathbb{I} \to X$ such that the following analogues of diagrams (4.2) hold:

$$\begin{tikzcd} X \otimes (X \otimes X) \arrow[r, "\simeq"] \arrow[d, "id\otimes *"'] & (X \otimes X) \otimes X \arrow[r, "*\otimes id"] & X \otimes X \arrow[d, "*"] \\ X \otimes X \arrow[rr, "*"'] & & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I} \otimes X \arrow[r] \arrow[dr, "\simeq"'] & X \otimes X \arrow[d, "*"] & X \otimes \mathbb{I} \arrow[l] \arrow[dl, "\simeq"] \\ & X & \end{tikzcd} . \tag{4.1.3}$$

For example, the category **Cat** has binary products and terminal objects. Therefore, the classical concept of monoid can be internalized in **Cat** (i.e, *the concept of monoid can be categorified by internalization*). The result is called a *monoidal category*. It is just a category **C** endowed with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and a distinguished object $\mathbb{I} \in \mathbf{C}$ such that diagrams (4.3) holds. In other words, a category is monoidal when it has a product (which is associative up to natural isomorphisms) and a distinguished object behaving as a neutral element up to isomorphisms with respect to this product (a naive version of this notion was used in Section 3.1).

Now, notice that, by the previous discussion, the concept of monoid can be enriched over any category with associative product and distinguished object. But these categories are just the monoidal categories! Therefore, the concept of monoid can be enriched over any monoidal category. This is a manifestation of another principle in category theory: the *microcosm principle*:

Microcosm Principle: *a suitable concept can be enriched over any of its categorifications by internalization.*

The interested reader can read more about this in [21, 125]). We end with two remarks:

1. when defining a monoidal category, the natural isomorphisms

$$a_{xyz} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z), \quad {}_x u : X \otimes 1 \simeq X \quad \text{and} \quad u_x : X \simeq 1 \otimes X,$$

   usually called *associators* and *unitors*, are **part of the definition**. This means that in order to know the whole monoidal structure we need to know explicitly the formula for the associators and unitors (i.e, it is not sufficient to ensure their existence);

2. generally we work with monoidal categories whose associators and unitors satisfy certain additional properties, called *coherence conditions*, which can be translated in terms of the commutativity of the diagrams below. Maybe the relevance of these conditions will not be transparent in this chapter, but certainly the reader will be convinced of their relevance at Section 6.3.

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "a_{(w\otimes x)yz}"] \arrow[d, "a_{wxy}\otimes id_z"'] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "a_{wx(y\otimes z)}"] & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{w(x\otimes y)z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "id_w\otimes a_{xyz}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes \mathbb{I}) \otimes Y \arrow[rr, "a_{xiy}"] \arrow[dr, "{}_x u\otimes id_y"'] & & X \otimes (\mathbb{I} \otimes Y) \arrow[dl, "id_x\otimes u_y"] \\
& X \otimes Y &
\end{tikzcd}$$

Morphisms

As the concept of monoid, the notion of homomorphism between monoids also admit a purely categorical characterization and, thus, can also be enriched and internalized. Indeed, a homorphism between a monoid X, say with multiplication $*$ and neutral element e, and other monoid X', say with multiplication $*'$ and neutral element e', is simply a function $f : X \to X'$ such that $f(x * y) = f(x) *' f(y)$ and $f(e) = e'$. As can be rapidly checked, these conditions can be translated in terms of the following commutative diagrams, so that *the notion of homomorphism of monoid can be internalized/enriched in/over the same ambient in which the concept of monoid can be internalized/enriched.*

$$\begin{tikzcd}
X \times X \arrow[r, "f\times f"] \arrow[d, "*"'] & X' \times X' \arrow[d, "*'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
X \arrow[rr, "f"'] & & X'
\end{tikzcd} \tag{4.1.4}$$

In the same way that in internalizing the concept of monoid in **Cat** we get the notion of monoidal category, when internalizing the notion of homomorphism between monoids in **Cat** we get the concept of *monoidal functor* between monoidal categories. These are functors $F : \mathbf{C} \to \mathbf{C}'$ between the underlying categories together with natural isomorphisms[1]

$$F_{xy} : F(X \otimes Y) \simeq F(X) \otimes' F(Y) \quad \text{and} \quad F_1 : F(1) \simeq 1' \tag{4.1.5}$$

commuting with associators and unitors. By "commuting with associators" here we mean that the following diagram must be commutative (similar diagram describes commutativity with unitors):

$$\begin{tikzcd}
F((X \otimes Y) \otimes Z)) \arrow[r, "F(a_{xyz})"] \arrow[d, "F_{(x\otimes y)z}"'] & F(X \otimes (Y \otimes Z)) \arrow[r, "F_{x(y\otimes z)}"] & F(X) \otimes' F(Y \otimes Z) \arrow[d, "id\otimes' F_{xy}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[r, "F_{xy}\otimes' id"'] & (F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "a'_{F(x)F(y)F(z)}"'] & F(X) \otimes' (F(Y) \otimes' F(Z))
\end{tikzcd}$$

On the other hand, exactly as the concept of monoid can be enriched in any monoidal category (producing monoid objects), the same can be done with the notion of homomorphism between monoids, producing *morphisms between monoid objects*, whose definition is very suggestive and characterized by the following diagrams totally analogous to (4.4).

$$\begin{tikzcd}
X \otimes X \arrow[r, "f\otimes f"] \arrow[d, "*"'] & X' \otimes X' \arrow[d, "*'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
X \arrow[rr, "f"'] & & X'
\end{tikzcd}$$

Therefore, we can build the subcategory $\mathbf{Mon} \subset \mathbf{Cat}$ of all monoidal categories and, for any monoidal category $\mathbf{C} \in \mathbf{Mon}$, a corresponding category $\mathrm{Mon}(\mathbf{C}, \otimes)$ of their monoid objects. It can be show, by a purely diagram chasing, that any monoidal functor $F : \mathbf{C} \to \mathbf{D}$ maps monoid objects into monoid objects, so that it induces a functor

$$F_M : \mathrm{Mon}(\mathbf{C}, \otimes) \to \mathrm{Mon}(\mathbf{D}, \otimes').$$

4.2 Examples

In this section we give concrete examples of monoids internalized on **Cat** (i.e monoidal categories) and monoids enriched over them (i.e monoid objects). Some useful examples of monoidal functors and morphisms between monoid objects also will be presented.

Example 4.1 (*classical monoids*)**.** The canonical example of monoidal category is **Set** endowed with the cartesian product bifunctor and whose distinguished object is some unit set. The associators a_{xyz} are given simply by the map assigning to any $((x, y), z)$ the corresponding $(x, (y, z))$. The unitors are the projections $(x, 1) \mapsto x$ and $(1, x) \mapsto x$. The monoids enriched over such monoidal structure are, of course, the classical monoids.

[1] Instead of requiring the existence of natural **isomorphisms**, we could required only the existence of natural **transformations**. In this case, we would obtain the notion of *lax monoidal functors*.

Example 4.2 (*topological and smooth monoids*)**.** Similarly, we can endow **Top** and **Diff** with the monoidal structure given by product of topological spaces and smooth manifolds. The associators and unitors are exactly those of the previous examples, except that in the present context they become continuous/smooth. A monoid object in **Top** is just a *topological monoid*: a topological space X with a structure of monoid such that the operation $X \times X \to X$ is continuous. Similarly, the monoid objects in **Diff** are the *smooth monoids*: smooth manifolds which also are monoids with a smooth operation.

The examples above can be generalized: any category $\mathbf{C}$ with binary products $X \times Y$ and a terminal object (say $1 \in \mathbf{C}$) has a canonical monoidal category structure, called *cartesian monoidal structure*, whose product is given by the binary product bifunctor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and whose distinguished object is the terminal object. The associativity of $\times$ up to natural isomorphisms follows from the universality of binary products. Similarly, the natural isomorphisms $1 \times X \simeq X$ and $X \simeq 1 \times X$ also come from universality.

Other examples of cartesian monoidal structure include:

Example 4.3 (*categorical monoids*)**.** Notice that, **Cat** being complete, it has a cartesian monoidal structure. As can be easily verified, the corresponding monoid objects are just the *strict monoidal categories*! In other words, the monoidal categories for which the associators and unitors are not isomorphisms, but indeed equalities.

Example 4.4 (*monoid of functors*)**.** Recall that if $\mathbf{D}$ is a category with some limit, then for any other category $\mathbf{C}$ the corresponding functor category $\mathrm{Func}(\mathbf{D};\mathbf{C})$ also has this limit. Specially, if $\mathbf{D}$ becomes endowed with a cartesian monoidal structure, then it induces a cartesian monoidal structure in each $\mathrm{Func}(\mathbf{D};\mathbf{C})$. The monoid objects in this case are functors F together with natural transformations $* : F \times F \Rightarrow F$ and $e : 1 \Rightarrow F$ fulfilling monoid-like diagrams.

Example 4.5 (*loop space*)**.** We introduced the loop space ΩX as the adjoint to the reduced suspension functor ΣX. As will be proved still in this section, if this adjoint exists, then for a large class of spaces it is just the space of loops $\mathbb{S}^1 \to X$ endowed with a very standard topology. Under this identification, we have canonical continuous applications

$$\# : \Omega X \times \Omega X \to \Omega X \quad \text{and} \quad \mathrm{cst} : * \to \Omega X,$$

respectively given by concatenation of loops and by the constant loop at the base point of X. More precisely, for $f, g : \mathbb{S}^1 \to X$ two loops in X, we define

$$f\#g = \begin{cases} f(2t), & 0 \leq t < 1/2 \\ g(1-2t), & 1/2 \leq t \leq 1 \end{cases}$$

where here we are taking $\mathbb{S}^1 \simeq [0,1]/\partial[0,1]$. Furthermore, if x_o is the base point of X, then $\mathrm{cst}(*) : \mathbb{S}^1 \to X$ is the constant function at x_o. These operations **does not** give a structure of topological monoid for ΩX. Indeed, $\#$ it is not associative, nor the neutral element property is satisfied by cst. On the other hand, they are fulfilled up to homotopy. This means that, after passing to the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$, the above operations make ΩX a monoid object respectively to the cartesian monoidal structure. In other words, ΩX is an example of *H-monoid*.

Note that a monoidal functor between cartesian monoidal categories is just a functor preserving products and terminal objects. Particularly, for any $Y \in \mathbf{C}$ the hom-functor h^Y is monoidal, because covariant representable functors preserve all limits. Therefore, if $X \in \mathbf{C}$ is a monoid object, then for any Y we have that $\mathrm{Mor}_{\mathbf{C}}(Y, X)$ is a monoid in $\mathbf{Set}$. We assert that the reciprocal is also valid.

Indeed, suppose that X is such that each hom-set $\mathrm{Mor}_{\mathbf{C}}(Y, X)$ has the structure of monoid, say with multiplication $*_Y$ and with neutral element e_Y. Then the rules $Y \mapsto m_Y$ and $Y \mapsto e_Y$ define natural transformations

$$m : h_X \times h_X \Rightarrow h_X \quad \text{and} \quad e : 1_X \Rightarrow h_X$$

fulfilling monoid-like diagrams, so that h_X is a monoid object in the functor category $\mathrm{Func}(\mathbf{D}; \mathbf{Set})$. It happens that, by Yoneda lemma (and by the fact that $h_X \times h_X \simeq h_{X \times X}$), these natural transformations are induced by maps

$$m : X \times X \to X \quad \text{and} \quad e : 1 \to X$$

fulfilling the same commutativity conditions. Therefore, they actually define a monoid object structure in X, as required. Summarizing we have been given a complete characterization of the monoid objects into a cartasian monoidal category: *an object $X \in \mathbf{C}$ is a monoid iff for any Y the set of morphisms $\mathrm{Mor}_{\mathbf{C}}(Y; X)$ is a monoid.*

Non-Cartesian Examples

Up to this point we worked out only with cartesian monoidal categories. The next examples, on the other hand, clarify that there many other classes of monoidal structures.

Example 4.6 (*algebraic monoids*)**.** Let R be a commutative ring and let $\mathbf{Mod}_R$ be the category of R-modules. As discussed in Chapter 2, it is a complete category, so that it has binary products and terminal objects. Therefore, it can be endowed with the corresponding cartesian monoidal struture. It happens that there is another notion of product between R-modules: the *tensor product*. It naturally extends to a bifunctor $\otimes$ (recall that we have a notion of tensor product between R-homomorphisms) which is associative up to isomorphisms. Furthermore, for any R-module X we have $R \otimes X \simeq X \simeq X \otimes R$ in a natural way, so that $\mathbf{Mod}_R$ becomes a monoidal category when endowed with $(\otimes, R)$. The corresponding monoids are R-modules X together with R-linear maps $X \otimes X \to X$ and $R \to X$ satisfying monoid-like diagrams. Now, recall that we have an isomorphism

$$\mathrm{Hom}_R(X \otimes Y, Z) \simeq \mathrm{Bil}_R(X \times Y, Z),$$

so that a R-linear map defined on a tensor product is the same as a bilinear map. Therefore, a map $X \otimes X \to X$ is precisely a bilinear operation on X. On the other hand, the map $R \to X$ is determined by its action on the unit $1 \in R$. Concluding, a *monoid object on* $(\mathbf{Mod}_R, \otimes)$ *is equivalently a unital and associative R-algebra.*

Remark. The last example reveals that the *same category may admit two non isomorphic monoidal structures*. Indeed, we introduced the cartesian monoidal structure and the tensor product structure on $\mathbf{Mod}_R$. We would like to observe that this fact is important, but it is

not a special feature of monoidal category theory. For instance, recall that analogous situations appear in other areas of math: the same set may have two non-homeomorphic topological structures, non-isomorphic group structures, etc. On the other hand, monoidal category theory has a peculiarity. In fact, recall that the associators/unitors also are part of the definition of a monoidal category. Therefore, the same category may have two different monoidal structures with the **same product**, but with **different associators/unitors**. This property (actually, a similar property) is fundamental in the study of supersymmetry, as will be discussed in the next chapter.

The next example generalizes the previous ones.

Example 4.7 (*graded monoids*)**.** A very similar monoidal structure can be introduced in the more general category $G\mathbf{Grad}_R$ of G-graded R-modules. Indeed in this case the tensor product bifunctor is defined on objects and on morphisms by

$$(X\otimes Y)_g = \bigoplus_{h+h'=g} X_h \otimes Y_{h'} \quad \text{and} \quad (f\otimes g)(x_h\otimes y_{h'}) = f(x_h)\otimes f(y_{h'}), \tag{4.2.1}$$

and the distinguished object $1 \in G\mathbf{Grad}_R$ satisfying $1\otimes X \simeq X \simeq X\otimes 1$ is just the image of the free module R under the canonical inclusion[2] $\delta : \mathbf{Mod}_R \hookrightarrow G\mathbf{Grad}_R$, which assigns to any module X the trivial G-graded module $\overline{X}$ condensated at X, i.e such that $\overline{X}_0 = X$ and $\overline{X}_g = 0$ for $g \neq 0$. If on one hand the monoids on $\mathbf{Mod}_R$ are just associative R-algebras, on the other *the monoids on $G\mathbf{Grad}_R$ correspond to the well known notion of G-graded R-algebras.* There is special interest in the $\mathbb{Z}_2$-graded algebras, which are called *superalgebras.* For instance, any $\mathbb{Z}$-graded algebra[3] induces a corresponding $\mathbb{Z}_2$-graded algebra by rewriting its direct sum decomposition into only two pieces: the odd and the even pieces. For our purposes, a fundamental example is the following:

- *cohomology of ring spectrum*. Recall that a generalized cohomology theory is a sequence of hom-functors in $\mathrm{Ho}(\mathbf{Top}_*)$ defined by a Ω-spectrum $\mathbb{E} = (E_n)$. In the next subsection we will prove that the cohomology **groups** $H^k(X;\mathbb{E})$ are indeed **groups**, while in Section 5.1 it will be seen that these are not only groups, but indeed abelian groups, so that the sum $H(X;\mathbb{E}) = \oplus_k H^k(X;\mathbb{E})$ is a graded abelian group. On the other hand, in Sections 1.2 and 1.3 we used the fact that for some cohomology theories (as ordinary cohomology, K-theory and cobordism) we can multiply cohomology classes of different degrees, meaning that $H(X;\mathbb{E})$ is a ring. Exactly as in $\mathbf{Mod}_R$, where we have a canonical non-cartesian monoidal structure given by the tensor product $\otimes$, in $\mathrm{Ho}(\mathbf{Top}_*)$ we can also introduce a non-cartesian structure given by the *smash product* $\wedge$, as will be discussed still in this section. As discussed in Section 2.3, spectra are the topological version of graded modules, so that it is expected that $\wedge$ induces a monoidal structure into $\mathrm{Ho}(\mathbf{Spec})$ analogously as $\otimes$ induces a monoidal structure into $G\mathbf{Grad}_R$. This is really the case, as we will show in Section 5.3. A monoid into $(\mathrm{Ho}(\mathbf{Spec}),\otimes)$ is called a *ring spectrum.* The fundamental fact

[2] We have the following analogy: thinking of a R-module as being a real number and a G-graded module as being the limit of a sequence of real functions, then the functor δ acts like the rule assigning to any number $x\in\mathbb{R}$ the Dirac delta δ_x centered at x.

[3] Identical argument holds for $\mathbb{N}$-graded algebras.

is that for any space X, the functor

$$[X,-] : (\mathrm{Ho}(\mathbf{Spec}), \wedge) \to (\mathbb{Z}\mathbf{Grad}_{\mathbb{Z}}, \otimes)$$

which assign to any spectrum $\mathbb{E}$ the corresponding graded abelian cohomology group $\mathrm{H}(X;\mathbb{E})$ of X maps ring spectrum into $\mathbb{Z}$-graded $\mathbb{Z}$-algebras. In other words, *the cohomology of a ring spectrum is always a $\mathbb{Z}$-graded algebra and, therefore, a superalgebra*. This justify why we know how to multiply classes in ordinary cohomology, K-theory and cobordism: the underlying spectra are ring spectra.

Recall that we can consider other graded versions of $\mathbf{Mod}_R$ rather than $G\mathbf{Grad}_R$; we can consider the full subcategory $G\mathbf{Mod}_R \subset \mathbf{Mod}_R$ (in which the morphisms does not need to preserve the grading, i.e does not need to have degree equal to zero) and the category $G\mathbf{DGrad}_R^{\alpha}$ of differential graded modules with a given degree α. In both cases, the previous construction can be effectively done, as will be explained below. More precisely, in each case we can also define a tensor product bifunctor having a "neutral element".

Example 4.8 (*full case*)**.** Let us analyze $G\mathbf{Mod}_R$ first. We start by observing that this category has the same objects as $G\mathbf{Grad}_R$, while its morphisms differ by the fact that here they are arbitrary homomorphisms. Therefore, we can define $\otimes$ identically on objects. In order to define it on morphisms, recall that arbitrary morphisms between graded modules can be written as a linear combination of morphisms having degree, so that we can look only to this class of generating morphism. Therefore, what we need to do is to enlarge the definition of $f \otimes g$ given above in order to incorporate morphisms with arbitrary degree. This can be done picking some symmetric morphism $\langle \cdot, \cdot \rangle : G \otimes G \to \mathbb{Z}_2$, called a *fundamental pairing*, putting

$$(f \otimes g)(x_h \otimes y_{h'}) = (-1)^{\langle \deg(f), \deg(g) \rangle} f(x_h) \otimes g(y_{h'}) \tag{4.2.2}$$

and extending it by linearity. For instance, in the case $G = \mathbb{Z}$ we can consider the pairing as being simply the rule taking two integers z, z' and forming the mod 2 class of $z \cdot z'$. The monoid objects of this new monoidal structure are very similar to the G-graded R-algebras, being usually called by the same name.

Example 4.9 (*differential case*)**.** Now, let us consider the differential graded situation. We will work only with chain/cochain complexes, which is the most interesting case for our objectives. We leave the general case to the reader. So, let (X_*, d) and (X'_*, d') be two chain complexes of R-modules. These differ from the $\mathbb{Z}$-graded R-modules by the existence of a map $d : X_* \to X_*$ of degree -1 such that $d^2 = 0$. So, the main idea is to define the tensor product $(X_*, d) \otimes (X'_*, d')$ is as some mixing of the last two examples: as the usual tensor product of modules (4.2.1) endowed with the tensor product (4.2.2) of maps $d \otimes d'$. However, this does **not** produce a chain complex, because

$$\deg(d \otimes d') = \deg(d) + \deg(d') = (-1) + (-1) = -2.$$

Notice that the most simple way to get a morphism of degree -1 by making use of d, d' and $\otimes$ is by considering the combination $d \otimes id + id \otimes d'$, because the identity map has degree zero. Its square is zero:

$$\begin{aligned}(d\otimes id+id\otimes d')^2 &= (d\otimes id)^2+(d\otimes id)\circ(id\otimes d')+(id\otimes d')\circ(d\otimes id)+(id\otimes d')^2\\ &= (d^2\otimes id)+(d\otimes d')+(d'\otimes d)+(id\otimes d'^2)\\ &= (d\otimes d')+(d'\otimes d)\\ &= (d\otimes d')-(d\otimes d')=0,\end{aligned}$$

so that this actually defines a chain complex. The morphisms $f:(X_*,d)\to(X'_*,d')$ between chain complexes are just morphisms between the underlying $\mathbb{Z}$-graded modules which commute with the differentials, i.e, such that $f\circ d=d'\circ f$. It can be seen that the tensor product of two morphisms of graded modules commuting with differentials d and d' commutes $d\otimes 1+1\otimes d'$. Therefore, the tensor product is well defined for chain maps, giving a monoidal structure to $\mathbf{Ch}_R$. Analogous construction produces a monoidal structure on the category $\mathbf{CCh}_R$ of cochain complexes (the unique difference is that now the maps have degree $+1$ instead of -1). In both cases the monoid objects are called *differential graded algebras*. Explicitly, this is a $\mathbb{Z}$-graded algebra A, say with product $*:A\otimes A\to A$, endowed with a map $d:A\to A$ of degree -1 (or $+1$, depending if we are working with chain or cochain complexes) such that $d^2=0$ and satisfying the graded Leibniz rule:

$$d(x*y)=dx*y+(-1)^k x*dy,$$

where k is degree of y. Examples to keep in mind are the following:

1. *de Rham cohomology*. The exterior algebra $\oplus_i\Lambda^i(M)$ of any manifold M has a canonical differential structure given the exterior derivative of differential forms. Consequently, de Rham cohomology is not only a graded algebra (as the cohomology defined by a ring spectrum), but indeed a differential graded algebra.

2. *Chevalley-Eilenberg algebra*. Recall that a Lie algebra $\mathfrak{g}$ is a vector space endowed with a bilinear map $[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (the *Lie bracket* of the algebra) which is anticommutative and satisfy the Jacobi identity

$$[x,[y,z]]+[z,[x,y]]+[y,[x,z]]=0, \tag{4.2.3}$$

measuring the non-associativity of the bracket. We have a contravariant functor $(-)^*:\mathbf{Vec}_{\mathbb{K}}\to\mathbf{Vec}_{\mathbb{K}}$ which assigns to any vector space V its dual V^* and to any k-linear map a corresponding dual. So, the bracket induces a dual bilinear operation

$$[-,-]^*:\mathfrak{g}^*\to\mathfrak{g}^*\times\mathfrak{g}^*,$$

here denoted d_1, which can be understood as a linear map from[4] $\Lambda^1(\mathfrak{g})$ to $\Lambda^2(\mathfrak{g})$ fulfilling an additional condition: the Jacobi identity. It happens that the operation

$$[-,[-,-]]+[-,[-,-]]+[-,[-,-]]:\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g} \tag{4.2.4}$$

is itself trilinear and anticommutative, so that its dual defines a map from $\Lambda^1(\mathfrak{g})$ to $\Lambda^3(\mathfrak{g})$. In this perspective, the Jacobi identity only says that this map is the null map. The operation

[4] If the reader is do not know the Grasmann algebra (also called the exterior algebra) of a vector space, see Example 5.5 for a review.

above involves a combination of "compositions of the bracket $[-,-]$", suggesting that the dual of (4.2.4) is also a combination of "compositions of d_1". So, there are good reasons to conjecture the existence of a differential graded algebra structure into the exterior algebra $\Lambda(\mathfrak{g})$ of $\mathfrak{g}$ whose differential d is obtained as an extension of d_1 and whose condition $d_2 \circ d_1 = 0$ is just the Jacobi identity. Such an algebra really exists: it is the *Chevalley-Eilenberg algebra* of $\mathfrak{g}$, usually denoted by $\mathrm{CE}(\mathfrak{g})$.

Example 4.10 (*Künneth theorem*)**.** Recall that for any ring R we can build the category $\mathbf{CCh}_R$ of cochain complexes over R. We have a sequence of functors $H^k : \mathbf{CCh}_R \to \mathbf{Mod}_R$, the *algebraic cohomology functors*, which can be simultaneously described by a unique functor $H : \mathbf{CCh}_R \to \mathbb{Z}\mathbf{Mod}_R$. By the last examples, this is a functor between monoidal categories whose products are both given by flavors of the tensor product. So, we can search for conditions under which H becomes a monoidal functor. This means that if X_* and Y_* are cochain complexes, then

$$H(X_* \otimes Y_*) \simeq H(X_*) \otimes H(Y_*)$$

and, therefore, the cohomology of the tensor product of the complexes can be known from the cohomology of the underlying complexes. A fundamental result on homological algebra, usually called the *Künneth theorem*, states that this is the case when each $H^k(Y_*)$ is a free R-module, which happens, for instance, when R is a field. So, *the algebraic cohomology functors with coefficients on a field are monoidal functors.*

We end this subsection with a more abstract example.

Example 4.11 (*span structures*)**.** From a category $\mathbf{C}$ with pullbacks we can build another interesting category $\mathrm{Span}(\mathbf{C})$. Its objects are just the objects of $\mathbf{C}$, while a morphism between X and Y is another object Z, called a *span*, endowed with morphisms $X \leftarrow Z \to Y$ in $\mathbf{C}$. The composition between two spans is obtained by the composition of the segmented arrows in the first diagram below, where the upper segmented arrows comes from the pushout of the continuous lines. We notice that, when defined in this way, $\mathrm{Span}(\mathbf{C})$ is **not** a category, because the composition is not associative. Indeed, the composition is obtained from pullbacks, but pullbacks are defined only up to isomorphisms. The problem can be remedied if we consider the morphisms as equivalence classes of spans, where two spans are considered equivalent when there is the segmented arrow turning commutative the second diagram below.

$$\begin{tikzcd}
 & & \mathrm{Pb} \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & Z \arrow[dl, dashed] \arrow[dr] & & Z' \arrow[dl] \arrow[dr, dashed] & \\
X & & Y & & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr] \arrow[dd, dashed] & \\
X & & Y \\
 & Z' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Now, recall that, as discussed in Section 2.1, a category have pullbacks when it has binary products and equalizers. In this case, we can simultaneously build $\mathrm{Span}(\mathbf{C})$ and endow $\mathbf{C}$ with the cartesian monoidal structure. We observe that this cartesian structure induces a (non-cartesian) monoidal structure on $\mathrm{Span}(\mathbf{C})$. Indeed, the product bifunctor $\otimes$ acts on objects exactly as the binary product. On the other hand, on morphisms (i.e on spans) $X \leftarrow Z \rightarrow Y$ and $X' \leftarrow Z' \rightarrow Y'$ it acts as shown in the first diagram below, where the arrows defining the product span were obtained from the universality of products, as in the second diagram (the distinguished arrows on the first diagram come from universality applied to the corresponding distinguished arrows on the second diagram).

$$
\begin{tikzcd}
 & Z \times Z' \arrow[dl, dashed] \arrow[dr, dotted] & \\
X \times X' & & Y \times Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & Z \times Z' \arrow[dl, dashed, shift right] \arrow[dl, dotted, shift left] \arrow[dr, dashed, shift left] \arrow[dr, dotted, shift right] & & \\
 & Z \arrow[dl, dashed] \arrow[dr, dotted] & & Z' \arrow[dl, dashed] \arrow[dr, dotted] & \\
X & & Y \quad X' & & Y'
\end{tikzcd}
$$

Remark. The last example may seems only an abstract construction without any physical application, but it is very important in the description of the pull-push approach to the quantization process [183, 159].

Comonoids

We introduced previously two different monoidal structures on the category of R-modules: the cartesian monoidal structure and the tensor monoidal structure. The next example clarifies that there is also a third structure that can be considered there.

Example 4.12 (*cocartesian algebraic monoids*)**.** The category $\mathbf{Mod}_R$ has another monoidal structure whose product is given by the direct sum bifunctor $\oplus$ and whose distinguished object is the trivial module. Recall that for any functor $F : \mathbf{C} \to \mathbf{D}$ having limit and colimit there is a canonical map $\operatorname{limit} F \to \operatorname{colimit} F$ obtained by gluing the diagrams (2.1). In $\mathbf{Mod}_R$ (and more generally in any abelian category) the morphism connecting finite products to finite coproducts is indeed an isomorphism. In particular, initial and terminal objects are isomorphic and $X \oplus Y \simeq X \times Y$ for any X, Y. Consequently, *this new monoidal structure on* $\mathbf{Mod}_R$ *is equivalent to the cartesian monoidal structure*. For instance, this means that both monoidal structure induces the same notion of monoid objects.

The last example can be generalized: any category with binary coproducts and an initial object has a canonical monoidal structure, called *cocartesian structure*, whose product is the binary coproduct bifunctor and whose distinguished object is precisely the initial object. So, we have cocartesian structures in **Set**, **Top**, **Diff** etc.

It happens that in **any** case, *the monoid objects in a cocartesian structure are trivial*. Indeed, if $\varnothing$ is a terminal object, then (from the definition of terminal object) for any object X we have a unique map $\varnothing \to X$, meaning that there is at most one monoid structure on each X. Such a structure actually exists and it is given by the *codiagonal map*[5] $X \oplus X \to X$. Furthermore, this construction extends to a functor $F : \mathbf{C} \to \mathrm{Mon}(\mathbf{C}; \oplus)$ which is an equivalence between $\mathbf{C}$ and

[5] We recall that the codiagonal map is the dual version of the diagonal map $X \to X \times X$. Indeed, recall that, by the universality of coproducts, if Z is an object for which we have morphisms $X \to Z$ and $Y \to Z$, then there exists a unique $X \oplus Y \to Z$ factoring the given morphisms. The codiagonal map is that universal map obtained from the canonical inclusions $\imath_0 : X \to X \oplus X$ and $\imath_1 : X \to X \oplus X$.

the category of monoid objects into the cocartesian structure. This means that *the **cocartesian** structure does not admit any interesting monoid*!

Remark. The last paragraph, together with Example 4.12 allows us to conclude that *there is no interesting monoid objects even in the **cartesian** structure of* $\mathbf{Mod}_R$. This fact can be used to give another proof that the cartesian structure and the structure given by the tensor product are not equivalent: if they were, then they would induce the same notion of monoid object. But this is not the case: the cartesian structure has only trivial monoids, while the monoid objects of the tensor product structure are the associative algebras.

The problem with searching monoid objects into cocartesian monoidal structures is clear: monoids are **covariant** objects, while the cocartesian structure is defined by coproducts, which are **contravariant**. This suggests that in order to get nontrivial entities into cocartesian structures we need to consider some dual version of the notion of monoid objects. Notice that the concept of "monoid object" into an arbitrary monoidal category $\mathbf{C}$ is given by an object X together with morphisms $X \otimes X \to X$ and $1 \to X$, called *multiplication* and *unity*, which are required to satisfy the commutativity conditions (4.3). Thanks to the duality principle, this data can be dualized producing the notion of *comonoid*. This is an object X together with morphisms $X \to X \otimes X$ and $X \to 1$, called *comultiplication* and *counit*, satisfying commutativy conditions obtained from (4.3) by reverting the arrows. We can define a complete analogous notion of *morphism between comonoids*, which fits into a category $\mathrm{CoMon}(\mathbf{C}, \otimes)$.

As can be verified in concrete examples, *a cocartesian structure really contains nontrivial comonoid objects.* On the other hand, following the above philosophy, it is expected that cartesian structures admit only trivial comonoids. Indeed, any X has a unique comonoid object structure, whose comultiplication is the diagonal map $X \to X \times X$ and whose counity is the unique map $X \to 1$ (recall that in the cartesian structure 1 is a terminal object). This fact can be translated in terms of an equivalence $\mathrm{CoMon}(\mathbf{C}, \times) \simeq \mathbf{C}$.

Summarizing, *a **cartesian** monoidal structures have only trivial **comonoids** and, dually, that **cocartesian** structures have only trivial **monoids**.*

Recall that the monoids in a cartesian structure admits a complete characterization: X is a monoid objects iff for any Y the hom-set $\mathrm{Mor}_{\mathbf{C}}(Y; X)$ is a monoid. This was proven by making use of the covariant version of the Yoneda lemma and the preservation of limits from representable functors. We have a totally dual version of these results, allowing us to get a complete characterization of comonoid objects into cocartesian structure: *X is comonoid iff for any Y the hom-set $\mathrm{Mor}_{\mathbf{C}}(X; Y)$ is a monoid.* A consequence of this characterization is the following:

Example 4.13 (*reduced suspension*)**.** Let $F \rightleftharpoons G$ be a pair of adjoint functors from $\mathbf{C}$ to itself, so that for any two objects $X, Y \in \mathbf{C}$ we have a natural bijection

$$\mathrm{Mor}_{\mathbf{C}}(F(X); Y) \simeq \mathrm{Mor}_{\mathbf{C}}(X; G(Y)).$$

Suppose that $\mathbf{C}$ has both cartesian and cocartesian monoidal structures (which happens, for instance, if $\mathbf{C}$ is finitely complete and cocomplete). In this case, as a direct consequence of the characterization of monoids/comonoids into cartesian/cocartesian structures we see that the following assertion are equivalent:

1. for any Y the object $G(Y)$ is a monoid in the cartesian structure;

2. for any X the object $F(X)$ is a comonoid in the cocartesian structure.

An example of this data is for $\mathbf{C}$ given by the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$ and $F \rightleftharpoons G$ given by the adjunction $\Sigma \rightleftharpoons \Omega$ between the reduced suspension and the loop space functors[6]. From Example 4.5 we known that ΩX is always a monoid object into the cartesian structure. Consequently, we conclude that for any X its reduced suspension ΣX is a *H-comonoid*, i.e, a comonoid object into the cocartesian structure. In particular, because $\mathbb{S}^n \simeq \Sigma \mathbb{S}^{n-1}$ we conclude that *each sphere $\mathbb{S}^n$, with $n > 0$, is a H-comonoid.*

Remark. If (E_n) is a sequence of CW-complexes defining a **suspension spectrum**, then we have $E_{n+1} \simeq \Sigma E_n$ and the same argument used in the last example reveals that each E_n, **with** $n > 0$, is a H-comonoid. As a consequence, all the homotopy groups $\pi_n(X)$, **with** $n > 0$, have the structure of monoid. On the other hand, if (E_n) defines a **Ω-spectrum**, then $E_n \simeq \Omega E_{n+1}$ and therefore each E_n (even E_0) is a H-monoid. Consequently, every generalized cohomology group is, indeed, a monoid. The existence of these algebraic structures was firstly commented in Section 2.1. There were also commented that, for $n \geq 2$, both homotopy groups and cohomology groups are, indeed, **abelian** monoids. This more stronger fact will be proved in the next chapter, where we will discuss the notion of "abelian monoids" and "abelian comonoids" into certain "symmetric monoidal categories".

He have been show that comonoid objects in cartesian monoidal structures are trivial, while in cocartesian structures they can be totally classified. Now, let us explore some examples of comonoids into monoidal structures which are not cartesian/cocartesian.

Example 4.14 (*coalgebras*)**.** While monoids into $(\mathbf{Mod}_R, \otimes)$ are R**-algebras**, the comonoids correspond to the well known concept of R**-coalgebras**. These are R-modules X endowed with a comultiplication map $X \to X \otimes X$ and a counit map $R \to X$ fulfilling comonoid-like diagrams. For instance, in the same way that any set is a trivial comonoid with the diagonal map and with the counit given by the constant map $X \to *$, we can always give a trivial coalgebra structure to any vector space. Indeed, exactly as in the context of cartesian structures, its comultiplication is also a diagonal map $X \to X \otimes X$, given by $x \mapsto x \otimes x$, and its counit is also a constant map $X \to R$, now such that $x \mapsto 1$, where $1 \in R$ is the multiplicative unity of R. Difference: in **Set** (or, more generally, in any cartesian monoidal category) the set $*$ is a terminal object, so that this trivial comonoid structure is the **only one** that can be given. On the other hand, R is **not** a terminal object for $\mathbf{Mod}_R$, so that we may have nontrivial comonoids (i.e, there may exist nontrivial coalgebras). Similarly, comonoids into the category of G-graded modules $(G\mathbf{Grad}_R, \otimes)$ are usually known as G**-graded coalgebras**, and so on.

Example 4.15 (*cospan structures*)**.** If $\mathbf{C}$ is a category with pushouts, then we can dualize the construction in Example 4.11 in order to produce the category $\mathrm{CoSpan}(\mathbf{C})$ of *cospans.* Particularly, if $\mathbf{C}$ has binary coproducts and an initial object (instead of products and terminal object as assumed in Example 4.11), then its cocartesian structure induces a corresponding monoidal structure on $\mathrm{CoSpan}(\mathbf{C})$ in a totally analogous way. For example, **Diff** has binary coproducts

[6] Recall that the homotopy category is very poor of limits/colimits, but arbitrary products and coproducts exist there, being given by the homotopy class of the underlying products and coproducts in $\mathbf{Top}_*$.

(given by the disjoint union) and an initial object (given by the empty manifold), but it does not have pushouts, as discussed in Example 2.4. Therefore, a priori we cannot build the category of "cospans manifolds". On the other hand, recall that the gluing along boundaries is a well defined smooth process. Therefore, if $\mathbf{C}_n \subset \partial\mathbf{Diff}_n$ is the subcategory whose objects are smooth n-manifolds with boundary and whose morphisms are just inclusions of the boundaries, then pushouts exist on $\mathbf{C}_n$, allowing us to build the category $\mathrm{CoSpan}(\mathbf{C}_n)$, which is (up to some details) just $\mathbf{Cob}_n$. This is a way to understand that *the category of cobordism has a canonical monoidal structure whose product bifunctor is the disjoint union.*

Remark. In many moments we saw that quantum theories of p-branes are **certain types** of functors $\mathbf{Cob}_{p+1} \to \mathbf{Vec}_{\mathbb{C}}$. Thanks to the last example we have a canonical monoidal structure on any category of cobordisms. Now we can say that a topological quantum theory of p-branes is **precisely** a monoidal functor $(\mathbf{Cob}_{p+1}, \sqcup) \to (\mathbf{Vec}_{\mathbb{C}}, \otimes)$. A motivation to this definition was given in the Introduction and will be more detailed explored in Section 14.1. There we will see that the functorial and the monoidal properties are very natural generalizations of the standard axioms for quantum mechanics.

We end this subsection with the following remark.

Remark. Recall that the duality principle relies on the existence of a functor $(-)^{op} : \mathbf{Cat} \to \mathbf{Cat}$ relating any categorical construction on a category $\mathbf{C}$ with a corresponding construction on the opposite category $\mathbf{C}^{op}$. We have seen that comonoids are the dual notion of monoids. So, it is natural to expect that this dual relation are obtained from the duality principle. Indeed, given a monoidal category $(\mathbf{C}, \otimes)$, the duality principle immediately says that the dual category $\mathbf{C}^{op}$ has an induced monoidal structure whose product is just $\otimes^{op}$. We can then easily verify that *the monoids on $\mathbf{C}^{op}$ are exactly the comonoids on $\mathbf{C}$*. More precisely, there is a canonical isomorphism

$$\mathrm{Mon}(\mathbf{C}^{op}, \otimes^{op})^{op} \simeq \mathrm{CoMon}(\mathbf{C}, \otimes). \tag{4.2.5}$$

Smash

Now, after the previous digression, let us return to discuss examples of noncartesian monoidal structures. We start by presenting an "almost monoidal structure".

Example 4.16 (*smash structure*)**.** The category $\mathbf{Top}_*$ has both cartesian and cocartesian structures, but there is also another monoidal structure that can be introduced. The idea is to consider the new product as being something between the product $X \times Y$ and the coproduct $X \vee Y$. More precisely, given based spaces (X, x_o) and (Y, y_o) we define their *smash product* $X \wedge Y$ as the quotient space $X \times Y / X \vee Y$ endowed with the canonical base point: the equivalence class $x_o \wedge y_o$ of the pair (x_o, y_o). Because the morphisms of $\mathbf{Top}_*$ are required to preserve the base points, the smash product acts naturally on any pair $f : X \to Y$ and $f' : X' \to Y'$ producing a new morphism

$$f' \wedge f : X' \wedge X \to Y' \wedge Y, \quad \text{given by} \quad (f' \wedge f)(x' \wedge x) = f'(x') \wedge f(x).$$

Therefore, $\wedge$ extends to a bifunctor. As can be easily verified, it has a canonical neutral element object: the sphere $\mathbb{S}^0 \simeq * \sqcup *$ (i.e, the trivial space $*$ viewed as a based space). Consequently, in order to conclude that $\wedge$ induces a monoidal structure on $\mathbf{Top}_*$ it is enough to verify that it is associative up to homeomorphism. It happens that this is **not** possible. Indeed, there exist spaces X, Y, Z for which there is no homeomorphism $(X \wedge Y) \wedge Z \simeq X \wedge (Y \wedge Z)$! Counterexamples are given in the Introduction of [145]. For instance, there it is proved that for $X = \mathbb{N} = Y$ with the discrete topology and $Z = \mathbb{R}$ with the usual topology the required homeomorphism really cannot exist.

We notice that the smash product construction generalizes. Indeed, recall that $X \wedge Y$ was obtained from the product $X \times Y$ by collapsing $X \vee Y$ into a point. On the other hand, by construction, the coproduct $X \vee Y$ of $\mathbf{Top}_*$ is obtained from the coproduct $X \sqcup Y$ in $\mathbf{Top}$ by gluing x_o into y_o (i.e, by gluing the based points). Therefore we can see $X \wedge Y$ equivalently as obtained from $X \times Y$ by collapsing $X \times x_o \sqcup Y \times y_o$. But collapsing is just a pushout, so that we can do a similar construction in any monoidal category $\mathbf{C}$ with pushouts, coproducts and terminal objects, getting a bifunctor $\wedge : \mathbf{C}_* \times \mathbf{C}_* \to \mathbf{C}_*$.

More precisely, for any $X, Y \in \mathbf{C}_*$ (i.e for any pair of maps $* \to X, Y$ where $* \in \mathbf{C}$ is a terminal object) we define $X \wedge Y$ as the pushout presented in the first diagram below. If $f, f' \in \mathbf{C}_*$ are two morphisms we define $f' \wedge f$ as the morphism obtained from universality of pushouts, as presented in the second diagram below.

$$
\begin{tikzcd}
X \wedge Y & X \otimes Y \arrow[l] \\
* \arrow[u] & (X \otimes *) \oplus (Y \otimes *) \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \wedge Y & Y' \otimes Y \arrow[l] & & \\
* \arrow[u] & X' \wedge X \arrow[ul, dashed, "f' \wedge f"'] & X' \otimes X \arrow[l] \arrow[ul, bend right, "f' \otimes f"'] \\
& * \arrow[ul, bend left] \arrow[u] & (X' \otimes *) \oplus (X \otimes *) \arrow[l] \arrow[u]
\end{tikzcd}
$$

In the topological context the smash product has $\mathbb{S}^0 \simeq * \sqcup *$ as neutral object. In the general context the smash product also has a neutral object, given analogously by $S^0 := * \oplus *$. On the other hand, as the last example shows, associativity generally fails and, consequently, *in the general case the smash product do not defines a monoidal structure.* Even so, it is a very natural candidate for a monoidal structure and, therefore, it is interesting to search for conditions under which the smash product becomes associative and consequently defines a genuine monoidal structure.

The idea is the following: observe that the definition of $X \wedge Y$ is a pushout involving the product $X \otimes Y$, the terminal object $*$ and coproducts in $\mathbf{C}$. We know that colimit functors (being representable) preserve other colimits. Particularly, coproducts preserve pushouts and, therefore, preserve the smash product. So, if for any Y the functor $- \otimes Y : \mathbf{C} \to \mathbf{C}$ also preserve colimits (and, therefore the smash product construction), then the following diagram, obtained from the definition of $X \wedge Y$ in the case $X = X_1 \wedge X_2$, will be commutative[7].

$$
\begin{tikzcd}
(X_1 \wedge X_2) \wedge Y & (X_1 \wedge X_2) \otimes Y \arrow[l] & (X_1 \otimes X_2) \otimes Y \arrow[l] \\
* \arrow[u] & (X_1 \wedge X_2) \oplus Y \arrow[l] \arrow[u] & (X_1 \oplus X_2) \oplus Y \arrow[l] \arrow[u]
\end{tikzcd}
$$

[7]Here we used an abuse of notation writing $X \oplus Y$ instead of $(X \otimes *) \oplus (Y \otimes *)$.

But now, recall that both $\otimes$ and $\oplus$ are associative up to isomorphism, so that the above diagram induces the diagram below. But this diagram is exactly that defining $X_1 \wedge (X_2 \wedge Y)$! So, by uniqueness of pushouts we have $(X_1 \wedge X_2) \wedge Y \simeq X_1 \wedge (X_2 \wedge Y)$. In conclusion, *when* $-\otimes Y$ *preserves (finite) colimits for any* Y, *then the smash product is associative up to isomorphism and, therefore, defines a genuine monoidal structure on* $\mathbf{C}_*$.

$$\begin{tikzcd} (X_1 \wedge X_2) \wedge Y & X_1 \otimes (X_2 \otimes Y) \arrow[l] \\ * \arrow[u] & X_1 \oplus (X_2 \oplus Y) \arrow[l] \arrow[u] \end{tikzcd}$$

A way to ensure that a functor preserves colimits is requiring that it has an adjoint. Therefore, by requiring the existence of an adjoint $[Y,-]$ for each $-\otimes Y$ we are ensuring that the smash product $\wedge$ really defines a monoidal structure. A monoidal category for which these adjoints exist is called *closed* (the object $[Y,Z] \in \mathbf{C}$ is called the *internal hom-object* between Y and Z). So, in other words, $(\mathbf{C}, \otimes)$ is *closed* precisely when we have natural bijections

$$\mathrm{Mor}_{\mathbf{C}}(X \otimes Y, Z) \simeq \mathrm{Mor}_{\mathbf{C}}(X, [Y;Z]).$$

We would like to observe that, when a monoidal category is closed, not only the smash product defines a monoidal structure on $\mathbf{C}_*$, but this new structure is itself closed. This means that for any pointed object Y the functor $-\wedge Y$ also has a left adjoint $[Y,-]_*$, which is given by the following pullback (see construction 4.19 of [63] and Construction 3.3.14 of [172]):

$$\begin{tikzcd} {[Y,Z]_*} \arrow[r] \arrow[d] & {[*,Z]} \arrow[d] \\ * \arrow[r] & {[Y,Z]} \end{tikzcd}$$

Now, after the above discussion we can return to the last example and ask: *what is the problem with the smash product on* $\mathbf{Top}_*$? The following example clarifies that in this case the non-associativity of $\wedge$ has a purely topological nature.

Example 4.17 (**Set** *is cartesian closed*)**.** For any three given sets we have natural bijections

$$\alpha : \mathrm{Mor}_{\mathbf{Set}}(X, \mathrm{Mor}_{\mathbf{Set}}(Y,Z)) \simeq \mathrm{Mor}_{\mathbf{Set}}(X \times Y, Z) \tag{4.2.6}$$

defined by $\alpha(f)(x,y) = f(x)(y)$. Therefore, when endowed with the cartesian monoidal structure, the category of sets is closed. Particularly, the internal hom-object $[X,Y]$ is just the usual hom-set $\mathrm{Mor}_{\mathbf{Set}}(X,Y)$. Consequently, the corresponding smash product makes $\mathbf{Set}_*$ a genuine monoidal category. So, forgetting the topology, the smash product on $\mathbf{Top}_*$ becomes associative.

On the other hand, we observe that for any two given topological spaces X and Y there is a canonical topology that can be put in the set $\mathrm{Mor}_{\mathbf{Top}}(X,Y)$ of continuous maps between them: the *compact-open topology*. This is the topology whose fundamental neighborhoods are the sets $V(K,U)$, where $K \subset X$ is compact and $U \subset X$ is open, of all continuous functions $f : X \to Y$ that maps K into U, i.e, such that $f(K) \subset U$. Let $\mathrm{Map}(X,Y)$ be the corresponding topological space. Example 4.16 shows that **Top** is not cartesian closed, so that the space $\mathrm{Map}(X, Y)$ **cannot be** the internal hom-object in **Top**, but the last example reveals that **it should be**.

So, we can search for subcategories $\mathscr{C} \subset$ **Top** such that $\mathrm{Map}(Y, Z) \in \mathscr{C}$ and such that restricted to it the functor $\mathrm{Map}(Y, -)$ becomes the left adjoint to $- \times Y$ and, therefore, for which the smash product defines a monoidal structure on $\mathscr{C}_*$. As discussed in Section 2.1, **Top** is complete and cocomplete, so that we also would like to search for subcategories $\mathscr{C}$ preserving these good properties. On the other hand, from the homotopy theory viewpoint, the most important class of topological spaces are the CW-complexes, so that it is naturalto restrict our search for subcategories $\mathscr{C}$ containing CW-complexes. A subcategory satisfying all these requirements is called *a convenient category of topological spaces*. Surprisingly, these very nice subcategories really exist! Indeed, as proven by Ronnie Brown and Steenrod in [33, 34, 196], the category of Hausdorff and compactly generated topological spaces is convenient. See Chapter 8 of [88] and Chapter 5 of [141].

Warning. From this point on, we will work only with convenient spaces. Therefore, "topological space" will be synonymous of "compactly generated Hausdorff space".

In order to motivate the mistrusting reader, let us give a more concrete consequence of working with a convenient category of topological spaces. Indeed, up to this point we have used the fact that the reduced suspension functor Σ **has** an adjoint Ω. But we have not given a **explicit form** for ΩX. When working with convenient spaces this can be done. More precisely, in some moments we assumed that "the loop space is just the space of loops". Now we are read to prove this assertion. In fact, as can be easily verified, we have that $\Sigma X \simeq X \wedge \mathbb{S}^1$. Therefore, $\Sigma \simeq - \wedge \mathbb{S}^1$. But $- \wedge \mathbb{S}^1$ (and therefore Σ) has adjoint $\mathrm{Map}_*(\mathbb{S}^1, -)$. By the uniqueness of adjoints up to isomorphisms we conclude effectively that *the loop space is the space of loops*.

We end this subsection with the next example, which clarifies that the notion of closed monoidal structure is *independent of the construction of smash products*. Indeed, it is interesting by itself, because it ensures very nice properties to the product $- \otimes Y$ as, for instance, the preservation of colimits.

Example 4.18 (*tensor product*)**.** Recall that, when R is commutative, we have a canonical R-module structure on the hom-set $\mathrm{Mor}_{\mathbf{Mod}_R}(X, Y)$ between two R-modules. Let $\mathrm{Hom}_R(X, Y)$ be the corresponding module. The set of bilinear maps $X \times Y \to Z$ also has a canonical R-module structure, which we will denote by $\mathrm{Bil}_R(X \times Y, Z)$. Now, notice that the bijections (4.2.6) preserve these linear structures, so that we have natural isomorphisms

$$\mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, Z)) \simeq \mathrm{Bil}_R(X \times Y, Z) \simeq \mathrm{Hom}_R(X \otimes Y, Z),$$

where the second isomorphism comes from the universal property of the tensor product. Therefore, when endowed with the tensor product monoidal structure, $\mathbf{Mod}_R$ is closed and the internal hom-object is just $\mathrm{Hom}_R(Y, Z)$ (i.e, the usual hom-set considered as R-module with the canonical structure). A similar result is also valid for the category of G-graded R-modules and for the category of differential G-graded R-modules. Particularly, it is valid for superalgebras and for chain/cochain complexes.

4.3 H-Spheres

In the last section we concluded that ΣX is always a H-cospace. As an immediate consequence, each sphere $\mathbb{S}^n$ have the structure of H-cospace, because $\mathbb{S}^n \simeq \Sigma \mathbb{S}^{n-1}$ for each $n > 0$. A very **natural** question is: *which spheres are H-monoids?* We observe that this question is **deeper** than it is **natural**. Indeed, as will be clear in the next chapter, the H-monoid structure introduces many cohomological constraints. For instance, we will see that the cohomology groups of a H-monoid must have the structure of a *Hopf algebra*, meaning that they have compatible nontrivial structures of algebra and coalgebra.

It is easy to see that for $n = 0, 1, 3$ the sphere $\mathbb{S}^n$ is a H-monoid. In fact, in these cases we can embed $\mathbb{S}^n$ respectively as the unitary vectors of the real division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively given by real numbers, complex numbers and quaternions. Each of these algebras are associative and have unity, so that restricting their products to the corresponding sphere we get the required H-monoid structure[8].

There is another canonical division algebra over $\mathbb{R}$: *the algebra $\mathbb{O}$ of octonions*, which is 8-dimensional. Unfortunately, this algebra is not associative, so that $\mathbb{S}^7$ (considered as its set of unitary vectors) does not have an induced H-monoid structure. But the octonions algebra is unital, so that $\mathbb{S}^7$ can be endowed with an non-associative operation having neutral element up to homotopy. In other words, this sphere is an "almost H-monoid" in the sense that only associativity up to homotopy fails. These "almost H-monoids" are also called *H-spaces*. Therefore we conclude that for $n = 0, 1, 3, 7$ the corresponding sphere $\mathbb{S}^n$ has a canonical H-space structure. The nontrivial fact is that *there is no other value of n for which the corresponding sphere can be endowed with a H-space structure*. This result is one of the jewels of complex K-theory[9].

This theorem has several consequences. For instance, it imply that *only for $n = 0, 1, 3, 7$ the sphere $\mathbb{S}^n$ is paralellizable*[10] (i.e, its tangent bundle $T\mathbb{S}^n$ is trivial or, equivalently, there are exactly n vector fields that are l.i. in each point)[11]. We cannot miss the opportunity to outline the proof of this very classical and beautiful result. It is based on the following four steps, which will be schematically presented each in the following (for more details, see the original reference [3] or any textbook covering K-theory and its applications, like as [141, 200, 11, 109]):

---

[8] We observe that in these cases the spheres are not only H-monoids, but actually Lie groups.

[9] The result was firstly proven by Adam in [1] by making use of (secondary) operations in ordinary cohomology, so that this result is **not** necessarily about K-theory. It happens that the K-theoretic background clarify many ideas and simplify drastically the proof, now given by primary operations on K-theory: the Adam's operations. This version is due to Adam and Atiyah in [3]. On the other hand, K-theory gives a more accurate result, stating exactly the maximal number of l.i. vector fields existing in $\mathbb{S}^n$ in terms of n. This is also due to Adams in [2].

[10] This is an early result, proven independently by Kervaire in [111] and by Bott-Milnor in [31]. Hovever, we we notice that, despite being proved without Adam's operations, both works make use of a version of Bott-periodicity, which is the fundamental result in K-theory.

[11] In order to conclude that the parallelizability of the canonical spheres imply a structure of H-space we make explicit use of the ambient space in which they are embedded and of the fact that they are exactly the set of unitary vectors. Since the works [154, 155, 112] of Milnor and Kervaire we known that there exist exotic structures in $\mathbb{S}^7$, i.e, there are spaces Σ^7 which are homeomorphic to the canonical sphere without being diffeomorphic to it. We could ask if these exotic spheres also are H-spaces. The answer is **affirmative**. Indeed, it can be shown that a homotopy sphere (i.e, any space Σ^n weakly homotopic to the canonical sphere $\mathbb{S}^n$) is parallelizable iff $n = 0, 1, 3, 7$ and in this case the parallel structure also induce a H-space structure (see [192]). Based in this fact, the reader could ask if the correspondence between parallelizability and existence of H-space structure is valid in the context arbitrary manifolds instead of only spheres. The answer is **no**: in [??] it is given examples of parallelizable manifolds which cannot be endowed with any H-space structure.

1. any map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ defines a numerical invariant $h(f)$, the *Hopf invariant* of f, obtained by making use of ordinary cohomology;

2. starting with a H-space structure on $\mathbb{S}^{n-1}$, a certain construction, called *Hopf construction*, produces a map with Hopf invariant equal to one;

3. using Chern character we transfer the above steps from ordinary cohomology to K-theory;

4. calculations with Adam's operations on K-theory reveals that a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ has odd Hopf invariant iff $n = 2, 4, 8$.

Hopf Invariant

Recall that, thanks to the gluing law for pushouts discussed in Section 2.1, we can build the reduced suspension of a based space X by making use of any continuous map $f : X \to Y$. Indeed, we first take the cone C_f for which we have a canonical inclusion $\imath : Y \hookrightarrow C_f$. Then, collapsing $\imath(Y)$ into a point we get ΣX. The sequence

$$X \xrightarrow{f} Y \xrightarrow{\imath} C_f \xrightarrow{\pi} \Sigma X$$

is precisely the *homotopy fiber sequence of* f. So, particularly, given a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ between spheres we have a corresponding homotopy fiber sequence

$$\mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n \xrightarrow{\imath} C_f \xrightarrow{\pi} \Sigma\mathbb{S}^{2n-1} \simeq \mathbb{S}^{2n}. \tag{4.3.1}$$

We notice that using the given map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ we can also build a CW-complex $C(f)$ with one 0-cell, one n-cell and one $2n$-cell. Indeed, because we have only one zero cell, the starting space $C(f)_0$ is a point. Therefore, there is exactly one map $\mathbb{S}^n \to C(f)_0$ which we use to glue the n-cell into $C(f)_0$, producing $C(f)_n$. On the other hand, we can use the given map f to glue a $2n$-cell into $C(f)_n$, ending the construction of $C(f)$. Thanks to the fact that the cone is invariant by homotopy we have that $C(f)$ and C_f are homotopic.

We would like to calculate the ordinary cohomology groups of the cone C_f. The last homotopy equivalence says that it is enough to compute the cohomology of the CW-complex $C(f)$. Because we have only one nontrivial n-cell and only one nontrivial $2n$-cell, we conclude that the ordinary cohomology of X with coefficients in an abelian group G is given by

$$H^k(C(f), G) := [C(f), K(k; G)] \simeq \begin{cases} G, & k = n \text{ or } k = 2n \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.2}$$

As will be discussed in the next chapter, we can introduce a monoidal structure in the category of spectrum, so that we can talk of *ring spectrum*: the monoid objects in this monoidal structure. When a generalized cohomology comes from a ring spectrum $\mathbb{E}$, the operation on the spectrum determines operations between the cohomology groups. As we will see, the Eilenberg-Mac Lane spectrum is a ring spectrum, so that we have products in ordinary cohomology. Let $\cup$ denote this product. Thanks to the structure of the cohomology groups (4.3.2) the unique nontrivial multiplication is

$$\cup : H^n(C(f), G) \times H^n(C(f), G) \to H^{2n}(C(f), G).$$

We will be interested only in the $G = \mathbb{Z}$ case. In this situation, the whole cohomology of $C(f)$ is determined by two integers: α and β which are generators for $H^n(C(f), \mathbb{Z}) \simeq \mathbb{Z}$ and $H^{2n}(C(f), \mathbb{Z}) \simeq \mathbb{Z}$, respectively. Therefore, the product $\cup$ is determined by the unique integer $h(f)$ satisfying $\alpha \cup \alpha = h(f) \cdot \beta$. This is the *Hopf invariant of f.*

Hopf Construction

Recall that in order to build the reduced suspension ΣX we start with the product $X \times I$, for which we have two canonical inclusions $\imath_0 : X \hookrightarrow X \times I$ and $\imath_1 : X \to X \times I$. We then take the pushout of each of them with the collapsing map $X \to *$. For instance, replacing one of these collapsing maps by an arbitrary map $f : X \to Y$ we get the cone C_f.

Now, observe that there is also another analogous constructions that can be applied when X is a product space $X = A \times B$. Indeed, in this case we replace each collapsing maps $X \to *$ by the canonical projections $A \times B \to A$ and $A \times B \to B$. The resultant space is called the *join between A and B* and is denoted by $A * B$. Given a map $\varphi : A \times B \to Z$ and selected a pair (a_o, b_o) we can build evaluation maps $ev_{b_o} : A \to Z$ and $ev_{a_o} : B \to Z$ such that $ev_{b_o}(a) = \varphi(a, b_o)$ and $ev_{a_o}(b) = \varphi(a_o, b)$. The universality of pushouts defining $A * B$ and ΣZ then gives a map

$$H(\varphi) : A * B \to \Sigma Z,$$

called the *Hopf construction* for φ with respect to the pair (a_o, b_o).

Let us suppose that $\varphi : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the multiplication giving a structure of H-space on $\mathbb{S}^{n-1}$ with neutral element e. Then we can use this distinguished element e to do the Hopf construction for φ with respect to the pair (e, e). The result is a map $H_\varphi : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, where we used the identifications $\mathbb{S}^k * \mathbb{S}^l \simeq \mathbb{S}^{k+l+1}$ and $\Sigma \mathbb{S}^k \simeq \mathbb{S}^{k+1}$. Therefore, we can calculate the Hopf invariant $h(H_\varphi)$, which is $+1$ or -1.

Chern Character

The classical Adam's argument [1] dealt with secondary cohomology operations on ordinary cohomology. Doing an intricate computation, he showed that *a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ admits an odd Hopf invariant only if $n = 0, 2, 4, 8$*. Therefore, *the Hopf-construction implies that only for these values of n can $\mathbb{S}^{n-1}$ be endowed with a H-space structure*. With the development of K-theoreric methods by Atiyah and Adam, a new proof of this obstruction could be obtained using much more simple computations .

Recall that the Hopf invariant was defined on ordinary cohomology, so that in order to use K-theoretical methods we need some process connecting these two cohomology theories. Let us explain how this can be done. Recall that, as discussed in Section 1.3, we have a canonical characteristic class $\mathrm{ch}_X : KU(X) \to H(X, \mathbb{Q})$, the *Chern character*, connecting complex K-theory and ordinary cohomology with rational coefficients. Both spectra underlying these cohomologies are ring spectra, so that both cohomologies have products, which are preserved by ch_X.

Now, for a given map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ let us consider the first diagram below, whose upper and lower rows were obtained applying K-theory and rational/integer ordinary cohomology functors on the fiber homotopy sequence of f (recall that cohomology functors are contravariant, so that the directions of the arrows must be reversed). The non-identified vertical arrows are the canonical natural transformation $H(-; \mathbb{Z}) \to H(-; \mathbb{Q})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (recall that this inclusion induces a morphism $B\mathbb{Z} \to B\mathbb{Q}$ between the corresponding classifying spaces and, therefore, between the Eilenberg-Mac Lane spaces $K(k,\mathbb{Z}) \to K(k,\mathbb{Q})$; the natural transformation is then obtained from the Yoneda lemma). This diagram is commutative, because the Chern character is also a natural transformation.

$$\begin{tikzcd}
KU(\mathbb{S}^{2n}) \arrow[r] \arrow[d, "\mathrm{ch}"'] & KU(C_f) \arrow[r] \arrow[d, "\mathrm{ch}"'] & KU(\mathbb{S}^n) \arrow[d, "\mathrm{ch}"] \\
H(\mathbb{S}^{2n},\mathbb{Q}) \arrow[r] & H(C_f,\mathbb{Q}) \arrow[r] & H(\mathbb{S}^n,\mathbb{Q}) \\
H(\mathbb{S}^{2n},\mathbb{Z}) \arrow[r] \arrow[u] & H(C_f,\mathbb{Z}) \arrow[r] \arrow[u] & H(\mathbb{S}^n,\mathbb{Z}) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{ch}"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{ch}"'] & \mathbb{Z} \arrow[d, "\mathrm{ch}"] \\
\mathbb{Q} \arrow[r] & \mathbb{Q}\oplus\mathbb{Q} \arrow[r] & \mathbb{Q} \\
\mathbb{Z} \arrow[r] \arrow[u] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[u] & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

When n is odd, the Hopf invariant $h(f)$ is always equal to zero, so that we assume n even, say equal to $2m$. In this case, Bott-periodicity says that $KU(\mathbb{S}^{2m}) \simeq KU(\mathbb{S}^2) \simeq \mathbb{Z}$. Now, recall that for any abelian group G, the only nontrivial ordinary cohomology group is $H^k(\mathbb{S}^k, G) \simeq G$. On the other hand, each of the left horizontal arrows in the first diagram above is injective and each of the right horizontal is surjective. Together, these facts imply that the first diagram is equivalent to the second. The Hopf invariant $h(f)$ is precisely on the lower $\mathbb{Z}\oplus\mathbb{Z}$ term, corresponding to ordinary cohomology. But the commutativity of the diagram and the fact that the Chern character preserves products allow us to lift $h(f)$ to the upper $\mathbb{Z}\oplus\mathbb{Z}$ term which is given by K-theory, meaning that we can move from ordinary cohomology to K-theory, as required.

<u>Adam's Operations</u>

Recall that in ordinary cohomology we have canonical operations $Sq^k : H^n_{\text{sing}} \Rightarrow H^{n+k}_{\text{sing}}$, the *Steenrod square operations*, which can be totally characterized by a list of properties. In complex K-theory we also have internal operations $\psi^k : KU \Rightarrow KU$, called *Adam's operations*, which are uniquely characterized by the following properties:

1. ψ^k_X are ring homomorphisms;
2. $\psi^k_X \cdot \psi^l_X = \psi^{kl}_X = \psi^l_X \cdot \psi^k_X$;
3. $\psi^k_X(x) = x^k$, if $x \in KU(X)$ is the class of a line bundle;
4. $\psi^k_X(x) \equiv x^k \bmod k$, if k is prime;
5. $\psi^k_{\mathbb{S}^{2n}}(x) = k^n x$ for any class of the sphere $\mathbb{S}^{2n}$.

We will not give the construction of these operations here. Instead we will show that *these axiomatic properties imply that a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ has odd Hopf invariant only if $n = 2, 4, 8$, ending the proof that only $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ admit a H-space structure.* This is a purely arithmetic calculation, whose essence we extracted from p. 327-329 of [9] and p. 212 of [141]. Indeed, let us supposed that $h(f)$ is odd. In this case, as discussed previously, n must be even, so that we write $n = 2m$.

From the last diagram, $h(f)$ is the relation between the image β of a class $b_{2n} \in KU(\mathbb{S}^{2n})$ by π^* and the preimage α of a class $b_n \in KU(\mathbb{S}^n)$ by $\imath^*$. From property (5.) above we have

$\psi^k(b_{2n}) = k^{2m} b_{2n}$ and $\psi^k(b_n) = k^m b_n$. So, by the naturality of Adam's operations we get $\psi^k(\beta) = k^{2m}\beta$ and

$$\imath^*(\psi^k(\alpha)) = \psi^k(\imath^*(\alpha)) = \psi^k(b_n) = k^m b_n = k^m \imath^*(\alpha) = \imath^*(k^m\alpha),$$

so that

$$\imath^*(\psi^k(\alpha) - k^m\alpha) = 0, \quad \text{and, therefore,} \quad \psi^k(\alpha) = k^m\alpha + \sigma(k)\beta \tag{4.3.3}$$

for some $\sigma(k) \in \mathbb{Z}$. On the other hand, from property (4.) we have $\psi^2(\alpha) \equiv \alpha^2 \mod 2$. By definition of $h(f)$, it is the number such that $\alpha^2 = h(f)\beta$. Therefore, $\psi^2(\alpha) - h(f)\beta$ is even. From (4.3.3) applied to $k = 2$ we then conclude that $\sigma(2)$ and $h(f)$ must have the same parity, so that $\sigma(2)$ is odd. For any odd k, by the linearity of Adam's operations,

$$\begin{aligned} \psi^k(\psi^2(\alpha)) &= \psi^k(2^m\alpha + \sigma(2)\beta) \\ &= 2^m\psi^k(\alpha) + \sigma(2)\psi^k(\beta) \\ &= 2^m k^m\alpha + (\sigma(k) + \sigma(2)k^{2m})\beta \end{aligned}$$

and, analogously,

$$\psi^2(\psi^k(\alpha)) = 2^m k^m\alpha + (\sigma(2) + \sigma(k)2^{2m})\beta.$$

Therefore, by property (.2), the equality of the β coefficients imply

$$\sigma(k) + \sigma(2)k^{2m} = \sigma(2) + \sigma(k)2^{2m}, \quad \text{i.e,} \quad 2^m(2^m - 1)\sigma(k) = k^m(k^m - 1)\sigma(2).$$

Consequently, because $\sigma(2)$ is odd, 2^m divides $k^m - 1$ for every k. Taking $k = 3$ we see that the unique solutions are $m = 1, 2, 4$ and, therefore, $n = 2m = 2, 4, 8$, ending the proof.

Chapter 5

Commutative Monoids

In order to get more abstract and powerful concepts, in the present chapter we continue applying the categorification process in both of its incarnations: as internalization into **Cat** and as enrichment over **Cat**. More specifically, if the last chapter was about categorification of the classical concept of monoid, this chapter is about categorification of monoids fulfilling additional conditions, as commutativity and existence of inverses.

We start in the first section by internalizing these entities on **Cat**, getting the notion of *symmetric monoidal category*. By the microcosm principle, the obtained structures defines exactly the context in which we can talk of commutative monoid objects. After defining them, we present some concrete examples.

In the same way as associators and unitors are part of the data defining "monoidal category", in order to define a symmetric monoidal category we need to specify certain *braidings*. As an example of this fact, we show that the category of superalgebras endowed with the monoidal structure given by the tensor product admits two different symmetric structures, allowing us to talk about *commutative superalgebras* and *graded-commutative* superalgebras. We then discuss that this difference lies at the heart of supersymmetry and that it gives one more reason to work with **extended** quantum field theories instead of with the usual quantum field theories given by functors on cobordism categories.

Ending the first section we show that in the presence of braidings, the category of monoid objects acquires a natural monoidal structure, allowing us to talk of "monoid objects into the category of monoid objects". The *Eckmann-Hilton duality* shows that both they are equivalent to commutative monoid objects. Using this fact we justify why the homotopy groups $\pi_n(X)$ have abelian group structure **only for** $n \geq 2$, while the cohomology groups $H^n(X;\mathbb{E})$ with coefficients in any spectra $\mathbb{E}$ have abelian group structure for **every** n.

In Section 5.2 we study the "comonoid objects into the category of monoid objects" and the "monoid objects into the category of comonoid objects". These are the so called *bimonoids*. Special classes are given by the *Hopf monoids*, which can be understood as "categorified groups". We discuss many examples which are important in mathematics and in physics. Indeed, from the mathematical viewpoint we discuss Hopf algebras, which have cohomology groups of H-spaces and Steenrod operations as examples. From the physical viewpoint we try to justify why supersymmetry is the most general kind of symmetry that can be considered in a system of particles, which will be proven in Section 8.4.

We start Section 5.3 by noticing many similarities between the category **Spec** of spectra and the category $\mathbf{CCh}_R$ of cochain complexes, leading us to think of a spectra as a "graded topological space" in the same way as a cochain complex is a special kind of graded module. Both categories $\mathbf{Mod}_R$ and $\mathbf{Top}_*$ have closed symmetric monoidal structures, respectively given by the tensor product $\otimes_R$ and by the smash product. The product $\otimes_R$ extends to a symmetric monoidal structure on $\mathbf{CCh}_R$, so that, following the above analogy, it is natural to expect that **Spec** also becomes a symmetric monoidal category in a similar fashion. We prove a theorem, due to Lewis [123], which states that such a structure does not exist! On the other hand, we see that $\wedge$ induce a well defined symmetric monoidal structure in the stable homotopy category $\mathrm{Ho}(\mathbf{Spec})$. The corresponding monoid and commutative monoid objects are called *ring spectrum* and *commutative ring spectrum*.

We end the chapter by giving a proof that the cohomology groups $H^k(X;\mathbb{E})$ with coefficients in a commutative ring spectrum fits into a graded-commutative superalgebra, meaning that we know how to multiply cohomology classes.

5.1 Braidings

In the last chapter we categorified the concept of monoid. In this section we would like to categorify the notion of *commutative monoid*. Recall that this is just a monoid X for which we have $x*y = y*x$ for any $x, y \in X$. This condition can be characterized by the commutativity of the following diagram, where $b(x,y) = (y,x)$:

$$\begin{tikzcd} X \times X \arrow[rr, "b"] \arrow[dr, "*"'] & & X \times X \arrow[dl, "*"] \\ & X & \end{tikzcd}$$

As can be easily verified, exactly as happens for usual monoids, the commutative monoids really can be internalized in any category $\mathbf{H}$ having binary products and a terminal object, so that they can be internalized into **Cat** (i.e, they really can be categorified by internalization). The result is what is known as a *symmetric monoidal category*[1]. This is just a monoidal category $\mathbf{C}$ whose product $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is commutative up to isomorphisms. This means that a symmetric monoidal category has not only associators a_{xyz} and unitors ${}_xu$ and u_x, but also *braidings* $b_{xy} : X \otimes Y \simeq Y \otimes X$.

Remark. In Secion 4.1 we commented that we generally work with monoidal categories whose associators and unitors satisfy additional commutative conditions: the *coherence conditions*. In the context of symmetric monoidal categories, some coherence conditions are also usually required. The most common are those given by the following commutative diagram together with an analogous version of it, obtained by replacing the associators a_{xyz} by its inverses a^{-1}_{xyz} and the products $b_{xy} \otimes id$ by $id \otimes b_{xy}$ (the importance of these coherence condition will become clear in

[1]In the literature these categories are called ***braided*** *monoidal categories*. The name **symmetric** monoidal category is reserved to braided monoidal categories whose braidings b_{xy} satisfy the additional property $b_{xy} \circ b_{xy} = id$. Following this nomenclature, in the scope of this text, all important braided monoidal categories are symmetric, but the property $b_{xy} \circ b_{xy} = id$ itself will not be relevant. This justifies our definition.

Section 7.3):

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "a_{xyz}"] \arrow[d, "b_{xy}\otimes id"'] & X\otimes (Y\otimes Z) \arrow[r, "b_{x(y\otimes z)}"] & (Y\otimes Z)\otimes X \arrow[d, "a_{yzx}"] \\
(Y\otimes X)\otimes Z \arrow[r, "a_{yxz}"'] & Y\otimes (X\otimes Z) \arrow[r, "id\otimes b_{xz}"'] & Y\otimes (Z\otimes X)
\end{tikzcd}$$

By the microcosm principle, we can enrich the notion of commutative monoid over any of its categorification by internalization. In other words, we can talk of *commutative monoid objects into any symmetric monoidal category.* By the duality principle, into any symmetric monoidal category we can also define a dual notion of *commutative comonoid object*.

Finally, the notion of homomorphism between comonoids can be internalized into **Cat**, defining morphisms between symmetric monoidal categories and, therefore, a corresponding subcategory **SymMon** $\subset$ **Mon**. These morphisms are simply monoidal functors between the underlying monoidal categories which commute with the braidings. Similarly, the homomorphisms between commutative comonoids enrich to a notion of morphism between commutative monoid/comonoid objects into a fixed symmetric monoidal category $\mathbf{C}$, producing categories

$${}_c\mathrm{Mon}(\mathbf{C},\otimes) \quad \text{and} \quad {}_c\mathrm{CoMon}(\mathbf{C},\otimes).$$

Examples

Now, in order to clarify the ideas, let us give some examples of symmetric monoidal categories and commutative monoids/comonoids.

Example 5.1 (*classical commutative monoids*)**.** Recall that **Set** has a canonical cartesian monoidal structure whose product is given by the cartesian product bifunctor $\times$. We notice that this structure becomes naturally symmetric with the braidings $b_{xy} : X\times Y \to Y\times X$, such that $b_{xy}(x,y) = (y,x)$. As expected, the classical notion of commutative monoid is recovered as the commutative monoids object into this symmetric monoidal structure. On the other hand, recall that we have only trivial comonoids on $(\mathbf{Set},\times)$ in the sense that any set X can be endowed with a unique comonoid structure. The multiplication is given by the diagonal map $\Delta : X \to X\times X$, which actually commutes with the braiding b_{xx}. Therefore, *this trivial comonoid structure is a commutative comonoid structure.* A totally analogous discussion holds if we consider on **Set** the cocartesian monoidal structure given by the disjoint union bifuctor $\sqcup$. In fact, this structure is also symmetric for some canonical braidings $b_{xy} : X\sqcup Y \to Y\sqcup X$, but now the nontrivial objects are the commutative comonoids instead of the commutative monoids.

Example 5.2 (*cartesian/cocartesian commutative monoids/comonoids*)**.** We notice that the braidings $b_{xy} : X\times Y \to Y\times X$ and $b'_{xy} : X\sqcup Y \to Y\sqcup X$ discussed in the last example are obtained from universality of binary products and binary coproducts, so that they can be defined into any cartesian/cocartesian monoidal structure. This means that the above discussion generalizes to this more ample context.

Example 5.3 (*smash*)**.** Recall that for a closed monoidal category $\mathbf{C}$, the corresponding based category $\mathbf{C}_*$ inherits a canonical closed monoidal structure given by the smash product $\wedge$. Here we would like to observe that, if $\mathbf{C}$ is symmetric, then the induced structure into $\mathbf{C}_*$ is also symmetric. Recalling that the smash product is defined by a pushout, the main idea is to consider the braidings $\overline{b}_{xy}$ into $\mathbf{C}_*$ as coming from the braidings b_{xy} of $\mathbf{C}$ by universality of pushouts. This is really the case, as shown in the diagram below. In it, the left and the right squares are respectively the pushout squares defining the smash products $X \wedge Y$ and $Y \wedge X$. Using the segmented arrows and the universality of $X \wedge Y$ we get the map $\overline{b}_{xy}$. This map is indeed an isomorphism. Its inverse $\overline{b}_{xy}^{-1}$ is obtained from the dotted arrows and the universality of $Y \wedge X$.

$$
\begin{tikzcd}
X \wedge Y \arrow[rrr, dashed, bend left, "\overline{b}_{xy}"] & X \otimes Y \arrow[l] \arrow[r, dashed, shift left, "b_{xy}"] & Y \otimes X \arrow[l, dotted, shift left, "b_{xy}^{-1}"] \arrow[r] & Y \wedge X \\
* \arrow[u] & X \oplus Y \arrow[l] \arrow[u] \arrow[r, dashed, shift left, "\simeq"] & Y \oplus X \arrow[l, dotted, shift left, "\simeq"] \arrow[u] \arrow[r] & * \arrow[u]
\end{tikzcd}
$$

Example 5.4 (*commutative algebras/coalgebras*)**.** The tensor product $X \otimes Y$ between R-modules is commutative up to isomorphisms. Indeed, the rule $x \otimes y \to y \otimes x$ extends linearly to an isomorphism. It happens that the bifunctor $\otimes$ defines a monoidal structure on $\mathbf{Mod}_R$, so that if we take the above isomorphisms as the braidings, then this monoidal structure will be symmetric. While the monoid/comonoid objects on $(\mathbf{Mod}_R, \otimes)$ are (associative) algebras/coalgebras, the commutative monoids/comonoids are (associative and) commutative algebras/coalgebras.

A monoidal structure is defined not only by the product bifunctor and by the neutral object, but also by the associators and unitors. This means that a priori we can introduce two different monoidal structures into the same category, which have the same product and the same neutral object, but differing on the associators and unitors. Similarly, the braidings are part of the data defining "symmetric monoidal structure", so that *a priori we can introduce two different symmetric structures into the same monoidal category, differing only on the braidings.* As a consequence, a priori we may have monoid objects which are commutative relatively to a symmetric structure but which are not relatively to some other. The next example is a manifestation of these facts.

Example 5.5 (*commutative superalgebras*)**.** Given a abelian group G, we have the corresponding category $G\mathbf{Grad}_R$ of G-graded R-modules, whose morphisms are required to preserve the grading. The tensor product can be naturally defined in this category by the rule (4.2.1), producing a monoidal structure. For two given graded modules $X = \oplus X_h$ and $Y = \oplus Y_{h'}$, we get isomorphisms

$$b_{xy} : X \otimes Y \to Y \otimes X \quad \text{by putting} \quad b_{xy}(x_h \otimes y_{h'}) = y_{h'} \otimes x_h,$$

so that the monoidal structure is symmetric when endowed with the braidings b_{xy}. The commutative monoids/comonoids are called *commutive G-graded algebras/coalgebras.* On the other hand, recall that when selected a fundamental pairing $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{Z}_2$ we can extend the bifunctor $\otimes$ from $G\mathbf{Grad}_R$ to $G\mathbf{Mod}_R$ (where the morphisms are arbitrary R-module homomorphisms) by adding a term like $(-1)^{\langle \deg(f), \deg(g) \rangle}$ in the action of $\otimes$ on morphisms, as presented in (4.2.2). Instead of adding this term on morphisms, we could added it on objects, using this fact to get a different braiding for $G\mathbf{Grad}_R$. More precisely, with a fundamental pairing on hands we can define new isomorphisms

$$b'_{xy} : X \otimes Y \to Y \otimes X \quad \text{by} \quad b'_{xy}(x_h \otimes y_{h'}) = (-1)^{\langle h,h' \rangle} y_{h'} \otimes x_h$$

and, therefore, a new symmetric monoidal structure on $G\mathbf{Grad}_R$ whose underlying monoidal structure is exactly that given by the tensor product. The commutative monoids/comonoids in this new symmetric structure are known as the *graded-commutative G-graded algebras/coalgebras*. So, for instance, a graded-commutative G-graded R-algebra can be identified with a G-graded R-module $X = \oplus X_h$ endowed with a associative and unit multiplication

$$m : X \otimes X \to X \quad \text{such that} \quad m(x_h \otimes x'_{h'}) = (-1)^{\langle h,h' \rangle} x'_{h'} \otimes x_h.$$

Recall that $\mathbb{Z}_2$-graded algebras are superalgebras, so that graded-commutative $\mathbb{Z}_2$-graded R-algebras are usually called *graded-commutative superalgebras over R*. The standard examples to have in mind are the following (their physical relevance will be discussed in the sequence):

1. *Symmetric and Grassman algebras*. Let X be a R-module. Taking its tensor powers we can build a $\mathbb{N}$-graded module $T(X)$ such that $T(X)_n = X^{\otimes n}$, which becomes a graded algebra (the *tensor algebra* of X) when endowed with the tensor This graded algebra **is not** commutive nor graded-commutative. In other words, $T(X)$ is a monoid into $(\mathbb{N}\mathbf{Grad}_R, \otimes)$ which is **not** necessarily a commutative monoid with respect to both bradings b_{xy} and b'_{xy} introduced above. This happens exactly because the equalities

$$x \otimes y = y \otimes x \quad \text{and} \quad x \otimes y = (-1)^{n \cdot m} y \otimes x$$

   are **not** valid for arbitrarily given elements $x \in X^{\otimes n}$ and $y \in X^{\otimes m}$, with $n, m \in \mathbb{N}$. On the other hand, it is easy to see that the elements satisfying some of the last conditions defines graded subalgebras of $T(X)$, respectively denoted by $\mathrm{Sym}(X)$ and $\Lambda(X)$. These are the *symmetric algebra* and the *Grassman* (or *exterior*) *algebra* of X. By construction, they are respectively commutative and graded-commutative. Both algebras are $\mathbb{N}$-graded so that by splitting them into even-odd pieces we get $\mathbb{Z}_2$-graded algebras. In other words, *the symmetric algebra is a commutative superalgebra, while the Grassman algebra is a graded-commutative superalgebra*. The product $\mathrm{Super}(X) = \mathrm{Sym}(X) \otimes \Lambda(X)$ is also graded-commutative, being called the *superalgebra* of X. As particular cases, we see that both de Rham algebra of a manifold and Chevalley-Eilenberg algebra of a Lie algebra (introduced in Example 4.9) are graded-commutative. Indeed, both are flavors of Grasmann algebras.

2. *Clifford algebras* (a non-example). Recall that, as discussed in the remarks on Example 1.12, to any vector space V endowed with a non-degenerated quadratic form q we can associate an algebra $\mathcal{C}\ell(V,q)$, called the *Clifford algebra*. As will be seen in Section 11.1, such an algebra becomes equipped with a canonical $\mathbb{Z}_2$-grading $\mathcal{C}\ell(V,q) \simeq \mathcal{C}\ell(V,q)^0 \oplus \mathcal{C}\ell(V,q)^1$ and, therefore, it is a superalgebra. On the other hand, *there is **no** symmetric structure in $\mathbb{Z}_2\mathbf{Grad}_{\mathbb{K}}$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, turning $\mathcal{C}\ell(V,q)$, with $1 < \dim V < \infty$, commutative*. This is due to the classification of real/complex Clifford algebras (sketched in Section 11.1) and to the classification of symmetric structures that can be introduced in the category of superalgebras (proved in Section.

3. *Cohomology rings*. As briefly discussed in Example 4.7 (and as will be more concretely discussed in Section 5.3), the smash product should induce a monoidal structure into the homotopy category $\mathrm{Ho}(\mathbf{Spec})$ of spectra and the cohomology groups with coefficients in

a monoid object (i.e, in a "ring spectrum") should fit into a superalgebra. Now, because the smash product is symmetric, it is natural to expect that the induced smash product on spectra is also symmetric, allowing us to talk of "commutative ring spectrum". Furthermore, it is also natural to expect that the functors

$$[X, -] : (\mathrm{Ho}(\mathbf{Spec}), \wedge) \to (\mathbb{Z}\mathbf{Grad}_{\mathbb{Z}}, \otimes) \tag{5.1.1}$$

becomes not only monoidal, but indeed symmetric monoidal, meaning that the cohomology of a "commutative ring spectrum" is a "commutative super algebra". It happens that such an assertion depends of the symmetric extension of the monoidal structure $(\mathbb{Z}\mathbf{Grad}_{\mathbb{Z}}, \otimes)$. As discussed above, there are at least two of these extensions. So, we can ask: what is the correct braidings that turn 5.1.1 into a symmetric monoidal functor. As a consequence of the "minus" signal appearing into the Barrat-Puppe sequence (2.2), if the superalgebra has a "commutative product", then it must be *graded-commutative*.

We end this subsection by presenting some remarks which explain the physical importance of the last examples.

1. *bosons together with fermions imply a superalgebra of states*. Recall that a quantum theory for particles is a monoidal functor $(\mathbf{Cob}_1, \sqcup) \to (\mathbf{Vec}_{\mathbb{C}}, \otimes)$. Therefore, the quantum states of a system composed by a single particle is described by a vector space $\mathcal{H}$ and a system with n identical particles is described by $\mathcal{H}^{\otimes n}$. More generally, a system with an arbitrary number of particles should be described by the tensor algebra $T(\mathcal{H})$. On the other hand, if the reader have some knowledge of quantum mechanics he certainly knows that quantum particles have additional degrees of freedom. One example is the spin, which allows us to classify the particles as **bosons** (having integer spin) or **fermions** (having half-integer spin). But the states of arbitrary particles are described by $T(\mathcal{H})$, so that it is expected that bosons and fermions define different subalgebras of $T(\mathcal{H})$. This is really the case. Indeed, Pauli's exclusion principle states that *in a system with many **electrons**, two of them cannot be in the same nontrivial state*. This can be formalized in the following way: if $x = x_1 \otimes x_2 \otimes ....$ is a state of a system in which the ith and the jth particles are electrons, then the condition $x_i = x_j$ imply $x = 0$. In our context, this is the same as requiring

$$x_1 \otimes x_2 \otimes ... \otimes x_i \otimes ... \otimes x_j \otimes ... = (-1) \cdot (x_1 \otimes x_2 \otimes ... \otimes x_j \otimes ... \otimes x_i \otimes ...),$$

allowing us to rewrite Pauli's principle in the following way: *the states of a system of electrons are vectors of the Grassman algebra* $\Lambda(\mathcal{H})$. Electrons are examples of fermions. An extension of this principle to other fermions is given by the *spin-statistics theorem* ([207, 148, 215]). Systems of bosons, on the other hand, can have degenerated states, meaning that *the states of a system of bosons are vectors of the symmetric algebra* $\mathrm{Sym}(\mathcal{H})$. Therefore, *a general quantum system, containing both bosons and fermions, should be described by the superalgebra* $\mathrm{Super}(\mathcal{H})$.

2. *Clifford superalgebra of internal degrees of freedom*. We discussed above that the existence of internal degrees of freedom for the **quantum** particle imply that the algebra of states must be a superalgebra. It happens that quantum theories generally come from classical theories by quantization, so that we need to have a description of "internal degrees of freedom" for

classical particles too. As will be discussed in Chapter 11, this is done by requiring that the spacetime M in which the classical particle moves has a *spin-structure*, meaning that the frame bundle FM is structure over the universal cover $\mathrm{Spin}(n)$ of $SO(n)$. For instance, in the 3-dimensional case, we have $\mathrm{Spin}(3) = SU(2)$, which is the usual group associated with the spin in quantum mechanics. We know that (at least over manifolds) the universal cover always exist, but in principle there is no canonical way to build it. In our present context, thanks to Clifford superalgebra structure, we have a concrete presentation for the group of internal degrees of freedom $\mathrm{Spin}(n)$.

3. *cohomology superalgebras as supersymmetric field theories*. The usual definition of topological quantum field theory as a monoidal functor $(\mathbf{Cob}_{p+1}, \sqcup) \to (\mathbf{Vec}_{\mathbb{C}}, \otimes)$ predicts that any system of particles is described by the tensor algebra $T(\mathcal{H})$. On the other hand, if the particles are supposed to have spin, then standard results on quantum mechanics (Pauli's exclusion principle and the spin-statistics theorem) imply that the quantum system should be described by the sub-superalgebra $\mathrm{Super}(\mathcal{H}) \subseteq T(\mathcal{H})$. This is another problem with the given 1-categorical definition of topological quantum theories. It relies on the fact that we are considering cobordisms between manifolds. But manifolds are objects modeled over commutative variables, while we need commutative **and** anticommutative variables, i.e, we need to work with *supermanifolds*. We could try to define "supercobordism between supermanifolds", getting a category $\mathrm{Super}\mathbf{Cob}_{p|s}$ and then consider "super quantum field theories" as functors $\mathrm{Super}\mathbf{Cob}_{p|s} \to \mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}$ from supercobordisms to super-vector spaces. This is done in [43, 198], where the authors show that the $(1|1)$ theory is classified by the superalgebra of K-theory (a generalized cohomology theory). They also conjecture that the $(2|1)$ theories are also classified by another generalized cohomology theory called *topological modular theory*.

Eckmann-Hilton

In Section 4.2 we proved that the homotopy groups $\pi_n(X)$, with $n > 0$, and every generalized cohomology group $H^n(X; \mathbb{E})$ are monoids. Here we will see that the cohomology groups and are, indeed, **abelian** monoids. We will also prove that, for arbitrary spaces, the homotopy groups are abelian only for $n \geq 2$, but in the case of topological groups (or, more generally, H-monoids), the fundamental group $\pi_1(X)$ is abelian too. All these facts can be justified by the same simple result: *commutative monoids are precisely monoids in the category of monoids.*

We start by recalling that for any monoidal category $\mathbf{C}$ we can build the corresponding category $\mathrm{Mon}(\mathbf{C}, \otimes)$ of monoid objects. We would like to observe that, *if* $\mathbf{C}$ *is symmetric, then the category* $\mathrm{Mon}(\mathbf{C}, \otimes)$ *carries a natural monoidal structure*. Indeed, the product bifunctor $\otimes$ extends to another bifunctor

$$\otimes_M : \mathrm{Mon}(\mathbf{C}, \otimes) \times \mathrm{Mon}(\mathbf{C}, \otimes) \to \mathrm{Mon}(\mathbf{C}, \otimes)$$

which takes two monoid objects (say X and Y, with products $* : X \otimes X \to X$ and $*' : Y \otimes Y \to Y$, as well as with unities $1 \to X$ and $1 \to Y$) and returns a monoid object structure on the product $X \otimes Y$, whose multiplication is given by the map

$$(X \otimes Y) \otimes (X \otimes Y) \xrightarrow{\simeq} (X \otimes X) \otimes (Y \otimes Y) \xrightarrow{*\otimes *'} X \otimes Y$$

and whose neutral element is given by the map

$$1 \xrightarrow{\simeq} 1 \otimes 1 \longrightarrow X \otimes Y ,$$

where the symmetry of $\mathbf{C}$ was used to define the first of these maps. The neutral object in this monoidal structure is just the monoid object 1 of $\mathbf{C}$, endowed with the trivial monoid object structure, whose multiplication is the unitor $1 \otimes 1 \simeq 1$ and whose neutral element is the identity map $id_1 : 1 \to 1$.

Now, there are two kinds of entities that can be considered. Being $\mathbf{C}$ a symmetric monoidal category, we can consider the commutative monoid objects into it, while being $\mathrm{Mon}(\mathbf{C}, \otimes)$ a monoidal category, we can consider the monoid objects into it. The surprising fact is that both notions are the same! More precisely, for any symmetric monoidal category $\mathbf{C}$ we have an equivalence

$$\mathrm{Mon}(\mathrm{Mon}(\mathbf{C}, \otimes), \otimes_M) \simeq {}_{\mathrm{c}}\mathrm{Mon}(\mathbf{C}, \otimes).$$

This is the content of the so called *Eckmann-Hilton argument*, which will be briefly explained now. Details can be founded in [61, 19, 8]. First of all, recall that a monoid object in $\mathrm{Mon}(\mathbf{C}, \otimes)$ is just a monoid object $(X, *, e)$ in $\mathbf{C}$, where $*$ is the multiplication and e is the unity, together with morphisms of monoid objects

$$m : (X, *, e) \otimes_M (X, *, e) \to (X, *, e) \quad \text{and} \quad u : (1, \simeq, id_1) \to (X, *, e)$$

making commutative certain "monoid-like" diagrams. So, a monoid object on $\mathrm{Mon}(\mathbf{C}, \otimes)$ is just a monoid object on $\mathbf{C}$ with additional commutative conditions on certain diagrams. We can see that these additional conditions say precisely that the underlying monoid is commutative.

Remark. Thanks to relation (4.2.5) we can use the duality principle to get a totally analogous version of the Eckmann-Hilton argument for comonoids, meaning that there is the following isomorphism (the reader is invited to verify the details):

$$\mathrm{CoMon}(\mathrm{CoMon}(\mathbf{C}, \otimes), \otimes_M) \simeq_c \mathrm{CoMon}(\mathbf{C}, \otimes).$$

Now we can explain what was promised: the additional structures in the sets defining the homotopy groups and the generalized cohomology groups. Both cases can be justified by noting that, for any $X, Y \in \mathbf{Top}_*$

1. $[\Sigma^2 X, Y]$ *is a commutative monoid.* Indeed, recall that ΣZ is a comonoid into $(\mathrm{Ho}(\mathbf{Top}_*), \vee)$ for any Z, so that $[\Sigma Z, Y]$ is always a monoid into $(\mathbf{Set}, \times)$. Particularly, for $Z = \Sigma X$ we conclude that $[\Sigma^2 X, Y]$ is a monoid into $\mathrm{Mon}(\mathbf{Set}, \times)$ and, therefore, a commutative monoid by the Eckmann-Hilton argument. Therefore, if $\mathbb{E}$ is a suspension spectra of CW-complexes, then $E_n \simeq \Sigma^2 E_{n-2}$ for any $n \geq 2$, so that $[E_n, Y]$ is always a commutative monoid. In particular, $\pi_{n \geq 2}(X)$ is a commutative monoid;

2. $[X, \Omega^2 Y]$ *is a commutative monoid.* This follows directly from the adjuntion between the loop space funtor Ω and the reduced suspension functor Σ, together with the last result. So, if $\mathbb{E}$ is now a Ω-spectrum, then $E_n \simeq \Omega^2 E_{n+2}$ for any n, so that the corresponding cohomology groups $H^n(X, \mathbb{E}) = [X, E_n]$ are commutative monoids for arbitrary n.

5.2 Hopf

In a monoidal category $\mathbf{C}$ we can talk about monoid and comonoid objects. In special, if $\mathbf{C}$ is symmetric, then the corresponding categories $\mathrm{Mon}(\mathbf{C},\otimes)$ and $\mathrm{CoMon}(\mathbf{C},\otimes)$ of monoids and comonoids have induced monoidal structures. Therefore, in these cases we can consider monoids into the category of monoids and comonoids into the category of comonoids. The Eckmann-Hilton argument gives isomorphisms

$$\mathrm{Mon}(\mathrm{Mon}(\mathbf{C},\otimes),\otimes_M) \simeq {}_{\mathrm{c}}\mathrm{Mon}(\mathbf{C},\otimes) \quad \text{and} \quad \mathrm{CoMon}(\mathrm{CoMon}(\mathbf{C},\otimes),\otimes_M) \simeq {}_{\mathrm{c}}\mathrm{CoMon}(\mathbf{C},\otimes),$$

clarifying that these iterated entities are nothing else than the initial entity (say monoids or comonoids) in its commutative version.

But we could also considered **monoid** objects into the category of **comonoids** and **comonoids** into the category of **monoids**. In this case both would be the same as an object $X \in \mathbf{C}$ endowed with compatible structures of monoid and comonoid. Such a entity is called a *bimonoid* on $(\mathbf{C},\otimes)$ and, supposing that (X,m,u) and (X,w,v) are the underlying monoid and comonoid structures, the compatibility between them can be formally described by the following requirement: if

$$m : X \otimes X \to X \quad \text{and} \quad u : 1 \to X$$

are the multiplication and the unit giving to X the structure of monoid object, then we require that these morphisms are indeed morphisms between comonoids, where here we are considering X with the comonoid structure (X,w,v), the product $X \otimes X$ with the comonoid structure $(X,w,v) \otimes_M (X,w,v)$ and 1 with the trivial comonoid structure.

The most interesting bimonoids are those for which we have a morphism $\mathrm{inv} : X \to X$ such that the first diagram below is commutative (we also assume the commutativity of a totally analogous diagram, obtaining replacing $id \otimes \mathrm{inv}$ by $\mathrm{inv} \otimes id$). These are the *Hopf monoids*. For instance, recall that endowing **Set** with the cartesian monoidal structure (i.e, whose product is the binary product and whose neutral object is a unit set), any object becomes a comonoid in a trivial way, so that any monoid is indeed a bimonoid. Therefore, a Hopf monoid for $(\mathbf{Set},\times)$ is just a monoid X endowed with a map $\mathrm{inv} : X \to X$ making commutative the second diagram below and its analogous. But these commutativity conditions only means that

$$m(x,\mathrm{inv}(x)) = e = m(\mathrm{inv}(x),x)$$

for any x, i.e that each x has an inverse $\mathrm{inv}(x)$. Therefore, a *Hopf monoid into* $(\mathbf{Set},\times)$ *is just a classical group*!

$$\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, "w"'] & 1 \arrow[r, "u"] & X \\
X \otimes X \arrow[rr, "id\otimes inv"'] & & X \otimes X \arrow[u, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, "\Delta"'] & 1 \arrow[r, "u"] & X \\
X \times X \arrow[rr, "id\times inv"'] & & X \times X \arrow[u, "m"']
\end{tikzcd}$$

Remark. In the last chapter the notion of "monoid object into a monoidal category" was obtained as the categorification by enrichment of the usual concept of monoid. Similarly, at the beginning of this chapter, the concept of commutative monoid into a symmetric monoidal category" was obtained as the categorification of "commutative monoid". Notice that when looking to the monoid objects and commutative monoid objects into $(\mathbf{Set}, \times)$ we recovered precisely the starting notions of "monoid" and "commutative monoid". The last paragraph shows that Hopf monoids into $(\mathbf{Set}, \times)$ are just groups. Therefore, this suggest that *the notion of "Hopf monoid" is simply the categorification of the usual notion of "group"*.

Remark. Any monoidal functor maps monoid objects into monoid objects and comonoid objects into comonoid objects[2]. However, it is not true that any monoidal functor maps Hopf monoids into Hopf monoids. The reason is intuitively clear: in order to define Hopf monoids we **need** a structure of **symmetric** monoidal category, but an arbitrary monoidal functor need not to preserve this symmetric structure. With the same argument it is easy to be convinced that *a **symmetric** monoidal functor really maps Hopf monoids into Hopf monoids*.

Examples in Mathematics

Let us see some examples of Hopf monoids.

Example 5.6 (*cartesian/cocartesian Hopf monoids*)**.** The discussion presented above for Hopf monoids into $(\mathbf{Set}, \times)$ extends trivially to any cartesian monoidal category and, by duality, to any cocartesian monoidal category. Indeed, let $\mathbf{C}$ be a cartesian monoidal category. Then any object has a trivial structure of comonoid, so that any monoid becomes a bimonoid in a trivial way. Therefore, a Hopf monoid is just a monoid X together with a morphism $\mathrm{inv} : X \to X$ behaving as a map that "gives inverses to each element of X". Observe that **this only an analogy**, because if $\mathbf{C}$ is not concrete, then its objects are not sets and, therefore, we lose the notion of "elements". However, in the context of concrete categories this works perfectly well. For instance, a Hopf monoid on **Top** or **Diff** is respectively a topological group or a Lie group. The Hopf monoids on $(\mathrm{Ho}(\mathbf{Top}_*), \times)$ are called *H-groups*, while the Hopf monoids on $(\mathrm{Ho}(\mathbf{Top}_*), \vee)$ are the *H-cogroups*.

Example 5.7 (*bicartesian Hopf monoids*)**.** What happens if a category $\mathbf{C}$ has biproducts? Well, in this case both cartesian and cocartesian structures are equivalent, so that any given object has a trivial structure of monoid/comonoid and, therefore, a trivial structure of bimonoid, whose multiplication/comultplication is given by the codiagonal/diagonal. There is a unique morphism $\mathrm{inv} : X \to X$ simultaneously compatible with both Δ and ∇: the identity id_X. Therefore, *in a category with biproducts, any object becomes trivially a Hopf monoid*. This is the case, for instance, of the category $\mathbf{Mod}_R$.

Example 5.8 (*Hopf algebras and graded Hopf algebras*)**.** On the other hand, as discussed previously, the category $\mathbf{Mod}_R$ has a nontrivial closed symmetric monoidal structure given by tensor product $\otimes$. The monoids are the R-algebras and the comonoids are the R-coalgebras. The bimonoids are then R-modules with compatible structures of algebra and coalgebra. These are the *R-bialgebras*. The corresponding Hopf monoid are R-bialgebras whose multiplication is "invertible" in the sense of the last diagrams. A totally analogous situation holds in the more general

[2]Here we have to say that this fact is a consequence of the definition of "monoidal functor" that we given. Indeed, recall that for us a monoidal functor satisfy $F(X \otimes Y) \simeq F(X) \otimes F(Y)$ and $F(1) \simeq 1$. If instead we have only transformations $F(X \otimes Y) \to F(X) \otimes F(Y)$ and $F(1) \to 1$ we say that it is *lax monoidal*, while if there are natural transformations in the opposite direction we say that it is *oplax monoidal*. The general result is that any lax (resp. oplax) monoidal functor maps monoid (resp. comonoid) objects into monoid (res. comonoid) objects. See, for instance, Proposition 3.29 of [8].

monoidal category $G\mathbf{Grad}_R$ of G-graded R-modules. The corresponding Hopf monoids are the *G-graded Hopf algebras.* Concrete examples to have in mind are the following:

1. cohomology of H-spaces. Some generalized cohomology theories have presentations in terms of the algebraic cohomology of (bounded) cochain complexes. More precisely, for some spectra $\mathbb{E}$ defining a cohomology theory $H_{\mathbb{E}} : \mathrm{Ho}(\mathbf{Top}_*) \to \mathbb{Z}\mathbf{Grad}_R$ there is a functor E such that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{Top}_*) \arrow[r, "E"] \arrow[rr, bend right, "H_{\mathbb{E}}"'] & \mathrm{Ho}(\mathbf{Ch}_R^+) \arrow[r, "H"] & \mathbb{Z}\mathbf{Grad}_R \end{tikzcd}$$

   Thanks to Künneth formula, when R is a field, the algebraic cohomology functor $H : \mathrm{Ho}(\mathbf{Ch}_R^+) \to \mathbb{Z}\mathbf{Mod}_R$ becomes monoidal, as argued in Example 4.10. So, if E is also monoidal, then $H_{\mathbb{E}}$ itself is monoidal and, therefore, it maps Hopf monoids into Hopf monoids. In other words, it maps H-groups into $\mathbb{Z}$-graded Hopf algebras. When $\mathbb{E}$ is a ring spectra, much more is true: *the cohomology group of any* ***connected*** *H-space (not necessarily an H-group) is a graded Hopf algebra.* Indeed, notice that under the hypothesis $H_{\mathbb{E}}$ take values into the category of $\mathbb{Z}$-graded algebras, as discussed in Example 4.7. Now, being every $H_{\mathbb{E}}^k$ representable and contravariant, they map monoid objects into comonoid objects, so that if X is an H-space, the comodule structure of each $H^k(X;\mathbb{E})$ couple to the graded algebra structure of $H(X;\mathbb{E})$ producing a structure of graded bialgebra. Because X is **connected**, we have $H^0(X;\mathbb{E}) \simeq \mathbb{K}$. A $\mathbb{Z}$-graded $\mathbb{K}$-algebra $A = \oplus_i A_i$ such that $A_0 \simeq \mathbb{K}$ is also called connected, so that we have been proved that the cohomology of a H-space fits into a connected bialgebra. It happens that *every connected graded bialgebra has an unique extension to a Hopf algebra structure, so that if X is a H-space and $\mathbb{E}$ a ring spectrum, then $H(X;\mathbb{E})$ is indeed a Hopf algebra.* A typical situation occurs when $\mathbb{E}$ is the Eilenberg-Mac Lane spectrum of a field $\mathbb{K}$ (i.e, when $H_{\mathbb{E}}^k$ is the ordinary cohomology with coefficients on $\mathbb{K}$). In this case, the functors E giving the cochain complex presentation is simply that assigns to any topological space X its corresponding singular cocomplex and its monoidal property is given by the so called Eilenberg-Zilber theorem. We notice that, by Milnor's uniqueness theorem, this then holds to any generalized cohomology theory satisfying $H(*,\mathbb{E}) \simeq \mathbb{K}$.

2. Steenrod operations. Despite the generalized cohomology groups of a H-monoid, there is another example of Hopf algebra arising from Algebraic Topology: the algebra of Steenrod operations. This is due to the work [152] of John Milnor. This result and the fact that the cohomology of certain spaces (as the classifying spaces MO, BO and BU) has a Hopf algebra structure allow us to use the theory of Hopf algebra in order to give much more concrete/clear/short proof of important results as Thom theorem on cobordisms and Bott-periodicty. This is the content of the (very well written) Chapter 21 of [146].

3. tensor algebra. Now, let us a give a more concrete examples. We start by recalling that, for a commutative R, the tensor algebra $T(X)$ of any R-module X is a $\mathbb{N}$-graded algebra. We assert that this algebra has an unique natural extension to a Hopf algebra structure. In order to get a coalgebra structure, we need to define the comultiplication $w : T(X) \to T(X) \otimes T(X)$ and the counit $v : T(X) \to R$. We notice that $T(X)^0 \simeq R$ at the same time that $\oplus_{i\geq 1} T(X)^i$ is freely generated by X. Therefore, it is enough to define w on elements

$x \in X$ and on the unity $1 \in R$. The map w must preserve the degree, so that $\deg w(x) = 1$ and $\deg w(1) = 0$, implying

$$w(x) = x \otimes 1 + 1 \otimes x \quad \text{and} \quad w(1) = 1 \otimes 1.$$

On the other hand, recall that $\overline{R} = R \oplus 0 \oplus 0...$ is the neutral object of the monoidal structure $(\mathbb{N}\mathbf{Grad}_R, \otimes)$, so that there is an unique graded morphism $v : T(X) \to \overline{R}$ given by $v(x) = 0$ and $v(1) = 1$. A direct computation shows that w and v makes the tensor algebra $T(X)$ a connected graded bialgebra and, therefore, a Hopf algebra.

4. <u>*universal enveloping algebra*</u>. A Lie algebra $\mathfrak{g}$ is, by definition, **non-associative**. Given an **associative** algebra A, say with multiplication $*$, we can always deform $*$ by defining $[x, y] := x * y - y * x$ in order to get a Lie algebra structure. In concise terms, we have a canonical functor

$$\mathcal{L} : \mathrm{Mon}(\mathbf{Vec}_{\mathbb{K}}; \otimes) \to \mathbf{LieAlg}_{\mathbb{K}}.$$

Not every Lie algebra comes from the deformation of an associative algebra, meaning that F is not essentially surjective and, therefore, the lifting problem below does not admit any solution.

$$\begin{tikzcd}
 & \mathrm{Mon}(\mathbf{Vec}_{\mathbb{K}}; \otimes) \arrow[d, "\mathcal{L}"] \\
\mathbf{LieAlg}_{\mathbb{K}} \arrow[ur, dashed, "U"] \arrow[r, "id"'] & \mathbf{LieAlg}_{\mathbb{K}}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{L} \circ U' \\
id \arrow[ur, Rightarrow, "\xi'"] \arrow[r, Rightarrow, "\xi"'] & \mathcal{L} \circ U \arrow[u, Rightarrow, "\mathcal{L} \circ u"']
\end{tikzcd}$$

Now, recall from the discussion in Sections 1.3 and 2.1, when a lifting problem of functors does not have solution, we can always consider the best left/right approximation, which are the *Kan lifts*. Indeed, these are given by a pair (U, ξ), where

$$U : \mathbf{LieAlg} \to \mathrm{Mon}(\mathbf{Vec}_{\mathbb{K}}; \otimes) \quad \text{and} \quad \xi : id \Rightarrow \mathcal{L} \circ U,$$

such that for any other pair (U', ξ') there exists a unique $u : U \Rightarrow U'$ such that the second diagram above commutes. We say that $U(\mathfrak{g})$ is the *universal enveloping algebra* of the Lie algebra $\mathfrak{g}$. It is always a Hopf algebra. Indeed, a model to $U(\mathfrak{g})$ is given by the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal I generated by all elements

$$x \otimes y - y \otimes x - [x, y].$$

It happens that this ideal is, in some sense, invariant by the comultiplication w of $T(\mathfrak{g})$, so that the quotient $U(\mathfrak{g}) = T(\mathfrak{g})/I$ has an induced structure of connected graded bialgebra and, therefore, of Hopf algebra.

5. <u>*Grassman Algebra*</u>. In Example 5.5 we introduced the Grassman algebra of a R-module as the subalgebra $\Lambda(X)$ of the tensor algebra $T(X)$ which is generated by the graded-commutative elements. By the isomorphism theorem of algebras, we can identify $\Lambda(X)$ as the quotient of $T(X)$ by the ideal generated by all elements $x \otimes y - (-1)^{n \cdot m} y \otimes x$. It happens that this ideal is also invariant by the comultiplication w of $T(X)$, so that the Grassman algebra also have an induced Hopf algebra structure.

Examples on Physics

In the last subsection we presented some examples of Hopf monoids which are specially interesting when studying pure mathematics. But we would like to observe that Hopf monoid objects are not important only for abstract mathematics. Indeed, they are also very important in physics, as clarified by the next examples.

Example 5.9 (*Connes-Kreimer approach to renormalization*)**.** Recall that in Section 3.3 we discussed that there are two ways to describe the interactions between particles: internally and externally. In the internal approach the abstract motions are described by certain singular manifolds (the Feynman graphs). At the singularities some physical quantities can acquire infinity values, so that the Feynman graphs can be divided into two classes: those which have physical meaning and those which do not have. So, in principle, a physical theory describing non-physical graphs cannot be used in order to get predictions. But it is expected that some infinities are not exactly infinities, but only bad definitions of the parameters of the theory. A process that identifies the graphs with "false infinities" and that redefines the parameters in order to get a well defined theory is called a *renormalization scheme*. One approach, due to Connes and Kreimer [46, 47, 48], characterizes the renormalization as given precisely by a Hopf algebra (for an overview, see [165]). This will be briefly discussed in Chapter 15. The construction is extended by making use of abstract Hopf monoid in the final chapter of [8].

Example 5.10 (*3d topological quantum field theories*)**.** As commented previously, quantum theories are monoidal functors $U : \mathbf{Cob}_{p+1} \to \mathbf{Vec}_{\mathbb{C}}$. These functors are natural sources of smooth invariants, so that we are interested in strategies to build them. There is a construction, called *Reshetikhin-Turaev construction*, which builds these functors for $n = 3$ starting with some *modular tensor category*. But the standard examples of these categories arises as the representation theory of quantum groups. These quantum groups, on the other hand, are certain parametrized family of Hopf algebras (see [25, 68]). This will be discussed in Chapter 14.

There is also a third example clarifying the role of Hopf algebras in physics which we would like to explain in more detail. We start by recalling that, as discussed in the last section, the quantum particles have internal degrees of freedom, which allows us to classify them as as bosons or fermions. Furthermore, a system describing both bononic and fermionic particles is generally described by the superalgebra $\mathrm{Super}(\mathcal{H}) = \mathrm{Sym}(\mathcal{H}) \otimes \Lambda(\mathcal{H})$ of some Hilbert space $\mathcal{H}$. Therefore, such a system may have two kinds of infinitesimal symmetry: those acting on bosons/fermions **separately** or those acting on both **simultaneously**.

More precisely, in the canonical context of quantum field theory of particles, as will be discussed in more details in Chapter 14, the functor $U : \mathbf{Cob}_1 \to \mathbf{Vec}_{\mathbb{C}}$ is always induced by a distinguished operator $\hat{H} : \mathcal{H} \to \mathcal{H}$, called the *Hamiltonian operator*. Symmetries are then interpreted as other operators commuting with $\hat{H}$. So, it is natural to suppose that, under the presence of internal degrees of freedom, the symmetries are also described by operators $\hat{O}$ acting on $\mathrm{Super}(\mathcal{H})$ commuting with the Hamiltonian operator. In this context, the two kinds of symmetries are respectively described by operators $\hat{O}$ which are tensor products $\hat{O}_{\mathrm{bos}} \otimes \hat{O}_{\mathrm{ferm}}$ of operators

$$\hat{O}_{\mathrm{bos}} : \mathrm{Sym}(\mathcal{H}) \to \mathrm{Sym}(\mathcal{H}) \quad \text{and} \quad \hat{O}_{\mathrm{ferm}} : \Lambda(\mathcal{H}) \to \Lambda(\mathcal{H}),$$

and by those that cannot be written as such products. This second kind of operators generally mixes the bosonic and fermionic sectors, meaning that they map bosonic/fermionic states into fermionic/bosonic states. The symmetries of the first kind are called *usual symmetries* while those of the second kind are called *supersymmetries*. Now, natural questions are the following:

1. *why consider systems of particles exhibiting supersymmetry rather than usual symmetries?*

2. *what are the most general kind of symmetries that can be considered in particle physics?*

As discussed in Example 3.5, the Standard Model does not predict certain experimental results and, therefore, it need to be considered as an effective theory. The extended theory must have "extended symmetry", meaning that it should have all the symmetries of the Standard Model and (possibly) many others. The symmetries of the Standard Model are described by Lie groups and, therefore, by their Lie algebras. There are essentially two classes of symmetries: those given by Poincaré algebra and those associated with the gauge group of the interaction, meaning that the algebra of symmetries is of the form $\mathrm{Pnc}(4) \oplus \mathfrak{g}$.

One of the fundamental objects in perturbative quantum field theory is the *S-matrix* [207, 148, 215], which we can think as the sum over the Feynman diagrams of the theory. The Feynman diagrams (and, therefore, the S-matrix) comes from the Lagrangian, so that they also are invariant by $\mathrm{Pnc}(4) \oplus \mathfrak{g}$. Because we are trying to do new fundamental physics, the idea is to search for extensions of the algebra $\mathrm{Pnc}(4) \oplus \mathfrak{g}$ which also make invariant the S-matrix and which cannot be decomposed as $\mathrm{Pnc}(4) \oplus \mathfrak{h}$ for some Lie algebra $\mathfrak{h}$. Coleman-Mandula theorem [45] says that such a Lie algebra extension does not exist.

A **possible** way to avoid to the Coleman-Mandula theorem is to search for superalgebra extensions of $\mathrm{Pnc}(4) \oplus \mathfrak{g}$ (instead of usual algebra extensions). The very impressive is that this is not only a possible solution, but it is indeed the **unique** solution! This assertion is motivated by *Deligne's theorem on tensor categories* [54, 55, 163]. Indeed, recall that a system of quantum particles is described by a superalgebra $\mathrm{Super}(\mathcal{H})$ and, therefore, by an object of the symmetric monoidal category $(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)$. Deligne's theorem imply that there exist a Hopf superalgebra G whose category of representations on $\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}$ is equivalent to $\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}$. So, the whole system of quantum particles (including the possible symmetries) is described by representation theory of a superalgebra. Consequently, *supersymmetry is the most general kind of symmetry that can be considered.*

For a very clear discussion on this topic, see [186].

Remark. Deligne's theorem can be understood as a high generalization of the classical Wigner's theorem [208] dating 1930's, which states that a system of quantum particles with group of symmetries G and whose space of states is a Hilbert space $\mathcal{H}$, is totally classified by the irreducible unitary/anti-unitary representations of G into $\mathcal{H}$.

Remark. The previous fact (that supersymmetry is the most general kind o symmetry that can be considered in a system of particles) could be formalized without Deligne's theorem. The idea is the following: by the previous discussion, it is natural to suppose that a system of bosonic/fermionic particles is described by a superalgebra $\mathrm{Super}(\mathcal{H}) = \mathrm{Sym}(\mathcal{H}) \otimes \Lambda(\mathcal{H})$, which is a monoid on $(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)$. Therefore, the collection of symmetries on a such system should have the structure of "group internal to $\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}$", i.e of Hopf monoid on this monoidal category. But the notion of Hopf monoid depends of the **symmetric** monoidal structure and, therefore, of a choice of braidings. So, the different braidings on $(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)$ will produce the different flavors of symmetries that can be considered into a system of bosonic/fermionic particles. In Example 5.5

we showed that there are at least two of these braidings: one trivial, describing usual symmetries, and other nontrivial, whose Hopf monoids are the Hopf superalgebras, describing supersymmetry. The fundamental fact is that *these are the unique braidings*! This follows from a homotopical calculation on a topological space associated to the monoidal category $(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)$, as will be done in Chapter 8. We would like to observe, on the other hand, that Deligne's theorem is more general at least in two aspects:

1. it holds not only for the category of $\mathbb{Z}_2$-graded complex modules, but for a large class of tensor category;

2. it not only classify the possible symmetries of a system of particles, but also the space of states.

Frobenius

We end this section by recalling that a Hopf monoid is an object that has monoid and comonoid structures which are compatible in certain sense. These compatibility conditions are motivated by the usual group structure, but in principle we could considered other different conditions, getting different enriched objects. Another example of usual compatibility conditions are those given by the diagrams below, whose corresponding enriched objects are called *Frobenius objects.*

$$
\begin{tikzcd}
(X\otimes X)\otimes X \arrow[d, "a_{xxx}"'] & X\otimes X \arrow[l, "w\otimes id"'] \arrow[r, "id\otimes w"] \arrow[d, "m"] & X\otimes (X\otimes X) \arrow[d, "a^{-1}_{xxx}"] \\
 & X \arrow[d, "w"] & \\
X\otimes (X\otimes X) \arrow[r, "id\otimes m"'] & X\otimes X & (X\otimes X)\otimes X \arrow[l, "m\otimes id"]
\end{tikzcd}
$$

These kind of objects are also very important in physics. Indeed, as will be discussed in Chapter 14, the sphere $\mathbb{S}^1$ is a Frobenius object in the category $\mathbf{Cob}_2$ of 2-cobordism. Not only this: we will also show that the whole monoidal structure of $\mathbf{Cob}_2$ is generated by $\mathbb{S}^1$, considered as a Frobenius object! Consequently, giving a monoidal functor $U : \mathbf{Cob}_2 \to \mathbf{Vec}_{\mathbb{C}}$ is the same as giving a Frobenius object into $\mathbf{Vec}_{\mathbb{C}}$. In other words, *a topological quantum field theory for strings is exactly the same as a* (*commutative and finite dimensional*) *Frobenius algebra.*

5.3 Spectrum

Since Section 2.3 we have been observed that spetra and (unbounded) cochain complexes are very similar entities. Indeed, both are composed by a sequence of objects X_n connected by structural maps $\sigma_n : X_n \to X_{n+1}$ fulfilling some additional condition. Furthermore, in both cases the morphisms between the entities are sequences $f_n : X_n \to X'_n$ commuting with the structural maps, i.e, such that $\sigma'_n \circ f_n = f_{n+1} \circ \sigma_n$. In other words, while cochain complex are "graded linear spaces" and cochain maps are "linear maps preserving the grading", we can think of a spectrum as some kind of "graded topological space" whose morphisms are "continuous maps preserving the grading".

The similarities does not stop here. Indeed, the condition imposed to the structural maps of the cochain complex allows us to define certain invariants: the *algebraic cohomology groups* $H^k(X_*)$. Similarly, for spectra we have the *stable homotopy groups* $\pi_k^S(\mathbb{X})$. In both cases, the class W of morphisms inducing isomorphisms into each invariant can be used in order to localize the original theory, producing a well behaved homotopy theory.

Here we would like to observe that it is natural to expect the existence of another similarity between the categories $\mathbf{CCh}_R$ and **Spec**. Indeed, as commented above, $\mathbf{CCh}_R$ is some "graded version" of $\mathbf{Mod}_R$. In the category of R-modules we have a canonical noncartesian monoidal structure given by the tensor product $\otimes_R$, whose neutral object is R. This structure is symmetric and closed. Similarly, recall that **Spec** is a "graded version" of $\mathbf{Top}_*$. It happens that this category (or at least a convenient subcategory of topological spaces) also has a canonical noncartesian monoidal structure: the *smash product* $\wedge$, of which the sphere $\mathbb{S}^0$ is the neutral object. This structure is also symmetric and closed.

As explained in Example 4.7 and Example 4.9, $\otimes_R$ induces a corresponding monoidal structure into $\mathbf{CCh}_R$ whose product is R, trivially regarded as a cochain complex. Furthermore, the obtained structure is closed and symmetric at least in two different ways. So, it is natural to expect that, in a totally analogous way, the smash product $\wedge$ also induce a closed symmetric monoidal structure into **Spec** whose neutral object is the sphere $\mathbb{S}^0$, trivially regarded as a spectrum. The natural way to regard a space X as a spectrum is by considering its *suspension spectrum* $\Sigma^\infty X$, whose nth term is $\Sigma^n X$ and whose structural maps are the maps $\Sigma^n X \to \Sigma^{n+1}X$ obtained by applying Σ^n to the canonical embedding $X \to \Sigma X$. Note that $\Sigma^\infty \mathbb{S}^0$ is precisely the *sphere spectrum* $\mathbb{S} = (\mathbb{S}^n)$. Therefore, summarizing, *it is natural to expect the existence of closed symmetric smash product on spectra whose neutral object is the sphere spectrum.*

Recall that at level of cochain complexes, the tensor product was defined (as detailed discussed in Example 4.9) by

$$(X_* \otimes X'_*)_k = \bigoplus_{i+j=k} X_i \otimes X'_j, \quad \text{with differential} \quad D = d \otimes id + id \otimes d'.$$

We notice that $\oplus$ is the coproduct of $\mathbf{Mod}_R$. Therefore, the immediate idea is to mimic this construction, defining the smash product between two spectra $\mathbb{X}$ and $\mathbb{Y}$ as

$$(\mathbb{X} \wedge \mathbb{Y})_k := \bigvee_{i+j=k} X_i \wedge Y_j, \tag{5.3.1}$$

where $\vee$ is the wedge sum (i.e, the coproduct of $\mathbf{Top}_*$). In order to turn this into a genuine spectrum we have to give maps

$$\sigma_k : \Sigma(\mathbb{X} \wedge \mathbb{Y})_k \to (\mathbb{X} \wedge \mathbb{Y})_{k+1}, \quad \text{i.e,} \quad \sigma_k : \bigvee_{i+j=k} \Sigma(X_i \wedge Y_j) \to \bigvee_{i+j=k+1} X_i \wedge Y_j, \tag{5.3.2}$$

where we used that Σ preserve colimits (because it has an adjoint Ω). Notice that the expression

$$D = d \otimes id + id \otimes d'$$

was fixed (up to trivial ambiguities) by the necessity that $\deg(D) = -1$, meaning that the tensor product $X_* \otimes X'_*$ has **canonical** structural maps turning it a cochain complex. This is **not** the case of the product (5.3.1) with respect to the structural maps (5.3.2).

Indeed, in order to fix (5.3.2) we have to led with many non-trivial ambiguities. First of all, noticing that $\Sigma X \simeq \mathbb{S}^1 \wedge X$ and that the smash product is commutative up to homeomorphisms (because it defines a symmetric monoidal structure in $\mathbf{Top}_*$), then the term

$$\Sigma(X_i \wedge Y_j) \quad \text{can be considered as} \quad (X_i \wedge \mathbb{S}^1) \wedge Y_j \quad \text{or as} \quad X_i \wedge (\mathbb{S}^1 \wedge Y_j).$$

In principle, this could be seem only a "first order ambiguity", but notice that it propagates, because the maps σ_l, with $l < k$, also depends of choices like this, revealing that σ_k has not only an ambiguity given by Σ_2 (the permutation group of two elements), but indeed by Σ_k (the permutation group of k-elements), which has $k!$ elements. Therefore, the number of ambiguities to be considered grown very fast when $k \to \infty$.

There are essentially two ways to avoid these ambiguities, as we pass to discuss:

1. insisting in the definition of spectra as a sequence of spaces, as will be done in the next subsection;

2. giving a more well behaved notion of spectra. Indeed, instead insisting with the notion of "spectrum" as a sequence of spaces with maps, we could redefine them in order to incorporate all possible ambiguities appearing in (5.3.2). There are many approaches and the most known are called *symmetric spectra, orthogonal spectra and* $\mathbb{S}$*-modules*. We refer the reader to [187, 101, 64, 142]. A very well written survey is [65]. See also [67, 143]

Lewis's Obstruction

Let us insist in the usual definition of spectra. In this case, in order to get a smash product on **Spec**, instead of considering $\mathbb{X} \wedge \mathbb{Y}$ as in (5.3.1) we could consider a more simple expression, involving not a coproduct of many terms, but a single term. For instance, we could take

$$(\mathbb{X} \wedge \mathbb{Y})_k := X_i \wedge Y_{k-i} \tag{5.3.3}$$

for an arbitrary i. In this case, notice that the ambiguities disappear, because one times fixed $\Sigma(X \wedge Y) \simeq (\Sigma X) \wedge Y$ we have canonical maps

$$\sigma_k : \Sigma(\mathbb{X} \wedge \mathbb{Y})_k \to (\mathbb{X} \wedge \mathbb{Y})_{k+1}, \quad \text{given by} \quad \sigma_k = \sigma_k^X \wedge id_{Y_{k-i}}.$$

This is not the only possibility. Indeed, we could also define

$$(\mathbb{X} \wedge \mathbb{Y})_k := \begin{cases} X_n \wedge Y_n, & k = 2n \\ X_n \wedge \Sigma Y_n, & k = 2n+1, \end{cases} \tag{5.3.4}$$

with structural maps

$$\sigma_{2k} : (\Sigma X_n) \wedge Y_n \to X_n \wedge (\Sigma Y_n) \quad \text{and} \quad \sigma_{2k+1} : (\Sigma X_n) \wedge (\Sigma Y_n) \to X_{n+1} \wedge Y_{n+1},$$

respectively given by the canonical isomorphism

$$\sigma_{2k} : (\Sigma X_n) \wedge Y_n \simeq \Sigma(X_n \wedge Y_n) \simeq X_n \wedge (\Sigma Y_n) \quad \text{and by} \quad \sigma_{2k+1} = \sigma_n^X \wedge \sigma_n^Y.$$

We notice, however, that both definitions of "smash product of spectra" given above produce a well defined (and equivalent) monoidal structure *only on the* ***homotopy category*** *of spectra* Ho(**Spec**). Indeed, in order to be genuine monoidal structure in **Spec** the smash product should satisfy

$$(\mathbb{X}\wedge\mathbb{Y})\wedge\mathbb{Z}\simeq\mathbb{X}\wedge(\mathbb{Y}\wedge\mathbb{Z})\quad\text{and}\quad\mathbb{S}\wedge\mathbb{X}\simeq\mathbb{X}\simeq\mathbb{X}\wedge\mathbb{S}$$

for any spectra. In particular, we would have

$$(\mathbb{S}\wedge\mathbb{S})\wedge\mathbb{S}\simeq\mathbb{S}\wedge(\mathbb{S}\wedge\mathbb{S})\quad\text{and}\quad\mathbb{S}\wedge\mathbb{S}\simeq\mathbb{S}\simeq\mathbb{S}\wedge\mathbb{S}. \tag{5.3.5}$$

Now, remember that the structural maps of the sphere spectrum are given by the homeomorphisms $\sigma_k:\Sigma\mathbb{S}^k\simeq\mathbb{S}^{k+1}$, so that the isomorphism (5.3.5) should be obtained from the limit $k,l,m\to\infty$ of the first commutative diagram below, where $\varphi_{kl}:\mathbb{S}^k\wedge\mathbb{S}^l\simeq\mathbb{S}^{k+l}$. It happens that this diagram is **not** commutative! For instance, if $k,l,m=0$ both constructions produce $\mathbb{S}^0$ with different base points. On the other hand, it is commutative **up to homotopy**, because the final maps of the diagram have the same degree[3].

$$\begin{tikzcd}
(\mathbb{S}^k\wedge\mathbb{S}^l)\wedge\mathbb{S}^m \arrow[rr, "\simeq"] \arrow[d, "\varphi_{kl}\wedge id"'] & & \mathbb{S}^k\wedge(\mathbb{S}^l\wedge\mathbb{S}^m) \arrow[d, "id\wedge\varphi_{lm}"] \\
\mathbb{S}^{k+l}\wedge\mathbb{S}^m \arrow[r, "\varphi_{(k+l)m}"'] & \mathbb{S}^{k+l+m} & \mathbb{S}^k\wedge\mathbb{S}^{l+m} \arrow[l, "\varphi_{k(l+m)}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}^k\wedge\mathbb{S}^l \arrow[r, "b_{kl}"] \arrow[d, "\varphi_{kl}"'] & \mathbb{S}^l\wedge\mathbb{S}^k \arrow[d, "\varphi_{lk}"] \\
\mathbb{S}^{k+l} \arrow[r, "(-1)^{kl}"'] & \mathbb{S}^{l+k}
\end{tikzcd} \tag{5.3.6}$$

The situation here is very similar to the problem with the concatenation of loops. Indeed, recall that, as discussed in Example 4.5, for any X its loop space ΩX becomes equipped with a canonical product $\#:\Omega X\times\Omega X\to\Omega X$ defined in $\mathbf{Top}_*$, which does not define a monoid structure on X, because the product is associative only up to homotopy, but it defines in the homotopy category Ho($\mathbf{Top}_*$).

We would have the same problem if we try to make $\wedge$ a **symmetric** monoidal product. Indeed, the existence of braidings $b_{xy}:\mathbb{X}\wedge\mathbb{Y}\simeq\mathbb{Y}\wedge\mathbb{X}$ would imply, in particular, the existence of braidings $b:\mathbb{S}\wedge\mathbb{S}\simeq\mathbb{S}\wedge\mathbb{S}$, which should be obtained taking the limit $k,l\to\infty$ at maps $(-1)^{kl}:\mathbb{S}^{k+l}\to\mathbb{S}^{l+k}$ given by reverting the coordinates, meaning that the second diagram above should be commutative. But it is commutative only up to homotopy.

Remark. A complete prove that the above products really introduce equivalent symmetric monoidal structure Ho(**Spec**) is long and boring. Details can be founded in part III of [4] and in Section 2.2 of [137]. See also Section 2.2 of [214] and Chapter 8 of [200]. An important consequence is that the homotopy category of spectra is additive, meaning that for any two spectra $\mathbb{X}$ and

[3] Let X and Y be spaces such that $H^n(X;\mathbb{Z})\simeq\mathbb{Z}\simeq H^n(Y;\mathbb{Z})$ for some n. Let 1_x and 1_y be the respective generators of the cohomology groups under these isomorphisms. Recall that the nth degree of a map $f:X\to Y$ is the number $\deg(f)$ such that $f^*(1_y)=\deg(f)\cdot 1_x$. It is clearly a homotopy invariant, i.e, homotopic maps have the same nth degree. Hopf theorem asserts the reciprocal when $Y=\mathbb{S}^n$ and X is (homotopic to) a compact oriented manifold. The original article is [98]. For expositions, see [??,??]. We notice that the result can be understood as a direct consequence of the Thom-Pontryagin theorem for cobordisms with framing. The fundamental step of ...... In other words,

$$[M,\mathbb{S}^n]\simeq\operatorname{Iso}(\mathbf{Cob}_0^{\mathrm{fram}})\simeq\pi_0^S(\mathbb{S})=\lim_{k\to\infty}\pi_k(\mathbb{S}^k)\simeq\lim_{k\to\infty}\mathbb{Z}\simeq\mathbb{Z},$$

where the integer corresponding to a homotopy class $[f]$ is just its degree.

$\mathbb{Y}$, the set of morphisms $\mathrm{Mor}_{\mathrm{Ho}(\mathbf{Spec})}(\mathbb{X};\mathbb{Y})$ has a structure of abelian group. Furthermore, the smash product is bilinear with respect to these structures.

We could think of the non-definition of a genuine monoidal structure in **Spec** as pathologies of the products (5.3.3) and (5.3.4), and then asking: *is there some symmetric monoidal structure in* **Spec** *whose neutral object is* $\mathbb{S}$*?* By a result usually called as *Lewis's obstruction theorem*, such a monoidal structure (fulfilling some very natural condition) does not exist!

In order to state Lewis's theorem in a more precise formulation, we notice that in general the functors Σ and Ω are **not** monoidal with respect to the smash product of spaces, because we have

$$\Sigma(X\wedge Y)\simeq(\Sigma X)\wedge Y \quad \text{instead of} \quad \Sigma(X\wedge Y)\simeq(\Sigma X)\wedge(\Sigma Y),$$

and similarly for Ω. On the other hand, recall that for any Y we have a canonical maps $Y\to\Sigma Y$ and $\Omega Y\to Y$. Therefore, composing these maps with the equivalences above we get maps

$$\Sigma(X\wedge Y)\to(\Sigma X)\wedge(\Sigma Y) \quad \text{and} \quad (\Omega X)\wedge(\Omega Y)\to\Omega(X\wedge Y),$$

revealing that Σ and Ω are, respectively, lax monoidal and colax monoidal. Iterating Σ and Ω and taking the limit $n\to\infty$ we then expect to get lax/oplax monoidal functors in the category of spectra. In other words, it is natural to expect that the correct "smash product of spectra" becomes equipped with lax/oplax monoidal functors

$$\Sigma^\infty:(\mathbf{Top}_*,\wedge)\to(\mathbf{Spec},\wedge) \quad \text{and} \quad \Omega^\infty:(\mathbf{Spec},\wedge)\to(\mathbf{Top}_*,\wedge).$$

With this in mind we can state Lewis's theorem [123][4]:

Theorem 5.1 (Lewis's obstruction theorem)**.** *There is no subcategory $\mathcal{S}$ of* **Spec** *in which the following conditions hold simultaneously:*

1. *it becomes endowed with a closed symmetric monoidal structure;*
2. *whose neutral element object is the sphere spectrum $\mathbb{S}$;*
3. *for which we have a lax/oplax adjunction $\Sigma^\infty\rightleftharpoons\Omega^\infty$;*
4. *such that $\Omega^\infty\Sigma^\infty X\simeq\mathrm{colim}\Omega^n\Sigma^n X$ for any space X .*

This theorem could be seen very surprisingly, but it is indeed very natural. In fact, notice that the homotopy commutativity of diagrams (5.7) is not a property of any "smash product of spectra", but indeed of the spheres (and, therefore, of the sphere spectrum). It happens that, together the conditions (3.) and (4.) above imply that, if $(\mathcal{S},\wedge,\mathbb{S})$ is the the symmetric monoidal structure ensured by conditions (1.) and (2.), then the action of $\wedge$ on $\mathbb{S}$ must be given by the limit of the diagrams (5.7), which are commutative only up to homotopy. Following [123], let us give a formal proof. For a discussion on the consequences of the theorem, see [66].

[4] We enunciate the result somewhat differently from the one presented in the original article.

Proof. Suppose that such a category exists. Then $\mathbb{S} = \Sigma^\infty \mathbb{S}^0$ is a commutative monoid object into this structure. As a consequence, $\Omega^\infty \mathbb{S}$ should be a commutative topological monoid. It is a classical result of Topology that any path connected commutative topological monoid is weakly homotopy equivalent to a product of Eilenberg-Mac Lan spaces. Therefore, the path component P of the identity in $\Omega^\infty \Sigma^\infty \mathbb{S}^0$ should be equivalent to a product of Eilenberg-Mac Lane spaces, which is false. Indeed, because by (.4)

$$\begin{aligned} \pi_{k\geq 1}(X) &\simeq \pi_k(\Omega^\infty \Sigma^\infty \mathbb{S}^0) \\ &\simeq \pi_k(\mathrm{colim}\Omega^n \Sigma^n \mathbb{S}^0) \\ &\simeq \mathrm{colim}\pi_k(\Omega^n \mathbb{S}^n) \\ &\simeq \pi_k^S(\mathbb{S}), \end{aligned}$$

i.e, the homotopy groups of P are the stable homotopy groups of spheres, which generally have different structure for different values of k. On the other hand, if $\Pi_i K(G_i, i)$ is a product of Eilenberg-Mac Lane spaces, then

$$\begin{aligned} \pi_k(\Pi_i K(G_i, i)) &\simeq \Pi_i \pi_k(K(G_i, i)) \\ &\simeq \Pi_i G_i \end{aligned}$$

is a product of the same abelian monoids for any k. □

Multiplicative Cohomology

Despite the nonexistence of a nice smash product on **Spec**, we have a well defined symmetric monoidal structure on Ho(**Spec**), so that we can study monoid objects there. They are called *ring spectra*, while their commutative version are called *commutative ring spectra*.

Therefore, intrinsically a (commutative) ring spectrum is a spectrum $\mathbb{E}$ endowed with spectra morphisms $m : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$ and $u : \mathbb{S} \to \mathbb{E}$ fulfilling (commutative) monoid-like diagrams up to homotopy. But, in order to explicit this definition, we need to select a model to the smash product $\wedge$ in Ho(**Spec**). For instance, if we select the model (5.3.3), then a ring spectrum is represented by maps

$$m_k : E_i \wedge E_{k-i} \to E_k, \quad \text{and} \quad u_k : \mathbb{S}^k \to E_k$$

for some i, fulfilling the homotopy-commutativity conditions. On the other hand, if we choose the model (5.3.4), then a ring spectrum pass to be represented by the following maps, satisfying the required commutativity conditions:

$$\begin{cases} m_k : E_n \wedge E_n \to E_k, & \text{if } k = 2n \\ m_k : E_n \wedge \Sigma E_n \to E_k, & \text{if } k = 2n+1 \end{cases} \quad \text{and} \quad u_k : \mathbb{S}^k \to E_k.$$

Let us see some examples.

Example 5.11 (*sphere*)**.** The sphere spectrum is, by construction, the fundamental example of commutative ring spectrum. The multiplication $m_k : \mathbb{S}^i \wedge \mathbb{S}^{k-i} \to \mathbb{S}^k$ is the canonical homeomorphism $\mathbb{S}^i \wedge \mathbb{S}^{k-i} \simeq \mathbb{S}^k$ and the unit $u_k : \mathbb{S}^k \to \mathbb{S}^k$ is the identity map. The commutative conditions up to homotopy are given by diagrams (5.7). More concisely, the sphere spectrum is a commutative ring spectrum because it is the neutral object of a symmetric monoidal category.

Example 5.12 (*Eilenberg-Mac Lane*)**.** Let G be a discrete abelian group. Then we have the corresponding Eilenberg-Mac Lane spectrum $K(G,n) \simeq B^n G$, introduced in Example 1.11. This spectrum is a ring spectrum. Indeed, we have canonical maps

$$m_{k+l} : K(G,k) \wedge K(G,l) \to K(G,k+l) \quad \text{and} \quad u_k : \mathbb{S}^k \to K(G,k).$$

In order to get the first, recall that B preserve products and that we have a projection map $\pi : G \to BG$ (and, therefore, a projection map $\pi_k : G \to B^k G$), so that $* : G \times G \to G$ induces the diagram below.

$$B^k G \times B^l G \xrightarrow{B^k \pi_l \times B^l \pi_k} B^k(B^l G) \times B^l(B^k G) \xrightarrow{\simeq} B^{k+l}(G \times G) \xrightarrow{B^{k+l}*} B^{k+l} G$$

We notice that the projection $\pi \times \pi : G \times G \to BG \times BG$ maps each par (e,g) or (g,e) into the same point, so that it pass to the quotient, being defined on $G \wedge G$. Consequently, the composition above becomes defined on $B^k G \wedge B^l G$, giving the required map m_{k+l}. The second map u_k, on the other hand, comes from the image of the constant map at the neutral element $e \in G$ from the following isomorphism:

$$[\mathbb{S}^k; K(G,k)] \simeq [\mathbb{S}^0; \Omega^k B^k G] \simeq [\mathbb{S}^0; G].$$

Example 5.13 (*Bott*)**.** Recall the complex K-theory spectrum $\mathbb{KU}$, given by $(\mathbb{KU})_n \simeq \Omega^n BU$. We assert that this spectrum is a ring spectrum. Indeed, recall the Bott periodicity, which states that this spectrum is periodic of period equal to 2. Consequently, the only nonequivalent terms are $(\mathbb{KU})_0 \simeq BU \times \mathbb{Z}$ and $(\mathbb{KU})_1 \simeq U$, so that by making use of the model (5.3.3) we have to build maps

$$\begin{aligned} m : (BU \times \mathbb{Z}) \wedge (BU \times \mathbb{Z}) &\to BU \times \mathbb{Z} \\ m' : (BU \times \mathbb{Z}) \wedge U &\to BU \times \mathbb{Z} \\ m'' : U \wedge U &\to BU \times \mathbb{Z}, \end{aligned} \tag{5.3.7}$$

corresponding to the multiplication, and maps

$$u_0 : \mathbb{S}^0 \to BU \times \mathbb{Z} \quad \text{and} \quad u_1 : \mathbb{S}^1 \to U,$$

describing the unit. Recall that $KU^1(\mathbb{S}^1) \simeq \mathbb{Z}$ and, by Bott periodicity,

$$[\mathbb{S}^0; BU \times \mathbb{Z}] \simeq [\mathbb{S}^2; BU \times \mathbb{Z}] = KU^0(\mathbb{S}^2) \simeq \mathbb{Z},$$

so that we can define u_1 and u_0 as the maps arising from the unit of $\mathbb{Z}$ from these isomorphisms. We outline the construction of (5.3.7). We start by noticing that we have canonical applications $\alpha_{kl} : U(k) \times U(l) \to U(kl)$, given by the tensor product of matrices. Applying B, taking the colimit and $- \times \mathbb{Z}$, we get

$$(BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \to BU \times \mathbb{Z},$$

which pass to the quotient, defining the map m. In order to define m'' we proceed similarly: taking the colimit of α_{kl} we get $U \times U \to U$; composing with the projecting map $U \to BU$ and with $BU \hookrightarrow BU \times \mathbb{Z}$ we obtain m''.

Example 5.14 (*Thom*)**.** As introduced at the last subsection of Section 1.2, we another very usual spectrum, the *Thom spectrum*. We recall that $\mathbb{MO}_n$ is the Thom space $T(V(n))$ of the canonical n-vector bundle $V(n)$. Let us see that it is a ring spectrum. We need to build maps

$$m_{k+l} : \mathbb{MO}_k \wedge \mathbb{MO}_l \to \mathbb{MO}_{k+l} \quad \text{and} \quad u_k : \mathbb{S}^k \to \mathbb{MO}_k.$$

The multiplications can be obtained from the direct sum of matrices

$$O(k) \times O(l) \to O(k+l)$$

by noticing that this induce a corresponding bundle morphism is $b : V(k) \oplus V(l) \to V(k+l)$ and that the Thom space construction maps direct sum into smash product:

$$T(V(k)) \wedge T(V(l)) \xrightarrow{\simeq} T(V(k) \oplus V(l)) \xrightarrow{T(b)} T(V(k+l))$$

In order to define u_k, recall that we can define Thom spectrum for other sequence of groups rather than $O(k)$. For instance, we can do this for the sequence in which all groups are trivial, whose corresponding corresponding Thom spectrum is just the sphere spectrum, i.e, $(\mathbb{M}*)_k \simeq \mathbb{S}^k$. It happens that, because we have a canonical map $* \to O(k)$ for each k, we get an induced map between the corresponding Thom spectra, which we identify with $u_k : \mathbb{S}^k \to \mathbb{MO}_k$.

In some parts of the text we used that singular cohomology, complex K-theory and cobordism are graded rings, meaning that we know how to multiply classes in these cohomology theories. Now we can explain why this happens: *it is exactly because the underlying spectra, as described in the examples above, are given by ring spectra*. Indeed, we have the following:

Proposition 5.1. *The cohomology theory represented by a Ω-spectrum $\mathbb{E}$ is such that $H(X;\mathbb{E})$ has a graded ring structure natural in X iff $\mathbb{E}$ is a ring spectrum. Furthermore, the graded ring is graded commutative iff $\mathbb{E}$ is commutative.*

Proof. Given a space X and a natural $k \in \mathbb{N}$, let $\Sigma^{\infty-k}X$ be the *k-reduced suspension spectrum* of X, defined by

$$(\Sigma^{\infty-k}X)_n = \begin{cases} \Sigma^{n-k}X, & \text{if } n \geq k \\ *, & \text{otherwise,} \end{cases}$$

with the obvious structural maps. Notice that if X is given by $E_k \simeq \Sigma^k\Omega^\infty\mathbb{E}$ for some spectrum $\mathbb{E}$, then $\Sigma^{\infty-k}X \simeq E_k$. This rule rule is actually a functor $\Sigma^{\infty-k} : \mathrm{Ho}(\mathbf{Top}_*) \to \mathrm{Ho}(\mathbf{Spec})$. Given spaces X, Y and maps $f : X \to E_k$ and $g : Y \to E_l$ we get the following morphism

$$\alpha^{kl}_{xy} : [X, E_k] \times [Y, E_l] \to [X \times Y; E_k \wedge E_l], \quad \text{with} \quad \alpha^{kl}_{xy}(f,g) = \Sigma^{k+l}(\Omega^\infty(f^{\infty-k} \wedge f^{\infty-l})). \tag{5.3.8}$$

Because $\mathbb{E}$ is a Ω-spectrum, its cohomology groups are abelian, meaning that each set above, appearing in the domain and in the codomain of α_{xy}, is an abelian group. Recall that, despite we have not discussed here, the homotopy category of spectra is additive, i.e, the space of morphisms between any two spectra is an abelian group, and the smash product is bilinear. Therefore, we see from (5.3.8) that α_{xy} is bilinear, so that it extends to a linear map defined on the tensor product. Now, let us suppose that $\mathbb{E}$ is a ring spectrum with multiplication $m : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$. So, for any given space X, by making use of (5.3.3), from m_{k+l} and α_{xx}^{kl} the diagonal map $\Delta : X \times X \to X$ we get induced bilinear maps

$$[X, E_k] \otimes [X, E_l] \longrightarrow [X \times X, E_k \wedge E_l] \longrightarrow [X \times X, E_{k+l}] \longrightarrow [X, E_{k+l}],$$

which gives the structure of graded ring to $H(X; \mathbb{E})$. Certainly, if $\mathbb{E}$ is commutative, then the corresponding graded ring is graded commutative. The multiplicative neutral element comes from the unit $u : \mathbb{S} \to \mathbb{E}$ of $\mathbb{E}$. Indeed, because $\mathbb{S}$ is ring spectrum, $H(X; \mathbb{S})$ is a graded ring and u induces a graded morphism $u^* : H(X; \mathbb{S}) \to H(X; \mathbb{E})$. Therefore, the multiplicative unit of $H(X; \mathbb{E})$ is just the image under this morphism of the unit of $H(X; \mathbb{S}) = \oplus_i [X, \mathbb{S}^i]$, which is given by $c \oplus 0 \oplus ... \oplus 0$, where $c \in [X, \mathbb{S}^0]$ is the class of the constant map at base point of $\mathbb{S}^0$. This proves that the cohomology of ring Ω-spectrum has the structure of graded unital ring. The reciprocal is a consequence of the so called Brown representability theorem. See [??]. $\square$

Chapter 6

Abstract Categories

In the last two chapters we applied the enrichment/internalization process to very simple concepts (as the concepts of monoid and commutative monoid) and we concluded that the resulting notions are very abstract and useful, meaning that different classical concepts are now unified into a unique categorified concept. The present chapter is about categorification of **languages**. More precisely, in the first section we apply enrichment and internalization to the notion of category, producing the concepts of *enriched categories* and *internal categories* defined on some ambient $\mathbf{H}$. We will also enrich/internalize functors, getting *enriched functors* and *internal functors*. Together, both notions fit into categories $\mathbf{Cat}(\mathbf{H})$ and $\mathbf{Cat}_{\mathbf{H}}$ describing $\mathbf{H}$-*enriched categorical language* and $\mathbf{H}$-*internal categorical language*.

Recall that the categorification process was naively introduced in Section 2.3 as some kind of way to pass from classical logic to categorical logic, so that iterating the process we would get more and more abstract languages. Observe that this is an inductive process and, by the above discussion, we have two ways to do the "induction step": by internalizing and by enriching. So, we can ask: which is the "correct" step?

The requirement of new abstract languages comes from the axiomatization problem of physics, so that by "correct language" we mean that they can be applied to this problem. With this in mind, in Section 6.3 we show that *enriched category theory is the correct background language*. Indeed, we start by showing that many classical results are no longer generalized in the internalization context. More precisely, we show that the axiom of choice admits a purely categorical characterization, so that we can analyze its validity in any category $\mathbf{H}$. The fact is that it generally fails and, consequently, no results proven using it can be directly internalized on $\mathbf{H}$. On the other hand, we show that the failure of the axiom of choice does not affect the enriched categorical language, at the same time that the notions of limit, Kan extensions, etc, admit a natural enriched version.

Second 6.2 clarify that, despite the above assertion, internal language is also useful in physics. Indeed, there we see that internal language plays a central role in the description of the configuration space of classical theories of **particles**. We also discuss that configurations spaces of gauge theories for **strings** cannot be described by internal language, but it could be by "higher internal language". More precisely, we give a more concrete justification to the naive idea that "string theory" is "categorified particle theory", presented in Section 2.4. We also conjecture the existence of some kind of "categorified Lie theory" and we present two examples of what could be their role in physics.

6.1 Categorifying

In this section we will talk about *categorification of the concept of category.* As in the previous chapters, we start by applying the internalization process. The first step is then to give a totally categorical characterization of the concept of category. Indeed, as will be explained now, the notion of category admits two of these characterizations.

We start by observing that a category can be understood as a "monoid with many objects". More precisely, if $\mathbf{C}$ is a category with only one object $*$, then its set of morphisms $\mathrm{Mor}_{\mathbf{C}}(*,*)$ has a structure of monoid whose multiplication is just the composition operation and whose neutral element is determined by the identity map $id_* : * \to *$. Reciprocally, for any given monoid X we can define a corresponding category $\mathbf{B}X$ which has only one object $*$ and whose set of morphisms $\mathrm{Mor}_{\mathbf{C}}(*,*)$ is just X. Evidently, this correspondence extends to an equivalence $\mathrm{Mon} \simeq *_{\mathbf{Cat}}$ between the category Mon of monoids and the category $*_{\mathbf{Cat}}$ of all categories with only one object.

Therefore an arbitrary category really can be understood as a "monoid with many objects". But monoids admit a totally categorical characterization, so that it is expected the same for arbitrary categories. It happens that there are **two ways** to see a category as a "monoid with many objects", which will imply the existence of two categorical characterizations to the concept of category.

The idea is the following: in the equivalence $\mathrm{Mon} \simeq *_{\mathbf{Cat}}$ we have only one object, so that all morphisms necessarily belong to the same set. In other words, *when we have only one object we do not need to specify the source and the target of morphisms.* But when we have arbitrary objects, source/target information is relevant and we need to incorporate it. This can be done in two different ways: dividing the morphisms into sets $\mathrm{Mor}_{\mathbf{C}}(X;Y)$ parametrized by their source and target (as done in our formal definition of categories at the beginning of Chapter 1), or combining all morphisms into a single set $\mathrm{Mor}(\mathbf{C})$ and adding new functions s and t responsible for specifies the source and target of each morphism.

In order to distinguish these situations, we will say that in the first case we have a *category with hom-sets*, while in the second we have a *category with source/target.* Both approaches will produce a complete categorical characterization of the notion of category which reproduce the characterization of the monoids in the "one object limit". This will allow us to internalize and enrich the notion of category into other categories, producing abstract notions as *internal categories* and *enriched categories.*

Remark. Exactly as associators/unitors/braidings are part of the data defining a monoidal structure, the choice of a categorical characterization is part of the data defining internalization/enrichment. This means that the **same** concept having **different** categorical characterizations can be categorified in different ways, producing **different** abstract concepts. As we have been discussing, *the concept of category admits two different characterizations, so that it is natural to expect that each of them will produce different categorified concepts.* This is really the case, as will become clear in the next subsections.

Categories With Hom-Sets

Let us start by analyzing the first approach, i.e, let us apply internalization/enrichment into the concept of "category with hom-sets". In order to get the categorical characterization, notice that a category with hom-sets is composed by the following data: a set $\mathrm{Ob}(\mathbf{C})$ of objects, for any two objects a corresponding set of morphisms, for any three objects an associative composition $\circ_{xyz}$ and for any object X a distinguished morphism $id_X : X \to X$ satisfying $id_X \circ f = f$ and $g \circ id_X = g$. The associativity of compositions can be translated in terms of commutative diagrams involving binary products in **Set**, as presented below[1].

$$\begin{tikzcd}
(X,Y)\times((Y,Z)\times(Z,W)) \arrow[r, "\simeq"] \arrow[d, "id\times\circ_{yzw}"'] & ((X,Y)\times(Y,Z))\times(Z,W) \arrow[r, "\circ_{xyz}\times id"] & (X,Z)\times(Z,W) \arrow[d, "\circ_{xzw}"] \\
(X,Y)\times(Y,W) \arrow[rr, "\circ_{xyw}"'] & & (X,W)
\end{tikzcd}$$

On the other hand, the identities (which are distinguished endomorphisms) can be understood as functions $id_X : 1 \to \mathrm{Mor}_{\mathbf{C}}(X,X)$, where 1 is a terminal object in **Set**, satisfying the commutativity conditions presented below.

$$\begin{tikzcd}
1\times(X,Y) \arrow[r, "id_X\times id"] \arrow[dr, "\simeq"'] & (X,X)\times(X,Y) \arrow[d, "\circ_{xxy}"] \\
& (X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,Y)\times(Y,Y) \arrow[d, "\circ_{xyy}"'] & (X,Y)\times 1 \arrow[l, "id\times id_Y"'] \arrow[dl, "\simeq"] \\
(X,Y) &
\end{tikzcd}$$

So, any category with hom-sets really can be characterized by categorical information (objects, morphisms, binary products and terminal object) satisfying additional commutativity conditions and, therefore, internalization and enrichment applies.

- *internalization*. By definition, we can **internalize** a concept in any category having **the same** categorical information used to characterizes it. So, it seems that we can internalize the notion of category with hom-sets in **any** category with binary products and terminal object. We would like to observe that **this is not the case**. Indeed, when listing the defining data of a category with hom-sets we have to consider entities parametrized by the set of objects. In fact, we have to consider "for any two objects a set of morphisms", "for any three objects a notion of composition", and so on. Therefore, the concept of **element** is also part of the data characterizing categories with hom-sets. Consequently, to internalize them into a category **H**, such a category really needs to have binary products and terminal object, but we also need that its objects belong to a set (possibly endowed with further structure, of course). In other words, *the notion of category with hom-sets can be internalized only into the* ***concrete*** *categories with binary products and terminal objects*. Therefore, internal categories with hom-sets are not very abstract objects, but only usual categories with some further properties. It is for this reason that this notion is not useful in the literature.

- *enrichment*. Observe, on the other hand, that the set of objects does not appear explicitly in the previous diagram, so that when applying the enrichment process[2] to the notion of

[1] For simplicity, we have written (X,Y) instead of $\mathrm{Mor}_{\mathbf{C}}(X,Y)$.
[2] Indeed, recall that enrichment is about internalization of diagrams.

category with hom-sets we will not find the problems described above. Specially, *we can enrich the notion of category with hom-sets over any monoidal category.* In explicit terms, if $(\mathbf{H}, \otimes, 1)$ is monoidal, then a category with hom-sets enriched over $\mathbf{H}$ is an entity $\mathbf{C}$ given by the following data:

1. a **set** of objects $\mathrm{Ob}(\mathbf{C})$;
2. for any two elements of $\mathrm{Ob}(\mathbf{C})$ a corresponding object $\mathbf{H}_{\mathbf{C}}(X, Y)$ of $\mathbf{H}$;
3. for any three elements a morphism in $\mathbf{H}$ abstracting the compositions
$$\circ_{xyz} : \mathbf{H}_{\mathbf{C}}(X, Y) \otimes \mathbf{H}_{\mathbf{C}}(Y, Z) \to \mathbf{H}_{\mathbf{C}}(X, Z);$$
4. for any element a morphism $id_x : 1 \to \mathbf{H}_{\mathbf{C}}(X, X)$ presenting the identities, such that the following analogues of the previous diagrams are commutative[3]:

$$\begin{tikzcd}
(X,Y)\otimes((Y,Z)\otimes(Z,W)) \arrow[r, "\simeq"] \arrow[d, "id\otimes\circ_{yzw}"'] & ((X,Y)\otimes(Y,Z))\otimes(Z,W) \arrow[r, "\circ_{xyz}\otimes id"] & (X,Z)\otimes(Z,W) \arrow[d, "\circ_{xzw}"] \\
(X,Y)\otimes(Y,W) \arrow[rr, "\circ_{xyw}"'] & & (X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes(X,Y) \arrow[r, "id_X\otimes id"] \arrow[dr, "\simeq"'] & (X,X)\otimes(X,Y) \arrow[d, "\circ_{xxy}"] \\
& (X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,Y)\otimes(Y,Y) \arrow[d, "\circ_{xyy}"'] & (X,Y)\otimes 1 \arrow[l, "id\otimes id_Y"'] \arrow[dl, "\simeq"] \\
(X,Y) &
\end{tikzcd}$$

Remark. Notice that when $\mathbf{H}$ is a concrete category with binary products and terminal objects, then the concept of "category internalized into $\mathbf{H}$" almost coincides with the concept of "category enriched over the cartesian monoidal category $(\mathbf{H}, \times, 1)$", explaining why for categories with hom-sets the enrichment process is much more fruitful.

Now, exactly as in Section 4.1 "homomorphism between monoids" could be enriched giving the notion of "morphisms between monoid objects into a monoidal category", we can give a totally categorical characterization to the usual notion of "functor between categories with hom-sets" in such a way that it can be enriched producing the concept of "enriched functor between enriched categories". Indeed, if $\mathbf{C}$ and $\mathbf{D}$ are categories with hom-sets enriched over the same monoidal category $(\mathbf{H}, \otimes, 1)$, then a *enriched functor* between them is a rule $F : \mathbf{C} \to \mathbf{D}$ given by the following data:

1. a function $F : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$;

2. for any two objects X and Y, a morphism of $\mathbf{H}$
$$F_{xy} : \mathbf{H}_{\mathbf{C}}(X, Y) \to \mathbf{H}_{\mathbf{D}}(F(X), F(Y)),$$

[3] Again we have written (X, Y) instead of $\mathbf{H}_{\mathbf{C}}(X, Y)$. Furthermore, the equivalences $\simeq$ in the first/second diagram are given by associators/unitors of the monoidal category $\mathbf{H}$.

such that the following diagrams[4] (describing the preservation of compositions and identities) are commutative.

$$\begin{tikzcd}
(X,Y)\otimes(Y,Z) \arrow[r, "\circ_{xyz}"] \arrow[d, "F_{xy}\otimes F_{yz}"'] & (X,Z) \arrow[d, "F_{xz}"] \\
(F(X),F(Y))\otimes(F(Y),F(Z)) \arrow[r, "\circ_{F(x)F(y)F(z)}"'] & (F(X),F(Z))
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "id_x"'] \arrow[dr, "id_{F(x)}"] & \\
(X,X) \arrow[rr, "F_{xx}"'] & & (F(X),F(X))
\end{tikzcd}$$

For any monoidal category $\mathbf{H}$ we can then define the corresponding category $\mathbf{Cat}(\mathbf{H})$ of categories with hom-sets enriched over $\mathbf{H}$ and enriched functors between them. Recall that any category defines an area of math. If $\mathbf{Cat}$ describes category theory, then $\mathbf{Cat}(\mathbf{H})$ describes $\mathbf{H}$-*enriched category theory*. We may ask if this new theory is really abstract. For instance, we can ask if the very powerful categorical notions of natural transformations, limits, ends, etc., also have enriched versions. Fortunately, thanks to the works of Kelly, Eilengerg, Day and others, all these concepts actually can be enriched, making enriched categorical language a very abstract language! This fact will become more clear is the next section. See [52, 62, 59, 110, 172].

Categories With Source/Target

Now, let us analyze the other characterization of the notion of category, given by *categories with source/target*. Here, instead of using families of hom-sets, compositions and identities parametrized by the set of objects, the idea is to consider a single set $\mathrm{Mor}(\mathbf{C})$ of all morphisms, a single function $\circ$ representing all compositions (in the sense that $\circ(f,g) = f\circ g$) and a single function id representing all identities (i.e, such that $id(X) = id_X$), together with two functions $s,t : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$ assigning to any morphism f corresponding objects $s(f)$ and $t(f)$ which will be interpreted as their source and target. For instance, it is assumed that

$$s(id(X)) = X = t(id(X)), \quad s(g\circ f) = s(f) \quad \text{and} \quad t(g\circ f) = t(g). \tag{6.1.1}$$

However, when working with this approach we have to be careful. In fact, surely the composition map $\circ(f,g) = f\circ g$ takes values in the set of morphisms, but which is their domain? Maybe the reader would assume that the the composition map is defined on the product $\mathrm{Mor}(\mathbf{C})\times\mathrm{Mor}(\mathbf{C})$. This is **not** generally true. Indeed, if it would be so, then the composition between **any two** given morphisms should be well defined, but this is **not** the case: the composition $g\circ f$ can be done only if the source of g coincides with the target of f. In other words, only if $s(g) = t(f)$. Therefore, $\circ$ is actually defined on the subset of the product $\mathrm{Mor}(\mathbf{C})\times\mathrm{Mor}(\mathbf{C})$ in which $s(\pi_1(f,g)) = t(\pi_2(f,g))$. But this is just the pullback between s and t!

Finally, we notice that the associativity of the compositions and the "neutral element property" of the identities correspond to the commutativity of the diagrams below, where the segmented arrows come from the universality of pullbacks. On the other hand, it is easy to see that conditions (6.1.1) also correspond to commutative diagrams.

$$\begin{tikzcd}
\mathrm{pb}(\mathrm{pb}(s,t),s) \arrow[r, dashed, "\simeq"] \arrow[d, dashed] & \mathrm{pb}(\mathrm{pb}(s,t),t) \arrow[r, dashed] & \mathrm{pb}(s,t) \arrow[d, "\circ"] \\
\mathrm{pb}(s,t) \arrow[rr, "\circ"'] & & \mathrm{Mor}(\mathbf{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pb}(s\circ id,t) \arrow[r, dashed] \arrow[dr, "\pi_1"'] & \mathrm{pb}(s,t) \arrow[d, "\circ"] & \mathrm{pb}(s,t\circ id) \arrow[l, dashed] \arrow[dl, "\pi_2"] \\
& \mathrm{Mor}(\mathbf{C}) &
\end{tikzcd}$$

[4]As usually we have written (X,Y) instead of $\mathbf{H}_{\mathbf{C}}(X,Y)$.

The discussion above reveals that categories with source/target also have a characterization in terms of purely categorical data (objects, morphisms and pullbacks) satisfying commutativity conditions. Consequently, internalization and enrichment apply.

- *internalization*. If $\mathbf{H}$ has pullbacks, then a *category with source/target internal to* $\mathbf{H}$ is an entity $\mathbf{C}$ defined by the following data:

  1. objects $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ of $\mathbf{H}$, also respectively denoted by $\mathbf{C}_0$ and $\mathbf{C}_1$;
  2. source and target maps, represented by morphisms $s, t : \mathbf{C}_1 \to \mathbf{C}_0$ in $\mathbf{H}$;
  3. morphisms $id : \mathbf{C}_0 \to \mathbf{C}_1$ and $\circ : \mathrm{pb}(s,t) \to \mathbf{C}_1$ representing the identity maps and the compositions, which are compatible with the source/target maps in the sense that the diagrams below (corresponding to the conditions (6.1.1)) are commutative. We also require the associativity of $\circ$ and the neutral element property of id by postulating the commutativity of a analogous version of the last diagrams.

$$
\begin{tikzcd}
\mathrm{Ob}(\mathbf{C}) \arrow[r, "id"] \arrow[dr, "id"'] & \mathrm{Mor}(\mathbf{C}) \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
 & \mathrm{Ob}(\mathbf{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pb}(s,t) \arrow[dr, "id"] \arrow[drr, bend left, "\circ"] \arrow[ddr, bend right, "\circ"'] & & \\
 & \mathrm{pb}(s,t) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathbf{C}) \arrow[d, "t"] \\
 & \mathrm{Mor}(\mathbf{C}) \arrow[r, "s"'] & \mathrm{Ob}(\mathbf{C})
\end{tikzcd}
$$

- *enrichment*. Recall that enrichment is about "internalization of diagrams". Therefore, we can enrich a concept $\mathcal{P}$ in any category in which the diagrams constraining the categorical information of $\mathcal{P}$ make sense. This means that a priori we can enrich $\mathcal{P}$ over a category $\mathbf{H}$ which does not has all the categorical information characterizing $\mathcal{P}$, but only that appearing in the commutative diagrams. This explains why the enrichment of a category with hom-sets produces a entity in which $\mathrm{Ob}(\mathbf{C})$ remains a **set** instead of a **object** of $\mathbf{H}$: this set does not appear explicitly in the characterizing diagrams. On the other hand, in the context of categories with source/target the situation changes. Indeed, looking at the previous diagrams we see that $\mathrm{Ob}(\mathbf{C})$ appears explicitly on the description of conditions (6.1.1). Therefore, after enrichment, the set $\mathrm{Ob}(\mathbf{C})$ will be an object of the enriching ambient. On the other hand, in the same way as the binary product functor was replaced by an arbitrary associative bifunctor in the enrichment of the concept of monoid (or in the enrichment of categories with hom-sets), in the enrichment of categories with source/target we could try to replace the pullback rule by another suitable rule P. Then, we could enrich the concept of category with source/target in any pair $(\mathbf{H}, P)$. Indeed, a "category with source/target enriched over $(\mathbf{H}, P)$" would be given by objects $\mathbf{C}_0, \mathbf{C}_1 \in \mathbf{H}$ and morphisms

$$s, t : \mathbf{C}_1 \to \mathbf{C}_0, \quad \circ : P(s,t) \to \mathbf{C}_1 \quad \text{and} \quad id : \mathbf{C}_0 \to \mathbf{C}_1$$

  such that analogues of the previous diagrams, obtained replacing pb by P, holds. This last requirement imposes many constraints on P, meaning that P should have the same universal property as pb. But by the uniqueness of pullbacks up to isomorphisms, this imply that $P(f,g) \simeq \mathrm{pb}(f,g)$ for any two morphisms. Therefore, we can enrich in any category $\mathbf{H}$ with pullbacks and the enriched concept is exactly the same that a category with source/target internal to $\mathbf{H}$.

Conclusion. *In the context of categories with source/target, internalization and enrichment produce the same abstract concept. This explains why the expression "enriched category with source/target" is not usual in the literature.*

For categories with hom-sets we have seen that the notion of functor admits a categorical characaterization, allowing us to define "enriched functors between enriched categories with hom-sets", fitting into a category $\mathbf{Cat}(\mathbf{H})$ of categories and functors enriched over $\mathbf{H}$. For categories with source/target the situation is not different. Indeed, the notion of functor between categories with source/target maps also has a purely categorical characterization, so that it can also be internalized in any category $\mathbf{H}$ with pullbacks producing "internalized functors between internal categories with source/target".

More precisely, if $\mathbf{C}$ and $\mathbf{D}$ are categories with source/target internal to $\mathbf{H}$, then a *internal functor* between them is a rule determined by morphisms

$$F_0 : \mathbf{C}_0 \to \mathbf{D}_0 \quad \text{and} \quad F_1 : \mathbf{C}_1 \to \mathbf{D}_1$$

in $\mathbf{H}$, describing the fact that functors map objects into objects and morphisms into morphisms, such that the diagrams below are commutative. The first of them means that a functor preserves the source and the target of any morphism, while the second and the third ones mean that compositions and identities are also preserved.

$$\begin{tikzcd} \mathbf{C}_1 \arrow[r, "F_1"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathbf{C}_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \mathbf{C}_0 \arrow[r, "F_0"'] & \mathbf{C}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C}_0 \arrow[r, "F_0"] \arrow[d, "id"'] & \mathbf{C}_0 \arrow[d, "id"] \\ \mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{C}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb}(s,t) \arrow[r, "\mathrm{pb}(F_1,F_1)"] \arrow[d, "\circ"'] & \mathrm{pb}(s,t) \arrow[d, "\circ"] \\ \mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{C}_1 \end{tikzcd}$$

The notion of internal functor between categories with source/target internal to $\mathbf{H}$ fits into a category $\mathbf{Cat}_{\mathbf{H}}$, which describes $\mathbf{H}$-*internal category theory*. So, we can ask if this new language really is abstract (recall that, as commented, $\mathbf{H}$-enriched category theory is abstract, so that it is natural to ask if the same holds in the internalized case). The answer is **no**. Indeed, we expect that a language abstracting classical mathematics admits the classical results as particular cases. But, as will be discussed in Section 6.3, if we try to use $\mathbf{Cat}_{\mathbf{H}}$ as a background language, then many fundamental results are no longer valid.

For instance, in many internal contexts we lose the corresponding analogues of the axiom of choice (and therefore, any of their several equivalent formulations). This has many important implications. Indeed, this means that each classical theorem whose proof makes use of the axiom of choice cannot be directly abstracted to the internal context.

Remark. In the discussion above we concluded that when trying to enrich the notion of category with source/target over some category $\mathbf{H}$, we discovered that it must be given by very similar data to that defining a internal category with source/target on $\mathbf{H}$. The unique different is that instead of using the existence of pullbacks $\mathrm{pb}(s,t)$ in $\mathbf{H}$ we may use other suitable rules $P(s,t)$ satisfying the same **universal conditions** as $\mathrm{pb}(s,t)$. The uniqueness of pullbacks up to isomorphisms then forces $P \simeq \mathrm{pb}$, meaning that in this context enrichment and internalization are equivalent. On the other hand, recall that, as commented in the last subsection, for $(\mathbf{H}, \otimes, 1)$ a suitable monoidal category, the corresponding $\mathbf{H}$-enriched category theory is abstract, meaning that we can talk about $\mathbf{H}$-limits and, specially, $\mathbf{H}$-pullbacks. Particularly, in these cases we have a notion of $\mathbf{H}$-universality. Therefore, replacing **universality** by **H-universality**, we may have rules P which are similar to pullbacks but that need not be equivalent to them. Consequently, for suitable monoidal categories $(\mathbf{H}, \otimes, 1)$ we really can do "theory of categories with source/target enriched over $\mathbf{H}$". Many aspects of this theory were developed in Aguiar's PhD Thesis [6].

Summarizing

Before presenting examples of internal/enriched categories, let us summarize the discussion at the previous subsections. Indeed, there we have seen that:

1. the concept of category admits two different categorical characterization: as *category with hom-sets* and as *category with source/target*;

2. the internalization of categories with hom-sets is not useful. On the other hand, they can be enriched over any monoidal category $(\mathbf{H}, \otimes, 1)$, producing the category $\mathbf{Cat}(\mathbf{H})$ which describes $\mathbf{H}$-*enriched category theory*. For a large class of $\mathbf{H}$, the corresponding theory is very abstract and powerful, as will be explained in Section 6.3.

3. in the usual context, internalization and enrichment of categories with source/target are equivalent and both can be applied on any category $\mathbf{H}$ with pullbacks. We have a corresponding category $\mathbf{Cat}_{\mathbf{H}}$ describing $\mathbf{H}$-*internal category theory*. Such a theory is generally poor in properties. For instance, it generally does not admits an "axiom of choice", as will be discussed in Section 6.3.

Warning. Thanks to the above points, there is no interest on internal category with hom-sets and on enriched category with source/target, Therefore, from this moment on, by *"enriched category"* we will mean "enriched category with hom-sets". Similarly, by *"internal category"* we will mean "internal category with source/target".

Remark. In many parts of the text we mentioned that category theory itself is not abstract enough to axiomatize all laws of physics. So, we are searching for higher abstract languages. The discussion above reveals that *internal category theory is not useful as a background language to attack Hilbert's sixth problem*, because generally it is not abstract enough. On the other hand, thanks to its abstractness, enriched category theory seems a very natural language to attack Hilbert's problem. For instance, recall that in Section 2.3 we predicted that categorification produces notions of "n-categories" which was used in Section 2.4 in order to get many insights in physics. In the next section we will see that categorification by enrichment of the concept of category reproduces precisely the conjectured 2-categories. Then, iterating the process we will get higher category theory, which by construction will be very abstract.

Warking. The last remark **cannot** be used to conclude that internalization is not useful in physics. Indeed, it only means that internal category theory is not useful as a **fundamental background language**. The difference will be clear in the next section.

Remark. Recall that a groupoid is a category in which any morphism is an isomorphism. This additional condition can be easily written in terms of the commutativity of a diagram, so that we can enrich and internalize the notion of "groupoid" into the same ambient in which the notion of "category" could be internalized/enriched. In other words, we can talk of "groupoid internal to any category with pullbacks" and of "groupoid enriched over any monoidal category". For instance, if $\mathbf{H}$ has pullbacks, then a groupoid internal to $\mathbf{H}$ is just a category $\mathbf{C}$ internal to $\mathbf{H}$ endowed with a morphism $\mathrm{inv} : \mathbf{C}_1 \to \mathbf{C}_1$ such that the diagrams below commute. We also need a diagrams analogous to the first of them, obtained replacing $id \times \mathrm{inv}$ with $\mathrm{inv} \times id$.

$$
\begin{tikzcd}
\mathbf{C}_1 \arrow[r] \arrow[d, "s"'] & \mathrm{pb}(t,t) \arrow[r, "id\times\mathrm{inv}"] & \mathrm{pb}(t,s) \arrow[d, "\circ"] \\
\mathbf{C}_0 \arrow[rr, "id"'] & & \mathbf{C}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}_1 \arrow[r, "\mathrm{inv}"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \mathbf{C}_1 \arrow[dl, shift right, "s"'] \arrow[dl, shift left, "t"] \\
\mathbf{C}_0 &
\end{tikzcd}
$$

Remark. Notice that to any category $\mathbf{C}$ internal to $\mathbf{H}$ we can associate a diagram into $\mathbf{H}$, as below. The same holds for internal groupoids, but we need to add the morphism $\mathrm{inv} : \mathbf{C}_1 \to \mathbf{C}_1$ into the diagram.

$$
\begin{tikzcd}
\mathrm{pb}(s,t) \arrow[r, "\circ"] & \mathbf{C}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbf{C}_0 \arrow[r, "id"] & \mathbf{C}_1
\end{tikzcd}
$$

6.2 Examples

Let us finally give some simple examples of enriched/internal categories. We will also talk of "internal sheaves", giving an idea about their role in the unification problem of physics. More specific examples will be discussed in the other parts of the text.

Enriched Categories

We start by presenting examples of enriched categories.

Example 6.1 (*additive and abelian categories*)**.** For a given commutative ring R, we have the corresponding monoidal category $(\mathbf{Mod}_R, \otimes)$ of R-modules, so that we can consider categories enriched over it, which are called *R-categories*. By definition, they are entities $\mathbf{C}$ defined by giving a set $\mathrm{Ob}(\mathbf{C})$ of objects, for any two objects a R-module $\mathrm{Hom}_R(X, Y)$ of morphisms, for any three objects a linear composition

$$\circ : \mathrm{Hom}_R(X, Y) \otimes \mathrm{Hom}_R(Y, Z) \to \mathrm{Hom}_R(X, Z)$$

and for any object a linear map $R \to \mathrm{Hom}_R(X, X)$, which is determined by their image of the unity $1 \in R$. Therefore, a R-category can be identified with an usual category whose hom-sets become endowed with a R-module structure such that the composition maps are bilinear. There is special interest in those enriched categories for which finite limits/colimits exist. These are the *R-additive categories*. For instance, in additive categories products and coproducts are always isomorphic. Because additive categories are assumed to have finite limits/colimits, any morphism $f : X \to Y$ has kernel/cokernel. The universality of pullbacks and pushouts then gives a canonical morphism

$$f_* : \ker(\mathrm{coker}\,(f)) \to \mathrm{coker}(\ker(f)).$$

There is further interest in the additive categories, usually called *abelian categories*, in which any induced morphism f_* is an isomorphism. The standard example of abelian category is $\mathbf{Mod}_R$ itself. In it, the domain of f_* is simply the quotient $X/\ker(f)$, while their codomain is the image $f(X)$. Therefore, in this case saying that f_* is an isomorphism is the same as saying that the first isomorphism theorem holds. So, in the abstract context we can say that a *abelian category is a additive category in which we have a first isomorphism theorem*. Notice that, in order to define "chain complexes" and "cochain complexes" (as done in Section 1.11) all we need is the notion of "null map" (for $d^2 = 0$ makes sense), while in order to build the homology and cohomology groups we need the notions of kernel and cokernel. Therefore, we can try to develop Homological Algebra in any additive category. The fundamental results of classical Homological Algebra can be obtained in this abstract context when the additive category is indeed an abelian category. So, *we can also think of abelian category as an abstract ambient to develop Homological Algebra*. This approach was firstly developed by Grothendieck in his famous "Tôhoku paper" [90].

Example 6.2 (*differential graded categories*)**.** Instead of enriching over $(\mathbf{Mod}_R, \otimes, R)$ we could considered categories enriched over the monoidal category $(G\mathbf{Grad}_R, \otimes)$ of G-graded R-modules or over the monoidal category $(\mathbf{CCh}_R, \otimes)$ of chain/cochain complexes. In the first case, the corresponding enriched categories are called *G-graded R-categories*, while in the second they are called *differential graded R-categories*. Recall that the topological (or nonlinear) analogue of a cochain complex is given by a spectrum. We have some models of the symmetric monoidal category $(\mathbf{Spec}, \wedge, \mathbb{S})$ of spectra, as discussed in Section 5.3. A category enriched over them is called a *spectral category*. It can be understood as a "nonlinear version" of the differential graded category.

Example 6.3 (*closed monoidal structure*)**.** We notice that the standard examples of categories enriched over $\mathbf{Mod}_R$, $G\mathbf{Grad}_R$ or $\mathbf{CCh}_R$ are precisely these categories themselves. This happens because in these cases the enrichment ambient $(\mathbf{H}, \otimes, 1)$ is a **closed** monoidal category. Indeed, any closed monoidal category can be trivially enriched over itself by considering the hom-objects $\mathbf{H}(X, Y)$ as being simply the internal hom-object $[X, Y]$. So, not only the previous categories are enriched over themselves, but also any convenient category of topological spaces $\mathscr{C} \subset \mathbf{Top}$ is topologically enriched, because it is cartesian closed. Additionally, $\mathscr{C}_*$ can be enriched over itself with respect to the closed monoidal structure given by the smash product.

Internal Categories

We now give examples of internal categories.

Example 6.4 (*linear categories*)**.** Recall that $\mathbf{Mod}_R$ is complete, so that it has pullbacks and, therefore, we can talk of categories internal to $\mathbf{Mod}_R$. These are usually known as the *R-linear categories*. In the context of vector spaces (i.e, when R is a field $\mathbb{K}$), these are called *2-vector spaces*[5]. So, explicitly, a 2-vector space is given by a vector space $\mathbf{C}_0$ of objects, a vector space $\mathbf{C}_1$ of morphisms, linear source/target maps $s, t : \mathbf{C}_1 \to \mathbf{C}_0$, linear composition

[5] The reader must pay attention here, sinse there is no universal notion of "2-vector space" in the literature. The definition used here corresponds to what some authors calls *Baez-Crans 2-vector spaces*. On [124], for instance, a 2-vector space is defined as a category enriched over $\mathbf{Vec}_{\mathbb{K}}$ (which for us are $\mathbb{K}$-categories), while in [19, 20] it is used the same definition adopted here.

$\circ : \mathrm{Pb}(s,t) \to \mathbf{C}_1$ and a linear identity map $id : \mathbf{C}_0 \to \mathbf{C}_1$ satisfying all the compatibility conditions described previously[6]. The interesting fact is that *this set of data is overdetermined*: together, the source/target and the identity maps determine the composition (see Lemma 6, p. 9-11, [22]).

Example 6.5 (*2-groups and crossed modules*)**.** The category **Grp** of groups also has pullbacks, so that we can also consider categories internal to **Grp**. These are called 2-*groups* analogously as internal categories to $\mathbf{Vec}_{\mathbb{K}}$ are 2-vector spaces. Certainly, any $\mathbb{Z}$-linear category is a 2-group, because a $\mathbb{Z}$-module is the same as an abelian group. Explicitly, a 2-group is given by a group of objects $\mathbf{C}_0$, a group of morphisms $\mathbf{C}_1$, source/target group homorphisms $s,t : \mathbf{C}_1 \to \mathbf{C}_0$, composition homomorphism $\circ : \mathrm{Pb}(s,t) \to \mathbf{C}_1$ and identity $id : \mathbf{C}_0 \to \mathbf{C}_1$. As in linear case, the data defining a 2-group is overdetermined, meaning that we only need to know the source/target and the identity maps. The standard ways to get 2-groups are the following (for details about them, see [26] and the final chapter of [131]):

1. *categorical groups*. Recall that $(\mathbf{Cat}, \times, \mathbf{1})$ is a symmetric monoidal category, so that we can consider its Hopf monoids (i.e, the group objects into **Cat**). These are monoidal categories $(\mathbf{C}, \otimes, 1)$ with inversion functor $\mathrm{inv} : \mathbf{C} \to \mathbf{C}$. Notice that viewing $\mathbf{C}$ as a category with source/target, the actions of $\otimes$ and inv on objects/morphisms induce group structures on $\mathbf{C}_0$ and $\mathbf{C}_1$. The functoriality of $\otimes$ means that the composition and the source/target maps are indeed group homomorphisms, so that we have exactly a 2-group structure. Reversing the process we see that any 2-group can be obtained in this way, so that we have an equivalence
$$\mathbf{Cat}_{\mathrm{Hopf}(\mathbf{Set})} \simeq \mathrm{Hopf}(\mathbf{Cat}).$$

2. *crossed modules*. A R-module can be understood as an action $R \times G \to G$ of a ring R onto an abelian group G, which preserves multiplications and units. More generally, given an arbitrary (not necessarily commutative) group H we can define H-*modules* as b actions $\alpha : H \times G \to G$ onto another arbitrary group. A special case is that in which $H = \mathrm{Aut}(G)$ is the automorphism group of G. We then have an action by evaluation $(f,x) \mapsto f(x)$. This case is special because we also have a canonical embedding $\delta : G \to H$, assigning to each element x the left translation $\delta(x) = \ell_x$, which is compatible with the action and with the multiplication, in the sense that $\alpha(\delta(x), y) = x * y$ and $\delta(\alpha(f,x)) = f \circ \delta(x)$, as described by the diagrams below. Note that these conditions only means $\ell_x(y) = x * y$ and $\ell_{f(x)} = f \circ \ell_x$. A H-module $\alpha : H \times G \to G$ which becomes endowed with a morphism $\delta : G \to H$ satisfying these commutative conditions is called a *crossed H-module* (we also say that the module was *twisted by* δ).
$$\begin{tikzcd} G \times G \arrow[rr, "\delta \times id"] \arrow[dr, "*"'] & & H \times G \arrow[dl, "\alpha"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} H \times G \arrow[r, "\alpha"] \arrow[d, "id \times \delta"'] & G \arrow[d, "\delta"] \\ H \times H \arrow[r, "\circ"'] & H \end{tikzcd}$$
Now, we notice that any crossed module (H, G, α, δ) defines a 2-groups $\mathbf{C}$ whose group of objects is $\mathbf{C}_0 = H$ and whose group of morphisms is the *semi-direct product* $\mathbf{C}_1 = H \ltimes G$

[6] Notice the difference between the data defining R-categories and R-linear categories.

induced by the action α. In other words, it is the set $H \times G$ with the group structure given by $(f, x) \cdot (g, y) = (f \circ g, x * \alpha(f, y))$. The source and target maps $s, t : \mathbf{C}_1 \to \mathbf{C}_0$ are respectively given by the projection $s = \pi_1$ on the first argument and by the composition $t = \delta \circ \alpha$. The fact that t really is a group homomorphism follows from the commutativity of the second diagram above (note that t is precisely its diagonal). We observe that the identity $id : \mathbf{C}_0 \to \mathbf{C}_1$ is almost determined by this same diagram. Indeed, the condition $s(id(f)) = f$ means $\pi_1(id(f)) = f$, so that $id(f) = (f, x)$ for some x. On the other hand, using the above diagram, the condition $t(id(f)) = f$ implies $x \in \ker \delta$. Therefore, a natural choice for the identity map is $id(f) = (f, e)$, where $e \in G$ is the neutral element. Notice that, because the data defining a 2-groups is overdetermined, if a composition map $\circ : \mathrm{ps}(s, t) \to \mathbf{C}_1$ exists, then it must be by a explicitly expression. Now, the first diagram above says precisely that this rule is a homomorphism and, therefore, this completes the construction of the 2-group associated to the given crossed module. Reciprocally, if $\mathbf{C}$ is some 2-group, we get a crossed module (H, G, α, δ) by considering H as the morphism object $\mathbf{C}_1$ and G as the kernel of the source map $s : \mathbf{C}_1 \to \mathbf{C}_0$. The $\delta : G \to H$ map is simply the restriction of the target map t to $\ker(s)$. Finally, as $\ker(s) \subset \mathbf{C}_1$ is a normal subgroup, it acts naturally on $\mathbf{C}_1$ by conjugation and we take α as this action. Such correspondence between 2-groups and crossed modules extends to an equivalence $\mathbf{Cat}_{\mathbf{Grp}} \simeq \mathrm{CrossMod}$. For full details, see [167].

Remark. The first part of the last example can be generalized to any nice internalization ambient. More precisely, let $\mathbf{H}$ be any category with pullbacks admitting a cartesian monoidal structure (for instance, any category with finite limits). This structure automatically induces a cartesian monoidal structure on $\mathbf{Cat}_{\mathbf{H}}$ just as the cartesian products on **Set** induces binary products on **Cat**. Therefore, we can consider Hopf monoids into $\mathbf{Cat}_{\mathbf{H}}$. On the other hand, notice that the category $\mathrm{Hopf}(\mathbf{H})$ has the same limits as $\mathbf{H}$, so that it has pullbacks and, therefore, it is an internalization ambient[7]. The categories internal to $\mathrm{Hopf}(\mathbf{H})$ are called *2-groups internal to* $\mathbf{H}$. Following identical steps applied in the first part of the last example we conclude that *categories internal to group objects are equivalent to group objects into internal categories*:

$$\mathbf{Cat}_{\mathrm{Hopf}(\mathbf{H})} \simeq \mathrm{Hopf}(\mathbf{Cat}_{\mathbf{H}}).$$

Remark. In the last remark we used the fact that the cartesian monoidal structure on $\mathbf{H}$ induces a corresponding cartesian structure on $\mathbf{Cat}_{\mathbf{H}}$, but much more is true. Indeed, as proved in [28], if the internalization ambient has finite limits and is cartesian closed, then the same properties holds in $\mathbf{Cat}_{\mathbf{H}}$.

Remark. The second part of the last example can also be generalized. Indeed, it is clear that it makes perfectly in any cartesian monoidal category $\mathbf{H}$, allowing us to build the category $\mathrm{CrossMod}_{\mathbf{H}}$ of crossed modules internal to $\mathbf{H}$, such that

$$\mathrm{CrossMod}_{\mathbf{H}} \simeq \mathbf{Cat}_{\mathrm{Hopf}(\mathbf{H})}.$$

[7] In order to conclude that $\mathrm{Hopf}(\mathbf{H})$ has limits, recall that, as discussed in Section 4.2, for a **cartesian** monoidal structure the group objects into $\mathbf{H}$ are in 1-1 correspondence with the objects $X \in \mathbf{H}$ such that $\mathrm{Mor}_{\mathbf{H}}(X, Y)$ is a group for any Y. Now, if $F : \mathbf{C} \to \mathbf{H}$ is a functor such that each $F(X)$ is a group object, then, thanks to the cocompleteness of **Grp**,

$$\mathrm{Mor}_{\mathbf{H}}(\lim F, Y) \simeq \mathrm{colim}\, \mathrm{Mor}_{\mathbf{H}}(F(X), Y)$$

is also a group, so that $\lim F$ is a group object.

This is useful for categories like **Diff** and **Top**, whose cartesian monoidal structure has non-trivial monoids. On the other hand, this is not useful for algebraic categories, because, as discussed in Example 4.12, they generally have trivial monoid objects. So, it is natural to ask if the above construction can be generalized to the algebraic context. In [106] the author shows that "crossed modules" can be internalized in "semi-abelian categories" (and, therefore, in abelian categories), producing the equivalence between the category of these internalized crossed modules and the category of categories internal to $\mathbf{H}$.

In Example 6.3 we showed that a closed monoidal category can be naturally enriched over itself. We would like to observe, however, that the same does not hold for internalization. More precisely, *given a category* $\mathbf{H}$ *with pullbacks, there is no canonical way to internalize it into itself.* The reason for this discrepancy follows from the fact that in the enrichment process only the hom-sets acquire new structure, but in the internalization process both the hom-sets and the objects set acquire new structure[8].

For instance, there is no canonical way to internalize $\mathbf{Vec}_{\mathbb{K}}$ into itself, because there is no canonical way to assign to the collection $\mathrm{Ob}(\mathbf{Vec}_{\mathbb{K}})$ of all vector spaces a linear structure turning the source and target maps linear. In other words, there is no canonical way to sum two vector spaces and to multiply a vector space by an element of $a \in \mathbb{K}$.[9]. Similarly, there is no canonical topology which can be introduced on the collection $\mathrm{Ob}(\mathbf{Top})$ of all topological spaces, so that there is no canonical way to internalize $\mathbf{Top}$ into itself.

On the other hand, the next example clarifies that any object $X \in \mathbf{H}$ of a category with pullbacks can be considered a category internal to $\mathbf{H}$ in a canonical (and trivial) way.

Example 6.6 (*canonical embedding*)**.** For any category $\mathbf{H}$ with pullbacks there is a canonical embedding $\mathrm{disc} : \mathbf{H} \to \mathbf{Cat}_{\mathbf{H}}$. Indeed, any object X defines a $\mathbf{H}$-internal category $\mathrm{disc}(X)$, called the *discrete category of* X, whose diagrammatic representation is the following:

$$
X \xrightarrow{id_X} X \underset{id_X}{\overset{id_X}{\rightrightarrows}} X \xrightarrow{id_X} X
$$

Evidently, any $f : X \to Y$ induces a corresponding internal functor $\mathrm{disc}(f)$ between $\mathrm{disc}(X)$ and $\mathrm{disc}(Y)$ which in objects and in morphisms is given by f, completing the definition of disc.

Internal Groupoids

Here we will give some examples of special internal categories: the *internal groupoids.*

Example 6.7 (*canonical embedding*)**.** By the last example, any object $X \in \mathbf{H}$ of a category with pullbacks can be naturally regarded as a trivial $\mathbf{H}$-internal category. We notice that this internal category is, indeed, an internal groupoid. In fact, recall that all structure maps (s, t, $\circ$ and id) of $\mathrm{disc}(X)$ are given by the identity $id_X : X \to X$ morphism. Therefore, the identity id_X is itself a rule that assigns to any morphism an inverse, turning $\mathrm{disc}(X)$ an internal groupoid.

[8] As will become clear in the next section, this is also the fundamental reason why internalized category theory is not generally a high abstract language, while enriched category theory is.

[9] We can "sum" two vector spaces X and Y by considering its direct sum $X \oplus Y$, but this operation is commutative and associative only up to isomorphism. Furthermore, the operation $\oplus$ does not has a globally defined inverse $\ominus$.

Example 6.8 (*underlying groupoid*)**.** Every usual category $\mathbf{C}$ defines a groupoid $\mathbf{C}_{pd}$ by forgetting the morphisms which are not isomorphisms. Because functors preserve isomorphisms, this automatically gives a functor $(-)_{pd} : \mathbf{Cat} \to \mathbf{Gpd}$. This construction can be directly extended to the internal case, giving a functor $(-)_{pd} : \mathbf{Cat}_{\mathbf{H}} \to \mathbf{Gpd}_{\mathbf{H}}$. In practice, for any given internal category $\mathbf{H}$ this functor only replaces the object of morphisms $\mathbf{C}_1$ by the corresponding object of isomorphisms $\mathbf{C}_1^{iso}$. We leave the details to the reader.

Example 6.9 (*Lie groupoids*)**.** The category $\mathbf{Diff}$ does no has arbitrary pullbacks, as explained in Example 2.4. Consequently, we cannot consider categories/groupoids internal to $\mathbf{Diff}$. But, recall that pullbacks between *transversal maps* exist on $\mathbf{Diff}$. So, the subcategory $\mathbf{Diff}_{sub} \subset \mathbf{Diff}$ of smooth manifolds and submersions has pullbacks and, therefore, is an internalization ambient. A groupoid internal to $\mathbf{Diff}_{sub}$ is called a *Lie groupoid*. The category of Lie groupoids is usually denoted by $\mathbf{LieGpd}$. So, explicitly, a Lie groupoid is an entity given by a smooth manifold $\mathbf{C}_0$ of objects, a smooth manifold $\mathbf{C}_1$ of morphisms, together with smooth submersions

$$s,t : \mathbf{C}_1 \to \mathbf{C}_0, \quad \circ : \mathrm{pb}(s,t) \to \mathbf{C}_1, \quad id : \mathbf{C}_0 \to \mathbf{C}_1 \quad \text{and} \quad \mathrm{inv} : \mathbf{C}_1 \to \mathbf{C}_1 \tag{6.2.1}$$

such that the groupoid-like diagrams are satisfied. Notice that in the one object limit (i.e, when $\mathbf{C}_0$ is the trivial manifold with only one point) we recover the usual notion of Lie group.

Example 6.10 (*internal 2-groups*)**.** In Example 6.5 we concluded that, for suitable internalization ambient $\mathbf{H}$, the categories internal to $\mathrm{Hopf}(\mathbf{H})$ (i.e 2-groups internal to $\mathbf{H}$) are the same as group objects on $\mathbf{Cat}_{\mathrm{Hopf}(\mathbf{H})}$. It happens that both groups into $\mathbf{Cat}_{\mathrm{Hopf}(\mathbf{H})}$ and categories internal to $\mathrm{Hopf}(\mathbf{H})$ are automatically $\mathbf{H}$-internal groupoids, so that we have equivalences

$$\mathbf{Gpd}_{\mathrm{Hopf}(\mathbf{H})} \simeq \mathbf{Cat}_{\mathrm{Hopf}(\mathbf{H})} \simeq \mathrm{Hopf}(\mathbf{Cat}_{\mathbf{H}}) \simeq \mathrm{Hopf}(\mathbf{Gpd}_{\mathbf{H}}),$$

which can be used in order to get many examples of internal groupoids. For instance, any 2-group internal to $\mathbf{Diff}_{sub}$ (i.e, any group object into $\mathbf{LieGpd}$, here called a *Lie 2-group*) defines a groupoid internal to the category of Lie groups. So, explicitly, a Lie 2-group is just a Lie groupoid whose smooth manifolds of objects $\mathbf{C}_0$ and of morphisms $\mathbf{C}_1$ are Lie groups, and whose structure maps (6.2.1) are morphisms of Lie groups.

Example 6.11 (*deloopings*)**.** Any group $(G, *, e, \imath)$ internal to a cartesian monoidal category with pullbacks can be delooped to a groupoid $\mathbf{B}G$ internal to $\mathbf{H}$. Indeed, we define the object of objects as $\mathbf{B}G_0 = *$ and the objects of morphisms as $\mathbf{B}G = G$. The source/target $s,t : \mathbf{B}G_1 \to \mathbf{B}G_0$ are the unique map $G \to 1$ (recall that 1 is a terminal object). Therefore, $\mathrm{pb}(s,t) \simeq G \times G$ and the composition $\circ : G \times G \to G$ can be taken as the multiplication $*$ of G. Similarly, the morphisms $id : \mathbf{B}G_0 \to \mathbf{B}G_1$ and $\mathrm{inv} : \mathbf{B}G_1 \to \mathbf{B}G_1$ are considered as the "neutral element" of G and the "inverses" of G. Summarizing, the internal groupoid $\mathbf{B}G$ is that diagrammatically represented by

$$\begin{tikzcd} G \times G \arrow[r, "*"] & G \arrow[r, shift left] \arrow[r, shift right] & 1 \arrow[r, "e"] & G \end{tikzcd}$$

Example 6.12 (*Picard 2-group*)**.** From any monoidal category $(\mathbf{H}, \otimes, 1)$ with pullbacks we can a full subcategory $\mathrm{Pic}(\mathbf{H}) \subset \mathbf{H}$ by considering only the objects $X \in \mathbf{H}$ for which there is some $X^{-1} \in \mathbf{H}$ and isomorphisms $X \otimes X^{-1} \simeq 1 \simeq X^{-1} \otimes X$. With the same operations, $(\mathrm{Pic}(\mathbf{H}), \otimes, 1)$ becomes a monoidal category, which fits into the structure of groupoid, called the *Picard groupoid* of $\mathbf{H}$. Notice that the set $\mathrm{Pic}(\mathbf{H})_0$ of objects acquires a group structure when endowed with $\otimes$ and 1. Similarly, $\otimes$ also give a group structure to the set of morphisms $\mathrm{Pic}(\mathbf{H})_0$, so that $\mathrm{Pic}(\mathbf{H})$ is indeed a groupoid internal to the category of groups, i.e, a 2-group.

Locality

Now, we recall the discussion on sheaves at the end of Section 2.2. There we concluded that any category $\mathbf{H}$ can be embedded into a complete/cocomplete functor category by the Yoneda embedding. But such a category of functors is not the most natural way to describe the objects of $\mathbf{H}$. For instance, as discussed later, when $\mathbf{H} = \mathbf{Diff}$ these functors do not capture the "local nature" of smooth manifolds. The solution was then obtained by constructing a Grothendieck topology and looking to the functors (called *sheaves*) preserving it.

More precisely, we recall that such a Grothendieck topology J on $\mathbf{H}$ is simply a way to assign to any object X a family of morphisms $\pi : U \to X$, called *covering families*, which satisfy some conditions, including stability under pullbacks. These conditions fits into the first diagram below, called the Cech diagram of the cover and represented by $\check{C}(U)$. Taking the colimit of the Cech diagram we get an arrow $u : \mathrm{colim}\check{C}(U) \to X$. A functor $F : \mathbf{H}^{op} \to \mathbf{Set}$ is sheaf precisely when, for any object X and any cover $U \to X$, the corresponding arrow $F(u)$ is an isomorphism.

$$
\begin{tikzcd} U \times_X U \times_X U \arrow[r] & U \times_X U \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & U \arrow[r, "\pi"] & X \end{tikzcd}
$$

Now, notice that the last diagram $\check{C}(U)$ is exactly the diagrammatic presentation of a groupoid internal to $\mathbf{H}$. This is the so called *Cech groupoid* of the cover. The theory of sheaves and stacks[10] is much more related with the theory of internal groupoids than only by the Cech groupoid structure on the covertings. Indeed, we observe that *any groupoid* $\mathbf{C}$ *internal to* $\mathbf{H}$ *defines a stack in* $\mathbf{H}$ *with respect to any Grothendieck topology* J, as follows. We have a functor $[-, \mathbf{C}] : \mathbf{H}^{op} \to \mathbf{Gpd}$ which assigns to each $X \in \mathbf{H}$ the groupoid $[X, \mathbf{C}]$ whose diagrammatic representation is obtained by applying $\mathrm{Mor}_{\mathbf{H}}(X, -)$ in the diagrammatic representation of $\mathbf{C}$. On the other hand, for a fixed topology J we have a left adjoint $\mathscr{L}$ to the inclusion

$$
\imath : \mathrm{Stack}(\mathbf{H}, J) \hookrightarrow \mathrm{Func}(\mathbf{H}^{op}; \mathbf{Grpd}),
$$

so that we can define the stack associated to $\mathbf{C}$ as the stackfication of $[-, \mathbf{C}]$ (i.e, as its image under $\mathscr{L}$). The stacks which are equivalent to those defined by some $\mathbf{H}$-internal groupoid are called *geometric* $\mathbf{H}$*-stacks*. Specially, the smooth stacks coming from Lie groupoids are called *differential stacks*.

Remark. Surely, if two internal groupoids $\mathbf{C}$ and $\mathbf{D}$ are isomorphic (as objects of $\mathbf{Gpd}_{\mathbf{H}}$), then they induce equivalent stacks. On the other hand, the reciprocal is no longer valid for a general internalization ambient $\mathbf{H}$. Indeed, it depends explicitly on the axiom of choice, which (as will be discussed in the next section) may fail in $\mathbf{H}$. This suggests that, *when the axiom of choice fail, the internal functors are not the correct morphisms between internal groupoids*. Indeed, in the next section we will see that in this context the correct morphisms are the *Morita morphisms*, meaning that if two stacks coming from internal groups are equivalent, them the underlying internal groups are Morita equivalent.

Geometric Stacks and Physics

Here we would like to explain the role of geometric stacks (specially differential stacks) in physics. We start by recalling that a classical theory of physics is determined by the set of

[10] Recall that a stack is just a sheaf taking values into the category of groupoids.

minimizing configurations of an action functional $S : \mathrm{Fields}(\Sigma) \to \mathbb{R}$, defined on a set $\mathrm{Fields}(\Sigma)$ which depends of the choice of an abstract motion. For instance, if we are interested only in free motions inside a ambient space M, then $\mathrm{Fields}(\Sigma)$ is the space of smooth maps $\varphi : \Sigma \to M$. In this situation the rule $\Sigma \mapsto \mathrm{Fields}(\Sigma)$ extends to the functor

$$\mathrm{Fields} : \mathbf{Diff}^{op} \to \mathbf{Set} \quad \text{given by} \quad \mathrm{Fields}(\Sigma) = \mathrm{Mor}_{\mathbf{Diff}}(\Sigma, M).$$

Because any set can be trivially regarded as a groupoid, this functor can be considered as taking values in **Gpd**. We notice that the corresponding functor $\mathrm{Fields} : \mathbf{Diff}^{op} \to \mathbf{Gpd}$ is a smooth stack: the smooth stack defined by M regarded as the discrete Lie groupoid $\mathrm{disc}(M)$. Therefore, *in this special case the space of configurations extends to a differential stack*. Another case in which we led to the same conclusion is when Fields is the functor assigning to any manifold Σ the groupoid of G-bundles over Σ. By the classification theorem of G-bundles this functor is precisely the smooth stack induced by the dellooped Lie groupoid $\mathbf{B}G$. The surprising fact is that this result extends to **any** interesting classical physical theory of **particles**. More precisely, as will be proved in Chapter 11, *each interesting configuration space for **particles** can be extended to a differential stack* $\mathbf{Diff}^{op} \to \mathbf{Gpd}$.

The last conclusion about the structure of classical theories for **particles** reveals that internal categorical language (specially smooth categorical language) is very useful to describe physics[11]. Now, we would like to explain why the internal language is not abstract enough to axiomatize all physical laws (i.e, why it is not the correct background language to do physics).

We start by noticing that, the fact that configuration spaces for **particles** are given by smooth stacks $\mathrm{Fields} : \mathbf{Diff}^{op} \to \mathbf{Gpd}$ means that, for any good cover $U_i \hookrightarrow \Sigma$, the corresponding $\mathrm{Fields}(\Sigma)$ is totally determined by the collection of all $\mathrm{Fields}(U_i)$ subjected to some compatibility conditions on the intersections $U_i \cap U_j$. On the other hand, when doing gauge theory for **strings** we meet geometrical structures on the ambient manifold M which are **not** determined by conditions on $U_i \cap U_j$. Indeed, we also need to take into account conditions on the "intersections between intersections" $U_i \cap U_j \cap U_k$, and so on.

It happens that these kind of objects **cannot** be smooth stacks. Indeed, stacks are defined on the Cech groupoid $\check{C}(U_i)$, which only consider elements $x_i \in U_i$ (as objects) and $x_{ij} \in U_i \cap U_j$ (as morphisms). So, these entities should be certain "higher smooth stacks", defined on a "Cech higher groupoid", which do not stop on the morphisms, having $x_{ijk} \in U_i \cap U_j \cap U_k$ as 2-morphisms, and so on.

The standard examples of smooth stacks are those induced by Lie groupoids. Therefore, it is natural to expect that these "higher smooth stacks" comes from "higher Lie groupoids". In the next sections we will see that enrichment is the natural abstract language to describe "higher categories", in which we will could internalize groupoids, getting these "higher Lie groupoids" and, therefore, these "higher smooth stacks". Thus, *if on one hand internal category theory do not axiomatize all physical laws, enriched category theory alone is also not the correct language. What we need is **internalization into higher enriched languages**.*

[11]Maybe this purely mathematical conclusion is very abstract to motivate the physicist reader. On the other hand, we have to say that this characterization step is crucial. For instance, it is only for this kind of theories that the pull-push approach to quantization can be applied.

Categorified Lie Theory

Examples 6.9 and 6.11 suggest the existence of a "categorified Lie theory". More precisely, we obtained two different generalizations of the concept of Lie groups: *Lie groupoids and Lie 2-groups.* So, we can ask if, in the same way as the geometric structure of a Lie group is captured by the algebraic structure of its associated Lie algebra, the "higher geometric" structure of Lie groupoids and Lie 2-groups can be described by some associated "Lie algebroid" and "Lie 2-algebra". This is really the case, as will be more generally discussed in Section 10.3.

Indeed, a Lie groupoid is a "many objects version" of a Lie group, so that the idea is to consider a "Lie algebroid" as a certain kind of a "many objects version" of a Lie algebra. On the other hand, a Lie 2-group is a group object into the category of categories internal to **Diff**, so that the main idea is to consider a Lie 2-algebra as a "Lie algebra object" into the category of categories internal to $\mathbf{Vec}_{\mathbb{C}}$ (i.e, Lie algebra objects into the category of 2-vector spaces). Furthermore, just as any Lie group G has an associated Lie algebra $\mathfrak{g}$, any Lie groupoid/Lie 2-group induces a corresponding Lie algebroid/Lie 2-algebra. More precisely, the functor

$$\mathcal{L}:\mathbf{LieGrp}\rightarrow\mathbf{LieAlg}$$

assigning to any Lie group its Lie algebra can be categorified in order to get functors

$$\mathcal{L}:\mathbf{LieGpd}\rightarrow\mathbf{LieAld}\quad\text{and}\quad\mathcal{L}:\mathbf{Lie2Grp}\rightarrow\mathbf{Lie2Alg}$$

assigning to each Lie groupoid its corresponding Lie algebroid and to any Lie 2-group the associated Lie 2-algebra.

Recall that not every Lie algebra is the Lie algebra of a Lie group (in other words, not every Lie algebra is *integrable*). Indeed, by the *third Lie theorem*, such a correspondence exists only in the *finite dimensional* case. This condition means that the functor $\mathcal{L}$ is essentially surjective on the subcategory $\mathbf{FinLieAlg}\subset\mathbf{LieAlg}$ of finite dimensional Lie algebras. On the other hand, the *second Lie theorem* states that for simply connected Lie groups, $\mathcal{L}$ is fully-faithful. Consequently, we have an equivalence

$$\mathbf{SimplyLieGrp}\simeq\mathbf{FinLieAlg}$$

between the category of simply connected Lie groups and finite dimensional Lie algebras.

Certainly, the analogous conclusion will **fail** for the case of Lie groupoids and Lie 2-groups, because in order to obtain the above isomorphism we used that a functor is an equivalence iff it is fully-faithful and essentially surjective. It happens that this characterization is explicitly dependent of the axiom of choice, which (as commented later and as will be discussed in detail in the next section) is no longer valid in many internal categories. Even so, we can search for integrability conditions under which a given Lie algebroid/Lie 2-algebra always comes from some Lie groupoid/Lie 2-group. This situation has a strong physical appeal, as clarified by the following examples:

1. higher gauge groups. As commented in the last subsection and as will become more clear in Chapters 11-13, gauge theories of strings are described by "higher Lie groups". So, it is natural to search for conditions under which this higher group can actually be described by their infinitesimal "higher Lie algebra".

2. geometric quantization of Poisson manifolds. As briefly commented in Section 3.3, there are several approaches to the quantization process. Two of them are the *geometric quantization* and the *deformation quantization*. The first applies to classical theories whose space of configurations (i.e, the space where the action functional is defined) has the structure of a symplectic manifold, while the second applies more generally to Poisson manifolds. So, a natural question is about the existence of some "**extended** geometric quantization" in the context of Poisson manifolds. How to proceed? Any Poisson manifold can be naturally regarded as a Lie algebroid which in some good cases can be "integrated" in order to get a Lie groupoid, which behaves much like a "many objects version of a symplectic manifold", called *symplectic Lie groupoid*. Therefore, the searched for "**extended** geometric quantization" should be some kind of "**higher** geometric quantization". As will be discussed in Chapter 16, this process can be realized by pull-push quantization in complex K-theory.

6.3 Abstractness

In the last section we applied the internalization and the enrichment process to the concept of category. We have seen that if $\mathbf{H}$ is a category with pullbacks, then we can do $\mathbf{H}$-*internal category theory,* and if $(\mathbf{H},\otimes,1)$ is monoidal, then we can do $\mathbf{H}$-*enriched category theory*. The main advertising for internalization/enrichment is the production of high abstract mathematics from classical mathematics.

In this section we would like to analyze if internal/enriched category theory really are abstract. We start by showing that generally internal category theory is **not** abstract. More precisely, we will see that we can talk about the "axiom of choice" in any category and that it generally fails in the internalization ambient $\mathbf{H}$, producing pathological internal languages. In the sequence we will see that this problem can be ignored if we replace internal functors by another class of morphisms between internal categories: the *anafunctors*. We discuss that this replacement is at the heart of homotopy theory.

On the other hand, we show that enriched category theory is naturally abstract and useful in the sense that we can talk about enriched limits, enriched Kan extensions, and so on, at the same time that the pathologies appearing in the $\mathbf{H}$-internal language automatically disappear in the enriched language.

Internalized Categories

Recall that when doing axiomatic set theory we usually assume the *axiom of choice*. In its more familiar formulation it states that:

Axiom of Choice (*classical version*) *If* $(X_i)_{i\in I}$ *if a family of sets parametrized over an* ***arbitrary*** *set* I *of indexes, then it is possible to build a new set* S *by selecting an element* x_i *in each* X_i.

The above axiom has a purely categorical characterization. Notice that the family $(X_i)_{i\in I}$ is, by definition, just the same as the disjoint union $X=\sqcup_{i\in I}X_i$. This disjoint union can be equivalently considered as a surjective function $\pi : X\rightarrow I$ whose fiber $\pi^{-1}(i)$ is X_i. Under this equivalence, building a set S by selecting $x_i\in X_i$ is the same as giving a function $s:I\rightarrow X$ such that $\pi(s(i))=i$. Therefore, the axiom can be written in the following equivalent way:

Axiom of Choice (*intermediary version*) *In* **Set** *any surjective morphism has a section.*

This new formulation seems purely categorical, except for the fact that the notion of "surjective morphism" is not canonical. In **Set**, the surjective maps are just the same as *epimorphisms*: functions f with the property that $g \circ f = h \circ f$ imply $g = h$. This condition makes perfect sense in any category, so that we could try to use epimorphisms as a model to the notion of "surjective morphisms".

In concrete categories $\mathbf{C} \subset \mathbf{Set}$ the morphisms are functions satisfying additional properties. Therefore, in this case it is expected that the correct model to "surjective morphisms" be given simply by surjective maps satisfying the additional properties. But there are concrete categories whose epimorphisms are **not** surjective maps satisfying properties, so that the notion of "epimorphism" is not the correct model to "surjective morphisms". For instance, in the category **Rng** of rings, the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism, but it is not a surjective function.

Therefore, we need another characterization of "surjectivity". In order to get it, notice that the surjective functions are just the coverings into the canonical Grothendieck topology of **Set**, so that the axiom of choice can also be interpreted in the following way:

Axiom of Choice. (*abstract version*) *In the site* $(\mathbf{Set}, J)$ *any covering has a section.*

This abstract formulation is finally completely categorical, so that we can talk of the validity of the "axiom of choice" in any site. For instance, in **Top** with the corresponding Grothendieck topology given by continuous surjections, the corresponding axiom is no longer valid, because there are continuous surjective maps without any global section. Indeed, for any topological group G, a G-principle bundle $\pi : P \to X$ has a section iff it is trivial. The axiom also fails in $\mathbf{Mod}_R$ with the Grothendieck topology of surjective homomorphisms, because in this case[12] the existence of a section for a surjective $f : X \to Y$ is equivalent to the existence of an isomorphism $X \simeq \ker f \oplus Y$, which generally is not true. But this is true when $R = \mathbb{K}$ is field, because in this case $X \simeq \ker f \oplus \operatorname{img} f$, by the rank-nullity theorem.

We notice that in *losing the axiom of choice we automatically lose any result whose proof depends explicitly on this axiom.* Specially, when doing category theory, the axiom of choice is crucial in the proof that a given functor $F : \mathbf{C} \to \mathbf{D}$ is an equivalence iff it is fully faithful[13] and essentially surjective[14]. More explicitly, *the axiom of choice on* $(\mathbf{Set}, J)$*, where J is the canonical topology, is equivalent to the statement that a* **Set***-internal functor is an equivalence between* **Set***-internal categories iff it is fully-faithful and essentially surjective.* Therefore, if the axiom of choice fails in a site $(\mathbf{H}, J)$, then we cannot get a analogous characterization for **H**-internal functors (i.e, we cannot conclude that an **H**-internal functor is an equivalence iff it is fully faithful and essentially surjective).

Remark. The above paragraph makes sense. Notice that in order to ask if a fully-faithful and essentially surjective **internal functor** is always an equivalence between **internal categories** we need to have these internal notions. In other words, we need to verify if internalization applies

[12]This is consequence of the *splitting lemma*, a fundamental result in Homological Algebra. You can read about this in any text on the subject, as [??,??].

[13]Recall that a functor is called *full* (resp. *faithful*) when it is surjective (resp. injective) on morphisms. Then, a *fully faithful* functor is one which is bijective on morphisms.

[14]Recall that a functor is called *essentially surjective* if it is surjective up to isomorphisms on objects. More precisely, if for any $Y \in \mathbf{D}_0$ there is $X \in \mathbf{C}_0$ such that $F(X) \simeq Y$.

to the usual notions of fully-faithful functor and essentially surjective functors, producing the corresponding internal versions. This is really the case, as explained below.

- *fully faithful property*. Recall that a usual functor $F : \mathbf{C} \to \mathbf{D}$ is fully faithful precisely when it is bijective on morphisms. In the source/target approach, this means that for any $g \in \mathbf{D}_1$ there is a unique $f \in \mathbf{C}_1$ such that $F_1(f) = g$ and whose source/target are respectively $F_0(s(f))$ and $F_0(t(f))$. Equivalently, the first diagram below is a limiting diagram. Observe that this limit diagram is indeed a sequence of pullback squares, so that it makes sense in any internalization ambient $\mathbf{H}$ and, therefore, the notion of fully faithful functors extends naturally to the internal context.

- *essentially surjectivity*. A usual functor is essentially surjective if it is surjective up to isomorphisms on objects. There is no problem with the notion of "isomorphisms" in any internalization ambient $\mathbf{H}$, but in order to talk about "surjective maps" we need to make a choice of Grothendieck topology J, so that what we actually define is "J-essentially surjective internal functors". Indeed, given a $\mathbf{H}$-internal functor $F : \mathbf{C} \to \mathbf{D}$, we say that it is *$J$-essentially surjective* when the segmented arrow presented in the second diagram below belongs to J. There, $\mathbf{D}_1^{iso}$ is the object discussed in Example 6.8. When $\mathbf{H}$ is $\mathbf{Set}$ and J is the topology given by the surjective functions we recover the classical definition of essentially surjective functor.

$$
\begin{tikzcd}
 & \mathbf{C}_1 \arrow[dl, "s"'] \arrow[d, "F_1"] \arrow[dr, "t"] & \\
\mathbf{C}_0 \arrow[d, "F_0"'] & \mathbf{D}_1 \arrow[dl, "s"] \arrow[dr, "t"'] & \mathbf{C}_0 \arrow[d, "F_0"] \\
\mathbf{D}_0 & & \mathbf{D}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{C}_0 \times_{\mathbf{D}_0} \mathbf{D}_1^{iso} \arrow[dl] \arrow[d] \arrow[dr, dashed] & \\
\mathbf{C}_0 \arrow[d, "F_0"'] & \mathbf{D}_1^{iso} \arrow[dl, "s"] \arrow[dr, "t"'] & \mathbf{D}_0 \arrow[d, "id"] \\
\mathbf{D}_0 & & \mathbf{D}_0
\end{tikzcd}
$$

Remark. The above conclusion that the failure of axiom of choice on $\mathbf{H}$ imply the loss of the characterization of internal equivalences suggest that *internal categorical language depends strongly on the internalization ambient*. This seems natural. Indeed, in order to pass from the concept of category to the concept of $\mathbf{H}$-internal category, **all** the set theoretic structure defining the category is replaced by analogues in $\mathbf{H}$, so it is natural to expect that the behavior of the internalization ambient $\mathbf{H}$ will reflect much of the behavior of $\mathbf{Cat}_{\mathbf{H}}$. On the other hand, when enriching a category, only the set of morphisms is replaced by an object of $\mathbf{H}$, so that a enriched category has one half of set theoretic structures and one half of enriched structures, meaning that *enriched categorical language is less sensible to the ambient than internal categorical language.* For instance, as will be discussed in the next subsection, this fact imply that we can **always** (even in the absence of the axiom of choice) characterize an equivalence between enriched categories as a fully faithful and essentially surjective enriched functor.

Now we would like to explain that, if in a internalization ambient $\mathbf{H}$ the axiom of choice fails, we can modify the category $\mathbf{Cat}_{\mathbf{H}}$ in order to get a new category in which any fully-faithful and essentially surjective internal functors are equivalences between internal categories. This is done by "adding formal inverses" to any internal functor with these properties. Indeed, fixed some

Grothendieck topology J in $\mathbf{H}$, if W_J is the class of fully-faithful and J-essentially surjective internal functors, then, "localizing" $\mathbf{Cat}_{\mathbf{H}}$ with respect to W_J we get the universal category in which each element of W_J is an isomorphism.

Let us see that in good cases the localized category $\mathbf{Cat}_{\mathbf{H}}[W_J^{-1}]$ has a very simple presentation. We start by recalling that this "localization approach" was used in other parts of the text. Indeed, when studying classical homotopy theory we have seen that $\mathbf{Top}_*$ is very complex. Then we passed to the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$, but this is also very complex. It happens that in the homotopical context we have natural invariants: the homotopy groups π_n. With them on hand we could define the class W of the *weak homotopy equivalences*, which are continuous functions f such that each $\pi_n(f)$ is an isomorphism. Localizing with respect to them we got the derived category $\mathbf{Top}_*[W^{-1}]$, which is much more tractable.

In this topological context, we also have a very special feature: there is a class of topological spaces (the class of CW-complexes) for which Whitehead's theorem holds, meaning that for them being a homotopy equivalence and being a weak homotopy equivalence is exactly the same. Furthermore, any topological space X is weak homotopically equivalent to a corresponding CW-complex ΓX. These facts allow us to give a concrete construction of $\mathbf{Top}_*[W^{-1}]$ by replacing each continuous based map $f : X \to Y$ by a pair $(\hat{f}, r)$, where $\hat{f} : \Gamma X \to Y$ is a continuous map and $r : \Gamma X \simeq X$ is a weak homotopy equivalence. In other words, in this case the localized category is equivalent to a category of special spans of CW-complexes, whose left leg belongs to W, as in the first diagram below.

$$
\begin{tikzcd}
 & Z \arrow[dl, squiggly, "W\ni"'] \arrow[dr] & \\
X & & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathbf{E} \arrow[dl, squiggly, "W_J\ni"'] \arrow[dr] & \\
\mathbf{C} & & \mathbf{D}
\end{tikzcd}
$$

Returning to the $\mathbf{H}$-internal context, it can be proven that, under some conditions over W_J (satisfying the so called *calculus of fractions*), an analogous concrete construction can be done with $\mathbf{Cat}_{\mathbf{H}}[W_J^{-1}]$. This means that it is equivalent to $\mathrm{Ana}_{(\mathbf{H},J)}$, whose objects are internal categories and whose morphisms are the spans (called *anafunctors*), presented in the second diagram above. For details, see [173, 77].

Enriched Categories

In the previous subsections we concluded that we can do some kind of "well behaved homotopy theory" with internal categories, but $\mathbf{H}$-internal category theory itself is not a source of new abstract concepts. Indeed, we have no direct and useful notions of "internal limit", "internal Kan extension", "internal end", and so on. Furthermore, the internal language is generally problematic when used as background language, because if the axiom of choice fails in the internalization ambient, we immediately lose many very useful conditions as, for instance, the assertion that "a functor is an equivalence iff it is fully faithful and essentially surjective".

Here we will show that for enriched category theory the situation is totally different. Indeed, we will show that enriched category theory not only is very abstract (in the sense that we have very natural notions of "enriched limit", "enriched Kan extension", etc), but it is also a nice background language, meaning that the sentence "a functor is an equivalência iff it is fully faithful and essentially surjective" holds in **any** internalization ambient **H** (even if the axiom of choice fails into **H**).

Remark. As commented later and as will become clear in the next paragraphs, this difference between the behavior of internal category theory and enriched category theory comes from a ingenuous fact: *in internal categories all information resided in the internalization ambient, while in enriched categories only one half is about the enrichment ambient.*

Let us see that the usual characterization of equivalences as "essentially surjective" and "fully-faithful" functors also holds in the enriched context. We start by noticing that both notions can be enriched without any problem. Indeed, for a monoidal category $(\mathbf{H}, \otimes, 1)$, because the objects any **H**-enriched category belongs to a **set**, we can define a *essentially surjective enriched functor* $F : \mathbf{C} \to \mathbf{D}$ between **H**-enriched categories as a usual enriched functor such that

$$F_0 : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$$

is surjective up to isomorphisms. Similarly, we say that it is *fully-faithful* when, for any two objects $X, Y \in \mathbf{C}$ the corresponding morphism

$$F_{xy} : \mathbf{H_C}(X, Y) \to \mathbf{H_D}(F(X), F(Y))$$

is an isomorphism of **H**. Now, notice that, because F_{xy} are assumed isomorphisms, they have inverse, so that in order to prove that an enriched functor is an equivalence iff it is essentially surjective and fully faithful we only need to build an inverse for F_0.

In the non-enriched context, this is done by making use of the axiom of choice, but this immediately applies here into the enriched context, because on objects any enriched functor acts as a usual function between sets! Therefore, *even if the axiom of choice fails into the enrichment ambient* **H***, the required result is valid.*

The fact that the objects of an enriched category fit into a set was crucial in order to conclude that the failure of the axiom of choice into **H** does not affect **H**-enriched category theory. Let us see that this fact also allow us to prove that enriched categorical language is a very abstract language. More precisely, let us see that the notions of natural transformation, Kan extensions and limits can be naturally enriched over any monoidal category. We refer the reader to [110, 59, 172] for more details.

Recall that a natural transformation $\xi : F \Rightarrow G$ between usual functors $F : \mathbf{C} \to \mathbf{D}$ is a family of morphisms $\xi_X : F(X) \to G(X)$ parametrized by the **set of objects** of **C**, such that the diagram below is commutative for any element $f \in \mathrm{Mor}_{\mathbf{C}}(X; Y)$ of the **set of morphisms** from X to Y.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\xi_X"'] & F(Y) \arrow[d, "\xi_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

When defined in this way, the notion of natural transformation **cannot** be directly enriched over any monoidal category **H** because, in principle, despite we have a set of objects in any enriched category, generally we only have an object $\mathbf{H_C}(X, Y)$ of morphisms (which is not a set if **H** is not concrete). However, this problem can be easily avoided by noticing that a morphism $f : X \to Y$ can be equivalently described by a function $1 \to \mathrm{Mor}_{\mathbf{C}}(X, Y)$, where 1 is a unit set.

In this new characterization, the notion of "natural transformation" really can be enriched, producing the notion of *enriched natural transformation between enriched functors*. Indeed, given **H**-enriched functors $F, G : \mathbf{C} \to \mathbf{D}$, we define an *enriched transformation* between them as a family $\xi_X : 1 \to \mathbf{H}_{\mathbf{D}}(F(X), G(X))$, parametrized by the set of objects of **C**, such that the diagram below[15] is commutative for any $f : 1 \to \mathbf{H}_{\mathbf{C}}(X, Y)$.

$$\begin{tikzcd}
\mathbf{C}(X,Y) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & 1 \otimes \mathbf{C}(X,Y) \arrow[r, "\xi_X \otimes G_{x,y}"] & \mathbf{D}(F(X),G(X)) \otimes \mathbf{D}(G(X),G(Y)) \arrow[d, "\circ_{F(x)G(x)G(y)}"] \\
\mathbf{C}(X,Y) \otimes 1 \arrow[r, "F_{xy} \otimes \xi_Y"'] & \mathbf{D}(F(X),F(Y)) \otimes \mathbf{D}(F(Y),G(Y)) \arrow[r, "\circ_{F(x)F(y)G(y)}"'] & \mathbf{D}(F(X),G(Y))
\end{tikzcd}$$

Now, recall that (as introduced in Section 2.1) given two usual functors $F : \mathbf{A} \to \mathbf{D}$ and $\imath : \mathbf{A} \to \mathbf{C}$, the *left Kan extension* of F with respect to $\imath$ is the universal left approximation to the extension problem below. In other words, it is a pair (L, ξ), where $L : \mathbf{C} \to \mathbf{D}$ is a functor and $\xi : F \Rightarrow L \circ \imath$ is a natural transformation which is universal in the sense that for any other pair (L', ξ') there exists a unique $u : L' \Rightarrow L$ such that the second diagram below commutes.

$$\begin{tikzcd}
& \mathbf{D} \\
\mathbf{A} \arrow[ur, "F"] \arrow[r, "\imath"'] & \mathbf{C} \arrow[u, "L"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& L' \circ \imath \\
F \arrow[ur, Rightarrow, "\xi'"] \arrow[r, Rightarrow, "\xi"'] & L \circ \imath \arrow[u, Rightarrow, "u \circ \imath"']
\end{tikzcd}$$

We notice that this concept (and its dual version) can be immediately enriched over any monoidal category $(\mathbf{H}, \otimes, 1)$, giving the notions of "left/right enriched Kan extension" of a **H**-enriched functor respectively to other. So, *enriched categorical language is abstractly enough to abstract the very general notion of Kan extensions and, therefore, of limits and colimits.*

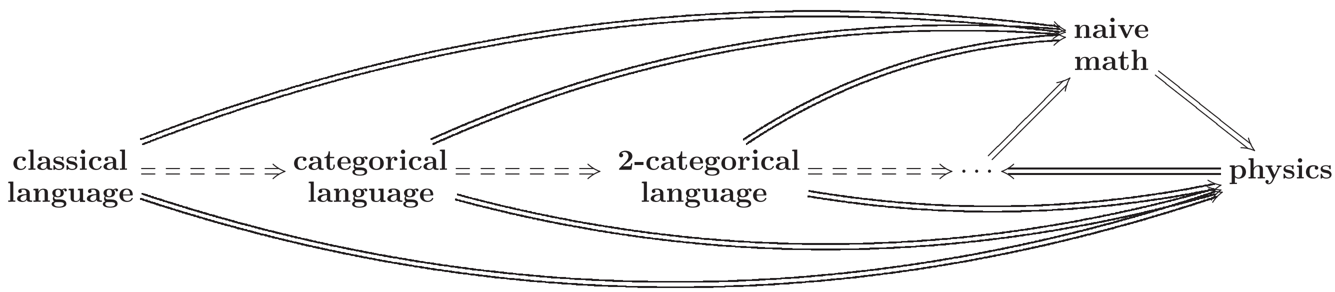
[15]We have been written $\mathbf{C}(X, Y)$ in order to abbreviate $\mathbf{H}_{\mathbf{C}}(X, Y)$.

Part III

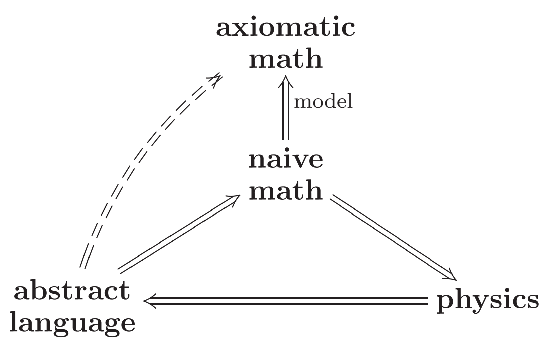
Higher Categorical Language

Prologue

In order to attack Hilbert's sixth problem we have to fix a background language and study the relation between the corresponding naive mathematics and the foundations of physics. In the first part we discussed that the correct background language should be obtained by iterating certain "categorification process", as in the following diagram:



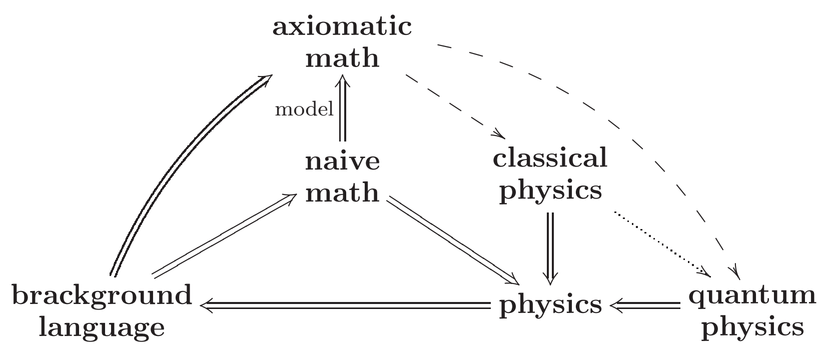
In the second part we formalized this "categorification process", so that now we can search a lifting from "naive mathematics" to "axiomatic mathematics", as below. This is what will be done in this part.



Indeed, recall that in the last chapter we concluded that there are two ways to categorify the concept of "category": by internalization and by enrichment, which give, respectively, "internalized category theory" and "enriched category theory". While the first produce a language that has no good properties (as the abstract axiom of choice), the second seems to be promised. With this in mind, in Chapter 7 we iterate the process of "categorification by enrichment", getting ∞-category theory.

In the other chapters we show that ∞-categorical language (also called *higher categorical language*) really is a very nice abstract background language, in the sense that any interesting classical concept admits a "higher categorical analogue", i.e, it can be internalized into some class of ∞-categories. More precisely, in Chapter 8 we study ∞-category theory properly and we show that the class of $(\infty,1)$-*categories* is a natural context in which we can internalize homotopy theory. In Chapter 9 we see that stable homotopy theory can also be internalized in such class of $(\infty,1)$-categories. Finally, in Chapter 10 we study the *geometric ∞-topos*, which constitute a very general ambient to internalize (differential) geometry.

Having developed a very nice abstract background language, we can effectively attack Hilbert's sixth problem. This will be done in Parts IV and V. Indeed, in Part IV we will show that the concept of "classical physics" can be internalized into any geometric ∞-topos and in Part V we will discuss that the same is valid for "quantum physics". Furthermore, we will study a process that attempt to connect both axiomatizations of classical and quantum physics, meaning that we will complete the "Hilbert's sixth problem diagram".



Warning. This will be the most abstract part of the text. The reader who survive to it will not encounter any difficulty in the remaining parts.

Chapter 7

Construction

In the last chapter we concluded that "enriched categorical language" generalizes classical categorical language, as required by Hilbert's sixth problem. Motivated by this fact, in the present chapter we will iterate the enrichment process with the objective to build a very abstract class of languages.

Indeed, we start Section 7.1 by showing that for any monoidal category $(\mathbf{H}, \otimes, 1)$ the associated category $\mathbf{Cat}(\mathbf{H})$ of all $\mathbf{H}$-enriched categories and all enriched functors is itself a monoidal category, allowing us to consider "categories enriched over $\mathbf{Cat}(\mathbf{H})$", which will be the base case of our induction process. The resultant entities are composed not only by objects and 1-morphisms, but also for any two 1-morphisms a corresponding "space of morphisms between 1-morphisms", called 2-*morphisms*. These "categories enriched over categories" are also known as 2-*categories*. We discuss that the standard way to build a 2-category is when we have an "homotopy category" associated to some category. In this case, the 2-morphisms are just (isomorphic classes of) homotopies between 1-morphisms.

With the base case on hand, we show that we also have the induction step, allowing us to define n-categories for each value of n. They fit into a category $n\mathbf{Cat}(\mathbf{H})$ and we prove that varying n we get an inductive system internal to $\mathbf{Cat}$. Thanks to the cocompleteness of $\mathbf{Cat}$ this system has colimit, which we call the *category of ∞-categories enriched over* $\mathbf{H}$. We also see that for each n the category $n\mathbf{Cat}(\mathbf{H})$ is itself an $(n+1)$-category and we use this fact to give a vast generalization of the Weakening Principle, implying that the obtained ∞-categorical language is very abstract, as desired.

On the other hand, in Section 7.2 we discuss that, despite the the abstractness of this new language, many intuitive examples which should be ∞-category are not. The problem is that, while in the definition of ∞-category the compositions are assumed strictly associative, in the examples they are only associative up to higher morphisms. We then modify our definition in order to incorporate the examples, but we conclude that the obtained definition is impracticable! This reveals that we need a more careful examination of the higher categorical notions, leading to many realizations (also called *presentations*) of them. Unfortunately, they will not be discussed here. We refer the reader to [127] for the standard of those approaches.

7.1 Iterating

Recall that the notion of "category" (as category with hom-sets) can be enriched over any monoidal category $(\mathbf{H}, \otimes, 1)$, producing the concept of "$\mathbf{H}$-enriched category". As discussed in Section 6.1, this is an entity $\mathbf{C}$ composed by a set $\mathrm{Ob}(\mathbf{C})$ of objects, for any two objects X, Y an object $\mathbf{H}_{\mathbf{C}}(X, Y) \in \mathbf{H}$ of morphisms, for three objects a composition-morphism

$$\circ_{xyz} : \mathbf{H}_{\mathbf{C}}(X, Y) \otimes \mathbf{H}_{\mathbf{C}}(Y, Z) \to \mathbf{H}_{\mathbf{C}}(X, Z),$$

which fulfill "associativity-type" diagrams, and for any X a distinguished $id_x : 1 \to \mathbf{H}_{\mathbf{C}}(X, X)$ satisfying commutative diagrams which translate the "neutral element" property.

The notion of "functor between categories" can also be enriched over $\mathbf{H}$, giving "enriched functors between $\mathbf{H}$-enriched categories". Indeed, if $\mathbf{C}$ and $\mathbf{D}$ are $\mathbf{H}$-enriched categories, then an enriched functor between them is specified by a function

$$F_0 : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D}),$$

mapping objects into objects, and for any two objects a corresponding morphism

$$F_{xy} : \mathbf{H}_{\mathbf{C}}(X, Y) \to \mathbf{H}_{\mathbf{D}}(F_0(X), F_0(Y)),$$

in such a way that the compositions $\circ_{xyz}$ and the identity morphisms id_x are preserved. This means that $\mathbf{H}$-enriched categories and enriched functors fits into a category $\mathbf{Cat}(\mathbf{H})$.

Here we would like to notice that, if $(\mathbf{H}, \otimes, 1)$ is a concrete category freely generated by sets (i.e, if there is a forgetful functor $\imath : \mathbf{H} \to \mathbf{Set}$ which admits a left adjoint $\jmath : \mathbf{Set} \to \mathbf{H}$), then its monoidal structure $(\otimes, 1)$ induce a monoidal structure $(\otimes_{\mathbf{H}}, \mathbf{1}_{\mathbf{H}})$ on $\mathbf{Cat}(\mathbf{H})$. Indeed, for two given $\mathbf{H}$-enriched categories $\mathbf{C}$ and $\mathbf{D}$ we define $\mathbf{C} \otimes_{\mathbf{H}} \mathbf{D}$ as the $\mathbf{H}$-enriched category whose set of objects is the product

$$\jmath(\mathrm{Ob}(\mathbf{C})) \otimes \jmath(\mathrm{Ob}(\mathbf{D})),$$

regarded as a set, whose objects of "morphisms between products" are given by "products between morphisms", i.e,

$$\mathbf{H}_{\mathbf{C} \otimes_{\mathbf{H}} \mathbf{D}}(X \otimes Y, X' \otimes Y') := \mathbf{H}_{\mathbf{C}}(X, X') \otimes \mathbf{H}_{\mathbf{D}}(Y, Y'),$$

and whose compositions and identities are defined analogously. The neutral object of $\mathbf{Cat}(\mathbf{H})$ respectively to this new product is the trivial $\mathbf{H}$-enriched category $\mathbf{1}_{\mathbf{H}}$, whose set of objects is 1 (regarded as a set), whose object of morphisms also is 1, whose identity morphism is just the identity map $1 \to 1$ and whose composition is the canonical isomorphism $1 \otimes 1 \simeq 1$.

Because $(\mathbf{Cat}(\mathbf{H}), \otimes_{\mathbf{H}}, \mathbf{1}_{\mathbf{H}})$ is monoidal, we can now consider categories enriched over it! In other words, *for a fixed monoidal category* $(\mathbf{H}, \otimes, 1)$ *we can consider categories enriched over the category of all* $\mathbf{H}$*-enriched categories!* These are very abstract entities $\mathbf{C}$ composed by a set of objects $\mathrm{Ob}(\mathbf{C})$, for any two objects a corresponding $\mathbf{H}$-enriched category of morphisms

$$\mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Y) \in \mathbf{Cat}(\mathbf{H}), \tag{7.1.1}$$

for any three objects an enriched functor

$$\circ_{xyz} : \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Y) \otimes_{\mathbf{H}} \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(Y, Z) \to \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Z) \tag{7.1.2}$$

and for any object a identity functor

$$id_x : \mathbf{1_H} \to \mathbf{Cat(H)_C}(X, X), \tag{7.1.3}$$

such that every usual associativity-type and neutral element type diagram are satisfied. More explicitly, such a entity is determined by the following data:

1. a set of objects;
2. for any two objects a set, whose elements we call of 1-*morphisms between* X *and* Y. In the sequence we will write simply $1\mathrm{Mor}_{\mathbf{C}}(X, Y)$ in order to denote this set;
3. for any two 1-morphisms $f, g : X \to Y$ between two fixed objects a corresponding object of "morphisms between 1-morphisms", which will be denoted by $\mathbf{2H}_{xy}(f, g)$;
4. for any three 1-morphisms between two fixed objects a composition of "morphisms between 1-morphisms" (which we call of 2-*morphisms*) given by a morphism
$$\bullet_{fgh} : \mathbf{2H}_{xy}(f, g) \otimes \mathbf{2H}_{xy}(g, h) \to \mathbf{2H}_{xy}(f, h);$$
5. for any three objects a notion of "composition between 1-morphisms" given by a function
$$\circ_{xyz} : 1\mathrm{Mor}_{\mathbf{C}}(X, Y) \otimes 1\mathrm{Mor}_{\mathbf{C}}(X, Y) \to 1\mathrm{Mor}_{\mathbf{C}}(X, Y);$$
6. for any three morphisms between different objects another "composition of 2-morphisms", presented by a morphism
$$\circ_{fgh} : \mathbf{2H}_{xy}(f, g) \otimes \mathbf{2H}_{yz}(g, h) \to \mathbf{2H}_{xz}(f, h)$$
7. for each object and for each 1-morphism between objects a function and a morphism
$$id_x : 1 \to 1\mathrm{Mor}_{\mathbf{C}}(X, X) \quad \text{and} \quad id_f : 1 \to \mathbf{2H}(f, f),$$

which together satisfy many *compatibility conditions*, associativity-type and neutral element-type diagrams. Notice that in this correspondence the data (2.), (.3) and (4.) correspond to (7.1.1), while (5.) and (6.) correspond to (7.1.2) and (6.) corresponds to (7.1.3). The "compatibility conditions" referred above are those describe the functoriality of the rules (7.1.2) and (7.1.3). So, for instance, they imply

$$(a \bullet_{fgh} b) \circ_{fgh} (a' \bullet_{fgh} b') = (a \circ_{fgh} a') \bullet_{fgh} (b \circ_{fgh} b'),$$

meaning that the two composition laws for 2-morphisms are, in some sense, compatible.

Examples

Before proceeding to the full abstraction, attempting to illustrate the last concept, let us present some concrete examples.

Example 7.1 (*bimodules*)**.** Let **Rng** be the category of rings with unity. To any object R we can associate two other categories L$\mathbf{Mod}_R$ and R$\mathbf{Mod}_R$, of left and right R-modules. Let us see that, putting together all these categories, we can define a 2-category 2**Rng**. Its objects will be rings, a 1-morphism $X : R \to S$ between two rings will be a (R, S)-bimodule ${}_RX_S$ (i.e, a left R-module X which is also right S-module in such way that the two sctructures are compatible) and its 2-morphisms $\xi : {}_RX_S \to_{R'} X'_{S'}$ will be morphisms of bimodules (i.e, abelian groups homomomorphisms $\xi : X \to X'$ which preserve simultaneously both left and right actions in the sense that $\xi(r \cdot x \cdot s) = r \cdot \xi(x) \cdot s$). In order to complete the definition of this 2-category we need to define compositions between 1-morphisms and between 2-morphisms, as well as identity maps. So, starting with three objects R, S, T and given bimodules ${}_RX_S$ and ${}_SY_T$ describing 1-morphisms $X : R \to S$ and $Y : S \to T$, the main idea is to define a third 1-morphism $Y \circ_{rst} X : R \to T$ as the tensor bimodule $({}_RX_S) \otimes_S ({}_SY_T)$. It happens that the compositions must be associative, but the tensor product is associative only up to isomorphisms, so that we actually have to define $Y \circ_{rst} X : R \to T$ as the *isomorphism class* of the tensor bimodule $({}_RX_S) \otimes_S ({}_SY_T)$. Given a ring R, its identity $id_r : R \to R$ is just R regarded as a (R, R)-bimodule. Furthermore, if $\xi : {}_RX_S \Rightarrow_R Y_S$ and $\zeta : {}_RY_S \Rightarrow_R Z_S$ are two 2-morphisms between 1-morphisms parametrized by the same objects, we define their composition $\zeta \circ_{xyz} \xi$ as the bimodule morphism whose underlying abelian group homomorphism is precisely the composition $\zeta \circ \xi$ of the underlying abelian group homomorphisms. It is clearly associative. On the other hand, if were now given 2-morphisms $\xi : {}_RX_S \Rightarrow_S Y_T$ and $\zeta : {}_SY_T \Rightarrow_S Z_T$ for consecutive objects, we define its composition $\zeta \bullet_{xyz} \xi$ as the product $\zeta \otimes \xi$. It is associative because we are working with isomorphism classes of tensor products. The identity 2-morphism of a 1-morphism $X : R \to S$ is the bimodule morphism $id_x :_R X_S \Rightarrow_R X_S$ whose underlying abelian group homomorphism is the identity map.

Two remarks on the last example:

1. we proved that 2**Ring** is a 2-category enriched over the cartesian $(\mathbf{Set}, \times, 1)$. We notice, however, that it is indeed enriched over $(\mathbf{Ab}, \otimes, \mathbb{Z})$. In fact, for any two given 1-morphisms, say $X : R \to S$ and $X' : R' \to S'$, the corresponding set of 2-morphisms $2\mathrm{Mor}^{rs}_{\mathbf{2Ring}}(X; Y)$, which is the set of bimodule morphism, is naturally an abelian group with the operation of sum of morphisms;

2. the example can be generalized. Observe that **Ring** is the category of monoid objects of the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$. We can reproduce the example if we consider other ambient symmetric monoidal category $(\mathbf{H}, \otimes, 1)$. Indeed, if $R, S \in \mathrm{Mon}(\mathbf{H}, \otimes)$ is a monoid object, we have the notion of (R, S)-bimodule object and the notion of morphism between bimodules, so that we can build a category $\mathrm{BiMod}_{(R,S)}(\mathbf{H}, \otimes)$. Because $\mathbf{H}$ was assumed symmetric, its monoidal structure $(\otimes, 1)$ induce a corresponding monoidal structure $(\otimes_M, 1)$ in $\mathrm{Mon}(\mathbf{H}, \otimes)$, as discussed in Section 5.1. This structure induce, in turn, bifunctors

$$\otimes_B : \mathrm{BiMod}_{(R,S)}(\mathbf{H}, \otimes) \times \mathrm{BiMod}_{(S,T)}(\mathbf{H}, \otimes) \to \mathrm{BiMod}_{(R,T)}(\mathbf{H}, \otimes)$$

for any three given monoid objects. We can then define a 2-category $2\mathbf{H}$ analogously as we defined $2\mathbf{Ring}$: by considering the set of objects as the set of monoid objects, the set of 1-morphisms between two monoid objects as the set of (isomorphic classes of) bimodule objects over them, and 2-morphisms as morphisms of bimodule objects. The composition of 1-morphisms is the action of the product $\otimes_B$ in objects. The composition of 2-morphisms is the action of $\otimes_B$ in morphisms.

Example 7.2 (*homotopy category*)**.** Let $\mathscr{C} \subset \mathbf{Top}$ be a convenient category of topological spaces. As discussed in Section 4.2, this category is cartesian closed, so that by Example 6.3 it is enriched over itself. Consequently, by considering the quotient topology, the homotopy category $\mathrm{Ho}(\mathscr{C})$ also becomes enriched over **Top**. We can join the categories $\mathscr{C}$ and $\mathrm{Ho}(\mathscr{C})$ in order to obtain a 2-category $2\mathbf{Top}$ fully enriched over $(\mathbf{Top}, \times, *)$. Indeed, we consider topological spaces as objects, continuous maps as 1-morphisms and hotomopies between continuous maps as 2-morphisms. The composition of 1-morphisms are the usual composition of functions. Therefore, the identity of each space X is just the identity map id_x. We can think of a homotopy between two functions as a continuous path H_t connecting them[1]. So, given three morphisms $f, g, h : X \to Y$ and two homotopies $H : f \Rightarrow g$ and $H' : g \Rightarrow h$ we define its composition $H' \bullet_{xyz} H$ as that corresponding to the concatenation of paths $H'_t \# H_t$, as defined in Example 4.5. Explicitly,

$$(H' \bullet_{fgh} H)(t,x) = \begin{cases} H(2t,x), & 0 \leq t \leq 1/2 \\ H'(2t-1,x), & 1/2 < t \leq 1. \end{cases}$$

Furthermore, given homotopies $H : f \Rightarrow g$ and $H' : g \Rightarrow h$, where f, g and h are functions between consecutive spaces, we define its composition

$$H' \circ_{fgh} H : g \circ_{xyz} f \Rightarrow h \circ_{yzw} g \quad \text{as} \quad (H' \circ_{fgh} H)(t,x) = H'(t, H(t,x)).$$

The identity of a 1-morphism f should be the trivial homotopy, induced by the constant path at f. But, we have a problem: the concatenation of paths is not associative at time that the constant path is not a neutral element for this operations. Indeed, we only have associativity and neutral element property up to path homotopies. Therefore, in order to get a 2-category we have to consider homotopy class of homotopies as 2-morphisms. Now, that $2\mathbf{Top}$ is a 2-category is clear. Let us explain why it is fully enriched over **Top**. Putting the compact-open topology in the sets of 1-morphisms they become naturally topological spaces in such a way that the composition of 1-morphisms are continuous. Therefore, as a 1-category, $2\mathbf{Top}$ is enriched over **Top**. In order to get the enrichment also in the 2-categorical context, recall that the sets of 2-morphisms are quotient spaces of sets of homotopies. But homotopies are, itself, continuous maps. So, we can first take the compact-open topology and then the quotient topology.

Example 7.3 (*algebraic homotopy category*)**.** Recall that, as discussed in Section 1.2, we have a homotopy theory for chain complexes which is very similar to the homotopy theory for CW-complexes. On the other hand, by definition, CW-complexes belong to any convenient category of topological spaces. The last example shows that a convenient category of topological spaces $\mathscr{C}$ (and, in particular, the subcategory of CW-complexes) define a 2-category $2\mathbf{Top}$ enriched over

[1] This makes perfect sense in the compact-open topology, because we are working in the category of convenient topological spaces.

Top by putting together $\mathscr{C}$ and Ho($\mathscr{C}$). Therefore, it is natural to expect that for any ring R we also have a 2-category $2\mathbf{Ch}_R$ obtained putting together $\mathbf{Ch}_R$ and its homotopy category Ho($\mathbf{Ch}_R$). This is really the case. Indeed, we consider chain complexes as objects, chain maps as 1-morphisms and algebraic homotopies as 2-morphisms. The composition of 1-morphisms is just the usual composition (the identity of a cochain complex is the identity chain map). In order to define the compositions between 2-morphisms we recall that a chain homotopy $H : f \Rightarrow g$ between chain maps $f, g : X_* \to Y_*$ is a sequence of homomorphisms $H_n : X_n \to Y_{n+1}$ such that

$$f_n - g_n = \partial_n^Y \circ H_n + H_{n-1} \circ \partial_{n-1}^X.$$

In the topological case, we defined the composition of homotopies by first considering a topology in the set of morphisms, showing that homotopies can be regarded as paths in the space of maps and then taking the composition as concatenation of paths. So, the idea is to try the same here. Because the monoidal category $(\mathbf{Ch}_R, \otimes, \overline{R})$ is closed, it follows that $\mathbf{Ch}_R$ is enriched over itself, so that for the set of chain maps between two cochain complexes can be naturally regarded as cochain complex. The next step is to define "path in a cochain complex". A path between two points x and y of topological space X is simply a continuous function $\gamma : I \to X$, where here I is the internal $[0,1]$, such that $\gamma(0) = x$ and $\gamma(1) = y$. This last conditions correspond precisely to the commutativity of the following diagram:

$$\begin{tikzcd}
 & I \arrow[d, "\gamma"] & \\
* \arrow[ur, "0"] \arrow[r, "x"'] & X & * \arrow[ul, "1"'] \arrow[l, "y"]
\end{tikzcd}$$

Therefore, introducing the notion of "internal object" in $\mathbf{Ch}_R$ we could define an *algebraic path* in X_* as a chain map $I_* \to X_*$. One can see $[0,1]$ as a CW-complex with two 0-cells (the points 0 and 1) and one 1-cell connecting them. The algebraic analogue of this data is as below, where $R \oplus R$ is in degree zero.

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow R \xrightarrow{id \oplus (-id)} R \oplus R \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

It can be directly verified that giving a chain homotopy $H : f \Rightarrow g$, with $f, g : X_* \to Y_*$, is the same as giving an algebraic path $H : I_* \to \mathbf{Ch}_R(X_*, Y_*)$ connecting f and g, in the sense that the diagram below commutes.

$$\begin{tikzcd}
 & I_* \arrow[d, "H"] & \\
\overline{R} \arrow[ur, "0"] \arrow[r, "f"'] & \mathbf{Ch}_R(X_*, Y_*) & \overline{R} \arrow[ul, "1"'] \arrow[l, "g"]
\end{tikzcd}$$

So, in order to conclude that $2\mathbf{Ch}_R$ actually is a 2-category we have to prove that the notion of "concatenation of paths" can be abstracted to the algebraic context. But this can be easily done, because the concatenation of paths $\gamma : x \to y$ and $\lambda : y \to z$ is characterized as the path $\gamma \# \lambda$ making commutative the diagram below and this diagram has an evident algebraic analogue.

As in the topological case, the concatenation is associative only up to homotopy, so that replacing the space of algebraic homotopies by the space of algebraic homotopy classes of algebraic homotopies we finally get the desired 2-category structure. It is enriched over $\mathbf{Ch}_R$ because the space of homotopies can be regarded as a chain complex and this additional structure passes to the quotient.

Example 7.4 (*categories with path objects*)**.** We can imitate the last examples in order to build a 2-category $2\mathbf{H}$ starting with a monoidal category $(\mathbf{H}, \otimes, 1)$ enriched over itself in which we have the notion of "path in an object X" corresponding to a class of morphisms $\gamma : I \to X$, where I some kind of "interval object", and the notion of "concatenation of paths". Indeed, with the notion of "paths" we consider the "space of paths" of any object X, denoted by $\mathrm{Path}(X)$, which can be regarded as an object of $\mathbf{H}$ (because $\mathbf{H}$ is assumed enriched over itself). We then define a *homotopy* between two morphisms $f, g : X \to Y$ as a path in the object of morphisms $\mathbf{H}(X; Y)$. The homotopy relation is an equivalence relation, so that we can define the "homotopy category" $\mathrm{Ho}(\mathbf{H})$ of $\mathbf{H}$ as the category whose objects are objects of $\mathbf{H}$ and whose morphisms are homotopy classes of morphisms of $\mathbf{H}$. Finally, we define the desired 2-category $2\mathbf{H}$ by putting together the structures of $\mathbf{H}$ and $\mathrm{Ho}(\mathbf{H})$. Indeed, it is the entity whose objects are objects of $\mathbf{H}$, whose 1-morphisms are morphisms of $\mathbf{H}$ and whose 2-morphisms are homotopy classes of homotopies between 1-morphisms in the sense above. The composition of 1-morphisms is the composition of $\mathbf{H}$ and the non-obvious composition of 2-morphisms are that induced by the assumed notion of "concatenation of paths". For instance, as a particular example of this very general situation, we obtain that not only $\mathbf{Top}$ and $\mathbf{Ch}_R$, but also the category $\mathbf{Spec}$ of sequencial spectra induces a 2-category $2\mathbf{Spec}$.

n-Categories

Here we notice that not only the notion of category, but also the concept of "functor between categories" can be enriched over the monoidal category $(\mathbf{Cat}(\mathbf{H}), \otimes_{\mathbf{H}}, \mathbf{1}_{\mathbf{H}})$. Indeed, given two $\mathbf{Cat}(\mathbf{H})$-enriched categories $\mathbf{C}$ and $\mathbf{D}$, a enriched functor between them is explicitly determined by the following data:

1. a function F_0 mapping objects into objects;
2. for any two objects a function F_{xy} mapping 1-morphisms into 1-morphisms;
3. for any two 1-morphisms a corresponding morphism F_{fg} mapping "morphisms between 1-morphisms" into "morphisms between 1-morphisms";

such that each composition law (of 1-morphisms or of "morphisms between 1-morphisms") and each identity is preserved. Therefore, we have the category $\mathbf{Cat}(\mathbf{Cat}(\mathbf{H}))$ of categories enriched over $\mathbf{Cat}(\mathbf{H})$, which we denote by $2\mathbf{Cat}(\mathbf{H})$.

It becomes monoidal by applying the same process used to build a monoidal structure on the category of all $\mathbf{H}$-enriched categories from the monoidal structure of $\mathbf{H}$. Therefore, we can now consider categories enriched over $2\mathbf{Cat}(\mathbf{H})$! These are entities defined by a set of objects, a set of 1-morphisms between objects, a set of 2-morphisms and an object of "morphisms between 2-morphisms", which we call of 3-*morphisms*. In addition, we have one composition law for 1-morphisms, two composition laws for 2-morphisms and three composition laws for 3-morphisms, which are compatible and associative. Finally, to each 1-morphism, 2-morphism and 3-morphism we also have a corresponding identity morphism that fulfill the neutral element property. The reader is invited to explicit the data (and the commutative diagrams satisfied by them) defining this kind of entity in same way as we done for categories enriched over $\mathbf{Cat}(\mathbf{H})$.

The notion of "functor" also enriches over $2\mathbf{Cat}(\mathbf{H})$, defining a category

$$\mathbf{Cat}(2\mathbf{Cat}(\mathbf{H})) = \mathbf{Cat}(\mathbf{Cat}(\mathbf{Cat}(\mathbf{H}))),$$

which we denote by $3\mathbf{Cat}(\mathbf{H})$. This category is itself monoidal, so that in it the notions of "category" and "functor" can be internalized, producing $4\mathbf{Cat}(\mathbf{H})$, and so on. In other words, by induction we can define

$$n\mathbf{Cat}(\mathbf{H}) = \mathbf{Cat}(\mathbf{Cat}(\mathbf{Cat}(...\mathbf{Cat}(\mathbf{H}))))$$

for every n! An object of this category is an entity containing a set of objects, a set of 1-morphisms, and so on, up to a set of $(n-1)$-morphisms, together with an object of n-morphisms. For each $1 \leq k \leq n$ we have exactly k different composition laws for k-morphisms, which are associative and mutually compatible in a functorial sense. Additionally, to each object and to each k-morphisms we have an identity morphism fulfilling the neutral element property.

An enriched functor between two of these entities is just a sequence of rules, mapping objects into objects and k-morphisms into k-morphisms in such a way that every composition and every identity is preserved.

Remark. Because the data defining it, we usually say that an $(n-1)\mathbf{Cat}(\mathbf{H})$-enriched category is an $\mathbf{H}$-*enriched* n-*category*. Therefore, by an $\mathbf{H}$-enriched n-category we mean an entity that has k-morphisms, for $0 \leq k \leq n$, where here "0-morphisms" means "objects". Furthermore, for $k = 0, ..., n-1$ the data belongs to $\mathbf{Set}$ (i.e, we have a **set** of objects, a **set** of 1-morphisms, a **set** of 2-morphisms, and so on, up to $(n-1)$-morphisms) and for $k = n$ it belongs to $\mathbf{H}$ (i.e, we have a **object** of n-morphisms). In the special case in that $(\mathbf{H}, \otimes, 1)$ is $(\mathbf{Set}, \times, 1)$, every information belongs to $\mathbf{Set}$ and we say simply that we have an n-*category*.

∞-Categories

For a given (concrete and freely generated) monoidal category $(\mathbf{H}, \otimes, 1)$, in the last subsection we obtained the notion of "n-category enriched over $\mathbf{H}$" by iterating the enrichment process. We notice that each of these n-categories can be trivially regarded as a $(n+1)$-category by adding a trivial object of $(n+1)$-morphisms.

More precisely, for a given n-category $\mathbf{C}$ enriched over $\mathbf{H}$ we define a corresponding $(n+1)$-category $\imath_n\mathbf{C}$, whose set of objects is the set of objects of $\mathbf{C}$, whose sets of 1-morphisms are also the sets of 1-morphisms of $\mathbf{C}$, and so on, up to sets of $(n-1)$-morphisms. The sets of n-morphisms of $\imath_n\mathbf{C}$ are the objects of n-morphisms of $\mathbf{C}$, regarded as sets (i.e, are their images under the inclusion functor $\imath : \mathbf{H} \hookrightarrow \mathbf{Set}$). Finally, the objects of $(n+1)$-morphisms of $\imath_n\mathbf{C}$ are all given by the neutral object 1, all composition laws for n-morphisms are $1 \otimes 1 \simeq 1$ and all identity morphisms are equal to $id_1 : 1 \to 1$. The construction extends to enriched functors, defining a functor

$$\imath_n : n\mathbf{Cat}(\mathbf{H}) \to (n+1)\mathbf{Cat}(\mathbf{H}).$$

Therefore, we can build the sequence below. It is a sequence internal to **Cat**, which is a cocomplete category. Consequently, this sequence has a colimit, meaning that we can define the notion of **H**-*enriched* ∞-*categories!* More precisely, the colimit is a category $\infty\mathbf{Cat}(\mathbf{H})$ whose objects we call of ∞-*categories enriched over* **H** and whose morphisms we call *enriched* ∞-*functors.*

$$0\mathbf{Cat}(\mathbf{H}) \longrightarrow 1\mathbf{Cat}(\mathbf{H}) \longrightarrow 2\mathbf{Cat}(\mathbf{H}) \longrightarrow 3\mathbf{Cat}(\mathbf{H}) \longrightarrow \cdots \tag{7.1.4}$$

So, concretely, an **H**-enriched ∞-category **C** is composed by the following data:

1. a set $\mathrm{Ob}(\mathbf{C})$ of objects;

2. for any two fixed objects X and Y, a set $1\mathrm{Mor}_{\mathbf{C}}(X;Y)$ of 1-morphisms $f : X \to Y$ between them;

3. for any two 1-morphisms $f, g : X \to Y$, a set $2\mathrm{Mor}^{xy}_{\mathbf{C}}(f;g)$ of 2-morphisms $\xi : f \Rightarrow g$ between them;

4. for any two 2-morphisms $\xi, \zeta : f \Rightarrow g$ between the same 1-morphisms $f, g : X \to Y$, a corresponding set $3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\xi;\zeta)$ of 3-morphisms $\varphi : \xi \Rrightarrow \zeta$ between them;

5. and so on;

6. for any three fixed objects a function (where here the sets of 1-morphisms are first considered as freely generated objects of **H**, so that the product $\otimes$ can be done, and in the sequence the result is regarded as a set)
$$\circ_{xyz} : 1\mathrm{Mor}_{\mathbf{C}}(X;Y) \otimes 1\mathrm{Mor}_{\mathbf{C}}(Y;Z) \to 1\mathrm{Mor}_{\mathbf{C}}(X;Z)$$
fulfilling associativity-type diagrams;

7. for any three 1-morphisms between the same objects, a function (in the same sense as above)
$$\circ_{fgh} : 2\mathrm{Mor}^{xy}_{\mathbf{C}}(f;g) \otimes 2\mathrm{Mor}^{xy}_{\mathbf{C}}(g;h) \to 2\mathrm{Mor}^{xy}_{\mathbf{C}}(f;h)$$
satisfying associativity-type diagrams;

8. for any three 1-morphisms between consecutive objects a function
$$\bullet_{fgh} : 2\mathrm{Mor}^{xy}_{\mathbf{C}}(f;g) \otimes 2\mathrm{Mor}^{xy}_{\mathbf{C}}(g;h) \to 2\mathrm{Mor}^{xy}_{\mathbf{C}}(f;h)$$
fulfilling associativity-type diagrams and which is compatible with $\circ_{fgh}$ in a functorial way;

9. for three 2-morphisms between the same 1-morphisms, which in turn are defined between the same objects, a function
$$\circ_{\xi\zeta\eta} : 3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\xi;\zeta) \otimes 3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\zeta;\eta) \to 3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\xi;\eta),$$
satisfying associativity-type diagrams;

10. for three 2-morphisms between consecutive 1-morphisms defined between the same objects, a function
$$\bullet_{\xi\zeta\eta} : 3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\xi;\zeta) \otimes 3\mathrm{Mor}^{gh_{xy}}_{\mathbf{C}}(\zeta;\eta) \to 3\mathrm{Mor}^{fh_{xy}}_{\mathbf{C}}(\xi;\eta),$$
fulfilling associativity-type diagrams and being compatible with $\circ_{\xi\zeta\eta}$ in a functorial way;

11. for three 2-morphisms between consecutive 1-morphisms defined between consecutive objects, a function
$$\star_{\xi\zeta\eta} : 3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\xi;\zeta) \otimes 3\mathrm{Mor}^{gh_{yz}}_{\mathbf{C}}(\zeta;\eta) \to 3\mathrm{Mor}^{fh_{xz}}_{\mathbf{C}}(\xi;\eta),$$
satisfying associativity-type diagrams and being compatible with $\bullet_{\xi\zeta\eta}$ (and, therefore, with $\circ_{\xi\zeta\eta}$) in a functorial way;

12. and so on;

13. for any object, any 1-morphism, any 2-morphism, etc., a corresponding map
$$id_x : 1 \to 1\mathrm{Mor}_{\mathbf{C}}(X;X), \quad id_f : 1 \to 2\mathrm{Mor}^{xy}_{\mathbf{C}}(f;f), \quad id_\xi : 1 \to 3\mathrm{Mor}^{fg_{xy}}_{\mathbf{C}}(\xi;\xi),$$
and so on, satisfying neutral element-type diagrams respectively to the previous composition laws.

Remark. In the data defining a **H**-enriched n-category only that about n-morphisms belongs to **H**; each other data (say about objects, 1-morphism, etc.) belongs to **Set**. Therefore, when taking the (co)limit $n \to \infty$, we get an entity defined by data belonging exclusively to **Set**, as above. We notice, however, that all this data could be considered internal to **H**. Indeed, we have the functor $\jmath : \mathbf{Set} \hookrightarrow \mathbf{H}$, allowing us to replace each set by the object freely generated by it. Therefore, we can think of a **H**-enriched ∞-category as an entity with a set of objects, and objects of k-morphisms, for each $k > 0$, such that the composition laws and the identity maps are indeed morphisms of **H**.

Recall that in order to get the notion of ∞-category we started by building a functor
$$\imath_n : n\mathbf{Cat}(\mathbf{H}) \to (n+1)\mathbf{Cat}(\mathbf{H}),$$
which assigns to each enriched n-category $\mathbf{C}$ a corresponding enriched $(n+1)$-category $\imath_n\mathbf{C}$, and then we considered the colimit of the sequence internal to **Cat** defined by it. This functor was build by making use of the inclusion $\imath : \mathbf{H} \hookrightarrow \mathbf{Set}$. It happens that the inclusion has the left adjoint $\jmath : \mathbf{Set} \to \mathbf{H}$, so that it is natural to expect a dual construction of the notion of "**H**-enriched ∞-category". More precisely it is natural to expect the existence of functors
$$\jmath_n : (n+1)\mathbf{Cat}(\mathbf{H}) \to n\mathbf{Cat}(\mathbf{H}),$$
induced by $\jmath$, which will define the sequence below in such a way that its limit is a category equivalent to $\infty\mathbf{Cat}(\mathbf{H})$.
$$\cdots \longrightarrow 2\mathbf{Cat}(\mathbf{H}) \longrightarrow 1\mathbf{Cat}(\mathbf{H}) \longrightarrow 0\mathbf{Cat}(\mathbf{H}) \tag{7.1.5}$$

We assert that the left adjoint $\jmath$ really induce functors $\jmath_n$ (that the limit of the sequence defined by them reproduces or not the notion of enriched ∞-category will be discussed later). Indeed, if $\mathbf{C}$ is a $(n+1)$-category enriched over $\mathbf{H}$, we define a corresponding n-category $\jmath_n\mathbf{C}$ by considering the same set of objects that $\mathbf{C}$, the same sets of 1-morphisms, and so on, up to $(n-1)$-morphisms. The objects of n-morphisms of $\jmath_n\mathbf{C}$ are the objects of $\mathbf{H}$ freely generated by the sets of n-morphisms of $\mathbf{C}$.

Furthermore, following the same strategy, if $F:\mathbf{C}\to\mathbf{D}$ is a enriched functor between two $(n+1)$-categories enriched over $\mathbf{H}$ we can build a functor$\jmath_n F$ between the corresponding n-categories $\jmath_n\mathbf{C}$ and $\jmath_n\mathbf{D}$. It can be directly verified that this rule is functorial, so that it defines a functor $\jmath_n$. This means that the sequence (7.1.5) can be defined and, because **Cat** is complete, its limit exists. The question is: *is this limit the same as the colimit of* (7.1.4)*?*

This will be the case if the functors $\jmath_n$ and $\imath_n$, defining the corresponding sequences, are adjoints. It happens that in the general situation they are not: given an $(n+1)$-category $\mathbf{C}$ and an n-category $\mathbf{D}$, an enriched functor $F:\jmath_n\mathbf{C}\to\mathbf{D}$ is given by a rule F_0 mapping objects into objects, for any two objects a function mapping 1-morphisms between the given objects into 1-morphisms, and so on, up to $(n-1)$-morphisms. For any two $(n-1)$-morphisms φ,ψ we also have a morphism

$$n\mathbf{H}_{\jmath_n\mathbf{C}}(\varphi,\psi)\to n\mathbf{H}_{\mathbf{D}}(F(\varphi),F(\psi))$$

between the corresponding objects of n-morphisms. Here we recall that, by definition, the object of n-morphisms of $\jmath_n\mathbf{C}$ is that freely generated by the set of n-morphisms of $\mathbf{C}$. In other words, the morphism above is indeed of the form

$$\jmath(n\mathbf{H}_{\mathbf{C}}(\varphi,\psi))\to n\mathbf{H}_{\mathbf{D}}(F(\varphi),F(\psi)). \tag{7.1.6}$$

On the other hand, an enriched functor $G:\mathbf{C}\to\imath_n\mathbf{D}$ is determined by a function between objects, a function between 1-morphisms, and so on, up to $(n-1)$-morphisms. Additionally, for any two $(n-1)$-morphisms ξ,ζ we have a map

$$n\mathbf{H}_{\mathbf{C}}(\xi,\zeta)\to n\mathbf{H}_{\imath_n\mathbf{D}}(G(\xi),G(\zeta)) \tag{7.1.7}$$

between the sets of n-morphisms and, for any two n-morphisms φ,ψ, a morphism

$$(n+1)\mathbf{H}_{\mathbf{C}}(\xi,\zeta)\to(n+1)\mathbf{H}_{\imath_n\mathbf{D}}(G(\xi),G(\zeta)) \tag{7.1.8}$$

between the corresponding objects of $(n+1)$-morphisms. By definition, the set of n-morphisms of $\imath_n\mathbf{D}$ is obtained by forgetting the additional structure of the object of n-morphisms of $\mathbf{D}$, while the object of $(n+1)$-morphisms is given by the neutral element object 1. Therefore, (7.1.7) is explicitly given by

$$n\mathbf{H}_{\mathbf{C}}(\xi,\zeta)\to\imath(n\mathbf{H}_{\mathbf{D}}(G(\xi),G(\zeta))) \tag{7.1.9}$$

while (7.1.8) is indeed given by

$$(n+1)\mathbf{H}_{\mathbf{C}}(\xi,\zeta)\to 1. \tag{7.1.10}$$

When comparing the data defining enriched functors $\jmath_n\mathbf{C}\to\mathbf{D}$ and $\mathbf{C}\to\imath_n\mathbf{D}$ we see that they coincide for objects, for 1-morphisms, and so on, up to $(n-1)$-morphisms. Furthermore, thanks to the adjunction $\jmath\rightleftharpoons\imath$, the functions (7.1.6) and (7.1.9) are in bijections, so that the data coincides for n-morphisms too. We notice that this data completely defines $\jmath_n\mathbf{C}\to\mathbf{D}$, but in order to define $\mathbf{C} \to \imath_n \mathbf{D}$ we need to consider the morphisms (7.1.10) between the objects of $(n+1)$-morphisms.

Therefore, they are precisely these morphisms that make different the data defining functors $\jmath_n \mathbf{C} \to \mathbf{D}$ and $\mathbf{C} \to \imath_n \mathbf{D}$. In other words, *they are precisely these additional morphisms that imply the nonexistence of an adjunction* $\jmath_n \rightleftharpoons \imath_n$. But, if $1 \in \mathbf{H}$ is indeed a terminal object, then the data encoded into the morphisms (7.1.10) is trivial, so that in this case the adjunction $\jmath_n \rightleftharpoons \imath_n$ actually exists.

Conclusion. In general both dual ways to define "∞-categories enriched over $(\mathbf{H}, \otimes, 1)$" does not coincide. But, if $1 \in \mathbf{H}$ is a terminal object (which happens, for instance, when the monoidal structure in question is cartesian), then such dual definitions coincide. In particular, the notion of ∞-category can be defined by two dual ways.

Warning. Due to the conclusion above, from this point we will work only with ∞-categories enriched over monoidal categories $(\mathbf{H}, \otimes, *)$, where $* \in \mathbf{H}$ is a terminal object. The fundamental example will be $\mathbf{H} = \mathbf{Set}$ with its cartesian structure, meaning that our canonical objects of study will be simply ∞-categories and ∞-functors between them.

Remark. In the following we will write $n\mathrm{S}\mathbf{Cat}$ (instead of $n\mathbf{Cat}(\mathbf{Set})$) to denote the category of n-categories and $\infty\mathrm{S}\mathbf{Cat}$ to denote the category of ∞-categories (i.e, the limit or colimit of the sequence of $n\mathrm{S}\mathbf{Cat}$). Here, "S" reads *strict* and its meaning will become clear in the next section.

Higher Kan Extensions

Starting with a monoidal category $(\mathbf{H}, \otimes, *)$, in the last subsections, by iterating the enrichement process and then taking its limit/colimit, we constructed a language of "∞-categories". It seems natural to believe that this new language is very abstract and powerful. If this is the case, then any useful concept of classical mathematics will have a higher version internal to $\infty\mathbf{Cat}(\mathbf{H})$. Many classical concepts can be obtained as Kan extensions, so that a good "test" is to verify if the notion of "Kan extensions" admits a higher version in the language of ∞-categories. Let us see that such a "higher Kan extensions" really exists.

Recall that, as discussed in Section 6.3 the notion of "Kan extensions" makes sense in any category in which we have the notion of Kan extension can be enriched over any monoidal category $(\mathbf{H}, \otimes, 1)$, giving the notion of $\mathbf{H}$-*enriched Kan extensions* of *enriched functors* between $\mathbf{H}$-enriched categories. Indeed, this is due to the fact that the notion of "natural transformation" can itself be enriched over $\mathbf{H}$ and in order to define "Kan extension" we only need the notion of "natural transformation".

When iterating the enrichement process we will get not only the usual notion of "enriched natural transformation", but also the notion of "higher enriched natural transformation", which will be used to define the desired "higher enriched Kan extensions". Indeed, because we have "$\mathbf{H}$-enriched natural transformations", the category $\mathbf{Cat}(\mathbf{H})$ is enriched over itself, i.e, the category of all $\mathbf{H}$-enriched categories is indeed a $\mathbf{H}$-enriched 2-category. So, replacing $\mathbf{H}$ with $\mathrm{Cat}(\mathbf{H})$ we conclude that

$$2\mathrm{Cat}(\mathbf{H}) = \mathrm{Cat}(\mathrm{Cat}(\mathbf{H}))$$

is enriched over itself, i.e, the category os all $\mathbf{H}$-enriched 2-categories is a $\mathbf{H}$-enriched 3-category. Inductively, we then get that $n\mathrm{Cat}(\mathbf{H})$ is indeed a $(n+1)$-category and, taking the limit (or the colimit), that $\infty\mathrm{Cat}(\mathbf{H})$ is an enriched ∞-category.

In Section 1.3 we saw that one of the fundamental principles of categorical language is the "Weakening Principle". It is based on the existence of the notion of "natural transformation". We recall its contents:

Weakening Principle: *Any concept/result defined/obtained by making use only of commutative diagrams of functors can be abstracted/weakened by replacing the commutativity condition by commutativity up to natural transformations.*

It was by applying this principle to the concept of "extension" that we obtained the notion of "Kan extensions". Similarly, because we have "enriched natural transformations" we can enrich the Weakening principle, getting an "enriched Weakening Principle". We can understood the usual "enriched Kan extensions" as arising from the application of this enriched principle to the notion of "extensions".

When saying that $2\mathbf{Cat}(\mathbf{H})$ is a enriched 3-category we are saying that we have objects (the enriched categories), morphisms (enriched functors), 2-morphisms (enriched natural transformations) and, in addition, 3-morphisms (which we call 2-natural transformations). This means that if now we have a diagram of enriched natural transformations we can replaced commutativity by commutativity up to 2-natural transformations. Repeating this idea and taking the limit we then conclude that in enriched ∞-categorical language we have the following principle:

Higher Enriched Weaking Principle: *Any concept/result defined/obtained by making use only of commutative diagrams of k-morphisms into an enriched ∞-category can be abstracted/weakened by replacing the commutativity condition by commutativity up to $(k+1)$-morphisms.*

By applying this principle to the notion of "extension" we then get the desired "higher enriched Kan extensions". More precisely, recall that if $F:\mathbf{A}\to\mathbf{D}$ is a enriched functors between enriched categories, then a enriched left Kan extension of F respectively to other enriched functor $\imath:\mathbf{A}\to\mathbf{C}$ is a pair (L,ξ) given by an enriched functor $L:\mathbf{C}\to\mathbf{D}$ and an enriched natural transformation $\xi : F\Rightarrow L\circ\imath$, which is universal, in the sense that the second diagram below commutes.

$$
\begin{tikzcd}
 & \mathbf{D} \\
\mathbf{A} \arrow[ur, "F"] \arrow[r, "\imath"'] & \mathbf{C} \arrow[u, "L"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & L'\circ\imath \\
F \arrow[ur, Rightarrow, "\xi'"] \arrow[r, Rightarrow, "\xi"'] & L\circ\imath \arrow[u, Rightarrow, "u\circ\imath"']
\end{tikzcd}
$$

If now all data belongs to $2\mathbf{Cat}(\mathbf{H})$, by the Higher Weakening Principle we can abstract this concept by replacing the commutativity of the second diagram with commutativity up to 2-natural transformations fulfilling additional universality conditions. These additional conditions, in turn, are given by analogous new commutative diagram of 2-natural transformations. But, if we are in $3\mathbf{Cat}(\mathbf{H})$ then we would be able to apply the Higher Weaking Principle one more time, replacing the commutativity with commutativity up to 3-natural transformations satisfying universal conditions, and so on.

Therefore, given two enriched ∞-functors $F:\mathbf{A}\to\mathbf{D}$ and $\imath:\mathbf{A}\to\mathbf{C}$ we can formally define the *higher enriched left Kan extension* of F with respect to $\imath$ as a pair (L,ξ), where $L:\mathbf{C}\rightarrow\mathbf{D}$ is a enriched ∞-functor, $\xi:F\Rightarrow L\circ\imath$ such that for any other (L',ξ') there exists a unique pair (u,φ), where $u:L'\Rightarrow L$ is an enriched transformations and $\varphi:\xi'\Rrightarrow(u\circ\imath)\circ\xi$ is a 2-natural transformation, such that if (u',φ') is another pair, there there exists a one more pair (v,α), with $v:u'\Rrightarrow u$, etc.

Intuition

In order to get some feeling on the notion of "k-transformation", let us explicit it in the case in which the enrichment ambient $(\mathbf{H},\otimes,1)$ is just $(\mathbf{Set},\times,*)$. In other words, let us to say explicitly what is a higher natural transformations between ∞-functors.

Before extrapolating to this higher categorical context, we start by trying to understand more concretely the role of the natural transformations in the usual 1-categorical language.

When interested in the enrichment process, we work with the concept of category as "category with hom-sets". Recall, however, that we have the equivalent characterization as "category with source and target maps". In the following, it will be more interesting to think of a category in this second approach.

So, given an arbitrary category $\mathbf{C}$, let us denote its set of objects by $\mathbf{C}_0$ and its set of morphisms by $\mathbf{C}_1$. All that we can define in usual 1-category theory must involve such sets and some relations between them. There are essentially four ways to relate them:

1. by functions $a_{00}:\mathbf{C}_0\rightarrow\mathbf{C}_0$ (that maps objects into objects);

2. by functions $a_{01}:\mathbf{C}_0\rightarrow\mathbf{C}_1$ (that maps objects into morphisms);

3. by functions $a_{11}:\mathbf{C}_1\rightarrow\mathbf{C}_1$ (mapping morphisms into morphisms);

4. by functions $a_{10}:\mathbf{C}_1\rightarrow\mathbf{C}_0$ (mapping morphisms into objects)

But recall that in order to describe the structure of $\mathbf{C}$ we have to take into account the source and target maps, which are functions $s,t:\mathbf{C}_1\rightarrow\mathbf{C}_0$, meaning that in the list above the data (4.) should be discarded. So, we ask: which kind of concept the remaining functions describe in category theory?

By definition, a *functor* between 1-categories is given by a rule that maps objects into objects and morphisms into morphisms satisfying certain coherence conditions. Therefore, the functions $a_{00}:\mathbf{C}_0\rightarrow\mathbf{C}_0$ and $a_{11}:\mathbf{C}_1\rightarrow\mathbf{C}_1$ corresponds together to the concept of functor. On the other hand, a *natural transformation* is a rule assigning morphisms to objects in a coherent way, so that the functions $a_{01}:\mathbf{C}_0\rightarrow\mathbf{C}_1$ correspond to the concept of natural transformation.

We can codify this conclusion by considering the matrix (a_{ij}), with $0\leq i,j\leq 1$. Indeed, diagonal describes the concept of functor, the lower triangular part corresponds to source and target maps and the upper triangular part describes the notion of natural transformation. It happens that this characterization of natural transformation can be directly extended to the context of ∞-categories!

More precisely, notice that when applying the enrichment process and taking the limit $n\rightarrow\infty$, we get the notion of ∞-category in the "hom-sets" perspective (recall the data defining a ∞-category presented in the last subsections). But, in the same way as a 1-category, we can also describe a ∞-category in the "source-target" approach. In it we have the set $\mathbf{C}_0$ of objects and all k-morphisms (for $k > 0$) between different $(k-1)$-morphisms are grouped into the same *set of k-morphisms* $\mathbf{C}_k$. In order to identify the domain and codomain of each k-morphisms we need functions $s_k, t_k : \mathbf{C}_k \to \mathbf{C}_{k+1}$. The composition laws are defined in some pullback of these functions and we also have the identity maps, all this satisfying some commutative diagrams which describe, for instance, that the compositions are associative and that the identities are "neutral elements".

Following the same discussion for the case of 1-categories we see that any construction in the theory of n-categories (and consequently in the theory of ∞-categories by taking the limit $n \to \infty$) is described by making use of functions $a_{ij} : \mathbf{C}_i \to \mathbf{C}_j$, with $0 \leq i, j \leq n$. Similarly to the 1-categorical case, the diagonal of the matrix (a_{ij}), i.e, the data composed by functions a_{kk} which maps objects into objects and k-morphisms into k-morphisms, correspond to the concept of n-functor between n-categories. Furthermore, the lower triangular part are compositions of source and target maps. So, it seems natural to use each element a_{ij} of the upper triangular part to define the desired higher transformations between n-functors.

We notice, however, that if this were the case, then there would exist $n^2/2 - n = n(n/2-1)$ different notions of "transformations between n-functors". More precisely, we would have usual natural transformations, and then natural transformations between natural transformations, which we call 2-natural transformations, and so on up to $n(n/2-1)$-natural transformations. In particular, this would imply that $n\mathrm{S}\mathbf{Cat}$ is a $(n(n/2-1)+1)$-category. But, by the very discussion in the last subsection, we known that such a category is indeed a $(n+1)$-category. This suggests that *not all terms in the upper triangular part are really fundamental, but only n of them*.

One way to select the n fundamental is as follows. Notice that a rule between $\mathbf{C}_1$ and $\mathbf{C}_3$ can be regarded as the composition of a rule $\mathbf{C}_1 \to \mathbf{C}_2$ with a rule $\mathbf{C}_2 \to \mathbf{C}_3$. By generalizing this we see that any term in the upper triangular part can be decomposed as a compositions of terms in the upper diagonal. This means that we can consider the n terms of the upper diagonal as the fundamental. In this perspective, k-transformations should be rules $\xi : \mathbf{C}_{k-1} \to \mathbf{C}_k$ assigning k-morphisms to $(k-1)$-morphisms.

On the other hand, the terms of the upper diagonal (and, therefore, every term in the upper triangular part) is pulled back to the first line a_{0i}, with $0 < i \leq n$. Indeed, Notice that a given $\mathbf{C}_k \to \mathbf{C}_l$ can be replaced by $\mathbf{C}_0 \to \mathbf{C}_l$ after composition with $\mathbf{C}_0 \to \mathbf{C}_k$. Here, a k-transformation is then a rule $\xi : \mathbf{C}_0 \to \mathbf{C}_k$ assigning k-morphisms to objects.

7.2 Rigidness

As discussed in Example 1.4, we usually define a n-manifold as a topological which locally looks like the euclidean space $\mathbb{R}^n$. But this definition is rigid: there are some entities which intuitively should be n-manifolds but that do not are in the sense above. For instance, this is the case of the disc $\mathbb{D}^n$. Indeed, the disc cannot be modeled over the euclidean space because $\mathbb{R}^n$ has a *single flavor* of open sets, while $\mathbb{D}^n$ has *two flavors*: those intersecting the boundary $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$ and those which are totally contained in the open ball $B_1(0)$.

In order to incorporate these intuitive examples, the standard approach is to redefine the notion of manifold by replacing the canonical model over with the objects are modeled. Doing this we get the concept of *manifold with boundary*, which is as interesting as the starting concept of manifold, but abstract enough in order to incorporate examples as the disc $\mathbb{D}^n$.

It happens that there are other examples that intuitively should be manifolds but which do not are manifolds nor manifolds with boundary. They are the entities which have not only two kinds of open sets, but indeed three (or four, five, or even a finite amount of them). For example, these kind of entities appear as the objects modeling (some kinds of) Feynman diagrams, as discussed in Section 3.3. Another time we are forced to redefine the notion of "manifold" in order to incorporate these new examples. This really can be done, obtaining *orbifolds*.

The above discussion is a manifestation of the following very general idea: assume that we are trying to model some phenomena/idea that we have very clear in our mind but which a priori is not so clear mathematically. We start by giving a first attempt of definition, having some examples in mind. So, whenever given an object which intuitively should satisfy our definition we verify if it is really satisfied. In the failure, we need to rethink the starting definition in order to incorporate the desired example.

In the last subsections, by iterating the enrichement process and then taking the limit (or colimit) $n \to \infty$, we obtained the notion of "∞-category enriched over a monoidal category $(\mathbf{H}, \otimes, 1)$". Furthermore, we saw that this ∞-categorical language is much more abstract that usual categorical language, in the sense that the concept of "Kan extension" (and all other notions that derive from it) lifts to the concept of "higher Kan extension".

Now, it is time to analyze if this obtained powerful notion of "∞-category" really is the "correct model" for that concept. Indeed, we have a very clear idea of what should be a "∞-category" in our mind and, thinking in examples which sgould satisfy this idea, we will verify if they actually satisfy the present definition of ∞-category.

Example 7.5 (*cobordism*)**.** Given $p \in \mathbb{N}$, in Example 1.5 we introduced the category $\mathbf{Cob}_{p+1}$, whose objects are compact p-manifolds and whose morphisms are cobordisms between them. It happens that any manifold can be regarded as a cobordism $\varnothing \to \varnothing$, meaning that any object of $\mathbf{Cob}_{p+1}$ can be regarded as a morphism of $\mathbf{Cob}_{p}$. This motivate us to unify both categories into the same entity, having $(p-1)$-manifolds as objects, p-cobordisms as morphisms and $(p+1)$-cobordisms as 2-morphisms. Recursively, this leads us to imagine an entity $\mathbf{Cob}(p+1)$, which intuitively should be a $(p+1)$-category, whose objects are 0-manifolds, whose 1-morphisms are 1-cobordisms between 0-manifolds, and so on, up to $(p+1)$-cobordisms between p-manifolds. The physical interest in this hypothetical entity was discussed in Section 2.4.

Example 7.6 (*homotopy categories*)**.** In Example 7.2 we saw that joining $\mathbf{Top}$ and its homotopy category $\mathrm{Ho}(\mathbf{Top})$ we get a 2-category $2\mathbf{Top}$ enriched over $\mathbf{Top}$. We recall that it has topological spaces as objects, continuous maps as 1-morphisms and homotopy classes of continuous maps as 2-morphisms. Now, notice that homotopies are itself continuous functions, so that we can talk of "homotopies between homotopies", also called 2-*homotopies*. Indeed, if $f, g : X \to Y$ are continuous functions and $H, H' : X \times I \to Y$ are homotopies between them, then a homotopy between H and H' is simply a continuous function $\xi : (X \times I) \times I \to Y$ satisfying

$$\xi(x, t, 0) = H(x, t) \quad \text{and} \quad \xi(x, t, 1) = H'(x, t).$$

Therefore, it is natural to imagine an entity $3\mathbf{Top}$, which intuitively should be a 3-category enriched over $\mathbf{Top}$, whose objects are topological spaces, whose 1-morphisms are continuous maps, whose 2-morphisms are homotopies and whose 3-morphisms are homotopy classes of 2-homotopies. But 2-homotopies are itself continuous functions, so that we can take homotopies between them, which we call 3-*homotopies*, allowing us to imagine a 4-category 4**Top**. So, inductively we expect the existence of a n-category n**Top** and, taking the limit $n \to \infty$, of a ∞-category ∞**Top**.

Example 7.7 (*fundamental groupoid*)**.** Recall that we have a functor $\pi_1 : \mathrm{Ho}(\mathbf{Top}_*) \to \mathbf{Grp}$ assigning to any base topological space (X, x) its fundamental group. Such invariant depends explicitly of the base point x. We can try to collect all invariants $\pi_1(X, x)$, for $x \in X$, into a single entity independent of base points. Indeed, the idea is to define an entity $\Pi(X)$ which has elements $x \in X$ as objects and elements of $\pi_1(X, x)$ as morphisms $x \to x$. Recall that an element $[\gamma] \in \pi_1(X, x)$ is a homotopy class of loops $\gamma : \mathbb{S}^1 \to X$. These loops can be identified with paths $\gamma : I \to X$ such that $\gamma(0) = x = \gamma(1)$. Therefore, this lead us to define a category $\Pi(X)$ whose objects are the base points of X and whose morphisms $x \to y$ are homotopy classes of paths linking these two points (the composition is given by concatenation of paths and the identity of x is the constante path, such that $\gamma(t) = x$ for all t). It happens that, as discussed in the previous examples, homotopy is a continuous map, so that we can take homotopies between homotopies, and so on. So, it is natural to imagine that $\Pi(X)$ regarded as a ∞-category.

Example 7.8 (*monoidal categories*)**.** As discussed in Section 6.1, in any category $\mathbf{C}$, the associativity of the composition and the existence of identity maps imply that for each object X the set $\mathrm{Mor}_{\mathbf{C}}(X; X)$ endomorphisms is, indeed, a monoid. Therefore, if the category $\mathbf{C}$ has a single object, say $*$, then it is totally characterized by the monoid $\mathrm{Mor}_{\mathbf{C}}(*, *)$. This means that *monoids are just the same as categories with only one object.* But now recall that 2-categories are, by definition, categorification of the concept of category, at the same time as monoidal categories are categorification of the concept of monoid. So, it is expected that *monoidal categories are the same as* 2*-categories with only one object.*

After presented these very natural and examples, we need to verify if our current definition of n-category (as iterated enrichment of the concept of category) and of ∞-category (as the limit/colimit of the definition of n-category) is large enough in order to incorporate them.

Let us start by analyzing Example 7.5. More precisely, let us verify if the entity 3**Top** defined there is (or not) a 3-category in the current sense. The objects of this hypothetical 3-category are topological spaces, the 1-morphisms are continuous maps, the 2-morphisms are **homotopies** and the 3-morphisms are homotopy classes of 2-homotopies. Compare this data with that defining 2**Top**, which is a genuine 2-category: objects are topological spaces, 1-morphisms are continuous maps and 2-morphisms are **homotopy classes of homotopies**.

Notice the difference: in 3**Top** the 2-morphisms are homotopies, while in 2**Top** we consider homotopy classes of them. This may seems only a technical conditions, but it is really crucical. Indeed, in both entities it is assumed that the compositions are given by usual compositions of homotopies. It happens that these compositions are generally not associative; they are only associative up homotopies between homotopies! Similarly, it is assumed that the identities are the trivial homotopies, but these satisfy the neutral element property only up to homotopies between homotopies. Therefore, *while* 2**Top** *is genuine enriched* 2*-category,* 3**Top** *is* ***not*** *a* 3*-category.*

Notice that (homotopy classes of) homotopies between homotopies (i.e, 2-homotopies) are supposed to be the 3-morphisms of 3**Top**. So, the explicit problem with this example is the following: *compositions of* 2*-morphisms are associative only up to* 3*-morphisms and the* 2*-identities satisfy the neutral element property up to* 3*-morphisms.* It can be checked that the same kind of problem appears in each of the previous examples!

For instance, a monoidal category is not the same as a 2-category with a single object $*$, because in our definition of 2-categories the compositions of 2-morphisms are strictly associative. On the other hand, by definition, a monoidal category is endowed with a product $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is **not** strictly associative, but only associative up to natural isomorphisms.

Despite all these counterexamples, here is one example in the current sense:

Example 7.9 (*arrow category*)**.** For any category $\mathbf{C}$ we can associate a corresponding ∞-category $\infty\mathrm{Sq}(\mathbf{C})$, as follows. Its objects 1-morphisms are just the objects and morphisms of $\mathbf{C}$. Given morphisms $f : X \to Y$ and $f' : X' \to Y'$, the 2-morphisms $\xi : f \Rightarrow f'$ are commutative squares, as below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Downarrow\xi"] & Y \arrow[d] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

Notice that a square has two dimensions: vertical and horizontal, which will correspond to the two compositions of 2-morphisms. Indeed, if $f, g, h : X \to Y$, we compose 2-morphisms $\xi : f \Rightarrow g$ and $\zeta : g \Rightarrow h$ "horizontally", as in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Downarrow\xi"] & Y \arrow[d] \\
X \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\Downarrow\xi'"] & Y \arrow[d] \\
X \arrow[r, "h"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\xi'\circ\xi"] & Y \arrow[dd] \\
& \\
X \arrow[r, "h"'] & Y
\end{tikzcd}$$

However, if given consecutive 1-morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we compose ξ and η "vertically", as below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Downarrow\xi"] & Y \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\Downarrow\xi'"] & Z \arrow[d] \\
Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & W
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "g\circ f"] \arrow[d] \arrow[dr, phantom, "\Downarrow\xi'\bullet\xi"] & Z \arrow[d] \\
Y \arrow[r, "h\circ g"'] & W
\end{tikzcd}$$

It is clear that both compositions are strictly associative and that the identity of $f : X \to Y$ is the trivial square $id_f : f \Rightarrow f$ whose vertical arrows are id_X and id_Y. The 3-morphisms $\varphi : \xi \Rrightarrow \xi'$ between two squares $\xi : f \Rightarrow g$ and $\xi' : f' \Rightarrow g'$ are cubes whose upper face is ξ and whose lower face is ξ', as below.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] \arrow[dl, phantom, "\Downarrow \xi"] & \\
& Z \arrow[rr, "g"'] \arrow[dd] & & W \arrow[dd] \\
X' \arrow[rr, "f'"] \arrow[dr] & & Y' \arrow[dr] \arrow[dl, phantom, "\Downarrow \xi'"] & \\
& Z' \arrow[rr, "g'"'] & & W'
\end{tikzcd}
$$

Now we have three dimensions, which will be used to define three different composition laws between cubes. Inductively we define an n-category $n\mathrm{Sq}(\mathbf{C})$ whose k-morphisms are k-squares in $\mathbf{C}$. Here, a 0-square is an object, a 1-square is a morphism, a 2-square is a commutative square of morphisms, a 3-square is a commutative cube of morphisms, and so on. Finally, taking the limit $n \to \infty$ we then get the desired ∞-category. We notice that the k-morphisms of $\infty\mathrm{Sq}(\mathbf{C})$ are commutative diagrams of morphisms of $\mathbf{C}$. Functors preserve commutative diagrams, so that any $\mathbf{C} \to \mathbf{D}$ extends to a corresponding ∞-functor $\infty\mathrm{Sq}(\mathbf{C}) \to \infty\mathrm{Sq}(\mathbf{D})$, giving a functor

$$\infty\mathrm{Sq} : \mathbf{Cat} \to \infty\mathrm{S}\mathbf{Cat}.$$

This functor is actually an embedding, so that *the ∞-square construction is a way to embed categorical language into ∞-categorical language.*

Redefining

The previous examples shows that the notion of enriched ∞-category is rigid. Following the philosophy of the last subsection we need to redefine the concept of ∞-category in order to incorporate these very natural examples. Each of them fail to be a ∞-category in the current sense exactly because the composition laws of k-morphisms and the identity k-morphisms do not satisfy associativity and neutral element property strictly, but only up to $(k+1)$-morphisms.

Therefore, the solutions seems pretty simple: in order to incorporate them as "∞-categories" we only need to redefine ∞-category by replacing the requirement of strict commutativity of the associativity-type and neutral element-type diagrams by the requirement of *weak commutativity* (i.e, commutativity up to higher morphisms). In other words, we need to work with *weak ∞-categories.*

Remark. This explain why we used a "S" in $\infty\mathrm{S}\mathbf{Cat}$ to denote the category of all ∞-categories: because they are *strict ∞-categories* in the sense that associativity and neutral element property are strictly satisfied.

Let us try to formalize this desired notion of "weak ∞-category". Recall that we obtained the concept of strict ∞-category by first defining 2-category explicitly, n-category by induction and ∞-category by the limit $n \to \infty$. Therefore, in the weak context the idea is to follow the same strategy, so that we start by defining "weak 2-category" explicitly.

For a given monoidal category $(\mathbf{H}, \otimes, 1)$, tecall that a $\mathbf{H}$-enriched 2-category (as was defined previously) is just as a usual category $\mathbf{C}$, but whose set of morphisms between any two objects replaced by a category enriched over $\mathbf{H}$, whose composition laws replaced by enriched bifunctors

$$\circ_{xyz} : \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Y) \otimes_{\mathbf{H}} \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(Y, Z) \to \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Z)$$

and whose identity maps also are replaced by enriched funtors

$$id_x : \mathbf{1_H} \to \mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Y),$$

such that "associativity-type" and "neutral element-type" diagrams are strictly satisfied. More explicitly, the associativity is described by the following diagram[2]:

$$\begin{tikzcd}
(X,Y) \otimes ((Y,Z) \otimes (Z,W)) \arrow[r, "\simeq"] \arrow[d, "id \otimes \circ_{xzw}"'] & ((X,Y) \otimes (Y,Z)) \otimes (Z,W) \arrow[r, "\circ_{xyz} \otimes id"] & (X,Z) \otimes (Z,W) \arrow[d, "\circ_{xzw}"] \\
(X,Y) \otimes (Y,W) \arrow[rr, "\circ_{xyw}"'] & & (X,W),
\end{tikzcd}$$

while the neutral element property is described by the following diagrams:

$$\begin{tikzcd}
\mathbf{1_H} \otimes (X,Y) \arrow[r, "id_x \otimes id"] \arrow[dr, "\simeq"'] & (X,X) \otimes (X,Y) \arrow[d, "\circ_{xxy}"] \\
& (X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,Y) \otimes (Y,Y) \arrow[d, "\circ_{xyy}"'] & (X,Y) \otimes \mathbf{1_H} \arrow[l, "id \otimes id_y"'] \arrow[dl, "\simeq"] \\
(X,Y) &
\end{tikzcd}$$

Therefore, the associativity of compositions and the "neutral element property" of the identities are described by commutative diagrams of enriched functors. This means that, by the Enriched Weakening Principle, the concept of 2-category can be weakened by replacing these commutative diagrams by the existence of enriched natural transformations satisfying, itself, certain commutativity conditions. Indeed, the strict associativity is now replaced by the existence of enriched transformations

$$\xi_{xyzw} : (\circ_{xzw}) \circ (\circ_{xyz} \otimes id) \Rightarrow (\circ_{xyw}) \circ (id \otimes \circ_{yzw}),$$

while the strict "neutral element property" is replaced by the existence of enriched transformations

$$\varphi_{xy} : (\circ_{xxy}) \circ (id_x \otimes id) \Rightarrow id \quad \text{and} \quad \phi_{xy} : id \Rightarrow (\circ_{xyy}) \circ (id \otimes id_y).$$

There are many possibilities for the additional commutativity conditions required to be satisfied by the transformations above. For each of them we will obtain a different concept of "weak 2-category". For instance, we could require simply that they are enriched natural isomorphisms, giving the following:

Definition 7.1 (*weak 2-categories - first version*)**.** *A weak 2-category enriched over a monoidal category* $(\mathbf{H}, \otimes, 1)$ *is an entity* $\mathbf{C}$ *composed of the following data:*

[2] We used (X, Y) as an abbreviation for $\mathbf{Cat}(\mathbf{H})_{\mathbf{C}}(X, Y)$.

1. *a set of objects;*
2. *for any two objects a* **H***-enriched category of morphisms;*
3. *for any object a distinguished functor* id_x*;*
4. *for any three objects a enriched bifunctor* $\circ_{xyz}$*,*

in such way that there are enriched natural isomorphisms ξ_{xyzw}*,* φ_{xy} *and* ϕ_{xy} *as present above.*

With the concept of enriched weak 2-category on hand we define enriched weak n-categories inductively and, by taking the limit/colimit $n \to \infty$, we get enriched ∞-categories. A priori, it seems that the problem with the initial notion of strict ∞-categories was finally solved. Indeed, it seems that each of the previous examples are, in fact, ∞-categories in this new sense. But this is **not** the case, as will be discussed in the next subsection.

Coherence

In the last subsection we reformulated the notion of n-category in order to incorporate the desired examples discussed previously. However, the given new definition remains problematic, as will be discussed here.

We start by observing that when we **think** of associativity of an operation $*$ in a set X (and similarly of the neutral elements of such an operation) we have in mind the "removal of all brackets" of expressions with **arbitrary number** of elements of X, as the following:

$$((x * y) * z) * w, \quad (x * (y * z)) * ((w * a) * b), \quad \text{etc.} \tag{7.2.1}$$

That the brackets of a given expression really can be removed is translated in terms of the commutativity of a certain diagram. So, in principle, when we **think** of associativity we have in mind that a **huge** amount of diagrams commute.

On the other hand, when we **say** that an operation is associative we only require that brackets of expressions involving three elements, as $(x * y) * z$ and $x * (y * z)$, can be effectively removed. So, the **definition** of associativity require the commutativity of a **single** diagram. However, there is no contradiction between what we **think** and what we **do**: *the strict commutativity of the diagram involving exactly three elements implies the strict commutativity of each diagram involving arbitrary number of elements.*

Now, the problem with the definition of enriched weak ∞-category obtained in the last subsection is clear. Indeed, similar to the strict case, when we **think** of a weak category we have in mind the removal (up to higher morphisms) of brackets in any expression, such as (7.2.1). Therefore we have in mind the weak commutativity of a **huge** number of diagrams. On the other hand, in the **given definition** we only required that brackets in expressions with three k-morphisms can be removed, which is translated in the weak commutativity of **only one** diagram. But, unlike the strict case, here there is no obvious relation between what we **think** and what we **do**: *weak commutativity of this single diagram does not imply the weak commutativity of diagrams with arbitrary number of* k*-morphisms.*

Therefore, we conclude that if on the one hand the starting definition of strict ∞-category is too rigid, on the other the definition of weak ∞-category obtained in the last subsection is too weak. So, in order to correct this new problem, we can try to redefine the concept of weak 2-category (and, consequently, of weak n-category and of weak ∞-category) in the following way:

Definition 7.2 (*weak* 2*-categories - second version*)**.** *A weak* 2*-category enriched over a monoidal category* $(\mathbf{H}, \otimes, 1)$ *is an entity* $\mathbf{C}$ *composed of the following data:*

1. *a set of objects;*
2. *for any two objects an* $\mathbf{H}$*-enriched category of morphisms;*
3. *for any object a distinguished functor* id_x*;*
4. *for any three objects an enriched bifunctor* $\circ_{xyz}$*,*

in such a way that every "associativity-type" diagram and each "neutral element-type" diagram, involving any number of objects, commutes up to enriched natural isomorphisms.

The new notion of weak ∞-category induced from the above definition of weak 2-category certainly fixes the problem presented by the initial definition. We notice however, that a new problem is created: suppose that we are trying to prove that one of the motivating examples for the notion of enriched weak ∞-category is, indeed, a weak ∞-category in the sense of the last definition. So, we have to verify, one by one, that each "associativity-type" and each "neutral element-type" diagram of k-morphisms commutes up to $(k+1)$-morphisms, for every k.

Notice that the amount of "associativity-type" diagrams of 1-morphims are parametrized by (arbitrary products of) the set of objects. More generally, the "associativity-type" diagrams of k-morphisms are parametrized by products of the set of l-morphisms, for every $l < k$. So, we really have to verify a **very huge** amount of conditions! In order to be more explicit, let us assume that we are trying to prove that certain entity is a weak 2-category, say with set of objects, 1-morphisms and 2-morphisms respectively given by $\mathbf{C}_0$, $\mathbf{C}_1$ and $\mathbf{C}_2$. Then, the number "associativity-type" conditions to be verified has at least the order of the set $\prod_{\mathbf{C}_0} \mathbf{C}_1$, and this number grows very fast for $n > 2$. Indeed, for $n = 3$ it has at least the order of

$$\prod_{\prod_{\mathbf{C}_0} \mathbf{C}_1} \mathbf{C}_2.$$

Therefore, if at least one of the set of k-morphisms is uncountable, then we number of conditions to be verified is also uncountable! This means that, *although formal, the last definition of enriched weak* ∞*-category is impracticable!*

This motivate us to look at values of n for which the concept of weak n-category is *coherent*. This means that there are **few** associativity-type and neutral element-type diagrams of k-morphisms, with $0 \leq k \leq n$, which are really **fundamental**, in the sense that the weak commutativity of one of them does not imply the weak commutativity of the others. By "few" we mean that they constitute a space which is weakly contractible in some nontrivial topology. If n is coherent, we say that the fundamental diagrams describe *coherence conditions*.

It can be show that for $n = 2, 3, 4$ the concept of weak n-category (in the sense of the last definition) is coherent. For instance, for $n = 2$, it is a classical result that there are only two fundamental associative type and only two fundamental neutral element type diagrams of 2-morphisms. It is due to Power in [168], based in a particular case proved by Mac Lane (see [131] for an exposition and [133] for the classical reference). The case $n = 3$ is due to Gordon, Power and Street [83], while the case $n = 4$ is due to Trimble [206].

Remark. Recall that, after introduced the notion of "monoidal category" in Section 3.1 we saw that in the literature it is usual to work with *coherent monoidal categories*, which are assumed to satisfy some additional commutativity conditions. commutativity conditions are precisely the coherence conditions appearing in the Mac Lane theorem for weak 2-categories, allowing us to conclude that *a coherent monoidal category is precisely a weak* 2*-category with only one object*.

Remark. Another situation where the notion of "coherence" appears is in the study of Kan extensions: as discussed in Section 2.1, in order to prove that a category $\mathbf{C}$ is complete or cocomplete, a priori we need to verify the existence of Kan extensions which are parametrized by the collection of functors taking values in $\mathbf{C}$. This collection is very huge, because in principle the functors could have arbitrary domain. But, thanks to existence theorem of limits and the reconstruction of Kan extensions by ends, it is enough to analyze the collection of functors defined in two different categories, corresponding to products and equalizers.

As commented in [20], the definition of weak 3-category, including the coherence conditions, takes 6-pages and the definition of equivalence of such entities takes more 16-pages! The data needed to explicit a weak 4-category is also more extensive! Indeed, the original manuscripts [206] of Trimble in which the full data defining a weak 4-category takes incredible 51-pages!

This shows that the number of coherence conditions becomes fully impracticable when n grows, suggesting that there are values of n for which the corresponding notion of weak n-category is not coherent. So, *although we have a very lucid idea of what a n-weak category should be, or large values of n we do not have a canonical definition which is at the same time rigorous and practicable.*

We end with a remark.

Remark. Besides the problem of determining if a given value of n is or not "coherent" we can also ask if it is "strictficable". This is a very strong condition. Indeed, we say that n is *strictficable* (or *rectificable*) if it is coherent and any weak n-category is equivalent to a strict n-category. There is a classical theorem of Mac Lane which states that any coherent monoidal category is equivalent to a strict one, which means that any weak 2-category with only one object is equivalent to a strict 2-category with only one object (see [131] and [??]). The result extends to the case of arbitrary weak 2-categories [??], so that $n = 2$ is strictficable. It is known that $n > 2$ is **not** strictificable (see [193]). On the other hand, it is an open conjecture of Simpson that every coherent n is "semi-strictificable" in the sense that any weak ∞-category is equivalent to a weak ∞-category whose "neutral element-type" diagrams are the only which hold weakly. See [195].

7.3 Presentations

Recall that at this moment we have a very lucid idea of the concept of weak category, but unfortunately we do not know how to write this idea in precise and practicable mathematical terms. In order to be more precise, we defined weak n-categories in satisfactory rigorous terms but we concluded that the given definition is impracticable in the sense that there are values of n for which the correspondent concept of n-category is not coherent. So, the question is: *how to* *proceed?*

In order to get some felling, we observe an analogy between higher category theory and the theory of ordinary differential equations (ODE). We start by recalling that the standard problem in ODE's is to study the solutions of the system defined by a smooth vector field $X : M \to TM$ (here we recall that an *integral curve* for X is simply a path $x : I \to M$ such that $x'(t) = X(x(t))$ for any $t \in I$). This problem really makes sense: we have a unique solution of the ODE defined by X starting in every given point $x_o \in M$.

Although we have existence and uniqueness of solutions, the theorem does not say how to explicit the function $x(t)$ defining them. This led us to divide the vector fields in two categories: those (called *integrable*) whose integral curves can be written explicitly, and the others, whose integral curves cannot be written explicitly.

We can try to estimate the integrability of a system by looking at its number of *degrees of freedom*. In order to define this number, we notice that many ODE's arise as equations of motions of N particles moving into some n-dimensional ambient space M. In these cases, the vector field X can be interpreted as the force acting on the particles. The system may have *conservation laws*. For instance, if the force is conservative (i.e, if it is a gradient-like vector field), then we have the conservation of mechanical energy and the conservation of linear momentum in each direction. The *number of degrees of freedom* of the system is the difference $d = N.n - l$, where l is the number of "independent" conservation laws. We know that if $d \leq 1$, then the system is always integrable. Furthermore, when d grows, the system becomes non-integrable.

As an example, let us consider a system with k particles $p_1, ..., p_k$ subjected to the gravitational force. As discussed in the Chapter 3, this problem can be analyzed by two different perspectives: the *internal* and the *external* perspective. In the external perspective we start by fixing one of the k particles (say p_1). It is then embedded into some ambient space M, in the sense that we take paths $\varphi : I \to M$) and we consider an action S defined on such embeds. This action should contains information about the interaction between the distinguished particle p_1 and the other $k-1$ particles. The physically interesting configurations are exactly those minimize S, which means that they satisfy a system of differential equations: the *equations of motion*.

When the ambient space is $\mathbb{R}^n$ and when the gravitational interaction is modeled by the newtonian gravity, then the equations of motion are just Newton's second law "$F = m \cdot a$" for the particle p_1 subjected to the force given Newton's law for gravitation. In other words, they are

$$\frac{d^2 x_1^a}{dt^2} = -G \sum_{i=2}^{k-1} \frac{m_i}{|r_0 - r_i|^2}, \quad \text{with } a = 1, ..., n,$$

where $x_1(t) = (x_1^1(t), ..., x_1^n(t))$ are the coordinates of the particle p_1 and $r_i(t) = \| x_i \|$ is the distance from the ith particle to the origin. This is a decoupled system **second order** ordinary differential equations. The equations are certainly nonlinear, but they are *quasilinear*. This means that order reduction applies and by defining $v_1^j = dx_1^j/dt$ we get a new system of $2n$ equations of first order totally equivalent to the previous one. Therefore, the existence theorem can be applied, so that solutions exist for any initial data.

We ask about the integrability of the system. We considered a **single** particle p_1 moving into $\mathbb{R}^n$ subjected to a **conservative force** given by the gravitational interaction of the other particles. Due to this conservative property, we have many conservations laws, as energy conservation, linear momentum conservation in each axe of $\mathbb{R}^n$ and angular momentum convervation around each axe.

Therefore, we have $d = 1\cdot n-(1+n+n) = -n-1$ degrees of freedom and the system is integrable in all dimension n, because for any n we will have $d \leq 1$.

On the other hand, what happens if we need to describe the movement of the k particles **simultaneously**? In this case the equations of motions are given by the following system of coupled second order equations:

$$\frac{d^2 x_j^a}{dt} = -G \sum_{i_j=1, i_j\neq j}^{k} \frac{m_{i_j}}{|r_j - r_{i_j}|^2}, \quad \text{with } a = 1, ..., n \text{ and } j = 1, ..., k.$$

We have the same number of conservation laws, but now we are considering all the k particles interacting mutually. Therefore, the number of degrees of freedom of the system writes $d = k\cdot n - (1+n+n) = n(k-2)-1$ and we can analyze in which conditions we have integrability by requiring $d \leq 1$, which implies $(k-2)n \leq 2$. The canonical physical situations are of particles moving into $\mathbb{R}^3$, so that let us look to the case $n = 3$. In this situation the system will be integrable iff $3k \leq 8$ and, therefore, iff $k < 3$. In other words, *a system of particles mutually interacting gravitationally in $\mathbb{R}^3$ is integrable iff the number of particles is less than three.* Particularly, a system with three particles in non-integrable. This a very shocking example, usually known as the *three body problem.* It says, for example, that we cannot known explicitly the function that determines the time evolution of the solar system!

Returning to the analogy between higher category theory and ODE, the notion of weak n-category corresponds to the integral curves of a vector field whose system has $d = n$ degrees of freedom. Indeed, for any n we have a precise definition of weak n-category and, similarly, we have the existence and uniqueness of solutions. Furthermore, there are values of n for which the notion of n-category is coherent (meaning in the analogy that the system is integrable), as well as there are other values for which the notion of weak n-category cannot be written explicit (meaning, in the analogy, that the system is non-integrable). The known examples of coherent n-categories happens for small n (corresponding to small degrees of freedom) and we lose coherence when n grows (exactly which happens with the integrability of a system when the number of degrees of freedom grows).

Established the relation between higher category theory and the theory of ODE's, in order to get some direction in order to answer our previous question, we can look at its analogue in the domain of ODE's. So, let us ask: *how to proceed in order to study a non-integrable system?* The answer is given by the *qualitative theory of ODE's*, which study the solutions of a system without knowing them explicitly. Therefore, the main idea is to proceed doing some kind of *qualitative theory of higher categories.*

The first step in the qualitative theory of ODE's is to replace the attention from the smooth vector field X to the foliation induced in M by its integral curves (the so called *phase space*). This replacement is very interesting because there are properties of the phase space which are *intrinsic* to the manifold and, therefore, must be satisfied by *any* vector field. For instance, the Poincaré-Hopf theorem states that the local behavior of *any vector field* near isolated singularities is determined by the Euler characteristic $\chi(M)$ of M.

Consequently, following the analogy, the approach to qualitative higher category theory should be replacing the attention from the precise definition of weak n-category to the lucid intuitive idea that we have about it, and searching for intrinsic properties of the notion of weak category, which must be satisfied by any precise definition that can be build. The next step is then to build definitions (also called *presentations* to the concept of weak category) satisfying the intuitive properties.

dynamical system	higher category theory
system with d degrees of freedom	weak d-category definition
integrable systems	coherent categories
phase space	intuitive idea of weak categories
intrinsic properties	intuitive properties
smooth vector field realizing intrinsic properties	presentation to weak category

Table 7.1: Higher Category Theory vs. ODE's

There are many different presentations to the notion of weak ∞-category, most of them based in combinatorial and homotopical aspects of the intuitive notion of weak ∞-category. Unfortunately, none of them will be discussed here. For a discussion on (and comparison between) them, see [42, 120, 121].

Chapter 8

Abstract Theory

In the last chapter we concluded that there is a natural and canonical way to build a formal language of weak n-categories, for $0 \leq n \leq \infty$, but unfortunately this language is not coherent for large n. We then discussed strategies to get presentations of the concept of ∞-category. Each of these presentations produce a different "model" to Higher Category Theory, meaning that one time fixed a presentation, the corresponding definitions, theorems, etc, will be explicitly dependent of this choice.

In the present chapter (and in the remaining parts of the text), instead working in a fixed presentation we will use our intuition behind the notion of weak ∞-category in order to develop Higher Category Theory intrinsically (i.e, independently of the any presentation)[1]. This means that we will do **naive** mathematics, so that *all results which will be presented here must be formally understood as* ***conjectures***. In order to turn them into theorems we need to select a presentation and show that the naive proof makes sense there. We are, however, in a comfortable situation. Indeed, in the last years, thank to works of Jabob Lurie and others, all that will be discussed here were formalized in at least one presentation.

In the first two sections we discuss how usual 1-category theory can be abstracted to the context of ∞-categories. More precisely, we discuss that the notions of functors, natural transformations, limits, and so on, admit a ∞-categorical version. In special, we see that exactly as category theory, which is based in three principles (the duality principle, the relativity principle and the weakening principle), ∞-category theory can also be developed in higher analogues of them. By making use of the ∞-version of the weakening principle we will be able to discuss a ∞-version of the categorification process, allowing us to talk of monoidal ∞-categories, ∞-monoid objects, and so on. We will also see that the notion of ∞-category can itself be ∞-enriched and ∞-internalized in certain abstract ambient ∞-category. As a consequence, we will be able to define ∞-sites and ∞-stacks. We then discuss that *the ∞-stacks in $(\mathbf{Diff}_{sub}, J)$ are the "higher stacks" required in Section 6.2 in order to describe the space of fields in string physics.*

There is special interest in the $(\infty, 1)$-*categories*. These are ∞-categories whose k-morphisms, with $k > 1$, are invertible up to $(k+1)$-morphisms. These "invertibility" condition imply that, if $\mathbf{C}$ is a $(\infty, 1)$-category, then it can be "truncated" in each level $k > 1$, allowing us to study every 1-morphism $f : X \to Y$ by its sequence of truncations $\tau_k f$, with $k = 2, 3, ...$ In special, by

[1] Similar "almost naive" approach was also considered, for instance, in the first part of the HoTT book [??] and in [35].

analyzing the truncations of $X \to *$ and $* \to X$, where here $*$ is a terminal object, we will get information about X. These "reconstructive processes" correspond to the *Postnikov tower* and to the *Whitehead tower* of X, which we study in the first part of Section 8.3.

As will become clear, the theory of $(\infty, 1)$-categories in much closely to homotopy theory. Thinking in this way, we will end this chapter presenting a hypothesis (which in a presentation becomes a theorem) asserting that *the homotopy theory of* ∞*-groupoids is just the homotopy theory of* **Top**. In other words, *the* $(\infty, 1)$*-categories* $\infty\mathbf{Gpd}$ *and* **Top** *are equivalents!* We will then try to convince the reader that this hypothesis affects deeply the foundations of math and of physics in the same proportion as the axiom of choice. Indeed, we will prove that the Homotopy Hypothesis imply the classical and seminal result of Thom on oriented cobordism theory, which was discussed in Section 1.2. We will also prove that it also imply that the monoidal category $(\mathbb{Z}_2\mathbf{Grad}, \otimes)$ has exactly two braidings, which means (as discussed in Section 5.2) that *supersymmetry is the most general kind of symmetry which can be considered in a quantum physical system.*

Remark. The fundamental definitions/results presented here are formalized in Lurie's works [127, 125, 130] and in the second part of Schreiber's work [182]. The prove that the $(\mathbb{Z}_2\mathbf{Grad}, \otimes)$ has exactly two braidings was based in the first part of [182] and in Kapranov's works [??,??].

8.1 General

A ∞*-category* $\mathbf{C}$ has objects, morphisms, 2-morphisms, and so on. The n-morphisms can be composed in n-different ways, which are associative up to $(n+1)$-morphisms. A ∞*-functor* between ∞-categories is a rule $F : \mathbf{C} \to \mathbf{D}$ mapping objects into objects, morphisms into morphisms and so on, in such a way that each composition of n-morphisms is preserved. A n*-category* is a ∞-category whose $k > n$ morphisms are only identities. We have the $(n+1)$-category $n\mathbf{Cat}$ of all n-categories and a ∞-category $\infty\mathbf{Cat}$ of all ∞-categories, i.e, the categories of n categories is enriched over itself. A 2-morphism in $\infty\mathbf{Cat}$ (i.e, a map between ∞-functor) is called a natural ∞-transformation, while a 3-morphism is called a 2-natural ∞-transformation, and so on.

We say that $\mathbf{C}$ is a *sub* ∞*-category* of another ∞-category $\mathbf{D}$ when each object and each k-morphism of $\mathbf{C}$ is also in $\mathbf{D}$. The most important examples of ∞-categories to have in mind are those presented in Section 7.2. Some of these examples include ∞-categories for which each k-morphism $\xi : f \Rightarrow g$ has an inverse $\eta : g \Rightarrow f$ up to $(k+1)$-morphisms. They are called ∞*-groupoids*.

Recall that, as discussed in Chapter 1, the tradicional categorical language is based in three principles: the *duality principle*, the *relativity principle* and the *weakening principle*. In the context of ∞-categorical language the same holds. Indeed, we have a ∞*-duality principle*, meaning that there is a ∞-functor

$$(-)^{op} : \infty\mathbf{Cat} \to \infty\mathbf{Cat}$$

assigning to any ∞-category the corresponding entity with the same objects but with inverted morphisms, 2-morphisms, etc. We also have the ∞*-relativity principle*, which gives ∞-functorial ∞-slice categories in the sense that for any $\mathbf{C}$ we have ∞-functors $\mathbf{C}/$ and $/\mathbf{C}$ from $\mathbf{C}$ to $\infty\mathbf{Cat}$. Finally, we have a ∞*-weakening principle* under which any concept defined on a ∞-category using only commutative diagrams can be weakened by replacing equality of k-morphisms by the existence of $(k+1)$-morphisms between them, satisfying some universality condition.

For instance, in Chapter 1 we used the weakening principle in order to define equivalence between categories, adjunctions between functors, etc. Here, analogously, we can make use of the ∞-weakening principle in order to define ∞-equivalence between ∞-categories and adjunctions between ∞-functors. Indeed, recall that two 1-categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent* when there exist functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ together with natural isomorphisms $G \circ F \simeq id_{\mathbf{C}}$ and $F \circ G \simeq id_{\mathbf{D}}$. In other words, when there exist natural transformations

$$\xi : G \circ F \Rightarrow id_{\mathbf{C}}, \quad \xi' : id_{\mathbf{C}} \Rightarrow G \circ F, \quad \eta : id_{\mathbf{D}} \Rightarrow F \circ G \quad \text{and} \quad \eta' : F \circ G \Rightarrow id_{\mathbf{D}},$$

such that $\xi' \circ \xi = id_F$ and $\eta \circ \eta' = id_G$. If $\mathbf{C}$ and $\mathbf{D}$ are 2-categories, then we have the notion of natural 2-transformation, so that we can replace the last equalities by the existence of natural 2-isomorphisms, i.e, natural 2-transformations

$$\alpha : \xi' \circ \xi \Rrightarrow id_F, \quad \alpha'' : id_F \Rrightarrow \xi' \circ \xi, \quad \beta : id_G \Rrightarrow \eta \circ \eta' \quad \text{and} \quad \beta'' : \eta \circ \eta' \Rrightarrow id_G$$

fulfilling some equalities. Similarly, if the underlying categories are 3-categories these equalities could be replaced by the existence of natural 3-isomorphisms, and so on. With the notion of equivalence between ∞-category on hand we can define adjunction between ∞-functors. In fact, we say that F and G are ∞-adjoint when there exist natural equivalence between the ∞-categories

$$\infty\mathbf{Cat}_{\mathbf{D}}(F(X); Y) \simeq \infty\mathbf{Cat}_{\mathbf{C}}(X; G(Y)).$$

The ∞-weakening principle also allow us to talk of *weak ∞-limits* and, by the ∞-duallity principle, about *weak ∞-colimits*. These are something as limits of 1-morphisms but whose diagrams are commutative only up to 2-morphisms fulfilling universality conditions expressed in terms of diagrams which are commutative up to 3-morphisms, and so on. By the requirement of universality, ∞-limits are unique up to ∞-isomorphisms.

For instance, given two 1-morphisms f and g we can talk about their *weak ∞-pullbacks* (and, by the ∞-duallity principle, about *weak ∞-pushouts*). It is given by a square as in the first diagram below which is commutative up to a 2-morphism (in the sense that there is $\xi : f \circ \pi_1 \Rightarrow g \circ \pi_2$) and universal, meaning that for any other square commuting up to a 2-morphism ξ' (as in the second diagram), there is a unique 1-morphism $u : \mathrm{pb}' \to \mathrm{pb}$ such that the whole diagram commutes up to 2-morphisms (third diagram) satisfying some condition involving ξ' up to 3-morphisms, and so on.

$$\begin{tikzcd} \mathrm{pb} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \arrow[dl, Rightarrow, "\xi"'] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb}' \arrow[drr, bend left, "\pi_1'"] \arrow[ddr, bend right, "\pi_2'"'] & & \\ & \arrow[r, phantom, "\Swarrow \xi'"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pb}' \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\pi_1'"] \arrow[ddr, bend right, "\pi_2'"'] & & \\ & \mathrm{pb} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \arrow[dl, Rightarrow, "\xi"'] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

We say that a ∞-category is *weak ∞-complete/∞-cocomplete* (or simply ∞-complete/∞-cocomplete) when it has all weak ∞-limits/∞-colimits. For instance, any category which is complete and cocomplete can be trivially regarded as a ∞-category weak complete/cocomplete.

Remark. In Section 2.1 we have seen that arbitrary limits can be reconstructed from products and equalizers. Dually, we can build arbitrary colimits from coproducts and coequalizers. Much more than only existence results, these facts give to us a concrete way to **compute** any limit/colimit. This computability is a problem with our naive approach to higher categories. Indeed, in almost all time we will not need to compute ∞-limits and ∞-colimits, but in some few moments these computations will be important. In them, we need to breakdown our model-free approach by making use of a fixed presentation of higher categories in order to effectively compute ∞-limits and ∞-colimits.

Weak Categorification, I

In the same way as we can enrich and internalize concepts in usual categories we can enrich and internalize concepts in ∞-categories. In other words, in the previous chapters we studied enrichment/internalization whose ambient of enrichment/internalization was an usual 1-category $\mathbf{H}$. Now we can study the case in which $\mathbf{H}$ is, indeed, an ∞-category. The difference is that here we have the ∞-weakening principle, which allow us to weak the obtained enriched/internalized notions, as summarized in the following diagram. More precisely, *the result of weak categorification will be just the concept obtained from usual categorification, but with commutative diagrams replaced by diagrams which commute up to* 2*-morphisms satisfying conditions up to* 3*-morphisms, and so on.*

$$\begin{tikzcd}
\begin{matrix}\textbf{classical}\\ \textbf{concept}\end{matrix} \arrow[r, Rightarrow, "\text{internalization}", "\text{enrichment}"'] \arrow[rr, Rightarrow, bend right, "\text{weak categorification}"'] & \begin{matrix}\textbf{abstract}\\ \textbf{concept}\end{matrix} \arrow[r, Rightarrow, "\infty-\text{weakening}", "\text{principle}"'] & \begin{matrix}\textbf{abstract}\\ \textbf{weakened}\\ \textbf{concept}\end{matrix}
\end{tikzcd}$$

So, for example, as discussed in the previous chapters, the classical concept of monoid can be internalized into $\mathbf{Cat}$, giving the notion of *monoidal category*. It is a category $\mathbf{C}$ endowed with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and with a distinguished object $1 \in \mathbf{C}$ such that $\otimes$ is associative up to isomorphisms and 1 behaves as a “neutral element”. Furthermore, this concept of “monoidal category” is coherent in the sense that the additional hypothesis of some few commutative diagrams (up to isomorphisms) imply the commutativity (up to isomorphisms) of every associativity-type and every neutral element-type diagrams.

Similarly, we can internalize the concept of monoid into the ∞-category $\infty\mathbf{Cat}$. The result will be a ∞-category $\mathbf{C}$ endowed with a ∞-functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and with a distinguished object $1 \in \mathbf{C}$ such that identical diagrams of “coherent monoidal category” hold. Now we can apply the ∞-weakening principle, getting the notion of *∞-monoidal ∞-category.* Indeed, observe that in the usual monoidal category the coherence conditions are diagrams of functors which commute up to natural isomorphisms. This commutativity “up to natural isomorphisms” is translated in the commutativity of diagrams of natural transformations. But, in the context of monoidal ∞-categories we require that these diagrams are commutative only up to higher natural transformations, and so on.

On the other hand, as discussed in the Chapter 6, the concept of monoid can be enriched precisely over any monoidal category. This fact was a manifestation of the *microcosm principle*, which states that a suitable classical concept can be enriched over any of its categorification by internalization. Here, in the higher categorical context, it is expected something similar. Indeed, we expect some "∞-microcosm principle", meaning that *any nice classical concept can be* ***weakly enriched*** *over any of its* ***weak categorifications***. For instance, the concept of monoid should be enriched over any ∞-monoidal category $\mathbf{C}$. The result would be like a usual monoid object $(X, *, e)$ in a monoidal category, but now the product $* : X \otimes X \to X$ would no longer be strictly associative, but indeed associative up to higher morphisms, motivating us to call them as ∞*-monoid objects*.

Furthermore, similar discussion should be valid for commutative monoids, allowing us to talk of ∞*-symmetric monoidal categories* and *commutative* ∞*-monoid objects* into them. The ∞-duality principle should then give the notions of ∞*-comonoid objects* and *commutative* ∞*-comonoid objects*. The next step should be build a ∞-monoidal structure into the category $\infty\mathrm{Mon}(\mathbf{C}, \otimes)$ of ∞-monoid objects, getting something as a ∞*-Eckmann-Hilton duality*

$$\infty\mathrm{Mon}(\infty\mathrm{Mon}(\mathbf{C}, \otimes), \otimes_M) \simeq \infty_c\mathrm{Mon}(\mathbf{C}, \otimes) \tag{8.1.1}$$

and a dual version for ∞-comonoids, meaning that iterating ∞-monoids or ∞-comonoids we would get their commutative version. So, taking crossed terms (i.e, ∞-comonoids into ∞-monoids and vice versa) we would get isomorphisms

$$\infty\mathrm{Mon}(\infty\mathrm{CoMon}(\mathbf{C}, \otimes), \otimes_M) \simeq \infty\mathrm{CoMon}(\infty\mathrm{Mon}(\mathbf{C}, \otimes), \otimes_M),$$

producing the notion of ∞*-bimonoid objects*. Then we could study ∞*-Hopf objects* into a symmetric monoidal category, which would fit into a ∞-category $\infty\mathrm{Hopf}(\mathbf{C}, \otimes)$. All these facts really are valid in a usual presentation for ∞-categories, as can be seen in Chapter 3 of [125] and in [21].

Example 8.1 (A_∞*-spaces and* E_∞*-spaces*)**.** Recall that the products of the category **Top** passes to the homotopy context, becoming defined in $\mathrm{Ho}(\mathbf{Top})$. This suggest that the cartesian monoidal structure on **Top** (viewed as a usual 1-category) refines to a cartesian ∞-monoidal structure. This is really the case. The corresponding ∞-monoid/commutative ∞-monoid objects are usually known as A_∞*-spaces*/E_∞*-spaces*. The standard examples are the following:

1. *looping spaces and suspensions*. As discussed in Section 4.2, for any based topological space X, its loop space ΩX has a H-monoid structure (i.e, it is a monoid in the cartesian monoidal structure of $\mathrm{Ho}(\mathbf{Top}_*)$). The multiplication $m : \Omega X \times \Omega X \to \Omega X$ is given by the hotomopy class of concatenation of loops. Notice that this operation is more generally defined on $\mathbf{Top}_*$. However, in this level it is not associative/unital, but only associative and unital up to homotopy. So, if we take homotopies into account (i.e, viewing $\mathbf{Top}_*$ as a ∞-category), then the concatenation of loops induces a genuine ∞-monoid object structure into the loop space. In other words, for any X the corresponding ΩX is a A_∞-space. Similarly, the coproducts of $\mathbf{Top}_*$ refine to a monoidal ∞-structure in such a way that ΣX becomes a ∞-comonoid object;

2. *iterated looping spaces and suspensions*. For any X the double looping space $\Omega^2 X = \Omega(\Omega X)$ is a H-space in the category of H-spaces and, therefore, by the Eckmann-Hilton argument, it is a commutative H-space. Similarly, the double suspension $\Sigma^2 X$ is a commutative H-cospace. Thanks to the ∞-version (8.1.1) of the Eckmann-Hilton argument, it follows that $\Omega^2 X$ and $\Sigma^2 X$ are indeed E_∞-spaces and E_∞-cospaces.

Example 8.2 (*Künneth theorem*)**.** Künneth theorem is valid at the level of homology/cohomology, but the Eilenberg-Zilber maps is defined at the level of cochain complexes, but there it is commutative only up to homotopies. This means that the Künneth theorem is indeed a higher categorical phenomena.

Example 8.3 (E_∞*-ring spectrum*)**.** In Section 5.3 we discussed that, restricting to a convenient category of topological spaces, the smash product $\wedge$ induces a smash product on the homotopy category of spectra Ho(**Spec**), allowing us to of about *ring spectrum*, which are the representing objects for multiplicative generalized cohomology theories. Lewis's obstruction theorem says that the smash product cannot be lifted to a symmetric monoidal structure on **Spec** fulfilling some natural properties. But, notice that, because cohomology theories are homotopical invariants, in order to get a multiplicative structure, we only need a ring structure holding up to homotopy, so that we only need a structure of ∞-monoidal ∞-category on **Spec**. As will be discussed in the next chapter, the smash product actually refines to such a ∞-monoidal structure on **Spec**, whose ∞-monoid objects will be called E_∞*-ring spectrum*.

We end this subsection with the following remark.

Remark. We defined above a symmetric monoidal n-category as a monoidal n-category whose product n-functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is commutative up to higher morphisms. We can ask if there is some characterization of this commutative structure in the spirit of the Eckmann-Hilton argument. More precisely, notice that, thanks to the Eckmann-Hilton argument, we can define a commutative monoid as an object X endowed with an operation $* : X \otimes X \to X$ satisfying symmetry conditions **or** as an object X endowed with two operations $*_1, *_2 : X \otimes X \to X$ satisfying compatibility condition. So, we can ask: *can we define a **symmetric** monoidal n-category as a n-category endowed with certain number of compatible monoidal n-structures?* As conjectured in [19, 20] and fully proved in [125], $k = n + 2$ compatible monoidal structures are enough. More precisely, in [19, 20] it is presented a category $n\mathbf{Cat}_k$ of *k-tuply monoidal n-category*, which is some kind of n-category $\mathbf{C}$ endowed with k operations $\otimes_1, ..., \otimes_k : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ satisfying compatibility conditions, and whose morphisms are functors preserving all the k operations. Notice that for $k' \geq k$ we have a forgetful functor $\imath : n\mathbf{Cat}_{k'} \hookrightarrow n\mathbf{Cat}_k$ given by forgetting the additional operations $\otimes_{k'+1}, ..., \otimes_k$. The authors discuss that when k grows (for a fixed n) the underlying structure will becoming more commutative, meaning that the inclusion $\imath$ itself will becoming more and more fully-faithful, and they conjecture that for $k \geq n + 2$ the inclusion is completely fully-faithful and, therefore, an equivalence. The conjecture was known as the *Stabilization Hypothesis*. It imply, for instance, that a **symmetric** monoidal 1-category is a monoidal category with $k \geq 1 + 2$ compatible operations. If we have only $k = 1 + 1$ compatible operations, we get a **braided** monoidal structure, which is "almost" symmetric.

Weak Categorification, II

Have been analyzed the weak categorification of usual concepts as monoids, commutative monoids and groups, the next step is to study the weak categorification of their "many object versions", i.e, categories and groupoids. In the usual categorification context (developed in the previous chapters), we started by giving two different categorical characterizations to the concept of category (and, similarly, to the concept of groupoid): as *categories with hom-sets* and as *categories with source/target*. We then concluded that enrichment is useful only for categories with hom-sets, while internalization is useful only for categories with source/target. Specially, we have seen that categories with hom-sets can be enriched over any monoidal category and categories with source/target can be internalized into any category with pullbacks.

Consequently, categories with hom-sets can be particularly enriched over ∞-monoidal categories, while categories with source/target can be internalized on ∞-categories with ∞-pullbacks. In this case, we can apply the ∞-weakening principle in order to get more abstract concepts, which we will call simply as $\mathbf{H}$-*enriched* and $\mathbf{H}$-*internal categories*, for $\mathbf{H}$ a ∞-monoidal ∞-category or a ∞-category with ∞-pullbacks, respectively. These will be exactly the same as usual categories enriched over monoidal categories or internalized into categories with pullbacks. The only difference is that strictly commutative diagrams are now replaced by diagrams which commute up to 2-morphisms satisfying some conditions up to 3-morphisms, and so on.

Explicitly, if $(\mathbf{H}, \otimes, 1)$ is a ∞-monoidal category, then a *category enriched over* $\mathbf{H}$ is defined by the following data:

1. a collection $\mathrm{Ob}(\mathbf{C})$ of objects;
2. for any two elements of $\mathrm{Ob}(\mathbf{C})$ a corresponding object $\mathbf{H}_{\mathbf{C}}(X, Y)$ of $\mathbf{H}$;
3. for any three elements a 1-morphism in $\mathbf{H}$ abstracting the compositions
$$\circ_{xyz} : \mathbf{H}_{\mathbf{C}}(X, Y) \otimes \mathbf{H}_{\mathbf{C}}(Y, Z) \to \mathbf{H}_{\mathbf{C}}(X, Z)$$
4. for any element a 1-morphism $id_x : 1 \to \mathbf{H}_{\mathbf{C}}(X, X)$, such that the diagrams below (which characterizes the usual enriched categories) are **no longer commutative**, but there are
   (a) for any four elements of $\mathrm{Ob}(\mathbf{C})$ a corresponding 2-morphism
   $$\alpha_{xyzw} : (\circ_{xzw}) \circ (\circ_{xyz} \otimes id) \circ (\simeq) \Rightarrow (\circ_{xyw}) \circ (id \otimes \circ_{yzw})$$
   (b) for any two elements other 2-morphisms
   $$\mu_{xy} : (\circ_{xxy}) \circ (id_x \otimes id) \Rightarrow (\simeq) \quad \text{and} \quad \nu_{xy} : (\circ_{xyy}) \circ (id \otimes id_y) \Rightarrow (\simeq)$$
   satisfying analogous associativity-type and neutral element-type diagrams, which in turn are not commutative, but there are 3-morphisms which also will satisfy some diagrams up to 4-morphisms, and so on.

$$\begin{tikzcd}
(X,Y)\otimes((Y,Z)\otimes(Z,W)) \arrow[r, "\simeq"] \arrow[d, "id\otimes\circ_{yzw}"'] & ((X,Y)\otimes(Y,Z))\otimes(Z,W) \arrow[r, "\circ_{xyz}\otimes id"] & (X,Z)\otimes(Z,W) \arrow[d, "\circ_{xzw}"] \\
(X,Y)\otimes(Y,W) \arrow[rr, "\circ_{xyw}"'] & & (X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes(X,Y) \arrow[r, "id_X\otimes id"] \arrow[dr, "\simeq"'] & (X,X)\otimes(X,Y) \arrow[d, "\circ_{xxy}"] \\
& (X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X,Y)\otimes(Y,Y) \arrow[d, "\circ_{xyy}"'] & (X,Y)\otimes 1 \arrow[l, "id\otimes id_Y"'] \arrow[dl, "\simeq"] \\
(X,Y)
\end{tikzcd}$$

Similarly, when **H** is a ∞-category with weak ∞-pullbacks we can explicit the definition of a *category with source/target internal to* **H**. Indeed, this will be an entity **C** determined by the following data:

1. objects $\mathbf{C}_0$ and $\mathbf{C}_1$ of **H**, describing the objects and the morphisms of **C**;
2. 1-morphisms $s, t : \mathbf{C}_1 \to \mathbf{C}_0$ of **H** representing the source and the target maps;
3. 1-morphisms $id : \mathbf{C}_0 \to \mathbf{C}_1$ and $\circ : \infty\mathrm{pb}(s,t) \to \mathbf{C}_1$ corresponding to the identities and the composition maps, such that the following diagrams (which previously were used to describe the associativity of the compositions and the neutral element property of the identities are **no longer commutative**, but there are 2-morphisms between them, which satisfy conditions up to 3-morphisms, and so on.

$$\begin{tikzcd}
\infty\mathrm{pb}(\infty\mathrm{pb}(s,t),s) \arrow[r, dashed] \arrow[d, dashed] & \infty\mathrm{pb}(\infty\mathrm{pb}(s,t),t) \arrow[r, dashed] & \infty\mathrm{pb}(s,t) \arrow[d, "\circ"] \\
\infty\mathrm{pb}(s,t) \arrow[rr, "\circ"'] & & \mathbf{C}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\infty\mathrm{pb}(s \circ id, t) \arrow[r, dashed] \arrow[dr, "\pi_1"'] & \infty\mathrm{pb}(s,t) \arrow[d, "\circ"] & \infty\mathrm{pb}(s, t \circ id) \arrow[l, dashed] \arrow[dl, "\pi_2"] \\
& \mathbf{C}_1 &
\end{tikzcd}$$

Remark. In the definition of internal category to **H**, we also require the **strict commutativity** of the diagrams below. So, in the context of internal ∞-categories we could required that these diagrams commute only up to higher morphisms, as for the other diagrams, but this is unusual, because these diagrams only say that the identity maps are automorphisms and that the source and target of the compositions are well defined.

$$\begin{tikzcd}
\mathbf{C}_0 \arrow[r, "id"] \arrow[dr, "id"'] & \mathbf{C}_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
& \mathbf{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\infty\mathrm{pb}(s,t) \arrow[dr, "id"] \arrow[drr, bend left, "\circ"] \arrow[ddr, bend right, "\circ"'] & & \\
& \infty\mathrm{pb}(s,t) \arrow[r] \arrow[d] & \mathbf{C}_1 \arrow[d, "t"] \\
& \mathbf{C}_1 \arrow[r, "s"'] & \mathbf{C}_0
\end{tikzcd}$$

Weak Categorification, III

Up to this point we enriched/internalized **classical** concepts into **higher categories** and we used the higher structure of the ambient of enrichment/internalization in order to get more a weak (i.e, more general) version of the resultant notions. Now, thanks to this higher structure of the ambient we will be able to enrich and internalize more categorical entities. For instance, we can now try to enrich/internalize the notions of ∞-category and ∞-groupoids.

As always, the starting point is to give a (higher) categorical characterization. Exactly as the concept of 1-category, the notion of ∞-category admits two of such characterizations: as

∞*-categories with k-hom-sets* and as ∞*-categories with source/target*. In the first case, we think of a ∞-category $\mathbf{C}$ as being composed by

1. a collection $\mathrm{Ob}(\mathbf{C})$ of objects;

2. for any two objects X, Y a collection $1\mathrm{Mor}(X, Y)$ of 1-morphisms;

3. for any two morphisms $f, g : X \to Y$ between the same objects a collection $2\mathrm{Mor}(f, g)$ of 2-morphisms, and so on;

4. for any three 1-morphisms a composition law which is associative up to 2-morphisms;

5. for any three 2-morphisms, two different composition laws, which are at the same time compatible and associative up to 3-morphisms, and so on;

6. for any object X a distinguished 1-morphism $id_X : X \to X$ (equivalently, a distinguished map $1 \to 1\mathrm{Mor}(X, X)$) which satisfy the neutral element property up to 2-morphisms;

7. for any 1-morphism f a distinguished map $id_f : 1 \to 2\mathrm{Mor}(f, f)$ satisfying neutral element property up to 3-morphisms, and so on.

Notice that, in this perspective, because we have composition laws for every k-morphism, for each k the corresponding collection of k-morphisms appears in the diagrams describing associativity and the neutral element property. Enrichment is about categorification of diagrams, so that in order to enrich the above data over an ambient $(\mathbf{H}, \otimes, 1)$ we need to replace each collection of k-morphisms by an object of $\mathbf{H}$. On the other hand, the collections of k-morphisms of $\mathbf{C}$ also appear as **parameters** for the compositions and for the identities, so that after enrichment they must remain having elements and, therefore, they must remain being collections, but now with some further additional structure. Summarizing: *we can enrich the notion of* ∞*-category with k-hom-sets over any* ***concrete*** *monoidal* ∞*-category.* Explicitly, if $(\mathbf{H}, \otimes, 1)$ is a concrete monoidal ∞-category, then a ∞*-category enriched over* $\mathbf{H}$ is an entity composed by

1. a collection $\mathrm{Ob}(\mathbf{C})$ of objects of $\mathbf{C}$;

2. for any two $X, Y \in \mathrm{Ob}(\mathbf{C})$ an object $1\mathbf{H}(X, Y)$ of $\mathbf{H}$, meaning the "1-morphisms of $\mathbf{C}$";

3. for any two f, g an object $2\mathbf{H}(f, g)$ of $\mathbf{H}$, describing the "2-morphisms of $\mathbf{C}$", and so on;

4. for any three objects X, Y, Z of $\mathbf{C}$ a presentation of the "composition law for 1-morphisms of $\mathbf{C}$" by a 1-morphism
$$\circ_{XYZ} : 1\mathbf{H}(X, Y) \otimes 1\mathbf{H}(Y, Z) \to 1\mathbf{H}(X, Z),$$
of $\mathbf{H}$, which is associative up to higher morphisms of $\mathbf{H}$;

5. for any three f, g, h "1-morphisms of $\mathbf{C}$" two composition laws presented by two 1-morphisms of $\mathbf{H}$, which satisfy associativity up to higher morphisms of $\mathbf{H}$, and so on;

6. for any object $X \in \mathrm{Ob}(\mathbf{C})$ a distinguished 1-morphism $id_X : 1 \to 1\mathbf{H}(X, X)$ of $\mathbf{H}$ which satisfy the neutral element property up to higher morphisms of $\mathbf{H}$;

7. for any "1-morphism f of $\mathbf{C}$" a 1-morphism $id_f : 1 \to 2\mathbf{H}(f, f)$ of $\mathbf{H}$ satisfying neutral element property up to higher morphisms of $\mathbf{H}$, and so on.

Remark. Exactly as any usual category is trivially enriched over $(\mathbf{Set}, \times, 1)$, any ∞-category is enriched over the cartesian monoidal ∞-category $(\infty\mathbf{Cat}, \times, \mathbf{1})$.

Now, let us analyze the second perspective. In it, a ∞-category is supposed to have "source and target maps" for any k-morphisms, so that no one collection of k-morphism is used as a parameter[2]. More precisely, is this characterization a ∞-category $\mathbf{C}$ is given by:

1. a sequence $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2....$ of collections of objects, 1-morphisms, 2-morphisms, and so on;
2. for any k source and target maps $s_k, t_k : \mathbf{C}_k \to \mathbf{C}_{k-1}$;
3. for any k a sequence $\circ_k^i$, with $i = 1, ..., k$, of composition maps defined in some pullback between the source and target maps, which is associative up to higher morphisms;
4. for any k identity maps $id_k : \mathbf{C}_k \to \mathbf{C}_{k+1}$ which satisfy the neutral element property up to higher morphisms;
5. a list of compatibility conditions between the compositions/identities and the source and target maps, meaning that the source and target of compositions and identities are well defined. For instance, we require that $s_{k+1} \circ id_k = t_{k+1} \circ id_k$ be the identity function.

The above data can be internalized in any ambient $\mathbf{H}$ with pullbacks. Indeed, this data can be understood as a huge diagram in $\mathbf{Set}$, such that the only categorical structure used is the pullback appearing in the domain of the composition maps. Therefore, by the ∞-weakening principle, *the notion of ∞-category with source/target can be internalized into any ∞-category with ∞-pullbacks.* Explicitly, an *internal ∞-category into* $\mathbf{H}$ is a huge diagram in $\mathbf{H}$ composed by

1. a sequence $\mathbf{C}_0, \mathbf{C}_1, ...$ of objects of $\mathbf{H}$, representing the "object of objects of $\mathbf{C}$", the "object of 1-morphisms of $\mathbf{C}$", and so on;
2. for each k corresponding 1-morphisms $s_k, t_k : \mathbf{C}_k \to \mathbf{C}_{k-1}$ of $\mathbf{H}$ describing the "source and target maps";
3. for each k a sequence of 1-morphisms $\circ_k^i$ of $\mathbf{H}$, with $i = 1, ..., k$, representing the "composition maps", which are defined in the ∞-pullback between the source/target maps and which are associative up to higher morphisms;
4. for each k, 1-morphisms $id_k : \mathbf{C}_k \to \mathbf{C}_{k+1}$ of $\mathbf{H}$, describing the "identity maps", which satisfy the neutral element property up to higher morphisms;
5. a list of compatibility conditions between the compositions/identities and the source/target maps.

[2]Recall that, as discussed in the end of the last chapter, this naive characterization is the main idea used to build presentations for the notion of ∞-category.

Remark. The same discussion above holds for the concept of ∞-groupoid. Indeed, it also admit two characterizations. The first can also be enriched over any concrete monoidal ∞-category, while the second can also be internalized in any ∞-category with ∞-pullbacks. For instance, the internal ∞-groupoid will be a huge internal diagram exactly as a ∞-category, but with some additional arrows $\mathrm{inv}_k : \mathbf{C}_k \to \mathbf{C}_k$ describing the rules that assign to any k-morphism its inverse.

Remark. We can also apply the discussion above to the concept of ∞-functor between ∞-categories, producing the notion of ∞-functors between $\mathbf{H}$-enriched/$\mathbf{H}$-internal categories, which fits into categories $\infty\mathbf{Cat}(\mathbf{H})$ and $\infty\mathbf{Cat}_{\mathbf{H}}$. Furthermore, the notions of higher natural transformations also can be internalized/enriched, giving ∞-structures to these categories. We will write $\infty\mathbf{Gpd}(\mathbf{H})$ and $\infty\mathbf{Gpd}_{\mathbf{H}}$ to the corresponding full sub ∞-category of $\mathbf{H}$-enriched/$\mathbf{H}$-internal ∞-groupoids.

Examples

Here we will discuss some examples of internal ∞-categories/∞-groupoids. They are essentially the higher categorical version of the examples of internal 1-categories/1-groupoids presented in Section 6.2.

Example 8.4 (*canonical embedding*)**.** Let $\mathbf{H}$ be a ∞-category. Any object $X \in \mathbf{H}$ can be regarded as a ∞-category internal to $\mathbf{H}$ in a trivial way: by considering all objects of k-morphism equal to X and all the structural 1-morphisms (i.e, all source/target 1-morphisms t_k, s_k, all identity 1-morphisms id_k and all compositions $\circ_k^i$) equal to the identity map $id_X \in \mathbf{H}$. This construction extends naturally to an embedding $\infty\mathrm{disc} : \mathbf{H} \hookrightarrow \infty\mathbf{Cat}_{\mathbf{H}}$. Notice that the ∞-category associated to X is, indeed, a internal ∞-groupoid whose inversion maps inv_k are all equal to id_X. As a huge diagram, $\mathrm{disc}(X)$ is given as below, where all arrows are identities.

$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X \rightrightarrows X \longrightarrow X,$$

Example 8.5 (*underlying ∞-groupoid*)**.** Recall that any $\mathbf{H}$-internal category $\mathbf{C}$ admits a canonical underlying internal groupoid $\mathbf{C}_{pd}$ obtained by forgetting all morphisms which are not isomorphisms. Analogously, each ∞-category $\mathbf{C}$ has a underlying ∞-groupoid $\mathbf{C}_{pd}$ obtained by forgetting the k-morphisms which have not inverses up to $(k+1)$-morphisms.

Example 8.6 (*internal ∞-groups*)**.** As discussed in Example 6.5, for a cartesian monoidal category $(\mathbf{H}, \times)$ with pullbacks, the corresponding category $\mathbf{Gpd}_{\mathbf{H}}$ of $\mathbf{H}$-groupoids has a inhering cartesian monoidal structure and the category $\mathrm{Hopf}(\mathbf{H})$ of group objects into $\mathbf{H}$ has at last the same limits as $\mathbf{H}$. Therefore, we can talk of group objects into $\mathbf{Gpd}_{\mathbf{H}}$ and of groupoids internal to $\mathrm{Hopf}(\mathbf{H})$. The fundamental fact is that these two notions coincide, meaning that we have a canonical isomorphism

$$\mathrm{Hopf}(\mathbf{Gpd}_{\mathbf{H}}) \simeq \mathbf{Gpd}_{\mathrm{Hopf}(\mathbf{H})}. \tag{8.1.2}$$

The same is valid in the higher categorical context. More precisely, if $\mathbf{H}$ is a ∞-category with ∞-pullbacks, binary ∞-products and a terminal object, then the underlying cartesian ∞-monoidal structure induces a cartesian ∞-monoidal structure on $\infty\mathbf{Gpd}_{\mathbf{H}}$ and $\infty\mathrm{Hopf}(\mathbf{H})$ get the sane ∞-limits which are in $\mathbf{H}$. Furthermore, the above isomorphism refines to a ∞-equivalence

$$\infty\mathrm{Hopf}(\infty\mathbf{Gpd}_{\mathbf{H}}) \simeq \infty\mathbf{Gpd}_{\infty\mathrm{Hopf}(\mathbf{H})}. \tag{8.1.3}$$

Recall that the objects of the categories appearing in the isomorphism (8.1.2) were called 2-groups internal to $\mathbf{H}$, which motivate us to call the objects in (8.1.3) of ∞-*groups internal to* $\mathbf{H}$. The ∞-groups internal to $\mathbf{Set}$ (trivially regarded as a discrete ∞-category) are called ∞-*groups*. Concrete examples to have in mind are the following:

1. *automorphism ∞-group*: given a ∞-category $\mathbf{C}$, any object X defines a ∞-groupoid $\mathrm{Aut}(X)$, whose objects are the automorphisms of X (i.e, 1-morphisms $X \to X$ invertible up to 2-morphisms), whose 1-morphisms are automorphisms $\xi : f \Rightarrow f$ of automorphisms $f : X \to X$, and so on. The operation giving the structure of ∞-group is just the composition of morphisms. This construction extends to a ∞-functor
$$\mathrm{Aut} : \mathbf{C} \to \infty\mathrm{Hopf}(\infty\mathbf{Gpd}).$$
Indeed, notice that the ∞-groupoid $\mathrm{Aut}(X)$ can be viewed as a full sub ∞-category of $\mathbf{C}_{pd}$ with only one object X. It happens that both rules $(-)_{pd}$ and "taking the one object version" are functorial, so that Aut is the composition of two functors and, therefore, is functorial;

2. *Lie ∞-groups*: the 2-groups internal to $\mathbf{Diff}_{sub}$ were called Lie 2-grops. Analogously, regarding these categories as ∞-categories, the ∞-groups internal to them are called *Lie ∞-groups*. Explicitly, a Lie ∞-group is a sequence of Lie groups $G_0, G_1, ...$, endowed with smooth submersions $s_k, t_k, \circ_k^i$ and id_k, fulfilling the previous diagram.

3. *Picard ∞-group*: recall that to any monoidal category $(\mathbf{H}, \otimes, 1)$ we can associate a groupoid $\mathrm{Pic}(\mathbf{H})$, the *Picard groupoid of* $\mathbf{H}$, obtained by forgetting the objects $X \in \mathbf{H}$ which have no inverse with respect to $\otimes$, i.e, for which there is no Y such that $X \otimes Y \simeq 1 \simeq Y \otimes X$. This internal groupoid is indeed a 2-group, whose Hopf monoid structure is given simply by $\otimes$. In the higher categorical context, we have a direct analogous result. In fact, if $(\mathbf{H}, \otimes, 1)$ is now a ∞-monoidal ∞-category, then it defines a ∞-groupoid $\infty\mathrm{Pic}(\mathbf{H})$, which becomes a ∞-group when endowed with the product $\otimes$. This is the *Picard ∞-group* of $\mathbf{H}$. If $\mathbf{H}$ is an n-category, then $\infty\mathrm{Pic}(\mathbf{H})$ is an n-group.

Example 8.7 (*Internal nerve*)**.** In Example 8.4 we saw that any object in a ∞-category $\mathbf{H}$ can be naturally regarded as a ∞-category internal to $\mathbf{H}$. Here we will see that this is indeed a particular case of a more general construction which assign to any 1-category $\mathbf{C}$ internal to $\mathbf{H}$ a corresponding internal ∞-category $\mathrm{N}(\mathbf{C})$, called the *internal nerve of* $\mathbf{C}$. More precisely, we will build a functor $\mathrm{N} : \mathbf{Cat_H} \to \infty\mathbf{Cat_H}$ such that $\infty\mathrm{disc} = \mathrm{N} \circ \mathrm{disc}$, where $\mathrm{disc} : \mathbf{H} \to \mathbf{Cat_H}$ is the usual canonical embedding, introduced in Example 6.7. The construction is pretty simple. Indeed, we put the object of objects $\mathrm{N}(\mathbf{C})_0$ as $\mathbf{C}_0$, the object of 1-morphisms $\mathrm{N}(\mathbf{C})_1$ as $\mathbf{C}_1$, the object of 2-morphisms $\mathrm{N}(\mathbf{C})_2$ as $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$ (i.e, as the pullback between the source and target maps $s, t : \mathbf{C}_1 \to \mathbf{C}_0$), the object of 3-morphisms as the iterated pullback $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$, and so on. All the source/target s_k, t_k, all identities id_k and all composition laws $\circ_k^i$ comes from universality of pullbacks. As a huge diagram, the nerve of $\mathbf{C}$ is represented by
$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \rightrightarrows \mathbf{C}_0 \longrightarrow \mathbf{C}_1,$$
A concrete example to have in mind is the following:

- <u>*Nerve of delloped ∞-group*</u>. Recall the discussion on nonabelian cohomology in Section 1.2. There we assigned to any topological group G a corresponding topological space BG, the *classifying space of* G, which was the representing object for nonabelian cohomology with coefficients in G. The construction of BG has two steps. In the first we consider sequences of products $G^n = G \times ... \times G$ linked by arrows $G^n \to G^{n-1}$, while in the second these arrows are used in order to glue each G^n, producing the desired CW-complex BG. We notice that to any G we can assign an internal groupoid $\mathbf{B}G$, called the *delloping of* G, such that $\mathbf{B}G_0 = *$ and $\mathbf{B}G_1 = G$. Observe that $G \times G$ is just $\mathbf{B}G_1 \times_{\mathbf{B}G_0} \mathbf{B}G_1$, so that the first step in the building of BG is to take the nerve of the delloped groupoid $\mathbf{B}G$. This construction applies more generally to any internal ∞-group.

We end with the following remark. It will be fundamental in the proof (in Section 8.3) that supersymmetry is the most general kind of symmetry which can be considered in a system of quantum particles.

Remark. As introduced in Example 8.6, a ∞-group G is a Hopf ∞-monoid object in the cartesian monoidal ∞-category of ∞-groupoids. So, it is particularly a ∞-monoid in the category of all ∞-categories and, therefore, a ∞-monoidal ∞-category. We say that G is a *braided/symmetric ∞-group* when it is braided/symmetric as a monoidal ∞-category. By the Stabilization Hypothesis, this is equivalent to say that G has two/three compatible ∞-monoidal ∞-structures. Notice that G has two compatible monoidal ∞-structures iff $\mathbf{B}G$ the ∞-groupoid is indeed a ∞-group, meaning that $\mathbf{B}(\mathbf{B}G) \equiv \mathbf{B}^2 G$ is well defined. Similarly, G has three monoidal structures iff $\mathbf{B}^2 G$ is also a ∞-group and, therefore, $\mathbf{B}^3 G$ is well defined. In other words, *a ∞-group is braided/symmetric iff it has double/triple delooping.*

8.2 Locality

As an application of the existence of this new kind of weak categorification process, here we will see that the notion of Grothendieck topology also extends to the higher categorical context, allowing us to talk of ∞-stacks and ∞-geometric stacks. Indeed, recall that a Grothendieck topology in a usual category $\mathbf{C}$ with pullbacks was defined as a rule J assigning to any object $X \in \mathbf{C}$ a family $J(x)$ of morphisms $\pi : U \to X$, called *coverings of* X, which contains all isomorphisms and which are stable under pullbacks.

So, similarly, we can define a ∞-*Grothendieck topology* on a ∞-category $\mathbf{H}$ by simply replacing "pullbacks" by "weak ∞-pullbacks", i.e, as a rule J assigning to any object X a family $J(x)$ of 1-morphisms $\pi : U \to X$, also called *coverings* of X, such that all isomorphism $Y \simeq X$ belongs to $J(x)$ and such that if $\pi : U \to X$ is a covering and $f : Y \to X$ is any 1-morphism, then the weak ∞-pullback $f^*U \to U$ is a covering. The pair $(\mathbf{H}, J)$ is called a ∞-*site*.

With the notion of ∞-site on hand, the new step is to define ∞-*sheaves*. In order to do this, recall that for a usual site $(\mathbf{H}, J)$, a sheaf with values into a category $\mathbf{C}$ is simply a functor $F : \mathbf{H}^{op} \to \mathbf{C}$ such that for any cover $U \to X$ the corresponding internal Cech groupoid $\check{C}(U)$ is such that the canonical morphism

$$F(X) \to \lim F(\check{C}(U))$$

is an isomorphism. We notice that this morphism can be obtained by making use of the Yoneda embedding/Yoneda lemma and of the fact that colimits are converted into limits by contravariant representable functors. More precisely, we start by applying the Yoneda embedding $h_{-} : \mathbf{H} \to \mathrm{Func}(\mathbf{H}^{op}; \mathbf{C})$ to the canonical morphism $\operatorname{colim} \check{C}(U) \to X$. On the other hand, by the Yoneda lemma an by the fact that contravariant hom-functors maps colimits into limits we get the required morphism:

$$F(X) \simeq \mathrm{Nat}(X, F) \longrightarrow \mathrm{Nat}(\operatorname{colim} \check{C}(U), F) \simeq \lim \mathrm{Nat}(\check{C}(U), F) \simeq \lim F(\check{C}(U))$$

We already know that ∞-presentable functor preserve ∞-limits. The Yoneda lemma and, consequently, the Yoneda embedding also generalize to this higher. Indeed, by the discussion in Section 6.3 we have Yoneda lemma for enriched categories. By induction it is valid for **strict** ∞-category. The fact that it is also valid for **weak** ∞-categories can be seen in [127]. So, in order to define ∞-sheaves we only need to replace the Cech groupoid by a more abstract ∞-groupoid. Recall that the Cech groupoid of a covering $\pi : U \to X$ is simply that given by the diagram

$$U \times_X U \times_X U \longrightarrow U \times_X U \rightrightarrows U \longrightarrow U$$

where $U \times_X U$ is the pullback between π and π. Therefore, given a ∞-covering $\pi : U \to X$, the main idea in order to define its *Cech ∞-groupoid* is to take the internal ∞-groupoid $\check{C}_\infty(U)$ defined by the following iterated ∞-pullbacks, which is just the internal nerve $\mathrm{N}(\check{C}(U))$.

$$\cdots \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} U \times_X U \times_X U \times_X U \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} U \times_X U \times_X U \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} U \times_X U \rightrightarrows U \longrightarrow U,$$

With all that we need on hand we can finally define ∞-sheaves. In fact, if $(\mathbf{H}, J)$ is a ∞-site, then a ∞-*sheaf* in a ∞-category $\mathbf{C}$ internal to $\mathbf{H}$ is an internal ∞-functor $F : \mathbf{H}^{op} \to \mathbf{C}$ such that for any object X and cover $\pi : U \to X$ in $J(x)$, the morphism represented below is an isomorphism.

$$F(X) \simeq \mathrm{Nat}(F, X) \longrightarrow \mathrm{Nat}(F, \operatorname{colim} \check{C}_\infty(U)) \simeq \lim \mathrm{Nat}(F, \check{C}_\infty(U)) \simeq \lim F(\check{C}_\infty(U))$$

In the 1-categorical context, a sheaf with values in the category of (internal) groupoids was called a *stack*. Here, analogously, a ∞-sheaf with coefficients into $\infty\mathbf{Gpd}$ (resp. $\infty\mathbf{Gpd}_{\mathbf{H}}$) is called a ∞-*stack* (resp. an internal ∞-*stack*). We have seen that any $\mathbf{H}$-internal groupoid G induces a canonical internal stack $F : \mathbf{H}^{op} \to \mathbf{Gpd}_{\mathbf{H}}$ by applying the internal hom functor into the diagram defining $\mathbf{G}$. The stacks that can be obtained in this way were called $\mathbf{H}$-*geometric stacks*. Similarly, any ∞-groupoid internal to a ∞-category $\mathbf{H}$ defines a ∞-stack in $\mathbf{H}$, called $\mathbf{H}$-*geometric ∞-stacks*.

In the special case in which $\mathbf{H}$ is $\mathbf{Diff}_{sub}$ or, trivially regarded as a ∞-site, the ∞-stacks are called smooth ∞-stacks and the geometric ∞-stacks are called *differentiable ∞-stacks*. The fact that they are "geometrical" here means that they are determined by some ∞-Lie groupoid.

∞-Topos

If $(\mathbf{H}, J)$ is a ∞-site, then we can form the ∞-category $\infty\mathrm{Stack}(\mathbf{H}, J)$ of ∞-stacks. Such an ∞-category is usually called a ∞-*topos* (or *Grothendieck ∞-topos* or even *Grothendieck-Rezk-Lurie ∞-topos*) analogously as the category of sheaves on a site is called a Grothendieck topos.

Recall that, as discussed in Section 2.2, the usual Grothendieck topos on a site $(\mathbf{H}, J)$ can be regarded as a localization of the category of functors $\mathrm{Func}(\mathbf{H}^{op}, \mathbf{Set})$ at the class of the Cech diagrams, meaning that the inclusion

$$\imath : \mathrm{Shv}(\mathbf{H}, J) \hookrightarrow \mathrm{Func}(\mathbf{H}^{op}, \mathbf{Set}) \quad \text{has an adjoint} \quad \mathscr{L} : \mathrm{Func}(\mathbf{H}^{op}, \mathbf{Set}) \to \mathrm{Shv}(\mathbf{H}, J),$$

which assign to any functor F its corresponding sheafication $\mathscr{L}(F)$, such that finite limits are preserved. For ∞-stacks, analogous condition holds. Indeed, the ∞-category of ∞-stacks can be understood $\infty\mathrm{Stack}(\mathbf{H}, J)$ as a localization of the category of ∞-functors $\infty\mathrm{Func}(\mathbf{H}^{op}; \infty\mathbf{Gpd})$ at Cech nerves. The adjoint $\mathscr{L}_\infty$ of the inclusion $\imath$ is now called the *∞-stackfication*.

In this ∞-categorical context we also have analogous Giraud's characterization axioms. Indeed, as can be seen in Section 6.1.5 of [127], a ∞-category is a Grothendieck ∞-topos iff it satisfy the following properties:

1. it is ∞-cocomplete and there is a collection S of objects which generates all the others, i.e, any $X \in \mathbf{C}$ can be written as a ∞-colimit indexed in S (this condition imply the existence of ∞-limits);

2. for any 1-morphism $f : X \to Y$, the induced morphism $f/\mathbf{C} : X/\mathbf{C} \to Y/\mathbf{C}$ between the over categories preserves all colimits;

3. for any two objects X and Y, the ∞-pullback between the inclusions $X \hookrightarrow X \oplus Y \hookleftarrow Y$ is equivalent to the initial object;

4. not only ∞-groups internal to $\mathbf{H}$ has deelooping, but all ∞-groupoid internal to $\mathbf{H}$.

Let us explore some examples to keep in mind.

Example 8.8 (*∞-groupoids and topological spaces*)**.** By the characterization above we conclude that both $\infty\mathbf{Gpd}$ and $\mathbf{Top}$ are indeed ∞-topos. On the other hand, this could be verified directly by exhibiting a explicit ∞-site $(\mathbf{H}, J)$ such that $\infty\mathrm{Stack}(\mathbf{H}, J) \simeq \infty\mathbf{Gpd}$, and similarly for $\mathbf{Top}$. Indeed, let us consider the trivial category $\mathbf{1}$ with only one object $*$ and whose morphisms and higher morphisms are only identities. It is a ∞-site with the trivial Grothendieck topology J whose coverings are just the identities. In this topology, any ∞-functor $F : \mathbf{1}^{op} \to \infty\mathbf{Gpd}$ is immediately an ∞-stack. But such a functor is totally determined by the ∞-groupoid $F(*)$. Consequently, in order to give a ∞-stack on the ∞-site $(\mathbf{1}, J)$ is just the same as giving a ∞-groupoid. In other words, $\infty\mathrm{Stack}(\mathbf{1}, J) \simeq \infty\mathbf{Gpd}$. As will discussed later, we have an equivalence $\infty\mathbf{Gpd} \simeq \mathbf{Top}$, so that the same trivial ∞-site can be used in order to give a ∞-topos structure on the ∞-category of topological spaces.

Example 8.9 (*slice ∞-topos*)**.** If $\mathbf{H}$ is a ∞-topos, then for any A the slice ∞-category $\mathbf{H}/A$ is also a ∞-topos. We can get this result by observing that the "higher Giraud's axioms" are satisfied (see Section 6.3.5 of [127]). On the other hand, let us give a ∞-site presentation to this fact. More precisely, let $(\mathbf{C}, J)$ be some ∞-site such that $\mathbf{H} \simeq \infty\mathrm{Stack}(\mathbf{C}, J)$. Let $\jmath$ be the Yoneda embedding and suppose that the ∞-site J is such that $\jmath(X)$ is a ∞-stack relatively to J (in this case, we say that J is *subcanonical*). We will prove that

$$\infty\mathrm{Stack}(\mathbf{C}, J)/\jmath(X) \simeq \infty\mathrm{Stack}(\mathbf{C}/X, J/X) \quad \text{for any} \quad X \in \mathbf{C}, \tag{8.2.1}$$

where here J/X is the ∞-site in $\mathbf{C}/X$, induced from the ∞-site J in X, whose coverings $\pi : u \Rightarrow f$ of each $f \in \mathbf{C}/X$ are such that $\pi : U \to X$ belongs to $J(x)$, as in the diagram below. We notice that this condition immediately imply that $u : U \to A$ belongs to $J(a)$. For instance, when $\mathbf{C} = \mathbf{Top}$, the commutativity condition $u = \pi \circ f$ means that u is the pre-image of a covering by a continuous function and, therefore, it also is a covering. The ∞-site J/A is that under the projection $\pi : \mathbf{C}/A \to \mathbf{C}$ we recover the initial ∞-site J.

$$
\begin{tikzcd}
U \arrow[rr, "\pi"] \arrow[dr, "u"'] & & X \arrow[dl, "f"] \\
& A &
\end{tikzcd}
$$

In order to get (8.2.1), recall that for any ∞-site the corresponding category of ∞-stacks can be understood of a localization of the category of ∞-functors at the coverings, so that we have the adjunctions presented in the diagram below by the continuous arrows. Now, suppose that we have an equivalence given by upper segmented arrows. So, we get the lower dotted arrows, which constitute the desired equivalence.

$$
\begin{tikzcd}
\infty\mathrm{Func}(\mathbf{C}^{op};\infty\mathbf{Gpd})/\jmath(X) \arrow[r, dashed, shift left, "F"] & \infty\mathrm{Func}(\mathbf{C}/X^{op};\infty\mathbf{Gpd}) \arrow[l, dashed, shift left] \arrow[d, shift left, "\mathscr{L}_\infty"] \\
\infty\mathrm{Stack}(\mathbf{C},J)/\jmath(X) \arrow[u, shift left, "\imath/\jmath(X)"] \arrow[r, dotted, shift left] & \infty\mathrm{Stack}(\mathbf{C}/X, J/X) \arrow[l, dotted, shift left] \arrow[u, shift left, "\imath"]
\end{tikzcd}
\tag{8.2.2}
$$

We assert that there exist a unique upper equivalences, so that we really have (8.2.1). Indeed, for a fixed X we have two canonical maps, as presented below by continuous arrows: $\jmath_X$ is just the Yoneda embedding for the ∞-category $\mathbf{C}/X$, while $\overline{\jmath}$ is defined by $\overline{\jmath}(X \to Y) = \jmath(X) \to \jmath(Y)$. By Corollary 5.1.6.12 of [127], if a ∞-functor F as in (8.2.2) preserve ∞-colimits and is an extension (as in diagram below), then it is automatically an equivalence. But, notice that if F is an extension, then it must satisfy $F \circ \overline{\jmath} = \jmath$. Because Yoneda embedding is full, this actually totally characterize F. But, the Yoneda embedding is limit-preserving, so that F preserve colimits, ending the proof.

Physics

In the end of Section 6.2 we commented that the space of configurations of **particle** physics is totally axiomatized by (super)differential stacks, but in order to axiomatize the space of fields of **string** physics "higher stacks" would be needed. These "higher stacks" are just the geometric ∞-stacks introduced above. It happens that a classical theory of physics **is not determined** only by its space of fields: we also have to consider an action functional $S_\Sigma : \mathrm{Fields}(\Sigma) \to \mathbb{R}$. Notice that, being a Lie group, $\mathbb{R}$ induces a Lie ∞-groupoid $\mathbf{B}\mathbb{R}$ which determines a geometric ∞-stack. As will be discussed in Chapters 12 and 13, any interesting action, say defined in the space of fields described by a ∞-stack $\mathrm{Fields} : \mathbf{Diff}^{op}_{sub} \to \infty\mathbf{Gpd}$, is just a lifting of Fields from $\infty\mathbf{Gpd}$ to $\infty\mathbf{Gpd}/\mathbf{B}\mathbb{R}$, as presented in the first diagram below. This lifting assigns to any manifold Σ a corresponding ∞-functor $S_\Sigma : \mathrm{Fields}(\Sigma) \to \mathbf{B}\mathbb{R}$, so that it can also be understood as a ∞-natural transformation between Fields and the ∞-functor constant in $\mathbf{B}\mathbb{R}$, as in the second diagram.

$$
\begin{tikzcd}
 & \infty\mathbf{Gpd}/\mathbf{B}\mathbb{R} \arrow[d, "\mathbb{R}"] \\
\mathbf{Diff}^{op}_{sub} \arrow[r, "\mathrm{Fields}"'] \arrow[ur, dashed, "S"] & \infty\mathbf{Gpd}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Diff}^{op}_{sub} \arrow[d, bend right=50, "\mathrm{Fields}"'] \arrow[d, bend left=50, "\mathrm{cst}_{\mathbf{B}\mathbb{R}}"] \arrow[d, phantom, "\overset{S}{\Longrightarrow}"] \\
\infty\mathbf{Gpd}
\end{tikzcd}
$$

Therefore, it seems that the language determined by the ∞-topos of smooth ∞-stacks is sufficiently abstract in order to give a complete axiomatization of classical physics. There exists, however, a further detail which we can be taken into account. In fact, as commented in Section 2.4 and also in Chapter 3, it is a general postulate of classical physics that, *in this lower level of energy, not all configurations occur in the nature, but only that minimize the action functional.* Therefore, in order to identify the observable configurations we need to analyze the critical locus $dS = 0$. But, in order to do this, we need a notion of "derivative" of the action functional.

The ∞-topos of smooth-∞-stacks is very well behaved, characterizing it as a *cohesive ∞-topos*. It happens that in general there is no canonical way to define the "derivative" of an arbitrary object into a cohesive ∞-topos, so that dS is generally not defined. Fortunately, as will be discussed in Chapter 10, any cohesive ∞-topos can be embedded into a *differential cohesive ∞-topos*, in which the abstract notion of derivative always exists. For the ∞-topos of smooth ∞-stacks, this embedding can be obtained by replacing manifolds by *formal manifolds*, meaning that a very natural candidate to a unifying language to classical physics is that described by the ∞-topos of formal-smooth ∞-stacks.

Remark. We take the moment to explain another thing. As commented above, the action functional is generally a ∞-transformation $S : \mathrm{Fields} \Rightarrow \mathrm{cts}_{\mathbf{B}\mathbb{R}}$. The terminal object of $\infty\mathbf{Gpd}_{\mathbf{Diff}}$ is the ∞-groupoid $\mathbf{1}$ whose space of objects, morphisms and higher morphisms are all given by the trivial manifold $*$. Therefore, because $\mathbb{R}$ is contractible[3], we have that $\mathbf{B}\mathbb{R} \simeq \mathbf{1}$ and, consequently, *the space of action functionals defined on any space of configurations is homotopically trivial.*

Remark. In the process of quantization we generally make use of the *exponentiated action functional* $e^{\frac{i}{\hbar}S_{\Sigma}} \in \mathbb{S}^1$, where here $e^{it} : \mathbb{R} \to \mathbb{S}^1$, with $\mathbb{S}^1 \simeq U(1)$, is the complex exponentiation and $\hbar$ is the Plank's constant (introduced in Section 3.2), which is the effective parameter of quantum theories. Differently of $\mathbf{B}\mathbb{R}$, the ∞-groupoid $\mathbf{B}U(1)$ is **not** homotopically trivial. This means that, if we replace $\mathbb{R}$ by $U(1)$ in the discussion of the last remark, we will conclude that the space of transformations $\mathrm{Fields} \Rightarrow \mathrm{cts}_{\mathbf{B}U(1)}$ is not trivial. *This suggest that the fundamental physical notion (or at least the more mathematically interesting notion) is of "exponentiated action functional" instead of "action functional".*

[3] Formally, here we need to use also the Homotopy Hypothesis, which will be sketched in the next section.

8.3 Homotopy

In this section we will focus on the ∞-categories $\mathbf{C}$ for which there is some n such that every k-morphism $\mu : f \Rightarrow g$, with $k > n$, is an isomorphism up to $(k+1)$-morphisms, i.e, there exists another k-morphism $\nu : g \Rightarrow f$ together with $(k+1)$-isomorphisms $\alpha : \nu \circ \mu \Rightarrow id_f$ and $\beta : id_g \Rightarrow \mu \circ \nu$. These are called *$(\infty, n)$-categories.* For instance, for $n = 0$ this is exactly the notion of ∞-groupoid.

We notice that if $\mathbf{C}$ is a $(\infty, n+1)$-category, then forgetting its objects and considering the entity composed only by its 1-morphisms, 2-morphisms and so on, we get a (∞, n)-category. In the language of weak enrichment this means that a $(\infty, n+1)$-category $\mathbf{C}$ can be understood as a usual category weakly enriched over the ∞-monoidal ∞-category $(\infty, n)\mathbf{Cat}$ of all (∞, n)-categories, considered as a full sub-∞-category of $\infty\mathbf{Cat}$.

If $\mathbf{C}$ is an ∞-category we get a k-category by forgetting each l-morphism for $l > k$. In other words, for any k there is a **trivial** forgetful functor $\infty\mathbf{Cat} \to k\mathbf{Cat}$. Part of the interest in (∞, n)-category is due to the fact that in such cases we have **nontrivial** functors $\tau_{\leq k} : (\infty, n)\mathbf{Cat} \to k\mathbf{Cat}$, called *truncation functors*, for each $k \geq n$.

Indeed, by definition in a (∞, n)-category each k-morphism, with $k > l$, in invertible. This means that the relation "*two k-morphisms are equivalent if there is a $(k+1)$-morphism between them*" is symmetric when $k \geq n$. It is also transitive because the composition of k-morphisms is associative up to $(k+1)$-morphisms and it is reflexive because the identity k-morphisms satisfy the "neutral element property" up to $(k+1)$-morphisms. Therefore, in a (∞, n)-category we have an equivalence relation $\simeq_k$ in each set of k-morphisms, allowing us to define quotient categories $\mathbf{C}/\simeq_k$, here denoted by $\mathbf{C}_k$, for which we have the projection functor $\mathbf{C} \to \mathbf{C}_k$. More precisely, we define $\mathbf{C}_k$ as the k-category which has the same objects, 1-morphisms, 2-morphisms, and so on up to $(k-1)$-morphisms than $\mathbf{C}$, but we replace the set of k-morphism with its quotient space by $\simeq_k$. So, the category $\mathbf{C}_k$ is some kind of *"truncation at level k"* of $\mathbf{C}$. This construction extends to ∞-functors between (∞, n)-categories, giving the required $\tau_{\leq k}$.

The main interest in the existence of these truncations is the possibility of study inductively a k-morphism $\xi : f \Rightarrow g$, with $k \geq n$, by its truncated versions. With this in mind, we see that there is further interest in $(\infty, 1)$-categories, because for them this approach can be applied in every k-morphism, for **any** k. In special, if the $(\infty, 1)$-category has a terminal object $*$, the strategy above can be applied into two dual cases: for 1-morphisms $* \to X$ and $X \to *$, giving information on the object X. These strategies are known as *Postnikov/Whitehead approximations of X.*

Remark. By the paragraphs above, the interest in $(\infty, 1)$-categories is evident. However, there are some examples of (∞, n)-categories, with $n \neq 1$, which are also relevant in this approach to Hilbert's sixth problem. They include:

1. *cobordisms.* In the end of Section 2.4, we introduced the $p+1$-category $\mathbf{Cob}(p+1)$ whose objects are 0-manifolds, whose 1-morphisms are cobordisms, whose 2-morphisms are cobordisms between cobordisms, and so on up to cobordisms between p-manifolds. We notice that this entity admits a natural refinement to a $(\infty, p+1)$-category. Indeed, the $(p+1)$-morphisms of $\mathbf{Cob}(p+1)$ are $(p+1)$-manifolds, so that we define the collection of $(p+2)$-morphisms between two $(p+1)$-morphisms as the space of diffeomorphisms between the underlying manifolds. Furthermore, the $(p+3)$-morphisms between diffeomorphisms are defined as smooth isotopies, while higher morphisms are isotopies between

isotopies, and so on. In order to motivate the relevance of this additional structure, recall that an extended quantum theory of dimension p was defined as a monoidal ∞-functor $U : (\mathbf{Cob}(p+1), \sqcup) \to (\mathbf{C}, \otimes)$. Baez and Dolan conjectured in [20] the Cobordism Hypothesis (which classify all these quantum theories) for this class of objects, without the additional categorical structure in $\mathbf{Cob}(p+1)$, but Lurie's proof [124] makes explicit use of the existence of this additional data;

2. <u>*higher spans*</u>. As discussed in Example 4.11, for any category $\mathbf{C}$ we can assign a category $\mathrm{Span}(\mathbf{C})$ of spans in $\mathbf{C}$. Recall that the first attempt to define this category is by taking objects of $\mathbf{C}$ as objects and spans as morphisms, but this does not defines a category, because the composition of spans is well defined only up to isomorphisms of $\mathbf{C}$. In order to avoid this problem, we redefined the morphisms of $\mathrm{Span}(\mathbf{C})$ as equivalence classes of spans: two spans $X \leftarrow Z \to Y$ and $X \leftarrow Z' \to Y$ are were considered equivalent when there exist some span $Z \leftarrow Q \to Z'$ making commutative the first diagram below. So, in the present higher categorical language, it seems more natural to regard $\mathrm{Span}(\mathbf{C})$ as a 2-category, whose objects are objects of $\mathbf{C}$, whose morphisms are spans and whose 2-morphisms between spans are diagrams as below. Now, as can be verified, the diagram below is a span in the 1-category spans, so that as a 2-category, $\mathrm{Span}(\mathbf{C})$ has objects of $\mathbf{C}$ as objects, spans as 1-morphisms and spans in the category of spans (here called 2-*spans*) as 2-morphisms. So, if $\mathbf{C}$ is not only a 1-category, but indeed a ∞-category, then we can try to define a (∞, n)-category $\mathrm{Span}_n(\mathbf{C})$ inductively, whose k-morphisms, with $k \leq n$, are k-spans. This definition was sketched in [124] (around the page 59) and fully formalized in [92]. This kind of (∞, n)-categories will be important in the construction of a general version of pull-push quantization in Chapter 17.

<u>Postnikov</u>

Let $\mathbf{C}$ be an $(\infty, 1)$-category and suppose that it has a terminal object $*$. Then we can apply the idea discussed in the last subsection to the (unique) morphism $X \to *$, giving a inductive way to study X. More precisely, in this case the inductive approach consists in searching a decreasing sequence of 1-morphisms $X_i \to X_{i-1}$ of $\mathbf{C}$, such that:

1. it starts at X, i.e, there is an equivalence $\infty\mathrm{colim} X_i \simeq X$;

2. the object X_i belongs to the i-th truncation of X, meaning $X_i \simeq \tau_{\leq i}(X)$;

3. the sequence ends at $*$, i.e, $X_0 \simeq *$.

We call such a sequence a *Postnikov presentation* (or *Postnikov tower* or even *Postnikov system*) for X. In order to clarify the ideas, let us see how this tower behaves in $\mathbf{Top}$.

Example 8.10 (*classical Postnikov towers*)**.** In the $(\infty, 1)$-category of topological spaces, the higher morphisms are homotopies between homotopies, and so on, so that the k-th truncation $\tau_{\leq k} : \mathbf{Top} \to \mathbf{Top}_k$ is the functor which assign to any space X its *kth homotopy type*. Note that the starting point of the Postnikov tower is a space equivalent to X and the ending point is a space equivalent to $*$ (i.e, a contractible space). Without loss of generality we can work with CW-complexes. This is very interesting, because (by Whitehead's theorem) the kth homotopy type of a CW-complex X can be identified with a space X_k such that $\pi_k(X_i) \simeq \pi_k(X)$ for $k \leq i$. Therefore, a way to present the Postnikov tower of a space X is to start by taking its CW-replacement and then searching for a decreasing sequence of continuous maps (which can be replaced by fibrations) $X_i \to X_{i-1}$ obtained by "killing" the homotopy groups from the left to the right, meaning

$$\pi_k(X_i) \simeq \begin{cases} \pi_k(X), & k \leq i \\ 0, & k > i. \end{cases}$$

For instance, for $i = 1$ the only nontrivial homotopy groups of X_1 are π_0 and π_1, which coincide with the homotopy groups of X. So, if X is assumed path connected, then $\pi_0(X) \simeq 0$ and the only nontrivial homotopy group of X_1 is $\pi_1(X_1) \simeq \pi_1(X)$. In other words, we have been determined X_1 as the Eilenberg-Mac Lane space $K(\pi_1(X), 1)$! Consequently, any map $Y \to X_1$ gives a cocycle in $H^1(Y, \pi_1(X))$. In the general case (i.e, for $k > 1$), we can show that the first ∞-pullback below is equivalent to the Eilenberg-Mac Lane space $K(\pi_i(X), i)$. Notice that for $i = 1$ we recover the above equivalence, as in the second diagram.

$$\begin{tikzcd} \infty\mathrm{pb} \arrow[r] \arrow[d] & * \arrow[d] \\ X_i \arrow[r] & X_{i-1} \end{tikzcd} \qquad \begin{tikzcd} \infty\mathrm{pb} \arrow[r] \arrow[d, "\simeq"'] & * \arrow[d] \\ X_1 \arrow[r] & * \end{tikzcd}$$

Let us spend some words on the last example. By universality of pullbacks, for each i we get a sequence of maps, as presented below. By the equivalence $\infty\mathrm{pb} \simeq K(\pi_i(X), i)$ these maps corresponds to cohomology classes $\kappa(i, j) \in H^i(X_{i+j}, \pi_i(X))$, which are called the *Postnikov classes* of the tower and, in some sense, contain all that we need to know about the tower.

$$\begin{tikzcd} \cdots \arrow[r] & X_{i+2} \arrow[r] \arrow[rrr, bend left] \arrow[drr, bend right] \arrow[rr, dotted] & X_{i+1} \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr] & K(\pi_i(X), i) \arrow[r] \arrow[d] & * \arrow[d] \\ & & & X_i \arrow[r] & X_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

It happens that these Postnikov classes depend of the intermediate objects X_k. Following the section 4.3 of [91] we would like to explain that there is some special situation in which the whole tower can be explicitly reconstructed (up to equivalences) exclusively in terms of invariants of X.

As will be introduced in Section 9.3 (and as was implicitly used in Section in order to build the Barrat-Puppe sequence (2.2)), a ∞-pullback diagram as below is called a *fiber sequence*, while the result is called the *homotopy fiber of f* (in other words, the homotopy fiber is something as the ∞-kernel of f). Notice that, given a fiber sequence $\infty\mathrm{pb} \to X \to Y$, by universality of pullbacks, the canonical map $\Omega X \to X$ lifts to the homotopy fiber.

A fiber sequence is *principal* when there is another fiber sequence $\infty\mathrm{pb}' \to Y' \to X'$, together with vertical weak equivalences, as in the second diagram below. In this case, by universality of

∞-pullbacks, the third diagram is also a pullback square.

$$\begin{tikzcd} \infty\mathrm{pb} \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \infty\mathrm{pb} \arrow[r] \arrow[d, "\simeq"'] & X \arrow[r, "f"] \arrow[d, "\simeq"'] & Y \arrow[d, "\simeq"'] & \\ \Omega X' \arrow[r] & \infty\mathrm{pb}' \arrow[r] & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

By the last example, for any space X, its Postnikov tower induces a sequence of fiber sequences. Let us suppose that all of them are principal (by the theorem 4.69 of [91] this happens iff $\pi_1(X)$ act trivially in each $\pi_n(X)$ and, in particular, if X is simply connected). Therefore, because the Eilenberg-Maclane space constitute a Ω-spectrum, the first vertical equivalence imply that $X' \simeq K(\pi_i(X), i+1)$. Consequently, each X_i is equivalent to the homotopys fiber of the map f'. In special, now we have more canonical classes, defined by maps

$$K(\pi_i(X), i) \to K(\pi_i(X), i+1)$$

obtained as the composition of the dotted arrows below, which are called the *fundamental Postnikov invariants* of X.

$$\begin{tikzcd} K(\pi_i(X), i) \arrow[r] \arrow[d] & * \arrow[d] \\ X_i \arrow[r] & X_{i-1} \end{tikzcd} \qquad \begin{tikzcd} K(\pi_i(X), i) \arrow[r, dashed] \arrow[d, "\simeq"'] & X_i \arrow[r, dashed] \arrow[d, "\simeq"'] & X_{i-1} \arrow[d, dashed, "\simeq"'] & \\ \Omega X' \arrow[r] & \infty\mathrm{pb}' \arrow[r] & X' \arrow[r] & Y' \end{tikzcd}$$

Remark. As will be discussed in the next subsections, this relation between Postnikov towers and cohomology classes refines to the general case of $(\infty, 1)$-categories and it is the fundamental step in the study of Obstruction Theory, which leads with question as "*under which conditions a given morphism admits a nontrivial extension/lifting?*"

Whitehead

In the last subsection we used the structure of $(\infty, 1)$-category in order to study inductively a given object X by the sequence of its truncations that start at X and end at $*$. In other words, we applied the inductive method of truncations to the 1-morphism $X \to *$. We could try to do the dual situation. More precisely, we could apply the truncation method in order to study X by 1-morphisms $* \to X$. This imply searching an increasing sequence $X^i \to X^{i-1}$ such that

1. it starts at $*$, i.e, we have an equivalence $\infty\mathrm{colim}\, X^i \simeq *$;

2. the object X^i belongs to the i-th truncation of $*$, meaning $X^i \simeq \tau_{\leq i}(*)$;

3. it ends at X, i.e, $X^0 \simeq X$.

Such a sequence is called a *Whitehead presentation* (*Whitehead tower* or *Whitehead system*) for the object X. Let us see that in the $(\infty, 1)$-category **Top** we have (as for Postnikov towers) a natural way to work/build Whitehead presentations.

Example 8.11 (*Classical Whitehead*)**.** We start exactly as in the Postnikov case: by assuming that we are working with fibrations and CW-complexes. So, the i-th truncation of $*$ is given by a space X_i such that $\pi_k(X^i) \simeq 0$ when $i \leq k$. Because the sequence now ends at X, we fix the homotopy type of X^i by requiring $\pi_k(X^i) \simeq \pi_k(X)$ if $i > k$. In other words, the idea is to build each X^i by "killing" homotopy groups from the right to the left. More precisely, the idea is to build the spaces X^i in the "reverse" order: knowing the homotopy groups of X we build X^1 by killing $\pi_1(X)$, and then we build X^2 by killing $\pi_2(X)$, and so on. For instance, X^1 need to be simply connected, but $\pi_i(X^1) \simeq \pi_i(X)$ for $i > 1$. We are searching for cases in which the map $X^1 \to X$ is a fibration (whose standard examples are projections of fiber bundles). Under mild conditions[4] there is a bundle $X^1 \to X$ with these desired homotopical properties: the *universal covering space of* X. So, we can think of the other maps $X^n \to X$ as "n-connected universal covers" of X. Because the spaces X^i should constitute some kind of "dual Postnikov tower", we could try to use the Postnikov tower of X in order to build X^i. Indeed, recall that the 0-connected coverings $Y \to X$ of a space X are classified by (the conjugacy classes of) $\pi_1(X)$. So, we expect to get these n-connected covers $X^n \to X$ by analyzing each $\pi_i(X)$ with $i \leq n+1$, i.e, by analyzing the $(n+1)$th homotopy type of X, which can be modeled by the term X_{n+1} in the Postnikov tower of X. It can be shown that the map $X^{n+1} \to X$ really can be obtained as the homotopy fiber of $X \to X_{n+1}$, as presented in the diagram below (this diagram will be called the *Whitehead-Postnikov diagram of* X). See [??] for further details.

8.4 Hypothesis

In many moments of this chapter (and in some other parts of the text) we said that there is an equivalence between the $(\infty,1)$-categories **Top** and ∞**Gpd**. This equivalence is known as the *Grothendieck Homotopy Hypothesis*. A formal proof that $\mathbf{Top} \simeq \infty\mathbf{Gpd}$ depends explicitly of the presentation selected to model the notions of "∞-groupoids" and "$(\infty,1)$-categories". Even so, we would like to give (in our present naive context) a rough idea of how this equivalence behave.

We start by recalling that, as discussed in Example ??, to any topological space X we can assign its fundamental ∞-groupoid $\Pi(X)$. This is the ∞-groupoid whose objects are points of X, whose 1-morphisms $x \to y$ are paths $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$, whose 2-morphisms $\xi : \gamma \Rightarrow \gamma'$ are homotopies preserving the base points, and so on. This defined the rule Π on objects. We notice that it extends to a ∞-functor $\Pi : \mathbf{Top} \to \infty\mathbf{Gpd}$.

Indeed, if $f : X \to Y$ is a continuous function (i.e, a 1-morphism in the ∞-category **Top**) then we get a ∞-functor $\Pi(f) : \Pi(X) \to \Pi(Y)$ between ∞-groupoids (which are the 1-morphism in the ∞-category ∞**Gpd**) given by $\Pi(f)(x) = f(x)$ on objects, $\Pi(f)(\gamma) = f \circ \gamma$ on morphisms, and so on, so that Π really maps 1-morphisms into 1-morphisms. In the same way, given a homotopy $h : f \Rightarrow g$ between functions $f, g : X \to Y$ (i.e a 2-morphism in **Top**) we define a ∞-natural transformation $\Pi(h) : \Pi(f) \Rightarrow \Pi(g)$ as the rule that to any object $x \in \Pi(X)$ assigns the 1-morphism $\Pi(h)_x$ (i.e, the path) between $\Pi(f)(x) = f(x)$ and $\Pi(g)(x) = g(x)$ given by $\Pi(h)_x(t) = h_t(f(x))$, and also a 2-morphism (i.e, a homotopy between paths), and so on, all done in a natural way.

The statement of Homotopy Hypothesis is that the fundamental ∞-groupoid functor Π has

[4] Such as being locally path connected and locally simply connected, which happens for instance the space is locally contractible.

an inverse. In order to motivate the existence of this inverse, let us see that the ∞-groupoid $\Pi(X)$ contains all (homotopical) information about the topological space X. More precisely, we will see that $\Pi(X)$ contains the homotopy groups $\pi_n(X,x)$ for any $n \geq 0$ and for any base point x, so that if X is a CW-complex, then its homotopy type is totally described by $\Pi(X)$.

The idea is the following: because we are working with ∞-groupoids (which are the same as $(\infty,0)$-categories), we have the truncations functors $\tau_k : \infty\mathbf{Gpd} \to k\mathbf{Gpd}$ for any $k \geq 0$, where $0\mathbf{Gpd}$ is just $\mathbf{Set}$. In particular, we can define τ_k^x as τ_k, but retaining information only about the object x. On the other hand, for any x we have the functor $h^x : \mathbf{Cat}_k \to \mathbf{Cat}_{k-1}$ assigning to any other object y the $(k-1)$-category of morphism $x \to y$. Repeating the process k times we get a functor ${}_kh^x : \mathbf{Cat}_k \to \mathbf{Set}$ assigning to any object x the set of k-morphisms between $(k-1)$-morphisms, between $(k-2)$-morphisms, and so on up to 1-morphisms between x and other object.

We assert that, for any $x \in X$ and any k we have ${}_kh_x(\tau_k^x(\Pi(X))) = \pi_k(X,x)$. This assertion can be formally obtained from finite induction, but in order to explain what are happening, let us analyze some particular cases for small k. Indeed, if $k=1$, then for any $x \in X$ the 1-groupoid $\tau_1^x(\Pi(X))$ is such that the only object is x and whose morphisms are homotopy classes of maps $\gamma : I \to X$, satisfying $\gamma(0) = x = \gamma(1)$. The functor ${}_kh_x$ retains only the set of these morphisms which is just $\pi_1(X,x)$. Indeed, such a path is the same as a map $\gamma : I \to X$ such that $\gamma(\partial I) = x$, which is equivalently a based map $\gamma : I/\partial I \to (X,x)$. But $I/\partial I \simeq \mathbb{S}^1$.

For $k=2$, we have the 2-groupoid $\tau_2^x(\Pi(X))$, with only one object x, whose morphisms are paths $\gamma : I \to X$ on x and whose 2-morphisms are homotopy classes of homotopies between these paths, preserving the base point. So, they are equivalently homotopy classes of bases maps $h : I \times I/\partial(I \times I) \to (X,x)$, i.e hotomopy classes of based functions $\mathbb{S}^2 \to (X,x)$. The functor ${}_2h^x$ retains the set of these classes, which is precisely the second homotopy group $\pi_2(X,x)$. Now, the general induction argument becomes evident and we prefer to omit it.

Convinced that the fundamental ∞-groupoid construction retains the homotopical information of any topological space, let us see that the ∞-functor Π has an inverse $|\cdot| : \infty\mathbf{Gpd} \to \mathbf{Top}$, which assign to any ∞-groupoid $\mathbf{C}$ a corresponding space $|\mathbf{C}|$, called the *geometric realization* of $\mathbf{C}$, such that $|\Pi(X)| \simeq X$ or, without loss of generality, that $\pi_k(|\Pi(X)|) \simeq \pi_k(X)$ for every $k \geq 0$.

We start by recalling that any ∞-category $\mathbf{C}$ (in particular any ∞-groupoid) can be understood in the "source/target perspective" as a huge diagram into $\mathbf{Set}$. This diagram is composed by a sequence of sets $\mathbf{C}_0, \mathbf{C}_1, ...$ (corresponding to the collections of objects, 1-morphisms, and so on) connected by source/target arrows $\mathbf{C}_{i+1} \to \mathbf{C}_i$, together with arrows $\mathbf{C}_i \to \mathbf{C}_{i+1}$ describing the identities and other arrows describing compositions, which are supposed to satisfy some additional commutative conditions. So, the most natural way to get functors $\infty\mathbf{Gpd} \to \mathbf{Top}$ is first search for ways to turn any ∞-groupoid a diagram internal to $\mathbf{Top}$ (i.e a topological ∞-groupoid) and then define the geometric realization as the colimit of this new diagram. In other words, the idea is to build some special funtor $\imath : \infty\mathbf{Gpd} \to \infty\mathbf{Gpd}_{\mathbf{Top}}$ and define $|\mathbf{C}|$ as $\operatorname{colim} \imath(\mathbf{C})$.

There is a trivial way to do this: putting into each $\mathbf{C}_i$ the discrete topology, for which the source/target maps, etc, becomes immediately continuous. However, we notice that this is strategy **does not** produce an inverse for Π. Indeed, supposing the opposite, we need to have $\pi_k(F(\Pi(X))) \simeq \pi_k(X)$ for every X. It happens that, for each groupoid $\mathbf{C}$ the resultant space $F(\mathbf{C}) = \operatorname{colim}\mathbf{C}_i$ is discrete and, therefore, $\pi_k(F(\mathbf{C})) = 0$ for $k > 0$. But, if X is not discrete, then we may have $\pi_k(X) = \pi_k(F(\Pi(X))) \neq 0$ for $k > 0$, contradicting the hypothesis.

This show that seeing each $\mathbf{C}_i$ as a trivial topological space is not a good idea. Instead of this we could see them as *parameters* of certain homotopically trivial spaces. More precisely, the new idea is to chose a sequence of homotopically trivial topological spaces D_n and define $\imath : \infty\mathbf{Gpd} \to \infty\mathbf{Gpd}_{\mathbf{Top}}$ as the assignment that to any $\mathbf{C}$ associate the topological ∞-groupoid $\imath(\mathbf{C})$ whose space of k-morphisms is $\imath(\mathbf{C})_k = \mathbf{C}_k \times D_k$, where $\mathbf{C}_k$ is endowed with the discrete topology and, therefore, the product $\mathbf{C}_k \times D_k$ is understood as a family of copies of D_k parametrized by the set $\mathbf{C}_k$. Because each D_n is homotopically trivial, we have canonical maps $D_n \to D_{n-1}$ and $D_n \to D_{n+1}$ which are used to build $\imath(\mathbf{C})_k \to \imath(\mathbf{C})_{k+1}$ and $\imath(\mathbf{C})_k \to \imath(\mathbf{C})_{k-1}$.

For instance, we could consider $D_n = I \times ... \times I$, $D_n = \Delta^n$ or even $D_n = \mathbb{D}^n$. For this last choice, the ∞-topological groupoid $\imath(\mathbf{C})$ is simply a sequence of k-cells parametrized by the set of k-morphisms $\mathbf{C}_k$ and then the colimit $\operatorname{colim} \imath(\mathbf{C})$ simply glue all these cells, producing a CW-complex.

The above construction seems a natural candidate to an inverse to Π (i.e, to the geometric realization process) and it really is.

Example 8.12 (*classifying spaces*)**.** Recall that in Section 1.2, at the discussion on nonabelian cohomology, we have been commented that for any topological group G we can associate a topological space BG, called the *classifying space* of G. Such a space is just $|\mathbf{B}G|$, i.e, the geometric realization of the delloped groupoid.

Remark. Note that $\Pi(X)$ is defined for any topological space (not only for CW-complexes). Similarly, we can define $|\mathbf{C}|$ for arbitrary ∞-category (not only for ∞-groupoids). On the other hand, it is only after restricting to CW-complexes and ∞-groupoids that $|\cdot| \rightleftharpoons \Pi$ becomes one the inverse of the other.

Properties

Here we will discuss some important properties of the geometric realization functor $|\cdot|$. Directly from the definition we see that it preserve finite limits. In particular, it is cartesian ∞-monoidal, so that it maps ∞-monoid objects of $\infty\mathbf{Cat}$ into ∞-monoid objects of $\mathbf{Top}$. In other words, it maps ∞-monoidal ∞-categories into A_∞-spaces. Similarly, it maps symmetric ∞-monoidal ∞-categories (i.e, commutative ∞-monoid objects of $\infty\mathbf{Cat}$) into E_∞-spaces (which are the commutative ∞-monoids of $\mathbf{Top}$).

Furthermore, as any symmetric monoidal functor, the geometric realization $|\cdot|$ also maps ∞-Hopf objects of $\infty\mathbf{Cat}$ (i.e, ∞-groups) into ∞-Hopf objects of $\mathbf{Top}$ (i.e, ∞-loop spaces, as discussed in Example 8.1.3). We can ask if there is some other class of symmetric ∞-monoidal ∞-categories, less restrictive than ∞-groups, such that it is mapped under $|\cdot|$ into ∞-loop spaces. The answer is **affirmative**. The idea is the following: recall that a topological space X is a ∞-loop space iff $\pi_0(X)$ is a group. So, due to the Homotopy Hypothesis, for a given symmetric ∞-monoidal ∞-category $\mathbf{C}$, the corresponding $|\mathbf{C}|$ is a ∞-loop space iff

$$\pi_0(|\mathbf{C}|) \simeq \mathrm{Iso}_0(\mathbf{C}) = \tau_0(\mathbf{C}) \quad \text{is a group.}$$

Notice that this is immediately satisfied when $\mathbf{C}$ is a ∞-group, because in this case the collection of objects has, itself, a group structure. However, this group structure in $\mathbf{C}_0$ is not necessary in order to get a group structure into $\mathrm{Iso}_0(\mathbf{C})$. Indeed, because $\mathbf{C}$ is assumed a symmetric ∞-monoidal ∞-category, the product $\otimes$ and the neutral object 1 induce an abelian monoid structure into $\mathrm{Iso}_0(\mathbf{C})$, so that what we need is an inverse for each object $X \in \mathbf{C}$. These should be given by an object X^{-1} endowed with morphisms

$$\mu : X \otimes X^{-1} \to 1 \quad \text{and} \quad \nu : 1 \to X^{-1} \otimes X$$

which becomes isomorphisms into the truncated category. This means that there should exist 2-morphisms

$$\alpha : \nu \circ \mu \Rightarrow id_{X \otimes X^{-1}} \quad \beta : id_1 \Rightarrow \mu \circ \nu \tag{8.4.1}$$

fulfilling conditions up to 3-morphisms, and so on. In the case of ∞-groups, the morphisms μ and ν are actually isomorphisms, so that α, β and the higher morphisms between them are all identities.

We could, however, work with other conditions on the 2-morphisms (8.4.1) and as well as on the higher morphisms. For instance, we could require some adjoint relation up to higher morphisms. More precisely, we could require that the diagrams below are commutative up to 2-morphisms (replacing the α and β above), which in turn fulfill conditions up to 3-morphisms, and so on.

$$\begin{tikzcd} X \arrow[r, "id\otimes\nu"] \arrow[dr, "id"'] & X \otimes X^{-1} \otimes X \arrow[d, "\mu\otimes id"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X^{-1} \arrow[r, "\nu\otimes id"] \arrow[dr, "id"'] & X^{-1} \otimes X \otimes X^{-1} \arrow[d, "id\otimes\mu"] \\ & X \end{tikzcd}$$

An object X for which there exists μ and ν as above is called *dualizable*; we say that X^{-1} is a *dual* for X. If a ∞-monoidal ∞-category is such that every object is dualizable, we say that is has *dual objects*. So, we have been concluded that *the geometric realization of a ∞-monoidal ∞-category with dual objects is a ∞-loop space*. See Section 2.3 of [124] and the commentary before Theorem 2.5.10, p. 51 of the same reference.

Remark. The class of monoidal categories with dual objects really is more large than the class of ∞-groups. Indeed, in the monoidal category $(\mathbf{Vec}_{\mathbb{K}}, \otimes)$, an object is dualizable iff is finite-dimensional and, in this case, the dual X^{-1} of X is just its linear dual X^*. Therefore, the subcategory of finite dimensional $\mathbb{K}$-vector spaces has dual objects, but it is not a 2-group, as commented in Section 6.2.

Implications

The "proof" of the Homotopy Hypothesis given in the last subsections is merely formal. Indeed, recall that in order to turn it into a real "proof" we need to make a choice of some presentation of ∞-categories and then show that the argument used here makes sense in this presentation. On the other hand, here we would like to convince the reader that there are good reasons to believe that the Homotopy Hypothesis is correct independently of the presentation selected.

More precisely, we will show that the Homotopy Hypothesis affects deeply both mathematics and physics in such a way that it can be compared with the axiom of choice. This is the reason of the qualification "hypothesis": it is a too deep assertion that it is expect that any good presentation of higher category theory should be such that we have an equivalence $\infty\mathbf{Gpd} \simeq \mathbf{Top}$.

The situation is similar to what happens in physics with the "energy conservation law". Indeed, in any known physical theory it is a theorem and, one time created some new model to a certain physical phenomena, the immediate step is to verify if we have conservation of energy. In this perspective, while "energy conservation" is a filter to new physical theories, the Homotopy Hypothesis enter as a filter to new presentation of higher category theory.

- **Mathematical Implication** (*recovering Thom-Pontryagin theorem on cobordism*). Here we will show that the Homotopy Hypothesis imply that the classical result (discussed in the end Section 1.2), that there is a spectrum $\mathbb{E}$ such that, for any n, the set of cobordism classes of oriented n-manifolds is just the nth stable homotopy groups of $\mathbb{E}$. In fact, let $\mathbf{Cob}(\infty) = \mathrm{colim}_{n\to\infty}\mathbf{Cob}(n)$ be the ∞-category whose objects are 0-manifolds, whose 1-morphisms are 1-cobordisms, whose n-morphisms are n-cobordisms, and so on. Let $\mathbf{Cob}^{or}(\infty)$ be the sub ∞-category of $\mathbf{Cob}(\infty)$ obtained retaining only the **oriented** manifolds. This entity is indeed an ∞-groupoid: each n-morphism $\Sigma : M \to N$ is an **oriented** cobordism between **oriented** manifolds. Its inverse is just the cobordism obtained by reverting the orientations. By the Homotopy Hypothesis, this ∞-groupoid is equivalent to a topological space X: the geometric realization of $\mathbf{Cob}^{or}(\infty)$, which actually is a CW-complex and, therefore, has the homotopy type determined by its homotopy groups. Each category $\mathbf{Cob}(n)$ is symmetric monoidal and has duals, so that (by the discussion in the last subsection) X is actually a ∞-loop space. Therefore, there is a spectrum $\mathbb{E}$ such that $X = \Omega^\infty E_\infty$. Thus, because the homotopy groups preserve inductive colimits, we get the desired result:

$$\begin{aligned}
\mathrm{Iso}_n(\mathbf{Cob}^{or}(\infty)) &\simeq \pi_n(X) \\
&\simeq \pi_n(\Omega^\infty E_\infty) \\
&\simeq \pi_n(\lim_k \Omega^k E_k) \\
&\simeq \lim_k \pi_{n+k}(E_k) = \pi_n^S(\mathbb{E})
\end{aligned}$$

- **Physical Implication** (*supersymmetry is the most general kind of symmetry*). As another implication of the Homotopy Hypothesis we will show that the monoidal category $(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)$ of complex super vector spaces admits precisely two braidings: the commutative and the graded-commutative. By the discussion in Section 5.2, this fact has a very physical appealing: it imply that *in a quantum theory of particles describing both bosons and fermions, the most general kind of symmetry that can be introduced is supersymmetry.* This will be done in two steps. In the first we will discuss the problem in the general context and we will show that it can be replaced by a purely homotopical question. In the second step we will attack the homotopical problem in the specific case $(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)$.

  1. *rewriting the problem*. Let us start by considering any monoidal category $(\mathbf{C}, \otimes, 1)$. As discussed in Examples 6.12 and 8.6.3, we have the corresponding Picard 2-group $\mathrm{Pic}(\mathbf{C}, \otimes)$, which is a 2-group and, therefore, a group object in the category of 1-groupoids. By the Homotopy Hypothesis, we have a corresponding 1-homotopy type $X = |\mathrm{Pic}(\mathbf{C})|$ (which can be understood as a CW-complex whose only nontrivial homotopy groups are π_0 and π_1) with a group structure, meaning that it has a delooping BX. Two isomorphic monoidal structures on $\mathbf{C}$ have isomorphic Picard 1-groupoids and, therefore, their corresponding homotopy 1-types are equivalent. So, *determining the homotopy type of X we are determining the maximal number of non-isomorphic monoidal structures on* $\mathbf{C}$. This is not exactly which we are interested. Indeed, we

are interested in the isomorphic classes of **symmetric** monoidal structures on $\mathbf{C}$. The fundamental remark is that if $(\mathbf{C}, \otimes, 1)$ is endowed with a **symmetric** structure, the Stabilization Hypothesis imply that $\mathrm{Pic}(\mathbf{C}, \otimes)$ is a 4-group, so that by the Homotopy Hypothesis $X = |\mathrm{Pic}(\mathbf{C})|$ is now a 3-homotopy type (i.e, space with $\pi_i(X) = 0$ for $i > 3$), which has deloopings $B^3 X$. Therefore, determining the homotopy structure of this 3-homotopy type (or of its delooping $B^3 X$) we are determining the possible **symmetric** monoidal structures that can be introduced on $\mathbf{C}$. But this can be done by looking to the Postnikov tower of $B^3 X$, which in turn is determined by the fundamental Postnikov invariants.

2. *attacking the problem*. Returning to our concrete problem, following the discussion of the last topic, we need to determine the Postnikov invariants of $B^3 X$ for $X = |\mathrm{Pic}(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}, \otimes)|$, i.e, we need to compute the homotopy classes of maps

$$K(\pi_i(B^3 X); i) \to K(\pi_i(B^3 X); i+1). \tag{8.4.2}$$

We notice that it is enough to do the computations for only one i. Indeed, let us suppose that we have been done the computations for certain i. Then, for any $j \leq i$,

$$\pi_i(B^3 X) = [\mathbb{S}^i; B^3 X] \simeq [\Sigma^{i-j}\mathbb{S}^j; B^3 X] \simeq [\mathbb{S}^j; \Omega^{i-j} B^3 X] = \pi_j(\Omega^{i-j} B^3 X).$$

On the other hand, because the Eilenberg-Mac Lane constitute a Ω-spectrum, by applying Ω^j, any map as (8.4.2) induce a corresponding

$$K(\pi_j(\Omega^{i-j} B^3 X); i-j) \to K(\pi_j(\Omega^{i-j} B^3 X); i-j+1).$$

Now, recall that if $\rho : G \to H$ is a group homomorphism, it induce morphisms between the corresponding Eilenberg-Mac Lane spaces $K(G; k) \to K(H; k)$ for every k. Therefore, the canonical $\Omega^{i-j} B^3 X \to B^3 X$ induce maps

$$K(\pi_j(B^3 X); i-j) \to K(\pi_j(B^3 X); i-j+1)$$

representing Postnikov invariants of degree $i-j$. For $j > i$ we proceed in a similar way. So, let us work with $i = 3$. In this case, we have[5] $\pi_3(B^3 X) \simeq \pi_0(X) \simeq \mathbb{Z}_2$, so that we have to compute $[K(\mathbb{Z}_2; 3), K(\mathbb{Z}_2; 4)]$. We assert that this space is contained in $\mathbb{Z}_2$, implying that there are at most two distinct symmetric structures on $\mathrm{Pic}(\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}})$ and, therefore, on $\mathbb{Z}_2\mathbf{Grad}_{\mathbb{C}}$. This can be proved of different ways. For instance, by the Yoneda lemma, the above maps corresponds bijectively to natural transformations between $H^3(-, \mathbb{Z}_2)$ and $H^4(-, \mathbb{Z}_2)$. In other words, they are operations in ordinary cohomology with coefficients on $\mathbb{Z}_2$ and, therefore, are determined by the Steenrod operations Sq^1. So, by making us of the axiomatic properties of these operations (as done in Section 4.3, where we used the axiomatic properties of Adam's operations in order to proof the Adam-Atiyah theorem) we could infer the required result. We will

[5]The second equivalence comes from the definition of the Picard groupoid. Indeed, it is the set of $\mathbb{Z}_2$-graded vector spaces $V = V_0 \oplus V_1$ such that there is $W = W_0 \oplus W_1$ for which $V \otimes W \simeq \mathbb{K} \oplus 0$. By dimension analysis we conclude that, up to isomorphism, there are only two possible configurations satisfying this condition: $V \simeq \mathbb{K} \oplus 0$ and $V \simeq 0 \oplus \mathbb{K}$.

take a shortcut. Indeed, if there is a map $K(\mathbb{Z}_2;3) \to K(\mathbb{Z}_2;4)$ then there is also a map $\Omega^3 K(\mathbb{Z}_2;3) \to \Omega^3 K(\mathbb{Z}_2;4)$ and, consequently, because the Eilenberg-Mac Lane spaces are a Ω-spectrum, a map $\mathbb{Z}_2 \to K(\mathbb{Z}_2;1)$. Applying the loop space functor one more time we get $\Omega\mathbb{Z}_2 \to \mathbb{Z}_2$. It happens that $\Omega\mathbb{Z}_2 \simeq \mathbb{Z}_2$ and, because we are working with based maps, there are only two possible $\mathbb{Z}_2 \to \mathbb{Z}_2$.

Remark. In the mathematical motivation, we concluded that the Homotopy Hypothesis imply that the oriented cobordism classes are the stable homotopy of **some** spectrum $\mathbb{E}$. But the Thom-Pontryagin theorem says more: it **specify** the spectrum as the sphere spectrum $\mathbb{E} = \mathbb{S}$. The central question is: *could we recover the whole theorem using only the Homotopy Hypothesis?* Notice that $\mathbf{Cob}(\infty)$ was defined as the limit $\mathrm{colim}_{n\to\infty}\mathbf{Cob}(n)$ and, because the geometric realization functor preserve ∞-colimits (recall that it has an ∞-adjoint) we have

$$|\mathbf{Cob}^{or}(\infty)| = \mathrm{colim}_{n\to\infty}|\mathbf{Cob}^{or}(n)|,$$

so that in order to determine explicitly the spectrum, we need to calculate explicitly the n-homotopy types $|\mathbf{Cob}^{or}(n)|$, which is an information that **cannot** be accessed only by the Homotopy Hypothesis, meaning that we need something more. This "something more" is precisely the Cobordism Hypothesis (firstly commented in Section 2.4 and which will be sketched in Appendix B). It states that the ∞-category $\mathbf{Cob}^{or}(n)$ is freely generated by only one object: the trivial manifold $*$. Thus, because $|\mathbf{Cob}^{or}(\infty)|$ is a ∞-loop space, we conclude that the underlying spectrum is generated by the point $*$. But this is just the sphere spectrum! Summarizing:

$$\begin{matrix}\textbf{Homotopy}\\ \textbf{Hypothesis}\end{matrix} + \begin{matrix}\textbf{Cobordism}\\ \textbf{Hypothesis}\end{matrix} \Longrightarrow \begin{matrix}\textbf{Thom's}\\ \textbf{Work}\end{matrix}$$

Chapter 9

Stable Theory

In Section 2.3, in order to convince the reader that there must exist languages which are more abstract that categorical language, we presented many similarities between **Spec** and $\mathbf{CCh}_R$, suggesting the existence of some "abstract stable homotopy theory" of which both categories are only particular examples. In the last two chapters we developed the basic structures of higher category theory and we showed that the language of $(\infty, 1)$-categories is sufficiently abstract in order to incorporate abstract homotopy theory. For instance, we discussed that we can talk of "n-truncations" of a object in the same way as we can talk of the "n-th homotopy type" of a topological space. In the present chapter we will see that this $(\infty, 1)$-categorical context is also abstract enough to develop the conjectured abstract **stable** homotopy theory.

Recall that, despite the similarities between **Spec** and $\mathbf{CCh}_R$, in Section 5.3 we showed that they have a deep difference: while the tensor product $\otimes$ of $\mathbf{Mod}_R$ induces a symmetric monoidal structure in $\mathbf{CCh}_R$, the smash product in $\mathbf{Top}_*$ induce a symmetric monoidal structure in the homotopy category $\mathrm{Ho}(\mathbf{Spec})$. Furthermore, Lewis's obstruction theorem states that this structure **cannot** be lifted to a symmetric monoidal structure in **Spec**. In few words, the main objective of Sections 9.1 and 9.2 is to proof that, when passing to the higher categorical context, this difference disappear: *we actually have a canonical notion of "smash product" in* **Spec**, *now considered as a* $(\infty, 1)$*-category*.

More precisely, we start in Section 9.1 by presenting the notion of *stable* ∞*-category* which incorporate the fundamental common properties of **Spec** and $\mathbf{CCh}_R$. We then show that to any $(\infty, 1)$-category $\mathbf{C}$ we can associate a corresponding stable ∞-category $\mathrm{Stab}(\mathbf{C})$, in a very similar way as **Spec** is obtained from $\mathbf{Top}_*$. Concretely, will see that the notion of Ω-spectrum makes sense internal to any ∞-category and that they fit into a stable ∞-category.

The section 9.2 is on the functoriality of the *stabilization rule* $\mathbf{C} \mapsto \mathrm{Stab}(\mathbf{C})$. Indeed, as will be discussed there, this rule is generally **not** functorial. The problem is that there is no canonical way to assign to any ∞-functor $F : \mathbf{C} \to \mathbf{D}$ a corresponding $\mathrm{Stab}(\mathbf{C}) \to \mathrm{Stab}(\mathbf{D})$ such that compositions are preserved. On the other hand, it is always possible approximate F by a functor $P_1 F$ for which Stab becomes functorial. So, the problem of determining the functoriality of Stab can be attacked by searching for conditions under which $F \simeq P_1 F$. Furthermore, if a functor F does not satisfy the desired conditions (and, therefore, we do not have $F \simeq P_1 F$), we can study the iterated (excisive) approximation $P_2 F = P_1(P_1 F)$, and so on, giving the diagram below. When its colimit converges to F (i.e, when $\mathrm{colim}_n P_n F \simeq F$), we recover the functoriality of the stabilization rule.

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow F$$

We can think of the approximation P_nF as the n-th derivative D_nf of a smooth function, so that the sequence above is the higher categorical analogue of Taylor's series and then the problem of determining the functoriality of Stab becomes similar to the problem of determining if a given smooth map is or not analytic. This analogy is in the heart of the *Goodwillie calculus*. For our purposes, the fundamental application of this calculus is the proof that any colimit preserving monoidal structure $\otimes$ on $\mathbf{C}$ induces an essentially unique monoidal structure $\wedge_{\otimes}$ on spectra $\mathrm{Stab}(\mathbf{C})$. Particularly, this imply that the smash product on spaces actually induce an unique monoidal structure into the $(\infty,1)$-category of Ω-spectra, as desired.

Recall that, as explored in Section 1.2, Ω-spectra are the representing objects for generalized cohomology theories, which are one of the most prominent invariants of (stable) homotopy theory. Therefore, because the notion of Ω-spectra makes sense in any $(\infty,1)$-category, it is natural to expect that the notion of "generalized cohomology theory" can also be internalized into the ∞-categorical context. In fact, in Section 9.3 we see that *there exists a notion of abstract cohomology theory internal to any $(\infty,1)$-category, which abstract not only the generalized cohomology theories, but all examples of cohomology theories discussed Section 1.2!* In special, for G a ∞-group internal to ∞-category $\mathbf{H}$ we can talk of its *abstract nonabelian cohomology*, whose cocycles can be used to define *G-principal ∞-bundles* in $\mathbf{H}$.

Finally, in Section 9.4, we discuss obstruction theory in the higher categorical context. There we see that the concept of orientation of a bundle with respect to some generalized cohomology theory can be extended to a notion of *orientation of ∞-bundles with respect to abstract cohomology theories*. We then comment that the "orientability condition" is a fundamental step in order to build a (pull-push) quantization scheme.

Remark. The fundamental references for Sections 9.1 and 9.2 are [125, 129, 126, 10]. A good discussion in the spirit of Section 9.3 is in Section 3.6.9 of [182] and in [126]. Section 9.4 was based in [145, 143, 10].

Warning. The theory of orientation for A_∞-ring objects is also in construction (see the recent papers [??,??]), so that *some results discussed here maybe are not valid in this level of generality*. They are valid, however, for E_∞-ring objects, as formalized in [??]. As far as our acknowledgement goes, the role of the commutativity condition (i.e, the necessity of working with E_∞-rings instead of with A_∞-rings) in the construction of a quantization scheme presently is not understood.

9.1 Stabilization

Recall that, as discussed in Section 1.2, the categories $\mathbf{CW}_*$ and $\mathbf{CCh}_R^+$ have very similar homotopical properties. On the other hand, there is a primary difference between them: while in both we have canonical *reduced suspension functor* Σ and *loop space functor* Ω, the cochain complex are "stable" under these functors while CW-complexes are not. This could be fixed by replacing CW-complexes by *spectra* or *Ω-spectra*, which are objects naturally stable under suspensions or loopings. This motivate us to conjecture the existence of a more general abstract "stable homotopy theory" for which the homotopy theory of spectra and of unbounded cochain complexes are particular cases. In this subsection we will outline the construction of such a theory.

The starting point is to build Σ and Ω axiomatically. Indeed, given any $(\infty,1)$-category $\mathbf{C}$ with null object $*$ and finite ∞-limits/∞-colimits we define the *reduced suspension* ΣX and the *loop space object* ΩX of an object $X \in \mathbf{C}$ as the ∞-pullback/∞-pushout presented below. We say that $\mathbf{C}$ is *stable* when for any X the canonical maps $\Omega X \to X$ and $X \to \Sigma X$, here obtained from universality, are ∞-isomorphism.

$$
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Sigma X & * \arrow[l] \\
* \arrow[u] & X \arrow[l] \arrow[u]
\end{tikzcd}
$$

This requirement has several consequences. We list some of them:

1. *loop and suspension are dual equivalances*. It immediately imply that the ∞-functors Σ, Ω are equivalences, being one the the inverse of the other.

2. *every object is deloopable*. As discussed in Example 8.7, any ∞-group internal to a ∞-category $\mathbf{C}$ is deloopable. If $\mathbf{C}$ is additionally a ∞-topos, then one of the higher Giraud's axioms ensures that not only ∞-groups, but indeed every ∞-groupoid internal to $\mathbf{C}$ can be deloopable. If $\mathbf{C}$ is now a stable ∞-category, then **any** object $X \in \mathbf{C}$ has a delooping, which is given by the suspension ΣX.

3. *triangulated homotopy category*. As will be discussed in Section 9.3, internal to any $(\infty,1)$-category we have an abstract notion of cohomology, which can be computed by making use of certain fiber sequences. The homotopy category of stable categories are *triangulated*, which means that the fiber sequences are simple and, therefore, the computation of cohomology is more easy.

Spectrum

Here we would like to explain that we can always associate to any $(\infty,1)$-category $\mathbf{C}$ a corresponding stable $(\infty,1)$-category $\mathrm{Stab}(\mathbf{C})$. More precisely, we will see that in any $(\infty,1)$-category we have the notion of *sequencial spectrum object* which fits into a stable ∞-category. This will be very useful. Indeed, recall that the spectra in $\mathbf{Top}_*$ are the representing objects of generalized cohomology theories. Therefore, abstracting the notion of "spectrum" we are immediately abstracting the notion "generalized cohomology theories".

So, let $\mathbf{C}$ be a ∞-category. We define a *sequencial prespectrum object* in $\mathbf{C}$ as an increasing sequence $\mathbb{X} = (X_n)$ of objects together with structural 1-morphisms $\Sigma X_n \to X_{n+1}$. We say that the sequencial prespectrum is a *suspension spectrum* (or a Σ-*spectrum*) if the structural 1-morphisms are ∞-isomorphisms. Similarly, we say that they are *loop spectrum* (or Ω-*spectrum*) if the adjoint structural maps $X_{n+1} \to \Omega X_n$ are ∞-isomorphism. A 1-morphism $f : \mathbb{X} \to \mathbb{Y}$ between two sequencial prespectra $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbf{C}$ is a sequence of 1-morphisms $f_n : X_n \to Y_n$ of

$\mathbf{C}$ such that each square below is commutative up to 2-morphisms of $\mathbf{C}$.

$$\begin{tikzcd} \Sigma X_n \arrow[r] \arrow[d, "\Sigma(f_n)"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r] & Y_{n+1} \end{tikzcd}$$

Similarly, we define a 2-morphism $h : f \Rightarrow g$ between two 1-morphisms $f, g : \mathbb{X} \to \mathbb{Y}$ as a sequence of 2-morphisms h_n in $\mathbf{C}$ such that each analogous higher square is commutative up to 3-morphisms of $\mathbf{C}$, and so on. Inductively we get a $(\infty, 1)$-category $\mathrm{Spec}(\mathbf{C})$ of prespectra, of which we have two sub-∞-categories $\Omega\mathrm{Spec}(\mathbf{C})$ and $\Sigma\mathrm{Spec}(\mathbf{C})$ of loop spectrum objects and suspension spectrum objects.

Alternatively (and more formally), following [126, 125] we can define the ∞-category of prespectra in $\mathbf{C}$ as the ∞-category of ∞-functors[1] $\mathbb{X} : \mathbb{Z}\times\mathbb{Z} \to \mathbf{C}$ such that $\mathbb{X}(n, m) \simeq *$ if $n \neq m$ (we write $\mathbb{X}(n)$ instead of $\mathbb{X}(n, n)$). Indeed, notice that this condition means that each of these functors are equivalent to a sequence of weakly commutative squares (presented below) and by universality of ∞-pullbacks/∞-pushouts such diagrams induce adjoint 1-morphisms $\mu_n : \Sigma\mathbb{X}(n) \to X(n + 1)$ and $\nu_{n+1} : \mathbb{X}(n) \to \Omega\mathbb{X}(n + 1)$, as in the second diagram below. So, in this perspective the ∞-categories of Ω-spectra and Σ-spectra in $\mathbf{C}$ are just the subcategories of $\infty\mathrm{Func}(\mathbb{Z} \times \mathbb{Z}; \mathbf{C})$ such that the corresponding maps ν_{n+1} and μ_n are, respectively, ∞-isomorphisms.

$$\begin{tikzcd} \mathbb{X}(n,n) \arrow[r] \arrow[d] & \mathbb{X}(n,n+1) \simeq * \arrow[d] \\ * \simeq \mathbb{X}(n+1,n) \arrow[r] & \mathbb{X}(n+1,n+1) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X}(n) \arrow[dr, dashed, "\nu_{n+1}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \Omega\mathbb{X}(n+1) \arrow[r] \arrow[d] & * \arrow[d] \\ & * \arrow[r] & \mathbb{X}(n+1) \end{tikzcd}$$

Recall that a category of ∞-functors between ∞-categories $\mathbf{C}'$ and $\mathbf{C}$ has the same ∞-limits and ∞-colimits as $\mathbf{C}$. Therefore, in this new perspective to the notion of prespectrum object, we conclude that $\mathrm{Spec}(\mathbf{C})$ has at least the ∞-limits/∞-colimits existing in $\mathbf{C}$. In particular, for any prespectrum $\mathbb{X}$, we can build its reduced suspension and its loop space, which will be denoted by $\overline{\Sigma}\mathbb{X}$ and $\overline{\Omega}\mathbb{X}$. Despite the endofunctors $\overline{\Sigma}$ and $\overline{\Omega}$, there are also ∞-functors Σ^∞ and Ω^∞, respectively defined by $\Sigma^\infty(X)(n) = \Sigma^n X$ and $\Omega^\infty(\mathbb{X}) = \mathbb{X}(0)$, getting the weakly commutative diagram below.

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "\Sigma^\infty"] \arrow[d, shift right, "\Sigma"'] & \mathrm{Spec}(\mathbf{C}) \arrow[l, shift left, "\Omega^\infty"] \arrow[d, shift left, "\overline{\Sigma}"] \\ \mathbf{C} \arrow[r, shift right, "\Sigma^\infty"'] \arrow[u, shift right, "\Omega"'] & \mathrm{Spec}(\mathbf{C}) \arrow[l, shift right, "\Omega^\infty"'] \arrow[u, shift left, "\overline{\Omega}"] \end{tikzcd}$$

[1]Here we are considering $\mathbb{Z}$ as the ∞-category trivially produced by the 1-category given by the poset $(\mathbb{Z}, \leq)$, where $\leq$ is the usual order in the set of integer numbers. More precisely, $\mathbb{Z}$ has integer numbers as objects, there is a 1-morphism $n \to m$ iff $n \leq m$ and all higher morphisms are trivial.

The weak commutativity follows from the fact that the functor $\overline{\Sigma}$ stabilizes when restricted to the full sub-∞-category of suspension spectrum objects, while $\overline{\Omega}$ stabilizes at loop spectrum objects. On the other hand, it is not clear if both ∞-functors stabilizes over **all** prespectra. In other words, *it is not expected that* $\mathrm{Spec}(\mathbf{C})$ *becomes a stable* ∞*-category*.

In order to get a complete stable structure the immediate idea is restrict the attention on the suspension spectra **or** into loop spectra after showing that one of them can always be generated by the other. In fact, it can be show that *every prespectrum induces a canonical loop spectrum, so that the correct stable category associated to* $\mathbf{C}$ *will be* $\Omega\mathrm{Spec}(\mathbf{C})$. More precisely, the category of Ω-spectrum objects can be understood as a localization of the category of prespectrum objects, meaning that the inclusion we have an adjunction $\imath : \Omega\mathrm{Spec}(\mathbf{C}) \rightleftharpoons \mathrm{Spec}(\mathbf{C}) : \mathscr{L}$, where $\mathscr{L}$ preserve finite ∞-limits.

This result could be obtained abstractly (i.e, as a purely existence theorem, without specifying who $\mathscr{L}$ is). On the other hand, under the mild assumptions over $\mathbf{C}$, say when it is a ∞-topos, we can do a explicitly construction[2]. Indeed, the functor $\mathscr{L}$ is such that for any prespectrum $\mathbb{X}$ it assign the Ω-spectrum $\mathscr{L}(\mathbb{X})(n) = \mathrm{colim}_k \Omega^k \mathbb{X}(n+k)$ whose structural morphisms $\Omega\mathscr{L}(\mathbb{X})(n+1) \simeq \mathscr{L}(\mathbb{X})(n)$ is induced by

$$
\begin{aligned}
\Omega\mathscr{L}(\mathbb{X})(n+1) &= \Omega \lim_k \Omega^k \mathbb{X}((n+1)+k) \\
&\simeq \lim_k \Omega^{k+1} \mathbb{X}(n+(k+1)) \\
&\simeq \mathscr{L}(\mathbb{X})(n).
\end{aligned}
$$

We usually say that $\mathscr{L}(\mathbb{X})$ is the *spectrification* of $\mathbb{X}$ (see Section 8 of [126] for details on the proof that $\mathscr{L}$ really is a left-adjoint to $\imath$). With this adjunction on hand, the above diagram can be completed in order to get the diagram below.

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, shift left, "\Sigma^\infty"] \arrow[d, shift right, "\Sigma"'] & \mathrm{Spec}(\mathbf{C}) \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "L"] \arrow[d, shift left, "\overline{\Sigma}"] & \Omega\mathrm{Spec}(\mathbf{C}) \arrow[l, hook', shift left, "\imath"] \arrow[d, shift left, "\overline{\Sigma}"] \\
\mathbf{C} \arrow[u, shift right, "\Omega"'] \arrow[r, shift right, "\Sigma^\infty"'] & \mathrm{Spec}(\mathbf{C}) \arrow[l, shift right, "\Omega^\infty"'] \arrow[u, shift left, "\overline{\Omega}"] \arrow[r, shift right, "L"'] & \Omega\mathrm{Spec}(\mathbf{C}) \arrow[l, hook, shift right, "\imath"'] \arrow[u, shift left, "\overline{\Omega}"]
\end{tikzcd}
$$

Now, let us prove that $\Omega\mathrm{Spec}(\mathbf{C})$ really is a stable ∞-category. It is clear that any Ω-spectrum object X is invariant by $\overline{\Omega}$. So, we need prove that it is also invariant by $\overline{\Sigma}$. Indeed, for any Y we have

$$
\mathrm{Mor}_{\Omega\mathrm{Spec}(\mathbf{C})}(Y, \overline{\Sigma}X) \simeq \mathrm{Mor}_{\Omega\mathrm{Spec}(\mathbf{C})}(\overline{\Omega}Y, X) \simeq \mathrm{Mor}_{\Omega\mathrm{Spec}(\mathbf{C})}(Y, X),
$$

implying $\overline{\Sigma}X \simeq X$ by the ∞-Yoneda lemma (more precisely, the result follows from the uniqueness of the representing object of a representable ∞-functor). Summarizing, we have been obtained the following conclusion.

Conclusion. *To any* ∞*-topos* $\mathbf{C}$ *with finite* ∞*-limits/*∞*-colimits and a null object we can associate a stable* ∞*-category: the category of* Ω*-spectrum objects of* $\mathbf{C}$*. Furthermore, for any* $\mathbf{C}$ *we*

[2] When $\mathbf{C}$ is a ∞-topos as above, it is also usual to write $T\mathbf{C}$ instead of $\Omega\mathrm{Spec}(\mathbf{C})$. In this case, $T\mathbf{C}$ is also a ∞-topos, called the *tangent* ∞*-topos* of $\mathbf{C}$. See Section 4.1 of [182].

have an ∞-adjunction between it and its stabilization, given by the functors $\imath \circ \Omega^{\infty}$ and $\Sigma^{\infty} \circ \mathscr{L}$, which will be denoted by $\overline{\Omega}^{\infty}$ and $\overline{\Sigma}^{\infty}$.

Remark. Due to the conclusion above we will use $\mathrm{Stab}(\mathbf{C})$ to denote the stable ∞-category associated to $\mathbf{C}$.

9.2 Functoriality

A natural question is on the functoriality of the previous construction. Indeed, we can ask if the the rule assigning to any $(\infty, 1)$-category the corresponding stable ∞-category of Ω-spectrum objects admits an extension to a ∞-functor

$$\mathrm{Stab} : \infty\mathbf{Cat} \to \infty\mathbf{Stab},$$

where $\infty\mathbf{Stab} \subseteq \infty\mathbf{Cat}$ is the full sub∞-category of stable $(\infty, 1)$-categories. Here we would like to explain that in the general case the answer is **negative**. Indeed, in order to define the above functor we need to build some map assigning to any ∞-functor $F : \mathbf{C} \to \mathbf{D}$ a corresponding ∞-functor

$$\mathrm{Stab}(F) : \Omega\mathrm{Stab}(\mathbf{C}) \to \mathrm{Stab}(\mathbf{D})$$

so that compositions and identities are preserved up to natural ∞-isomorphisms. The canonical idea is to define $\mathrm{Stab}(F)$ by making use of the adjoint ∞-functors $\overline{\Omega}^{\infty}$ and $\overline{\Sigma}^{\infty}$, as in the first diagram below.

$$\begin{tikzcd}
\mathbf{C} \arrow[d, "F"'] & \mathrm{Stab}(\mathbf{C}) \arrow[l, "\overline{\Omega}^{\infty}"'] \arrow[d, dashed, "\mathrm{Stab}(F)"] \\
\mathbf{D} \arrow[r, "\overline{\Sigma}^{\infty}"'] & \mathrm{Stab}(\mathbf{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C} \arrow[d, "F"'] & \mathrm{Stab}(\mathbf{C}) \arrow[l, "\overline{\Omega}^{\infty}"'] \arrow[d, dashed, "\mathrm{Stab}(F)"] \\
\mathbf{D} & \mathrm{Stab}(\mathbf{D}) \arrow[l, "\overline{\Omega}^{\infty}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C} \arrow[d, "F"'] \arrow[r, "\overline{\Sigma}^{\infty}"] & \mathrm{Stab}(\mathbf{C}) \arrow[d, dashed, "\mathrm{Stab}(F)"] \\
\mathbf{D} \arrow[r, "\overline{\Sigma}^{\infty}"'] & \mathrm{Stab}(\mathbf{D})
\end{tikzcd}$$

Identities are certainly preserved by this rule. The problem is that arbitrary compositions **are not** preserved, because there is no canonical way to split $\mathrm{Stab}(G \circ F)$ into two pieces. On the other hand, there two special situation in which this problem does not appear:

1. restricting to the subcategory $\infty\mathbf{Cat}_{\mathrm{lim}}$ of ∞-functors which commute with finite ∞-limits;
2. restricting to the subcategory $\infty\mathbf{Cat}_{\mathrm{colim}}$ ∞-functors which commute with finite ∞-colimits.

Indeed, in the first case the functors will commute with $\overline{\Sigma}^{\infty}$, while in the second it will commute with $\overline{\Omega}^{\infty}$, so that in each case we can define $\mathrm{Stab}(F)$ respectively as the dotted map in the second and third diagrams above. In the general case (i.e without restricting to some very particular subcategory), however, the problem remains and the stabilization is not functorial.

Notice that the problem with functoriality involves the existence of a morphism $\mathrm{Stab}(F)$ which are required to satisfy strict commutativity condition. Thanks to the ∞-weakening principle, instead of attacking the problem directly, we can consider approximate solutions and, in particular, we can look for the "best approximate solution". So, for any ∞-functor $F : \mathbf{C} \to \mathbf{D}$

we are looking for a pair $(\mathrm{Stab}(F), \varphi)$, where $\mathrm{Stab}(F)$ is a ∞-functor between the corresponding categories of Ω-spectra and

$$\varphi : F \circ \overline{\Omega}^{\infty} \Rightarrow \overline{\Omega}^{\infty} \circ \mathrm{Stab}(F)$$

is a ∞-natural transformation, which is universal in the sense that, for any other pair $(\mathrm{Stab}'(F), \varphi')$ there is a unique $u : \mathrm{Stab}(F) \Rightarrow \mathrm{Stab}'(F)$ such that the diagram below commutes up to higher morphisms.

$$\begin{tikzcd} & \overline{\Omega} \circ \mathrm{Stab}'(F) \\ F \circ \overline{\Omega} \arrow[ur, Rightarrow, "\varphi'"] \arrow[r, Rightarrow, "\varphi"'] & \overline{\Omega} \circ \mathrm{Stab}(F) \arrow[u, Rightarrow, "\overline{\Omega}\circ u"'] \end{tikzcd}$$

Now we can search for conditions under which this best approximation exists. As can be seen in Section 6.2.1 of [125], this is the case of ∞-functors between ∞-topos[3] preserving **few** colimits (say inductive ∞-colimits). In Section 6.2.2 is then proved that under these hypothesis, the rule $F \mapsto \mathrm{Stab}(F)$ really becomes functorial[4], generalizing the situations discussed previously, where to get functoriality we had to require preservation of **all** ∞-colimits.

Remark. By the adjunction between $\overline{\Sigma}^{\infty}$ and $\overline{\Omega}^{\infty}$ we could be defined such a best approximation in a dual (and equivalent) way by making use of the infinity suspension functor instead of the infinity loop space functor.

We could attack the problem of the existence of the best "spectral approximation" for F without any explicit mention of spectra! Indeed, following some kind of "inverse strategy" we can try to identify ∞-functors between categories of spectra as ∞-functors between usual categories satisfying additional conditions. So, under this identification, we expect to get a more concrete analysis of the functoriality of Stab.

Indeed, above we used $\overline{\Sigma}^{\infty}$ and $\overline{\Omega}^{\infty}$ in order to produce functors between spectra from functors between ∞-categories. Let us do the opposite: by making of the adjunction $\overline{\Sigma}^{\infty} \rightleftharpoons \overline{\Omega}^{\infty}$ we get functors between categories from functors between the corresponding spectra. More precisely, given $S : \mathrm{Stab}(\mathbf{C}) \to \mathrm{Stab}(\mathbf{D})$ we associate to it a ∞-functor

$$\mathrm{Dstab}(S) : \mathbf{C} \to \mathbf{D} \quad \text{defined as} \quad \mathrm{Dstab}(S) = \overline{\Omega}^{\infty} \circ S \circ \overline{\Sigma}^{\infty}.$$

Restricting to the ∞-functors on spectra that preserve ∞-colimits (meaning that we can commute them with the infinity suspension functor) this rule extends itself to a ∞-functor

$$\mathrm{Dstab} : \infty\mathrm{Func}_{\mathrm{colim}}(\mathrm{Stab}(\mathbf{C}); \mathrm{Stab}(\mathbf{D})) \to \infty\mathrm{Func}(\mathbf{C}; \mathbf{D}),$$

which is fully-faithful in the context of ∞-topos. Therefore, the functor DStab is an equivalence over its image, meaning that we can replace any ∞-functor on spectra by a ∞-functor between

[3] More precisely, this is valid for which [125] calls *differentiable ∞-categories.*

[4] The proof of this result is based on a relation between functors of spectra and excisive functors, as will be discussed below.

the underlying categories satisfying some additional conditions.

$$
\begin{tikzcd}
\infty\mathrm{Func}_{\mathrm{colim}}(\mathrm{Stab}(\mathbf{C});\mathrm{Stab}(\mathbf{D})) \arrow[r, "\mathrm{Dstab}"] \arrow[dr, dashed] & \infty\mathrm{Func}(\mathbf{C};\mathbf{D}) \\
 & \infty\mathrm{Exc}(\mathbf{C};\mathbf{D}) \arrow[u, hook]
\end{tikzcd}
$$

In more details, notice that for any S the corresponding ∞-functor $\mathrm{Dstab}(S)$ maps ∞-pushouts into ∞-pullbacks. This is due essentially because in a stable ∞-category being a ∞-pushout square is the as being a ∞-pullback square, and because S and $\overline{\Sigma}^{\infty}$ preserve ∞-colimits, while $\overline{\Omega}^{\infty}$ preserve ∞-limits. A ∞-functor satisfying this property (of mapping ∞-pushouts into ∞-pullbacks) is called *excisive*. Therefore, the functor Dstab factors as in the diagram above, where $\infty\mathrm{Exc}(\mathbf{C};\mathbf{D})$ is the category of excisive ∞-functors.

Reciprocally, it can be show that any excisive ∞-functor $F : \mathbf{C} \to \mathbf{D}$ preserving some additional colimits (say null objects and inductive ∞-colimits) is induced by some functor S between spectra, so that we have an equivalence

$$\infty\mathrm{Func}_{\mathrm{colim}}(\mathrm{Stab}(\mathbf{C});\mathrm{Stab}(\mathbf{D})) \simeq \infty\mathrm{Exc}_*(\mathbf{C};\mathbf{D}), \tag{9.2.1}$$

where the right hand-side is the full sub-∞-category of excisive functors which preserve the additional colimits. Therefore, given $F : \mathbf{C} \to \mathbf{D}$, in order to get its best "spectral approximation", instead of searching for a universal pair $(\mathrm{Stab}(F), \varphi)$, as presented previously, we can now look for an universal pair (P_1F, ξ), where P_1F is an excisive functor (preserving the additional colimits) and $\xi : F \Rightarrow P_1F$ is a natural transformation.

The advantage of this new approach is that there is a canonical way to assign to any F a corresponding pair (P_1F, ξ_F), so that the final work is to verify if this pair really is universal (i.e, if the corresponding approximation really is the best possible). Indeed, this follows from the fact that not only the image of Dstab is generated by excisive functors, but the whole category of ∞-functors. More precisely, the inclusion

$$\imath : \infty\mathrm{Exc}(\mathbf{C};\mathbf{D}) \hookrightarrow \infty\mathrm{Func}(\mathbf{C};\mathbf{D}) \quad \text{has an adjoint} \quad P_1 : \infty\mathrm{Func}(\mathbf{C};\mathbf{D}) \to \infty\mathrm{Exc}(\mathbf{C};\mathbf{D}),$$

so that for any F we define P_1F as its image by P_1 and the transformation ξ_F as the counit of the adjunction at F.

We commented previously that in Section 6.2.2 of [125] it is proved that, if $\mathbf{D}$ is well behaved (say a ∞-topos), then, when restricted to a very large class of functors, the rule $F \mapsto \mathrm{Stab}(F)$ becomes functorial. Before the discussion above we can finally say that in Section 6.2.2 Lurie actually prove that, under these hypothesis, *the canonical pair* (P_1F, ξ_F) *really is universal*.

Remark. The relation between functors preserving colimits and excisive functors preserving the terminal object is more stronger than that given by the equivalence 9.2.1. Indeed, it can be show that if $\mathbf{D}$ is stable (in particular, if it is the category of Ω-spectrum objects of another ∞-category) then for any $\mathbf{C}$ (not necessarily stable) we have an equivalence

$$\infty\mathrm{Func}_{\mathrm{colim}}(\mathbf{C};\mathbf{D}) \simeq \infty\mathrm{Exc}_*(\mathbf{C};\mathbf{D}). \tag{9.2.2}$$

Following Sections 9 and 10 of [126], let us see that this fact has an interesting consequence, which allows us to get information on the stable homotopy theory of an arbitrary ∞-category by studying the stable homotopy theory of topological spaces. In fact, let $\mathbf{Top}_*^{\text{fin}} \subset \mathbf{Top}_*$ be the smallest full sub-∞-category which is closed under finite ∞-colimits. In other words, this is the category such that for any other ∞-category $\mathbf{D}$ with null object we have an equivalence

$$\infty\mathrm{Func}_{\mathrm{colim}}(\mathbf{Top}_*^{\text{fin}};\mathbf{D}) \simeq \mathbf{D}$$

given by evaluation at the point $*$. Therefore, thanks to (9.2.2) we conclude that

$$\begin{aligned}\mathrm{Stab}(\mathbf{D}) &\simeq \infty\mathrm{Func}_{\mathrm{colim}}(\mathbf{Top}_*^{\text{fin}};\mathrm{Stable}(\mathbf{D})) \\ &\simeq \infty\mathrm{Exc}_*(\mathbf{Top}_*^{\text{fin}};\mathrm{Stable}(\mathbf{D})) \\ &\simeq \infty\mathrm{Exc}_*(\mathbf{Top}_*^{\text{fin}};\mathbf{D}),\end{aligned} \tag{9.2.3}$$

where the last equivalence is obtained by composition with the functor $\overline{\Omega}^{\infty}$. So, *for any* $\mathbf{D}$ *its stabilization is equivalent to some category of functors from topological spaces to* $\mathbf{D}$.

Goodwillie Calculus

In the last subsection we studied the relation between the functoriality of the stabilization Stab and the properties of the approximation of a ∞-functor F by its canonical excisive ∞-functor P_1F. More precisely, we have seen that if this excisive approximation is the best possible for any F, then Stab is functorial. We discussed some conditions under which this actually happens. But, a question remains: what can we do if such conditions are **not** satisfied?

Notice that in this case the excisive approximation P_1F is not the best possible, so that the main idea is to search for some refinement of P_1F. Here we will give a very brief outline of the fact any ∞-functor F admits not only the excisive approximation P_1F, but indeed a sequence P_nF of more refined excisive approximations which become equipped with canonical natural transformations $\xi_F^n : P_nF \to P_{n-1}F$. So, if there is some n such that the corresponding n-excisive approaximation is universal, then each P_mF collapses into P_nF (because P_nF is now the best approximation) and the stabilization process becomes functorial.

The idea is the following: if P_1F is about excisive functors, then P_nF should be about *n-excisive functors*. But the excisive ∞-functors are, by definition, those mapping ∞-pushout squares into ∞-pullback squares. Therefore, n-excisive functors should map *∞-pushout n-squares* into *∞-pullback n-squares*. Finally, because we obtained $\xi_F : P_1F \Rightarrow F$ as the counit of a canonical adjunction, it is natural to build ξ_F^n following an analogous process. Together, all these insights will give the required sequence of excisive approximations, as can be seen in [82, 129, 125]. For instance, let us see how to formalize the notion of *n-excisive functors*.

We start by recalling that, as commented in Example ??, to any 1-category $\mathbf{C}$ we can assign a **strict** ∞-category $\infty\mathrm{Sq}(\mathbf{C})$ whose k-morphisms are **commutative** k-squares of morphism of $\mathbf{C}$. Now, if $\mathbf{C}$ is actually a ∞-category, then we can make use of the ∞-weakening principle and define a **weak** version of the ∞-category $\infty\mathrm{Sq}(\mathbf{C})$. This is done by working with k-squares of 1-morphisms of $\mathbf{C}$ which are commutative up to 2-morphisms of $\mathbf{C}$, which in turn satisfy some condition up to 3-morphisms, and so on.

Notice that a ∞-pullback square is something of a terminal object of $\infty\mathrm{Sq}(\mathbf{C})$, while the ∞-pushout squares behave as initial objects. For every $n \in \mathbb{N}$ we define another ∞-category $\infty\mathrm{Sq}^n(\mathbf{C})$, obtained from $\infty\mathrm{Sq}(\mathbf{C})$ by forgetting the $(n+1)$-category of objects, 1-morphisms, and so on up to $(n-1)$-morphisms. In other words, $\infty\mathrm{Sq}^n(\mathbf{C})$ is the ∞-category whose objects are weak commutative n-squares in $\mathbf{C}$, whose 1-morphisms are $(n+1)$-squares, and so on. A *∞-pullback/∞-pushout n-square* is a terminal object/initial object in $\infty\mathrm{Sq}^n(\mathbf{C})$.

We then define the category $\mathrm{Exc}^n(\mathbf{C};\mathbf{D})$ of *n-excisive ∞-functors* as the full sub ∞-category of those ∞-functors $F:\mathbf{C}\to\mathbf{D}$ which map ∞-pushout n-squares into ∞-pullback n-squares. The fundamental fact is that the inclusion of this category into the ∞-category of all ∞-functors has an adjoint

$$P_n:\infty\mathrm{Func}(\mathbf{C};\mathbf{D})\to\mathrm{Exc}^n(\mathbf{C};\mathbf{D}),$$

so that the counit of this adjunction gives a canonical transformation $\varphi^n_F:P_nF\to F$. It happens that, by construction, every n-excisive functor is also m-excisive for any $m\geq n$ (essentially because any m-square can be build by gluing n-squares), so that the transformation φ^n_F factors by a transformation $\xi^n_F:P_nF\to P_{n-1}F$ which is that giving the n-excisive approximation of F.

Remark. Recall that homotopy theory and differential calculus have many similarities, as presented in Section 1.2. The above disscussion on n-excisive functors can be understood as an extension of this analogy to the context of abstract stable homotopy theory. Indeed, the first excisive approximation P_1F is usually called the *categorical derivative* of F, so that P_nF is some kind of *nth derivative* and the sequence

$$\cdots\longrightarrow P_{n+1}\longrightarrow P_n\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow F$$

is the "higher categorical Taylor series of F". Because of this analogy the study of the n-excisive approximations is usually known as the *Goodwillie calculus of functors*. See [??] for the original references in the context of classical stable homotopy theory, [??] for the higher categorical approach discussed above and [??] for more one the analogy with differential calculus.

Smash

Up to this point we worked with ∞-monoidal ∞-categories **or** with stable ∞-categories. Now it is time of mix these two classes of objects: we will pass to discuss *stable ∞-monoidal ∞-categories*. These are just ∞-categories which are simultaneously stable and ∞-monoidal (without any compatibility condition between both structures). Our main objective here is to give an outline of the following result, which appear in a slightly different version as Theorem ?? in [??]:

- If $(\mathbf{C},\otimes,1)$ is ∞-monoidal ∞-category whose product $\otimes$ preserve ∞-colimits, then, up to ∞-equivalences, its stabilization $\mathrm{Stab}(\mathbf{C})$ has a unique (symmetric) ∞-monoidal structure whose neutral object is the free spectrum $\overline{\Sigma}^{\infty}1$. Therefore, it becomes a stable ∞-monoidal ∞-category in a canonical way.

Before sketching this result, let us digress on its mathematical meaning. We start by recalling that (as discussed in Section 4.2) the cartesian monoidal structure on any convenient category $\mathscr{C}\subset\mathbf{Top}$ of topological space induces a symmetric monoidal structure in category $\mathscr{C}_*$ given by the *smash product* $\wedge$. In Section 5.3, on the other hand, we showed that $\wedge$ induces a natural symmetric monoidal structure onto the **homotopy category** of spectra $\mathrm{Ho}(\mathbf{Spec})$, given by the smash product of spectra $\wedge_{\mathbb{S}}$, whose neutral element is the homotopy class of the sphere spectrum $\mathbb{S}$: the suspension spectrum generated by the point $*$.

Then, many reasons led us to ask if this monoidal structure given by the "smash product of spectra" can be lifted to the actual category of spectra **Spec**. The answer was given by the Lewis obstruction theorem: this lifting **cannot** be done. But, notice that in the previous chapters we have been regarded $\mathscr{C}_*$ and **Spec** as 1-categories. The result above imply that, if now we regard **Spec** as the $(\infty, 1)$-category given by the stabilization of $\mathscr{C}_*$, then the smash product actually can lifted to spectra in a unique way! As a motivation, recall that Lewis's obstruction theorem is based on the fact that the smash product fulfill nice properties only up to homotopy, so that it is natural to expect the obstructions disappear in the higher categorical context, where all homotopies are taken into account.

Summarizing, the previous result together with Lewis's obstruction theorem give the following assertion:

Theorem 9.1. *For any monoidal ∞-category $(\mathbf{C}, \otimes, 1)$ whose product preserve ∞-colimits, the corresponding category of spectra has a canonical smash product. Furthermore, in the general case this induced ∞-monoidal structure does not gives a 1-monoidal structure.*

Sketch of the proof. The second affirmative is a direct consequence of Lewis's theorem. The idea of the first affirmative is pretty simple. Indeed, recall that, as discussed in the last subsection, if $F : \mathbf{C} \to \mathbf{D}$ preserve ∞-colimits, then the stabilization $\mathrm{Stab}(F)$ is well defined. The same is valid in the context of ∞-functors with many variables (see Section 6.1.3 of [125]). So, under the hypothesis the derivative of the product $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ exists. By definition, it is a ∞-functor

$$\wedge_\otimes : \mathrm{Stab}(\mathbf{C}) \times \mathrm{Stab}(\mathbf{C}) \to \mathrm{Stab}(\mathbf{C})$$

satisfying some universal property, which ensures its uniqueness. In order to see that it really defines a monoidal structure, notice that, as discussed in Section 4.2, if $(\mathbf{C}, \otimes, *)$ is any monoidal category, the preservation of colimits by $\otimes$ imply the associativity of the smash product in $\mathbf{C}_*$. Here, in an analogous way, the preservation of ∞-colimits by $\otimes$ imply that the "smash product on spectra" $\wedge_\otimes$ is associative up to higher morphisms. Finally, let us prove that the free spectrum $\mathbb{S}_1 = \overline{\Sigma}^\infty 1$ satisfy the neutral element property up to equivalences. So, given any spectrum object we need to verify that $\mathbb{X} \wedge_\otimes \mathbb{S}_1 \simeq \mathbb{X}$. The case in which $\mathbb{X} = \overline{\Sigma}^\infty X$ for some X is immediate: by the relation between $\wedge_\otimes$ and $\otimes$, we have

$$\mathbb{X} \wedge_\otimes \mathbb{S}_1 \simeq (\overline{\Sigma}^\infty X) \wedge_\otimes (\overline{\Sigma}^\infty 1) \simeq \overline{\Sigma}^\infty (X \otimes 1) \simeq \overline{\Sigma}^\infty X = \mathbb{X}.$$

In Sections 1.4.3 and 6.2.4 of [125], the general case makes explicit use of the relation between the category of spectrum objects in $\mathbf{C}$ and the category of excisive functors $\mathscr{C}_* \to \mathbf{C}$, obtained in (9.2.3). Here we will give an alternative approach. Indeed, notice that the neutral object $1 \in \mathbf{C}$ is the same as a ∞-functor $e : \mathbf{1} \to \mathbf{C}$ whose image is 1, where $\mathbf{1}$ is the ∞-category with only one object and whose k-morphisms are identities. This functor has derivative because it preserve ∞-colimits. Therefore, there is $\mathrm{Stab}(e) : \mathrm{Stab}(\mathbf{1}) \to \mathrm{Stab}(\mathbf{C})$ such that $\overline{\Sigma}^\infty_{\mathbf{C}} \circ e \simeq \mathrm{Stab}(e) \circ \overline{\Sigma}^\infty_{\mathbf{1}}$, so that $\mathrm{Stab}(e)$ is the same as the spectrum object $\overline{\Sigma}^\infty_{\mathbf{C}} 1$. The neutral element property follows from the functoriality of the stabilization. □

We end with some technical remarks:

Remark. In the definition of stable monoidal ∞-category we **do not** require any compatibility between the monoidal and the stable structures. For instance, it would be natural to require that both Σ and Ω becomes ∞-**monoidal** ∞-functors, but this condition does not appear in the previous definition. We notice that such a compatibility is indeed a **consequence** of the stable properties of Σ and Ω. In fact, by the stability we have $\Sigma 1 \simeq 1$ and $\Omega 1 \simeq 1$, meaning that the units are preserved. Furthermore,

$$\Sigma(X \otimes Y) \simeq X \otimes Y \simeq (\Sigma X) \otimes (\Sigma Y)$$

and similarly for the loop space ∞-functor, so that both Σ and Ω are monoidal ∞-functors. Analogous argument allows us to conclude that if the underlying monoidal structure is braided/symmetric then these ∞-functors are also braided/symmetric.

Remark. In the topological context $\mathbf{C} = \mathscr{C}_*$, the ∞-monoid objects into the symmetric ∞-monoidal structure on spectra **Spec** given by the smash product $\wedge_{\mathbb{S}}$ are usually called A_∞*-ring spaces*, while the commutative ∞-monoid objects are called E_∞*-ring spaces*. In the abstract context (i.e for $(\mathbf{C}, \otimes, 1)$ an arbitrary monoidal ∞-category in the hypothesis of the last theorem), we adopt a similar nomenclature: the ∞-monoid objects and the commutative ∞-monoid objects into $(\mathrm{Stab}(\mathbf{C}), \wedge_\otimes, \mathbb{S}_1)$ are respectively called A_∞*-ring objects* and E_∞*-ring objects*. The corresponding categories will be denoted by

$$A_\infty\mathrm{Ring}(\mathbf{C}, \otimes) \quad \text{and} \quad E_\infty\mathrm{Ring}(\mathbf{C}, \otimes)$$

instead of by the more complicated notations (independently, the notation for the unit $\mathbb{S}_1$ is maintained, being called the *sphere spectrum*)

$$\infty\mathrm{Mon}(\mathrm{Stab}(\mathbf{C}), \wedge_\otimes, \mathbb{S}_1) \quad \text{and} \quad \infty\mathrm{Mon}_c(\mathrm{Stab}(\mathbf{C}), \wedge_\otimes, \mathbb{S}_1).$$

Remark. Recall that for any ∞-monoid object R in a ∞-monoidal ∞-category $(\mathbf{C}, \otimes, 1)$ we have the corresponding ∞-category $\infty\mathrm{Mod}_R(\mathbf{C}, \otimes)$ of ∞-module objects over R. If the structure $(\mathbf{C}, \otimes, 1)$ is symmetric, then it induces monoidal structure $\otimes_R$ in $\infty\mathrm{Mod}_R(\mathbf{C}, \otimes)$. So, in particular, for a given A_∞-ring object $\mathbb{A}$ in a symmetric ∞-monoidal ∞-category in the hypothesis of Theorem 9.1, we have a corresponding ∞-monoidal structure in the ∞-category of ∞-module spectrum over $\mathbb{A}$. We usually say that $(\wedge_\otimes)_\mathbb{A}$ is the *derived smash product* with respect to $\mathbb{A}$ and we write $\wedge_\mathbb{A}$ when the ∞-monoidal structure $(\otimes, 1)$ of $\mathbf{C}$ can be omitted without loss of understanding.

9.3 Cohomology

In Section 1.2 we discussed many examples of invariants arising from functors. Most of them were flavors of cohomology theories. Indeed, there we discussed nonabelian cohomology, generalized cohomology, twisted cohomology algebraic cohomology, abelian sheaf cohomology, algebraic K-theory, etc. We note that for some of these cohomology theories, the functor defining them is simply a hom-functor on the (homotopy category of) some $(\infty, 1)$-category. For instance,

1. *nonabelian cohomology* with coefficients in a continuous group G is given by $[-, BG]$, defined on the homotopy category of **Top**;

2. *generalized cohomology* (as ordinary cohomology, complex K-theory and oriented cobordism) with coefficients into a Ω-spectrum $\mathbb{E} = (E_n)$ is given by $[-, E_n]$, defined on the homotopy category of $\mathbf{Top}_*$.

Other cohomology theories, on the other hand, despite have been defined on the homotopy category of some $(\infty, 1)$-category, they are given by functors which seems much less natural than hom-functors. Indeed,

1. *twisted generalized cohomology* is given by functors Γ^n, defined on the homotopy category of $\mathbf{Top}_*$, which assigns to any topological space X the space of sections of some bundle over X whose fiber belongs to some Ω-spectra $\mathbb{E}$;

2. *abelian sheaf cohomology* of a space X was defined as the (derived functor of the) global sections functor $\Gamma(X, -)$, assigning to any sheaf F on X the object $F(X)$;

3. *algebraic cohomology* was defined on the homotopy category of (unbounded) cochain complexes $\mathbf{CCh}_R$ as the functor H^k that assigns to any cochain complex (X^*, d) the corresponding quotient $\ker(d^{k+1})/\mathrm{img}(d^k)$.

In addition we have the fact that the name "cohomology groups" is unnatural, because the different cohomology theories take values in different categories, not necessarily in the category of groups. For instance, if G is not abelian, then the nonabelian cohomology with coefficients in G is not a group, but only a set. On the other hand, as discussed in Section 5.1, independently of the Ω-spectrum $\mathbb{E}$, the generalized cohomology functors of $\mathbb{E}$ take values in $\mathbf{Ab}$; if $\mathbb{E}$ is a ring spectrum, then the theory take values into the category of graded rings, as discussed in Section 5.3. Furthermore, for a given ring R, the algebraic cohomology always take values into the category of R-modules.

Summarizing, up to this point there is a "zoo" of many different cohomology theories, defined by different classes of functors and taking values into different algebraic categories. Here we would like to explain that this "zoo" has indeed only one "cage". In other words, as we will see, *there is a very simple* (*and, therefore, very abstract*) *notion of cohomology of which all examples above are only particular cases.* By making use of this new abstract definition of cohomology new interesting examples will appear, giving new powerful invariants.

Warning. Let $(\infty, 1)\mathbf{Cat}$ be the category of all $(\infty, 1)$-category. As discussed in Section 8.3, for any $k \geq 1$ we have the k-truncation functor $\tau_{\leq k}$, which assigns to any $(\infty, 1)$-category $\mathbf{C}$ the k-category $\tau_{\leq k}\mathbf{C}$. For $k = 1$, the 1-category $\tau_{\leq 1}\mathbf{C}$ is obtained by replacing the ∞-groupoids of morphisms between two objects X and Y by the set of equivalence classes of 1-morphisms $f, g : X \to Y$ by relation $f \simeq g$ iff there exist some 2-morphism $h : f \Rightarrow g$. For the purposes of the next subsections, we will say that $\tau_{\leq 1}\mathbf{C}$ is the *homotopy category* of $\mathbf{C}$, writing $\mathrm{Ho}(\mathbf{C})$ to denote it. We will also write π_0 instead of $\tau_{\leq 1}$ in order to denote the projection functor.

Unification

Let $\mathbf{H}$ be a $(\infty, 1)$-category which we will assume ∞-complete and ∞-cocomplete. Following something as Section 3.6.9 of [182], for any two given objects $X, A \in \mathbf{H}$ we define the *abstract cohomology group* (or *the abstract* 0*th cohomology group*) *of* X *with coefficients on* A as the set of equivalence classes

$$H^0(X; A) := \pi_0(\infty\mathbf{Gpd}(X, A)).$$

This immediately gives a representable functor

$$H^0(-; A) : \mathrm{Ho}(\mathbf{H}) \to \mathbf{Set},$$

which we call the *abstract cohomology theory* (or *the abstract* 0*th cohomology theory*) *with coefficients on* A. So, in more explicit terms, $H^0(-;A)$ is just the representable functor of $\mathrm{Ho}(\mathbf{H})$ classified by A. Or, in other words, it is just the image of A under the Yoneda embedding.

Depending of the characteristics of the coefficient object A, the abstract cohomology theory defined by it will acquire new properties/structures. For instance, if A is a ∞-monoid, commutative ∞-monoid or a ∞-group object in the cartesian ∞-monoidal structure of $\mathbf{H}$, then for any X the corresponding 0-th cohomology groups $H^0(X,A)$ will have induced monoid, commutive monoid or group structures. Furthermore, if A has iterated deloopings $\mathbf{B}^k A$, say for $k=0,1,...,n$, then we can define not only the 0-th cohomology group with coefficient in A, but also *k-th cohomology groups* for $i=0,1,...,n$. Indeed, we put

$$H^k(X;A) := H^0(X;\mathbf{B}^k A) := \pi_0(\infty\mathbf{Gpd}(X,\mathbf{B}^k A)).$$

In particular, if $\mathbf{B}^k A$ exists for any k, then A is ∞-loop space object, meaning that there is a spectrum object $\mathbb{A}=(A_n)$ of $\mathrm{Stab}(\mathbf{H})$ such that $A=\overline{\Omega}^{\infty}\mathbb{A}$ and, therefore, that $\mathbf{B}^k A\simeq A_k$. In other words, we have

$$H^k(X;A)\simeq H^0(X;A_k)$$

and in this case each kth abstract cohomology group is automatically an abelian group. Additionally, if it happens that $\mathbb{A}$ be a A_∞-ring object or a E_∞-ring object, then (exactly as done in Section 5.3 for ring spectra and commutative ring spectra) we get products

$$H^k(X;A)\times H^l(X;A)\to H^{k+l}(X;A)$$

giving to the abstract cohomology groups a structure of graded ring. In this case we say that we have a *multiplicative abstract cohomology theory with coefficients in A* (or in the underlying spectrum object $\mathbb{A}$).

Now it is easy to understand how the very different flavors of cohomology theories discussed in Section 1.2 (and listed in the last subsection) are only particular cases of abstract cohomology, as defined above. Indeed,

1. *nonabelian cohomology*. As commented in Example 8.12, if G is a topological group, its classifying space BG is modeled by the geometric realization $|\mathbf{B}G|$ of the delooped ∞-groupoid. This means that under the homotopy hypothesis we can regard the classified space BG as $\mathbf{B}G$, so that usual nonabelian cohomology with coefficients in G is just 1th abstract cohomology $H^1(X;G)=[X;\mathbf{B}G]$ for $\mathbf{H}$ the ∞-topos $\infty\mathbf{Gpd}$.

2. *generalized cohomology*. Similarly, if $\mathbb{E}$ is a topological spectrum, then usual generalized cohomology defined by it is just abstract cohomology for $\mathbf{H}$ the ∞-topos $\mathbf{Top}_*\simeq\infty\mathbf{Gpd}_*$ with coefficients in the ∞-loop space $\Omega^\infty\mathbb{E}$ and its deloopings.

3. *twisted generalized cohomology*. In an arbitrary category $\mathbf{C}$, giving a section $s:Y\to X$ for a morphism $f:X\to Y$ is just the same as giving a morphism $s:id_Y\to f$ in the slice category $\mathbf{C}/Y$. This fact can be used in order to recover twisted generalized cohomology of X with coefficients into some spectrum $\mathbb{E}$ as abstract cohomology in the slice ∞-topos $\infty\mathbf{Gpd}_*/X$.

4. <u>algebraic cohomology</u>: We start by recalling that the categories $\mathbf{Ch}_R^+$ and $\mathbf{CCh}_R^+$ are stable, meaning that any object has all deloopings, given by the suspension functor, which in this case is just the shifting functor: the unique difference from chain complexes to cochain complexes is the direction that we are shifting. For any k, let $\mathbf{B}^k R$ be the **chain** complex whose only nontrivial term is the kth, being R. In other words, it is the kth delooping of R. Equivalently, it is the chain complex such that

$$H_i(\mathbf{B}^k R) \simeq \begin{cases} R, & i = k \\ 0, & i \neq k. \end{cases}$$

Notice that $\mathbf{B}^k R$ is some "algebraic analogue" of the Eilenberg-Mac Lane spaces with coefficients in R. Now, a direct computation shows that, for any **cochain** complex X_*, the chain maps $X^* \to \mathbf{B}^k R$ from the dual **chain** complex to the Eilenberg-Mac lane complex is in bijection with the elements of $\ker d_{k+1}$, while the chain homotopies are in bijection with the elements of $\operatorname{img} d_k$. Therefore,

$$\pi_0(\infty\mathbf{Gpd}_{\mathbf{Ch}_R}(X^*, \mathbf{B}^k R)) \simeq \ker d_{k+1}/\operatorname{img} d_k = H^k(X_*),$$

showing that algebraic cohomology is just abstract cohomology in the $(\infty, 1)$-category of chain complexes.

Abstraction

In the following we will see that the notions of *cocycles* and *characteristic classes* makes sense in any abstract cohomology theory. We will also see that we can talk of *objects classified by cocycles*, generalizing how nonabelian cohomology classifies principal bundles.

We start by recalling that in algebraic cohomology (say with coefficients in a ring R) a cocycle of a cochain complex X_* is an element $x \in X_k$ such that $dx = 0$. In other words, *a cocycle is precisely an element that descends to a class of the cohomology group* H_R^k. In nonabelian cohomology, a cocycle is given by a family of functions $g_{ij} : U_i \cap U_j \to G$ satisfying the conditions $g_{ij} \cdot g_{jk} = g_{ik}$ and $g_{ii} = e$. In other words, as discussed in Section 6.2, it is a functor $\check{C}(U_i) \to \mathbf{B}G$ which, indeed, represent a morphism $X \to \mathbf{B}G$ and, therefore, a function $X \to BG$. This, in turn, induces *an element of the cohomology group* $[X, BG]$. Therefore, in both cases, the cocycles are just elements of the cohomology groups. Motivated by this fact, given objects $X, A \in \mathbf{H}$ in any $(\infty, 1)$-category, we define a *cocycle of X with coefficients in A* as just an element of the abstract cohomology $H^0(X; A)$.

As discussed in Section 1.3, if $\mathbb{E}$ is a Ω-spectrum representing a generalized cohomology theory, then a universal characteristic class with coefficients in $\mathbb{E}$ is a natural transformation $\xi : \mathrm{Bun}_G \Rightarrow H_{\mathbb{E}}^n$, where Bun_G is the functor assigning to any space X the isomorphism class of G-principal bundles over X. By the classication theorem of G-bundles, $\mathrm{Bun}_G \simeq [-, BG]$ and we see that *a characteristic class is just* a *map from nonabelian cohomology to some generalized cohomology.* By the Yoneda lemma, these transformations are into 1-1 correspondence with the morphisms $BG \to E_n$ between the representing objects.

So, motivated by the last situation, for $A, B \in \mathbf{H}$ objects in a $(\infty, 1)$-category, we define a *characteristic class* between the corresponding abstract cohomology theories $H^0(-; A)$ and $H^0(-; B)$ as a morphism $\xi : A \to B$ (or, equivalently, by the ∞-Yoneda lemma, as a ∞-natural transformation between the cohomology functors). In other words, *a characteristic class is a morphism between abstract cohomology theories.*

Recall that the equivalence $[-, BG] \simeq \mathrm{Bun}_G$ is obtained by the pullback construction over some universal G-bundle. More precisely, it is given by associating to any function $f : X \to BG$ the first pullback presented below, where $EG \to BG$ is some G-bundle whose total space is weakly contractible (i.e all homotopy groups are trivial). The construction need to be homotopy invariant, so that the pulback is indeed a ∞-pullback. This means that EG can be replaced by the trivial space $*$ and, therefore, we see that the equivalence $[-, BG] \simeq \mathrm{Bun}_G$ is given by taking the homotopy fiber (i.e, the ∞-kernel) of f, as in the ∞-pullback below. In other words, *any cocycle $f : X \to BG$ in nonabelian cohomology classifies a G-bundle and this classification is obtained by taking the homotopy fiber of f.*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd} \qquad \begin{tikzcd} \infty\mathrm{pb} \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

Now, notice that the ∞-pullback above makes sense for any cocycle $f : X \to A$ in any abstract cohomology theory, motivation us to define the *object classified by f* as being its homotopy fiber. This allow us to define "higher bundles", as in the following example:

Example 9.1 (∞-*bundle*)**.** If G is a ∞-group object in a $(\infty, 1)$-category $\mathbf{H}$ (or, more generally, if it is a ∞-groupoid in a ∞-topos), then the delooping $\mathbf{B}G$ exists and the object classified by a cocycle $X \to \mathbf{B}G$ in nonabelian abstract cohomology is called a *G-principal ∞-bundle over X*. If G is a topological group trivially regarded as a ∞-group and X is a topological space regarded as a discrete ∞-groupoid, then a G-principal ∞-bundle over X is the same as a usual G-principal bundle over X.

Associated Bundles

In the last subsection we showed that the usual notions of cocycles and characteristic classes refine to analogous notions in abstract cohomology theory. We then used this new notions in order to define principal ∞-bundles. Here we will see that the process of constructing associated bundles can also be refined to associated ∞-bundles. This will be used in the next section in order to show that the Thom isomorphism and the Poincaré duality, which are classical results in usual cohomology, can be lifted to the context of abstract cohomology.

The strategy is to give a purely category characterization of the associated bundle construction in such a way that all concepts can be naturally abstracted. So, we start by recalling the construction, which was sketched in Example 1.7. Let G be a group acting into a space F. By an action we mean a map $\rho : G \times F \to F$ fulfilling the usual conditions or, equivalently, a group homomorphism $\rho : G \to \mathrm{Aut}(F)$, which induces a map $\overline{\rho} : BG \to B\mathrm{Aut}(F)$. Therefore, given a G-bundle, say classified by $f : X \to BG$, the obvious idea to get a G-bundle over X with fiber F is to consider the bundle classified by the composition $f \circ \overline{\rho}$, as in the first diagram below.

$$\begin{tikzcd} \infty\mathrm{pb} \arrow[rr] \arrow[d] & & * \arrow[d] \\ X \arrow[r, "f"'] & BG \arrow[r, "\overline{\rho}"'] & B\mathrm{Aut}(F) \end{tikzcd}$$

We have, however, at least two problems with this construction:

1. if the action is not effective, i.e, if $\rho : G \to \mathrm{Aut}(F)$ is not injective, then the obtained bundle is not structured over G as desired, but only over $\mathrm{Aut}(F)$;

2. the typical fiber of the obtained bundle is the automorphism group $\mathrm{Aut}(F)$ and not F itself.

The first problem imply that we need to replace $\overline{\rho}$ in the diagram below, while the second problem imply that the map $* \to B\mathrm{Aut}(F)$ should also be replaced. So, the next tentative is to consider the ∞-pullback below, where $F//\rho$ is some space depending on F and ρ, which becomes endowed with a canonical projection onto BG.

$$
\begin{tikzcd}
\infty\mathrm{pb} \arrow[r] \arrow[d] & F//\rho \arrow[d] \\
X \arrow[r] & BG
\end{tikzcd}
$$

In order to get some felling on what this space should be, notice that if we consider the simple case in which $F = G$ and $\rho : G \to \mathrm{Aut}(G)$ is the action by left multiplication $g \mapsto \ell_g$, we expect to recover the starting G-bundle. So, we need to build a space $G//\rho$, depending both of G and ρ, which becomes equipped with a canonical projection onto BG and such that $G//\rho \simeq *$. The obvious choice is to consider $G//\rho$ as the quotient space of G by the action ρ. This really give a trivial space, but it does not be endowed with a canonical projection onto BG. We can get this projection by adding a new variable, i.e, by considering $G//\rho$ as the quotient of $G \times G$ by the relation $(g, h) \sim (g', h')$ iff $g' = g$ and $h' = L_g(h)$.

Physics

In Section 6.2 we commented that the configuration space (i.e, the space of fields) of *any interesting classical theory of* ***particles*** *can be regarded as a smooth stack*: a functor $\mathrm{Fields} : \mathbf{Diff}^{op}_{sub} \to \mathbf{Gpd}$ such that for any manifold Σ and any open covering $U_i \hookrightarrow \Sigma$ the corresponding "space of fields on Σ" satisfy compatibility conditions in the intersections $U_i \cap U_j$ which allow us to study it by studying "the space of fields along U_i".

We also commented that in the case of **strings** (or higher dimensional objects) there are interesting classical theories which make use of fields having the property that, in order to recover them when given a covering $U_i \hookrightarrow \Sigma$, we need compatibility conditions not only in the immediate intersections $U_i \cap U_j$, but also in the secondary intersections $U_i \cap U_j \cap U_k$. This means that, *in order to incorporate them into an unified axiomatic language, we need to have the notion of "higher stacks"*. These "higher stacks" were then formalized in Section 8.2, being given by smooth ∞-stacks $\mathrm{Fields} : \mathbf{Diff}^{op}_{sub} \to \infty\mathbf{Gpd}$.

Now, with the language of abstract cohomology on hand we can give a cohomological interpretation to the facts above. Indeed, recall that in the Sctions 6.2 and 8.2 we emphasized that the smooth ∞-stacks describing classical physical theories are **geometric**, which means that they are always induced by some ∞-Lie groupoid. In other words, we always have

$$
\mathrm{Fields} = \mathrm{Mor}_{\infty\mathrm{Stack}(\mathbf{Diff})}(-, \mathbf{Fields})
$$

for some $\mathbf{Fields} \in \infty\mathbf{Gpd}_{\mathbf{Diff}}$. Consequently, composing with the first order truncation functor π_0 we see that *any classical field theory can be understood as an abstract cohomology theory in the* ∞*-topos* $\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J)$.

Recall that, as commented in Section 2.4, it is expected that gauge theory of strings should be described by some geometry of "higher bundles", defined on "higher groups". Let us see that the above interpretation led us to give a precise meaning to this fact. Indeed, for a geometric smooth ∞-stack Fields representing a classical physical theory, every cocycle $[f] : \Sigma \to \mathbf{Fields}$ in the corresponding abstract cohomology classifies, by Example 9.1, precisely **Fields**-principal ∞-bundles over Σ.

9.4 Obstruction

As an application of the abstract notions of abstract cohomology and characteristic classes, let us study lifting/extension problems in an arbitrary $(\infty, 1)$-category. We start by recalling that when $\mathbf{H}$ is a usual 1-category, we say a 1-morphism $f : X \to Y$ can be *lifted* with respect to other 1-morphism $\pi : P \to Y$ when there exists some $\hat{f}$ making commutative the first diagram below. In this case we say that it is a *lifting of* f *with respect to* π. Similarly, we say that $f : A \to Y$ can be *extended* with respect to $\imath : A \to X$ when there exists $\tilde{f}$ such that the second diagram below is commutative and in this case this map is called an *extension of* f *with respect to* $\imath$.

$$
\begin{tikzcd}
 & P \arrow[d, "\pi"] \\
X \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Y \\
A \arrow[ur, "f"] \arrow[r, "\imath"'] & X \arrow[u, dashed, "\tilde{f}"']
\end{tikzcd}
$$

Now, let us suppose that the ambient category $\mathbf{H}$ is not only a 1-category, but indeed a ∞-category. So, by the ∞-weakening principle the lifting/extension problems can be weakened by replacing strictly commutative diagrams by diagrams commutative up to higher morphisms. More precisely, in this context we say that the 1-morphism $f : X \to Y$ has a ∞-*lifting with respect to* $\pi : P \to Y$ when there is $\hat{f}$ such that the first diagram above is commutative up to higher morphisms, i.e, we have some 2-isomorphism $\xi : f \simeq \hat{f} \circ \pi$ satisfying some condition up to 3-morphisms, and so on. In a totally dual fashion we define ∞-*extensions* of 1-morphisms.

Here we would like to explain that the problems of existence of ∞-liftings and ∞-extensions have a purely cohomological nature. More precisely, we will see that a given 1-morphism f can be ∞-lifted/∞-extended only if some cocycle associated to it is trivial. We usually say that the triviality of this cocycle is an *obstruction* to the existence of the ∞-lifting/∞-extension. In this sense, what we will develop here is some kind of *obstruction theory*.

Remark. In Section 2.1 we studied the extension problem in the 2-category **Cat**, getting the notions of Kan extension. Consequently, we can apply obstruction theory in order to get obstructions to the existence of Kan extensions. In other words, *to determine if a category is complete/cocomplete is a cohomological problem.*

Notice that the ∞-lifting problem and the ∞-extension problem are dual, so that studying some of them we can get analogous results by applying the ∞-duality principle. Because of this, we will analyze only the ∞-lifting problem. So, let $f : X \to Y$ be a 1-morphism and let

$\pi : P \to Y$ be some 1-morphism respectively to which we are searching for a lifting, as in the first diagram below. The idea is pretty simple: in the classical situations where lifting problems are considered, P is some bundle (say a G-principal bundle). But G-bundles are classified by nonabelian cohomology with coefficients in G. This means that in the typical cases P is the fiber of some other map $c : Y \to Y'$, as in the second diagram below. Because $*$ is a terminal object, there is a unique morphism $0 : X \to *$. So, by the universality of ∞-pullbacks, the ∞-lifting $\hat{f}$ exists iff the exterior diagram is ∞-commutative (meaning $c \circ f \simeq 0$) as in the third diagram.

$$
\begin{tikzcd}
 & P \arrow[d, "\pi"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[r] \arrow[d, "\pi"] & * \arrow[d] \\
X \arrow[r, "f"'] & Y \arrow[r, "c"'] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[r] \arrow[d, "\pi"] & * \arrow[d] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\hat{f}"] \arrow[urr, bend left, "0"] & Y \arrow[r, "c"'] & Y'
\end{tikzcd}
$$

Now, let us interpret this conclusion in terms of the cohomological language introduced in the last subsection. Indeed, the 1-morphism f induces a cocycle $[f]$ in the abstract cohomology group $H^0(X; Y)$. By the ∞-Yoneda lemma, the morphism $c : Y \to Y'$ corresponds to an universal characteristic class $\mathbf{c} : H^0(-; Y) \to H^0(-; Y')$, so that the $c \circ f$ corresponds to the evaluated class $\mathbf{c}([f])$. Therefore, $c \circ f \simeq 0$ means precisely that the cocycle $\mathbf{c}([f])$ is trivial.

Conclusion. *Let* $\mathbf{H}$ *be a* ∞*-category with finite* ∞*-limits/*∞*-colimits. If a morphism* $\pi : P \to Y$ *is part of a fiber sequence, say with* $c : Y \to Y'$*, then a given* $f : X \to Y$ *has a* ∞*-lifting with respect to* π *if, and only if, the characteristic class* $\mathbf{c}([f])$ *is trivial.*

Example

An interesting situation in which obstruction theory can be concretely applied is in the Whitehead tower of an object X. Indeed, recall that, as discussed in Section 8.3, if $\mathbf{H}$ is an $(\infty, 1)$-category (say ∞-complete and ∞-cocomplete, with terminal object $*$), we can use the truncation $\tau_{\leq n}$ in order to study any object X by analyzing the iterated truncations of the maps $X \to *$ and $* \to X$. The obtained sequences are respectively called *Postnikov tower* and *Whitehead tower* for X.

In Examples 8.11 and 8.12 we saw that, for the ∞-topos **Top**, these towers have a very concrete meaning. Indeed, the respective nth term in the Postnikov and in the Whitehead towers are spaces X_n and X^n whose weak homotopy type is determined by the conditions

$$
\pi_i(X_n) \simeq \begin{cases} \pi_i(X), & i \leq n \\ 0, & i > n \end{cases} \quad \text{and} \quad \pi_i(X^n) \simeq \begin{cases} 0, & i \leq n \\ \pi_i(X), & i > n \end{cases}.
$$

This characterization was then used to reconstruct the Whitehead tower is terms of the Postnikov tower. Indeed, we commented that the nth term X^n is exactly the **fiber** of the canonical map $X \to X_n$ in the Postnikov tower, meaning that for any n the first diagram below is a ∞-pullback. So, given a continuous map $f : A \to X$ we can study the problem of lifting from

Y to Y^n by making use of obstruction theory, as in the third diagram below.

$$
\begin{tikzcd}
X^n \arrow[d] \arrow[rrr] & & & * \arrow[dddd] \\
X^{n-1} \arrow[d] & & & \\
\vdots \arrow[d] & & & \\
 & & & \\
X \arrow[r] & \cdots \arrow[r] & X_{n+1} \arrow[r] & X_n
\end{tikzcd}
=
\begin{tikzcd}
X^n \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] & X_n
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X^n \arrow[r] \arrow[d] & * \arrow[d] \\
A \arrow[r, "f"'] \arrow[ur, dashed, "\hat{f}"] \arrow[urr, bend left, "0"] & X \arrow[r] & X_n
\end{tikzcd}
$$

The situation will be more clear when working with a concrete example.

Example 9.2 (*$O(n)$-bundles*)**.** Let us take $X = BO(n)$, where $O(n)$ is the group of orthogonal $n \times n$ real matrices. Thus

$$
\begin{aligned}
\pi_1(X) &\simeq [\mathbb{S}^1, BO(n)] \\
&\simeq [\Sigma\mathbb{S}^0, BO(n)] \\
&\simeq [\mathbb{S}^0, \Omega BO(n)] \\
&\simeq \pi_0(O(n)) \simeq \mathbb{Z}_2
\end{aligned}
$$

and, therefore, $X_1 \simeq K(\mathbb{Z}_2, 1)$. A map $c : BO(n) \to K(\mathbb{Z}_2, 1)$ represent a class in $H^1(BO(n); \mathbb{Z}_2)$, which is totally determined by the Stiefel-Whitney class w_1. On the other hand, each $f : X \to BO(n)$ is a cocycle in nonabelian cohomology with coefficients on $O(n)$, which classify $O(n)$-bundles (i.e, rank n real vector bundles) over X. Finally, by a direct computation we have $X^1 \simeq BSO(n)$, so that a lifting of f from X to X^1 corresponds to lifting from a $O(n)$-structure to a $SO(n)$-structure. But a $SO(n)$-bundle is precisely a orientable rank n bundle. Therefore, *given a $O(n)$-bundle over X, say classified by a map f, it is orientable iff its first Stiefel-Whitney class $w_1(f)$ vanishes.*

Remark. We could do a analogous analysis for X equal to $BU(n)$ instead of equal to $BO(n)$, where here $U(n)$ is the group of unitary $n \times n$ complex matrices. Despite the similarities between the definitions of $O(n)$ and $U(n)$, the final conclusion would be **totally different**. This comes from the fact that $U(n)$ is **connected**, while $O(n)$ is not. Consequently,

$$\pi_1(BU(n)) \simeq \pi_0(U(n)) \simeq 0,$$

so that in this case $X_1 \simeq K(0, 1)$ is the trivial space, meaning that the cohomology group $H^0(-, X_1)$ has only trivial cocycles. Particularly, any $c : BU(n) \to X_1$ is necessarily trivial and, therefore, there is no obstructions to lift a given $f : X \to BU(n)$ to $Y^1 \simeq BSU(n)$. In other words, *given a $U(n)$-bundle, it is always possible to replace its structural group by $SU(n)$.* Summarizing: while a **real** bundle is orientable iff $w_1(f) = 0$, any **complex** bundle is orientable.

Orientation

Here, following [10, 145], we will see that the concept of orientation of vector bundles presented in Section 5.3 can be vastly abstracted to the the context of ∞-bundles. This new abstract notion of orientation has a very physical meaning, as will be discussed later. Indeed, the orientatibility condition of ∞-bundles can be identified with quantum anomaly cancelations and, therefore, as a necessary condition to build a (pull-push) quantization scheme.

We start by noticing that, if $\mathbf{Rng}$ is the usual category of (associative and unital) rings, then we have a canonical functor $GL_1 : \mathbf{Rng} \to \mathbf{Grp}$ which assigns to any ring R its group $GL_1(R)$ of units (i.e, the group elements which have multiplicative inverse). For instance, if R is a field $\mathbb{K}$, then $GL_1(\mathbb{K}) = \mathbb{K} - 0$, while if R is the ring $M(n, \mathbb{K})$ of $n \times n$ matrices with coefficients into a field $\mathbb{K}$, then $GL_1(M(n, \mathbb{K})) = GL_1(n, \mathbb{K})$ is just the group of invertible matrices. There two fundamental remarks on the functor GL_1:

1. for any ring R we have a canonical inclusion morphism $GL_1(R) \hookrightarrow R$, which indeed fits into natural transformation $\xi : GL_1 \Rightarrow id_{\mathbf{Rng}}$;

2. it has an adjoint $\mathbb{Z}[-] : \mathbf{Grp} \to \mathbf{Rng}$, which assigns to any group G the group-ring $\mathbb{Z}[G]$ with coefficients in $\mathbb{Z}$.

Le us see that the structures above refine to the context of abstract stable homotopy theory. More precisely, let $(\mathbf{H}, \otimes, 1)$ be a symmetric monoidal ∞-category. As discussed in Section 9.2, if the product $\otimes$ preserves ∞-colimits in both arguments, then the corresponding category $\mathrm{Stab}(\mathbf{H})$ of Ω-spectrum objects acquires a canonical symmetric monoidal ∞-structure given by the smash product $\wedge_\otimes$ on spectra and whose neutral object is the sphere spectrum $\mathbb{S}_1 = \overline{\Sigma}^\infty 1$. We assert that in the case when $\mathbf{H}$ is concrete, we have analogous adjunction

$$A_\infty\mathrm{Ring}(\mathbf{H}, \otimes, 1) \underset{\mathbb{S}_1[-]}{\overset{GL_1^\infty}{\rightleftarrows}} \infty\mathrm{Hopf}(\mathbf{H}, \otimes, 1).$$

We will construct GL_1^∞ and outline the existence of the adjunction. For details see [10]. Recall that for any spectrum $\mathbb{A}$ in $\mathrm{Stab}(\mathbf{H})$ we can associate an object $\overline{\Omega}^\infty \mathbb{A} \in \mathbf{H}$ and, truncating at the first level (i.e, by passing to the homotopy category), we get another object $\pi_0(\overline{\Omega}^\infty \mathbb{A}) \in \mathrm{Ho}(\mathbf{H})$. By construction, $\overline{\Omega}^\infty \mathbb{A}$ is a ∞-loop object and, therefore, a commutative ∞-Hopf object in the cartesian structure (i.e, an abelian ∞-group object). Consequently, $\pi_0(\overline{\Omega}^\infty \mathbb{A})$ is a an abelian group object in the homotopy category.

Now, if $\mathbb{A}$ is indeed a A_∞-ring object, then, independently of the previous structures, $\overline{\Omega}^\infty \mathbb{A}$ is an A_∞-space and, consequently, $\pi_0(\overline{\Omega}^\infty \mathbb{A})$ is a monoid object. Therefore, joining both structures we see that $\overline{\Omega}^\infty \mathbb{A}$ is a ∞-ring object, so that $\pi_0(\overline{\Omega}^\infty \mathbb{A})$ is a ring object in $\mathrm{Ho}(\mathbf{H})$. Because $\mathbf{H}$ is concrete, this structure then forgets to a usual ring. This allow us to apply GL_1. Particularly, we can build the ∞-pullback below which we define as $GL_1^\infty(\mathbb{A})$. Because each entity in the pullback is a ∞-group, the result is also a ∞-group. Finally, the construction is functorial because each step is functorial.

$$\begin{tikzcd}
\infty\mathrm{pb} \arrow[r] \arrow[d] & \overline{\Omega}^\infty \mathbb{A} \arrow[d] \\
GL_1(\pi_0(\overline{\Omega}^\infty \mathbb{A})) \arrow[r] & \pi_0(\overline{\Omega}^\infty \mathbb{A})
\end{tikzcd} \tag{9.4.1}$$

With the ∞-functor GL_1^∞ on hand we are ready to introduce the notion of orientation in abstract generalized cohomology theory. Indeed, let $\mathbb{A}$ be a A_∞-ring object, representing the coefficients of an abstract generalized cohomology theory in $\mathbf{H}$, and let $GL_1^\infty(\mathbb{A})$ be the internal ∞-group constructed above. Applying $\overline{\Sigma}^\infty$ we then get a ∞-group on the stable ∞-category $\mathrm{Stab}(\mathbf{H})$, so that the delooping $\mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{A})$ is well defined. From Example 9.1, the morphisms

$$f : \mathbb{X} \to \mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{A})$$

for a given Ω-spectrum $\mathbb{X}$ classify ∞-principal $\overline{\Sigma}^\infty GL_1^\infty(\mathbb{A})$-bundles over $\mathbb{X}$.

We have a canonical action of $GL_1^\infty(\mathbb{A})$ on $\overline{\Omega}^\infty\mathbb{A}$, in the same way as $GL_1(R)$ acts naturally into the underlying ring R. By the adjunction between $\overline{\Omega}^\infty$ and $\overline{\Sigma}^\infty$, this action induces an action of $\overline{\Sigma}^\infty GL_1^\infty(\mathbb{A})$ over $\mathbb{A}$. Consequently, by the discussion in the subsection "Associated Bundles", for any bundle classified by a morphism ζ as above, we have an associated ∞-line bundle $\mathbb{X}^\zeta$ whose fiber is $\mathbb{A}$. This is called the $\mathbb{A}$-*Thom spectrum of* f, denoted by $M_{\mathbb{A}}f$.

Now, let us consider the above construction for the case $\mathbb{A} = \mathbb{S}_1$. Recall that $\mathbb{S}_1$ is an initial object in $\mathrm{A}_\infty\mathrm{Ring}(\mathbf{C})$, so that for any A_∞-ring $\mathbb{A}$, we have a canonical $c : \mathbb{S} \to \mathbb{A}$. Therefore, each f induces a corresponding morphism

$$\begin{tikzcd} \mathbb{X} \arrow[r, "f"] \arrow[rr, bend right, "f_{\mathbb{A}}"'] & \mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{S}) \arrow[r, "c"] & \mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{A}), \end{tikzcd}$$

which classifies a $GL_1^\infty(\mathbb{A})$ ∞-bundle and, by the previous construction, a $\mathbb{A}$-line ∞-bundle. We say that $f_{\mathbb{A}}$ *is orientable* (or that the Thom bundle Mf is $\mathbb{A}$-*orientable*) when the induced $\mathbb{A}$-line bundle is trivial. In this case, a choice of trivialization is called an $\mathbb{A}$-*orientation of* Mf. By our discussion on obstruction theory, $Mf_{\mathbb{A}}$ is trivial iff if $f_{\mathbb{A}} \simeq 0$, i.e, iff $\mathbf{c}([f]) = 0$. This identify the space of orientations as the space of liftings in the diagram below, which is equivalently the space of sections of $Mf_{\mathbb{A}}$.

$$\begin{tikzcd} & \infty\mathrm{pb} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathbb{X} \arrow[ur, dashed] \arrow[urr, bend left, "0"] \arrow[r, "f"] \arrow[rr, bend right, "f_{\mathbb{A}}"'] & \mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{S}) \arrow[r, "c"] & \mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{A}) \end{tikzcd}$$

It happens that this space of liftings is equivalent to the space of morphisms $Mf \to \mathbb{A}$, as can be seen in the Corollary 2.12 at page 8 of [10]. This is the generalization of a classical result of May, Quinn, Ray and Tornehave presented in [142].

Conclusion. *if Mf is $\mathbb{A}$-orientable then we have the following equivalence which we call the generalized Thom isomorphism:*

$$\mathrm{Mor}_{\mathrm{A}_\infty\mathrm{Ring}(\mathbf{C},\otimes)}(Mf, \mathbb{A}) \simeq \Gamma(Mf_{\mathbb{A}}) \simeq \mathrm{Mor}_{\mathrm{Stab}(\mathbf{C},\otimes)}(\mathbb{X}; \mathbb{A}). \tag{9.4.2}$$

We end this subsection noticing that the above abstract notion of orientation really contains as particular case the usual concept of orientation of vector bundles discussed in Section 5.3.

Recall that, for a given ring spectrum $\mathbb{A}$ describing a multiplicative cohomology theory, a rank n bundle $E \to X$ is called $\mathbb{A}$-*orientable* when the $\mathbb{A}$-cohomology of the Thom space $\mathrm{Th}(E)$ is equivalent to the $\mathbb{A}$-cohomology of the base space X, i.e, when for each k we have a canonical isomorphism $[X, A_k] \simeq [\mathrm{Th}(E), A_{k+n}]$, called *Thom isomorphism*.

A choice of this isomorphism corresponds to an $\mathbb{A}$-*orientation of* E. In order to motivate the connection between this concrete picture and the previous abstract picture, we notice that the Thom isomorphisms can be condensate into a unique isomorphism in $\pi_0\mathrm{Stab}(\mathscr{C}_*)$:

$$\mathrm{Mor}_{\pi_0\mathrm{Stab}(\mathscr{C}_*)}(\overline{\Sigma}^{\infty-n}\mathrm{Th}(E), \mathbb{A}) \simeq \mathrm{Mor}_{\pi_0\mathrm{Stab}(\mathscr{C}_*)}(\overline{\Sigma}^{\infty}X, \mathbb{A}), \tag{9.4.3}$$

which is much similar to (9.4.2). So, in order to get the desired conclusion we need to prove that any vector bundle $E \to X$ induce a corresponding map

$$\mu : \overline{\Sigma}^{\infty}X \to \mathbf{B}\overline{\Sigma}^{\infty}GL_1^{\infty}(\mathbb{S}) \tag{9.4.4}$$

whose Thom spectrum Mf coincides with the spectrum $\overline{\Sigma}^{\infty-n}\mathrm{Th}(E)$ generated by the Thom space. This will be done into two steps:

1. *showing that any bundle induces a map as in (9.4.4)*. We start by noticing that, from the adjunction between $\overline{\Omega}^{\infty}$ and $\overline{\Sigma}^{\infty}$, in order to give as as in (9.4.4) it is enough to give a dual map $\overline{\mu} : X \to \mathbf{B}GL_1^{\infty}(\mathbb{S})$. Let O be the colimit $\mathrm{colim}_n O(n)$. As discussed in Example 1.14, we have a canonical map $J_\infty : O \to \overline{\Omega}^{\infty}\mathbb{S}$ induced from the J_n-homomorphisms. So, from the universality of ∞-pullbacks of $GL_1^{\infty}(\mathbb{S})$ we then get a map $J : O \to GL_1^{\infty}(\mathbb{S})$. Now, recalling that a vector bundle is classified by a map $f : X \to \mathbf{B}O(n)$, we can define $\overline{\mu}$ as the composition below.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rrr, bend right, "\overline{\mu}"'] & \mathbf{B}O(n) \arrow[r, "\mathrm{colim}"] & \mathbf{B}O \arrow[r, "\mathbf{B}J"] & \mathbf{B}GL_1^{\infty}(\mathbb{S}) \end{tikzcd}$$

2. *proving that the Thom spectrum of the induced map coincides with the spectra of the Thom space*. We will give only an indicatives that this should really happens. A formal proof can be seen in [??]. See also [??]. Recall that the Thom space of $E \to X$ is obtained as the quotient space $D(E)/S(E)$ of the disk bundle by the sphere bundle of E relatively to some riemmanian metric in the fibers. $\mathrm{Th}(E)$ is indeed the total space of a bundle over X (the Thom bundle) whose fibers are quotients $D(E_x)/S(E_x)$ and, therefore, are n-spheres. So, the Thom space construction takes a n-plane bundle over X and gives a $\mathbb{S}^n$-bundle over X. Notice that the bundle classified by $\overline{\mu}$ has fiber $\mathbb{S}^n$, so that the the process of taking the compositions in the diagram below and then considering the induced bundle has essentially the same effect over the starting bundle as the Thom space construction.

Duality

In the last subsection we saw that the notion of orientability in abstract cohomology allow us to build an abstract version of the Thom spectrum and, consequently, of the Thom isomorphism. Despite the Thom isomorphism, there is another isomorphism in classical cohomology which is a central tool in Algebraic Topology: the *Poincaré duality*. Here we would like to discuss that this result can also be abstracted to the general context of abstract cohomology theories.

We recall that the standard context in which we have a Poincaré duality is for singular cohomology of compact manifolds. Indeed, if M is a compact oriented n-dimensional manifold, then the classical Poincaré duality states that for every k there is a canonical isomorphism

$$H^k(M;\mathbb{R}) \simeq H_{n-k}(M;\mathbb{R}), \tag{9.4.5}$$

where the left-hand side is singular cohomology, while the right-hand side singular homology. So, *in order to produce a Poincaré duality, the first step is to have not only a cohomology theory, but also a dual homology theory.*

We generally study singular homology by making use of the existence of a cochain complex presentation for singular cohomology , allowing us to consider the dual chain complex, as commented in Example 5.8.1. We notice, however, that singular homology can also be formulated in terms of purely spectrum data, without the necessity of a cochain complex presentation. Indeed, while real singular cohomology is the cohomology of the Eilenberg-Mac Lane spectrum $K(\mathbb{R},n)$, singular homology of a space X can be regarded as (see [??])

$$H_k(X;\mathbb{R}) \simeq \lim_{n\to\infty} \pi_k(X \wedge K(\mathbb{R}, n+k)). \tag{9.4.6}$$

In other words, the singular homology groups of X are the stable homotopy groups of the induced spectrum $X \wedge K(\mathbb{R},n)$. In order to better understand the Poincaré duality (9.4.5) in the language of spectra, we notice that both singular cohomogy and homology can be understood as functors on the category **Spec**. Indeed, regarding a space X as a free spectrum $\overline{\Sigma}^{\infty}X$, we have

$$[\overline{\Sigma}^{\infty}X, \overline{\Sigma}^{k}\mathbb{HR}] \simeq [X, \overline{\Omega}^{\infty}\overline{\Sigma}^{k}\mathbb{HR}] \simeq [X, \mathbb{HR}_k] = [X, K(\mathbb{R},k)] = H^k(X;\mathbb{R}),$$

where $\mathbb{HR}$ is the real Eilenberg-Mac Lane spectrum. Similarly, by making use of (9.4.6) we get

$$[\overline{\Sigma}^{k}\mathbb{S}, \overline{\Sigma}^{\infty}X \wedge \mathbb{HR}] \simeq H_k(X;\mathbb{R}).$$

Therefore, in the language of spectra, the classical Poincaré duality (9.4.5) establishes that for any compact oriented manifold M there are isomorphisms

$$[\overline{\Sigma}^{\infty}M, \overline{\Sigma}^{k}\mathbb{HR}] \simeq [\overline{\Sigma}^{n-k}\mathbb{S}, \overline{\Sigma}^{\infty}M \wedge \mathbb{HR}]. \tag{9.4.7}$$

With eyes in an abstract version of the Poincaré duality, let us add to the spectrum characterization above the *Atiyah duality theorem*. In order to state it, notice that, because the manifold M is assumed orientable, we have a Thom isomorphism (9.4.3), so that

$$\begin{aligned}
[\overline{\Sigma}^{\infty-n}\mathrm{Th}(TM), \overline{\Sigma}^{k}\mathbb{HR}] &\simeq [\overline{\Sigma}^{n-k}\mathbb{S}, \overline{\Sigma}^{\infty}M \wedge \mathbb{HR}]. && (9.4.8)\\
&\simeq [\mathbb{S}, \overline{\Sigma}^{\infty-n+k}M \wedge \mathbb{HR}] && (9.4.9)\\
&\simeq [\mathbb{S}, \overline{\Sigma}^{\infty-n}M \wedge \overline{\Sigma}^{k}\mathbb{HR}] && (9.4.10)
\end{aligned}$$

As discussed in Section 8.4, in any monoidal category $(\mathbf{C}, \otimes, 1)$ we have the notion of *dualizable object*. Indeed, we say that X is *dualizable* when there exists X^* together with 1-morphisms

$\mu : 1 \to X \otimes X^*$ and $\nu : X^* \otimes X \to 1$ fulfilling adjoint-type diagrams up to higher morphisms. Equivalently, if for any other A we have the following isomorphisms induced from μ and ν:

$$\mathrm{Mor}_{\mathbf{C}}(X, A) \simeq \mathrm{Mor}_{\mathbf{C}}(1, A \otimes X*) \quad \text{and} \quad \mathrm{Mor}_{\mathbf{C}}(X^*, A) \simeq \mathrm{Mor}_{\mathbf{C}}(1, X \otimes A).$$

In particular, we can study dualizable objects in the homotopy category of spectra with the canonical monoidal structure given by the smash product. In this case, a spectrum $\mathbb{X}$ is dualizable when there exists $\mathbb{X}^*$ and spectrum morphisms $\mathbb{S} \to \mathbb{X} \wedge \mathbb{X}^*$ and $\mathbb{X}^* \wedge \mathbb{X} \to \mathbb{S}$ such that for any other spectrum $\mathbb{A}$ we have induced isomorphisms

$$[\mathbb{X}, \mathbb{A}] \simeq [\mathbb{S}, \mathbb{A} \wedge \mathbb{X}^*] \quad \text{and} \quad [\mathbb{X}^*, \mathbb{A}] \simeq [\mathbb{S}, \mathbb{X} \wedge \mathbb{A}].$$

Now, looking at the second of these isomorphisms and comparing it with the characterization (9.4.8) of the Poincaré duality, it seems that $\overline{\Sigma}^{\infty - n} M$ is dualizable and with dual given by $\overline{\Sigma}^{\infty - n}\mathrm{Th}(TM)$. The Atiyah duality theorem asserts that this is really the case. Notice that reverting the steps of the explanation above we conclude that *Athiyah duality together with Thom isomorphism imply the classical Poincaré duality.* Summarizing,

$$\begin{matrix} \textbf{monoidal duality} \\ + \\ \textbf{orientability} \end{matrix} \Longrightarrow \textbf{Poincaré duality}$$

This is all we need to give a completely abstract version of Poincaré duality. Indeed, let $\mathbb{A}$ be a A_∞-ring object of the stable ∞-category $\mathrm{Stab}(\mathbf{C})$ of some ∞-monoidal ∞-category $(\mathbf{C}, \otimes, 1)$. We say that a Ω-spectrum object $\mathbb{X}$ is a $\mathbb{A}$-*Poincaré object of degree* n where there exists some

$$f : \mathbb{X} \to \mathbf{B}\overline{\Sigma}^{\infty} GL_1^{\infty}(\mathbb{S})$$

such that the corresponding Thom bundle Mf is $\mathbb{A}$-orientable and dualizable, with dual given by $\overline{\Omega}^n \mathbb{X}$. In this case, we have

$$\begin{aligned} [\mathbb{X}; \overline{\Sigma}^k \mathbb{A}] &\simeq [Mf; \overline{\Sigma}^k \mathbb{A}] \\ &\simeq [\mathbb{S}, \overline{\Omega}^n \mathbb{X} \wedge \overline{\Sigma}^k \mathbb{A}] \\ &\simeq [\mathbb{S}, \overline{\Omega}^{n-k} \mathbb{X} \wedge \mathbb{A}] \\ &\simeq [\overline{\Sigma}^{n-k} \mathbb{S}, \mathbb{X} \wedge \mathbb{A}], \end{aligned}$$

which is a direct abstraction of (9.4.7). Notice that the starting term is precisely the kth abstract cohomology of $\mathbb{X}$ with coefficients in $\mathbb{A}$. Therefore, if we define the *kth abstract homology group* $H_k(\mathbb{X}; \mathbb{A})$ of $\mathbb{X}$ with respect to $\mathbb{A}$ as $[\overline{\Sigma}^k \mathbb{S}, \mathbb{X} \wedge \mathbb{A}]$, the last equivalences translate into the following

Proposition 9.1 (abstract Poincaré duality)**.** *If $\mathbb{X}$ is an $\mathbb{A}$-Poincaré object of degree n, then for any $k \leq n$ we have a canonical isomorphism*

$$H^k(\mathbb{X}; \mathbb{A}) \simeq H_{n-k}(\mathbb{X}; \mathbb{A}).$$

Remark. The primary interest in this abstract version of Poincaré duality is that it gives a direct approach to get *fiber integration* in abstract cohomology. More precisely, because cohomology is contravariant, for any morphism $f : X \to Y$, when applying the cohomology functors we obtain induced morphisms in the opposite direction $f^* : H^k(Y; A) \to H^k(X; A)$. A *fiber integration process* for A-cohomology is some kind of process that take $f : X \to Y$ and return a map in the unnatural direction $f_! : H^k(X; A) \to H^{k+l}(X; A)$, possibly with a shifting on the degree. The subscript "!" appears in the literature in order to emphasize the fact that it is a great surprise when the map $f_!$ exists. Other usual names for $f_!$ are *Gysin map* and *Umkehr map*. Now, let us see how Poincaré duality imply the existence of fiber integration. Let $f : \mathbb{X} \to \mathbb{Y}$ be a map between spectrum objects and suppose that both $\mathbb{X}$ and $\mathbb{Y}$ are $\mathbb{A}$-Poincaré objects of the same degree n. Then, for any k we define $f_!$ as the composition below, where the isomorphisms correspond to the Poincaré duality for each object and f_* is the induced morphism in homology.

$$\begin{tikzcd} H^k(\mathbb{X}; \mathbb{A}) \arrow[r, "\simeq"] \arrow[rrr, bend right, "f_!"'] & H_{n-k}(\mathbb{X}; \mathbb{A}) \arrow[r, "f_*"] & H_{n-k}(\mathbb{Y}; \mathbb{A}) \arrow[r, "\simeq"] & H^k(\mathbb{Y}; \mathbb{A}) \end{tikzcd}$$

Physics

As discussed in the Introduction, the natural approach to Hilbert's sixth problem is based in the axiomatization of classical/quantum physics and in the building of some link between them, corresponding to the a notion of *quantization*.

Up to this point we commented that a quantum theory of p-branes can be axiomatized as a monoidal functor $U : (\mathbf{Cob}_{p+1}, \sqcup) \to (\mathbf{D}, \otimes)$ from the category of p-cobordisms to some monoidal category and that a classical theory can be axiomatized as a pair $(\mathrm{Fields}, e^{iS})$, given by a geometric smooth ∞-stack $\mathrm{Fields} : \mathbf{Diff}^{op} \to \infty\mathbf{Gpd}$ and by a lifting to $\infty\mathbf{Gpd}/\mathbf{B}U(1)$. The quantization process can be axiomatized (as quantization by pull-push) at least over the class of all **orientable** and **dualizable** classical theories (see [159] and the Part 6 of [182]).

$$\begin{tikzcd} & \mathrm{Span}(\infty\mathbf{Gpd}/\mathbf{B}U(1)) \arrow[d, "U(1)"] \\ \mathbf{Cob}_{p+1} \arrow[ur, dashed, "e_p^{iS}"] \arrow[r, "\mathrm{Fields}_p"'] & \mathrm{Span}(\infty\mathbf{Gpd}) \end{tikzcd}$$

The idea is cute. Indeed, a quantization scheme should be some kind of rule $\mathscr{Q}$ assigning to any suitable classical theory $(\mathrm{Fields}, e^{iS})$ a corresponding quantum theory $\mathscr{Q}_S : (\mathbf{Cob}_p, \sqcup) \to (\mathbf{D}, \otimes)$. A cobordism $\Sigma : \Sigma_0 \to \Sigma_1$ between p-manifolds can be understood as a cospan[5], so that each pair $(\mathrm{Fields}, e^{iS})$ induces a functor Fields_p and a lifting e_p^{iS}, as above. Concretely, the lifting e_p^{iS}

[5] Recall the definitions of span and cospan given in Examples ?? and ??.

assigns to any cobordism $\Sigma : \Sigma_0 \to \Sigma_1$ a span in $\infty\mathbf{Gpd}/\mathbf{B}U(1)$ as schematized below.

$$
\begin{tikzcd}
 & \Sigma & \\
\Sigma_0 \arrow[ur, hook, "\imath_0"] & & \Sigma_1 \arrow[ul, hook', "\imath_1"']
\end{tikzcd}
\overset{e_p^{iS}}{\Longrightarrow}
\begin{tikzcd}
 & \mathrm{Fields}(\Sigma) \arrow[dl, "\mathrm{Fields}(\imath_0)"'] \arrow[dr, "\mathrm{Fields}(\imath_1)"] & \\
\mathrm{Fields}(\Sigma_0) \arrow[rr, Rightarrow, "e^{iS}"] \arrow[dr, "\chi_0"'] & & \mathrm{Fields}(\Sigma_1) \arrow[dl, "\chi_1"] \\
 & \mathbf{B}U(1) &
\end{tikzcd}
$$

Because Fields is representable, Fields_p and e^{iS} are monoidal. Therefore, given any monoidal functor (which in physics plays the role of the *path integral*, as will be discussed in Chapter 16)

$$\int : \mathrm{Span}(\infty\mathbf{Gpd}/\mathbf{B}U(1)) \to \mathbf{D},$$

the composition $\int e_p^{iS} : \mathbf{Cob}_{p+1} \to \mathbf{D}$ yields another monoidal functor and, consequently, a quantum theory of p-branes. It happens that generally there is no canonical choice of $\int$, but *this distinguished path integral exists when the classical space of fields* Fields *is* ***orientable*** *and* ***dualizable*** *with respect to some abstract generalized cohomology theory represented by a E_∞-spectrum* $\mathbb{E}$. More precisely, notice that any morphism of ∞-groups $\rho : U(1) \to GL_1^\infty(\mathbb{E})$ induces a morphism $\overline{\rho}$ between the corresponding delooped groupoids and, by composition, we have the dotted arrows in the diagram below[6].

$$
\begin{tikzcd}
 & \mathrm{Fields}(\Sigma) \arrow[dl, "\imath_0"'] \arrow[dr, "\imath_1"] \arrow[ddd, dotted, bend right=60, "\jmath_0"'] \arrow[ddd, dotted, bend left=60, "\jmath_1"] & \\
\mathrm{Fields}(\Sigma_0) \arrow[rr, Rightarrow, "e^{\frac{i}{\hbar}S}"] \arrow[dr, "\chi_0"] \arrow[ddr, dashed, "\overline{\rho}\circ\chi_0"'] & & \mathrm{Fields}(\Sigma_1) \arrow[dl, "\chi_1"'] \arrow[ddl, dashed, "\overline{\rho}\circ\chi_1"] \\
 & \mathbf{B}U(1) \arrow[d, "\overline{\rho}"] & \\
 & \mathbf{B}GL_1^\infty(\mathbb{E}) &
\end{tikzcd}
$$

Now, applying $\overline{\Sigma}^\infty$ in each of these maps, we get morphisms taking values into $\mathbf{B}\overline{\Sigma}^\infty GL_1^\infty(\mathbb{E})$, which in turn give (by the discussion of the last two subsections) $\mathbb{E}$-line ∞-bundles. Let us write E_j (resp. E^j) to denote the ∞-bundles induced by $\imath_j$ (resp. τ^j). Therefore, the diagram above provides the first diagram below. Applying the section functor Γ we then get the second diagram, which belongs to the ∞-category

$$\infty\mathrm{Mod}_{\mathbb{E}}(\mathrm{Stab}(\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J))) \tag{9.4.11}$$

of smooth spectra which are $\mathbb{E}$-modules respectively to the derived smash product $\wedge_{\mathbb{E}}$. We notice that the dotted isomorphism in the first (and, consequently, in the second) diagram comes from

[6] We are writing τ_j, with $j = 0, 1$, in order to denote the compositions $\overline{\rho} \circ \chi_j \circ \mathrm{Fields}(\imath_j)$.

the existence of a 2-morphism $\xi : \jmath_0 \Rightarrow \jmath_1$ which is given essentially by $\overline{\rho} \circ e^{iS}$.

$$
\begin{tikzcd}
E_0 \arrow[r, "\imath^0"] \arrow[dd] & E^0 \simeq E^1 \arrow[d] & E_1 \arrow[l, "\imath^1"'] \arrow[dd] \\
& \mathrm{Fields}(\Sigma) \arrow[dl, "\imath_1"] \arrow[dr, "\imath_0"'] & \\
\mathrm{Fields}(\Sigma_0) & & \mathrm{Fields}(\Sigma_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(E_0) \arrow[r, "\Gamma(\imath^0)"] & \Gamma(E^0) \simeq \Gamma(E^1) \arrow[r, dashed, bend right, "\Gamma(\imath^1)!"'] & \Gamma(E_1) \arrow[l, "\Gamma(\imath^1)"']
\end{tikzcd}
$$

Finally, if $\Gamma(\imath^1)$ is dualizable as a 1-morphism in the ∞-monoidal category (9.4.11) and $\Gamma(E^1)$ is self-dual[7], then the dotted arrow in the second diagram above actually exists and a choice of it allow us to define the quantum theory associated to $(\mathrm{Fields}, e^{iS})$ as the following rule:

$$
\Big(\Sigma_0 \xrightarrow{\Sigma} \Sigma_1\Big) \xmapsto{\quad \int e^{iS} \quad} \Big(\Gamma(E_0) \xrightarrow{\Gamma(\imath^1)! \circ \Gamma(\imath^0)} \Gamma(E_1)\Big)
$$

From the last subsection, this dualizability conditions can be obtained when $\mathrm{Fields}(\Sigma_1)$ and $\mathrm{Fields}(\Sigma)$ are $\mathbb{E}$-Poincaré objects with respect to the maps $\jmath_1$ and $\overline{\rho} \circ \imath_1$. In this case, the classical theory is called *orientable* and *dualizable*.

Conclusion. *For any orientable and dualizable classical theory we can assign a corresponding quantum theory, which act on cobordisms as the space of sections of some line ∞-bundle.*

Remark. At first glance it may seem strange described the states of a quantum object as sections of a line bundle. But, this is a **very natural** fact. Indeed, for the reader knowing Quantum Mechanics, recall that there the quantum states are given by wave functions $\psi(x, t)$ subjected to the Schrödinger equation. So, they are certain complex functions $\psi : M \to \mathbb{C}$ satisfying some regularity condition. It happens that these complex functions are just sections of the trivial complex line bundle $\pi : M \times \mathbb{C} \to M$.

[7] Recall the definitions in Section 8.4 and the discussion in the last subsection.

Chapter 10

Cohesive Theory

In the previous chapters we have seen that the theory of ∞-categories is a natural ambient to develop homotopy theory and stable homotopy theory. In order to attack Hilbert sixth's problem we need a language in which not only homotopy theory, but also geometry can be internalized, i.e, we need a background language which models *homotopical geometry.* The main objective of this chapter is to present some candidates to these desired languages.

Up to this point, the most prominent ∞-categories that we worked are the ∞-*topos*. On the other hand, classical geometry is developed internal to sub-∞-categories of $\infty\mathbf{Gpd}$, as $\mathbf{Diff}$. Therefore, it is natural to expect that homotopical geometry should be described in certain classes of ∞-topos strongly related to $\infty\mathbf{Gpd}$.

We start this chapter in Section 10.1 by discussing the notion of "relation between two ∞-topos" and we show that any ∞-topos can naturally related with $\infty\mathbf{Gpd}$. This notion of "relation" is essentially given by an adjunction, so that we define a "strong relation" between ∞-topos as an adjunction such that the functors have themselves other adjuncts. We then introduce three classes of ∞-topos strongly related with $\infty\mathbf{Gpd}$, called *geometric ∞-topos*, and we explore some immediate examples. We also give a criterion to identify if a ∞-topos is geometric by looking at properties satisfied by the underlying ∞-site. The standard example of geometric ∞-topos fulfilling these properties is $\infty\mathrm{Stack}(\mathbf{Diff}, J)$.

In Section 10.2 we try to convince the reader that the language of geometric ∞-topos is really a natural candidate to model homotopical geometry. There we see, for instance, that internal to any ∞-topos $\mathbf{H}$ which is *cohesive* (in the sense that to every object $X \in \mathbf{H}$ we can associate *modalities* or *geometric homotopy types*) we have a canonical way to build abstract Whitehead towers from certain "geometric Postnikov tower" in the same way as we can obtain the classical Whitehead tower of a topological space from the underlying classical Postinikov tower. This will allow us to prove that if we start with a Lie group, then the nth stage in the Whitehead tower will be a Lie n-group.

We also show that in a cohesive ∞-topos every cohomology admits a "differential refinement", allowing us to discuss an abstract version of de Rham cohomology, which reduces to the classical nth de Rham cohomology in the ∞-topos of smooth ∞-stacks, with coefficients in $\mathbf{B}^n U(1)$. We prove that if $\mathbf{H}$ is cohesive, then its stabilization $\mathrm{Stab}(\mathbf{H})$ is also cohesive and we show that any Ω-spectrum object $\mathbb{E}$ fits in the middle of a "differential cohomology hexagon", constituted by the different modalities of $\mathbb{E}$. It is also proved that if $\mathbb{E}$ is the spectrification of $\mathbf{B}^n U(1)$, then we recover the ordinary differential cohomology hexagon given by the Deligne complex $\mathcal{D}^{n+1}$.

The chapter ends in Section 10.3. There we discuss that "cohesion" is not enough to model homotopical **differential** geometry. From the physical viewpoint this means that in an arbitrary cohesive ∞-topos the notions of "Euler-Lagrange equation" and "space of minimal solutions" are not well defined. From the mathematical viewpoint, we do not have, for instance, a canonical notion of "∞-algebroid" associated to an internal ∞-groupoid. In order to correct these problems we introduce the class of *differential cohesive ∞-topos.*

10.1 Geometry

Recall that a ∞-category $\mathbf{H}$ is a ∞-topos if, and only if, it is the localization of some category of ∞-functors. In other words, if, and only if, s there exists a ∞-category $\mathbf{C}$ together with an adjunction

$$\mathbf{H} \underset{\imath}{\overset{\mathscr{L}_\infty}{\rightleftarrows}} \infty\mathrm{Func}(\mathbf{C}, \infty\mathbf{Gpd}),$$

such that $\imath$ is an embedding and $\mathscr{L}_\infty$ preserves finite ∞-limits. Notice that, following this characterization , the entire category of ∞-functor can itself be regarded as a trivial ∞-topos for which both $\mathscr{L}_\infty$ and $\imath$ are the identity functors.

This motivates us to define a *morphism* between two ∞-topos $\mathbf{H}$ and $\mathbf{H'}$ as a pair of adjoint ∞-functors $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H'} : \mathscr{G}$. When the ∞-functor $\mathscr{G}$ preserve finite ∞-limits, it is useful to say that the morphism is *geometric*. If in addition $\imath$ is an embedding, we say that we have a *geometric embedding*. Therefore, from the paragraph above, any ∞-topos $\mathbf{C}$ can be geometrically embedded into the trivial ∞-topos.

Remark. Because we are working in the context of ∞-topos, a ∞-functor $\mathbf{H} \to \mathbf{H'}$ has a left/right adjoint iff it preserves all ∞-limits/∞-colimits (Proposition ?? of [127]). Therefore, in order to give a morphism $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H'} : \mathscr{G}$ it is necessary and sufficient to specify some of the following equivalent data:

1. a ∞-functor $\mathscr{F} : \mathbf{H} \to \mathbf{H'}$ preserving ∞-limits;
2. a ∞-functor $\mathscr{G} : \mathbf{H'} \to \mathbf{H}$ preserving ∞-colimits.

Furthermore, given ∞-functors $\mathscr{F}$ and $\mathscr{G}$ as above, they define the same morphism iff they become equipped with canonical natural transformations $\mathscr{G} \circ \mathscr{F} \Rightarrow id_{\mathbf{H}}$ and $id_{\mathbf{H'}} \Rightarrow \mathscr{F} \circ \mathscr{G}$ fulfilling the triangle identities up to higher morphisms.

We can now define a category $\infty\mathbf{Topos}$, whose objects are ∞-topos and whose morphisms are those characterized above. Important examples of morphisms to keep in mind are the following:

Example 10.1 (*terminal geometric morphism*)**.** The (covariant) hom ∞-functors are the most natural ∞-limit preserving ∞-functors. So, for any object X in a ∞-topos $\mathbf{H}$ we have a ∞-topos morphism $\mathrm{Mor}_{\mathbf{C}}(X, -) : \mathbf{H} \rightleftharpoons \infty\mathbf{Gpd} : \mathscr{G}$. Following Proposition 2.2.5, p. 223 of [182] we assert that, for $X = *$ (where $*$ is a terminal object, whose existence is ensured by the higher Giraud's axioms), the corresponding $\mathscr{G}$ is precisely the ∞-stacktification of the ∞-functor cst_{-}, assigning to any ∞-groupoid G the ∞-functor cst_G, constant at G. Indeed,

$$
\begin{aligned}
\mathrm{Mor}_{\mathbf{H}}(\mathscr{L}_\infty(\mathrm{cst}_G);X) &\simeq \infty\mathrm{Nat}(\mathrm{cst}_G;\imath(X))\\
(*) &\simeq \infty\mathrm{Nat}(\mathrm{cst}_{\mathrm{colim}_G\mathbf{1}};\imath(X))\\
&\simeq \infty\mathrm{Nat}(\mathrm{colim}_G\,\mathrm{cst}_{\mathbf{1}};\imath(X))\\
&\simeq \lim_G \infty\mathrm{Nat}(\mathrm{cst}_{\mathbf{1}};\imath(X))\\
&\simeq \lim_G \mathrm{Mor}_{\mathbf{H}}(\mathscr{L}_\infty(\mathrm{cst}_{\mathbf{1}});X)\\
(**) &\simeq \lim_G \mathrm{Mor}_{\mathbf{H}}(*;X)\\
&\simeq \lim_G \mathrm{Mor}_{\infty\mathbf{Gpd}}(\mathbf{1};\mathrm{Mor}_{\mathbf{H}}(*;X))\\
&\simeq \mathrm{Mor}_{\infty\mathbf{Gpd}}(\mathrm{colim}_G\mathbf{1};\mathrm{Mor}_{\mathbf{H}}(*;X))\\
(***) &\simeq \mathrm{Mor}_{\infty\mathbf{Gpd}}(G;\mathrm{Mor}_{\mathbf{H}}(*;X)),
\end{aligned}
$$

where in $(**)$ we used that the ∞-stacktification colimits (in particular, terminal objects), while in $(*)$ and $(***)$ we used that any ∞-groupoid G can be recovered as the ∞-colimit of the ∞-functor $G \to \infty\mathbf{Gpd}$ constant at the terminal object $\mathbf{1}$. A concrete case is the following:

1. *global section functor*. For a given topological space X, let us consider $\mathbf{H}$ as the 1-topos of sheaves on the site $(\mathrm{Open}(X), J)$, whose objects are open sets $U \subseteq X$ and whose coverings $\pi : \overline{U} \to U$ are induced by usual open coverings $U_i \hookrightarrow U$. In this case, the terminal object is just X, so that by Yoneda lemma the terminal object morphism is given by evaluation at X. It happens that there is a canonical class of sheaves on $(\mathrm{Open}(X), J)$: the sheaves of sections of bundles $E \to X$ over X. For them, "evaluating at X" is just the as considering the set of global sections $\Gamma(X;E)$ of the corresponding bundle $E \to X$. With this in mind, we also call the terminal object morphism of the "global section morphism", writing Γ in order to denote $\mathrm{Mor}_{\mathbf{H}}(*,-)$.

2. *discrete topology functor*. Let us consider $\mathbf{H} = \infty\mathbf{Gpd}$. Any set X can be reconstructed as the set of its elements, i.e, as the set of maps $1 \to X$, where 1 is a terminal object in **Set**. Analogously, any ∞-groupoid G is equivalent to the ∞-groupoid of ∞-functors $\mathbf{1} \to G$, so that $\mathrm{Mor}(\mathbf{1};G) \simeq G$. Similarly we show that $\mathrm{cst}_G \simeq G$. Therefore, *for the ∞-topos of ∞-groupoids the terminal object morphism is just the identity morphism.* By making use of the Homotopy Hypothesis we then identify the terminal object morphism for the case $\mathbf{H} = \mathbf{Top}$: it is given by $\Pi \rightleftharpoons |\cdot|$, where Π is the fundamental ∞-groupoid ∞-functor and $|\cdot|$ is the geometric realization ∞-functor. Notice that if G is a usual 1-group, then, by the definition of geometric realization, $|G|$ is just G regarded as a **discrete** topological space. Because of this, in the literature it is also usual to denote the ∞-stacktification $\mathscr{L}_\infty\mathrm{cst}_-$ of the constant functor by Disc, saying that it is the *discrete structure functor*.

Example 10.2 (*direct/inverse image*)**.** In the context of sheaves on a fixed space X there is another important class of examples of geometric morphisms. Indeed, any open map $f : X \to Y$ induces a pair of functors $f^* : \mathrm{Shv}(Y) \rightleftharpoons \mathrm{Shv}(X) : f_*$, respectively called *inverse image* and *direct image* functors, defined by

$$
f_*(G)(U) = G(f^{-1}(U)) \quad \text{and} \quad f^*(F)(V) = F(f(V)). \tag{10.1.1}
$$

We assert that they are adjoint:

$$\mathrm{Mor}_{\mathrm{Shv}(X)}(f^*(F);G) \simeq \mathrm{Mor}_{\mathrm{Shv}(Y)}(F;f_*(G)), \tag{10.1.2}$$

so that they really define a morphism between ∞-topos. Indeed, this follows from a direct inspection: a transformation $\xi : f^*(F) \Rightarrow G$ is a rule assigning to any open $V \subset X$ a map $\xi_V : F(f(V)) \to G(V)$, while a transformation $\varphi : F \Rightarrow f_*(G)$ is a family $\varphi_U : F(U) \to G(f^{-1}(U))$ parametrized by open sets $U \subset Y$. Notice that for $V = f^{-1}(U)$ both data coincide. This means that we have indeed an equality of sets instead only a bijection as required in (10.1.2)!

Remark. We have a morphism $f^* \rightleftharpoons f_*$ even if f is not open, but an arbitrary continuous map. In the construction above, we used that f is open in (10.1.1) in order to define f^*. Therefore, in the general case f^* need to be modified. Indeed, if f is only continuous, then for a given open set V, its image $f(V)$ need not to be open, but even so it can be "approximated" by open sets. There are two ways to do this approximation: by open sets $U \subset f(V)$ contained in $f(V)$ or by open sets $U \supset f(V)$ containing $f(V)$. In the first case the "best" approximation would be obtained taking the limit over U, while in the second it would be obtained from the colimit. Notice that we are searching for a f^* which is a left adjoint to f_*, so that it will appear in the first argument of $\mathrm{Mor}_{\mathrm{Shv}(X)}(-,-)$, which is well behaved with respect to colimits. This motivates us to define

$$f^*(F)(V) := \mathrm{colim}_U F(U).$$

A direct verification shows that in the general situation it really is a left adjoint for f_*.

Geometric ∞-Topos

In Example 10.1 above we have seen that any ∞-topos $\mathbf{H}$ becomes equipped with a nontrivial geometric morphism with values in $\infty\mathbf{Gpd}$ (the *terminal geometric morphism*), meaning that each ∞-topos is at least **minimally** related with $\infty\mathbf{Gpd}$. Here we will study some classes of ∞-topos which are **strongly** related with $\infty\mathbf{Gpd}$.

The reason of this interest is clear: the ∞-topos of ∞-groupoids constitute a very interesting **concrete** language, because many concepts and results make sense internal to it. For instance, from the Homotopy Hypothesis (discussed in Section 8.4), we have $\infty\mathbf{Gpd} \simeq \mathbf{Top}$, so that in this ∞-topos there is a nice homotopy theory, in which the (weak) homotopy type can be characterized by canonical invariants (the homotopy groups). Furthermore, it contains **Diff** as a sub-∞-category, showing that we can do differential geometry internal to it. Therefore, when studying ∞-topos $\mathbf{H}$ strongly related with $\infty\mathbf{Gpd}$ we expect to get a abstract powerful language in which many concepts and results that make sense internal to $\infty\mathbf{Gpd}$ can be defined **synthetically** in $\mathbf{H}$.

So, a natural question is the following: *how can we describe a "strong relation" between two ∞-topos $\mathbf{H}$ and $\mathbf{H}'$?* A relation between them is given by a (geometric) morphism, so that a "strong relation" should be a (geometric) morphism fulfilling additional conditions. Indeed, we say that a geometric morphism $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H}' : \mathscr{G}$ is *locally local* when the functor $\mathscr{F} : \mathbf{H} \to \mathbf{H}'$ has not only a left adjoint $\mathscr{G}$, but also a right adjoint $\overline{\mathscr{G}}$. Similarly, we say that it is *locally ∞-connected* when $\mathscr{G}$ has not only a right adjoint $\mathscr{F}$, but also a right adjoint $\overline{\mathscr{F}}$, as below.

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\mathscr{F}" description] & \mathbf{H}' \arrow[l, shift right=2ex, "\mathscr{G}"'] \arrow[l, shift left=2ex, "\overline{\mathscr{G}}"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H}' \arrow[r, "\mathscr{G}" description] & \mathbf{H} \arrow[l, shift right=2ex, "\overline{\mathscr{F}}"'] \arrow[l, shift left=2ex, "\mathscr{F}"] \end{tikzcd}$$

A geometric morphism which is simultaneously *locally local* and *locally ∞-connected* is called *locally cohesive*. This kind of relation between two ∞-topos is specially interesting because we can glue the diagrams above, producing the first diagram below. So, we see that any locally cohesive geometric morphism $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H}' : \mathscr{G}$ induces three distinguished endomorphisms of $\mathbf{H}$, usually called *modalities*, which pass in $\mathbf{H}'$ (see the second diagram below) meaning that they can be used to internalize in $\mathbf{H}$ concepts which a priori make sense only in $\mathbf{H}'$.

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\mathscr{F}" description] & \mathbf{H}' \arrow[l, shift right=2, "\mathscr{G}"'] \arrow[l, shift left=2, "\overline{\mathscr{G}}"] \arrow[r, "\mathscr{G}" description] & \mathbf{H} \arrow[l, shift right=2, "\overline{\mathscr{F}}"'] \arrow[l, shift left=2, "\mathscr{F}"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, "\mathscr{G}\circ\mathscr{F}" description] & \mathbf{H} \arrow[l, shift right=2, "\mathscr{G}\circ\overline{\mathscr{F}}"'] \arrow[l, shift left=2, "\overline{\mathscr{G}}\circ\mathscr{F}"] \end{tikzcd}$$

Remark. By the discussion in the last subsection, a morphism $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H}' : \mathscr{G}$ is locally local iff $\mathscr{F}$ preserves not only ∞-limits, but indeed ∞-colimits. Similarly, it is locally ∞-connected when $\mathscr{G}$ preserve ∞-limits. So, it is locally cohesive when both ∞-functors $\mathscr{F}$ and $\mathscr{G}$ preserve ∞-limits and ∞-colimits.

For a given $\mathbf{H}$, the definitions above can be particularized to the terminal geometric morphism $\Gamma : \mathbf{H} \rightleftharpoons \infty\mathbf{Gpd} : \mathrm{Disc}$, giving classes of ∞-topos strongly related with $\infty\mathbf{Gpd}$, as desired. Indeed, we say that $\mathbf{H}$ is *locally local*, *locally ∞-connected* or *locally cohesive* when the corresponding terminal geometric morphism has these respective properties. In this case, we usually write coDisc and Π instead of $\overline{\mathrm{Disc}}$ and $\overline{\Gamma}$. We say that $\mathrm{coDisc}(G)$ is the *codiscrete structure* of a ∞-groupoid and that $\Pi(X)$ is the *geometric fundamental ∞-groupoid* of $X \in \mathbf{H}$.

In the case of a locally cohesive ∞-topos, the modalities of the terminal geometric morphism also get new notations, as below, and new names: $\mathbf{\Pi}$ is the *shape modality*, $\flat$ is the *flat modality* and $\sharp$ is the *sharp modality*.

$$\begin{tikzcd} \mathbf{H} \arrow[r, "\mathrm{Disc}\circ\Gamma" description] & \mathbf{H} \arrow[l, shift right=2, "\mathrm{Disc}\circ\Pi"'] \arrow[l, shift left=2, "\mathrm{coDisc}\circ\Gamma"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H} \arrow[r, "\flat" description] & \mathbf{H} \arrow[l, shift right=2, "\mathbf{\Pi}"'] \arrow[l, shift left=2, "\sharp"] \end{tikzcd}$$

Let us present some examples.

Example 10.3 (*trivial case*)**.** As discussed in Example 10.1.2, in the case $\mathbf{H} = \infty\mathbf{Gpd}$ the terminal object geometric morphism $\Gamma \rightleftharpoons \mathrm{Disc}$ is just the identity morphism, so that both Γ and Disc have left/right adjoints, also given by the identity functor. Consequently, the ∞-topos of ∞-groupoids is trivially locally cohesive.

$$\begin{tikzcd} \infty\mathbf{Gpd} \arrow[r, shift left=3, "id"] \arrow[r, shift right=1, "id" description] & \infty\mathbf{Gpd} \arrow[l, shift right=1, "id" description] \arrow[l, shift left=3, "id"] \end{tikzcd}$$

Example 10.4 (*transitive structures*)**.** The "strong relation" is transitive. More precisely, if two morphisms $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H}' : \mathscr{G}$ and $\mathscr{G} : \mathbf{H}' \rightleftharpoons \mathbf{H}'' : \mathscr{H}$ are locally ∞-connected, locally local or locally cohesive, then the composition between them is cohesive too. This can be verified directly (as done in the diagram below for the case of cohesion) or indirectly (by making use of the last remark). As a consequence, the classes of locally ∞-connected, locally local and locally cohesive

∞-topos define subcategories of ∞**Topos**.

$$\begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, shift left=3, "\overline{\mathscr{F}}"] & \mathbf{H}' \arrow[l, shift left=1, "\mathscr{G}"] \arrow[r, shift left=3, "\overline{\mathscr{G}}"] \arrow[l, shift left=3, "\overline{\mathscr{G}}"] & \mathbf{H}'' \arrow[l, shift left=1, "\mathscr{H}"] \arrow[l, shift left=3, "\overline{\mathscr{H}}"] \end{tikzcd} \quad \begin{tikzcd} \mathbf{H} \arrow[r, shift left=1, "\mathscr{F}"'] & \mathbf{H}' \arrow[r, shift left=1, "\mathscr{G}"'] & \mathbf{H}'' \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, shift left=3, "\overline{\mathscr{G}}\circ\overline{\mathscr{F}}"] \arrow[r, shift right=1, "\mathscr{G}\circ\mathscr{F}"'] & \mathbf{H}'' \arrow[l, shift right=1, "\mathscr{G}\circ\mathscr{H}"'] \arrow[l, shift left=3, "\overline{\mathscr{G}}\circ\overline{\mathscr{H}}"] \end{tikzcd}$$

These subcategories are "stable under isomorphisms" in the sense that if a ∞-topos $\mathbf{H}$ is isomorphic (as a ∞-topos!) to a geometric ∞-topos $\mathbf{H}'$, then it is geometric too. Indeed, if a morphism $\mathscr{F} : \mathbf{H} \rightleftharpoons \mathbf{H}' : \mathscr{G}$ is an isomorphism in ∞**Topos**, then both $\mathscr{F}$ and $\mathscr{G}$ are equivalences in ∞**Cat**, so that the starting morphism is indeed locally cohesive. The result then follows from the last result, as explained in the diagram below. In particular, if $\mathscr{F} : \mathbf{H} \to \mathbf{H}'$ is an equivalence of ∞-categories and $\mathbf{H}'$ is geometric, then $\mathbf{H}$ is too. Consequently, by the last example and by the Homotopy Hypothesis it follows that the ∞-topos **Top** is locally cohesive.

$$\begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, shift left=3, "\mathscr{G}^{-1}"] \arrow[r, shift right=1, "\mathscr{F}"'] & \mathbf{H}' \arrow[l, shift right=1, "\mathscr{G}"'] \arrow[l, shift left=3, "\mathscr{F}^{-1}"] \arrow[r, shift left=3, "\Pi"] \arrow[r, shift right=1, "\Gamma"'] & \infty\mathbf{Gpd} \arrow[l, shift right=1, "\mathrm{Disc}"'] \arrow[l, shift left=3, "\mathrm{CoDisc}"] \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, shift left=3, "\Pi\circ\mathscr{G}^{-1}"] \arrow[r, shift right=1, "\Gamma\circ\mathscr{F}"'] & \infty\mathbf{Gpd} \arrow[l, shift right=1, "\mathscr{G}\circ\mathrm{Disc}"'] \arrow[l, shift left=3, "\mathscr{F}^{-1}\circ\mathrm{CoDisc}"] \end{tikzcd}$$

Example 10.5 (*functor ∞-topos*)**.** Recall that, as discussed in Example ??, if a ∞-category $\mathbf{H}$ is an ∞-topos, then for any other ∞-category $\mathbf{C}$ (with initial object), the corresponding ∞-category $\infty\mathrm{Func}(\mathbf{C};\mathbf{H})$ of ∞-functors is also a ∞-topos. Let us prove that if the starting category is in addition geometric, then the category of ∞-functors is geometric too. As a particular case, it will follows that *the stable ∞-category of a geometric ∞-topos is also a geometric ∞-topos*. Let us assume $\mathbf{C}$ not only with initial object $\varnothing$, but also with terminal object $*$. They can be identified with ∞-functors $\imath : \mathbf{1} \to \mathbf{H}$ and $\jmath : \mathbf{1} \to \mathbf{H}$, which are respectively the left and right adjoints to the unique $c : \mathbf{H} \to \mathbf{1}$. In turn, these functors induce ∞-functors

$$\imath^*, \jmath^* : \infty\mathrm{Func}(\mathbf{C};\mathbf{H}) \to \infty\mathrm{Func}(\mathbf{1};\mathbf{H}) \simeq \mathbf{H}$$

which are left and right adjoints to the corresponding c^*. Because $\mathbf{H}$ is ∞-complete and ∞-cocomplete, any ∞-functor $F : \mathbf{A} \to \mathbf{H}$, defined in any ∞-category $\mathbf{A}$, has both left and right ∞-Kan extensions respectively to any $\mathbf{A} \to \mathbf{D}$. Consequently, $\imath^*$ and $\jmath^*$ have both left and right adjoints. This means that $\infty\mathrm{Func}(\mathbf{C};\mathbf{H})$ is locally cohesive respectively to $\mathbf{H}$, as in the diagram below[1]. Consequently, because $\mathbf{H}$ is locally cohesive, the result follows from the last example.

$$\begin{tikzcd}[column sep=huge] \infty\mathrm{Func}(\mathbf{C};\mathbf{H}) \arrow[r, shift left=3, "\jmath^*"] \arrow[r, shift right=1, "\imath^*"'] & \mathbf{H} \arrow[l, shift right=1, "c^*"'] \arrow[l, shift left=3, "\imath_*"] \end{tikzcd}$$

Geometric ∞-Site

Remembering that a ∞-topos $\mathbf{H}$ is a ∞-category of ∞-stacks over some ∞-site $(\mathbf{C}, J)$, imposing conditions on the ∞-site we expect to get ∞-topos with more properties. With this in mind we can try to search for conditions under which the ∞-topos becomes geometric. This strategy is fruitful, as exemplified by the proposition below.

[1] Here, $\imath_*$ denotes the left adjoint of $\imath^*$.

Proposition 10.1. *Let $(\mathbf{C}, J)$ be a ∞-site with finite coproducts and a terminal object. Given any covering $\pi : U \to X$ in $J(x)$, let us consider the following conditions on the corresponding Cech ∞-groupoid:*

1. *for any k, the collection of k-morphisms can be written as a coproduct of (the image of) representable functors;*
2. *the geometric realization of* $\operatorname{colim} \check{C}_\infty(U)$ *is (weakly) contractible;*
3. *the geometric realization of the canonical morphism below is a (weak) homotopy equivalence:*

$$\mathrm{Mor}_{\mathbf{H}}(*, \check{C}_\infty(U)) \to \mathrm{Mor}_{\mathbf{H}}(*, X).$$

Then, if (1.) is satisfied, the underlying ∞-topos of ∞-stacks is ∞-local. Furthermore, if (2.) and (3.) are satisfied we get a ∞-connected ∞-topos. Finally, if all conditions are satisfied together, we obtain a cohesive ∞-topos.

Proof. This is essentially Propositions ??, ?? and ?? of [182]. There, the proof is given by making use of the presentation of ∞-categories as quasi-categories, i.e, in the same spirit of Lurie's work. □

Before giving examples of geometric ∞-topos that can be obtained from this proposition, let us present two remarks:

1. *the conditions (1.)-(3.) are **not** necessary in order to have a cohesive ∞-topos*. Indeed, for a given topological space X, let $\mathrm{Shv}(X)$ be topos of sheaves on X, trivially regarded as a ∞-topos. As reviewed in Example 10.1, it is the ∞-topos of sheaves on the site $(\mathrm{Open}(X), J)$, whose objects are open sets $U \subseteq X$ and whose coverings $\pi : \overline{U} \to U$ are induced by usual open coverings $U_i \hookrightarrow U$. The kth term in the Cech ∞-groupoid $\check{C}_\infty(U_i)$ is given by coproducts of intersections $\sqcup_{i_1,...,i_k} U_{i_1} \cap ... \cap U_{i_k}$. Each topological space Y can regarded as the set of morphisms from the trivial space, i.e, $Y \simeq \mathrm{Mor}_{\mathbf{Top}}(*, Y)$, so that the Cech ∞-groupoid of any covering is degreewise the coproduct of representable objects. This means that the condition (1.) of the last proposition is satisfied and, therefore, $\mathrm{Shv}(X)$ is a ∞-local ∞-topos. Notice, however, that the conditions (2.) and (3.) are not satisfied by arbitrary coverings, so that we cannot use Proposition 10.1 to conclude that $\mathrm{Shv}(X)$ is cohesive. But we can verify directly that it really is cohesive. Indeed, the codiscrete functor coDisc, i.e, the right adjoint to Γ, assign to any ∞-groupoid G the set of objects G_0, regarded as a codiscrete topological space. Furthermore, the geometric fundamental ∞-groupoid $\Pi(X)$ is the set $\pi_0(X)$ of path components of X, trivially regarded as a ∞-groupoid.
2. *nerve theorem on good open coverings*. On the other hand, if X is paracompact, then the ∞-site $(\mathrm{Open}(X), J)$ contains a sub-∞-site in which the conditions (2.) and (3.) are satisfied. It is obtained by restricting to the full sub-∞-category $\mathrm{Good}(X) \subset \mathrm{Open}(X)$ of contractible open subsets, endowed with the topology $J' \subset J$ induced by open coverings $U_i \hookrightarrow U$ such that each U_i and each intersection $U_{i_1} \cap ... \cap U_{i_k}$, for every k, is contractible (we usually say these are *good open coverings*). Now, that (2.) and (3.) are satisfied follows directly from the *Nerve theorem* (see Section 4.F of [91]), which states that, *if Y is paracompact, then*

for any good open covering $U_i \hookrightarrow Y$ we have a weak homotopy equivalence $|\check{C}_\infty(U_i)| \simeq Y$. Indeed, in order to get (2.) notice that, because we assumed X paracompact, each open subset $U \subset X$ is also paracompact. Therefore, for any $\pi : U_i \hookrightarrow U$ in $J'(u)$ the Nerve theorem applies. But U is contractible, so that $|\check{C}_\infty(U_i)| \simeq *$. Condition (3.) comes from analogous argument and from the fact that $\mathrm{Mor}_{\mathbf{Top}}(*, Y) \simeq Y$ for any topological space Y.

Now, let us use Proposition 10.1 to get examples of geometric ∞-topos.

Example 10.6 (*trivial cases*)**.** In the last subsection we proved that **Top** and ∞**Gpd** are cohesive ∞-topos without appealing to the proposition. Let us see, on the other hand, that this proposition could also be used to get the same conclusion. From Example 8.8, ∞**Gpd** can be regarded as the ∞-category of ∞-stacks on the trivial ∞-site $(\mathbf{1}, J)$, where $\mathbf{1}$ is the ∞-category with only one object and whose k-morphisms are identities. It immediately satisfy all conditions of the last proposition, implying that ∞**Gpd** is indeed a cohesive ∞-topos. From the Homotopy hypothesis (discussed in Section 8.4), we get that the ∞-topos of topological spaces, continuous maps, etc., is also cohesive.

Example 10.7 (*∞-topos of smooth ∞-stacks*)**.** Let $\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J)$ be the ∞-topos of smooth ∞-stacks. The coverings $\pi : U \to X$ in $J(x)$ are the induced maps $\pi : \sqcup_i U_i \to X$ for open covering $U_i \hookrightarrow X$. Therefore, the kth term in the Cech ∞-groupoid $\check{C}_\infty(U_i)$ is given by coproducts of intersections $\sqcup_{i_1,...,i_k} U_{i_1} \cap ... \cap U_{i_k}$ and, by argument analogous to that given in remark (1.) above, the ∞-topos of smooth ∞-stacks is ∞-local. Furthermore, exactly as happened for $\mathrm{Shv}(X)$, conditions (2.) and (3.) may not be satisfied by arbitrary coverings, so that *we cannot use directly Proposition 10.1 to conclude that* $\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J)$ *is cohesive*. On the other hand, in remark (2.) above we showed that we can obtain a sub-∞-site of $(\mathrm{Open}(X), J)$ in which (2.) and (3.) are satisfied by restricting to the paracompact contractible objects $U \in \mathrm{Open}(X)$ and to the good open coverings. Therefore, we get the same conclusion when restricting to paracompact contractible manifolds and good open coverings. More precisely, from the Nerve theorem we conclude that the ∞-topos $\infty\mathrm{Stack}(\mathbf{Diff}^{\mathrm{con}}_{sub}, J')$, where $\mathbf{Diff}^{\mathrm{con}}_{sub}$ is the ∞-category of contractible paracompact smooth manifolds and smooth submersions, and $J' \subset J$ is the sub-∞-site of good open coverings, is cohesive. Despite these similarities, there is a primary difference between $\mathbf{Diff}_{sub}$ and $\mathrm{Open}(X)$: here we are not working with arbitrary topological spaces, but indeed with **manifolds**. It happens that smooth manifolds are locally equivalent to $\mathbb{R}^n$, i.e, our starting category $\mathbf{Diff}_{sub}$ is generated by the particular dense subcategory $\mathbf{CartSp}_{sub} \subset \mathbf{Diff}^{\mathrm{con}}_{sub}$ whose objects are cartesian spaces and whose morphisms are smooth submersions $\mathbb{R}^n \to \mathbb{R}^k$. Therefore,

$$\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J) \simeq \infty\mathrm{Stack}(\mathbf{Cart}_{sub}, J')$$

implying that the ∞-topos of smooth ∞-stacks is cohesive.

Physics

As commented in Section 8.2 (and as will be discussed with more details in the next chapters), the ∞-topos $\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J)$ of smooth ∞-stacks produce a language which abstract enough to axiomatize classical physics for **bosonic** objects. Furthermore, in Section 9.4 we sketched how we can formulate a quantization process for systems described by pairs $(\mathrm{Fields}, e^{\frac{i}{\hbar}S})$ in $\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J)$.

Since the discovering of spinning particles, we need to describe physical systems containing not only bosonic, but also **fermionic** degrees of freedom. As commented in Sections 5.1, 5.2 and 8.4, in the quantum level, we can distinguish bosons from fermions by making use of Spin Statistics and Pauli's exclusion principle, which state that *bosonic states are totally symmetric and fermionic states are totally antisymmetric*. In other words, if $\psi_1, ..., \psi_n$ is a family of states, then they describe bosons or fermions exactly when $[\psi_i, \psi_j] = 0$ or $[\psi_i, \psi_j] = -[\psi_j, \psi_i]$.

It happens that, as will become more clear in the next part of the text, the ∞-topos of smooth ∞-stacks is **not** abstract enough to describe the usual classical theories for fermions! Therefore, we are led to a terrible conclusion: *all our work in searching for a useful language to attack Hilbert's sixth problem was for nothing!* Happily, this is not the case. Indeed, the essential problem with smooth ∞-stacks is that they are build over manifolds, but locally manifolds are described by coordinates $(x_1, ..., x_n)$ in $\mathbb{R}^n$, which totally commute (i.e, $[x_i, x_j] = 0$) and, therefore, describe only bosonic systems. So, the immediate idea is to work with entities which are not modelled over $\mathbb{R}^n$, but over $\mathbb{R}^n \otimes \Lambda(\mathbb{R}^m)$, meaning that locally there are commuting coordinates $(x_1, ..., x_n)$ and anticommuting coordinates $(\theta_1, ..., \theta_m)$. These are the *supermanifolds*. We have the ∞-topos

$$\infty\mathrm{Stack}(\mathbf{SuperDiff}_{sub}, J)$$

of super-smooth ∞-stacks and all that were discussed for smooth ∞-stacks remain valid in this new context. Furthermore, as will be seen in the next chapters, every usual classical theories for bosons and/or fermions can be described in this new ∞-topos.

If we are taking Hilbert's sixth problem seriously, we should consider the scare we take above as a warning: when choicing a proper language to axiomatize the physics we have to take into account that new physical discovers will be obtained in future, which may impact directly the choiced language, as the discovering of the spin have impacted the ∞-topos of smooth ∞-stacks. So, we need to look for languages which produce a "stable" axiomatization for physics in the sense that new physical discovers will not impact directly the axiomatization. This imply working *synthetically*.

What are the natural class of languages internal to which the physical theories have a synthetic axiomatization? Well, certainly it must contain the ∞-topos of (super) smooth ∞-stacks as a particular example, because it axiomatize (fermionic and) bosonic classical physics and quantization. On the other hand, this ∞-topos is build over the ∞-site $(\mathbf{Diff}_{sub}, J)$ and, by the Homotopy Hypothesis, $\mathbf{Diff}_{sub}$ embeds into $\infty\mathbf{Gpd}$. This suggest to consider languages given by ∞-topos strongly related with $\infty\mathbf{Gpd}$, as the classes of ∞-local, ∞-connected and cohesive ∞-topos. From Example 10.4 we known that $\infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J)$ is cohesive. Therefore, joining all these facts we are led to conclude that *the cohesive ∞-topos constitute a natural class of languages to formulate a synthetic axiomatization of physics.*

10.2 Applications

In the last section we introduced the classes of ∞-local, ∞-connected and cohesive ∞-topos as models for ∞-topos "strongly related" with $\infty\mathbf{Gpd}$, in the desire to get abstract languages in which concepts proper of $\infty\mathbf{Gpd}$ acquire a synthetic formulation. This really can be done:

1. Whitehead towers. As discussed in Section 8.3, in **Top** we can build Whitehead towers from Postnikov towers. This could be done thanks to the fact that the homotopy type

of CW-complexes is determined by their homotopy groups. We will see that in any ∞-connected ∞-topos $\mathbf{H}$ we can associate to every X a sequence of higher groupoids $\mathbf{\Pi}_n(X)$, taking the role of the fundamental n-groupoids of a topological space. This sequence fits into a "geometric Postnikov tower of X" allowing us to build a Whitehead tower of X in analogously as in **Top**.

2. *de Rham cohomology*. In the category **Diff** of smooth manifolds we have a canonical cohomology theory: the de Rham cohomology. We will prove that for any object A in a cohesive ∞-topos $\mathbf{H}$ induce another $\flat_{dR}A$ such that the abstract cohomology of X with coefficients in $\flat_{dR}A$ behaves as "abstract de Rham cohomology" of X.

3. *connections on ∞-bundles*. Despite de Rham cohomology, there is a more refined cohomology that can be considered in **Diff**: the Deligne cohomology. We will see that not only de Rham, but also Deligne cohomology can be internalized into any cohesive ∞-topos. This will allow us to give a synthetic formulation for the geometric notion of "connection on a G-bundle".

4. *differential cohomology*. Recall from the discussion in Section 1.2 that Deligne cohomology is a model for ordinary differential cohomology. Therefore, it is natural to expect that the synthetic formulation for "connection on G-bundle" gives rise to a synthetic formulation for "differential cohomology". We will show that this is really the case. More precisely, we will show that that any Ω-spectrum object in a ohesive ∞-topos fits in the middle of a hexagonal diagram analogous that characterizes ordinary differential cohomology.

For more details on these examples and for other structures arising in geometric ∞-topos, see [182].

Whitehead Tower

Let $\mathbf{H}$ be a ∞-connected ∞-topos. By definition, this means that the discrete ∞-functor

$$\mathrm{Disc} : \infty\mathbf{Gpd} \to \mathbf{H} \quad \text{has a left adjoint} \quad \Pi : \infty\mathbf{Gpd} \to \mathbf{H},$$

allowing us to consider the shape modality $\int : \mathbf{H} \to \mathbf{H}$, given by the composition $\mathrm{Disc} \circ \Pi$. Furthermore, recall that we have the truncation functors

$$\tau_k : (\infty, 1)\mathbf{Cat} \to k\mathbf{Cat}.$$

For a given object X, we define its *geometric homotopy n-type* X, denoted by $\mathbf{\Pi}_n(X)$, as the n-truncation of its shape. In other words,

$$\mathbf{\Pi}_n(X) := \tau_n \mathbf{\Pi}(X) = \tau_n \mathrm{Disc}(\Pi(X)).$$

Notice that in the ∞-topos **Top**, the geometric fundamental ∞-groupoid is just the fundamental ∞-groupoid, while the discrete ∞-functor is the geometric realization, so that the shape modality $|\Pi(X)|$ is equivalent to X. Consequently, the geometric homotopy n-type $\mathbf{\Pi}_n(X)$ is given by the usual homotopy n-type, justifying the adopted nomenclature. The homotopy n-type of a topological space is determined by the homotopy groups $\pi_i(X)$, with $i \leq n$. So, in the general case, we can think of $\mathbf{\Pi}_n(X)$ as containing some "geometric homotopy groups".

Independently of the intuition behind $\mathbf{\Pi}_n(X)$, they are exactly the intermediate steps in the abstract Postnikov tower of $\mathbf{\Pi}X \to *$, as below. Here we will see that it induce a Whitehead tower for X in the same way as the classical Postnikov tower of a topological space induce a corresponding Whitehead tower.

$$\cdots \longrightarrow \mathbf{\Pi}_2(X) \longrightarrow \mathbf{\Pi}_1(X) \longrightarrow \mathbf{\Pi}_0(X)$$

Indeed, recall that, as commented in Example 8.11, for a given topological space X, we get a model to its Whitehead tower by considering X^n as the homotopy fiber of the canonical map $X \to X_{n+1}$, where X_{n+1} is the $(n+1)$th term of the Postnikov tower of X. So, the main idea is to try to reproduce this construction in the abstract context: to define the nth term X^n of the Whitehead tower of an object $X \in \mathbf{H}$ as the homotopy fiber of $X \to \mathbf{\Pi}_{n+1}X$, as in the diagram below.

$$\begin{tikzcd} X^n \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & X_{n+1} \end{tikzcd} \qquad \begin{tikzcd} X^n \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & \mathbf{\Pi}_{n+1}(X) \end{tikzcd}$$

Here, however, we need to be careful: in the topological context, the map $X \to X_{n+1}$ is obtained from the fact that $\mathrm{colim} X_n \simeq X$, so that its existence depends explicit of the fact that X_{n+1} belongs to the Postnikov tower of X. It happens that in general the last diagram does **not** define a Postnikov tower of X, because we may not have $\mathbf{\Pi}_n(X) \simeq X$, so that in the abstract context there is **no** complete analogous of the map $X \to X_{n+1}$. On the other hand, we have a canonical $X \to \mathbf{\Pi}_k(X)$ for any k, arising from the counit of the adjunction $\Pi \rightleftharpoons \mathrm{Disc}$, so that we put X^n as the homotopy fiber of $X \to \mathbf{\Pi}_{n+1}(X)$, as in the diagram above.

$$\begin{tikzcd} X^{n-1} \arrow[r] \arrow[d] & \infty\mathrm{pb} \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & \mathbf{\Pi}_n(X) \arrow[r] & \mathbf{\Pi}_{n-1}(X) \end{tikzcd} \qquad \begin{tikzcd} X^n \arrow[r] \arrow[d, dashed] \arrow[dd, bend right] & * \arrow[d] & \\ X^{n-1} \arrow[r] \arrow[d] & \infty\mathrm{pb} \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & \mathbf{\Pi}_n(X) \arrow[r] & \mathbf{\Pi}_{n-1}(X) \end{tikzcd}$$

The maps $X^n \to X^{n-1}$ can be obtained from a direct analogy with the topological case: by making use of the maps $\mathbf{\Pi}_n(X) \to \mathbf{\Pi}_{n-1}(X)$, together with pasting law of ∞-pullbacks and universality. More precisely, notice that, by the pasting law of ∞-pullbacks, we can write X^{n-1} as the ∞-pullback in the first diagram above. Furthermore, X^n is identified with the upper pullback in the second diagram, so that we get $X^n \to X^{n-1}$ from universality.

In order to conclude that the obtained sequence X^n really gives a presentation for the abstract Whitehead tower of X we need to prove that $\mathrm{colim} X^n \simeq *$ and that $X^0 \simeq X$. The first condition follows from the dual property of the Postnikov tower for $\mathbf{\Pi}(X)$. In order to get the second condition we can indeed **define** $X^0 := X$. Indeed, the construction above only defines X^n for positive values, i.e, for $n > 0$.

de Rham

Let us discuss now a synthetic version of de Rham cohomology. As previously, let $\mathbf{H}$ be a ∞-connected ∞-topos, so that we have the shape modality $\mathbf{\Pi}$. For each X we have a canonical morphism $X \to \mathbf{\Pi}X$: the counit of the adjunction $\Pi \rightleftharpoons \mathrm{Disc}$ defining the modality.

In the last subsection we used homotopy fibers of (the truncations of) this morphism in order to get a geometric presentation of the Whitehead tower. Very surprisingly, the synthetic formulation of de Rham cohomology is obtained by making use of homotopy cofibers (instead of homotopy fibers) of this morphism! Indeed, for a given X we define its *de Rham refinement* as the homotopy cofiber below.

$$
\begin{tikzcd}
\mathbf{\Pi}_{dR}X & * \arrow[l] \\
\mathbf{\Pi}X \arrow[u] & X \arrow[l] \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\flat_{dR}A \arrow[r] \arrow[d] & * \arrow[d] \\
\flat A \arrow[r] & A
\end{tikzcd}
$$

For a fixed $A \in \mathbf{H}$, the *abstract de Rham cohomology of X with coefficients in X* is just the abstract cohomology of the de Rham refinement:

$$H_{dR}(X; A) := H^0(\mathbf{\Pi}_{dR}X; A) = \pi_0 \infty\mathbf{Gpd}(\mathbf{\Pi}_{dR}X, A). \tag{10.2.1}$$

In view of justifying why this can be used as a synthetic presentation for de Rham cohomology, recall that in a ∞-connected ∞-topos we have another modality besides $\mathbf{\Pi}$: the *flat modality* $\flat$, arising from the adjunction $\mathrm{Disc} \rightleftharpoons \Gamma$ (which actually exists in any ∞-topos). So, for any $A \in \mathbf{H}$ we can define a corresponding $\flat_{dR}A$ as in the second diagram above.

Let us prove that the induced functors $\mathbf{\Pi}_{dR}$ and $\flat_{dR}$ are adjoints (at least in the homotopy category), so that we could be defined "abstract de Rham cohomology of X with coefficients in A" equivalently as abstract cohomology of X with coefficients in $\flat_{dR}A$, meaning that the adjunction $\mathbf{\Pi}_{dR} \rightleftharpoons \flat_{dR}$ induce an isomorphism

$$H_{dR}(X; A) \simeq H^0(X, \flat_{dR}A). \tag{10.2.2}$$

In order to get the desired adjunction, let us see that to any given 1-morphism $X \to \flat_{dR}A$ we have a corresponding $\mathbf{\Pi}_{dR}X \to A$ and, reciprocally, that any $\mathbf{\Pi}_{dR}X \to A$ induce some morphism $X \to \flat_{dR}A$ in such a way that both processes become one the inverse of the other. Therefore, let $X \to \flat_{dR}A$ be a given 1-morphism. By composition with $\flat_{dR}A \to \flat A$ we get $X \to \flat A$, as in the first diagram below. From the adjunction $\mathbf{\Pi} \rightleftharpoons \flat$, this morphism corresponds to another $\mathbf{\Pi}X \to A$. Therefore, from universality of ∞-pushouts we get $\mathbf{\Pi}_{dR}X \to A$, as in the second diagram below (there, the distinguished arrows are exactly the same distinguished arrows of the first diagram).

$$
\begin{tikzcd}
 & \flat_{dR}A \arrow[r] \arrow[d] & * \arrow[d, squiggly] \\
X \arrow[ur] \arrow[r, dashed] & \flat A \arrow[r] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A & & \\
 & \mathbf{\Pi}_{dR}X \arrow[ul, dotted] & * \arrow[l] \arrow[ull, squiggly] \\
 & \mathbf{\Pi}X \arrow[u] \arrow[uul, dashed] & X \arrow[l] \arrow[u]
\end{tikzcd}
$$

Starting with a morphism $\mathbf{\Pi}_{dR}X \to A$ proceed in a totally dual fashion. Indeed, by composing with $\mathbf{\Pi}X \to \mathbf{\Pi}_{dR}X$ the given morphism induce a $\mathbf{\Pi}X \to A$, as in the first diagram below. From the adjunction $\mathbf{\Pi} \rightleftharpoons \flat$ we then get $X \to \flat A$. So, by making use of universality of ∞-pushouts as in the second diagram below, we get the desired $X \to \flat_{dR}A$.

$$
\begin{tikzcd}
 & \mathbf{\Pi}_{dR}X \arrow[dl] & * \arrow[l] \\
A & \mathbf{\Pi}X \arrow[l, dashed] \arrow[u] & X \arrow[l] \arrow[u, squiggly]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, squiggly, bend left] \arrow[ddr, dashed] & & \\
 & \flat_{dR}A \arrow[r] \arrow[d] & * \arrow[d] \\
 & \flat A \arrow[r] & A
\end{tikzcd}
$$

That both constructions define an adjunction $\mathbf{\Pi}_{dR} \rightleftharpoons \flat_{dR}$ comes directly from the fact that they were obtained from universality and from the adjunction $\mathbf{\Pi} \rightleftharpoons \flat$.

Following Section ?? of [182], let us finally justify why (10.2.1) can be regarded as a synthetic formalization of de Rham cohomology. Given a Lie ∞-group G we can consider iterated deloopings $\mathbf{B}^nG$ and so the induced geometric smooth ∞-stacks. Let us analyze $\mathbf{B}^nU(1)$. We would like to calculate its de Rham refinement $\flat_{dR}\mathbf{B}^nU(1)$ explicitly.

Notice that, because $U(1)$ is a 1-group, its deloopings only "shift" the starting structure. More precisely, as a Lie ∞-group $U(1)$ has manifold of objects given $U(1)$, trivial manifold of 1-morphisms, trivial manifold of 2-morphisms, and so on. On the other hand, $\mathbf{B}U(1)$ has trivial manifold of objects, $U(1)$ as manifold of 1-morphisms, trivial manifold of 2-morphisms, and so on. So, when applying $\mathbf{B}$ we are "shifting" $U(1)$ from 0-morphisms to 1-morphisms. Consequently, iterating $\mathbf{B}$ we are shifting $U(1)$ even more. Diagrammatically we have the following picture, where the right-hand side was obtained from the left one by repeated applications of $\mathbf{B}$:

$$
\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} * \substack{\longrightarrow\\ \longrightarrow} * \longrightarrow U(1)
\qquad\qquad
\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} * \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} U(1) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} * \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots \substack{\longrightarrow\\ \longrightarrow} * \longrightarrow *
$$

The associated geometric ∞-stack, here denoted by $\dot{\mathbf{B}}^nU(1)$, is the ∞-stackfication of the functor which assigns to any manifold X the ∞-groupoid diagrammatically presented by applying $C^\infty(X,-)$ at the diagram above. By definition, its flat modality is given by

$$
\flat\dot{\mathbf{B}}^nU(1) := \mathrm{Disc}(\Gamma(\dot{\mathbf{B}}^nU(1))).
$$

From the remark on "flat modality" at the end of the last section it follows that $\flat\dot{\mathbf{B}}^nU(1)$ is the ∞-stacktification of the constant functor at the ∞-groupoid $\mathbf{B}^nU(1)$. In order to get the desired de Rham refined, we need to compute the following ∞-pullback:

$$
\begin{tikzcd}
\flat_{dR}\dot{\mathbf{B}}^nU(1) \arrow[r] \arrow[d] & \flat\dot{\mathbf{B}}^nU(1) \arrow[d] \\
* \arrow[r] & \mathbf{B}^nU(1)
\end{tikzcd}
$$

From the discussion above we see that such ∞-pullback is identified with

$$\begin{tikzcd}
\flat_{dR}\dot{\mathbf{B}}^n U(1) \arrow[r] \arrow[d] & \cdots \overset{\vdots}{\rightrightarrows} * \overset{\vdots}{\rightrightarrows} U(1) \overset{\vdots}{\rightrightarrows} * \overset{\vdots}{\rightrightarrows} \cdots \arrow[d] \\
* \arrow[r] & \cdots \overset{\vdots}{\rightrightarrows} * \overset{\vdots}{\rightrightarrows} C^\infty(-,U(1)) \overset{\vdots}{\rightrightarrows} * \overset{\vdots}{\rightrightarrows} \cdots
\end{tikzcd}$$

In order to compute it, we recall the stable Dold-Kan correspondence

$$\mathrm{DK} : (\mathbf{CCh}_R, \otimes_R) \simeq \mathrm{Mod}_{\mathbb{H}R}(\mathbf{Spec}, \wedge_{\mathbb{H}R})$$

between the ∞-category of cochain complexes and the ∞-category of ∞-module objects in **Spec** over the Eilenberg-Mac Lane spectrum $\mathbb{H}R$, where R is a commutative ring. By composition, it induces a ∞-functor

$$\infty\mathrm{DK} : \infty\mathrm{Shv}(\mathbf{Diff}^{op}_{sub}; \mathbf{CCh}_{\mathbb{Z}}) \to \infty\mathrm{Shv}(\mathbf{Diff}^{op}_{sub}; \mathbf{Spec}).$$

As will be proved below in (10.2.9) the right-hand side belongs to the stable ∞-category of the cohesive ∞-topos of smooth ∞-stacks. Therefore, applying $\overline{\Omega}^\infty$ we get a functor

$$\infty\mathrm{Shv}(\mathbf{Diff}^{op}_{sub}; \mathbf{CCh}_{\mathbb{Z}}) \to \infty\mathrm{Stack}(\mathbf{Diff}_{sub}, J).$$

Both $\overline{\Omega}^\infty$ and ∞DK have left adjoints, so that they preserve ∞-limits. This means that if we identify smooth ∞-sheaves of cochain complexes that are mapped by $\overline{\Omega}^\infty \circ \infty\mathrm{DK}$ onto the objects defining the last ∞-pullback, then we can calculate this ∞-pullback by first computing it in the level of complexes. It happens that in the present case it is easy to identify the underlying complexes: we only need to replace $*$ with 0. More precisely, the ∞-pullback of sheaves of cochain complexes which will give the desired $\flat_{dR}\mathbf{B}^n U(1)$ is:

$$\begin{tikzcd}
\infty\mathrm{pb} \arrow[r] \arrow[d] & (\cdots \arrow[r] & 0 \arrow[r] & U(1) \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots) \\
(\cdots \to 0 \to \cdots) \arrow[r] & (\cdots \arrow[r] & 0 \arrow[r] & C^\infty(-,U(1)) \arrow[r] & 0 \arrow[r] & \cdots)
\end{tikzcd}$$

The final step is to notice that, because we are working with ∞-limits, we have the freedom to replace each object in the diagram with other in the same homotopy class. The homotopy category of cochain complexes is determined by the algebraic derived category, so that two objects are homotopic exactly when there is an isomorphism between their cohomology groups. On the other hand, recall that sheaves are local objects, so that the ∞-pullback above can is ultimately computed over good coverings of smooth manifolds. Therefore, by the Poincaré lemma the chain morphism below is indeed a homotopy equivalence.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & U(1) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & C^\infty(-,U(1)) \arrow[r, "d"'] & \Lambda^1 \arrow[r, "d"'] & \Lambda^2 \arrow[r, "d"'] & \cdots \arrow[r, "d"'] & \Lambda^n \arrow[r, "d"'] & 0 \arrow[r] & \cdots ,
\end{tikzcd}$$

so that we make this replacement in the last ∞-pullback. Recall that (as discussed in Sections 7.3 and 8.1), in the projective model structure presentation of the $(\infty,1)$-category $\mathbf{CCh}_R$ of cochain complexes, a morphism is a fibration precisely if it is degreewise an epimorphism. Furthermore, $\mathbf{CCh}_R$ is proper, so that if at least one morphism in a ∞-pullback is a fibration, then the ∞-pullback can be computed as a usual 1-pullback. The homotopy equivalence above is a fibration, so that when using it in the last ∞-pullback we will be able do a explicit computation, giving as result (see Proposition ?? of [182]):

$$\cdots \longrightarrow 0 \longrightarrow \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n \xrightarrow{d} 0 \longrightarrow \tag{10.2.3}$$

Therefore, *the de Rham refinement of the smooth ∞-stack $\mathbf{B}^n U(1)$ is just the image, via Dold-Kan correspondence, of the de Rham complex truncated at level n.* But notice that this truncated complex is the presentation of classical de Rham cohomology, for $n > 1$, in the language of sheave cohomology. On the other hand, $\flat_{dR}\mathbf{B}^n U(1)$ is the representing object for nth abstract de Rham cohomology. Consequently, *in the cohesive ∞-topos of smooth ∞-stacks, for any object X induced by a manifold, abstract de Rham cohomology coincides with classical de Rham cohomology:*

$$H^n_{dR}(X) \simeq H_{dR}(X; \mathbf{B}^n U(1)).$$

Remark. In the presentation (10.2.3) it is usual to agree that Λ^n is in degree zero. This will be important later.

Remark. We obtained a synthetic formulation for de Rham cohomology. We could ask if there is also a synthetic presentation for the de Rham complex. An approach is as follows. In order to talk of de Rham complex we need the notion of "closed form". In the classical context, de Rham cohomology is defined as the quotient of the space of closed forms, so that we have a universal projection from closed forms to de Rham cohomology. In our abstract context, for a given A, with the notion of de Rham cohomology on hand we can define the ∞-stack of *A-valued closed* as the universal object $\Lambda_{\mathrm{cl}}(-;A)$ with trivial k-morphisms (meaning that it lie in the 0th truncation) endowed with a morphism $\Lambda_{\mathrm{cl}}(-;A) \to \flat_{dR}A$ such that for any X the corresponding map

$$\Lambda_{\mathrm{cl}}(X;A) \to H_{dR}(X;A)$$

is a projection. In order to get de Rham complex we need indeed closed forms of different degrees, but with coefficients into a same object. This is easy to get from the case above by considering $A = \mathbf{B}^k G$. Finally, we need an "exterior derivative" d realizing de Rham cohomology as a quotient of closed forms by the image of d. So, for a given ∞-group G we define an *abstract exterior derivative with coefficients in* G as a sequence of morphisms

$$d_k : \flat_{dR}\mathbf{B}^k G \to \flat_{dR}\mathbf{B}^{k+1}G$$

which lifts to morphisms between closed forms (in the sense of the first diagram below) and such that for any X the pushout in the second diagram models $H^k_{dR}(X;G)$.

$$\begin{tikzcd}
\Lambda^k_{\mathrm{cl}}(-;G) \arrow[r, dashed] \arrow[d] & \Lambda^{k+1}_{\mathrm{cl}}(-;G) \arrow[d] \\
\flat_{dR}\mathbf{B}^k G \arrow[r, "d_k"'] & \flat_{dR}\mathbf{B}^{k+1}G
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{ps} & \ast \arrow[l] \\
\Lambda^{k+1}_{\mathrm{cl}}(X;G) \arrow[u] & \Lambda^k_{\mathrm{cl}}(X;G) \arrow[l, dashed] \arrow[u]
\end{tikzcd}$$

Connections

For any object A in a ∞-connected ∞-topos $\mathbf{H}$ we can consider its de Rham refinement $\flat_{dR}A$, as discussed in the last subsection. It is the homotopy fiber of the unit $\flat A \to A$ and, therefore, it becomes equipped with a canonical morphism $\flat_{dR}A \to \flat A$. Let us now consider the homotopy fiber of this new map. By the gluing property of ∞-pullbacks, this gives a model to the loop space of A, which itself becomes equipped with a distinguished map $\Omega A \to \flat_{dR}A$, as below.

$$\begin{tikzcd}
\infty\mathrm{pb} \arrow[r] \arrow[d] & \ast \arrow[d] \\
\flat_{dR}A \arrow[r] \arrow[d] & \flat A \arrow[d] \\
\ast \arrow[r] & A
\end{tikzcd}
=
\begin{tikzcd}
\infty\mathrm{pb} \arrow[r] \arrow[dd] & \ast \arrow[dd] \\
 & \\
\ast \arrow[r] & A
\end{tikzcd} \tag{10.2.4}$$

If A is now the delooping ∞-groupoid $\mathbf{B}G$ of a internal ∞-group G, then we have $\Omega\mathbf{B}G \simeq G$, so that the construction above gives a canonical morphism $\theta_G : G \to \flat_{dR}\mathbf{B}G$. In terms of abstract cohomology, this defines a class in the first abstract de Rham coholgy $H_{dR}(G;\flat_{dR}\mathbf{B}G)$.

Let us assume that G has a further delooping $\mathbf{B}^2G$. Let $\Lambda^2_{\mathrm{cl}}(-;\mathfrak{g})$ be a model for abstract G-valued closed 2-forms, as discussed in the end of the last subsection. In this case, the ∞-pullback below is called the *object of abstract G-connections*.

$$\begin{tikzcd}
\mathbf{B}G_{\mathrm{conn}} \arrow[r] \arrow[d] & \Lambda^2_{\mathrm{cl}}(-;\mathfrak{g}) \arrow[d] \\
\mathbf{B}G \arrow[r] & \flat_{dR}\mathbf{B}^2G
\end{tikzcd}$$

In order to explain the interest in this kind of object, let us compute $\mathbf{B}U(1)_{\mathrm{conn}}$ explicitly. The idea is to follow the same strategy used in the last subsection in order to show that abstract de Rham cohomology is a synthetic formulation of classical de Rham cohomology. Indeed, as observed in (10.2.3), $\flat_{dR}\mathbf{B}^2U(1)$ has a presentation as cochain complex. Furthermore, because $\mathfrak{u}(1) \simeq \mathbb{R}$, we see that the smooth ∞-stack $\Lambda^2_{\mathrm{cl}}(-,\mathfrak{u}(1))$ can be presented by the cochain complexes concentrated in the usual sheaf of closed 2-forms. Therefore, a presentation for $\mathbf{B}U(1)_{\mathrm{conn}}$ as a smooth ∞-sheaf of complexes is given by the following ∞-pullback, where we indicated the degree of each term and the parenthesis $[-$ and $-]$ imply that from that point the complex is trivial.

$$\begin{tikzcd}
 & \infty\mathrm{pb} \arrow[rrrr] \arrow[d] & & & & {[\overset{-1}{0}} \arrow[r] & \overset{0}{\Lambda^2_{\mathrm{cl}}} \arrow[r] \arrow[d] & \overset{1}{0}] \\
{[\underset{-3}{0}} \arrow[r] & \underset{-2}{C^\infty(-,U(1))} \arrow[r] & \underset{-1}{0}] \arrow[rr] & & {[\underset{-2}{0}} \arrow[r, "d"] & \underset{-1}{\Lambda^1} \arrow[r, "d"] & \underset{0}{\Lambda^2} \arrow[r, "d"] & \underset{1}{0}]
\end{tikzcd} \tag{10.2.5}$$

In positive degrees the morphisms are epimorphisms, so that this ∞-pullback can be computed as a 1-pullback, giving (see Proposition 4.4.91, p. 518 of [182]):

$$\cdots 0 \longrightarrow C^\infty(-,U(1)) \xrightarrow{d} \Lambda^1 \xrightarrow{d} 0 \longrightarrow \cdots$$

But recall the definition of Deligne cohomology in (??). Indeed, for a given n, the Deligne cohomology of a manifold X is the abelian sheaf cohomology of the sheaf of complexes $\mathcal{D}^{k+1}$ below. Therefore, we see that *the presentation of* $\mathbf{B}U(1)_{\mathrm{conn}}$ *by cochain complexes is given by the Deligne complex* $\mathcal{D}^2$.

$$\cdots 0 \longrightarrow C^\infty(-,U(1)) \xrightarrow{d} \Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k \longrightarrow 0 \longrightarrow \cdots$$

Having computed $\mathbf{B}U(1)_{\mathrm{conn}}$ explicitly, we can now say what is the theoretical interest in the object $\mathbf{B}G_{\mathrm{conn}}$. A G-bundle with connection ∇ over a manifold X, for $G = U(1)$, is classified precisely by cocycles in Deligne cohomology $H^2_{Dl}(X)$ and, therefore, in the abstract cohomology $H(X;\mathbf{B}U(1)_{\mathrm{conn}})$. So, for a ∞-group G in an arbitrary cohesive ∞-topos $\mathbf{H}$, we can regard a morphism $X \to \mathbf{B}G_{\mathrm{conn}}$ as a synthetic formulation for the notion of "G-principal ∞-bundle with connection over X".

Remark. If a ∞-group G has $n > 2$ deloopings, then we can define not only the "object of G-connections" $\mathbf{B}G_{\mathrm{conn}}$, but also $\mathbf{B}^kU(1)_{\mathrm{conn}}$ for every $1 \leq k < n$. Indeed, it is just the ∞-pullback below. In this case, the cochain complex presentation would be given by $\mathcal{D}^{k+1}$.

$$\begin{tikzcd}
\mathbf{B}^kG_{\mathrm{conn}} \arrow[r] \arrow[d] & \Lambda^{k+1}_{\mathrm{cl}}(-;\mathfrak{g}) \arrow[d] \\
\mathbf{B}^{k+1}G \arrow[r] & \flat_{dR}\mathbf{B}^{k+1}G
\end{tikzcd}$$

Differential Cohomology

We have seen that any abstract cohomology in a ∞-connected ∞-topos, with coefficients in an arbitrary object A, admits a "differential refinement" which can be understood as an abstract version of de Rham cohomology. If $A = \mathbf{B}^{k+1}G$ for some ∞-group G, then the de Rham refinement $\flat_{dR}\mathbf{B}^{k+1}G$ refines even more, defining $\mathbf{B}^kG_{\mathrm{conn}}$.

In the case $G = U(1)$ this is modeled by Deligne complex of degree $k+1$. Deligne cohomology $H^{k+1}_{Dl}(X)$ is the classical model to ordinary differential cohomology, as discussed in Section 1.2. Therefore, it seems natural to call the $H^0(X; \mathbf{B}^k G_{\mathrm{conn}})$ of *kth abstract differential nonabelian cohomology* of X with coefficients in G. But, since the works of Hopkins-Singer [99] and Simons-Sullivan , any model to the notion of "differential cohomology" should sitting in the middle of a "differential cohomology hexagon" of interlocking exact sequences.

Here we will show that, at least after stabilization, the groups $H^0(X; \mathbf{B}^k G_{\mathrm{conn}})$ actually satisfy this hexagon diagram. More precisely, following [35, 36, 183] we will show that any Ω-spectrum $\mathbb{E}$ in a cohesive ∞-topos $\mathbf{H}$ sits in the middle of a hexagon analogous to the "differential cohomology hexagon". We then sketch how to recover the diagram for Deligne cohomology from this abstract picture.

We start by recalling that, from Example 10.5, the cohesive structure of $\mathbf{H}$ induces a cohesive structure in $\mathrm{Stab}(\mathbf{H})$. So, we can consider $\flat_{dR}\mathbb{E}$. Recall that this is the homotopy fiber of the morphism $\flat\mathbb{E} \to \mathbb{E}$. Consequently, we can analyze the associated abstract fibration sequence:

$$\cdots \longrightarrow \overline{\Omega}\flat\mathbb{E} \longrightarrow \overline{\Omega}\mathbb{E} \longrightarrow \flat_{dR}\mathbb{E} \longrightarrow \flat\mathbb{E} \longrightarrow \mathbb{E}$$

For instance, the first four terms are just the maps appearing in diagram (10.2.4) for the case $A = \mathbb{E}$. Applying the geometric fundamental ∞-groupoid ∞-functor $\mathbf{\Pi}$ to the sequence above we get the commutative diagram below, where the vertical arrows are the counit of the adjunction $\Pi \rightleftharpoons \mathrm{Disc}$ defining $\mathbf{\Pi}$.

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{\Omega}\flat\mathbb{E} \arrow[r] \arrow[d] & \overline{\Omega}\mathbb{E} \arrow[r] \arrow[d] & \flat_{dR}\mathbb{E} \arrow[r] \arrow[d] & \flat\mathbb{E} \arrow[r] \arrow[d] & \mathbb{E} \arrow[d] \\
\cdots \arrow[r] & \mathbf{\Pi}\overline{\Omega}\flat\mathbb{E} \arrow[r] & \mathbf{\Pi}\overline{\Omega}\mathbb{E} \arrow[r] & \mathbf{\Pi}\flat_{dR}\mathbb{E} \arrow[r] & \mathbf{\Pi}\flat\mathbb{E} \arrow[r] & \mathbf{\Pi}\mathbb{E}
\end{tikzcd}$$

From the stability of $\mathrm{Stab}(\mathbf{H})$ we have $\overline{\Omega}\mathbb{X} \simeq \mathbb{X}$ for any $\mathbb{X}$, so that the sequence above is periodic. Let us look at its non-periodic part, presented below. The vertical distinguished arrow is an equivalence, because in any cohesive ∞-topos we have $\flat \circ \mathbf{\Pi} \simeq \flat$. Consequently, the non-periodic part reduces to a commutative square which is forced to be ∞-pullback square.

$$\begin{tikzcd}
\flat\mathbb{E} \arrow[r] \arrow[d, dashed] & \mathbb{E} \arrow[r] \arrow[d] & \flat_{dR}\mathbb{E} \arrow[d] \\
\mathbf{\Pi}\flat\mathbb{E} \arrow[r] & \mathbf{\Pi}\mathbb{E} \arrow[r] & \mathbf{\Pi}\flat_{dR}\mathbb{E}
\end{tikzcd} \tag{10.2.6}$$

Thanks to the stability of $\mathrm{Stab}(\mathbf{H})$, any $\mathbb{X}$ is infinitely deloopable and $\mathbf{B}\mathbb{X} \simeq \mathbb{X}$. So, we can identify the right upper horizontal arrow with the abstract Maurer-Cartan form of $\mathbb{E}$. Therefore, the diagram above collapses into the first diagram below. A totally dual argumentation will produce the second diagram as a ∞-pullback.

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\theta_{\mathbb{E}}"] \arrow[d] & \flat_{dR}\mathbb{E} \arrow[d] \\
\mathbf{\Pi}\mathbb{E} \arrow[r, "\mathbf{\Pi}(\theta_{\mathbb{E}})"'] & \mathbf{\Pi}\flat_{dR}\mathbb{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Pi}_{dR}\flat\mathbb{E} \arrow[r] \arrow[d] & \mathbf{\Pi}_{dR}\mathbb{E} \arrow[d] \\
\flat\mathbb{E} \arrow[r] & \mathbb{E}
\end{tikzcd}$$

Gluing these two diagrams at the common vertex $\mathbb{E}$ we get the diagram below. We assert that both upper and lower outer sequences are fibration sequences.

$$\begin{tikzcd}
 & \mathbf{\Pi}_{dR}\mathbb{E} \arrow[rr] \arrow[dr, dashed] & & \flat_{dR}\mathbb{E} \arrow[dr, dashed] & \\
\mathbf{\Pi}_{dR}\flat\mathbb{E} \arrow[ur, dashed] \arrow[dr, dotted] & & \mathbb{E} \arrow[ur, dashed, "\theta_{\mathbb{E}}"] \arrow[dr, dotted] & & \mathbf{\Pi}\flat_{dR}\mathbb{E} \\
 & \flat\mathbb{E} \arrow[rr] \arrow[ur, dotted] & & \mathbf{\Pi}\mathbb{E} \arrow[ur, dotted, "\mathbf{\Pi}(\theta_{\mathbb{E}})"'] &
\end{tikzcd} \tag{10.2.7}$$

Indeed, by rotating the second ∞-pullback square and using the fibration sequence (10.2.4) together with its dual version, we get the following diagram, where each square is a ∞-pullback. So, by gluing properties, the outer square is also a ∞-pullback square. Consequently, the distinguished sequences are fibrations sequences. But, by commutativity, these sequences are precisely the distinguished sequences of the last diagram.

$$\begin{tikzcd}
 & & * \arrow[dr, dashed] & & \\
 & \mathbf{\Pi}_{dR}\mathbb{E} \arrow[ur, dashed] \arrow[dr] & & \mathbf{\Pi}\mathbb{E} \arrow[dr, dashed] & \\
\mathbf{\Pi}_{dR}\flat\mathbb{E} \arrow[ur, dashed] \arrow[dr, dotted] & & \mathbb{E} \arrow[ur] \arrow[dr] & & \mathbf{\Pi}\flat_{dR}\mathbb{E} \\
 & \flat\mathbb{E} \arrow[ur] \arrow[dr, dotted] & & \flat_{dR}\mathbb{E} \arrow[ur, dotted] & \\
 & & * \arrow[ur, dotted] & &
\end{tikzcd} \tag{10.2.8}$$

We end with a remark.

Remark. The construction of the abstract hexagon above was firstly done in [36] in the cohesive ∞-topos of smooth ∞-stacks. The general case is due to Schreiber [183]. We advert the reader that in [36] the authors do not work explicitly with spectrum objects in the ∞-topos of smooth ∞-stacks. Instead, they work with "smooth sheaves of spectra", i.e, with ∞-sheaves $\mathbf{Diff}^{op} \to \mathbf{Spec}$. But both notions are equivalent:

$$\begin{aligned}
\infty\mathrm{Func}(\mathbf{Diff}^{op};\mathbf{Spec}) &\simeq \infty\mathrm{Func}(\mathbf{Diff}^{op};\infty\mathrm{Func}(\mathbb{Z}\times\mathbb{Z};\infty\mathbf{Gpd})) \\
&\simeq \infty\mathrm{Func}(\mathbf{Diff}^{op}\times(\mathbb{Z}\times\mathbb{Z});\infty\mathbf{Gpd}) \\
&\simeq \infty\mathrm{Func}((\mathbb{Z}\times\mathbb{Z})\times\mathbf{Diff}^{op};\infty\mathbf{Gpd}) \\
&\simeq \infty\mathrm{Func}(\mathbb{Z}\times\mathbb{Z};\infty\mathrm{Func}(\mathbf{Diff}^{op};\infty\mathbf{Gpd})).
\end{aligned} \tag{10.2.9}$$

Deligne

One time built the abstract differential cohomology hexagon, let us show that we can recover the canonical hexagon for the Deligne complex as a particular case. More precisely, we will obtain a spectrum object $\mathbb{E}$ in the cohesive ∞-topos smooth ∞-stacks $\mathbf{H}$ such that the corresponding abstract differential cohomology hexagon (10.2.6) is exactly that captured by Deligne cohomology.

Recalling the stable Dold-Kan correspondence, which establishes a ∞-monoidal equivalence

$$\mathrm{DK} : (\mathbf{CCh}_R, \otimes_R) \simeq \mathrm{Mod}_{\mathbb{H}R}(\mathbf{Spec}, \wedge_{\mathbb{H}R}),$$

we get the functor

$$\infty\mathrm{DK} : \infty\mathrm{Shv}(\mathbf{Diff}^{op}_{sub}; \mathbf{CCh}_{\mathbb{Z}}) \to \infty\mathrm{Shv}(\mathbf{Diff}^{op}_{sub}; \mathbf{Spec})$$

taking values into $\mathrm{Stab}(\mathbf{H})$, due to (10.2.9). Therefore, the Deligne complex $\mathcal{D}^{n+1}$ is mapped under ∞DK onto a Ω-spectrum, which is the obvious candidate to the desired $\mathbb{E}$ (notice that, as obtained previously, the ∞-loop space of $\mathbb{E}$ is exactly $\mathbf{B}^n U(1)_{\mathrm{conn}}$).

Because ∞DK preserve ∞-limits, it is enough to do the calculations at the level of cochain complexes. In other words, it is enough to compute each term in the diagram below

$$\begin{tikzcd} & \mathbf{\Pi}_{dR}\mathcal{D}^{n+1} \arrow[rr] \arrow[dr] & & \flat_{dR}\mathcal{D}^{n+1} \arrow[dr] & \\ \mathbf{\Pi}_{dR}\flat\mathcal{D}^{n+1} \arrow[ur] \arrow[dr] & & \mathcal{D}^{n+1} \arrow[ur] \arrow[dr] & & \mathbf{\Pi}\flat_{dR}\mathcal{D}^{n+1} \\ & \flat\mathcal{D}^{n+1} \arrow[rr] \arrow[ur] & & \mathbf{\Pi}\mathcal{D}^{n+1} \arrow[ur] & \end{tikzcd} \tag{10.2.10}$$

showing that they produce the usual differential cohomology diagram:

$$\begin{tikzcd} & \Lambda^n(X)/\mathrm{img}(d) \arrow[rr] \arrow[dr] & & \Lambda^{n+1}_{\mathrm{cl}}(X) \arrow[dr] & \\ H^n(X;\mathbb{R}) \arrow[ur] \arrow[dr] & & H^{n+1}_{Dl}(X) \arrow[ur] \arrow[dr] & & H^{n+1}(X;\mathbb{R}) \\ & H^n(X;U(1)) \arrow[rr] \arrow[ur] & & H^{n+1}(X;\mathbb{Z}) \arrow[ur] & \end{tikzcd} \tag{10.2.11}$$

We will indicate how the computations could be done in this particular case. We start by analyzing the diagonal

$$\flat\mathcal{D}^{n+1} \longrightarrow \mathcal{D}^{n+1} \longrightarrow \flat_{dR}\mathcal{D}^{n+1}. \tag{10.2.12}$$

Recall that the flat modality $\flat$ takes a ∞-stack F and gives the ∞-stack $\flat F$ constant and equal to $F(*)$. Therefore, the flat version of Deligne complex $\mathcal{D}^{n+1}$ is given by the constant smooth ∞-sheaf of complexes

$$\cdots \longrightarrow \underset{-n-2}{0} \longrightarrow \underset{-n-1}{U(1)} \longrightarrow \underset{-n}{0} \longrightarrow \cdots,$$

which under ∞DK is mapped precisely onto the (stabilization of the smooth ∞-stack induced by) the Eilenberg-Mac Lane space $K(U(1),n)$. The term $\flat_{dR}\mathcal{D}^{n+1}$ is, by definition, the ∞-pullback

$$\begin{tikzcd} \flat_{dR}\mathcal{D}^{n+1} \arrow[r] \arrow[d] & \flat\mathcal{D}^{n+1} \arrow[d] \\ * \arrow[r] & \mathcal{D}^{n+1}. \end{tikzcd}$$

It can be computed as a usual pullback by making use of Poincaré lemma in order to replace the constant complex $\flat\mathcal{D}^{n+1}$ as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & U(1) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & C^\infty(-,U(1)) \arrow[r] & \Lambda^1 \arrow[r] & \cdots \arrow[r] & \Lambda^n_{\mathrm{cl}} \arrow[r] & \Lambda^{n+1}_{\mathrm{cl}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

A direct computation then reveals that $\flat_{dR}\mathcal{D}^{n+1}$ is given by

$$\cdots \longrightarrow 0 \longrightarrow \Lambda^{n+1}_{\mathrm{cl}} \longrightarrow 0 \longrightarrow \cdots\,, \tag{10.2.13}$$

so that under stable Dold-Kan correspondence the starting sequence (10.2.12) becomes

$$\mathbf{B}^n U(1) \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}} \longrightarrow \Lambda^{n+1}_{\mathrm{cl}}.$$

Let us now study the ∞-pullback square below.

$$\begin{tikzcd}
\mathcal{D}^{n+1} \arrow[r] \arrow[d] & \flat_{dR}\mathcal{D}^{n+1} \arrow[d] \\
\mathbf{\Pi}\mathcal{D}^{n+1} \arrow[r] & \mathbf{\Pi}_{dR}\mathcal{D}^{n+1}
\end{tikzcd} \tag{10.2.14}$$

It happens that we actually have a ∞-pullback presentation for the Deligne complex, obtained in (10.2.5), given by the following diagram.

$$\begin{tikzcd}
 & \mathcal{D}^{n+1} \arrow[rrrr] \arrow[d] & & & & {[\overset{-1}{0}} \arrow[r] & \overset{0}{\Lambda^n_{\mathrm{cl}}} \arrow[r] \arrow[d] & {\overset{1}{0}]} \\
{[\underset{-n-2}{0}} \arrow[r] & \underset{-n-1}{C^\infty(-,U(1))} \arrow[r] & {\underset{-n}{0}]} \arrow[r] & {[\underset{-n-1}{0}} \arrow[r] & \underset{-n}{\Lambda^1} \arrow[r] & \cdots \arrow[r] & \underset{0}{\Lambda^n} \arrow[r] & {\underset{1}{0}]}
\end{tikzcd}$$

Furthermore, from (10.2.13) the upper right term in the last diagram is exactly $\flat_{dR}\mathcal{D}^{n+1}$, so that the lower arrow should be at least a pullback of $\mathbf{\Pi}\mathcal{D}^{n+1} \to \mathbf{\Pi}\flat_{dR}\mathcal{D}^{n+1}$. We assert that the diagram below is the required presentation, leading us to identify the image under Dold-Kan correspondence of $\mathbf{\Pi}\mathcal{D}^{n+1} \to \mathbf{\Pi}\flat_{dR}\mathcal{D}^{n+1}$ as $\mathbf{B}^{n+1}\mathbb{Z} \to \mathbf{B}^{n+1}\mathbb{R}$.

$$\begin{tikzcd}
{[\overset{-n-2}{0}} \arrow[r] & \overset{-n-1}{C^\infty(-,U(1))} \arrow[r] \arrow[d] & {\overset{-n}{0}]} \arrow[r] & {[\underset{-n-1}{0}} \arrow[r] & \underset{-n}{\Lambda^1} \arrow[r] & \cdots \arrow[r] \arrow[d] & \underset{0}{\Lambda^n} \arrow[r] & {\underset{1}{0}]} \\
{[\underset{-n-3}{0}} \arrow[r] & \underset{-n-2}{\mathbb{Z}} \arrow[r] & {\underset{-n-1}{0}]} \arrow[rr] & & {[\underset{-n-3}{0}} \arrow[r] & \underset{-n-2}{\mathbb{R}} \arrow[r] & {\underset{-n-1}{0}]} &
\end{tikzcd} \tag{10.2.15}$$

First of all notice that we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow U(1) \longrightarrow 0,$$

which induces a quasi-isomorphism between the cochain complexes

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & U(1) \arrow[r] & 0 \arrow[r] & \cdots , \end{tikzcd}$$

allowing us to replace the lower arrow in (10.2.15) with

$$[0] \longrightarrow [\underset{-n-2}{0} \longrightarrow \underset{-n-1}{U(1)} \longrightarrow \underset{-n}{0}],$$

which in turn, due to Poincaré lemma, can be replaced with

$$[0] \longrightarrow [\underset{-n-2}{0} \longrightarrow \underset{-n-1}{C^\infty(-,U(1))} \longrightarrow \underset{-n}{\Lambda^1} \longrightarrow \cdots \longrightarrow \underset{0}{\Lambda^n_{\mathrm{cl}}} \longrightarrow \underset{1}{0}].$$

Doing these replacements, a direct computation shows that the diagram (10.2.15) really is a ∞-pullback square, meaning that under Dold-Kan correspondence the ∞-pullback (10.2.14) is mapped onto

$$\begin{tikzcd} \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[r] \arrow[d] & \Lambda^{n+1}_{\mathrm{cl}} \arrow[d] \\ \mathbf{B}^{n+1}\mathbb{Z} \arrow[r] & \mathbf{B}^{n+1}\mathbb{R} \end{tikzcd}$$

From (10.2.8) we see that $\mathbf{\Pi}_{dR}\flat\mathcal{D}^{n+1}$ can be understood as the loop space of $\mathbf{\Pi}\flat_{dR}\mathcal{D}^{n+1}$, so that under Dold-Kan correspondence we have

$$\infty\mathrm{DK}(\mathbf{\Pi}_{dR}\flat\mathcal{D}^{n+1}) \simeq \Omega\mathbf{B}^{n+1}\mathbb{R} \simeq \mathbf{B}^n\mathbb{R}.$$

Finally, by making use of de Rham theorem and of the fact that the sequence

$$\mathbf{\Pi}_{dR}\flat\mathcal{D}^{n+1} \longrightarrow \mathbf{\Pi}_{dR}\mathcal{D}^{n+1} \longrightarrow \flat_{dR}\mathcal{D}^{n+1} \longrightarrow \mathbf{\Pi}\flat_{dR}\mathcal{D}^{n+1}$$

is a fibration sequence, we conclude that under ∞DK the term $\mathbf{\Pi}_{dR}\mathcal{D}^{n+1}$ is given by $\Lambda^n/\mathrm{img}(d)$. Therefore, joining all these results we see that the abstract hexagon diagram (10.2.10) is mapped by Dold-Kan correspondence onto

$$\begin{tikzcd} & \Lambda^n/\mathrm{img}(d) \arrow[rr] \arrow[dr] & & \Lambda^{n+1}_{\mathrm{cl}} \arrow[dr] & \\ \mathbf{B}^n\mathbb{R} \arrow[ur] \arrow[dr] & & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[ur] \arrow[dr] & & \mathbf{B}^{n+1}\mathbb{R} \\ & \mathbf{B}^n U(1) \arrow[rr] \arrow[ur] & & \mathbf{B}^{n+1}\mathbb{Z} \arrow[ur] & \end{tikzcd}$$

which really reproduces the concrete differential cohomology diagram (10.2.11) in the cohomology level.

Bibliography

[1] Adams, J. F., *On the nonexistence of elements of Hopf invariant one*, Bull. Amer. Math. Soc. Volume 64, Number 5 (1958), 279-282.

[2] Adams, J. F., *Vector Fields on Spheres*, Annals of Math., Vol. 75, No. 3, May, 1962.

[3] Adams, J. F., Atiyah, M. F., *K-Theory and the Hopf Invariant*, Quart. J. Math. Oxford (2), 17 (1966), 31-38.

[4] Adams, J. F., *Stable Homotopy and Generalised Homology*, University Of Chicago Press, 1995.

[5] Alvarez-Gaumé, L., *Supersymmetry and the Atiyah-Singer Index Theorem*, Comm. Math. Phys., 90, 161-173 (1983).

[6] Aguiar, M., *Internal Categories and Quantum Groups*, PhD thesis, Cornell University, 1997.

[7] Aguiar, M., Mahajan, S., *Hopf Algebras and Tensor Categories*, AMS, 2013.

[8] Aguiar, M., Mahajan, S., *Monoidal Functors, Species and Hopf Algebras*, AMS, 2010.

[9] Aguilar, M., Gitler, S., Prieto, C., *Algebraic Topology from a Homotopical Viewpoint*, Springer, 2002.

[10] Ando, M., Blumberg, A. J., et al., *Units of ring spectra and Thom spectra*, arXiv:0810.4535.

[11] Atiyah, M., *K-Theory,* Westview Press, 1994.

[12] Atiyah, M., Singer, I. M., *Index theory for skew-adjoint fredholm operators*, Inst. Hautes tudes Sci. Publ. Math. , (37):5–26, 1969.

[13] Atiyah, M., Bott, R., Shapiro, A., Raoul Bott, *Clifford modules*, Topology , 3(suppl.1):3–38, 1964.

[14] Atiyah, M., Segal, G., *Twisted K-theory*, arXiv:math/0407054.

[15] Ayala, D., *Geometric Cobordism Categories*, arXiv:0811.2280

[16] Ayala, D., Francis, J., *The cobordism hypothesis*, arXiv:1705.02240

[17] Baez, J. C., Lauda, A., *A Prehistory of n-Categorical Physics*, arXiv:0908.2469

[18] Baez, J. C., *Quantum Quandaries: a Category-Theoretic Perspective*, arXiv:quant-ph/0404040.

[19] Baez, J., Dolan, J., *Categorification*, arXiv:math/9802029

[20] Baez, J., Dolan, J., *Higher Dimensional Algebra and Topological Quantum Field Theories.* arXiv:q-alg/9503002

[21] Baez, J., Dolan, J., *Higher Algebra III: n-Categories and the Algebra of Opetopes*, arXiv:q-alg/9702014.

[22] Baez, J., Crans, A. S., *Higher-Dimensional Algebra VI: Lie 2-Algebras*, arXiv:math/0307263

[23] Baez, J., Schreiber, U., *Higher Gauge Theory: 2-Connections on 2-Bundles*, arXiv:hep-th/0412325

[24] Baez, J. C., Schreiber, U., *Higher Gauge Theory*, arXiv:math/0511710

[25] Bakalov, B., Kirillov, A., *Lectures on Tensor Categories and Modular Functors*, AMS, 2000.

[26] -Barker, M. F., *Group Objects and Internal Categories*, arXiv:math/0212065.

[27] Bartlett, B. H., *Categorical Aspects of Topological Quantum Field Theories*, arXiv:math/0512103.

[28] Bastiani A., Ehresmann, C., *Categories of sketched structure*s, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 13 no. 2 (1972), p. 104-214

[29] Bergner, J., *Models for* (∞, n)*-Categories and the Cobordism Hypothesis*, in Hisham Sati, Urs Schreiber (eds.) Mathematical Foundations of Quantum Field and Perturbative String Theory, AMS 2011

[30] Berline, N., Getzler, E., Vergne, M., *Heat Kernels and Dirac Operators*, Springer, 1996.

[31] Bott, R., Milnor, J. W., *On the parallelizability of spheres*, 1958.

[32] Bott, R., Tu, L. W., *Differential Forms in Algebraic Topology,* Springer, 1995

[33] Brown, R., *Ten topologies for* $X \times Y$, Quart. J.Math. (2) 14 (1963), 303–319.

[34] Brown, R., *Function spaces and product topologies*, Quart. J. Math. (2) 15 (1964), 238–250.

[35] Bunke, U., *Differential Cohomology*, arXiv:1208.3961

[36] Bunke, U., Nikolaus, T., Völkl, M., *Differential cohomology theories as sheaves of spectra*, arXiv:1311.3188

[37] Castellani, L., D'Auria, R., Fré, P., *Supergravity and Superstrings - A Geometric Perspective, vol. 1,2,3*, World Scientific, 1991

[38] Castellani, L., Catenacci, R., Grassi P. A, *The Geometry of Supermanifolds and New Supersymmetric Actions*, arXiv:1503.07886.

[39] Castellani, L., Catenacci, R., Grassi P. A, *Supergravity Actions with Integral Forms*, arXiv:1409.0192

[40] Castellani, L., Catenacci, R., Grassi P. A, *Hodge Dualities on Supermanifolds*, arXiv:1507.01421

[41] Castellani, L., Catenacci, R., Grassi P. A, *The Hodge Operator Revisited*, arXiv:1511.05105.

[42] Cheng, E., Lauda, A., *Higher Dimensional Categories: an illustrated guide book*,

[43] Cheung, P., *Supersymmetric field theories and cohomology*, arXiv:0811.2267

[44] Choquet-Bruhat, Y., DeWitt-Morette, C., *Analysis, Manifolds and Physics, Part II*, North Holland, 2000.

[45] Coleman, S., Mandula, J., *All Possible Symmetries of the S-Matrix*, Physical Review, 159(5), 1967, pp. 1251–1256.

[46] Connes, A., Kreimer, D., *Hopf Algebras, Renormalization and Noncommutative Geometry*, arXiv:hep-th/9808042

[47] Connes, A., Kreimer, D., *Renormalization in quantum field theory and the Riemann-Hilbert problem*, arXiv:hep-th/9909126

[48] Connes, A., Kreimer, D., *Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem*, arXiv:hep-th/9912092

[49] Costello, K., Gwilliam, O. , *Factorization algebras in perturbative quantum field theory*

[50] Dadam, F., Martins, Y. X., *Topologia, Geometria e Buracos Negros*, NEA, 2016.

[51] D'Auria, R., Fré, P., *Geometric Supergravity in D=11 and its hidden supergroup*, Nuclear Physics B201 (1982) 101-140

[52] Day, B.J., Kelly, G.M. *Enriched functor categories.* In: Reports of the Midwest Category Seminar III. Lecture Notes in Mathematics, vol 106. Springer, Berlin, Heidelberg, 1969.

[53] Deligne, P., Etingof, P., Freed, D. S., Jeffrey, L. C., Kazhdan D., Morgan, J. W., Morrison, D. R., Witten, E., *Quantum Fields and Strins: A Course for Mathematicians*, vol. 1 and 2, AMS, 1999.

[54] Deligne, P., *Catégorie Tensorielle*, Moscow Math. Journal 2 (2002) no. 2, 227-248.

[55] Deligne, P., Milde, J., *Tannakian categories*, 2012.

[56] DeWitt, B., *Supermanifolds*, Cambridge Monographs on Mathematical Physics, 1984

[57] tom Dieck, T., *Algebraic Topology*, EMS, 2008

[58] Donaldson, S., *An Application of Gauge Theory to Four Dimensional Topology*, Journal of Differential Geometry, 18 (2): 279–315, 1983.

[59] Dubuc, E. J., *Kan Extensions in Enriched Category Theory*, Springer,

[60] Dyson, F., *Divergence of perturbation theory in quantum electrodynamics*, Phys. Rev. 85, 631, 1952

[61] Eckmann, B., Hilton, P. J., *Group-Like Structures in General Categories I: Multiplications and Comultiplications*, Math. Annalen, 142, p. 227-255, 1962.

[62] Eilenberg, S., Kelly, G. M., *Closed categories*. Proc. Conf. on Categorical Algebra (La Jolla 1965), (Springer-Verlag 1966), 421–562.

[63] Elmendorf, A., Mandell, M., *Permutative categories, multicategories, and algebraic K-theory*, arXiv:0710.0082

[64] Elmendorf, A. D., Kriz, I., Mandell, M. A., May, J. P., *Rings, modules, and algebras in stable homotopy theory*, with an appendix by M. Cole, Mathematical Surveys and Monographs, 47, AMS, Providence, RI (1997)

[65] Elmendorf, A. D., *The Development of Structured Ring Spectra*, in Structured Ring Spectra, Cambridge Univesity Press, 2004.

[66] Elmendorf, A. D., *Compromises forced by Lewis's theorem*, in Structured Ring Spectra, Cambridge Univesity Press, 2004.

[67] Elmendorf, A. D., Kriz, I., May, J. P., *Modern Foundations for Stable Homotopy Theory.*

[68] Etingof, P., Gelaki, S., Nikshych, D., Ostrik, V., *Tensor categories*, AMS, 2015

[69] Fecko, M., *Differential Geometry and Lie Groups for Physicists*, Cambridge University Press, 2006.

[70] Fedosov, B., *Formal quantization*, Some Topics of Modern Mathematics and their Applications to Problems of Mathematical Physics (in Russian), Moscow (1985), 129-136.

[71] Fedosov, B., *Index theorem in the algebra of quantum observables*, Sov. Phys. Dokl. 34 (1989), 318-321.

[72] Fedosov, B., *A simple geometrical construction of deformation quantization*, J. Differential Geom. Volume 40, Number 2 (1994), 213-238.

[73] Freed, D. S., *Dirac Charge Quantization and Generalized Differential Cohomology*, arXiv:hep-th/0011220.

[74] Freed, D. S., *The cobordism hypothesis*, arXiv:1210.5100.

[75] Freed, D. S.; Uhlenbeck, K. K., *Instantons and four-manifolds*, Springer, 1984.

[76] Feshbach, M., Voronov, A. A., *A higher category of cobordisms and topological quantum field theory*, arXiv:1108.3349

[77] Gabriel, P., Zisman, M., *Calculus of fractions and homotopy theory,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer, New York (1967).

[78] Gelfand, S. I., Manin, Y. I., *Methods of Homological Algebra*, Springer, 2003.

[79] Getzler, E., *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem*, Comm. Math. Phys. Volume 92, Number 2 (1983), 163-178.

[80] Getzler, E., *A short proof of the local Atiyah–Singer index theorem*, Topology, 25: 111–117, 1988.

[81] Goldblatt, R., *Topoi: The Categorical Analysis of Logic*, Dover Publications, 2006.

[82] Goodwillie, T., *The differential calculus of homotopy functors*, ICM

[83] Gordon, Power, Street, *Coherence for tricategories*, Mem. Amer. Math Soc. 117 (1995) no 558

[84] Gotay, M. J., http://arxiv.org/abs/math-ph/9809011

[85] Gotay, M. J., Grabowski, J., Grundling, H. B., http://arxiv.org/abs/dg-ga/9706001

[86] Gotay, M. J., Grundling, H. B., http://arxiv.org/abs/dg-ga/9710024

[87] Gotay, M. J., *Functorial Geometric Quantization and Van Hove's Theorem*, Int. J. Th. Ph., vol. 19, n. 2, 1980.

[88] Gray, B., *Homotopy Theory, an Introduction to Algebraic Topology*, Academic Press, 1975.

[89] Groenewold, H. J., *On the Principles of elementary quantum mechanics*, Physica 12, pp. 405-460, 1946

[90] Grothendieck, A., *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. vol 9, n.2, 3, 1957, The Tohoku university, Sendai, Japan

[91] Hatcher, A., *Algebraic Topology*, Cambridge University Press, 2001.

[92] Haugseng, R., *Iterated spans and "classical" topological field theories*, arXiv:1409.0837

[93] Henriques, A., *Lecture by Michael Hopkins: the string orientation of tmf*, arXiv:0805.0743.

[94] Henriques, A., *A proof of Bott periodicity via Clifford algebras*, 2009

[95] Hilbert, D., Mathematical Problems, Bulletin of the American Mathematical Society, vol. 8, no. 10 (1902), pp. 437-479. Earlier publications (in the original German) appeared in Göttinger Nachrichten, 1900, pp. 253-297, and Archiv der Mathematik und Physik, 3dser., vol. 1 (1901), pp. 44-63, 213-237.

[96] Hirschhorn, P. S., *Model Categories and Their Localizations*, AMS, 2009

[97] Hoffnung, A. E., *Spans in 2-Categories: A monoidal tricategory*, arXiv:1112.056

[98] Hopf, H., *Abbildungsklassen n-dimensionalcr Mannigfaltigkeiten*, Math. Annalen 96 (1926), 209-224.

[99] Hopkins, M. J., Singer, I. M., *Quadratic functions in geometry, topology and M-theory*, arXiv:math/0211216

[100] Hovey, M., *Model Categories*, AMS, 2007.

[101] Hovey, M., Shipley, B., Smith, J., *Symmetric spectra*, J. Amer. Math. Soc. 13, 1 (2000), 149–208

[102] Huerta, J., *Division Algebras, Supersymmetry and Higher Gauge Theory*, PhD thesis, UCLA, Riverside, 2011.

[103] Hüsemoller, D., *Fibre Bundles*, Springer-Verlag, second edition

[104] Husemöller, D., Joachim, M., Jurco, B., Schottenloher M., *Basic Bundle Theory and K-Cohomology Invariants*, Springer-Verlag, 2008.

[105] J.-Freyd, T., *The formal path integral and quantum mechanics*, arXiv:1004.430

[106] Janelidze G., *Internal crossed modules*, Georgian Mathematical Journal 10 (2003) pp 99-114.

[107] Johnstone, P., *Topos theory*, London Math. Soc. Monographs 10, Acad. Press 1977, xxiii+367 pp.

[108] Karoubi, M., *Algèbres de Clifford et K-théorie*, Ann. Sci. École Norm. Sup. , 1(4):161–270, 1968.

[109] Karoubi, M, *K-theory*, Springer, 2008.

[110] Kelly, M., *Basic Concepts of Enriched Category Theory, Cambridge University Press,* 1982. Republished in: *Reprints in Theory and Applications of Categories*, No. 10 (2005) pp. 1-136.

[111] Kervaire, M. A., Non-Parallelizability of the n-Sphere for $n > 7$, Proc Natl Acad Sci U S A. 1958 Mar; 44(3): 280–283.

[112] Kervaire, M. A., Milnor, J. W., *Groups of homotopy spheres: I*, Annals of Mathematics. Princeton University Press. 77 (3): 504–537

[113] Kobayashi, S., Nomizu, L., *Foundations of Differential Geometry, Vol.1*, Wiley Classics Library, 1996.

[114] Kochman, S. O., *Bordism, Stable Homotopy and Adams Spectral Sequences*, AMS, 1996.

[115] Kono, A., Tamaki, D., *Generalized Cohomology*, AMS, 2006.

[116] Kontsevich, M., *Deformation quantization of Poisson manifolds, I*, arXiv:q-alg/9709040

[117] Labastida, J., Mariño, M., *Topological Quantum Field Theory and Four Manifolds*, Springer, 2005.

[118] Lang, S., *Algebra*, Springer-Verlag

[119] Lawson, H. B., Michelsohn, M., *Spin Geometry*, Princeton University Press, 1989.

[120] Leinster, T., *A Survey of Definitions of n-Categories*, Theory and Applications of Categories, 10, 2002, 1-70.

[121] Leinster, T., *Higher Operads, Higher Categories*, Cambridge University Press

[122] Lemos, A., N., *Convite à Física Matemática*, Editora Livraria da Física, 2013

[123] Lewis, L. G. Jr, *Is there a convenient category of spectra?* Journal of Pure and Applied Algebra 73 (1991) 233-2-M

[124] Lurie, J., *On the Classification of Topological Field Theories*, arXiv:0905.046.

[125] Lurie, J., *Higher Algebra*,

[126] Lurie, J., *Stable Infinity Categories*, arXiv:math/0608228

[127] Lurie, J., *Higher Topos Theory*, Princeton University Press, 2009

[128] Lurie, J., *Derived Algebraic Geometry III: Commutative Algebra*, arXiv:math/0703204

[129] Lurie, J., $(\infty, 2)$-*Categories and the Goodwillie Calculus I*, arXiv:0905.0462

[130] Lurie, J., E_k-*algebras* ....

[131] Mac Lane, S., *Categories for the Working Mathematician*, Springer, 1998.

[132] Mac Lane, S., Moerdijk, I., *Sheaves in Geometry and Logic*, Springer, 1992.

[133] Mac Lane, S., *Natural associativity and commutativity*, Rice University Studies 49, 28-46 (1963)

[134] Malkiewich, C., *The stable homotopy category.*

[135] -Maluga, T. A., Brans, C., *Exotic Smoothness and Physics*, World Scientific, 2007.

[136] Marcolli, M, *Seiberg-Witten Gauge Theory.*

[137] Margolis, H. R., *Spectra and the Steenrod Algebra*, ELSEVIER SCIENCE PUBLISHERS B.V, 1983.

[138] Mariño, M., *Chern-Simons theory, matrix models and topological strings*, Oxford University Press, 2005.

[139] Martins, Y. X., Biezuner, R. J., *On obstructions to the topology of the universe*, in preparation.

[140] Martins, Y. X., *Introdução à Teoria da Homotopia Abstrata*, to appear.

[141] May, J. P., *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999

[142] May, J. P., E_∞-*ring spaces and* E_∞-*ring spectra*, Springer-Verlag, Berlin, 1977.

[143] May, J. P., *What precisely are E_∞-ring spaces and E_∞-ring spectra*?, Geometry and Topology Monographs 16 (2009) 215–282

[144] May, J. P., *Stable Algebraic Topology, 1945-1966.*

[145] May, J. P., Sigurdsson, J., *Parametrized Homotopy Theory*, AMS 132, 2006.

[146] May, J. P., Ponto, K., *More Concise Algebraic Topology: Localization, Completion, and Model Categories*, University Of Chicago Press, 2012

[147] McLarty, C., *Elementary Categories, Elementary Toposes*, Oxford Logic Guides, 1996.

[148] de Melo, W., de Faria, E., *Mathematical Aspects of Quantum Field Theory*, Cambridge University Press, 2010.

[149] Moore, J. D., *Lectures on Seiberg-Witten Invariants*, Springer, 1991.

[150] Morgan, J. W., The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds, Princeton University Press, 1996.

[151] Mosher, R. E., Tangora, M. C., *Cohomology Opperations and Applications in Homotopy Theory*, Dover Publications, 2008.

[152] Milnor, J. W., *The Steenrod algebra and its dual*, Ann. of Math. 67 (1958), 150–171.

[153] Milnor, J. W., *On Axiomatic Homology Theory*, Pacific Journal of Mathematics, v. 12, n.1, 1962.

[154] Milnor, J. W., *On manifolds homeomorphic to the 7-sphere*, Annals of Mathematics, 64 (2): 399–405.

[155] Milnor, J. W., *Differentiable structures on spheres*, American Journal of Mathematics, 81 (4): 962–972

[156] Milnor, J., Moore, J., *On the structure of Hopf algebras*, Annals of Math. 81 (1965), 211-264.

[157] Nicolaescu, L. I., *Notes on Seiberg-Witten Theory*, AMS, 2000.

[158] Nicholson, W. K., *A Short Proof of Weddernburn-Artin Theorem*, New Zeland Journal of Mathematics Volume 22 (1993), 83-86

[159] Nuiten, J., *Cohomological quantization of local prequantum boundary field theory*, MSc thesis, Utrecht, August 2013

[160] Ohtsuki, T., *Quantum Invariants*, World Scientific, 2002.

[161] Oliveira, C. R., *Intermediate Spectral Theory and Quantum Dynamics*, Birkhäuser, 2000

[162] Ortin, T., *Gravity and Strings*, Cambridge University Press, 2004.

[163] Ostrik, V., *Tensor categories (after P. Deligne)*, arXiv:math/0401347

[164] Park, E., *Complex K-theory*, Cambridge University Press, 2008.

[165] Paugam, F., *Towards the Mathematics of Quantum Field Theory*, Springer, 2014.

[166] Paul S. Aspinwall Bridgeland, T., Craw, A., Douglas, M. R., Gross, M., Kapustin, A., Moore, G. W., Segal, G., Szendr?i, B., Wilson, P. M. H., *Dirichlet Branes and Mirror Symmetry*, AMS, 2009.

[167] Porst, S.-S, *Strict 2-Groups are Crossed Modules*, arXiv:0812.1464

[168] Power, A. J., *A general coherence result,* J. Pure Appl. Algebra 57 (1989), no. 2, 165–173.

[169] Rabin, J. M., Crane, L., *Global Properties of Supermanifolds*, Commun. Math. Phys. 100, 141-160 (1985)

[170] Reed, M., Simon, B., *Functional Analysis*, Academic Press, 1980

[171] Reed, M., Simon, B., *Fourier Analysis, Self-Adjointness*, Academic Press, 1975

[172] Riehl, E., *Categorical Homotopy Theory*, Cambridge University Press, 2014.

[173] Roberts, D. M., *Internal categories, anafunctors and localisations*, arXiv:1101.2363

[174] Rogers, A., *Supermanifolds, Theory and Applications*, World Scientific, 2007

[175] Rudyak, Y. B., *On Thom Spectra, Orientability and Cobordism*, Springer, 2008.

[176] Sachse, C., *A Categorical Formulation of Superalgebra and Supergeometry*, arXiv:0802.4067.

[177] Salamon, D., *Spin Geometry and Seiberg-Witten Invariants*, 1996.

[178] Sati, H., Schreiber, U. (editors), *Mathematical Foundations of Quantum Field Theory and Perturbative String Theory*, AMS, 2010.

[179] Sati, H., Schreiber, U., Stacheff, J., *L-infinity algebra connections and applications to String- and Chern-Simons n-transport*, arXiv:0801.3480

[180] Schommer-Pries, C. J., *The Classification of Two-Dimensional Extended Topological Field Theories,* arXiv:1112.1000.

[181] Schreiber, U., *What, and for what is Higher geometric quantization.*

[182] Schreiber, U., *Differential Cohomology in a Cohesive ∞-Topos*, arXiv:1310.7930.

[183] Schreiber, U., *Differential Cohomology in a Cohesive ∞-Topos*, v.2.

[184] Schreiber, U., Konrad, W., *Smooth Functors vs. Differential Forms*, arXiv:0802.0663

[185] Schreiber, U., Konrad, W., *Connections on Connections on Non-Abelian Gerbes and their Holonomy*, arXiv:0808.1923

[186] Schreiber, U., *Why Supersymmetry? Because of Deligne's theorem,* 2016. https://www.physicsforums.com/insights/supersymmetry-delignes-theorem/

[187] Schwede, S., *Symmetric Spectra.*

[188] Segal, G., *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 250, Kluwer Acad. Publ., Dordrecht, (1988), 165-171

[189] Seiberg, N.; Witten, E., *Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nuclear Phys. B, 426 (1): 19–52, 1994.

[190] Seiberg, N.; Witten, E., *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, Nuclear Phys. B, 431 (3): 484–550, 1994.

[191] Sharma, A., *Cobordism Hypothesis*, 2014.

[192] Shiraiwa, K., *A Note on Tangent Bundles*, Nagoya Mathematical Journal, Volume 29 March 1967 , pp. 259-267.

[193] Shulman, M. A., *Not every pseudoalgebra is equivalent to a strict one*, arXiv:1005.1520

[194] Simons, J., Sullivan, D., *Axiomatic Characterization of Ordinary Differential Cohomology*, Journal of Topology 1.1 (2008): 45-56.

[195] Simpson, C., *Homotopy types of strict 3-groupoids*, arXiv:math/9810059

[196] Steenrod, N., *A convenient category of topological spaces*, Michigan Math. Journal, 14, 1967.

[197] Sternberg, S., *Lectures on Differential Geometry*, Prentice-Hall, 1964.

[198] Stoltz, S., Teichner, P., *Supersymmetric field theories and generalized cohomology*, arXiv:1108.0189

[199] Strom, J., *Modern Classical Homotopy Theory*, AMS, 2011.

[200] Switzer, R. M., *Algebraic Topology: Homology and Homotopy*, Springer-Verlag, 1975

[201] Teleman, C., *Five Lectures on Topological Field Theory.* In: Alvarez Consul L., Andersen J., Mundet i Riera I. (eds) Geometry and Quantization of Moduli Spaces. Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, Cham

[202] Tuynman, G. M., *Supermanifolds and Supergroups, Basic Theory*, Kluwer Academic Publishers, 2004

[203] van Hove, L. , Mem. Acad. Roy. de Belgique, Classe des Sci. 26 (1951) No. 6.

[204] Taubes, C. H., *Seiberg Witten and Gromov Invariants for Symplectic 4-Manifold*, International Press, 2000.

[205] Taubes, C. H., *Metrics, Connections and Gluing Theorems*, AMS, 1996.

[206] Trimble T., *Notes on Tetracategories*, 1995. Available as math.ucr.edu/home/baez/trimble/tetracategories.html.

[207] Weinberg, S., *The Quantum Theory of Fields, vol.* 1, Cambridge University Press, 2005.

[208] Wigner, E., *On unitary representations of the inhomogeneous Lorentz group*, Annals of Mathematics, 40 (1939) 149–204

[209] Witten, E., *D-Branes And K-Theory*, arXiv:hep-th/9810188.

[210] Witten, E., *Topological quantum field theory*, Comm. Math. Phys., Volume 117, Number 3 (1988), 353-386.

[211] Witten, E., *Topological Sigma Models*, Comm. Math. Phys., Volume 118, Number 3 (1988), 411-449.

[212] Witten, E., *Monopoles and four-manifolds*, Mathematical Research Letters, 1 (6), 769–796, 1994.

[213] Witten, E., *Quantum Field Theory and the Jones Polynomial*, Comm. Math. Phys., 121, 351-399 (1989)

[214] Yuli, B. R., *On Thom Spectra, Orientability and Cobordism*, Springer, 2008.

[215] Zee, A., *Quantum Field Theory in a Nutshell*, Princeton University Press, 2010

[216] Zhao, Y., *Extended Topological Field Theories and the Cobordism Hypothesis*, postdoctor thesis