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Open Quantum System Approach to Neutral Kaon Interferometry

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This work is dedicated to those who
don't want to believe, but want to know.

"The good thing about science is that it is true whether or not you believe in it."

Neil deGrasse Tyson

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Abstract

Besides Heisenberg uncertainty, entanglement is another mechanism that can enforce complementarity in multipartite quantum systems. Analysing its effects in a bipartite system of two qubits, M. Jakob and J. Bergou had shown that the usual wave-particle duality relation, in interferometric systems, regarding which-way information and interference visibility, should be extended to a "triality" relation containing, in addition, a quantitative entanglement measure. Employing a model for the neutral K-meson propagation in free space in which its weak decay products are included as a second party, we study its effect in the neutral kaon interferometry. We show that a new quantitative triality relation can also be established in this case. The state of the total system remains pure in the course of dynamics, so, we can use the von Neumann entropy of a reduced party as a quantitative measure of entanglement. The other two quantities in the triality relation are the distinguishability between the decay products states corresponding to the two distinct kaon propagation modes K_S and K_L, and the wave-like path interference between these states. These two distinct modes are the analogues to the two separated paths in usual interferometric devices. The inequality obtained here can account for the complementarity between strangeness oscillations and lifetime information previously considered in the literature. Moreover, it allows us to visualise through the $K^0 - \overline{K^0}$ oscillations the fundamental role of entanglement in quantum complementarity.

Keywords: Complementarity. Entanglement. Neutral-Kaon. Interferometry.

Resumo

Ademais do princípio da incerteza de Heisenberg, o emaranhamento é outro mecanismo capaz de levar à complementaridade em sistemas quânticos com múltiplas partições. Analisando seus efeitos em um sistema bipartido de dois *qubits*, M. Jakob e J. Bergou mostraram que a habitual relação de dualidade onda-partícula em sistemas interferométricos, que leva em conta quantificadores de informação de caminho e visibilidade na interferência, deveria ser estendida a uma relação de "trialidade", contendo, também, um quantificador de emaranhamento. Empregando um modelo para a propagação do méson K neutro no espaço livre no qual seus produtos de decaimento são incluídos como uma segunda partição estudamos seu efeito na interferometria do káon neutro. Mostramos que uma nova relação de trialidade quantitativa pode ser estabelecida nesse caso. O estado total do sistema permanece puro ao longo da dinâmica e, assim, pudemos usar a entropia de von Neumann de uma partição reduzida como o quantificador de emaranhamento. As outras duas quantidades que entram na relação de trialidade são a *distinguibilidade* entre os estados de produtos de decaimento correspondentes aos distintos modos de propagação dos káons, K_S e K_L, e um quantificador de interferência de caminho entre esses dois estados. Esses dois distintos modos funcionam como análogos aos dois diferentes caminhos em sistemas interferométricos usuais. A inequação obtida aqui é consistente com a relação de complementaridade entre oscilações de estranheza e informação de "tempo de vida" previamente considerados na literatura. Além disso, nos permite visualizar através das oscilações $K^0 - \overline{K^0}$ o papel fundamental do emaranhamento no conceito de complementaridade.

Palavras-Chave: Complementaridade. Emaranhamento. Kaon neutro. Interferometria.

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Introduction

The neutral K-meson, or neutral kaon, is a very rich physical system encountered in Nature. The particle itself was discovered in 1947 in a cloud-chamber experiment conducted by Rochester and Butler [1] and, owing to its surprising appearance, already represented a disrupt in the paradigm of particle physics at the time. Apart from being unexpected, the particle first brought attention due to its long lifetime of, roughly, 10^{10} times greater than previously discovered particles [2]. To account for this new feature, a new quantum number, which came to be known as strangeness, had to be assigned for all particles.

The introduction of this strangeness number and, therefore, the discovery of neutral kaons, was essential for particle physics and posteriorly led to the development of the quark model and the Standard Model. Furthermore, this system is where several common phenomena in particle physics, such as particle mixing and oscillation, regeneration and CP-violation, were initially observed. Strangeness oscillations in kaons occurs because the strangeness eigenstates, in which the particles are produced, are not eigenstates of the weak interaction responsible for its decay. Rather, they are a linear combination of the short-lived and long-lived weak eigenstates [3]. Moreover, since these short-lived and long-lived states have decay products with different CP eigenvalues, they constitute the perfect set-up to the investigation of CP-violation, discovered in 1964. Likewise, they are also important in the investigation of other fundamental symmetries like time reversal symmetry T, and the combination CPT [4].

Lately, its value became obvious also in the field of quantum information. Entangled kaon pairs are one of the many set-ups where Bell inequalities are constantly being tested [5, 6, 7]. Beyond that, because an analogy between this system and an interferometer is easily made, they proved to be an entire new arena where quantitative statements of complementarity and quantum erasure can be examined [8, 9, 10, 11]. The aim of this master thesis is to show how a complementarity relation involving entanglement and ordinary interferometric quantities can be established when we use an open quantum system model for the evolution of kaons and its decay products.

This thesis is organised as follows.

Chapter 1 contains the main ideas related to the neutral K-meson system. We start with a brief review on its history: how it was discovered and its importance in the development of particle physics. In the following section we present the concept of strangeness and how neutral kaons are defined in terms of this quantum number. In Section 1.2 we discuss the concepts and properties of the parity and charge conjugation symmetries as well as their relation and consequences to neutral kaons. This will lead us to the introduction of their CP-states. Then, in the next section, we examine the phenomenon of CP-violation by neutral kaons, define their long and short lived mass states and summarise its properties - lifetimes, masses, quark content and decay rates - in a table. In the sequence, we describe the dynamics of kaons in the Wigner-Weisskopf approximation and discuss the phenomena of strangeness oscillation and regeneration.

Chapter 2 is devoted to introducing two basic concepts of quantum mechanics: complementarity and entanglement. We begin with a discussion on complementarity by analysing the particle-wave duality in Young's double slit experiment. In the next section, we present the quantitative complementarity relation of Greenberger and Yasin. The two entries in this quantitative statement are the path predictability, a particle-like property, and the wave-like property, fringe visibility [12]. Section 2.2 is dedicated to the concept of entanglement. We give a definition for it, discuss how it can be quantified for general pure states and in the specific case of two qubits. Its relation to complementarity is also explored. In the last section, we revisit the problem of quantitative statements of complementarity. We present Englert's inequality relating fringe visibility and which-way information for a system entangled with a which-way detector [13] and the Jakob and Bergou's quantitative complementarity relation for pure bipartite system of two qubits [14]. In this last relation, we see that the entanglement measure concurrence enters explicitly as one of the complementary quantities.

Open Quantum Systems is the topic of Chapter 3. We introduce its formalism: the dynamical map, the Kraus operator sum representation and the Lindblad master equation. Furthermore, we discuss these concepts properties. We, then, show how this approach can be used as a model for the dynamics of unstable particles. Particularly, we present the model developed by Caban *et. al.* for the time evolution of neutral pions and kaons [15], showing its compatibility with the phenomenological Wigner-Weisskopf prescription.

In Chapter 4 we study complementarity in the neutral kaon system. First we analyse the work of Bramon, Garbarino and Hiesmayr on the neutral kaon interferometry. We introduce its basic ideas, how it can be seen as an interferometric system, and its results - a quantitative complementarity relation [8, 9, 10]. Then we investigate it using an open quantum system model in which the kaon's weak decay products are treated as a second party, allowing us to the study the role of bipartite entanglement in this set-up [11]. As a result we get a triality relation similar to one obtained by Jakob and Bergou [14]. Here, the system's entropy of entanglement enters as one of the complementary quantities, along with distinguishability and an interference visibility defined in terms of the fidelity between the states associated with the two distinct propagation modes. Furthermore, we discuss how this new inequality accounts for the complementarity between strangeness and lifetime information in the neutral kaon system.

Finally, in Chapter 5, we conclude the thesis. We emphasise the role of entanglement in complementarity and discuss how its inclusion in a quantitative statement of complementarity refine the limits for the observed reduction in the interference visibility.

Moreover, in Appendix A, one finds a derivation for the dynamics of neutral kaons in the Wigner-Weisskopf approximation [16]. Appendix B specifies the properties of the dynamical map. It also contains a "coarse-grained" derivation of the Lindblad Master Equation [17]. Appendix C contains Choi's theorems for completely positive maps [18, 19], Sylvester's criterion [20] for positive Hermitian matrices and detailed calculations involved in the derivation of the open quantum system model for the neutral pion evolution [15].

1 The Neutral K Meson

By the beginning of 1947 physicists believed that the theory of elementary particle physics was essentially done. It was long known that matter was made of atoms with very small and very heavy nuclei, composed of bounded protons and neutrons, with electrons "orbiting" around them. Photons were known to be the mediators – the "messengers" – of the electromagnetic interaction. In the same way, the pions, which had been discovered in the beginning of that year [21], were seen as the mediators of the strong force that bounded the protons and neutrons together in the nuclei. At the time, Dirac's idea of the existence of antiparticles was confirmed by the long discovery of the positron. Although not yet detected, there was compelling evidence for the existence of neutrinos, such as Fermi's theory of beta decay [22] and the kink in the direction of the muon produced in the decay of the pion. These last particles, called mu-meson at the time, appeared as the only puzzle remaining. They seemed quite unnecessary in this overall scheme of things, but, it was possible to believe that there were not any major problems in elementary particle physics.

At the end of 1947, however, a publication by Rochester and Butler ended this comfortable state. By analysing a cloud-chamber photograph of cosmic rays showers penetrating a lead plate, they concluded that a new neutral particle was produced in this process [1]. In their experiment, the presence of this new particle is revealed when it decays into two charged particles that leaves a forked "V" track in the gas of the chamber. Further analysis showed that these charged particles were in fact charged pions. Here, then, was a new neutral particle – which we nowadays call neutral kaon – with at least twice the mass of the pion. Over the following years, several other particles, such as charged kaons, Λ, Σ's, Ξ's, and so on, were discovered, and they came to be collectively known as "strange" particles. Besides their unexpected appearance, another feature that contributed for the assignment of the name "strange" to these particles was the observation of their long lifetime. While particles decaying via the strong interaction have a lifetime of about $10^{-20}s$, these new particles were observed to have a lifetime of about $10^{-10}s$ [2]. This led Gell-Mann and Nishijima to propose a new property to all particles which could account for this decay inhibition – the so-called strangeness.

Since its discovery, the (neutral) kaon has became an unique source of fundamental discoveries in physics. It greatly contributed to the development of the *Eightfold Way* and the quark model, which were essential to the posterior establishment of the Standard Model [2]. It, also, constitutes the first system where particle oscillation, particle regeneration and violation of CP symmetry has been observed. In addition, it is a key system for current tests of time reversal symmetry, T, and the combined CPT symmetry [4]. More recently, they also brought the attention of quantum information physicists, proving to be suitable for tests of Bell inequalities [5, 6, 7], Bohr's complementarity – which we analyse in this master thesis – and quantum erasure [8, 9, 10, 11].

1.1 Strangeness

To explain the long lifetime of the strange particles, Gell-Mann and Nishijima had to assign to all particles a new quantum number, which we nowadays call strangeness [23, 24]. They stated that this new quantum number was conserved by any strong and electromagnetic processes but not conserved in weak processes. This way, the "strange particles" could not decay via strong or electromagnetic interactions. Therefore, they could only decay via the much slower weak interaction, thus providing these particles their long lifetime.

Presently, we define the strangeness of a particle as the difference in the number of strange antiquarks and quarks present in its composition, i.e.:

$$S = -(n_s - n_{\bar{s}}). \tag{1.1}$$

With the introduction of strangeness, we can define kaons as eigenvectors of the strangeness operator S. The neutral K mesons consist of a bounded state of a down quark and a strange antiquark and, hence, have strangeness equals to unity. Therefore, kaons (antikaons), represented as $|K^0\rangle$ ($|\overline{K^0}\rangle$), are eigenvectors of S with eigenvalue +1 (-1), that is,

$$\begin{aligned} S|K^0\rangle &= +|K^0\rangle, \\ S|\overline{K^0}\rangle &= -|\overline{K^0}\rangle, \end{aligned} \tag{1.2}$$

where $|K^0\rangle$ and $|\overline{K^0}\rangle$ form an orthonormal basis.

Since strangeness is conserved by strong and electromagnetic interactions, this basis is ideal when one wants to study the production of $K^0\overline{K^0}$ pairs from systems with initial null strangeness. Examples are production of this particles in strong processes, such as $p\bar{p}$ collisions, or in electromagnetic processes, such as e^+e^- collisions [25]. Likewise, this is the basis which one must deal with when detecting these particles by looking at its strong interaction with nucleons, which will project the initial arbitrary state of the kaon in one of the strangeness eigenvectors.

1.2 CP States

Until 1956 physicists believed that nature was ambidextrous; that is, it was taken for granted that the mirror image of any physical process represented another perfectly and equally possible process - which was called the "parity invariance". However, in the beginning of that year, C. N. Yang and T. D. Lee suggested that weak interactions might violate this law. Following this suggestion, in the next months, an experiment conducted by C. S. Wu with radioactive cobalt 60 nuclei undergoing beta decay proved them right [26]. The so-called parity invariance, the "mirror symmetry", was not preserved in these weak events.

In this experiment, by means of a near absolute zero temperature and a uniform magnetic field, the Co^{60} nuclei were aligned so that their spins pointed in a specific direction. Then, the directions of the electrons emitted by the beta decay of these nuclei were recorded [26]. If parity conservation was really present in this system, the electrons produced in this phenomenon would be emitted with no preferred direction of decay relative to the nuclear spin (or magnetic moment vector). But the result was that, most of the electrons originated from the decays were emitted in the direction opposite to the magnetic moment vectors of the nuclei - see Figure 1 - and, hence, the experiment established the first observation of violation of parity.

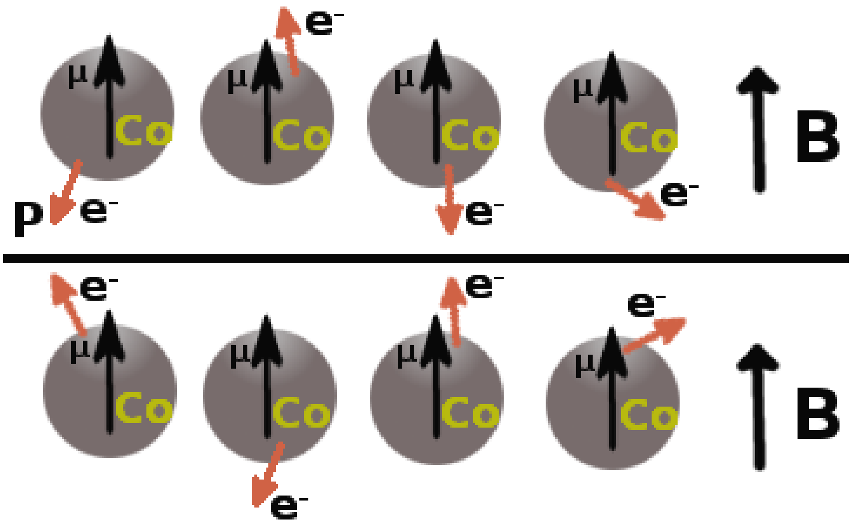

Figure 1 – Representation of Wu's results. Above the "P line" are the results obtained by Wu and her collaborators: the majority of the electrons emitted in the decays of several Co^{60} isotopes, represented by brown arrows, go to the direction opposite to that of their respective father nucleus magnetic moment. Since the electrons are emitted in these preferred directions, parity symmetry is violated. Below the "P line" we show how the parity conjugated version of these results would be.

Another, and even more severe, example of the breaking of this symmetry occurs in the weak decay of the pion [2]

$$\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu. \tag{1.3}$$

Measurements of this decay process show that all antineutrinos produced in the decay are right-handed[1]. The "handedness" of a particle describes the direction of its spin vector along the direction of motion. Therefore, "left-handed" means that the spin vector of the particle points always in the opposite direction of its momentum vector, while "right-handed" means that its spin vector always points in the same direction of its momentum vector. Under parity operation the neutrinos would change their handedness, thus, if there was parity conservation in these weak decays, half of the antineutrinos they produce would have to be left-handed, which is not observed, so these weak decay also violate parity conservation. In fact, it is noticed that violation of parity is practically the trademark of weak processes and can be easily found once you look for it [2].

The parity operator - P - action is to take each point and carry it into its diametrically opposite point relative to the origin of the coordinate system, i.e., $P|\Psi(\vec{r})\rangle = |\Psi(-\vec{r})\rangle$. If our system is, for example, a right hand, P turns it into an upside down and backwards left hand. This means that, when applied to a vector $\vec{x}$, P produces a vector with opposite sign, i.e., $P\vec{x} = -\vec{x}$. Furthermore, the operator P enables then a distinction between two kinds of vectors. If one applies it to a cross product of vectors, $\vec{z} = \vec{x} \times \vec{y}$, one gets $P\vec{z} = \vec{z}$. So, one can conclude that there is "ordinary" vectors, like $\vec{x}$ and $\vec{y}$, and other kind of vectors, like $\vec{z}$, which are called pseudovectors.

For kaons we have $P|K^0\rangle = -|K^0\rangle$, which expresses the fact that kaons are represented by pseudoscalar fields [2, 16].

Aside from parity, a second quantum number believed at the time to be conserved in nature was the "charge conjugation number". The effect of the charge conjugation operator C, is that it changes every particle into its antiparticle. This means this operator changes the sign of all internal quantum numbers of a particle - like electric charge, colour, flavour, magnetic moment, etc. - while it keeps mass, energy, momentum and spin the same. That is to say, for a particle p,

$$C|p\rangle = |\bar{p}\rangle. \tag{1.4}$$

The laws of electromagnetism, for example, are invariant under this transformation. If we change each charge q to a charge $-q$, thus reversing the direction of the electric and magnetic fields, the dynamics will preserve the same form. This statement can be easily verified in the classical version of electromagnetism. By making a simultaneous change in the signals of electromagnetic charges and fields we still get the same Lorentz Law, so the forces come out the same. In addition, charge conjugation symmetry also holds true in gravity and strong interactions.

Weak interactions, however, violate this C-symmetry. If we would apply the operator C in a neutrino, which are always left-handed[2], it would gives us a left-handed antineutrino.

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1 Considered here in the rest frame of the original pion.

2 Neutrinos were first believed to be massless, and, as consequence, only left-handed. But since that is

But antineutrinos, which are produced precisely by weak processes, are always right-handed, so charge conjugation is not a symmetry of the weak interactions.

Nevertheless, physicists were not inclined to abandon the concept of a "mirror symmetry" and so they came up with a new idea. If we apply the operator P to the above equation for the decay of negatively charged pions into a muon and a antineutrino, equation 1.3, we change the handedness of these particles – the right-handed muons and antineutrinos become left-handed ones. If, continuing, we apply the operator C, we get positively charged pions decaying into left-handed antimuons and neutrinos. So, the CP operator turns left-handed antimuons (and neutrinos) into right-handed muons (and antineutrinos). Hence, the combination of these two operations gives us exactly what we observe in nature. This realisation led physicists to believe that it was in fact these combined operations that was the true "mirror symmetry" that every one has supposed to hold for all physical phenomena, and so, it came to be called CP-symmetry.

The assumption of CP-symmetry, however, leads to unexpected consequences in the neutral K meson system, as was showed by Gell-Mann and Pais [3]. They considered the following decay of kaons

$$K^0 \longrightarrow \pi^- + \pi^+. \tag{1.5}$$

In this decay the pions are left in a definite relative angular momentum l, and therefore with a definite charge conjugation quantum number $C = (-1)^l$, which doesn't change under the charge conjugation operation. Pions are pseudoscalars under parity, so they change sign under this operation. Hence, for the two pions, $P|\pi^+\pi^-\rangle = (-1)^2|\pi^+\pi^-\rangle$, and we get a $(-1)^2$ factor. Under CP operation, thus, the above process, 1.5, becomes

$$\overline{K^0} \longrightarrow \pi^- + \pi^+. \tag{1.6}$$

Assuming the validity of CP-symmetry, this second decay must also occur and must leave the pions in the same state. Furthermore, since the reverse processes must also be possible, at least as a virtual process, the weak interaction induces the transition $K^0 \rightleftarrows \pi^+ + \pi^- \rightleftarrows \overline{K^0}$. To treat this novel situation, Gell-Mann and Pais introduced a change of representation to describe the decay process.

Because kaons are pseudoscalars, that is,

$$P|K^0\rangle = -|K^0\rangle \quad \text{and} \quad P|\overline{K^0}\rangle = -|\overline{K^0}\rangle, \tag{1.7}$$

and operator C makes

$$C|K^0\rangle = |\overline{K^0}\rangle \quad \text{and} \quad C|\overline{K^0}\rangle = |K^0\rangle; \tag{1.8}$$

not true, they do have mass, there must exist right-handed neutrinos. However, the ones produced in the decay of the pion are always left-handed in the reference frame under consideration.

we have

$$CP|K^0\rangle = -|\overline{K^0}\rangle \quad \text{and} \quad CP|\overline{K^0}\rangle = -|K^0\rangle. \tag{1.9}$$

Therefore, the kaons states preserving CP-symmetry are given by

$$\begin{aligned} |K_1^0\rangle &= \frac{|K^0\rangle + |\overline{K^0}\rangle}{2}, \\ |K_2^0\rangle &= \frac{|K^0\rangle - |\overline{K^0}\rangle}{2}, \end{aligned} \tag{1.10}$$

where $CP|K_1^0\rangle = |K_1^0\rangle$ and $CP|K_2^0\rangle = -|K_2^0\rangle$. Hereafter, they argued that the particles which we really observe in the laboratory (in decay processes) are not K^0 and $\overline{K^0}$ but rather these symmetric and antisymmetric linear combinations of them, i.e.: the K_1^0 and K_2^0. Thereby, we still have the K^0 and $\overline{K^0}$ as primary objects in production phenomena, but the decay process is better described in terms of the K_1^0 and K_2^0.

Moreover, since K_1^0 and K_2^0 have different CP number they must have quite different patterns of decay, which any decay mode possible for one, not being possible for the other. The K_1^0 can only decay into states with $CP = +1$ while K_2^0 can only decay into states with $CP = -1$. Accordingly, we must have completely unrelated rates of decay for these two states and, thus, two independent lifetimes. In a state of zero angular momentum, the two pion state produced in the decay of kaons, $|\pi^+\pi^-\rangle$ has $C = +1$, and therefore $CP = +1$, whereas, a three pion state, $|\pi^+\pi^-\pi^0\rangle$, has $CP = -1$. Hence, we have K_1^0 decaying into two pions only and K_2^0 decaying into three pions only. Now, when a particle system decays, the rate depends upon the excess of energy available [2], so the two pions decay occurs much faster than the one with three pions. If we start with a beam of K^0's, then, the K_1^0 component will quickly decay away leaving, down the line, only a K_2^0 beam. Near the source of production we shall have lots of two pions decay, but later on, we must expect only three pions events.

At the time of Gell-Man and Pais publication, the three pions decay mode of the kaons were unknown and their theory were considered baloney. Yet, soon after that, Lederman and his collaborators found the K_2^0 meson, which survived for much longer than the already known, due to its two pion decay mode, K_1^0. [27].

1.3 CP-violation and Mass States

It is easily noticed that the neutral K meson system constitutes the perfect experimental set-up to test the validity of CP-symmetry [2, 28]. By waiting long enough, so that the short-lived component of a sample decays away, we can produce an extremely pure beam of the long-lived species. If at this point we observe a two pion decay we must infer from it that CP has been violated. Such an experiment was conducted by Cronin et. al. in 1964 [28, 29] and showed that this, in fact, occurs. The result was of, roughly, one $\pi^+ + \pi^-$

pair produced in about 500 (long lived kaon) decays. Therefore, the short-lived, K_S, and long-lived, K_L, kaons are not absolute eigenstates of the CP operator, but rather, some admixture of them containing a small amount of K_2^0 and K_1^0, respectively. That is,

$$\begin{aligned} |K_S\rangle &= \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_1^0\rangle + \epsilon|K_2^0\rangle), \\ |K_L\rangle &= \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_2^0\rangle + \epsilon|K_1^0\rangle), \end{aligned} \tag{1.11}$$

where $|\epsilon| = (2.228 \pm 0.011) \times 10^{-3}$ [4] is the so-called complex CP-violating parameter, which represents a measure of the departure of these states from the CP ones. The K_S and K_L are known as the mass states because they are the true weak eigenstates with definite lifetime and, accordingly, can be properly used to determinate the particles time evolution.

This former mentioned type of CP-violation is known as indirect CP-violation and is caused by the mixing of K^0 and its antiparticle. But there is also another kind of CP-violation, the direct CP-violation effect, which can be seen in the semileptonic decay of K_L. Alternatively to the three pions mode, the K_L can also decay by the following semileptonic ones

$$\begin{aligned} K_L &\longrightarrow \pi^+ + e^- + \nu_e, \\ K_L &\longrightarrow \pi^- + e^+ + \bar{\nu}_e. \end{aligned} \tag{1.12}$$

The CP operator takes one mode into the other. So, in order to preserve CP-symmetry, we should have these two modes occurring with the same probability. However, experiments show that the decay producing positrons is a little bit more common than the one producing electrons, the difference being of $(3.32 \pm 0.06) \times 10^{-3}$ [4]. This outcome accounts for the first physical process observed to make an absolute distinction between matter and antimatter. Furthermore, it enables us to address two fundamental issues. First, it provides an unambiguous and convention-free definition to the positive charge: the positive charge is the charge of the lepton more often produced in the semileptonic decay of the long-lived neutral kaon. The other one, is the Baryonic Asymmetry problem. Since CP-violation provides an unequal treatment for particles and antiparticles, it might be the mechanism responsible for the verified dominance of matter over antimatter in the Universe [30]. It is worth noticing, likewise, that the breaking of CP-symmetry in weak phenomena, although a small effect, and not present in the other interactions, ended the last hopes for a general "mirror symmetry" in nature.

In the following table we summarise the properties of the neutral K mesons:

Name	(Neutral) Kaon	K-Short	K-Long
Symbol	K^0	K_S	K_L
Antiparticle	$\overline{K^0}$	Self	Self
Quark Content	$d\bar{s}$	$(d\bar{s} + s\bar{d})/\sqrt{2}$	$(d\bar{s} - s\bar{d})/\sqrt{2}$
Mass (MeV)	$m_{K^0} = 497.611$ ± 0.013	$m_S = m_{K^0} - \Delta m/2$	$m_L = m_{K^0} + \Delta m/2$
Lifetime (s)	—	$\tau_S = (0.8954 \pm 0.0004) \times 10^{-10}$	$\tau_L = (5.116 \pm 0.021) \times 10^{-8}$
Decay Modes (rate %)	—	$\pi^+ + \pi^-, (69, 20 \pm 0.05)\%$; $\pi^0 + \pi^0, (30, 69 \pm 0.05)\%$	$\pi^\pm + e^\mp + \nu_e$, $(40.55 \pm 0.011)\%$; $\pi^\pm + \mu^\mp + \nu_\mu$, $(27.04 \pm 0.07)\%$; $\pi^0 + \pi^0 + \pi^0$, $(19.52 \pm 0.12)\%$; $\pi^+ + \pi^- + \pi^0$, $(12.54 \pm 0.05)\%$; $\pi^+ + \pi^-$, $(1.967 \pm 0.010) \times 10^{-3}\% CPV$

Table 1 – In this table we present the main properties of the neutral K meson system. Here is included only the decay modes which correspond to percentages bigger than $10^{-3}\%$. We have $\Delta m = m_L - m_S = (3,484 \pm 0,0006) \times 10^{-12} MeV$. In the row "Quark Content", $d(\bar{d})$ and $s(\bar{s})$ stands for the quarks down(antidown) and strange(antistrange), respectively. All these data can be found in [4].

1.4 Dynamics of Kaons with Wigner-Weiskopff Approximation

We are now in position to turn our attention to dynamics in the the neutral kaon system. Since there are common decay modes – two or three pions – for both K^0 and $\overline{K^0}$, a transition $K^0 \rightleftarrows 2(3)\pi \rightleftarrows \overline{K^0}$ or mixing is possible [16, 28, 29]. That is, a K^0 can change to $\overline{K^0}$ in vacuum. Let H be the hermitian Hamiltonian containing the strong interaction, which is the one involved in the production of kaons, as well as the weak interaction, which induces mixing and decay. Because of this possible mixing and oscillations, we must have $\langle K^0|H|\overline{K^0}\rangle \neq 0$. Moreover, the time evolution of the neutral kaon system can be written as [16]

$$\begin{aligned} i\frac{d}{dt}|\psi(t)\rangle &= H|\psi(t)\rangle, \\ |\psi(t)\rangle &= a(t)|K^0\rangle + b(t)|\overline{K^0}\rangle + \sum_n c_n(t)|n\rangle. \end{aligned} \tag{1.13}$$

Here, $|n\rangle$ is any other state into which K^0 and $\overline{K^0}$ can decay. However, it is very difficult to solve this equation in general. Therefore, it is convenient to discard them and to consider only the parts associated to states representing the kaons, introducing an effective Hamiltonian H_{eff}, such that the two-component state function obeys a Schrödinger-like equation – see appendix A –

$$\begin{aligned} i\frac{d}{dt}|\psi(t)\rangle &= H_{eff}|\psi(t)\rangle, \\ |\psi(t)\rangle &= \alpha(t)|K^0\rangle + \beta(t)|\overline{K^0}\rangle. \end{aligned} \tag{1.14}$$

Furthermore, as we saw in the previous section, the true eigenstates of H_{eff} must be the K_S and K_L states. Thus, it is also convenient to make a rotation in $\psi(t)$ in order to write it as a linear combination of the states that diagonalises the effective Hamiltonian. That is, we rewrite $|\psi(t)\rangle = \alpha'(t)|K_S\rangle + \beta'(t)|K_L\rangle$, where we have (note that $\Gamma_k = 1/\tau_k$)

$$\begin{aligned} H_{eff}|K_S(t)\rangle &= (m_S - \frac{i}{2}\Gamma_S)|K_S\rangle, \\ H_{eff}|K_L(t)\rangle &= (m_L - \frac{i}{2}\Gamma_L)|K_L\rangle. \end{aligned} \tag{1.15}$$

Therefore, for the general state $|\psi(t)\rangle$,

$$|\psi(t)\rangle = \alpha_0 e^{-t(\Gamma_S/2+im_S)}|K_S\rangle + \beta_0 e^{-t(\Gamma_L/2+im_L)}|K_L\rangle, \tag{1.16}$$

with α_0 and β_0 constants such that $|\alpha_0|^2 + |\beta_0|^2 = 1$.

In this approximation – so-called Wigner-Weisskopf Approximation (WWA) – any transition to other states n – a decay product – is considered as loss of K_S, K_L (or K^0, $\overline{K^0}$) from this two-component system. Hence, we have their probability amplitude (in the kaon rest frame) decreasing with time

$$\langle K_{S,L}(t)|K_{S,L}(t)\rangle = e^{-t\Gamma_{S,L}}. \quad (1.17)$$

The WWA is a successful approach when used to analyse the phenomena of particle oscillation, regeneration, and even kaon interferometry. However, its intrinsic probability loss and the fact that it does not take into account the decay product can bring some difficulties, especially when one considers entangled systems [5, 15].

1.4.1 Strangeness Oscillation

Strangeness oscillation consists of the transition of kaon states to antikaon states, and vice-versa, allowed by the fact that strangeness is not conserved in weak interactions. In order to describe it mathematically, first we look at the relation between strangeness and mass states, which can be written as

$$\begin{aligned} |K_S\rangle &= p|K^0\rangle + q|\overline{K^0}\rangle \\ |K_L\rangle &= p|K^0\rangle - q|\overline{K^0}\rangle \end{aligned} \quad (1.18)$$

where

$$\begin{aligned} &\frac{p}{q} = \frac{1+\epsilon}{1-\epsilon}, \\ &|p|^2 + |q|^2 = 1, \\ &|p|^2 - |q|^2 = \delta_L = \langle K_S|K_L\rangle = \frac{2Re(\epsilon)}{1+|\epsilon|^2} \end{aligned} \quad (1.19)$$

Accordingly to the Wigner-Weisskopf prescription, we have $|K_{S,L}(t)\rangle = e^{-it\lambda_{S,L}}|K_{S,L}\rangle$, where $\lambda_{S,L} = m_{S,L} - \frac{i}{2}\Gamma_{S,L}$, thus

$$\begin{aligned} |K^0(t)\rangle &= g_+(t)|K^0\rangle + \frac{q}{p}g_-(t)|\overline{K^0}\rangle, \\ |\overline{K^0}(t)\rangle &= \frac{p}{q}g_-(t)|K^0\rangle + g_+(t)|\overline{K^0}\rangle, \\ g_\pm &= \frac{1}{2}(\pm e^{-i\lambda_S t} + e^{-i\lambda_L t}). \end{aligned} \quad (1.20)$$

The above equations shows that a $\overline{K^0}$ component appears in the originally K^0 state as time passes. Hence, if initially we have a beam of pure K^0, at a posterior time t, the probability of finding a $\overline{K^0}$ is given by

$$\begin{aligned} P(\overline{K^0}, t; K^0, 0) = |\langle\overline{K^0}|K^0(t)\rangle|^2 &= \left|\frac{q}{p}\right|^2 |g_-(t)|^2 \\ &= \frac{1}{4}\left|\frac{q}{p}\right|^2 [e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\Gamma t}\cos(\Delta m t)], \end{aligned} \quad (1.21)$$

where $\Delta m = m_L - m_S$, and $\Gamma = (\Gamma_S + \Gamma_L)/2$. This means that the beam oscillates with frequency given by their mass difference Δm. The visibility of the oscillations becomes very evident at times large enough, of the order of a few τ_S, so that all K_S mesons have died out, leaving only the K_L component in it. That is to say that, far from the production source, the $\overline{K^0}$ component will be detected, through its presence in the K_L meson, with equal probability as the K^0, even though, the initial beam, at $t = 0$, contained only this last meson.

One can also look for strangeness oscillations by analysing directly the expectation value of the strangeness operator. In the basis of its own eigenvectors, the operator S can be written as

$$S = |K^0\rangle\langle K^0| - |\overline{K^0}\rangle\langle\overline{K^0}| \tag{1.22}$$

Taking again as the initial state $\rho(0) = |K^0\rangle\langle K^0|$, we will have, at time t,

$$\begin{aligned}\rho(t) = |K^0(t)\rangle\langle K^0(t)| =& |g_+(t)|^2|K^0\rangle\langle K^0| + \left|\frac{q}{p}\right|^2 |g_-(t)|^2|\overline{K^0}\rangle\langle\overline{K^0}| + \\ &+ \left(\frac{q}{p}\right)^* g_+(t)g_-^*(t)|K^0\rangle\langle\overline{K^0}| + \left(\frac{q}{p}\right) g_+^*(t)g_-(t)|\overline{K^0}\rangle\langle K^0|\end{aligned} \tag{1.23}$$

thus

$$\begin{aligned}\langle S(t)\rangle = tr[S\rho(t)] &= |g_+(t)|^2 - \left|\frac{q}{p}\right|^2 |g_-(t)|^2 \\ &= \frac{e^{-\Gamma t}}{1+\delta_L}\left[\cos(\Delta m t) + \delta_L \cosh\left(\frac{1}{2}\Delta\Gamma t\right)\right].\end{aligned} \tag{1.24}$$

Experiments analysing these oscillation effects can be used to determine with great accuracy the mass difference Δm [4].

1.4.2 Regeneration

Regeneration is the phenomenon by which K_S's particles can be produced from a pure beam of K_L, by letting it interact with matter [16, 28, 31]. This is possible because K^0 and $\overline{K^0}$ interact differently with matter. Having $S = +1$, K^0 is habitually elastically scattered in the process $K^0 N \longrightarrow K^0 N$ or quasi-elastically scattered in the process $K^0 p \longrightarrow K^+ n$. Whereas, $\overline{K^0}$, having $S = -1$, can also produce hyperons by processes such as $\overline{K^0} p \longrightarrow \pi^+ + \Lambda$ and $\overline{K^0} n \longrightarrow \pi^- + \Sigma^+$, therefore having larger absorption rate.

Let us suppose that at $t = 0$ we have a pure beam of either K^0 or $\overline{K^0}$, thus we have also both states K_S and K_L. If then we wait a time long enough, of the order of $4.8\tau_S$, $\tau_S = 1/\Gamma_S$, so that all K_S die out by decaying into to two pions, we will have a pure K_L-beam. Now, letting this K_L-beam interact with matter, and writing the states after this passage through matter as

$$|K^0\rangle \longrightarrow f^0|K^0\rangle \quad \text{and} \quad |\overline{K^0}\rangle \longrightarrow \bar{f}^0|\overline{K^0}\rangle, \tag{1.25}$$

then, ignoring small CP-violation effects,

$$\begin{aligned} |K_L\rangle = (|K^0\rangle - |\overline{K^0}\rangle)/\sqrt{2} &\longrightarrow (f^0|K^0\rangle + \bar{f}^0|\overline{K^0}\rangle)/\sqrt{2} \\ &= \frac{1}{2}(f^0 + \bar{f}^0)|K_L\rangle + \frac{1}{2}(f^0 - \bar{f}^0)|K_S\rangle. \end{aligned} \tag{1.26}$$

Since the scattering amplitudes f^0 and $\bar{f}^0$, related to K^0 and $\overline{K^0}$, respectively, are different, $f^0 - \bar{f}^0 \neq 0$. Hence, a K_S component appears after the passage of this beam through matter. To verify that this phenomenon indeed takes place, one needs only to count how many two pions decay are occurring after the collision of the pure K_L-beam with the matter-target. If this number is larger than the small number of occurrences of the CP-violating two pions decay of the long-lived kaon, then, regeneration has taken place. In fact, CP-violation was discovered precisely in an experiment such as described [28, 29].

After this introduction to the features and phenomenology of the neutral K-meson we proceed to the discussion of the fundamental quantum mechanical concepts of complementarity and entanglement, which will be necessary in the posterior analysis of the neutral kaon interferometry.

2 Complementarity and Entanglement

2.1 Complementarity

The principle of complementarity states that a quantum system may possess properties that are equally real but mutually exclusive [13, 14, 32]. Its canonical illustration is given in the particle-wave duality observed in Young's double-slit experiment [17, 33, 34]. For this reason we shall describe and present the results of this experiment.

Consider an apparatus consisting of a source S emitting mono-energetic particles, which can be photons, electrons, neutrons, atoms, molecules, etc. These particles pass through a first screen A with a collimating slit, which restricts the aperture of the incoming particle beam. After that, they proceed to a second screen B with two holes separated by a distance a through which the particles may traverse. Finally, they impinge on a third screen C paved with detectors. See figure below.

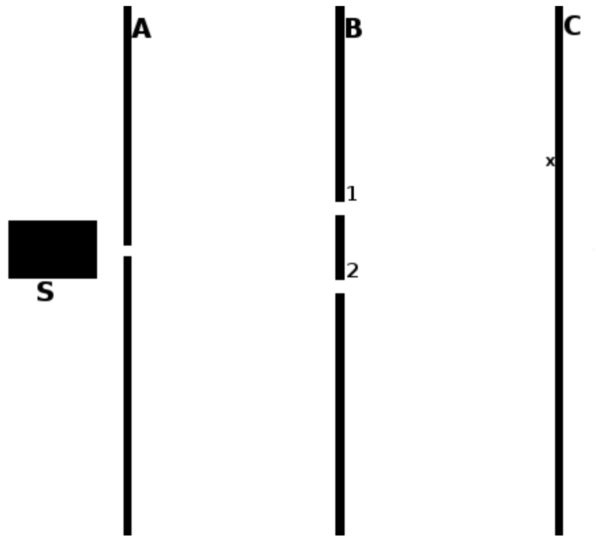

Figure 2 – The source S emits mono-energetic particles which pass through a collimating slit in screen A before reaching a second screen B with two slits, labelled by the numbers 1 and 2. One, then, may ask what is the probability of detecting these particles in the position x of a the screen C, which is paved with detectors.

Suppose we want to detect the impact at a distance x from the centre of the screen of incoming electrons[1]. It is noticed that the current arriving at x is not continuous, but corresponds to a rain of particles. If we make the intensity of the source S very low, the detector records pulses representing the arrival of individual particles on x. Moreover, by placing detectors through all over the screen C, with the source S weak enough so that

[1] The results described here are valid for all kinds of particles, we consider electrons for simplicity.

an electron is emitted only after the previously one has crossed the entire apparatus, we note that only one detector respond at each transit time. That is, one detector respond, then after a little time, another one, etc. There are never half responses or two detectors responding simultaneously; either one electron arrives at one detector or nothing happens. So, we can conclude that, the detector in x records the passage of a single corpuscular entity travelling through the apparatus. This is why we say electrons are particles.

However, after a large number of particles has been detected, what one observes on the screen C is a pattern of equidistant lines with a high density of impacts - bright fringes - separated by stripes where electrons hardly ever arrive - dark fringes. Assuming the emitted electrons have momentum p, their the de Broglie wavelength is $\lambda = h/p$, where h is Planck's constant. The angular separation of two adjacent bright fringes seen on C is $\theta = \lambda/a$. This is easily explained if one associates the beam of particles to a monochromatic wave. Then, the two holes appear as secondary sources radiating forward partial waves, which interfere constructively at bright fringes and interfere destructively at dark fringes. This reflects the wave properties of electrons. Therefore, although the discreteness of the detection events exhibits the corpuscular nature of the electrons, the final interference pattern shows an unmistakable wavy character. Furthermore, another puzzling aspect is that, if any of the holes on B is closed, no interference pattern is observed on the third screen.

But, what if we make the question: through which hole does a given electron passes in the second screen? In quantum physics such question only makes sense if one performs an experiment to determine the position of the electron when it crosses this second screen. In order to see through which hole the electron goes, we may place a source of light behind B and watch. Electrons scatter light, so if light is scattered behind hole 1, we may conclude that an electron passed through hole 1; and if it is scattered behind hole 2, we will know the electron crossed hole 2. Yet, to determine precisely whether the light scattering occurred behind hole 1 or 2 we must use light of wavelength smaller than a, the distance between the holes, because an image cannot be located in space with precision greater than the order of the wavelength of the probing light. Accordingly, then, each probing photon carries a momentum greater than h/a. This implies that each electron which scatters a photon will recoil with a fluctuating momentum component normal to the beam direction with magnitude $\Delta p > h/a$. This random recoil produces an angular fluctuation $\Delta\theta \approx \Delta p/p > h/ap = \lambda/a$, which magnitude is larger than the angular separation θ of the fringes seen on screen C, resulting in a blurring of the interference pattern. In other words, a mechanism designed to determine the path of the electron disturbs the system in at least the amount necessary to destroy the interference pattern. This results can be seen as an example of the consequences of Heisenberg uncertainty principle. To see wave-like interference, the particles must have a well-defined momentum, which is incompatible with the determination of their position when they pass through the holes.

In this experiment we say that the corpuscular and wave aspects of the particles are complementary. The manifestation of the wave or corpuscular nature of particles are mutually exclusive, with each one manifesting itself only when the experimental set-up is appropriate.

Heisenberg uncertainty principle is not the only physical mechanism that can enforce complementarity. As was showed by Scully *et. al.* [34], even when the apparatus measuring through which slit the particles traverses the second screen does not greatly disturb their momentum, the interference pattern is still washed out. However, in this case, entanglement is the mechanism responsible for this effect. We shall return to this discussion in the following sections.

2.1.1 Quantitative Complementarity

In the previous section we discussed the two extreme situations of the particle-wave duality. The cases in which the interference pattern is perfectly visible and no information about through which slit - "which-way" information - the particle has passed is available, and the one in which full which-way information is available but no interference pattern can be seen. However, cases in which partial information about wave-like and particle-like aspects can be measured are also possible and an inequality relating this two properties can be established [12, 13, 35]. In order to obtain this inequality we define two quantifiers related to these wave-like and particle-like properties. The one associated with the wave-like character, usually denoted by $\mathcal{V}_0$ and called "fringe visibility", quantifies the contrast of the interference pattern and is defined by

$$\mathcal{V}_0 \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}, \tag{2.1}$$

where $I_{\max}$ and $I_{\min}$ denote maximum and minimum intensity (in the interference pattern), respectively. The quantifier of which-way information, the particle-like property, is denoted by $\mathcal{P}$ and called the "path predictability". It measures the *a priori* knowledge of whether the "system" take a path or another. That is, if one bets on a more probable path, the probability of predicting the way correctly will be given by $(1 + \mathcal{P})/2$.

To illustrate their use and establish their relation, let us suppose a particle in a two-path interferometer - such as the double-slit - in a state given by

$$|\Psi(\phi)\rangle = a|\psi_1\rangle + be^{i\phi}|\psi_2\rangle, \tag{2.2}$$

where a and b are positive numbers with $a^2 + b^2 = 1$. The kets $|\psi_1\rangle$ and $|\psi_2\rangle$, with $\langle\psi_1|\psi_2\rangle = 0$, represent the states corresponding to the two spatially separated paths of the interferometer and ϕ the phase-difference between these paths. Then, the path predictability, $\mathcal{P}$, is defined as the modulus of the difference in the probabilities for the particle to take either of the two interferometric paths. These probabilities are usually denoted by ω_I and ω_{II}, and thus,

$$\mathcal{P} \equiv |\omega_I - \omega_{II}|. \tag{2.3}$$

For the state under consideration 2.2, hence, we have

$$\mathcal{P} = |a^2 - b^2|. \tag{2.4}$$

We may obtain the fringe visibility from the coefficient of the phase-dependent term in the expressions [8, 9]

$$I_\pm(\phi) = |\langle\psi_\pm|\Psi(\phi)\rangle|^2 = \frac{1}{2}(1 \pm \mathcal{V}_0 \cos(\phi)), \tag{2.5}$$

where the intensities, $I_\pm$, are associated to measurements on the symmetric and antisymmetric basis states

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle \pm |\psi_2\rangle). \tag{2.6}$$

Calculating these intensities for 2.2, we get

$$I_\pm = \frac{1}{2}(1 \pm 2ab\cos\phi), \tag{2.7}$$

and, therefore, the fringe visibility is

$$\mathcal{V}_0 = 2ab. \tag{2.8}$$

It follows immediately from equations 2.4 and 2.8 that

$$\mathcal{P}^2 + \mathcal{V}_0^2 = |a^2 - b^2|^2 + (2ab)^2 = (a^4 + 2a^2b^2 + b^4) = (a^2 + b^2)^2 = 1. \tag{2.9}$$

However, this equality is valid only when the system is in a pure state, in the general case (of mixed states), we must replace it by the inequality:

$$\mathcal{P}^2 + \mathcal{V}_0^2 \leqslant 1. \tag{2.10}$$

As an example, when the interferometer is symmetric, such as the double-slit considered previously with slits of the same size, we have $a = b = 1/\sqrt{2}$ and the two paths are taken with equal probability. Then, $\mathcal{P} = 0$, no a priori which-way knowledge is available; and the fringe visibility is maximal, $\mathcal{V}_0 = 1$.

The inequality 2.10 was first derived by Greenberger and Yasin [12] and is consistent with measurements made by Rauch *et. al.* with an asymmetric neutron interferometer [36]. Later on, several others similar relations were also proposed for various distinct situations [13, 14, 37, 38], but, to address some these other relation we shall first discuss another fundamental quantum phenomenon: entanglement.

2.2 Entanglement

Entanglement can be simply defined as a quantum correlation[2]. When two physical systems interact, a correlation of a quantum nature, which we call entanglement, may be established between them, persisting even after they no longer interact or after being spatially separated.

In practice, it is easier to define a *non-entangled* state. In order to do this, let $|\psi\rangle$ denotes the *pure quantum state* of a composite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If $|\psi\rangle$ can be written as a product state

$$|\psi\rangle = |\kappa_A\rangle \otimes |\varsigma_B\rangle, \tag{2.11}$$

where $\kappa_A \in \mathcal{H}_A$ and $\varsigma_B \in \mathcal{H}_B$, then we say that this state is *separable*. When a quantum state is non-separable, we then say that this state is entangled.

Two important examples of an entangled states are the maximally entangled two-qubit states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \tag{2.12}$$

where $|ij\rangle = |i_A\rangle \otimes |j_B\rangle$. The one with the minus sign can describe for example the state of the electron and nuclear spins in the ground state of the hydrogen. On the other hand, the one with the plus sign, may describe the polarisations of two photons emitted in opposite directions when an atom undergoes two cascading transitions from an excited state to an ground level, with both having zero angular momentum [17]. Now, if we measure the local observable on subsystem A associated with the projector $P = |0\rangle\langle 0|$, the state of this subsystem obviously collapses to $|0_A\rangle$. Moreover, in this case, the state of subsystem B is also instantly modified to $|1_B\rangle$[3]. The astonishing thing about entanglement is that this instantly collapse of the state of subsystem B occurs even when the parts A and B are spatially separated, no matter the distance[4]!

A quantitative statement of entanglement can be made for pure bipartite systems by means of the *entropy of entanglement*, defined as the von Neumann entropy of its reduced density matrices

$$E_V(|\psi\rangle) = S(\rho_A) = S(\rho_B), \tag{2.13}$$

where, for a operator ρ, with eigenvalues p_i, S is given by

$$S(\rho) = -tr[\rho \ln \rho] = -\sum_i p_i \ln p_i. \tag{2.14}$$

[2] We wish to keep things simple here. For more information on entanglement characterisation and quantification see for example [39].

[3] Of course the same logic applies when we decide to measure $|1_A\rangle\langle 1_A|$ in A. In this scenario, the state of B instantly collapses to $|0_B\rangle$.

[4] Here we are ignoring the effect of decoherence due to a possible coupling of the subsystems with an environment.

There are several other quantifiers of entanglement (see [39]). One that is particularly useful for a system of two qubits is the *concurrence* [40]. Denoting by ϱ the density matrix of the two-qubit system, the concurrence is defined as

$$C(\varrho) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\} \tag{2.15}$$

where the λ_i's are the eigenvalues of the matrix $\varrho(\sigma_y \otimes \sigma_y)\varrho^*(\sigma_y \otimes \sigma_y)$, with $\lambda_i > \lambda_{i+1}$. σ_y is the Pauli matrix and the star in ϱ^* denotes complex conjugation in the computational basis $|ij\rangle = |i\rangle \otimes |j\rangle, \quad i, j = 0, 1$.

Entanglement has a profound relation with complementarity. In the double-slit experiment discussed previously, any attempt made to acquire which-way information implies a coupling of the degrees of freedom of the particles travelling through the apparatus with a probing system. Hence, the particle and the probe become entangled by their interaction [17]. In a simple case we may write

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle|\alpha\rangle + |\psi_-\rangle|\beta\rangle), \tag{2.16}$$

where $|\psi_\pm\rangle$ are interfering orthogonal particle states and $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal final states of the probe (detector) containing information about the path of the particle. In the absence of the "which-way detector" the state of the particle can be regarded as $|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle$, and thus, its density operator is given by

$$\rho_p = \frac{1}{2}(|\psi_+\rangle\langle\psi_+| + |\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-|). \tag{2.17}$$

In this expression, it is the crossed terms, $|\psi_+\rangle\langle\psi_-|$ and $|\psi_-\rangle\langle\psi_+|$, that are the ones responsible for the observed interference pattern. However, in the presence of the which-way detector, the reduced density operator of the particle becomes

$$\rho_p = tr_D(|\psi\rangle\langle\psi|) = |\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-|, \tag{2.18}$$

and, therefore, the interference pattern is washed out as a consequence of its entanglement with the detector system. It is worth noticing that one does not even need to read the information about the path in order to destroy the interference, the simple entanglement of system and probe is enough to make it disappear [34]. If a system which can acquire information about the path of a particle is in place, the particles' reduced density operator is, automatically, given by 2.18, which means no interference, even if the information about the path is never put to use, or never read. That is, information about the path taken by the particle needs only to be available in order for the interference pattern to be destroyed.

After this brief discussion about entanglement we can proceed to other quantitative statements of complementarity.

2.2.1 Quantitative Complementarity Revisited

We have seen so far that it is possible to establish a quantitative relation between wave-like and particle-like aspects of a quantum system in an interferometer. Let us consider now the case in which the system in a symmetric interferometer is coupled to another physical system that is meant to serve as a which-way detector. Denoting by $\rho_D^{(in)}$ the initial state in which the detector is prepared, the combined initial total density operator is written as

$$\rho^{(in)} = \rho_Q^{(in)} \otimes \rho_D^{(in)}, \tag{2.19}$$

where $\rho_Q^{(in)}$ is the density operator of the system passing through the interferometer - also referred to as quanton. Let us denote by $|0\rangle$ and $|1\rangle$ the two possible paths the system can take. As a result of the interaction of the detector with the quanton, the detector state is transformed into

$$\rho_D^i = U_D^{(i)} \rho_D^{(in)} U_D^{(i)\dagger}, \quad i = 0, 1, \tag{2.20}$$

where $U_D^{(i)}$ are unitary operators affecting only the degrees of freedom of the detector. Hence, the unitary operator acting on the total system, quanton-detector, while the quanton pass through the interferometer is given by

$$U = |0\rangle\langle 0| \otimes U_D^{(0)} + |1\rangle\langle 1| \otimes U_D^{(1)}. \tag{2.21}$$

Therefore, the total system density operator is transformed into the final state

$$\begin{aligned} \rho^{(f)} = &\frac{1}{2}|0\rangle\langle 0| \otimes U_D^{(0)} \rho_D^{(in)} U_D^{(0)\dagger} + \frac{1}{2}|0\rangle\langle 1| \otimes U_D^{(0)} \rho_D^{(in)} U_D^{(1)\dagger} \\ &+ \frac{1}{2}|1\rangle\langle 0| \otimes U_D^{(1)} \rho_D^{(in)} U_D^{(0)\dagger} + \frac{1}{2}|1\rangle\langle 1| \otimes U_D^{(1)} \rho_D^{(in)} U_D^{(1)\dagger}. \end{aligned} \tag{2.22}$$

By tracing out the degrees of freedom of the quanton, we get the detector final state,

$$\rho_D^{(f)} = \frac{1}{2}\rho_D^{(0)} + \frac{1}{2}\rho_D^{(1)}. \tag{2.23}$$

In a 1996 paper [13], Englert showed that, in this set-up, which-way information that has become available due to the interaction between quanton and detector could be quantified by the *distinguishability* of the paths, $\mathcal{D}$. Mathematically, this number is given by the distance between $\rho_D^{(0)}$ and $\rho_D^{(1)}$ in the trace-class norm,

$$\mathcal{D} = \frac{1}{2}\left\| \rho_D^{(0)} - \rho_D^{(1)} \right\| = \frac{1}{2} tr_D\Big[\Big| \rho_D^{(0)} - \rho_D^{(1)} \Big| \Big]. \tag{2.24}$$

The distinguishability is the maximum difference of probabilities of the correct and incorrect decisions about the paths. This means that, the likelihood for guessing correctly the way taken by the quanton in the interferometer is given by $(1+\mathcal{D})/2$. When $\mathcal{D} = 0$ the paths cannot be distinguished at all, whereas when $\mathcal{D} = 1$ the paths of the quanton are completely distinguishable. Furthermore, Englert also shown that, the distinguishability obeys the inequality

$$\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1, \tag{2.25}$$

where the equality holds for pure bipartite quantum systems. Here, the fringe visibility $\mathcal{V}$ is given by

$$\mathcal{V} = \left| tr_D \left[U_D^{(0)} \rho_D^{(in)} U_D^{\dagger(1)} \right] \right|. \tag{2.26}$$

It is also worth noticing that, since

$$\mathcal{D}(\rho, \varrho) = \frac{1}{2} \|\rho - \varrho\| \leqslant \sqrt{1 - \mathcal{F}^2(\rho, \varrho)}, \tag{2.27}$$

where $\mathcal{F}(\rho, \varrho) = tr\left[\sqrt{\sqrt{\rho}\varrho\sqrt{\rho}}\right]$ is the *fidelity* between the two density operators ρ and ϱ, another good definition for a visibility is $\mathcal{V} = \mathcal{F}(\rho_D^{(0)}, \rho_D^{(1)})$ [11, 38].

Later on, Jakob and Bergou showed that for bipartite two-dimensional systems - qubits - the distinguishability is related to the entanglement measure concurrence by [14]

$$\mathcal{D}^2 = \mathcal{C}^2 + \mathcal{P}^2, \tag{2.28}$$

where $\mathcal{P}$ is the path predictability. They argued that, when the quanton is interacting with another system, the correlations established between them, i.e., entanglement, must also enters in a quantitative complementarity relation. Particularly, in the case of the two qubits, a "triality" relation may be stated as

$$\mathcal{C}^2 + \mathcal{P}^2 + \mathcal{V}_0^2 \leqslant 1, \tag{2.29}$$

which explicitly shows the role of entanglement in complementarity. That is, in contrast to the duality relation of single partite quantum systems, when the quanton is coupled to another party, there is a trade-off between three mutually exclusive but equally real quantities, corresponding to the usual which-way information and quantum interference, but also, in addition, entanglement.

Finally, there are yet other generalisations of inequality 2.25, like the ones proposed by Banaszek *et. al.* for quantons with internal degrees of freedom and by Tej *et. al.* for the case in which the detector is entangled with an ancilla. The interested reader can look at [37] and [38], respectively.

3 Open Quantum System

An open system S is a quantum system which is coupled (interact) with another quantum system – its environment E – and therefore is a subsystem of the combined total $S+E$. As a result, the state of the subsystem S will change as a consequence of its internal dynamics and of the interaction with its surroundings. In general, this interactions leads to certain system-environment correlations such that the resulting state changes of S can no longer be represented in terms of unitary, Hamiltonian dynamics. However, it is assumed that the combined system $S+E$ is still closed, therefore, following such dynamics scheme. In order to study the time evolution of the open quantum system, then, we must consider the total system.

For a closed quantum system, represented by a state vector $|\psi(t)\rangle$, the time evolution is given by the Schrödinger equation[1]

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{3.1}$$

where $H(t)$ is the Hamiltonian of the system. Formally, the solution of the Schrödinger equation may be written in terms of a unitary time-evolution operator $U(t,t_0)$, such that $U(t,t_0)U^\dagger(t,t_0) = \mathbb{1}$, which transforms the initial state, $|\psi(t_0)\rangle$, at some time t_0, into the final state, at time t,

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle. \tag{3.2}$$

The unitary time-evolution operator itself obeys an operator equation given by

$$i\frac{\partial}{\partial t}U(t,t_0) = H(t)U(t,t_0), \tag{3.3}$$

with the initial condition $U(t_0,t_0) = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. When the system is isolated[2], its Hamiltonian is time-independent and integration of 3.3 yields the usual expression

$$U(t,t_0) = e^{-iH(t-t_0)}. \tag{3.4}$$

Moreover, if the system under consideration is in a mixed state characterised by a density operator ρ, it is straightforward to derive its equation of motion. Assuming that, at some time t_0, the state of the system is described by

$$\rho(t_0) = \sum_i p_i|\psi_i(t_0)\rangle\langle\psi_i(t_0)|, \tag{3.5}$$

[1] We are setting the Planck's constant $\hbar$ equals to 1.

[2] An isolated system is a special kind of closed system, in which the Hamiltonian is time-independent. However, if the dynamics of a system can be formulated in terms of a possibly time-dependent Hamiltonian generator $H(t)$, then the system is said to be closed.

where p_i are positive weights and $|\psi_i(t_0)\rangle$ are normalised state vectors which evolve in time according to the Schrödinger equation 3.2. The state of the system, at time t, will be given by

$$\rho(t) = \sum_i p_i U(t,t_0)|\psi_i(t_0)\rangle\langle\psi_i(t_0)|U^\dagger(t,t_0) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0). \tag{3.6}$$

Differentiating this equation with respect to time, we get the following equation of motion – the so-called von Neumann equation

$$\frac{d}{dt}\rho(t) = -i[H(t),\rho(t)]. \tag{3.7}$$

Regarding now a combined total system $S+E$, let us denote by $\mathcal{H}_S$ and $\mathcal{H}_E$ the Hilbert spaces of system S and the environment E, respectively. Then, the total Hilbert space of $S+E$ will be given by the tensor product

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E. \tag{3.8}$$

The total Hamiltonian $H(t)$, can be expressed in the form

$$H(t) = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + H_I(t), \tag{3.9}$$

where H_S is the Hamiltonian of the open system S, H_E is the Hamiltonian of the environment E, and $H_I(t)$ is the Hamiltonian describing the interaction between the open system and the environment.

The quantity of central interest in the description of the open quantum system S is its reduced density operator ρ_S. If the total system is described by the density operator ρ, then, ρ_S is given by

$$\rho_S = tr_E[\rho], \tag{3.10}$$

where tr_E denotes partial trace over the degrees of freedom of the environment E.

Since the total system evolves unitarily we have

$$\rho_S(t) = tr_E[U(t,t_0)\rho(t_0)U^\dagger(t,t_0)], \tag{3.11}$$

where $U(t,t_0)$ is the unitary time-evolution operator of the total system. The equation of motion for the reduced density operator is obtained by taking the partial trace over the degrees of freedom of the environment on both sides of the von Neumann equation for the total system, which yields

$$\frac{d}{dt}\rho_S(t) = -itr_E[H(t),\rho(t)]. \tag{3.12}$$

We regard an open system S to be singled out by the fact that all measurements of interest refer to this subsystem. All observables referring to S have the form

$$A \otimes \mathbb{1}_E, \tag{3.13}$$

where A is an operator acting on the Hilbert space $\mathcal{H}_S$ and $\mathbb{1}_E$ is the identity operator in the Hilbert space of the environment $\mathcal{H}_E$. Thus, the expectation values of this operators acting on the open system's Hilbert space are determined by

$$\langle A \rangle = tr_S[A\rho_S], \tag{3.14}$$

where, similarly, tr_S denotes the partial trace over the open system's Hilbert space.

As we have seen, in a closed system, time-evolution is given by an unitary operator $U(t, t_0)$. For open systems, however, evolution is not unitary, hence, the transformation describing the change of the reduced system from the initial time $t = 0$ to some time $t > 0$ is made by a so-called dynamical map, which we express here by $V(t)$.

Let us suppose that at $t = 0$ the state of the total system $S + E$ is described by an uncorrelated tensor product state

$$\rho(0) = \rho_S(0) \otimes \rho_E, \tag{3.15}$$

where $\rho_S(0)$ is the initial state of the reduced system S and ρ_E represents some reference state of the environment. Then we can write

$$\rho_S(0) \longmapsto \rho_S(t) = V(t)\rho_S(0) \equiv tr_E[U(t, 0)(\rho_S(0) \otimes \rho_E)U^\dagger(t, 0)]. \tag{3.16}$$

Considering the reference environment state ρ_E and the final time t as fixed, this relation defines a map from the space $\mathcal{S}(\mathcal{H}_S)$ of density operators of the reduced system into itself,

$$V(t) : \mathcal{S}(\mathcal{H}_S) \longrightarrow \mathcal{S}(\mathcal{H}_S). \tag{3.17}$$

This map, also called superoperator, describes the state change of the open quantum system over the time t.

The dynamical map is completely characterised in terms of operators pertaining to the open system's Hilbert space $\mathcal{H}_S$. To show this, let us consider the spectral decomposition of the density operator of the environment ρ_E,

$$\rho_E = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|, \tag{3.18}$$

where $0 \leq \lambda_i \leq 1$, $\sum_i \lambda_i = 1$, and $\{|\phi_i\rangle\}$ form an orthonormal basis in $\mathcal{H}_E$. From equation 3.16, then, we get the following representation,

$$V(t)\rho_S(0) = \sum_{i,j} W_{i,j}(t)\rho_S(0)W^\dagger_{i,j}(t), \tag{3.19}$$

where the $W_{i,j}$, being operators in $\mathcal{H}_S$ – called Kraus operators –, are defined by

$$W_{i,j}(t) = \sqrt{\lambda_j}\langle\phi_i|U(t, 0)|\phi_j\rangle, \tag{3.20}$$

and have the property

$$\begin{aligned}\sum_{i,j} W_{i,j}^{\dagger}(t)W_{i,j}(t) &= \sum_{i,j}\lambda_j\langle\phi_j|U^{\dagger}(t,0)|\phi_i\rangle\langle\phi_i|U(t,0)|\phi_j\rangle = \sum_j \lambda_j\langle\phi_j|U^{\dagger}U|\phi_j\rangle \\ &= \sum_j \lambda_j\langle\phi_j|\mathbb{1}_{S+E}|\phi_j\rangle = \mathbb{1}_S,\end{aligned} \tag{3.21}$$

from which we can deduce that

$$tr_S[V(t)\rho_S(0)] = tr_S[\rho_S(0)] = 1. \tag{3.22}$$

If, at $t = 0$, the environment is in a pure state $\rho_E = |\phi_0\rangle\langle\phi_0|$, then, the above equations are simplified to

$$\begin{aligned}&W_k = \langle\phi_k|U(t,0)|\phi_0\rangle, \\ &\rho_S(t) \equiv V(t)\rho_S(0) = \sum_k W_k(t)\rho_S(0)W_k^{\dagger}, \\ &\sum_k W_k^{\dagger}(t)W_k(t) = \mathbb{1}_S.\end{aligned} \tag{3.23}$$

Since the dynamical map, $V(t)$, obeys 3.19, it must be a completely positive map (see Appendix C). Therefore, $V(t)$ represents a convex-linear, completely positive and trace-preserving quantum operation. Its properties are summarised in Appendix B.

The dynamical map $V(t)$ was introduced above for a fixed time $t \geq 0$. If we now allow t to vary, we get a one-parameter family $\{V(t)|t \geq 0\}$ of dynamical maps, with $V(0)$ being the identity map. Such a family describes the whole future time evolution of the open quantum system, which, in general, can be very complicated. Nonetheless, if the characteristic time scale of the environment is much smaller than the characteristic time scale of the system, $\tau_E \ll \tau_S$, then, the memory effects of the system about the environment are negligible[3] – a Markovian-type behaviour -, and, hence, the dynamical map $V(t)$ forms a semigroup

$$V(t_1)V(t_2) = V(t_1 + t_2), \quad t_1, t_2 \geq 0. \tag{3.24}$$

Now, given a quantum dynamical semigroup there exists[4] a linear map $\mathcal{L}$, the generator of the semigroup, which allows a representation in exponential form

$$V(t) = e^{\mathcal{L}t}. \tag{3.25}$$

From this representation we get a first-order differential equation for the reduced density operator of the open system,

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t), \tag{3.26}$$

[3] See B.2.

[4] Under certain mathematical conditions, see [41].

which is called the Markovian quantum master equation.

For the simple case of a finite-dimensional Hilbert space $\mathcal{H}_S$, with $\dim \mathcal{H}_S = N$, it can be shown that the most general form for the generator $\mathcal{L}$ of a quantum dynamical semigroup is [41] - also on B.2 -

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t) = -i[H, \rho_S(t)] - D[\rho_S(t)], \tag{3.27}$$

where in this equation, known as Lindblad master equation, H is the system's Hamiltonian and $D[\rho_S]$ is the so-called dissipator

$$D[\rho_S] = \frac{1}{2}\sum_{k}^{N^2-1} \lambda_k \Big(A_k^\dagger A_k \rho_S + \rho_S A_k^\dagger A_k - 2A_k \rho_S A_k^\dagger\Big). \tag{3.28}$$

A_k are called Lindblad-operators and λ_k are decoherence constants.

After this introduction on Open Quantum Systems formalism, we are ready to look at its use for the description of decaying systems.

3.1 An Open Quantum System Formulation for Unstable Particles

In 2005, Caban *et al.* [15] proposed a model to the time evolution of unstable particles regarding them as an open quantum system. The system under consideration is seen as a two-level system which can be found either in the particle state or in the vacuum state[5]. The system is considered as an open one because its decay products are treated as part of the environment.

The time evolution of this system is represented by the continuous one-parameter family of linear superoperators $V(t)$ such that

$$\rho(t) = V(t)\rho(0), \tag{3.29}$$

where $\rho(t)$ is the density operator of the system at the time t. This superoperators are trace-preserving completely positive maps and they form a one-parameter semigroup, i.e.:

$$\begin{aligned} tr[V(t)\rho(0)] = tr[\rho(0)] = 1, \quad V(t+t') = V(t)V(t'), \\ V(t)\otimes \mathbb{1}_n[\rho\otimes\varrho] \geq 0, \quad \forall t, t' \geq 0 \quad \text{and } \forall \varrho \in \mathbb{C}^n. \end{aligned} \tag{3.30}$$

First, we will describe the model for the case of the neutral pion. This case is simple and it illustrates the main ideas of this approach. Then, we proceed to the more complicated case of kaons.

[5] Here vacuum stands for a state with absence of this particle

3.1.1 The Open Quantum System Evolution for the Neutral Pion

We consider the (total) Hilbert space of the system to be constructed as a direct sum of the Hilbert space of the particle $\mathcal{H}_{\pi^0}$, spanned by the vector $|\pi^0\rangle$, and the Hilbert space of the vacuum $\mathcal{H}_0$, spanned by the vector $|0\rangle$. That is, we have $\mathcal{H} = \mathcal{H}_{\pi^0} \oplus \mathcal{H}_0$. We assume the decay process is Markovian, and not being described by an unitary-Hamiltonian evolution.

Let us represent the vectors $|\pi^0\rangle$ and $|0\rangle$ by

$$|\pi^0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.31}$$

The time evolution of the system is given by the map $V(t)$

$$\rho(t) = V(t)\rho(0), \tag{3.32}$$

where $\rho(t)$ is the density operator of the system.

Such a density operator can be decomposed as

$$\rho(t) = \rho_{11}(t)|\pi^0\rangle\langle\pi^0| + \rho_{12}(t)|\pi^0\rangle\langle 0| + \rho_{12}^*(t)|0\rangle\langle\pi^0| + \rho_{22}(t)|0\rangle\langle 0|. \tag{3.33}$$

We assume the time evolution of the state to be consistent with the phenomenological Wigner-Weisskopf approach, i.e.,

$$|\pi^0(t)\rangle = e^{-t(im+\Gamma/2)}|\pi^0\rangle, \tag{3.34}$$

where m and Γ are the mass and decay width of the pion, respectively.

Hence, accordingly with this equation, we get $\rho_{11}(t) = e^{-t\Gamma}\rho_{11}(0)$ and, since the map $V(t)$ must be trace-preserving, $\rho_{22}(t) = 1 - e^{-t\Gamma}\rho_{11}(0)$. Moreover, the linearity of the map allows us to write the $\rho_{12}(t)$ as the time-dependent linear combination of all the elements of the initial density operator, that is, we can write

$$\rho_{12}(t) = \sum_{i,j=1}^{2} A_{ij}(t)\rho_{ij}(0), \tag{3.35}$$

with initial conditions $A_{12}(0) = 1$ and all remaining A's equals to zero at $t = 0$.

Therefore, the action of the map $V(t)$ can be written in matrix form as follows:

$$\rho(t) = V(t)\rho(0) = \begin{pmatrix} e^{-t\Gamma}\rho_{11}(0) & \sum_{i,j=1}^{2} A_{ij}(t)\rho_{ij}(0) \\ \sum_{i,j=1}^{2} A_{ij}^*(t)\rho_{ij}^*(0) & \rho_{22}(t) \end{pmatrix}, \tag{3.36}$$

where $\rho_{22}(t) = \rho_{22}(0) + (1 - e^{-t\Gamma})\rho_{11}(0)$.

We then use Choi's theorem, described in Appendix C, to find the conditions under which this map is completely positive. This theorem states that a map $V(t)$ is completely positive if and only if its corresponding Choi's matrix, given by (see C for more details about the Choi's theorem and the calculations hereafter)

$$Choi(V) = \begin{pmatrix} e^{-t\Gamma} & A_{11}(t) & 0 & A_{12}(t) \\ A_{11}^*(t) & 1-e^{-t\Gamma} & A_{21}^*(t) & 0 \\ 0 & A_{21}(t) & 0 & A_{22}(t) \\ A_{12}^*(t) & 0 & A_{22}^*(t) & 1 \end{pmatrix}, \tag{3.37}$$

is (semi)positive. Hence, we must have

$$\begin{aligned} |A_{12}(t)|^2 &\leqslant e^{-t\Gamma}, \\ |A_{11}(t)|^2 &\leqslant (1-e^{-t\Gamma})(e^{-t\Gamma} - |A_{12}(t)|^2), \\ A_{21}(t) &= 0, \\ A_{22}(t) &= 0. \end{aligned} \tag{3.38}$$

The condition of semigroup leads to

$$\begin{aligned} A_{12}(t+t') &= A_{12}(t')A_{12}(t), \\ A_{11}(t+t') &= A_{11}(t')e^{-t\Gamma} + A_{12}(t')A_{11}(t). \end{aligned} \tag{3.39}$$

Now, taking into account the fact that superpositions between particle and vacuum are not observed in nature, we are led to a *superselection rule* [15, 42]: we can set $\rho_{21}(t) = 0$ for any time t, which implies $A_{11}(t) = 0$.

According to these conditions, the only possible solution for $A_{12}(t)$ is

$$A_{12}(t) = e^{-t(\Gamma/2 + i\mu)}, \tag{3.40}$$

where $\mu \geqslant 0$. Keeping consistency with equation 3.34 requires $\mu = m$. Thus, the density matrix describing the evolution of π^0 is

$$\rho(t) = \begin{pmatrix} e^{-t\Gamma}\rho_{11}(0) & 0 \\ 0 & 1 - e^{-t\Gamma}\rho_{11}(0) \end{pmatrix}. \tag{3.41}$$

Since the evolution of the system, 3.41, is given by a completely positive map, it can be written in the operator-sum representation

$$\rho(t) = \sum_k W_k(t)\rho(0)W_k^\dagger(t). \tag{3.42}$$

where the corresponding Kraus operators have the form:

$$\begin{aligned} W_0(t) &= e^{-t(\Gamma/2+im)}|\pi^0\rangle\langle\pi^0| + |0\rangle\langle 0|, \\ W_1(t) &= \sqrt{1-e^{-t\Gamma}}|0\rangle\langle\pi^0|. \end{aligned} \tag{3.43}$$

From the operator-sum representation we can find the master equation for this system in the Lindblad form —see B or [17, 41],

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}, \hat{\rho}(t)] - \frac{1}{2}\{\hat{L}_1^\dagger L_1, \hat{\rho}(t)\} + \hat{L}_1\hat{\rho}(t)\hat{L}_1^\dagger, \tag{3.44}$$

where the Hamiltonian and the Lindblad operator of the system are given, respectively, by

$$\begin{aligned} H &= m|\pi^0\rangle\langle\pi^0|, \\ L_1 &= \sqrt{\Gamma}|0\rangle\langle\pi^0|. \end{aligned} \tag{3.45}$$

When the initial state of the system is $\rho(0) = |\pi^0\rangle\langle\pi^0|$, the probability of detecting π^0 at the time t is given, as expected, by the Geiger-Nutall law

$$p(\pi^0;t) = tr\Big[\rho(t)|\pi^0\rangle\langle\pi^0|\Big] = e^{-t\Gamma}. \tag{3.46}$$

Now that the main ideas of this method are illustrated, we proceed to the kaons model.

3.1.2 Time Evolution of the Neutral Kaon

In the case of kaons, the Hilbert space of the total system is seen as the direct sum of the Hilbert space of the vacuum $\mathcal{H}_0$, spanned by the vector $|0\rangle$, and the Hilbert space of the particles $\mathcal{H}_{K^0}$, spanned by the vectors $|K^0\rangle$ and $|\overline{K^0}\rangle$. That is, $\mathcal{H} = \mathcal{H}_{K^0} \oplus \mathcal{H}_0$.

The most convenient way of analysing the evolution of the density operator of this system is by decomposing it in terms of the mass eigenstates $|K_S\rangle$ and $|K_L\rangle$. In this basis the time evolution follows the Wigner-Weisskopf phenomenological prescription $|K_\alpha(t)\rangle = e^{-t(im_\alpha+\Gamma_\alpha/2)}|K_\alpha\rangle$, with $\alpha = S, L$. Thus

$$\begin{aligned} \rho(t) = {} & \rho_{SS}(t)|K_S^0\rangle\langle K_S^0| + \rho_{SL}(t)|K_S^0\rangle\langle K_L^0| + \rho_{S0}(t)|K_S^0\rangle\langle 0| \\ & + \rho_{LS}^*(t)|K_L^0\rangle\langle K_S^0| + \rho_{LL}(t)|K_L^0\rangle\langle K_L^0| + \rho_{L0}(t)|K_L^0\rangle\langle 0| \\ & + \rho_{S0}^*(t)|0\rangle\langle K_S^0| + \rho_{L0}^*(t)|0\rangle\langle K_L^0| + \rho_{00}(t)|0\rangle\langle 0|, \end{aligned} \tag{3.47}$$

with $\rho_{SS}(t) = e^{-t\Gamma_S}\rho_{SS}(0)$ and $\rho_{LL}(t) = e^{-t\Gamma_L}\rho_{LL}(0)$[6].

Similarly to what was done in the case of pions, we can write the off-diagonal elements of the matrix representation of ρ as time-dependent linear combinations of all the elements of the initial density operator,

$$\begin{aligned} \rho_{SL}(t) &= \sum_{i,j=S,L,0} A_{ij}\rho_{ij}(0), \quad \rho_{S0}(t) = \sum_{i,j=S,L,0} B_{ij}\rho_{ij}(0), \\ \rho_{L0}(t) &= \sum_{i,j=S,L,0} C_{ij}\rho_{ij}(0), \quad \rho_{00}(t) \sum_{i,j=S,L,0} D_{ij}(t)\rho_{ij}(0). \end{aligned} \tag{3.48}$$

[6] For simplicity we will develop the kaon model ignoring CP-violation. Otherwise, the coefficients of the decomposition 3.47 would not represent the matrix elements of the density operator ρ in the basis $\{|K_S\rangle, |K_L\rangle\}$ (see [15]).

Then, the density matrix of the system can be written as

$$\rho(t) = V(t)\rho(0) = \begin{pmatrix} e^{-t\Gamma_S}\rho_{SS}(0) & \sum_{i,j=S,L,0} A_{ij}\rho_{ij}(0) & \sum_{i,j=S,L,0} B_{ij}\rho_{ij}(0) \\ \sum_{i,j=S,L,0} A^*_{ij}\rho^*_{ij}(0) & e^{-t\Gamma_L}\rho_{LL}(0) & \sum_{i,j=S,L,0} C_{ij}\rho_{ij}(0) \\ \sum_{i,j=S,L,0} B^*_{ij}(t)\rho^*_{ij}(0) & \sum_{i,j=S,L,0} C^*_{ij}(t)\rho^*_{ij}(0) & \sum_{i,j=S,L,0} D_{ij}(t)\rho_{ij}(0) \end{pmatrix}. \tag{3.49}$$

The superselection rules for the neutral kaon allows us set $\rho_{S0}(t) = \rho_{L0}(t) = 0$. This causes that the only nonvanishing B's and C's are $B_{\alpha 0}$ and $C_{\alpha 0}$ with $\alpha = S, L$. Moreover, since $\rho_{00}(t)$ must be real, $D_{SL}(t) = D^*_{LS}(t)$, $D_{S0}(t) = D^*_{0S}(t)$, $D_{L0}(t) = D^*_{0L}(t)$ and the other $D's$ must be real functions. The initial conditions are $A_{SL}(0) = B_{S0}(0) = C_{L0}(0) = 1$ and all other coefficients vanishing for $t = 0$. Furthermore, if

$$\rho(0) = |0\rangle\langle 0|, \quad \text{then}, \quad \forall t \geqslant 0 : \rho(t) = |0\rangle\langle 0|, \tag{3.50}$$

which implies $D_{00}(t) = 1$. We also have a condition for the trace

$$\begin{aligned} tr[\rho(t)] = \left(e^{-t\Gamma_S} + D_{SS}(t)\right)\rho_{SS}(0) + \left(e^{-t\Gamma_L} + D_{LL}(t)\right)\rho_{LL}(0) + 2Re(D_{SL}(t)\rho_{SL}(0)) \\ + 2Re(D_{S0}(t)\rho_{S0}(0)) + 2Re(D_{L0}(t)\rho_{L0}(0)) + \rho_{00}(0) = tr[\rho(0)], \end{aligned} \tag{3.51}$$

from which we get

$$\begin{aligned} &D_{SS}(t) = 1 - e^{-t\Gamma_S}, \\ &D_{LL}(t) = 1 - e^{-t\Gamma_L} \quad \text{and} \\ &D_{SL}(t) = D_{S0}(t) = D_{L0}(t) = 0. \end{aligned} \tag{3.52}$$

Therefore, the corresponding Choi's matrix is

$$Choi(V) = \begin{pmatrix} e^{-t\Gamma_S} & A_{SS}(t) & 0 & 0 & A_{SL}(t) & 0 & 0 & A_{S0}(t) & B_{S0}(t) \\ A^*_{SS}(t) & 0 & 0 & A^*_{LS}(t) & 0 & 0 & A^*_{0S}(t) & 0 & C_{S0} \\ 0 & 0 & D_{SS}(t) & 0 & 0 & D_{SL}(t) & B^*_{0S}(t) & C^*_{0S}(t) & D_{S0}(t) \\ 0 & A_{LS}(t) & 0 & 0 & A_{LL}(t) & 0 & 0 & A_{L0}(t) & B_{L0}(t) \\ A^*_{SL}(t) & 0 & 0 & A^*_{LL}(t) & e^{-t\Gamma_L} & 0 & A^*_{0L}(t) & 0 & C_{L0} \\ 0 & 0 & D^*_{SL}(t) & 0 & 0 & D_{LL}(t) & B^*_{0L}(t) & C^*_{0L}(t) & D_{L0}(t) \\ 0 & A_{0S}(t) & B_{0S}(t) & 0 & A_{0L}(t) & B_{0L}(t) & 0 & A_{00}(t) & B_{00}(t) \\ A^*_{S0}(t) & 0 & C_{0S}(t) & A^*_{L0}(t) & 0 & C_{0L}(t) & A^*_{00}(t) & 0 & C_{00}(t) \\ B^*_{S0}(t) & C^*_{S0}(t) & D^*_{S0}(t) & B^*_{L0}(t) & C^*_{L0}(t) & D^*_{L0}(t) & B^*_{00}(t) & C^*_{00}(t) & 1 \end{pmatrix} \Delta \tag{3.53}$$

In order for this matrix to be (semi)positive, the following conditions must be satisfied

$$
\begin{aligned}
|A_{SL}(t)|^2 &\leqslant e^{-t(\Gamma_S+\Gamma_L)},\\
|B_{S0}(t)|^2 &\leqslant e^{-t\Gamma_L},\\
|C_{L0}(t)|^2 &\leqslant e^{-t\Gamma_S},\\
|B_{S0}(t)|^2 e^{-t\Gamma_L} + |C_{L0}(t)|^2 e^{-t\Gamma_S} + |A_{SL}(t)|^2 - 2Re(A_{SL}(t)B^*_{S0}(t)C_{L0}(t)) &\leqslant e^{-t(\Gamma_S+\Gamma_L)},
\end{aligned}
\tag{3.54}
$$

and all other coefficients vanishing.

Now, taking into account the semigroup condition and the above equations, we get

$$
\begin{aligned}
A_{SL}(t) &= e^{-t[(\Gamma_S+\Gamma_L)/2+a-i\mu_{SL}]},\\
B_{S0}(t) &= e^{-t(\Gamma_S/2+b+i\mu_S)},\\
C_{L0}(t) &= e^{-t(\Gamma_L/2+c+i\mu_L)},
\end{aligned}
\tag{3.55}
$$

where a, b, c, μ_S, μ_L, $\mu_{SL} \geqslant 0$. However, we observe that the concrete time dependence of the functions $B_{S0}(t)$ and $C_{L0}(t)$ is irrelevant, due to the superselection rules. So, without loss of generality, we can choose $a = b = c = \lambda$ and $\mu_S = m_S$, $\mu_L = m_L$, from which we get $\mu_{SL} = \Delta m$. Hence, we can put

$$
\begin{aligned}
A_{SL}(t) &= e^{-t[(\Gamma_S+\Gamma_L)/2+\lambda-i\Delta m]},\\
B_{S0}(t) &= e^{-t(\Gamma_S/2+\lambda+im_S)} \qquad \text{and}\\
C_{L0}(t) &= e^{-t(\Gamma_L/2+\lambda+im_L)}.
\end{aligned}
\tag{3.56}
$$

Therefore, the density matrix describing the evolution of the kaon is given by

$$
\rho(t) = \begin{pmatrix}
e^{-t\Gamma_S}\rho_{SS}(0) & e^{-t(\Gamma+\lambda-i\Delta m)}\rho_{SL}(0) & 0\\
e^{-t(\Gamma+\lambda+i\Delta m)}\rho^*_{SL}(0) & e^{-t\Gamma_L}\rho_{LL}(0) & 0\\
0 & 0 & \rho_{00}(t)
\end{pmatrix},
\tag{3.57}
$$

where $\Delta m = m_L - m_S$, $\Gamma = (\Gamma_S + \Gamma_L)/2$, λ is some decoherence parameter and

$$
\rho_{00}(t) = (1 - e^{-t\Gamma_S})\rho_{SS}(0) + (1 - e^{-t\Gamma_L})\rho_{LL}(0) + \rho_{00}(0). \tag{3.58}
$$

The time evolution of the density operator given by equation 3.57 can be written in the operator-sum representation with the following Kraus operators:

$$
\begin{aligned}
W_0(t) &= e^{-t[(\Gamma_S+\lambda)/2+im_S]}|K_S\rangle\langle K_S| + e^{-t[(\Gamma_L+\lambda)/2+im_L]}|K_L\rangle\langle K_L| + e^{-t\lambda/2}|0\rangle\langle 0|,\\
W_1(t) &= \sqrt{1-e^{-t\Gamma_S}}|0\rangle\langle K_S|,\\
W_2(t) &= \sqrt{1-e^{-t\Gamma_L}}|0\rangle\langle K_L|,\\
W_3(t) &= e^{-t\Gamma_S/2}\sqrt{1-e^{-t\lambda}}|K_S\rangle\langle K_S|,\\
W_4(t) &= e^{-t\Gamma_L/2}\sqrt{1-e^{-t\lambda}}|K_L\rangle\langle K_L|,\\
W_5(t) &= \sqrt{1-e^{-t\lambda}}|0\rangle\langle 0|.
\end{aligned}
\tag{3.59}
$$

The density operator $\rho(t)$ obeys the Lindblad master equation 3.27 with the following Hamiltonian and Lindblad operators, respectively:

$$
\begin{aligned}
H &= m_S|K_S\rangle\langle K_S| + m_L|K_L\rangle\langle K_L|,\\
L_1 &= \sqrt{\Gamma_S}|0\rangle\langle K_S|,\\
L_2 &= \sqrt{\Gamma_L}|0\rangle\langle K_L|,\\
L_3 &= \sqrt{\lambda}|K_S\rangle\langle K_S|,\\
L_4 &= \sqrt{\lambda}|K_L\rangle\langle K_L|,\\
L_5 &= \sqrt{\lambda}|0\rangle\langle 0|.
\end{aligned}
\tag{3.60}
$$

From 3.57, it follows that the probabilities of detecting a $|K^0\rangle$ or a $|\overline{K^0}\rangle$, at time t, are given by

$$
\begin{aligned}
p_{K^0} &= tr[\rho(t)|K^0\rangle\langle K^0|] = \frac{1}{2}\{e^{-t\Gamma_S}\rho_{SS}(0) + e^{-t\Gamma_L}\rho_{LL}(0) + 2Re[e^{-t(\Gamma+\lambda-i\Delta m)}\rho_{SL}(0)]\},\\
p_{\overline{K^0}} &= tr[\rho(t)|\overline{K^0}\rangle\langle \overline{K^0}|] = \frac{1}{2}\{e^{-t\Gamma_S}\rho_{SS}(0) + e^{-t\Gamma_L}\rho_{LL}(0) - 2Re[e^{-t(\Gamma+\lambda-i\Delta m)}\rho_{SL}(0)]\}.
\end{aligned}
\tag{3.61}
$$

The expected value of the strangeness operator, given by $S = |K^0\rangle\langle K^0| - |\overline{K^0}\rangle\langle\overline{K^0}|$, is

$$
\langle S\rangle = tr[\rho(t)S] = e^{-t\Gamma_S}\rho_{SS}(0) + e^{-t\Gamma_L}\rho_{LL}(0) + 2Re[e^{-t(\Gamma+\lambda-i\Delta m)}\rho_{SL}(0)]. \tag{3.62}
$$

If the initial state is $|K^0\rangle$, we get

$$
\langle S\rangle = e^{-t(\Gamma+\lambda)}\cos(\Delta m t), \tag{3.63}
$$

in agreement with 1.24 (with $\delta_L = 0$ there, and $\lambda = 0$ here).

We are now in position to discuss the problem of kaon interferometry using an open quantum model for the system time evolution, which is the aim of this thesis.

4 Complementarity in the Kaon System

Complementarity in the neutral kaon system was first studied by A. Bramon, G. Garbarino and B. Hiesmayr in [8, 9, 10] using the Wigner-Weisskopf approach to the isolated free-kaon propagation. They argued that the kaon propagation in free-space mimics an interfering object in an double-slit experiment. As we have seen, these particles are produced by strong interactions in the strangeness states $|K^0\rangle$ and $|\overline{K^0}\rangle$, but as soon as that, they start to propagate in free-space in a coherent superposition of the $|K_S\rangle$ and $|K_L\rangle$ components. These two components evolve in time without oscillating into each other and can be regarded as analogues to the two separated paths in a usual interferometer. Likewise, in the kaon interferometry, the role of the interference fringes is replaced by the phenomenon of strangeness oscillation, since they are the complementary quantity related to the propagation states. That is, once the propagation state of the kaon is known, K_S or K_L, no strangeness oscillation can be observed.

Suppose we start at (proper) time $\tau = 0$ with a kaon produced in the state $|K^0\rangle$, then[1], for a future time τ, we have

$$|K^0(\tau)\rangle = \frac{1}{\sqrt{2}}(e^{-i\tau\lambda_S}|K_S\rangle + e^{-i\tau\lambda_L}|K_L\rangle), \tag{4.1}$$

where $\lambda_{S,L} = m_{S,L} - \frac{i}{2}\Gamma_{S,L}$ are those defined in section 1.4.1. Now, normalising to kaons surviving up to time τ, the previous state can be rewritten more suitably as

$$|K^0(\tau)\rangle = \frac{1}{\sqrt{1+e^{-\Delta\Gamma\tau}}}(|K_S\rangle + e^{-\tau(\Delta\Gamma/2+i\Delta m)}|K_L\rangle), \tag{4.2}$$

where $\Delta\Gamma = \Gamma_L - \Gamma_S < 0$ and $\Delta m = m_L - m_S$. Relating the present case with the discussion in section 2.1.1, we see that $|K_S\rangle$ and $|K_L\rangle$ play the role of the states $|\psi_1\rangle$ and $|\psi_2\rangle$, with the relative phase $\phi = -\Delta m\tau$. The probabilities for propagation via K_S and K_L of a initial K^0 surviving up to time τ are, therefore

$$\begin{aligned} \omega_S &\equiv |\langle K_S|K^0(\tau)\rangle|^2 = \frac{1}{1+e^{-\Delta\Gamma\tau}}, \\ \omega_L &\equiv |\langle K_L|K^0(\tau)\rangle|^2 = \frac{1}{1+e^{\Delta\Gamma\tau}}, \end{aligned} \tag{4.3}$$

from which we obtain the *width predictability*, analogue to the *path predictability* in the usual interferometers,

$$\mathcal{P}(\tau) \equiv |\omega_S - \omega_L| = \left|\tanh\left(\frac{1}{2}\Delta\Gamma\tau\right)\right|. \tag{4.4}$$

[1] Ignoring CP-violation effects. This approximation is justified because neutral kaons decaying after times bigger than $4.8\tau_S$ can be identified as K_L's with negligible error probability.

We can also write equation 4.2 in the strangeness basis $\{|K^0\rangle, |\overline{K^0}\rangle\}$ - which are, obviously, the natural substitutes for the basis $\{|\psi_+\rangle, |\psi_-\rangle\}$ in the above mentioned section - from which we have

$$|K^0(\tau)\rangle = \frac{1}{\sqrt{2}}\left[\frac{1+e^{-\tau(\Delta\Gamma/2+i\Delta m)}}{\sqrt{1+e^{-\Delta\Gamma\tau}}}|K^0\rangle + \frac{1-e^{-\tau(\Delta\Gamma/2+i\Delta m)}}{\sqrt{1+e^{-\Delta\Gamma\tau}}}|\overline{K^0}\rangle\right]. \tag{4.5}$$

Then, the fringe visibility of the $K^0 - \overline{K^0}$ oscillation is easily obtained from

$$\begin{aligned} |\langle K^0|K^0(\tau)\rangle|^2 &= \frac{1}{2}\left[1 + \frac{1}{\cosh(\Delta\Gamma\tau/2)}\cos(\Delta m\tau)\right] \equiv \frac{1}{2}\Big\{1 + \mathcal{V}_0\cos(\Delta m\tau)\Big\}, \\ |\langle \overline{K^0}|K^0(\tau)\rangle|^2 &= \frac{1}{2}\left[1 - \frac{1}{\cosh(\Delta\Gamma\tau/2)}\cos(\Delta m\tau)\right] \equiv \frac{1}{2}\Big\{1 - \mathcal{V}_0\cos(\Delta m\tau)\Big\}. \end{aligned} \tag{4.6}$$

Therefore

$$\mathcal{V}_0(\tau) = \frac{1}{\cosh\left(\frac{1}{2}\Delta\Gamma\tau\right)}. \tag{4.7}$$

Combining equations 4.4 and 4.7 one immediately gets a relation equivalent to the one proposed by Greenberger and Yasin discussed in section 2.1.1,

$$\mathcal{P}(\tau)^2 + \mathcal{V}_0(\tau)^2 = 1. \tag{4.8}$$

When $\tau = 0$, the probabilities for propagation in the K_S and K_L components are the same, $\omega_S = \omega_L = 1/2$. Hence, the width predictability vanishes, $\mathcal{P}(0) = 0$, and, conversely, the visibility of strangeness oscillations is maximal, $\mathcal{V}_0(0) = 1$. However, since their lifetimes are very different, $\Gamma_S \approx 579\Gamma_L$, they work like two paths of an asymmetric interferometer, and, therefore, for times $\tau > 0$, the probabilities for propagation in one state or another becomes quite distinct. Indeed, the larger τ is, the bigger the likelihood for propagation in the K_L component, once the K_S component leads to much faster decay. Thus, for initial K^0's surviving up to time τ, which-width information is obtained and accounted for in $\mathcal{P}(\tau)$. Whereas the complementary strangeness oscillation visibility $\mathcal{V}_0(\tau)$ is reduced, and almost completely disappears for $\tau \gtrsim 4.8\tau_S$, when the probability for a K_S survival is of a few per thousands.

Now that we have illustrate the basic ideas of the neutral kaon interferometry, we can proceed to the discussion and analysis of this problem using an open quantum system model for the kaons time evolution.

4.1 Open Quantum System Model to Neutral Kaon Interferometry

Now we revisit the problem of the neutral kaon interferometry of the previous section using an open quantum system model instead of the above Wigner-Weisskopf approach. The advantage of this model is that it allows us to include the specific decay products of the kaons as a second party and, consequently, to study the contribution of quantum correlations to its quantitative complementarity relation. The results discussed here have been published in [11].

Let us consider the total Hilbert space of the system to be the tensor product of the kaon (quanton) Hilbert space and the Hilbert space of the decay products states,

$$\mathcal{H} = \mathcal{H}_Q \otimes \mathcal{H}_P. \tag{4.9}$$

We take the kaon space state to be the direct sum $\mathcal{H}_Q = \mathcal{H}_{K^0} \oplus \mathcal{H}_0$ considered in section 3.1.2. That is, $\mathcal{H}_0$ is the Hilbert space spanned by the vector $|0_K\rangle$, representing the vacuum (absence of particle), whereas $\mathcal{H}_{K^0}$ is the usual kaon Hilbert space, spanned by the eigenstates of strangeness $|K^0\rangle$ and $|\overline{K^0}\rangle$. Under the assumption of CP-symmetry, the short and long lived kaons are given by

$$|K_S\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K^0}\rangle), \quad |K_L\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K^0}\rangle), \tag{4.10}$$

with $\langle K_S|K_L\rangle = 0$. Moreover, with this neglection of CP-violation, we consider that K_L never decays into two pions. Therefore, the states of the decay products can be labelled by its pion content. Indeed, the short-lived kaon, K_S always decays into two pions, which can be either a pair of charged pions $\pi^+ + \pi^-$ or a pair of neutral ones $\pi^0 + \pi^0$. On the other hand, long-lived kaons, K_L's, have several decay modes. They can decay into three pions, $\pi^+ + \pi^- + \pi^0$ or $\pi^0 + \pi^0 + \pi^0$, or —via its semileptonic decays —into $\pi^\pm + e^\mp + \nu_e$ or $\pi^\pm + \mu^\mp + \nu_\mu$. Hence, it is enough to identify the states of the decay products by the number of pions it contains. Thus, we can take the Hilbert space $\mathcal{H}_P$ as the space spanned by the orthonormalized vectors $|0_\pi\rangle$, $|\pi\pi\rangle$ and $|\widetilde{\pi\pi}\rangle$, representing states with no pions, two pions and one or three pions, respectively.

The physically meaningful initial states are those with a kaon and no pions, that is, factorised states of the form

$$|\Psi(0)\rangle = (\alpha|K_S\rangle + \beta|K_L\rangle + \gamma|0_K\rangle)|0_\pi\rangle, \tag{4.11}$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Because we do not expect to have superpositions between kaons and pions, we assume the evolution of the system to take place in the subspace $\mathcal{W} < \mathcal{H}$ spanned by $\{|0_K\rangle|0_\pi\rangle, |K_S\rangle|0_\pi\rangle, |K_L\rangle|0_\pi\rangle, |0_K\rangle|\pi\pi\rangle, |0_K\rangle|\widetilde{\pi\pi}\rangle\}$. Furthermore, we assume the system evolves according to the map

$$\begin{aligned}
|0_K\rangle|0_\pi\rangle &\longmapsto |0_K\rangle|0_\pi\rangle, \\
|K_S\rangle|0_\pi\rangle &\longmapsto e^{-\tau(\Gamma_S/2+im_S)}|K_S\rangle|0_\pi\rangle + \sqrt{1-e^{\tau\Gamma_S}}|0_K\rangle|\pi\pi\rangle, \\
|K_L\rangle|0_\pi\rangle &\longmapsto e^{-\tau(\Gamma_L/2+im_L)}|K_L\rangle|0_\pi\rangle + \sqrt{1-e^{\tau\Gamma_L}}|0_K\rangle|\widetilde{\pi\pi}\rangle.
\end{aligned} \tag{4.12}$$

This is equivalent to say that in the Kraus sum representation discussed in chapter 3.1, the evolution of the state is given by

$$\rho(\tau) = \sum_{i=0}^{1} W_i(\tau)\rho(0)W_i^\dagger(\tau), \quad \text{where} \quad \rho(0) = |\Psi(0)\rangle\langle\Psi(0)|, \quad \text{and}$$

$$\begin{aligned} W_0 &= |0_K\rangle\langle 0_K| \otimes \Big(|\pi\pi\rangle\langle\pi\pi| + |\pi\pi\rangle\langle\widetilde{\pi\pi}| + |0_\pi\rangle\langle 0_\pi|\Big) \\ W_1 &= \Big[e^{-\tau(\Gamma_S/2+im_S)}|K_S\rangle\langle K_S| + e^{-\tau(\Gamma_L/2+im_L)}|K_L\rangle\langle K_L|\Big] \otimes |0_\pi\rangle\langle 0_\pi| \\ &+ \sqrt{1+e^{-\tau\Gamma_S}}|0_K\rangle\langle K_S| \otimes |\pi\pi\rangle\langle 0_\pi| + \sqrt{1+e^{-\tau\Gamma_L}}|0_K\rangle\langle K_L| \otimes |\widetilde{\pi\pi}\rangle\langle 0_\pi|. \end{aligned} \tag{4.13}$$

Under these conditions, the density operator of the composite system $\rho(\tau)$, with the initial state $|\Psi(0)\rangle$, remains pure in the course of dynamics. That is, $\rho(\tau) = |\Psi(\tau)\rangle\langle\Psi(\tau)|$. Focusing our attention now in kaons produced in the strangeness eigenstate $|K^0\rangle$, we may take $\alpha = \beta = \frac{1}{\sqrt{2}}$ and $\gamma = 0$. Therefore, we have the completely positive, trace-preserving and Wigner-Weisskopf phenomenological description compatible evolution

$$\begin{aligned} \rho(\tau) &= |\Psi(\tau)\rangle\langle\Psi(\tau)|, \\ |\Psi(\tau)\rangle &= \frac{1}{\sqrt{2}}\Big\{\Big[e^{-\tau(\Gamma_S/2+im_S)}|K_S\rangle + e^{-\tau(\Gamma_L/2+im_L)}|K_L\rangle\Big] \otimes |0_\pi\rangle \\ &+ |0_K\rangle \otimes \Big[\sqrt{1-e^{-\tau\Gamma_S}}|\pi\pi\rangle + \sqrt{1-e^{-\tau\Gamma_L}}|\widetilde{\pi\pi}\rangle\Big]\Big\}. \end{aligned} \tag{4.14}$$

Tracing out the degrees of freedom of the pions space, we get the reduced kaon state

$$\begin{aligned} \rho_Q(\tau) = tr_P[\rho(\tau)] =& \frac{1}{2}\Big\{e^{-\tau\Gamma_S}|K_S\rangle\langle K_S| + e^{-\tau\Gamma_L}|K_L\rangle\langle K_L| \; + e^{-\tau(\Gamma-i\Delta m)}|K_S\rangle\langle K_L| \\ &+ e^{-\tau(\Gamma+i\Delta m)}|K_L\rangle\langle K_S| + (2-e^{-\tau\Gamma_S}-e^{-\tau\Gamma_L})|0_K\rangle\langle 0_K|\Big\}. \end{aligned} \tag{4.15}$$

We notice that this coincides with the kaon state evolution described in section 3.1.2 (with the same initial condition). Likewise, the reduced pionic state can be readily evaluated from $\rho(\tau)$, which yields

$$\begin{aligned} \rho_P(\tau) = tr_Q[\rho(\tau)] =& \frac{1}{2}\Big\{(1-e^{-\tau\Gamma_S})|\pi\pi\rangle\langle\pi\pi| + (1-e^{-\tau\Gamma_L})|\widetilde{\pi\pi}\rangle\langle\widetilde{\pi\pi}| + (e^{-\tau\Gamma_S}+e^{-\tau\Gamma_L})|0_\pi\rangle\langle 0_\pi| \\ &+ \sqrt{(1-e^{-\tau\Gamma_S})(1-e^{-\tau\Gamma_L})}\Big(|\pi\pi\rangle\langle\widetilde{\pi\pi}| + |\widetilde{\pi\pi}\rangle\langle\pi\pi|\Big)\Big\}. \end{aligned} \tag{4.16}$$

Now we are in position to apply our model to a quantitative analysis of complementarity in the neutral kaon system. Our aim is to establish a relation between interference visibility, width/lifetime information and the new feature allowed by the model bipartite's character: entanglement. In this analysis we can restrict ourselves to proper times in the interval $0 \leqslant \tau \leqslant \tau_0$, where $\tau_0 = 4.8\tau_S$. As discussed in the previous section, we can regard kaons decaying after τ_0 as long-lived kaons K_L with an error of only a few per thousands.

In order to quantify the bipartite entanglement of the system we may use the entropy of entanglement given by the von Neumann entropy S of the reduce states. It is easily shown that the eigenvalues of 4.15 and 4.16 are $\{0, x, 1-x\}$, where

$$x(\tau) \equiv \frac{e^{-\tau\Gamma_S} + e^{-\tau\Gamma_L}}{2}. \tag{4.17}$$

Therefore,

$$S = -x\ln x - (1-x)\ln(1-x). \tag{4.18}$$

As correlations between the quanton (kaon) and detector (pion states) are dynamically generated, information about the kaon's $\{|K_S\rangle, |K_L\rangle\}$ components leaks to the pionic subsystem. Hence, the natural quantifier of which width information available to be retrieved through the pionic state is the distinguishability

$$\mathcal{D}(\tau) = \frac{1}{2}\left\|\rho_P^{(S)}(\tau) - \rho_P^{(L)}(\tau)\right\|, \tag{4.19}$$

i.e., by the distance in the trace-class norm between the states corresponding to the distinct kaon propagation modes K_S and K_L,

$$\begin{aligned}\rho_P^{(S)}(\tau) &= 2\langle K_S|\rho(\tau)|K_S\rangle = e^{-\tau\Gamma_S}|0_\pi\rangle\langle 0_\pi|,\\ \rho_P^{(L)}(\tau) &= 2\langle K_L|\rho(\tau)|K_L\rangle = e^{-\tau\Gamma_L}|0_\pi\rangle\langle 0_\pi|.\end{aligned} \tag{4.20}$$

Therefore,

$$\mathcal{D}(\tau) = \frac{1}{2}\left| e^{-\tau\Gamma_S} - e^{-\tau\Gamma_L} \right|. \tag{4.21}$$

The interferometric visibility in this case may be quantified by the natural complementary of the trace distance, fidelity

$$\mathcal{V}(\tau) = \mathcal{F}\Big(\rho_P^{(S)}(\tau), \rho_P^{(L)}(\tau)\Big) = tr_P\left[\sqrt{\sqrt{\rho_P^{(S)}(\tau)}\rho_P^{(L)}(\tau)\sqrt{\rho_P^{(S)}(\tau)}}\right]. \tag{4.22}$$

The fidelity is an "overlap" measure, hence, may be used to quantify the visibility of quantum interference between $\rho_P^{(S)}$ and $\rho_P^{(L)}$. We have

$$\mathcal{V}(\tau) = e^{-\tau\Gamma}. \tag{4.23}$$

The time evolution of these three quantities can be seen in the figure 3. In the proper time interval $[0,\tau_0]$, the entropy of entanglement S and the distinguishability $\mathcal{D}$ are monotonically increasing functions, whereas the interference visibility $\mathcal{V}$ monotonically decreases.

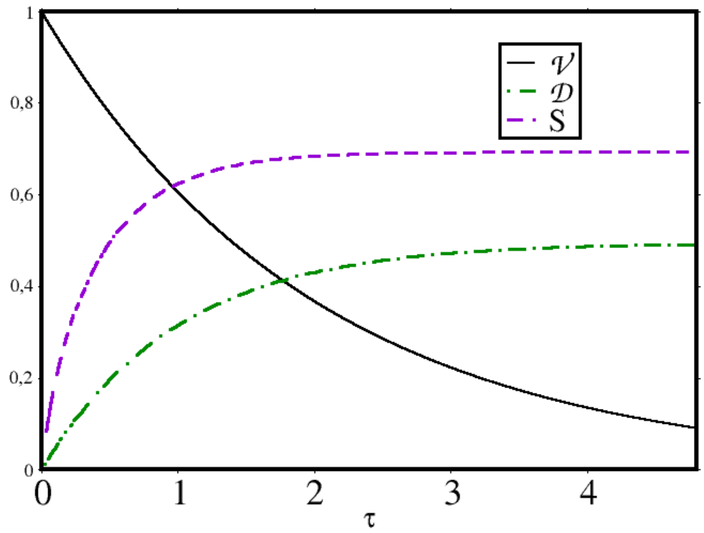

Figure 3 – The green dot-dashed line and the purple dashed line show the increase in $\mathcal{D}$ and S, respectively, whereas the solid black line shows the correspondingly decrease in $\mathcal{V}$. Proper times ranges from $\tau = 0$ to $\tau = 4.79\tau_S$.

We may compare the upper bounds for $\mathcal{V}^2$ given by $1-\mathcal{D}^2$ alone and by $1-\mathcal{D}^2-S^2$. We notice the importance of the inclusion of the quantitative entanglement measure S in this triality relation. Its presence makes the relation much more consistent with the actual reduction in the interference visibility $\mathcal{V}$ enforced by the quantum complementarity:

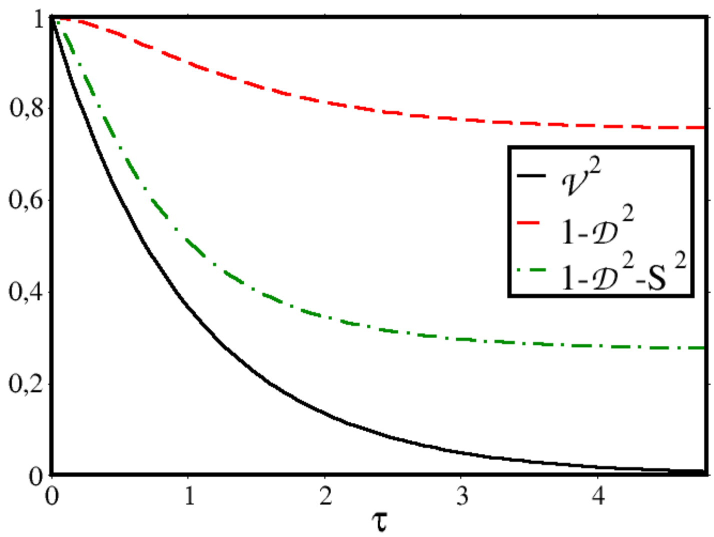

Figure 4 – Upper bounds for $\mathcal{V}^2$, the solid black line, given by $1-\mathcal{D}^2$, in the dashed red line, and $1-\mathcal{D}^2-S^2$, in the dot-dashed green line.

As pointed out by Jakob and Bergou [14], in bipartite system complementarity must relate the particle-like and wave-like properties of one system with a entanglement measure. Here we have

$$\mathcal{D}^2 + \mathcal{V}^2 = e^{-\tau(\Gamma_S+\Gamma_L)} + \frac{(e^{-\tau\Gamma_S} - e^{-\tau\Gamma_L})^2}{4} = x(\tau)^2, \tag{4.24}$$

such that

$$\mathcal{V}^2 + \mathcal{D}^2 + S^2 = x^2 + [x\ln(x) + (1-x)\ln(1-x)]^2 \leqslant 1. \tag{4.25}$$

In the figure 5 one can see the time evolution of the overall quantity $\mathcal{V}^2 + \mathcal{D}^2 + S^2$, compared to its upper bound 1.

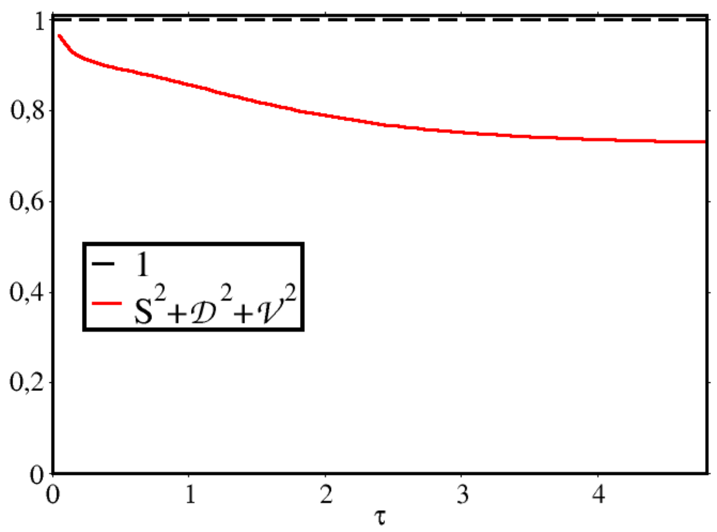

Figure 5 – The solid red line shows $\mathcal{V}^2 + \mathcal{D}^2 + S^2$. The upper bound 1 is shown in the black dashed line.

This inequality also accounts for the complementarity between strangeness oscillation and lifetime information. The visibility of $K^0\overline{K^0}$ oscillation must be defined here as the quantity $\mathcal{V}_0(\tau)$ such that

$$tr_Q\Big[|K^0\rangle\langle K^0|\rho_Q(\tau)\Big] = F(\tau)\{1 + \mathcal{V}_0(\tau)\cos(\Delta m\tau)\}. \tag{4.26}$$

Direct calculation gives

$$\mathcal{V}_0(\tau) = \frac{2e^{-\tau\Gamma}}{e^{-\tau\Gamma_S} + e^{-\tau\Gamma_L}} = \frac{\mathcal{V}(\tau)}{x}. \tag{4.27}$$

Now, the ratio between its derivative and that of the fidelity visibility

$$\frac{d\mathcal{V}_0}{d\tau}\bigg/\frac{d\mathcal{V}}{d\tau} = \frac{1}{x}\left(1 - \frac{\Gamma_S e^{-\tau\Gamma_S} + \Gamma_L e^{-\tau\Gamma_L}}{\Gamma(e^{-\tau\Gamma_S} + e^{-\tau\Gamma_L})}\right) = \frac{2}{x^2}\left(\frac{\Gamma_L e^{-\tau\Gamma_S} + \Gamma_S e^{-\tau\Gamma_L}}{\Gamma}\right) > 0. \tag{4.28}$$

Both these quantities decrease in time, therefore, the increase of lifetime information as measured by $\mathcal{D}(\tau)$ forces not only $\mathcal{V}$, but also the oscillation visibility $\mathcal{V}_0$ to decrease in time. Of course, here, $\mathcal{V}_0 = 1/\cosh(\Delta\Gamma\tau/2)$, the same as showed in the previous section.

5 Conclusion

Entanglement plays a crucial role in quantum mechanics complementarity. Here we have shown that this can be clearly illustrated and discussed in the kaon-antikaon oscillating system. We considered a bipartite open quantum system model where a single neutral kaon and its weak decay products are taken into account. From an interferometric point of view, the kaon is treated as the interfering object, the quanton, and lifetime/which-width information plays the role of which-way information. This is similar to what is made in previous works on the subject of kaon interferometry by Bramon, Garbarino and Hiesmayr [8, 9, 10]. We verified that, as entanglement correlations between these two parties are established, lifetime information leaks and becomes available in the pionic state. Accordingly, as entanglement is generated and lifetime information is acquired, the visibility of which-way interference is reduced. The interplay between these quantities was proved to be governed by the "triality" relation:

$$\mathcal{V}^2 + \mathcal{D}^2 + S^2 \leqslant 1. \tag{5.1}$$

Qualitatively, this inequality is similar to the one proposed by Jakob and Bergou in their analysis of wave-particle duality in two-qubits bipartite systems [14].

We have shown, figure 4.1, that the inclusion of the entanglement measure in the quantitative complementarity relation is very important. In it, we can see that the upper bound for $\mathcal{V}^2$ given by $1 - \mathcal{D}^2 - S^2$, that is, including the calculated entanglement quantity, is much sharper and consistent with the actual reduction in $\mathcal{V}$ than the one given by $1 - \mathcal{D}^2$.

We have also shown how our inequality accounts for the complementarity between strangeness oscillations and lifetime information in the interval relevant for the analysis. This demonstrates consistency with the previous analysis of complementarity in the neutral kaon system, and with the general principle that visibility of any quantum interference phenomenon whatsoever must reduce when which-way information becomes available [34].

Bibliography

1 ROCHESTER, G. D.; BUTLER, C. C. Evidence for the Existence of New Unstable Elementary Particles. *Nature*, v. 160, p. 855–857, 1947.

2 GRIFFITHS, D. *Introduction to Elementary Particles*. [S.l.]: Wiley, 2008. (Physics textbook).

3 GELL-MANN, M.; PAIS, A. Behavior of neutral particles under charge conjugation. *Phys. Rev.*, v. 97, p. 1387–1389, Mar 1955.

4 PATRIGNANI, C. et al. Review of Particle Physics. *Chin. Phys.*, C40, n. 10, p. 100001, 2016.

5 BERTLMANN, R. A.; HIESMAYR, B. C. Bell inequalities for entangled kaons and their unitary time evolution. *Phys. Rev. A*, v. 63, p. 062112, May 2001.

6 BERTLMANN, R.; GRIMUS, W.; HIESMAYR, B. Bell inequality and CP violation in the neutral kaon system. *Physics Letters A*, v. 289, n. 1–2, p. 21 – 26, 2001.

7 BRAMON, A.; NOWAKOWSKI, M. Bell inequalities for entangled pairs of neutral kaons. *Phys. Rev. Lett.*, v. 83, p. 1–5, Jul 1999.

8 BRAMON A., G. G.; HIESMAYR, B. C. Quantitative duality and neutral kaon interferometry. *Eur. Phys. J. C*, v. 32, n. 3, p. 377–380, 2004.

9 BRAMON, A.; GARBARINO, G.; HIESMAYR, B. C. Quantitative complementarity in two-path interferometry. *Phys. Rev. A*, v. 69, p. 022112, Feb 2004.

10 BRAMON, A.; GARBARINO, G.; HIESMAYR, B. C. Quantum marking and quantum erasure for neutral kaons. *Phys. Rev. Lett.*, v. 92, p. 020405, Jan 2004.

11 SOUZA, G. de et al. Open-quantum-systems approach to complementarity in neutral-kaon interferometry. *Phys. Rev. A*, v. 94, p. 062108, Dec 2016.

12 GREENBERGER, D. M.; YASIN, A. Simultaneous wave and particle knowledge in a neutron interferometer. *Physics Letters A*, v. 128, n. 8, p. 391 – 394, 1988.

13 ENGLERT, B.-G. Fringe visibility and which-way information: An inequality. *Phys. Rev. Lett.*, v. 77, p. 2154–2157, Sep 1996.

14 JAKOB, M.; BERGOU, J. A. Quantitative complementarity relations in bipartite systems: Entanglement as a physical reality. *Optics Communications*, v. 283, n. 5, p. 827 – 830, 2010.

15 CABAN, P. et al. Unstable particles as open quantum systems. *Phys. Rev. A*, v. 72, p. 032106, Sep 2005.

16 NAGASHIMA, Y.; NAMBU, Y. *Elementary Particle Physics: Quantum Field Theory and Particles*. [S.l.]: Wiley, 2011.

17 HAROCHE, S.; RAIMOND, J. *Exploring the Quantum: Atoms, Cavities, and Photons*. [S.l.]: OUP Oxford, 2006. (Oxford Graduate Texts).

18 CHOI, M.-D. Completely positive linear maps on complex matrices. *Linear Algebra and its Applications*, v. 10, n. 3, p. 285 – 290, 1975.

19 BHATIA, R. *Positive Definite Matrices*. [S.l.]: Princeton University Press, 2009. (Princeton Series in Applied Mathematics).

20 HORN, R.; JOHNSON, C. *Matrix Analysis*. [S.l.]: Cambridge University Press, 2012. (Matrix Analysis).

21 LATTES, C. M. G.; OCCHIALINI, G. P. S.; POWELL, C. F. Observations on the Tracks of Slow Mesons in Photographic Emulsions. 1. *Nature*, v. 160, p. 453–456, 1947.

22 FERMI, E. Versuch einer theorie der β-strahlen. i. *Zeitschrift für Physik*, v. 88, n. 3, p. 161–177, 1934.

23 GELL-MANN, M. Isotopic spin and new unstable particles. *Phys. Rev.*, v. 92, p. 833–834, Nov 1953.

24 NAKANO, T.; NISHIJIMA, K. Charge Independence for V-particles. *Prog. Theor. Phys.*, v. 10, p. 581–582, 1953.

25 BRAMON, A.; GARBARINO, G.; HIESMAYR, B. C. Quantum Mechanics with Neutral Kaons. *Acta Phys. Polon.*, B38, p. 2763–2776, 2007.

26 WU, C. S. et al. Experimental test of parity conservation in beta decay. *Phys. Rev.*, v. 105, p. 1413–1415, Feb 1957.

27 LANDE, K. et al. Observation of long-lived neutral v particles. *Phys. Rev.*, v. 103, p. 1901–1904, Sep 1956.

28 CRONIN J. W., G. M. S. CP symmetry violation. *Physics Today*, v. 35, p. 38, Jul 1982.

29 CHRISTENSON, J. H. et al. Evidence for the 2π decay of the K_2^0 meson. *Phys. Rev. Lett.*, v. 13, p. 138–140, Jul 1964.

30 WILCZEK, F. The cosmic asymmetry between matter and antimatter. *Sci. Am.*, v. 243N6, p. 60–68, 1980.

31 PAIS, A.; PICCIONI, O. Note on the decay and absorption of the θ^0. *Phys. Rev.*, v. 100, p. 1487–1489, Dec 1955.

32 BOHR, N. The Quantum Postulate and the Recent Development of Atomic Theory. *Nature*, v. 121, n. 3050, p. 580–590, 1928.

33 FEYNMAN, R.; HIBBS, A.; STYER, D. *Quantum Mechanics and Path Integrals*. [S.l.]: Dover Publications, 2010. (Dover Books on Physics).

34 SCULLY, M. O.; ENGLERT, B.-G.; WALTHER, H. Quantum optical tests of complementarity. *Nature*, v. 351, n. 6322, p. 111–116, May 1991.

35 WOOTTERS, W. K.; ZUREK, W. H. Complementarity in the double-slit experiment: Quantum nonseparability and a quantitative statement of bohr's principle. *Phys. Rev. D*, v. 19, p. 473–484, Jan 1979.

36 SUMMHAMMER, J.; RAUCH, H.; TUPPINGER, D. Stochastic and deterministic absorption in neutron-interference experiments. *Phys. Rev. A*, v. 36, p. 4447–4455, Nov 1987.

37 BANASZEK, K. et al. Quantum mechanical which-way experiment with an internal degree of freedom. *Nature Communications*, v. 4, n. 2594, oct 2013.

38 TEJ, J. P. et al. Quantum which-way information and fringe visibility when the detector is entangled with an ancilla. *Phys. Rev. A*, v. 89, p. 062116, Jun 2014.

39 ADESSO, G. *Entanglement of Gaussian states*. Tese (Doutorado) — Salerno U., 2007.

40 COFFMAN, V.; KUNDU, J.; WOOTTERS, W. K. Distributed entanglement. *Phys. Rev. A*, v. 61, p. 052306, Apr 2000.

41 BREUER, H.; PETRUCCIONE, F. *The Theory of Open Quantum Systems*. [S.l.]: Oxford University Press, 2002.

42 BALLENTINE, L. *Quantum Mechanics: A Modern Development*. [S.l.]: World Scientific, 1998.

43 ARFKEN, G.; WEBER, H. *Mathematical methods for physicists*. [S.l.]: Elsevier Acad. Press, 2008.

44 GAEMERS, K.; VISSER, T. Deviations from exponential decay in quantum mechanics. *Physica A: Statistical Mechanics and its Applications*, v. 153, n. 2, p. 234 – 251, 1988.

Appendix

APPENDIX A – The Wigner-Weiskopff Approximation for Neutral Kaons

Here we describe the time evolution of the neutral kaons in the Wigner-Weisskopf approximation. We closely follow the arguments of [16].

We consider that the system is described by the Hamiltonian

$$H = H_0 + H_W, \tag{A.1}$$

where H_0 is the Hamiltonian of the strong interaction and the free part of the weak interaction, while H_W is the part of the weak interaction which induces transition and is assumed to be a small perturbation to H_0. The eigenstates of H_0 consist of the two degenerate discrete states $|K\rangle$, where $K = 1 \equiv K^0$ or $K = 2 \equiv \overline{K^0}$, and a series of continuum states $|n\rangle$, where $n = 2\pi, 3\pi, \pi^{\pm}l^{\mp}\nu, ...$, to which the meson K can decay weakly.

$$\begin{aligned} H_0|K\rangle &= E_0|K\rangle \\ H_0|n\rangle &= E_n|n\rangle. \end{aligned} \tag{A.2}$$

We consider the initial state $t = 0$ as a superposition of the discrete states $|K\rangle$.

$$|t = 0\rangle = a_1(0)|1\rangle + a_2(0)|2\rangle \equiv a_1(0)|K^0\rangle + a_2(0)|\overline{K^0}\rangle. \tag{A.3}$$

The time evolution of this vector state is given by the Schrödinger equation

$$i\frac{d}{dt}|t\rangle_S = (H_0 + H_W)|t\rangle_S \tag{A.4}$$

where the subscript S denotes that the states are considered in the Schrödinger picture. To work in the interaction picture we make the following changes

$$\begin{aligned} |t\rangle \equiv |t\rangle_I &= e^{iH_0t}|t\rangle_S, \\ H_W(t) &= e^{iH_0t}H_We^{-iH_0t}. \end{aligned} \tag{A.5}$$

This state, then, obeys the equation

$$i\frac{d}{dt}|t\rangle = H_W(t)|t\rangle. \tag{A.6}$$

We have

$$\begin{aligned} |t = 0\rangle &= |t = 0\rangle_S \\ |t\rangle &= a_1(t)|1\rangle + a_2(t)|2\rangle + \sum_n b_n(t)|n\rangle. \end{aligned} \tag{A.7}$$

It follows from it that

$$
\begin{aligned}
i\frac{da_k(t)}{dt} &= \sum_{K'=1,2} \langle K|H_W|K^{'}\rangle a_{K'}(t) + \sum_n e^{i(E_0-E_n)t}\langle K|H_W|n\rangle b_n(t),\\
i\frac{db_m(t)}{dt} &= \sum_{K'=1,2} e^{i(E_m-E_0)t}\langle m|H_W|K^{'}\rangle a_{K'}(t) + \sum_n e^{i(E_m-E_n)t}\langle m|H_W|n\rangle b_n(t)
\end{aligned}
\tag{A.8}
$$

The first Wigner-Weisskopf's approximation is to regard the decay products of the K mesons - most pions and muons - as stable particles. This is done by imposing the initial conditions

$$
a_1(0) = a_1, \quad a_2(0) = a_2, \quad b_n(0) = 0. \tag{A.9}
$$

This is equivalent to discarding the second term in the equation for $b_n(t)$ in A.8. With this approximation we can solve A.8 and obtain

$$
\begin{aligned}
b_n(t) &= -i\sum_{k=1,2}\int_0^t dt^{'}\, e^{i(E_n-E_0)t^{'}}\langle n|H_W|k\rangle a_k(t^{'}),\\
a_k(t) &= a_k - \sum_{k^{'}=1,2}\int_0^t dt^{'}\,\langle k|H_W|k^{'}\rangle a_{k^{'}}(t^{'}) -\\
&- \sum_{n,k^{'}}\int_0^t dt^{'}\int_0^{t^{'}} dt^{''}\, e^{i(E_0-E_n)(t^{'}-t^{''})}\langle k|H_W|n\rangle\langle n|H_W|k^{'}\rangle a_{k^{'}}(t^{''}).
\end{aligned}
\tag{A.10}
$$

We can solve the equations for $a_k(t)$ by means of a Laplace transform. Define a new variable

$$
\bar{a}_k(s) = \int_0^\infty dt e^{-st} a_k(t). \tag{A.11}
$$

Inserting it in Eq. A.10 we get

$$
\begin{aligned}
\bar{a}_k(t) &= \frac{a_k}{s} - \frac{i}{s}\sum_{k^{'}} W_{kk^{'}}(s)\bar{a}_{k^{'}}(s),\\
W_{kk^{'}}(s) &= \langle k|H_W|k^{'}\rangle + \sum_n \frac{\langle k|H_W|n\rangle\langle n|H_W|k'\rangle}{E_0 - E_n + is}.
\end{aligned}
\tag{A.12}
$$

Using the matrix notation

$$
\begin{aligned}
a(0) &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix}, \quad \bar{a}(s) = \begin{bmatrix} \bar{a}_1(s) \\ \bar{a}_2(s) \end{bmatrix},\\
W &= \begin{bmatrix} W_{kk^{'}}(s) \end{bmatrix},
\end{aligned}
\tag{A.13}
$$

equations A.12 can be put in the form

$$
\begin{aligned}
&\left[s + iW(s)\right]\bar{a}(s) = a(0)\\
&\therefore \bar{a}(s) = \left[s + iW(s)\right]^{-1} a(o)
\end{aligned}
\tag{A.14}
$$

Then we can apply the inverse Laplace transform to get

$$a(t) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds e^{st}\bar{a}(s) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty} ds e^{st}\left[s+iW(s)\right]^{-1}a(0) \tag{A.15}$$

where the integration must be done along the vertical line $Re(s) = s_0$ in the complex plane, such that s_0 is greater than the real part of all singularities of $[s + iW(s)]^{-1}$. Except for those points where $E_0 - E_n + is = 0$, which are on the imaginary axis, $W(s)$ is analytic, therefore, we can set $s_0 = \epsilon > 0$, with ϵ infinitesimally small.

Now, when $H_W = 0$, $W(s) = 0$; and the singularity in the integrand of A.15 is a pole at $s = 0$. So, the main contribution to the integral comes from the vicinity of $s = 0$. When H_W is switched on, this pole moves, but it will stay close to the imaginary axis, since we regard it as a small perturbation. Moreover, $W(s)$ is expected to vary appreciably only when s changes by amounts of $\sim E_0 - E_n$, i.e. over a range of energy determined by the strong interaction H_0. The second approximation of the Wigner-Weisskopf approach is to replace $W(s)$ in the vicinity of $s = 0$, $Re(s) > 0$, by a constant

$$\begin{aligned} &W(s) \to W \equiv \lim_{\epsilon\to 0} W(\epsilon) \\ &W_{kk'} = \langle k|H_W|k'\rangle + \sum_n \frac{\langle k|H_W|n\rangle\langle n|H_W|k'\rangle}{E_0 - E_n + i\epsilon} \end{aligned} \tag{A.16}$$

This approximation is valid for t very long compared with strong interaction scale. Then using the residue theorem to evaluate A.15 [43], we obtain

$$a(t) = \frac{1}{2\pi i}\oint ds \frac{e^{st}}{s+iW}a(0) = e^{-iWt}a(0), \qquad t > 0 \tag{A.17}$$

Denoting $\Phi(t) = e^{-iH_0 t}a(t)$ and returning to the Schrödinger picture, we get a Schrödinger-like equation for it, where the Hamiltonian is replaced by the so-called mass matrix Λ

$$i\frac{d}{dt}\Phi(t) = \Lambda\Phi(t), \tag{A.18}$$

where Λ is defined as

$$\begin{aligned} &\Lambda = H_0 + W \equiv M - \frac{i}{2}\Gamma \\ &M = \frac{1}{2}(\Lambda + \Lambda^\dagger), \quad \Gamma = i(\Lambda - \Lambda^\dagger) \end{aligned} \tag{A.19}$$

Both M and Γ are Hermitian, but Λ is not. Denoting the eigenstates and eigenvalues of the mass matrix as $|\lambda\rangle$ and λ, respectively,the time evolution of these states are given by

$$\begin{aligned} &|\lambda(t)\rangle = e^{-i\lambda t}|\lambda(0)\rangle, \\ &\lambda = m_\lambda - \frac{i}{2}\Gamma_\lambda, \end{aligned} \tag{A.20}$$

and represents exponentially decaying states with mean lifetime $\tau = 1/\Gamma_\lambda$.

It is worth noting that exponential decay law is only an approximation. It can be shown that this approximation fails both for very short and for very long times [42, 44].

APPENDIX B – The Dynamical Map Properties and The Lindblad Master Equation

B.1 Properties of the Dynamical Map

Here we present the properties of the dynamical map $V(t)$. For more details we recommend [17, 41].

1. The dynamical map is trace-preserving:
   We have
$$
\begin{aligned}
V(t)\rho(0) &= \sum_k W_k(t)\rho(0)W_k^\dagger(t), \\
\sum_k W_k^\dagger(t)W_k(t) &= \mathbb{1}.
\end{aligned}
\tag{B.1}
$$
   Hence,
$$
tr[V(t)\rho(0)] = tr\Big[\sum_k W_k(t)\rho(0)W^\dagger\Big] = tr\Big[\sum_k W_k^\dagger(t)W_k(t)\rho(0)\Big] = tr[\rho(0)]. \tag{B.2}
$$

2. It preserves hermiticity:
$$
\Big(V(t)\rho(0)\Big)^\dagger = \Big(\sum_k W_k\rho(0)W_k^\dagger\Big)^\dagger = \sum_k W_k\rho^\dagger(0)W_k^\dagger = V(t)\rho(0). \tag{B.3}
$$

3. $V(t)$ is a convex linear map:
   This means
$$
V(t)\sum_i p_i\rho_i = \sum_i p_i V(t)\rho_i, \tag{B.4}
$$
   where $\sum_i p_i = 1$ is a convex sum.

4. The dynamical map $V(t)$ is completely positive:
   Let $V(t)$ be a positive map $V(t)[\rho] \geq 0$ for all $\rho \geq 0$ and for all $t \geq 0$ on a finite dimensional Hilbert space $\mathcal{H}$. Then the map $V(t)$ is completely positive if the extension
$$
V_n(t) = V(t) \otimes \mathbb{1}_n \tag{B.5}
$$
   defined on $\mathcal{H} \otimes \mathbb{C}^n$ for all n is positive, i.e.:
$$
V_n(t)[\rho \otimes \varrho] = V(t)[\rho] \otimes \varrho \geq 0 \tag{B.6}
$$
   for all $\rho \in \mathcal{H}$ and for all $\varrho \in \mathbb{C}^n$.
   Let us consider two systems A and B, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively,

which has been entangled at some time in the past through some unspecified interaction. The global system can be described by the density operator ρ_{AB} on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Suppose the dynamical map V_A affects only A. The superoperator acting on ρ_{AB} is, thus, $V_A \otimes \mathbb{1}_B$, where $\mathbb{1}_B$ is the identity operator in $\mathcal{H}_B$. Completely positivity is important because it ensures that the complete quantum map acting on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ preserve the positivity of ρ_{AB}, that is, if V_A is completely positive, then

$$\langle \phi_{AB} | V_A \otimes \mathbb{1}_B | \phi_{AB} \rangle \geq 0, \quad \forall |\phi_{AB}\rangle \in \mathcal{H}_{AB}, \tag{B.7}$$

for any system B and any ρ_{AB}. When the map $V(t)$ has a Kraus sum representation, it is, consequently, completely positive (see C and [17, 41]).

B.2 Derivation of the Lindblad Master Equation

Here we make a derivation of the Lindblad master equation. The arguments presented here are based on the "coarse-grained description" and closely follow the ones made in [17]. For a more technical derivation see [41].

We want to derive a first-order differential equation for the density operator ρ describing an open quantum system S coupled to an environment E. Hence, we expect to get an incremental evolution of ρ, $\rho(t+\delta t) - \rho(t) = \delta t d\rho/dt$. Therefore, we require δt to be very short compared to the characteristic time τ_S of evolution of ρ. Mathematically, the exact evolution must be recovered when we go to the limit $\delta t \to 0$. However, this limit is not physically achievable and we must keep δt coarse-grained, allowing a quasi-continuous time evolution of the system's observables expectation values, but renouncing a too precise description at very short times.

We start by observing that the state of the total system $S + E$ can be written at time t in the following form

$$\rho_{S+E}(t) = \rho(t) \otimes [\rho_E + \delta\rho_E(t)] + \delta\rho_{S+E}(t), \tag{B.8}$$

where the density operator ρ_E describes a steady state of the environment at time t. $\delta\rho_E(t)$ describes a fluctuation around this state due to its prior interaction with S, and $\delta\rho_{S+E}(t)$ accounts for a contribution due to the entanglement between $S + E$, generally present as consequence of their coupling in the past of t.

This exact expression implies two difficulties. First, the line of reasoning developed to get a quantum dynamical map linking linearly $\rho(t)$ to $\rho(t+\delta t)$ is based on the assumption that, at the initial time, S and E are uncorrelated, being expressed, therefore, by a simple tensor product. Second, even if a map for the initial state B.8 exists, it would, a priori, depend upon the fluctuating initial state of the environment. This way, the associated Kraus operators, W_k, would be functions of t, leading to unwanted time-varying coefficients in the master equation.

However, it can be shown that the evolution made over not too small a time interval δt, can be computed as if S and E were not entangled at the beginning of this interval, with E, then, being in its steady state. This result is based on the Markov approximation, which we will discuss in qualitative terms in the sequence.

Normally, the environment E is a large system with energy levels spanning a wide range $\hbar\Delta\omega$. The correlation time of the observables O_E of E are very short, of the order of $\tau_E = 1/\Delta\omega$. For time intervals smaller than τ_E, S and E undergo a coherent evolution, because the observables of E appearing in the interaction Hamiltonian H_I, remains nearly constant. During the next τ_E time interval, the phase relations between S and E are lost and a new coherent evolution starts again. This means that, the environment fluctuation $\delta\rho(t)$, and the $S - E$ correlation $\delta\rho_{S+E}(t)$, have a very short correlation time, of the order of τ_E, thus, products of two matrix elements of these quantities taken at times differing by more than τ_E vanish on average.

The Markov condition of a very short memory time is, then,

$$\tau_E \ll \tau_S. \tag{B.9}$$

When the Markov condition is satisfied, the system's evolution can be analysed with two different time 'units'. The very short memory time τ_E, setting the time scale of its fluctuations and correlations, and the long time τ_S, measuring the dynamical evolution of its observables.

The short memory implies that the initial conditions defined by $\delta\rho_E(t)$ and $\delta\rho_{S+E}(t)$ influence $\rho(t^{'})$ up to $t^{'} \approx t + \tau_E$ only. Furthermore, $\delta\rho_E(t)$ depends on the state of S just before t, thus, on this short time scale, $\rho(t^{'})$ depends not only on $\rho(t)$, but also, through $\delta\rho_E(t)$, on $\rho(t^{''})$, at times $t^{''}$ comprised between $t - \tau_E$ and t. If we choose the time interval $\delta t \gg \tau_E$, the transient evolution of ρ over the very small τ_E, at the beginning of the time interval, becomes entirely negligible compared to that which is produced by the fluctuations and correlations appearing in the system from $t^{'} \approx t + \tau_E$ up to $t + \delta t$. That is, discarding the initial environment fluctuations and system-environment correlations at time t, does not appreciably change the value of $\rho(t + \delta t)$. Therefore it is legitimate, under these Markov conditions, to replace B.8 by

$$\rho_{S+E}(t) = \rho(t) \otimes \rho_E. \tag{B.10}$$

With the choice of a finite time interval δt, such that $\tau_E \ll \delta t \ll \tau_S$, we can apply repetitively infinitesimal quantum maps to the successive values ρ at multiples δt, obtaining a "coarse-grained description" of the system's dynamics. On the one hand, δt must be course enough to be insensitive to the fast fluctuations affecting the system's evolution at the τ_E time scale. On the other hand, it must be fine enough to provide quasi-continuous description of the evolution of the density operator over times of the order of τ_S.

With the total state of the system at time t given by B.10 we can write the evolution of ρ from t to $t + \delta t$ as

$$\rho(t+\delta t)V_{\delta t}[\rho(t)] = \sum_k = W_k(\delta t)\rho(t)W_k^\dagger. \tag{B.11}$$

Since $V_{\delta t}[\rho(t)] = \rho(t) + O(\delta t)$, where $O(\delta t)$ is a first-order contribution in δt, one of the Kraus operators must be of the order $\sim \mathbb{1} + O(\delta t)$, and all the other of order $\sim O(\delta t)$. Hence, we may write

$$\begin{aligned} W_0 &= \mathbb{1} - i\Big(K - iH\Big)\delta t + O(\delta t^2), \\ W_k &= \sqrt{\delta t} A_k, \quad k > 0, \end{aligned} \tag{B.12}$$

where K and H are hermitian operators and A_k are the so-called Lindblad operators.

Therefore, we have

$$\begin{aligned} \rho(t+\delta t) &= W_0\rho(t)W_0^\dagger + \sum_{k\neq 0} W_k\rho(t)W_k^\dagger \\ &= \Big(\mathbb{1} + (K - iH)\delta t\Big)\rho(t)\Big(\mathbb{1} + (K + iH)\delta t\Big) + \delta t \sum_{k\neq 0} A_k\rho(t)A_k^\dagger \\ &= \rho(t) - i[H, \rho(t)]\delta t + \delta t\Big(\{K, \rho(t)\} + \sum_{k\neq 0} A_k\rho(t)A_k^\dagger\Big). \end{aligned} \tag{B.13}$$

Now, from the normalization condition on the Kraus operators we get

$$\begin{aligned} \sum_k W_k^\dagger W_k = \Big(\mathbb{1} + (K + iH)\delta t\Big)\Big(\mathbb{1} + (K - iH)\delta t\Big) + \delta t \sum_{k\neq 0} A_k^\dagger A_k = \mathbb{1} \\ \therefore K = -\frac{1}{2}\sum_{k\neq 0} A_k^\dagger A_k. \end{aligned} \tag{B.14}$$

Thus,

$$\lim_{\delta t\to 0} \frac{\rho(t+\delta t) - \rho(t)}{\delta t} \equiv \frac{d}{dt}\rho(t) = -i[H, \rho(t)] - D[\rho(t)], \tag{B.15}$$

where the dissipator D is given by

$$D[\rho] = -\frac{1}{2}\sum_{k\neq 0}\Big(A_k^\dagger A_k\rho + \rho A_k^\dagger A_k - 2A_k\rho(t)A_k^\dagger\Big). \tag{B.16}$$

APPENDIX C – Choi's Theorem and Sylvester's Criterion

C.1 Choi's Theorem

Here we only state Choi's theorems for completely positive maps, for more details and proofs see [18, 19].

Choi's Theorem 1. Let $\Phi : \mathbb{C}^n \mapsto \mathbb{C}^m$ be a linear positive map. Then, Φ is *completely positive* if and only if Φ is of the form

$$\Phi(A) = \sum_{k}^{nm} W_k^{\dagger} A W_k, \tag{C.1}$$

for all $A \in \mathbb{C}^n$ and where $W_k \in \mathbb{C}^{n\times m}$.

Choi's Theorem 2. Let $\Phi : \mathbb{C}^n \mapsto \mathbb{C}^m$ be a linear map. The following are equivalent:

1. Φ is m-positive.

2. The *Choi matrix* of Φ with operator entries

$$Choi(\Phi) = \Big(\mathbb{1}_n \otimes \Phi\Big)\Big(\sum_{ij} E_{ij} \otimes E_{ij}\Big) = \sum_{ij} E_{ij} \otimes \Phi(E_{ij}) \in \mathbb{C}^{nm}, \tag{C.2}$$

   where $E_{ij} \in \mathbb{C}^{nn}$ is the matrix with 1 in the ij-th entry and 0's elsewhere, is positive.

3. Φ is completely positive.

C.2 Sylvester's Criterion

Here we only present the theorem which establish an easy computational method to verify if a Hermitian matrix is positive semidefinite. For more details and proofs see [20].

Let $M = (m_{ij})$ be a $n \times n$ Hermitian matrix.

Definition 1: A $k \times k$ submatrix M_{IJ} is constructed from M as follows: take a subset of k column indices $J = \{j_1, j_2, ..., j_k\}$ and a subset of k row indices $I = \{i_1, i_2, ..., i_k\}$ and define M_{IJ} by

$$M_{IJ} = \begin{bmatrix} m_{i_1 j_1} & m_{i_1 j_2} & \cdots & m_{i_1 j_k} \\ m_{i_2 j_1} & m_{i_2 j_2} & \cdots & m_{i_2 j_k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{i_k j_1} & m_{i_k j_2} & \cdots & m_{i_k j_k} \end{bmatrix}. \tag{C.3}$$

If $I = J$, then we say that the submatrix is principal. The associated minor is the determinant of the matrix M_{IJ}.

Theorem *(Sylvester's criterion for positive-semidefiniteness)*. A $n \times n$ Hermitian matrix $M = (m_{ij})$ is positive-semidefinite if and only if all its principal minors are nonnegative.

C.3 The Pion Evolution as an Example

Here we detail some of the calculations involved in the development of the open quantum system formulation of the time evolution of π^0.

The most general form for the action of the map $V(t)$ giving the time evolution of the system is

$$\rho(t) = V(t)\rho(0) = \begin{pmatrix} e^{-t\Gamma}\rho_{11}(0) & \sum_{i,j=1}^{2} A_{ij}(t)\rho_{ij}(0) \\ \sum_{i,j=1}^{2} A_{ij}^{*}(t)\rho_{ij}^{*}(0) & \rho_{22}(t) \end{pmatrix}, \tag{C.4}$$

where $\rho_{22}(t) = \rho_{22}(0) + (1 - e^{-t\Gamma})\rho_{11}(0)$.

The Choi's matrix of $V(t)$ is given by

$$Choi(V) = \Big(\mathbb{1}_2 \otimes V\Big)\Big(\sum_{ij} E_{ij} \otimes E_{ij}\Big) = \sum_{ij} E_{ij} \otimes V(t)[E_{ij}], \tag{C.5}$$

where E_{ij} is the matrix with 1 in the ij-th entry and 0's elsewhere. Thus,

$$
\begin{aligned}
Choi(V) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes V(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes V(t) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes V(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes V(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-t\Gamma_S} & A_{11}(t) \\ A_{11}^*(t) & 1-e^{-t\Gamma} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & A_{12}(t) \\ A_{21}^*(t) & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & A_{21}(t) \\ A_{12}^*(t) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & A_{22}(t) \\ A_{22}^*(t) & 1 \end{pmatrix},
\end{aligned} \tag{C.6}
$$

that is,

$$
Choi(V) = \begin{pmatrix} e^{-t\Gamma} & A_{11}(t) & 0 & A_{12}(t) \\ A_{11}^*(t) & 1-e^{-t\Gamma} & A_{21}^*(t) & 0 \\ 0 & A_{21}(t) & 0 & A_{22}(t) \\ A_{12}^*(t) & 0 & A_{22}^*(t) & 1 \end{pmatrix}. \tag{C.7}
$$

The Choi's theorem states that the map $V(t)$ is completely positive if and only if its Choi matrix is (semi)positive. Moreover, Sylvester's criterion states that a Hermitian matrix, such as $Choi(V)$, is (semi)positive if and only if all its principal minors are nonnegative. Some of the principal submatrices (the ones which does not trivially satisfy the condition of nonnegative of its minors) of $Choi(V)$ are (using the *Definition 1* stated above)

$$
\begin{aligned}
&\text{For } I = J = 1,4 \Rightarrow \begin{pmatrix} e^{-t\Gamma} & A_{12}(t) \\ A_{12}^*(t) & 1 \end{pmatrix}, \quad \text{For } I = J = 2,3 \Rightarrow \begin{pmatrix} 1-e^{-t\Gamma} & A_{21}^*(t) \\ A_{12}(t) & 0 \end{pmatrix}, \\
&\text{For } I = J = 3,4 \Rightarrow \begin{pmatrix} 0 & A_{22}(t) \\ A_{22}^*(t) & 1 \end{pmatrix}, \quad \text{For } I = J = 1,2,4 \Rightarrow \begin{pmatrix} e^{-t\Gamma} & A_{11}(t) & A_{12}(t) \\ A_{11}^*(t) & 1-e^{-t\Gamma} & 0 \\ A_{12}^*(t) & 0 & 1 \end{pmatrix},
\end{aligned} \tag{C.8}
$$

from which we get the conditions for the coefficients A_{ij} appearing in the text

$$\begin{aligned}
&\text{for } I=J=1,4: \quad e^{-t\Gamma}-|A_{12}(t)|^2 \geqslant 0 \Rightarrow |A_{12}(t)|^2 \leqslant e^{-t\Gamma}\\
&\text{for } I=J=2,3: \quad -|A_{21}(t)|^2 \geqslant 0 \Rightarrow A_{21}(t)=0,\\
&\text{for } I=J=3,4: \quad -|A_{22}(t)|^2 \geqslant 0 \Rightarrow A_{22}(t)=0,\\
&\text{and for } I=J=1,2,4: \quad (1-e^{-t\Gamma})(e^{-t\Gamma}-|A_{12}(t)|^2)-|A_{11}(t)|^2 \geqslant 0 \Rightarrow\\
&\qquad \Rightarrow |A_{11}(t)|^2 \leqslant (1-e^{-t\Gamma})(e^{-t\Gamma}-|A_{12}(t)|^2).
\end{aligned} \tag{C.9}$$

We can also verify that the Kraus operators

$$\begin{aligned}
W_0(t) &= e^{-t(\Gamma/2+im)}|\pi^0\rangle\langle\pi^0| + |0\rangle\langle 0|,\\
W_1(t) &= \sqrt{1-e^{-t\Gamma}}|0\rangle\langle\pi^0|,
\end{aligned} \tag{C.10}$$

give the correct density matrix $\rho(t)$.

These Kraus operators are written in matrix form as

$$W_0(t) = \begin{pmatrix} e^{-t(\Gamma/2+im)} & 0 \\ 0 & 1 \end{pmatrix}, \quad W_1(t) = \begin{pmatrix} 0 & 0 \\ \sqrt{1-e^{-t\Gamma}} & 0 \end{pmatrix}, \tag{C.11}$$

and, therefore,

$$\begin{aligned}
\rho(t) &= V(t)\rho(0) = \sum_{k=0}^{1} W_k(t)\rho(0)W_k^\dagger(t)\\
&= \begin{pmatrix} e^{-t(\Gamma/2+im)} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_{11}(0) & 0 \\ 0 & 1-\rho_{11}(0) \end{pmatrix}\begin{pmatrix} e^{-t(\Gamma/2-im)} & 0 \\ 0 & 1 \end{pmatrix}\\
&+ \begin{pmatrix} 0 & 0 \\ \sqrt{1-e^{-t\Gamma}} & 0 \end{pmatrix}\begin{pmatrix} \rho_{11}(0) & 0 \\ 0 & 1-\rho_{11}(0) \end{pmatrix}\begin{pmatrix} 0 & \sqrt{1-e^{-t\Gamma}} \\ 0 & 0 \end{pmatrix}\\
&= \begin{pmatrix} e^{-t\Gamma}\rho_{11}(0) & 0 \\ 0 & 1-\rho_{11}(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (1-e^{-t\Gamma})\rho_{11}(0) \end{pmatrix}\\
&= \begin{pmatrix} e^{-t\Gamma}\rho_{11}(0) & 0 \\ 0 & 1-e^{-t\Gamma}\rho_{11}(0) \end{pmatrix}.
\end{aligned} \tag{C.12}$$

We stress that this is the same reasoning applied in the development of the time evolution in the kaon case, although the steps there require more work.