UNIVERSIDADE FEDERAL DE MINAS GERAIS
Insituto de Ciências Exatas
Programa de Pós-graduação em Física

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ANOTHER BRICK IN THE HIGGS:
the composite Higgs scenario

Belo Horizonte
2024

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Dissertation presented to the Graduate Program in Physics of the Federal University of Minas Gerais as a partial requirement for obtaining the title of Master of Science in Physics.

Supervisor: Prof. Dr. Gláuber Carvalho Dorsch

Belo Horizonte
2024

Dados Internacionais de Catalogação na Publicação (CIP)

P436a Pereira, Vítor Guilherme de Souza.
Another brick in the Higgs: the composite Higgs scenario / Vítor Guilherme de Souza Pereira. – 2024.
112 f. : il.

Orientador: Gláuber Carvalho Dorsch.
Dissertação (mestrado) – Universidade Federal de Minas Gerais, Departamento de Física.
Bibliografia: f. 109-112.

1. Partículas elementares. 2. Bósons de Higgs. 3. Teoria de grupos. I. Título. II. Dorsch, Gláuber Carvalho. III. Universidade Federal de Minas Gerais, Departamento de Física.

CDU – 539.12 (043)

Ficha catalográfica elaborada por Romário Martins Ribeiro – CRB6 3595
Biblioteca Professor Manoel Lopes de Siqueira – Departamento de Física - UFMG

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ATA

ATA DA SESSÃO DE ARGUIÇÃO DA 738ª DISSERTAÇÃO DO PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA, DEFENDIDA POR VÍTOR GUILHERME DE SOUZA PEREIRA orientado pelo professor Gláuber Carvalho Dorsch, para obtenção do grau de **Mestre, área de concentração Física**. Às 14:00 horas de vinte e cinco de outubro de dois mil e vinte e quatro reuniu-se a Comissão Examinadora, composta pelos professores **Gláuber Carvalho Dorsch** (Orientador - Departamento de Física/UFMG), **Bruce Lehmann Sánchez Vega** (Departamento de Física/UFMG) e **Mario Sérgio Carvalho Mazzoni** (Departamento de Física/UFMG), para dar cumprimento ao Artigo 37 do Regimento Geral da UFMG, submetendo o bacharel **VÍTOR GUILHERME DE SOUZA PEREIRA** à arguição de seu trabalho de dissertação, que recebeu o título de **"Another Brick in the Higgs: the composite Higgs scenario".** O candidato fez uma exposição oral de seu trabalho durante aproximadamente 50 minutos. Após esta, os membros da comissão prosseguiram com a sua arguição, e apresentaram seus pareceres individuais sobre o trabalho, concluindo pela aprovação do candidato.

Belo Horizonte, 25 de outubro de 2024.

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Documento assinado eletronicamente por **Vítor Guilherme de Souza Pereira**, **Usuário Externo**, em 09/12/2024, às 16:49, conforme horário oficial de Brasília, com fundamento no art. 5º do Decreto nº 10.543, de 13 de novembro de 2020.

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To those who came before me, who fought for freedom, dignity, and justice. May this work honor your legacy and illuminate the path for those who will come after me.

ACKNOWLEDGMENT

Many people helped make my stay in Belo Horizonte as enjoyable as possible. I am incredibly grateful to João Vítor, Titú, Alan, Matheus Aleluia, Matheus Cardoso, Julia, Guilherme, Catarina, Douglas, Joás, Bruno, João Carvalho, João Tolentino, Daniel, Isaac, Gabriel, Alvaro, Jullie, Késsia, Agatha, Rafael, Carol, Deivid, Michele, Paulinho, Vítor Monteiro, Bruce, Thaisa, Lucas, Vitão, Maria Vitória, Luiz, Will, Gustavo, Emílio, Hugo, and Diego. My special thanks go to Maria Clara, an outstanding companion and partner over the past two years. To Diogo, my best friend, thank you for your support, friendship, and help during tough times.

I also want to express my gratitude to the Astroparticle and Particle Physics group at IIP in Natal, led by Prof. Farinaldo and composed of José, Letícia, Ricardo, Yohan, Jacinto, and Lucia. Your expertise and guidance in the final stages of this work have been instrumental, and I am genuinely thankful for your warm reception and assistance.

My family provided unconditional support throughout my master's studies, for which I am deeply grateful. I want to thank Mariazita, Paulo Henrique, Ana Carolina, Zélia, Edinei, Nilceia, Luciana, Chico, Sirlene, Vinícius, and many others for their unwavering support. To Camila, my partner during this journey, I am incredibly fortunate to have had you by my side; this sacrifice is also yours, thank you for the incredible support! I also want to thank my kids (cats), Pandora, Eros, and Isis, for helping me relax when I came home stressed about work. To my parents, Efigênio and Dilene, I owe everything; this (potential) master's degree is dedicated to both of you, my source of strength and determination.

Lastly, but not the least, I thank Gláuber, my supervisor, for your time and patience. I am honored to have been your student, and you have become a role model for the physicist I aspire to be, both as a scientist, teacher, and mentor. Thank you for trusting me to develop this project with you; it was a rewarding experience.

Criança morta, 1944, Candido Portinari [1].

"O essencial é conhecer a técnica íntima de sua arte para, com esses elementos, encontrar a sua feição original."

"The essential is to know the intimate technique of your art in order to, with these elements, find your original feature."

(Candido Portinari)

RESUMO

O Modelo Padrão (SM) da física de partículas teve numerosos sucessos na previsão de resultados experimentais. Apesar disso, a maioria dos físicos de partículas acreditam que o SM não é a teoria final para as interações fundamentais, mas uma teoria efetiva válida apenas em alguma escala de baixa energia. Isso significa que deveria haver física além do SM (BSM). Uma das razões para esta crença é o problema da hierarquia, ou seja, porque a massa de Higgs é muito inferior à escala de Planck. Existem algumas soluções para este problema, como supersimetria, dimensões extras e o Higgs composto. O Higgs composto é uma das soluções mais exploradas para o problema de hierarquia atualmente. Esse modelo estende as simetrias do SM e gera o campo de Higgs como um bóson pseudo-Nambu-Goldstone composto, gerado por uma quebra de simetria. Essa quebra pode gerar diversos campos de Higgs, como singletos e dubletos. Esta dissertação faz uma abordagem abrangente do Modelo Composto de Higgs, com foco principal na construção do modelo e no Higgs Composto Mínimo como principal exemplo. O primeiro capítulo define e discute o Problema da Hierarquia. O segundo capítulo fornece uma revisão completa do mecanismo de Higgs, começando pelos fundamentos da invariância da simetria de calibre e progredindo até a quebra de simetria eletrofraca e as massas dos bósons de calibre. O capítulo três investiga as discussões formais em torno do mecanismo de Higgs a partir de uma perspectiva da teoria de grupos. Finalmente, o capítulo quatro apresenta detalhadamente a construção do Higgs composto, com o modelo mínimo servindo como exemplo concreto.

Palavras-chave: física de partículas; modelo padrão; problema da hierarquia; bóson de higgs; higgs composto.

ABSTRACT

The Standard Model (SM) of particle physics has had numerous successes in predicting the results of experiments. Despite this, most particle theorists believe it is not the final theory for the fundamental interactions but an effective theory valid only at some low energy scale. This means that there should be physics beyond the SM (BSM). One of the reasons for this belief is the hierarchy problem, i.e., why the Higgs mass is so much lower than the Planck scale. There are a few solutions to this problem, such as supersymmetry, extra dimensions, and the Composite Higgs. The composite Higgs is one of the most explored solutions to the hierarchy problem nowadays. This model extends the SM symmetries and generates the Higgs field as a composite pseudo-Nambu-Goldstone Boson generated by a symmetry breaking. This breaking can generate several Higgs fields, such as singlets and doublets. This thesis takes a introductory approach to the Composite Higgs Model, with a primary focus on the model's construction and the Minimal Composite Higgs as a prime example. The first chapter defines and discusses the Hierarchy Problem. The second chapter provides a thorough review of the Higgs mechanism, starting from the basics of gauge symmetry invariance and progressing to Electroweak Symmetry Breaking and the masses of gauge bosons. Chapter three delves into the formal discussions surrounding the Higgs mechanism from a group theory perspective. Finally, chapter four presents the composite Higgs construction in detail, with the minimal composite Higgs serving as a concrete example.

Keywords: particle physics; standard model; hierarchy problem; higgs boson; composite higgs.

SUMMARY

Chapter 1

INTRODUCTION

1.1 Why should we go beyond?

No one can deny that the SM has had an immense number of successful prediction, in excellent agreement with experimental results. However, some observations can not be explained using only the SM, such as dark matter, dark energy, neutrino oscillations, and the hierarchy problem. The SM is not a final theory of fundamental particles and their interactions; it is "just" an effective theory, i.e., there is no reason for it to be valid at a high energy limit, and we expect it not to be!

In order to understand why the SM is an effective theory, one can simply recall that a complete description of gravity is absent in this theory. More precisely, a partial description of quantum gravity at a perturbative level can be used when the gravitational field is not too intense. However, this approach breaks down at the Planck mass scale $E \gtrsim 4\pi M_P \simeq 10^{19}$ GeV, when gravity becomes significant even at the quantum level. Some new physics must appear at last at scales of M_P order to overcome the perturbative limit, giving rise to a "true" elementary quantum gravity theory. We can denote by $\Lambda_{\rm SM}$ the maximum energy scale up to which the Standard Model is a reliable effective theory, and we then have $\Lambda_{\rm SM} \lesssim M_P$.

The example of Quantum Gravity discussed earlier illustrates the effective nature of the Standard Model (SM). However, it does not imply that the theory's cutoff is necessarily at the Planck scale. In fact, we expect the cutoff to be much lower than the Planck scale, because otherwise one would have to face other difficulties related to electroweak symmetry breaking, as explained in what follows.

As an effective description, the SM Lagrangian is calculated at the scale $\Lambda_{\rm SM}$ by integrating out the contributions from higher energy dynamics. The surviving terms in the Lagrangian are those that respect fundamental principles such as the $SU(3)_C \times$

$SU(2)_L \times U(1)_Y$ (the SM gauge group), and Lorentz invariance. However, beyond these invariances, there is little guidance regarding the precise form the Lagrangian should take. One characteristic we do know must be satisfied is that all terms in the Lagrangian must have the correct dimension, denoted as $[\mathcal{L}] = E^4$, with $\Lambda_{\rm SM}$ being the only relevant scale. Consequently, any SM operator must be proportional to $1/(\Lambda_{\rm SM})^{d-4}$, with d being the operator's energy dimension, enabling us to classify operators based on their energy dimensions. When constructing models with predictive power, it is crucial to consider the dimensions of operators. Operators with dimension $d > 4$ correspond to non-renormalizable operators which must be suppressed by some factors of the scale of $\Lambda_{\rm SM}$. Operators with dimension $d = 4$ describe most aspects of SM fields and define a renormalizable theory. However, it is worth examining operators with dimension $d = 2$ more closely.

In the SM, there is only one operator with dimension $d = 2$, the Higgs mass term. Since each Higgs field has a dimension of one in energy, the operator must be *enhanced* by Λ^2_{SM}[2], in order to have the correct dimension. Mathematically, this can be expressed as

$$c\Lambda^2_{\rm SM} H^\dagger H, \tag{1.1.1}$$

where "c" represents a coefficient. The Higgs mass term holds a significant role in Electroweak Symmetry Breaking (EWSB), as it controls the scale of EWSB and, consequently, the mass of all the particles interacting with the Higgs. Following the discovery of the Higgs boson, we know that its mass, m_H, is approximately 125 GeV. This mass value leads to the mass term being $\mu^2 = m_H^2/2 = (85 \text{ GeV})^2$. If we assume that the cutoff of the SM model lies at the Planck scale, we encounter a lack of naturalness in the Higgs mass term

$$c = \frac{\mu^2}{\Lambda^2_{\rm SM}} \sim 10^{-28} \lll 1. \tag{1.1.2}$$

There is no apparent reason for this to occur, and it gives rise to what is known as the hierarchy problem[1]. In the next section, we will discuss this issue in more detail, shedding light on why we should expect new physics below the Planck scale.

1.2 The Hierarchy Problem

In the preceding section, we briefly touched upon the hierarchy problem without delving into its intricacies. Nevertheless, it holds significant relevance as the driving force behind this dissertation. In this section, we aim to elucidate the origin of this problem and

[1] The naturalness problem, and therefore the Hierarchy problem, may be just a pseudo-problem, meaning that it can be just a coincidence that the parameter is in the way it is. There are also exciting anthropic explanations, which may involves a Multiverse hypothesis, [3, 2] that do not need new physics to solve the problem.

present potential resolutions. As one might surmise, our primary focus will revolve around the Composite Higgs Model as a viable solution; alternative approaches are explored in the literature (see e.g. [4, 3]).

To better understand the hierarchy issue, let us consider a scenario where humanity possesses complete knowledge of the final nature theory, encompassing all interactions and particles. This all-encompassing theory, which includes the electroweak interaction and the electroweak symmetry breaking (EWSB) delivered by the Higgs boson, must accurately predict the mass-term of the Higgs field, denoted as μ^2. The Higgs boson mass can be expressed as $m_H^2 = 2\mu^2$. Assuming the Higgs mass depends on energy and on the other parameters "p_{true}" of this fundamental theory, one has[2]

$$m_H^2 = \int_0^\infty dE \frac{dm_H^2}{dE}(E; p_{true}), \tag{1.2.1}$$

where, the integration is performed over a range of energy values, which is necessary to account for contributions across the entire energy scale. This integration should encompass the energy scale from zero to infinity or at least extend up to a very high cutoff imposed by this hypothetically more fundamental theory.

The continuous integration region allows us to divide the integral into two parts. The first part involves integrating up to the cutoff energy of the Standard Model (SM), while the second part covers the remaining energy range. Mathematically

$$\begin{aligned} m_H^2 &= \int_0^{\Lambda_{\rm SM}} dE \frac{dm_H^2}{dE}(E; p_{true}) + \int_{\Lambda_{\rm SM}}^\infty dE \frac{dm_H^2}{dE}(E; p_{true}) \\ &= \delta_{\rm SM} m_H^2 + \delta_{\rm BSM} m_H^2. \end{aligned} \tag{1.2.2}$$

The term $\delta_{\rm BSM} m_H^2$ represents an unknown contribution from energies equal to or higher than $\Lambda_{\rm SM}$. On the other hand, the term $\delta_{\rm SM} m_H^2$ originates from virtual particles below the cutoff energy, and their behavior is assumed to be well described by the SM. Before identifying the specific BSM theory, we cannot make any definitive statements about $\delta_{\rm BSM} m_H^2$. However, we can estimate $\delta_{\rm SM} m_H^2$ by considering the diagrams depicted in Figure (1.1), where only the most relevant contributions are considered, allowing us to derive a reasonable approximation. This calculation gives rise to [2, 3, 4]

$$\delta_{\rm SM} m_H^2 = \frac{3y_t^2}{8\pi^2}\Lambda_{\rm SM}^2 - \frac{3g_W^2}{8\pi^2}\left(\frac{1}{4} + \frac{1}{8\cos^2\theta_W}\right)\Lambda_{\rm SM}^2 - \frac{3\lambda}{8\pi^2}\Lambda_{\rm SM}^2, \tag{1.2.3}$$

where y_t is the Yukawa coupling, g_W is a coupling related to the electroweak theory, θ_W is the Weinberg angle, and λ is the self coupling of the Higgs field[2]. The previous contributions arise from the top quark, gauge bosons and Higgs self-coupling, respectively.

[2] All of these parameters are going to be explained in details in the following chapters.

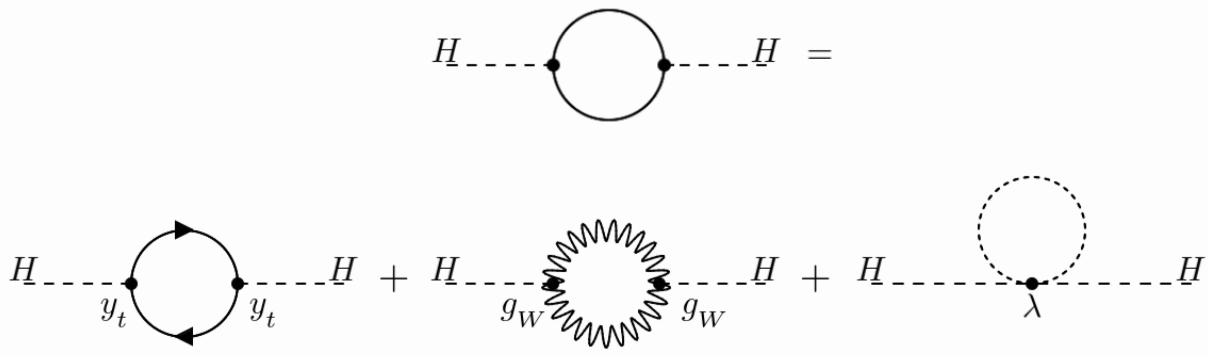


Figure 1.1: Diagramically representation of the Higgs boson mass corrections, delivered by the top quark, gauge boson, and self-coupling, respectively. The y_t is the Yukawa coupling between the Higgs boson and the quark top, the g_W is the Higgs coupling to the electroweak gauge bosons, and the λ is the Higgs self coupling.

Quantum loop correction to the Higgs mass due to the SM goes quadratically with the Λ_{SM}, and, therefore, if $\Lambda_{SM} \gg 125$ GeV, the corrections would be bigger than the Higgs mass itself since $m_H = 125$ GeV. The hierarchy problem becomes evident from this point of view. We must recognize that we are dealing with this calculation solely in the low-energy limit to address this issue. Considering the low-energy regime, we can overcome the divergence problem associated with the quadratic divergences.

Now, the hierarchy problem becomes evident. The "true" mass term receives two contributions that are entirely unrelated due to the stark difference in their energy scales. Suppose the value of $\Lambda_{\rm SM}$ is large. In that case, the second contribution must be just as significant as the first one, with an opposite sign, to reproduce the Higgs boson's light mass in the SM, as shown in Eq. (1.2.2). In other words, $\delta_{\rm BSM} m_H^2 \backsimeq -\delta_{\rm SM} m_H^2$. A measure of fine-tuning can quantify this cancellation between the two contributions, denoted as Δ [2]:

$$\Delta \geq \frac{\delta_{\rm SM} m_H^2}{m_H^2} = \frac{3y_t^2}{8\pi^2}\left(\frac{\Lambda_{\rm SM}}{m_H}\right)^2 \backsimeq \left(\frac{\Lambda_{\rm SM}}{450 \text{ GeV}}\right)^2. \tag{1.2.4}$$

The dominance of the top contribution is considered, as the Yukawa coupling plays a significant role compared to the other terms present in Eq. (1.2.3).

The problem becomes even more evident when we consider a scenario where the SM is assumed to be valid at very high energy scales, i.e., $\Lambda_{\rm SM} \gtrsim 10^{15}$. In this case, the fine-tuning measure is estimated to be $\Delta \backsimeq 10^{24}$. This implies that in order to achieve the required cancellation in the "true" theory description for the Higgs boson mass m_H, the two contributions must match up to the $24th$ decimal place. However, expecting such a precise cancellation between two completely disconnected contributions is highly unlikely. In essence, a theory plagued by this issue loses its predictive power as it would necessitate an experiment capable of determining the two mass contributions with an accuracy of

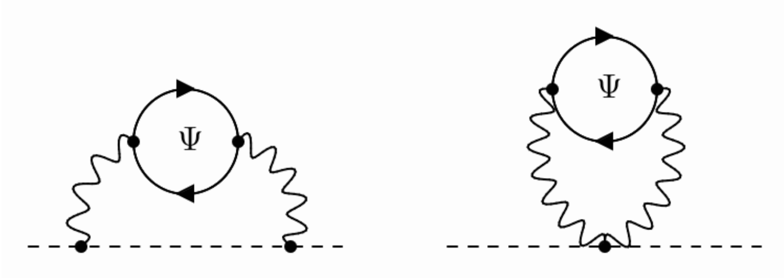


Figure 1.2: Diagrams representing the Higgs mass-term contribution due to two-loop corrections delivered by a new heavy fermionic particle coupled to the Higgs indirectly via SM gauge bosons.

at least 24 decimal places to estimate m_H [3]. Achieving such a level of precision may be practically impossible, leading to an inability to comprehend the microscopic origin of EWSB [2, 4].

The hierarchy problem is not dependent on the chosen renormalization scheme[4], and it goes beyond the issue of canceling divergences. Instead, it arises from the fundamental challenge of reconciling the vastly different energy scales of electroweak and UV scales [5]. Introducing any coupling between the Higgs boson and new physics reintroduces the quadratic dependence on the energy scale at which the new physics manifests. For instance, let us consider a new scalar particle that arises from new physics at a scale m_S and couples to the Higgs boson through a four-point interaction. This new scalar interaction leads to a further contribution in the Lagrangian, $\Delta\mathcal{L} \supset \lambda_S |H|^2|S|^2$, resulting in a correction to the Higgs mass, as given by the equation [4]

$$\delta m_H^2 = \frac{\lambda_S}{16\pi^2}\left[\Lambda_{\rm UV}^2 + 2m_S^2 \ln\left(\frac{\Lambda_{\rm UV}}{m_S}\right) + (\text{higher order cont.})\right]. \tag{1.2.5}$$

By considering the first term inside the bracket[6], we observe that the Higgs mass receives quadratic contributions due to the presence of new physics scales.

The quadratic sensitivity of the Higgs mass persists even when the new states are not directly coupled to the Higgs but interact with other fields in the Standard Model. To illustrate this point, let us consider a pair of heavy fermions, denoted as Ψ, that

[3] The Hierarchy problem can be viewed as merely a pseudo-problem, suggesting that any cancellation may merely be a coincidence, leaving us with no actual issue. However, humanity is at a standstill because we are unsure of the next steps in the scientific exploration of the physical fundamental nature of the universe. We face several challenges within the Standard Model, and it is essential to investigate all possible options to solve them. Therefore, the Hierarchy problem must be thoroughly examined to help us discover new directions in BSM physics.

[4] The claim that the hierarchy problem is a mere renormalization issue or disappears in dimensional regularization is fallacious in both cases, we are not going to this topic, but the reader can go to [5] to have more information.

[5] Here, we denote the UV scales as high energy scales; the UV means Ultraviolet. IR often denotes low energy scales, meaning Infra Red [6, 7].

[6] Repare, there is no physical reason to do it but it is done for the sake of argument.

are charged under the SM group but do not have direct couplings to the Higgs. Due to the interaction between the gauge bosons and these new fermions, the Higgs mass term will acquire "indirect" contributions from diagrams depicted in Figure (1.2). These contributions lead to [4]

$$\delta m_H^2 \sim \left(\frac{g^2}{16\pi^2}\right)^2 \left[a\Lambda_{\text{UV}}^2 + 48 m_F^2 \ln\left(\frac{\Lambda_{\text{UV}}}{m_F}\right) + (\text{higher order cont.})\right], \tag{1.2.6}$$

which has the same form of Eq. (1.2.5), showing again the quadratic sensitivity to a new physics.

With the arguments presented, we can rewrite the hierarchy problem as follows: "The hierarchy problem is the issue that the Higgs mass m_H is sensitive to any high scales in the theory, even if it only indirectly couples to the Standard Model" [4]. This lack of naturalness can be addressed by constraining the scale of new physics to $\Lambda_{\text{SM}} \sim$ few TeV, which results in a fine-tuning parameter of $\Delta \sim 10$. This implies a cancellation of only one digit between the SM and BSM quantum corrections. Consequently, this new physics should be within the reach of current or near-future collider experiments[7]. In this scenario, the SM cutoff eliminates the large loop contribution from above the TeV scale.

Once the issue involving the Hierarchy problem is clear, an attentive reader may ask: why do we attribute the problem only to the Higgs boson mass but not to the fermions and gauge bosons?. The reason is that fermions and gauge bosons have protections against large quantum corrections, which are suppressed by their small tree-level mass parameter. The fermion protection arises from the chiral symmetry in the massless limit. For gauge bosons, the protection is due to the restoration of gauge symmetry in the massless limit [4]. This protection leads to logarithmic corrections in the cutoff scale of the theory, Λ,

$$\Delta m_e \sim m_e \ln\left(\frac{\Lambda}{m_e}\right) \tag{1.2.7}$$

$$\Delta M_W^2 \sim M_W^2 \ln\left(\frac{\Lambda}{M_W}\right). \tag{1.2.8}$$

This absence of quadratic divergences in the loop correction of fermion masses is a good hint towards the idea of Composite Higgs Model (CHM). Instead of being an elementary field, suppose the Higgs is a composite of a new bunch of heavy fermions, or gauge bosons, delivered by a new strong sector. In that case, the divergences do not appear in a quadratic power. The composite Higgs we are going to deal with is about this assumption. In the following chapters, we will treat this topic more deeply.

[7] Therefore, the naturalness is restored.

Chapter 2

THE HIGGS MECHANISM

This chapter is dedicated to elucidating the Higgs Mechanism within the SM. The mechanism serves the purpose of upholding a fundamental principle known as local gauge invariance, which we shall revisit and explore in the first section bellow.

2.1 Gauge theories

We embark on our exploration of the Higgs Mechanism by delving into gauge symmetries, a concept that underpins the foundation of the Standard Model of particle physics. Symmetries are so crucial in particle physics that it is possible to resume all interactions as a resulting junction of symmetries and dynamics, as Figure 2.1 shows. There are two types of symmetries that are foundations of the Standard Model: space-time and gauge symmetries. Here, we will delve into just the gauge case. Extensive literature exists on this topic, with notable references including [8, 9, 10] for a comprehensive overview, and [11, 6] for a more in-depth and didactic approach. We will start our discussion by the most straightforward case we can: the abelian model.

2.1.1 Abelian gauge theory

In particle physics, quantum field theory, or even classical mechanics, symmetries or invariances in physical systems are associated with the conservation of specific quantities[1]. For instance, a system that exhibits translation invariance leads to the conservation of momentum. In contrast, invariance under time displacements and rotations correspond to energy conservation and angular momentum, respectively.

[1] In this context, it is assumed that the reader possesses a certain familiarity with the concepts of field theory and particle physics. Prior knowledge of these subjects will significantly aid in comprehending the following discussion.

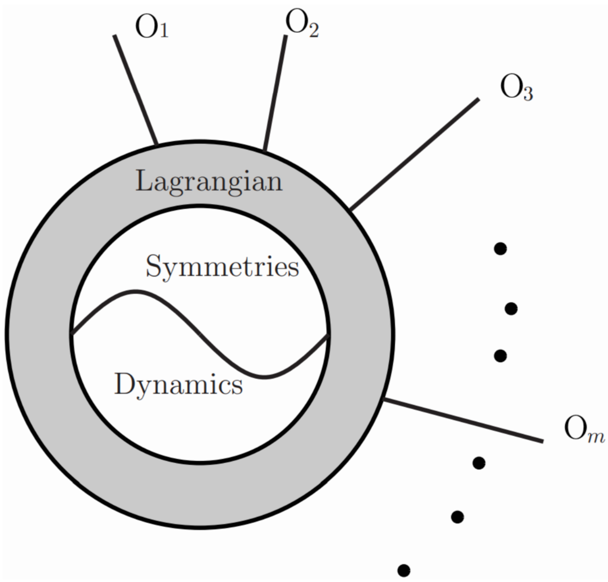


Figure 2.1: The basic structure of the SM. The symmetries and dynamics lays in the core of the theory, then, putting together in the Lagragian, giving rise to the interactions representated by the "O's". Image from [12].

These conservation laws are well-established principles derived from Noether's theorem. To provide an understanding of this theorem, we will briefly discuss it. Our starting point is writing down the system's action.

$$S = \int d^4x \mathcal{L}\left(\phi_i, \partial_\mu \phi_i\right). \tag{2.1.1}$$

Consider the field behaving like $\phi_i \to \phi_i + \delta_\epsilon \phi_i$ under some transformation, linearly depending on N parameters ϵ_A. If the system remains unchanged under the transformation, the action must change at most by a total derivative [13]

$$\delta_\epsilon S = \int d^4x \partial_\mu W^\mu \tag{2.1.2}$$

with W^μ being a linear function of ϵ_A. Under this assumption, it is possible to verify the existence of a conserved current [7], defined by

$$j^\mu(\epsilon) = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} \delta_\epsilon \phi_i - W^\mu, \tag{2.1.3}$$

respecting

$$\partial_\mu j^\mu = \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_i}\right] \delta_\epsilon \phi_i \approx 0. \tag{2.1.4}$$

Therefore, we have N conserved currents for each transformation given by ϵ_A. This is Noether's theorem! It is essential to say that the conservation holds on-shell, i.e., where the classical equations of motions apply [14]. The transformations considered before were global, i.e., the transformation parameter does not depend on the space-time coordinates. We can define a quantity called charge by using the current

$$Q(\epsilon) = \int d^3x j^0(\epsilon). \tag{2.1.5}$$

Integrating Eq. 2.1.4 over the three-space

$$\begin{aligned} 0 &= \int d^3x \partial_\mu j^\mu = \int d^3x \partial_0 j^0 + \int d^3x \vec{\nabla} \cdot \vec{j} = \partial_0 \int d^3x j^0 + \oint_S \vec{j} \cdot d\vec{s} \\ &= \partial_0 Q + \text{surface term}, \end{aligned} \tag{2.1.6}$$

where the divernce theorem was applied. If the fields goes to zero in the infinity, the surface term goes to zero, leading to [15]

$$\dot{Q}(\epsilon) = 0. \tag{2.1.7}$$

Conservation of a certain charge is then a direct consequence of the existence of symmetry in the physical system.

Noether's theorem establishes a profound connection between symmetries and conservation laws, stating that "every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conservation law". As we progress further, we will draw upon concepts from group theory, making it crucial to understand symmetries within our context clearly. We will discuss this in more detail in what follows

In field theory there are two types of symmetries: internal symmetry and space-time symmetry[2]. Unlike space-time symmetries, internal symmetries do not involve fields with varying space-time properties. There are two types of internal symmetries: global and local symmetries. To illustrate what a global symmetry is, let us consider the Lagrangian for the free electron, denoted by the field ψ, which is described by the following Lagrangian[3]

$$\mathcal{L}_{free} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi. \tag{2.1.8}$$

Here, ψ represents a four-component spinor, with each component representing an independent field variable, and the γ_μ denotes the conventional Dirac matrices. For further reference on the notation used, please consult the notation and convention adopted.

One can perform a phase transformation on the field, given by

$$\psi(x) \to e^{i\alpha}\psi(x), \tag{2.1.9}$$

[2] As discussed in the refs. [11, 13]

[3] By applying the Euler-Lagrange equation in this Lagrangian, the world wide famous Dirac equation is obtained [7].

where α is a real constant. This transformation belongs to a family of phase transformations represented by the unitary Abelian group $U(1)$, denoted as $U(\alpha) = e^{i\alpha}$ [6]. The $U(1)$ group comprises all possible phase transformations parametrized by the real constant α. The $U(1)$ gauge transformation does not induce any change in the electron's Lagrangian, Eq. (2.1.8). We can demonstrate the invariance by considering the transformed Lagrangian, denoted as $\mathcal{L}'_{free}$, as follows

$$\mathcal{L}'_{free} = \bar{\psi}'\gamma^{\mu}\partial_{\mu}\psi' - m\bar{\psi}'\psi', \tag{2.1.10}$$

where ψ' is the transformed field, i.e., $\psi' = U(\alpha)\psi$. Expanding this expression, we find

$$\begin{aligned} \mathcal{L}'_{free} &= e^{-i\alpha}\bar{\psi}\gamma^{\mu}(e^{i\alpha}\partial_{\mu}\psi) - me^{-i\alpha}\bar{\psi}(e^{i\alpha}\psi) \\ &= e^{-i\alpha}e^{i\alpha}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - me^{-i\alpha}e^{i\alpha}\psi\bar{\psi}. \end{aligned} \tag{2.1.11}$$

Since $e^{i\alpha}$ and $e^{-i\alpha}$ cancel out in each term, we obtain

$$\mathcal{L}'_{free} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi. \tag{2.1.12}$$

Notice that the transformed Lagrangian, $\mathcal{L}'_{free}$, is equivalent to the original Lagrangian, $\mathcal{L}_{free}$, indicating that the Lagrangian remains invariant under the $U(1)$ gauge transformation, confirming that there is an $U(1)$ internal symmetry.

When the transformation parameter, α, remains constant, it indicates that the field undergoes the same transformation at all points in space-time, referred to as a global symmetry. Now, let us consider the scenario where these transformations become local, meaning that the transformation parameter becomes a function of position, $\alpha \to \alpha(x)$. In this case, different points in space-time experience distinct transformations. If we apply a local $U(1)$ transformation to the field

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \tag{2.1.13}$$

the Lagrangian takes on the form

$$\mathcal{L}'_{free} = \bar{\psi}'\gamma^{\mu}\partial_{\mu}\psi' - m\bar{\psi}'\psi' \tag{2.1.14}$$

$$= e^{-i\alpha(x)}\bar{\psi}\gamma^{\mu}\partial_{\mu}(e^{i\alpha(x)}\psi) - me^{-i\alpha(x)}\bar{\psi}(e^{i\alpha(x)}\psi). \tag{2.1.15}$$

Since the transformation parameter depends on the position, it will not be "transparent" to the derivative, thus

$$\partial_{\mu}(e^{i\alpha(x)}\psi) = e^{i\alpha(x)}\partial_{\mu}\psi + ie^{i\alpha(x)}\psi\partial_{\mu}\alpha. \tag{2.1.16}$$

This new expression will break the $U(1)$ invariance of the lagrangian, explicitly

$$\begin{aligned} \mathcal{L}'_{free} &= e^{-i\alpha(x)}e^{i\alpha(x)}\bar{\psi}\gamma^\mu\partial_\mu\psi - me^{-i\alpha(x)}e^{i\alpha(x)}\psi\bar{\psi} + ie^{-i\alpha(x)}e^{i\alpha(x)}\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha \\ &= \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi\right) + i\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha. \end{aligned} \tag{2.1.17}$$

The term between parenthesis is just the original Lagrangian, therefore

$$\mathcal{L}'_{free} = \mathcal{L}_{free} + i\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha, \tag{2.1.18}$$

and we see that there is an extra term spoiling the Lagrangian gauge invariance.

What occurs when we attempt to restore gauge invariance in the local U(1) transformation? This question is intriguing, as imposing local gauge invariance does not have a direct physical significance. Still, intriguing consequences can emerge from doing so. To embark on this task, we can recognize that the additional term in Eq. (2.1.18) originates from the derivative. Hence, modifying the derivative is the most straightforward approach to restoring gauge invariance. We denote this modified derivative operator as D_μ, which must satisfy the following condition to preserve the Lagrangian's invariance,

$$D_\mu\psi \to e^{i\alpha(x)}D_\mu\psi. \tag{2.1.19}$$

Because it transforms in the same way as the field, we call this operator a covariant derivative. We must introduce in the derivative a new field with transformation properties that eliminate the extra term. The construction that accomplishes this is given by

$$D_\mu \equiv \partial_\mu - ieA_\mu. \tag{2.1.20}$$

Here, the "auxiliary" field A_μ is introduced as a clever technique to restore gauge symmetry. Its purpose is to be constructed in a way that ensures the field undergoes $U(1)$ transformations, transforming like

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha. \tag{2.1.21}$$

The subsequent transformation rule for the field A_μ, referred to as a gauge field, effectively eliminates the term that compromises local gauge invariance. Additionally, including the covariant derivative and Eq. (2.1.21) introduces novel characteristics to the Lagrangian. To observe this, let us combine all the components of the newly gauge-transformed Lagrangian in Eq. (2.1.8),

$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\gamma_\mu D^\mu\psi - m\bar{\psi}\psi \\ &= \bar{\psi}\gamma_\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi \\ &= \mathcal{L}_{free} - ie\bar{\psi}\gamma_\mu A_\mu\psi. \end{aligned} \quad (2.1.22)$$

By examining this equation, we can confirm its local gauge invariance. The transformed free Lagrangian includes an additional term, Eq. (2.1.18), which will be nullified as a result of the gauge field transformation,

$$\begin{aligned}\mathcal{L}'_{free} &= \mathcal{L}_{free} + i\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha - ie\bar{\psi}\gamma_\mu\left(A_\mu + \frac{1}{e}\partial_\mu\alpha\right)\psi \\ &= \mathcal{L}_{free} + i\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha - i\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha - ie\bar{\psi}\gamma_\mu A_\mu\psi \\ &= \mathcal{L}_{free} - ie\bar{\psi}\gamma_\mu A_\mu\psi. \end{aligned} \quad (2.1.23)$$

The additional term in the last equation signifies that the field A_μ plays a significant physical role rather than a mere mathematical artefact. It will engage with the electron field, leading to observable effects. We can also introduce a kinetic term[4] that is invariant under the field transformation, Eq. (2.1.21), to impart dynamics to this field A_μ. The sole entity that fulfills all the requirements associated with the field A_μ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.1.24)$$

The term above corresponds to the field strength tensor, which appears in the covariant formulation of electromagnetism. We can identify the field A_μ as the photon field. Interestingly, through the imposition of local gauge invariance, a new field emerges, and in the case of the electron, we obtain the electromagnetic field as a bonus. This showcases the remarkable potential of gauge symmetry. By demanding local gauge invariance, we can construct an interacting field theory. The final Lagrangian we obtain for the electron is the Quantum Electrodynamics Lagrangian (QED)

$$\mathcal{L}_{QED} = \bar{\psi}\left(i\gamma_\mu D^\mu - m\right)\psi - ie\bar{\psi}\gamma_\mu A_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (2.1.25)$$

This is the construction of the QED as an abelian gauge theory[5].

In the Lagrangian formulation of Quantum Field Theory (QFT), the term associated with mass is quadratic in the field. In the case of the QED Lagrangian, the electron's mass term is represented by $(m\bar{\psi}\psi)$, which is absent in the Lagrangian for the photon

[4] To be considered acceptable, this dynamical term must maintain both gauge and Lorentz invariance.
[5] The term "abelian" is used to describe the $U(1)$ group's character, which refers to its commutative nature. In an abelian group, the order of the group elements does not affect the result of their multiplication.

field. It is widely understood that the photon is massless[6]. However, even if it did possess mass, it would not be possible to describe it within a gauge invariant Lagrangian. A mass term of the form

$$\mathcal{L}_{\gamma-mass} = \frac{1}{2}m^2 A^\mu A_\mu \tag{2.1.26}$$

violates local gauge invariance since

$$A^\mu A_\mu \to \left(A_\mu + \frac{1}{e}\partial_\mu \alpha\right)\left(A^\mu + \frac{1}{e}\partial^\mu \alpha\right) \neq A^\mu A_\mu. \tag{2.1.27}$$

Indeed, local gauge symmetry leads to the emergence of a massless field. This intriguing characteristic motivates us to introduce the Higgs field, as we will delve into and explain further in the subsequent chapters.

2.1.2 Non-Abelian gauge theories

Now, let us move on to the discussion of non-abelian gauge theories. Non-abelian gauge theories are built upon gauge groups that are not abelian, such as $SO(N)$ and $SU(N)$. These theories are necessary to describe self-interacting fields, which abelian theories cannot capture adequately.

We will use the theory of the strong interaction among colour charges as an example. This will lead us to the well-known theory of Quantum Chromodynamics (QCD). The starting point is to recognize that each quark can be found in three possible states, namely it can possess a red, green or blue colour charges. Then the quark field can be arranged as a 3-plet of color fields denoted as q_j described by a lagrangian

$$\mathcal{L}_{free} = \bar{q}_j \left(i\gamma_\mu \partial^\mu - m\right) q_j. \tag{2.1.28}$$

Next we postulate that the strong interaction does not distinguish between these three colours, i.e. there is no preferred direction in colour space. We can then perform a "rotation" in this space without affecting the dynamics. Since this field redefinition must not change total probabilities (i.e. it must not alter the norm of physical states), it must be described by an unitary operator. We thus postulate that the theory must be symmetric under SU(3) transformations given by

$$q(x) \to Uq(x) \equiv e^{i\alpha_a(x)T_a} q(x), \tag{2.1.29}$$

where here, U represents an arbitrary 3×3 unitary matrix, and a summation over repeated Latin indices is applied. The T_a, with $a = 1, ..., 8$, denotes the eight $SU(3)$ generators, and

[6] Actually, the photon mass is subject to an upper limit, as indicated by $M_\gamma < 10^{-18} eV$ [11]. However, this value is minimal because the photon can be considered massless in all practical scenarios.

α_a is the local group parameter.

To proceed from Eq. (2.1.29) towards local gauge invariance, we must consider an infinitesimal transformation and expand the U matrix in Taylor series [6] to apply the group action on the field. Under this consideration, we can write the unitary matrix U up to the first order as

$$U \equiv e^{i\alpha_a(x)T_a} \backsimeq 1 + i\alpha_a(x)T_a, \tag{2.1.30}$$

where $\alpha_a(x)$ are now infinitesimal transformation parameters. Fields and derivatives will obey

$$\begin{aligned} q &\rightarrow (1 + i\alpha_a T_a)q & (2.1.31) \\ \partial_\mu q &\rightarrow (1 + i\alpha_a T_a)\partial_\mu q + iT_a q \partial_\mu \alpha_a. & (2.1.32) \end{aligned}$$

It is clear that similar to the $U(1)$ case, the last term hampers gauge invariance. Motivated by our previous discussion, we are enticed to introduce a gauge field similarly, leading to the covariant derivative

$$D_\mu = \partial_\mu + igT_a G^a_\mu, \tag{2.1.33}$$

and imposing a field transformation

$$G^a_\mu \rightarrow G^a_\mu - \frac{1}{g}\partial_\mu \alpha^a, \tag{2.1.34}$$

to eliminate the additional term. It is crucial to emphasize that the two indices in the field correspond to the Lorentz index, denoted as $\mu = 0, ..., 3$, and the group index, represented by $a = 1, ..., 8$. Consequently, we have eight distinct 4-plet gauge fields that can be identified as the gluon fields, the strong force mediator.

Putting all the pieces together in the transformed Lagrangian, we obtain

$$\begin{aligned} \mathcal{L}' &= \bar{q}(i\gamma_\mu D^\mu - m)q \\ &= \bar{q}(i\gamma_\mu \partial^\mu - m)q - g(\bar{q}\gamma^\mu T_a q)G^a_\mu. & (2.1.35) \end{aligned}$$

This Lagrangian is not invariant due to the part in parenthesis in the last term. This arises from the non-abelian nature of the $SU(3)$ group, i.e., the group generators do not commute among themselves. To have a better view of how the last term transforms, we can apply the transformation

$$\begin{aligned} \bar{q}\gamma^\mu T_a q &\rightarrow [(1 - i\alpha_a T_b)\bar{q}](\gamma^\mu T_a)[(1 + i\alpha_a T_b)\bar{q}] \\ &\rightarrow (\bar{q}\gamma^\mu T_a q) + i\alpha_b \bar{q}\gamma^\mu (T_a T_b - T_b T_a)q \\ &\rightarrow (\bar{q}\gamma^\mu T_a q) - f_{abc}\alpha_b(\bar{q}\gamma^\mu T_c q), & (2.1.36) \end{aligned}$$

where we used the commutation relations between the group generators [6]

$$[T_a, T_b] = i f_{abc} T_c. \tag{2.1.37}$$

We can now see the root of the problem, and we introduce another term to compensate for the non-abelian behavior of the generator. The modification is made on the field; it will transform like

$$G^a_\mu \rightarrow G^a_\mu - \frac{1}{g}\partial_\mu \alpha_a - f_{abc}\alpha_b G^c_\mu, \tag{2.1.38}$$

where the last term ensures that 2.1.35 is gauge invariance.

At the end of the day, what we have is the Lagrangian for Quantum Chromodynamics (QCD), which, including a dynamical term to the gauge field, assumes the form

$$\mathcal{L}_{QCD} = \bar{q}(i\gamma_\mu\partial^\mu - m)q - g(\bar{q}\gamma^\mu T_a q)G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a, \tag{2.1.39}$$

where

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f_{abc} G^b_\mu G^c_\nu. \tag{2.1.40}$$

Once again, we observe a modification compared to the abelian model. The strength tensor now contains an additional term[7]. When we compute the product of the strength tensor, cubic and quartic terms involving the gauge field arise,

$$\mathcal{L}_{QCD} \supset g\text{``}G^3\text{''} + g^2\text{``}G^4\text{''}. \tag{2.1.41}$$

These terms introduce self-interactions, as depicted in Figure 2.2. Consequently, a non-abelian theory is inherently a self-interacting theory.

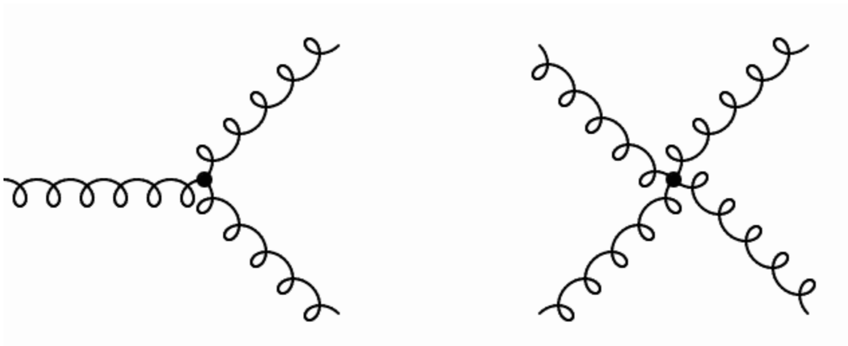


Figure 2.2: Self-interaction of the gluon in QCD. The first diagram is due to the cubic term in G_μ, and the second arises from the quartic term.

Both the non-abelian and abelian gauge theories encounter a similar challenge: introducing a mass term disrupts the gauge symmetry. We do have to introduce some

[7] The invariant strength tensor is constructed, not just assumed to be a determined form. Reference [11] reasonably explains the construction of such a quantity in a pretty general approach.

new mechanism to give mass to the gauge bosons; in the case of photons, it is okay not to have a mass term[8]. However, in the gauge boson in electroweak theory, it is different; it possesses a measurable mass. How can we escape this apparent "nightmare" for the gauge theories? The Higgs[9] has the answer!

2.2 Spontaneous Symmetry Breaking

The previous Section provided a comprehensive overview of local gauge symmetry and its implications. A remarkable feature is the fascinating emergence of a gauge field that mediates interactions when enforcing the theory's invariance. In fact, we consider local gauge invariance to be a fundamental characteristic of nature.

However, a significant issue arises in the absence of an invariant mass term for the gauge bosons. This becomes particularly relevant as we observe that gauge bosons have mass in Weak and Strong interactions. Consequently, a fundamental question emerges. How can we generate mass for the gauge bosons to account for the observed masses in the weak and strong interactions? The answer lies in breaking the symmetry. The idea of breaking the gauge symmetry might sound paradoxical, considering it is a fundamental feature of nature. Nevertheless, two distinct methods exist to achieve this: explicit and spontaneous symmetry breaking.

Explicit symmetry breaking is achieved by introducing a new term, by hand, to the Lagrangian, a therm that breaks the symmetry, so that the total Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_{sym} + \mathcal{L}_{breaking}. \tag{2.2.1}$$

However, this method has notable drawbacks. Firstly, we lose a crucial guide when constructing a Lagrangian theory by abandoning gauge symmetry. Symmetries provide valuable insights into determining the appropriate form of the terms in the Lagrangian. Disregarding this principle can lead to ambiguity and uncertainty in the theory's formulation.

Secondly, a mass term introduced by hande, precesely because it breaks the symmetry, can lead to problematic unrenormalizable divergences in the weak and strong interactions [6]. These divergences can render the theory mathematically inconsistent and hinder its predictive power.

For these reasons, explicit symmetry breaking, in this context[10], proves to be a

[8] Since it does not have a measurable mass.

[9] The physicist, not the particle.

[10] We will see that explicit symmetry breaking is a fundamental tool to construct the Composite Higgs Model, but it has a different approach than here.

less promising approach and may not be worth the effort. Therefore, it becomes evident that spontaneous symmetry breaking presents a viable solution to our problems[11]. This is indeed the case, as we will see!

2.2.1 Spontaneously breaking of a global $U(1)$ symmetry

Spontaneous symmetry breaking, also known as hidden symmetry, occurs when the Lagrangian remains symmetric under certain group transformations, while the physical vacuum breaks this symmetry [9]. To illustrate this process, let us consider a simple case with one complex scalar field described by the free Lagrangian

$$\mathcal{L}_{free} = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi. \tag{2.2.2}$$

This equation exhibits global $U(1)$ symmetry, i.e., $\phi \to e^{i\alpha}\phi$. Within this Lagrangian, there are both a dynamical and a mass term. Now, we can extend this by including a self-interaction term. The simplest term that respects all the symmetries of the free Lagrangian is

$$\mathcal{L}_{int} = -\lambda \left(\phi^\dagger \phi\right)^2. \tag{2.2.3}$$

It is crucial to emphasize that the potential

$$V(\phi) = \mu^2 \left(\phi^\dagger \phi\right) + \lambda \left(\phi^\dagger \phi\right)^2, \tag{2.2.4}$$

must be bounded below. This essential condition ensures that the system possesses a stable ground state. Without this lowest energy level, the system could have negative energy and exhibit unstable behavior [6]. Incorporating this interaction term, the complete equation becomes

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \left[\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2\right]. \tag{2.2.5}$$

An important feature can be applied here. The complex scalar field can be parametrized by two real fields φ_1 and φ_2, such that

$$\phi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right), \tag{2.2.6}$$

$$\phi^\dagger = \frac{1}{\sqrt{2}}\left(\varphi_1 - i\varphi_2\right). \tag{2.2.7}$$

Therefore, the Lagrangian in Eq.(2.2.5) can be rewritten as

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi_1\right)^2 + \frac{1}{2}\left(\partial_\mu \varphi_2\right)^2 - \frac{\mu^2}{2}\left(\varphi_1^2 + \varphi_2^2\right) - \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2. \tag{2.2.8}$$

[11] Sadly, not all of our life problems, just the mass term issue.

In quantum field theories, particles are viewed as excited states of the field. These excitations manifest as quantized fluctuations around the field's lowest energy state, known as the vacuum state. The field's value in this state is called the vacuum expectation value (VEV). To ascertain the particle spectra, we expand the potential around its minimum at the lowest energy state [9], expressed as

$$\begin{aligned} V(\varphi_1,\varphi_2) &= V(\langle\varphi_1\rangle,\langle\varphi_2\rangle) + \sum_{a=1,2}\left(\frac{\partial V}{\partial\varphi_a}\right)\Bigg|_{VEV}(\varphi_a-\langle\varphi_a\rangle) \\ &+ \frac{1}{2}\sum_{a,b=1,2}\left(\frac{\partial^2 V}{\partial\varphi_a\partial\varphi_b}\right)\Bigg|_{VEV}(\varphi_a-\langle\varphi_a\rangle)(\varphi_b-\langle\varphi_b\rangle)+\dots\,, \end{aligned} \tag{2.2.9}$$

where $\langle\phi\rangle = (\langle\varphi_1\rangle\ \langle\varphi_2\rangle)^T$ is the VEV of ϕ. One can readily observe that the second term of the expansion becomes zero when the requirement of VEV being a minimum is imposed[12]. The mixed derivative of this term can be identified as the mass matrix

$$\frac{\partial^2 V}{\partial\varphi_a\partial\varphi_b} = m^2_{ab}, \tag{2.2.10}$$

which can then be diagonalized to identify the system's mass-eigenstates.

The parameters μ and λ in Eq. (2.2.5) govern the physical properties of the system's minimum. When $\lambda > 0$ and $\mu^2 > 0$, the system exhibits a non-degenerate minimum state, as illustrated on the left side of Figure 2.3. In this scenario, there is no breaking of the symmetry. The unique minimum occurs at $\langle\varphi_1\rangle = \langle\varphi_2\rangle = 0$, and the mass matrix is diagonal

$$m^2_{ab} = \begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix}, \tag{2.2.11}$$

leading to fields with equal mass, with mass eingenvalues of m.

Another possibility for the parameter's value is that we can make $\lambda > 0$ and $\mu^2 < 0$. In this case, the potential becomes

$$V(\varphi_1,\varphi_2) = -\frac{\mu^2}{2}\left(\varphi_1^2+\varphi_2^2\right) + \frac{\lambda}{4}\left(\varphi_1^2+\varphi_2^2\right)^2. \tag{2.2.12}$$

The potential minimum is obtained by requiring the first derivatives to be zero,

$$\left(\frac{\partial V}{\partial\varphi_1}\right)\Bigg|_{VEV} = -\mu^2\langle\varphi_1\rangle + \lambda\langle\varphi_1\rangle\left(\langle\varphi_1\rangle^2+\langle\varphi_2\rangle^2\right) = 0, \tag{2.2.13}$$

$$\left(\frac{\partial V}{\partial\varphi_2}\right)\Bigg|_{VEV} = -\mu^2\langle\varphi_2\rangle + \lambda\langle\varphi_2\rangle\left(\langle\varphi_1\rangle^2+\langle\varphi_2\rangle^2\right) = 0. \tag{2.2.14}$$

[12] Actually, the first derivative only ensure that we have a sadle point, and not a mininum but, remember, we said that the potential is bounded bellow, what ensure that there are a global minimum.

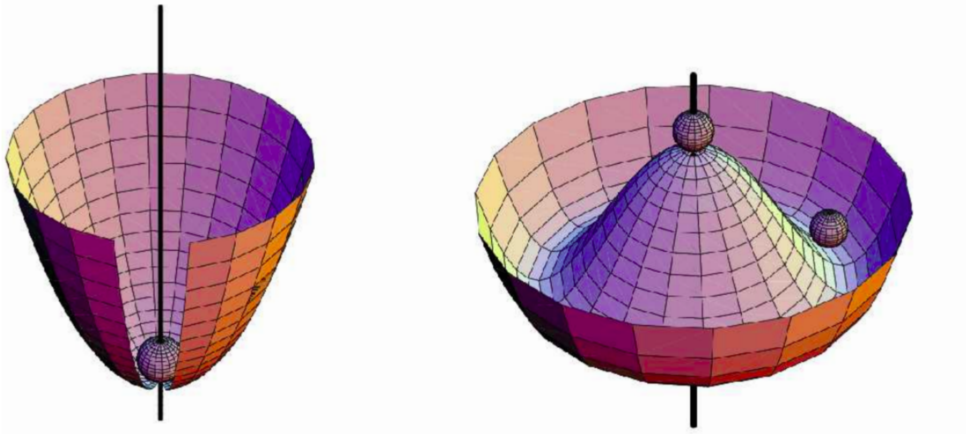


Figure 2.3: **Left**: Here $\mu^2 > 0$, and no symmetry breaking takes place. **Right**: The parameter μ changes, $\mu^2 < 0$, and the potential assumes a shape where the vacuum degenerates; choosing one minimum between many equal ones breaks the symmetry. Image from [8].

The solutions for these equations, apart from the trivial one where the VEV is zero, are

$$\langle\varphi_1\rangle^2 + \langle\varphi_2\rangle^2 = \frac{\mu^2}{\lambda} = v^2. \tag{2.2.15}$$

This equation can be rewritten in terms of the field ϕ

$$\langle\phi^\dagger\phi\rangle^2 = \frac{\mu^2}{2\lambda} = \frac{v^2}{2}, \tag{2.2.16}$$

Figure 2.3, right, illustrates the new potential's structural representation. The VEV now forms a circle with a radius denoted by v, as defined in the last equation. This value results in the spontaneous breaking of the original $U(1)$ symmetry, as the potential must select a specific vacuum state from the range of possibilities.

Figure 2.4 shows a simple analogy that can clarify this concept. Imagine applying pressure to a symmetrically positioned needle along its central axis. Initially, the needle maintains its configuration. However, if the force becomes sufficiently intense, the needle will bend in a particular direction, causing a breaking in its cylindrical symmetry. The newly acquired state shares an energy configuration with all other directions, yet predicting the exact direction of the final state becomes impossible.

Now let us look into what happens with the mass matrix. The second derivatives

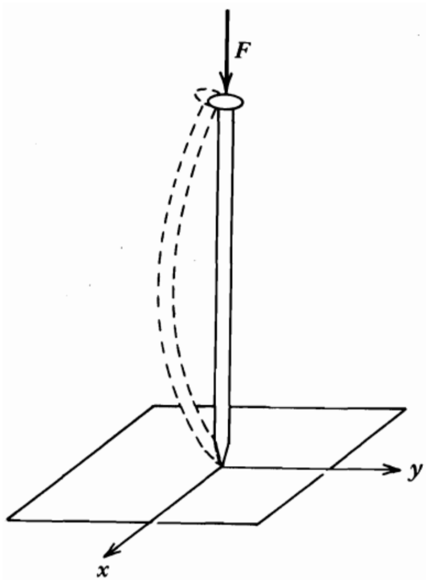


Figure 2.4: Demonstration of spontaneous symmetry breaking in bending a needle. Image from [6].

of Eq. (2.2.12) with respect to the real fields are given by

$$\frac{\partial^2 V}{\partial \varphi_1^2} = -\mu^2 + \lambda\left(\varphi_1^2 + \varphi_2^2\right) + 2\lambda\varphi_1^2. \tag{2.2.17}$$

$$\frac{\partial^2 V}{\partial \varphi_2^2} = -\mu^2 + \lambda\left(\varphi_1^2 + \varphi_2^2\right) + 2\lambda\varphi_2^2. \tag{2.2.18}$$

$$\frac{\partial^2 V}{\partial \varphi_1 \partial \varphi_2} = 2\lambda\varphi_1\varphi_2. \tag{2.2.19}$$

Since all the vacuum states are symmetric, we can choose any configuration without losing generality or physical consequences. Here we pick $\langle\varphi_1\rangle = v$ and $\langle\varphi_2\rangle = 0$. The mass matrix becomes

$$m_{12}^2 = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.2.20}$$

We now realize that the point $\left\langle\phi^\dagger\phi\right\rangle = 0$ does not represent the true minimum. In order to study the field fluctuations, we can translate the field to the real vacuum state, i.e., $\left\langle\phi^\dagger\phi\right\rangle = v^2$. Two new fields can parameterize the fluctuations around the minima, namely ξ and η, as in

$$\phi(x) = \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\xi(x)\right] \tag{2.2.21}$$

One represents the real, and the other the imaginary parts of the fluctuations. It is possible to rewrite the latter equation by using imaginary exponential and disregarding second and higher order terms in ξ and η, i.e.

$$\phi = \left(v + \frac{\eta(x)}{\sqrt{2}}\right) e^{i\xi(x)/\sqrt{2}v}. \tag{2.2.22}$$

Thus, we have a radial and an angular direction in the field vacuum, as shown in Figure 2.5. With the new parameterization, we can write the Lagrangian as follows

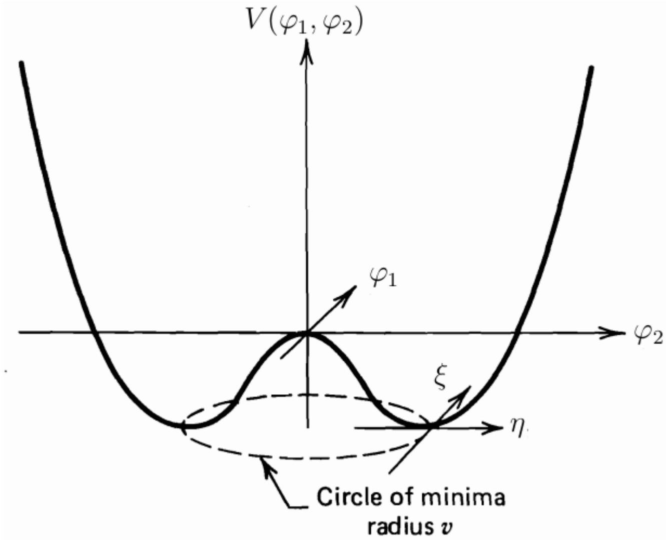


Figure 2.5: Potential structure for a complex scalar field with $\lambda > 0$ and $\mu < 0$. Image from [6] with modifications.

$$\mathcal{L} = \left[\frac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right) + \mu^2\eta^2\right] + \frac{1}{2}\left(\partial_\mu \xi \partial^\mu \xi\right) - \lambda\sqrt{\frac{\mu^2}{\lambda}}\eta\left(\eta^2+\xi^2\right) - \frac{\lambda}{4}\left(\eta^2+\xi^2\right)^2 + \frac{\mu^4}{4\lambda}. \quad (2.2.23)$$

The first term in the Lagrangian is nothing more than the Lagrangian for a massive scalar field η (Klein-Gordon Lagrangian), with mass $m_\eta = \sqrt{2\mu^2}$. The second term is a kinect term for a massless field ξ. Meanwhile, the third and fourth are interacting terms for the two fields and a constant term that does not play any role in the model[13]. The η-particle mass can be viewed as a consequence of a restoring force against oscillations in the radial direction, i.e., the potential in that direction is not free to change. On the other hand, the massless nature of the ξ-particle is a consequence of an $SO(2)$ invariance in the Lagrangian; there is no restoring force against angular oscillations, and you do not have to pay any energy cost to create that particle in that direction [11].

Let us pause briefly; breathe and ask yourself, what happened here? I.e., we started from the Lagrangian for a complex scalar field and ended up with a weird Lagrangian describing massless particles. What is going on? We did nothing but change the notation to the real degrees of freedom of the system, which allowed us to realize the Lagrangian properties more accurately. The Lagrangian must describe the same physical system. Thus, the rise of a massless scalar field should not be due to a mere choice of notation. Indeed, this is a perfect example of Goldstone's theorem, which states that massless scalars

[13] Since the Lagrangian must be integrated to obtain the action, any constant terms will disappear during this process, because it will turn to be surface terms that are neglegible if the field goes to zero in the infinity, which is the case. Therefore, we can ignore it here without any further issues.

occur whenever a continuous symmetry of a physical system is "spontaneously broken" [6]. These scalar particles are known as Nambu-Goldstone Bosons. The Goldstone theorem will be discussed in the next chapter.

Spontaneous symmetry breaking can look like an useless tool by now due to the issues with the Nambu-Goldstone particles. However, it will be helpful to generate mass for other particles, as discussed in the following section. Let us just keep going and study the case of spontaneously broken gauge symmetries.

2.3 The Abelian Higgs Model

We have seen the spontaneous breaking of global gauge symmetry in the last section, but what happens if the spontaneously broken symmetry is local? Well, let us see!

We start with the simplest example: a local $U(1)$ symmetry. Eq. (2.2.5) describes a complex scalar field; the first step is to make it invariant under a local $U(1)$ gauge transformation, namely

$$\phi \to e^{i\alpha(x)}\phi. \tag{2.3.1}$$

Just as in section 2.1, the partial derivative in the Lagrangian must be replaced by the covariant one,

$$D_\mu \equiv \partial_\mu - iqA_\mu, \tag{2.3.2}$$

and the gauge field transforms as

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha. \tag{2.3.3}$$

Putting all this information together and introducing a dynamical term to the gauge field, one has

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\dagger D^\mu\phi - \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2, \tag{2.3.4}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.3.5}$$

Once again

$$\phi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right), \tag{2.3.6}$$

$$\phi^\dagger = \frac{1}{\sqrt{2}}\left(\varphi_1 - i\varphi_2\right). \tag{2.3.7}$$

When $\mu^2 > 0$, we have just the QED Lagrangian for a charged scalar particle of mass μ, plus an extra self-interaction term and the field A_μ remains massless.

For $\mu^2 < 0$, spontaneous symmetry breaking appears. The potential has a continuum of absolute minima, corresponding to a continuum of degenerate vacua at

$$\langle \phi \rangle^2 = -\frac{\mu^2}{2\lambda} = \frac{v}{2}. \tag{2.3.8}$$

Here, the same procedure done in the last section can be used, i.e., shift the fields to rewrite the Lagrangian in terms of perturbations around the true vacuum. Therefore, the Lagrangian with perturbations around the true vacuum is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\partial_\mu \eta - qA_\mu \xi\right]^2 + \frac{1}{2}\left[\partial_\mu \xi + qA_\mu \left(v + \eta\right)\right]^2 \\ &\quad -\lambda \left[v\eta + \frac{\eta^2 + \xi^2}{2}\right]^2 . \end{aligned} \tag{2.3.9}$$

Doing the math and working out all the terms, we get

$$\begin{aligned} \mathcal{L} &= \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(q^2 v^2\right) A_\mu A^\mu}_{Massive\ Vector\ Field} + \underbrace{\frac{1}{2}\left(\partial_\mu \eta\right)^2 - \frac{1}{2}\left(2\lambda v^2\right)\eta^2}_{Massive\ Scalar\ Field} + \underbrace{\frac{1}{2}\left(\partial_\mu \xi\right)^2}_{Nambu-Goldstone\ Boson} \\ &\quad + \underbrace{\left[q^2 A_\mu A^\mu \left(v\eta + \frac{\eta^2}{2} + \frac{\xi^2}{2}\right) - \frac{\lambda}{4}\left(\eta^2 + \xi^2\right)^2 - \lambda v \eta \left(\eta^2 + \xi^2\right)\right]}_{Interacting\ Term} \\ &\quad + \underbrace{\left[qvA_\mu \partial^\mu \xi + qA_\mu \left(\eta \partial^\mu \xi - \xi \partial^\mu \eta\right)\right]}_{Kinetic\ Mixing\ Terms} . \end{aligned} \tag{2.3.10}$$

As expected from the previous section, the η-field now has a mass corresponding to the radial fluctuation. The gauge field A_μ seems to have acquired a mass, as shown in the first term in the Lagrangian. Nevertheless, the ξ-field, which we identify as the Nambu-Goldstone particle, is again present with some weird mix of terms involving its derivatives and the other fields. These terms can be represented in Feynman diagrams as in Figure 2.6, and we see that they would naively correspond to a scalar field being converted to a vector field. These kind of terms represent off-diagonal contributions, that shows us to beware in interpreting the Lagrangian. Indeed, the particle spectrum in $\mathcal{L}$ does not represent the correct one. A proper choice of gauge can reveal the true degrees of freedom

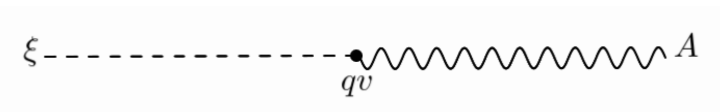


Figure 2.6: Mixing interaction between the gauge boson and the Goldstone field.

of the model. To do this, note that at first order

$$\phi = \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\xi(x)\right] = \frac{1}{\sqrt{2}}\left[v + \eta(x)\right] e^{i\xi(x)/v}. \tag{2.3.11}$$

Using the freedom of gauge transformation, we can choose $\alpha(x) = -\xi(x)/v$ in Eq. (2.3). The gauge transformation becomes

$$A_\mu \to A_\mu - \frac{1}{gv}\partial_\mu \xi. \tag{2.3.12}$$

The field transforms as

$$\phi \to e^{-i\xi(x)/v}\phi = \frac{1}{\sqrt{2}}\left(v + \eta(x)\right). \tag{2.3.13}$$

Since the Lagrangian is locally gauge invariant, we put all the pieces together

$$\begin{aligned} \mathcal{L} \;=\; & \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(q^2v^2\right)A_\mu A^\mu}_{Massive\ Vector\ Field} + \underbrace{\frac{1}{2}\left(\partial_\mu\eta\right)^2 - \frac{1}{2}\left(2\lambda v^2\right)\eta^2}_{Massive\ Scalar\ Field} \\ & + \underbrace{\left[q^2 A_\mu A^\mu\left(v\eta + \frac{\eta^2}{2}\right) - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4\right]}_{Interacting\ Term}, \end{aligned} \tag{2.3.14}$$

plus an irrelevant constant [11].

In our choice of gauge, the Nambu-Goldstone was removed, and we ended up with an η-field with mass $m^2 = 2\lambda v^2$, and a massive vector field A_μ. The ξ-particle is no longer in the Lagrangian, and the boson field has become massive. In fact, fixing the gauge has explicitly shown us that the formerly ξ-field has become the longitudinal component of the massive vector field A_μ. After the symmetry breaking, we had four degrees of freedom: two scalars and two helicity states of the massless gauge field. After the symmetry breaking, we end up with a massive scalar particle, η, plus three helicity states of the massive gauge field A_μ, the same degrees of freedom, but in a different configuration. It is usual to say that the massless Nambu-Goldtone boson was "eaten" by the massless photon to become a massive vector boson [6]. The "new" massive boson η is known as the Higgs boson, and this process of generating mass by dynamical means is known as the Higgs mechanism. The gauge we have used is known as the unitary gauge, or U-gauge [11]. In that gauge, only states that appear in the Lagrangian are the physical ones. The unitarity of the S-matrix and the complete set of Feynman rules for the model are evident in this gauge [11].

Peter Higgs and others who worked on the issue of mass in gauge theory had an excellent motivation behind generating mass via spontaneous symmetry breaking [16]. In order to see this argument, consider a free massless $U(1)$ theory, in which the gauge boson propagator goes like $\sim ip^{-2}$ [6]. By introducing an interaction with some constant background field with a strength $i\Sigma$, the propagator will change. The 1 particle irreducible

$(1PI)$ interaction for this model can be computed, as pictured in Figure 2.7 [7], by

$$\begin{aligned}\underbrace{\frac{i}{p^2}}_{Free\ Theory} &+ \underbrace{\frac{i}{p^2}(-i\Sigma)\frac{i}{p^2}}_{1PI} + \underbrace{\frac{i}{p^2}(-i\Sigma)\frac{i}{p^2}(-i\Sigma)\frac{i}{p^2}}_{(1PI)^2} + \dots \\ &= \frac{i}{p^2}\left[1 + \frac{\Sigma}{p^2} + \left(\frac{\Sigma}{p^2}\right)^2 + \left(\frac{\Sigma}{p^2}\right)^3 + \dots\right] \\ &= \frac{i}{p^2}\sum_{i=0}^{\infty}\left[\frac{\Sigma}{p^2}\right]^n = \frac{i}{p^2}\left(1 - \frac{\Sigma}{p^2}\right)^{-1} \qquad (2.3.15)\end{aligned}$$

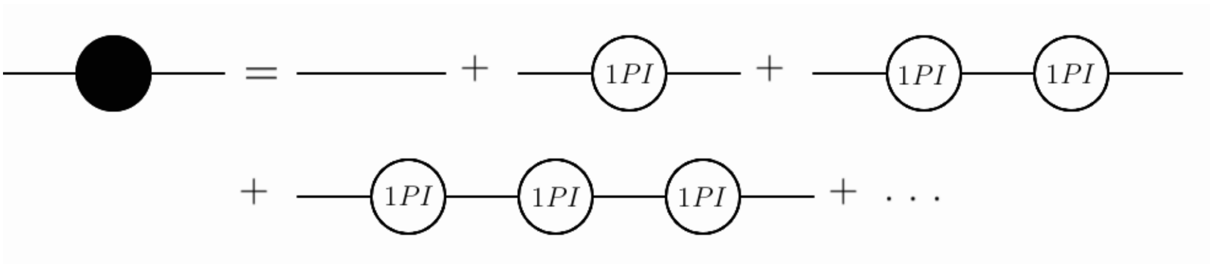


Figure 2.7: The propagator of the interaction theory can be obtained by computing all the $1PI$ insertions.

If the background field strength is given by $-i\Sigma = iq^2\langle\phi\rangle^2$, the expression before can be rewritten as

$$\frac{i}{p^2} + \frac{i}{p^2}(-i\Sigma)\frac{i}{p^2} + \frac{i}{p^2}(-i\Sigma)\frac{i}{p^2}(-i\Sigma)\frac{i}{p^2} + \dots = \frac{i}{p^2 - q^2\langle\phi\rangle^2}, \qquad (2.3.16)$$

where the q is again the electric charge. Surprise, surprise, the interaction between the massless gauge boson with a background field has shifted the mass pole of the A-field to $p^2 = m_A^2 = q^2\langle\phi\rangle^2$. And more: by giving mass, a new degree in the polarization in the field arises [16]. Is it not shocking? This event indicates that generating mass dynamically to the gauge boson is possible [16]. Once there is the Yang-Mills theory, the Goldstone Theorem, and the indication that the gauge field mass can be generated by interacting with another field, the puzzle of giving mass to the gauge bosons has a natural solution in the Higgs Mechanism.

The model discussed in this section, of course, is not observed in nature. There is no observation of a massive photon, at least until now. The $U(1)$ model was only a toy model to explain how the mechanism works. We are now able to extend it to a non-abelian model, which, by the way, applies to the entire Standard Model of particle physics.

2.4 The Glashow-Weinberg-Salam Model

After discussing the Abelian Higgs, it is straightforward to consider extending the approach to a non-Abelian case. Indeed, this step is crucial in order to construct the electroweak model, known as the Weinberg-Salam model.

2.4.1 Non-Abelian Higgs Model

The discussion of the non-abelian model will be constructed for the $SU(2)$ group, one of the fundamental groups of the electroweak interaction. We start by taking the Lagrangian for a complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 . \tag{2.4.1}$$

The field ϕ is a doublet of $SU(2)$, which explicitly is given by

$$\phi = \begin{pmatrix} \phi_\alpha \\ \phi_\beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} . \tag{2.4.2}$$

At this point, it is not difficult to notice that the Lagrangian is invariant under $SU(2)$ phase transformations, such as

$$\phi \to \phi' = e^{i\alpha_a \tau_a/2} \phi, \tag{2.4.3}$$

where $a = 1, 2, 3$ and τ_a are the three groups generators.

To promote the symmetry to a local one, we must take $\alpha_a = \alpha_a(x)$ and, as done before, the covariant derivative must be introduced,

$$D_\mu = \partial_\mu + ig\frac{\tau_a}{2} W_\mu^a . \tag{2.4.4}$$

Here we have three new fields, $W_\mu^a(x)$, due to the "gauging" of the $SU(2)$ group. Consider infinitesimal gauge transformations

$$\phi(x) \to \phi'(x) = \left(1 + \frac{i}{2} \boldsymbol{\alpha} \cdot \boldsymbol{\tau}\right) \phi(x), \tag{2.4.5}$$

where the bold symbol is an abbreviation for three components, i.e., $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ and similarly to the other entities. The gauge fields will transform as

$$\mathbf{W}_\mu \to \mathbf{W}_\mu - \frac{1}{g} \partial_\mu \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \mathbf{W}_\mu . \tag{2.4.6}$$

Here, the difference between an Abelian gauge field transformation, Eq. (2.3.12), and a non-abelian one is evident. The extra term of the latter equation is due to the vectorial nature of the field $\mathbf{W}_\mu$ under $SU(2)$ transformations; it is rotated even if $\boldsymbol{\alpha}$ is independent of x [6]. Putting the covariant derivative in the Lagrangian leads to

$$\begin{aligned}\mathcal{L} &= D_\mu\phi^\dagger D^\mu\phi - V(\phi) - \frac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} \\ &= \left(\partial_\mu\phi + i\frac{g}{2}\boldsymbol{\tau}\cdot\mathbf{W}_\mu\phi\right)^\dagger\left(\partial^\mu\phi + i\frac{g}{2}\boldsymbol{\tau}\cdot\mathbf{W}^\mu\phi\right) - V(\phi) - \frac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu}, \quad (2.4.7)\end{aligned}$$

where

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2, \quad (2.4.8)$$

and the gauge field kinetic term is given by

$$\mathbf{W}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}_\mu\times\mathbf{W}_\nu. \quad (2.4.9)$$

Again, the last term in the equation above is a consequence of the non-abelian nature of the $SU(2)$ group; that is, it occurs because the group generators $\boldsymbol{\tau}$'s do not commute among themselves [6].

The potential structure, Eq. (2.4.8), reminds us of the already studied potential where the μ parameter led to two different physical situations. The $\mu^2 > 0$ case, where there is no symmetry breaking, and the exciting case of $\mu^2 < 0$ which leads to the $SU(2)$ symmetry breaking. In the latter, the potential has a minimum located at a finite value ϕ, where

$$\phi^\dagger\phi \equiv \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) = -\frac{\mu^2}{2\lambda}. \quad (2.4.10)$$

Of course, the vacuum manifold at which the potential is minimized is invariant under $SU(2)$ transformations. As done before, the appropriate way to study the symmetry breaking is expanding the field $\phi(x)$ around a particular minimum, namely

$$\begin{aligned}\phi_1 &= \phi_2 = \phi_4 = 0, \\ \phi_3^2 &= -\frac{\mu^2}{\lambda} \equiv v^2. \quad (2.4.11)\end{aligned}$$

Since a simple rotation can connect all the vacua points, there is no physical implication in choosing a particular state. The system's choice of one specific VEV is the symmetry's spontaneous breaking [6].

We develop the model around its minimum. To do it, we write the field in terms of fluctuations, $\xi_{1,2,3}(x)$ and $h(x)$ as [6, 17]

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_2 + i\xi_1 \\ (v+h(x)) - i\xi_3\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1+i\xi_3/v & i(\xi_1 - i\xi_2)/v \\ i(\xi_1+i\xi_2)/v & 1-i\xi_3/v\end{pmatrix}\begin{pmatrix}0 \\ v+h(x)\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\left[\mathbf{1} + \frac{i}{v}\boldsymbol{\tau}\cdot\boldsymbol{\xi}(x)\right]\begin{pmatrix}0 \\ v+h(x)\end{pmatrix} \\ &\simeq \frac{1}{\sqrt{2}}e^{i\boldsymbol{\tau}\cdot\boldsymbol{\xi}(x)/v}\begin{pmatrix}0 \\ v+h(x)\end{pmatrix}.\end{aligned} \tag{2.4.12}$$

It is possible to get rid of the exponential in the last equation by doing a gauge transformation, just as we did in the abelian case by using the U-gauge, i.e.

$$\phi \to \phi_0 = e^{-i\boldsymbol{\tau}\cdot\boldsymbol{\xi}(x)/v}\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v+h(x)\end{pmatrix}. \tag{2.4.13}$$

The ξ-fields are the ones that will be absorbed by the gauge fields; therefore, they do not appear in the final Lagrangian.

Based on the current situation where the Goldstones are no longer present in the Lagrangian, our focus now shifts to the mass terms of the gauge bosons. These bosons are expected to have acquired mass due to the symmetry-breaking process. In order to accomplish this task we can substitute ϕ_0 of the Eq. (2.4.13) into the Lagrangian in Eq. (2.4.7). The term of interest is [6]

$$\begin{aligned}\left|i\frac{g}{2}\boldsymbol{\tau}\cdot\mathbf{W}_\mu\phi\right|^2 &= \left(i\frac{g}{2}\boldsymbol{\tau}\cdot\mathbf{W}_\mu\phi\right)^\dagger\left(i\frac{g}{2}\boldsymbol{\tau}\cdot\mathbf{W}^\mu\phi\right) \\ &= \frac{g^2}{8}\left|\begin{pmatrix}W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & W_\mu^3\end{pmatrix}\begin{pmatrix}0 \\ v\end{pmatrix}\right|^2 \\ &= \frac{g^2v^2}{8}\left[\left(W_\mu^1\right)^2 + \left(W_\mu^2\right)^2 + \left(W_\mu^3\right)^2\right].\end{aligned} \tag{2.4.14}$$

the symbol $|\ |^2$ is an abbreviation of $(\ )^\dagger(\ )$. Upon comparing the equation above to the gauge boson mass term, $\frac{1}{2}M^2B_\mu^2$, it becomes evident that the mass is $M_W = \frac{1}{2}gv$. As a result, the Lagrangian describes a massive scalar field, $h(x)$ (the Higgs field), and three massive gauge fields. The scalar degrees of freedom, originating from the Goldstones, subsequently became the longitudinal polarizations of the massive vector bosons.

The Higgs mechanism offers a pathway to introduce masses to gauge bosons. Explicitly adding the mass term to the Lagrangian, however, renders the theory unrenormalizable and leads to a loss of predictive power. However, if the symmetry is spontaneously broken, it remains present but is “hidden” by the ground state choice, allowing the model to retain its renormalizability and predictive power.

We can now extend the Higgs mechanism to a “real” theory that describes nature, as we do in the following subsection.

2.4.2 The electroweak symmetry breaking

With all the tools we have lectured about, we are in the position of going forward and formulating the Higgs mechanism so that the gauge bosons W and Z become massive and the photon remains without mass. The model where this is done is called the Glashow-Weinberg-Salam model (GWS)[14], and it unifies two of the four known fundamental forces of nature, the electromagnetic and weak forces, at an energy of $v = 246$ GeV. This model has has been shown to be renormalizable by Gerardus ’t Hooft and Martinus Veltman [20, 21].

We introduce four real scalar fields ϕ_i to achieve our goals. Based on the phenomenology of the electromagnetic and weak interactions [6], it was known that the gauge symmetry corresponding to these interactions is a product of two simple groups of the $SU(2)_L \times U(1)_Y$ group[15]. For a scalar field, we have

$$\mathcal{L} = \left| \left(i\partial_\mu - ig\boldsymbol{T} \cdot \mathbf{W}_\mu - ig' \frac{Y}{2} B_\mu \right) \phi \right|^2 - V(\phi), \tag{2.4.15}$$

where $\boldsymbol{T}$ are the three generators of $SU(2)_L$, Y is the $U(1)_Y$ generator, and the potential $V(\phi)$ is the one in Eq. (2.4.8). This Lagrangian has the same structure as before, plus the $U(1)$ group term. The scalar field must be constructed as a group’s representation to develop a group invariant model. The most economical choice possible is an isospin doublet with weak hypercharge $Y = 1$ [6]

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{2.4.16}$$

[14] Some textbooks refer to this model just as Weinberg-Salam model [6, 11]. What can be unfair to Glashow, who gave significant contributions to the field, as in [18], and won the Nobel prize together with Weinberg and Salam. By the way, his supervisor, Julian Schwinger, gave the idea for his Nobel prize work during his Ph.D. studies. There is an excellent Glashow interview [19] about this topic on YouTube.

[15] The Y stands for hypercharge. The L stands for “left”, indicating the left-chiral nature of particles and how they transform under the group action. It is a way of categorizing particles based on their behavior under the weak interaction. So, particles not carrying a charge under these groups are often called “right-handed” particles. The presence of $SU(2)_L$ in the Standard Model means that only particles with left-chiral properties interact via the weak interaction, while right-handed particles do not. For further information about this topic, please consult the references [6, 11, 7, 21].

where

$$\phi^+ \equiv \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}, \quad (2.4.17)$$

$$\phi^0 \equiv \frac{(\phi_3 + i\phi_4)}{\sqrt{2}}. \quad (2.4.18)$$

Here, the "+" superscript signifies that the doublet component carries an electric charge, whereas the "0" indicates that the field equation is electrically neutral.

Now, it is time to break the symmetry to generate mass for the gauge bosons, but not the photon, of course. To induce a spontaneous symmetry breaking, we demand, as before, the parameter $\mu^2 < 0$ and $\lambda > 0$ and choose a particular vacuum expectation value ϕ_0. One appropriate choice is

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (2.4.19)$$

Why is this a good VEV choice? As one may know, for a vacuum state to be symmetric (invariant) under a group, i.e., $G\phi_0 = \phi_0$, where G is the group transformation [4, 11]. However, we desire the symmetry breaking to generate the boson's masses. Let us see what happens when we apply the group generators to the vacuum state

$$\begin{aligned}
T^1\phi_0 &= \tau^1\phi_0 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0, \ \textit{broken!} \\
T^2\phi_0 &= \tau^2\phi_0 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -iv \\ 0 \end{pmatrix} \neq 0, \ \textit{broken!} \\
T^3\phi_0 &= \tau^3\phi_0 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 \\ -v \end{pmatrix} \neq 0, \ \textit{broken!} \\
Y\phi_0 &= Y_\phi\phi_0 = +1\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0, \ \textit{broken!}
\end{aligned}$$

Where we have used that $\boldsymbol{T} = (1/2)\boldsymbol{\tau}$, the $\boldsymbol{\tau}$ being the Pauli matrices [6]. We also use that the scalar field has hypercharge of +1, by construction [21]. In the previous equations, the field VEV is clearly not symmetric; it breaks all the generators of both groups. Nonetheless, a combination of T^3 and Y is not broken [21], indeed

$$\begin{aligned}
\left(T^3 + \frac{Y}{2}\right)\phi_0 &= \frac{1}{2}\left(\tau^3 + Y\right)\phi_0 = \frac{1}{2\sqrt{2}} \begin{pmatrix} Y_\phi + 1 & 0 \\ 0 & Y_\phi - 1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0, \ \textit{unbroken!} \quad (2.4.20)
\end{aligned}$$

Hence, the generator $Q = \left(T^3 + \frac{Y}{2}\right)$ is a system symmetry. This Q is the electric charge operator, also known as the Gell-Mann–Nishijima relation for the electric charge. This operator gives rise to the $U(1)_{EM}$, i.e., it is the group generator, representing the electromagnetic group, and the Q is the (electric) charge conserved[16]. This result is astonishing; the scalar field's VEV breaks the $SU(2)_L \times U(1)_Y$ to the electromagnetic group, in other words

$$SU(2)_L \times U(1)_Y \to U(1)_{EM}. \tag{2.4.21}$$

Breaking the symmetry gives mass to the gauge bosons and more. It keeps a group intact, which explains why the charge is conserved[17] and how the photon remains massless. If you wonder what would happen if the field's VEV were differed, the answer is almost nothing! The thing is, it is always possible to rotate the vacuum state to another one. Hence, the physics should not be different by just changing the VEV. Indeed, when a rotation is performed in the Eq. (2.4.19), there will be no breaking in the $U(1)_{EM}$; in fact, what will change is the Gell-Mann–Nishijima relation, leading to another definition for the electric charge. Therefore, any VEV state will be electrically neutral and preserve the electromagnetic group [21, 22].

The gauge boson masses are obtained by substituting the vacuum expectation value ϕ_0 in the Lagrangian, whose the relevant term is [6]

$$\begin{aligned}
&\left|\left(-ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{W}_\mu - i\frac{g'}{2}B_\mu\right)\phi\right|^2 \\
=\ & \frac{1}{8}\left|\begin{pmatrix} gW^3_\mu + g'B_\mu & g\left(W^1_\mu - iW^2_\mu\right) \\ g\left(W^1_\mu + iW^2_\mu\right) & -gW^3_\mu + g'B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2 \\
=\ & \frac{1}{8}v^2 g^2\left[\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2\right] + \frac{1}{8}v^2\left(g'B_\mu - gW^3_\mu\right)\left(g'B^\mu - gW^{3\mu}\right) \\
=\ & \left(\frac{1}{2}v^2 g^2\right)^2 W^+_\mu W^{-\mu} + \frac{1}{8}v^2\begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} \\
=\ & \frac{1}{8}v^2\left[g\left(W^3_\mu\right)^2 - 2gg'W^3_\mu B^\mu + g'^2 B^2_\mu\right]
\end{aligned} \tag{2.4.22}$$

where $W^\pm = \left(W^1 \mp W^2\right)/\sqrt{2}$ are the mass eingenstates. Therefore, as the mass term must be $M^2_W W^+W^-$, we have

$$M_W = \frac{1}{2}vg \tag{2.4.23}$$

There are some mixing terms between the B_μ and the W^3, i.e., off-diagonal terms, meaning that the actual field states are not the mass eigenstates. Nevertheless, it is possible to

[16] Remember, by Noether theorem, the presence of a symmetry in the system implies in a conserved charge [11].

[17] A pretty good day on the job, I would say!

rewrite this mixing term in a more convenient form

$$\begin{aligned}\frac{1}{8}v^2\left[g\left(W_\mu^3\right)^2 - 2gg'W_\mu^3 B^\mu + g'^2 B_\mu^2\right] &= \frac{1}{8}v^2\left[gW_\mu^3 - g'B_\mu\right]^2 \\ &+ \; 0\left[g'W_\mu^3 + gB_\mu\right]^2.\end{aligned} \tag{2.4.24}$$

Now, it is clear that the terms inside the brackets are the two orthogonal mass eigenstates, one with eigenvalue zero and the other with eigenvalue $\frac{1}{8}v^2$. By naming the new orthogonal states as the physical fields Z_μ and A_μ, we can identify the Eq. (2.4.24) with

$$\frac{1}{2}M_Z^2 Z_\mu^2 + \frac{1}{2}M_A^2 A_\mu^2, \tag{2.4.25}$$

therefore, upon normalizing the fields [6]

$$A_\mu = \frac{g'W_\mu^3 + gB_\mu}{\sqrt{g^2 + g'^2}} \qquad \text{with} \quad M_A = 0, \tag{2.4.26}$$

$$Z_\mu = \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}} \qquad \text{with} \quad M_Z = \frac{1}{2}v\sqrt{g^2 + g'^2}. \tag{2.4.27}$$

Of course, at this point, the astute reader must have guessed that the A-field is the photon field, massless, and the Z-field is another massive gauge boson field. The $M_A = 0$ is achieved by construction; we have made our choices to make it possible. Therefore, it is not a prediction but a consistency check of the model.

By introducing the world-famous Weinberg angle, θ_W, defined as [6]

$$\frac{g'}{g} = \tan\theta_W, \tag{2.4.28}$$

we can express the A- and Z-fields as

$$\begin{aligned} A_\mu &= \cos(\theta_W)B_\mu + \sin(\theta_W)W_\mu^3, \\ Z_\mu &= -\sin(\theta_W)B_\mu + \cos(\theta_W)W_\mu^3. \end{aligned} \tag{2.4.29}$$

One can also express the W-mass using the Weinberg angle by combining Eqs. (2.4.23) and (2.4.27), leading to

$$\frac{M_W}{M_Z} = \cos\theta_W. \tag{2.4.30}$$

The inequality $M_Z \neq M_W$ originates from the mixing between W_μ^3 and B_μ. When $\theta_W = 0$ there is no mixing, and the Z- and W-mass equality is restored. The ratio M_W/M_Z is a standard model's prediction and, thus, a path to test it. Another SM prediction is the ρ parameter, which is an expression of the gauge bosons masses and the Weinberg angle, which, respects the following relation

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1. \tag{2.4.31}$$

It turns out that ρ is the parameter that specifies the relative strength of the neutral and charged current weak interaction. Experimentally, the parameter value is [23]

$$\rho_{exp} = 1.01013 \pm 0.00005, \tag{2.4.32}$$

a good agreement with the theory, mainly when we think that no loop corrections were considered in here. As we will see, the custodial symmetry is responsible for this parameter being equal to 1 in the SM. In extensions of the SM, it is desirable to have this symmetry to protect this parameter from significant corrections and avoid discrepancies from experimental data.

Until now, we have just worked on the exciting terms for our symmetry-breaking discussion in Eq. (2.4.15), i.e., the mass terms. However, other terms leads to physical consequences as well [11]. Also, we have not discussed the Higgs field during the symmetry-breaking process. Indeed, we focused only on the vacuum state, Eq. (2.4.19). To see the terms involving the Higgs field, we should repeat the steps we took in the non-abelian Higgs model and introduce a perturbation around the minimum, namely

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{2.4.33}$$

where the $h(x)$ is the Higgs field[18]. By substituting this doublet into the Lagrangian in Eq. (2.4.15), one obtains the interacting terms between the gauge bosons and the Higgs, together with the kinetic term to the Higgs boson. Putting all the pieces that contain the Higgs field together, including the potential, one gets [17]

$$\mathcal{L}_{Higgs} = \frac{1}{2}\partial_\mu h \partial^\mu h - \lambda v^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4, \tag{2.4.34}$$

where we have used that $v^2 = \mu^2/\lambda$. As a result, the Higgs mass is

$$M_h^2 = 2\lambda v^2 = -2\mu^2, \tag{2.4.35}$$

the parameter λ is the Higgs self-coupling. We will turn to this expression later during the composite Higgs discussion. It is necessary to stress that the Higgs mass is not a Standard Model prediction; we do not know the parameter λ, but there are some physical constraints, such as vacuum stability and the perturbative limit to theory [17].

[18] Or, as said before, the perturbation in the radial direction in the potential

2.4.3 The quark masses

Until now, we have said nothing about the fermions masses. The fact is that the Higgs mechanism is also needed to give them mass. The weak interaction is chiral and maximally party-violating [24], meaning that the SU(2) gauge bosons only couple to left-handed particles and right-handed antiparticles [25], i.e., particles that have the group charge. This is an experimental observation that was taken into account to construct the SM. Of course, there is also the $SU(3)$ group, but we will not delve into the strong interaction since it plays no role in the composite Higgs model construction; the interested reader can take a look at Ref. [26] for an excellent and pedagogical discussion about the symmetry of the strong interaction group.

In the Standard Model, the left-handed fermions, leptons, and quarks form pairs of SU(2) fundamental representations. They can be collected in three generations of SU(2) doublet pairs

$$L^i = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \tag{2.4.36}$$

$$Q^i = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \tag{2.4.37}$$

where $i = 1, 2, 3$ index the generations. The leptons $(e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)$ are the electron, tau, muon, and their neutrinos, respectively. The quarks (d, u, s, c, b, t) are the down, up, strange, charm, bottom, and top, respectively. The left-handed fermions transform as Weyl spinors[19]. The right-handed fermions are

$$e^i_R = \{e_R, \mu_R, \tau_R\}, \quad \nu^i_R = \{\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}\}, \tag{2.4.38}$$

$$u^i_R = \{u_R, c_R, t_R\}, \quad d^i_R = \{d_R, s_R, b_R\}. \tag{2.4.39}$$

these right-handed particles are uncharged under the SU(2). The right-handed neutrinos have not yet been observed but were included for completeness. In fact, it is an open question in particle physics where the neutrino masses come from; the interested reader can read this at reference [28]. In practice, the fields behave under the SM group action as

$$SU(2)_L : \quad \begin{cases} L \to L' = e^{i\alpha_a(x)\frac{\tau_a}{2}} L \\ R \to R' = R \end{cases}, \tag{2.4.40}$$

$$U(1)_Y : \quad \begin{cases} L \to L' = e^{i\beta(x)} L \\ R \to R' = e^{i\beta(x)} R \end{cases}. \tag{2.4.41}$$

[19] Left-handed particles transforms as $\left(\frac{1}{2}, 0\right)$ and right-handed weyl spinors transforms as $\left(0, \frac{1}{2}\right)$ under the $SU(2)_L \times SU(2)_R$. A pretty good treatment of the Lorentz group and its representations can be found in Refs. [13, 27].

Here L and R represents all the left- and right-handed fermions, respectively. Due to this behavior, we can construct a table with electroweak SM charges, Table 2.1.

Field	L^i	e_R	ν_R	Q^i	u_R	d_R	H
$SU(2)_L$	**2**	**1**	**1**	**2**	**1**	**1**	**2**
$U(1)_Y$	$-\frac{1}{2}$	-1	0	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$

Table 2.1: Standard model fermions. In the first line, the numbers indicate the representations, where **2** represents the fundamental representation of $SU(2)_L$ and **0** is a singlet of the same group. The second line is the hypercharge.

Using the information collected so far, it is possible to write down the interacting terms between the fermions and the gauge bosons, together with the kinetic terms. The procedure is standard: replace the partial derivative with the covariant one, with some necessary modification, and the result is

$$\begin{aligned}\mathcal{L}_{int} &= i\bar{L}^i\left(\partial\!\!\!/ - ig\frac{\tau^a}{2}W\!\!\!\!/^a - i\frac{g'}{2}Y_L B\!\!\!/\right)L_i + i\bar{Q}^i\left(\partial\!\!\!/ - ig\frac{\tau^a}{2}W\!\!\!\!/^a - i\frac{g'}{2}Y_Q B\!\!\!/\right)Q_i \\ &+ i\bar{e}_R^i\left(\partial\!\!\!/ - i\frac{g'}{2}Y_e B\!\!\!/\right)e_R^i + i\bar{\nu}_R^i\left(\partial\!\!\!/ - i\frac{g'}{2}Y_\nu B\!\!\!/\right)\nu_R^i \\ &+ i\bar{u}_R^i\left(\partial\!\!\!/ - i\frac{g'}{2}Y_u B\!\!\!/\right)u_R^i + i\bar{d}_R^i\left(\partial\!\!\!/ - i\frac{g'}{2}Y_d B\!\!\!/\right)d_R^i. \end{aligned} \tag{2.4.42}$$

The symbols with a hash are called slash, and means $\partial\!\!\!/ = \gamma^\mu\partial_\mu$, where the γ_μ are the Dirac matrices. The Y_Q and Y_L are the hypercharge of the left-handed fields, and Y_e, Y_ν Y_u and Y_d for the right-handed fields' hypercharges [24].

Cutting to the chase, we now go to the mass. For reasons that will be clear later, we will focus our discussion only on the quark masses. If one considers the Dirac mass term to the fermion [7]

$$\mathcal{L}_{mass} \supset -m\left(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L\right), \tag{2.4.43}$$

it is evident, due to Eq. (2.4.40), that this term is not invariant under local $SU(2)$ gauge transformation. The chiral nature of the electroweak interaction is the criminal guilty of this crime. There is no way to construct an operator to couple with the left-handed particle having the correct quantum numbers by using only the leptons and gauge bosons. Therefore, we must include the Higgs boson to save the day, once more.

Focusing on the quarks and including all three generations, we have

$$\mathcal{L}_{mass} = -Y_{ij}^d\overline{Q}^i\phi d_R^j - Y_{ij}^u\overline{Q}^i\tilde{\phi}u_R^j + h.c. \tag{2.4.44}$$

where $\tilde{\phi} \equiv i\tau_2\phi^\star$ and $h.c.$ represents the hermitian conjugate terms. The Y's in the last equation are the Yukawa three-dimensional matrices. After the Higgs acquires its VEV, one gets

$$\mathcal{L}_{mass} = -\frac{v}{\sqrt{2}} Y^d_{ij} \overline{d}^i_L d^j_R - \frac{v}{\sqrt{2}} Y^u_{ij} \overline{u}^i_L u^j_R + h.c. = -\frac{v}{\sqrt{2}} \left[\overline{d}_L Y_d d_R + \overline{u}_L Y_u u_R \right] + h.c. \quad (2.4.45)$$

The last equation is in a matricial form. To find the masses, the matrix must be diagonalized, leading to [24]

$$\mathcal{L}_{mass} = -\frac{v}{\sqrt{2}} y^d \, \overline{d}_L d_R - \frac{v}{\sqrt{2}} y^u \, \overline{u}_L u_R + h.c.. \quad (2.4.46)$$

Each quark acquires a mass given by

$$m^{(d,u)} = -\frac{v}{\sqrt{2}} y^{(d,u)}. \quad (2.4.47)$$

This is the quark mass we were looking for. Keep this expression in mind; we will get back to it soon!

Much more can be discussed about the fermionic sector of the electroweak interaction, such as the mixing between the generations, but for the CH debate present in this text, what was presented is sufficient to proceed to the next topic.

2.5 Explicit symmetry breaking: A first sight

Before finishing this chapter, it is worth examining the behavior of the potential described in Eq. (2.4.8) by introducing a perturbation term that explicitly breaks the underlying symmetry. The potential, then, becomes

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2 - \epsilon \phi_3, \quad (2.5.1)$$

where the parameter $\epsilon > 0$ and the field ϕ is a doublet given by Eq. (2.4.2). The new term changes the potential's vacuum configuration. To see that we can differenciate the potential with respect to the fields and make it equal to zero, i.e., we are looking at the sadle point of the potential. By doing this procedure, we get

$$\left(\frac{\partial V}{\partial \phi_k}\right) = \left[\mu^2 + 2\lambda \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)\right] \phi_k = 0, \quad (2.5.2)$$

$$\left(\frac{\partial V}{\partial \phi_3}\right) = \left[\mu^2 + 2\lambda \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)\right] \phi_3 - \epsilon = 0, \quad (2.5.3)$$

where $k = 1, 2, 4$. The field configuration that satisfies both equations is the vacuum state [29]. Substituting the usual VEV configuration, Eq. (2.4.11), the relations above will turn to

$$\left(\frac{\partial V}{\partial \phi_k}\right)\Big|_{VEV} = 0, \tag{2.5.4}$$
$$\left(\frac{\partial V}{\partial \phi_3}\right)\Big|_{VEV} = \left[\mu^2 + 2\lambda v^2\right] v - \epsilon = 0, \tag{2.5.5}$$

thus

$$\left[\mu^2 + 2\lambda v^2\right] = \frac{\epsilon}{v}. \tag{2.5.6}$$

Notice that in the limit of $\epsilon \to 0$ we recover the usual VEV configuration. The field can be expressed in terms of fluctuations around the minimum[20]

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i\xi_2 \\ (v + h(x)) - i\xi_3 \end{pmatrix}. \tag{2.5.7}$$

Putting this into the potential one finds

$$\begin{aligned} V &= \frac{\mu^2}{2}\left[\xi_1^2 + \xi_2^2 + \xi_3^2 + (v+h)^2\right] + \frac{\lambda}{4}\left[\xi_1^2 + \xi_2^2 + \xi_3^2 + (v+h)^2\right]^2 - \epsilon\,(v+h) \\ &= \frac{\mu^2}{2}\left(\xi_1^2 + \xi_2^2 + \xi_3^2 + h^2\right) + \frac{\lambda}{4}\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) v^2 + \text{interation terms.} \\ &= \frac{1}{2}\left(\mu^2 + 2\lambda v^2\right)\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) + \frac{\mu^2}{2} h^2 + \text{interation terms.} \end{aligned} \tag{2.5.8}$$

using Eq. (2.5.6), we reach

$$V = \frac{1}{2}\left(\frac{\epsilon}{v}\right)\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) + \frac{\mu^2}{2} h^2 + \text{interation terms.} \tag{2.5.9}$$

This result is surprising. By adding a perturbation that breaks explicitly the symmetry, the Golstones modes have acquired a mass

$$m_\xi = \sqrt{\frac{\epsilon}{v}}. \tag{2.5.10}$$

The Nambu-Goldtones become pseudo Nambu-Goldtone Bosons (pNGB) [29]. Why dis this happen? Remember that in the previous sections, we identified the Nambu-Goldstones as a perturbation on the angular direction in the potential. Due to vacuum degeneracy, no energy cost is associated with moving along the valley of the potential, suggesting that the mass term in this direction becomes zero [3]. Adding the ϵ term breaks the vacuum degeneracy. Indeed, Eq. (2.5.5) shows that the perturbation has lifted the vacuum

[20] We already have done this procedure in Eq. (2.4.12), but now, we are not going to the U-gauge once we want to see what the perturbation effect into the Goldtones modes

manifold. Figure 2.8 depicts the potential's format with and without explicit symmetry breaking. Now, there is a cost moving along the angular direction, and, therefore, the ξ-fields became massive.

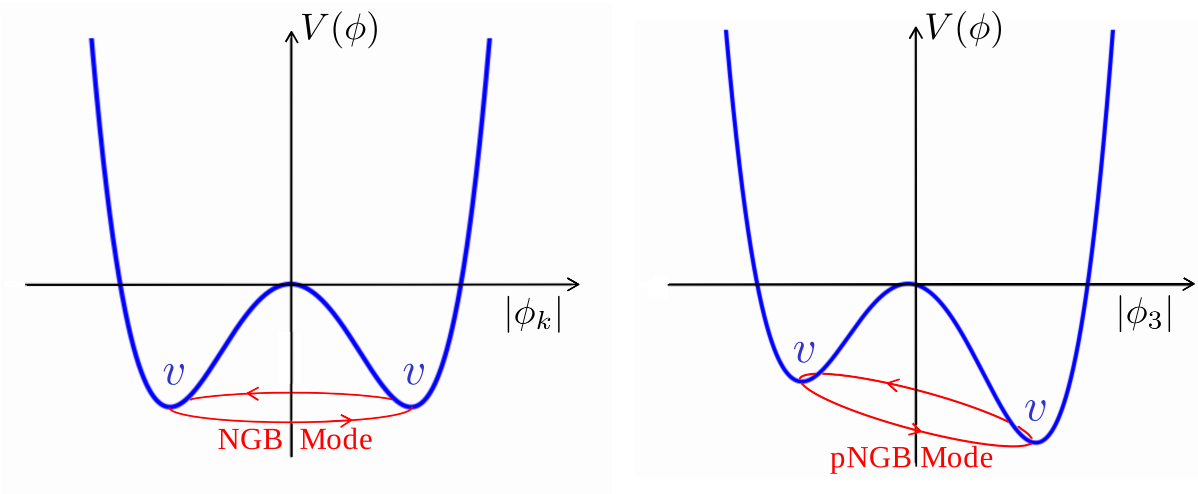


Figure 2.8: **Left**: There is no explicit symmetry breaking, therefore, the vacuum is degenerate. This situation is what happens in the fields ϕ_k. **Right**: The vacuum has been lifted, and the symmetry is explicitly broken, turning the NGB into pNGB. Image inspired by reference [30].

Despite the exciting effect of generating mass to the gauge bosons, the breaking term in the Lagrangian brings an unwanted effect. Calculating quantum corrections to observables in terms of Goldstone-boson loop diagrams will generate nonanalytic corrections in the symmetry-breaking parameter such as $\epsilon \ln(\epsilon)$ [29]. The procedure of explicitly breaking symmetry will be used in the Composite Higgs construction. Then, we must ensure that nonanalytic terms will not appear in the model.

Now, we are ready to enter the composite Higgs model. Hold on tight. The journey is just beginning!

Chapter 3

GROUP THEORY STRUCTURE

During this section we are going through a small stroll into group theory. The entire Composite Model is built mainly on group theory arguments. Therefore, it is helpful to discuss this foudations; of course, some group theory knowledge is desirable in order to understand the arguments, but it is not required. Indeed, most of the references do not cover this topic [2, 31]. Hence, it is possible to comprehend the model without this study.

3.1 Goldstone theorem

In chapter two, we have used the statement of Goldstone's theorem; now, we will discuss it more deeply. It is crucial not just in the Higgs mechanism but also in the Composite Higgs models.

Let ϕ be an n-tuplet of fields describing a theory, which transforms in some representation[1] of a global symmetry Lie group $\mathcal{G}$. In an interacting theory, these fields are subject to a potential $V(\phi_i)$, in which the minima define the vacuum manifold

$$V_0 \equiv \{\phi_0 \mid V(\phi_0) = V_{min}\}\,. \tag{3.1.1}$$

This manifold is the crucial aspect of the spontaneous symmetry breaking. If it is a single point, there is no symmetry breaking, but if the manifold is not trivial, the symmetry is spontaneously broken since the field must choose one of the minima points. Moreover, if this is the case, any transformation of the symmetry group $\mathcal{G}$, will move the vacuum configuration from one point to another within the vacuum manifold. If ϕ_0, $\phi_0' \in V_0$, then there is a $g \in \mathcal{G}$ such that

$$\phi_0' = g\phi_0. \tag{3.1.2}$$

[1] It is assumed that the reader is familiar with group theory. For any questions or uncertainties about this topic, please refer to Ref. [32, 33, 34]

It may happen that some elements of the group will keep the original state unchanged. We will denote by $\mathcal{H}$ this subgroup that leaves the vacuum invariant. If we consider a point ϕ_0, the group $\mathcal{H}$ is defined as the set of elements in $\mathcal{G}$ that leave ϕ_0 unchanged

$$\mathcal{H} \equiv \{h \in \mathcal{G}|\ h\phi_0 = \phi_0\}. \tag{3.1.3}$$

The field ϕ is called the order parameter of the symmetry $\mathcal{G}$.

By taking advantage of these definition, Goldstone's Theorem can be stated as follows.

Theorem 3.1.1 (*Goldstone's theorem*)**.** If a continuous and global symmetry $\mathcal{G}$ is spontaneously broken to $\mathcal{H}$, then there will be massless particles called Goldstone bosons. The number of these Goldstone bosons is given by

$$dim\,(\mathcal{G}/\mathcal{H}) = dim\mathcal{G} - dim\mathcal{H}.$$

To show how this theorem arises, supose the field ϕ is in an N dimensional represention of $\mathcal{G}$, and that it has compotents $\phi = (\phi_1, ..., \phi_i)$ with $i = 1, ..., N$. Applying the exponential map to the field

$$\phi \to \phi' = e^{i\alpha^A T^A}\phi = \left[1 + i\alpha^A T^A + ...\right]\phi \simeq \phi + i\alpha^A T^A \phi, \tag{3.1.4}$$

where we considered the α^A as an infinitesimal parameter and, thus, considered terms only up to the first order in the expansion. Each component of the field will change by

$$\delta\phi_i = i\alpha^A \left(T^A\right)_i^j \phi_j, \tag{3.1.5}$$

with the $\left(T^A\right)_i^j$ being the group generators in the representation taken. Putting the transformed field in the potential and expanding it around $\delta\phi = 0$ up to the first order, we have

$$\delta V(\phi + \delta\phi) = V(\phi) + i\alpha^A \frac{\partial V}{\partial \phi_i} \left(T^A\right)_i^j \phi_j. \tag{3.1.6}$$

The potential is symmetric under group transformations, therefore, it must remain unchanged, leading to

$$\delta V(\phi + \delta\phi) - V(\phi) = i\alpha^A \frac{\partial V}{\partial \phi_i} \left(T^A\right)_i^j \phi_j = 0. \tag{3.1.7}$$

Now, take the derivative with respect to ϕ_j,

$$\left[\frac{\partial V}{\partial \phi_i} \left(T^A\right)_i^j + \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \left(T^A\right)_i^k \phi_k\right] = 0, \tag{3.1.8}$$

where the factor α^A was dropped down since it is arbitrary. At the VEV, the first term goes to zero, since it is a minimum of the potential, while the remaining term is

$$\left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)\bigg|_{\phi=\phi_0}\left(T^A\phi_0\right)_i = 0, \tag{3.1.9}$$

for $A = 1, ..., dim\mathcal{G}$. The second derivative can be identified as the mass matrix and, thus,

$$M^{ij}\left(T^A\phi_0\right)_i = 0. \tag{3.1.10}$$

There are three situations where the equation above is satisfied. First, the vacuum state itself can be zero, $\phi_0 = 0$; in this case, there is no symmetry breaking, and no particles are expected. Secondly, $\left(T^A\phi_0\right)_i = 0$ is null, meaning that, for the generators that annihilate the vacuum, which is the case of the $\mathcal{H}$ generators no particle is created. Last but not least, for the case $\left(T^A\phi_0\right)_i \neq 0$, we have $M^{ij} = 0$, i.e. states with mass eigenstates equal to zero are created. These are the Goldstone bosons!

This is proof of the theorem at the level of classical field theory, which shall be enough for our purpose of motivating the theorem. The reader interested in the quantum treatment can consult Ref. [7]. We, now, go to see where the Higgs boson lives[2].

3.2 The Higgs house

This section intend to show the group theory foundation of the Higgs mechanism. We start our discussion considering a physical system described by a Lagrangian with spontaneous symmetry breaking, i.e., the Lagrangian is invariant under a Lie group $\mathcal{G}$, but the vacuum state is invariant just under a subgroup $\mathcal{H}$ of $\mathcal{G}$ [29]. The breaking pattern will generate $n = dim[\mathcal{G}] - \dim[\mathcal{H}]$ Nambu-Goldstones. These massless particles will be described by n fields, ϕ_i, that are real functions on Minkowski space, M^4. As done before, the fields can be arranged in an n-vector

$$\vec{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix}. \tag{3.2.1}$$

The vector defines an n-dimensional vector space[3]

$$V \equiv \left\{\phi : M^4 \to \mathbb{R}^n | \phi_i : M^4 \to \mathbb{R} \ \text{ continuous}\right\}. \tag{3.2.2}$$

[2] The boson, not the physicist.
[3] Which is also a manifold.

Now, we define a map $\widetilde{\varphi}$ that associates, in a unique way, to each pair $(g, \phi) \in \mathcal{G} \times V$ an element $\widetilde{\varphi}_V(g, \phi) \in V$, videlicet

$$\begin{aligned} \widetilde{\varphi} : \mathcal{G} \times V &\rightarrow V \\ (g, \phi) &\mapsto \widetilde{\varphi}_V . \end{aligned} \tag{3.2.3}$$

Here we are following the notation from the references [35, 36], where the barred arrow denotes the action of a function at the element level, i.e., while $f : A \rightarrow B$ indicates a function which has a domain in A and a codomain B, the $f : A \mapsto B$ indicates that a particular function f associates to an argument A a value B.

The map described in Eq. (3.2.3) aims to define the operation of the group $\mathcal{G}$ on the vector space V, known as a group action. This operation is a homomorphism from one group to another and satisfies the conditions [29]

$$\widetilde{\varphi}(e, \phi) = \phi \quad \forall \ \phi \in V \tag{3.2.4}$$

$$\widetilde{\varphi}(g_1, \widetilde{\varphi}(g_2, \phi)) = \widetilde{\varphi}(g_1 g_2, \phi) \quad \forall \ g_1, g_2 \in \mathcal{G} \text{ and } \forall \ \phi \in V. \tag{3.2.5}$$

The latter equation is the criterion for a map to be a homomorphism, ensuring that the group structure is preserved during the process. We now choose as the origin of the vector space V the system's vacuum state ϕ_0. Using the group action, we can make a definition we will use later[4].

Definition 3.2.1 (*Orbit*)**.** Given a set $\mathcal{X}$ on which a group $\mathcal{G}$ acts and $x \in \mathcal{X}$, the set of all images defined as

$$\mathcal{O}(x) = \{g \cdot x \mid g \in \mathcal{G}\} ,$$

is the orbit of $\mathcal{X}$ [33].

The Figure 3.1 shows an orbit of the group action. Since the system's ground state is invariant under the subgroup $\mathcal{H}$, the $\widetilde{\varphi}$ must map the vacuum to itself for all the subgroup elements. Reformulating $\widetilde{\varphi} : (h, \phi_0) \mapsto \phi_0$ for all $h \in \mathcal{H}$. We can define concept of stabilizer as follows.

Definition 3.2.2 (*Stabilizer*)**.** Given a set $\mathcal{X}$ which a group $\mathcal{G}$ acts on and $x \in \mathcal{X}$, the subgroup defined as

$$\mathcal{G}_x = \{g \cdot x = x \mid g \in \mathcal{G}\} ,$$

is the stabilizer of x [33].

[4] We will have to use some definitions taken from the references with modifications in some cases to make them more reliable. According to [32], "A good definition should be precise, economical, and capture a simple, intuitive idea. If in addition, it is easy to work with, so much the better."; a thought we will work to follow!

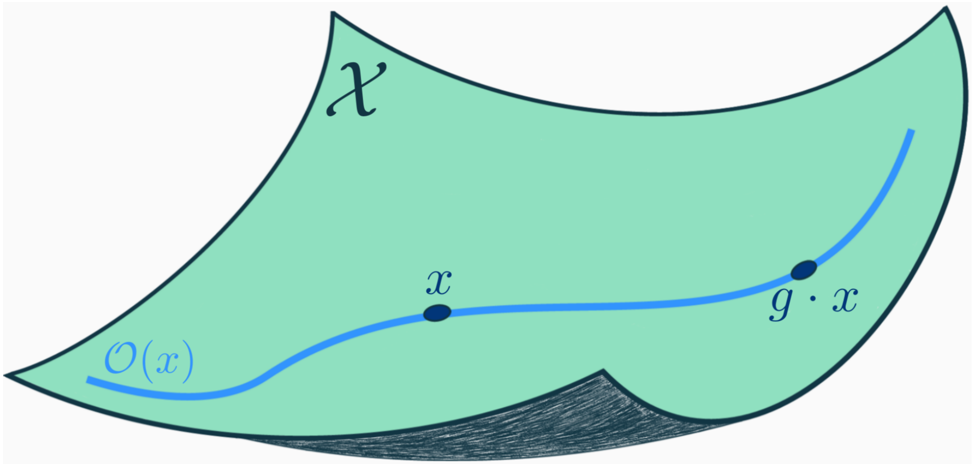


Figure 3.1: The orbit $\mathcal{O}(x)$ of a group action.

The stabilizer is also called little group. Clearly, $\mathcal{H}$ is the stabilizer of ϕ_0.

Given a subgroup $\mathcal{H}$ of $\mathcal{G}$, we define the left coset as

Definition 3.2.3 (*Left Coset*)**.** If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and $g \in \mathcal{G}$, the subset of $\mathcal{G}$ constructed as

$$g\mathcal{H} = \{g \cdot h \mid h \in \mathcal{H}\}$$

is called the left coset of $\mathcal{H}$ [33].

The definition of the right coset is straightforward, as the name suggests. All the left cosets together form a group defined as follows.

Definition 3.2.4 (*Quotient Group*)**.** If $\mathcal{H}$ is a normal subgroup, then the set of all left cosets of $\mathcal{H}$

$$\mathcal{G}/\mathcal{H} = \{g\mathcal{H} \mid g \in \mathcal{G}\},$$

forms a group called the quotient, or factor, group [33]. The group operation is defined by

$$g_1\mathcal{H} \cdot g_2\mathcal{H} = g_1 \cdot g_2\mathcal{H}.$$

A normal subgroup is one satisfying the property of $g\mathcal{H}g^{-1} = \mathcal{H}$ for all $g \in \mathcal{G}$. It is crucial to stress that the quotient group is not contained in $\mathcal{G}$[5]. The $\mathcal{G}/\mathcal{H}$ is a group where each element is by itself a subset of $\mathcal{G}$. Figure 3.2 illustrates this structure. By looking at this image, it is also possible to see that the cosets of the factor group either completely overlap or are completely disjoint [29]; the pink contours do not intersect each other.

The reader must be asking about the reason for all these definitions. It turns out that the NGBs live in the coset $\mathcal{G}/\mathcal{H}$; what we are going to do is use the definition

[5] Indeed, the elements of each coset in the quotient group are contained within $\mathcal{G}$. However, the elements of $\mathcal{G}/\mathcal{H}$ represent distinct cosets, meaning these elements are not individual elements of $\mathcal{G}$, but rather sets of elements.

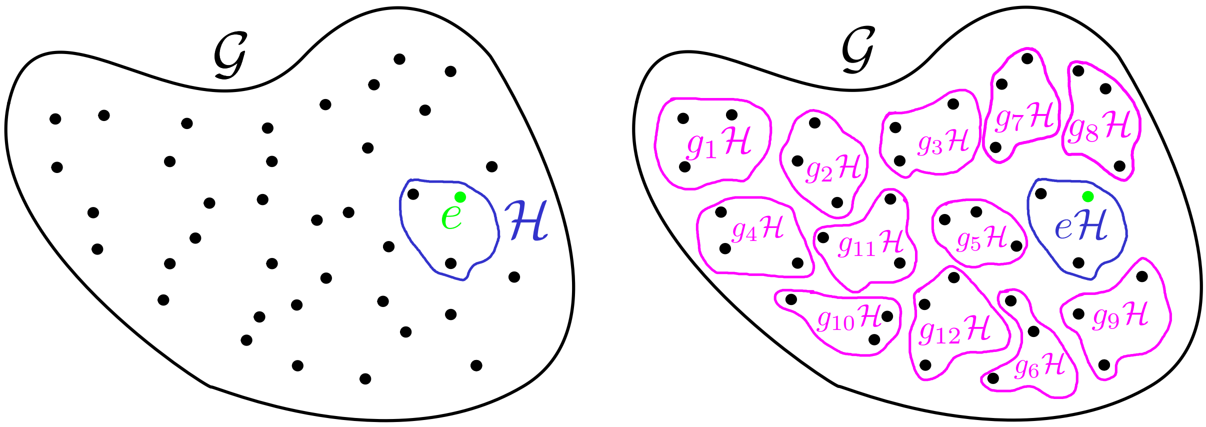


Figure 3.2: Illustration of the quotient group. **Left:** Pictorial representation of the group $\mathcal{G}$, its elements represented by the black. The $\mathcal{H}$ subgroup is pictured by the blue contour that contains the unitary element, e, as the green dot. **Right:** The elements of the group $\mathcal{G}/\mathcal{H}$ are represented by the pink contours. Each left coset $g_i\mathcal{H}$ represents an element of the quotient group, where $g_i \in \mathcal{G}$. Note that the subgroup $\mathcal{H}$ is itself the group's unity element.

presented before to prove this statement. We start this task by applying the group action (3.2.3) in the quotient group. In this sense we have a new map, namely

$$\begin{aligned} \varphi_{\mathcal{G}/\mathcal{H}} : \mathcal{G}/\mathcal{H} \times V &\rightarrow V \\ (gh, \phi) &\mapsto \varphi. \end{aligned} \tag{3.2.6}$$

This group action has the interesting property of mapping the origin onto the same vector in $\mathbb{R}^n$ under all elements of a given coset $g\mathcal{H} \in \mathcal{G}/\mathcal{H}$ [29]

$$\varphi(gh, \phi_0) = \varphi(g, \varphi(h, \phi_0)) = \varphi(g, \phi_0) \;\; \forall \;\; g \in \mathcal{G} \;\text{ and }\; h \in \mathcal{H}; \tag{3.2.7}$$

where we have used Eq. (3.2.5) and the fact that the vacuum state is invariant under the action of any element of $\mathcal{H}$. This feature ensures that the map is surjective, i.e., all the points in the vacuum manifold have a correspondent group action elements. The image of the group action is the entire codomain [36]. Consider now two elements $g_1, g_2 \in \mathcal{G}$, with $g_1 \notin g_2\mathcal{H}$. Assuming that $\varphi(g_1, \phi_0) = \varphi(g_2, \phi_0)$, one can show that[29]

$$\begin{aligned} \phi_0 = \varphi(e, \phi_0) = \varphi(g_1^{-1}g_1, \phi_0) &= \varphi(g_1^{-1}, \varphi(g_1, \phi_0)) \\ &= \varphi(g_1^{-1}, \varphi(g_2, \phi_0)) = \varphi(g_1^{-1}g_2, \phi_0). \end{aligned} \tag{3.2.8}$$

Here, we have reached a contradiction. The last equation is true only if $g_1^{-1}g_2 \in \mathcal{H}$, or, equivalently, $g_2 \in g_1\mathcal{H}$, which contradicts the assumption of $g_2 \notin g_1\mathcal{H}$ and, thus, $\varphi(g_1, \phi_0) \neq \varphi(g_2, \phi_0)$. In conclusion, the map is a one-to-one relationship, i.e., an injection

[36]. With the two properties before, we can state that the mapping is bijective [36] and, hence, can be inverted on the $\varphi(g, \phi_0)$ image. By taking advantage of the definitions of orbit, Defn. (3.2.1), and stabilizer, Defn. (3.2.2), the bijection relationship can be established in a much more straightforward manner through the Orbit-Stabilizer Theorem, stated as

Theorem 3.2.1 (*Orbit-Stabilizer theorem*)**.** For each point $x \in \mathcal{X}$, there is a bijection between the orbit $\mathcal{O}(x)$ and set of left cosets of the stabilizer $\mathcal{G}_x$ [32]. The bijection is given by

$$g \cdot x \longmapsto g\mathcal{G}_x.$$

How the theorem applies in our case may need to be clarified, but things get better when we call a spade a spade. We are considering the origin ϕ_0, thus, our orbit is given by $g \cdot \phi_0$ for $g \in \mathcal{G}$. The stabilizer of ϕ_0 is the subgroup $\mathcal{H}$, therefore, the bijection we have is

$$g \cdot \phi_0 \longmapsto g\mathcal{H}. \tag{3.2.9}$$

the $g \cdot \phi_0$ is, clearly, an element of V, hence, there is a bijection between the Goldstones and the factor group. All the definitions and properties above were to conclude that there exists an isomorphic mapping[6] between the factor group $\mathcal{G}/\mathcal{H}$ and the vector space V, i.e. the NGB's fields. While two isomorphic groups may exhibit differences in the characteristics of their elements, they share identical structures regarding subgroups, cosets, classes, and other related properties. Crucially, isomorphic groups inherently possess identical representations [37]; they are the same group for practical effects! The Goldstones, indeed, live in the coset $\mathcal{G}/\mathcal{H}$.

Distinct group elements $gh_1, gh_2, gh_3, ... \in \mathcal{G}$ are associated to the same element in $\mathcal{G}/\mathcal{H}$, as can be seen in Figure 3.2. We can choose a representative element of $\mathcal{G}$ for each element in $\mathcal{G}/\mathcal{H}$, denoted by $\hat{g}$. Hence, any group element $g \in \mathcal{G}$ can be written as $g = \hat{g}h$, for some $h \in \mathcal{H}$. For a compact and path-connected group [4], each element $\hat{g}$ and h can be expressed by using the exponential map, so that we have [37]

$$g = \hat{g}(\xi)h(\alpha) \equiv e^{i\xi_A T^A} e^{i\alpha_a T^a}, \tag{3.2.10}$$

where (ξ_A, α_a) are the group parameters [34], which may be promoted to fields, and the (T^A, T^a) are the generators of $\mathcal{G}$ and $\mathcal{H}$, respectively [4]. In terms of the isomorphism we established, each ϕ is uniquely associated with an element of $\mathcal{G}/\mathcal{H}$, i.e., a coset $\hat{g}h$ with an appropriate $\hat{g}$. Since the $\hat{g}$ is the representative of the group $\mathcal{G}$ we say that the g is the

[6] An isomorphic map is a map that is homomorphic and, in addition, has a one-to-one relation between the domain and the image. This injective property ensures the invertibility of the map.

representative of the coset $\hat{g}\mathcal{H}$ such that

$$\phi = \varphi(g, \phi_0) = \varphi(\hat{g}h, 0). \tag{3.2.11}$$

Using the maps defined before, we have

$$\varphi(g', \phi) = \varphi(g', \varphi(\hat{g}h, 0)) = \varphi(g'\hat{g}h, 0) = \varphi(f, \phi_0) = \phi', \quad f \in g'(\hat{g}\mathcal{H}). \tag{3.2.12}$$

This result shows us that in order to obtain an element ϕ' from another ϕ, we multiply the left coset $\hat{g}\mathcal{H}$ by an element g'. This is the indication that we can "travel" from different cosets by multiplying them by elements of the group $\mathcal{G}$

$$\begin{array}{ccc} \phi & \xrightarrow{g'} & \phi' \\ \downarrow & & \downarrow \\ \hat{g}\mathcal{H} & \xrightarrow{g'} & g'\hat{g}\mathcal{H} \end{array}$$

This process uniquely defines the Goldstone bosons' transformation properties within an appropriate selection of variables that parameterize the elements of the quotient $\mathcal{G}/\mathcal{H}$ [29].

3.3 Custodial symmetry

In Chapter 2 we have mentioned custodial symmetry as the principle behind protecting the ρ parameter. In this section, we will delve into this topic more deeply. Refs. [3, 25, 38] inspired the following discussion. For a general approach about the custodial symmetry in a model with multiple Higgs doublets, please see Ref. [39]. The custodial symmetry is discussed in the framework of Composite Higgs models in [40, 41][7].

After all our discussion about the gauge bosons and the electroweak sector, we are led to the Lagrangian in Eq. (2.4.15), namely

$$\mathcal{L}_{Higgs} = (D_\mu\phi)^\dagger D^\mu\phi - \mu^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2. \tag{3.3.1}$$

The Higgs Lagrangian is invariant under the SM $SU(2)_L \times U(1)_Y$ group. Consider, as we did before, the Higgs doublet

$$H = \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{3.3.2}$$

From now on, to explicitly indicate that we are dealing with the Higgs boson, the notation for the doublet will be H.

[7] Paper [41] is a classic in Composite Higgs Models.

Expanding the potential in terms of the doublet four components, we get

$$V(\phi) = \mu^2 \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) + \lambda \left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)^2 . \tag{3.3.3}$$

All four field components have the same interactions and couplings. Here, the accidental symmetry comes up. You are right if you noticed this is symmetric under $SO(4)$ transformations! It happens that the Higgs four-plet spans a four-dimensional vector space, where the field components can be used as the basis for this space since they are orthogonal. As shown in Ref. [2], the $SO(4)$ group is locally isomorphic to $SU(2)_L \times SU(2)_R$, which expressed as $SO(4) \simeq SU(2)_L \times SU(2)_R$. We can, therefore, put the Higgs doublet in a representation of the left and right groups

$$\Phi = \left(\tilde{H}, H\right) = \begin{pmatrix} \phi^{0*} & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix} . \tag{3.3.4}$$

This is a bi-doublet where $\tilde{H}$ and H is each one a doublet of $SU(2)_L$. They are rotated by an $SU(2)_L$ transformation, delivered by U_L from the left side of the bi-doublet

$$SU(2)_L : \, \Phi' \to U_L \Phi, \tag{3.3.5}$$

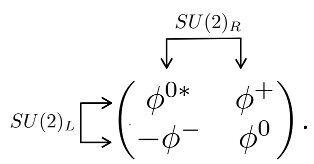
while an $SU(2)_R$ transformation U_R acts on the right side

$$SU(2)_R : \, \Phi' \to \Phi U_R^\dagger . \tag{3.3.6}$$

Therefore, both groups act in the bi-doublet as

$$SU(2)_L \times SU(2)_R : \, \Phi' \to U_L \Phi U_R^\dagger . \tag{3.3.7}$$

This transformation acts explicitly on the Higgs matrix entries as follows



By using the bi-doublet notation, the Lagrangian is expressed as

$$\mathcal{L}_{Higgs} = \frac{1}{2} \text{Tr} \, |D_\mu \Phi|^2 - V(\Phi), \tag{3.3.8}$$

with

$$V(\Phi) = \mu^2 \text{Tr} \, \left(\Phi^\dagger \Phi\right) + \lambda \left[\text{Tr} \, \left(\Phi^\dagger \Phi\right)\right]^2 . \tag{3.3.9}$$

This potential is invariant under the $SO(4)$ transformations, in fact, the lagrangian is invariant under global $SO(4)$ transformations, i.e., when there is no covariant derivative and no gauge bosons

$$\mathcal{L}_{global} = \text{Tr}\,\left[\partial_\mu \Phi^\dagger \partial^\mu \Phi\right] - V(\Phi). \tag{3.3.10}$$

However, going to the covariant derivative, i.e., coupling the Higgs boson to the gauge bosons, the above Lagrangian becomes not invariant (or variant) under global group transformations. The fact is that when we gauge the electroweak group, we break the symmetry by not gauging the entire group. To ilustrate this, we write down the covariant derivative applied to the fields

$$D_\mu \Phi = \left(\partial_\mu \Phi - i\frac{g}{2}\tau_a W^a_\mu \Phi + i\frac{g'}{2}\Phi\tau^3 B_\mu\right). \tag{3.3.11}$$

Under global $SU(2)_L$ transformations, the B_μ does not change, it is an $SU(2)_R$ field, and the W gauge bosons transforms as:

$$\tau_a W^a_\mu \to U_L \tau_a W^a_\mu U_L^\dagger, \tag{3.3.12}$$

therefore, using Eq. (3.3.5), we have

$$\tau_a W^a_\mu \Phi \to \left(U_L \tau_a W^a_\mu U_L^\dagger\right)(U_L\Phi) = U_L \tau_a W^a_\mu \Phi, \tag{3.3.13}$$

which leaves the lagrangian invariant under arbitrary $SU(2)_L$ transformations. However, the story changes when we consider the right group. Under arbitraty global $SU(2)_R$ transformation, the W's fields does not suffer any action, just as the B_μ (it has a trivial transformation under $U(1)_Y$ and no transformation under the remain elements of the $SU(2)_R$ group), leading a behaviour under $SU(2)_R$ action described by

$$\Phi\tau^3 B_\mu \to \Phi U_R^\dagger \tau^3 B_\mu \neq \Phi\tau^3 B_\mu U_R^\dagger, \tag{3.3.14}$$

which does not lead to and $SU(2)_R$ invariance, i.e., the term $\frac{i}{2}g' B_\mu$ in the Lagrangian breaks the custodial symmetry, because we privilege third generator of the $SU(2)_R$ group, T^3. It is analogous to introducing a vector in a three-dimensional space since it must point to a direction, the spherical symmetry is broken, and the system now has a privileged direction. The situation is depicted in Figure 3.3. If we consider the limit where $g' = 0$, the full symmetry, $SO(4)$, of the entire Lagrangian is acquired. In this scenario, the covariant derivative becomes

$$D_\mu = (\partial_\mu - ig\boldsymbol{T}\cdot\mathbf{W}_\mu). \tag{3.3.15}$$

And the kinectic term in the lagragian behaves under $SU(2)_L \times SU(2)_R$ transformation as

$$\begin{aligned}
\mathrm{Tr}\ \left[(D_\mu\Phi)^\dagger D^\mu\Phi\right] &\rightarrow \mathrm{Tr}\ \left[\left|\partial_\mu\left(U_L\Phi U_R^\dagger\right) - ig\left(U_L\boldsymbol{T}\cdot\mathbf{W}_\mu U_L^\dagger\right)\left(U_L\Phi U_R^\dagger\right)\right|^2\right] \\
&= \mathrm{Tr}\ \left[\left|U_L\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)U_R^\dagger\right|^2\right] \\
&= \mathrm{Tr}\ \left\{\left[U_L\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)U_R^\dagger\right]^\dagger\left[U_L\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)U_R^\dagger\right]\right\} \\
&= \mathrm{Tr}\ \left[U_R\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)^\dagger U_L^\dagger U_L\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)U_R^\dagger\right] \\
&= \mathrm{Tr}\ \left[U_R\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)^\dagger\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)U_R^\dagger\right] \\
&= \mathrm{Tr}\ \left[\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)^\dagger\left(\partial_\mu\Phi - ig\boldsymbol{T}\cdot\mathbf{W}_\mu\Phi\right)\right] \\
&= \mathrm{Tr}\ \left[(D_\mu\Phi)^\dagger D^\mu\Phi\right], \qquad (3.3.16)
\end{aligned}$$

where we have used the propertie of trace which states that $\mathrm{Tr}\,(P^{-1}AP) = \mathrm{Tr}\,(A)$ [42]. Thus, in the limit as $g' \rightarrow 0$, the Higgs sector of the Standard Model exhibits a global symmetry $SU(2)_L \times SU(2)_R$. This symmetry was not imposed when the model was built, therefore, it is called accidental symmetry. Since global $SU(2)_R$ is broken by the small parameter, it is said that this symmetry is approximate, or softly broken [43].

After the Higgs gets its VEV, the fully $SO(4)$ symmetry is broken to the $SO(3)$. To see the breaking in action, consider the VEV in the bi-doublet representation

$$\langle\Phi\rangle = \frac{1}{2}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}. \qquad (3.3.17)$$

this expression cleary breaks the chiral symmetry

$$U_L\langle\Phi\rangle \neq \langle\Phi\rangle \quad \langle\Phi\rangle U_R^\dagger \neq \langle\Phi\rangle. \qquad (3.3.18)$$

However, the case where $SU(2)_L = SU(2)_R$ remains intact

$$U_L\langle\Phi\rangle U_L^\dagger = U_L\left(\frac{v}{\sqrt{2}}\mathbb{1}\right)U_L^\dagger = \left(\frac{v}{\sqrt{2}}\right)U_L U_L^\dagger = \frac{v}{\sqrt{2}}\mathbb{1} = \langle\Phi\rangle\,. \qquad (3.3.19)$$

This is the custodial symmetry! The case $SU(2)_L = SU(2)_R$ is known as $SU(2)_V$, the vectorial or diagonal group. When discussing representation, the bidoublet is a $\mathbf{2}\otimes\mathbf{2}$ representation of $SU(2)_L \times SU(2)_R$ group; when the symmetry breaking, $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$, takes place, the representation goes like $\mathbf{2}\otimes\mathbf{2} \rightarrow \mathbf{3}\oplus\mathbf{1}$. Therefore, we have a triplet given by ϕ^{0*}, ϕ^+ and ϕ^-, which are the goldstones eaten by the gauge bosons, and a singlet, ϕ^0, the physical Higgs.

Since the coupling g' is small, the SM has an approximate custodial $SU(2)_V$

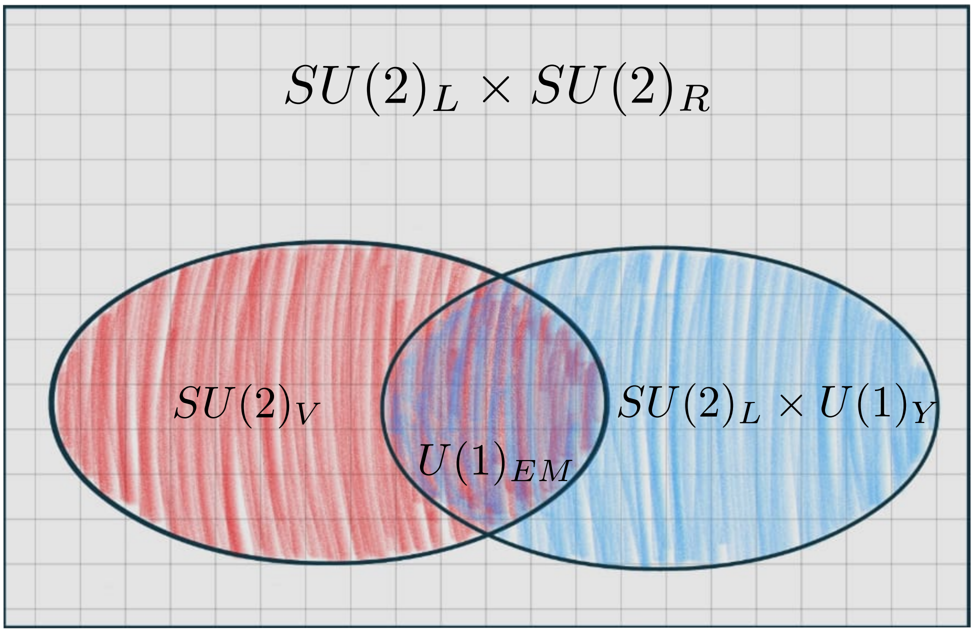


Figure 3.3: Group theory of the Higgs sector of the SM. The Higgs potential has a global $SU(2)_L \times SU(2)_R$ symmetry, shown as the gray ruled part. This symmetry is explicitly broken to the local EW Standard Model group ($SU(2)_L \times U(1)_Y$), represented in blue, by gauging only a part of the $SU(2)_R$ group. When the EWSB takes place, the global symmetry $SU(2)_L \times SU(2)_R$ is broken to the diagonal (or vectorial) group $SU(2)_V$, where $SU(2)_L = SU(2)_R$, represented by the red part of the image. The $SU(2)_V$ is the custodial symmetry. The remaining $U(1)_{EM}$ is the intersection of the vectorial and SM group. Image inspired by Weinberg in Ref. [44].

symmetry leading to the protection of the ρ parameter at tree level. The protection extends even at the loop level, where the leading corrections to the parameter due to the Higgs boson, Figure 3.4, is given by [25]

$$\hat{\rho} \approx 1 - \frac{11 G_F M_Z^2 \sin^2\theta_W}{24\sqrt{2}\pi^2} \ln \frac{M_h^2}{M_Z^2} \,, \tag{3.3.20}$$

with G_F being the Fermi constant [6]. When $g' \to 0$, $\sin^2\theta_W \to 0$, the correction goes to zero, protecting the parameter.

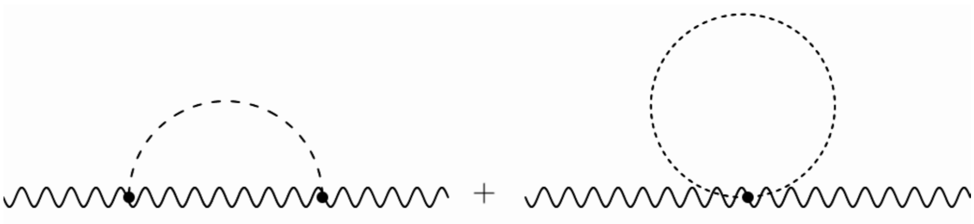


Figure 3.4: Corrections to the W and Z boson masses due to Higgs loops.

The discussion in this section was intended to show why custodial symmetry is necessary and desired in BSM models. Extending the SM is typically a tricky business because the model is extremely successful in numerous predictions, including the so-called electroweak precision tests. Any SM extension must agree with these tests[8]. If the model satisfies this custodial symmetry, one at least can be recomforted that constraints on the ρ parameter are unlikely to be violated.

[8] It is like changing the wheel of a moving car, which should continue to move (and the car is on fire).

Chapter 4

THE COMPOSITE HIGGS MODEL

In this chapter we will go through the Composite Higgs model. I expect to elucidate how the model is built, including some formal aspects, and how it works. As a concrete example, we will discuss the Minimial Composite Higgs Model (MCHM), the simplest reliable model that mimics the Standard Model.

As a starting point, we will briefly discuss the model's main idea and how it is constructed.

4.1 The main idea

The composite Higgs scenario is a simple and graceful solution to the hierarchy problem. As the name suggests, we will suppose that the Higgs is not a point-like particle but a composite state with a finite size, namely l_H. This bound state will be a product of a new confining interaction, i.e., a new strong sector characterized by a confinement scale $m_* = 1/l_H$ above the TeV scale. The finite size of the Higgs will affect the mass corrections in Eq. (1.2.1). At low energy levels, the quanta (photon) do not have the resolution to "see" what is inside the Higgs, the composite state behaves as an elementary particle, and the integrand in Eq. (1.2.1) grows linearly with E, just like in the SM. The finite size effects become apparent as E approaches and surpasses m_*. Similar to the scenario where a proton is hit by a virtual photon with a wavelength smaller than the proton radius, the composite Higgs is transparent to high-energy quanta, causing a decline in the integrand [2]. Therefore, the linear SM behaviour is replaced by a peak around $E \sim m_*$, followed by a quick decline. The Higgs mass generation phenomenon gets localized at $m_* = 1/l_H$, and m_H is insensitive to much higher energies.

Since we need a bound state, a new strong interaction must be included[1]. At least

[1] One can ask why we do not use the good and old QCD sector to generate the Higgs as a compound

a part of the new composite sector must take place in a strongly-coupled non-perturbative regime. The composite sector emerges from an even more fundamental theory at a very high scale $\Lambda_{UV} \gg$ TeV, whose precise value will not matter to us given that the whole point of the construction is precisely to make the EW scale insensitive to it [2]. There is also the need to include the SM particles in the model, i.e., all the particles we know as elementary ones. Since the gauge fields must be elementary, the new sector must respect the SM gauge symmetry, and, therefore, the symmetry group $\mathcal{G}$ of the new interaction must contain at least one $SU(2)_L \times U(1)_Y$ subgroup.

Imposing a new symmetry group means introducing new effects, particles, dynamics and even new symmetry breaking. Since we are going to construct a new strong sector, QCD will serve as an inspiration for this new model, more specifically the theory of pions[2]. As said before, we will use a strong sector respecting a global symmetry $\mathcal{G}$ as a starting point. The Higgs must arise as a composite state from this sector, described effectively as a NGB. Therefore, the symmetry must be broken spontaneously to a subgroup $\mathcal{H}_{global}$, leading to $\dim(\mathcal{G})-\dim(\mathcal{H}_{global})$ massless Goldtone bosons living in the coset $\mathcal{G}/\mathcal{H}$. In order to generate mass to the Higgs, it must become a pNGB, which is accomplished by introducing explicit symmetry breaking, delivered by the interaction of the new sector to the elementary sector. Figure (4.1) illustrates the structure discussed. There is also another source of an explicitly breaking of $\mathcal{G}$ by gauging a subgroup $\mathcal{H}_{gauge}$, leading to $\dim(\mathcal{H}_{gauge})$ gauge bosons. Among the NGB, a total of $\dim(\mathcal{H}_{gauge})-\dim(\mathcal{H}_{gauge} \cap \mathcal{H}_{global})$ must be absorbed to give mass to the gauge bosons; the remaining $\dim(\mathcal{G})$ - $\dim(\mathcal{H}_{gauge} \cup \mathcal{H}_{global})$ are the pNGB of the theory. The group $\mathcal{H} = \mathcal{H}_{gauge} \cap \mathcal{H}_{global}$ is broken, being the one containing the SM electroweak group [45], so that it is gauged and preserved by the strong dynamics [4], as is shown in Figure (4.2).

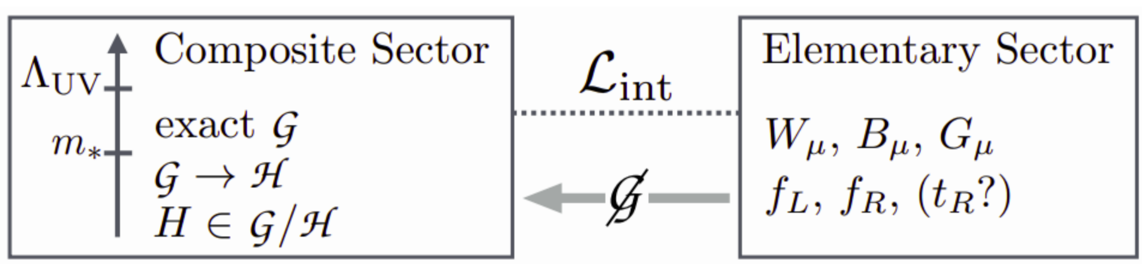


Figure 4.1: The basic composite scenario structure. The $(t_R?)$ indicates the possibility of the top quark being a part of the composite sector. Image from [2].

As we already know, a viable model requires the inclusion of $\mathcal{G}_{EW}$ in $\mathcal{H}_{gauge} \cap \mathcal{H}_{global}$. Thus, the simplest group one can choose is $\mathcal{G}_{EW} = \mathcal{H}_{gauge} = \mathcal{H}_{global}$. In order to have the

state. If the Higgs were a composite state of QCD, we would have reached an energy scale to "break" it, i.e., we already reached a resolution that would be sufficient to see inside a hypothetical composite Higgs confined by QCD.

[2] We are not going to discuss this model here, but the interested reader can take a look at the references [4, 5] for an excellent approach to the topic.

Higgs as a pNBG, we must have of dim($\mathcal{G}$)− dim($\mathcal{H}_{global}$) ≥ 4. The simplest group choice that can be made is $\mathcal{G} = SU(3)$. However, this choice brings large corrections to the ρ parameter due to the lack of a custodial symmetry [45]. By introducing the custodial symmetry[3] as an extension of the subgroup $\mathcal{H}_{global}$, namely $\mathcal{H}_{global} = SU(2)_L \times SU(2)_R \simeq SO(4)$, the issue with the ρ parameter is solved. The minimal choice for the global symmetry in this scenario is $\mathcal{G} = SO(5)$, leading to the emergence of the Higgs as the only pNGB.

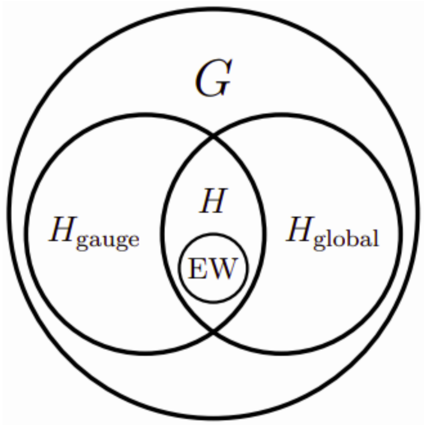


Figure 4.2: Symmetry Breaking pattern of the Composite Higgs scenario. Image from [4].

It may be worth mentioning that the concept of CHM is inspired by the successful understanding of pion dynamics based on chiral symmetry breaking in QCD. In the two-flavor version of QCD, the global symmetry is approximately $\mathcal{G} = SU(2)_L \times SU(2)_R$. The quark condensate $\langle \bar{q}q \rangle$ spontaneously breaks this symmetry to $\mathcal{H} = SU(2)_{L+R}$. The three Nambu-Goldstone bosons in the coset $\mathcal{G}/\mathcal{H}$ are identified as the world-famous pions, described by a Lagrangian where the global symmetry is realized non-linearly [48]. We will not enter deeper into this discussion; despite its obvious importance, the interested reader can find a good pedagogical approach in the references [3, 4].

One needs more than the entire group structure treated so far to ensure a realistic model; one also needs to guarantee the necessary non-derivative interactions to make one of the Goldostones a realistic Higgs candidate. This is provided by a mechanism called vacuum misalignment, which we will discuss in the next section.

[3] It may seem that the custodial symmetry is mandatory to the composite models, this is not really true. Indeed, the custodial symmetry brings protection to the oblique parameters [4]. However, there are other paths to address this issue, such as imposing extra symmetries to the model. An interesting example of a model that does not have custodial symmetry is discussed in [46, 47].

4.2 Vacuum misalignment

One of the most essential concepts behind the composite Higgs scenario is a mechanism by which the composite Higgs boson effectively behaves like an elementary one. This mechanism is called vacuum misalignment, which shifts the unbroken group, $\mathcal{H} \to \mathcal{H}'$, and breaks the electroweak group to $U(1)_{EM}$, as shown in Figure (4.3). To delve into this concept, we must enter into the group theory basis of the model.

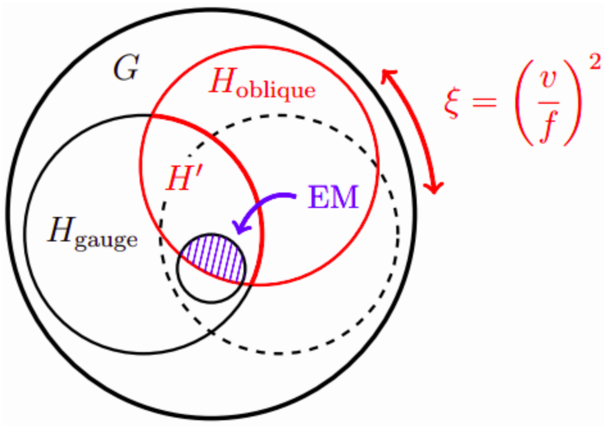


Figure 4.3: Representation shift due to the vacuum misalignment. The ξ is the degree of misalignment, representing the squared ratio of the electroweak symmetry breaking VEV and the $\mathcal{G} \to \mathcal{H}$ scale. Image from [4].

We start our discussion by remembering that a spontaneous symmetry breaking occurs not due to an explicit term in the lagrangian, but by the system's dynamic by istself, and can be diagramatically expressed by $\mathcal{G} \to \mathcal{H}$. In order to generate this symmetry-breaking, we assume that the vacuum state of the composite sector is not invariant under the entire new composite group but, instead, invariant just under the subgroup $\mathcal{H} \subset \mathcal{G}$. We choose a base of linearly independent generators in the Lie algebra of $\mathcal{G}$ to study this symmetry breaking. Taking the group generators, T^A, and splitting them into the broken $(A = \hat{a} = \hat{1}, ..., \dim[\mathcal{G}/\mathcal{H}])$ and unbroken $(A = a = 1, ..., \dim[\mathcal{H}])$ leads to

$$\{T^A\} = \{T^a, \widehat{T}^{\hat{a}}\}. \tag{4.2.1}$$

The Lie algebra of the subgroup $\mathcal{H}$ is generated by the set $\{T^a\}$. By choosing the vacuum state of the system as $\vec{F}$, the group generators must satisfy

$$T^a \vec{F} = 0, \tag{4.2.2}$$

$$\widehat{T}^{\hat{a}} \vec{F} \neq 0, \tag{4.2.3}$$

hence, the unbroken generator preserves the vacuum state, whereas the broken one does not and form a linearly independent set of vectors over the real numbers [2].

Any choice of the basis in the algebra of $\mathcal{G}$ would be equally good. Once the group is broken by gauging the electroweak group, the reference system must be such that $\mathcal{G}_{EW}$ is embedded in $\mathcal{H}$, i.e., the SM group generators must be included among the $\{T^a\}$'s generators. Therefore, we do not have to entail many assumptions about the $\mathcal{G} \to \mathcal{H}$ breaking pattern; in such a way, we have the freedom to choose any $\mathcal{G}$ group as long as it contains the $\mathcal{G}_{EW}$ as its subgroup[4]. Table (4.1) shows several symmetry choices to the Composite Higgs model.

$\mathcal{G}$	$\mathcal{H}$	C	N_G	$\mathbf{r}_{\mathcal{H}} = \mathbf{r}_{\mathrm{SU(2)\times SU(2)}}$ $(\mathbf{r}_{\mathrm{SU(2)\times U(1)}})$
SO(5)	SO(4)	✓	4	$\mathbf{4} = (\mathbf{2},\mathbf{2})$
SU(3) × U(1)	SU(2) × U(1)		5	$\mathbf{2}_{\pm\mathbf{1/2}} + \mathbf{1}_\mathbf{0}$
SU(4)	Sp(4)	✓	5	$\mathbf{5} = (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$
SU(4)	$[\mathrm{SU}(2)]^2 \times \mathrm{U}(1)$	✓*	8	$(\mathbf{2},\mathbf{2})_{\pm\mathbf{2}} = 2\cdot(\mathbf{2},\mathbf{2})$
SO(7)	SO(6)	✓	6	$\mathbf{6} = 2\cdot(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$
SO(7)	G_2	✓*	7	$\mathbf{7} = (\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2})$
SO(7)	SO(5) × U(1)	✓*	10	$\mathbf{10}_\mathbf{0} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2})$
SO(7)	$[\mathrm{SU}(2)]^3$	✓*	12	$(\mathbf{2},\mathbf{2},\mathbf{3}) = 3\cdot(\mathbf{2},\mathbf{2})$
Sp(6)	Sp(4) × SU(2)	✓	8	$(\mathbf{4},\mathbf{2}) = 2\cdot(\mathbf{2},\mathbf{2})$
SU(5)	SU(4) × U(1)	✓*	8	$\mathbf{4}_{-\mathbf{5}} + \bar{\mathbf{4}}_{+\mathbf{5}} = 2\cdot(\mathbf{2},\mathbf{2})$
SU(5)	SO(5)	✓*	14	$\mathbf{14} = (\mathbf{3},\mathbf{3}) + (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1})$
SO(8)	SO(7)	✓	7	$\mathbf{7} = 3\cdot(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$
SO(9)	SO(8)	✓	8	$\mathbf{8} = 2\cdot(\mathbf{2},\mathbf{2})$
SO(9)	SO(5) × SO(4)	✓*	20	$(\mathbf{5},\mathbf{4}) = (\mathbf{2},\mathbf{2}) + (\mathbf{1}+\mathbf{3},\mathbf{1}+\mathbf{3})$
$[\mathrm{SU}(3)]^2$	SU(3)		8	$\mathbf{8} = \mathbf{1}_\mathbf{0} + \mathbf{2}_{\pm\mathbf{1/2}} + \mathbf{3}_\mathbf{0}$
$[\mathrm{SO}(5)]^2$	SO(5)	✓*	10	$\mathbf{10} = (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{2})$
SU(4) × U(1)	SU(3) × U(1)		7	$\mathbf{3}_{-\mathbf{1/3}} + \bar{\mathbf{3}}_{+\mathbf{1/3}} + \mathbf{1}_\mathbf{0} = 3\cdot\mathbf{1}_\mathbf{0} + \mathbf{2}_{\pm\mathbf{1/2}}$
SU(6)	Sp(6)	✓*	14	$\mathbf{14} = 2\cdot(\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3}) + 3\cdot(\mathbf{1},\mathbf{1})$
$[\mathrm{SO}(6)]^2$	SO(6)	✓*	15	$\mathbf{15} = (\mathbf{1},\mathbf{1}) + 2\cdot(\mathbf{2},\mathbf{2}) + (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$

Table 4.1: The first two columns depict patterns of symmetry breaking $\mathcal{G} \to \mathcal{H}$ in some Lie groups. The third column indicates whether the custodial symmetry is present in the model. The fourth column provides the dimension of the coset. In contrast, the fifth column lists the representations of the NGBs under $\mathcal{H}$ and $SO(4) \simeq SU(2)_L \times SU(2)_R$. in the case of a lack of custodial symmetry, the representation showed is the one from $SU(2)_L \times U(1)_Y$. When there are more than two $SU(2)$ groups in $\mathcal{H}$ and multiple potential decompositions, the highest number of bi-doublets is presented. Table from [49].

Despite the vast number of constructions possible, the set of different composite Higgs models with N Nambu-Goldstone bosons is finite. This holds for the following reasons. Given that the number of subgroups $\mathcal{H}$ of a group $\mathcal{G}$ is finite, the number of

[4] The freedom here is incredible; however, ensuring that the model aligns with the experiments is crucial. Furthermore, the model can exhibit multi-step symmetry breaking patterns, as $\mathcal{G} \to \mathcal{H}' \to \mathcal{H}''$. Such models are not very common due to the requirement of introducing an additional NGB field at each step of the symmetry breaking process, which complicates the construction significantly.

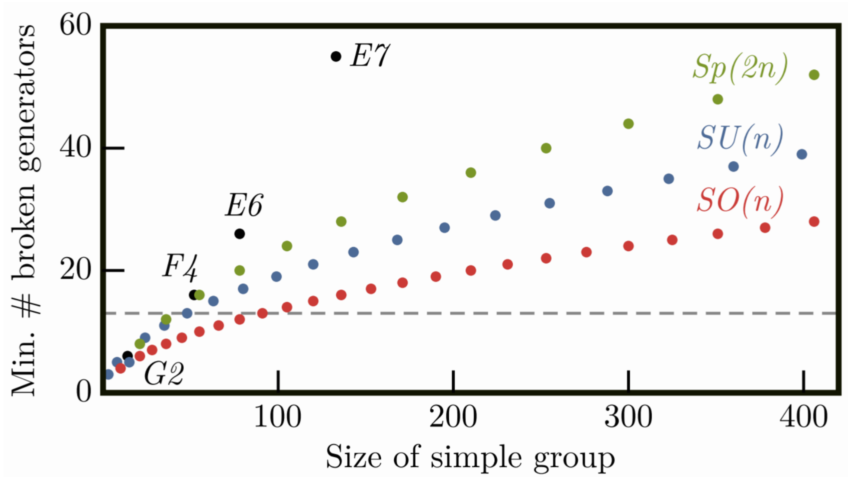


Figure 4.4: Size of same simple groups $\mathcal{G}$ and the minimum number of broken generators. The $G2$, $F4$, $E6$, and $E7$ are the five exceptional simple Lie groups. Image from [48].

cosets $\mathcal{G}/\mathcal{H}$ with N generators is also finite for a fixed group $\mathcal{G}$. An astute reader might imagine that by increasing the size of $\mathcal{G}$, there could be an infinite set of group-subgroup pairs with precisely the same number of broken generators. Well... not at all. As the size of $\mathcal{G}$ grows, examining its maximal subgroups reveals that the minimum number of broken generators also increases. For simple groups, this is visually depicted in Figure 4.4[5]; for the group families $SU(n)$, $Sp(2n)$, and $SO(n)$, the trend shown in the figure holds for arbitrarily large n. More generally, when the group is semi-simple, $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \ldots$, the maximal subgroups are either the maximal subgroup of one of the factors $\mathcal{G}_i$ or the diagonal subgroup of two equal factors, i.e., $\mathcal{G}_i \times \mathcal{G}_i \to \mathcal{G}_i$. Given that a finite number of groups generate a precise number of Nambu-Goldstone Bosons, a natural question arises: which groups are responsible for this generation? Table 4.2 provides the answer, detailing the groups that produce up to eight NGBs.

As said in the last chapter, the Goldstones are the parameters of local transformations in the direction of the broken generators $\{\widehat{T}^{\hat{a}}\}$. In the case of the Higgs mechanism, these directions are the angular direction in the potential. Hence, we are led to the already known Ansatz [2][6]

$$\vec{\Phi}(x) = e^{i\theta_{\hat{a}}\widehat{T}^{\hat{a}}}\,\vec{F}, \tag{4.2.4}$$

where the $\vec{\Phi}$ is a scalar field describing the theory of symmetry breaking. In the exponential, the $\theta_{\hat{a}}$ are fields, one for each broken generator, among which we must identify the four real components of at least one Higgs doublet. Extra components can be present as other scalars enlarging the Higgs sector. Of course, the Higgs VEV breaks the electroweak symmetry down to the electromagnetic group once we have to reproduce the SM.

[5] For further information about the exceptional simple Lie groups, please take a look at [50].
[6] Even though the reference [2] calls it an Ansatz, it has an explication lying in the group theory, as we will clarify it later on.

SU(2) × SU(2) content	$r_{\mathcal{H}}$	$\mathcal{H}$	$\mathcal{G}/\mathcal{H}$
$(\mathbf{2},\mathbf{2})$	$(\mathbf{2},\mathbf{2})$	SU(2) × SU(2)	$\mathrm{SO}(5)/\mathrm{SU}(2)^2$
$(\mathbf{2},\mathbf{2})+(1,1)$	$(\mathbf{2},\mathbf{2})+(1,1)$	SU(2) × SU(2)	$\mathrm{SO}(5)\times\mathrm{U}(1)/\mathrm{SU}(2)^2$
	$\mathbf{5}$	SO(5)	SU(4)/SO(5)
$(\mathbf{2},\mathbf{2})+(1,\mathbf{3})$	$(\mathbf{2},\mathbf{2})+(1,\mathbf{3})$	SU(2) × SU(2)	$\mathrm{SU}(2)\times\mathrm{SO}(5)/\mathrm{SU}(2)^2$
	$\mathbf{7}$	G_2	$\mathrm{SO}(7)/G_2$
	$\mathbf{7}$	SO(7)	SO(8)/SO(7)
	$(1,\mathbf{2},\mathbf{2})+(\mathbf{3},1,1)$	$\mathrm{SU}(2)^3$	$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(5)/\mathrm{SU}(2)^3$
$2\times(\mathbf{2},\mathbf{2})$	$2\times(\mathbf{2},\mathbf{2})$	SU(2) × SU(2)	$\mathrm{SU}(4)/(\mathrm{SU}(2)^2\times\mathrm{U}(1))$
	$\mathbf{4}+\bar{\mathbf{4}}$	SU(4)	SU(5)/SU(4) × U(1)
	$\mathbf{8}$	SO(7)	–
	$(\mathbf{2},\mathbf{4})$	SU(2) × SO(5)	Sp(6)/(SU(2) × SO(5))
	$(1,\mathbf{2},\mathbf{2})+(\mathbf{2},1,\mathbf{2})$	$\mathrm{SU}(2)^3$	–
	$\mathbf{8}_v$	SO(8)	SO(9)/SO(8)
	$(1,\mathbf{2},1,\mathbf{2})+(\mathbf{2},1,\mathbf{2},1)$	$\mathrm{SU}(2)^4$	$\mathrm{SO}(5)^2/\mathrm{SU}(2)^4$
$(\mathbf{2},\mathbf{2})+2\times(1,1)$	$(\mathbf{2},\mathbf{2})+2\times(1,1)$	SU(2) × SU(2)	$\mathrm{SO}(5)\times\mathrm{U}(1)^2/\mathrm{SU}(2)^2$
			$\mathrm{SU}(2)\times\mathrm{SO}(5)/(\mathrm{SU}(2)^2\times\mathrm{U}(1))$
	$1+\mathbf{5}$	SO(5)	SU(4) × U(1)/SO(5)
	$\mathbf{6}$	SU(4)	SO(7)/SU(4)
$(\mathbf{2},\mathbf{2})+(1,1)+(1,\mathbf{3})$	$1+\mathbf{7}$	G_2	$\mathrm{SO}(7)\times\mathrm{U}(1)/G_2$
	$(1,1)+(\mathbf{2},\mathbf{2})+(1,\mathbf{3})$	$\mathrm{SU}(2)^2$	$\mathrm{SU}(2)\times\mathrm{SO}(5)\times\mathrm{U}(1)/\mathrm{SU}(2)^2$
	$\mathbf{8}$	SO(7)	–
	$1+\mathbf{7}$	SO(7)	SO(8) × U(1)/SO(7)
	$(\mathbf{2},\mathbf{4})$	SU(2) × SO(5)	Sp(6)/(SU(2) × SO(5))
	$(1,\mathbf{5})+(\mathbf{3},1)$	SU(2) × SO(5)	$\mathrm{SU}(2)^2\times\mathrm{SU}(4)/(\mathrm{SU}(2)\times\mathrm{SO}(5))$
	$(1,\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{2},1)$	$\mathrm{SU}(2)^3$	–
	$(1,1,1)+(1,\mathbf{2},\mathbf{2})+(\mathbf{3},1,1)$	$\mathrm{SU}(2)^3$	$\mathrm{SU}(2)^2\times\mathrm{SO}(5)\times\mathrm{U}(1)/\mathrm{SU}(2)^3$
	$\mathbf{8}_v$	SO(8)	SO(9)/SO(8)
	$(1,1,\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{2},1,1)$	$\mathrm{SU}(2)^4$	$\mathrm{SO}(5)^2/\mathrm{SU}(2)^4$
$(\mathbf{2},\mathbf{2})+3\times(1,1)$	$(\mathbf{2},\mathbf{2})+3\times(1,1)$	$\mathrm{SU}(2)^2$	$\mathrm{SO}(5)\times\mathrm{U}(1)^3/\mathrm{SU}(2)^2$
			$\mathrm{SU}(2)\times\mathrm{SO}(5)/\mathrm{SU}(2)^2$
	$2\times 1+\mathbf{5}$	SO(5)	$\mathrm{SU}(4)\times\mathrm{U}(1)^2/\mathrm{SO}(5)$
			SU(2) × SU(4)/(SO(5) × U(1))
	$1+\mathbf{6}$	SU(4)	SO(7) × U(1)/SU(4)
	$\mathbf{7}$	SO(7)	SO(8)/SO(7)
	$(1,\mathbf{2},\mathbf{2})+(\mathbf{3},1,1)$	$\mathrm{SU}(2)^3$	$\mathrm{SU}(2)^2\times\mathrm{SO}(5)/\mathrm{SU}(2)^3$
$(\mathbf{2},\mathbf{2})+4\times(1,1)$	$(\mathbf{2},\mathbf{2})+4\times(1,1)$	$\mathrm{SU}(2)^2$	$\mathrm{SO}(5)\times\mathrm{U}(1)^4/\mathrm{SU}(2)^2$
			$\mathrm{SU}(2)\times\mathrm{SO}(5)\times\mathrm{U}(1)/\mathrm{SU}(2)^2$
			$\mathrm{SU}(2)^2\times\mathrm{SO}(5)/(\mathrm{SU}(2)^2\times\mathrm{U}(1)^2$
	$3\times 1+\mathbf{5}$	SO(5)	$\mathrm{SU}(4)\times\mathrm{U}(1)^3/\mathrm{SO}(5)$
			SU(2) × SU(4)/SO(5)
	$2\times 1+\mathbf{6}$	SU(4)	$\mathrm{SO}(7)\times\mathrm{U}(1)^2/\mathrm{SU}(4)$
			SU(2) × SO(7)/(SU(4) × U(1))
	$1+\mathbf{7}$	SO(7)	SO(8) × U(1)/SO(7)
	$(1,\mathbf{5})+(\mathbf{3},1)$	SU(2) × SO(5)	$\mathrm{SU}(2)^2\times\mathrm{SU}(4)/(\mathrm{SU}(2)\times\mathrm{SO}(5))$
	$(1,1)+(1,\mathbf{5})+(\mathbf{2},1)$	SU(2) × SO(5)	–
	$(1,1,1)+(1,\mathbf{2},\mathbf{2})+(\mathbf{3},1,1)$	$\mathrm{SU}(2)^3$	$\mathrm{SU}(2)^2\times\mathrm{SO}(5)\times\mathrm{U}(1)/\mathrm{SU}(2)^3$
	$2\times(\mathbf{2},1,1)+(1,\mathbf{2},\mathbf{2})$	$\mathrm{SU}(2)^3$	$\mathrm{SO}(5)\times\mathrm{SU}(3)/(\mathrm{SU}(2)^3\times\mathrm{U}(1))$
	$\mathbf{8}_v$	SO(8)	SO(9)/SO(8)
	$(1,1,\mathbf{2},\mathbf{2})+(\mathbf{2},\mathbf{2},1,1)$	$\mathrm{SU}(2)^4$	$\mathrm{SO}(5)^2/\mathrm{SU}(2)^4$

Table 4.2: Composite Higgs Models with up to eight Nambu-Goldstone Bosons (NGBs) are organized according to their scalar field content. The symmetries permitted in the scalar sector from an IR perspective are specified. Table from [48].

The composite sector, in isolation, has a spontaneous symmetry breaking pattern $\mathcal{G} \to \mathcal{H}$ leading to NGB's θ fields, with no potential, arbitrary vacuum state, and, consequently, massless. Their VEV is unobservable since any constant configuration θ corresponds to an equivalent vacuum that can be reached by applying a $\mathcal{G}$ transformation like $\exp[-i\langle\theta_{\hat{a}}\rangle\widehat{T}^{\hat{a}}]$. Moreover, it is possible to eliminate any VEV, $\langle\theta_{\hat{a}}\rangle$, by a redefinition of the fields of the type $\vec{\Phi} \to \exp[-i\langle\theta_{\hat{a}}\rangle\widehat{T}^{\hat{a}}]\vec{\Phi}$. The subgroup $\mathcal{H}$ is not broken in this situation, so the EWSB does not happen [2, 30, 31], as illustrated geometrically on the left of Figure (4.5).

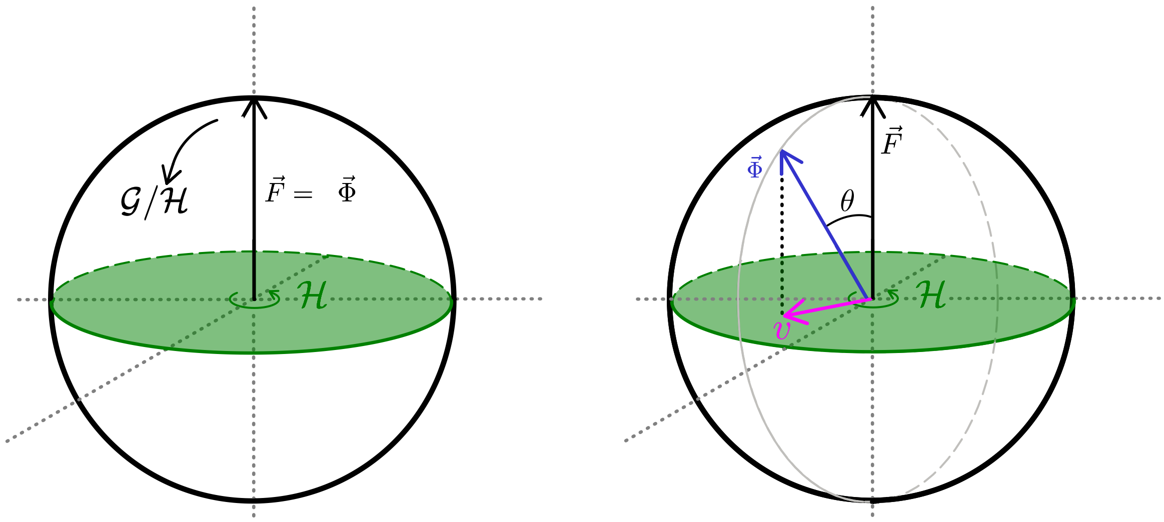


Figure 4.5: Representation of the Vacuum misalignment mechanism. **Left:** The composite sector in isolation leads to a vacuum state of the field $\vec{\Phi}$ aligned with the reference vacuum $\vec{F}$ and there is no EWSB. **Right:** When the interaction between elementary and composite sectors is introduced, the true vacuum state is shifted by an angle θ with respect to the reference vacuum, the Higgs acquires a non-zero VEV, and the EWSB takes place. The electroweak VEV is the projection of the vacuum into the $\mathcal{H}$ space. Image adapted from the references [30, 31].

The situation changes when the elementary sector is considered. The symmetry is explicitly broken due to the gauging of the electroweak group and couplings to the SM fermions[7]. Now that the fields θ acquire a potential, the VEV is not arbitrary and the field acquires a mass. It is no longer possible to perform a group transformation to eliminate these fields; they have become observable. In this case, the $\mathcal{G}_{EW} \subset \mathcal{H}$ is broken by shifting the true vacuum $\langle\vec{\Phi}\rangle$ with respect to the $\mathcal{H}$-preserving vacuum $\vec{F}$, leading to a vacuum misalignment, as shown on the right of Figure (4.5). It is expected that the entire EWSB effect, including masses of SM particles, is controlled by the projection of $\langle\vec{\Phi}\rangle$ onto the $\mathcal{H}$ plane [2]. Therefore, the EWSB scale will be determined by

$$v = f\sin(\langle\theta\rangle), \tag{4.2.5}$$

[7] Saying that a symmetry is explicitly broken is the same as saying that it is not an exact symmetry.

where $f = \left|\left\langle \vec{\Phi} \right\rangle\right|$ represents the scale of spontaneous breaking in $\mathcal{G} \to \mathcal{H}$.

It is possible to have a large value of $\langle\theta\rangle$. However, composite Higgs construction gains significance and diverges from technicolor[8] when the misalignment angle is small, i.e., $\langle\theta\rangle \approx 1$, creating a gap between the scale f and the EWSB scale v. This gap is widely expressed as [31]

$$\xi \equiv \frac{v^2}{f^2}. \tag{4.2.6}$$

As $\xi \to 0$, with v fixed, the composite sector decouples from low-energy physics by sending its typical scale f to infinity. Only the Goldstone boson Higgs persists in the spectrum in this limit, and all other bound states decouple. The theory consistently converges to the Standard Model as ξ tends to zero, rendering the composite Higgs effectively elementary. Experimental validations of the SM, especially its accurate portrayal of electroweak precision physics, can be systematically restored with a small enough ξ. The composite model is, therefore, attractive when the misalignment angle is small, leading to the condition.

$$\xi \equiv \frac{v^2}{f^2} = \sin^2(\langle\theta\rangle) \ll 1. \tag{4.2.7}$$

This condition is not automatically satisfied [2]; one must be concerned about it and introduce mechanisms to ensure that Eq. (4.2.7) is valid. Some options exist to make it happen, such as collective symmetry breaking or imposing extra symmetries to the model, such as a $\mathbb{Z}_2$ symmetry [4]. We are not going through this topic during our discussions, but instead, an assumption to be checked later on.

Now, we will discuss a composite Higgs model's construction, to exemplify how it can be done.

4.3 The Minimal Composite Higgs Model

We will use the Minimal Composite Higgs Model (MCHM)[9] to illustrate how to build the Higgs as a pNGB. As said before, the assumptions we have to make about the group structure of the new sector are pretty mild; it must just have the SM Electroweak group as its subgroup and a 4-plet to be identified as the Higgs field. The MCHM is built on the basis of an $SO(5)$ group broken into the $SO(4)$ subgroup. As one can check [2], the $SO(4)$ is locally isomorphic to the chiral group $SU(2)_L \times SU(2)_R$ group, and, therefore, contains the EW group. The discussion starts with the construction of the Higgs and the gauge bosons Lagrangian using the linear σ-model.

[8] For a good, and crash, review about the technicolor see [51].

[9] Here the term "minimal" refers to the minimal construction which is endowed with the Custodial $SO(4)_C$ Symmetry [4, 3].

4.3.1 Linear realization

As a starting point for our discussion, we identify the group structure of the model. Once we adopt the $SO(5) \to SO(4)$ breaking pattern and $SO(4) \simeq SU(2)_L \times SU(2)_R$, the $SU(2)_L$ group can be identified as the SM one, and the third generator of $SU(2)_R$ as the SM hypercharge, i.e., $Y = t^3_R$ [10]. As one may have noticed, the $SO(5)$ has 10 generators, acting on the fundamental **5** representation, conveniently taken to be $T^A = \{T^a, \widehat{T}^i\}$, $A = 1, ..., 10$, where [2]

$$T^a = \left\{ T^\alpha_{\mathbf{5}L} = \begin{pmatrix} T^\alpha_L & 0 \\ 0 & 0 \end{pmatrix} \; , \; T^\alpha_{\mathbf{5}R} = \begin{pmatrix} T^\alpha_R & 0 \\ 0 & 0 \end{pmatrix} \right\}, \tag{4.3.1}$$

$$\left(\widehat{T}^i\right)_{IJ} = -\frac{i}{\sqrt{2}} \left(\delta^i_I \delta^5_J - \delta^i_J \delta^5_I\right), \tag{4.3.2}$$

where, in the first equation, $a = 1, ..., 6$ and $\alpha = 1, 2, 3$; while, in the second one, $i = 1, ..., 4$ and $I, J = 1, ..., 5$. These generators are normalized as

$$Tr\left[T^A T^B\right] = \delta^{AB}. \tag{4.3.3}$$

The T^a, the first six generators, span the unbroken $SO(4)$ sub-algebra and are defined in Appendix B, Eqs. (B.22) and (B.23). The $\widehat{T}^i$ are the broken generators associated with the NGB's. We have a total of four broken generators, leading to four Goldstones transforming as $\mathbf{4} = (\mathbf{2}, \mathbf{2})$ of the unbroken group.

The T^A generators of $SO(5)$ have the structure [3]

$$T^A = \left(\begin{array}{c|c} \mathcal{H} & \mathcal{G}/\mathcal{H} \\ \hline \mathcal{G}/\mathcal{H} & \end{array}\right). \tag{4.3.4}$$

It is now clear that the vacuum satisfying Eqs. (4.2.2) and (4.2.3) is

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f \end{pmatrix}. \tag{4.3.5}$$

We are interested in the Nambu-Goldstone bosons, so there is nothing fairer than to expand

[10] This assumption will bring some issues when we consider the fermionioc particles, but, for now, it is enough consider just this structure.

the field around fluctuations in the directions of the broken generators, as explained before, leading to

$$\vec{\Phi} = e^{i\frac{\sqrt{2}}{f}\Pi_i(x)\widehat{T}^i} \begin{pmatrix} \vec{0} \\ f+\sigma(x) \end{pmatrix}. \quad (4.3.6)$$

The $\vec{\Pi}$ are the four NGB's and the $\sigma(x)$ is a resonance of the new composite sector, and the $\vec{0}$ is a 4-vector with zero in all the components. The Goldstone matrix is defined as being

$$U[\Pi] = e^{i\frac{\sqrt{2}}{f}\Pi_i(x)\widehat{T}^i}. \quad (4.3.7)$$

As demonstrated in the Appendix C, the equation above can be expressed in a more convenient manner as

$$U[\Pi] = \begin{pmatrix} \mathbb{1}_{4\times4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix}, \quad (4.3.8)$$

where $\Pi = \sqrt{\vec{\Pi}^T\vec{\Pi}}$, and $\mathbb{1}_{4\times4}$ is the unit matrix in four dimensions. Applying the last relation to Eq. (4.3.6) we are left with

$$\vec{\Phi} = (f+\sigma)\begin{pmatrix} \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ \cos\left(\frac{\Pi}{f}\right) \end{pmatrix}. \quad (4.3.9)$$

One can see that the unbroken generators' action on the $\vec{\Phi}$ leads to an rotation of the fourplet $\vec{\Pi}$

$$\vec{\Phi} \to e^{i\alpha_a T^a}\vec{\Phi} \Leftrightarrow \vec{\Pi} \to e^{i\alpha_a T^a}\vec{\Pi}. \quad (4.3.10)$$

We now turn our sight to the model's Lagrangian. Since we are considering a real scalar field, even as an effective theory, we have [2]

$$\mathcal{L}_C = \frac{1}{2}\partial_\mu\vec{\Phi}^T\partial^\mu\vec{\Phi} - \frac{g_*^2}{2}\left(\vec{\Phi}^T\vec{\Phi} - f^2\right)^2, \quad (4.3.11)$$

which we will study in the perturbative weakly-coupled regime $g_* < 4\pi$. This Lagrangian must be invariant under $SO(5)$ transformations acting on the field $\vec{\Phi}$ as

$$\vec{\Phi} \to g\cdot\vec{\Phi}, \quad g = e^{\alpha_A T^A} \in SO(5). \quad (4.3.12)$$

However, after the field acquires a VEV, the symmetry is broken to the $SO(4)$ subgroup. The minimization condition of the potential is, clearly, $\langle\vec{\Phi}^T\rangle\langle\vec{\Phi}\rangle = f^2$. Introducing Eq.

(4.3.9) into the Lagrangian, we are led to

$$\begin{aligned}\mathcal{L}_C &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{(g_* f)^2}{2}\sigma^2 - \frac{g_*^2 f}{2}\sigma^3 - \frac{g_*^2}{8}\sigma^4 \\ &+ \frac{1}{2}\left(1+\frac{\sigma}{f}\right)^2\left[\frac{f^2}{\Pi^2}\sin^2\left(\frac{\Pi}{f}\right)\partial_\mu\vec{\Pi}^T\partial^\mu\vec{\Pi} + \frac{f^2}{4\Pi^4}\left(\frac{\Pi^2}{f^2} - \sin^2\left(\frac{\Pi}{f}\right)\right)\partial_\mu\Pi^2\partial^\mu\Pi^2\right].\end{aligned} \tag{4.3.13}$$

Here we see that the resonance mass is

$$m_* = g_* f. \tag{4.3.14}$$

Typically, the resonances from a strong sector lies in the same range of the symmetry breaking scale, namely f, which is much larger than the EW scale. Some exciting properties arise from the Lagrangian. By expanding around $\Pi = 0$, it is easy to see that there is an infinite set of local interactions between an arbitrary number of Goldstone fields but only two derivatives. Every interaction, i.e. leg insertion, is weighted by the symmetry-breaking factor f due to the NGB entering the Goldstone matrix as $\vec{\Pi}/f$. Analogously to the Pion in QCD, the f is called the "Higgs decay constant" [2].

The 4-plet $\vec{\Pi}$ can be expressed in terms of the two Higgs Doublet components

$$H = \begin{pmatrix} h_u \\ h_d \end{pmatrix}, \tag{4.3.15}$$

by using the Eq. (B.31), as

$$\vec{\Pi} = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\left(h_u - h_u^\dagger\right) \\ h_u + h_u^\dagger \\ i(h_d - h_d^\dagger) \\ h_d + h_d^\dagger \end{pmatrix}. \tag{4.3.16}$$

As shown in Apenddix B, the Higgs doublet can be expressed as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \Pi_2 + i\Pi_1 \\ \Pi_4 - i\Pi_3 \end{pmatrix}. \tag{4.3.17}$$

The theory is of course also invariant under four non-linearly realized transformations associated with the broken generators [2]. Using that

$$\Pi^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 + \Pi_4^2, \tag{4.3.18}$$

we can write

$$|H| = \left(H^\dagger H\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\Pi_2 - i\Pi_1 & \Pi_4 + i\Pi_3\end{pmatrix}\begin{pmatrix}\Pi_2 + i\Pi_1 \\ \Pi_4 - i\Pi_3\end{pmatrix}\right]^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{2}}\left(\Pi_1^2 + \Pi_2^2 + \Pi_3^2 + \Pi_4^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\Pi, \tag{4.3.19}$$

and

$$\left(\partial_\mu \vec{\Pi}\right)^2 = 2\partial_\mu H^\dagger \partial^\mu H, \tag{4.3.20}$$

so that the Lagrangian, ignoring the resonances[11], becomes

$$\mathcal{L}_C \supset \frac{f^2}{2|H|^2}\sin^2\left(\frac{\sqrt{2}|H|}{f}\right)\partial_\mu H^\dagger \partial^\mu H + \frac{f^2}{8|H|^4}\left[2\frac{|H|^2}{f^2} - \sin^2\left(\frac{\sqrt{2}|H|}{f}\right)\right]\left(\partial_\mu |H|^2\right)^2. \tag{4.3.21}$$

As the endpoint, we have reached the Lagrangian for the Higgs dynamics and self-interaction. We still have to introduce the gauge bosons and fermions to the model to mimic the Standard model. However, before that, we will see how the CCWZ construction can be used to construct the composite model. The following sections will discuss the physical consequences of the Higgs Lagrangian.

4.3.2 Gauge sector

The new composite sector must contain the SM group and, hence, its particles, including the gauge bosons, discussed in the last chapter. In this section, we will introduce these particles by gauging the SM group, as we have done before.

The procedure of gauging a group is to promote the ordinary derivative to a covariant one in the Lagrangian. Since we want the SM group, we gauge its generators, i.e., the generators of $SU(2)_L \times U(1)_Y$. The $SO(4)$ is locally isomorphic to the chiral group $SU(2)_L \times SU(2)_R$ [2], which indicates to us how to choose the generators to be gauged: the entire $SU(2)_L$ group generators and, in some way, the generator of $U(1)_Y$ as one among the generators of $SU(2)_R$. The generator chosen is the third generator of the right chiral group, a choice made due to the transformation behaviour of the NGB under the group generators, the only one that has the right hypercharge and a proper transformation rule [2, 5] . Putting all this information together, we are led to

$$\partial_\mu \vec{\Phi} \to D_\mu \vec{\Phi} = \left(\partial_\mu - igW_\mu^\alpha T_L^\alpha - ig' B_\mu T_R^3\right)\vec{\Phi}. \tag{4.3.22}$$

[11] The resonances are strongly suppressed at low energy levels by the decay constant. Therefore, we can ignore them here without any further concern

The gauging process also leads us to elementary kinetic terms for the gauge bosons

$$\mathcal{L}_G = -\frac{1}{4} W^{\alpha}_{\mu\nu} W^{\mu\nu}_{\alpha} - \frac{1}{4} B^{\mu\nu}_{\mu\nu}. \tag{4.3.23}$$

At low energies, the EW gauge bosons propagators and self-interaction vertices are equal to the SM [2]. The fields, $(W^{\alpha}_{\mu}, B_{\mu})$, and the coupling strength, (g, g'), are the SM ones. Since we have chosen to gauge not the entire group but the SM generators, the group symmetry $\mathcal{G}$ is explicitly broken.

In Eq. (4.3.21) we faced the Lagrangian for the Higgs doublet, not for the field $\vec{\Phi}$. Hence, how should the covariant derivative be applied to the Higgs doublet? Inasmuch as the Higgs sector in the composite model must be the SM one in low energies, the covariant derivative is expected to be the same as in the SM. Indeed, this is precisely the case here. The change from the ordinary derivative to the covariant in the Higgs doublet is [2, 52]

$$\partial_{\mu} H \rightarrow D_{\mu} H = \left(\partial_{\mu} - i g W^{\alpha}_{\mu} \frac{\tau_{\alpha}}{2} - i g' B_{\mu} \frac{\mathbb{1}}{2} \right) H, \tag{4.3.24}$$

where the τ_{α} are the Pauli matrices. Ignoring the resonances, we have

$$\mathcal{L}_C \supset \frac{f^2}{2|H|^2} \sin^2 \left(\frac{\sqrt{2}|H|}{f} \right) (D_{\mu} H)^{\dagger} D^{\mu} H + \frac{f^2}{8|H|^4} \left[2\frac{|H|^2}{f^2} - \sin^2 \left(\frac{\sqrt{2}|H|}{f} \right) \right] \left(\partial_{\mu} |H|^2 \right)^2 . \tag{4.3.25}$$

The last term remains unchanged; the derivative is still ordinary because $|H|^2$ is a singlet [5]. The consequences of the new sector Lagrangian to the gauge bosons can be observed by computing the physical couplings. We do this job through the Higgs VEV in the unitary gauge

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ V + h(x) \end{pmatrix}, \tag{4.3.26}$$

where the $h(x)$ describes the physical Higgs fluctuations. Putting the VEV into the term $(D_{\mu} H)^{\dagger} D^{\mu} H$ leads to

$$(D_{\mu} \langle H \rangle)^{\dagger} D^{\mu} \langle H \rangle = \frac{1}{2} \begin{pmatrix} 0 \\ V + h(x) \end{pmatrix}^{\dagger} \left(\partial_{\mu} - i g W^{\alpha}_{\mu} \frac{\tau_{\alpha}}{2} - i g' B_{\mu} \frac{\mathbb{1}}{2} \right)^2 \begin{pmatrix} 0 \\ V + h(x) \end{pmatrix}.$$

It is important to notice that

$$|H|^2 = \frac{1}{2} (V + h(x))^2 , \tag{4.3.27}$$

$$\partial_{\mu} |H|^2 = (V + h(x))\, \partial_{\mu} h. \tag{4.3.28}$$

There is just one term in which the covariant derivative, and consequently, the gauge bosons, appear. Taking this first element of the Lagrangian (4.3.25) and considering only the relevant terms, we are left with

$$\frac{f^2}{2|H|^2}\sin^2\frac{\sqrt{2}|H|}{f}|D_\mu H|^2 \supset \sin^2\frac{V+h}{f}\left[\frac{g^2f^2}{8}\begin{pmatrix}0\\1\end{pmatrix}^T\begin{pmatrix}\frac{g'}{g}B_\mu+W_\mu^3 & W_\mu^1-iW_\mu^2\\ W_\mu^1+iW_\mu^2 & \frac{g'}{g}B_\mu+W_\mu^3\end{pmatrix}^2\begin{pmatrix}0\\1\end{pmatrix}\right.$$
$$\left.+\frac{f^2}{2\left(V+h\right)^2}\left(\partial_\mu h\right)^2\right]. \tag{4.3.29}$$

As a result we get

$$\frac{f^2}{2|H|^2}\sin^2\frac{\sqrt{2}|H|}{f}|D_\mu H|^2 = \sin^2\frac{V+h(x)}{f}\left[\frac{g^2f^2}{4}\left(W_\mu^+W^{-\mu}+\frac{Z_\mu Z^\mu}{2\cos\theta_W}\right)\right.$$
$$\left.+\frac{f^2}{2\left(V+h(x)\right)^2}\left(\partial_\mu h\right)^2\right], \tag{4.3.30}$$

where we have used the SM eigenstates and the Weinberg angle, as we did in the last chapter, namely [53]

$$W^\pm = \frac{(W^1\mp W^2)}{\sqrt{2}}, \tag{4.3.31}$$

$$Z_\mu = \frac{gW_\mu^3-g'B_\mu}{\sqrt{g^2+g'^2}}, \tag{4.3.32}$$

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}. \tag{4.3.33}$$

Observe that from Eq. (4.3.29) to Eq. (4.3.30), the left and right side's relations have changed, i.e., we passed from "$\subset$" to "$=$". The reason is that Eq. (4.3.29) has mixing terms involving the gauge bosons and the Higgs bosons derivative. However, when we use the eigenstates, these terms are not present anymore [5, 52]. Therefore, we have an equality relationship.

Finally, the entire Lagrangian in (4.3.25) becomes

$$\mathcal{L}_C \supset \sin^2\frac{V+h(x)}{f}\left[\frac{g^2f^2}{4}\left(W_\mu^+W^{-\mu}+\frac{Z_\mu Z^\mu}{2\cos\theta_W}\right)+\frac{f^2}{2\left(V+h(x)\right)^2}\left(\partial_\mu h\right)^2\right]$$
$$+\frac{f^2}{2\left(V+h(x)\right)^2}\left[\frac{\left(V+h(x)\right)^2}{f^2}-\sin^2\frac{V+h(x)}{f}\right]\left(\partial_\mu h\right)^2, \tag{4.3.34}$$

which can be rewritten as

$$\mathcal{L}_C \supset \frac{g^2f^2}{4}\sin^2\frac{V+h(x)}{f}\left(W_\mu^+W^{-\mu}+\frac{Z_\mu Z^\mu}{2\cos\theta_W}\right)+\frac{1}{2}\left(\partial_\mu h\right)^2. \tag{4.3.35}$$

The sine can be expanded by using Taylor expansion around $h = 0$, leading to

$$\begin{aligned}\sin^2\frac{V+h}{f} &= \sin^2\left(\frac{V}{f}\right)\\ &+ 2\cos\left(\frac{V}{f}\right)\sin\left(\frac{V}{f}\right)\frac{h}{f} + \frac{1}{2!}\left[-2\sin^2\left(\frac{V}{f}\right) + 2\cos^2\left(\frac{V}{f}\right)\right]\left(\frac{h}{f}\right)^2\\ &+ \frac{1}{3!}\left[-4\cos\left(\frac{V}{f}\right)\sin\left(\frac{V}{f}\right) - 4\sin\left(\frac{V}{f}\right)\cos\left(\frac{V}{f}\right)\right]\left(\frac{h}{f}\right)^3 + \ldots\\ &= \sin^2\left(\frac{V}{f}\right) + \frac{1}{2}\sin\left(\frac{2V}{f}\right)\frac{h}{f} + \cos\left(\frac{2V}{f}\right)\left(\frac{h}{f}\right)^2 - \frac{2}{3}\sin\left(\frac{2V}{f}\right)\left(\frac{h}{f}\right)^3 + \ldots\end{aligned} \tag{4.3.36}$$

Substituting the last expression into the Lagrangian, the first term of the expansion gives rise to the gauge bosons mass terms, given by

$$\mathcal{L}_C \supset \frac{g^2 f^2}{4}\sin^2\left(\frac{V}{f}\right)\left(W^+_\mu W^{-\mu} + \frac{Z_\mu Z^\mu}{2\cos\theta_W}\right) \tag{4.3.37}$$

therefore

$$M_W = M_Z\cos\theta_W = \frac{gf}{2}\sin\frac{V}{f} = \frac{gv}{2}, \tag{4.3.38}$$

where we have used

$$v = f\sin\frac{V}{f}, \tag{4.3.39}$$

with $v \simeq 246$ GeV the EW VEV. From Eq. (2.4.31), we can see that at tree level, the relation $\rho = 1$ holds in the minimal composite models. This is because the custodial symmetry is present in the model after the EWSB; the Higgs VEV breaks the symmetry to the custodial symmetry, $SO(4) \to SO(3)_c \simeq SU(2)_c$, like in the SM.

In order to write down the interacting terms, it is convenient to rewrite Eq. (4.3.36) by using the definition of $\xi = v^2/f^2$ and considering the misalignment angle small. The result is

$$\sin^2\frac{V+h}{f} = \left(\frac{v}{f}\right)^2\left[1 + 2\sqrt{1-\xi}\frac{h}{v} + (1-2\xi)\left(\frac{h}{v}\right)^2 - \frac{4}{3}\xi\sqrt{1-\xi}\left(\frac{h}{v}\right)^3 + \ldots\right]. \tag{4.3.40}$$

The interacting terms between the Higgs and the gauge bosons become

$$\mathcal{L}_{int} = \left[1 + 2\sqrt{1-\xi}\frac{h}{v} + (1-2\xi)\left(\frac{h}{v}\right)^2 + \ldots\right]\left(M_W^2 W^+_\mu W^{-\mu} + \frac{M_Z^2}{2}Z_\mu Z^\mu\right). \tag{4.3.41}$$

Consequently, the hVV vertex deviation with respect to the SM is [2]

$$\kappa_V \equiv \frac{g_{hVV}^{CH}}{g_{hVV}^{SM}} = \sqrt{1-\xi} < 1, \tag{4.3.42}$$

where $g_{hVV}^{SM} = 2m_V/v$. The vertex with two Higgsses is changed by

$$\kappa_{VV} \equiv \frac{g_{hhVV}^{CH}}{g_{hhVV}^{SM}} = 1 - 2\xi. \tag{4.3.43}$$

Hence, the composite model brings observable deviations from the SM. Due to the current LHC measures status, a SM coupling deviation up to around 20% is allowed, giving a bound of $\xi \leq 0.4$ [30], which, in terms of the composite sector scale reads $f \gtrsim 389$ GeV. The deviations vanish as the parameter $\xi \to 0$ while keeping v constant by doing $f \to \infty$. In this scenario, the composite Higgs essentially behaves like an elementary particle.

With the gauge sector clarified, we turn our eyes to the fermionic sector.

4.3.3 Fermionic sector

After discussing the gauge sector in the composite model, it is natural to aim to describe the fermions in this model. Despite the noble goal, our journey could be easier. Unlike the gauge construction, we have yet to determine how the fermions must be embedded in the model. The reason is that we need to learn how they transform under the groups $\mathcal{G}$ and $\mathcal{H}$. Since the most significant fermionic contribution to the Higgs potential comes from the top and bottom quarks [2, 47], they will be our example during the discussion.

With no background information about how to start the job proposed, one could think of using the SM Yukawa interaction fields,

$$\mathcal{L}_{Yukawa} = -\overline{q}_L y_t \tilde{H} t_R - \overline{q}_L y_b H b_R + \dots, \tag{4.3.44}$$

to construct the new model Lagrangian, i.e.

$$\mathcal{L}_{CompYuk} = \mathcal{L}_{CompYuk}\left(H, q_L, t_R, b_R\right). \tag{4.3.45}$$

Here, the q_L is the third generation of quarks, $q_L = (t_L, b_L)^T$, $\tilde{H} = i\tau_2 H^*$ and $\overline{q}_L = q_L^\dagger \gamma^0$, γ^0 being a Dirac matrix[12][25]. The guess made in Eq. (4.3.45) is not correct; the SM fermions do not have a well-defined transformation rule under the composite groups, $\mathcal{G}$ and $\mathcal{H}$; hence, it is not known how to construct invariant terms to the Lagrangian. Moreover,

[12] Take a look at the notation and convention section.

the fermions have a dependence on the representation. Thus, the NGB matrix must be such that constructing the interacting terms involving the Higgs and the fermions is possible. We need to change the guess in order to have a consistent theory.

The issue remains even if one takes the fermion field to a linear representation of $\mathcal{G}$, namely

$$q_L \to \underline{Q}_L, \quad t_R \to \underline{T}_R, \quad b_R \to \underline{B}_R, \tag{4.3.46}$$

because the symmetry is broken to $\mathcal{H}$; therefore, the fields must also be in a representation of the latter. However, ensuring that the field is embedded in a representation of the unbroken group is not enough since the quarks should be fundamental degrees of freedom in energies scales above the symmetry breaking, indicating that they must be some representation of $\mathcal{G}$.

Let us address the issue of how to construct invariant terms to the Lagrangian. For this, remember that a field $\underline{\Upsilon}$ of a theory that respects the symmetry $\mathcal{G}$ transforms according to the group action defined in the last section, namely

$$\underline{\Upsilon} \to \underline{\Upsilon}^{(g)} = \tilde{\varphi}(g, \underline{\Upsilon}), \quad \forall \ g \in \mathcal{G}. \tag{4.3.47}$$

Now, let us define a new field as [2]

$$\Upsilon \equiv \tilde{\varphi}(U^{-1}[\Pi], \underline{\Upsilon}). \tag{4.3.48}$$

Here the U^{-1} is the inverse Goldstone matrix defined in the previous section. By acting with a group element g on this field, the transformation must be in both elements of the group action since the Goldstone matrix also has transformations under the symmetry group, i.e.

$$\begin{aligned}
\tilde{\varphi}\left(g, \Upsilon\right) = \Upsilon^{(g)} &= \tilde{\varphi}\left(U^{-1}[\Pi^{(g)}], \underline{\Upsilon}^{(g)}\right) \\
&= \tilde{\varphi}\left(U^{-1}[\Pi^{(g)}], \tilde{\varphi}(g, \underline{\Upsilon})\right) \quad \text{(as per Eq. (4.3.47))} \\
&= \tilde{\varphi}\left(U^{-1}[\Pi^{(g)}]g, \underline{\Upsilon}\right) \qquad \text{($\tilde{\varphi}$ is homomorphism)} \\
&= \tilde{\varphi}\left(\left(hU^{-1}\left[\Pi\right]g^{-1}\right)g, \underline{\Upsilon}\right)\right) \quad \text{(Goldstone transformation rule)} \\
&= \tilde{\varphi}\left(hU^{-1}\left[\Pi\right], \underline{\Upsilon})\right) = \tilde{\varphi}\left(h, \tilde{\varphi}(U^{-1}[\Pi], \underline{\Upsilon})\right) \\
&= \tilde{\varphi}\left(h, \Upsilon\right),
\end{aligned} \tag{4.3.49}$$

where we have used the inverse of the Goldstone matrix transformation rule, that is

$$U^{-1}\left[\Pi^{(g)}\right] = hU^{-1}\left[\Pi\right]g^{-1}. \tag{4.3.50}$$

As a conclusion, Eq. (4.3.49) showed us that under $\mathcal{G}$ transformations the field defined in

Eq. (4.3.48) behaves like

$$\Upsilon \to \Upsilon^{(g)} = \tilde{\varphi}\left(h, \Upsilon\right) = \Upsilon^{(h)}, \quad \forall \;\; h \in \mathcal{H}. \tag{4.3.51}$$

An action of $\mathcal{G}$ results in a transformation under $\mathcal{H}$ due to the factor g^{-1} present in the Goldstone inverse matrix transformation rule that compensates the term of the $\mathcal{G}$ transformation [2]. With this result in hands, we use it to define a new fermion field as

$$Q_L \equiv \tilde{\varphi}(U^{-1}[\Pi], \underline{Q}_L), \tag{4.3.52}$$

which, clearly, transforms under $\mathcal{G}$ as

$$Q_L \to Q_L^{(g)} = \tilde{\varphi}(h, Q_L) = Q_L^{(h)} \tag{4.3.53}$$

The same idea can be applied to the right-handed fermions. Dressing the new fields with the inverse Goldstone matrix, leads to entities with a well-defined transformation rule, and, hence, one can use this properties to build the Lagrangian

$$\mathcal{L}_{fermionic} = \mathcal{L}_{fermionic}\left(Q_L, T_R, B_R\right). \tag{4.3.54}$$

We still have to face the fermion representation issue. Once the composite sector has, a priori, no requirement about the particle representation, the fermions are not necessarily embedded into the exact representation of the symmetry group $\mathcal{G}$. We must write down the explicit expression of the fields in order to determine the allowed operators. We have the freedom to choose the fermion representations. Therefore, we are going to discuss the simpliest one, the fundamental representation. Valuable treatments of other representations are present in [2, 5].

When a general group $SO(N)$ with fundamental representation denoted by $\square^{[\mathbf{N}]}$, is broken to a subgroup $SO(N-1)$, the fundamental representation decomposes as [43]

$$\square^{[\mathbf{N}]} \to \square^{[\mathbf{N-1}]} \oplus \mathbf{1}^{[\mathbf{N-1}]}, \tag{4.3.55}$$

where the $\mathbf{1}^{[\mathbf{N-1}]}$ is a singlet[13] of $SO(N-1)$. Therefore, we are able to decompose a fundamental representation of $SO(5)$ into representations of $SO(4)$ as

$$Q_L^{[\mathbf{5}]} = \underset{\cdot}{Q}_L^{[\mathbf{4}]} + \ldots \;, \tag{4.3.56}$$

where the $\underset{\cdot}{Q}_L^{[\mathbf{4}]}$ is a singlet representation of the unbroken group[14]. Of course, the same

[13] Any tensorial representation will have at least one singlet in its decomposition; of course, there is not too much to talk about the representation of the remaining elements of the decomposition [5].

[14] Here is important to stress that the $\underset{\cdot}{Q}_L^{[\mathbf{4}]}$ is a incomplet fiveplet of $SO(5)$, it has zero at some

argument is applied to the other fermions, in our case B_R and T_R. Hence, the Lagrangian must, for sure, contain at least the singlet representation

$$\mathcal{L}_{fermionic} \supset -y_t\lambda_t\overline{Q}^{[\mathbf{4}]}_{L} T^{[\mathbf{4}]}_{R} - y_b\lambda_b\overline{Q}^{[\mathbf{4}]}_{L} B^{[\mathbf{4}]}_{R}. \tag{4.3.57}$$

We have no clue about the remnant fields in the decomposition in Eq. (4.3.56), indicated by the dots; nonetheless, we know how to build at least one invariant operator. Our task will be embedding the fields in the fundamental representation, identifying the singlet component, and using it to write down the interactions.

After defining the decomposition of the representation, we must ensure that the fields in Eq. (4.3.57) can represent the SM field. We do it by studying how the field decomposes under the $\mathcal{G}_{EW}$. Taking advantage of Eq. (4.3.55), we see that a fundamental representation of $SO(5)$, $\square^{[\mathbf{5}]}$, decomposes, under the $SO(4)$ group, as

$$\square^{[\mathbf{5}]} \to \square^{[\mathbf{4}]} \oplus \mathbf{1}^{[\mathbf{4}]}, \tag{4.3.58}$$

with $\square^{[\mathbf{4}]}$ being in the fundamental representation of $SO(4)$ and $\mathbf{1}^{[\mathbf{4}]}$ the singlet. However, the unbroken group is isomorphic to the chiral group, $SO(4) \simeq SU(2)_L \times SU(2)_R$. As discussed in [2] the fundamental representation of $SO(4)$ behaves as $(\mathbf{2},\mathbf{2})$ of $SU(2)_L \times SU(2)_R$, indicating the decomposition under the chiral group

$$\square^{[\mathbf{4}]} \to \mathbf{2} \oplus \mathbf{2}, \tag{4.3.59}$$

where $\mathbf{2}$ is the fundamental representation of $SU(2)_{L,R}$. We have defined the third generator of the right group as the hypercharge operator, $Y = t^3_R$, therefore, the fundamental representation of $SO(4)$ decomposes as two doublets with opposite charges, $\pm 1/2$, as shown in [2] . Rewriting the last equation with the respective hypercharge leads to

$$\square^{[\mathbf{4}]} \to \mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}. \tag{4.3.60}$$

At the end of the day, we have

$$\square^{[\mathbf{5}]} \to \square^{[\mathbf{4}]} \oplus \mathbf{1}^{[\mathbf{4}]} \to \mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}} \oplus \mathbf{1}^{[\mathbf{4}]}_0, \tag{4.3.61}$$

where the 0 indicates that the singlet has no hypercharge. In the SM, the decomposition of the quarks under the $\mathcal{G}_{EW}$ are $\mathbf{2}_{1/6}$, $\mathbf{1}_{2/3}$, $\mathbf{1}_{-1/3}$ for the q_L, t_R, b_R, respectively. Therefore, we must face a severe issue; the quarks can not be embedded into the representations of the $SO(5)$ satisfyng the correct quantum number. To solve this problem, we add a new unbroken group to the model, namely $U(1)_X$, and make the hypercharge a combination of

components.

the third generator of $SU(2)_R$ and the new group generator

$$Y = T_R^3 + X. \tag{4.3.62}$$

Since the new group remains intact, the breaking pattern becomes

$$SO(5) \times U(1)_X \to SO(4) \times U(1)_X. \tag{4.3.63}$$

The new group plays no role in the symmetry breaking; no extra Goldstone appears. Moreover, the $U(1)_X$ group commutes with the $SO(5)$ [2], and thus, all the SM gauge bosons are neutral under the new group. Extra interacting terms must be added when fields have a X charge. For the fundamental representation, the choice of charge $X = 2/3$ under $U(1)_X$ will result in the decomposition [2]

$$\square^{[\mathbf{5}]}_{\frac{2}{3}} \to \square^{[\mathbf{4}]}_{\frac{2}{3}} \oplus \mathbf{1}^{[\mathbf{4}]}_{\frac{2}{3}} \to \mathbf{2}_{\frac{7}{6}} \oplus \mathbf{2}_{\frac{1}{6}} \oplus \mathbf{1}^{[\mathbf{4}]}_{\frac{2}{3}} \ . \tag{4.3.64}$$

The second and third elements of the decomposition have the correct quantum numbers of the q_L and t_R, respectively. Therefore, we can embed them into these representations. Nonetheless, the b_R can not be one of the representations shown above due to the wrong hypercharge; the solution is to change the X charge. In this case, we choose $X = -1/3$, decomposing as

$$\square^{[\mathbf{5}]}_{-\frac{1}{3}} \to \square^{[\mathbf{4}]}_{-\frac{1}{3}} \oplus \mathbf{1}^{[\mathbf{4}]}_{-\frac{1}{3}} \to \mathbf{2}_{\frac{5}{6}} \oplus \mathbf{2}_{\frac{1}{6}} \oplus \mathbf{1}^{[\mathbf{4}]}_{-\frac{1}{3}} \ , \tag{4.3.65}$$

the last term being adequate to embed the b_R. As an endpoint to the representation issue, we have the t_R being embedded in $\square^{[\mathbf{5}]}_{\frac{2}{3}}$, b_R in $\square^{[\mathbf{5}]}_{-\frac{1}{3}}$, and q_L in $\square^{[\mathbf{5}]}_{\frac{2}{3}}$ when coupled to the t_R, or q_L in $\square^{[\mathbf{5}]}_{-\frac{1}{3}}$ when coupled to the b_R.

It is helpful to take a moment to clarify the notation used above to ensure everything is as straightforward as possible. The $2/3$ and $-1/3$ values are in the first terms before the arrow and in both terms after the arrow in Eqs. (4.3.64) and (4.3.65), represent the charge under the $U(1)_X$ group. The subscripts on the last three terms on the right-hand side of these equations refer to the total hypercharge, given by Eq. (4.3.62). Specifically, these subscripts result from summing $2/3$, $-1/3$, and the values shown in Eq. (4.3.61). One way to interpret this is to consider that before the $SO(4)$ group is decomposed, there is no T_R^3 generator with charge in the representation, as the group is still $SO(5)$[15].

With all the ingredients collected, it is now possible to construct the invariants. Thus, the next step is to build them explicitly. We approach first the top quark construction. The starting point is embedding the fermions in an incomplete, linear representation of

[15] While the notation here might appear confusing at first glance, rest assured, it truly is!

$SO(5)$. For the right top, we have [2]

$$\underline{T}_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ t_R \end{pmatrix}. \tag{4.3.66}$$

Since the elements of the unbroken group are in block-diagonal form, with the first 4×4 block made of an $SO(4)$ rotation and "1" in the remaining entry, the choice made for the representation is clearly a singlet under the unbroken group [16]. Moving forward, we need to embed the q_L into $\square^{[\mathbf{5}]}_{-\frac{2}{3}}$, it also must be a doublet of $SU(2)_L$. As shown in Ref. [2] , the chiral group representations can be written in the bidoublet form, i.e., a $(\mathbf{2}, \mathbf{2})$ representation, where, as in Eq. (B.43), the bidoublet can be expressed in terms of two complex doublets, namely

$$\Psi \equiv (\Psi_-, \, \Psi_+) = \begin{pmatrix} \Psi^u_- & \Psi^u_+ \\ \Psi^d_- & \Psi^d_+ \end{pmatrix}. \tag{4.3.67}$$

Since it transforms like a doublet under $SU(2)_L$ and has the correct $T^3_R = -1/2$ charge, we choose the Ψ_- to be q_L,

$$q_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix} = \Psi_- = \begin{pmatrix} \Psi^u_- \\ \Psi^d_- \end{pmatrix}. \tag{4.3.68}$$

Hence, by inspection $\Psi^u_- = t_L$, $\Psi^d_- = b_L$ and $\Psi^u_+ = \Psi^d_+ = 0$. Putting these outcomes in Eq. (B.45) the fourplet of $SO(4)$ is

$$q_L^{[\mathbf{4}]} = \frac{1}{\sqrt{2}} \begin{pmatrix} -ib_L \\ -b_L \\ -it_L \\ t_L \end{pmatrix}. \tag{4.3.69}$$

Embedding the fourplet into an incomplete **5**-plet of $SO(5)$ leads to

$$\underline{Q}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} -ib_L \\ -b_L \\ -it_L \\ t_L \\ 0 \end{pmatrix} = \begin{pmatrix} q_L^{[\mathbf{4}]} \\ 0 \end{pmatrix}. \tag{4.3.70}$$

[16] Remember, the underline indicates a linear representation of G, and the result of embedding the t_R in this representation.

In order to finally write down the interacting terms, we "dress" the quarks representations of $SO(5)$ with the Goldstone matrix [17]. By Eq. (4.3.52), the left-chiral quarks will become

$$\begin{aligned} Q_L &= \tilde{\varphi}\left(U^{-1}[\Pi], \underline{Q}_L\right) = \begin{pmatrix} \mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix} \begin{pmatrix} q_L^{[\mathbf{4}]} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \left\{\mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2}\right\}\cdot q_L^{[\mathbf{4}]} \\ \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi}\cdot q_L^{[\mathbf{4}]} \end{pmatrix}. \end{aligned} \tag{4.3.71}$$

We have used Eq. (C.26) for the inverse Goldstone matrix. When constructing the invariant, we will use the $\overline{Q}_L$, which is given as

$$\begin{aligned} \overline{Q}_L &= \tilde{\varphi}^T\left(U^{-1}[\Pi], \underline{\overline{Q}}_L\right) = \underline{\overline{Q}}_L U[\Pi] = \begin{pmatrix} \overline{q}_L^{[\mathbf{4}]} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix} \\ &= \begin{pmatrix} \overline{q}_L^{[\mathbf{4}]}\cdot\left\{\mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2}\right\} \\ \sin\left(\frac{\Pi}{f}\right)\overline{q}_L^{[\mathbf{4}]}\cdot\frac{\vec{\Pi}}{\Pi} \end{pmatrix}. \end{aligned} \tag{4.3.72}$$

The simbol "T" in the group action is just to indicate that the action has changed, since the conjugate field is not a column vector, but a line vector instead, so the Goldstone matrix action and form must be different. Clearly, the above representation can be split in two others

$$\underset{\sim}{\overline{Q}}_L = \overline{q}_L^{[\mathbf{4}]}\cdot\left\{\mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2}\right\}, \tag{4.3.73}$$

$$\underset{\cdot}{\overline{Q}}_L = \sin\left(\frac{\Pi}{f}\right)\overline{q}_L^{[\mathbf{4}]}\cdot\frac{\vec{\Pi}}{\Pi}. \tag{4.3.74}$$

These entities belong to the representations

$$\underset{\sim}{Q}_L \in \mathbf{4}_{\frac{2}{3}}^{[\mathbf{4}]}, \tag{4.3.75}$$

$$\underset{\cdot}{Q}_L \in \mathbf{1}_{\frac{2}{3}}^{[\mathbf{4}]}. \tag{4.3.76}$$

Dressing now the right top quark with the Goldstone matrix leads to

$$\begin{aligned} T_R &= \tilde{\varphi}\left(U^{-1}[\Pi], \underline{T}_R\right) = \begin{pmatrix} \mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix} \begin{pmatrix} \vec{0} \\ t_R \end{pmatrix} \\ &= \begin{pmatrix} -t_R\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} \\ t_R\cos\left(\frac{\Pi}{f}\right) \end{pmatrix}. \end{aligned} \tag{4.3.77}$$

[17] They must be appropriately dressed to join the party, right?

Making the same procedure before, the T_R is split as

$$\underset{\sim}{T}_R = -t_R \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi}, \tag{4.3.78}$$

$$\underset{\cdot}{T}_R = t_R \cos\left(\frac{\Pi}{f}\right), \tag{4.3.79}$$

belonging to

$$\underset{\sim}{T}_R \in \mathbf{4}^{[\mathbf{4}]}_{\frac{2}{3}}, \tag{4.3.80}$$

$$\underset{\cdot}{T}_R \in \mathbf{1}^{[\mathbf{4}]}_{\frac{2}{3}}. \tag{4.3.81}$$

As discussed before, we always have the singlet representation in the decomposition of fundamental representations; for this reason, we chose this representation to build the invariants. However, one may ask if the choice for the $\mathbf{4}^{[\mathbf{4}]}$ would lead to a different term and, even more, to a different physics. Well, no! Here, both representations are not independent of each other; this can be seen as

$$\begin{aligned}\left(\overline{\underset{\sim}{Q}}_L\right)_i \left(\underset{\sim}{T}_R\right)^i + \overline{\underset{\cdot}{Q}}_L \underset{\cdot}{T}_R &= \left(\overline{Q}_L\right)_I \left(T_R\right)^I = \left(\overline{\underline{Q}}_L U\left[\Pi\right]\right)_I \left(U^{-1}\left[\Pi\right]\underline{T}_R\right)^I \\ &= \left(\overline{\underline{Q}}_L\right)_I \left(\underline{T}_R\right)^I = 0,\end{aligned} \tag{4.3.82}$$

since $\underline{T}_R$ and $\underline{Q}_L$ were constructed to be orthogonal. As a consequence, it does not matter what representation we choose. For the sake of simplicity, we select the singlet representation. The invariant is, then

$$\begin{aligned}\overline{\underset{\cdot}{Q}}_L \underset{\cdot}{T}_R &= \left[\sin\left(\frac{\Pi}{f}\right)\bar{q}_L^{[\mathbf{4}]}\cdot\frac{\vec{\Pi}}{\Pi}\right] t_R \cos\left(\frac{\Pi}{f}\right) = \frac{t_R}{2\Pi}\sin\left(\frac{2\Pi}{f}\right)\bar{q}_L^{[\mathbf{4}]}\cdot\vec{\Pi} \\ &= \frac{t_R}{2\sqrt{2}\Pi}\sin\left(\frac{2\Pi}{f}\right)\begin{pmatrix} i\bar{b}_L & -\bar{b}_L & i\bar{t}_L & \bar{t}_L\end{pmatrix}\begin{pmatrix}\Pi_1\\ \Pi_2\\ \Pi_3\\ \Pi_4\end{pmatrix} \\ &= \frac{t_R}{2\sqrt{2}\Pi}\sin\left(\frac{2\Pi}{f}\right)\left[\bar{b}_L\left(i\Pi_1-\Pi_2\right)+\bar{t}_L\left(i\Pi_3+\Pi_4\right)\right] \\ &= \frac{t_R}{2\sqrt{2}\Pi}\sin\left(\frac{2\Pi}{f}\right)\begin{pmatrix}\bar{t}_L & \bar{b}_L\end{pmatrix}\begin{pmatrix} i\Pi_3+\Pi_4\\ i\Pi_1-\Pi_2\end{pmatrix} \\ &= \frac{1}{2\Pi}\sin\left(\frac{2\Pi}{f}\right)\bar{q}_L\tilde{H}t_R,\end{aligned} \tag{4.3.83}$$

where in the last step Eq. (B.28) was used[18]. By replacing the Π, just like in Eq. (4.3.19),

[18] The convention being used for the matrix gamma is the Eq. (A.1) .

the expression before can be rewritten as

$$\overline{Q}_L T_R = \frac{1}{2\sqrt{2}|H|} \sin\left(\frac{2\sqrt{2}|H|}{f}\right) \overline{q}_L \tilde{H} t_R. \tag{4.3.84}$$

One question may appear after reading the equation above; the t_R is a singlet under the $SO(4)$ group, but the $\overline{q}_L$ is a doublet under the $SU(2)_L$ and, therefore, is not a singlet under the unbroken group. Confused? Here, the product $\overline{q}_L\tilde{H}$ is a singlet; in this sense, the Higgs operator acts on the left quark doublet, making the result a singlet. Introducing the couplings and constants the generalized Yukawa Lagrangian for the top is

$$\begin{aligned} \mathcal{L}_Y^t &= -y_t f \overline{Q}_L T_R + h.c. \\ &= -\frac{y_t f}{2\sqrt{2}|H|} \sin\left(\frac{2\sqrt{2}|H|}{f}\right) \overline{q}_L \tilde{H} t_R + h.c.\ , \end{aligned} \tag{4.3.85}$$

where the y_t is the composite Yukawa coupling, a free parameter, and f is the breaking energy scale. By letting the Higgs[19] make its magic, the symmetry is spontaneously broken by the VEV, which is given by $\tilde{H} \to \langle\tilde{H}\rangle = \left(V/\sqrt{2}, 0\right)^T$ since the Π_4 is the SM Higgs. After the EWSB, we are led to

$$\begin{aligned} \mathcal{L}_Y^t &= -\frac{y_t f}{2\sqrt{2}|\langle H\rangle|} \sin\left(\frac{2\sqrt{2}|\langle H\rangle|}{f}\right) \overline{q}_L \langle\tilde{H}\rangle t_R + h.c. \\ &= -\frac{y_t f}{2V} \sin\left(\frac{2V}{f}\right) \begin{pmatrix} \bar{t}_L & \bar{b}_L \end{pmatrix} \begin{pmatrix} \frac{V}{\sqrt{2}} \\ 0 \end{pmatrix} t_R + h.c. \\ &= -\frac{y_t f}{2\sqrt{2}} \sin\left(\frac{2V}{f}\right) \bar{t}_L t_R + h.c. \\ &= -\frac{y_t f}{\sqrt{2}} \sin\left(\frac{V}{f}\right) \cos\left(\frac{V}{f}\right) \bar{t}_L t_R + h.c. \\ &= -\frac{y_t f}{\sqrt{2}} \sqrt{\xi}\sqrt{1-\xi}\,\bar{t}_L t_R + h.c. \\ &= -y_t f \sqrt{\frac{\xi\left(1-\xi\right)}{2}}\,\bar{t}_L t_R + h.c. \end{aligned} \tag{4.3.86}$$

Eq. (4.3.39) were used together with the expression $\xi = v^2/f^2$. By comparing with Eq. (2.4.47), the equation above makes evident the quark mass parameter

$$m_t = y_t f \sqrt{\frac{\xi\left(1-\xi\right)}{2}}. \tag{4.3.87}$$

Having found the top mass, we can analyze the interaction with the SM Higgs. We introduce the mass obtained into the Lagrangian, but we go to the unitary gauge where

[19] The particle, not the physicist, of course.

the potential has fluctuations around its minimum, i.e., $\langle H\rangle = V + h(x)$. As a result, one gets

$$\begin{aligned}
\mathcal{L}_Y^t &= -\frac{y_t f}{2\sqrt{2}|\langle H\rangle|}\sin\left(\frac{2\sqrt{2}|\langle H\rangle|}{f}\right)\overline{q}_L\langle\tilde{H}\rangle t_R + h.c. \\
&= -\frac{m_t}{2|\langle H\rangle|}\frac{1}{\sqrt{\xi\,(1-\xi)}}\sin\left(\frac{2\sqrt{2}|\langle H\rangle|}{f}\right)\overline{q}_L\langle\tilde{H}\rangle t_R + h.c. \\
&= -\frac{\sqrt{2}m_t}{2\,(V+h)}\frac{1}{\sqrt{\xi\,(1-\xi)}}\sin\left(\frac{2\,(V+h)}{f}\right)\begin{pmatrix}\bar{t}_L & \bar{b}_L\end{pmatrix}\begin{pmatrix}\frac{V+h}{\sqrt{2}}\\ 0\end{pmatrix} t_R + h.c. \\
&= -\frac{m_t}{2}\frac{1}{\sqrt{\xi\,(1-\xi)}}\sin\left(\frac{2\,(V+h)}{f}\right)\bar{t}_L t_R + h.c.
\end{aligned} \tag{4.3.88}$$

Small perturbation around the minimum means that the h is small. Hence, we can expand the sine in the previous equation as Taylor series around $h = 0$ [54]

$$\begin{aligned}
\sin\frac{2\,(V+h)}{f} &= \sin\left(\frac{2V}{f}\right) + \cos\left(\frac{2V}{f}\right)\frac{2h}{f} - \frac{1}{2}\sin\left(\frac{2V}{f}\right)\left(\frac{2h}{f}\right)^2 + \dots \\
&= 2\sin\left(\frac{V}{f}\right)\cos\left(\frac{V}{f}\right) + 2\frac{h}{f}\left[1 - 2\sin^2\left(\frac{V}{f}\right)\right] \\
&- 4\frac{h^2}{f^2}\sin\left(\frac{V}{f}\right)\cos\left(\frac{V}{f}\right) + \dots \\
&= 2\sqrt{\xi}\sqrt{1-\xi} + 2\frac{h}{f}\,(1-2\xi) - 4\frac{h^2}{f^2}\sqrt{\xi}\sqrt{1-\xi} + \dots \\
&= 2\sqrt{\xi\,(1-\xi)} + 2\frac{h}{v}\sqrt{\xi}\,(1-2\xi) - 4\frac{h^2}{v^2}\xi\sqrt{\xi\,(1-\xi)} + \dots
\end{aligned} \tag{4.3.89}$$

Returning to the Lagrangian

$$\begin{aligned}
\mathcal{L}_Y^t &= -\frac{m_t}{2}\frac{\bar{t}_L t_R}{\sqrt{\xi\,(1-\xi)}}\left[2\sqrt{\xi\,(1-\xi)} + 2\frac{h}{v}\sqrt{\xi}\,(1-2\xi)\right. \\
&- \left. 4\frac{h^2}{v^2}\xi\sqrt{\xi\,(1-\xi)} + \dots\right] + h.c. \\
&= -m_t\bar{t}_L t_R\left[1 + \frac{h}{v}\frac{1-2\xi}{\sqrt{1-\xi}} - 2\frac{h^2}{v^2}\xi + \dots\right] + h.c. \\
&= -m_t\bar{t}t - k_t^{\mathbf{5}}\frac{m_t}{v}h\bar{t}t - c_t^{\mathbf{5}}\frac{m_t}{v}h^2\bar{t}t + \dots
\end{aligned} \tag{4.3.90}$$

This is the composite Lagrangian for the top quark, which provides the top mass and interactions with the SM Higgs. The term $h\bar{t}t$ is a SM-like interaction but with a modified coupling $k^{\mathbf{5}}$. Expressing it in terms of the composite and the SM couplings leads to

$$k_t^{\mathbf{5}} = \frac{1-2\xi}{\sqrt{1-\xi}}, \tag{4.3.91}$$

where the superscript **5** indicates we have embedded the fermions in the fundamental representation. It is essential to highlight that this modification is not unique to the model; it depends on the representation in which we embed the fermions. A different choice of representation would have led to a different coupling modification [2]. The third term in the top Lagrangian is absent in the SM, whose coefficient is

$$c_t^{\mathbf{5}} = -2\xi. \tag{4.3.92}$$

In the elementary Higgs limit, when $\xi \to 0$, the coupling modification reduces to the SM parameters, $k_t^{\mathbf{5}} = 1$ and $c_t^{\mathbf{5}} = 0$. More about these modifications and their consequences will be said in the following sections.

Moving forward to the bottom quark[20], we must have the right-handed one as a singlet under the unbroken group. The procedure is almost the same done with the top quark, with the difference in the $U(1)_X$ charge. Taking into account the decomposition pattern in Eq. (4.3.65), the bottom can be expressed as

$$\underline{B}_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_R \end{pmatrix}, \tag{4.3.93}$$

just in the same fashion as the top. There is a difference with the left-handed spinor; now, we must embed it in a representation that decomposes under $SO(4)$ as a T_R^3 eigenstate of $+1/2$; thus, it is chosen to match with the Ψ_+ in Eq. (B.43)

$$q_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix} = \Psi_+ = \begin{pmatrix} \Psi_+^u \\ \Psi_+^d \end{pmatrix}. \tag{4.3.94}$$

Substituting $t_L = \Psi_+^u$, $b_L = \Psi_+^d$ and $\Psi_-^u = \Phi_+^d = 0$ in Eq. (B.45), the embedded representation of $SO(4)$ is

$$q_L^{[\mathbf{4}]} = \frac{1}{\sqrt{2}} \begin{pmatrix} -it_L \\ t_L \\ ib_L \\ b_L \end{pmatrix}. \tag{4.3.95}$$

Putting the left fourplet into an $SO(5)$ is, once more, straightforward when we take a look

[20] I realize that I may be redundant here, as the explanation for the bottom quark closely mirrors that of the top quark, with only minor differences. However, I prefer to be repetitive rather than risk making the information unclear or confusing for the reader.

at the broken and unbroken generators, namely

$$\underline{Q}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} -it_L \\ t_L \\ ib_L \\ b_L \\ 0 \end{pmatrix} = \begin{pmatrix} q_L^{[\mathbf{4}]} \\ 0 \end{pmatrix}. \tag{4.3.96}$$

Dressing the bottom singlet with the Goldstone matrix, we have

$$\begin{aligned} B_R &= \tilde{\varphi}\left(U^{-1}[\Pi], \underline{B}_R\right) = \begin{pmatrix} \mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right] \frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix} \begin{pmatrix} \vec{0} \\ b_R \end{pmatrix} \\ &= \begin{pmatrix} -B_R \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} \\ b_R\cos\left(\frac{\Pi}{f}\right) \end{pmatrix}, \end{aligned} \tag{4.3.97}$$

the same math as of the right top quark. As before, we choose the singlet of the decomposition

$$\underset{\cdot}{B}_R = b_R \cos\left(\frac{\Pi}{f}\right), \qquad \underset{\cdot}{B}_R \in \mathbf{1}^{[\mathbf{4}]}_{-\frac{1}{3}} \tag{4.3.98}$$

One will find the same expression of Eq. (4.3.71) when dressing the q_L, i.e.,

$$\overline{Q}_L = \begin{pmatrix} \overline{q}_L^{[\mathbf{4}]} \cdot \left\{\mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2}\right\} \\ \sin\left(\frac{\Pi}{f}\right)\overline{q}_L^{[\mathbf{4}]}\cdot\frac{\vec{\Pi}}{\Pi} \end{pmatrix}. \tag{4.3.99}$$

but with a different left fourplet. The singlet one is, evidently

$$\underset{\cdot}{Q}_L = \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi}\cdot q_L^{[\mathbf{4}]}, \qquad \underset{\cdot}{Q}_L \in \mathbf{1}^{[\mathbf{4}]}_{-\frac{1}{3}} \tag{4.3.100}$$

The singlet is the only representation used to construct the invariant; thus, we are not going into the other presentation for the reasons we have already discussed. The invariant turns out to be

$$\begin{aligned}
\overline{\underset{\cdot}{Q}}_L \underset{\cdot}{B}_R &= \left[\sin\left(\frac{\Pi}{f}\right)\bar{q}_L^{[\mathbf{4}]}\cdot\frac{\vec{\Pi}}{\Pi}\right] b_R\cos\left(\frac{\Pi}{f}\right) \\
&= -\frac{b_R}{\sqrt{2}\Pi}\sin\left(\frac{\Pi}{f}\right)\cos\left(\frac{\Pi}{f}\right)\begin{pmatrix} i\bar{t}_L & \bar{t}_L & -i\bar{b}_L & \bar{b}_L\end{pmatrix}\begin{pmatrix}\Pi_1\\ \Pi_2\\ \Pi_3\\ \Pi_4\end{pmatrix} \\
&= \frac{b_R}{2\sqrt{2}\Pi}\sin\left(\frac{2\Pi}{f}\right)\begin{pmatrix}\bar{t}_L & \bar{b}_L\end{pmatrix}\begin{pmatrix} i\Pi_1+\Pi_2\\ -i\Pi_3+\Pi_4\end{pmatrix} \\
&= \frac{1}{2\sqrt{2}|H|}\sin\left(\frac{2\sqrt{2}|H|}{f}\right)\bar{q}_L H b_R. \qquad (4.3.101)
\end{aligned}$$

Hence, the composite bottom Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_Y^b &= -y_b f\overline{\underset{\cdot}{Q}}_L \underset{\cdot}{B}_R + h.c. \\
&= -\frac{y_b f}{2\sqrt{2}|H|}\sin\left(\frac{2\sqrt{2}|H|}{f}\right)\bar{q}_L H b_R + h.c.\ . \qquad (4.3.102)
\end{aligned}$$

After the EWSB the Higgs acquires its VEV, leading to

$$\begin{aligned}
\mathcal{L}_Y^b &= -\frac{y_b f}{2\sqrt{2}}\sin\left(\frac{2V}{f}\right)\bar{b}_L b_R + h.c. \\
&= -\frac{y_b f}{\sqrt{2}}\sin\left(\frac{V}{f}\right)\cos\left(\frac{V}{f}\right)\bar{b}_L b_R + h.c. = -y_b f\sqrt{\frac{\xi\,(1-\xi)}{2}}\bar{b}_L b_R + h.c. \\
&= -m_b\bar{b}b, \qquad (4.3.103)
\end{aligned}$$

where the new bottom mass is

$$m_b = y_b f\sqrt{\frac{\xi\,(1-\xi)}{2}}. \qquad (4.3.104)$$

In the unitary gauge and considering fluctuations around the potential's minimum, the Lagrangian turns out to be

$$\begin{aligned}
\mathcal{L}_Y^b &= -\frac{y_b f}{2\sqrt{2}}\sin\left(\frac{2\,(V+h(x))}{f}\right)\bar{b}_L b_R + h.c. \\
&= -\frac{m_b}{2\sqrt{\xi\,(1-\xi)}}\sin\left(\frac{2\,(V+h)}{f}\right)\bar{b}_L b_R + h.c.. \qquad (4.3.105)
\end{aligned}$$

Expanding in Taylor series as was done in Eq. (4.3.89)

$$\begin{aligned}\mathcal{L}_Y^b &= -\frac{m_b\bar{b}b}{2\sqrt{\xi\,(1-\xi)}}\left[2\sqrt{\xi\,(1-\xi)}+2\frac{h}{v}\sqrt{\xi}\,(1-2\xi)-4\frac{h^2}{v^2}\xi\sqrt{\xi\,(1-\xi)}+\dots\right]\\ &= -m_b\bar{b}b-k_b^{\mathbf{5}}\frac{m_b}{v}h\bar{b}b-c_b^{\mathbf{5}}\frac{m_t}{v}h^2\bar{b}b+\dots\end{aligned} \tag{4.3.106}$$

Where the couplings are

$$k_b^{\mathbf{5}} = \frac{1-2\xi}{\sqrt{1-\xi}}, \tag{4.3.107}$$

$$c_b^{\mathbf{5}} = -2\xi, \tag{4.3.108}$$

just as for the top quark. Please do not think that this is a general feature of composite Higgs models. It results from embedding both fermions in the same $SO(5)$ representation. If we have chosen to embed them in different representations, the story would be different, e.g., if the top had been taken to be in the **5** and the bottom in the spinorial **4** of $SO(5)$ the couplings would not be equal, $k_b = k_b^{\mathbf{4}} \neq k_t^{\mathbf{5}} = k_t$. The dependence on just one parameter ξ is also a coincidence of this specific case, which comes about because there was only one invariant operator for both top and bottom quarks. There exist representations for which the dependence is not of this type [2]. Since the construction done before makes use of the fundamental representation, it is often refered as $\mathrm{MCHM}_{\mathbf{5}}$ [30, 31].

As the above discussion suggests, the choice of representation directly affects the physics of the new model therefore, there is almost an unlimited number of variations and combinations to construct the model. The choice of the group on which the new sector is based is also arbitrary. All the freedom in the construction leads to a rich and vast number of viable models that may solve not just the hierarchy issue but other unsolved problems in the SM, such as baryon asymmetry, dark matter puzzle, Higgs vacuum stability, and many others.

4.3.4 Partial compositeness paradigm

Until now, we have discussed only the elementar sector of the composite scenario. However, two issues arise promptly. First, the expression we found as corrections to the SM vertices tells very little about the composite Higgs model. Of course, there is a deviation from the values expected by the SM prediction, but experiments would not be able to tell if this discrepancy is due to the new strong sector or any other BSM scenario [30]. Secondly, we have described the theory so far as an effective field theory, and at high energy scales, there is a need to consider the effects of the resonances from the new sector. Since their masses are expected to be higher than the TeV scale, only the terms interacting with the elementary sector are relevant to the discussion. The mixing terms also solve the first issue; they bring signatures that may differ from other models.

There are two methods for introducing mixing. The most straightforward approach would be to introduce a new kind of Yukawa coupling, as in [5]

$$\mathcal{L}_{billinear} = -y_t \bar{Q}_L \mathcal{O}_S^t T_R - y_b \bar{Q}_L \mathcal{O}_S^b B_R + h.c., \tag{4.3.109}$$

where the $\mathcal{O}_S^b$ and $\mathcal{O}_S^t$ are scalar operators. These operators are constructed with bricks[21] (fields) in the strong sector. This kind of operator brings interactions called bilinear [3] in the same fashion as technicolor models. The issue with this solution is that the operators' dimension is 1, allowing invariants in the form of

$$\mathcal{L}_{operator} = M_t^2 \left(\mathcal{O}_S^t\right)^2 + M_b^2 \left(\mathcal{O}_S^b\right)^2. \tag{4.3.110}$$

This messes up the entire construction we have done here since it reintroduces the hierarchy problem. This is the wrong path!

Using linear interactions is the correct manner of introducing the mixing between the composite and elementary sectors. The composite operator, made of field present in the new sector, couples linearly with only one field at the time, Figure 4.6 illustrates the bilinear and linear interactions. Putting this idea in mathematical language, we have[5]

$$\mathcal{L}_{linear} = y_L \bar{Q}_L \mathcal{O}_F^L - y_{tR} \bar{T}_R \mathcal{O}_F^{tR} - y_{bR} \bar{B}_R \mathcal{O}_F^{bR} + h.c. \tag{4.3.111}$$

Here, y's couplings are dimensionless coefficients, and the operators have a dimension of 5/2 in energy. This is the partial compositeness paradigm [31]. Fermionic operators create particle states with identical quantum numbers to those of elementary fermions. These operators can be related to resonances of the new strong sector, known as the fermion partner. Therefore [2]

$$\begin{aligned} \left\langle 0 \left| \mathcal{O}_F^L \right| \tilde{Q} \right\rangle &\neq 0, \\ \left\langle 0 \left| \mathcal{O}_R^L \right| \tilde{Q} \right\rangle &\neq 0, \end{aligned}$$

where $\tilde{Q}$ and $\tilde{T}$ are fermionic resonances. When the Higgs acquires the EW VEV, the fermions and their partners will mix.

When in four-dimensional space-time[22], the dimensional analysis shows that the

[21] Reminding you of the title of this thesis.
[22] The dimensional analises can be made for any theory, even in higher dimensions than four. However, the dimension of the fields, couplings, constants, and so on, can be different in higher dimensions.

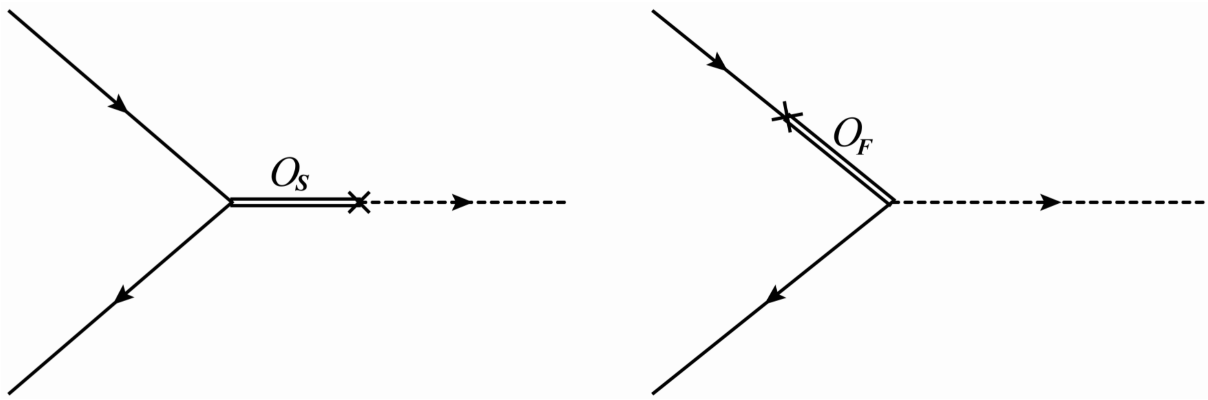


Figure 4.6: Types of interactions between the elementar and composite sectors. **Left:** bilinear interaction, the composite operator is scalar and reintroduce the hierarchy problem. **Right:** linear interaction, the composite operator is fermionic and there is no hierachy issues coming from it. Image from [30].

fermionic operator must be of the form

$$\mathcal{O}_F^L \to \frac{f}{4\pi}\Psi \tag{4.3.112}$$

with Ψ the fermionic resonance[23]. Writing down the Lagrangian for this field, together with mass terms

$$\mathcal{L}_{linear} = -f\left[\frac{y_L}{4\pi}\bar{Q}_L\Psi_L + \frac{y_{tR}}{4\pi}\bar{T}_R\Psi_{tR} + \frac{y_{bR}}{4\pi}\bar{B}_R\Psi_{bR} + h.c.\right] - f\bar{\Psi}\Psi. \tag{4.3.113}$$

The resonances can be integrated out at low energy levels, and the Yukawa transmutes to approximately [2]

$$y_t \sim \frac{y_L y_{tR}}{4\pi}, \tag{4.3.114}$$

$$y_b \sim \frac{y_L y_{bR}}{4\pi}. \tag{4.3.115}$$

Partial compositeness introduces new interactions and contributions that can be experimentally measured. It is crucial for developing a UV-complete model.

⋆ ⋆ ⋆ ⋆ ⋆

We could calculate the Higgs masses and parameters from the new strong sector, but this is beyond the scope of our current discussion. These calculations must be performed at the loop level, and the Coleman-Weinberg potential must be determined [2, 7]. Detailed methodologies can be found in Refs. [30, 5, 52]. The primary objective of this dissertation is to demonstrate the composite construction and the symmetry-breaking pattern.

[23] The $1/4\pi$ is just a normalization factor, do not let it bother you.

Chapter 5

CONCLUSIONS

This dissertation provides a fundamental understanding of the Standard Model and Higgs physics, focusing on the principles of the Composite Higgs Model. It also delves into the role of group theory in particle physics and its significance in advancing our understanding of these concepts.

At the heart of our analysis lies the examination of the Minimal Composite Higgs Model (MCHM), serving as the focal point of our inquiry. Through meticulous scrutiny of its intricacies, we uncover its potential to address the limitations of the Standard Model, thereby offering fresh insights into the mechanics of electroweak symmetry breaking.

In essence, this dissertation contributes another brick to the ongoing construction of our understanding of the Composite Higgs Model and its implications for particle physics. As we continue to unravel the mysteries of the universe, the Composite Higgs Model stands as a promising avenue for exploring new physics beyond the Standard Model paradigm.

Appendix A

CONVENTIONS AND NOTATIONS

The text follows, basically, the same notation of Ref. [13].

Since this is a dissertation that discuss a high energy physics' topic, we will use natural units

$$\hbar = c = 1.$$

The metric signature is

$$\eta_{ab} = (diag)(1, -1, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The greek indices are assumed to have the values of $\mu = 0, ..., 3$ while spatial indices are denoted by Latin letters, $i, j, ... = 1, 2, 3$, unless explicitly mentioned otherwise, in both cases.

The Dirac matrices used are the ones in the standard chiral representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

With the matrices in hands, we define the Dirac conjugation

$$\bar{\psi} = \psi^\dagger \gamma^0.$$

Pauli matrices will appear for a while during the text, they are

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also define

$$\tau^\mu = (1, \tau^i), \qquad \bar{\tau}^\mu = (1, -\tau^i).$$

The derivatives are denoted by

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2.$$

We will also use the slash notation, developed by Feynman, given by:

$$\not{A} = A_\mu \gamma^\mu, \tag{A.1}$$

in particular

$$\not{\partial} = \gamma_\mu \partial^\mu.$$

In the case of any further convetion and notation be used, it will be explained during the text.

Appendix B

A LITTLE BIT ABOUT THE $SO(4)$ GROUP

Our intention here is to make some definitions of the SO(4) group that is usefull in the composite caculations.

As already said a couple of times, there is an isomorphism between the $SO(4)$ and the chiral group, i.e.,

$$SO(4) \simeq SU(2)_L \times SU(2)_R, \tag{B.1}$$

meaning that they have the same algebra [32, 33, 37]. Therefore, it is possible to construct a presentation of the chiral group using the representation of the $SO(4)$. In order to do it, consider a $\vec{\Pi}$ four-plet in the fundamental representation of $SO(4)$, a $(\mathbf{2}, \mathbf{2})$ representation of $SU(2)_L \times SU(2)_R$ is constructed as [2]

$$\Sigma = \frac{1}{\sqrt{2}} \left(i\tau_\alpha \Pi^\alpha + \mathbb{1}_{2\times 2} \Pi^4 \right) = \frac{1}{\sqrt{2}} \bar{\tau}_i \Pi^i, \tag{B.2}$$

where $\alpha = 1, 2, 3$ and $i = 1, 2, 3, 4$. Following the Section of notations, we have that τ are the Pauli matrices and

$$\bar{\tau}_i = \{ i\tau_\alpha, \mathbb{1}_{2\times 2} \}. \tag{B.3}$$

The matrix Σ is also called a pseudo-real bidoublet, the pseudo reality comes from the fact that

$$\Sigma^* = \tau_2 \Sigma \tau_2, \tag{B.4}$$

and since it is a representation of the chiral group, it transforms as usual

$$\Sigma \to U_L \Sigma U_R^\dagger. \tag{B.5}$$

The $SU(2)$ group has the algebra defined by the Lie brackets[1],

$$\left[t^{\alpha}, t^{\beta}\right] = i\epsilon^{\alpha\beta\gamma} t^{\gamma}, \tag{B.6}$$

where the greek letters runs from one up to three, ϵ are the Levi-Civita symbol, and $t^{\alpha} = \tau^{\alpha}/2$ are the group's generators [34]. Turning to the $SO(4)$ group, they can be separated in two different $SU(2)$ set of generators, namely $t^{a} = \{t^{\alpha}_{L}, t^{\alpha}_{R}\}$. The two sets of generators obey the $SU(2)_L \times SU(2)_R$ algebra [2]

$$\left[t^{\alpha}_{L}, t^{\beta}_{L}\right] = \epsilon^{\alpha\beta\gamma} t^{\gamma}_{L}, \tag{B.7}$$

$$\left[t^{\alpha}_{R}, t^{\beta}_{R}\right] = \epsilon^{\alpha\beta\gamma} t^{\gamma}_{R}, \tag{B.8}$$

$$\left[t^{\alpha}_{L}, t^{\beta}_{R}\right] = 0. \tag{B.9}$$

To see how the field transforms under the group transformation, we make use of the exponential map

$$U_L = e^{i\delta^{L}_{\alpha} t^{\alpha}_{L}} \tag{B.10}$$

$$U_R = e^{i\delta^{R}_{\alpha} t^{\alpha}_{R}} \tag{B.11}$$

being the δ_{α} the infinitesimal transformation parameter. Applying the transformation in the field, we have, at the first order

$$U_L \Sigma = e^{i\delta^{L}_{\alpha} t^{\alpha}_{L}} \Sigma = \left(\mathbb{1}_{2\times 2} + i\delta^{L}_{\alpha} t^{\alpha}_{L}\right) \Sigma = \Sigma + i\delta^{L}_{\alpha} t^{\alpha}_{L} \Sigma \tag{B.12}$$

$$\Sigma U^{\dagger}_{R} = \Sigma e^{-i\delta^{R}_{\alpha} t^{\alpha}_{R}} = \Sigma \left(\mathbb{1}_{2\times 2} + i\delta^{R}_{\alpha} t^{\alpha}_{R}\right) = \Sigma - i\delta^{R}_{\alpha} \Sigma t^{\alpha}_{R} \tag{B.13}$$

Therefore, in the first order, the change in the field induced by the transformation is

$$\delta_L \Sigma = i\delta^{L}_{\alpha} t^{\alpha}_{L} \Sigma = i\delta^{L}_{\alpha} \frac{\tau^{\alpha}}{2} \Sigma, \tag{B.14}$$

$$\delta_R \Sigma = -i\delta^{R}_{\alpha} \Sigma t^{\alpha}_{R} = -i\delta^{R}_{\alpha} \Sigma \frac{\tau^{\alpha}}{2}. \tag{B.15}$$

To the fourplet, the transformation acts as

$$\delta_L \vec{\Pi} = i\delta^{L}_{\alpha} T^{\alpha}_{L} \vec{\Pi}, \tag{B.16}$$

$$\delta_R \vec{\Pi} = i\delta^{R}_{\alpha} T^{\alpha}_{R} \vec{\Pi}, \tag{B.17}$$

just in case the reader has forgotten, here $\vec{\Pi}$ has four components, so it is not a usual three-dimensional vector, and the T^{α} has not had the same form of t^{α}, since they are

[1] By the way, this is the exact algebra than the $SO(3)$ group, and, therefore, they are isomorphic to each other.

different generators. With this in hands, we can find the form of the generators, in order to do it we put Eq. (B.2) in Eq. B.14, namely,

$$\begin{aligned}\delta_L\Sigma &= \delta_L\left(\frac{1}{\sqrt{2}}\bar{\tau}^i\Pi^i\right) = \frac{1}{\sqrt{2}}\bar{\tau}^i\delta_L\Pi^i \\ &= \frac{1}{\sqrt{2}}\bar{\tau}^i\left(i\delta^L_\alpha T^\alpha_L\vec{\Pi}\right)^i = \frac{i}{\sqrt{2}}\delta^L_\alpha\bar{\tau}^i\left(T^\alpha_L\right)_{ij}\Pi^j,\end{aligned} \tag{B.18}$$

from the first to the second line, we have used the transformation in Eq. (B.16). The equation before holds for any infinitesimal transformation parametrized by δ^L_α. Due to Eq. (B.14) together with the result we got above, we have

$$\delta_L\Sigma = i\delta^L_\alpha\frac{\tau^\alpha}{2}\Sigma = \frac{i}{\sqrt{2}}\delta^L_\alpha\bar{\tau}^j\left(T^\alpha_L\right)_{ji}\Pi^i, \tag{B.19}$$

replacing the Σ by the Eq. B.2, leads to

$$\frac{i}{\sqrt{2}}\delta^L_\alpha\frac{\tau^\alpha}{2}\bar{\tau}_j\Pi^j = \frac{i}{\sqrt{2}}\delta^L_\alpha\bar{\tau}^i\left(T^\alpha_L\right)_{ij}\Pi^j. \tag{B.20}$$

Therefore, by inspection

$$\bar{\tau}^i\left(T^\alpha_L\right)_{ij} = \frac{\tau^\alpha}{2}\bar{\tau}_j. \tag{B.21}$$

This equation can be inverted solved in terms of the generator, leading to

$$\left(T^\alpha_L\right)_{ij} = \frac{1}{4}\mathrm{Tr}\left[\bar{\tau}^\dagger_i\tau^\alpha\bar{\tau}_j\right] = -\frac{i}{2}\left[\epsilon_{\alpha\beta\gamma}\delta^\beta_i\delta^\gamma_j + \left(\delta^\alpha_i\delta^4_j - \delta^\alpha_j\delta^4_i\right)\right]. \tag{B.22}$$

The same procedure can be done to the right generator, T^α_R, leading to

$$\left(T^\alpha_R\right)_{ij} = \frac{1}{4}\mathrm{Tr}\left[\bar{\tau}_i\tau^\alpha\bar{\tau}^\dagger_j\right] = -\frac{i}{2}\left[\epsilon_{\alpha\beta\gamma}\delta^\beta_i\delta^\gamma_j - \left(\delta^\alpha_i\delta^4_j - \delta^\alpha_j\delta^4_i\right)\right]. \tag{B.23}$$

Since these generators span a vectorial space, they must satisfy the completeness and normalization relations, namely

$$\mathrm{Tr}\left[T^\alpha_L T^\beta_L\right] = \mathrm{Tr}\left[T^\alpha_R T^\beta_R\right] = \delta^{\alpha\beta}, \qquad \mathrm{Tr}\left[T^\alpha_L T^\beta_R\right] = 0, \tag{B.24}$$

$$\sum_{\alpha=1}^{3}\left[\left(T^\alpha_L\right)_{ij}\left(T^\alpha_L\right)_{kl} + \left(T^\alpha_R\right)_{ij}\left(T^\alpha_R\right)_{kl}\right] = -\frac{1}{2}\left(\delta_{ik}\delta^{jl} - \delta_{il}\delta^{jk}\right), \tag{B.25}$$

$$\sum_{\alpha=1}^{3}\left[\left(T^\alpha_L\right)_{ij}\left(T^\alpha_L\right)_{kl} - \left(T^\alpha_R\right)_{ij}\left(T^\alpha_R\right)_{kl}\right] = -\frac{1}{2}\epsilon_{ijkl}, \tag{B.26}$$

where ϵ_{ijkl} is the four dimensional Levi-Civita anti-symmetric tensor.

Once the generators of the $SO(4)$ groups are defined, we turn ourselves to putting the Goldstones into a group representation. As said in the text, in the minimal composite

Higgs models, the $SU(2)_L$ group is identified to be the SM left-handed group, while the $U(1)_Y$ is identified as the being the group generated by the third generator of $SU(2)_R$, where we are disregarding the $U(1)_X$, since the Higgs boson is a singlet under this group. We, thus, need a $(\mathbf{2},\mathbf{2})$ representation of the chiral group and a hypercharge of 1/2. The goldstone matrix, Σ, has precisely the properties we need [2]; this can be easier seen by noting that the matrix can be written as

$$\Sigma = \left(\tilde{H}, H\right) = \begin{pmatrix} h_d^* & h_u \\ -h_u^* & h_d \end{pmatrix}, \tag{B.27}$$

where

$$\tilde{H} = i\tau^2 H^*, \tag{B.28}$$

and $H = (h_u \;\; h_d)^T$, writing down the components index notation

$$\Sigma_{\hat{i}\hat{j}} = \delta_{1\hat{j}}\tilde{H}_{\hat{i}} + \delta_{2\hat{j}}H_{\hat{i}}, \tag{B.29}$$

with the hat letters varing from one until two, i.e., $\hat{i} = 1, 2$. By using Eq. (B.2) we have, that

$$\begin{aligned} \Sigma &= \frac{1}{\sqrt{2}}\left(i\tau_\alpha\Pi^\alpha + \mathbb{1}_{2\times2}\Pi^4\right) \\ &= \frac{1}{\sqrt{2}}\left[i\begin{pmatrix} 0 & \Pi^1 \\ \Pi^1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i\Pi^2 \\ i\Pi^2 & 0 \end{pmatrix} + \begin{pmatrix} \Pi^3 & 0 \\ 0 & -\Pi^3 \end{pmatrix}\right] + \frac{1}{\sqrt{2}}\begin{pmatrix} \Pi^4 & 0 \\ 0 & \Pi^4 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \Pi^4 + i\Pi^3 & i\Pi^1 + \Pi^2 \\ i\Pi^1 - \Pi^2 & \Pi^4 - i\Pi^3 \end{pmatrix}. \end{aligned} \tag{B.30}$$

Therefore, by inspection of the equation above and Eq. (B.27) we get

$$H = \begin{pmatrix} h_u \\ h_d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i\Pi^1 + \Pi^2 \\ \Pi^4 - i\Pi^3 \end{pmatrix}. \tag{B.31}$$

A relevant question may appear: how does the Σ matrix transform under the chiral group? Let us start by applying a transformation of the left group

$$\Sigma \to U_L\Sigma, \tag{B.32}$$

using Eq. (B.29), we can write it down by using indices

$$\Sigma_{\hat{i}\hat{j}} \to (U_L)_{\hat{i}\hat{k}}\,\Sigma_{\hat{k}\hat{j}} = (U_L)_{\hat{i}\hat{k}}\left(\delta_{1\hat{j}}\tilde{H}_{\hat{k}} + \delta_{2\hat{j}}H_{\hat{k}}\right) \tag{B.33}$$

$$= \delta_{1\hat{j}}((U_L)_{\hat{i}\hat{k}}\,\tilde{H}_{\hat{k}} + \delta_{2\hat{j}}\,(U_L)_{\hat{i}\hat{k}}\,H_{\hat{k}}, \tag{B.34}$$

to respect the transformation low of the group, both $\tilde{H}$ and H transform a doublet of $SU(2)_L$ [5]. To determine the behavior of the Goldstone matrix under the $SU(2)_R$ group, we use the generators in Eq. (B.23) and the transformation law in Eq. (B.17), by starting with the first generator

$$i\delta_1^R T_R^1 \vec{\Pi} = -\frac{\delta_1^R}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = \frac{\delta_1^R}{2}\begin{pmatrix} -\Pi_4 \\ +\Pi_3 \\ -\Pi_2 \\ +\Pi_1 \end{pmatrix}. \tag{B.35}$$

Such a transformation will triguer a change in H, which can be calculated with help of Eq. (B.21),

$$\delta H = \frac{1}{\sqrt{2}}\begin{pmatrix} i\delta\Pi^1 + \delta\Pi^2 \\ \delta\Pi^4 - i\delta\Pi^3 \end{pmatrix} = \frac{i\delta_1^R}{2\sqrt{2}}\begin{pmatrix} \Pi^4 + i\Pi^3 \\ i\Pi^1 - \Pi^2 \end{pmatrix} = -\frac{i\delta_1^R}{2}\tilde{H}, \tag{B.36}$$

therefore, the bidoublet does not have a well-defined transformation rule under the first generator of the $SU(2)_R$ group. For T_R^2 we have

$$i\delta_2^R T_R^2 \vec{\Pi} = -\frac{\delta_2^R}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = -\frac{\delta_2^R}{2}\begin{pmatrix} +\Pi_3 \\ +\Pi_4 \\ -\Pi_1 \\ -\Pi_2 \end{pmatrix}, \tag{B.37}$$

leading to

$$\delta H = \frac{1}{\sqrt{2}}\begin{pmatrix} i\delta\Pi^1 + \delta\Pi^2 \\ \delta\Pi^4 - i\delta\Pi^3 \end{pmatrix} = \frac{\delta_2^R}{2\sqrt{2}}\begin{pmatrix} -\Pi^4 - i\Pi^3 \\ -i\Pi^1 + \Pi^2 \end{pmatrix} = -\frac{\delta_2^R}{2}\tilde{H}, \tag{B.38}$$

thus, the second generator also does not lead to any well-defined transformation rule. For the last generator, the third one, we get

$$i\delta_2^R T_R^3 \vec{\Pi} = -\frac{\delta_3^R}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = -\frac{\delta_3^R}{2}\begin{pmatrix} -\Pi_2 \\ +\Pi_1 \\ +\Pi_4 \\ -\Pi_3 \end{pmatrix}, \tag{B.39}$$

and, then

$$\delta H = \frac{1}{\sqrt{2}}\begin{pmatrix} i\delta\Pi^1 + \delta\Pi^2 \\ \delta\Pi^4 - i\delta\Pi^3 \end{pmatrix} = \frac{\delta_3^R}{2\sqrt{2}}\begin{pmatrix} i\Pi^2 - \Pi^2 \\ \Pi^3 + i\Pi^4 \end{pmatrix} = -i\frac{\delta_2^R}{2}H, \tag{B.40}$$

therefore, the reason why identify the third generator of $SU(2)_R$ as the hypercharge becomes evident. The H has the correct hypercharge to be the SM Higgs boson, i.e., $+1/2$.

It is equivalente to say that the bidoublet, or fourplet, decomposes as

$$\mathbf{4} = (\mathbf{2}, \mathbf{2}) \to \mathbf{2}_{1/2}, \tag{B.41}$$

under the SM $SU(2)_L \times U(1)_Y$ subgroup.

We have found the representation in which we desire to put the Goldstones on, which includes the Higgs in the SM correct representation. The procedure must be followed for the fermions as well. We must consider complex $SO(4)$ fourplet when dealing with these fields. In the same fashion, we did in Eq. (B.2), the complex Ψ can be expressed as

$$\Psi = \frac{1}{\sqrt{2}}\left(\psi^4 + i\tau_\alpha \psi^\alpha\right) = \frac{1}{\sqrt{2}}\bar{\tau}_i \psi^i, \tag{B.42}$$

or, explicitly,

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi^4 + i\psi^3 & \psi^2 + i\psi^1 \\ -\psi^2 + i\psi^1 & \psi^4 - i\psi^3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Psi^u_- & \Psi^u_+ \\ \Psi^d_- & \Psi^d_+ \end{pmatrix} = \left(\Psi_-, \, \Psi_+\right), \tag{B.43}$$

where ψ^i is a complex component of the representation $(\mathbf{2}, \mathbf{2})$ of the chiral group. Differently from the Goldstones case, these bidoublet does not satisfy the pseudo-reality condition in (B.4); because of this, we dub it as complex doublet $(\mathbf{2}, \mathbf{2})_\mathbf{c}$ [2]. Under the $SU(2)_L \times U(1)_Y$ each of the "doublets", of Ψ, has opposite hypercharges, represented by the minus and plus signal in the last two equations. Therefore, the representation decomposes as

$$\mathbf{4_c} = (\mathbf{2}, \mathbf{2})_\mathbf{c} \to \mathbf{2}_{+1/2} \oplus \mathbf{2}_{-1/2}. \tag{B.44}$$

This $\mathbf{4_c}$ representation of $SO(4)$, i.e. a fourplet, can be construct using (B.43),

$$\vec{\psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\Psi^u_+ - i\Psi^d_- \\ \Psi^u_+ - \Psi^d_- \\ i\Psi^d_+ - i\Psi^u_- \\ \Psi^d_+ + \Psi^u_- \end{pmatrix}. \tag{B.45}$$

The constructions and definitions presented here may look vague and nonsensical. However, as we will see, they will be helpful when constructing the Minimal Composite Higgs model.

Appendix C

GOLDSTONE MATRIX

We desire to show that the Goldstone matrix can be expressed as Eq. (4.3.8). We start our task by writing the broken generators, Eq. (4.3.2), explicitly

$$\widehat{T}^1 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{T}^2 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{C.1}$$

$$\widehat{T}^3 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad \widehat{T}^4 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{C.2}$$

We use these matrices to rewrite the argument in the exponential

$$i\frac{\sqrt{2}}{f}\Pi_i(x)\widehat{T}^i = i\frac{\sqrt{2}}{f}\left[\frac{\Pi_1(x)}{\sqrt{2}}\begin{pmatrix}0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0\end{pmatrix}+\frac{\Pi_2(x)}{\sqrt{2}}\begin{pmatrix}0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0\end{pmatrix}\right.$$

$$\left.+\frac{\Pi_3(x)}{\sqrt{2}}\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&i&0&0\end{pmatrix}+\frac{\Pi_4(x)}{\sqrt{2}}\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&i&0\end{pmatrix}\right]=$$

$$=\frac{1}{f}\begin{pmatrix}0&0&0&0&\Pi_1\\0&0&0&0&\Pi_2\\0&0&0&0&\Pi_3\\0&0&0&0&\Pi_4\\-\Pi_1&-\Pi_2&-\Pi_3&-\Pi_4&0\end{pmatrix}=\frac{\pi}{f}. \tag{C.3}$$

Replacing in the exponential and expanding as Taylor series, we have [54]

$$e^{i\frac{\sqrt{2}}{f}\Pi_i(x)\widehat{T}^i} = e^{\frac{\pi}{f}} = 1+\frac{\pi}{f}+\frac{1}{2!}\left(\frac{\pi}{f}\right)^2+\frac{1}{3!}\left(\frac{\pi}{f}\right)^3+\ldots=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\pi}{f}\right)^n. \tag{C.4}$$

There is a recursion relation of the matrix multiplication. To obtain this formula, we start by calculating the multiplications

$$\pi^2=\pi\pi=-\begin{pmatrix}\Pi_1^2&\Pi_1\Pi_2&\Pi_1\Pi_3&\Pi_1\Pi_4&0\\\Pi_2\Pi_1&\Pi_2^2&\Pi_2\Pi_3&\Pi_2\Pi_4&0\\\Pi_3\Pi_1&\Pi_3\Pi_2&\Pi_3^2&\Pi_3\Pi_4&0\\\Pi_4\Pi_1&\Pi_4\Pi_2&\Pi_4\Pi_3&\Pi_4^2&0\\0&0&0&0&\Pi^2\end{pmatrix}, \tag{C.5}$$

$$\pi^3=\pi\pi^2=-\Pi^2\begin{pmatrix}0&0&0&0&\Pi_1\\0&0&0&0&\Pi_2\\0&0&0&0&\Pi_3\\0&0&0&0&\Pi_4\\-\Pi_1&-\Pi_2&-\Pi_3&-\Pi_4&0\end{pmatrix}=\Pi^2\pi, \tag{C.6}$$

where

$$\Pi^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 + \Pi_4^2. \tag{C.7}$$

Using the equations before, we find

$$\pi^4 = \pi^3\pi = -\Pi^2, \tag{C.8}$$
$$\pi^5 = \pi^4\pi = -\Pi^2\pi^3 = \Pi^4\pi, \tag{C.9}$$
$$\pi^6 = \pi^5\pi = \pi^4\pi^2, \tag{C.10}$$
$$\pi^7 = \Pi^4\pi^3 = -\Pi^6\pi, \tag{C.11}$$

therefore, one concludes that

$$\pi^{2n+1} = (-1)^n \Pi^{2n}\pi, \tag{C.12}$$
$$\pi^{2(n+1)} = (-1)^2 \Pi^{2n}\pi^2, \tag{C.13}$$

with $n = 0, 1, 2, 3, \dots$. Making use of the last relation, the series in (C.4) can be split into terms of odd and even powers, as in [52]

$$\begin{aligned} e^{\frac{\pi}{f}} &= \mathbb{1} + \sum_{n=0}^{\infty} \frac{(-1)^n \Pi^{2n}}{(2n+1)! f^{2n+1}}\pi + \sum_{n=0}^{\infty} \frac{(-1)^n \Pi^{2n}}{(2n+2)! f^{2(n+1)}}\pi^2 \\ &= \mathbb{1} + \frac{\pi}{\Pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\Pi}{f}\right)^{2n+1} + \left(\frac{\pi}{\Pi}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{[2(n+1)]!}\left(\frac{\Pi}{f}\right)^{2(n+1)}. \end{aligned} \tag{C.14}$$

The first term can be easily identified as the Taylor series of the sine [54]

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{C.15}$$

The second term needs to be massaged a little in order to make it recognizable,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{[2(n+1)]!} x^{2(n+1)} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} x^{2n} = (-1) \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 - \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right] \\ &= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 - \cos x. \end{aligned} \tag{C.16}$$

Putting all the pieces together yields

$$
\begin{aligned}
e^{\frac{\pi}{f}} &= \mathbb{1} + \frac{\pi}{\Pi}\sin\left(\frac{\Pi}{f}\right) + \left(\frac{\pi}{\Pi}\right)^2\left[1-\cos\left(\frac{\Pi}{f}\right)\right] \\
&= \mathbb{1} + \frac{1}{\Pi}\sin\left(\frac{\Pi}{f}\right)\begin{pmatrix} 0 & 0 & 0 & 0 & \Pi_1 \\ 0 & 0 & 0 & 0 & \Pi_2 \\ 0 & 0 & 0 & 0 & \Pi_3 \\ 0 & 0 & 0 & 0 & \Pi_4 \\ -\Pi_1 & -\Pi_2 & -\Pi_3 & -\Pi_4 & 0 \end{pmatrix} \\
&\quad - \frac{1}{\Pi^2}\left[1-\cos\left(\frac{\Pi}{f}\right)\right]\begin{pmatrix} \Pi_1^2 & \Pi_1\Pi_2 & \Pi_1\Pi_3 & \Pi_1\Pi_4 & 0 \\ \Pi_2\Pi_1 & \Pi_2^2 & \Pi_2\Pi_3 & \Pi_2\Pi_4 & 0 \\ \Pi_3\Pi_1 & \Pi_3\Pi_2 & \Pi_3^2 & \Pi_3\Pi_4 & 0 \\ \Pi_4\Pi_1 & \Pi_4\Pi_2 & \Pi_4\Pi_3 & \Pi_4^2 & 0 \\ 0 & 0 & 0 & 0 & \Pi^2 \end{pmatrix}, \qquad \text{(C.17)}
\end{aligned}
$$

This equation can be simplified by using the definition of the vector $\vec{\Pi}$, and making some operations on it, namely

$$
\vec{\Pi} = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix}, \qquad \text{(C.18)}
$$

$$
\vec{\Pi}^T = \begin{pmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \end{pmatrix}, \qquad \text{(C.19)}
$$

$$
\vec{\Pi}\cdot\vec{\Pi}^T = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix}\cdot\begin{pmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \end{pmatrix} = \begin{pmatrix} \Pi_1^2 & \Pi_2\Pi_1 & \Pi_3\Pi_1 & \Pi_4\Pi_1 \\ \Pi_1\Pi_2 & \Pi_2^2 & \Pi_3\Pi_2 & \Pi_4\Pi_2 \\ \Pi_1\Pi_3 & \Pi_2\Pi_3 & \Pi_3^2 & \Pi_4\Pi_3 \\ \Pi_1\Pi_4 & \Pi_2\Pi_4 & \Pi_3\Pi_4 & \Pi_4^2 \end{pmatrix}. \qquad \text{(C.20)}
$$

With these results in hands, the first two terms in Eq. (C.17) can be expressed as

$$
\mathbb{1} + \frac{1}{\Pi}\sin\left(\frac{\Pi}{f}\right)\begin{pmatrix} 0 & 0 & 0 & 0 & \Pi_1 \\ 0 & 0 & 0 & 0 & \Pi_2 \\ 0 & 0 & 0 & 0 & \Pi_3 \\ 0 & 0 & 0 & 0 & \Pi_4 \\ -\Pi_1 & -\Pi_2 & -\Pi_3 & -\Pi_4 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{4\times 4} & \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & 1 \end{pmatrix}, \qquad \text{(C.21)}
$$

meanwhile the third term becomes

$$-\frac{1}{\Pi^2}\left[1-\cos\left(\frac{\Pi}{f}\right)\right]\begin{pmatrix} \Pi_1^2 & \Pi_1\Pi_2 & \Pi_1\Pi_3 & \Pi_1\Pi_4 & 0 \\ \Pi_2\Pi_1 & \Pi_2^2 & \Pi_2\Pi_3 & \Pi_2\Pi_4 & 0 \\ \Pi_3\Pi_1 & \Pi_3\Pi_2 & \Pi_3^2 & \Pi_3\Pi_4 & 0 \\ \Pi_4\Pi_1 & \Pi_4\Pi_2 & \Pi_4\Pi_3 & \Pi_4^2 & 0 \\ 0 & 0 & 0 & 0 & \Pi^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & 0 \\ 0 & -\left[1-\cos\left(\frac{\Pi}{f}\right)\right] \end{pmatrix}. \tag{C.22}$$

Putting all the pieces together, brings us to the final result

$$U[\Pi] = \begin{pmatrix} \mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix}. \tag{C.23}$$

It is worth stressing that this equation holds for any $SO(N) \to SO(N-1)$ symmetry breaking with the generators in the fundamental representation [2, 5]. It is not hard to see it. The general form of the broken generators for symmetry breaking patterns of $SO(N) \to SO(N-1)$ is

$$\left(\widehat{T}^i\right)_{IJ} = -\frac{i}{\sqrt{2}}\left(\delta_I^i\delta_J^N - \delta_J^i\delta_I^N\right), \tag{C.24}$$

where $i = 1, ..., N-1$ and $I, J = 1, ..., N$. Thus, substituting in the Goldstone matrix argument

$$i\frac{\sqrt{2}}{f}\Pi_i(x)\widehat{T}^i = \frac{1}{f}\begin{pmatrix} 0 & \cdots & 0 & \Pi_1 \\ & \ddots & & \Pi_2 \\ \vdots & & \vdots & \vdots \\ & & & \Pi_{N-1} \\ -\Pi_1 & \cdots & -\Pi_{N-1} & 0 \end{pmatrix}. \tag{C.25}$$

This is the same matrix of the Eq. (C.3), but in $N \times N$ dimesion. The calculations will be the same, but for N dimensions, leading to the same format of´ the Goldstone matrix.

Since we have arrived this far, it is worth also mentioning the inverse of the Goldstone matrix. Since it is orthogonal, its transpose [2],

$$U^{-1}[\Pi] = \begin{pmatrix} \mathbb{1}_{4\times 4} - \left[1-\cos\left(\frac{\Pi}{f}\right)\right]\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & -\sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}}{\Pi} \\ \sin\left(\frac{\Pi}{f}\right)\frac{\vec{\Pi}^T}{\Pi} & \cos\left(\frac{\Pi}{f}\right) \end{pmatrix}. \tag{C.26}$$

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