Special Distributions Determined By Their Singular Scheme and Residues

PhD Thesis

Departament of Mathematics
Federal University of Minas Gerais

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Special distributions determined by their singular scheme and residues

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RESUMO

O objetivo desta Tese é estudar as Distribuições Holomorfas de codimensão um e grau d em $\mathbb{P}^3$ que são especiais ao longo de uma curva suave e irredutível $\mathscr{C} \subset \mathbb{P}^3$. Primeiramente, definimos o resíduo de uma distribuição $\mathscr{F}$ ao longo de $\mathscr{C}$. Este resíduo é determinado via resíduo de Grothendieck em pontos isolados e pode ser interpretado como a contribuição numérica que a curva $\mathscr{C}$ oferece ao ser deformada em pontos singualres. O segundo objetivo é caracterizar tais distribuições através de seu esquema singular.

Palavras-Chave: Distribuições Holomorfas, Distribuições Especiais, Resíduos.

Abstract

The aim of this Thesis is to study codimension one Holomorphic Distributions on $\mathbb{P}^3$ of degree d which are special along the an irredicible smooth curve $\mathscr{C} \subset \mathbb{P}^3$. Firstly, we define residue of a distribution $\mathscr{F}$ along $\mathscr{C}$. This residue is determined via the Grothendieck's residues at singular points and can be interpreted as a numerical contribution offered by $\mathscr{C}$ when deformed into singular points. Secondly, we characterize these distributions by their singular scheme.

Key words: Holomorphic Distribution, Special Distribution, Residues.

Contents

INTRODUTION

In this thesis, we will study the codimension one holomorphic distributions $\mathscr{F}$ of degree d in $\mathbb{P}^3$ which singular scheme contains a smooth irreducible curve $\mathscr{C}$. The main objectives of this work are: to define the residue $\mathscr{F}$ along $\mathscr{C}$ and characterize such distributions via its singular scheme.

In the chapter 1, we describe briefly the tools used in the proofs of the results of this work, always indicating the references for more details.

In the chapter 2, we define the main object of study of this work which are the *special Holomorphic Distributions along the curve* $\mathscr{C} \subset \mathrm{Sing}(\mathscr{F})$. We opened a parenthesis to talk about the main motivation that led us to study this type of distribution. Let $\mathscr{F}$ be a holomorphic foliation by curves on $\mathbb{P}^3$ of degree d induced by global section $X \in H^0(\mathbb{P}^3, T_{\mathbb{P}^3}(d-1))$. If the singular scheme of $\mathscr{F}$ contains only isolated closed points $\{p_1, \ldots, p_k\}$, then, the number $n_{\mathscr{F}} = \sum_{i=1}^{k} \mu(\mathscr{F}, p_i)$, where $\mu(\mathscr{F}, p_i)$ denotes Milnor's number of $\mathscr{F}$ in each singularity is known (see [16]), to namely, $n_{\mathscr{F}} = d^3 + d^2 + d + 1$. On the other hand, suppose now that the singular scheme of $\mathscr{F}$ be the following disjoint union

$$\mathrm{Sing}(\mathscr{F}) = \mathscr{C} \cup \{p_1, \ldots, p_k\}.$$

In this case, is it possible to get $n_{\mathscr{F}}$? The answer is yes and was answered by Gilcione Nonato Costa when the foliation of $\mathbb{P}^3$ is special along $\mathscr{C}$ (see [5]). Let $\pi : \widetilde{\mathbb{P}^n} \longrightarrow \mathbb{P}^n$ be the blow up morphism of $\mathbb{P}^n$ along $\mathscr{C}$. We recall that a foliation by curves $\mathscr{F}$ on $\mathbb{P}^n$ is special along $\mathscr{C}$ if the foliation $\widetilde{\mathscr{F}}$ on $\widetilde{\mathbb{P}^n}$ obtained by $\mathscr{F}$ via π has only isolated singularities, and the exceptional divisor E is an invariant set of $\widetilde{\mathscr{F}}$. The demonstration consists of exploding $\mathbb{P}^3$ and obtaining a foliation on $\widetilde{\mathbb{P}^3}$ whose singular scheme contains only isolated points. In this case, the author showed that $n_{\mathscr{F}}$ is the difference between the total number of singularities of $\widetilde{\mathscr{F}}$ in $\widetilde{\mathbb{P}^3}$ and the number of singularities of $\widetilde{\mathscr{F}}$ over exceptional divisor E, because $\pi : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ is biholomorphism in the complement of E. In order to calculate all these numbers he also used the Baum-Bott formula (again, [16]) relates that the number of singularities of a foliation in $\mathbb{P}^3$ with the Chern classes of the objects involved.

By dualizing the situation described above, Joanoulou (see [9], Proposition 2.26) showed that the singular scheme of a codimension one foliation $\mathscr{F}$ of degree d on $\mathbb{P}^3$ induced by global section $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2))$ always has a codimension two component and therefore, the foliation will never be special along this component. On the other hand, non-integrable distributions on $\mathbb{P}^3$ can be special along curves similary to foliations by curves. At this point, the question is:

Is it possible to dualize the work of [5]? That is, what we could say about $n_{\mathscr{F}}$ when $\mathscr{F}$ is a distribution? It is possible to define the residue of $\mathscr{F}$ along $\mathscr{C}$?

Izawa's (see, [20], Theorem 2.2) papper plays a very important role in this part of the work, which tells us how we can get the total sum of residues from $\mathscr{F}$ in each isolated singularity. Thus, with the aid of this theorem, we define a residue of distribution $\mathscr{F}$ along $\mathscr{C}$ and obtain an upper bound for this number. Morever, we give a upper bound to $n_{\mathscr{F}}$. As not every global section $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2))$ induces a special distribution along curve, to get this upper bound, we need a result, which we call the Lemma of Perturbation (see Lemma 2.14).

Lema of Perturbation. Let $\mathscr{F}$ be a non-integrable codimension one distribution on $\mathbb{P}^3$ of degree d whose singular set is the following disjoint union of proper closed subsets

$$\mathrm{Sing}(\mathscr{F}) = \mathscr{C} \cup \{p_1, \ldots . p_n\},$$

where $\mathscr{C}$ is a smooth irreducible curve and closed points $p_1, \ldots, p_n$.
Then there exists a one-parameter family of holomorphic distributions, denoted by $\mathscr{F}_t$, defined on $\mathbb{P}^3$ with $t \in \mathbb{D} = D(0, \varepsilon)$, for $\varepsilon > 0$ sufficiently small such that

1) $\mathscr{F}_0 = \mathscr{F}$ e $\deg(\mathscr{F}_t) = \deg(\mathscr{F})$, $\forall t \in \mathbb{D}$,

2) $\mathscr{C} \subset \mathrm{Sing}(\mathscr{F}_t)$, $\forall t \in \mathbb{D}$,

3) $\mathscr{F}_t$ is special along $\mathscr{C}$, $\forall t \in \mathbb{D} \setminus \{0\}$,

4) $\mathrm{mult}_{\mathscr{C}}(\mathscr{F}_t) = \mathrm{mult}_{\mathscr{C}}(\mathscr{F})$ if $\mathscr{F}$ is dicritical or non dicritical,

5) $\mathrm{mult}_E(\pi^*\mathscr{F}_t) = \begin{cases} \mathrm{mult}_E(\pi^*\mathscr{F}), & \text{if } \mathscr{F} \text{ is non dicritical,} \\ \mathrm{mult}_E(\pi^*\mathscr{F}) - 1, & \text{if } \mathscr{F} \text{ is dicritical.} \end{cases}$

We will use the Perturbation Lemma to prove our first result:

Theorem I. Let $\mathscr{F}$ be a non integrable codimension one distribution on $\mathbb{P}^3$ of degree d whose singular set is a disjoint union of proper closed subsets

$$\mathrm{Sing}(\mathscr{F}) = \mathscr{C} \cup \{p_1, \ldots, p_n\},$$

where $\mathscr{C}$ is a smooth irreducible curve and closed points $p_1, \ldots, p_n$. Then

i)

$$\sum_{i=1}^{n} \mathrm{Res}(\mathscr{F}, p_i) \geq d^3 + 2d^2 + 2d - (\ell+3)\,\chi(\mathscr{C}) + \deg(\mathscr{C})\Big[(d+2)(3-3\ell^2) + 4\ell(\ell+1)\Big] + \left(3\ell - \ell^3\right)\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right) - N_G,$$

ii)

$$\operatorname{Res}(\mathscr{F},\mathscr{C}) \leq (\ell+3)\,\chi(\mathscr{C}) - \deg(\mathscr{C})\Big[(d+2)(3-3\ell^2)+4\ell(\ell+1)\Big]$$
$$-\big(3\ell-\ell^3\big)\big(\chi(\mathscr{C})-4\deg(\mathscr{C})\big)+N_G,$$

where $\deg(\mathscr{C})$, $\chi(\mathscr{C})$, denote respectively , the degree and Euler's characteristic of $\mathscr{C}$, ℓ is the order of annulment over an exceptional divisor and N_G denote the number of embedded closed points of $\mathscr{C}$ counted with multipilicities.

In chapter 3, motivated by the work of Araújo and Corrêa where the authors characterize distributions in $\mathbb{P}^n$ uniquely determined by their singular scheme, (see [6], theorems 1.2 and 1.4) we sought the characterization of special distributions via their singular scheme. In order, to achieve this characterization, we need to guarantee the vanishing of the certain cohomology groups and, for that, the following tools played important roles in this part of the work : the projection formula, which allows us to analyze the cohomology groups of the exploded variety $\widetilde{\mathbb{P}^3}$ through the cohomology groups of the base of the morphism of blow up, this is advantageous, because at the base, we have some results that help us in vanishing such cohomology groups, as for example, the Bott formula, and the regularity of Castelnuovo-Mumford. Thus we obtain such a characterization through the following result:

Theorem II. Let $\mathscr{F}_1$ be a non integrable codimension one holomorphic distributions on $\mathbb{P}^3$ of degree d such that its singular locus has just one non-zero dimensional component which is integral and non-degenerated somooth curve $\mathscr{C} \subset \mathbb{P}^3$. Assume that $\mathscr{F}_1$ is special along $\mathscr{C}$. Let $\pi : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ along $\mathscr{C}$ and E be the exceptional divisor. If $\mathscr{F}_2$ is another non integrable codimension one distribution of degree d on $\mathbb{P}^3$ and furthermore the following conditions occurs:

i) $\deg(\mathscr{C}) \geq 2$,

ii) $d \geq 2\deg(\mathscr{C})$,

iii) $\operatorname{Sing}(\mathscr{F}_1) \subset \operatorname{Sing}(\mathscr{F}_2)$,

iv) $\operatorname{Sing}(\widetilde{\mathscr{F}_1}|_E) \subset \operatorname{Sing}(\widetilde{\mathscr{F}_2}|_E)$,

v) $\ell = \operatorname{mult}_E(\widetilde{\mathscr{F}_1}) = 1$ or 2.

Then $\mathscr{F}_1 = \mathscr{F}_2$.

CHAPTER 1

PRELIMINARES

In this chapter, we will provide a brief description of the tools used in all this work. For more details, we indicate the references.

1.1 Castelnuovo-Mumford Regularity

The reference for this section is [17], [19], [7] and [13].

The Cartan-Serre-Grothendieck theorems imply that all the cohomological subtleties that may be associated with a coherent sheaf F on a projective space $\mathbb{P}^n$ disappear after twisting by a sufficiently high multiple of the hyperplane line bundle. Specifically, for $m >> 0$:

- The higher cohomology groups of $F(m)$ vanish,
- $F(m)$ is generated by global sections,
- The maps $H^0(\mathbb{P}^n, F(m)) \otimes H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \longrightarrow H^0(\mathbb{P}^n, F(m+k))$ are surjectives for all $k > 0$.

Castelnuovo-Mumford regularity gives a quantitative measure of how much one has to twist in order that these properties take effect. As we shall see, regularity is also well-adapted to arguments involving vanishing theorems.

Theorem 1.1. *(Grothendieck) Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathscr{F}$ on X, we have $H^i(X, \mathscr{F}) = 0$.*

Proof. See [19], Theorem 2.7. ■

Theorem 1.2. *(Serre) Let X be a projective scheme over a noetherian ring A, and let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X over Spec(A). Let $\mathscr{F}$ be a coherent sheaf on X. Then:*

a) For each $i \geq 0$, $H^i(X,\mathscr{F})$ is a finitely generated A-module.

b) There is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X,\mathscr{F}(n)) = 0$.

Proof. See [19], Theorem 5.2. ■

Definition 1.3. *Let $\mathscr{F}$ be a coherent sheaf on the projective space $\mathbb{P}^n$, and let m be an integer. One says that F is m-regular in the sense of Castelnuovo-Mumford if*

$$H^i(\mathbb{P}^n,\mathscr{F}(m-i)) = 0,$$

for all $i > 0$.

By Theorems 1.1 and 1.2 with $m = n_0 + n$ we have that $H^i(\mathbb{P}^n,\mathscr{F}(m-i)) = 0$ for all $i > 0$. Therefore there is an integer m such that $\mathscr{F}$ is m-regular.

Definition 1.4. *(Regularity of a sheaf) The Castelnuovo-Mumford regularity reg$(\mathscr{F})$ of a coherent sheaf $\mathscr{F}F$ on $\mathbb{P}^n$ is the least integer m for which $\mathscr{F}$ is m-regular.*

Theorem 1.5. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. If $\mathscr{F}$ is m-regular, then*

a) $\mathscr{F}$ is s-regular for all $s \geq m$,

b) The natural homomorphism

$$H^0(\mathscr{F}(s)) \otimes H^0(\mathscr{O}_{\mathbb{P}^n}(1)) \longrightarrow H^0(\mathscr{F}(s+1)),$$

is surjective for all $s \geq m$.

Proof. See [17], Theorem 1.8.3. ■

(Bott's Formula)

$$h^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n-p}{k}\binom{k-1}{p}, & \text{for} \quad q=0,\ 0\leq p\leq n,\ k>p, \\ 1, & \text{for} \quad k=0,\ 0\leq p=q\leq n, \\ \binom{-k+p}{-k}\binom{-k-1}{n-p}, & \text{for} \quad q=n,\ 0\leq p\leq n,\ k<p-n, \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.6. *1) The line bundle $\mathcal{O}_{\mathbb{P}^3}(k)$ is $(-k)$-regular.*

2) The ideal sheaf $\mathcal{I}_L \subset \mathcal{O}_{\mathbb{P}^n}$ of a linear subspace $L \subset \mathbb{P}^n$ is 1-regular.

3) $\Omega^1_{\mathbb{P}^n}$ is 2-regular by Bott's formula.

We next use previous Theorem 1.5 to show that at least for vector bundles, regularity has pleasant tensorial properties.

Proposition 1.7. *(Regularity of tensor products). Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, and let E be a locally free sheaf on $\mathbb{P}^n$. If $\mathcal{F}$ is m-regular and E is ℓ-regular, then $E \otimes \mathcal{F}$ is $(\ell + m)$-regular.*

Proof. See [17], Proposition 1.8.9. ■

Corollary 1.8. *(Wedge and symmetric products). If E is an m-regular locally free sheaf, then the p-fold tensor power T^pE is (pm)-regular. In particular, $\bigwedge^p E$ and S^pE are likewise (pm)-regular.*

Proof. See [17], Corollary 1.8.10. ■

We will now define Mumford's regularity for a subvariety of a projective space.

Definition 1.9. *(Regularity of a projective subvariety). We say that a subvariety (or subscheme) $X \subset \mathbb{P}^n$ is m-regular if its ideal sheaf $\mathcal{I}_X$ is. The regularity of X is the regularity reg$(\mathcal{I}_X)$ of its ideal sheaf.*

Then we close this section with a definition that will be widely used in chapter three.

Theorem 1.10. *(Regularity for curves). Let $\mathcal{C} \subset \mathbb{P}^n$ be an irreducible (but possibly singular) reduced curve of degree d. Assume that $\mathcal{C}$ is nondegenerate, i.e. that it doesn't lie in any hyperplanes. Then $\mathcal{C}$ is (d+2-n)-regular.*

Proof. See [17], Theorem 1.8.46. ■

1.2 Grothendieck Residue

The reference for this section is [13].

In this section we will be interested in maps $f:\mathbb{C}^n \longrightarrow \mathbb{C}^m$ and in its germs at points. Let $U \subset \mathbb{C}^n$ be a domain (open and connected set). Recall that if $m = 1$, then a differentiable function $f:U \longrightarrow \mathbb{C}$ is holomorphic provided $f'(z)$ exists for every for all $z \in U$.
If we identify $\mathbb{C} \simeq \mathbb{R}^2$, $z = x+iy$, $\bar{z} = x-iy$, $f(z) = u+iv$ and introduce the derivations:

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}),$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

Then, f holomorphic is equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Definition 1.11. *Let $p \in \mathbb{C}^n$. A map germ (smooth or holomorphic) or germ at p is an equivalence class of maps (smooth or holomorphic) where two maps are equivalent if they agree on a neighborhood of p. We adopt $f:(\mathbb{C}^n,p) \longrightarrow (\mathbb{C}^m,q)$ to denote the germ of f at p with $f(p) = q$.*

We denote by $|z|$ the hermitian norm in $\mathbb{C}^n$, $|z| = \sqrt{\sum_{j=1}^{n} z_j.\overline{z_j}}$.

Definition 1.12. *Let $f:(\mathbb{C}^n,p) \longrightarrow (\mathbb{C}^n,0)$ be a germ of holomorphic map with $f^{-1}(0) = \{p\}$. The index or Poincarè Hopf index of f at p, denoted by $I_p(f)$, is the degree of the smooth map*

$$\frac{f}{|f|}: S_{\varepsilon}^{2n-1}(p) \longrightarrow S_1^{2n-1}(p),$$

where $S_{\varepsilon}^{2n-1}(p)$ is the euclidean sphere of radius $\varepsilon > 0$, $S_{\varepsilon}^{2n-1}(p) = \{z \in \mathbb{C}^n : \ |z-p| = \varepsilon\}$ and $S_1^{2n-1}(p)$ is the unit sphere centered at $0 \in \mathbb{C}^n$.

Remark 1.13. *If ε is sufficiently small then the index is well defined and it does not depend on ε.*

An important result is

Proposition 1.14. *Let $f:(\mathbb{C}^n,p) \longrightarrow (\mathbb{C}^n,0)$ is the germ of a biholomorphism, then $I_p(f) = 1$.*

Proof. See [13], Proposition 2.1.13. ∎

Example 1.15. *Let $f(z_1,z_2)=(z_1^2,z_1+z_2^3)$. Then $f^{-1}(0)=\{0\}$ and the index $I_0(f)$ is given by the number of solutions of the equations $z_1^2=\zeta_1$ and $z_1+z_2^3=\zeta_2$ where $0<|(\zeta_1,\zeta_2)|<<\varepsilon$. Thus $I_0(f)=6$.*

Let $f=(f_1,\dots,f_n):U\longrightarrow V$ be a finite holomorphic map of multiplicity μ and $g\in\mathcal{O}(U)$. Let us recall the multiplicity of a zero of a holomorphic function of one variable. Suppose $f:U\longrightarrow\mathbb{C}$ is a holomorphic function defined in a neighborhood $U\subset\mathbb{C}$ of a point ζ and such that $f(\zeta)=0$. Expanding f in power series around ζ we get

$$f(z)=\sum_{\mu=k}^{\infty}a_k(z-\zeta)=(z-\zeta)^{\mu}g(z),$$

where $a_\mu=g(\zeta)\neq0$, g is holomorphic and $g(z)=\sum_{\mu+j}^{\infty}(z-\zeta)^j$. The number $\mu=\mu(f,\zeta)$ is the multiplicity of the zero ζ of f. Now, suppose ζ is a regular value of f and let $f^{-1}(\zeta)=\{\xi_1,\dots\xi_\mu\}$.

Consider the sum

$$\sum_{i=1}^{\mu}\frac{g(\xi_i)}{\det Jf(\xi_i)},$$

where

$$Jf(\xi_i)=\det\Big(\frac{\partial f_k}{\partial z_j}(\xi_k)\Big)_{1\le k,j\le n}.$$

Definition 1.16. *The residue at 0 of g relative to f is the limit*

$$Res_0(g,f)=\lim_{\zeta\to0}\sum_{i=1}^{\mu}\frac{g(\xi_i)}{det\,Jf(\xi_i)}.$$

The next result shows that the above limit exists.

Theorem 1.17. *Let $\varepsilon=(\varepsilon_1,\dots\varepsilon_n)$, $\varepsilon_i>0$ and consider the n-real cycle $\Gamma_\varepsilon=\{z\in U,|f_i(z)|=\varepsilon_i,\ 1\le i\le n\}$ with orientation prescribed by the n-form $darg f_1\wedge\dots,\wedge darg f_n$. If ε is suffciently close to 0 then,*

$$Res_0(g,f)=\Big(\frac{1}{2\pi i}\Big)^n\int_{\Gamma_\varepsilon}\frac{g\,dz_1\dots,dz_n}{f_1\cdots f_n}.$$

Proof. See [13], Theorem 3.2.2. ■

Let us see now some properties of residues. Again, for more details, see [13].

Property 1. If $a,b\in\mathbb{C}$ and $g,h\in\mathcal{O}(U)$ then

$$\mathrm{Res}_0(ag+bh,f)=a\mathrm{Res}_0(g,f)+b\mathrm{Res}_0(h,f).$$

Property 2. $\mathrm{Res}_0(\det(Df), f) = I_0(f)$.

Property 3. If f is a biholomorphism, then

$$\mathrm{Res}_0(g,f) = \frac{g(0)}{\det Df(0)}.$$

Property 4. If $g \in \mathfrak{I}_f$ the ideal generated by $f_1, \cdots, f_n$ then $\mathrm{Res}_0(g,f) = 0$.

Example 1.18. *Consider the open sets U, V of $\mathbb{C}^2$ containing the origin and $f : U \longrightarrow V$ a holomorphic map defined by $f = (f_1, f_2) = (z_2 + z_1^2, z_1^2 + z_2^2)$. Let us calculate the $Res_0(g,f)$, where $g = Jf$.*

Consider the polydisk $\Delta(0,\varepsilon)$ such that $\overline{\Delta(0,\varepsilon)} \cap f^{-1}(0) = \{0\}$. For a change of variable we have:

$$Res_0(g,f) \quad = \frac{1}{(2\pi i)^2}\int_{\Gamma_{(0,0)}} \frac{g dz_1 \wedge dz_2}{f_1, f_2} = \frac{1}{(2\pi i)^2}\int_{\Gamma_{(0,0)}} \frac{g}{detJ(f_1,f_2)}.\frac{df_1 \wedge df_2}{f_1.f_2}.$$

Note that in this case, $f_2 = (z_1 - iz_2)(z_1 + iz_2)$. Then:

$$Res_0(g,f) \quad = \frac{1}{(2\pi i)^2}\int_{\Gamma_{(0,0)}} \left(\frac{g}{detJ(f_1,f_2)} \frac{df_1 \wedge d(z_1 - iz_2)}{f_1.(z_1 - iz_2)} + \frac{g}{detJ(f_1,f_2)} \frac{df_1 \wedge d(z_1 + iz_2)}{f_1.(z_1 + iz_2)} \right).$$

Denoting $g = J(f_1, f_2)$, we get:

$$Res_0(g,f) \quad = \frac{1}{(2\pi i)^2}\int_{\Gamma^1_{(0,0)}} \frac{df_1 \wedge d(z_1 - iz_2)}{f_1.(z_1 - iz_2)} + \frac{1}{(2\pi i)^2}\int_{\Gamma^2_{(0,0)}} \frac{df_1 \wedge d(z_1 + iz_2)}{f_1.(z_1 + iz_2)} = 1 + 1 = 2,$$

where $\Gamma^1_{(0,0)} = \{|f_1| = \varepsilon_1, |z_1 - iz_2| = \varepsilon_2\}$ and $\Gamma^2_{(0,0)} = \{|f_1| = \varepsilon_1, |z_1 + iz_2| = \varepsilon_2\}$.

1.3 Blowing Up Submanifolds

The reference for this section is [7].

In this section, we will write about Blow up or quadratic transformations of a polydisc along a coordinate plane.

Let Δ be a n-dimensional polydisc with holomorphic coordinates $z_1, \ldots, z_n$ and $V \subset \Delta$ be the locus $z_1 = \cdots = z_k = 0$. Let $[l_1, \ldots, l_k]$ be the homogeneous coordinates on $\mathbb{P}^{k-1}$ and set

$$\widetilde{\Delta} \subset \Delta \times \mathbb{P}^{k-1},$$

be the smooth variety defined by the relations

$$\widetilde{\Delta} = \{(z,[l]),\ z_i l_j = z_j l_i,\ \ 1 \leq i,j \leq k\}.$$

The projection $\pi : \widetilde{\Delta} \longrightarrow \Delta$ on the first factor is an isomorphism away from V, while the inverse image of a point $z \in V$ is a projective space $\mathbb{P}^{k-1}$. The manifold $\widetilde{\Delta}$ together with the map $\pi : \widetilde{\Delta} \longrightarrow \Delta$ is called blow up or quadratic transformation of Δ along V. The inverse image $E = \pi^{-1}(V)$ is called the exceptional divisor of the blow up.

The set $\widetilde{\Delta}$ has a natural structure of n-dimensional complex manifold. For each $j \in \{1,2,\ldots,k\}$, let $U_j = \{[l_1,\ldots l_k] : l_j \neq 0\} \subset \mathbb{P}^{k-1}$ be the standard open cover, then

$$\widetilde{U_j} = \{(z,[\zeta]) \in \widetilde{\Delta},\ [\zeta] \in U_j\},$$

with holomorphic coordinates $\sigma(\zeta_1,\ldots,\zeta_n) = (z_1,\ldots,z_n)$ give by:

1) $z_i = \zeta_i$ for $i = j$ or $i > k$,

2) $z_i = \zeta_i \zeta_j$ for $i = 1,\ldots,\hat{j},\ldots,k$.

The coordinates $\zeta \in \mathbb{C}^n$ are affine coordinates on each fiber $\pi^{-1}(p) \simeq \mathbb{P}^{k-1}$.

We can generalize this construction. Let $S \subset M$ be a submanifold of dimension $n-k$. Let $\{\phi_\alpha, U_\alpha\}$ be a collection of local charts covering S and $\phi_\alpha : U_\alpha \longrightarrow \Delta_\alpha$, where Δ_α is a n-dimensional polydisc. We may suppose that $V_\alpha = \phi_\alpha(X \cap U_\alpha)$ is given by $z_1 = \cdots = z_k = 0$. Let $\pi_\alpha : \widetilde{\Delta}_\alpha \longrightarrow \Delta_\alpha$ be the blow up of $\widetilde{\Delta}_\alpha$ along V_α. Then, we have isomorphisms

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}[\phi_\alpha(U_\alpha \cap U_\beta)] \longrightarrow \pi_\beta^{-1}[\phi_\beta(U_\alpha \cap U_\beta)],$$

and using them, we can patch together the blow up $\widetilde{\Delta}_{\pi_\alpha}$ to form a manifold $\widetilde{\Delta} = \cup_{\pi_{\alpha\beta}} \widetilde{\Delta}_\alpha$ with the map $\pi : \widetilde{\Delta} \longrightarrow \cup \widetilde{\Delta}_\alpha$.

Finally, since π is an isomorphism away from the exceptional divisor, we can take $\widetilde{M} = (M - S) \cup_\pi \widetilde{\Delta}$, together with the map $\pi : \widetilde{M} \longrightarrow M$, extending π on $\widetilde{\Delta}$ and the identity on $M - S$, is called the blow up of M along S. The blow up has the following properties :

1) The exceptional divisor E is fiber over S with fiber $\mathbb{P}^{k-1}$. Indeed, $\pi|_E : E \longrightarrow S$ is naturally identified with the projectivization $\mathbb{P}(N_{S/M})$ of the normal bundle $N_{S/M}$ of S in M. If M is an algebraic threefold and S a regular compact curve, the exceptional divisor E will be a ruled surface,

2) For any variety $Y \subset M$, we may define the proper transform $\widetilde{Y}$ in the blow up $\widetilde{M_S}$ to be the closure in $\widetilde{M_S}$ of the inverse image

$$\pi^{-1}(Y - S) = \pi^{-1}(Y) - E,$$

of Y away from the exceptional divisor E. The intersection $\widetilde{Y} \cap E \subset \mathbb{P}(N_{S/M})$ corresponds to the image in $N_{S/M}$ of the tangent cones $T_p(Y) \subset T_p(M)$ to Y at point of $Y \cap S$. In particular, for $Y \subset M$, a divisor we have

$$\widetilde{Y} = \pi^{-1}(Y) - mE,$$

where $m = \mathrm{mult}_S(Y)$ is the multiplicity of Y at a generic point of S.

1.3.1 The Cohomology of Blow up

let $\rho_F : F \longrightarrow S$ be a complex vector bundle with transition functions

$$\{g_{\alpha\beta}\} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

We write F_p for the fiber over p. The projectivization of F,

$$\rho_F : \mathbb{P}(F) \longrightarrow S,$$

is by definition the fiber bundle whose fiber at a point p in S is the projective space $\mathbb{P}(F_p)$ and whose transition functions

$$\overline{g_{\alpha\beta}} : U_\alpha \cap U_\beta \longrightarrow \mathbb{P}GL(r, \mathbb{C}),$$

are induced from $g_{\alpha\beta}$. Thus a point of $\mathbb{P}(F)$ is a line ℓ_p in the fiber F_p. On $\mathbb{P}(F)$ there are several tautological bundles : the pullback $\pi_F^{-1}F$, the universal bundle also called the tautological line bundle $T \subset \rho_F^{-1}(F)$ and the universal quotient bundle Q. The cohomology ring $H^*(\mathbb{P}(F))$ is via the pullback map

$$H^*(S) \xrightarrow{\rho_F^*} H^*(\mathbb{P}(F)),$$

an algebra over the ring $H^*(S)$. A complete description of $H^*(\mathbb{P}(F))$ is given in these terms by following proposition:

Proposition 1.19. *For S any compact oriented C^∞ manifold, $F \longrightarrow S$ any complex vector bundle of rank r, the cohomology ring $H^*(\mathbb{P}(F))$ is generated as an $H^*(S)$-algebra by the Chern class $\zeta = c_1(T)$ of tautological bundle, with the single relation*

$$\zeta^r - \rho_F^* c_1(F)\zeta^{r-1} + \cdots + (-1)^{r-1}\rho_F^* c_{r-1}(F)\zeta + (-1)^r \rho_F^* c_r(F) = 0.$$

Proof. See [7] page 606. ■

Now, we give a brief description of Chern class of a blow up. For more details, again, we recommend [7].

Our objective is to compare $c(T\widetilde{M})$ with $\pi^* C(TM)$. Let $i : S \longrightarrow M$, $j : E \longrightarrow \widetilde{M}$ be the inclusions. We write $N = N_{S/M}$ and $c(M)$, $c(\widetilde{M})$ and $c(S)$ for $c(TM)$, $c(T\widetilde{M})$ and $c(TS)$ respectively. Then, we have that:

Theorem 1.20 (Porteous). *With the above notation and $\zeta = c_1(T)$, we have:*

$$c(\widetilde{M}) - \pi^* c(M) = j_*(\pi_E^* c(S).\alpha),$$

where

$$\alpha = \frac{1}{\zeta}\sum_{i=0}^{r}[1-(1-\zeta)(1+\zeta)^i]\pi_E^* c_{r-i}(N).$$

In this expression, the term in brackets as a polynomial in ζ and α is the polynomial on obtains after formally dividing by ζ and r is the rank of N.

Proof. The proof can be found in [8]. ■

Can be shown, for example, (see [6], section 2.2) that

$$c_1(\widetilde{\mathbb{P}^3}) = 4\pi^*\mathbf{h} - \mathbf{E}. \tag{1.1}$$

$$c_2(\widetilde{\mathbb{P}^3}) = 6\pi^*\mathbf{h}^2 - \mathbf{E}^2 - \pi_E^* c_1(T\mathscr{C})\cdot\mathbf{E}. \tag{1.2}$$

$$c_3(\widetilde{\mathbb{P}^3}) = 4\pi^*\mathbf{h}^3 - \pi^* c_2(N)\cdot\mathbf{E} - \pi^* c_1(N)\cdot\mathbf{E.}^2 + \mathbf{E}^3. \tag{1.3}$$

where $\mathbf{h}$ is the hyperplane class on $\mathbb{P}^3$.

CHAPTER 2

SPECIAL DISTRIBUTIONS

In this chapter, we will define the main object of study of this work, which are Special Holomorphic Distributions along a curve of singularities.

2.1 Holomorphic Distributions

The reference for this section is [15].

Definition 2.1. *Let X be smooth a complex manifold.*

i) A codimension r distribution $\mathscr{F}$ on X is given by an exact sequence

$$\mathscr{F} : 0 \longrightarrow T_{\mathscr{F}} \xrightarrow{\varphi} T_X \xrightarrow{\pi} N_{\mathscr{F}} \longrightarrow 0, \tag{2.1}$$

where $T_{\mathscr{F}}$ is a coherent sheaf of rank $s := dim(X) - r$ and $N_{\mathscr{F}}$ is a torsion free sheaf. The sheaves $T_{\mathscr{F}}$ and $N_{\mathscr{F}}$ are called respectivelly the tangent and the normal sheaves of $\mathscr{F}$.

ii) $Sing(\mathscr{F}) = \{x \in X, (N_{\mathscr{F}})_x$ is not a free $\mathscr{O}_{X,x}$ module$\}$ is the singular set of the distribution $\mathscr{F}$.

Definition 2.2. *A foliation is an integrable distribution, that is, a distribution*

$$\mathscr{F} : 0 \longrightarrow T_{\mathscr{F}} \xrightarrow{\varphi} T_X \xrightarrow{\pi} N_{\mathscr{F}} \longrightarrow 0,$$

whose tangent sheaf is closed under the Lie Bracket of vector fields, i.e, $[\varphi(T_{\mathscr{F}}),\varphi(T_{\mathscr{F}})] \subset \varphi(T_{\mathscr{F}})$.

Remark 2.3. *When $r = 1$, the normal sheaf being a torsion free sheaf of rank 1, must be the twisted ideal sheaf $I_{Z/X} \otimes det(T_X) \otimes det(T_{\mathscr{F}})^{\vee}$ of a closed subscheme $Z \subset X$ of codimension at least 2, which is precisely the singular scheme of $\mathscr{F}$.*

2.2 Codimension one Distributions on $\mathbb{P}^3$

We are interested in codimension on distributions on $\mathbb{P}^3$ of degree d. In this case, by remark 2.3 we have $N_{\mathscr{F}} = \mathscr{I}_{Z/\mathbb{P}^3}(d+2)$ where Z is the singular scheme of $\mathscr{F}$. Therefore, the sequence 2.1 now reads

$$\mathscr{F} : 0 \longrightarrow T_{\mathscr{F}} \xrightarrow{\varphi} T_X \xrightarrow{\pi} \mathscr{I}_{Z/\mathbb{P}^3}(d+2) \longrightarrow 0, \tag{2.2}$$

where $T_{\mathscr{F}}$ is a rank 2 reflexive sheaf.
A codimension one distribution of degree d on $\mathbb{P}^3$ can also represented by a section $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2))$ given by the dual of the morphism $\pi : T_{\mathbb{P}^3} \longrightarrow \mathscr{I}_{Z/\mathbb{P}^3}(d+2)$.
On the other hand, such section yields a sheaf map $\omega : \mathscr{O}_{\mathbb{P}^3} \longrightarrow \Omega^1_{\mathbb{P}^3}(d+2)$. Taking duals, we get a cosection

$$\omega^{\vee} : \left(\Omega^1_{\mathbb{P}^3}(d+2)\right)^{\vee} = T_{\mathbb{P}^3}(-(d+2)) \longrightarrow \mathscr{O}_{\mathbb{P}^3},$$

whose image is the ideal sheaf $\mathscr{I}_{Z/\mathbb{P}^3}$ of the singular scheme. The kernel of $\omega^{\vee}$ is the tangent sheaf of distribution $\mathscr{F}$ twisted by $\mathscr{O}_{\mathbb{P}^3}(-(d+2))$. From this point of view the integrability condition is equivalent to $\omega \wedge d\omega = 0$.
The 1-form ω can be written down in homogeneous coordinates

$$\omega = \sum_{i=0}^{3} P_i dz_i,$$

with $P_i \in H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(d+1))$, where $[z_0 : z_1 : z_2 : z_3]$ are the homogeneous coordinates of $\mathbb{P}^3$.
In addition, the coefficients P_i must satisfiy the condition

$$i_R\omega = \sum_{i=0}^{3} z_i P_i = 0,$$

where i_R denotes the inner product of the vector field $R = \sum_{i=0}^{3} z_i \dfrac{\partial}{\partial z_i}$ with differential forms.

Before we start talking about Special Distributions, let us first talk about motivation for this work. Let $\mathscr{F}$ be a codimension one distribution on $\mathbb{P}^3$ of degree d induced by global section $\omega \in H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+2))$.

Proposition 2.4. *For integrable ω, the singular set must contain a codimension two component.*

Proof. See [9], Proposition 2.6. ■

The Jouanolou's Proposition is not valid for distributions, i.e, there are codimension one distributions on $\mathbb{P}^3$ such that the singular scheme has only isolated points, as shown in the example below.

Example 2.5 (see [15], example 8.2)**.** *Consider the distribution $\mathscr{F}$ on $\mathbb{P}^3$ induced by*

$$\omega = (\xi_0^2 + \xi_1^2 + \xi_2^2)d\xi_3 - (\xi_3\xi_0 + \xi_2\xi_1)d\xi_0 + (\xi_2\xi_0 - \xi_3\xi_1)d\xi_1 - \xi_3\xi_2 d\xi_2.$$

The singular scheme of $\mathscr{F}$ is $\left\{2[i:-1:0:0], 2[i:1:0:0], [0:0:0:1]\right\}$.

In the above example, using the Grothendieck residue Theorem, we can determine the residue at each point, and hence, the sum of these residues for such distribution.
This is just what Izawa's Theorem tells us. More precisely

Theorem 2.6 (Izawa's Theorem)**.** *Let ω be a codimension one singular distribution with $\mathscr{G}$ rank one locally free subsheaf of Ω_X and $(f_1^j, \ldots, f_n^j)$ a local coefficients of ω near p_j. Then we have*

$$\int_X c_n(\Omega_X \otimes \mathscr{G}^\vee) = \sum_{j=1}^{k} Res_{p_j} \begin{bmatrix} df_1^j & \wedge & \ldots & \wedge\, df_n^j \\ & f_1^j & \ldots & f_n^j \end{bmatrix},$$

where $\mathrm{Sing}(\mathscr{F}) = \{p_1, \ldots, p_k\}$ and the residue above is the Grothendieck Residue.

Proof. See [20], Theorem 2.2. ■

Returning to the Example 2.5, in the affine open set $U_3 = \left\{[\xi] \in \mathbb{P}^3 \;\; \xi_3 \neq 0\right\}$ with coordinates (z_i), where $z_i = \dfrac{\xi_i}{\xi_3}$ for $i = 0, 1, 2$. We have

$$\omega = -(z_0 + z_2 z_1)dz_0 + (z_2 z_0 - z_1)dz_1 - z_2 dz_2. \tag{2.3}$$

where $f_0 = -z_0 - z_2 z_1 \quad f_1 = z_2 z_0 - z_1$ and $f_2 = -z_2$.
Let us to consider the germ $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^3, 0)$ defined by $f = (f_0, f_1, f_2)$ and note that $f^{-1}(0) = \{0\}$. The Jacobian matrix is given by

$$Df = \begin{pmatrix} -1 & -z_2 & -z_1 \\ z_2 & -1 & z_0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, the Jacobian is : $J(Df) = -(1 + z_2^2)$.

As $J(Df)(0) = -1$, then f is a germ of a biholomorphism.
Therefore, by Proposition 1.14 and the Property (2) of residues, we have $\mathrm{Res}_0(J(Df), f) = 1$.

The points $(\pm i, 0, 0)$ belong to the open affine set $U_0 = \left\{[\xi] \in \mathbb{P}^3, \xi_0 \neq 0\right\}$. In this case, the 1-form is given by

$$\omega = (1 + z_1^2 + z_2^2) dz_3 + (z_2 - z_3 z_1) dz_1 - z_3 z_2 dz_2. \tag{2.4}$$

Consider the germ $h : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^3, 0)$ defined by $h_1 = z_2 - z_3 z_1 \quad h_2 = -z_3 z_2$ and $h_3 = 1 + z_1^2 + z_2^2$. In this case, the Jacobian matrix is given by

$$Dh = \begin{pmatrix} -z_3 & 1 & -z_1 \\ 0 & -z_3 & -z_2 \\ 0 & 2z_2 & 0 \end{pmatrix}.$$

$J(Dh) = -2(z_1^2 z_3 + z_2^2 z_3 + z_2 z_1)$. The residue is

$$\mathrm{Res}(J(Dh), f) = \frac{1}{(2\pi i)^3} \int_\Gamma \frac{2(z_1^2 z_3 + z_2^2 z_3 + z_2 z_1) dz_1 \wedge dz_2 \wedge dz_3}{(z_2 - z_3 z_1)(1 + z_1^2 + z_2^2) z_3 z_2} \tag{2.5}$$

where Γ is real 3-cycle around singularity $(-i, 0, 0)$. Let us make a translation $z_1 \longrightarrow z_1 + i$ to calculate the residue at the origin.

$$\frac{2((z_1 + i)^2 z_3 + z_2^2 z_3 + z_2(z_1 + i))}{(z_2 - z_3(z_1 + i))(1 + (z_1 + i)^2 + z_2^2) z_3 z_2} = \frac{2(-z_3 + i z_2 + 2i z_1 z_3 + z_1^2 z_3 + z_2^2 z_3 + z_1 z_2}{(z_2 - z_3(z_1 + i))(1 + (z_1 + i)^2 + z_2^2) z_3 z_2}. \tag{2.6}$$

by simplifying the equation (2.6), we have the following integrals

$$\begin{aligned}\mathrm{Res}(J(Dh),f) &= \frac{1}{(2\pi i)^3}\int_\Gamma\left(\frac{2}{(z_2-iz_3-z_1z_3)z_2}+\frac{2}{(z_2-iz_3-z_1z_3)(2iz_1+z_1^2+z_2^2)z_2}\right) \\ &+\frac{1}{(2\pi i)^3}\int_\Gamma\left(\frac{2}{(z_2-iz_3-z_1z_3)(2iz_1+z_1^2+z_2^2)z_3}+\frac{2z_1}{(z_2-iz_3-z_1z_3)(2iz_1+z_1^2+z_2^2)z_3}\right).\end{aligned} \tag{2.7}$$

The value of each triple integral (2.7) is respectively: $0,1,1,0$. Therefore $\mathrm{Res}(J(Dh),f)=2$. Similarly, after the translation $z_1 \longrightarrow z_1-i$, we calculate the residue at the point $(i,0,0)$ which is also equal to 2. Thus,

$$\sum_{i=1}^{3}\mathrm{Res}(\omega,p_i)=2+2+1=5.$$

After this example we can ask: What happens if the singular scheme of a distribution contains a curve? What can we say about the residue of the distribution along this curve?

Motivated by these questions, we seek the answer to a certain type of distribution, namely the special distributions along a curve.

2.3 Special Distributions along a Regular Curve

In this section, we will define the main object of study of this work, which are Special Holomorphic Distributions along a curve of singularities.

Let $\mathscr{F}$ be a non-integrable codimension one distribution on $\mathbb{P}^3$ of degree d whose singular set is the following disjoint union of proper closed subsets

$$\mathrm{Sing}(\mathscr{F})=\mathscr{C}\cup\{p_1,\cdots,p_n\},$$

where $\mathscr{C}$ is a smooth irreducible curve and isolated points $p_1,\cdots p_n$.

Notice that, once the curve $\mathscr{C}$ is regular, by the local submersion Theorem, for each point $p\in\mathscr{C}$, there is an open $U\subset\mathbb{C}^3$ and a biholomorphism $\varphi:U\longrightarrow V\subset\mathbb{C}^3$ such that the image of $\mathscr{C}$ for φ is given by $\mathscr{C}\cap U=\{z_1=z_2=0\}$. So, if f is a function that vanish over $\mathscr{C}$ curve, we can write

$$f(z)=z_1f_1(z)+z_2f_2(z). \tag{2.8}$$

If f_1 and f_2 also vanish on the z_3-axis, we can apply (2.8) again to all of them. Thus, the function f can be written as

$$f(z) = z_1^2 f_{2,0}(z_1,z_2,z_3) + z_1 z_2 f_{1,1}(z_1,z_2,z_3) + z_2^2 f_{0,2}(z_1,z_2,z_3). \tag{2.9}$$

By repeating this process until we find some function $f_{i,j}$ that does not cancel along the z_3-axis, we can then write:

$$f(z) = \sum_{i+j=m} z_1^i z_2^j f_{i,j}(z), \tag{2.10}$$

in that for some pair i,j, $f_{i,j}(0,0,z_3) \neq 0$.

Definition 2.7. *The number m in* (2.10) *is called multiplicity of f over $\mathscr{C}$ and will be denoted for $mult_{\mathscr{C}}(f)$.*

Remark 2.8. *The $mult_{\mathscr{C}}(f)$ is independent of the coordinate system choosen (see [5]).*

Thus, in the open $V \subset \mathbb{C}^3$ the distribution $\mathscr{F}$ is given for 1-form

$$\omega = P(z)dz_1 + Q(z)dz_2 + R(z)dz_3, \tag{2.11}$$

with $\text{mult}_{\mathscr{C}}(P) = m$, $\text{mult}_{\mathscr{C}}(Q) = n$ and $\text{mult}_{\mathscr{C}}(R) = p$.

$$\begin{cases} P(z) &=& z_1^m P_0(z) + z_1^{m-1} z_2 P_1(z) + \cdots z_1 z_2^{m-1} P_{m-1}(z) + z_2^m P_m(z), \\ Q(z) &=& z_1^n Q_0(z) + z_1^{n-1} z_2 Q_1(z) + \cdots z_1 z_2^{n-1} Q_{n-1}(z) + z_2^n Q_n(z), \\ R(z) &=& z_1^p R_0(z) + z_1^{p-1} z_2 R_1(z) + \cdots z_1 z_2^{p-1} R_{p-1}(z) + z_2^p R_p(z). \end{cases} \tag{2.12}$$

Definition 2.9. *The algebraic multiplicity of the distribution $\mathscr{F}$ over $\mathscr{C}$, denoted by $mult_{\mathscr{C}}(\mathscr{F})$ will be the minimum of the numbers m, n and p.*

Lemma 2.10. *Let $\mathscr{F}$ be a codimension one distribution on $\mathbb{P}^3$ with $\mathscr{C} \subset Sing(\mathscr{F})$ where $\mathscr{C}$ is a regular curve. Then, for each point $p \in \mathscr{C}$ exists a neighborhood U of p and a holomorphic coordinates $w = (w_1, w_2, w_3) \in \mathbb{C}^3$ such that $w(0) = 0 \in \mathbb{C}^3$ and $U \cap \mathscr{C} = \{w_1 = w_2 = 0\}$ and $\mathscr{F}$ is defined in U by the following* 1*-form:*

$$\omega = \sum_{i=1}^{3} P_i dw_i,$$

where P_i are given as in (2.12) *with:*

i) $mult_{\mathscr{C}}(P_i) = mult_{\mathscr{C}}(\mathscr{F}) = m_1$ for $i = 1,2$,

ii) $mult_{\mathscr{C}}(P_3) = m_3$,

iii) $m_1 \leq m_3$.

Proof. For a holomorphic change of coordinates, we know that the curve $\mathscr{C}$ can be locally given by $z_1 = z_2 = 0$. In this neighborhood, the distribution $\mathscr{F}$ is induced by the following 1-form:

$$\omega = \sum_{i=1}^{3} Q_i dz_i,$$

Let $A = (a_{i,j}) \in \mathrm{GL}(3,\mathbb{C})$ be a matrix such that $a_{i3} = 0$ for $1 \leq i \leq 2$. Consequently $B = A^{-1} = (b_{i,j})$ has the same property, i.e, $b_{i3} = 0$ for $1 \leq i \leq 2$. Thus, the linear transformation $z = Aw$ preserves the w_3-axis. Then

$$\begin{aligned} A_*\omega &= Q_1 \circ A(w)(a_{11}dw_1 + a_{12}dw_2) + Q_2 \circ A(w)(a_{21}dw_1 + a_{22}dw_2) \\ &\quad + Q_3 \circ A(w)(a_{31}dw_1 + a_{32}dw_2 + a_{33}dw_3). \end{aligned} \tag{2.13}$$

By doing $Q_i \circ A(w) = P_i$, we have

$$A_*\omega = (a_{11}P_1 + a_{21}P_2 + a_{33}P_3)dw_1 + (a_{12}P_1 + a_{22}P_2 + a_{32}P_3)dw_2 + a_{33}P_3dw_3. \tag{2.14}$$

Adjusting some coefficients, if necessary, the distribution $\mathscr{F}$ is induced by

$$\omega = \sum_{i=1}^{3} P_i dw_i,$$

where each P_i satisfies the conditions of the Lemma. ■

Let us do the blowing up of $\mathbb{P}^3$ along $\mathscr{C}$, using the chart σ_0 in $\widetilde{V}_0$ with coordinates $(u_1,u_2,u_3) \in \mathbb{C}^3$, such that : $\sigma_0(u_1,u_2,u_3) = (u_1,u_1u_2,u_3) = (z_1,z_2,z_3)$. So:

$$\begin{aligned} P(u_1,u_1u_2,u_3) &= \sum_{i=0}^{m} u_1^{m-i}(u_1u_2)^i P_i(u_1,u_1u_2,u_3), \\ &= u_1^m \sum_{i=0}^{m} u_2^i P_i(u_1,u_1u_2,u_3). \end{aligned} \tag{2.15}$$

Notice that:

$$P_i(u_1,u_1u_2,u_3) = P_i(0,0,u_3) + u_1\widetilde{P}_i(u_1,u_2,u_3) = p_i(u_3) + u_1\widetilde{P}_i(u). \tag{2.16}$$

By replacing (2.16) in (2.15), we have

$$\begin{aligned}
P(u_1,u_1u_2,u_3) &= u_1^m \sum_{i=0}^{m} u_2^i \Big[p_i(0,0,u_3)+u_1\widetilde{P}_i(u)\Big], \\
&= u_1^m \Big[\sum_{i=0}^{m} u_2^i p_i(u_3)+u_1\sum_{i=0}^{m} u_2^i \widetilde{P}_i(u)\Big], \\
&= u_1^m \Big[\sum_{i=0}^{m} u_2^i p_i(u_3)+u_1 P_1(u)\Big], \qquad (2.17)
\end{aligned}$$

where $P_1(u)=\sum_{i=0}^{m} u_2^i \widetilde{P}_i(u)$.
Similarly, we have

$$Q(u_1,u_1u_2,u_3) = u_1^n \Big[\sum_{i=0}^{n} u_2^i q_i(u_3)+u_1 Q_1(u)\Big]. \qquad (2.18)$$

$$R(u_1,u_1u_2,u_3) = u_1^p \Big[\sum_{i=0}^{p} u_2^i r_i(u_3)+u_1 R_1(u)\Big]. \qquad (2.19)$$

Finally, since that $z_2=u_1u_2$ and $dz_2=u_1du_2+u_2du_1$, we have:

$$\begin{aligned}
\pi^*(\omega) = &\Big(u_1^m\Big[\sum_{i=0}^{m} u_2^i p_i(u_3)+u_1P_1(u)\Big]\Big)du_1+\Big(u_1^n\Big[\sum_{i=0}^{p} u_2^i q_i(u_3)+u_1Q_1(u)\Big]\Big)\Big(u_1du_2+u_2du_1\Big) \\
&+\Big(u_1^p\Big[\sum_{i=0}^{p} u_2^i r_i(u_3)+u_1R_1(u)\Big]\Big)du_3, \qquad (2.20)
\end{aligned}$$

$$\begin{aligned}
\pi^*(\omega) = &\Big(u_1^m\Big[\sum_{i=0}^{m} u_2^i p_i(u_3)+u_1P_1(u)\Big]\Big)du_1+\Big(u_1^{n+1}\Big[\sum_{i=0}^{p} u_2^i q_i(u_3)+u_1Q_1(u)\Big]\Big)du_2 \\
&+\Big(u_1^n u_2\Big[\sum_{i=0}^{p} u_2^i q_i(u_3)+u_1Q_1(u)\Big]\Big)du_1+\Big(u_1^p\Big[\sum_{i=0}^{p} u_2^i r_i(u_3)+u_1R_1(u)\Big]\Big)du_3, \qquad (2.21)
\end{aligned}$$

$$\begin{aligned}
\pi^*(\omega) = &\Big(u_1^m\Big[\sum_{i=0}^{m} u_2^i p_i(u_3)+u_1P_1(u)\Big]+u_1^n u_2\Big[\sum_{i=0}^{p} u_2^i q_i(u_3)+u_1Q_1(u)\Big]\Big)du_1 \\
&+\Big(u_1^{n+1}\Big[\sum_{i=0}^{p} u_2^i q_i(u_3)+u_1Q_1(u)\Big]\Big)du_2+\Big(u_1^p\Big[\sum_{i=0}^{p} u_2^i r_i(u_3)+u_1R_1(u)\Big]\Big)du_3. \qquad (2.22)
\end{aligned}$$

Notice that, in this chart σ_0, the exceptional divisor $E=\pi^{-1}(\mathscr{C})$ is given by $u_1=0$ and that $\pi^*(\omega)$ vanish over E. The distribution induced by $\mathscr{F}$ via π, what we denoted by $\widetilde{\mathscr{F}}$ is given by $\pi^*(\omega)$ after the division of 2.22 by an adequate power of u_1.

By Lemma 2.10, we can suppose that $m = n \leq p$. So

First case: Non-dicritical curve of singularities

Se $n = m$ and $\sum_{i=0}^{n} u_2^i \Big[[p_i(u_3) + u_2 q_i(u_3) \Big] \not\equiv 0$,
We have to analyze the following sub-cases:

a) $p = n$.

Dividing (2.22) by u_1^n, we have:

$$\tilde{\omega} = \left(\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3) \Big] + u_1 \Big(u_2 Q_1(u) + P_1(u) \Big) \right) du_1$$
$$+ u_1 \left(\sum_{i=0}^{p} u_2^i q_i(u_3) + u_1 Q_1(u) \right) du_2 + \left(\sum_{i=0}^{p} u_2^i r_i(u_3) + u_1 R_1(u) \right) du_3. \tag{2.23}$$

The singularities over the exceptional divisor, $E = \{u_1 = 0\}$, is the set of zeros of the equations below

$$\begin{cases} \sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3) \Big] = 0, \\ \sum_{i=0}^{p} u_2^i r_i(u_3) = 0. \end{cases} \tag{2.24}$$

Generically, since there are two equations with two unknowns, the system (2.24) has isolated singularities over E.

b) $p > n$.

Dividing (2.22) by u_1^n, we have

$$\widetilde{\omega} = \left(\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3) \Big] + u_1 \Big(u_2 Q_1(u) + P_1(u) \Big) \right) du_1$$
$$+ u_1 \left(\sum_{i=0}^{p} u_2^i q_i(u_3) + u_1 Q_1(u) \right) du_2 + u_1^{p-n} \left(\sum_{i=0}^{p} u_2^i r_i(u_3) + u_1 R_1(u) \right) du_3. \tag{2.25}$$

In this case, the singularities over $E = \{u_1 = 0\}$ are determined by the set of zeros of the following equation:

$$\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3) \Big] = 0. \tag{2.26}$$

in this case, over E, there will be curves of singularities.

Second case: Dicritical curve of singularities.

If $n = m$ and $\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3) \Big] \equiv 0.$
From (2.22), we have

$$\begin{aligned} \pi^* \widetilde{\omega} = {} & u_1^{n+1} \Big(P_1(u) + u_2 Q_1(u) \Big) du_1 + u_1^{n+1} \Big(\sum_{i=0}^{n} u_2^i q_i(u_3) + u_1 Q_1(u) \Big) du_2 \\ & + u_1^p \Big(\sum_{i=0}^{n} u_2^i r_i(u_3) + u_1 R_1(u) \Big) du_3. \end{aligned} \tag{2.27}$$

Now let us to analyze at the following sub cases

a) $p = n + 1$.

Dividing (2.27) by u_1^p, we have

$$\widetilde{\omega} = \Big(P_1(u) + u_2 Q_1(u) \Big) du_1 + \Big(\sum_{i=0}^{n} u_2^i q_i(u_3) + u_1 Q_1(u) \Big) du_2 \tag{2.28}$$

$$+ \Big(\sum_{i=0}^{n} u_2^i r_i(u_3) + u_1 R_1(u) \Big) du_3. \tag{2.29}$$

On exceptional divisor, the singularities are the solutions of the system of equations below

$$\begin{cases} P_1(u) + u_2 Q_1(u) = 0, \\ \sum_{i=0}^{n} u_2^i q_i(u_3) = 0, \\ \sum_{i=0}^{n} u_2^i r_i(u_3) = 0. \end{cases} \tag{2.30}$$

b) $p = n$.

In this case, the singularities over $E = \{u_1 = 0\}$ are determined by the set zeros of the following equation

$$\sum_{i=0}^{n} u_2^i r_i(u_3) = 0. \tag{2.31}$$

On the exceptional divisor E will be curves of singularities.

c) $p > n+1$.

Dividing (2.27) by u_1^{n+1}, we have

$$\begin{aligned}\widetilde{\omega} = &\Big(P_1(u) + u_2 Q_1(u)\Big)du_1 + \Big(\sum_{i=0}^{n} u_2^i q_i(u_3) + u_1 Q_1(u)\Big)du_2 \\ &+ u_1^{p-(n+1)}\Big(\sum_{i=0}^{n} u_2^i r_i(u_3) + u_1 R_1(u)\Big)du_3.\end{aligned} \tag{2.32}$$

The singularities on the exceptional divisor are the solutions of the following equation system

$$\begin{cases} P_1(u) + u_2 Q_1(u) = 0, \\ \sum_{i=0}^{n} u_2^i q_i(u_3) = 0. \end{cases} \tag{2.33}$$

Again, generically, the system (2.33) has isolated singularities.

Definition 2.11. *Let $\mathscr{F}$ be a non integrable codimension one distribution on $\mathbb{P}^3$ whose singular set*

$$Sing(\mathscr{F}) = \mathscr{C} \cup \{p_1, \cdots, p_n\},$$

where $\mathscr{C}$ is a smooth irreducible curve and isolated points $p_1, \cdots, p_n$. We say that $\mathscr{F}$ is

I) ***Non dicritical****, if $n \leq m$ and $\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3)\Big] \not\equiv 0$,*

II) ***Dicritical****, if $n = m$ and $\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3)\Big] \equiv 0$.*

The sum $\sum_{i=0}^{n} u_2^i \Big[p_i(u_3) + u_2 q_i(u_3)\Big]$ is called tangent cone.

The following show us a special distribution along a curve.

Example 2.12. *Let $\mathscr{F}$ be a codimension one distribution and degree m on $\mathbb{P}^3$ induced by $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(m+2))$ where on the affine open set: $U_3 = \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3; z_3 \neq 0\right\}$, we write*

$$\omega = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz,$$

$$P(x,y,z) = A_m(x,y) + (\alpha y + \beta z)G_m(x,y),$$

$$Q(x,y,z) = B_m(x,y) + (\gamma z - \alpha x)G_m(x,y),$$

$$R(x,y,z) = C_m(x,y) - (\beta x + \gamma y)G_m(x,y),$$

and

$$A_m(x,y) = \sum_{i=0}^{m} a_i x^{m-i} y^i,$$

$$B_m(x,y) = \sum_{i=0}^{m} b_i x^{m-i} y^i,$$

$$C_m(x,y) = \sum_{i=0}^{m} c_i x^{m-i} y^i,$$

$$G_m(x,y) = \sum_{i=0}^{m} g_i x^{m-i} y^i,$$

$\alpha, \beta, \gamma \in \mathbb{C}$ and A_m, B_m, C_m, G_m are homogeneous polynomials of degree m and $gdc(A_m, B_m, C_m, G_m) = 1$.

Let us note that $\mathscr{C} = \{x = y = 0\} \subset Sing(\mathscr{F})$. By doing the blown-up of $\mathbb{P}^3$ along $\mathscr{C}$, in *the chart $\sigma(u_1,u_2,u_3)=(u_1,u_1u_2,u_3)=(x,y,z)$, we have*

$$\begin{aligned}
\pi^*(\omega) &= \left[A_m(u_1,u_2u_1)+(\alpha u_2u_1+\beta u_3)G_m(u_1,u_2u_1)\right]du_1\\
&\quad+\left[B_m(u_1,u_2u_1)+(\gamma u_3-\alpha u_1)G_m(u_1,u_2u_1)\right](u_2du_1+u_1du_2)\\
&\quad+\left[C_m(u_1,u_2u_1)-(\beta u_1+\gamma u_2u_1)G_m(u_1,u_2u_1)\right]du_3,\\
\pi^*(\omega) &= u_1^m\left[a(u_2)+(\alpha u_2u_1+\beta u_3)g(u_2)\right]du_1\\
&\quad+u_1^{m+1}\left[b(u_2)+(\gamma u_3-\alpha u_1)g(u_2)\right]du_2\\
&\quad+u_2u_1^m\left[b(u_2)+(\gamma u_3-\alpha u_1)g(u_2)\right]du_1\\
&\quad+u_1^m\left[c(u_2)-u_1(\beta+\gamma u_2)g(u_2)\right]du_3,\\
\pi^*(\omega) &= u_1^m\left[a(u_2)+(\alpha u_1+\beta u_3)g(u_2)+u_2b(u_2)+u_2(\gamma u_3-\alpha)g(u_2)\right]du_1\\
&\quad+u_1^{m+1}\left[b(u_2)+(\gamma u_3-\alpha u_1)g(u_2)\right]du_2\\
&\quad+u_1^m\left[c(u_2)-u_1(\beta+\gamma u_2)g(u_2)\right]du_3,\\
\widetilde{\omega} &= \left[a(u_2)+u_2b(u_2)+u_3(\beta+\gamma u_2)g(u_2)\right]du_1\\
&\quad+u_1\left[b(u_2)+(\gamma u_3-\alpha u_1)g(u_2)\right]du_2\\
&\quad+\left[c(u_2)-u_1(\beta+\gamma u_2)g(u_2)\right]du_3.
\end{aligned}$$

where:

$$a(u_2)=\sum_{i=0}^{m}a_iu_2^i, \tag{2.34}$$

$$b(u_2)=\sum_{i=0}^{m}b_iu_2^i, \tag{2.35}$$

$$c(u_2)=\sum_{i=0}^{m}c_iu_2^i, \tag{2.36}$$

$$g(u_2)=\sum_{i=0}^{m}g_iu_2^i, \tag{2.37}$$

are polynomials of degree m.

The singularities on the exceptional divisor $E=\{u_1=0\}$, are the system solutions below.

$$\left\{\begin{array}{rcl} a(u_2)+u_2b(u_2)+u_3(\beta+\gamma u_2)g(u_2) &=& 0,\\ c(u_2) &=& 0. \end{array}\right. \tag{2.38}$$

From the second system equation (2.38)*, we have* $c(u_2) = 0$*, then* $u_2 = u_2^i$ *with* $i = 1, 2, \dots m$*. By replacing this in the first equation of* (2.38) *we get*

$$u_3^i = -\frac{a(u_2^i) - u_2^i b(u_2^i)}{(\beta + \gamma u_2^i) g(u_2^i)}$$

with $u_2^i \neq \dfrac{-\beta}{\gamma}$ *and* $\gamma \neq 0$*. Thus, we have* m *solutions on the exceptional divisor. Let us also note that we can not have simultaneously*

$$\begin{cases} u_1 & = & 0, \\ b(u_2)(\gamma u_3 - \alpha) g(u_2) & = & 0. \end{cases} \tag{2.39}$$

In fact, suppose that worth the equalities of (2.39) *are valid. So we would have* $u_3^i = -\dfrac{b(u_2^i)}{\gamma g(u_2^i)}$ *with* $i = 1, 2, \dots m$*. As* $u_2 = u_2^i$*, by replacing this in the first equation of* (2.38) *we have*

$$a(u_2^i) + u_2^i \frac{-b(u_2^i)}{\gamma g(u_2^i)} (\beta + \gamma u_2^i) g(u_2^i) = 0, \tag{2.40}$$

$$\gamma a(u_2^i) + \cancel{\beta \gamma u_2^i b(u_2^i)} - \gamma \beta b(u_2^i) - \cancel{\beta \gamma u_2^i b(u_2^i)} = 0, \tag{2.41}$$

$$\gamma (a(u_2^i) - \beta b(u_2^i)) = 0. \tag{2.42}$$

As $\gamma \neq 0$*, we must have*

$$a(u_2^i) + \beta b(u_2^i) = 0. \tag{2.43}$$

The last equation has no solution, since we are assuming by hypothesis that

$$gdc(a(u_2), b(u_2), c(u_2), g(u_2)) = 1.$$

Let us now to find out the singularities in the complementary to the exceptional divisor E*. For this, let us analyze the following system of equations.*

$$\begin{cases} a(u_2) + u_2 b(u_2) + u_3 (\beta + \gamma u_2) g(u_2) & = & 0, \\ b(u_2) + (\gamma u_3 - \alpha u_1) g(u_2) & = & 0, \\ c(u_2) - u_1 (\beta + \gamma u_2) g(u_2) & = & 0. \end{cases} \tag{2.44}$$

From the second and third equations of (2.44) *we have respectively that*

$$u_3 = -\frac{a(u_2) + u_2 b(u_2)}{(\beta + \gamma u_2)g(u_2)}. \tag{2.45}$$

$$u_1 = \frac{c(u_2)}{(\beta + \gamma u_2)g(u_2)}. \tag{2.46}$$

By replacing (2.45) *and* (2.46) *in the first equation of* (2.44)*, and simplifying*

$$\beta b(u_2) - \gamma a(u_2) - \alpha c(u_2) = 0. \tag{2.47}$$

Therefore we have on the complementary E m solutions.

As codim(Sing(ω)) ≥ 2*, by Hartog's Theorem, ω it extends to holomorphic* 1*-form on* $\mathbb{P}^3$*, which we will continue to call ω.*

Now, doing $x = \frac{z_0}{z_3}$*,* $y = \frac{z_1}{z_3}$ *e* $z = \frac{z_2}{z_3}$*, we have*

1. $dx = \dfrac{z_3 dz_0 - z_0 dz_3}{z_3^2}$,
2. $dy = \dfrac{z_3 dz_1 - z_1 dz_3}{z_3^2}$,
3. $dz = \dfrac{z_3 dz_2 - z_2 dz_3}{z_3^2}$.

Thus:

$$\begin{aligned}\omega = &\Big[z_3 A_m(z_0,z_1) + (\alpha z_1 + \beta z_2)G_m(z_0,z_1)\Big]dz_0 + \Big[z_3 B_m(z_0,z_1) + (\gamma z_2 - \alpha z_0)G_m(z_0,z_1)\Big]dz_1 \\ &+ \Big[z_3 C_m(z_0,z_1) - (\beta z_0 + \gamma z_1)G_m(z_0,z_1)\Big]dz_2 - \Big[z_0 A_m(z_0,z_1) + z_1 B_m(z_0,z_1) + z_2 C_m(z_0,z_1)\Big]dz_3.\end{aligned}$$

Let us now look at the singularities in the hyperplane at infinity $H_\infty = \Big\{z_3 = 0\Big\}$ *relative to the* U_3*.*
In this case, writing : $z_3 = 0$*,* $z_2 = 1$*,* $u = \frac{z_0}{z_2}$*,* $v = \frac{z_1}{z_2}$ *e* $w = \frac{z_3}{z_2}$*:*

$$\begin{aligned}\omega = &\Big[(\alpha v + \beta)G_m(u,v)\Big]du + \Big[(\gamma - \alpha u)G_m(u,v)\Big]dv \\ &- \Big[u A_m(u,v) + v B_m(u,v) + C_m(u,v)\Big]dw.\end{aligned}$$

Using the chart: $\sigma_1(v_1,v_2,v_3)=(v_1,v_1v_2,v_3)=(u,v,w)$, *we have*

$$\pi^*\omega = v_1^m\Big[(\beta+\gamma v_2)g_m(v_2)\Big]dv_1 + v_1^{m+1}\Big[(\gamma-\alpha v_1)g_m(v_2)\Big]dv_2$$
$$-v_1^m\Big[v_1a_m(v_2)+v_2b_m(v_2)+c_m(v_2)\Big]dv_3.$$

$$\pi^*\omega = \Big[(\beta+\gamma v_2)g_m(v_2)\Big]dv_1 + v_1\Big[(\gamma-\alpha v_1)g_m(v_2)\Big]dv_2$$
$$-\Big[v_1a_m(v_2)+v_2b_m(v_2)+c_m(v_2)\Big]dv_3.$$

On the exceptional divisor whose equation is $v_1=0$, *we have*

$$\left\{\begin{array}{ccc}(\beta+\gamma v_2)g_m(v_2) & = & 0,\\ c_m(v_2) & = & 0.\end{array}\right. \tag{2.48}$$

The system has no solution, because the second equation gives us $v_2=v_2^i$ *with* $i=1,2,\cdots,m$, *i.e,* m *roots. In the first equation, we have a polynomial degree* $m+1$ *in the variable* v_2.

Now, for $v_1\neq 0$, *we have:*

$$v_1=-\frac{c_m(v_2)}{a_m(v_2)+b_m(v_2)}.$$

So we have m *solutions* $v_1=v_1^i$, *with* $i=1,2,\cdots,m$.

As $p(u_2)=a(u_2)+u_2b(u_2)+u_3(\beta+\gamma u_2)g(u_2)\not\equiv 0$, *and* $Sing(\mathscr{F})=\{p_1,\cdots,p_{3m}\}$ ω *induces a special distribution* $\mathscr{F}$ *along the curve* $\mathscr{C}=\{z_0=z_1=0\}$.

Thus, we can define the main object of study of our work, which are the special distributions along a curve of singularities.

Definition 2.13. *Let* $\mathscr{F}$ *be a non-integrable codimension one distribution on* $\mathbb{P}^3$ *of degree* d *whose singular set is the following disjoint union of proper closed subsets*

$$Sing(\mathscr{F})=\mathscr{C}\cup\{p_1,\ldots,p_n\},$$

where $\mathscr{C}$ *is a smooth irreducible curve and closed points* $p_1,\ldots,p_n$. *Let* $\pi:\widetilde{\mathbb{P}^3}\longrightarrow\mathbb{P}^3$ *be the blow up morphism of* $\mathbb{P}^3$ *along* $\mathscr{C}$ *with exceptional divisor* E. *We say that a distribution* $\mathscr{F}$ *is special along* $\mathscr{C}$ *if*

1) $\mathscr{F}$ *is non-dicritical at* $\mathscr{C}$*,*

2) The singular set of the distribution $\widetilde{\mathscr{F}}$ *induced by* $\mathscr{F}$ *via* π*, denoted b* $Sing(\widetilde{\mathscr{F}})$ *has only isolated singularities.*

We have seen before that not every global section $\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2))$ induces a special distribution.
In The next Lemma, one of the main results of this chapter, will show how we can obtain a special distribution along a curve, preserving the invariants of the original distribution.

Lemma 2.14 (Perturbation Lemma). *Let* $\mathscr{F}$ *be a non-integrable codimension one distribution on* $\mathbb{P}^3$ *of degree d which singular set is the following disjoint union of proper closed subsets*

$$Sing(\mathscr{F}) = \mathscr{C} \cup \{p_1, \dots, p_n\},$$

where $\mathscr{C}$ *is a smooth irreducible curve and closed points* $p_1, \dots, p_n$*.*
Then there exists an one-parameter family of holomorphic distribuions $\mathscr{F}_t$ *defined in* $\mathbb{P}^3$ *with* $t \in \mathbb{D} = D(0, \varepsilon)$*, for* $\varepsilon > 0$ *sufficiently small such that*

1) $\mathscr{F}_0 = \mathscr{F}$ *e* $deg(\mathscr{F}_t) = deg(\mathscr{F})$*,* $\forall t \in \mathbb{D}$*,*

2) $\mathscr{C} \subset Sing(\mathscr{F}_t)$*,* $\forall t \in \mathbb{D}$*,*

3) $\mathscr{F}_t$ *is special along* $\mathscr{C}$*,* $\forall t \in \mathbb{D} \setminus \{0\}$*,*

4) $mult_{\mathscr{C}}(\mathscr{F}_t) = mult_{\mathscr{C}}(\mathscr{F})$*,*

5)

$$mult_E(\pi^* \mathscr{F}_t) = \begin{cases} mult_E(\pi^* \mathscr{F}), & if \quad \mathscr{F} \text{ is not dicritical}, \\ mult_E(\pi^* \mathscr{F}) - 1, & if \quad \mathscr{F} \text{ is dicritical}. \end{cases}$$

$\forall t \in \mathbb{D} \setminus \{0\}$.

Proof. Let $\mathscr{F}$ be a distribution induced by global section
$\omega \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2))$ and $\mathscr{C} = Z(f, g)$ a theoretic complete intersection set, where f and g are two polynomials. Since $\mathscr{C}$ is smooth curve for each $p \in \mathscr{C}$ there is an open set $U \subset U_i$ for some affine open set U_i of $\mathbb{P}^3$ such that

$$df(z) \wedge dg(z) \neq 0,$$

for all $z \in \mathscr{C} \cap U$.

Therefore, without loss of generality, we can admit that:

$$\frac{\partial f(z)}{\partial z_1}\frac{\partial g(z)}{\partial z_2}-\frac{\partial f(z)}{\partial z_2}\frac{\partial g(z)}{\partial z_1}\neq 0, \tag{2.49}$$

for all $z=(z_1,z_2,z_3)\in U$.

Let $F:U\longrightarrow V\subset\mathbb{C}^3$ be a locally biholomorphism defined as follows

$$F(z)=(f(z),g(z),z_3)=(u,v,w).$$

So the image of $F(\mathscr{C}\cap U)=\mathscr{C}_V=\{u=v=0\}$ is the w-axis restricted to V. We can then describe the pushforward of $F_*\mathscr{F}$ in V as follows

$$\theta=F_*(\mathscr{F})=F_*(\omega)=L(u,v,w)du+M(u,v,w)dv+N(u,v,w)dw. \tag{2.50}$$

where L, M and N written as in (2.12), and $\mathrm{mult}_{\mathscr{C}}(L)=\ell_1$, $\mathrm{mult}_{\mathscr{C}}(M)=m_1$ and $\mathrm{mult}_{\mathscr{C}}(N)=n_1$.

Let us build a small perturbation θ_t for the 1-form θ. By Lemma 2.10 we can assume that $m=\ell_1=m_1\leq n_1$. Thus

$$\begin{aligned}\theta_t&=\theta+t\left[A_m(u,v)+(\alpha v+\beta w)G_m(u,v)\right]du\\&\quad+t\left[B_m(u,v)+(\gamma w-\alpha u)G_m(u,v)\right]dv\\&\quad+t\left[C_m(u,v)-(\beta u+\gamma v)G_m(u,v)\right]dw.\end{aligned} \tag{2.51}$$

where $A_m=\sum_{i=0}^{m}a_iu^{m-i}v^i$, $B_m=\sum_{i=0}^{m}b_iu^{m-i}v^i$, $C_m=\sum_{i=0}^{m}c_iu^{m-i}v^i$ and $G_m=\sum_{i=0}^{m}g_iu^{m-i}v^i$ are homogeneous polynomials of degree m and $\mathrm{gdc}(A_m,B_m,C_m,G_m)=1$ and $\alpha,\beta,\gamma\in\mathbb{C}$ and also $a_i,b_i,c_i\in\mathbb{C}$. By construction $\mathscr{C}_V\subset\mathrm{Sing}(\theta_t)$ for all $t\in\mathbb{D}$.
Let $\pi:\widetilde{U}\longrightarrow U$ be the blowing up of U centered on w-axis with exceptional divisor E. In the chart $\sigma_1(s)=(u,v,w)$ and also using (2.15), (2.19) and (2.18) for the functions L, M and N we have that

$$\begin{aligned}\pi^*\widetilde{\theta}_t&=\left(s_1^m\left[\sum_{i=0}^{m}s_2^iL_i(s_3)+s_1L_1(s)\right]\right)ds_1+\left(s_1^m\left[\sum_{i=0}^{m}s_2^iM_i(s_3)+s_1M_1(s)\right]\right)(s_2ds_1+s_1ds_2)\\&\quad+\left(s_1^{n_1}\left[\sum_{i=0}^{n_1}s_2^iN_i(s_3)+s_1N_1(s)\right]\right)ds_3+t\left[s_1^m a(s_2)+(\alpha s_1+\beta s_3)s_1^m g(s_2)\right]ds_1\\&\quad+t\left[s_1^m b(s_2)+(\gamma s_3-\alpha s_1)g(s_2)\right](s_1ds_2+s_2ds_1)+t\left[s_1^m c_{n_1}(s_2)-s_1^{m+1}(\beta+\gamma s_2)g(s_2)\right]ds_3.\end{aligned}$$

where $a(s_2)=\sum_{i=0}^{m} a_i s_2^i,\quad b(s_2)=\sum_{i=0}^{m} b_i s_2^i,\quad c(s_2)=\sum_{i=0}^{m} c_i s_2^i,\quad g(s_2)=\sum_{i=0}^{m} g_i s_2^i.$

$$\begin{aligned}\pi^*\widetilde{\theta}_t = &\left(s_1^m\left[\sum_{i=0}^{m} s_2^i L_i(s_3)+s_1L_1(s)\right]+s_1^m s_2\left[\sum_{i=0}^{m_1} s_2^i M_i(s_3)+s_1M_1(s)\right]\right.\\ &\left.+ts_1^m\left[(a(s_2)+s_2b(s_2))+s_3(\gamma s_2+\beta)g(s_2)\right]\right)ds_1\\ &+\left(s_1^{m+1}\left[\sum_{i=0}^{m} s_2^i M_i(s_3)+s_1M_1(s)\right]+ts_1^{m+1}\left[b(s_2)+(\gamma s_3-\alpha s_1)g(s_2)\right]\right)ds_2\\ &+\left(s_1^{n_1}\left[\sum_{i=0}^{n_1} s_2^i N_i(s_3)+s_1N_1(s)\right]+ts_1^m\left[c(s_2)-s_1(\beta+\gamma s_2)g(s_2)\right]\right)ds_3.\end{aligned}\tag{2.52}$$

$$\begin{aligned}\pi^*\widetilde{\theta}_t = s_1^m&\left(\sum_{i=0}^{m}\left(s_2^i(L_i(s_3)+s_2M_i(s_3)\right)+s_1\left(L_1(s)+s_2M_1(s)\right)\right.\\ &\left.+t\left[\left(a(s_2)+s_2b(s_2)\right)+s_3\left(\gamma s_2+\beta)g(s_2\right)\right]\right)ds_1\\ &+s_1^{m+1}\left(\left[\sum_{i=0}^{m} s_2^i M_i(s_3)+s_1M_1(s)\right]+t\left[b(s_2)+(\gamma s_3-\alpha s_1)g(s_2)\right]\right)ds_2\\ &+\left(s_1^{n_1}\left(\sum_{i=0}^{n_1}(N_i(s_3)+s_1N_1(s)\right)+ts_1^m\left[c(s_2)-s_1(\beta+\gamma s_2)g(s_2)\right]\right)ds_3.\end{aligned}\tag{2.53}$$

Dividing (2.53) by s_1^m, we have

$$\begin{aligned}\widetilde{\theta}_t = &\left[\sum_{i=0}^{m} s_2^i\big(L_i(s_3)+ta_i)+s_2(M_i(s_3)+tb_i\big)+s_1\big(L_1(s)+s_2M_1(s)\big)+ts_3\big(\gamma s_2+\beta\big)g(s_2)\right]ds_1\\ &+s_1\left(\left[\sum_{i=0}^{m} s_2^i M_i(s_3)+s_1M_1(s)\right]+t\left[b(s_2)+(\gamma s_3-\alpha s_1)g(s_2)\right]\right)ds_2\\ &+\left(s_1^{n_1-m}\left(\sum_{i=0}^{n_1}(N_i(s_3)+s_1N_1(s)\right)+t\left[c(s_2)-s_1(\beta+\gamma s_2)g(s_2)\right]\right)ds_3.\end{aligned}$$

In order to determine the singular set of $\widetilde{\theta}_t$ we have two situations to consider. To namely non-dicritical and dicritical curves of singularities.

First Situation: Non-dicritical curve of singularities.

If $m=m_1$ e $\left[\sum_{i=0}^{m} L_i(s_3)+s_2M_i(s_3)\right]\not\equiv 0.$

We have two sub-cases

a) $m = n_1$.

The singularities on the exceptional divisor $E = \{s_1 = 0\}$ are the solutions of the system of equations below

$$\begin{cases} \sum_{i=0}^{m} s_2^i\Big((L_i(s_3)+ta_i)+s_2(M_i(s_3)+tb_i)\Big)+ts_3\big(\gamma s_2+\beta\big)g(s_2)\Big)=0, \\ \sum_{i=0}^{m}\Big(N_i(s_3)+tc_i(s_2)\Big)=0. \end{cases} \tag{2.54}$$

Generically, since there are two equations and two unknowns, the system (2.54) has isolated singularities over E.

b) $n_1 > m$.

$$\begin{cases} \sum_{i=0}^{m}\Big(L_i(s_3)+s_2M_i(s_3)\Big)s_2^i+t\Big[(a(s_2)+s_2b(s_2))+s_3(\gamma s_2+\beta)g(s_2)\Big]=0, \\ tc(s_2)=0. \end{cases} \tag{2.55}$$

where $a(s_2)$, $b(s_2)$, $c(s_2)$ and $g(s_2)$ are polynomials of degree m in variable s_2 written as (2.34), (2.35), (2.36) and (2.37), respectively.

Analogous to the previous item, in a generic manner, on the exceptional divisor E we have only isolated singularities. The other case, $m > n_1$ is discussed in a manner analogous to this.

<u>Second Situation</u>: Dicritical curve of singularities.

If $m = m_1$ e $\left[\sum_{i=0}^{m} L_i(s_3)+s_2M_i(s_3)\right] \equiv 0$.

Again, we will analyze the following two sub-cases:

a) $n_1 > m$.

$$\begin{cases} \sum_{i=0}^{m} s_2^i\Big((L_i(s_3)+ta_i)+s_2(M_i(s_3)+tb_i)\Big)+s_3\Big(\gamma s_2+\beta\Big)g(s_2)\Big)=0 \\ tc_m(s_2)=0 \end{cases} \tag{2.56}$$

But, in this situation , for appropriate choices of A_m, B_m and C_m, we have

$$\sum_{i=0}^{m} s_2^i\Big((L_i(s_3)+ta_i)+s_2(M_i(s_3)+tb_i\Big)=t\Big(a(s_2+s_2b(s_2)\Big)$$

i.e. $\widetilde{\theta_t}$ is non-dicritical distribution for all $t \in \mathbb{D}$.

From the first equation of (2.56), we have

$$s_3 = f(s_2) = -\frac{\Big(a_m(s_2) + s_2 b_m(s_2)\Big)}{(\beta + \gamma s_2) g_m(s_2)}. \tag{2.57}$$

The second equation of (2.56) gives m-solutions $s_2 = s_2^j$, with $j = 1, 2, \cdots, m$. Therefore, on the exceptional divisor, we generically have only isolated singularities.

b) $n_1 = m$.

In this case, the singularities over the exceptional divisor E are given by solutions of the system

$$\begin{cases} t\Big[(a_m(s_2) + s_2 b_m(s_2)) + s_3(\gamma s_2 + \beta) g_m(s_2)\Big] = 0, \\ \sum_{i=0}^{m} (s_2^i N_i(s_3) + t c_i(s_2)) = 0. \end{cases} \tag{2.58}$$

From the first equation of (2.58), we have again, the equation (2.57).
Now, by replacing (2.57) in the second equation of the same system, we get the following analytic function:

$$\Psi(s_2) = \sum_{i=0}^{m} (s_2^i N_i(f(s_2) + t c_i(s_2)),$$

which zero set contains only isolated points.

In case it is necessary to disturb the coefficients, we can assume that ω_t is special along $\mathscr{C}$. Let $\omega_t = F^*\theta_t$ be a polynomial 1-form on U. By Hartog's Theorem we can extend it to all $\mathbb{P}^3$.

$$
\begin{aligned}
\omega_t = & \left(\sum_{i=0}^{\ell_1} f(z)^{\ell_1 - i} g(z)^i L_i(f(z), g(z), z_3)\right)\left(\sum_{i=1}^{3} \frac{\partial f}{\partial z_i} dz_i\right) \\
& + t\left[A_{n_1}(f(z), g(z)) + (\alpha g(z) + \beta z) G_{n_1}(f(z), g(z))\right]\left(\sum_{i=1}^{3} \frac{\partial f}{\partial z_i} dz_i\right) \\
& + \left(\sum_{i=0}^{m_1} f(z)^{m_1 - i} g(z)^i M_i(f(z), g(z), z)\right)\left(\sum_{i=1}^{3} \frac{\partial g}{\partial z_i} dz_i\right) \\
& + t\left[B_{n_1}(f(z), g(z)) + (\gamma z_3 - \alpha f(z)) G_{n_1}(f(z), g(z))\right]\left(\sum_{i=1}^{3} \frac{\partial g}{\partial z_i} dz_i\right) \\
& + \left(\sum_{i=0}^{n_1} f(z)^{n_1 - i} g(z)^i N_i(f(z), g(z), z_3)\right) dz_3 \\
& + t\left[C_{n_1}(f(z), g(z)) - (\beta f(z) + \gamma g(z)) G_{n_1}(f(z), g(z))\right] dz_3.
\end{aligned} \tag{2.59}
$$

By construction the scalars α, β and γ are chosen in order to $\deg(\mathscr{F}_t) = \deg(\mathscr{F})$, for $t \in \mathbb{D} \setminus \{0\}$, showing (1). The affirmative (2) is immediate. Shrinking ε if necessary we can admit that ω_t is special along $\mathscr{C}$ suffice for that a perturbation of the coefficients a_i, b_i and c_i.

Also by constrution we have $\mathrm{mult}_{\mathscr{C}}(\mathscr{F}) = \mathrm{mult}_{\mathscr{C}}(\mathscr{F}_t)$ and $\mathrm{mult}_E(\pi^*\mathscr{F}) = \mathrm{mult}_E(\pi^*\mathscr{F}_t)$ if $\mathscr{F}$ is non-dicritical and $\mathrm{mult}_E(\pi^*\mathscr{F}) - 1 = \mathrm{mult}_E(\pi^*\mathscr{F}_t)$ if $\mathscr{F}$ is dicritical distribution.

■

In the $\widetilde{\mathbb{P}^3}$ the normal bundle of the distribution $\widetilde{\mathscr{F}}$ is determined by

$$
N^{\vee}_{\widetilde{\mathscr{F}}} \simeq \pi^*(N^{\vee}_{\mathscr{F}}) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(\ell E). \tag{2.60}
$$

where the symbol $\vee$ denotes the dual of a sheaf and ℓ is the order annulment of the pullback distribution $\pi^*(\mathscr{F})$ at E and π the blow up morphism with exceptional divisor E.

Lemma 2.15. *Let $\mathscr{F}$ be a non integrable codimension one distribution on $\mathbb{P}^3$ of degree d whose singular set is a disjoint union of proper subsets $Sing(\mathscr{F}) = \mathscr{C} \cup \{p_1, \cdots, p_n\}$. If $\mathscr{F}$ is special along $\mathscr{C}$ then*

$$\sum_{i=1}^{n} Res(\mathscr{F},p_i) \geq d^3+2d^2+2d-(\ell+3)\chi(\mathscr{C})+deg(\mathscr{C})\Big[(d+2)(3-3\ell^2)+4\ell(\ell+1)\Big]$$
$$+\left(3\ell-\ell^3\right)\left(\chi(\mathscr{C})-4deg(\mathscr{C})\right).$$

where $deg(\mathscr{C})$ and $\chi(\mathscr{C})$ denote, the degree, Euler characteristic of the curve $\mathscr{C}$ respectivelly and ℓ is the annulment order of $\pi^*\mathscr{F}$ on the exceptional divisor E.

Proof. By (2.60), we have

$$N^{\vee}_{\widetilde{\mathscr{F}}} \simeq \pi^*(N^{\vee}_{\mathscr{F}})\otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(\xi E).$$

By Lemma 2.14 (of Perturbation) item (5), we can write

$$\xi = \begin{cases} \ell = \mathrm{mult}_E(\pi^*\mathscr{F}), & \text{if } \mathscr{F} \text{ is non-dicritical,} \\ \ell - 1 = \mathrm{mult}_E(\pi^*\mathscr{F}), & \text{if } \mathscr{F} \text{ is dicritical.} \end{cases}$$

The dicritical case, just replace ℓ by $\ell-1$.

Let $\pi:\widetilde{\mathbb{P}^3}\to\mathbb{P}^3$ be blow-up morphism of $\mathbb{P}^3$ along $\mathscr{C}$. By hypothesis, ω is special along $\mathscr{C}$ and by definition, we have that $\mathrm{Sing}(\widetilde{\omega})$ is finite. By Theorem 2.6 (of Izawa's) we can write

$$\sum_{\widetilde{p_i}\in \mathrm{Sing}(\widetilde{\mathscr{F}})} \mathrm{Res}(\widetilde{\omega},\widetilde{p_i}) = \int_{\widetilde{\mathbb{P}^3}} c_3(\Omega^1_{\widetilde{\mathbb{P}^3}}\otimes\mathscr{L}). \tag{2.61}$$

$$\begin{aligned}\mathscr{L} &\simeq N_{\widetilde{\mathscr{F}}}\\ &\simeq N_{\widetilde{\mathscr{F}}} \simeq \pi^*(\mathscr{O}_{\mathbb{P}^3}(d+2))\otimes\mathscr{O}_{\widetilde{\mathbb{P}^3}}(-\ell E).\end{aligned} \tag{2.62}$$

But, as $\pi:\widetilde{\mathbb{P}^3}\backslash E \longrightarrow \mathbb{P}^3\backslash\mathscr{C}$ is a biholomorphism, we have

$$\sum \mathrm{Res}(\omega,p_i) = \sum_{p_i\notin E}\mathrm{Res}(\widetilde{\omega},\widetilde{p_i}) = \sum\mathrm{Res}(\widetilde{\omega},\widetilde{p_i}) - \sum_{p_i\in E}\mathrm{Res}(\widetilde{\omega},\widetilde{p_i}). \tag{2.63}$$

Let $\widetilde{\mathscr{F}_1} = \widetilde{\mathscr{F}|_E}$ be restriction of distribution $\widetilde{\mathscr{F}}$ over the exceptional divisor E.
The annihilator (see [21], page 178) of $\widetilde{\mathscr{F}_1}$ is defined by

$$\widetilde{\mathscr{F}_1^{(a)}} = \{v\in\mathfrak{X}(T_E);\langle v,\widetilde{\omega}\rangle = 0\}, \tag{2.64}$$

we therefore have the following inclusion

$$\mathrm{Sing}(\widetilde{\omega})|_E \subset \mathrm{Sing}(\widetilde{\mathscr{F}_1^{(a)}}). \tag{2.65}$$

Indeed, if $p \in \mathrm{Sing}(\widetilde{\omega})|_E$ then $\widetilde{\omega}(p) = 0$, thus $\langle v(p), \widetilde{\omega}(p)\rangle = 0$.
So:

$$\sum_{\substack{i=1 \\ p_i \in E}}^{n} \mathrm{Res}(\widetilde{\omega}, \widetilde{p_i}) \leq \int_E c_2(\Omega_E^1 \otimes \mathscr{L}). \tag{2.66}$$

So we can rewrite the equation, (2.63) as

$$\sum_{i=1}^{n} \mathrm{Res}(\omega, p_i) \geq \int_{\widetilde{\mathbb{P}^3}} c_3(\Omega^1_{\widetilde{\mathbb{P}^3}} \otimes \mathscr{L}) - \int_E c_2(\Omega_E^1 \otimes \mathscr{L}). \tag{2.67}$$

We now calculate each the integrals above. For this, we use the following equalities that can be found at [6].

I) $\int_{\widetilde{\mathbb{P}^3}} \pi^*\mathbf{h}^3 = \int_{\mathbb{P}^3} \mathbf{h}^3 = 1$,

II) $\int_{\widetilde{\mathbb{P}^3}} \pi^*\mathbf{h}^2 \cdot \mathbf{E} = \int_E \pi^*\mathbf{h}^2 = \int_{\mathscr{C}} \mathbf{h}^2 = 0$,

III) $\int_{\widetilde{\mathbb{P}^3}} \pi^*\mathbf{h} \cdot \mathbf{E}^2 = \int_E \pi^*\mathbf{h} \cdot \mathbf{E} = (-1)\int_{\mathscr{C}} \mathbf{h} = -\deg(\mathscr{C})$,

IV) $\int_{\widetilde{\mathbb{P}^3}} \mathbf{E}^3 = \int_E \mathbf{E}^2 = \chi(\mathscr{C}) - 4\deg(\mathscr{C}) = (2 - 2g - 4\deg(\mathscr{C}))$, where g denotes the genus of the curve $\mathscr{C}$.

$$c_3\left(\Omega^1_{\widetilde{\mathbb{P}^3}} \otimes \mathscr{L}\right) = c_3(\Omega^1_{\widetilde{\mathbb{P}^3}}) + c_2(\Omega^1_{\widetilde{\mathbb{P}^3}}).c_1(\mathscr{L}) + c_1(\Omega^1_{\widetilde{\mathbb{P}^3}}).c_1^2(\mathscr{L}) + c_1^3(\mathscr{L}). \tag{2.68}$$

From equation (2.60) we can write

1) $c_1(\mathscr{L}) = (d+2)\pi^*\mathbf{h} - \ell\mathbf{E}$,

2) $c_1^2(\mathscr{L}) = (d+2)^2\pi^*\mathbf{h}^2 - 2(d+2)\ell\pi^*\mathbf{h}\cdot\mathbf{E} + \ell^2\mathbf{E}^2$,

3) $c_1^3(\mathscr{L}) = (d+2)^2\pi^*\mathbf{h}^3 - 3(d+2)^2\ell\pi^*\mathbf{h}^2.\mathbf{E} + 3(d+2)\ell^2\pi^*\mathbf{h}\cdot\mathbf{E}^2 - \ell^3\mathbf{E}^3$.

$$\int_{\widetilde{\mathbb{P}^3}} c_1^3(\mathscr{L}) = (d+2)^3\int_{\widetilde{\mathbb{P}^3}} \pi^*\mathbf{h}^3 - 3(d+2)^2\ell\int_{\widetilde{\mathbb{P}^3}} \pi^*\mathbf{h}^2\cdot\mathbf{E} + 3(d+2)\ell^2\int_{\widetilde{\mathbb{P}^3}} \pi^*\mathbf{h}\cdot\mathbf{E}^2 - \ell^3\int_{\widetilde{\mathbb{P}^3}} \mathbf{E}^3,$$
$$\int_{\widetilde{\mathbb{P}^3}} c_1^3(\mathscr{L}) = (d+2)^3 - 3(d+2)\ell^2\deg(\mathscr{C}) - \ell^3(2 - 2g - 4\deg(\mathscr{C})). \tag{2.69}$$

From (1.1) we get $c_1(\widetilde{\mathbb{P}^3})$. The next term of (2.68) is given by

$$c_1(\Omega_{\widetilde{\mathbb{P}^3}})\cdot c_1^2(\mathscr{L}) = \Big[4\pi^*\mathbf{h}-\mathbf{E}\Big]\cdot\Big[(d+2)^2\pi^*\mathbf{h}^2 - 2\ell(d+2)\pi^*\mathbf{h}\cdot\mathbf{E}+\ell^2\cdot\mathbf{E}^2\Big]. \tag{2.70}$$

By integrating each term of the equation (2.70) , we have

$$\int_{\widetilde{\mathbb{P}^3}} c_1(T_{\widetilde{\mathbb{P}^3}})\cdot c_1^2(\mathscr{L}) = 4(d+2)^2 - 4\ell^2\deg(\mathscr{C}) - (2-2g-4\deg(\mathscr{C}))\,\ell^2 - 2\ell(d+2)\deg(\mathscr{C}). \tag{2.71}$$

Now, integrating $c_2(T_{\widetilde{\mathbb{P}^3}})\cdot c_1(\mathscr{L})$ and using (1.2) for $c_2(\widetilde{T\mathbb{P}^3})$, we have

$$c_2(T_{\widetilde{\mathbb{P}^3}})\cdot c_1(\mathscr{L}) = \Big[6\pi^*\mathbf{h}^2\cdot\mathbf{E}^2 - \pi_E^* c_1(T_{\mathscr{C}})\cdot\mathbf{E}\Big]\cdot\Big[(d+2)\pi^*\mathbf{h}-\ell\cdot\mathbf{E}\Big]. \tag{2.72}$$

we have

$$\int_{\widetilde{\mathbb{P}^3}} c_2(T_{\widetilde{\mathbb{P}^3}})\cdot c_1(\mathscr{L}) = 6(d+2)+(d+2)\deg(\mathscr{C})+\ell(2-2g-4\deg(\mathscr{C}))-\ell(2-2g) \tag{2.73}$$

Finally, using the (1.3) and integrating $c_3(T_{\widetilde{\mathbb{P}^3}})$, we get

$$\int_{\widetilde{\mathbb{P}^3}} c_3(\widetilde{T\mathbb{P}^3}) = 4+\chi(\mathscr{C}). \tag{2.74}$$

Therefore

$$\begin{aligned}\int_{\widetilde{\mathbb{P}^3}} c_3\left(\Omega^1_{\widetilde{\mathbb{P}^3}}\otimes\mathscr{L}\right) &= d^3+2d^2+2d+\left(\ell+\ell^2-\ell^3\right)\left(\chi(\mathscr{C})-4\deg(\mathscr{C})\right)-(\ell+1)\,\chi(\mathscr{C})\\ &\quad +(d+2)\deg(\mathscr{C})\left(2\ell+-3\ell^2+1\right)+4\ell^2\deg(\mathscr{C}).\end{aligned} \tag{2.75}$$

In order to determine the singularities on the exceptional divisor E we will computation the following integral

$$\int_E c_2\left(\Omega^1_E\otimes\mathscr{L}\right) = \int_E c_2\left(\Omega^1_E\right)+\int_E c_1\left(\Omega^1_E\right)c_1(\mathscr{L})+\int_E c_1(\mathscr{L})^2. \tag{2.76}$$

We know that $c_1(T_{\widetilde{\mathbb{P}^3}}) = c_1(T_{\widetilde{\mathscr{F}}})+c_1(N_{\widetilde{\mathscr{F}}})$. Now, restricting the exceptional divisor, we can write

$$c_1(T_{\widetilde{\mathbb{P}^3}})|_E = c_1(T_E)+c_1(N_E) = c_1(T_E)+[E] = c_1(T_E)+E.$$

So,

$$\pi^* c_1(T_{\mathbb{P}^3}) - \mathbf{E} = c_1(T_E) + \mathbf{E},$$

and

$$c_1(T_E) = 4\pi^*\mathbf{h} - 2\mathbf{E}. \tag{2.77}$$

From the Whitney Formula we have

$$\begin{aligned} c(T_{\widetilde{\mathbb{P}^3}}) &= c(T_E)\cdot c(E) \\ &= (1 + c_1(T_E) + c_2(T_E))(1 + c_1(E)). \end{aligned} \tag{2.78}$$

Then

$$\begin{aligned} c_2(T_{\widetilde{\mathbb{P}^3}}) &= c_1(T_E)\cdot \mathbf{E} + c_2(T_E) \\ c_2(T_E) &= \pi^* c_2(T_{\widetilde{\mathbb{P}^3}})F - \pi^* c_1(T_{\mathscr{C}})\cdot \mathbf{E} - 4\pi^*\mathbf{h}\cdot\mathbf{E} + 2\mathbf{E}^2 \\ &= \pi^* c_2(T_{\widetilde{\mathbb{P}^3}}) - \pi^* c_1(T_{\mathscr{C}})\cdot \mathbf{E} - 4\pi^*\mathbf{h}\cdot\mathbf{E} + \mathbf{E}^2. \end{aligned} \tag{2.79}$$

Integrating each term of equation (2.79), we have

$$\int_E c_2(T_E) = 6\int_E \pi^*\mathbf{h}^2 - \int_E \pi^*(T_{\mathscr{C}})\cdot\mathbf{E} - 4\int_E \mathbf{h}\cdot\mathbf{E} + \int_E \mathbf{E}^2. \tag{2.80}$$

Then

$$\int_E c_2(T_E) = \chi(\mathscr{C}) - 4\deg(\mathscr{C}) + \chi(\mathscr{C}) + 4\deg(\mathscr{C}) = 2\chi(\mathscr{C}). \tag{2.81}$$

From (2.77) we can write

$$c_1(T_E)\cdot c_1(\mathscr{L}) = \left(4\pi^*\mathbf{h} - 2\mathbf{E}\right)\cdot\left((d+2)\pi^*\mathbf{h} - \ell\mathbf{E}\right). \tag{2.82}$$

Now by integrating each terms of the equations (2.82) and using (1) we have

$$\int_E c_1(T_E)\cdot c_1(\mathscr{L}) = 4(d+2)\int_E \pi^*\cdot\mathbf{h}^2 - 4\ell\int_E \pi^*\mathbf{h}\cdot\mathbf{E} - 2(d+2)\int_E \pi^*\mathbf{h}\cdot\mathbf{E} + 2\ell\int_E \mathbf{E}^2. \tag{2.83}$$

So

$$\int_E c_1(T_E) = 4\ell\deg(\mathscr{C}) + 2(d+2)\deg(\mathscr{C}) + 2\ell\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right). \tag{2.84}$$

$$c_1^2(\mathscr{L}) = (d+2)^2\pi^*\mathbf{h}^2 - 2(d+2)\ell\pi^*\mathbf{h}\cdot\mathbf{E} + \ell^2\mathbf{E}^2. \tag{2.85}$$

By integrating each term of the equation (2.85), and again using (1) we have

$$\int_E c_1^2(\mathscr{L}) = 2\ell(d+2)\deg(\mathscr{C}) + \ell^2\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right). \tag{2.86}$$

So from the equations (2.85) and (2.83) we get the equality

$$\begin{aligned}\int_E c_2(\Omega_E^1\otimes\mathscr{L}) &= 2\chi(\mathscr{C}) - 4\deg(\mathscr{C}) - 2(d+2)\deg(\mathscr{C}) - 2\ell\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right)\\ &\quad + 2\ell(d+2)\deg(\mathscr{C}) + \ell^2\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right).\end{aligned} \tag{2.87}$$

Therefore, from (2.67) we get the inequality

$$\begin{aligned}\sum_{i=1}^n \mathrm{Res}(\mathscr{F},p_i) \geq d^3 + 2d^2 + 2d - (\ell+3)\chi(\mathscr{C}) + \deg(\mathscr{C})\Big[(d+2)(3-3\ell^2) + 4\ell(\ell+1)\Big]\\ + \left(3\ell - \ell^3\right)\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right).\end{aligned}$$

■

2.4 Non-Special Holomorphic Distributions along a Regular Curve

The main result of this section will tell us that we can display an upper bound for the residue of a singular holomorphic distribution along a curve. Before enunciating this result, we shall define this residue. After that, we will demonstrate that the residue of a distribution $\mathscr{F}$ along a curve of singularities is well defined.

Definition 2.16. *Let $\mathscr{F}$ be a codimension one distribution on $\mathbb{P}^3$ of degree d which singular set is a disjoint union of proper closed subsets*

$$Sing(\mathscr{F}) = \mathscr{C} \cup \{p_1,\ldots,p_n\},$$

where $\mathscr{C}$ is a smooth irreducible curve and closed points $p_1,\ldots,p_n$. We define the residue of $\mathscr{F}$ along $\mathscr{C}$ by

$$Res(\omega,\mathscr{C}) := \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} p_i^t \in \mathscr{C}} Res(\omega_t, p_i^t), \tag{2.88}$$

where ω_t is a generic perturbation of ω.

Proposition 2.17. *The residue $Res(\mathscr{F},\mathscr{C})$ is well defined.*

Proof. Let ω_1^t and ω_2^t be two generic perturbations of ω with $\deg(\omega_1^t) = \deg(\omega_2^t) = d$ such that:

$$\mathrm{Sing}(\omega_1^t) = \{p_1^t, \ldots, p_k^t\},$$

$$\mathrm{Sing}(\omega_2^t) = \{q_1^t, \ldots, q_s^t\}.$$

So, we can write

$$\begin{aligned} \mathrm{Res}(\omega, p_i) :&= \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} p_i^t \notin \mathscr{C}} \mathrm{Res}(\omega_1^t, p_i^t) \\ &= \sum \mathrm{Res}(\omega_1^t, p_i^t) - \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} p_i^t \in \mathscr{C}} \mathrm{Res}(\omega_1^t, p_i^t). \end{aligned} \tag{2.89}$$

We can also write

$$\begin{aligned} \sum \mathrm{Res}(\omega, p_i) :&= \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} q_j^t \notin \mathscr{C}} \mathrm{Res}(\omega_2^t, q_j^t) \\ &= \sum \mathrm{Res}(\omega_2^t, q_j^t) - \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} q_j^t \in \mathscr{C}} \mathrm{Res}(\omega_2^t, q_j^t). \end{aligned} \tag{2.90}$$

Subtracting equations (2.89) and (2.90), we have

$$\lim_{t\to 0} \sum_{\lim\limits_{t\to 0} q_j^t \in \mathscr{C}} \mathrm{Res}(\omega_2^t, q_j^t) = \lim_{t\to 0} \sum_{\lim\limits_{t\to 0} p_i^t \in \mathscr{C}} \mathrm{Res}(\omega_1^t, p_i^t). \tag{2.91}$$

By Izawa's Theorem we have

$$\sum \mathrm{Res}(\omega_1^t, p_i^t) = \sum \mathrm{Res}(\omega_2^t, q_j^t) = d^3 + 2d^2 + 2d.$$

since by hypothesis we have $\deg(\omega_1^t) = \deg(\omega_2^t) = d$. Thus, the residue along curve $\mathscr{C}$ is well defined.

∎

Theorem 2.18. *Let $\mathscr{F}$ be a non integrable codimension one distribution on $\mathbb{P}^3$ of degree d which singular set is a disjoint union of proper closed subsets*

$$Sing(\mathscr{F}) = \mathscr{C} \cup \{p_1, \ldots, p_n\},$$

where $\mathscr{C}$ is a smooth irreducible curve and closed points $p_1, \ldots, p_n$. Then

i)

$$\sum_{i=1}^{n} Res(\mathscr{F}, p_i) \geq d^3 + 2d^2 + 2d - (\ell+3)\,\chi(\mathscr{C}) + deg(\mathscr{C})\Big[(d+2)(3-3\ell^2) + 4\ell(\ell+1)\Big] + \left(3\ell - \ell^3\right)\left(\chi(\mathscr{C}) - 4deg(\mathscr{C})\right) - N_G,$$

ii)

$$Res(\mathscr{F}, \mathscr{C}) \leq (\ell+3)\,\chi(\mathscr{C}) - deg(\mathscr{C})\Big[(d+2)(3-3\ell^2) + 4\ell(\ell+1)\Big] - \left(3\ell - \ell^3\right)\left(\chi(\mathscr{C}) - 4deg(\mathscr{C})\right) + N_G,$$

where $deg(\mathscr{C})$, $\chi(\mathscr{C})$, denote respectively , the degree and Euler characteristic of $\mathscr{C}$ and N_G the number of embedded closed points of $\mathscr{C}$ counted with multipilicities.

Proof. Let $\mathscr{F}$ be a codimension one distribution on $\mathbb{P}^3$ of degree d induced by global section $\omega \in H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(d+2))$ and $\pi : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ the blow up morphism of $\mathbb{P}^3$ along $\mathscr{C}$ with exceptional divisor E. Suppose that $\mathscr{F}$ is non-dicritical. The dicritical case, just replace ℓ by $\ell - 1$.

Again, we have two cases to consider.

i) First case: If $\mathscr{F}$ is special along $\mathscr{C}$, then the result follows from lemma 2.15 doing $N_G = 0$.

Second case : If $\mathscr{F}$ is not special along $\mathscr{C}$, then there exists a special distribution along $\mathscr{C}$ induced by ω_t for all $t \in \mathbb{D}^*$. Then

$$\sum \mathrm{Res}(\omega, p_i) = \sum_{\lim_{t\to 0} p_i^t \notin \mathscr{C}} \mathrm{Res}(\omega_t, p_i^t). \tag{2.92}$$

Since that $\pi : \widetilde{\mathbb{P}^3} \setminus E \longrightarrow \mathbb{P}^3 \setminus \mathscr{C}$ is a biholomorphism, we can write

$$\sum_{\lim_{t\to 0} \widetilde{p_i^t} \notin \mathscr{C}} \mathrm{Res}(\omega_t, p_i^t) = \sum_{\lim_{t\to 0} \widetilde{p_i^t} \notin E} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}). \tag{2.93}$$

$$\sum_{\lim_{t\to 0} \widetilde{p_i^t} \notin E} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}) = \sum_{\widetilde{p_i^t} \in \mathrm{Sing}(\widetilde{\omega_t})} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}) - \sum_{\widetilde{p_i^t} \in E} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}) - \sum_{\substack{\lim_{t\to 0} \widetilde{p_i^t} \in E \\ \widetilde{p_i^t} \notin E}} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}). \tag{2.94}$$

As ω_t is special along $\mathscr{C}$, from (2.61) e (2.66), we get

$$\sum_{\widetilde{p_i^t} \in \mathrm{Sing}(\widetilde{\omega_t})} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}) = \int_{\widetilde{\mathbb{P}^3}} c_3(\Omega_{\widetilde{\mathbb{P}^3}} \otimes \mathscr{L}). \tag{2.95}$$

$$\sum_{\widetilde{p_i^t} \in E} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}) \leq \int_E c_2(\Omega_{\widetilde{\mathbb{P}^3}} \otimes \mathscr{L}). \tag{2.96}$$

Using (2.67) (2.93), (2.95) , (2.96) and doing

$$N_G = \sum_{\substack{\lim_{t\to 0} \widetilde{p_i^t} \in E \\ \widetilde{p_i^t} \notin E}} \mathrm{Res}(\widetilde{\omega_t}, \widetilde{p_i^t}). \tag{2.97}$$

we get the inequality

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{Res}(\mathscr{F}, p_i) \geq{}& d^3 + 2d^2 + 2d + \left(\ell + \ell^2 - \ell^3\right)\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right) - (\ell+1)\chi(\mathscr{C}) \\ &+ (d+2)\deg(\mathscr{C})\left(2\ell - 3\ell^2 + 1\right) + 4\ell^2 \deg(\mathscr{C}) - N_G. \end{aligned}$$

ii) Let ω_t be a generic perturbation of ω such that $\deg(\omega) = \deg(\omega_t)$. Now by Izawa's Theorem, we can write

$$\sum_{p_i^t \in \operatorname{Sing}(\omega_t)} \operatorname{Res}(\omega_t, p_i^t) = d^3 + 2d^2 + 2d. \tag{2.98}$$

We have two possibilities: $\lim_{t\to 0} p_i^t \not\in \mathscr{C}$ or $\lim_{t\to 0} p_i^t \in \mathscr{C}$.

By definition (2.16) we can write

$$\operatorname{Res}(\omega, \mathscr{C}) = \lim_{t\to 0} \sum_{\lim_{t\to 0} p_i^t \in \mathscr{C}} \operatorname{Res}(\omega_t, p_i^t). \tag{2.99}$$

If $\lim_{t\to 0} p_i^t \not\in \mathscr{C}$ then $\lim_{t\to 0} p_i^t = p_i$ for some i.

$$\sum_{i=1}^{k} \operatorname{Res}(\omega, p_i) = \lim_{t\to 0} \sum_{\lim_{t\to 0} p_j^t = p_i} \operatorname{Res}(\omega_t, p_j^t). \tag{2.100}$$

By adding the equations (2.99) and (2.100), we have

$$\operatorname{Res}(\mathscr{F}, \mathscr{C}) + \sum \operatorname{Res}(\omega, p_i) = \lim_{t\to 0} \sum_{\substack{j=1 \\ p_j^t \in \operatorname{Sing}(\omega_t, p_j^t)}}^{n} \operatorname{Res}(\omega_t) = d^3 + 2d^2 + 2d. \tag{2.101}$$

Thus:

$$\operatorname{Res}(\mathscr{F}, \mathscr{C}) = d^3 + 2d^2 + 2d - \sum_{i=1}^{k} \operatorname{Res}(\omega, p_i). \tag{2.102}$$

by the previous item (i), we can write

$$\begin{aligned} \operatorname{Res}(\mathscr{F}, \mathscr{C}) \leq (\ell+3)\chi(\mathscr{C}) - \deg(\mathscr{C})\Big[(d+2)(3-3\ell^2) + 4\ell(\ell+1)\Big] \\ - \left(3\ell - \ell^3\right)\left(\chi(\mathscr{C}) - 4\deg(\mathscr{C})\right) + N_G. \end{aligned}$$

we get, then the desired inequality, thus showing the item (ii).

■

Example 2.19 (Non-Dicritical Case). *Consider the non-integrable distribution $\mathscr{F}$ on $\mathbb{P}^3$ induced by*

$$\omega = (z_0^2 + z_1^2)dz_3 - z_3(z_0dz_0 + z_1dz_1) + z_1(z_0dz_2 - z_2dz_0).$$

See example 8.4 of [15].

First, as we can see the curve $\mathscr{C} = \left\{z_0 = z_1 = 0\right\} \subset Sing(\mathscr{F})$.

In the affine open set $U_3 = \left\{[z] \in \mathbb{P}^3,\ z_3 \neq 0\right\}$ with coordinates $x = \dfrac{z_0}{z_3}$, $y = \dfrac{z_1}{z_3}$ and $z = \dfrac{z_2}{z_3}$, the distribution is rewritten as

The singular set of $\mathscr{F}$ is given by following system

$$\omega = (-x - yz)dx - ydy + xydz.$$

$$\begin{cases} -x - yz = 0, \\ y = 0, \\ xy = 0. \end{cases}$$

It is not hard to see that $Sing(\mathscr{F}) \cap U_3$ is only the z-axis.

Now we looking for the singularities in the hyperplane at infinity $H_\infty = \{z_3 = 0\}$.

$$\begin{cases} z_0^2 + z_1^2 = 0, \\ z_1 = 0. \end{cases}$$

Thus, with $z_2 = 1$, we have $p = [0:0:1:0] \in Sing(\mathscr{F})$ with multiplicity equal to 2 and there are no other singularities. We also note that $p \in \mathscr{C}$. So

$$Sing(\mathscr{F}) = \left\{z_0 = z_1 = 0\right\} \cup \left\{2[0:0:1:0]\right\}.$$

Let $\pi : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ be the blow up centered at $\mathscr{C}$.

In the chart $\sigma_1(u) = (u_1, u_1u_2, u_3) = (x, y, z)$ we have

$$\begin{aligned}
\pi^*\omega &= (-u_1 - u_1u_2u_3)du_1 - u_1u_2\Big(u_2du_1 + u_1du_2\Big) + u_1^2u_2du_3 \\
&= (-u_1 - u_1u_2u_3 - u_1u_2^2)du_1 - u_1^2u_2du_2 + u_1^2u_2du_3 \\
&= u_1(-1 - u_2u_3 - u_2^2)du_1 - u_1^2u_2du_2 + u_1^2u_2du_3 \\
\widetilde{\omega} &= (-1 - u_2u_3 - u_2^2)du_1 - u_1u_2du_2 + u_1u_2du_3.
\end{aligned} \tag{2.103}$$

As $p(u_1,u_2,u_3) = 1 + u_2u_3 + u_2^2 \not\equiv 0$, $\mathscr{F}$ is non-dicritical distribution.
The singular set of $\mathscr{F}$ restricted to the exceptional divisor $E = \{u_1 = 0\}$ is given by

$$-1 - u_2u_3 - u_2^2 = 0,$$

i.e. there are no isolated singularities in E. Consequently ω does not induce a special distribution in $\mathbb{P}^3$ along $\mathscr{C}$.
By Lemma of Perturbation there exists a special distribution $\mathscr{F}_t$ along $\mathscr{C}$ induced by following 1-form :

$$\begin{aligned}
\omega_t = \omega &+ t\Big[A_1(x,y) + \big(\alpha y + \beta z\big)G_1(x,y)\Big]dx \\
&+ t\Big[B_1(x,y) + \big(\gamma z - \alpha x\big)G_1(x,y)\Big]dy \\
&+ t\Big[C_1(x,y) - \big(\beta x + \gamma y\big)G_1(x,y)\Big]dz.
\end{aligned}$$

for all $t \in D(0,\varepsilon)$. Thus

$$\begin{aligned}
\omega_t = &\Big(\big(-x - yz\big) + t\Big[A_1(x,y) + \big(\alpha y + \beta z\big)G_1(x,y)\Big]\Big)dx \\
&+ \Big(-y + t\Big[B_1(x,y) + \big(\gamma z - \alpha x\big)G_1(x,y)\Big]\Big)dy \\
&+ \Big(xy + t\Big[C_1(x,y) - \big(\beta x + \gamma y\big)G_1(x,y)\Big]\Big)dz.
\end{aligned}$$

where A_1, B_1, C_1 and G_1 are homogeneous polynomials of degree one and non identically null with $gdc(A_1,B_1,C_1,G_1) = 1$ and $\alpha,\beta,\gamma \in \mathbb{C}$.
In the chart σ_1, we have

$$\pi^*\omega_t = \Big(-u_1 - u_1u_2u_3 + t\Big[u_1a(u_2) + u_1\big(\alpha u_1u_2 + \beta u_3\big)g(u_2)\Big]\Big)dx$$
$$+\Big(-u_1u_2 + t\Big[u_1b(u_2) + u_1\big(\gamma u_3 - \alpha u_1\big)g(u_2)\Big]\Big)\Big(u_2du_1 + u_1du_2\Big)$$
$$+\Big(u_2u_1^2 + t\Big[u_1c(u_2) - u_1^2(\beta + \gamma\lambda)g(u_2)\Big]\Big)dz. \tag{2.104}$$

where $a(u_2) = A_1(1, u_2)$, $b(u_2) = B_1(1, u_2)$ *and* $c(u_2) = C_1(1, u_2)$. *in this way*

$$\pi^*\omega_t = u_1\Big(-1 - u_2u_3 - u_2^2 + t\Big(a(u_2) + u_2b(u_2)\Big) + tu_3\Big(\beta + \lambda\gamma\Big)g(u_2)\Big)du_1$$
$$+u_1^2\Big(-u_2 + t\Big(b(u_2) + \Big(\gamma u_3 - \alpha u_1\Big)g(u_2)\Big)du_2$$
$$+u_1\Big(u_1u_2 + t\Big(c(u_2) - u_1\Big(\beta + \gamma\lambda\Big)\Big)du_3. \tag{2.105}$$

Dividing (2.105) *by* u_1, *we get*

$$\widetilde{\omega_t} = \Big(-1 - u_2u_3 - u_2^2 + t\Big(a(u_2) + u_2b(u_2)\Big) + tu_3\Big(\beta + \lambda\gamma\Big)g(u_2)\Big)du_1$$
$$+u_1\Big(-u_2 + t\Big(b(u_2) + \Big(\gamma u_3 - \alpha u_1\Big)g(u_2)\Big)du_2$$
$$+\Big(u_1u_2 + t\Big(c(u_2) - u_1\Big(\beta + \gamma\lambda\Big)\Big)du_3. \tag{2.106}$$

In this case, the singularities over the exceptional divisor are given by the set of zero of the following equations system

$$\begin{cases} \Big(-1 - u_2u_3 - u_2^2 + t\Big(a(u_2) + u_2b(u_2)\Big) + tu_3\Big(\beta + \lambda\gamma\Big)g(u_2)\Big) = 0, \\ tc(u_2) = 0. \end{cases}$$

From the second equation, we have $u_2 = u_2^1$, *because* $t \neq 0$ *and since* $c(u_2)$ *is an affine linear function. By replacing this in the first equation, we have:*

$$u_3^1 = \frac{\Big(u_2^1 + 1\Big) - t\Big[a(u_2^1) + u_2^1b(u_2^1)\Big]}{t\Big(\beta + \gamma\lambda\Big)g(u_2^1) - u_2^1}$$

Thus, there exists at least one singularity over the exceptional divisor defined by $(0, u_2^1, u_3^1)$.

Let us suppose that $u_1 \neq 0$. In this case, the singular set of $\mathscr{F}_t$ is given by

$$\begin{cases} \Big(-1-u_2u_3-u_2^2+t\Big(a(u_2)+u_2b(u_2)\Big)+tu_3\Big(\beta+\lambda\gamma\Big)g(u_2)\Big)=0, \\ u_1\Big(-u_2+t\Big(b(u_2)+\Big(\gamma u_3-\alpha u_1\Big)g(u_2)\Big)=0, \\ \Big(u_1u_2+t\Big(c(u_2)-u_1\Big(\beta+\gamma\lambda\Big)\Big)=0. \end{cases}$$

From the third and first equations, respectively, of the system above, we can write

$$u_1\Big[u_2-t\Big(\beta+\gamma u_2\Big)g(u_2)\Big]=-tc(u_2). \tag{2.107}$$

$$u_3\Big[u_2-t\Big(\beta+\gamma u_2\Big)g(u_2)\Big]=-1-u_2^2+t\Big(a(u_2)+u_2b(u_2)\Big). \tag{2.108}$$

Now, from the second equation, we have

$$t\Big(\gamma u_3-\alpha u_1\Big)=u_2-tb(u_2). \tag{2.109}$$

From (2.107) *and* (2.108)*, we get*

$$tg(u_2)\Big[-\gamma-\gamma u_2^2+\gamma t\Big(a(u_2+u_2b(u_2)\Big)-\alpha tc(u_2)\Big]=\Big(u_2-tb(u_2)\Big)\Big[u_2-t\Big(\beta+\gamma u_2\Big)g(u_2)\Big]. \tag{2.110}$$

From the equality in (2.110) *we get a polynomial of degree two in the variable u_2. Therefore, in the complement of the exceptional divisor E we have two singularities which we will call p_1^t and p_2^t.*
Since ω_t is a polinomyal 1-form, the Hartogs' Theorem allows us to extend ω_t to all $\mathbb{P}^3$. So,

$$\begin{aligned} \omega_t = & \Big(-z_0z_3-z_1z_2+t\Big[z_3A_1(z_0,z_1)+\Big(\alpha z_1+\beta z_2\Big)G_1(z_0,z_1)\Big]\Big)dz_0 \\ & +\Big(-z_1z_3+t\Big[z_3B_1(z_0,z_1)+\Big(\gamma z_2-\alpha z_0\Big)G_1(z_0,z_1)\Big]\Big)dz_1 \\ & +\Big(z_0z_1+t\Big[z_3C_1(z_0,z_1)-\Big(\beta z_0+\gamma z_1\Big)G_1(z_0,z_1)\Big]\Big)dz_2 \\ & +\Big(z_0^2+z_1^2-t\Big[z_0A_1(z_0,z_1)+z_1B_1(z_0,z_1)+z_2C_1(z_0,z_1)\Big]\Big)dz_3. \end{aligned} \tag{2.111}$$

We will looking for the singularities in the hyperplane at infinity.

a) *let us do* $z_3 = 0$ *e* $z_2 = 1$.

In this case, we have

$$\begin{cases} -z_1 + t\Big(\alpha z_1 + \beta\Big)G_1(z_0, z_1) = 0, \\ t\Big(\gamma - \alpha z_0\Big)G_1(z_0, z_1) = 0, \\ z_0^2 + z_1^2 - t\Big[z_0 A_1(z_0, z_1) + z_1 B_1(z_0, z_1) + C_1(z_0, z_1)\Big] = 0. \end{cases}$$

As $G_1(z_0, z_1) \not\equiv 0$ *from the second equation we have* $z_0 = \dfrac{\gamma}{\alpha}$. *So, replacing this in the first and third equations of the system, we will obtain two equations of the second degree in the variable* z_1. *Generically these equations do not have the same roots. Therefore this system has no solution.*

b) *Let us do* $z_3 = z_2 = 0$ *and* $z_1 = 1$.

$$\begin{cases} \alpha t g(z_0) = 0, \\ z_0 - t\Big(\beta z_0 + \gamma)g(z_0) = 0, \\ \Big(1 - t a_0\Big)z_0^2 - t\Big(a_1 + b_0\Big)z_0 + t b_1 + 1 = 0. \end{cases}$$

Where $g(z_0) = G_1(z_0, 1)$, $A_1(z_0, 1) = a_0 z_0 + a_1$ *and* $B_1(z_0, 1) = b_0 z_0 + b_1$. *For sufficiently small* t *and fixed* a_0, a_1, b_0 *and* b_1 *the third equation has no solution. So this system has no solution.*

c) $z_3 = z_2 = z_1 = 0$ *and* $z_0 = 1$.

In this case, it is not difficult to see that the point $p = [1 : 0 : 0 : 0] \not\in Sing(\mathscr{F}_t)$.

Thus

$$Sing(\omega_t) = \mathscr{C} \cup \Big\{p_1^t, p_2^t\Big\}. \tag{2.112}$$

We know that $Sing(\mathscr{F}) = \Big\{z_0 = z_1 = 0\Big\} \cup \Big\{2[0 : 0 : 1 : 0]\Big\}$ *and* $\pi : \widetilde{\mathbb{P}^3} \setminus E \longrightarrow \mathbb{P}^3 \setminus \mathscr{C}$ *is a biholomorphism. Therefore, from the* (2.112) *we concluded that* p_1^t *and* p_2^t *are embedding closed points of distribution* $\mathscr{F}$, *so* $N_G = 2$, *because when* $t \longrightarrow 0$ *we have* $\omega_t \longrightarrow \omega$.
Thus, we have $deg(\mathscr{F}) = 1$, $\ell_t = mult_E(\pi^*\mathscr{F}_t) = mult_E(\pi^*\mathscr{F}) = 1$ *for all* $t \in \mathbb{D} \setminus \{0\}$, $\chi(\mathscr{C}) = 2$.
By Theorem 2.18 we get

$$Res(\mathscr{F}, \mathscr{C}) \leq 6. \tag{2.113}$$

Example 2.20 (Dicritical Case)**.** *Consider the non-integrable distribution $\mathscr{F}$ induced by*

$$\begin{aligned}\omega =& \left[\mu z_1 z_3^2 + a_0 z_0^2 z_3 + (a_1 z_3 + a_3 z_2) z_0 z_1 + (a_2 z_3 + a_4 z_2) z_1^2\right] dz_0\\ &+ \left[-\mu z_0 z_3^2 + (b_0 z_3 - a_3 z_2) z_0^2 + (b_1 z_3 - a_4 z_2) z_0 z_1 + b_2 z_1^2 z_3\right] dz_1\\ &+ \left[c_0 z_0^2 z_3 + c_1 z_0 z_1 z_3 + c_2 z_1^2 z_3\right] dz_2\\ &+ \left[-a_0 z_0^3 - a_1 z_0^2 z_1 - a_2 z_0 z_1^2 - b_0 z_0^2 z_1 + b_2 z_1^3 - z_2\left(c_0 z_0^2 + c_1 z_0 z_1 + c_2 z_1^2\right)\right] dz_3.\end{aligned}$$

with $a_i \in \mathbb{C}$, $b_j \in \mathbb{C}$, $c_k \in \mathbb{C}$ and $0 \neq \mu \in \mathbb{C}$ and $i = 0, \ldots, 4$ and $k, j = 0, 1, 2$ and $gdc(a(\lambda), b(\lambda), c(\lambda)) = 1$ where $a(\lambda) = \sum_{i=0}^{2} a_i \lambda^i$ $\quad b(\lambda) = \sum_{i=0}^{2} b_i \lambda^i$ and $c(\lambda) = \sum_{i=0}^{2} c_i \lambda^i$.

In the open affine set $U_3 := \left\{[z] \in \mathbb{P}^3 : z_3 \neq 0\right\}$, with coordinates $x = \dfrac{z_0}{z_3}$, $y = \dfrac{z_1}{z_3}$ and $z = \dfrac{z_2}{z_3}$ the distribution is write as

$$\begin{aligned}\omega =& \left[\mu y + a_0 x^2 + (a_1 + a_2 z) x y + (a_3 + a_4 z) y^2\right] dx\\ &+ \left[-\mu x + (b_0 - a_2 z) x^2 + (b_1 - a_4 z) x y + b_2 y^2\right] dy + \left[c_0 x^2 + c_1 x y + c_2 y^2\right] dz.\end{aligned}$$

First note that $mult_{\mathscr{C}}(\mathscr{F}) = 1$. Also, note that the curve $\mathscr{C} = \left\{z_0 = z_1\right\} \subset Sing(\mathscr{F})$. The singular set of $\mathscr{F}$ is obtained, solving the following equations of system.

$$\begin{cases} \mu y + a_0 x^2 + (a_1 + a_2 z) x y + (a_3 + a_4 z) y^2 = 0, \\ -\mu x + (b_0 - a_2 z) x^2 + (b_1 - a_4 z) x y + b_2 y^2 = 0, \\ c_0 x^2 + c_1 x y + c_2 y^2 = 0. \end{cases} \tag{2.114}$$

Since by hypothesis $gdc(a(\lambda), b(\lambda), c(\lambda)) = 1$ on the affine chart U_3, there are no other singularities but the z-axis.

Let us now look at the singularities in the hyperplane at infinity $H_\infty = \{z_3 = 0\}$.

First*: Let us do $z_2 = 1$ and $z_3 = 0$.*

$$\begin{cases} a_3 z_0 z_1 + a_4 z_1^2 = 0,, \\ -a_3 z_0^2 - a_4 z_0 z_1 = 0 \\ -a_0 z_0^3 - a_1 z_0^2 z_1 - a_2 z_0 z_1^2 - b_0 z_0^2 z_1 + b_2 z_1^3 - \left(c_0 z_0^2 + c_1 z_0 z_1 + c_2 z_1^2\right) = 0. \end{cases} \tag{2.115}$$

Making $z_1 = \lambda z_0$ and considering $z_0 \neq 0$ and $z_1 \neq 0$, we have:

$$\begin{cases} z_0^2\left(a_3\lambda + a_4\lambda^2\right) = 0, \\ z_0^2(-a_3 - a_4\lambda) = 0, \\ z_0^2\left[z_0\left(-a_0 - a_1\lambda - a_2\lambda^2 - b_0\lambda + b_2\lambda^3\right) - \left(c_0 + c_1\lambda + c_2\lambda^2\right)\right] = 0. \end{cases} \tag{2.116}$$

From the second equation of (2.116)*, we have*

$$\lambda = -\frac{a_3}{a_4}.$$

From the third equation, we get:

$$z_0 = \frac{c_2(\lambda)}{p_3(\lambda)},$$

where $c_2(\lambda) = c_0 + c_1\lambda + c_2\lambda^2$ and $p_3(\lambda) = -a_0 - (a_1 + b_0)\lambda - a_2\lambda^2 + b_2\lambda^3$. Thus, we have point $p_1 = [z_0 : \lambda z_0 : 1 : 0]$

***Second**: Let us do $z_3 = z_2 = 0$ and $z_1 = 1$.*

$$a_0 z_0^3 + (a_3 + b_0) z_0^2 + a_2 z_0 - b_2 = 0. \tag{2.117}$$

If $a_0 \neq 0$, the singular set of $\mathscr{F}$ contains three more points, p_2, p_3, p_4, which are the roots of the equation (2.117)*. However, if $a_0 = 0$, we have two roots plus the point $p_4 = [1:0:0:0]$, which in this case will also be an element of Sing($\mathscr{F}$). So, the singular set is the following union :*

$$Sing(\mathscr{F}) = \mathscr{C} \cup \{p_1, \ldots, p_4\}.$$

Again, considering the open affine set U_3 and by blowing up along the curve $\mathscr{C}$ and using the chart $\sigma_1(\varsigma_1, \varsigma_2, \varsigma_3) = (\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3) = (x, y, z)$, we have

$$\begin{aligned} \pi^*\omega = &\left[\mu\varsigma_2\varsigma_1 + a_0\varsigma_1^2 + (a_1 + a_3\varsigma_3)\varsigma_2\varsigma_1^2 + (a_2 + a_4\varsigma_3)\varsigma_2^2\varsigma_1^2\right] d\varsigma_1 \\ &+ \left[-\mu\varsigma_1 + (b_2 - a_3\varsigma_3)\varsigma_1^2 + (b_1 - a_4\varsigma_3)\varsigma_2\varsigma_1^2 + b_2\varsigma_2^2\varsigma_1^2\right]\left(\varsigma_2 d\varsigma_1 + \varsigma_1 d\varsigma_2\right) \\ &+ \varsigma_1^2\left[c_0 + c_1\varsigma_2 + c_2\varsigma_2^2\right] d\varsigma_3. \end{aligned} \tag{2.118}$$

$$\begin{aligned}\pi^*\omega = \Big[&\mu\varsigma_2\varsigma_1 + \varsigma_1^2\Big(a_0 + (a_1 + a_3\varsigma_3)\varsigma_2 + (a_2 + a_4\varsigma_2)\varsigma_2^2\Big) - \mu\varsigma_1\varsigma_2 \\ &+ \varsigma_1^2\Big((b_2 - a_3\varsigma_3)\varsigma_2 + (b_1 - a_4\varsigma_3)\varsigma_2^2 + b_2\varsigma_2^3\Big]d\varsigma_1 \\ &+ \varsigma_1^2\Big[-\mu + (b_2 - a_4\varsigma_3)\varsigma_1 + (b_1 - a_4\varsigma_3)\varsigma_2\varsigma_1 + b_2\varsigma_1\varsigma_2^2\Big]d\varsigma_2 + \varsigma_1^2\Big[c_0 + c_1\varsigma_2 + c_2\varsigma_2^2\Big]d\varsigma_3.\end{aligned} \tag{2.119}$$

$$\begin{aligned}\pi^*\omega = \varsigma_1^2&\Big[a_0 + (a_1 + b_2)\varsigma_2 + (a_2 + b_1)\varsigma_2^2 + b_2\varsigma_2^3\Big]d\varsigma_1 \\ &+ \varsigma_1^2\Big[-\mu + (b_2 - a_4\varsigma_3)\varsigma_1 + (b_1 - a_4\varsigma_3)\varsigma_1\varsigma_2 + b_2\varsigma_2^2\varsigma_1\Big]d\varsigma_2 + \varsigma_1^2\Big[c_0 + c_1\varsigma_2 + c_2\varsigma_2^2\Big]d\varsigma_3.\end{aligned} \tag{2.120}$$

Dividing the equation (2.120) *by* ς_1^2*, we have:*

$$\begin{aligned}\tilde{\omega} = &\big(a_0 + (a_1 + b_2)\varsigma_2 + (a_2 + b_1)\varsigma_2^2 + b_2\varsigma_2^3\big)d\varsigma_1 \\ &+ \big(-\mu + (b_2 - a_4\varsigma_3)\varsigma_1 + (b_1 - a_4\varsigma_3)\varsigma_1\varsigma_2 + b_2\varsigma_2^2\varsigma_1\big)d\varsigma_2 + \big(c_0 + c_1\varsigma_2 + c_2\varsigma_2^2\big)d\varsigma_3.\end{aligned} \tag{2.121}$$

The solutions of the following system below are the singularities of $\tilde{\omega}$ *on the exceptional divisor* $E = \{\varsigma_1 = 0\}$.

$$\begin{cases} a_0 + (a_1 + b_2)\varsigma_2 + (a_2 + b_1)\varsigma_2^2 + b_2\varsigma_2^3 = 0, \\ -\mu + (b_2 - a_4\varsigma_3)\varsigma_1 + (b_1 - a_4\varsigma_3) = 0, \\ c_0 + c_1\varsigma_2 + c_2\varsigma_2^2 = 0. \end{cases}$$

we generically can choose the coefficients a_i*,* b_i *and* c_i *so that the system above has no solution. Thus, over the exceptional divisor, we have no singularities.*
Again, in the hyperplane at infinity $H_\infty = \{z_3 = 0\}$ *we have*

$$\begin{aligned}\omega = &\big(a_3z_0z_1 + a_4z_1^2\big)dz_0 - \big(a_3z_0^2 + a_4z_0z_1\big)dz_1 \\ &- \Big[a_0z_0^3 - a_3z_0^2z_1 - a_2z_0z_1^2 - b_0z_0^2z_1 + b_2z_1^3 - \big(c_0z_0^2 + c_1z_0z_1 + c_1z_1^2\big)\Big]z_3.\end{aligned} \tag{2.122}$$

Making the explosion along the curve $\mathscr{C} = \{z_0 = z_1 = 0\}$*, with* $z_1 = uz_0$*, we have:*

$$\begin{aligned}
\pi^*\omega &= z_0^2\left(a_3u+a_4u^2\right)dz_0 - z_0^2(a_3+a_4u)(udz_0+z_0du)\\
&\quad + z_0^2\Big[z_0\left(-a_0-(a_3+b_0)u-a_3u^2+b_2u^3\right)-\left(c_0+c_1u+c_2u^2\right)\Big]dz_3\\
\pi^*\omega &= z_0^2\Big[\left(a_3u+a_4u^2\right)-\left(a_3u+a_4u^2\right)\Big]dz_0 - z_0^3(a_3+a_4u)du\\
&\quad + z_0^2(z_0p(u)-c(u))dz_3\\
\pi^*\omega &= -z_0^3(a_3+a_4u)du + z_0^2(z_0p(u)-c(u))dz_3\\
\widetilde{\omega} &= z_0(a_3+a_4u)du + (z_0p(u)-c(u))dz_3. \qquad (2.123)
\end{aligned}$$

where $p(u) = -a_0-(a_3+b_0)u-a_3u^2+b_2u^3$ *and* $c(u) = c_0+c_1u+c_2u^2$.

So, over exceptional divisor $E = \{z_0 = 0\}$ *we have* $c(u) = 0$, *thus* $u = u_i$, $i = 1,2$. *Then, two singularities.*
As $\pi : \widetilde{\mathbb{P}^3}\setminus E \longrightarrow \mathbb{P}^3\setminus\mathscr{C}$ *is a biholomorphism, we have*

$$Sing(\widetilde{\mathscr{F}}) = \{p_1, p_2, p_3, p_4, \tilde{p_5}, \tilde{p_6}\}.$$

Let ω_t *be a small perturbation of* ω. *By Lemma of Perturbation,* ω_t *is special along* $\mathscr{C}$ *for all* $t\in\mathbb{D}\setminus\{0\}$.

1) $\ell_t = mult_E(\pi^*\mathscr{F}_t) = mult_E(\pi^*\mathscr{F}) = 2-1 = 1$, *for all* $t\in D(0,\varepsilon)\setminus\{0\}$.

2) $deg(\mathscr{F}) = deg(\mathscr{F}_t) = 2$, *for all* $t\in D(0,\varepsilon)\setminus\{0\}$.

So by item (1) *above and by Lemma of Perturbation item* (5) *we have that* $\mathscr{F}$ *is a dicritical distribution.*
By Theorem 2.18 and replacing items (1) *and* (2) *above in* (2.75) *and* (2.87), *we have*

Remark 2.21. *In this case, notice that, over exceptional divisor, we have*

$$\sum_{p_i^t\in E} Res(\widetilde{\mathscr{F}_t}, \widetilde{p_i^t}) = \int_E c_2(\Omega_E^1\otimes L) = 2.$$

$$4 = \sum_{i=1}^{4}\mathrm{Res}(\mathscr{F}, p_i) = 18-2-N_G \Longrightarrow N_G = 12$$

which results

$$\mathrm{Res}(\mathscr{F},\mathscr{C}) = 16.$$

CHAPTER 3

SPECIAL DISTRIBUTION DETERMINED BY THEIR SINGULAR SCHEME

Our goal in this section is to study the special distributions that are uniquely determined by their singular scheme for a certain fixed annulment order. The references are [3], [7] and [19].

The following results will allow us to demonstrate the main result of this chapter.

Remark 3.1. *For any holomorpic vector bundle E of rank r;*

$$\bigwedge^{k} E \simeq \bigwedge^{r-k} E^{\vee} \otimes det(E).$$

By the previous Remark, we can write:

Remark 3.2.

$$\begin{aligned} \bigwedge^{2} T_{\mathbb{P}^3} &\simeq \bigwedge^{3-2} T^{\vee}_{\mathbb{P}^3} \otimes det(T_{\mathbb{P}^3}), \\ &\simeq \Omega^1_{\mathbb{P}^3} \otimes \omega^{\vee}_{\mathbb{P}^3}, \\ &\simeq \Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4). \end{aligned} \tag{3.1}$$

Remark 3.3.

$$\bigwedge^{3} T_{\mathbb{P}^3} \simeq \mathscr{O}_{\mathbb{P}^3}(4). \tag{3.2}$$

Proposition 3.4 (Projection Formula)**.** *If $f:(X,\mathcal{O}_X)\longrightarrow(Y,\mathcal{O}_Y)$ is a morphism of ringed spaces, if* F *is an $\mathcal{O}_X$-module, and if* G *is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*\left(\mathrm{F}\otimes_{\mathcal{O}_X}f^*\mathrm{G}\right)\simeq f_*(\mathrm{F})\otimes_{\mathcal{O}_Y}\mathrm{G}$.*

Proposition 3.5. *Let $f:X\longrightarrow Y$ be an affine morphism of schemes with X noetherian and let $\mathscr{F}$ be a quasi-coherent sheaf on X. If $R^if_*\mathscr{F}=0$ for all $i>0$, then*

$$H^i(X,\mathscr{F})\simeq H^i(Y,f_*\mathscr{F}),$$

for each $i\geq 0$.

Proof. See [4] proposition 3.26. ∎

Remark 3.6 (Direct Image by Blow up)**.** *Let $\pi:\widetilde{\mathbb{P}^3}\longrightarrow\mathbb{P}^3$ be the blowing-up along a irreducible smooth curve $\mathscr{C}\subset\mathbb{P}^3$ with exceptional divisor E. For this purpose, the following result is valid:*

$$\pi_*\mathcal{O}_{\widetilde{\mathbb{P}^3}}(-nE)=\mathscr{I}^{\otimes n}_{\mathscr{C}/\mathbb{P}^3}.$$

where $\mathscr{I}_{\mathscr{C}/\mathbb{P}^3}$ is the ideal sheaf of $\mathscr{C}$.

For more details, see [10].

Making $E=\Omega^1_{\mathbb{P}^n}$ in remark (3.1), we have

Proposition 3.7. *(Vector Bundle Isomorphism)*

$$(\Omega^p_{\mathbb{P}^n})^\vee\simeq\Omega^{n-p}_{\mathbb{P}^n}\otimes(\Omega^n_{\mathbb{P}^n})^\vee.$$

In our case, as we are blowing up $X=\mathbb{P}^3$ along the curve C, the dualizing sheaf of $\widetilde{X}$ is given by:

Lemma 3.8 (Dualizing Sheaf)**.**

$$\omega_{\widetilde{\mathbb{P}^3}}\simeq\pi^*(\omega_{\mathbb{P}^3})\otimes\mathcal{O}_{\widetilde{\mathbb{P}^3}}((k-1)E),$$

where k is the codimension of center of blow up.

In our case, as we are blowing up $X=\mathbb{P}^3$ along the curve C, the dualizing sheaf of $\widetilde{X}$ is given by: $\omega_{\widetilde{\mathbb{P}^3}}\simeq\pi^*(\omega_{\mathbb{P}^3})\otimes\mathcal{O}_{\widetilde{\mathbb{P}^3}}(E)$.

In the proof of our main result, we are going to use the following results.

Lemma 3.9 (see [1], Lemma 1.4). *Let $X \subset M$ be a smooth codimension e subvariety of a smooth variety M. Let $f : P = Bl_X \longrightarrow M$ be the blowing-up of M along of X and let $E \subset P$ be the exceptional divisor. If $0 \leq t \leq e-1$, then:*

$$H^i(P, f^*F \otimes \mathcal{O}_P(tE)) \simeq H^i(M, F),$$

for all i and for any locally free sheaf F on M.

Theorem 3.10 ([2], Theorem 1.1). *Let X be a smooth projective variety and $\mathcal{E}$ and $\mathcal{G}$ be locally free sheaves on X of rank e and g, respectively. Let $\varphi : \mathcal{E} \longrightarrow \mathcal{G}$ be a generically surjective morphism. Denote by $\omega_\varphi \in H^0(X, \bigwedge^g(\mathcal{E}^*) \otimes det(\mathcal{G}))$ the associated global section and by Z its zero scheme. Suppose that the following conditions hold:*

i) Z has pure codimension : $e-g+1$.

ii) For every $i \in \{1, \dots, e-g\}$

$$H^i(X, \bigwedge^g(\mathcal{E}^*) \bigwedge^{i+1} \mathcal{E} \otimes S_i(\mathcal{G}^*)) = 0.$$

If $\omega \in H^0(X, \bigwedge^g(\mathcal{E}^*) \otimes det(\mathcal{G}))$ is such that $\omega|_Z = 0$, then there is an endomorphism $\alpha \in End(\bigwedge^g(\mathcal{E}^*))$ such that $\omega = \alpha \circ \omega_\varphi$.

Let us state our main result.

Theorem 3.11. *Let $\mathcal{F}_1$ be a non integrable codimension one holomorphic distributions on $\mathbb{P}^3$ of degree d such that its singular locus has just one non-zero dimensional component which is integral and non-degenerated somooth curve $\mathcal{C}$. Assume that $\mathcal{F}_1$ is special along $\mathcal{C}$. Let $\pi : \widetilde{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along $\mathcal{C}$ and E the exceptional divisor. If $\mathcal{F}_2$ is another non integrable codimension one distribution of degree d on $\mathbb{P}^3$ and furthermore the following conditions are satisfied :*

i) $deg(\mathcal{C}) \geq 2$,

ii) $d \geq 2deg(\mathcal{C})$,

iii) $Sing(\mathcal{F}_1) \subset Sing(\mathcal{F}_2)$,

iv) $Sing(\widetilde{\mathcal{F}_1}|_E) \subset Sing(\widetilde{\mathcal{F}_2}|_E)$,

v) $\ell = mult_E(\widetilde{\mathcal{F}_1}) = 1$ or 2, *Then* $\mathscr{F}_1 = \mathscr{F}_2$ *.*

Proof. Let $d = \deg(\mathscr{F}_1) = \deg(\mathscr{F}_2)$. The distributions $\mathscr{F}_1$ and $\mathscr{F}_2$ are induced, respectively, by $\omega_1, \omega_2 \in H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(d+2))$. Similarly the distributions $\widetilde{\mathscr{F}_1}$ and $\widetilde{\mathscr{F}_2}$ are induced, respectively, by $\widetilde{\omega}_1, \widetilde{\omega}_2 \in H^0(\widetilde{\mathbb{P}^3}, \pi^*\Omega^1_{\mathbb{P}^3} \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(d+2) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-\ell E))$.
Let us show that

$$\widetilde{\omega}_1 = \lambda \widetilde{\omega}_2. \tag{3.3}$$

Let $\operatorname{Sing}(\widetilde{\omega_1}) = Z(\widetilde{\omega_1})$ and $\operatorname{Sing}(\widetilde{\omega_2}) = Z(\widetilde{\omega_2})$.
Firstly, by biholomorphism $\pi : \widetilde{\mathbb{P}^3} \setminus E \longrightarrow \mathbb{P}^3 \setminus \mathscr{C}$, we have the following inclusion:

$$Z(\widetilde{\omega_1}) \subset Z(\widetilde{\omega_2}). \tag{3.4}$$

Now, by item (iv), we have that inclusion above is valid in $\widetilde{\mathbb{P}^3}$.
In $\widetilde{\mathbb{P}^3} \setminus E$ the morphism of locally free sheaves $\varphi_{\widetilde{\omega_1}} : \pi^*(T_{\mathbb{P}^3}) \longrightarrow N_{\widetilde{\mathscr{F}}}$ induced by global section $\widetilde{\omega_1}$ is surjective and by definition, the degeneracy scheme of $\varphi_{\widetilde{\omega_1}}$ it is given by $\operatorname{Sing}(\varphi_{\widetilde{\omega_1}}) = Z(\widetilde{\omega_1})$.

By hypothesis, as ω_1 is special along $\mathscr{C}$, then $Z(\widetilde{\omega_1})$ is a nonempty and zero-dimensional scheme. By taking in theorem (3.10), $\mathscr{E} = \pi^*(T_{\mathbb{P}^3})$ and $\mathscr{G} = N_{\widetilde{\mathscr{F}}}$ whose ranks are 3 and 1, respectively, we can write:

$$\operatorname{Codim}(\operatorname{Sing}(\varphi_{\widetilde{\omega_1}})) = \operatorname{Codim}(Z(\widetilde{\omega_1})) = \operatorname{rank}(\pi^*(\mathscr{E})) - \operatorname{rank}(\mathscr{G}) + 1 = 3 - 1 + 1 = 3. \tag{3.5}$$

Now, let us show that the following vanishing of cohomology each other out:

$$H^i(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^\vee \otimes \bigwedge^{i+1} \pi^*(T_{\mathbb{P}^3}) \otimes S_i(N^\vee_{\widetilde{\mathscr{F}}}))$$

where $S_i(N^\vee_{\widetilde{\mathscr{F}}})$ denotes the its i-th symmetric power, with $i = 1, 2$.
By hypothesis, item (v); we are interested in the case $\ell = 1, 2$. Let us make, then, both cases.

First Case: $\ell = 1$.

In this case, we have:

$$\begin{aligned} N^\vee_{\widetilde{\mathscr{F}}} &\simeq \pi^*(N^\vee_{\mathscr{F}}) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E), \\ &\simeq \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E). \end{aligned} \tag{3.6}$$

a) $\underline{\text{i = 1}}$.

Note also that:

$$\begin{aligned} S_1(N_{\mathscr{F}}^{\vee}) &= N_{\mathscr{F}}^{\vee}, \\ &= \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E). \end{aligned} \tag{3.7}$$

In order to simplify the notation, we do:

$H^1 = H^1(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^2 \pi^*(T_{\mathbb{P}^3}) \otimes S_1(N_{\mathscr{F}}^{\vee}))$.

$$\begin{aligned} H^1 &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^2 \pi^*(T_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \pi^*(\bigwedge^2 T_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(2-d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)). \end{aligned} \tag{3.8}$$

where in the second isomorphism of (3.8), we use the Remark 3.2. By Lemma 3.9 we can write:

$$\begin{aligned} H^1 &\simeq H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(2-d)), \\ &\simeq H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(2-d)). \end{aligned} \tag{3.9}$$

By Bott's formula, $H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(2-d)) = 0$.

b) $\underline{i = 2}$.

In this case we have

$$\begin{aligned} S_2(N_{\mathscr{F}}^{\vee}) &= (N_{\mathscr{F}}^{\vee}) \otimes (N_{\mathscr{F}}^{\vee}), \\ &= \pi^*(\mathscr{O}_{\mathbb{P}^3}(-2(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E). \end{aligned} \tag{3.10}$$

$H^2 = H^2(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^3 \pi^*(T_{\mathbb{P}^3}) \otimes S_2(N_{\mathscr{F}}^{\vee}))$.

$$
\begin{aligned}
H^2 &\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^\vee \otimes \bigwedge^3 \pi^*(T_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-2(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)),\\
&\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(4)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-2(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)),\\
&\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \mathscr{O}_{\mathbb{P}^3}(-2d))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)).
\end{aligned}
\tag{3.11}
$$

where in the second isomorphism of (3.11), we use the Remark 3.3. In this case, we cannot use Lemma (3.9), because $t = 2$. We will use Serre's duality (see [3]).
So,

$$
H^2 \simeq H^1\left(\widetilde{\mathbb{P}^3}, \left[\pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(-2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E))\right]^\vee \otimes \omega_{\widetilde{\mathbb{P}^3}}\right). \tag{3.12}
$$

Let us do it now, $F = \left[\pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(-2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E))\right]$. Thus

$$
\begin{aligned}
F^\vee &= \left[\pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \mathscr{O}_{\mathbb{P}^3}(-2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E))\right]^\vee,\\
&= \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(-2d))^\vee \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)^\vee,\\
&= \pi^*((\Omega^1_{\mathbb{P}^3})^\vee \otimes \mathscr{O}_{\mathbb{P}^3}(-2d))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)^\vee.
\end{aligned}
\tag{3.13}
$$

By Proposition (3.7) we have

$$
(\Omega^1_{\mathbb{P}^3})^\vee \simeq \Omega^2_{\mathbb{P}^3} \otimes (\Omega^3_{\mathbb{P}^3})^\vee. \tag{3.14}
$$

We know that

$$
\begin{aligned}
\Omega^3_{\mathbb{P}^3} &= \bigwedge^3 \Omega^1_{\mathbb{P}^3} = \det(\Omega^1_{\mathbb{P}^3}),\\
&= \omega_{\mathbb{P}^3} = \mathscr{O}_{\mathbb{P}^3}(-4).
\end{aligned}
\tag{3.15}
$$

Therefore

$$
(\Omega^3_{\mathbb{P}^3})^\vee = \mathscr{O}_{\mathbb{P}^3}(4). \tag{3.16}
$$

From (3.14) we have

$$(\Omega^1_{\mathbb{P}^3})^\vee \simeq \Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4). \tag{3.17}$$

From (3.13) and (3.14) , the isomorphism (3.12) can be written like this

$$\begin{aligned} H^2 &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4) \otimes \mathscr{O}_{\mathbb{P}^3}(2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-2E) \otimes \pi^*(\omega_{\mathbb{P}^3}) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4) \otimes \mathscr{O}_{\mathbb{P}^3}(2d)) \otimes \pi^*\mathscr{O}_{\mathbb{P}^3}(-4)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4) \otimes \mathscr{O}_{\mathbb{P}^3}(2d)) \otimes \mathscr{O}_{\mathbb{P}^3}(-4)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3}(2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-E)), \end{aligned} \tag{3.18}$$

It follows from the projection formula that and by Lemma 3.9, we have

$$\begin{aligned} H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3}(2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-E)) &\simeq H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(2d) \otimes \mathscr{I}_{\mathscr{C}}) \\ &\simeq H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}(2d)). \end{aligned} \tag{3.19}$$

We have that

Remark 3.12.
- *$\Omega^1_{\mathbb{P}^3}$ is 2-regular by Bott's formula,*
- *$\Omega^2_{\mathbb{P}^3} := \bigwedge^2 \Omega^1$ is 4-regular by Corollary 1.8,*
- *$\mathscr{I}_{\mathscr{C}}$ is $(deg(\mathscr{C})-1)$-regular by Theorem 1.10.*

Therefore $\Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}$ is $(\deg(\mathscr{C})+3)$-regular by Proposition 1.7. Now, by definition of Castelnuovo-Mumford's regularity, the vanishing of (3.19) is thus obtained since that:

$$H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}(2d)) = H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}((2d+1)-1)) = 0,$$

because

$$\begin{aligned} 2d+1 &\geq (\deg(\mathscr{C})+3), \\ 2d &\geq (\deg(\mathscr{C})+2), \\ d &\geq \frac{1}{2}(\deg(\mathscr{C})+2). \end{aligned} \tag{3.20}$$

Second Case: $\ell = 2$.

In This case, we have

$$\begin{aligned} N^{\vee}_{\widetilde{\mathscr{F}}} &\simeq \pi^*(N^{\vee}_{\mathscr{F}}) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E), \\ &\simeq \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E). \end{aligned} \tag{3.21}$$

a) $\underline{i=1}$.

Again, making $H^1 = H^1(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^2 \pi^*(T_{\mathbb{P}^3}) \otimes S_1(N^{\vee}_{\widetilde{\mathscr{F}}}))$ and from item (a) of the previous case, we have

$$\begin{aligned} H^1 &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^2 \pi^*(T_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \pi^*(\bigwedge^2 T_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)), \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(2-d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(2E)). \end{aligned} \tag{3.22}$$

Again, by Proposition 3.7 we can write

$$\begin{aligned} (\Omega^2_{\mathbb{P}^3})^{\vee} &\simeq \Omega^1_{\mathbb{P}^3} \otimes (\Omega^3_{\mathbb{P}^3})^{\vee}, \\ &\simeq \Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4). \end{aligned} \tag{3.23}$$

From (3.23) and again for the Serre's duality

$$\begin{aligned} H^1 &\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4) \otimes \mathscr{O}_{\mathbb{P}^3}(d-2)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-2E) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-4)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)), \\ &\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(d-2)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-E)). \end{aligned} \tag{3.24}$$

By projection formula

$$H^2(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(d-2)) \otimes \mathscr{I}_{\mathscr{C}}) = H^2(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}(d-2)). \tag{3.25}$$

According to the remark 3.12 we have $\Omega^1_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}$ is $(\deg(\mathscr{C})+1)$-regular.
So, for $H^2(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3} \otimes \mathscr{I}_{\mathscr{C}}(d-2)) = 0$, we must have

$$d \geq (\deg(\mathscr{C})+1). \tag{3.26}$$

b) $\underline{i=2}$.
In this case, we note that

$$S_2(N^{\vee}_{\widetilde{\mathscr{F}}}) \simeq \pi^*(\mathscr{O}_{\mathbb{P}^3}(-2(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(4E). \tag{3.27}$$

Simirlarly, we take $H^2 = H^2(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^3 \pi^*(T_{\mathbb{P}^3}) \otimes S_2(N^{\vee}_{\widetilde{\mathscr{F}}}))$.

$$\begin{aligned} H^2 &\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3})^{\vee} \otimes \bigwedge^3 \pi^*(T_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-2(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(4E)) \\ &\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3}) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(4)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-2(d+2))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(4E)) \\ &\simeq H^2(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(-2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(4E)). \end{aligned} \tag{3.28}$$

By Serre's duality:

$$\begin{aligned} H^2 &\simeq H^1(\widetilde{\mathbb{P}^3}, [\pi^*(\Omega^1_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(-2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(4E))]^{\vee} \otimes \omega_{\widetilde{\mathbb{P}^3}}) \\ H^2 &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(4)) \otimes \mathscr{O}_{\mathbb{P}^3}(2d))) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-4E)) \otimes \pi^*(\mathscr{O}_{\mathbb{P}^3}(-4)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(E)) \\ &\simeq H^1(\widetilde{\mathbb{P}^3}, \pi^*(\Omega^2_{\mathbb{P}^3} \otimes \mathscr{O}_{\mathbb{P}^3}(2d)) \otimes \mathscr{O}_{\widetilde{\mathbb{P}^3}}(-3E)). \end{aligned} \tag{3.29}$$

By projection formula and by remark 3.6, we can write.

$$H^2 \simeq H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}^{\otimes 3}_{\mathscr{C}}(2d)). \tag{3.30}$$

We already know that $\Omega^2_{\mathbb{P}^3}$ is 4-regular and by Proposition 1.7 we have that $\mathscr{I}^{\otimes 3}_{\mathscr{C}}$ is $3(\deg(\mathscr{C})-1) = (3\deg(\mathscr{C})-3)$-regular. So, $\Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}^{\otimes 3}_{\mathscr{C}}$ is $(3\deg(\mathscr{C})+1)$-regular.
So, for $H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}^{\otimes 3}_{\mathscr{C}}(2d)) = H^1(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3} \otimes \mathscr{I}^{\otimes 3}_{\mathscr{C}}((2d+1)-1))$, we must have

$$\begin{aligned} 2d+1 &\geq (3\deg(\mathscr{C})+1), \\ d &\geq \frac{3}{2}\deg(\mathscr{C}). \end{aligned} \tag{3.31}$$

By equation (3.4) we have: $\widetilde{\omega_2}|_{Z(\widetilde{\omega_1})} = 0$. Thus, the conclusion of theorem (3.10) allows us to write:

$$\widetilde{\omega_1} = \alpha \circ \widetilde{\omega_2}. \tag{3.32}$$

where $\alpha \in \mathrm{End}(\pi^*(T_{\mathbb{P}^3})^\vee)$.

$$\begin{aligned}
\alpha \in \mathrm{End}(\pi^*(T_{\mathbb{P}^3})^\vee) &\simeq H^0(\widetilde{\mathbb{P}^3}, (\pi^*(T_{\mathbb{P}^3})^\vee)^\vee \otimes \pi^*(T_{\mathbb{P}^3})^\vee),\\
&\simeq H^0(\widetilde{\mathbb{P}^3}, \pi^*(T_{\mathbb{P}^3}) \otimes \pi^*(T_{\mathbb{P}^3})^\vee),\\
&\simeq H^0(\mathbb{P}^3, T_{\mathbb{P}^3} \otimes T^\vee_{\mathbb{P}^3}),\\
&\simeq H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}) \simeq \mathbb{C}.
\end{aligned} \tag{3.33}$$

where the penultimate isomorphism in (3.33) was obtained by the projection formula. Then, there exist $\lambda \in \mathbb{C}^*$ such that $\widetilde{\omega_1} = \lambda \widetilde{\omega_2}$.

By projection, we have $\omega_1 = \lambda\omega_2$ and therefore $\mathscr{F}_1 = \mathscr{F}_2$. ■

The next example shows that a hypothesis about the degree of the curve can not be removed.

Example 3.13. *As we saw in example 2.12 the following global section, $\omega \in H^0\left(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(d+2)\right)$ given by:*

$$\begin{aligned}
\omega = &\Big[z_3 A_m(z_0,z_1) + (\alpha z_1 + \beta z_2) G_m(z_0,z_1)\Big] dz_0 + \Big[z_3 B_m(z_0,z_1) + (\gamma z_2 - \alpha z_0) G_m(z_0,z_1)\Big] dz_1\\
&+ \Big[z_3 C_m(z_0,z_1) - (\beta z_0 + \gamma z_1) G_m(z_0,z_1)\Big] dz_2 - \Big[z_0 A_m(z_0,z_1) + z_1 B_m(z_0,z_1) + z_2 C_m(z_0,z_1)\Big] dz_3.
\end{aligned} \tag{3.34}$$

induces a special distribution $\mathscr{F}$ on $\mathbb{P}^3$ along curve $\mathscr{C} = \left\{[z] \in \mathbb{P}^3;\ z_0 = z_1 = 0\right\}$.

Let $\mathscr{F}_t$ be a codimension one holomorphic distribution of degree one on $\mathbb{P}^3$ with $t \in \mathbb{C}$ induced by:

$$
\begin{aligned}
\omega = &\left[z_3(z_0+(-1-t)z_1)+\left(\frac{(4-t)}{11}z_1+\frac{2}{11}z_2\right)\left(-\frac{88}{3}z_0-\frac{44}{3}z_1\right)\right]dz_0\\
&+\left[z_3(tz_0+2z_1)+\left(-\frac{3}{11}z_2-\frac{(4-t)}{11}z_0\right)\left(-\frac{88}{3}z_0-\frac{44}{3}z_1\right)\right]dz_1\\
&+\left[z_3(3z_0-2z_1)-\left(\frac{2}{11}z_0-\frac{3}{11}z_2\right)\left(-\frac{88}{3}z_0-\frac{44}{3}z_1\right)\right]dz_2\\
&-\left[z_0(z_0+(-1-t)z_1)+z_1(tz_0+2z_1)+z_2(3z_0-2z_1)\right]dz_3. \qquad (3.35)
\end{aligned}
$$

Comparing (3.35) *with* (3.34)*, its not hard to see that distribution $\mathscr{F}_t$ is special along $\mathscr{C}$ for all $t \in \mathbb{C}$.*

One can check that Sing($\mathscr{F}_t$) is the disjoint union of the curve $\mathscr{C} = \left\{[z] \in \mathbb{P}^3; z_0 = z_1 = 0\right\}$ and the closed points $p_1 = [1:1:-2:4]$, $p_2 = [-7:14:11:0]$ for all $t \in \mathbb{C}$. Thus, we obtain a one-parameter family of pairwise distinct distributions, with the same singular scheme for all $t \in \mathbb{C}$.

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