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Hopf algebras and polynomial graph invariants

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RESUMO

Começamos com uma breve introdução ao polinômio cromático e de Tutte de grafos. Então damos uma introdução às álgebras de Hopf. Em combinatória, álgebras de Hopf surgem visto que muitos objetos combinatórios tem associados a eles operações de união (levando a uma estrutura de multiplicação) e a decomposição (levando a uma estrutura de comultiplicação). Em seguida, discutimos alguns resultados clássicos de Tutte em $V-$ e $W-$ funções e polinômios de Tutte. Mostramos que o polinômio cromático é o único morfismo na categoria de álgebras de Hopf de uma álgebra de Hopf de grafos para uma álgebra de Hopf de polinômios, que é um resultado de Foissy. Em seguida, apresentamos um método algébrico de Hopf de Schmitt por sistemas de Whitney, demonstrando que o polinômio cromático de um grafo G pode ser determinado examinando apenas os subgrafos de G duplamente conexos, que é um resultado de Whitney. O último capítulo é sobre a conjectura de reconstrução de Ulam e Kelly. Aqui mostramos que certos argumentos de contagem usados por Kocay e Tutte são essencialmente idênticos a um argumento de contagem usado por Schmitt em seus métodos de álgebra de Hopf para o polinômio cromático.

ABSTRACT

We begin with a brief introduction to the chromatic and Tutte polynomial of graphs. We then give an introduction to Hopf algebras. In combinatorics, Hopf algebras arise since many combinatorial objects have associated with them operations of union (leading to multiplicative structure) and decomposition (leading to a comultiplicative structure). We then discuss some classic results of Tutte on $V-$ and $W-$ functions and Tutte polynomials. We show that the chromatic polynomial is the unique morphism in the category of Hopf algebras from a Hopf algebra of graphs to a Hopf algebra of polynomials, which is a result of Foissy. We then present a Hopf algebraic method of Schmitt for Whitney systems, demonstrating that the chromatic polynomial of a graph G can be determined by examining only the doubly connected subgraphs of G, which is a result of Whitney. The last chapter is on the reconstruction conjecture of Ulam and Kelly. Here we show that certain counting argument used by Kocay and Tutte are essentially identical to a counting argument used by Schmitt in his Hopf algebra methods for the chromatic polynomial.

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Introduction

The chromatic polynomial was originally introduced by Birkhoff in the context of the four-colour problem. One of the fundamental properties of the chromatic polynomial is its contraction-deletion expansion which expresses the chromatic polynomial of a graph in terms of the chromatic polynomials of graphs obtained by deletion or contraction of a single edge, thus allowing a recursive calculation of the chromatic polynomial. Tutte showed that there is a larger class of polynomial invariants satisfying the deletion-contraction property; the polynomial is now known as the Tutte polynomial. Whitney and Tutte made significant contributions to the theory of chromatic and Tutte polynomials. Both the chromatic and the Tutte polynomial have been extensively studied over the past century. The well known Jones polynomial in knot theory is a specialisation of the Tutte polynomial. The partition functions in statistical mechanics are also related to the Tutte polynomial. Recently other generalisations of the polynomials have been studied, notably, the chromatic symmetric function and the symmetric Tutte polynomial, both introduced by Stanley. Two questions about these invariants (among other numerous important questions) are of interest: Are there efficient algorithms to compute these invariants? What are some classes of graphs for which an invariant is complete? (For example, a famous question of Bollobás and Riordan asks if almost all graphs are uniquely determined by their Tutte polynomial.)

Hopf algebras arose in the work of Hopf in topology [7]. In informal terms, a Hopf algebra is an algebra (a vector space or a module with multiplication) which is also a coalgebra (has comultiplication), with multiplication and comultiplication interacting in a certain way. Hopf algebras have many applications in mathematical physics (where a certain class of Hopf algebras is called quantum groups). In the 1970s, Gian-Carlo Rota found many combinatorial examples of Hopf algebras, and encouraged a study of combinatorial Hopf algebras. Examples of multiplication and comultiplication arise naturally when we study combinatorial objects. For example, multiplication may be thought of as taking a disjoint union of two objects (or combining combinatorial objects in some way to obtain objects of the same type) and comultiplication may be thought of as decomposing an object into pairs of objects in many ways.

Reconstruction is one of the great outstanding problems in graph theory. It was proposed by Ulam and Kelly in the 1940s (possibly somewhat earlier). In a reconstruction type problem, the main question has the following form: given a combinatorial structure we may easily construct isomorphism class of its substructures; but given its substructures, can we construct the original structure uniquely (up to isomorphism)? Many variations of the original conjecture of Ulam and Kelly have been studied, with nearly 700 papers published on these problems.

In this dissertation, we study the three themes described above, and make an attempt to understand connections between them.

In Chapter 1, we present definitions in graph theory, and basic results about the chromatic polynomial and the Tutte polynomial.

In Chapter 2, we give an introduction to Hopf algebras which are bialgebras with an additional map (the coinverse map) satisfying the coinverse property.

In Chapter 3, we give an exposition of a classic paper of Tutte [23]. Tutte states that a V-function can take arbitrary values on one-vertex graphs with loops.

In Chapter 4, we study a result of Foissy [10] that the chromatic polynomial is the unique morphism in the category of Hopf algebras from a Hopf algebra of graphs to a Hopf algebra of polynomials.

In Chapter 5, we present the paper of Schmitt [19]. We define a category of objects called Whitney system that is a more general configuration of graphs and matroids. We present Whitney's subgraph expansion theorem using Hopf algebras of graphs.

In Chapter 6, we present the problem of reconstruction and prove results about reconstruction of invariants that depend of enumerating spanning subgraphs; for example, reconstructibility of the number of spanning trees, the number of Hamiltonian cycles, the characteristic polynomial and the Tutte polynomial.

Chapter 1

The chromatic and Tutte polynomials of graphs

We begin by defining graph theoretic terminology and notation. The remaining sections develop the theory of the chromatic polynomial and the Tutte polynomial. We discuss some simple ways of calculating the chromatic polynomial, and calculate the chromatic polynomial for a few examples. We define the Tutte polynomial based on the rank generating polynomial. The material in this chapter is based on the books by Biggs [2], Bollobás [3] and Bondy [5].

1.1 Graph theoretic background and notation

Definition 1.1.1 (Hypergraphs and graphs)**.** A **hypergraph** H is a triple (V,E,I), where V is the set of **vertices** of H, often denoted by $V(H)$, and E is the set of **edges** of H, often denoted by $E(H)$, and I is a map $I\colon E \to 2^V \setminus \{\emptyset\}$, called the **incidence relation**. For $e \in E$, the vertices in $I(e)$ are said to be **incident** with e.

If I is not injective, then H is said to have **multiedges**. If $|I(e)| = 1$, then e is called a **loop**. If $|I(e)| \leq 2$ for all $e \in E$, then H is called a **graph**. If H is a graph and has no multiedges or loops, then H is called a **simple graph**. A **simple edge** is an edge e such that $|I(e)| = 2$ and there is no other edge f with $I(f) = I(e)$.

A **subgraph** of a graph G is another graph formed from a subset of the vertices and edges of G. An induced subgraph of a graph is a subgraph formed from a subset of vertices and from all of the edges that have both endpoints in the subset. A subgraph is spanning when it includes all of the vertices of the given graph.

Definition 1.1.2. The **null-graph** is the graph with empty vertex and edge sets; we denote it by $\Phi = (\phi,\phi)$. An **empty graph** is a graph with empty edge set. A **vertex-graph** is a graph with one vertex and no edges. A **loop-graph** is a graph with one vertex and one loop.

Definition 1.1.3 (Directed graphs)**.** A **directed graph** is analogously defined with the difference that I is a map $I\colon E \to V \cup V^2$. Edges of a directed graph are called **arcs**, and if $e \in E$ and $I(e) = (u,v)$, then v is called the **head** or the **terminal vertex** of e, and u is called the **tail** or the **initial vertex** of e.

We sometimes use the term **multigraph** for a graph to emphasise that the graph may have multiedges or loops. But such a usage is redundant; **unless stated explicitly, by 'graphs' we mean graphs that allow multiedges and loops**.

When the incidence relation is injective, we may omit the incidence relation, and define a graph or a hypergraph to be a pair (V,E), where each edge in E is a non-empty subset of V.

Definition 1.1.4 (Isomorphism of simple graphs)**.** Let G and H be graphs. An **isomorphism** of G to H is a bijection $f : V(G) \to V(H)$ such that $\{u,v\} \in E(G)$ if and only if $\{f(u),f(v)\} \in E(H)$. An **isomorphism class** of graphs is an equivalence class of graphs under the isomorphism relation.

Definition 1.1.5 (Complete graph)**.** A complete graph is a simple graph in which there is an edge joining every pair of vertices. A complete graph on n vertices is denoted by K_n.

Definition 1.1.6 (Edge-deleted subgraph)**.** Given a graph G and an edge e of G, the graph $G - e$ obtained by deleting e, leaving all vertices and all other edges intact, is called an edge-deleted subgraph of G.

Definition 1.1.7 (Vertex-deleted subgraph)**.** Given a graph G and a vertex v of G, the graph $G - v$ obtained by deleting v and all edges of G that are incident with v is called a vertex-deleted subgraph of G.

Definition 1.1.8 (Regular graph)**.** The **degree** of a vertex v in a graph G is the number of edges of G that are incident with v, where each loop is counted as two edges. A graph G is k-regular if the degree of each vertex is k. A **regular graph** is one that is k-regular for some k.

Definition 1.1.9 (Homeomorphism)**.** Two graphs are **homeomorphic** if they can both be obtained from the same graph by inserting or removing vertices of degree two.

Definition 1.1.10 (Partitions of a set)**.** A **partition** of a set S is a family $\mathcal{F} \subseteq 2^S$ such that the sets in $\mathcal{F}$ are non-empty and mutually disjoint, and their union is S. The elements of $\mathcal{F}$ are called **blocks** of the partition. Let $n,k \in \mathbb{N}$. The **Stirling number of the second kind**, denoted by $S(n,k)$, is the number of partitions of an n-element set in k blocks.

Proposition 1.1.11. *The null set $\emptyset$ has a single partition $\mathcal{F} = \emptyset$; this partition has 0 blocks; hence $S(0,0) = 1$. (Empty set is not a block of the partition since, by definition, all elements of $\mathcal{F}$ are non-empty sets.)*

Definition 1.1.12 (Paths and cycles)**.** Let $G = (V,E)$ be a graph. A **path** in G is a sequence $v_1,e_1,v_2,e_2,...,v_k$, where v_i are distinct vertices and each e_i is an edge incident with vertices v_i and v_{i+1}. A **null path** has no vertices. A **cycle** in G is a sequence $v_1,e_1,v_2,e_2,...,v_k,e_k,v_{k+1} = v_1$, where $v_i, i = 1,\ldots,k$ are distinct vertices, and each e_i is an edge incident with vertices v_i and v_{i+1}.

Definition 1.1.13 (Connectivity)**.** Let $G = (V,E)$ be a graph. It is **connected** if for all $u,v \in V$, there is a path from u to v. A graph that is not connected is **disconnected**.

Definition 1.1.14 (Separable graph)**.** The graph G is separable if it is either disconnected or can be disconnected by removing one vertex, called cut-vertex.

Let $k \in \mathbb{N}$. Let $G := (V,E,I)$ be a simple graph. Then G is k-**connected** if it has at least $k+1$ vertices and for all $S \subseteq V$ such that $|S| = k-1$, the graph $G - S$ is connected. The **connectivity** of G is the maximum integer k for which G is k-connected.

A multigraph is 2-connected if it has no loops and the underlying simple graph is 2-connected. For $k \geq 3$, a multigraph is k-connected if its underlying simple graph is k-connected. (The underlying simple graph is the graph obtained by replacing all multiedges by simple edges.)

Let $\sim$ be an equivalence relation on V defined by: for all $x,y \in V$, $x \sim y$ if and only if there exists a path in G from x to y. The subgraphs induced by the equivalence classes of $\sim$ are called the components of G.

We make some observations about the null graph. They are needed to make certain algebraic operations unambiguous.

Proposition 1.1.15. *Let $G = (V,E) = \Phi$.*

1. *G has 0 components. (We have $V/{\sim} = \emptyset$. Note that Φ is not a component of G since $V/{\sim} = \emptyset$ implies that there is no $U \in V/{\sim}$ such that $G \mid U = \Phi$.)*

2. *G is connected.*

3. *G has rank 0 and corank 0 (see Definition 1.1.22).*

Remark 1.1.16. By our definition of a connected graph, for the null graph, it is vacuously true that for all $u,v \in V$, there exists a path from u to v, hence it is connected. According to some authors, the null graph is neither connected nor disconnected. See [15] for more discussion about issues related to the null graph. Also, a single vertex graph with a loop is connected.

We make some observations about connectivity.

Proposition 1.1.17. *Let $G = (V,E)$ be a graph.*

1. *If G has a single vertex, then it is connected. (If v is the vertex, then we have a unique path starting and ending at v, with no edges.)*

2. *If G is connected, then it is either a null-graph or a single vertex graph or a 1-connected graph.*

3. *If G is disconnected, then it has at least 2 vertices and has connectivity 0.*

Definition 1.1.18 (Blocks of graphs)**.** Let $G := (V,E,I)$ be a multigraph. Define a relation $\sim$ on E as follows: for all $e \in E$, $e \sim e$; for all $e_1,e_2 \in E$ distinct, $e_1 \sim e_2$ if there is a cycle in G that contains the edges e_1 and e_2. The relation $\sim$ is an equivalence relation. A **block** of G is either a subgraph consisting of an isolated vertex in G or a subgraph consisting of the edges in an equivalence class of $\sim$ and their incident vertices. An edge that belongs to a single-edge block is called an **isthmus** or a **bridge**.

The following theorem summarises basic facts about blocks.

Theorem 1.1.19. *Let $G := (V,E,I)$ be a multigraph.*

1. *If G is the null graph, then it has no blocks.*
2. *If G is a single-vertex graph, them it has exactly one block.*
3. *G is a union of its blocks.*
4. *If a block of G contains at least 2 edges with distinct sets of incident vertices, then it is a 2-connected subgraph of G.*
5. *Two blocks can have at most one vertex in common.*

Definition 1.1.20 (Incidence matrix)**.** Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. We orient each edge of G arbitrarily (see example 1.1.23). The incidence matrix $\mathbf{D}$ of G, with respect to given orientation of G, is an $n \times m$ matrix (d_{ij}) whose entries are

$$d_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the head of } e_j; \\ -1, & \text{if } v_i \text{ is the tail of } e_j; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1.1.21. *The incidence matrix $\mathbf{D}$ of G has rank $n - k(G)$ where $k(G)$ denote the number of connected components of G and n the number of vertices of G. In particular, the rank and the co-rank do not depend on the orientation.*

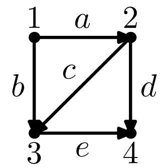
Note that if the graph G has only one vertex with n loops then the incidence matrix is null and the rank of G is zero.

Definition 1.1.22 (Rank and co-rank)**.** The rank of G and the co-rank of G are, respectively,

$$r(G) = n - k(G); \quad s(G) = m - n + k(G),$$

where n denote the number of vertices of G, m the number of edges of G and $k(G)$ the number of connected components of G.

Example 1.1.23. The rank of the following graph is 3 and the co-rank is 2.



$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

Figure 1.1: Shows a graph and its incidence matrix.

1.2 The chromatic polynomial

Definition 1.2.1. A **colour-partition** of a graph G is a partition of $V(G)$ into subsets, called colour classes, such that no simple edge of G has both its end vertices in the same colour class.

Definition 1.2.2 (Proper vertex colouring)**.** A proper vertex colouring of G is defined to be an assignment of colours to the vertices, with the property that adjacent vertices have different colours.

Definition 1.2.3 (Chromatic polynomial)**.** Let G be a graph with n vertices. Let $m_r(G)$ denote the number of distinct colour-partitions of $V(G)$ into r colour-classes. Define $u_r \in \mathbb{Q}[u]$ by $u_r := u(u-1)\cdots(u-r+1)$. The **chromatic polynomial** of G is defined by

$$C(G;u) = \sum_{r=1}^{n} m_r(G)u_r.$$

Proposition 1.2.4. *If s is a natural number, then $C(G;s)$ is the number of vertex-colourings of G using colours from a set of cardinality s.*

Proof. Note that every vertex-colouring of G in which exactly r colours are used gives rise to a colour-partition into r colour-classes. Conversely, for each colour-partition into r colours we can assign s colours to the colour-classes in $s(s-1)...(s-r+1)$ ways. Therefore equality is true. $\square$

Example 1.2.5. The chromatic polynomial of the complete graph K_n is given by $C(K_n;u) = u(u-1)\dots(u-n+1)$. If G is a graph with connected components $G_1,\dots,G_k$, then

$$C(G;u) = C(G_1;u)C(G_2;u)\cdots C(G_k;u).$$

Let e be an edge in a graph G. The graph G/e constructed from $G - e$ (see Definition 1.1.6) by identifying the two vertices incident with e in G, then deleting e, is said to be obtained by contracting e.

Proposition 1.2.6. *[The deletion-contraction expansion] The chromatic polynomial satisfies the relation*

$$C(G;u) = C(G-e;u) - C(G/e;u).$$

Proof. The set of all vertex colourings of G may be partitioned into two disjoint sets: colourings in which the ends of e are coloured differently, and colourings in which the ends of e have the same colour. The first set is in bijective correspondence with the proper colourings of G. The second set is in bijective correspondence with the colourings of G/e. Therefore $C(G-e;s) = C(G;s) + C(G/e;s)$, for each natural number s. Hence we have $C(G;u) = C(G-e;u) - C(G/e;u)$ by the fundamental property of polynomials that any two polynomials agreeing at infinitely many points are equal. □

The deletion-contraction has the following immediate consequence for trees. The chromatic polynomial of trees can also be computed by direct enumeration of the number of proper colourings with k colours for any $k \geq 1$.

Corollary 1.2.7. *If T is a tree with n vertices then $C(T;u) = u(u-1)^{n-1}$.*

Let G_1 and G_2 be graphs. We define the **join** $G_1 + G_2$ of G_1 and G_2 to be the graph with vertex set

$$V(G_1 + G_2) = V(G_1) + V(G_2)$$

and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{\{x,y\} | x \in V(G_1), y \in V(G_2)\}.$$

Proposition 1.2.8. *The numbers of colour-partitions of $G = G_1 + G_2$ are given by*

$$m_i(G) = \sum_{j+l=i} m_j(G_1) m_l(G_2).$$

Proof. Since every vertex of G_1 is adjacent to every vertex of G_2, every colour-class of vertices in G is either a colour-class in G_1 or a colour-class in G_2. Now the result follows. □

Corollary 1.2.9. *The chromatic polynomial of the join $G_1 + G_2$ is given by*

$$C(G_1 + G_2; u) = C(G_1; u) \circ C(G_2; u),$$

where the "$\circ$" signifies that we write each polynomial in the form $\sum m_i u_i$ and multiply as if u_i were the power u^i.

Definition 1.2.10 (Rank polynomial)**.** The rank polynomial of a general graph G is a two-variable polynomial in indeterminates x and y given by by

$$R(G; x, y) := \sum_{S \subseteq E(G)} x^{r\langle S \rangle} y^{s\langle S \rangle},$$

where $r\langle S \rangle$ and $s\langle S \rangle$ are the rank and co-rank of the subgraph $\langle S \rangle$ of G, and $\langle S \rangle$ is the subgraph of G consisting of the set of edges in S and the vertices incident with edges in S.

If we write $R(G; x, y) = \sum \rho_{rs} x^r y^s$, then ρ_{rs} is the number of subgraphs of G with rank r and co-rank s, and we say that the matrix (ρ_{rs}) is the rank matrix of G.

Theorem 1.2.11. *The chromatic polynomial of a graph G with n vertices has an expansion in terms of subgraphs as follows:*

$$C(G; u) = \sum_{S \subseteq E(G)} (-1)^{|S|} u^{n - r\langle S \rangle}$$

The demonstration can be found in Biggs [2].

Corollary 1.2.12. *The chromatic polynomial and the rank polynomial of a general graph G with n vertices are related by the identity*

$$C(G; u) = u^n R(G; -u^{-1}, -1). \tag{1.1}$$

If the chromatic polynomial is

$$C(G; u) = b_0 u^n + \ldots + b_{n-1} u + b_n$$

then the coefficients b_i can be expressed in terms of the entries in the rank matrix, as follows: $(-1)^i b_i = \sum_j (-1)^j \rho_{ij}$.

Proof. By definition and the previous theorem,

$$\begin{aligned} u^n R(G; -u^{-1}, -1) &= u^n \Big(\sum_{S \subset E(G)} (-u^{-1})^{r\langle S \rangle} (-1)^{s\langle S \rangle} \Big) \\ &= \sum_{S \subset E(G)} (-1)^{|S|} u^{n-r\langle S \rangle} \\ &= C(G; u) \end{aligned}$$

proving Equation 1.1. In terms of the coefficients, we have

$$\begin{aligned} \sum_i b_i u^{n-i} &= C(G; u) \\ &= u^n R(G; -u^{-1}, -1) \\ &= u^n \sum_{r,s} \rho_{rs} (-u)^{-r} (-1)^s \\ &= \sum_r \sum_s (-1)^{r+s} \rho_{rs} u^{n-r}, \end{aligned}$$

which implies the result. □

The formula for the coefficients expresses b_i as an alternating sum of the entries in the i-th row of the rank matrix.

1.3 The Tutte polynomial

Definition 1.3.1 (Rank generating polynomial)**.** Let $G = (V, E)$ be a graph. We say that its rank generating polynomial is given by

$$S(G; x, y) = \sum_{S \subseteq E(G)} x^{r\{E\} - r\{S\}} y^{s\{S\}}$$

Here we write $\{S\}$ for the graph (V, S).

Remark 1.3.2. We have $S(G) = S(G; x, y) = \sum_{S \subseteq E(G)} x^{k\{S\} - k\{E\}} y^{s\{S\}}$, where $k\{S\}$ is the number of components of S.

Theorem 1.3.3. *Let $G = (V,E)$ be a graph with $e \in E(G)$. Then,*

$$S(G;x,y) = \begin{cases} (x+1)S(G-e;x,y), & \text{if } e \text{ is a bridge;} \\ (y+1)S(G-e;x,y), & \text{if } e \text{ is a loop;} \\ S(G-e;x,y) + S(G/e;x,y), & \text{otherwise.} \end{cases}$$

Furthermore, $S(E_n;x,y) = 1$ for the empty n-vertex graph E_n, $n \geq 1$.

Proof. Let $G' = G - e$, $G'' = G/e$ be and write r' and s' for the rank and co-rank functions in G', and r'' and s'' for those in G''.

If $e \in E(G)$ and $S \subseteq E(G) - e$ then $r\{S\} = r'\{S\}$, $s\{S\} = s'\{S\}$, $r\{E\} - r\{S \cup e\} = r''\{E-e\} - r''\{S\} = r(G'') - r''\{S\}$,

$$r\{E\} = \begin{cases} r'\{E-e\} + 1, & \text{if } e \text{ is a bridge;} \\ r'\{E-e\}, & \text{otherwise.} \end{cases}$$

and

$$s\{S \cup e\} = \begin{cases} s''\{S\} + 1, & \text{if } e \text{ is a loop;} \\ s''\{S\}, & \text{otherwise.} \end{cases}$$

Let us split $S(G;x,y)$ as follows: $S(G;x,y) = S_0(G;x,y) + S_1(G;x,y)$ where $S_0(G;x,y) = \sum\limits_{S \subseteq E(G); e \notin S} x^{r\{E\}-r\{S\}} y^{s\{S\}}$ and $S_1(G;x,y) = \sum\limits_{S \subseteq E(G); e \in S} x^{r\{E\}-r\{S\}} y^{s\{S\}}$. Thus,

$$\begin{aligned} S_0(G;x,y) &= \sum_{S \subseteq E-e} x^{r\{E\}-r\{S\}} y^{s\{S\}} \\ &= \begin{cases} \sum\limits_{S \subseteq E(G')} x^{r'\{E-e\}+1-r'\{S\}} y^{s'\{S\}}, & \text{if } e \text{ is a bridge;} \\ \sum\limits_{S \subseteq E(G')} x^{r'\{E-e\}-r'\{S\}} y^{s'\{S\}}, & \text{otherwise.} \end{cases} \\ &= \begin{cases} xS(G-e;x,y), & \text{if } e \text{ is a bridge;} \\ S(G-e;x,y), & \text{otherwise.} \end{cases} \end{aligned}$$

$$
\begin{aligned}
S_1(G;x,y) &= \sum_{S\subseteq E-e} x^{r\{E\}-r\{S\cup e\}} y^{s\{S\cup e\}} \\
&= \begin{cases} \displaystyle\sum_{S\subseteq E(G'')} x^{r(G'')-r''\{S\}} y^{s''\{S\}+1}, & \text{if } e \text{ is a loop;} \\ \displaystyle\sum_{S\subseteq E(G'')} x^{r(G'')-r''\{S\}} y^{s''\{S\}+1}, & \text{otherwise.} \end{cases} \\
&= \begin{cases} yS(G/e;x,y), & \text{if } e \text{ is a loop;} \\ S(G/e;x,y), & \text{otherwise.} \end{cases}
\end{aligned}
$$

Hence,

$$
S(G) = \begin{cases} xS(G-e) + S(G/e), & \text{if } e \text{ is a bridge;} \\ yS(G/e) + S(G-e), & \text{if } e \text{ is a loop;} \\ S(G-e) + S(G/e), & \text{otherwise.} \end{cases}
$$

and

$$
S(G) = \begin{cases} (x+1)S(G-e), & \text{if } e \text{ is a bridge;} \\ (y+1)S(G-e), & \text{if } e \text{ is a loop;} \\ S(G-e) + S(G/e), & \text{otherwise.} \end{cases}
$$

□

We define the Tutte polynomial of a graph in terms of its rank polynomial.

Definition 1.3.4 (Tutte polynomial)**.** The Tutte polynomial of a graph G is defined in terms of the rank generating polynomial as the polynomial

$$T_G(x,y) = S(G;x-1,y-1).$$

Proposition 1.3.5. *Let $G = (V,E)$ be a graph with $e \in E(G)$. Then, $T_{E_n}(x,y) = 1$ and*

$$
T_G = \begin{cases} xT_{G-e}, & \textit{if } e \textit{ is a bridge;} \\ yT_{G-e}, & \textit{if } e \textit{ is a loop;} \\ T_{G-e} + T_{G/e}, & \textit{otherwise.} \end{cases}
$$

Proof. Using Definition 1.3.4 apply Theorem 1.3.3. □

Remark 1.3.6. T is also the unique function on graphs such that:

1. if G has b bridges, l loops and no other edges then $T(G) = x^b y^l$;

2. if G has no bridges or loops then there is an edge $e \in E(G)$ such that $T_G = T_{G-e} + T_{G/e}$.

1.4 The universal form of the Tutte polynomial

Let $\mathcal{G}$ be the set of all isomorphism class of finite multigraphs, for convenience, we will consider the elements of $\mathcal{G}$ to be graphs rather than isomorphism class of graphs. Also, we shall refer to the elements of $\mathcal{G}$ as graphs.

We will show that the Tutte polynomial is easily lifted to a more general polynomial. For this, we see that

$$\deg_x T_G(x,y) = \max\{r(G) - r\{S\} \mid S \subseteq E(G)\} = r(G)$$

and

$$\deg_y T_G(x,y) = \max\{s\{S\} \mid S \subseteq E(G)\} = s(G).$$

Theorem 1.4.1. *There is a unique map $U : \mathcal{G} \to \mathbb{Z}[x,y,\alpha,\sigma,\tau]$ such that*

$$U(E_n) = U(E_n; x,y,\alpha,\sigma,\tau) = \alpha^n,$$

and for every $n \geq 1$, and for every $e \in E(G)$ we have

$$U(G) = \begin{cases} xU(G-e), & \text{if } e \text{ is a bridge;} \\ yU(G-e), & \text{if } e \text{ is a loop;} \\ \sigma U(G-e) + \tau U(G/e), & \text{otherwise.} \end{cases}$$

Furthermore,

$$U(G) = \alpha^{k(G)} \sigma^{s(G)} \tau^{r(G)} T_G\left(\frac{\alpha x}{\tau}, \frac{y}{\sigma}\right). \tag{1.2}$$

We call the polynomial U above the **universal polynomial of graphs**. If R is a commutative ring and $x,y,\alpha,\sigma,\tau \in R$ then there is a unique map $\mathcal{G} \to R$ satisfying the conditions of the theorem. The Tutte polynomial is just U evaluated at $\alpha = \sigma = \tau = 1$.

The polynomial U is multiplicative in the following sense: If G_1 and G_2 are vertex disjoint graphs then

$$U(G_1 \cup G_2) = U(G_1)U(G_2).$$

In fact, $U(G_1 \cup G_2) = \alpha^{k(G_1)+k(G_2)}\sigma^{s(G_1)+s(G_2)}\rho^{r(G_1)+r(G_2)}T_{G_1}T_{G_2}$. If G_1 and G_2 share one vertex then

$$U(G_1 \cup G_2) = \frac{U(G_1)U(G_2)}{\alpha}.$$

Indeed, $U(G_1 \cup G_2) = \alpha^{k(G_1)+k(G_2)-1}\sigma^{s(G_1)+s(G_2)}\rho^{r(G_1)+r(G_2)}T_{G_1}T_{G_2}$.

Remark 1.4.2. 1. If L is a loop graph then $U(L) = \alpha y$;

2. $U(K_2) = \alpha^2 x$;

3. $U(K_1) = \alpha$.

Thus, U is determined by this multiplicativity property together with the conditions above.

Theorem 1.4.3. *The chromatic polynomial $C(G;u)$ of a graph G is a specialisation of the Tutte polynomial:*

$$C(G;u) = (-1)^{r(G)}u^{k(G)}T_G(1-u,0).$$

Proof. The chromatic polynomial has the following properties:

$$C(E_n;u) = u^n,$$

and for every edge $e \in E(G)$,

$$C(G;u) = C(G-e;u) - C(G/e;u).$$

We can see the above properties by Equation 1.1 and Proposition 1.2.6.

If e is a loop, then

$$\begin{aligned} C(G;u) &= C(G-e;u) - C(G-e;u) \\ &= 0. \end{aligned}$$

If e is a bridge, then

$$\begin{aligned} C(G;u) &= C(G-e;u) - \frac{1}{x}C(G-e;u) \\ &= \frac{x-1}{x}C(G-e;u). \end{aligned}$$

Thus, by previous theorem,

$$\begin{aligned} C(G;u) &= U(G,\frac{u-1}{u},0,u,1,-1) \\ &= (-1)^{r(G)}u^{k(G)}T_G(1-u,0). \end{aligned}$$

$\square$

Chapter 2

Hopf algebras

The purpose of this chapter is to provide an introduction to Hopf algebras, which are bialgebras with an additional map (the coinverse map) satisfying the coinverse property. We assume basic facts about groups, rings, fields, vector spaces and modules. Contents of this chapter are based on [25], [12] and [8].

Throughout this chapter, K is a field, and $\otimes = \otimes_K$ denotes tensor product over K.

2.1 Tensor products

Let $M_1, M_2, \cdots, M_n$ and A be a collection of R-modules, where R is a commutative ring with unity.

Definition 2.1.1. A function $f\colon M_1 \times M_2 \times \cdots \times M_n \to A$ is R-n-linear if for all i, $1 \leq i \leq n$ and all $a_i, b_i \in M_i$, $r \in R$,

1. $f(a_1, a_2, \ldots, a_i + b_i, \ldots, a_n) = f(a_1, a_2, \ldots, a_i, \ldots, a_n) + f(a_1, a_2, \ldots, b_i, \ldots, a_n)$,
2. $f(a_1, a_2, \ldots, ra_i, \ldots, a_n) = rf(a_1, a_2, \ldots, a_i, \ldots, a_n)$.

An R-bilinear map is an R-2-linear map.

Definition 2.1.2 (Tensor product)**.** A tensor product of $M_1, M_2, \cdots, M_n$ over R is an

R-module $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ together with an R-n-linear map

$$f\colon M_1 \times M_2 \times \cdots \times M_n \to M_1 \otimes M_2 \otimes \cdots \otimes M_n$$

so that for every R-module A and R-n-linear map $h\colon M_1 \times M_2 \times \cdots \times M_n \to A$ there exists a unique R-module map $\widehat{h}\colon M_1 \otimes M_2 \otimes \cdots \otimes M_n \to A$ for which

$$\widehat{h}f = h.$$

Let $F\langle M_1 \times M_2 \times \cdots \times M_n\rangle$ denote the free R-module on the set $M_1 \times M_2 \times \cdots \times M_n$. Let J be the submodule of $F\langle M_1 \times M_2 \times \cdots \times M_n\rangle$ generated by quantities of the form

$$(a_1,a_2,\cdots,a_i+b_i,\cdots,a_n)-(a_1,a_2,\cdots,a_i,\cdots,a_n)-(a_1,a_2,\cdots,b_i,\cdots,a_n),$$

$$(a_1,a_2,\cdots,ra_i,\cdots,a_n)-r(a_1,a_2,\cdots,a_i,\cdots,a_n),$$

for all i, $1 \leq i \leq n$, and $a_i, b_i \in M_i$, $r \in R$. Let

$$\iota\colon M_1 \times M_2 \times \cdots \times M_n \to F\langle M_1 \times M_2 \times \cdots \times M_n\rangle$$

be the natural inclusion map and let

$$s\colon F\langle M_1 \times M_2 \times \cdots \times M_n\rangle \to F\langle M_1 \times M_2 \times \cdots \times M_n\rangle/J$$

be the canonical surjection. Let $f = s\iota$. Then the quotient space $F\langle M_1 \times M_2 \times \cdots \times M_n\rangle/J$ together with the map f is a tensor product.

Proposition 2.1.3. *$F\langle M_1 \times M_2 \times \cdots \times M_n\rangle/J$ together with the map f is a tensor product of $M_1, M_2, \cdots, M_n$ over R.*

The proof can be found in Underwood [25]. As a consequence of the previous proposition, we write

$$F\langle M_1 \times M_2 \times \cdots \times M_n\rangle/J = M_1 \otimes M_2 \otimes \cdots \otimes M_n.$$

The tensor product has some interesting properties.

Proposition 2.1.4. *Let M_1, M_2, M_3 be R-modules and let N_1 be an R-submodules of M_1 and let N_2 be an R-submodule of M_2. Then there is an isomorphism of R-modules*

- $M_1/N_1 \otimes M_2/N_2 \cong (M_1 \otimes M_2)/(N_1 \otimes M_2 + M_1 \otimes N_2)$,
- $M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3$.

Proposition 2.1.5. *Let $M_1, M_2, \cdots, M_n, N_1, N_2, \cdots, N_n$ be R-modules and let $f_i \colon M_i \to N_i$, for $1 \leq i \leq n$, be R-module maps. There exists a unique map of R-modules*

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \colon M_1 \otimes M_2 \otimes \cdots \otimes M_n \to N_1 \otimes N_2 \otimes \cdots \otimes N_n$$

such that

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = f_1(a_1) \otimes f_2(a_2) \otimes \cdots \otimes f_n(a_n)$$

for all $a_i \in M_i$.

Corollary 2.1.6. *Let K be a field and let V_i, $1 \leq i \leq n$, be a finite set of vector spaces over K. Then*

$$V_1^* \otimes V_2^* \otimes \cdots V_n^* \subseteq (V_1 \otimes V_2 \otimes \cdots \otimes V_n)^*.$$

2.2 Algebras and coalgebras

Definition 2.2.1 (Algebra)**.** A K**-algebra** is a triple (A, m, λ), where A is a vector space over K, and $m \colon A \otimes A \to A$ and $\lambda \colon K \to A$ are K-linear maps that satisfy the following conditions.

1. (Multiplication is associative.) The following diagram commutes:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{Id} \otimes m"] \arrow[d, "m \otimes \mathrm{Id}"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

where the map Id is the identity map. Equivalently, for all $a, b, c \in A$, we have

$$m(\mathrm{Id} \otimes m)(a \otimes b \otimes c) = m(m \otimes \mathrm{Id})(a \otimes b \otimes c).$$

2. The following diagram commutes:

$$\begin{tikzcd}
A \otimes K \arrow[r, "\mathrm{Id} \otimes \lambda"] \arrow[d, "s_2"] & A \otimes A \arrow[dl, "m"] \\
A & K \otimes A \arrow[l, "s_1"'] \arrow[u, "\lambda \otimes \mathrm{Id}"']
\end{tikzcd}$$

The map $s_1 \colon K \otimes A \to A$ is defined by $r \otimes a \mapsto ra$, and the map $s_2 \colon A \otimes K \to A$ is defined by $a \otimes r \mapsto ra$. Equivalently, we have for all $r \in K$ and for all $a \in A$,

$$(\mathrm{Id} \otimes \lambda)(a \otimes r) = ra = m(\lambda \otimes \mathrm{Id})(r \otimes a).$$

The map m is called the **multiplication (or product) map**, and the map λ is called the **unit map**.

Remark 2.2.2. We can also define an algebra A as a module over a commutative ring R with a multiplication map and unit map. For most parts of the dissertation, we use Definition 2.2.1

Definition 2.2.3. A K**-algebra** is a ring A with unity 1 together with a ring homomorphism $\lambda \colon K \to A$ which satisfies $\lambda(r)a = a\lambda(r)$ for $a \in A$ and $r \in R$. Then A is a vector space over K with scalar multiplication given by

$$ra := \lambda(r)a := a\lambda(r),$$

for $r \in K$, $a \in A$.

Proposition 2.2.4. *The definition 2.2.1 is equivalent the definition 2.2.3.*

Definition 2.2.5 (Commutative algebra)**.** A K-algebra (A, m, λ) is **commutative** if $m\tau = m$, where τ denotes the **twist map** defined as $\tau(a \otimes b) := b \otimes a$ for $a, b \in A$.

Coalgebras are in some sense dual to algebras - they are obtained by reversing the arrows in the definition of algebras.

Definition 2.2.6 (Coalgebra)**.** A K**-coalgebra** is a triple (C, Δ, ϵ) consisting of a vector space C over K, and K-linear maps $\Delta\colon C \to C \otimes C$ and $\epsilon\colon C \to K$ that satisfy the following conditions.

1. The following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{Id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{Id}"] & C \otimes C \otimes C \end{tikzcd}$$

   Equivalently, for all $c \in C$, we have $(\mathrm{Id} \otimes \Delta)\Delta(c) = (\Delta \otimes \mathrm{Id})\Delta(c)$.

2. The following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "1 \otimes -"] \arrow[d, "- \otimes 1"] \arrow[dr, "\Delta"] & K \otimes C \\ C \otimes K & C \otimes C \arrow[l, "\mathrm{Id} \otimes \epsilon"] \arrow[u, "\epsilon \otimes \mathrm{Id}"] \end{tikzcd}$$

   Here the maps $- \otimes 1$ and $1 \otimes -$ are defined by $c \mapsto c \otimes 1$ and $c \mapsto 1 \otimes c$, respectively. Equivalently, for all $c \in C$, we have

$$(\epsilon \otimes \mathrm{Id})\Delta(c) = 1 \otimes c \text{ and}$$
$$(\mathrm{Id} \otimes \epsilon)\Delta(c) = c \otimes 1.$$

The map Δ is called the **comultiplication (or coproduct) map**, and the map ϵ is called the **counit map**. The first condition above says that the comultiplication is **coassociat-**

ive.

Sometimes we use a notation introduced by Sweendler [21] to simplify long expressions involving comultiplication. We write

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}.$$

Definition 2.2.7 (Cocommutative coalgebra)**.** A K-coalgebra (C, Δ, ϵ) is **cocommutative** if $\tau(\Delta(c)) = \Delta(c)$ for all $c \in C$.

Example 2.2.8 (Polynomial ring)**.** The polynomial ring $K[x]$ is a K-algebra with $m\colon K[x] \otimes K[x] \to K[x]$ given by ordinary polynomial multiplication and $\lambda\colon K \to K[x]$ defined as $r \mapsto r1$, for all $r \in K$. Moreover, $K[x]$ may be given a K-coalgebra structure in two ways, with K-linear maps defined on $\{1, x, x^2, ...\}$ as follows:

1. $\Delta_1(x^m) = \displaystyle\sum_{i=0}^{m} \binom{m}{i} x^i \otimes x^{m-i}$ and $\epsilon_1(x^m) = \delta_{0,m}$, where δ function is defined as $\delta_{i,j} := 1$ if $i = j$, and 0 otherwise.
2. $\Delta_2(x^m) = x^m \otimes x^m$ and $\epsilon_2(x^m) = 1$.

(The maps are extended linearly to all polynomials.)

Definition 2.2.9 (Homomorphisms and isomorphisms)**.**

1. Let (A, m_A, λ_A) and (B, m_B, λ_B) be K-algebras. A K**-algebra homomorphism** from A to B is a map of additive groups $\phi\colon A \to B$ for which

$$\begin{aligned} \phi(1_A) &= 1_B, \\ \phi(m_A(a \otimes b)) &= m_B(\phi(a) \otimes \phi(b)), \\ \phi(\lambda_A(r)) &= \lambda_B(r), \end{aligned}$$

   for $a, b \in A$, $r \in K$. In particular, for A to be a subalgebra of B we require that $1_A = 1_B$.

2. Let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be K-coalgebras. A K-linear map $\phi\colon C \to D$ is a

K**-coalgebra homomorphism** if

$$\begin{aligned}(\phi \otimes \phi)\Delta_C(c) &= \Delta_D(\phi(c)),\\ \epsilon_C(c) &= \epsilon_D(\phi(c))\end{aligned}$$

for all $c \in C$.

A homomorphism ϕ is an **isomorphism** if ϕ is injective and surjective.

2.3 Morphisms and quotient structures in algebras and coalgebras

Definition 2.3.1 (Ideal)**.** Let (A,m,λ) be a K-algebra. A ideal in A is a subspace I of A with $m(A \otimes I) \subset I$ and $m(I \otimes A) \subset I$.

Proposition 2.3.2. *Let (A, m, λ) be a K-algebra. Let I be an ideal of A. Then the quotient space A/I is a K-algebra.*

Proof. We need to define a multiplication map $\widehat{m}$ and a unit map $\widehat{\lambda}$ of A/I.

Let $s\colon A \to A/I$ denote the canonical quotient map. The composition $s \circ m\colon A \otimes A \to A/I$ is a map of K-vector spaces defined as $(s \circ m)(a \otimes b) := ab + I$. Note that $I \otimes A + A \otimes I$ is a subspace of $A \otimes A$. Let $a \otimes b + c \otimes d \in I \otimes A + A \otimes I$ for $a, d \in I$ and $b, c \in A$. Since I is an ideal, $m(a \otimes b + c \otimes d) = ab + cd \in I$, we have $I \otimes A + A \otimes I \subseteq \ker(s \circ m)$. Thus by the universal mapping property for kernels, there is a map of vector spaces

$$\overline{s \circ m}\colon (A \otimes A)/(I \otimes A + A \otimes I) \to A/I$$

defined as

$$\overline{s \circ m}(a \otimes b + (I \otimes A + A \otimes I)) := ab + I.$$

There is an isomorphism of vector spaces

$$\widehat{\beta}\colon A/I \otimes A/I \to (A \otimes A)/(I \otimes A + A \otimes I)$$

given by

$$\widehat{\beta}((a + I) \otimes (b + I)) := a \otimes b + (I \otimes A + A \otimes I).$$

Let $\overline{m} = \overline{s \circ m} \circ \widehat{\beta}$ be defined by $(a+I) \otimes (b+I) \mapsto ab + I$. One can check that $\overline{m}$ satisfies the associative property since m does.

For the unit map of A/I, let $\widehat{\lambda} = s \circ \lambda$. Then $\widehat{\lambda}$ satisfies the unit property. Thus $(A/I, \widehat{m}, \widehat{\lambda})$ is a K-algebra. $\square$

Definition 2.3.3 (Coideal)**.** Let C be a K-coalgebra. A subspace $I \subset C$ is a **coideal** of C if $\Delta(I) \subset I \otimes C + C \otimes I$ and $\epsilon(I) = 0$.

Proposition 2.3.4. *Let $I \subset C$ be a coideal of C. Then the quotient space C/I is a K-coalgebra.*

Definition 2.3.5 (Quotient algebra and quotient coalgebra)**.** If I is an ideal of A, then the K-algebra A/I is the quotient algebra of A by I. If I é a coideal of C, then the coalgebra C/I is the quotient coalgebra of C by I.

Example 2.3.6. For $m \geq 1$, let I be the subspace of $K[x]$ generated by the basis $\{x^m - 1, x^{m+1} - 1, ...\}$. For $i \geq 0$, $\Delta_2(x^i - 1) = (x^i - 1) \otimes 1 + 1 \otimes (x^i - 1) + (x^i - 1) \otimes (x^i - 1)$. Hence $\Delta_2(I) \subset I \otimes K[x] + K[x] \otimes I$. Also, $\epsilon_2(I) = 0$, so I is a coideal of $K[x]$. The quotient coalgebra $K[x]/I$ is a vector space of dimension m on the basis $\{1, x, x^2, ..., x^{m-1}\}$.

Definition 2.3.7 (Grouplike elements)**.** Let C be a K-coalgebra. A non-zero element c of C for which $\Delta(c) = c \otimes c$ is a **grouplike element** of C.

In the next proposition, we show an interesting property.

Proposition 2.3.8. *Let $\phi \colon C \to D$ be a homomorphism of K-coalgebras. If c is a grouplike element of C, then $\phi(c)$ is a grouplike element of D.*

Proof. By hypothesis, $\Delta_D(\phi(c)) = (\phi \otimes \phi)\Delta_C(c)$. Since c is grouplike, we have

$$(\phi \otimes \phi)\Delta_C(c) = (\phi \otimes \phi)(c \otimes c) = \phi(c) \otimes \phi(c).$$

Therefore $\phi(c)$ is grouplike. $\square$

2.4 Duality

In this section we consider the linear duals of algebras and coalgebras. We show that if (C, Δ, ϵ) is a coalgebra, then (C^*, m, λ) is an algebra, where C^* is the linear dual of C, and the maps m and λ are induced by the transpose of Δ and ϵ, respectively. However, if A is an algebra, then A^* is not always a coalgebra. In general, if A is an algebra then $A^\circ \subset A^*$, called the finite dual (see definition 2.4.3) , is a coalgebra.

Proposition 2.4.1. *If (C, Δ, ϵ) is a coalgebra, then C^* is an algebra.*

Proof. To show that C^* is a K-algebra we construct a multiplication map m and a unit map λ, and show that they satisfy the associative and unit properties, respectively.

The transpose of Δ is a K-linear map $\Delta^* \colon (C \otimes C)^* \to C^*$ defined as

$$\Delta^*(\psi)(c) := \psi(\Delta(c)) \text{ for } \psi \in (C \otimes C)^* \text{ and } c \in C.$$

Since $C^* \otimes C^* \subseteq (C \otimes C)^*$ (see Corollary 2.1.6), Δ^* restricts to a K-linear map $m \colon C^* \otimes C^* \to C^*$ defined as

$$m(f \otimes g)(c) = \Delta^*(f \otimes g)(c) = (f \otimes g)(\Delta(c)).$$

We can verify that map m satisfies the associative property.

The transpose of the counit map of C is $\epsilon^* \colon K^* \to C^*$ defined as $\epsilon^*(f)(c) := f(\epsilon(c))$ for $f \in K^*$, $c \in C$. Identifying $K = K^*$, we have $\epsilon^* \colon K \to C^*$ defined as $\epsilon^*(r)(c) := r(\epsilon(c)) = r\epsilon(c)$, for $r \in K$, $c \in C$. Define $\lambda = \epsilon^*$. The map λ satisfies the unit property.

Thus (C^*, m, λ) is an algebra. □

Definition 2.4.2 (Cofinite ideal)**.** Let (A, m, λ) be a K-algebra. An ideal I of A is **cofinite** iff the quotient space A/I is finite dimensional. Let f be an element of A^* and let S be a subset of A. Then f vanishes on S if $f(s) = 0$ for all $s \in S$.

Definition 2.4.3 (Finite dual)**.** Let A be a K-algebra. The finite dual A° of A is the subspace of A^* defined as $A^\circ = \{f \in A^* \mid f$ vanishes on some ideal $I \subset A$ of finite codimension $\}$.

Example 2.4.4. Let $e_i \in K[x]^*$ be defined as $e_i(x^j) = \delta_{i,j}$ for $i,j \geq 0$. Then $e_i \in K[x]^\circ$.

In fact, e_i vanishes on the ideal (x^{i+1}) and $\dim(K[x]/(x^{i+1})) = i + 1$.

The following two propositions can be found in Underwood [25].

Proposition 2.4.5. *If A is finite dimensional as a K-vector space, then $A^\circ = A^*$.*

Proposition 2.4.6. *If A is an algebra, then A° is a coalgebra.*

2.5 Bialgebras

Definition 2.5.1 (Bialgebra)**.** A K-bialgebra is a K-vector space B together with maps m, λ, Δ, ϵ that satisfy the following conditions:

1. (B, m, λ) is a K-algebra and (B, Δ, ϵ) is a K-coalgebra,
2. Δ and ϵ are homomorphisms of K-algebras.

Since B is a K-algebra the tensor product $B \otimes B$ has the structure of a K-algebra with multiplication $m_{B\otimes B} \colon (B \otimes B) \otimes (B \otimes B) \to B \otimes B$ defined by

$$m_{B\otimes B}((a \otimes b) \otimes (c \otimes d)) = ac \otimes bd \tag{2.1}$$

for a,b,c,$d \in B$. The unit map $\lambda_{B\otimes B} \colon K \to B \otimes B$ is given as

$$\lambda_{B\otimes B}(r) = \lambda_B(r) \otimes 1_B$$

for $r \in K$.

Definition 2.5.2 (Primitive element)**.** Let B be a bialgebra. An element $b \in B$ is a primitive element of B if $\Delta(b) = 1 \otimes b + b \otimes 1$.

Definition 2.5.3 (Monoid)**.** A monoid is a pair (S,b) where S is a set and $b : S \times S \to S$ is a binary operation, for x,$y \in S$ denote $b(x,y) = xy$, which satisfy the following properties:

1. (associativity) for any x,y,$z \in S$ we have $(xy)z = x(yz)$.
2. (identity) there exists such $e \in S$, that for any $x \in S$ we have $ex = xe = x$.

Example 2.5.4 (Monoid bialgebra)**.** Let K be a field, let S be a monoid. Then the monoid ring KS is a K-algebra with multiplication map $m \colon KS \otimes KS \to KS$ defined as $m(a \otimes$

$b) := ab$ and unit map $\lambda\colon K \to KS$ given by $\lambda(r) := r$ for all $a,b \in S$, $r \in K$. Moreover, KS is a K-coalgebra with comultiplication map $\Delta\colon KS \to KS \otimes KS$ being the map defined by

$$\Delta(\sum_{s \in S} r_s s) := \sum_{s \in S} r_s (s \otimes s),$$

and counit map $\epsilon\colon KS \to K$ defined by

$$\epsilon(\sum_{s \in S} r_s s) = \sum_{s \in S} r_s.$$

We can verify that Δ and ϵ are homomorphisms of K-algebras, and so KS is a K-bialgebra.

Example 2.5.5 (Polynomial bialgebra with x grouplike)**.** Let $K[x]$ be the K-algebra and K-coalgebra given by Example 2.2.8 with comultiplication Δ_2 then $(K[x], m, \lambda, \Delta_2, \epsilon_2)$ is a bialgebra.

Example 2.5.6 (Polynomial bialgebra with x primitive)**.** Let $K[x]$ be the K-algebra and K-coalgebra given by Example 2.2.8 with comultiplication Δ_1 then $(K[x], m, \lambda, \Delta_1, \epsilon_1)$ is a bialgebra.

Definition 2.5.7 (Bialgebra homomorphism)**.** Let B, B' be bialgebras. A K-linear map $\phi\colon B \to B'$ is a bialgebra homomorphism if ϕ is both an algebra and coalgebra homomorphism.

The following proposition can be found in Underwood [25].

Proposition 2.5.8. *Suppose the polynomial algebra $K[x]$ is given the structure of a K-bialgebra. Then there is some $z \in K[x]$ so that $K[z] = K[x]$, and z is either grouplike or z is primitive.*

By the above proposition, the bialgebra structures on $K[x]$ given in examples 2.5.5 and 2.5.6 are the only bialgebra structures on $K[x]$ up to algebra isomorphism.

Definition 2.5.9 (Biideal)**.** Let B a K-bialgebra. A biideal I is a K-subspace of B that is both an ideal and a coideal.

Proposition 2.5.10. *Let $I \subset B$ be a biideal of B. Then B/I is a K-bialgebra.*

Proof. From Proposition 2.3.2, we have that B/I is a K-algebra. By Proposition 2.3.4, B/I is a K-coalgebra. One notes that $\Delta_{B/I}$ is an algebra map since Δ is an algebra map. Moreover, $\epsilon_{B/I}$ is an algebra map since that property holds for ϵ. $\square$

Proposition 2.5.11. *If B is a bialgebra, then B° is a bialgebra.*

2.6 Hopf Algebras

A K-Hopf algebra H is a K-bialgebra with an additional map called the **coinverse (or antipode)** satisfying the coinverse (or antipode) property.

Definition 2.6.1 (Hopf algebra)**.** A K-Hopf algebra is a bialgebra $H = (H, m, \lambda, \Delta, \epsilon)$ over K together with a K-linear map $S\colon H \to H$ that satisfies

$$m(\mathrm{Id} \otimes S)\Delta(h) = \lambda(\epsilon(h)) = m(S \otimes \mathrm{Id})\Delta(h)$$

for all $h \in H$. The map S is the coinverse (or antipode) map.

Example 2.6.2. Let G be a finite group. Let KG be the monoid bialgebra of Example 2.5.4. Define a coinverse map $S\colon KG \to KG$ by

$$S(\tau) = \tau^{-1},$$

for $\tau \in G$. Then KG is a K-Hopf algebra.

Example 2.6.3. Let $K[x]$ be the polynomial bialgebra with x primitive (Example 2.5.6). Define the coinverse map $S\colon K[x] \to K[x]$ by $S(x^i) = (-x)^i$, for $i \geq 0$. Then $K[x]$ is a K-Hopf algebra.

A Hopf algebra H is **commutative** if it is a commutative algebra; H is **cocommutative** if it is a cocommutative coalgebra.

Definition 2.6.4 (Convolution)**.** Let C be a K-coalgebra and let A be a K-algebra. Let $\mathrm{hom}(C, A)$ denote the collection of linear transformations $\phi\colon C \to A$. On $\mathrm{hom}(C, A)$ we can define a multiplication as follows. For $f, g \in \mathrm{hom}(C, A)$, $a \in C$,

$$(f * g)(a) = m(f \otimes g)\Delta(a).$$

This multiplication is called convolution.

Proposition 2.6.5. *Let C be a K-coalgebra and let A be a K-algebra. Then $\mathrm{hom}(C, A)$ together with convolution $*$ is a monoid.*

Proof. 1. (Associativity) For $f,g,h \in \hom(C,A)$, we have

$$\begin{aligned}(f*(g*h))(a) &= m(f\otimes(g*h))\Delta(a)\\ &= \sum_{(a)} f(a_{(1)})(g*h)(a_{(2)})\\ &= \sum_{(a)} f(a_{(1)} \sum_{(a_{(2)})} g(a_{(2)_{(1)}})h(a_{(2)_{(2)}})\\ &= \sum_{(a)} f(a_{(1)})g(a_{(2)})h(a_{(3)}) \text{ (using Sweedler notation).}\end{aligned}$$

Now, by the coassociativity of Δ, we have

$$\begin{aligned}\sum_{(a)} f(a_{(1)})g(a_{(2)})h(a_{(3)}) &= \sum_{(a)}\sum_{(a_{(1)})} f(a_{(1)_{(1)}})g(a_{(1)_{(2)}})h(a_{(2)})\\ &= \sum_{(a)}(f*g)(a_{(1)})h(a_{(2)})\\ &= m((f*g)\otimes h)\Delta(a)\\ &= ((f*g)*h)(a),\end{aligned}$$

and so $*$ is associative.

2. (Identity element) The map $\lambda\epsilon$ serves as an identity element in $\hom(C,A)$. In fact, for $\phi \in \hom(C,A)$, $a \in C$,

$$\begin{aligned}(\lambda\epsilon*\phi)(a) &= m(\lambda\epsilon\otimes\phi)\Delta(a)\\ &= \sum_{(a)} \lambda(\epsilon(a_{(1)}))\phi(a_{(2)})\\ &= \sum_{(a)} \epsilon(a_{(1)})\lambda(1_K)\phi(a_{(2)})\\ &= \sum_{(a)} \epsilon(a_{(1)})1_A\phi(a_{(2)})\\ &= \sum_{(a)} \phi(\epsilon(a_{(1)})a_{(2)})\\ &= \phi(a),\end{aligned}$$

using the counit property and notation of Sweedler. Thus $\lambda\epsilon*\phi = \phi$. Similarly

$\phi * \lambda\epsilon = \phi$.

Therefore, $\hom(C, A)$ is a monoid under $*$. $\square$

Let H be a K-Hopf algebra. Let $\hom(H, H)$ be the monoid under convolution $*$. Then, we can verify that

$$S * \mathrm{Id} = \lambda\epsilon = \mathrm{Id} * S.$$

In other words, S is an inverse of Id under $*$.

Proposition 2.6.6. *Let H be a K-Hopf algebra with coinverse S. Then the following properties hold.*

1. $S(ab) = S(b)S(a)$ *for all* a,$b \in H$,
2. $S(1) = 1$.

Proposition 2.6.7. *Let H be a K-Hopf algebra with coinverse S. If H is cocommutative, then* $S^2 = \mathrm{Id}$.

Definition 2.6.8 (Hopf ideal)**.** Let H be a K-Hopf algebra. A Hopf ideal I is a biideal that satisfies $S(I) \subset I$.

Proposition 2.6.9. *Let $I \subset H$ be a Hopf ideal of H. Then H/I is a K-Hopf algebra.*

Definition 2.6.10 (Homomorphism of Hopf algebras)**.** Let H and H' be K-Hopf algebras. A bialgebra homomorphism $\phi\colon H \to H'$ is a homomorphism of Hopf algebras if $\phi(S(a)) = S'(\phi(a))$ for all $a \in H$. The Hopf homomorphism ϕ is an isomorphism of Hopf algebras if ϕ is a bijection.

Proposition 2.6.11. *Let H be a finite dimensional vector space over the field K. Then H is a K-Hopf algebra if and only if H^* is a K-Hopf algebra.*

Definition 2.6.12 (Graded module)**.** A graded K-module V is one with a K-module direct sum decomposition $V = \bigoplus_{n\geq 0} V_n$. Elements x in V_n are called homogeneous of degree n.

One endows tensor products $V \otimes W$ of graded K-modules V, W with graded module structure in which $(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j$.

Definition 2.6.13 (Graded map)**.** A K-linear map $\psi\colon V \to W$ between two graded K-modules is called graded if $\psi(V_n) \subset W_n$ for all n. Say that a K-algebra (coalgebra, bialgebra) is graded if it is a graded K-module and all of the relevant structure maps $(\lambda, \epsilon, m, \Delta)$ are graded. Say that a graded module V is connected if $V_0 \cong K$.

The following two propositions can be found in Grinberg [12].

Proposition 2.6.14. *A connected graded bialgebra H has a unique antipode S, which is a graded map $S\colon H \to H$, endowing it with a Hopf structure.*

Theorem 2.6.15 (Takeuchi's formula)**.** *In a connected graded Hopf algebra H, the antipode has formula*

$$S = \sum_{k \geq 0} m^{(k-1)} f^{\otimes k} \Delta^{(k-1)} \tag{2.2}$$

where $f := \mathrm{Id} - \lambda\epsilon$, $f^{\otimes k} := f \otimes \cdots \otimes f$ (k times) and $m^{-1} f^{\otimes 0} \Delta^{-1} = \lambda\epsilon$.

2.7 Modules over algebras and comodules over coalgebras

Definition 2.7.1 (Module)**.** Let A be a K-algebra. A **left module** over A is a K-vector space M together with a linear map $\alpha\colon A \otimes M \to M$ such that $\alpha(a \otimes x) := ax$ and $1_A x = x$ for all $a \in A$ and $x \in M$, and $a(by) = (ab)y$ for all $a, b \in A$ and $y \in M$. That is, the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{Id} \otimes \alpha"] \arrow[d, "m \otimes \mathrm{Id}"] & A \otimes M \arrow[d, "\alpha"] \\ A \otimes M \arrow[r, "\alpha"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K \otimes M \arrow[r, "\lambda \otimes \mathrm{Id}"] \arrow[dr, "\sim"'] & A \otimes M \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$

A **right A-module** is defined similarly.

Definition 2.7.2 (Comodule)**.** Let C be a K-coalgebra. A **left comodule** over C is a K-vector space M together with a coaction map $\phi\colon M \to C \otimes M$ such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\phi"'] & C \otimes M \arrow[d, "\mathrm{Id} \otimes \phi"] \\ C \otimes M \arrow[r, "\Delta \otimes \mathrm{Id}"] & C \otimes C \otimes M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "1 \otimes -"'] & C \otimes M \arrow[d, "\epsilon \otimes \mathrm{Id}"] \\ & K \otimes M \end{tikzcd}$$

A **right comodule** is defined similarly.

Definition 2.7.3 (Comodule-bialgebra)**.** Let B be a bialgebra over a field K. A comodule-bialgebra over B is a bialgebra in the category of B-comodules.

To be precise, a comodule-bialgebra over B is a bialgebra H endowed with a linear map $\phi\colon H \to B \otimes H$ such that:

- ϕ is a left coaction, i.e., for all $h \in H$, we have

$$(\Delta_B \otimes \mathrm{Id})\phi(h) = (\mathrm{Id} \otimes \phi)\phi(h) \text{ and}$$
$$(\epsilon_B \otimes \mathrm{Id})\phi(h) = 1 \otimes h.$$

- The coproduct Δ_H and the counit ϵ_H are morphisms of left B-comodules, where the comodule structure on K is given by the unit map λ_B, and the comodule structure on $H \otimes H$ is given by $\tilde{\phi} = (m_B \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ \tau_{23} \circ (\phi \otimes \phi)$, where τ_{23} stands for the flip of the two middle factors, i.e., for all $h \in H$, we have

$$\tilde{\phi} \circ \Delta_H(h) = (\mathrm{Id} \otimes \Delta_H)\phi(h) \text{ and}$$
$$(\mathrm{Id} \otimes \epsilon_H)\phi(h) = \lambda_B \circ \epsilon_H(h)$$

- m_H and λ_H are morphisms of left B-comodules. This amounts to saying that ϕ is an algebra morphism. In other words, for all $k \in K$ and h_1,$h_2 \in H$, we have

$$\phi \circ m_H(h_1 \otimes h_2) = (\mathrm{Id} \otimes m_H)\tilde{\phi}(h_1 \otimes h_2)$$
$$(\mathrm{Id} \otimes m_H)(m_B \otimes \mathrm{Id} \otimes \mathrm{Id}) = m_B \otimes m_H$$
$$(\mathrm{Id} \otimes \lambda_H)\lambda_B(k) = \phi \circ \lambda_H(k).$$

Definition 2.7.4 (Comodule-Hopf algebra)**.** A comodule-bialgebra H is a comodule-Hopf algebra if H is a Hopf algebra with antipode S such that $(\mathrm{Id} \otimes S)\phi(h) = \phi \circ S(h)$ for all $h \in H$.

Chapter 3

Algebraic aspects of the Tutte polynomial

In this chapter we give an exposition of a classic paper of Tutte [23]. Tutte defines a ring of graphs, and defines an ideal generated by the set of all deletion-contraction relations. He then shows that each coset of the ideal contains a single graph which has only vertices and loops. We also discuss topological invariants of graphs proposed by Tutte.

3.1 An algebra of graphs

Let $\mathcal{G}$ be the class of all isomorphism classes of finite graphs. We allow multiedges and loops. We make $\mathcal{G}$ a commutative monoid by defining multiplication in $\mathcal{G}$ by $G_1G_2 := G_1 \uplus G_2$ for any two graphs G_1 and G_2 in $\mathcal{G}$. Here $G_1 \uplus G_2$ denotes the disjoint union of G_1 and G_2. To be precise, we take disjoint representative graphs in the isomorphism classes G_1 and G_2, construct their union, and then define the product to be the isomorphism class of the union. The identity element in the monoid is the null graph.

Definition 3.1.1 (Graphic form)**.** Let $\mathbb{Z}\mathcal{G}$ be the monoid algebra of $\mathcal{G}$ over $\mathbb{Z}$. We consider $\mathcal{G}$ to be a subset of $\mathbb{Z}\mathcal{G}$ by identifying each $G \in \mathcal{G}$ with $1G \in \mathbb{Z}\mathcal{G}$. Elements of $\mathbb{Z}\mathcal{G}$ are finite linear combinations of isomorphism classes of graphs, with coefficients in $\mathbb{Z}$;

they are called **graphic forms**.

If $e \in E(G)$ then we say that e is a link if it is not a loop.

Definition 3.1.2 (W-function)**.** Let A be an additive abelian group. A function $w\colon \mathcal{G} \to A$ is called a W-function if $w(G) - w(G-e) - w(G/e) = 0$ for all $G \in \mathcal{G}$ for all links e in G. We say that a module morphism $w\colon \mathbb{Z}\mathcal{G} \to A$ (considering A as a $\mathbb{Z}$-module) is a W-function if its restriction to $\mathcal{G}$ is a W-function.

Definition 3.1.3 (V-function)**.** Let A be a commutative ring with identity. A function $w\colon \mathcal{G} \to A$ is called a V-function if it is a W-function and is multiplicative, i.e., $w(G_1G_2) = w(G_1)w(G_2)$ for all $G_1, G_2 \in \mathcal{G}$. A V-function from $\mathbb{Z}\mathcal{G}$ to A is a linear extension of a V-function from $\mathcal{G}$ to A.

Note that a V-function $w\colon \mathbb{Z}\mathcal{G} \to A$ is an algebra morphism (considering A to be a $\mathbb{Z}$-algebra.)

Example 3.1.4. The function $w\colon \mathcal{G} \to \mathbb{Z}$, where $w(G)$ equals the number of spanning trees in G, is a W-function.

Example 3.1.5. Let $w(G) := (-1)^{V(G)}C(G;n)$, where $C(G;n)$ is the value of the chromatic polynomial at $x = n$ (i.e., the number of proper colourings of G with n colours) and $V(G)$ is the number of vertices. Then $w\colon \mathcal{G} \to \mathbb{Z}$ is a W-function since the chromatic polynomial satisfies the deletion-contraction property.

Definition 3.1.6 (W-forms)**.** Let W be the submodule of $\mathbb{Z}\mathcal{G}$ generated by the set

$$\{G - (G-e) - (G/e) \mid G \in \mathcal{G}, e \in E(G), \text{ where } e \text{ is a link}\}.$$

The elements of W are called W**-forms**.

Lemma 3.1.7. *W is an ideal of $\mathbb{Z}\mathcal{G}$.*

Proof. If G is a graph and e is a link in G, and G_1 is another graph disjoint with G, then $GG_1 = G \uplus G_1$, and $(G-e)G_1 = (G \uplus G_1) - e$, and $(G/e)G_1 = (G \uplus G_1)/e$. Hence, by linearity, if $X \in W$ and $Y \in \mathbb{Z}\mathcal{G}$, then $XY \in W$. Thus W is an ideal of $\mathbb{Z}\mathcal{G}$. $\square$

We write $R := \mathbb{Z}\mathcal{G}/W$.

Theorem 3.1.8. *A single-valued function $w\colon \mathcal{G} \to A$ is a W-function (V-function) if and only if it has the form $w = \overline{w} \circ \pi$, where $\overline{w}\colon R \to A$ is a module morphism (an algebra morphism) and $\pi\colon \mathbb{Z}\mathcal{G} \to R$ is the canonical projection map.*

Proof. If $w\colon \mathbb{Z}\mathcal{G} \to A$ is a W-function, then $W \subseteq \ker(w)$, and by the mapping property of modules (the proof can be found in Artin [1]) there is a unique module morphism $\overline{w}\colon R \to A$ such that $w = \overline{w} \circ \pi$. Conversely, if $\overline{w}\colon R \to A$ is a module morphism, then $w = \overline{w} \circ \pi$ is a W-function.

If $w\colon \mathbb{Z}\mathcal{G} \to A$ is a V-function (hence also an algebra morphism), then there is a unique algebra map $\overline{w}\colon R \to A$ such that $w = \overline{w} \circ \pi$. Conversely, if $\overline{w}\colon R \to A$ is an algebra morphism, then $w = \overline{w} \circ \pi$ is a V-function. Since $\overline{w}$ and π are algebra morphisms, w is also an algebra morphism, hence is multiplicative. Since $\overline{w}(W) = 0$, we have $W \subseteq \ker(w)$, which implies that w is a W-function. Since w is multiplicative and is a W-function, it is also a V-function. □

For each $X \in \mathbb{Z}\mathcal{G}$ let $[X]$, denote the coset of X mod W. Let y_r denote any graph having just one vertex and r loops; we call these graphs **elementary graphs**. Note that y_r has rank $r(y_r) = 0$, number of components $k(y_r) = 1$ and corank $s(y_r) = r$.

We now come to the main theorem of the chapter.

Theorem 3.1.9. *If $G \in \mathcal{G}$ then $[G]$ can be expressed as a polynomial*

$$P[G] = P([G]; [y_0],[y_1],[y_2],...)$$

in the $[y_i]$ such that

1. *$P[G]$ has no constant term,*
2. *the coefficients of $P[G]$ are non-negative integers,*
3. *the degree of $P[G]$ is $|V(G)|$,*
4. *$P[G]$ involves no term y_i with i greater than $s(G)$,*
5. *if G is connected and has no bridge e such that for some component G_0 of $G - e$, $s(G_0) = 0$; then $P[G]$ has the form $[y_p] + [Q]$, where $p = s(G)$ and $[Q]$ is a polynomial in those $[y_i]$ for which i is less than p.*

Proof. By induction on $|E(G)|$. If $|E(G)| = 0$, then G is the product of $|V(G)|$ elementary graphs isomorphic to y_0. Thus, we write the polynomial by

$$P[G] = [y_0]^{|V(G)|},$$

and the theorem is true for G.

Assume that the theorem is true for all connected graphs having fewer than some finite number n of edges. For the case G is not connected, we can obtain $P[G]$ satisfying the theorem by multiplying together the polynomials of its components. Let G be any graph having just n edges. If G is connected, then either $V(G) = 1$ and $P[G] = [y_n]$, so the theorem is true for G; or else G contains a link e and

$$P[G] = P([G-e] + [G/e]).$$

Since $G - e$ and G/e have each fewer edges than G and so by inductive hypothesis the theorem is true for them. Note that Propositions 1 to 4 are true for G because $V(G-e) = V(G)$, $s(G) \geq s(G-e)$ and $s(G) = s(G/e)$.

Now suppose that G is connected and has no bridge e such that for some component G_0 of $G-e$, $s(G_0) = 0$. Then G/e also satisfies these conditions and by hypothesis

$$P[G/e] = [y_p] + [Q],$$

where p is $s(G)$ and $[Q]$ denotes another polynomial (not always the same polynomial), in those $[y_i]$ for which $i < p$. We have $P[G-e] = [Q]$ if e is not a bridge. Otherwise, $G-e$ is of the form G_0G_1. Since G satisfies the conditions 5, $s(G_0)$,$s(G_1) > 0$, and therefore since $s(G_0) + s(G_1) = s(G-e)$ we have $s(G_0)$,$s(G_1) < s(G-e)$. Hence $P[G - e] = P([G_0][G_1]) = [Q]$. Thus we conclude that $P[G] = [y_p] + [Q]$.

This completes the proof that the theorem is true for connected graphs. □

Corollary 3.1.10. *Any element $[X]$ of R can be expressed as a polynomial in the $[y_i]$ with integer coefficients and no constant term.*

3.2 The structure of R

Let S denote a spanning subgraph of a graph G. Let the number of components $T \subseteq S$ such that $s(T) = r$ be $i_r(S)$. We define a function $Z(G)$ of $\mathcal{G}$ by

$$Z(G) := \sum_{S \subseteq G} \prod_r z_r^{i_r(S)},$$

where the z_r are independents indeterminate over $\mathbb{Z}$.

Remark 3.2.1. $Z(G)$ involves a formal infinite product, but for a given S only a finite number of the $i_r(S)$ can be non-zero and so, $Z(G)$ is a polynomial in the z_i.

Definition 3.2.2. $R_0 := \mathbb{Z}[z_0,z_1,...]$.

Theorem 3.2.3. *$Z\colon \mathcal{G} \to R_0$ is a V-function.*

Proof. The subgraphs of G_1G_2 are the products of the subgraphs S_1 of G_1 with the subgraphs S_2 of G_2. We have

$$i_r(S_1S_2) = i_r(S_1) + i_r(S_2),$$

therefore,

$$\begin{aligned} Z(G_1G_2) &= \sum_{S_1 \subseteq G_1, S_2 \subseteq G_2} \prod_r z_r^{i_r(S_1)+i_r(S_2)} \\ &= (\sum_{S_1 \subseteq G_1} \prod_r z_r^{i_r(S_1)})(\sum_{S_2 \subseteq G_2} \prod_r z_r^{i_r(S_2)}) \\ &= Z(G_1)Z(G_2). \end{aligned}$$

Thus $Z(G)$ is multiplicative.

Suppose that e is a link of G. Let $\mathcal{H}$ be the set of all subgraphs of G containing e. Let $\mathcal{K}$ be the set of all subgraphs of G which do not contain e. The set $\mathcal{K}$ is equal to the set of subgraphs of $G - e$. Let $\mathcal{S}(G/e)$ be the set of all subgraphs of G/e. If $f\colon \mathcal{H} \to \mathcal{S}(G/e)$ is such that for every $S \in \mathcal{H}$ we have $f(S) = S/e$, then f is bijective. Note also that for every component T of S, we have $s(T/e) = s(T)$. Hence $i_r(S/e) = i_r(S)$ for all

r. Therefore,

$$\begin{aligned} Z(G) &= \sum_{S \subseteq G-e} \prod_r z_r^{i_r(S)} + \sum_{S \subseteq G/e} \prod_r z_r^{i_r(S)} \\ &= Z(G-e) + Z(G/e). \end{aligned}$$

So $Z(G)$ is a V-function, concluding the demonstration. $\square$

Theorem 3.2.4. $Z(y_r) = \sum_i \binom{r}{i} z_i$.

Proof. Each subgraph of y_r has just one component. Hence $Z(y_r)$ is a linear form in the z_r. The number of subgraphs S such that $s(S) = k$ is the number of subgraphs with $|E(S)| = k$, which is the number of ways of choosing k edges of r. $\square$

Lemma 3.2.5. $\sum_{i=0}^{r} (-1)^i \binom{r}{i} \binom{i}{j} = (-1)^r \delta_{rj}$.

Proof. Note that

$$\begin{aligned} x^r &= ((x-1)+1)^r \\ &= \sum_{i=0}^{r} \binom{r}{i} (x-1)^i \\ &= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{i} \binom{i}{j} x^j (-1)^{i-j}. \end{aligned}$$

Equalizing the coefficient of x^r on this equation, the result follows. $\square$

The next theorem gives the structure of the $\mathbb{Z}$-algebra R.

Theorem 3.2.6. *R is isomorphic to the $\mathbb{Z}$-algebra R_0.*

Proof. $Z(G)$ is a V-function with values R_0, by Theorem 3.1.8

$$Z(G) = \overline{w}([G])$$

where $\overline{w}\colon R \to R_0$ is an algebra morphism. Then we must prove that $\overline{w}$ is an isomorphism from R to R_0.

For this, let $[t_i]$ be the element of R defined by

$$[t_i] = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} [y_j] \tag{3.1}$$

By results 3.2.4 and 3.2.5, we have

$$\begin{aligned} \overline{w}([t_i]) &= Z(t_i) \\ &= \sum_{j=0}^{i} \sum_{k=0}^{j} (-1)^{i+j} \binom{i}{j} \binom{j}{k} z_k \\ &= \sum_{j=0}^{i} (-1)^{i} (-1)^{i} \delta_{ik} z_k \\ &= z_i \end{aligned}$$

Multiplying Equation 3.1 by $\binom{r}{i}$ and summing from $i = 0$ to $i = r$ gives

$$\begin{aligned} \sum_{i=0}^{r} \binom{r}{i} [t_i] &= \sum_{i=0}^{r} \sum_{j=0}^{i} (-1)^{i+j} \binom{r}{i} \binom{i}{j} [y_j] \\ &= [y_r] \end{aligned}$$

By the above conclusions (Corollary 3.1.10), since any element $[X]$ of R has a polynomial in the $[y_j]$ then has in the $[t_i]$. Moreover this expression is unique, otherwise there would be a polynomial relationship

$$P([t_i]) = 0$$

and therefore there would be a polynomial relationship

$$Q(z_i) = 0$$

but this would contradict the definition of the z_i, implying the result. □

Theorem 3.2.7. *Let x_0,x_1,... be an infinite sequence of connected graphs such that*

1. $x_0 \cong y_0$
2. $s(x_r) = r$

3. *x_r contains no bridge e such that for some component L_0 of $(x_r - e)$, $s(L_0) = 0$.*

Then any element $[X]$ of R has a unique expression as a polynomial in the $[x_i]$ with rational coefficients and no constant term.

3.3 Topologically invariant W-functions

Definition 3.3.1 (Subdivision)**.** Let $e := \{u,v\}$ be an edge of G. The graph obtained by subdividing e is the graph with vertex set $V(G) \cup \{w\}$ and edge set $(E(G) \setminus \{e\}) \cup \{\{u,w\}, \{v,w\}\}$.

We seek the condition that a W-function $w(G)$ shall be **topologically invariant** (see Definition 1.1.9), i.e., $w(G)$ shall be invariant under subdivision operations.

Let N denote the set of all elements of R which are of the form $[y_0][X] + [X]$. Note that N is ideal of R. Let $\{X\}$ denote that element of the quotient ring R/N which contains $[X]$.

Theorem 3.3.2. *A function $w\colon \mathcal{G} \to A$ is a topologically invariant W-function (V-function) if and only if it is of the form $k\{G\}$, where k is a homomorphism of the R/N into A.*

Proof. Suppose that e is any edge of L. Let M be obtained from L by subdividing e by a point p. Let us denote the new edges by f and g. We have

$$\begin{aligned} w(M) &= w(M-f) + w(M/f) \\ &= w((M-f)-g) + w((M-f)/g) + w(M/f) \\ &= w(p.(M-f)/g) + w((M-f)/g) + w(M/f). \end{aligned}$$

Hence p is a graph which consists solely of the vertice p ($p \cong y_0$). We have $M/f \cong L$ and $(M-f)/g \cong L_0$, where L_0 is the graph derived from L by suppressing e, hence

$$\begin{aligned} w(M) - w(L) &= w(y_0.L_0) + w(L_0) \\ &= \overline{w}([y_0][L_0] + [L_0]) \end{aligned}$$

Thus, the necessary and sufficient condition for the W-function (or V-function)

$w(L)$ to be topologically invariant is that $\overline{w}$ shall map all elements of R of the form $[y_0][L] + [L]$, and therefore all elements of N, on to the zero of A. □

Theorem 3.3.3. *Let x_0,x_1,... be an infinite sequence of connected graphs such that*

1. $x_0 \cong y_0$
2. $s(x_r) = r$
3. *x_r contains no bridge e such that for some component L_0 of $(x_r - e)$, $s(L_0) = 0$.*

Then any element $\{X\}$ of R/N has a unique polynomial in the $\{x_i\}$ with rational coefficients.

Chapter 4

Chromatic polynomials and bialgebras of graphs

We study the chromatic polynomial from a Hopf algebra perspective. We will present the following result: in the category of Hopf algebras, the chromatic polynomial is the only homomorphism from the Hopf algebra of graphs to the Hopf algebra of polynomials. This result was given by Foissy in the paper [10].

All vector spaces in this chapter are over $\mathbb{Q}$. We take K to be the field of rationals. We denote by m the ordinary multiplication in $K[x]$. We define two bialgebra structures on $K[x]$, with comultiplication and counit maps defined on the basis $\{1, x, x^2, \ldots\}$ as follows:

$$\Delta_1(x^m) = \sum_{i=0}^{m} \binom{m}{i} x^i \otimes x^{m-i}, \quad \epsilon_1(x^m) = \delta_{0,m};$$

$$\Delta_2(x^m) = x^m \otimes x^m, \quad \epsilon_2(x^m) = 1.$$

In this chapter, we denote by $\mathcal{G}$ the set of isomorphism classes of all simple finite graphs. Let $K\mathcal{G}$ be the free vector space generated by $\mathcal{G}$.

4.1 Hopf algebraic structures on graphs

We define on $K\mathcal{G}$ a commutative and associative multiplication $m : K\mathcal{G} \otimes K\mathcal{G} \to K\mathcal{G}$ such that $m(G \otimes H) := G \uplus H$. The unit map is given by $\lambda \colon K \to K\mathcal{G}$ such that $\lambda(r) := r1$ where 1 is the null graph. This gives $K\mathcal{G}$ an algebra structure. The algebra $K\mathcal{G}$ is isomorphic to the free commutative algebra generated by connected graphs.

In the following, we define two pairs of comultiplication and counit maps on $K\mathcal{G}$, one of them gives us a Hopf algebra, while the other gives a bialgebra.

Let G be a graph. For $I \subset V(G)$, let $G_{|I}$ denote the subgraph of G induced by I.

Proposition 4.1.1. *Let $\Delta_1 \colon K\mathcal{G} \to K\mathcal{G} \otimes K\mathcal{G}$ and $\epsilon_1 \colon KG \to K$ be defined by*

$$\Delta_1(G) := \sum_{V(G)=I\sqcup J} G_{|I} \otimes G_{|J},$$

where $I \sqcup J$ denotes disjoint union and

$$\epsilon_1(G) := \delta_{G,1},$$

for all $G \in \mathcal{G}$. (Here $\delta_{G,1}$ is 1 if G is the null graph, and 0 otherwise.) Then $(K\mathcal{G}, m, \lambda, \Delta_1, \epsilon_1)$ is a graded connected, commutative, cocommutative bialgebra.

Proof. 1. (Coassociativity) If G is a graph, and $J \subseteq I \subseteq V(G)$, then $(G_{|I})_{|J} = G_{|J}$. Hence

$$\begin{aligned}
(\Delta_1 \otimes \mathrm{Id})\Delta_1(G) &= (\Delta_1 \otimes \mathrm{Id}) \sum_{V(G)=I\sqcup J} G_{|I} \otimes G_{|J} \\
&= \sum_{V(G)=I\sqcup J; I=K\sqcup L} (G_{|I})_{|K} \otimes (G_{|I})_{|L} \otimes G_{|J} \\
&= \sum_{V(G)=K\sqcup L\sqcup J} G_{|K} \otimes G_{|L} \otimes G_{|J} \\
&= \sum_{V(G)=K\sqcup I; I=L\sqcup J} G_{|K} \otimes (G_{|I})_{|L} \otimes (G_{|I})_{|J} \\
&= (\mathrm{Id} \otimes \Delta_1)\Delta_1(G).
\end{aligned}$$

2. (Algebra homomorphism) Note that $V(GH) = V(G) \sqcup V(H)$, therefore,

$$\begin{aligned}
\Delta_1(m(G \otimes H)) &= \Delta_1(GH) \\
&= \sum_{V(G)=I\sqcup J; V(H)=K\sqcup L} GH_{|J\sqcup L} \otimes GH_{|I\sqcup K} \\
&= \sum_{V(G)=I\sqcup J; V(H)=K\sqcup L} G_{|J}H_{|L} \otimes G_{|I}H_{|K} \\
&= \Delta_1(G)\Delta_1(H) \\
&= m(\Delta_1(G) \otimes \Delta_1(H)),
\end{aligned}$$

where the last two lines denote the multiplication in $K\mathcal{G} \otimes K\mathcal{G}$, as defined in 2.1.

3. (Graded and connected) A graduation is given by $K\mathcal{G} = \oplus_{i\in\mathbb{N}} K\mathcal{G}_i$, where $\mathcal{G}_i$ is the set of graphs with i vertices.

   Since $\mathcal{G}_0 = \{\emptyset\}$, we have $K\mathcal{G}_0 \cong K$; hence $K\mathcal{G}$ is connected.

4. (counit) For all $G \in \mathcal{G}$,

$$\begin{aligned}
(\epsilon_1 \otimes \mathrm{Id})\Delta_1(G) &= (\epsilon_1 \otimes \mathrm{Id}) \sum_{V(G)=I\sqcup J} G_{|I} \otimes G_{|J} \\
&= \sum_{V(G)=I\sqcup J} \epsilon(G_{|I}) \otimes G_{|J} \\
&= 1 \otimes G.
\end{aligned}$$

   Similarly, $(\mathrm{Id} \otimes \epsilon_1)\Delta_1(G) = G \otimes 1$.

   We have that Δ_1 is homogeneous and cocommutative, concluding the proof. □

Since the bialgebra defined above is graded and connected, there exists an antipode (by Proposition 2.6.14), and hence it is also a Hopf algebra.

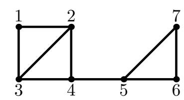
Example 4.1.2. $\Delta_1($  $)=$  $\otimes 1 + 1 \otimes$  $+3$  $\otimes$  $+3$  $\otimes$ 

Let V be a finite set. Let $\sim$ be an equivalence relation on V. Let $\pi : V \to V/\!\sim$ be the canonical projection.

Definition 4.1.3. Let G a graph. Let $\sim$ be an equivalence relation on $V(G)$.

1. (Contraction) The graph $G/\sim$ is defined by $V(G/\sim) := V(G)/\sim$ and
$$E(G/\sim) := \{\{\pi(x),\pi(y)\} \mid \{x,y\} \in E(G), \pi(x) \neq \pi(y)\}.$$
2. (Extraction) The graph $G\mid\sim$ is defined by $V(G\mid\sim) := V(G)$ and
$$E(G\mid\sim) := \{\{x,y\} \in E(G) \mid x \sim y\}.$$
3. We write $\sim \lhd G$ if $G_{|\pi^{-1}(x)}$ is connected for all x.

Example 4.1.4. Let G be the graph shown below.



For the equivalence relation $\sim$ on $V(G)$ defined by equivalence classes $\{1,2,3\},\{5,6,7\},\{4\}$, we have $G/\sim =$  and $G\mid\sim =$ .

Proposition 4.1.5. *Let $\Delta_2 \colon K\mathcal{G} \to K\mathcal{G} \otimes K\mathcal{G}$ and $\epsilon_2 \colon K\mathcal{G} \to K$ be defined by*
$$\Delta_2(G) := \sum_{\sim \lhd G} (G/\sim) \otimes (G\mid\sim)$$
and
$$\epsilon_2(G) := \begin{cases} 1, & \text{if } G \text{ is empty;} \\ 0, & \text{otherwise,} \end{cases}$$
for all $G \in \mathcal{G}$. Then $(K\mathcal{G}, m, \lambda, \Delta_2, \epsilon)$ is a bialgebra. The rank of G defines a grading on $K\mathcal{G}$. The bialgebra is not connected.

Proof. 1. (Coassociativity) Let G be a graph. If $\sim \lhd G$, then the connected components of $G/\sim$ are images of the canonical surjection of the connected components of G; the connected components of $G\mid\sim$ are the equivalence classes of $\sim$. If $\sim$ and $\sim'$ are two equivalences on G, then we shall write $\sim' \leq \sim$ if for all $x, y \in V(G)$,

$x \sim y$ implies $x \sim' y$. We have

$$\begin{aligned}(\Delta_2 \otimes \mathrm{Id}) \circ \Delta_2(G) &= \sum_{\sim \lhd G; \sim' \lhd G/\sim} (G/\sim)/\sim' \otimes (G/\sim) \mid_{\sim'} \otimes G \mid_{\sim} \\ &= \sum_{\sim,\sim' \lhd G; \sim' \leq \sim} (G/\sim)/\sim' \otimes (G/\sim) \mid_{\sim'} \otimes G \mid_{\sim} \\ &= \sum_{\sim,\sim' \lhd G; \sim' \leq \sim} (G/\sim') \otimes (G \mid_{\sim'})/\sim \otimes (G \mid_{\sim'}) \mid_{\sim} \\ &= \sum_{\sim \lhd G, \sim' \lhd G\mid_{\sim}} (G/\sim') \otimes (G \mid_{\sim'})/\sim \otimes (G \mid_{\sim'}) \mid_{\sim} \\ &= (\mathrm{Id} \otimes \Delta_2) \circ \Delta_2(G),\end{aligned}$$

where the second equality is valid provided that the equivalence relation $\sim'$ in $G/\sim$ and G are taken in a similar way.

2. (Algebra homomorphism) Let G, H be a graphs. Let $\sim$ be an equivalence on $V(GH) = V(G) \sqcup V(H)$. We put $\sim' = \sim_{|V(G)}$ and $\sim'' = \sim_{|V(H)}$. We have $\sim \lhd GH$ if and only if satisfies: $\sim' \lhd G$ and $\sim'' \lhd H$, if $x \sim y$, then $(x,y) \in V(G)^2$ or $(x,y) \in V(H)^2$. Note also that $(GH)/\sim = (G/\sim')(H/\sim'')$ and $(GH) \mid_{\sim} = (G \mid_{\sim'})(H \mid_{\sim''})$, so:

$$\begin{aligned}\Delta_2(m(G \otimes H) &= \Delta_2(GH) \\ &= \sum_{\sim' \lhd G; \sim'' \lhd H} (G/\sim')(H/\sim'') \otimes (G \mid_{\sim'})(H \mid_{\sim''}) \\ &= \Delta_2(G)\Delta_2(H) \\ &= m(\Delta_2(G) \otimes \Delta_2(H)),\end{aligned}$$

where the last two lines denote the multiplication in $K\mathcal{G} \otimes K\mathcal{G}$, as defined in 2.1.

3. (Counit) For all $x, y \in V(G)$ define: $x \sim_0 y$ if and only if $x = y$, and $x \sim_1 y$ if and only if x and y are in the same connected component of G. If $\sim \lhd G$, then $G/\sim$ is not empty except if $\sim = \sim_1$, and $G \mid_{\sim}$ is not empty except if $\sim = \sim_0$. Hence, if G is empty, then $\Delta_2(G) = G \otimes G$, and

$$(\epsilon_2 \otimes \mathrm{Id})\Delta_2(G) = (\epsilon_2 \otimes \mathrm{Id})(G \otimes G) = 1 \otimes G,$$

and
$$(\mathrm{Id} \otimes \epsilon_2)\Delta_2(G) = G \otimes 1.$$

Otherwise, denoting by n the number of vertices of G, and by k the number of connected components of G, we have:

$$\Delta_2(G) = \bullet^k \otimes G + G \otimes \bullet^n + \ker(\epsilon_2) \otimes \ker(\epsilon_2).$$

Hence ϵ_2 is the counit of Δ_2.

4. (Homogeneous) Let G be a graph with n vertices and k connected components. If $\sim \lhd G$, then $G/\sim$ has $|V/\sim|$ vertices and k connected components, so $r(G/\sim) = |V/\sim| - k$, and $G\,|\sim$ has n vertices and $|V/\sim|$ connected components, so $r(G\,|\sim) = n - |V/\sim|$. Hence,

$$r(G/\sim) + r(G\,|\sim) = |V/\sim| - k + n - |V/\sim| = r(G).$$

The bialgebra KG defined above is not connected because $r(G_0) = 0$ if and only if G_0 is empty, and thus $K\mathcal{G}_0 \ncong K$. $\square$

Example 4.1.6. $\Delta_2(\overset{\bullet}{\underset{\bullet}{|}}) = \bullet \otimes \overset{\bullet}{\underset{\bullet}{|}} + \overset{\bullet}{\underset{\bullet}{|}} \otimes \bullet\bullet$.

Proposition 4.1.7. *$(K\mathcal{G}, m, \lambda, \Delta_2, \epsilon_2)$ is not a Hopf algebra.*

Proof. Suppose that there exists an antipode function $S\colon K\mathcal{G} \to K\mathcal{G}$. Hence, by definition of antipode, we have

$$m(S \otimes \mathrm{Id})\Delta_2(\bullet) = m(S \otimes \mathrm{Id})(\bullet \otimes \bullet) = m(S(\bullet) \otimes \bullet),$$

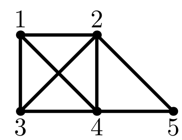
which cannot be $\lambda(\epsilon_2(\bullet)1)$ since $\lambda(\epsilon_2(\bullet)1)$ is the null graph, while the disjoint union of $S(\bullet)$ and $\bullet$ is not the null graph. $\square$

Now, $H := (K\mathcal{G}, m, \lambda, \Delta_2, \epsilon_2)/\langle \bullet - 1 \rangle := (K\mathcal{G}', m', \lambda', \Delta_2', \epsilon_2')$ becomes a graded connected bialgebra, hence a Hopf algebra with antipode S'. In effect, we identify the graph $\bullet$ with the null graph. Thus, $\bullet^n = 1$ and $r(G) = 0$ if and only if $G \cong 1$, concluding that H is connected.

Definition 4.1.8. Let G be a connected graph, and suppose that $G \neq \bullet$.

1. A **forest** of G is a set $\mathcal{F}$ of subsets of $V(G)$ such that:
   - $V(G) \in \mathcal{F}$.
   - If $I, J \in \mathcal{F}$, then $I \subseteq J$, or $J \subseteq I$, or $I \cap J = \emptyset$.
   - For all $I \in \mathcal{F}$, the graph $G_{|I}$ is connected and $G_{|I} \neq \bullet$.
2. The set of forests of G is denoted by $\mathbb{F}$.

Example 4.1.9. Let G be the graph shown below.



$\mathcal{F} := \{V(G), \{1,2,3\}, \{4,5\}, \{1,2\}\}$ is a forest of G.

Definition 4.1.10. Let $\mathcal{F} \in \mathbb{F}(G)$. For any $I \in \mathcal{F}$, the relation $\sim_I$ is an equivalence relation on I whose classes are the maximal elements (for the inclusion) of $\{J \in \mathcal{F} \mid J \subsetneq I\}$ (if this set is nonempty), and singletons. We put

$$G_{\mathcal{F}} := \prod_{I \in \mathcal{F}} (G_{|I})/\sim_I .$$

Example 4.1.11. If $G = \bullet\!\!-\!\!\bullet$, then $\mathcal{F} = \{\bullet\!\!-\!\!\bullet\}$ and $G_{\mathcal{F}} = \bullet\!\!-\!\!\bullet$.

Theorem 4.1.12. *For any connected graph G, $G \neq \bullet$, define*

$$S'(G) := \sum_{\mathcal{F} \in \mathbb{F}(G)} (-1)^{|\mathcal{F}|} G_{\mathcal{F}}.$$

Then S' is the antipode in H.

Proof. By induction on the number n of vertices of G. If $n = 2$, then $G = \bullet\!\!-\!\!\bullet$. Thus,

$$\begin{aligned}
0 &= \lambda' \epsilon_2'(G) \\
&= m'(S' \otimes \mathrm{Id})\Delta_2'(G) \\
&= m'(S' \otimes \mathrm{Id})(\bullet\!\!-\!\!\bullet \otimes 1 + 1 \otimes \bullet\!\!-\!\!\bullet) \\
&= S'(\bullet\!\!-\!\!\bullet) + \bullet\!\!-\!\!\bullet
\end{aligned}$$

Therefore, $S'(\mathbf{I}) = -\mathbf{I}$ and $\displaystyle\sum_{\mathcal{F}\in\mathbb{F}(G)} (-1)^{|\mathcal{F}|} G_{\mathcal{F}} = (-1)^1 \mathbf{I} = -\mathbf{I}$.

Suppose the assertion is valid for $n = k$. We will prove it for $k+1$. Let G be a connected graph such that $|V(G)| = k+1$. Then,

$$\begin{aligned}
0 &= \lambda' \epsilon_2'(G) \\
&= m'(\mathrm{Id} \otimes S')\Delta_2'(G) \\
&= m'(\mathrm{Id} \otimes S') \sum_{\sim \lhd G} (G/\sim) \otimes (G \mid\sim) \\
&= \sum_{\sim \lhd G} S'(G \mid\sim)(G/\sim).
\end{aligned}$$

Now, using the induction hypothesis, we have

$$\begin{aligned}
S'(G) &= -G - \sum_{\sim \lhd G} S'(G \mid\sim)(G/\sim) \\
&= -G - \sum_{\sim \lhd G; G/\sim = \{I_1,\ldots,I_k\}} \; \sum_{\mathcal{F}_i \in \mathbb{F}(G_{|I_i})} (-1)^{|\mathcal{F}_1| + \ldots + |\mathcal{F}_k|} (G/\sim)(G_{|I_1})_{\mathcal{F}_1} \ldots (G_{|I_k})_{\mathcal{F}_k} \\
&= -G - \sum_{\mathcal{F}\in\mathbb{F}(G); \mathcal{F} \neq \{G\}} (-1)^{|\mathcal{F}|-1} G_{\mathcal{F}} \\
&= \sum_{\mathcal{F}\in\mathbb{F}(G)} (-1)^{|\mathcal{F}|} G_{\mathcal{F}}
\end{aligned}$$

□

4.2 Cointeraction

Theorem 4.2.1. *With the coaction Δ_2, the bialgebras $(K\mathcal{G}, m, \lambda, \Delta_1, \epsilon_1)$ and $(K\mathcal{G}, m, \lambda, \Delta_2, \epsilon_2)$ are in cointeraction, that is, $(K\mathcal{G}, m, \lambda, \Delta_1, \epsilon_1)$ is a $(K\mathcal{G}, m, \lambda, \Delta_2, \epsilon_2)$-comodule bialgebra, or a Hopf algebra in the category of $(K\mathcal{G}, m, \lambda, \Delta_2, \epsilon_2)$-comodules. In other words:*

1. *$\Delta_2(1) = 1 \otimes 1$.*

2. *For all $a, b \in K\mathcal{G}$ we have $\Delta_2(ab) = \Delta_2(a)\Delta_2(b)$.*

3. *For all $a \in K\mathcal{G}$ we have $(\epsilon_1 \otimes \mathrm{Id})\Delta_2(a) = \epsilon_1(a)1$.*

4. $m^3_{2,4} \circ (\Delta_2 \otimes \Delta_2) \circ \Delta_1 = (\Delta_1 \otimes \mathrm{Id})\Delta_2$, *where:*

$$m^3_{2,4} : \begin{cases} K\mathcal{G} \otimes K\mathcal{G} \otimes K\mathcal{G} \otimes K\mathcal{G} \to K\mathcal{G} \otimes K\mathcal{G} \otimes K\mathcal{G} \\ \qquad a_1 \otimes b_1 \otimes a_2 \otimes b_2 \mapsto a_1 \otimes a_2 \otimes b_1 b_2 \end{cases}$$

Proof. The assertions 1 and 2 are true because Δ_2 is a linear map, and assertion 3 is immediate because is not necessary to count. Let us prove the 4th assertion. For any graph G, we have

$$\begin{aligned}(\Delta_1 \otimes \mathrm{Id}) \circ \Delta_2(G) &= \sum_{\sim \lhd G; V(G)/\sim = I \sqcup J} (G/\sim)_{|I} \otimes (G/\sim)_{|J} \otimes G \mid \sim \\ &= \sum_{V(G)=I' \sqcup J'; \sim' \lhd G_{|I}, \sim'' \lhd G_{|J}} (G_{|I'})/\sim' \otimes (G_{|J'})/\sim'' \otimes (G_{|I'}) \mid \sim' (G_{|J'})/\sim'' \\ &= m^3_{2,4} \circ (\Delta_2 \otimes \Delta_2) \circ \Delta_1(G).\end{aligned}$$

The second equality follows from $I' = \pi^{-1}(I)$, $I'' = \pi^{-1}(J)$, $\sim' = \sim_{|I'}$ and $\sim'' = \sim_{|J'}$. ☐

Definition 4.2.2 (Monoid of characters)**.** Let $M_{\mathcal{G}}$ be the set of algebra homomorphisms from $K\mathcal{G}$ to K; it is a monoid under the operation of convolution. It is called the **monoid of characters** of $K\mathcal{G}$.

To demonstrate the next theorem we need Theorem 4.2.1 and the results of the paper [11]

Theorem 4.2.3. *1. Let $\lambda \in M_{\mathcal{G}}$. It is an invertible element if and only if $\lambda(\bullet) \neq 0$.*

2. *Let B be a Hopf algebra. Let $E_{K\mathcal{G} \to B}$ be the set of Hopf algebra morphisms from $K\mathcal{G} \to B$. Then $M_{\mathcal{G}}$ acts on $E_{K\mathcal{G} \to B}$ by:*

$$\begin{cases} E \times M_{\mathcal{G}} \to E \\ \quad (\phi, \lambda) \mapsto (\phi \leftarrow \lambda) := m(\phi \otimes \lambda)\Delta_2 \end{cases}$$

3. *There exists a unique $\phi_0 \in E_{K\mathcal{G} \to K[x]}$ such that ϕ_0 is homogeneous and $\phi_0(\bullet) = x$. There exists a unique $\lambda_0 \in M_{\mathcal{G}}$ such that, for all $G \in \mathcal{G}$, $\phi_0(G) = \lambda_0(G)x^{|V(G)|}$. Moreover,*

$$\begin{cases} M_{\mathcal{G}} \to E_{K\mathcal{G} \to K[x]} \\ \quad \lambda \mapsto (\phi_0 \leftarrow \lambda) \end{cases}$$

is bijection.

4. *Let $\lambda \in M_{\mathcal{G}}$. There exists a the unique element $\phi \in E_{K\mathcal{G}\to K[x]}$ such that, for all $x \in K\mathcal{G}$, we have $\phi(x)(1) = \lambda(x)$. This morphism is $\phi_0 \leftarrow (\lambda_0^{-1} * \lambda)$.*

5. *There exists a unique morphism $\phi_1 : K\mathcal{G} \to K[x]$ such that:*

   - *ϕ_1 is a Hopf algebra morphism from $(K\mathcal{G}, m, \Delta_1)$ to $(K[x], m, \Delta_1)$.*
   - *ϕ_1 is a bialgebra morphism from $(K\mathcal{G}, m, \Delta_2)$ to $(K[x], m, \Delta_2)$.*

   This morphism is unique element of $E_{K\mathcal{G}\to K[x]}$ such that for all $x \in K\mathcal{G}$, we have $\phi_1(x)(1) = \epsilon_2(x)$. Moreover, $\phi_1 = \phi_0 \leftarrow \lambda_0^{-1}$.

4.3 The chromatic polynomial as a Hopf algebra morphism

We will determine ϕ_0 and ϕ_1.

Proposition 4.3.1. *For any graph G, we have $\phi_0(G) = x^{|V(G)|}$ and $\lambda_0(G) = 1$.*

Proof. Let $\psi \colon K\mathcal{G} \to \mathbb{Q}[x]$ be such that $\psi(G) = x^{|G|}$ for all $G \in \mathcal{G}$. It is a homogeneous algebra morphism. For any graph G on n vertices, we have

$$\begin{aligned}
(\psi \circ \psi)\Delta_1(G) &= (\psi \circ \psi) \sum_{V(G)=I\sqcup J} G_{|I} \otimes G_{|J} \\
&= \sum_{V(G)=I\sqcup J} x^{|I|} \otimes x^{|J|} \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i} \\
&= \Delta_1(x^n) \\
&= \Delta_1(\psi(G)).
\end{aligned}$$

So ψ is a Hopf algebra morphism. Since $\psi(\bullet) = x^{|\bullet|} = x$, by Theorem 4.2.3, $\psi = \phi_0$. ∎

Theorem 4.3.2. *The map $C : K\mathcal{G} \to K[x]$, where $C(G) = C(G, x)$ is the chromatic polynomial of G, is the morphism ϕ_1.*

Proof. By Equation 1.2.9, C is an algebra morphism. Let G be a graph. We consider:

$$\mathcal{V}(G,[k+l]) := \{f\colon V(G) \to [k+l] \mid f(i) \neq f(j); \{i,j\} \in E(G)\},$$

$$D_2 := \{(I,c',c'') \mid I \subseteq V(G), c' \in \mathcal{V}(G_{|I},[k]), c'' \in \mathcal{V}(G_{|V(G)\setminus I},[l]\},$$

where $k,l \in \mathbb{Z}^+$ and $[k] = \{1,...,k\}$. We define a map $\theta\colon \mathcal{V}(G,[k+l]) \to D_2$ by $\theta(c) = (I,c',c'')$, where $I = \{x \in V(G) \mid c(x) \in [k]\}$, and for all $x \in I$, $c'(x) = c(x)$, and for all $x \notin I$, $c''(x) = c(x) - k$. We define $\theta' : D_2 \to \mathcal{V}(G,[k+l])$ by $\theta(I,c',c'') = c$, where for all $x \in I$, $c(x) = c'(x)$, and for all $x \notin I'$, $c(x) = c''(x) + k$.

Both θ and θ' are well-defined; moreover $\theta \circ \theta' = \mathrm{Id}_{D_2}$ and $\theta' \circ \theta = \mathrm{Id}_{\mathcal{V}(G,[k+l])}$, so θ is bijection. By identifying $\mathbb{Q}[x] \otimes \mathbb{Q}[x]$ with $\mathbb{Q}[x,y]$, we have, for all $k,l \geq 1$,

$$\begin{aligned}
\Delta_1(C(G;k,l)) &= C(G,k+l) \\
&= |\mathcal{V}(G,[k+l])| \\
&= |D_2| \\
&= \sum_{I \subseteq V(G)} C(G_{|I},k)C(G_{|V(G)\setminus I},l) \\
&= (C \otimes C)(\sum_{V(G)=I\sqcup J} G_{|I} \otimes G_{|J})(k,l) \\
&= (C \otimes C)\Delta_1(G)(k,l).
\end{aligned}$$

Moreover, we have

$$\epsilon_1(G) = \epsilon_1(C(G)) = C(G)(0) = \begin{cases} 1, & \text{if } G \text{ is null;} \\ 0, & \text{otherwise.} \end{cases}$$

So $C \in E_{K\mathcal{G}\to\mathbb{Q}[x]}$.

For any graph G, we have

$$\begin{aligned}
C(G)(1) &= \begin{cases} 1, & \text{if } G \text{ is empty;} \\ 0, & \text{otherwise.} \end{cases} \\
&= \epsilon_2(G).
\end{aligned}$$

Hence C is also a coalgebra morphism and $\phi_1 = C(G,k)$.

By Theorem 4.2.3, it is the unique morphism from $K\mathcal{G}$ to $K[x]$. $\square$

Chapter 5

Whitney systems

In this chapter, we study another Hopf algebraic framework introduced by Schmitt [19]. Schmitt defines a category of objects, called Whitney systems, which are set systems that have minimal structure necessary to define the familiar notion of connectivity in graphs, matroids and other combinatorial structures.

Let the **type of a graph** be defined as the unordered list (multiset) of the isomorphism types of its blocks (see Definition 1.1.18). Let $\sigma := t_1, \ldots, t_k$ be a type, i.e., a graph of type σ has blocks $t_1, \ldots, t_k$. Given a graph G, let the number of subgraphs of type σ in G be denoted by n_σ. Whitney [26] showed that n_σ is a polynomial in $n_{t_1}, \ldots, n_{t_k}$ over rationals, and that the polynomial does not depend on G. Schmitt showed that Whitney's theorem is equivalent to a Hopf algebra structure theorem in the special case of a certain class of Whitney systems arising from graphs (see theorem 5.3.5).

5.1 Definitions and examples

Definition 5.1.1 (Whitney system)**.** A **Whitney system** is a pair $H = (V, \mathcal{C})$, where V is a set and $\mathcal{C}$ is a collection of non-empty subsets of V such that if $X, Y \in \mathcal{C}$ and $X \cap Y \neq \emptyset$, then $X \cup Y$ belongs to $\mathcal{C}$.

If $H = (V, \mathcal{C})$ is a Whitney system, then sometimes we write $V(H)$ for the underlying set V, and $\mathcal{C}(H)$ for the family of subsets $\mathcal{C}$. A Whitney system $H = (V, \mathcal{C})$ is called an empty Whitney system if $V = \emptyset$ and $\mathcal{C} = \emptyset$.

Definition 5.1.2 (Morphism)**.** A morphism $\varphi : H_1 \to H_2$ of Whitney systems is a function φ from $V(H_1)$ to $V(H_2)$ such that $\varphi(U) \in \mathcal{C}(H_2)$ whenever $U \in \mathcal{C}(H_1)$. We say that φ is a isomorphism if φ is a bijection such that $\varphi(U) \in \mathcal{C}(H_2)$ if and only if $U \in \mathcal{C}(H_1)$

The isomorphism class of an object H in the category of Whitney systems is denoted by $[H]$.

Definition 5.1.3 (Sum)**.** The sum of Whitney systems $H_1 = (V_1, \mathcal{C}_1)$ and $H_2 = (V_2, \mathcal{C}_2)$ is the Whitney systems (disjoint union) such that

$$H_1 + H_2 = (V_1 \sqcup V_2, \mathcal{C}_1 \sqcup \mathcal{C}_2).$$

Definition 5.1.4 (Restriction)**.** The restriction of a Whitney system H to a subset U of $V(H)$ is the Whitney systems given by

$$H_{|U} := (U, \{X \in \mathcal{C}(H) \mid X \subseteq U\}).$$

Definition 5.1.5. A Whitney system $H = (V, \mathcal{C})$ is **connected** if and only if $V \in \mathcal{C}$, or if $|V| = 1$. We say that $U \in V(H)$ is connected if and only if $H_{|U}$ is connected.

By definition, a Whitney system H is connected if and only if it is non-empty and cannot be written as the sum of two non-empty Whitney system.

If $|V(H)| = 1$ and $V(H) \in \mathcal{C}(H)$, then H is a called a **loop**; and $U \subseteq V(H)$ is a loop if $H_{|U}$ is a loop.

Given a Whitney system $H = (V, \mathcal{C})$, where V is finite, let σ_H denote the collection of maximal connected subsets of V. Note that σ_H is a partition of V. The elements of σ_H are called the connected components of H, and $H_{|U}$, for $U \in \sigma_H$ are called the **blocks** of H. The set of all blocks of H is denoted by $\beta(H)$. Thus any Whitney system H has the unique decomposition into blocks $H = \sum_{B \in \beta(H)} B$.

Example 5.1.6 (Whitney systems from graphs)**.** Let $G = (V, E)$ be a graph. If $U \subseteq V(G)$, then $G_{|U}$ is the subgraph of G induced by U. If $T \subseteq E(G)$, then $G_{|T}$ denotes the subgraph of G consisting of edges in T and vertices which are incident to edges in T.

Consider the sets:

$C_v := \{U \subseteq V \mid G_{|U} \text{ is connected}\}$,
$C_e := \{T \subseteq E \mid G_{|T} \text{ is connected}\}$,
$C_d := \{T \subseteq E \mid G_{|T} \text{ is doubly connected and } |U| > 1 \text{ where } U \subseteq V(G_{|T})\} \cup \{\{e\} \mid e \text{ is a loop of } G\}$.

The pairs $G_v = (V, C_v)$, $G_e = (E, C_e)$ and $G_d = (E, C_d)$ are Whitney systems.

Definition 5.1.7 (Matroid)**.** A matroid is an ordered pair $(S, \mathcal{B})$, consisting of a finite set S and a nonempty family $\mathcal{B}$ of subsets of S, called bases, which satisfy the following properties:

1. if $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there exists $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$;
2. if $B_1 \in \mathcal{B}$ and $B_2 \subseteq B_1$, then $B_2 \in \mathcal{B}$.

Example 5.1.8 (A Whitney system from a matroid)**.** Let $M = (S, \mathcal{B})$ be a matroid. An element e of a matroid M is a loop if it contained in no basis of M. If $U \subseteq S$, then $M_{|U} := (U, \{X \subseteq U \mid X \in \mathcal{B}\})$ denotes the restriction of M to U. Define $C_r = \{U \subseteq S \mid M_{|U} \text{ is connected and } |U| > 1\} \cup \{\{x\} \mid x \text{ is a loop of } M\}$. Then $M_r = (S, C_r)$ is Whitney system.

Example 5.1.9 (A Whitney system from a topology)**.** Let $\mathcal{T}$ be a topology on a set X, and let $\mathcal{T}_c$ be the collection of connected subsets of X. The pair $(X, \mathcal{T}_c)$ is a Whitney system.

5.2 Hopf algebras of Whitney Systems

Let $\mathcal{P}$ be a set of isomorphism types of finite Whitney system which is closed under the operations of sum and restriction. The set $\mathcal{P}$ is a commutative monoid with **product** defined by

$$[H_1][H_2] = [H_1 + H_2].$$

The identity element of the monoid is the empty Whitney system denoted by $[0]$.

Let $\mathcal{P}_0 \subseteq \mathcal{P}$ denote the set of isomorphism types of connected Whitney system in $\mathcal{P}$. Because $\mathcal{P}$ is closed under restriction, it follows that $\mathcal{P}$ is the free commutative monoid on the set $\mathcal{P}_0$. Every $[H]$ in $\mathcal{P}$ may be written as

$$[H] = [B_1][B_2]...[B_k]$$

where B_i is a block for all i.

Suppose K is a field of characteristic zero. Let $C(\mathcal{P})$ denote the **monoid algebra** of $\mathcal{P}$ over K. Note that, every element $x \in C(\mathcal{P})$ can be written as $\sum_{[H]\in\mathcal{P}} k_B[H]$, where only finitely many coefficients k_H are non-zero. By remark above, $C(\mathcal{P}) \simeq K[\mathcal{P}_0]$, where $K[\mathcal{P}_0]$ is the polynomial algebra over K with the set of indeterminates $\mathcal{P}_0$.

Define linear maps $\Delta : C(\mathcal{P}) \to C(\mathcal{P}) \otimes C(\mathcal{P})$ (**coproduct**) and $\epsilon : C(\mathcal{P}) \to K$ (**counit**) by

$$\Delta[H] = \sum_{U_1 \cup U_2 = V(H)} [H_{|U_1}] \otimes [H_{|U_2}]$$

and

$$\epsilon[H] = \begin{cases} 1, & \text{if } V(H) = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

for all $[H] \in \mathcal{P}$.

Next we verify that the maps Δ and ϵ satisfy the comultiplication and counit properties, respectively, making $C(\mathcal{P})$ a bialgebra.

$$\begin{aligned}(Id \otimes \Delta)\Delta[H] &= (Id \otimes \Delta) \sum_{U_1 \cup U_2 = V(H)} [H_{|U_1}] \otimes [H_{|U_2}] \\ &= \sum_{U_1 \cup U_2 = V(H), K_1 \cup K_2 = V(H_{|U_2})} [H_{|U_1}] \otimes [H_{|K_1}] \otimes [H_{|K_2}] \\ &= \sum_{U_1 \cup U_2 = V(H), T_1 \cup T_2 = V(H_{|U_1})} [H_{|T_1}] \otimes [H_{|T_2}] \otimes [H_{|U_2}] = (\Delta \otimes Id)\Delta[H]\end{aligned}$$

and

$$\begin{aligned}(\epsilon \otimes Id)\Delta[H] &= \sum_{U_1 \cup U_2 = V(H)} \epsilon[H_{|U_1}] \otimes [H_{|U_2}] \\ &= 1 \otimes [H].\end{aligned}$$

Analogically, we see that $(Id \otimes \epsilon)\Delta[H] = [H] \otimes 1$. Thus $C(\mathcal{P})$ is a bialgebra with coproduct Δ and counit ϵ. In fact, Δ is an algebra map,

$$\begin{aligned}\Delta([H_1][H_2]) &= \Delta[H_1 + H_2] \\ &= \sum_{U_1 \cup U_2 = V(H_1+H_2)} [(H_1 + H_2)_{|U_1}] \otimes [(H_1 + H_2)_{|U_2}] \\ &= (\sum_{U_1 \cup U_2 = V(H_1)} [(H_1)_{|U_1}] \otimes [(H_1)_{|U_2}])(\sum_{U_1 \cup U_2 = V(H_2)} [(H_2)_{|U_1}] \otimes [(H_2)_{|U_2}]) \\ &= \Delta[H_1]\Delta[H_2]\end{aligned}$$

For each $n \geq 1$, let I_n be the ideal of $C(\mathcal{P})$ generated by the set $\{[H] \in \mathcal{P} \mid |V(H)| \geq n\}$. Furthermore, the set of ideals $\{I_n : n \geq 1\}$ forms a local base at 0 for a topology on $C(\mathcal{P})$ which is a topological algebra. We use the notation $\widehat{C(\mathcal{P})}$ to the completion of $C(\mathcal{P})$ which is isomorphic to the algebra of formal power series $K[[\mathcal{P}_0]]$ the fact $C(\mathcal{P}) \simeq K[\mathcal{P}_0]$) and contains $C(\mathcal{P})$ as dense subalgebra.

The composition $C(\mathcal{P}) \to C(\mathcal{P}) \otimes C(\mathcal{P}) \hookrightarrow \widehat{C(\mathcal{P})} \otimes \widehat{C(\mathcal{P})}$ is continuous, and thus extends uniquely to a continuous (and coassociative) map $\widehat{\Delta} : \widehat{C(\mathcal{P})} \to \widehat{C(\mathcal{P})} \otimes \widehat{C(\mathcal{P})}$. Also, the counit $\epsilon : C(\mathcal{P}) \to K$ is continuous (with the discrete topology on K) and hence extends to a continuous map $\widehat{\epsilon} : \widehat{C(\mathcal{P})} \to K$. Thus, $\widehat{C(\mathcal{P})}$ is a bialgebra, with coproduct $\widehat{\Delta}$ and counit $\widehat{\epsilon}$.

Proposition 5.2.1. *$\widehat{C(\mathcal{P})}$ is a Hopf algebra with antipode $S : \widehat{C(\mathcal{P})} \to \widehat{C(\mathcal{P})}$ defined by $S(1) = 1$ and*

$$S[H] = \sum_{n \geq 1} \sum (-1)^n \prod_{i=1}^{n} [H_{|U_i}]$$

for all non-empty $[H] \in \mathcal{P}$, where the inner sum is over all ordered n-tuples $(U_1, U_2, ..., U_n)$ of non-empty subsets of $V(H)$ such that $\bigcup_{i=1}^{n} U_i = V(H)$.

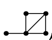
For $f \in \widehat{C(\mathcal{P})}^*$ (the dual algebra of $\widehat{C(\mathcal{P})}$) and $[H] \in \mathcal{P}$, let $\langle f, [H] \rangle$ denote the value of f on $[H]$. For any $[H]$ in $\mathcal{P}$, define $\delta_H \in \widehat{C(\mathcal{P})}^*$ by

$$\langle \delta_H, [G] \rangle = \begin{cases} 1, & \text{if } [G] = [H]; \\ 0, & \text{otherwise.} \end{cases}$$

for all $[G] \in \mathcal{P}$. For Whitney systems $H, H_1, ..., H_k$, define

$$(H : H_1, \cdots, H_k)$$

to be the number of ordered k-tuples $(U_1, ..., U_k)$ of subsets of $V(H)$, with union $V(H)$, such that $[H_{|U_i}] = [H_i]$, for all $1 \leq i \leq k$.

Example 5.2.2. For the graphs $H :=$ , $H_1 :=$ , and $H_2 :=$ , we have $(H : H_1, H_2) = 2$.

Proposition 5.2.3. *We have the following formula for the convolutional product in $\widehat{C(\mathcal{P})}^*$:*

$$\prod_{i=1}^{k} \delta_{[H_i]} = \sum_{[H] \in \mathcal{P}} (H : H_1, ..., H_k) \delta_H. \tag{5.1}$$

whenever $[H_1], ..., [H_k]$ belong to $\mathcal{P}$.

Proof. For all i, $\delta_{[H_i]} \in \hom(\widehat{C(\mathcal{P})}, K)$. Let m and Δ be the multiplication in K and comultiplication in $\widehat{C(\mathcal{P})}$, respectively. Then,

$$\delta_{H_1} * \cdots * \delta_{H_k} = m^{(k-1)} \circ (\delta_{H_1} \otimes \cdots \otimes \delta_{H_k}) \circ \Delta^{(k-1)}.$$

For arbitrary $[G]$, we write, using Sweedler notation,

$$(\prod_{i=1}^{k} \delta_{H_i})[G] = \sum_{(G)} \delta_{H_1}[G_1] \cdots \delta_{H_k}[G_k]$$

where $G_1, \cdots, G_k$ are Whitney systems such that $\cup_i V(G_i) = V(G)$, and the sum is over all such tuples $(G_1, \cdots, G_k)$ of induced Whitney systems of G. Therefore, the sum is equal $(G : H_1, \cdots, H_k)$ since only those tuples $(G_1, ..., G_k)$ contribute to the sum for which $G_i \cong H_i$ for all i. But,

$$(G : H_1, \cdots, H_k) = \sum_{[H]\in\mathcal{P}} (H : H_1, ..., H_k)\delta_H[G].$$

Now the result follows. □

Proposition 5.2.4. *Suppose that the field K is given the discrete topology.*

1. *$f \in \widehat{C(\mathcal{P})}^*$ is continuous if and only if there exists $n \in \mathbb{N}$ such that $f[H] = 0$ whenever $|V(H)| \geq n$.*
2. *there exists $n \in \mathbb{N}$ such that $f[H] = 0$ whenever $|V(H)| \geq n$ if and only if f is a finite linear combination of the functions δ_H.*

By Equation 5.1, the product of continuous elements of $\widehat{C(\mathcal{P})}^*$ is also continuous. The continuous dual algebra of $\widehat{C(\mathcal{P})}$ denoted by $C(\mathcal{P})'$ is the subalgebra of $\widehat{C(\mathcal{P})}^*$ generated by $\{\delta_H \mid [H] \in \mathcal{P}\}$.

If G, H are Whitney system where H has blocks $B_1, ..., B_k$ then let $c(G, H) := (G : B_1, ...B_k)$. If H has exactly k_B blocks isomorphic to B for all $[B] \in \mathcal{P}_0$, then $c(H, H) = \prod_{[B]\in\mathcal{P}_0} k_B!$.

Definition 5.2.5. We say that $[G] \leq [H]$ for all $[G], [H] \in \mathcal{P}$ if and only if $c(G, H) \neq 0$, or if $G = \emptyset$.

Proposition 5.2.6. *The relation $\leq$ defined above is a partial order on $\mathcal{P}$.*

Proof. If $c(G, H) \neq 0$, then there is a function $f : V(H) \to V(G)$ such that $[H|U] = [G|f(U)]$ for all connected $U \in \mathcal{C}(H)$ and f is surjection.

- (Reflective) Since $c(H, H) = \prod_{[B]\in\mathcal{P}_0} k_B! \geq 1$, we have $[H] \leq [H]$.
- (Transitive) Suppose $[G] \leq [H]$ and $[H] \leq [F]$. Thus, there are surjections $f_1 : V(H) \to V(G)$ and $f_2 : V(F) \to V[H]$ as above, hence for all U connected $[F|U] = [H|f_2(U)] = [G|f_1 \circ f_2(U)]$. Also, $f_1 \circ f_2$ is surjection, hence $[G] \leq [F]$.

- (Antisymmetric) Suppose $[G] \leq [H]$ and $[H] \leq [G]$ then there are $f_1 : V(G) \to V(H)$ and $f_2 : V(H) \to V(G)$ surjections such that $[H|U] = [G|f_2(U)]$ and $[G|W] = [H|f_1(W)]$ for all U, W connected, and thus $[G] = [H]$. □

If Q is a partially ordered set and $x \leq y$ in Q, then the set $[x, y] := \{z \in Q \mid x \leq z \leq y\}$ is called an interval in Q. The poset Q is said to be **locally finite** if all of its intervals are finite.

Definition 5.2.7 (Incidence algebra)**.** The incidence algebra over K of a locally finite the partially ordered set Q is the collection $I(Q)$ of all $f : I \to K$ with addition $(g + f)(x, y) = g(x, y) + f(x, y)$, scalar multiplication $(\lambda f)(x, y) = \lambda(f(x, y))$, and product (convolution) of f, g in $I(Q)$ defined by

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

for all $x \leq y$ in Q. The identity element e of $I(Q)$ is defined by $e(x, y) = \delta_{x,y}$ for all $x \leq y$ in Q.

Proposition 5.2.8. *A function $f \in I(Q)$ has a convolutional inverse if and only if $f(x, x) \neq 0$ for all $x \in Q$, in which case*

$$f^{-1}(x, x) = \frac{1}{f(x, x)}$$

$$f^{-1}(x, y) = \sum_{k \geq 0} \sum_{x = x_0 < \cdots < x_k = y} (-1)^k \frac{f(x_0, x_1) \cdots f(x_{k-1}, x_k)}{f(x_0, x_0) \cdots f(x_k, x_k)} \tag{5.2}$$

for all $x < y$ in Q.

Proof. By definition, f has a convolution inverse if and only if $1 = \delta_{x,x} = f * f^{-1}(x, x) = f(x, x)f^{-1}(x, x)$ if and only if $f(x, x) \neq 0$.

For $x < y$ in Q, we have

$$\begin{aligned}0 &= \delta_{x,y}\\ &= (f^{-1} * f)(x,y)\\ &= \sum_{x \le z \le y} f^{-1}(x,z) f(z,y)\\ &= \sum_{x \le z < y} f^{-1}(x,z) f(z,y) + f^{-1}(x,y) f(y,y).\end{aligned}$$

Thus,

$$f^{-1}(x,y) = -\frac{1}{f(x,y)} \sum_{x \le z < y} f^{-1}(x,z) f(z,y). \tag{5.3}$$

Now, we will prove Equation 5.2 by induction on $|[x,y]|$. If $|[x,y]| = 1$, then $x = y$ and $f^{-1}(x,x) = \frac{1}{f(x,x)}$. Suppose the assertion is valid for $|[x,y]| = k$. When $|[x,y]| = k+1$ and using Equation 5.3,

$$\begin{aligned}f^{-1}(x,y) &= -\frac{1}{f(y,y)} \sum_{x \le z < y} \Big(\sum_{k \ge 0} \sum_{x = x_0 < \cdots < x_k} (-1)^k \frac{f(x_0,x_1) \cdots f(x_{k-1},x_k)}{f(x_0,x_0) \cdots f(x_k,x_k)}\Big) f(z,y)\\ &= \sum_{k \ge 0} \sum_{x = x_0 < \cdots < x_{k+1} = y} (-1)^k \frac{f(x_0,x_1) \cdots f(x_k,x_{k+1})}{f(x_0,x_0) \cdots f(x_{k+1},x_{k+1})}.\end{aligned}$$

□

Theorem 5.2.9. *The linear map from $C(\mathcal{P})'$ onto the polynomial algebra $K[\mathcal{P}_0]$ defined by $\delta_B \to [B]$, for all $[B] \in \mathcal{P}_0$, is an algebra isomorphism.*

Proof. Let $f\colon C(P)' \to C(P)'$ be a linear map defined by $f(\delta_H) = \prod_{B \in \beta(H)} \delta_B$. By Equation 5.1, we have

$$f(\delta_H) = \sum_G c(G,H) \delta_G.$$

Let $g\colon C(P)' \to C(P)'$ be a linear map defined by

$$g(\delta_H) = \sum_G c^{-1}(G,H) \delta_G.$$

Here c^{-1} is the inverse of c in the incidence algebra $I(P)$ over K; it exists because $c(H,H) \neq 0$ for all H.

We show that $g(f(\delta_H)) = f(g(\delta_H)) = \delta_H$ for all H. For all H, we have

$$\begin{aligned}
g(f(\delta_H)) &= g\left(\sum_G c(G,H)\delta_G\right) \\
&= \sum_G c(G,H)g(\delta_G) && \because c(G,H) \text{ is a constant (in } K\text{) and } g \text{ is linear} \\
&= \sum_G c(G,H)\sum_{G'} c^{-1}(G',G)\delta_{G'} \\
&= \sum_{G'}\sum_G c^{-1}(G',G)c(G,H)\delta_{G'} \\
&= \sum_{G'} e(G',H)\delta_{G'} \\
&= \delta_H.
\end{aligned}$$

Here e is the identity of convolution in the incidence algebra $I(P)$. Similarly, $f(g(\delta_H)) = \delta_H$. Hence $f(g(x)) = g(f(x))$ for all x in $C(P)'$ (since $\{\delta_H \mid H \in P\}$ is a basis of $C(P)'$, and f and g are linear maps). Thus $f \circ g = g \circ f = \mathrm{Id}$ (which is the identity automorphism of $C(P)'$), implying that f and g are bijective, they are inverses of each other, and they are vector space automorphisms.

We have a natural bijective correspondence between finite subsets of P_0, monomials in $K[P_0]$ and P, given by $S \mapsto \prod_{B\in S} B \in K[P_0]$ and $S \mapsto \uplus_{B\in S} B \in P$ for all finite $S \in P_0$. Since $\{\delta_H \mid H \in P\}$ is a basis of $C(P)'$, the linear map $\psi\colon K[P_0] \to C(P)'$ defined by $\prod_{B\in S} B \mapsto \delta_H$, where $H := \uplus_{B\in S} B$ for all finite $S \subseteq P_0$, is a vector space isomorphism. Hence the composition $f \circ \psi$ is a vector space isomorphism. Moreover, $(f\circ\psi)(\prod_{B\in S} B) = \prod_{B\in S} \delta_B$ for all finite sets $S \subseteq P_0$, hence $f \circ \psi$ is multiplicative (homomorphism for ring multiplication). Hence $\psi \circ f$ is an algebra isomorphism. □

Since $g \circ f(\delta_H) = \delta_H$ (by proof of Theorem 5.2.9) we have:

$$\delta_H = \sum_{G\le H} c^{-1}(G,H) \prod_{B\in\beta(H)} \delta_B \tag{5.4}$$

Proposition 5.2.10.

$$\delta_H = c^{-1}(H,H)\left(\prod_{B\in\beta(H)}\delta_B - \sum_{G<H} c(G,H)\delta_G\right) \tag{5.5}$$

Proof. We have, using Equation 5.3,

$$\begin{aligned}
\delta_H &= \sum_{G\leq H} c^{-1}(G,H)\prod_{B\in\beta(H)}\delta_B \\
&= c^{-1}(H,H)\prod_{B\in\beta(H)}\delta_B + \sum_{G<H} c^{-1}(G,H)\prod_{B\in\beta(H)}\delta_B \\
&= c^{-1}(H,H)\prod_{B\in\beta(H)}\delta_B - \sum_{G<H} c^{-1}(H,H)\sum_{G'|G\leq G'<H} c^{-1}(G,G')c(G',H)\prod_{B\in\beta(H)}\delta_B \\
&= c^{-1}(H,H)\prod_{B\in\beta(H)}\delta_B - c^{-1}(H,H)\sum_{G'<H} c(G',H)\sum_{G|G\leq G'<H} c^{-1}(G,G')\prod_{B\in\beta(H)}\delta_B \\
&= c^{-1}(H,H)\prod_{B\in\beta(H)}\delta_B - c^{-1}(H,H)\sum_{G'<H} c(G',H)\delta_{G'} \\
&= c^{-1}(H,H)\left(\prod_{B\in\beta(H)}\delta_B - \sum_{G<H} c(G,H)\delta_G\right). \quad \square
\end{aligned}$$

The **dual Hopf algebra** of $\widehat{C(\mathcal{P})}$, denoted by $\widehat{C(\mathcal{P})}^{\circ} = \{f \in \widehat{C(\mathcal{P})}^{*} \mid \ker f$ contains a cofinite ideal$\}$, is the subalgebra of all elements of $\widehat{C(\mathcal{P})}^{*}$, with coproduct ψ given by restriction of the transpose of the product.

Proposition 5.2.11. *The algebra $C(\mathcal{P})'$ is contained in $\widehat{C(\mathcal{P})}^{\circ}$.*

Proof. For each $[H] \in \mathcal{P}$, we have $\delta_H(I) = 0$ whenever $I = \langle\{[B]^{k_B+1} \mid [B] \in \mathcal{P}_0\}\rangle$, where H has exactly k_B blocks which are isomorphic to B for all $[B] \in \mathcal{P}_0$, and this ideal is cofinite. $\square$

Thus $C(\mathcal{P})'$ is a Hopf algebra, with coproduct given by

$$\begin{aligned}\langle \psi(\delta_H), [G_1] \otimes [G_2] \rangle &= \langle \delta_H, [G_1][G_2] \rangle \\ &= \sum_{[H_1][H_2]=[H]} \langle \delta_{H_1}, [G_1] \rangle \langle \delta_{H_2}, [G_2] \rangle\end{aligned}$$

where line 1 is the definition of transpose. Therefore

$$\psi(\delta_H) = \sum_{[H_1][H_2]=[H]} \delta_{H_1} \otimes \delta_{H_2},$$

for all $[H] \in \mathcal{P}$.

Note that if $[B] \in \mathcal{P}_0$ we have $\psi(\delta_B) = \delta_B \otimes \epsilon + \epsilon \otimes \delta_B$, then δ_B is primitive. The antipode S' of $C(\mathcal{P})'$ is determined by $S'(\delta_B) = -\delta_B$ (since $C(\mathcal{P})' \cong K[\mathcal{P}_0]$ and δ_B is primitive). Moreover, by Theorem 5.2.9, $\{\delta_B : [B] \in \mathcal{P}_0\}$ is a basis for the space of primitive elements of $C(\mathcal{P})'$.

5.3 Restriction invariants and Hopf algebras

We now introduce another bialgebra structure on the monoid algebra of $\mathcal{P}$. Let $M(\mathcal{P})$ denote the monoid algebra of $\mathcal{P}$ over K, together with linear maps $\Theta : M(\mathcal{P}) \to M(\mathcal{P}) \otimes M(\mathcal{P})$ and $\alpha : M(\mathcal{P}) \to K$ given by

$$\Theta[H] = [H] \otimes [H]$$

and

$$\alpha[H] = 1,$$

for all $[H] \in \mathcal{P}$. $M(\mathcal{P})$ is a bialgebra with coproduct Θ and counit α.

Definition 5.3.1 (Algebra of invariants)**.** The dual algebra $M(\mathcal{P})^*$ is called the algebra of (K-valued) invariants on $\mathcal{P}$ and can be identified with algebra of all functions from $\mathcal{P}$ to K, under point-wise sum, product, and scalar multiplication.

Definition 5.3.2 (Algebra of restriction invariants)**.** Let $n_H : M(\mathcal{P}) \to K$ defined by

$n_H(G) =| \{U \subset V(G) \mid [G|U] = [H]\} |$. The subalgebra of $M(\mathcal{P})^*$ generated by the set $\{n_H \mid [H] \in \mathcal{P}\}$ is denoted by $M(\mathcal{P})'$, and is called the algebra of restriction invariants on $\mathcal{P}$.

Proposition 5.3.3. *The linear map $J : M(\mathcal{P}) \to C(\mathcal{P})$ defined by*

$$J[H] = \sum_{U \subset V(H)} [H_{|U}],$$

for all $[H] \in \mathcal{P}$, is a bialgebra isomorphism.

Proof. For all $[H] \in \mathcal{P}$,

$$\begin{aligned}
\Delta \circ J[H] &= \Delta\Big(\sum_{U \subseteq V(H)} [H_{|U}]\Big) \\
&= \sum_{U \subseteq V(H)} \sum_{U_1 \cup U_2 = U} [H_{|U_1}] \otimes [H_{|U_2}] \\
&= \sum_{U_1, U_2 \subseteq V(H)} [H_{|U_1}] \otimes [H_{|U_2}] \\
&= J[H] \otimes J[H] \\
&= (J \otimes J)\Theta[H]
\end{aligned}$$

and

$$\begin{aligned}
\epsilon(J[H]) &= \epsilon\Big(\sum_{U \subseteq V(H)} [H_{|U}]\Big) \\
&= \sum_{U \subseteq V(H)} \epsilon[H_{|U}] \\
&= 1 \\
&= \alpha[H].
\end{aligned}$$

For all $[H_1], [H_2] \in \mathcal{P}$,

$$\begin{aligned} J([H_1][H_2]) &= \sum_{U \subseteq V(H_1+H_2)} [(H_1+H_2)_{|U}] \\ &= \sum_{U_1 \subseteq V[H_1]} \sum_{U_2 \subseteq V[H_2]} [(H_1)_{|U_1}][(H_2)_{|U_2}] \\ &= J[H_1]J[H_2]. \end{aligned}$$

Hence J is a bialgebra homomorphism with inverse given by

$$J^{-1}[H] = \sum_{U \subseteq V(H)} (-1)^{|V(H) \setminus U|}[H_{|U}].$$

Therefore J is an isomorphism. $\square$

The composition $M(\mathcal{P}) \to^J C(\mathcal{P}) \hookrightarrow \widehat{C(\mathcal{P})}$ is an injective bialgebra map. Let J' denote the restriction to $C(\mathcal{P})'$ of the transpose of this map.

Corollary 5.3.4. *$J' : C(\mathcal{P})' \to M(\mathcal{P})'$ is an algebra isomorphism, which maps δ_H to n_H for all $[H]$ in $\mathcal{P}$.*

Proof. Since J is a coalgebra homomorphism we have J' is an algebra homomorphism, and for all $[G], [H] \in \mathcal{P}$,

$$\begin{aligned} \langle J'(\delta_H), [G] \rangle &= \langle \delta_H, J[G] \rangle \\ &= \sum_{U \subseteq V(G)} \langle \delta_H, [G_{|U}] \rangle \\ &= \langle n_H, [G] \rangle. \end{aligned}$$

$\square$

The next result is more general than a classic result of Whitney (see section 5.4).

Theorem 5.3.5. *The algebra map $K[\mathcal{P}_0] \to M(\mathcal{P})'$ defined by $[B] \to n_B$ for all $[B] \in \mathcal{P}_0$, is an isomorphism.*

Proof. By Theorem 5.2.9 there is an algebra isomorphism $f : K[\mathcal{P}_0] \to C(\mathcal{P})'$, and by Corollary 5.3.4 there is and algebra isomorphism $J' : C(\mathcal{P})' \to M(\mathcal{P})'$, hence $g = J' \circ f$ is an algebra isomorphism. □

By Corollary 5.3.4 and Equation 5.4 we have

$$n_H = \sum_{[G]\le[H]} c^{-1}(G,H) \prod_{B\in\beta(G)} n_B. \tag{5.6}$$

By Proposition 5.2.10, we have

$$n_H = \frac{1}{c(H,H)}[\prod_{B\in\beta(H)} n_B - \sum_{[G]<[H]} c(G,H)n_G].$$

For all $n \ge 1$, let $\hat{I}_n$ denote the image in $M(\mathcal{P})$ of the ideal $I_n \subseteq C(\mathcal{P})$, under the inverse isomorphism $J^{-1} : C(\mathcal{P}) \to M(\mathcal{P})$, i. e.,

$$\hat{I}_n = J^{-1}(I_n).$$

Proposition 5.3.6. *The set of ideals $\{\hat{I}_n \mid n \ge 1\}$ forms a local base at 0 for a topology on $M(\mathcal{P})$.*

Let $\widehat{M(\mathcal{P})}$ denote the completion of $M(\mathcal{P})$ (We have $\widehat{M(\mathcal{P})} \cong K[[\hat{\mathcal{P}}_0]]$, where $\hat{\mathcal{P}}_0 = \{J^{-1}[B] \mid [B] \in \mathcal{P}_0\}$). The composition $M(\mathcal{P}) \to^J C(\mathcal{P}) \hookrightarrow \widehat{C(\mathcal{P})}$ extends uniquely to a continuous Hopf algebra isomorphism

$$\hat{J} : \widehat{M(\mathcal{P})} \to \widehat{C(\mathcal{P})}.$$

The continuous dual of $\widehat{M(\mathcal{P})}$ can be identified with the **algebra of restriction invariants** $M(\mathcal{P})'$. Therefore $M(\mathcal{P})'$ is a Hopf algebra, and the restricted transpose map $J' : C(\mathcal{P})' \to M(\mathcal{P})'$ is a Hopf algebra isomorphism. Thus, the coproduct Ψ of $M(\mathcal{P})'$ is given by

$$\Psi(n_H) = \sum_{[H_1][H_2]=[H]} n_{[H_1]} \otimes n_{[H_2]}, \tag{5.7}$$

for all $[H] \in \mathcal{P}$.

Proposition 5.3.7. $\hat{I} = \hat{I}_1 \cdot \widehat{M(\mathcal{P})} = \ker\alpha$ *of* $\widehat{M(\mathcal{P})}$ *is an ideal.*

Proof. Suppose that $[G] \in I_1$, then

$$\begin{aligned}\alpha(J^{-1}[G]) &= \sum_{U \subseteq V} (-1)^{|V \setminus U|} \alpha[G_{|U}] \\ &= \sum_{U \subseteq V} (-1)^{|V \setminus U|} \\ &= 0\end{aligned}$$

because $V(G) \neq \emptyset$, and thus $\hat{I}_1 \subseteq \ker \alpha$ in $M(\mathcal{P})$. Suppose that $\sum_{i=1}^{n} a_i[G_i] \in \ker \alpha$ in $M(\mathcal{P})$. Then $\sum_{i=1}^{n} a_i \alpha[G_i] = 0$, which implies that $\sum_{i=1}^{n} a_i = 0$. Since $J(\sum_{i=1}^{n} a_i G_i) = \sum_{i=1}^{n} a_i \emptyset + \sum_{i=1}^{n} a_i \sum_{U \subseteq V(H); U \neq \emptyset} [(G_i)_{|U}]$, we have $\sum_{i=1}^{n} a_i[G_i] \in \hat{I}_1$. □

The function $\log : 1 + \hat{I} \to \hat{I}$ defined by

$$\log(x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (x-1)^n$$

is a bijection with inverse $\exp : \hat{I} \to 1 + \hat{I}$ given by

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}.$$

For $[H] \in \mathcal{P}$, define $f_H \in M(\mathcal{P})'$ by $\langle f_H, 0 \rangle = 0$ and $\langle f_H, [G] \rangle = \langle n_H, \log[G] \rangle$, for all $[G] \in \mathcal{P}$. Note that $\log[G]$ converges, because $[G] - 1$ is in the kernel of α, and thus $[G] \in 1 + \hat{I}$.

Proposition 5.3.8. *For all $[H] \in \mathcal{P}$,*

$$f_H = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1 \ldots [H_n] = [H]; [H_i] \neq 1} \prod_{i=1}^{n} n_{H_i}.$$

Proof. If $[G] \in \mathcal{P}$, we have

$$\begin{aligned}\langle f_H, [G]\rangle &= \langle n_H, \log[G]\rangle \\ &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \langle n_H, ([G]-1)^n\rangle \\ &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1]\cdots[H_n]=[H]} \prod_{i=1}^{n} \langle n_{H_i}, [G]-1\rangle\end{aligned}$$

by definition, and $n_{H_i}([G_i]-1) = n_{H_i}[G] - n_{H_i}(1) = n_{H_i}[G]$ if $[H_i] \neq 1$, from which the result follows. □

For all $[B] \in \mathcal{P}_0$, by Proposition 5.3.8, $f_B = n_B$. The inverse correspondence is given by

Proposition 5.3.9. *For all* $[H] \in \mathcal{P}$, $n_H = \sum_{n\geq 0} \frac{1}{n!} \sum_{[H_1 \ldots [H_n]=[H]} \prod_{i=1}^{n} f_{H_i}$.

Proposition 5.3.10. $\psi(f_{[H]}) = f_H \otimes \alpha + \alpha \otimes f_H$, *for all* $[H] \in \mathcal{P}$.

Proof. By definition of dual map, for all $[G_1], [G_2] \in \mathcal{P}$, we have

$$\begin{aligned}\langle \psi(f_H), [G_1]\otimes[G_2]\rangle &= \langle f_H, [G_1][G_2]\rangle \\ &= \langle n_H, \log([G_1][G_2])\rangle \\ &= \langle n_H, \log[G_1] + \log[G_2]\rangle \\ &= \langle f_H, [G_1]\rangle + \langle f_H, [G_2]\rangle \\ &= \langle f_H \otimes \alpha, [G_1]\otimes[G_2]\rangle + \langle \alpha \otimes f_H, [G_1]\otimes[G_2]\rangle \\ &= \langle f_H \otimes \alpha + \alpha \otimes f_H, [G_1]\otimes[G_2]\rangle.\end{aligned}$$

□

Thus f_H is an **additive invariant** for all $[H] \in \mathcal{P}$, that is, $\langle f_H, [G_1 + G_2]\rangle = \langle f_H, [G_1]\rangle + \langle f_H, [G_2]\rangle$ for all $[G_1], [G_2] \in \mathcal{P}$.

Proposition 5.3.11. *For all* $[H] \in \mathcal{P}$, *we have* $f_H = \sum_{[B]\in\mathcal{P}_0} c^{-1}(B, H) n_B$.

Proof. By Proposition 5.3.8 and Equation 5.6

$$f_H = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{H_1 \ldots [H_n]=[H];[H_i]\neq 1} \Big(\sum_{[G]\leq [H_i]} c^{-1}(G, H_i) \prod_{B\in\beta(G)} n_B \Big).$$

Since f_H is a primitive element of $M(\mathcal{P})'$ and all primitive elements of $M(\mathcal{P})'$ are the linear homogeneous polynomials in the $n'_B s$, all non-linear terms in the above expression cancel, and,

$$f_H = \sum_{[B]\in\mathcal{P}_0} c^{-1}(B, H) n_B.$$

□

This shows that f_H is equal to the sum of the linear terms in the expression for n_H as a polynomial in the $n'_B s$.

Corollary 5.3.12. *For all* $[G], [H] \in \mathcal{P}$, *we have*

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1 \ldots [H_n]=[H];[H_i]\neq 1} (G : H_1, \ldots, H_n) = \begin{cases} c^{-1}(G, H), & \text{if } G \in \mathcal{P}_0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Proposition 5.3.8 and expanding the product yields,

$$f_H = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1]\cdots[H_n]=[H];[H_i]\neq 1} \sum_{[G]\in\mathcal{P}} (G : H_1, \ldots, H_n) n_G,$$

which is equal to

$$\sum_{[G]\in\mathcal{P}} \Big(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1],\ldots,[H_n]=[H];[H_i]\neq 1} (G : H_1, \ldots, H_n) \Big) n_G$$

and comparing the above with the expression for f_H in Proposition 5.3.11 follows the result. □

5.4 A result of Whitney

Definition 5.4.1 (Cycle type)**.** The graphs $G = (V, E)$ and $H = (W, F)$ are said to have the same cycle type whenever there exists a bijection $f : E \to F$ such that $C \subseteq E$ is a cycle of G if and only if $f(C)$ is a cycle of H.

Note that if G and H have the same cycle type, then $r(G) = r(H)$ and $s(H) = s(G)$.

Proposition 5.4.2. *Let $\mathcal{G}$ be the set of all Whitney systems of the form G_d, where G is a graph (see Example 5.1.6) we have $[G_d] = [H_d]$ if and only if G and H have the same cycle type.*

Proof. Let $f \colon G_d \to H_d$ be an isomorphism. Then A is a minimal element in C_d (or G_d) if and only if $f(A)$ is a minimal element in C_d (or H_d). The minimal elements in G_d and H_d are precisely the edges sets of cycles and loops in G and H, respectively. Hence f is also a bijection that defines equality of cycle structures of G and H. Conversely, let $f \colon E(G) \to E(H)$ be a bijective map for which $A \subseteq E(G)$ is a cycle in G if and only if $f(A)$ is a cycle in H. Let $T \in C_d$ (or G_d). If $T = \{x\}$, then $f(x)$ is a loop in H, hence $f(T) \in C_d$(or H_d). If T is doubly connected, then for all $x, y \in T$ there is a cycle in G that contains x, y hence there is a cycle in H that contains $f(x)$, $f(y)$, hence $f(T)$ is doubly connected and $f(T) \in C_d$(or H_d). □

Definition 5.4.3. For all $i, j \geq 0$, the invariant $m_{ij} \in M(\mathcal{G})'$ is defined by

$$m_{ij} = \sum n_H,$$

where the sum is over all types $[H]$ having rank i and corank j.

Whitney showed that the chromatic polynomial of a graph $G = (V, E)$ is given by (see Theorem 1.2.11)

$$C(G, \lambda) = \sum_{i,j \geq 0} (-1)^{i+j} m_{ij}(G) \lambda^{|V|-i}.$$

The main results of Whitney that state the invariants m_{ij} can be expressed as polynomial with rational coefficients in the invariants n_B, for B doubly connected, and that the invariants n_B are algebraically independent over rationals. By Theorem 5.3.5, is special case $\mathcal{P} = \mathcal{G}$.

Proposition 5.4.4.
$$m_{ij}[G+H] = \sum_{i_1+i_2=i; j_1+j_2=j} m_{i_1 j_1}[G] m_{i_2 j_2}[H].$$

Proof. By Equation 5.7 and Definition 5.4.3, we have

$$\begin{aligned}
\psi(m_{ij}) &= \sum_{[H]\in\mathcal{G}_{ij}} \sum_{[H_1][H_2]=[H]} n_{H_1} \otimes n_{H_2} \\
&= \sum_{i_1+i_2=i, j_1+j_2=j} \Big(\sum_{[H_1]\in\mathcal{G}_{i_1 j_1}} n_{H_1} \Big) \otimes \Big(\sum_{[H_2]\in\mathcal{G}_{i_2 j_2}} n_{H_2} \Big) \\
&= \sum_{i_1+i_2=i, j_1+j_2=j} m_{i_1 j_1} \otimes m_{i_2 j_2},
\end{aligned}$$

where $\mathcal{G}_{ij}$ are all graphs of rank i and corank j. Thus,

$$\begin{aligned}
\langle m_{ij}, [A]+[B] \rangle &= \langle \psi(m_{ij}), [A]\otimes[B] \rangle \\
&= \langle \sum_{i_1+i_2=i, j_1+j_2=j} m_{i_1 j_1} \otimes m_{i_2 j_2}, [A]\otimes[B] \rangle \\
&= \langle \sum_{i_1+i_2=i, j_1+j_2=j} m_{i_1 j_1} m_{i_2 j_2}, [A]+[B] \rangle. \qquad \square
\end{aligned}$$

We write $f_{ij} := \sum f_H$, where the sum is over all block-types $[H]$ having rank i and corank j.

Proposition 5.4.5. $f_{ij}[G+F] = f_{ij}[G] + f_{ij}[F]$

Proof. Using the proof of Proposition 5.3.10,

$$\begin{aligned}
\langle f_{ij}, [G+F] \rangle &= \sum \langle f_H, [G+F] \rangle \\
&= \sum \langle f_H, [G] \rangle + \sum \langle f_H, [F] \rangle \\
&= \langle f_{ij}, [G] \rangle + \langle f_{ij}, [F] \rangle. \qquad \square
\end{aligned}$$

Proposition 5.4.6. $f_{ij} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{i_1+\cdots+i_n=i; j_i+\cdots+j_n=j; i_r, j_r \neq 0} \prod_{r=1}^{n} m_{i_r j_r}.$

Proof. By Proposition 5.3.8, we have

$$\begin{aligned} f_{ij} &= \sum_{[H]\in\mathcal{G}_{ij}} \Big(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1]\cdots[H_n]=[H];[H_i]\neq 1} \prod_{i=1}^{n} n_{H_i}\Big) \\ &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \Big(\sum_{[H]\in\mathcal{G}_{ij}} \sum_{[H_1]\cdots[H_n]=[H];[H_i]\neq 1} \prod_{i=1}^{n} n_{H_i}\Big) \\ &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{i_1+\cdots+i_n=i;j_i+\cdots+j_n=j;i_r,j_r\neq 0} \prod_{r=1}^{n} m_{i_r j_r}, \end{aligned}$$

where $\mathcal{G}_{ij}$ are all graphs of rank i and corank j. □

This formula for f_{ij} was originally proved by Whitney [26].

5.5 An application to Stirling numbers

Let one-point Whitney system be denoted by x (either a loop or non-loop). The set of types $\mathcal{P}$ consists of all non-negative integral powers of x, and is linearly ordered by degree. We have

$$\widehat{C(\mathcal{P})} \cong K[[x]],$$

with coproduct given by

$$\Delta(x) = x \otimes 1 + x \otimes x + 1 \otimes x.$$

The antipode of $\widehat{C(\mathcal{P})}$ is given by

$$S(x) = \sum_{n\geq 1} (-1)^n x^n = \frac{-x}{1+x}.$$

For $k \geq 0$, let $n_k = n_H$, where H is the Whitney system of type x^k. By Theorem 5.3.5, the algebra of restriction invariants $M(\mathcal{P})'$ is the polynomial algebra.

Proposition 5.5.1. *The expression for n_k as a polynomial in n_1 is $n_k = \binom{n_1}{k}$*

Proof. Since $\langle n_1, x^n\rangle = n$ and $\langle n_k, x^n\rangle = \binom{n}{k}$, we conclude that $n_k = \binom{n_1}{k}$. □

For all n and k, $c(x^k, x^n)$ is equal to the number of surjections from an n-element set onto a k-element set which is given by $k!S(n,k)$, where $S(n,k)$ is Stirling number of the second kind (see Definition 1.1.10).

Proposition 5.5.2. *We have*

$$c^{-1}(x^k, x^n) = \frac{s(n,k)}{n!},$$

where $s(n,k)$ denotes Stirling number of the first kind.

By Equation 5.6,

$$n_k = \sum_{r \leq k} c^{-1}(x^r, x^k) n_1^r \tag{5.8}$$

which is equivalent to the classical expression for falling factorials in terms of powers.

By Proposition 5.3.11,

$$\begin{aligned} f_k &= c^{-1}(x^1, x^k) n_1 \\ &= \frac{s(k,1)}{k!} n_1 \\ &= \frac{(k-1)!(-1)^{k-1}}{k!} n_1 \\ &= \frac{n_1(-1)^{k-1}}{k} \end{aligned}$$

Using Proposition 5.3.9, we obtain

$$\begin{aligned} n_k &= \sum_{r \geq 0} \frac{1}{r!} \sum_{k_1 + \ldots + k_r = k; k_i \neq 0} \prod_{i=1}^{k} \frac{(-1)^{k_i - 1}}{k_i} n_1 \\ &= \sum_{r \geq 0} \left(\frac{(-1)^{k-r}}{r!} \prod_{i=1}^{r} \frac{1}{k_i} \right) n_1^r \end{aligned}$$

Comparing this with the expression for n_k given in Equation 5.8 yields the identity for the Stirling numbers of the first kind.

$$s(k,r) = \frac{(-1)^{k-r} k!}{r!} \sum_{\substack{k_1 + \ldots + k_r = k \\ k_i \neq 0}} \prod_{i=1}^{r} \frac{1}{k_i}.$$

Chapter 6

Graph reconstruction problems

Some of the outstanding open problems in graph theory are the reconstruction conjectures. In the 1940s, Ulam and Kelly proposed the vertex reconstruction conjecture, which asserts that finite simple graphs can be uniquely constructed (up to isomorphism) from the collection (called the deck) of their unlabelled proper induced subgraphs. Variations of the conjecture have been studied by many researchers.

One of the directions of research on the conjecture of Ulam and Kelly is to demonstrate that strong invariants of graphs can be calculated from the deck. Tutte [24] proved fundamental results along these lines. In particular, Tutte showed that the rank polynomial, the number of spanning trees, the number of Hamiltonian cycles, and the characteristic polynomial can all be calculated given the deck of a graph. Kocay [18] gave an elegant counting argument to simplify the results of Tutte.

A motivation of this chapter came from the observation that Kocay's counting argument is essentially the same as that of Schmitt, Equation 5.1 in Chapter 5) . In fact, Thatte [22] used Kocay's argument to prove many other stronger results on reconstruction, as well as used it to show Whitney's subgraph expansion theorem. In this chapter, we present Kocay's lemma and its applications to reconstruction.

For a survey of graph reconstruction, we refer to [5] and [4] Tutte's results and a simplification due to Kocay are based on Tutte [24] and Kocay [18], respectively.

6.1 Graph reconstruction conjectures

Two graphs G and H on the same vertex set V are called *hypomorphic* if, for all $v \in V$, their vertex-deleted subgraphs $G - v$ and $H - v$ are isomorphic. This does not imply that G and H are themselves isomorphic. In fact, if $G = 2K_1$ and $H = K_2$, then G and H are hypomorphic but G and H are not isomorphic. However, these two graphs are the only known nonisomorphic but hypomorphic simple graphs. It was conjectured by Kelly and Ulam that there is no other such pair. This conjecture was reformulated by Harary (1964) in the more intuitive language of reconstruction.

Definition 6.1.1. A **reconstruction of a graph** G is any graph that is hypomorphic to G. A graph G is **vertex reconstructible** if every reconstruction of G is isomorphic to G.

In other words, G is vertex reconstructible if G can be reconstructed up to isomorphism from its (unlabelled) vertex-deleted subgraphs.

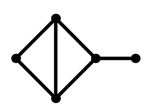


Figure 6.1: Graph G

$G - 1$ $G - 2$ $G - 3$ $G - 4$ $G - 5$

Figure 6.2: The collection of vertex-deleted subgraphs of G

Conjecture 6.1.2 (The vertex reconstruction conjecture)**.** *Every finite simple graph on at least three vertices is vertex reconstructible.*

An analogous edge reconstruction conjecture was proposed by Harary [13]. A graph is **edge recontructible** if it can be reconstructed up to isomorphism from its edge-deleted subgraphs. The only graphs that are known to be not edge reconstructible are the pair $K_{1,2} \uplus K_1$, and the pair $K_3 \uplus K_1$ and $K_{1,3}$.

Conjecture 6.1.3 (The edge reconstruction conjecture)**.** *Every finite simple graph on at least four edges is edge-reconstructible.*

One approach to the reconstruction conjecture is to show that it holds for various classes of graphs. A class of graphs is reconstructible if every member of the class is reconstructible. In the following two propositions, we summarise results on the vertex and the edge reconstruction of classes of graphs. We refer to Bondy [4] for more details.

Proposition 6.1.4. *Regular graphs, disconnected graphs, trees, outerplanar graphs, separable graphs without end vertices and unicyclic graphs are vertex reconstructible.*

Proposition 6.1.5. *All vertex reconstructible classes of graphs are edge reconstructible. Additionally, maximal planar graphs, bidegreed graphs, 4-connected planar graphs, chordal graphs, claw-free graphs, graphs (with sufficiently many vertices) containing Hamiltonian cycles are edge reconstructible.*

Because the edge-deleted subgraphs are in general much closer to the original graph than are the vertex-deleted subgraphs, it is intuitively clear that the edge reconstruction conjecture is no harder than the reconstruction conjecture. In fact, it can be proved that the vertex reconstruction conjecture implies the edge reconstruction conjecture. We present a proof of this result in Section 6.2

Another approach to the conjectures is to prove that specific parameters or invariants of graphs are reconstructible. We call a graphical parameter to be reconsctructible if the parameter takes the same value on all reconstructions of every graph G. We present such results in Section 6.3.

Theorem 6.1.6 (Tutte)**.** *The number of disconnected subgraphs of a given isomorphism class, the number of spanning trees, the number of unicyclic graphs, the number of Hamiltonian cycles, the characteristic polynomial, and the rank polynomial (hence, as a special case, the chromatic and the Tutte polynomials) are reconstructible invariants.*

We end this introduction by listing a few more important conjectures.

Infinite graphs are in general not vertex reconstructible. Take, for example, a tree T in which every vertex is of infinite degree, and another graph $T \uplus T$. The graphs T and $T \uplus T$ are clearly hypomorphic but not isomorphic. Halin proposed a modified vertex reconstruction conjecture as follows.

Conjecture 6.1.7 (Halin, 1970)**.** *Let G and H be undirected (finite or infinite) graphs. Let $f : V(G) \to V(H)$ be a bijection such that*

$$\forall v \in V(G), G - v \cong H - f(v).$$

Then G is isomorphic to a subgraph of H, and H is isomorphic to a subgraph of G.

Harary et al. [14] proposed the following conjecture.

Conjecture 6.1.8. *Let T_1 and T_2 be locally finite, hypomorphic trees. Then $T_1 \cong T_2$.*

The conjectures of Halin and Harary et al. were very recently settled negatively by Bowler et al. [6]

Another well known conjecture is due to Stanley [20]. Given a graph G and a vertex u of G, the graph obtained by switching u is the graph on the same vertex set obtained by deleting all edges incident with u and adding all non-edges at u. The vertex-switching deck of a graph G is the collection of unlabelled graphs obtained by switching each vertex of G. Stanley proposed the following conjecture.

Conjecture 6.1.9 (Stanley's vertex switching conjecture)**.** *Graphs on n vertices are vertex switching reconstructible provided n is not divisible by 4.*

The conjecture has been proved for triangle-free graphs by Ellingham and Royle [9].

6.2 From the edge-deck to the vertex-deck

Definition 6.2.1 (Deck)**.** For a graph G, the vertex-deck (edge-deck) of G is the collection of all vertex-deleted (edge-deleted) subgraphs of G. Each vertex-deleted subgraphs of G is called a card.

To prove that the edge reconstruction conjecture is weaker than the vertex reconstruction conjecture, we may consider line graphs: if two graphs have the same edge-deck then their line graphs have the same vertex deck. Thus solving the vertex reconstruction problem for the line graph of a graph G solves the edge reconstruction problem for the graph G. Moreover, we know that K_3 and $K_{1,3}$ are the only non-isomorphic graphs that have isomorphic line graph.

Here we present a different approach. We explicitly construct the vertex deck of a graph from its edge deck. We present a more concise proof of a result originally due to Hemminger [16].

Lemma 6.2.2. *If G is a graph without isolated vertices, then the vertex-deck of G can be constructed from its edge-deck.*

Proof. Let $G = (V,E)$, where $V := \{v_1, \dots, v_n\}$ and $E := \{e_1, \dots, e_m\}$. Let $\mathcal{D}_v := \{[G - v_i] \mid 1 \leq i \leq n\}$ be the vertex-deck of G. Let $\mathcal{D}_e := \{[G - e_i] \mid 1 \leq i \leq m\}$ be the edge-deck of G. (Both $\mathcal{D}_v$ and $\mathcal{D}_e$ are considered as multisets.)

Consider the collection $\mathcal{D}_{ev} := \{[G - e_i - v_j] \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. (Here the order of deletion - first the edge, then the vertex - is important.) Each graph in the vertex-deck $\mathcal{D}_v$ appears in the collection $\mathcal{D}_{ev}$: for each vertex v_i in G and for each edge e incident with v_i, we have $G - v_i = G - e - v_i$, and there are no isolated vertices in G. Moreover, if d_i is the degree of v_i in G and m_i is the multiplicity of $[G - v_i]$ in $\mathcal{D}_v$, then $[G - v_i]$ appears at least $m_i d_i$ times in $\mathcal{D}_{ev}$. Moreover, each graph in $\mathcal{D}_{ev}$ is in $\mathcal{D}_v$ or is an edge-deleted subgraph of a graph in $\mathcal{D}_v$ (or both).

Let $[H_1], \dots, [H_k]$ be the distinct graphs in $\mathcal{D}_{ev}$ ordered so that $|E(H_i)| \geq |E(H_j)|$ for $i < j$. Let μ_i be the multiplicity of H_i in $\mathcal{D}_{ev}$, m_i the multiplicity of H_i in $\mathcal{D}_v$ (note that m_i may be 0), $p_i = m - |E(H_i)|$ and n_{ij} the number of edge-deleted subgraphs of H_i that are isomorphic to H_j. We have $m_1 p_1 = \mu_1$. For all $j > 1$, we have $\mu_j = m_j p_j + \sum_{i<j} m_i n_{ij}$. We can thus calculate m_j recursively in terms of $m_i, i < j$. $\square$

6.3 Counting spanning subgraphs

Let $s(H,G)$ be the number of subgraphs of G that are isomorphic to H. For a tuple $\mathcal{F} := (F_1, \dots, F_k)$ of graphs, define the number of ways to decompose H into $\mathcal{F}$ (or the number of ways to cover H by $\mathcal{F}$) as

$$d(\mathcal{F}, H) := |\{(H_1, \dots, H_k) \mid \forall i \; H_i \subseteq H, H_i \cong F_i, \text{and} \cup_{i=1}^{k} H_i = H\}|.$$

Example 6.3.1. If $F_1 =$ , $F_2 =$ , and $H =$ , then $d(\mathcal{F}, H) = 3$.

Similarly, we can define the induced subgraph versions of the above notation as follows. Let $i(H, G)$ be the number of induced subgraphs of G that are isomorphic to H. Let $G \subseteq_i H$ denote that G is an induced subgraph of H. Let

$$\begin{aligned} c(\mathcal{F}, H) &:= |\{(H_1, \ldots, H_k) \mid \forall i\ H_i \subseteq_i H, H_i \cong F_i, \text{ and } \cup_{i=1}^k V(H_i) = V(H)\}| \\ &:= |\{(X_1, \ldots, X_k) \mid \forall i\ X_i \subseteq V(H), H[X_i] \cong F_i, \text{ and } \cup_{i=1}^k V(H_i) X_i = V(H)\}| \end{aligned}$$

The problem of graph reconstruction is completly solved in the case of trees, see Kelly [17].

Lemma 6.3.2 (Kelly's lemma)**.** *Let $v(G)$ and $v(H)$ be the number of vertices of G and H, respectively. If $v(H) < v(G)$, then*

$$s(H, G) = \frac{\sum_{v \in V(G)} s(H, G - v)}{v(G) - v(H)}.$$

Lemma 6.3.3 (Kocay's lemma)**.** *Let $\mathcal{F} := (F_1, \ldots, F_k)$ be a tuple of graphs, and let G be a graph. We have*

$$\prod_{i=1}^{k} s(F_i, G) = \sum_{H} d(\mathcal{F}, H) s(H, G),$$

where the sum is over all distinct (mutually non-isomorphic) H.

The induced subgraph version of the above lemma is the following.

Lemma 6.3.4. *Let $\mathcal{F} := (F_1, \ldots, F_k)$ be a tuple of graphs, and let G be a graph. We have*

$$\prod_{i=1}^{k} i(F_i, G) = \sum_{H} c(\mathcal{F}, H) i(H, G),$$

where the sum is over all distinct (mutually non-isomorphic) H.

Remark 6.3.5. Equation 5.1 is equivalent to Lemma 6.3.4.

Lemma 6.3.6 (Counting disconnected spanning subgraphs)**.** *Let G be a graph, and let $\mathcal{F} := (F_1, \ldots, F_k)$ be a tuple of non-empty connected graphs such that $k > 1$ and $\sum_i v(F_i) = v(G)$. Then $s(\uplus_i F_i, G)$ is reconstructible.*

Proof. We have

$$\begin{aligned}\prod_{i=1}^{k} s(F_i,G) &= \sum_{H} d(\mathcal{F},H)s(H,G) \\ &= \sum_{H|v(H)<v(G)} d(\mathcal{F},H)s(H,G) + \sum_{H|v(H)=v(G)} d(\mathcal{F},H)s(H,G).\end{aligned}$$

The term on the left and the first sum in the last line are obtained from Kelly's lemma; hence the second sum in the last line is reconstructible. The second summation consists of only one term - the one corresponding to $H \cong \uplus_i F_i$, which implies the result. □

Lemma 6.3.7 (Counting the number of spanning trees)**.** *Let G be a graph with at least 3 vertices. The total number of spanning trees in G is reconstructible.*

Proof. Let $v(G) = n > 2$, and let $\mathcal{F} := (F_1, \ldots, F_{n-1})$, where each F_i is isomorphic to K_2. By Kocay's lemma, we have

$$\begin{aligned}&\prod_{i=1}^{k} s(F_i,G) \\ &= \sum_{H|v(H)<v(G)} d(\mathcal{F},H)s(H,G) + \sum_{\substack{H|v(H)=v(G) \\ H \text{ is disconnected}}} d(\mathcal{F},H)s(H,G) + \sum_{\substack{H|v(H)=v(G) \\ H \text{ is connected}}} d(\mathcal{F},H)s(H,G).\end{aligned}$$

The term on the left and the first sum on the right are obtained from Kelly's lemma; the second sum on the right is known by Lemma 6.3.6. So the third term on the right can be calculated. A graph H contributes to the third sum if and only if H is a spanning tree, and for all spanning trees H, we have $d(\mathcal{F}, H) = (n-1)!$. Now $\sum_H s(H,G)$ over spanning trees H is the total number of spanning trees, which can now be calculated. □

Lemma 6.3.8 (Counting unicyclic graphs)**.** *Let G be a graph with at least 3 vertices. The total number of unicyclic spanning subgraphs of G which contain a cycle of length $k < v(G)$ is reconstructible.*

Proof. Let $v(G) = n > 2$, and let $\mathcal{F} := (F_1, \ldots, F_{n-1})$, where $F_1 \cong C_k, k < n$ (a cycle of length k), and $F_i \cong K_2$ for $i = 2, \ldots, n-k+1$. By Kocay's lemma, we have

$$\prod_{i=1}^{k} s(F_i,G)$$
$$= \sum_{H|v(H)<v(G)} d(\mathcal{F},H)s(H,G) + \sum_{\substack{H|v(H)=v(G)\\ H \text{ is disconnected}}} d(\mathcal{F},H)s(H,G) + \sum_{\substack{H|v(H)=v(G)\\ H \text{ is connected}}} d(\mathcal{F},H)s(H,G).$$

where the term on the left is obtained by Kelly's lemma; the first sum on the right is obtained by Lemma 6.3.2; the second sum on the right obtained by Lemma 6.3.6; in the third sum, H cannot be a graph other than a spanning unicyclic graph containing C_k. Thus $\sum_H s(H,G)$ over unicyclic graphs H that contain C_k can be calculated. (Note that $d(\mathcal{F},H)$ in the third sum is $(n-k)!$ for each unicyclic graph H that contains C_k.) □

Lemma 6.3.9 (Counting Hamiltonian cycles)**.** *Let G be a graph with at least 3 vertices. The total number of Hamiltonian cycles in G is reconstructible.*

Proof. Let $v(G) = n > 2$, and let $\mathcal{F} := (F_1, \ldots, F_n)$, where $F_i \cong K_2$ for all i. As in the above proofs, we write the equation in Kocay's lemma, and consider five cases of H in the summation: the contribution from H such that $v(H) < v(G)$ is obtained by Lemma 6.3.2; the contribution from H such that $v(H) = v(G)$ and H is disconnected is obtained by Lemma 6.3.6; the contribution from H such that $v(H) = v(G)$ and H is a spanning tree is obtained by Lemma 6.3.7; the contribution from H such that $v(H) = v(G)$ and H is a spanning unicyclic graph containing C_k for some $k < n$ is obtained by Lemma 6.3.8; the remaining contribution is from Hamiltonian cycles H, which can now be calculated. (Note that in the calculation of the contribution of spanning trees, the factor $d(\mathcal{F}, H)$ is the same for all spanning trees, and is equal to the number of bijections from a set with n elements to a set with $n-1$ elements. Similarly, in the calculation of the contribution from spanning unicyclic graphs and from Hamiltonian cycles, the factor $d(\mathcal{F}, H)$ is the same for all spanning unicyclic graphs and for all Hamiltonian cycles, and is equal to $n!$.) □

Lemma 6.3.10. *The Tutte polynomial is reconstructible.*

Proof. Since the Tutte polynomial is given by $T_G(x,y) = S(G; x-1, y-1)$, it is enough to prove the result for the rank generating polynomial. By definition, $S(G; x,y) =$

$\sum_{S \subseteq E(G)} x^{r\{E\}-r\{S\}} y^{s\{S\}}$. For all connected spanning subgraphs, we have $r\{E\} - r\{S\} = 0$ and $s\{S\} = |E(S)| - |V(G)| + 1$. Then, by Lemma 6.3.6, we know for each value of k, the number of connected spanning subgraphs S such that $|E(S)| = k$, as well as contribution from each type of disconnected spanning subgraphs. □

Lemma 6.3.11 (Lemma 7.3, Biggs [2]). *The coefficients of the characteristic polynomial are given by*

$$(-1)^i c_i = \sum (-1)^{r(S)} 2^{s(S)},$$

where the summation is over all subgraphs S of G such that each component of S is a single edge or a cycle, and S has i vertices.

Lemma 6.3.12. *The characteristic polynomial written as $P(G,\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$ is reconstructible.*

Proof. For $i < n$, subgraphs of each type with i vertices are counted using Kelly's lemma, and then c_i is calculated using Lemma 6.3.11. Now, c_n has contribution from hamiltonian cycles and from disconnected spanning subgraphs, therefore, c_n also can be calculated from Lemma 6.3.9 and Lemma 6.3.6. □

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