UNIVERSIDADE FEDERAL DE MINAS GERAIS
Instituto de Ciências Exatas
Programa de Pós-Graduação em Matemática

Adam Petzet Rudnik

A few Willmore-type inequalities in noncompact spaces

Belo Horizonte
2024

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Final Version

Dissertation presented to the Graduate Program in Mathematics of the Federal University of Minas Gerais in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Advisor: Ezequiel Rodrigues Barbosa
Co-Advisor: Celso dos Santos Viana

Belo Horizonte
2024

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R916f A few Willmore-type inequalities in noncompact spaces [recurso eletrônico] / Adam Petzet Rudnik. – 2024
1 recurso online (71 f. il.) : pdf.

Orientador: Ezequiel Rodrigues Barbosa.
Coorientador: Celso dos Santos Viana.

Tese (doutorado) - Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática.
Referências: f. 65-68

1. Matemática - Teses. 2. Desigualdades (Matemática) - Teses. 3. Variedades riemanianas – Teses. 4. Equações diferenciais - Teoria assintótica – Teses. 5. Curvatura – Teses. 6. Fluxo de Ricci – Teses. I. Barbosa, Ezequiel Rodrigues. II. Viana, Celso dos Santos. III. Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática. IV. Título.

CDU 51(043)

Ficha catalográfica elaborada pela bibliotecária Irénquer Vismeg Lucas Cruz CRB 6/819 - Universidade Federal de Minas Gerais - ICEx

ATA DA DUCENTÉSIMA DÉCIMA PRIMEIRA DEFESA DE TESE DE DOUTORADO DO ALUNO ADAM PETZET RUDNIK, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DO INSTITUTO DE CIÊNCIAS EXATAS DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA DIA 13 DE MAIO DE 2024.

Aos treze dias do mês de maio de 2024, às 15h00, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Adam Petzet Rudnik**, intitulada: "*A few Willmore-type inequalities in non compact spaces*" requisito final para obtenção do Grau de Doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Ezequiel Rodrigues Barbosa, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente, sem a presença do aluno, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 13 de maio de 2024.

PROF. DR. EZEQUIEL RODRIGUES BARBOSA
Orientador (UFMG)

PROF. DR. CELSO DOS SANTOS VIANA
Coorientador (UFMG)

PROF. DR. EMERSON ALVES MENDONÇA DE ABREU
Examinador (UFMG)

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PROF. DR. IVALDO PAZ NUNES
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PROF. DR. JOSÉ NAZARENO VIEIRA GOMES
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PROF. DR. MARCUS ANTÔNIO MENDONÇA MARROCOS
Examinador (UFAM)

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Ezequiel Barbosa, for his many suggestions and ideas during the doctorate. In particular, I thank him for suggesting part of the topic of this thesis. I am also grateful to my co-advisor, Professor Celso Viana, for his time, insights and the many discussions we shared. Most importantly, I would like to thank my wife, Rosalia Edith, for her encouragement, love, patience and also for reading and pointing out improvements in this manuscript. The good shape that this text find itself in today is due to her. Any remaining shortcoming is the province of its author. Finally, I am deeply thankful for my family. In particular, I would like to thank my father, Peter, and his wife, Zelia, for their immense support and companionship.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior — Brasil (CAPES) — Finance Code 001 in terms of a doctoral scholarship.

„Nur wer das Risiko eingeht, zu weit zu gehen,
kann möglicherweise herausfinden, wie weit man gehen kann.“

(T. S. Eliot)

RESUMO

Neste trabalho obtemos duas desigualdades de tipo Willmore para certas hipersuperfícies em variedades Riemannianas completas e não compactas. A primeira é uma desigualdade geométrica rígida para hipersuperfícies fechadas em variedades Riemannianas com curvatura de Ricci assintoticamente não negativa. A segunda desigualdade, também geométrica, está vinculada a ambientes Riemannianos com curvatura de Bakry-Émery-Ricci não negativa e cuja rigidez esta atualmente em desenvolvimento. Para este fim, utilizamos métodos usuais em teoria de comparação que são ambos provenientes das equações de Riccati e de Jacobi, e cujos elementos remetem ao trabalho de E. Heintze and H. Karcher [29], especialmente no que tange o crescimento de volume de tubos geodésicos em torno de hipersuperfícies. Além disso, esses métodos foram recentemente aplicados no trabalho de Wang [49] para simplificar significativamente a prova de uma desigualdade de tipo Willmore em variedades Riemannianas completas e não compactas e com curvature de Ricci não negativa, e que foi primeiramente revelada no trabalho de Agostiniani, Fagagnolo and Mazzieri [3].

Palavras-chave: curvatura assintoticamente não negativa; desigualdades tipo Willmore; razão de volume assintótico; curvatura de Bakry-Émery-Ricci.

ABSTRACT

In this work we obtain two Willmore-type inequalities for certain hypersurfaces in complete and noncompact Riemannian manifolds. The first is a sharp geometric inequality for closed hypersurfaces in Riemannian manifolds with asymptotically nonnegative Ricci curvature. The second concerns Riemannian manifolds with nonnegative Bakry-Émery Ricci curvature, whose sharpness is under current development. To accomplish this, we use standard comparison methods derived both from the Riccati and Jacobi equations, whose elements goes back to the work of E. Heintze and H. Karcher [29], specially in regards to volume growth of geodesic tubes around hypersurfaces. Moreover, these methods have been applied in a recent work by Wang [49] to greatly simplify the proof of the Willmore-type inequality in complete noncompact Riemannian manifolds of nonnegative Ricci curvature, which was first proved by Agostiniani, Fagagnolo and Mazzieri [3].

Keywords: asymptotically nonnegative curvature; Willmore-type inequality; asymptotic volume ratio; Bakry-Émery Ricci curvature.

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CHAPTER 1

INTRODUCTION

1.1 Overview in comparison geometry

Comparison Geometry can be traced back to the nineteenth century, but it did not take root until the 1930's, thorough the work of H. Hopf, Morse-Schoenberg, Myers and Synge. The real breakthrough, however, came in the 1950's with the pioneering work of Rauch and the foundational work of Alexandrov, Toponogov and Bishop [27].

A classical guideline to this geometric branch of Riemannian Geometry has been the comparison of the geometry of a given general manifold M with that of a simply connected model space M_H of constant sectional curvature H in the hope to obtain global properties from infinitesimal (=curvature) assumptions. A typical conclusion is that M retains particular geometrical properties of the model under the assumption that its sectional curvature is bounded between constants [14], [32]. The intuition behind all comparison theorems is that negative curvature forces geodesics to spread apart faster as you move away from a point, while positive curvature forces them to spread slower and eventually to begin converging.

The curvature tensor of the Riemannian manifold (M, g) is the fundamental local invariant of the metric tensor g which characterizes the geometry of the manifold. It has been playing a central role in understanding the local geometry of a manifold and its global geometry and topology. Assuming lower or upper bounds on the sectional curvature leads to plethora of information about the metric and distance functions. So it is natural to wonder if anything can be deduced if we weaken the hypothesis to other curvature quantities, such as Ricci or scalar curvature. A first phenomenon is that (in what concerns Comparison Geometry) Ricci curvature admits only lower bounds (check out [48]).

Assuming a lower bound on the Ricci curvature is the same as assuming a lower bound on the average of sectional curvatures taken over all 2-planes containing a preferred vector. This has led some mathematicians to consider special partial traces of the curvature tensor known as *intermediate Ricci curvature* [55], [53]. Intermediate Ricci curvature is a natural quantity that interpolates between sectional and Ricci curvature. In this context, modulo the discussion if there are many examples of manifolds satisfying certain partial positivity conditions on the curvature, it has been a particularly interesting subject to comparison geometers to investigate what geometric and topological results remain true if one replaces sectional by intermediate Ricci curvature (see [12]).

A fundamental truth in classical Comparison Geometry is that a lower bound on the Ricci curvature implies that the Riemannian measure is bounded above by the measure in the corresponding model space, so that not only Ricci curvature performs well alongside comparison theory but it also has exceptional relevance to differential geometry in the large. Aside from being the only nontrivial contraction of the full curvature tensor, the Ricci curvature is also special since it occurs in the theory of Ricci flow and the theory of General Relativity, as it appears in the Einstein equation relating the stress energy of matter to the curvature of spacetime (see [46]).

One of the most iconic theorems in Comparison Geometry is the classical Bishop-Gromov Volume Comparison Theorem and it is considered a major breakthrough for its innumerous applications and generalizations (see e.g. [50]). In particular, its generalization to asymptotic nonnegatively curved spaces not only has many applications to the topology and geometric structure of manifolds with lower curvature bounds (see e.g. Theorem 3.1. in [56]) and that of open manifolds (cf. [19]), but it also has a core relevance for this thesis: see Theorem A.3.1 at the appendix.

In this dissertation we use techniques from Comparison Geometry derived from the Riccati and the Jacobi equations, whose features go back to the work of E. Heintze and H. Karcher [29], specially in regards to volume growth of geodesic tubes around hypersurfaces, to bring about Willmore-type inequalities in complete and noncompact spaces with two curvature assumptions: first, we assume that the space has asymptotically nonnegative Ricci curvature; and second, that it has nonnegative Bakry-Émery Ricci curvature. In the next sections, we describe the context in which we are employing these comparison techniques.

Most notably, we argue that the corresponding simply connected model geometry for asymptotically nonnegative Ricci curvature (in a sense to be specified) is the Riemannian warped product

$$[0,\infty)\times_{\varrho} N \text{ with metric } g = dr\otimes dr + \varrho(r)^2 g_N,$$

for suitable choice of function $\varrho \in C^{\infty}\big([0,\infty),\mathbb{R}^{*}_{+}\big)$, with $\varrho' > 0$ on $(0,\infty)$, where (N, g_N) is simply connected and compact. This occurs for various reasons:

(μg) first, for suitable ϱ, the Ricci curvature is smallest in the radial direction;

(μg) second, in the radial direction, the Ricci curvature directly involves a Jacobi equation;

(μg) third, its geodesic tubes $\mathcal{T}_{\Gamma_0}(r)$, where $\Gamma_0 = \{0\}\times N^{n-1}$, have a well ordered volume growth;

1.2 Willmore-type inequalities

In the nineteen sixties, Willmore [54] introduced the concept of what is now know as the Willmore energy. In summary, the Willmore energy is a functional defined on compact surfaces in Euclidean space that measures how much a given surface deviates from being a round sphere. One of his original interests was to find the optimal immersion of a compact surface in three space of a given topological type. The Willmore energy has many generalizations from ambient spaces with more general curvature conditions to hypersurfaces in higher dimensional ambient manifolds, see [28], [36] and the references therein.

A classical theorem due to Willmore [54] asserts that any compact domain $D \subset \mathbb{R}^3$ with regular boundary $\partial D = \Sigma$ has

$$\int_\Sigma \left(\frac{\mathrm{H}}{2}\right)^2 d\sigma \geq 4\pi$$

where H is the mean curvature of Σ. Moreover, equality holds only for round spheres. Thus, inside the class of genus zero the inequality above is optimal.

It is now widely known that the Willmore inequality extends to $\mathbb{R}^n$, $n \geq 3$, to closed submanifolds of any codimension (see e.g. [15]). More generally, in 2020 Agostiniani, Fagagnolo and Mazzieri [3] established a sharp Willmore-type inequality for compact domains in noncompact Riemannian manifolds with nonnegative Ricci curvature. Their result reads as follows.

Theorem 1.2.1. [*Willmore-type inequality in nonnegatively curved spaces*]
Let (M, g) be a noncompact, complete n-dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. If $\Omega \subset M$ is a bounded and open subset with smooth boundary $\Sigma = \partial\Omega$ then

$$\int_{\partial\Omega} \left|\frac{H}{n-1}\right|^{n-1} d\sigma \geq AVR(g)|\mathbb{S}^{n-1}|,$$

where $AVR(g) \in (0, 1]$ is the asymptotic volume ratio of (M, g). Moreover, the equality holds if and only if $(M \setminus \Omega, g)$ is isometric to

$$\left([r_0, \infty) \times \Sigma, dr \otimes dr + \left(\frac{r}{r_0}\right)^2 g_\Sigma\right), \quad \textit{where} \ \ r_0^{n-1} = \left(\frac{|\Sigma|}{AVR(g)|\mathbb{S}^{n-1}|}\right).$$

In particular, Σ is a connected totally umbilic hypersurface with constant mean curvature.

The argument in [3] is based on PDE methods, and was acknowledged in a subsequent work by Wang [49] as beautiful; nonetheless, highly nontrivial. In his work, Wang obtained the same Willmore-type inequality for noncompact and complete Riemannian manifolds of nonnegative Ricci curvature and his argument is based on standard comparison methods in Riemannian geometry derived from the Riccati equation.

In a different direction, Brendle [9] studied the Isoperimetric problem for minimal hypersurfaces in $\mathbb{R}^{n+1}$ and obtained a Sobolev inequality which holds for any submanifold in Euclidean space. This established a long standing conjecture that states that the isoperimetric inequality holds for compact minimal hypersurfaces in place of domains. We note that Brendle's method, the Alexandrov-Bakelman-Pucci maximum principle, can be extended to more general ambient manifolds as in [10] and [31]. In particular, using Brendle's Sobolev inequalities in nonnegatively curved spaces together with Hölder's inequality is possible to obtain a weaker Willmore-type inequality. Briefly, if D is a compact domain with smooth boundary $\partial D = \Sigma$ in a complete noncompact Riemannian manifold (M, g) of nonnegative sectional curvature then, as shown in Appendix A.2, we have

$$\int_{\Sigma} \left| \frac{\vec{H}}{n-1} \right|^{n-1} d\sigma \geq \frac{|\mathbb{S}^{n-2}|}{n-1} AVR(g)$$

while $\alpha_{n-1} > \frac{\alpha_{n-2}}{n-1}$, where α_k is the k-dimensional area of the Euclidean unit sphere $\mathbb{S}^k$, $\vec{H}$ denotes the mean curvature vector of Σ, defined in (2.2) ahead, and $AVR(g)$ the asymptotic volume ratio of (M, g).

Now we can compare the Willmore's type inequality with the Sobolev inequalities in the above context. On the one hand, the overall extensiveness of these Sobolev inequalities regarding codimension and the flexibility on the function f are broader, not to mention that the claims on these inequalities concern submanifolds possibly with boundary. On the other hand, aside from an application of Hölder's inequality, the constant $AVR(g)|\mathbb{S}^{n-1}|$ appearing in the Willmore-type inequality is finer, while the hypothesis on the curvature is weaker and the rigidity statement is somehow stronger.

Willmore-type inequalities are a certain class of geometric inequalities named after the British mathematician T. J. Willmore. Generally speaking, these inequalities involve extrinsic and intrinsic geometric quantities such as curvature, area, volume, etc, related to the shape of submanifolds of fixed topological type in a wide range of Riemannian manifolds.

In this work we obtain two Willmore-type inequalities for the boundary of a compact domain in noncompact and complete Riemannian manifolds. The first concerns Riemannian manifolds with asymptotically nonnegative Ricci curvature, extending the main theorem in [3] to this ambient. The second treats those settings whose Bakry-Émery Ricci curvature is nonnegative.

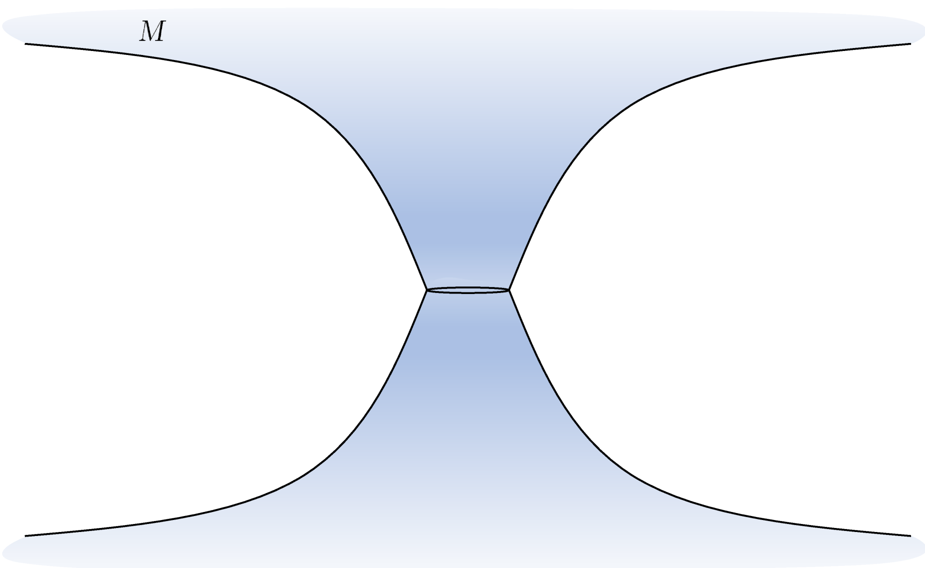
1.3 Willmore-type inequality for asymptotically nonnegative Ricci curvature

Back to the pursuit of extending Brendle's isoperimetric inequality, in 2022, Dong, Lin and Lu [21] extended Brendle's result to complete noncompact Riemannian manifolds of asymptotically nonnegative curvature. It is important to observe that the theorems obtained by Dong, Li and Lu do not yield our inequality, let aside the curvature assumptions. In particular, the equality cases analyzed in their work force the whole ambient manifold to be isometric with the Euclidean space, while our equality case either bends the ambient to be isometric with a Riemannian manifold of only one end and which is a truncated cone outside an open set Ω, or forces the ambient manifold to be isometric with a specific type of warped product outside an open set, see Theorem 1.3.1 for details. Moreover, in the first case, it is exactly the geometry of that set Ω that possibly prevents the ambient to be isometric with the Euclidean space, see Appendix A.1.

The class of asymptotic nonnegatively curved Riemannian manifolds was first studied by Abresch in the nineteen eighties, [1] and [2]. In the nineties, Zhu extended the classical Bishop-Gromov inequalities [26] in the case of complete Riemannian manifolds of nonnegative Ricci curvature to these manifolds [56]. It is interesting to notice that, according to [1], the class of asymptotic nonnegatively curved spaces includes the class of asymptotically flat manifolds (see Figure 1.1). In particular, it is expected that a Willmore-type inequality for these spaces gives a lower bound for the area of minimal hypersurfaces, provided that the asymptotic volume ratio does not vanish.

An "end" of a n-dimensional manifold is a region of the manifold diffeomorphic to $\mathbb{R}^n \setminus \mathbb{B}^n$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$. Our technique applies for manifolds with any number of disjoint ends, but when considering the rigidity cases, we will, for simplicity, consider only one end. We believe, however, that when dealing with finitely many ends we can apply our arguments to conclude analogous rigidity statements for each end.

Figure 1.1: A two-ended asymptotic nonnegatively curved Riemannian manifold



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A continuous and nonnegative function $\lambda : [0, \infty) \to \mathbb{R}^+$ is said to be an **associated function** if λ is nonincreasing. Associated functions play primary role in asymptotic nonnegatively curved spaces.

Recall that (M, g) is said to have ***asymptotically nonnegative*** Ricci (resp. sectional) curvature with base point p_0 and associated function λ if

$$\mathrm{Ric}_w \geq -(n-1)(\lambda \circ \mathrm{dist}_{p_0})(w) g_w \qquad \big(\text{resp. } \sec_w \geq -(\lambda \circ \mathrm{dist}_{p_0})(w)\big),$$

for $w \in M$ where $\mathrm{dist}_{p_0}(w)$ is the distance from w to p_0 and the associated function λ satisfy

$$b_0 \doteq \int_0^\infty t\lambda(t)dt < \infty \tag{1.1}$$

which implies

$$b_1 \doteq \int_0^\infty \lambda(t)dt < \infty. \tag{1.2}$$

Note that from the definition of the associated function λ, either λ vanishes identically and we write $\lambda = 0$ or there exists $\delta > 0$ such that $\lambda > 0$ on $[0, \delta)$, in which case $\lambda \neq 0$. In the following, we write $\lambda \neq 0$ meaning that λ does not vanish identically. Now, according to Lemma 2.1 in [56], the function

$$\Theta(r) = \frac{vol(B_{p_0}(r))}{\omega_n r^n}$$

is nonincreasing on $[0, \infty)$ and $0 \leq \Theta(r) \leq e^{(n-1)b_0}$, where $B_{p_0}(r)$ is the metric ball of radius r centered at p_0 and ω_n denotes the volume of the n-dimensional unit ball in Euclidean space, $\omega_n = |\mathbb{B}^n|$. Therefore, we may introduce the asymptotic volume ratio of (M, g) by

$$AVR(g) = \lim_{r \to \infty} \Theta(r) \tag{1.3}$$

Throughout this work we consistently assume that the dimension of the underlying ambient manifold is at least 3 and that it has positive asymptotic volume ratio, for otherwise our inequalities are trivial.

Theorem 1.3.1. [*Willmore-type inequality in asymptotic nonnegatively curved spaces*] *Let (M, g) be a noncompact, complete n-dimensional Riemannian manifold with asymptotically nonnegative Ricci curvature with base point p_0 and associated function λ. Let Ω be an open and bounded set with smooth boundary $\Sigma = \partial\Omega$, whose mean curvature is H. Then*

$$e^{(n-1)b_0} \int_\Sigma \left(\left| \frac{\mathrm{H}(x)}{n-1} \right| (1 + b_0) + b_1 \right)^{n-1} d\sigma(x) \geq AVR(g) |\mathbb{S}^{n-1}| \tag{1.4}$$

where $AVR(g)$ is the asymptotic volume ratio of g. In the following, we distinguish two cases.

- *(W**1**) If $\lambda = 0$ so that (M, g) has nonnegative Ricci curvature then equality holds iff $M \setminus \Omega$ is isometric to*

$$\left([r_0,\infty)\times\Sigma, dr\otimes dr+\left(\frac{r}{r_0}\right)^2 g_\Sigma\right),\ \ \textit{where}\ \ r_0^{n-1}=\left(\frac{|\Sigma|}{AVR(g)|\mathbb{S}^{n-1}|}\right).$$

In particular, Σ is a connected totally umbilic hypersurface with constant mean curvature;

(W2) If $\lambda\neq 0$ and equality holds then Σ is a totally umbilic hypersurface with nonnegative constant mean curvature on its connected components. If Σ is connected then equality in (1.4) holds if and only if $M\setminus\Omega$ is isometric to

$$\left([r_0,\infty)\times\Sigma, dr\otimes dr+\varrho(r)^2 g_\Sigma\right),\ \ \textit{where}\ \ r_0=dist_g(p_0,\Sigma), \tag{1.5}$$

and ϱ is the solution to the Jacobi equation $\varrho''(r)-\lambda(r)\varrho(r)=0$, over the interval $[r_0,\infty)$, with $\varrho(r_0)=1$, $\varrho'(r_0)=\frac{\mathrm{H}}{n-1}$ and satisfying the following property:

$$\lim_{r\to\infty}\frac{\varrho(r+r_0)}{e^{b_0}\left(\frac{H}{n-1}(1+b_0)+b_1\right)r}=1 \tag{1.6}$$

In particular, Ω is a geodesic ball centered at p_0.

Note that if $b_0=0$ in the inequality (1.4) above so that (M,g) has nonnegative Ricci curvature, we obtain

$$\int_\Sigma\left|\frac{\mathrm{H}}{n-1}\right|^{n-1}d\sigma\geq AVR(g)|\mathbb{S}^{n-1}|, \tag{1.7}$$

which is the main result in [3]. Therefore, the inequality (1.4) generalizes the Willmore-type inequality for noncompact complete Riemannian manifolds of nonnegative Ricci curvature to asymptotic nonnegatively curved spaces.

In a somewhat related direction, Bray studied the isoperimetric problem for surfaces in the three-dimensional Schwarzschild manifold. His interest was motivated by considerations in general relatively, regarding the positive mass theorem and the Penrose inequality. Earlier results of Schoen and Yau [47] and Ilmannen and Huisken [30] are historically key papers for understanding the general perspective on the subject. Since asymptotically flat manifolds are asymptotic (Ricci and sectional) nonnegatively curved spaces, it seems to the author that the present thesis supplements, somehow, these prior works.

The issue of the decay of the associated function λ is of delicate nature and it is related with various asymptotic behavior of quantities arising from the estimates obtained in our elementary inequalities, e.g. the volume element. In Section 3.2 we completely describe this matter and work out a sufficient condition for the inequality (1.4) to be strict depending solely on the decay of λ, see Proposition 3.2.3 for details.

The disadvantage of making comparison geometry in manifolds with asymptotically nonnegative curvature is caused by the dependence of the base point p_0. However, because of the decay condition on the curvature, it is expected that the behavior of these manifolds at infinity resemble the behavior of those manifolds with nonnegative curvature.

Next, we introduce a class of manifolds that include those with asymptotically nonnegative curvature. This new class will be useful to explore some specific types of warped product spaces to which we can extend the Willmore-type inequality from Theorem 1.3.1. Moreover, we investigate to what extent we may bring off the rigidity statement in the main theorem, not only for null-homologous hypersurfaces but also for a broader class of hypersurfaces.

Let (M, g) be a complete noncompact n-dimensional Riemannian manifold. We say that g has ***asymptotically* nonnegative*** Ricci (resp. sectional) curvature relatively to a k-dimensional submanifold $N^k \subset M$ with associated function λ if λ satisfies (1.1) and

$$\mathrm{Ric}_w \geq -(n-1)(\lambda \circ \mathrm{dist}_N)(w) g_w \quad \big(\text{resp. } \sec_w \geq -(\lambda \circ \mathrm{dist}_N)(w)\big),$$

for $w \in M$, where $\mathrm{dist}_N(w)$ is the distance from w to N. Observe that if N is a single point p_0, then the idea of asymptotic* nonnegative curvature reduces to the concept of asymptotic nonnegative curvature. In addition, it is not clear whether asymptotic* nonnegatively curved spaces possess a well defined notion of asymptotic volume ratio. We address this issue at Section 3.3 and restrict ourselves to spaces that have this property as defined there. For the statement of the next theorem, be mindful of Figure 1.2.

Theorem 1.3.2. [*Willmore-type inequality in asymptotic* nonnegatively curved spaces*]
Let (M, g) be a noncompact, complete n-dimensional Riemannian manifold with asymptotically nonnegative Ricci curvature relatively to a closed and connected hypersurface Γ_o and associated function λ, with $\lambda \neq 0$. Suppose that Γ_o separates M into two connected components: $M \setminus \Gamma_o = M^+ \cup M^-$, with M^+ unbounded. Let $\Sigma \subset M^+$ be a closed hypersurface homologous to Γ_o. Let $\Omega \subset M$ be the open set with the property $\partial\Omega = \Gamma_o \cup \Sigma$ and assume further that $|\Omega| < \infty$. Then*

$$e^{(n-1)b_0} \int_\Sigma \left(\left| \frac{\mathrm{H}(x)}{n-1} \right| (1 + b_0) + b_1 \right)^{n-1} d\sigma(x) \geq AVR^*(g) |\mathbb{S}^{n-1}| \tag{1.8}$$

where H is the mean curvature of Σ and $AVR^*(g)$ is the asymptotic volume ratio of g as defined in Digression 3.3.1. Moreover, if equality holds then Σ is a totally umbilic hypersurface whose mean curvature is constant on its connected components. In addition to that, if Σ is connected then equality in (1.8) holds iff $M^+ \setminus \Omega$ is isometric to*

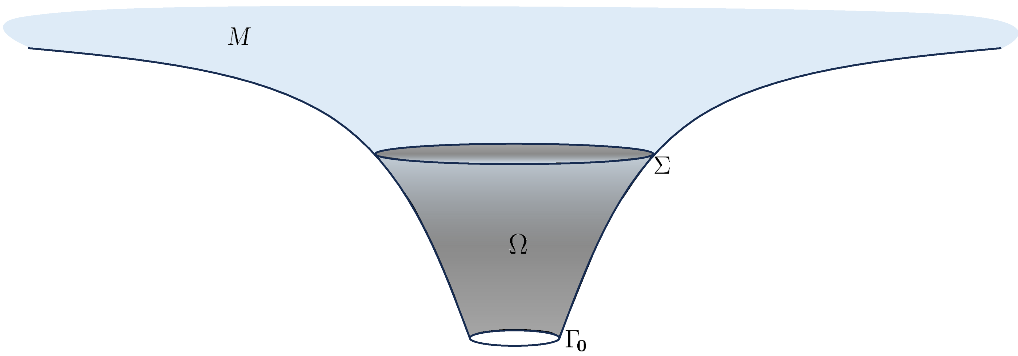
$$\Big([r_0, \infty) \times \Sigma, dr \otimes dr + \varrho(r)^2 g_\Sigma \Big), \quad \text{where } r_0 = dist_g(\Gamma_o, \Sigma), \tag{1.9}$$

and ϱ is the solution to the Jacobi equation $\varrho''(r) - \lambda(r)\varrho(r) = 0$, over $[r_0, \infty)$, with $\varrho(r_0) = 1$, $\varrho'(r_0) = \frac{\mathrm{H}}{n-1}$ and obeying to

$$\lim_{r\to\infty} \frac{\varrho(r)}{e^{b_0}\left(\frac{H}{n-1}(1+b_0) + b_1\right) r} = 1. \tag{1.10}$$

In particular, $\Gamma_{\boldsymbol{o}}$ is equidistant to Σ.

Figure 1.2: An open set Ω in M whose boundary is $\Gamma_{\mathbf{0}} \cup \Sigma$ and where $M^- = \emptyset$



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As an immediate application of our inequality we have the following estimate.

Corollary 1.3.3. [*Lower bound for the area of closed minimal hypersurfaces*]
As in Theorem 1.3.1 (resp. Theorem 1.3.2) let Σ be a closed hypersurface in a complete non-compact n-dimensional Riemannian ambient (M, g) with asymptotically (resp. asymptotically) nonnegative Ricci curvature with base point p_0 (resp. relatively to $\Gamma_{\boldsymbol{o}}$) and associated function λ, with $\lambda \neq 0$. If Σ is minimal and null-homologous (resp. homologous to $\Gamma_{\boldsymbol{o}}$) then,*

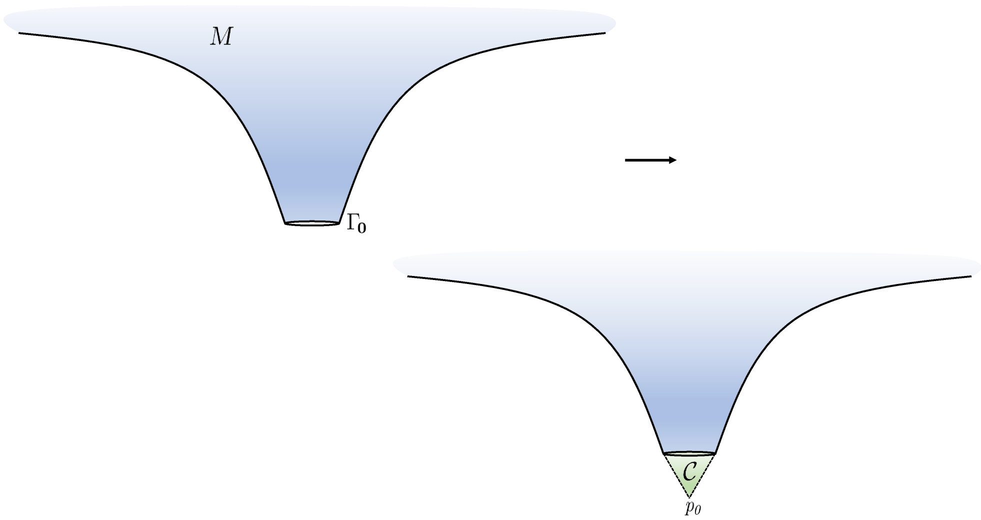
$$|\Sigma| \geq \frac{\mathcal{A}(g)|\mathbb{S}^{n-1}|}{(e^{b_0}b_1)^{n-1}}, \tag{1.11}$$

where $\mathcal{A}(g)$ is the asymptotic (resp. asymptotic) volume ratio of g.*

There is a nice geometrical idea behind the lower bound above. If b_1 approaches zero then the RHS in (1.11) becomes very large. Also, the associated function λ is approaching zero and in doing so, the whole Ricci tensor is gradually changing from being asymptotically nonnegative to being nonnegative. However, noncompact spaces of nonnegative Ricci curvature with Euclidean volume growth do not support closed minimal hypersurfaces, so it is expected that the area of Σ is getting larger, as formula (1.11) tells.

It is easy to check that if (M, g) has asymptotically nonnegative Ricci curvature with base point p_0 then (M, g) has asymptotically* nonnegative Ricci curvature relatively to K, for any compact submanifold K such that $p_0 \in K$. In fact, these concepts seems to be closely related, at least in codimension 1. Intuitively speaking, if (M,g) has asymptotically* nonnegative Ricci curvature relatively to $\Gamma_{\mathbf{0}}$, then produce a cone $\mathcal{C}$ whose boundary is $\Gamma_{\mathbf{0}}$ and whose tip is labelled as the base point p_0 so that the distance from any point $w \in M$ to p_0 is the distance from w to $\Gamma_{\mathbf{0}}$ plus the distance from $\Gamma_{\mathbf{0}}$ to p_0, which is constant, so that $(M \cup \mathcal{C}, \hat{g})$ has asymptotically nonnegative Ricci curvature with base point p_0, where $\hat{g}$ is the metric g properly extended to $M \cup \mathcal{C}$. In this line of thought, the question whether there are more general volume comparison theorems of Heintze-Karcher type in asymptotically nonnegative curved spaces remains open. The case where $N = \{p_0\}$ and the case where N is a closed geodesic were already inspected (see the remark after Corollary 2.1 in [56]). In the present work we treat the codimension one, closed and embedded case.

Figure 1.3: Gluing a cone $\mathcal{C}$ to an asymptotic* nonnegatively curved space M, with $\partial M = \partial \mathcal{C}$



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Next we apply our inequality (1.8) for a class of warped product manifolds that we now describe. Let (N, g_N) be a connected and compact Riemannian manifold of dimension $n-1 \geq 2$ such that

$$\mathrm{Ric}_N \geq (n-2)\rho g_N,$$

for some constant ρ. Furthermore, consider a positive and smooth function $h : [0,\infty) \to \mathbb{R}$ and define the quantities $\lambda_1, \lambda_2 : [0,\infty) \to \mathbb{R}$ by

$$\lambda_1(r) = \frac{h''(r)}{h(r)} \quad \text{and} \quad \lambda_2(r) = \frac{1}{(n-1)}\frac{h''(r)}{h(r)} - \left(\frac{n-2}{n-1}\right)\frac{\rho - h'(r)^2}{h(r)^2}.$$

In terms of the functions λ_1 and λ_2, assume that the function h satisfies the following conditions:

($\mathbf{\Lambda 1}$) Either we have $\lambda_1 > 0$ or $\lambda_2 > 0$ at some point.

($\mathbf{\Lambda 2}$) Assume that there exists a continuous monotone decreasing nonnegative function $\lambda :$ $[0, \infty) \to \mathbb{R}$ such that, for all $r \in [0, \infty)$, we have $\lambda(r) \geq \max\{\lambda_1(r), \lambda_2(r)\}$ and

$$b_0 \doteq \int_0^\infty t\lambda(t)dt < \infty. \tag{1.12}$$

($\mathbf{\Lambda 3}$) The function $[0, \infty) \ni r \mapsto \frac{h(r)}{r}$ is eventually nonincreasing.

($\mathbf{\Lambda 4}$) There exists $\tau_0 \geq 0$ such that $h'(r) \geq 0$ for all $r \in [\tau_0, \infty)$ and $\lim_{r\to\infty} h(r) = \infty$.

We now consider the product $M \doteq [0, \infty) \times N$ endowed with the Riemannian metric

$$g \doteq dr \otimes dr + h(r)^2 g_N. \tag{1.13}$$

Our main application of Theorem 1.3.2 is the following.

Corollary 1.3.4. [*Application to warped product spaces*]
Let (M, g) be a warped product manifold satisfying conditions ($\mathbf{\Lambda 1}$)—($\mathbf{\Lambda 3}$). Then, $\Gamma_{\boldsymbol{o}} = \{0\} \times N$ is a compact hypersurface whose area satisfies

$$\left|\Gamma_{\boldsymbol{o}}\right| \geq \frac{AVR^*(g)|\mathbb{S}^{n-1}|}{\left(e^{b_0}\left(\left|\frac{h'(0)}{h(0)}\right|(1+b_0)+b_1\right)\right)^{n-1}}. \tag{1.14}$$

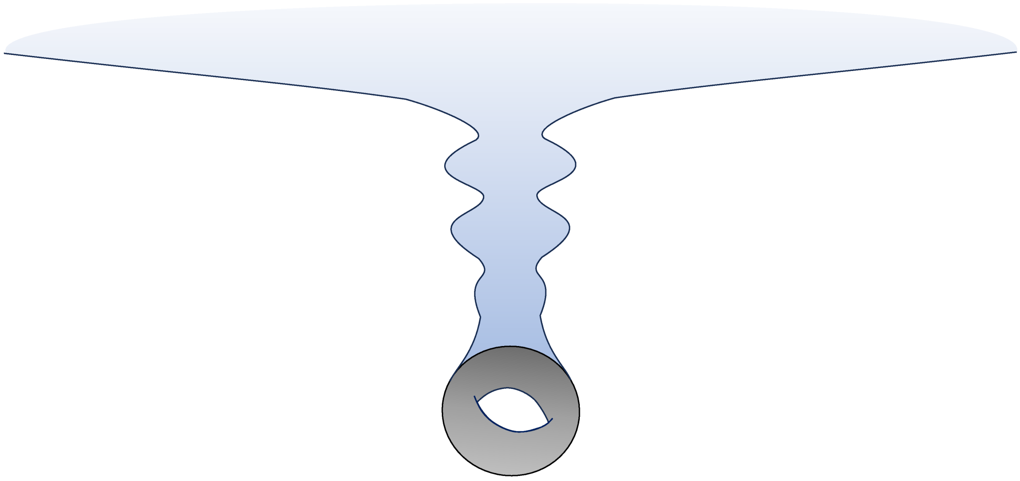
Let $\Sigma \subset M$ be a connected and closed hypersurface with mean curvature H*. If Σ is homologous to $\Gamma_{\boldsymbol{o}}$ then*

$$e^{(n-1)b_0} \int_\Sigma \left(\left|\frac{\mathrm{H}(x)}{n-1}\right|(1+b_0)+b_1\right)^{n-1} d\sigma(x) \geq AVR^*(g)|\mathbb{S}^{n-1}|. \tag{1.15}$$

Equality in (1.15) holds iff Σ is a slice $\{r_0\} \times N$, h is the solution to $h''(r) - \lambda(r)h(r) = 0$ over $[r_0, \infty)$, $h(r_0) = 1$, $h'(r_0) = \mathrm{H}/(n-1)$ and $\lim_{r\to\infty} h(r)/\left[e^{b_0}\left(h'(r_0)(1+b_0)+b_1\right)r\right] = 1$.

If hypothesis ($\mathbf{\Lambda 1}$) does not hold then (M, g) has $\mathrm{Ric}_M \geq 0$. Then, hypothesis ($\mathbf{\Lambda 1}$) suggests that the ambient does not have, possibly, nonnegative Ricci curvature. For instance, if (N, g_N) is Einstein, meaning that $\mathrm{Ric}_N = (n-2)\rho g_N$ then (M, g) does not have nonnegative Ricci curvature. The hypothesis ($\mathbf{\Lambda 2}$) guarantees that (M, g) has asymptotically* nonnegative Ricci curvature relatively to $\Gamma_{\mathbf{0}}$. The condition ($\mathbf{\Lambda 3}$) is there to ensure finiteness of the asymptotic* volume ratio of g and, finally, ($\mathbf{\Lambda 4}$) guarantees that $AVR^*(g)$ is positive. See Lemma 3.4.1 for details. It is interesting to bear in mind that the base manifold N may have whatever topology (see Figure 1.4).

Figure 1.4: Fizzy doughnut dropping into clean water



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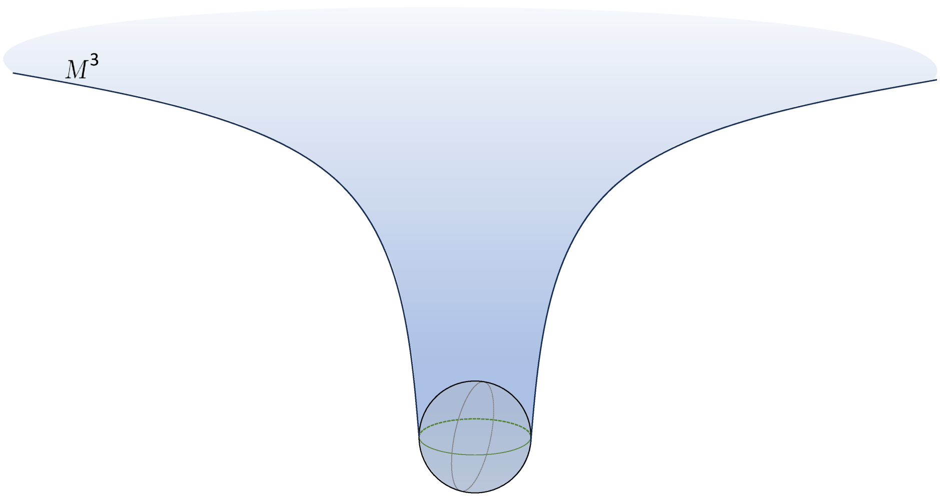
We point out that the Schwarzschild and the Reissner-Nordstrom manifolds meet the hypothesis of Corollary 1.3.4. We briefly recall the definition of these spaces here. Let $m > 0$ be a real number and write $\{s > 0 : 1 - ms^{2-n}\} = (\underline{s}, \infty)$. The Schwarzschild manifold (see Figure 1.5) is defined by

$$M_S = (\underline{s}, \infty) \times \mathbb{S}^{n-1}, \;\; g_S = \frac{1}{1 - ms^{2-n}} ds \otimes ds + s^2 g_{\mathbb{S}^{n-1}}. \tag{1.16}$$

Figure 1.5: The one-ended, three dimensional Schwarzschild space $(\underline{s}, \infty) \times \mathbb{S}^2$



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In Section 3.4 we discuss in depth this example: we show how to apply the inequality (1.14) to the horizon of the Schwarzschild space. The Reissner-Nordstrom manifold is defined by

$$M_R = (\underline{s}, \infty) \times \mathbb{S}^{n-1}, \;\; g_R = \frac{1}{1 - ms^{2-n} + q^2 s^{4-2n}} ds \otimes ds + s^2 g_{\mathbb{S}^{n-1}}, \tag{1.17}$$

where $m > 2q > 0$ are constants and $\underline{s}$ is defined as the larger of the 2 solutions of the equation $1 - ms^{2-n} + q^2 s^{4-2n} = 0$.

1.4 Willmore-type inequality for nonnegative Bakry-Émery Ricci tensor

As pointed out earlier, in this work we also obtain a Willmore-type inequality for smooth metric measure spaces. Formally, a smooth metric measure space is a triple $(M, g, e^{-f}\, dvol_g)$, where (M, g) is a complete Riemannian manifold together with a weighted volume form $e^{-f}\, dvol_g$, where f is a smooth function on M, sometimes called the potential function.

Weighted volume measures arise naturally from the study of conformal deformations of a Riemannian metric. Normally, given two conformally related Riemannian metrics, it is desirable to establish geometric and topological results relatively to the new metric and using data associated to the original metric. To accomplish this, it is interesting to have a concept of curvature associated to a weighted Laplacian. Such a concept has been introduced by Bakry and Émery [4].

The ***N-Bakry-Émery Ricci tensor*** is defined by

$$\mathrm{Ric}_f^N \doteq \mathrm{Ric} + D^2 f - \frac{1}{N} df \otimes df, \;\; \text{for} \;\; N > 0,$$

and the ∞-***Bakry-Émery Ricci tensor*** is defined by

$$\mathrm{Ric}_f \doteq \mathrm{Ric} + D^2 f, \;\; \text{so that} \;\; \mathrm{Ric}_f^N \doteq \mathrm{Ric}_f - \frac{1}{N} df \otimes df.$$

The Bakry-Émery Ricci tensor gives an analogue of the Ricci tensor for a Riemannian manifold with smooth measure. Therefore, it is interesting to investigate what geometric and topological results for the Ricci tensor extend to the Bakry-Émery Ricci tensor.

The Bakry-Émery Ricci tensor has an extension for diffusion operators [4] and nonsmooth metric measure spaces. The study of lower bounds on the Bakry-Émery Ricci tensor leads to geometrical and topological information on gradient Ricci solitons [41] [42], which takes valuable part in the theory of Ricci flow. Moreover, for N positive integer, the equation $\mathrm{Ric}_f^N = \lambda g$ corresponds to a warped product Einstein metric on $M \times_{e^{-\frac{f}{N}}} F^N$, see Proposition 9.106 in [7].

There are many references in the literature which work with volume growth of geodesic balls in relation to the corresponding model, [52], [44], [5]. This technique has been extensively used to obtain numerous theorems in geometric analysis and it is hard to emphasize enough the importance of its applications. In particular, it is quite natural to ask if there is an analogue of the classical volume comparison theorem of Heintze-Karcher type for tubular neighborhoods of a submanifold (see e.g. [40], pp. 179–184).

The purpose of the present study is to investigate the existence and sharpness of a Willmore-type inequality for nonnegative Bakry-Émery Ricci curvature on noncompact smooth metric measure spaces. To this end, we define a corresponding notion of asymptotic volume ratio and work with the volume of a geodesic tube around a compact domain in these spaces. We point out that the mean curvature (Laplacian) comparison results for hypersurfaces obtained here are not new and that we highly base our study on [52] and the techniques in [49]. For example, in Chapter 4 we prove the following

Theorem 1.4.1. [*Willmore-type inequality for nonnegative N-Bakry-Émery*]
Let (M, g) be a noncompact, complete n-dimensional Riemannian manifold with nonnegative N-Bakry-Émery Ricci curvature. Let Ω be an open and bounded set with smooth boundary $\Sigma = \partial\Omega$, whose mean curvature is H. *Then*

$$\int_{\Sigma} \left| \frac{\mathrm{H}_f(x)}{n+N-1} \right|^{n+N-1} e^{-f(x)} d\sigma(x) \geq f\text{-}AVR(g)|\mathbb{S}^{n+N-1}| \tag{1.18}$$

where f-$AVR(g)$ is the f-asymptotic volume ratio of g and $\mathrm{H}_f = \mathrm{H} - \langle \xi, grad_g f \rangle$.

As a result of Theorem 4.3.1, there are no f-minimal hypersurfaces $\Sigma = \partial\Omega$ in complete and non-compact spaces with nonnegative N-Bakry-Émery Ricci curvature whose f-asymptotic volume ratio does not vanish. In Chapter 4 we obtain similar statements for spaces whose ∞-Bakry-Émery Ricci curvature is nonnegative under additional assumptions on both the derivatives of the potential f and the weighted mean curvature H_f.

Conventions

In the following development, we assume that all manifolds are smooth and connected. Moreover, when treating subspaces of a given manifold M we refer to an embedded submanifold simply as a manifold and we explicitly declare when a submanifold is immersed.

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The Proof of the *Willmore-type inequality in asymptotic nonnegatively curved spaces*, Theorems 1.3.1 and 1.3.2, developed in Chapter 3 are reproduced from the original publication in the Arxiv [45] under the rights retained by the author.

CHAPTER 2

BACKGROUND

In this chapter we briefly establish the basic notations and preliminary tools for the remaining of the text. This chapter is intended only to establish the sign conventions, symbols and possibly some motivation to the theme we will be talking about along this work. It is not intended to give a review of basic Riemannian geometry, but only as a quick foreword to the basic elements we will be using here. For more detailed introductory texts the reader may refer to consolidated references on the field such as [43], [33] and the references therein.

2.1 Metrics, connection and curvature

On a smooth manifold M there exists many Riemannian metrics, but once a Riemannian metric g has been selected, M has been given a substantial amount of rigidity ([18], pp. 153). It is this rigidity imposed by the metric that distinguishes the subject of geometry from topology (see [23], [22]).

Given a Riemannian manifold (M, g), there exists a unique *Levi-Civita* connection ∇ (see e.g. [20], pp. 53 – 55) which is the tool that enables one to define the curvature of g. In its whole configuration, the ***Riemann curvature tensor*** is a $\binom{3}{1}$-tensor field that associates to each pair of vector fields $X, Y \in \mathfrak{X}(M)$ a mapping

$$R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

where ∇ is the unique Levi-Civita connection of g and $[X, Y]$ is the commutator of X and Y (see [43], pp. 33 – 35). The ***sectional curvature*** (at a given point $w \in M$) is a function on the space of two-planes $\Pi \subset T_w M$ defined by

$$\sec_M(u,v) = \langle R(u,v)v, u\rangle, \ \ \{u,v\} \text{ an orthonormal basis of } \Pi.$$

The ***Ricci curvature*** of g, denoted by Ric (or Ric_M, when is important to specify the underlying manifold) is the symmetric $\binom{2}{0}$-tensor field defined as the trace of the curvature tensor R on its first and last indices. Thus for $X, Y \in \mathfrak{X}(M)$

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\left(Z \mapsto R(Z,X)Y\right).$$

A Riemannian metric g is said to be an ***Einstein metric*** if its Ricci curvature fulfils $\mathrm{Ric} = \kappa g$, for some constant κ.

2.2 Submanifolds

Let N be a submanifold of M. Let νN be the ***normal bundle*** of N and indicate by $S\nu N$ its ***unit normal bundle***, that is, those vectors in νN of norm $\mathbb{1}$. We have the orthogonal decomposition

$$TM\big|_N = TN \otimes \nu N. \tag{2.1}$$

The symmetric bilinear ***second fundamental form of*** N is the $\binom{2}{1}$-tensor field defined by

$$\alpha(X,Y) = (\nabla_X Y)^{\perp}, \quad X, Y \in \mathfrak{X}(N)$$

and the tensor field of self-adjoint endomorphisms A_ζ of N, with $\zeta \in \nu N$, defined by $\langle A_\zeta X, Y\rangle = \langle \alpha(X,Y), \zeta\rangle$, where $X, Y \in \mathfrak{X}(N)$, is called the ***shape operator of N*** with respect to ζ. The ***mean curvature vector*** $\vec{H}$ ***of N*** at x is the normal vector defined by

$$\vec{H}(x) \doteq \sum_{i=1}^{\dim(N)} \alpha(X_i, X_i) \tag{2.2}$$

in terms of an orthonormal basis $X_1, ..., X_{\dim(N)}$ of T_xN. The submanifold N is said to be ***minimal*** if $\vec{H} = 0$. For more details, refer to [17]. Now let $f \in C^\infty(M)$ be a function. This function induces, in a natural manner, the weighted measure $e^{-f} dvol_g$ and, as consequence, we may form the triple $(M, g, e^{-f} dvol_g)$ which is referred to as a ***smooth metric measure space***. Analogously, we have an induced metric measure space $(N, i^*g, e^{-f} dvol_N)$, where $i : N \to M$ is the inclusion. The ***weighted mean curvature vector*** N is defined by

$$\vec{H}_f \doteq \vec{H} + (\mathrm{grad}_g f)^{\perp}$$

The immersed submanifold N is called ***f-minimal*** if its weighted mean curvature vector $\vec{H}_f$ vanishes identically. For more details, refer to [16]. From now on we specialize to submanifolds with codimension 1, i.e. hypersurfaces. Let $\Sigma \subset M$ be a hypersurface and fix a unit and normal vector $\zeta \in \nu_x\Sigma$ and define, respectively, the ***mean curvature*** H and the ***weighted mean curvature*** H_f of Σ at x relatively to ζ by

$$\mathrm{H}(x) = -\sum_{k=1}^{n-1} \langle \nabla_{X_k} X_k, \zeta \rangle,$$
$$\mathrm{H}_f(x) = \mathrm{H}(x) - \langle \mathrm{grad}_g f, \zeta \rangle$$

in terms of an orthonormal basis $X_1, ..., X_{n-1}$ of $T_x\Sigma$. An immersed hypersurface Σ in M is called a ***f-minimal hypersurface*** if $\mathrm{H}(x) = \langle \mathrm{grad}_g f, \zeta \rangle$.

2.3 Cut locus and distance

If $W \subset M$ is open then a function $g : W \to \mathbb{R}$ is said to be a (local) ***distance function*** if $|\mathrm{grad}_g g|_g = 1$. Distance functions can be thought of potential functions for an important class of geodesic vector fields and are the main object of this work.

Let Σ be, once and for all, a fixed embedded and closed hypersurface in (M, g). Denote by w the variable in M and by the letter x the variable in Σ. We either take Σ to be null-homologous, in which case $\partial\Omega = \Sigma$, for some open and bounded set Ω, or we take it to be homologous to some connected hypersurface Γ_0 in which case we write $\partial\Omega = \Sigma \cup \Gamma_0$, for some open set Ω (see Figure 2.1). In the latter, we also assume that $M \setminus \Gamma_0 = M^+ \cup M^-$ where M^+ and M^- are disjoint open subsets with $\Sigma \subset M^+$. Either way, we consider the oriented distance

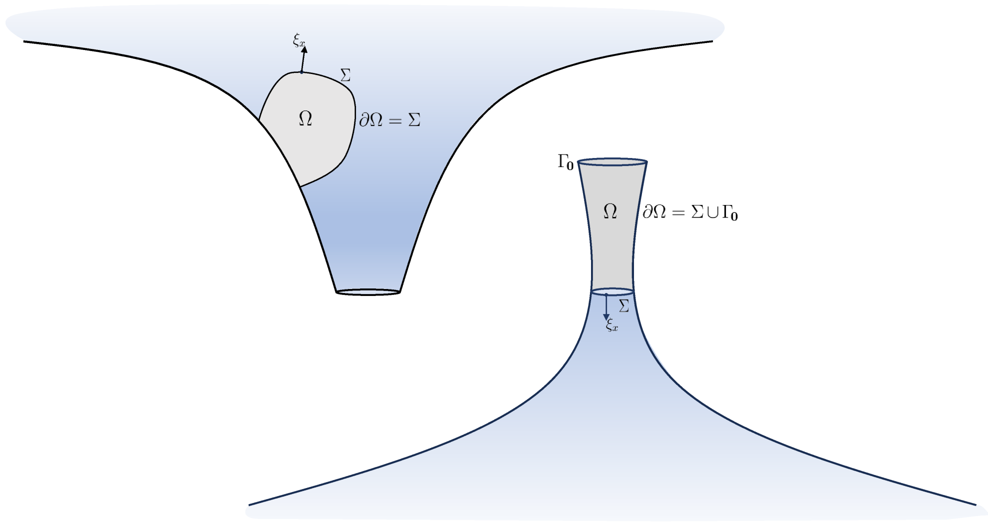
$$r(w) = \mathrm{dist}_g(w, \Sigma), \quad w \in M \setminus \Omega \quad (\text{or} \quad w \in M^+ \setminus \Omega). \tag{2.3}$$

Let $\xi : \Sigma \to TM$ be the outward-pointing (relatively to Ω — see Figure 2.1) unit, smooth and normal to Σ vector field. The ***normal exponential map*** is the restriction of the exponential map of M to $\nu\Sigma$ and it is denoted by $\exp^\perp$. Given $x \in \Sigma$, consider the geodesic $\gamma_{\xi_x}(t) = \exp_x t\xi_x$. Define the ***cut time of*** (x, ξ_x) by

$$\tau_c(\xi_x) \doteq \sup \{b > 0 : \; r(\gamma_{\xi_x}(b)) = b\} \in (0, \infty] \tag{2.4}$$

where $r = r(w)$ is the distance (2.3). We say that the geodesic γ_{ξ_x} realizes the distance in the interval $[0, \tau_c(\xi_x)]$. The ***cut locus of*** Σ, denoted by $C(\Sigma)$, is the image of $\{\tau_c(\zeta)\zeta : \zeta \in S\nu\Sigma \text{ and } \tau_c(\zeta) < \infty\}$ under the normal exponential map. As in the case where Σ is a point, the function $\tau_c : S\nu\Sigma \to \mathbb{R}$ is continuous. Since, for each $x \in \Sigma$, there is a unique unit outward-pointing normal vector ξ_x we can write $\tau_c(x)$ or $\tau_c(\xi_x)$ interchangeably and consider τ_c as a continuous function over Σ. Furthermore, the set $C(\Sigma)$ is closed and the normal exponential map

Figure 2.1: Hypersurfaces Σ belonging to different homology classes



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$$\exp^{\perp} : \{t\xi_x : x \in \Sigma \text{ and } 0 \leq t < \tau_c(x)\} \to \mathcal{U} \setminus C(\Sigma)$$

is a diffeomorphism, where $\mathcal{U} = M \backslash \Omega$ or $\mathcal{U} = M^+ \backslash \Omega$, depending on whether Σ is null-homologous or not. These assertions follow from standard arguments (see e.g. [43], pp. 139 – 141). As a result, there exists a diffeomorphism

$$\Phi : E \doteq \{(x, r) \in \Sigma \times [0, \infty) : r < \tau_c(x)\} \to \mathcal{U} \setminus C(\Sigma), \quad \Phi(x, r) = \exp_x r\xi_x. \tag{2.5}$$

We will routinely consider the pullback of the volume element in M induced by g to E via this diffeomorphism, that is, $(\Phi^* dvol_g)_{(x,r)} = \det J_{\xi_x}(r) d\sigma \wedge dr$, for $(x, r) \in E$. Opportunely, using Jacobi Fields, we will give a description of the coefficient function $\det J_\xi(t)$. For more details, refer to [6].

2.4 Jacobi tensor fields and the Riccati equation

Fix a geodesic $\gamma_\xi : I \to M$ with initial velocity $\xi (\in T_x\Sigma^\perp)$. A vector field J along the geodesic γ_ξ is called a **Jacobi Field** if it satisfies the Jacobi equation

$$D_t^2 J + R(J, \dot{\gamma}_\xi)\dot{\gamma}_\xi = 0,$$

where $D_t J$ is the usual covariant differentiation of J along γ_ξ. A normal Jacobi field J along γ_ξ is said to be ***transverse to*** Σ if *(i)* $J(0) \in T_x\Sigma$ and *(ii)* $D_t J(0) + A_\xi J(0) \in T_x\Sigma^\perp$ (see Figure 2.2). For $t > 0$, we say that $\gamma_\xi(t)$ is a ***focal point*** of Σ along the geodesic γ_ξ if there is a nontrivial transverse Jacobi field J along γ_ξ satisfying $J(t) = 0$. If there is such a $t > 0$, then we let $t_f(\xi)$ be the smallest such t and call $\gamma_\xi(t_f(\xi))$ the first focal point of Σ along γ_ξ. We have $\tau_c(\xi) \leq t_f(\xi)$, as in the case when Σ is a point. Notice that we may have $t_f(\xi) = \infty$. For more details on the matter refer to [20] (pp. 227 – 235).

Denote the space of vector fields along $\gamma = \gamma_\xi$ that are normal to the velocity field $\dot\gamma$ by $\mathfrak{X}^\perp(\gamma)$. A ***Jacobi tensor field*** $\mathbf{J}$ along γ is a smooth $\binom{1}{1}$-tensor field of endomorphisms $\mathfrak{X}^\perp(\gamma)^\perp \to \mathfrak{X}^\perp(\gamma)$ which satisfies the Jacobi equation

$$\mathbf{J}'' + R_{\dot\gamma}(\mathbf{J}) = 0$$

where $R_{\dot\gamma}$ is the endomorphism $X \mapsto R_{\dot\gamma}(X) = R(X, \dot\gamma)\dot\gamma$. Generally speaking, transverse Jacobi fields help us detect focal points of submanifolds, which correspond to the first moment where the determinant of certain Jacobi tensor fields vanish. The following construction sketches this situation and will be used throughout the text: let $(e_1, ..., e_{n-1}) \subset T_x\Sigma$ be an orthonormal basis consisting of eigenvectors of the shape operator A_ξ and let (E_α) be a parallel $(n-1)$ orthonormal frame along the geodesic γ such that $E_\alpha(0) = e_\alpha$. We then consider the operator $J_\xi = J_\xi(t)$ which takes the velocity $\dot\gamma$ to itself and maps $\mathcal{J}^\perp(\gamma)$ into $\mathcal{J}^\perp(\gamma)$ according to the rule (see Figure 2.2)

$$J_\xi(t)E_k(t) = J_k(t), \quad k = 1, ..., n-1 \tag{2.6}$$

where J_k is the Jacobi vector field with initial conditions $J_k(0) = e_k$ and $D_tJ_k(0) = \lambda_k e_k$, where the $\lambda_k's$ fulfill $A_\xi e_k = \lambda_k e_k$. Then, J_ξ is easily seen to be a Jacobi tensor field along γ. Moreover, the field of endomorphisms $U(t) \doteq J'_\xi J_\xi^{-1}(t)$, which is well defined over the interval $[0, t_f(\xi))$, satisfies $UJ_k = D_tJ_k$, $k = 1, ..., n-1$ and is a solution to the ***Riccati equation***

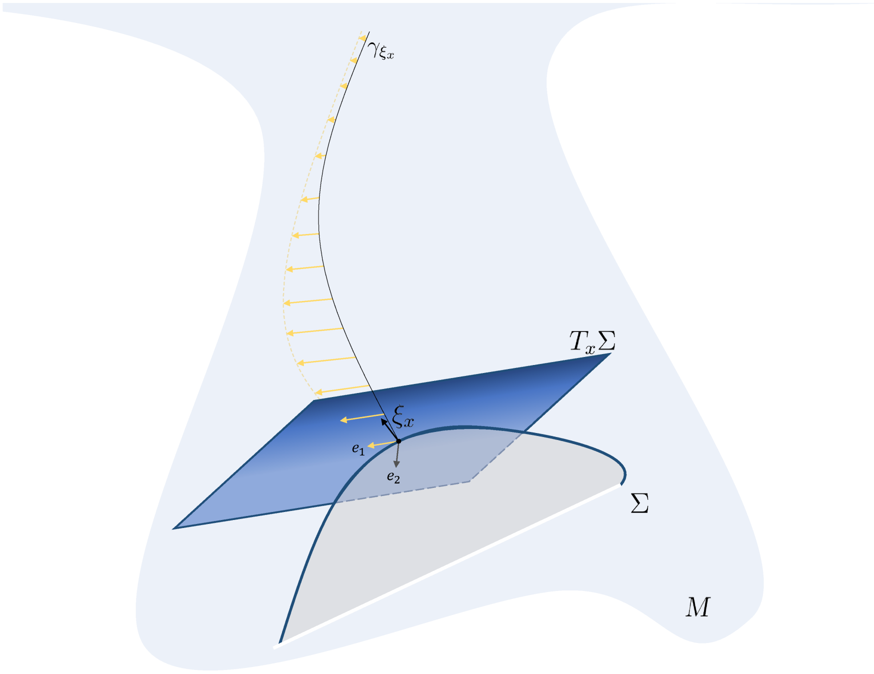
$$D_tU + U^2 + R_{\dot\gamma} = 0. \tag{2.7}$$

In this work, we will be frequently producing quantities relatively to the distance (2.3). One of the most important tensors associated to a function is its Hessian. In particular, it is well known that the Hessian operator of any local distance function satisfies the Riccati equation (2.7). It is easy to see that if $\mathcal{H}_r$ is the Hessian operator of (2.3), i.e., $D^2r(\cdot,\cdot) = \langle \mathcal{H}_r\cdot, \cdot\rangle$, where D^2r is the usual $\binom{2}{0}$ Hessian, then $\mathcal{H}_r = U$ along the integral curves of $\mathrm{grad}_g r$. Observe that the mean curvature is $\mathrm{H}(x) \doteq \mathrm{tr}\, U(0)$, or in other words, $\mathrm{H}(x) = \Delta r(0)$. By (2.2), we have

$$\vec{H} = -\mathrm{tr}\, U(0)\partial_r = -\mathrm{H}\partial_r$$

We say that Σ has ***mean convex boundary*** (MCB) if the mean curvature vector $\vec{H}$ is an inward pointing normal. The following example is a particular case of Example 4.3 in [6].

Figure 2.2: The geometry of a transverse Jacobi field



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Example 2.4.1. If $U = J'_\xi J_\xi^{-1}$, which is defined as long as $\det J_\xi > 0$, then set

$$u(t) \doteq \frac{1}{n-1} \operatorname{tr} U(t). \tag{2.8}$$

Since U satisfies (2.7) and covariant differentiation commutes with contractions, we have

$$\begin{aligned} u' &= \frac{1}{n-1} \operatorname{tr} D_t U \\ &= -\frac{1}{n-1} \operatorname{tr} U^2 - \frac{1}{n-1} \operatorname{tr} R_{\dot\gamma} \\ &\leq -\frac{1}{(n-1)^2} \big(\operatorname{tr} U\big)^2 - \frac{1}{n-1} \operatorname{tr} R_{\dot\gamma} \\ &= -u^2 - \frac{1}{n-1} \operatorname{Ric}(\dot\gamma, \dot\gamma), \end{aligned}$$

where we have used Schwarz inequality $(\operatorname{tr} U)^2 \leq (n-1)\operatorname{tr}(U^2)$. Now, if $\operatorname{Ric}(\dot\gamma, \dot\gamma) \geq (n-1)\kappa$, where $\kappa(t)$ is a smooth function defined on the domain of definition of U then

$$u'(t) \leq -u(t)^2 - \kappa(t). \tag{2.9}$$

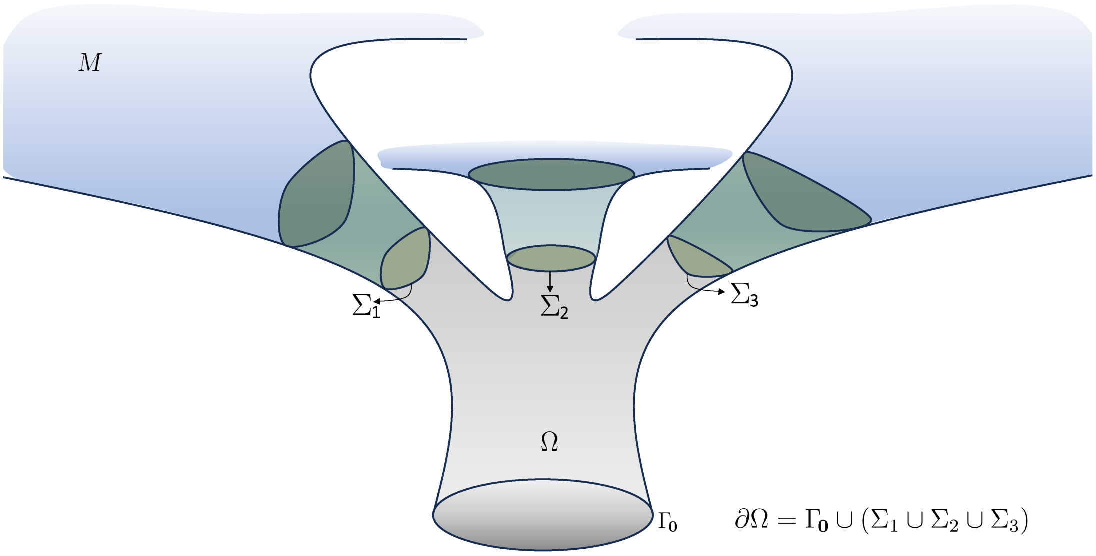
Equality in (2.9) holds if and only if $U = u\mathcal{P}_{\dot\gamma}$ and $\operatorname{Ric}(\dot\gamma, \dot\gamma) = (n-1)\kappa$ along γ, where $\mathcal{P}_{\dot\gamma}$ is the orthogonal projection onto $(\dot\gamma)^\perp$. Since U satisfies the Riccati equation, equality in (2.9) holds if and only if $U = u\mathcal{P}_{\dot\gamma}$ and $R_{\dot\gamma} = \kappa \mathcal{P}_{\dot\gamma}$. $\square$

2.5 Tubular neighborhoods and volume

In this essay we will frequently work with *volumen* of oriented tubular neighborhoods of hypersurfaces. Oriented meaning that a side is chosen, which we assume, by convention, to be the one given by the vector field ξ. That being said, if we denote this oriented tubular neighborhood by $\mathcal{T}^{+}_{\Sigma}(r)$, which is the set $\{w \in \mathcal{U} : \mathrm{dist}_g(w, \Sigma) < r\}$, then

$$vol(\mathcal{T}^{+}_{\Sigma}(r)) = \int_{\Sigma} \int_{0}^{r \wedge \tau_c(x)} \det J_{\xi_x}(r) dr d\sigma(x), \tag{2.10}$$

Figure 2.3: The oriented tube $\mathcal{T}^{+}_{\Sigma}(r)$ in a three-ended manifold with boundary $\Gamma_{\mathbf{0}}$



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where $r \wedge \tau_c(x)$ is the minimum between r and $\tau_c(x)$. A word about the asymptotic volume ratio is in order. Given a compact domain D, for example, in a Riemannian manifold (M, g) with asymptotically nonnegative curvature (Ricci or sectional) with base point p_0, denote the geodesic tube of radius r around D by $\mathcal{T}_D(r)$ so that $vol(\mathcal{T}_D(r)) = |D| + vol(\mathcal{T}^{+}_{\partial D}(r))$. Clearly, if $p_0 \in D$ we have $\Theta(r) \leq \frac{vol(\mathcal{T}_D(r))}{\omega_n r^n}$, for all $r > 0$, so that

$$AVR(g) \leq \lim_{r \to \infty} \frac{vol(\mathcal{T}_D(r))}{\omega_n r^n}. \tag{2.11}$$

It is not difficult to see that if $p_0 \notin D$ then, eventually, for sufficiently big radius r the set $B_{p_0}(r) \setminus \mathcal{T}_D(r)$ has small volume in comparison with r^n and, therefore, (2.11) also holds in this case.

CHAPTER 3

SPACES WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE

In this chapter we establish a Willmore-type inequality for asymptotic nonnegatively curved Riemannian manifolds. This chapter is organized as follows. In the next section we show some elementary inequalities that will play a pivotal role in this work. These inequalities are based on ODE methods that go back to Sturm, but which are quite trivial whence no reference is given. The proof of the main theorem, Theorem 1.3.1, occupies Section 3.2 and it is divided into two steps: the main inequality and the discussion of the rigidity statement. In Section 3.3, we develop the adaptation of the proof of the main theorem to the class of asymptotic* nonnegatively curved spaces, namely Theorem 1.3.2. Next, Section 3.4 is reserved to discuss how Corollary 1.3.4 may be obtained from Theorem 1.3.2, we postpone the converse of Theorems 1.3.1 and 1.3.2 to this section and give concrete examples of our inequality.

3.1 Elementary inequalities

Throughout this section we assume that λ is nontrivial, i.e., $\lambda \neq 0$ and we fix a point $x \in \Sigma$ and a unit speed geodesic γ_ξ that starts at x and realizes the distance (2.3), that is, $r(\gamma_\xi(t)) = t$ for t in some interval $[0, \alpha)$, where $\alpha > 0$. Hence, all quantities considered here depend only on the variable t and the point x is momentarily ignored. Also, from now on, by abuse of notation, the function λ is meant to be $(\lambda \circ \mathrm{dist}_{p_0} \circ \gamma_\xi)(t)$. We observe that if $H(x) \neq 0$ then all comparisons proved in this section are easier.

Let $J_\xi(t)$ be the Jacobi tensor field defined in (2.6) along the geodesic γ_ξ which, from now on, we denote simply by γ. Consider the function $\mathscr{J} : [0, t_f(\xi)) \to \mathbb{R}$ defined by

$$\mathscr{J}(t) \doteq \left(\det J_\xi(t)\right)^{1/(n-1)} \tag{3.1}$$

and compare it with 4.B.4 in [24]. Note that we may always arrange the basis $(e_1, ..., e_{n-1})$ of $T_x\Sigma$ so that $J_\xi(0) =$ the identity of $T_x\Sigma$.

Lemma 3.1.1. *Assume* $\mathrm{Ric}(\dot\gamma(t), \dot\gamma(t)) \geq -(n-1)(\lambda \circ \mathit{dist}_{p_0})(\gamma(t))$ *for all* t *in the domain of definition of* γ *and note that* $\det J_\xi > 0$ *on* $[0, t_f(\xi))$*. Then, the function* $\mathscr{J}$ *defined in (3.1)* *satisfies*

$$\begin{cases} \mathscr{J}'' - \lambda \mathscr{J} \leq 0; \\ \mathscr{J}(0) = 1, \quad \mathscr{J}'(0) = \mathrm{H}/(n-1). \end{cases} \tag{3.2}$$

where $\mathrm{H} = \mathrm{H}(x)$ *is the mean curvature of* Σ *at* x*. In addition to that, if* $U \doteq J'_\xi J_\xi^{-1}$ *then the equality* $\mathscr{J}'' - \lambda \mathscr{J} = 0$ *holds if and only if* $U = \frac{trU}{n-1}\mathcal{P}_{\dot\gamma}$ *and* $R_{\dot\gamma} = -\lambda \mathcal{P}_{\dot\gamma}$ *along* γ*, where* $\mathcal{P}_{\dot\gamma}$ *is the orthogonal projection onto the orthogonal complement of the velocity field* $\dot\gamma$*.*

Proof. Write $\mathscr{J}(t) = e^{\frac{1}{n-1}\log\det J_\xi(t)}$ and differentiate $\mathscr{J}$ to get

$$\begin{aligned} \mathscr{J}' &= \frac{1}{n-1}\left(\frac{d}{dt}\log\det J_\xi\right)\mathscr{J} \\ &= \frac{1}{n-1}(\operatorname{tr} U)\mathscr{J}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathscr{J}'' &= \left(-\frac{1}{n-1}\operatorname{tr} U^2 - \frac{1}{n-1}\operatorname{tr} R_{\dot\gamma}\right)\mathscr{J} + \frac{1}{n-1}(\operatorname{tr} U)\mathscr{J}' \\ &\leq -\frac{1}{(n-1)^2}(\operatorname{tr} U)^2\mathscr{J} - \frac{1}{n-1}\operatorname{Ric}(\dot\gamma,\dot\gamma)\mathscr{J} + \frac{1}{(n-1)^2}(\operatorname{tr} U)^2\mathscr{J} \\ &\leq (\lambda \circ \operatorname{dist}_{p_0})(\gamma)\mathscr{J}. \end{aligned}$$

The last statement follows from the Example 2.4.1. The initial conditions are clear. □

In order to avoid a notational inconvenience, we define the letter h to be

$$h \doteq \frac{\mathrm{H}}{n-1}. \tag{$\star$}$$

Lemma 3.1.2. *Let* y *and* j *be, respectively, the unique solutions to the following initial value problems (IVP)*

$$\begin{cases} y''(t) - \lambda(t)y(t) = 0, \\ y(0) = 1, \ \ y'(0) = |h|; \end{cases} \qquad \begin{cases} j''(t) = 0, \\ j(0) = 1, \quad j'(0) = |h|. \end{cases}$$

Then, $j \leq y$ *on* $[0,\infty)$*. Moreover, if* $\mathscr{J}$ *satisfies the conditions in (3.2) and* $\mathrm{H} = trU(0)$*, we have*

$$\mathscr{J} \leq y, \quad on \quad [0, t_f(\xi)).$$

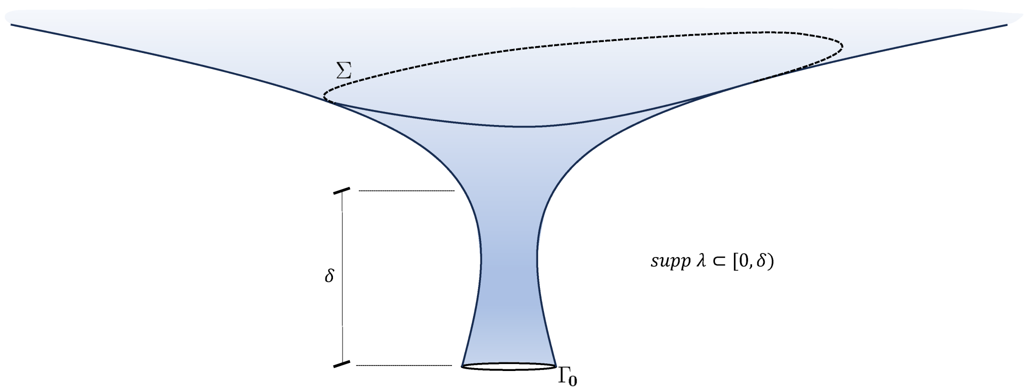
Proof. First, observe that y is a positive function. In fact, suppose there is a first time $T > 0$ for which $y(T) = 0$. Then, $y''(t) = \lambda(t)y(t) \geq 0$ on $[0, T]$ so that $y'(t) = |h| + \int_0^t \lambda(s)y(s)ds \geq 0$ and y is nondecreasing on $[0, T)$ which leads to a contradiction. This line of reasoning shows that $y'(t) = |h| + \int_0^t \lambda y \geq |h|$ whence $y(t) - 1 \geq t|h|$. This proves that $j \leq y$. To see the second statement, differentiate the quotient $\mathscr{J}/y$ to get

$$\left(\frac{\mathscr{J}(t)}{y(t)}\right)' = \frac{\mathscr{J}'(t)y(t) - \mathscr{J}(t)y'(t)}{y(t)^2} \leq 0,$$

since $\left(\mathscr{J}'y - \mathscr{J}y'\right)'(t) = \mathscr{J}''(t)y(t) - \mathscr{J}(t)y''(t) \leq 0$ and $(\mathscr{J}'y - \mathscr{J}y')(0) \leq 0$. Thus, $t \mapsto \mathscr{J}(t)/y(t)$ is nonincreasing and the fact that $(\mathscr{J}/y)(0) = 1$ yields $\mathscr{J} \leq y$. □

Remark 3.1.3. Here we bring the attention to the function $(\lambda \circ \mathrm{dist}_{p_0} \circ \gamma)$. If the fixed geodesic γ is getting far away from the base point p_0 then it may be the case that $(\mathrm{dist}_{p_0} \circ \gamma)$ is contained in a region where the associated function λ vanishes. For example, if the function λ has compact support and Ω is a geodesic ball centered at p_0 with radius sufficiently large so that $\mathrm{dist}_{p_0} \circ \gamma(t) \subset \mathbb{R} \setminus \mathrm{supp}(\lambda)$, for all t in the domain of γ, then all the comparisons in this sections are meaningless (see Figure 3.1). There are other, more general, situations in which this problem arises. We overcome this problem by assuming either that $\lambda(t) \neq 0$, for all $t \in \mathbb{R}^+$, or that Σ is contained in a region where $(\lambda \circ \mathrm{dist}_{p_0})(x)$ never vanishes, for all $x \in \Sigma$. The more general situations can be handled as in our proofs with mild modifications.

Figure 3.1: It is possible to choose a hypersurface "outside" the support of λ



Source: Created by the author.

Proposition 3.1.4. *Let y be the unique solution to the IVP in Lemma 3.1.2. Then*

$$\frac{y}{|h| + \int_0^t y\lambda} \leq t + \frac{1}{|h| + \int_0^t y\lambda}, \tag{3.3}$$

holds only for $t > 0$. Multiplying inequality (3.3) by λ, integrating on the interval $[\epsilon, t]$, for $0 < \epsilon < t$, and using the inequality $y \geq |h|t + 1$ from the previous lemma, we get

$$log\left(|h| + \int_0^t \lambda(s)y(s)ds\right) \leq \int_\epsilon^t s\lambda(s)ds + \int_\epsilon^t \frac{\lambda(s)ds}{|h| + \int_0^s \lambda(u)(|h|u+1)du} + log\left(|h| + \int_0^\epsilon \lambda(s)y(s)ds\right) \tag{3.4}$$

Proof. To prove the statements, we proceed as in [56]. Integrating the equation $y'' - \lambda y = 0$ we get

$$y'(t) = |h| + \int_0^t \lambda(s)y(s)ds, \tag{3.5}$$

which holds for all $t \geq 0$ so, integrating once again in $[0, t]$ with $t > 0$ and using that $\lambda \neq 0$, we get

$$\begin{aligned} y(t) - y(0) &= t|h| + \int_0^t \int_0^u \lambda(u)y(u)duds \\ &= t|h| + \int_0^t \left(\int_u^t \lambda(u)y(u)ds\right)du \\ &= t|h| + \int_0^t (t-u)\lambda(u)y(u)du \\ &\leq t|h| + t\int_0^t \lambda(u)y(u)du \end{aligned} \tag{3.6}$$

which gives

$$\frac{y(t) - 1}{|h| + \int_0^t \lambda(s)y(s)ds} \leq t$$

which is equivalent to (3.3). Next, multiplying it by λ we get

$$\frac{\lambda y}{|h| + \int_0^t y\lambda} \leq t\lambda + \frac{\lambda}{|h| + \int_0^t y\lambda}. \tag{3.7}$$

Now, from Lemma 3.1.2, we have $t|h| + 1 \leq y$ so that

$$\frac{\lambda(t)}{|h| + \int_0^t \lambda(s)y(s)ds} \leq \frac{\lambda(t)}{|h| + \int_0^t \lambda(s)(|h|s+1)ds}.$$

Thus, integrating (3.7) on $[\epsilon, t]$, for $0 < \epsilon < t$, and using the above inequality gives (3.4). □

Proposition 3.1.5. *Let y be the unique solution to the IVP in Lemma 3.1.2. Then*

$$y(t) \leq \left(e^{b_0}ft + 1\right), \quad for \ \ 0 \leq t < \infty \tag{3.8}$$

where $f = |h|(1 + b_0) + b_1$, and b_0 and b_1 are defined by (1.1) and (1.2) respectively.

Proof. Let $\epsilon > 0$ be a real number (a fixed parameter), and define an auxiliary function $\rho_\epsilon : [\epsilon, \infty) \to \mathbb{R}$ by

$$\rho_\epsilon(t) = \int_\epsilon^t s\lambda(s)ds + \int_\epsilon^t \frac{\lambda(s)ds}{|h| + \int_0^s \lambda(u)(|h|u+1)du} + \log\Big(|h| + \int_0^\epsilon \lambda(s)y(s)ds\Big).$$

We now distinguish two cases.

Case 1. — Assume first that $h \neq 0$. In this case we may choose the parameter to be $\epsilon = 0$. The heart of the matter is that we need to better control the second term on the right hand side of (3.4). Considering that term, integration by parts show us that

$$\begin{aligned}\int_0^t \left(\frac{1}{1+s|h|}\right)\frac{(1+s|h|)\lambda(s)}{|h| + \int_0^s \lambda(u)(|h|u+1)du}ds &= \frac{1}{1+t|h|}\log\left(|h| + \int_0^t \lambda(s)(|h|s+1)ds\right)\\ &- \log|h| - \int_0^t \frac{-|h|}{(1+s|h|)^2}\log\left(|h| + \int_0^s \lambda(u)(|h|u+1)du\right)ds\\ &\leq \frac{1}{1+t|h|}\log\left(|h| + \int_0^t \lambda(s)(|h|s+1)ds\right)\\ &- \log|h| + |h|\log f\int_0^t \frac{1}{(1+s|h|)^2}ds.\end{aligned}$$

Observe that $\rho_0'(t) \geq 0$ so that ρ_0 is nondecreasing and note that ρ_0 is bounded above by $b_0 + \log f$. Hence, by (3.5) and analysing inequality (3.4) with $\epsilon = 0$ and $h \neq 0$ we conclude that

$$y'(t) \leq e^{\rho_0(t)} \leq e^{b_0} f$$

holds for all $t \geq 0$.

Case 2. — We now suppose that $h = 0$. In this case, the function ρ_ϵ becomes

$$\rho_\epsilon(t) = \int_0^t s\lambda(s)ds + \log\left(\int_0^t \lambda(s)ds\right) - \log\left(\int_0^\epsilon \lambda(s)ds\right) + \log\left(\int_0^\epsilon \lambda(s)y(s)ds\right) - \int_0^\epsilon s\lambda(s)ds. \tag{3.9}$$

Again, $\rho_\epsilon' \geq 0$ and ρ_ϵ is nondecreasing. In addition, letting $t \to \infty$ we see from (3.9) that ρ_ϵ satisfies

$$\rho_\epsilon(t) \leq b_0 + \log b_1 + \log\left(\frac{\int_0^\epsilon \lambda(s)y(s)ds}{\int_0^\epsilon \lambda(s)ds}\right) - \int_0^\epsilon s\lambda(s)ds.$$

Therefore, by (3.5), after applying the exponential function, we may rewrite inequality (3.4) as

$$y'(t) \leq e^{\rho_\epsilon(t)} \leq \left(e^{b_0} b_1 \frac{1}{e^{\int_0^\epsilon s\lambda(s)ds}}\right) \frac{\displaystyle\int_0^\epsilon \lambda(s)y(s)ds}{\displaystyle\int_0^\epsilon \lambda(s)ds}, \tag{3.10}$$

which holds only for $t > 0$. Since the LHS of (3.10) does not depend on ϵ and it holds for all $\epsilon > 0$ small, we may take the limit on the RHS, if it exists, when $\epsilon \to 0$. But, the quotient of integrals above clearly satisfies

$$\lim_{\epsilon \to 0^+} \frac{1}{\displaystyle\int_0^\epsilon \lambda} \int_0^\epsilon \lambda y = \lim_{\epsilon \to 0^+} y(\epsilon) = 1.$$

Thus, letting $\epsilon \to 0$ in (3.10) gives $y' \leq e^{b_0} b_1$. As a result, regardless of the signal of h we have $y' \leq e^{b_0} f$. Integrating this differential inequality gives (3.8). $\square$

Remark 3.1.6. In view of Proposition 3.1.5, it is natural to wonder about the asymptotic behavior of $t \mapsto y(t)/(e^{b_0} f t + 1)$. As we will see later, it will play a crucial role in understanding the connection between the decay of λ and the rigidity statement. To understand this issue, we introduce the notation $\mathbb{X}(t) = (e^{b_0} f t + 1)$. By equations (3.6) and (3.5) we have

$$\begin{aligned}\left(\frac{y(t)}{\mathbb{X}(t)}\right)' &= \frac{e^{b_0} f}{\mathbb{X}(t)^2}\left[t\left(|h| + \int_0^t \lambda(s)y(s)ds\right) - y(t)\right] + \frac{1}{\mathbb{X}(t)^2}\left[|h| + \int_0^t \lambda(s)y(s)ds\right] \\ &= \frac{e^{b_0} f}{\mathbb{X}(t)^2}\left[-1 + \int_0^t u\lambda(u)y(u)du\right] + \frac{1}{\mathbb{X}(t)^2}\left[|h| + \int_0^t \lambda(s)y(s)ds\right].\end{aligned}$$

So, on the one hand, the signal of the above derivative depends on the factor $-1+\int_0^t u\lambda(u)y(u)du$. On the other hand, it depends on the numerator $y'\mathbb{X} - \mathbb{X}'y$ which is increasing as long as λ does not vanish since

$$\big(y'(t)\mathbb{X}(t) - \mathbb{X}'(t)y(t)\big)' = \lambda(t)y(t)\mathbb{X}(t) \geq 0.$$

In particular,

$$\left.\frac{d}{dt}\right|_{t=0}\left(\frac{y(t)}{\mathbb{X}(t)}\right) = |h| - e^{b_0} f < 0. \tag{3.11}$$

3.2 A sharp geometric inequality, *1st* case

In this section we give the proof of the Theorem 1.3.1. First we slightly deviate from the course of the proof to establish some notations and terminology.

Digression 3.2.1. We now look at the collective behavior of geodesics that start at the boundary $\Sigma = \partial\Omega$ and realize the distance $r(w) = \mathrm{dist}_g(w, \Sigma)$, for $w \in M \setminus \Omega$ to gather all the information developed so far into single maps. For each $x \in \Sigma$, define

$$\mathbb{X}_x(t) = e^{b_0} f(x)t + 1, \quad t \in \mathbb{R}, \tag{3.12}$$

where $f : \Sigma \to \mathbb{R}$ is the function $f(x) = |h|(x)(1 + b_0) + b_1$ and b_0 and b_1 are defined by (1.1) and (1.2) respectively. Remind that $h(x) = \mathrm{H}(x)/(n-1)$ as defined in ($\star$), where $\mathrm{H}(x)$ is the mean curvature of Σ at x. Recall that

$$\mathscr{J}_x(t) = \det J_{\xi(x)}(t)^{\frac{1}{n-1}}, \quad t \in [0, t_f(\xi_x)),$$

as defined in (3.1), where $\xi(x) = \xi_x$ is the outward, unit and normal vector at $x \in \Sigma$ and $t_f(\xi_x)$ is the first time the geodesic γ_{ξ_x} hits a focal point. We sometimes consider parameters as variables and write, e.g., $\mathbb{X}(x,t)$ instead of $\mathbb{X}_x(t)$ or $\mathscr{J}(x,t)$ in place of $\mathscr{J}_x(t)$. By Proposition 3.1.5 and Lemma 3.1.2 we have $\mathscr{J}_x^{n-1}(t) \leq \mathbb{X}_x^{n-1}(t)$, $t \in [0, t_f(\xi_x))$. Next, define the function $\hat{\theta} : \Sigma \times [0, \infty) \to \mathbb{R}$ by

$$\hat{\theta}(x,t) = \left(\frac{\mathscr{J}_x(t)}{\mathbb{X}_x(t)}\right)^{n-1}. \tag{3.13}$$

For each $x \in \Sigma$ the quantity $\hat{\theta}(x, \cdot)$ is clearly positive on $[0, t_f(\xi_x))$. Also, if y_x denotes the unique solution to $y_x''(t) - \lambda(t)y_x(t) = 0$, with suitable initial conditions, where $\lambda(t)$ is implicitly understood to be the function $\lambda(\mathrm{dist}_{p_0}(\gamma_{\xi_x}(t)))$, then by Proposition 3.1.5 we have

$$\hat{\theta}(x,t)^{\frac{1}{n-1}} \leq \frac{\mathscr{J}_x(t)}{y_x(t)}, \quad x \in \Sigma, \quad t \in [0, t_f(\xi_x)), \tag{3.14}$$

where $t \mapsto \mathscr{J}_x(t)/y_x(t)$ is monotone decreasing by Lemma 3.1.2. Moreover, for any $0 \leq a < b \leq \infty$ such that $b \leq \tau_c(x)$, where $\tau_c(x)$ is the cut time defined in (2.4), we shall consider the supremum of $\hat{\theta}(x, \cdot)$ for $t \in [a, b)$. That supremum is denoted by

$$\sup_{[a,b)} \hat{\theta}(x) \doteq \sup_{t \in [a,b)} \hat{\theta}(x,t) \leq 1, \quad x \in \Sigma.$$

Finally we define the function whose asymptotic behavior is key for this work. Let $\theta : \Sigma \times [0, \infty) \to \mathbb{R}$ be defined by

$$\theta(x,t)^{\frac{1}{n-1}} = \frac{y_x(t)}{\mathbb{X}_x(t)}.$$

In Remark 3.1.6 we analysed the first derivative of $\theta_x(t) = \theta(x,t)$. In particular, if $\lambda \neq 0$ then by (3.11) we have

$$\frac{\partial}{\partial t}\bigg|_{t=0} \theta(x,t)^{\frac{1}{n-1}} < 0, \tag{3.15}$$

so that for all $x \in \Sigma$, the function $t \mapsto \theta_x(t)^{\frac{1}{n-1}}$ is decreasing on a neighborhood of zero (Figure 3.2 describes the essentially three possibilities for the graph of θ, all of which fulfils $\theta(0) = 1$). $\square$

Next, we proceed with the proof of Theorem 1.3.1, which is divided into two steps. To prove inequality (1.4) we may assume, without loss of generality, that Ω has no hole, i.e., $M \setminus \Omega$ has no bounded component.

Proof of inequality (1.4). Suppose first that $\lambda = 0$. Then, by hypothesis, (M, g) has nonnegative Ricci curvature and, therefore, the argument in [49] goes through without modification to conclude inequality (1.4). We now assume that $\lambda \neq 0$. Thus, all the comparison arguments in Section 3.1 hold.

Denote by $\mathcal{T}_\Omega(R)$ the geodesic tube of radius R about Ω. Then, for any $R > 0$, we have

$$\begin{aligned}
vol(\mathcal{T}_\Omega(R)) &= |\Omega| + \int_\Sigma \int_0^{R\wedge\tau_c(x)} \det J_{\xi(x)}(t)\,dt\,d\sigma(x) \\
&\leq |\Omega| + \int_\Sigma \int_0^{R\wedge\tau_c(x)} \mathbb{X}(x,t)^{n-1}\,dt\,d\sigma(x) \\
&\leq |\Omega| + \int_\Sigma \int_0^{R} \left(e^{b_0} f(x)t + 1\right)^{n-1} dt\,d\sigma(x) \\
&= |\Omega| + e^{(n-1)b_0}\frac{R^n}{n}\int_\Sigma f(x)^{n-1}\,d\sigma(x) + \mathcal{O}(R^{n-1}),
\end{aligned}$$

where $R \wedge \tau_c$ denotes the minimum between R and τ_c. Dividing both sides of the inequality above by $\omega_n R^n = |\mathbb{S}^{n-1}|R^n/n$ and letting $R \to \infty$ we obtain

$$AVR(g) \leq e^{(n-1)b_0}\frac{1}{|\mathbb{S}^{n-1}|}\int_\Sigma f(x)^{n-1}\,d\sigma(x)$$

which implies (1.4).

Equality discussion. We first prove the case *(W1)*. If $\lambda = 0$ then (M, g) has nonnegative Ricci curvature. If equality in (1.4) holds, we have

$$\int_\Sigma \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} d\sigma = AVR(g)|\mathbb{S}^{n-1}|.$$

Then the argument in [49] goes through *ipsis litteris* to conclude this first part. Next, we move to the case *(W2)*. Suppose we have

$$e^{(n-1)b_0}\int_\Sigma\left(\left|\frac{\mathrm{H}}{n-1}\right|(1+b_0)+b_1\right)^{n-1}d\sigma=AVR(g)|\mathbb{S}^{n-1}|. \tag{3.16}$$

We must show that Σ is a totally umbilical hypersurface with constant mean curvature on components. Observe first that from the continuity of τ_c and from the inequalities in Proof of inequality (4) above we deduce $\tau_c \equiv \infty$ on Σ. Now, on the one hand, for all $x \in \Sigma$ and $R' > 0$, the hypothesis $\lambda \neq 0$ together with (3.14) ensures that

$$\sup_{[R',\infty)}\hat{\theta}(x)=\sup_{[R',\infty)}\left(\frac{\mathscr{J}_x(t)}{\mathbb{X}_x(t)}\right)^{n-1}\leq\sup_{[R',\infty)}\left(\frac{\mathscr{J}_x(t)}{y_x(t)}\right)^{n-1}=\left(\frac{\mathscr{J}_x(R')}{y_x(R')}\right)^{n-1}\leq 1. \tag{3.17}$$

On the other hand, for any $0 < R' < R$, we have

$$\begin{aligned}
vol(\mathcal{T}_\Omega(R)) &= |\Omega|+\int_\Sigma\int_0^R \det J_{\xi(x)}(t)dtd\sigma(x)\\
&=|\Omega|+\int_\Sigma\int_{R'}^R\hat{\theta}(x,t)\mathbb{X}(x,t)^{n-1}dtd\sigma(x)+\int_\Sigma\int_0^{R'}\det J_{\xi(x)}(t)dtd\sigma(x)\\
&\leq|\Omega|+\int_\Sigma\sup_{[R',R]}\hat{\theta}(x)\int_{R'}^R\left(e^{b_0}f(x)t+1\right)^{n-1}dtd\sigma(x)+\int_\Sigma\int_0^{R'}\det J_{\xi(x)}(t)dtd\sigma(x)\\
&=|\Omega|+e^{(n-1)b_0}\frac{R^n}{n}\int_\Sigma\sup_{[R',R]}\hat{\theta}(x)f(x)^{n-1}d\sigma(x)+\mathcal{O}(R^{n-1}).
\end{aligned}$$

Dividing both sides by $|\mathbb{B}^n|R^n/|\mathbb{S}^{n-1}| = R^n/n$ and letting $R\to\infty$, we get

$$|\mathbb{S}^{n-1}|AVR(g)\leq e^{(n-1)b_0}\int_\Sigma\sup_{[R',\infty)}\hat{\theta}(x)\left(\left|\frac{\mathrm{H}(x)}{n-1}\right|(1+b_0)+b_1\right)^{n-1}d\sigma(x), \tag{3.18}$$

for all $R' > 0$. Subtracting (3.16) from (3.18), one easily deduces that for almost every $x \in \Sigma$ and $R' > 0$

$$\sup_{[R',\infty)}\hat{\theta}(x)-1=0.$$

Hence, according to (3.17), we have $\mathscr{J}_x(R')/y_x(R') = 1$ for almost every $x \in \Sigma$ and for all $R' > 0$. By continuity, this holds everywhere. In particular, $\hat{\theta} \equiv \theta$. The initial conditions on $\mathscr{J}_x(t)$ and $y_x(t)$ return $|\mathrm{H}| = \mathrm{H} \geq 0$. Inspecting the comparison argument above and reasoning as in Lemma (3.1.1), we get that

(E1) $\mathcal{H}_r = \frac{\Delta r}{n-1}\mathcal{P}_{\partial_r}$;

(E2) $\mathrm{Ric}(\partial_r, \partial_r) = -(n-1)(\lambda \circ \mathrm{dist}_{p_0})$,

holds along $\Phi([0,\infty)\times\Sigma)$, where dist_{p_0} in *(E2)* above is evaluated at the integral curves of $\mathrm{grad}_g r$. It follows from the first equation *(E1)* that Σ is umbilic. It follows from the second equation that $\xi_x = \partial_r\big|_x$ must be an eigenvector of the Ricci tensor of (M,g) since $\mathrm{Ric} \geq -(n-1)(\lambda \circ \mathrm{dist}_{p_0})g$, for $x \in \Sigma$. Therefore $\mathrm{Ric}(\xi_x, e_k) = 0$, where $\{e_1, ..., e_{n-1}\}$ is an orthonormal basis for the tangent space $T_x\Sigma$. Using the Codazzi equations, we deduce that the curvature tensor of (M,g) satisfies $R(e_i, e_j, e_k, \xi) = \frac{1}{(n-1)}\big(e_i \mathrm{H}\delta_{jk} - e_j\mathrm{H}\delta_{ik}\big)$ so that

$$0 = \mathrm{Ric}(e_j, \xi) = \frac{(n-2)}{n-1} e_j \mathrm{H}.$$

Consequently, H is locally constant. This establishes the first claim. Next, Suppose Σ is connected. Write $\mathcal{V} = \{t\xi_x : x \in \Sigma,\ t \in [0,\infty)\}$ and for each $t \in [0,\infty)$, define $\Sigma_t^* = r^{-1}(t) \cap \exp^{\perp}(\mathcal{V})$, which is the part of the level sets consisting of regular points, and because $\tau_c \equiv \infty$, it is equal to $r^{-1}(t)$ itself. Moreover, since $r^{-1}[a,b]$ is compact and contains no critical points of r we conclude, by Morse theory, that the $\Sigma_t^{*\prime}s$ are connected, whence M has only one end.

For each $t \in [0,\infty)$, let $\mathrm{H}[t]$ be the mean curvature of Σ_t^* and let φ be the unique solution to $\varphi'(t) - \frac{\mathrm{H}[t]}{n-1}\varphi(t) = 0$, with initial condition $\varphi(0) = 1$, which is well posed as long as $r^{-1}(t)$ is connected. It is easy to see that φ is increasing and $\therefore$ positive (cf. Lemma 3.1.2). Now, let us show that φ satisfies a Jacobi type equation and that Σ is a geodesic sphere centered at p_0. To obtain these claims, observe that the function $r \mapsto \frac{\varphi'(r)}{\varphi(r)}$ satisfies a Riccati type equation since, by the Bochner formula, we have

$$\begin{aligned} \frac{1}{2}\Delta|\nabla r|^2 &= |D^2 r|^2 + \langle \mathrm{grad}_g r, \mathrm{grad}_g \Delta r\rangle + \mathrm{Ric}(\mathrm{grad}_g r, \mathrm{grad}_g r) \\ &= \frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}\Delta r - (n-1)(\lambda \circ \mathrm{dist}_{p_0})(\gamma_{\xi_x}(r)) \\ &= (n-1)\left(\left(\frac{\varphi'}{\varphi}\right)' + \left(\frac{\varphi'}{\varphi}\right)^2 - (\lambda \circ \mathrm{dist}_{p_0})\big(\gamma_{\xi_x}(r)\big)\right) = 0, \end{aligned} \tag{3.19}$$

for all $x \in \Sigma$ and each integral curve γ_{ξ_x} of $\mathrm{grad}_g r$. By uniqueness of solutions to (3.19) with given initial data, the function $\Sigma \ni x \mapsto \mathrm{dist}_{p_0}(x)$ can only be constant. Therefore, Ω is a geodesic ball centered at p_0 with radius $r_0 = \mathrm{dist}_g(p_0, \Sigma)$. Moreover, $M \setminus \Omega$ is easily seen to be locally a warped product, since defining $u : M \setminus \Omega \to \mathbb{R}$ by

$$u(w) \doteq \int_0^{r(w)} \varphi(t)dt, \tag{3.20}$$

gives $\mathrm{grad}_g u = \varphi(r)\mathrm{grad}_g r$ so that $D^2 u = \varphi' g$ which fulfils a characterization of warped product with 1-dimensional factor which goes back to Brinkmann (see e.g. [13], pp. 192 – 194, or the original [11]). Next, we globalize this result.

Claim. — *The manifold $M \setminus \Omega$ is isometric to* $\big([r_0,\infty)\times\Sigma, dr\otimes dr + \varrho(r)^2 g_\Sigma\big)$. In fact, since

Φ is a diffeomorphism from $[0, \infty) \times \Sigma$ onto $M \setminus \Omega$, the pulled back metric takes the form

$$\Phi^* g = dr \otimes dr + g_r,$$

where $r \mapsto g_r$ is a r-dependent family of metrics over Σ with $g_0 = g_\Sigma$. By equation *(E1)* again, using appropriate coordinates $\{x^1, ..., x^{n-1}\}$ on Σ we may deduce that

$$\frac{1}{2}\frac{\partial}{\partial r}(g_r)_{kl} = \frac{\Delta r}{n-1}(g_r)_{kl} = \frac{\varphi'}{\varphi}(g_r)_{kl}$$

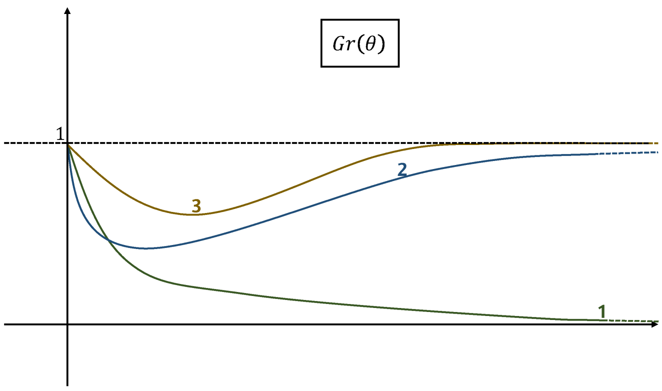
Therefore, $g_r = \varphi(r)^2 g_\Sigma$. Applying the translation $\varrho(r) = \varphi(r - r_0)$, $r \geq r_0$ it is straightforward to check that ϱ satisfies the required conditions. In particular, the identity (1.6) is a consequence of Proposition 3.2.3 together with the fact that $\sup\{\hat{\theta}(x,t) : t \in [R', \infty)\} = 1$. $\square$

Remark 3.2.2. Observe that Σ being a geodesic sphere has no role in the proof above. Thus, it is reasonable to contest this fact in view of the following reason. A priori, it can be the case that λ has compact support and $\{\text{dist}_{p_0}(x) : x \in \Sigma\} \subset \mathbb{R}^+ \setminus \text{supp}(\lambda)$ so that $\Sigma \ni x \mapsto \lambda(\text{dist}_{p_0}(x)) \equiv 0$. In this case, neither the solution φ to the equation $\varphi''(t) = 0$ nor its translation ϱ satisfy the limit in (1.6). We conclude that the set $\{\text{dist}_{p_0}(x) : x \in \Sigma\}$ must intersect the interior of $\text{supp}(\lambda)$.

3.2.1 The decay of λ

Here we address the issue of the decay of the associated function λ, especially regarding the equality case in the main theorem and in relation with the quantities defined in Digression 3.2.1. Throughout the next discussion, bear in mind Figure 3.2, in which we sketch the essentially three distinct behavior the function θ may have.

Figure 3.2: The essentially three possibilities for the graph of θ



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Proposition 3.2.3. *Suppose we have equality in (1.4). If $\lambda \neq 0$ then for every $x_0 \in \Sigma$ it does not exist an open neighborhood $\mathcal{W}$ of x_0 in Σ such that $t \mapsto \theta(x,t)^{\frac{1}{n-1}}$ does not have critical points for all $x \in \mathcal{W}$. In particular, if for some $x_0 \in \Sigma$, we have*

(D.1) $\frac{\partial}{\partial t}\theta(x_0,t)^{\frac{1}{n-1}} = 0$ *for all* $t \geq \tau$ *for some* $\tau > 0$ *then* λ *has compact support;*

(D.2) $\frac{\partial}{\partial t}\theta(x,t)^{\frac{1}{n-1}} > 0$, *for all* t *big enough, then* $\lim_{r\to\infty} \frac{\partial}{\partial t}\theta(x,r)^{\frac{1}{n-1}} = 0$.

Proof. Assume we have equality in the inequality (1.4). Suppose for the sake of contradiction that there exists $x_0 \in \Sigma$ and some open neighborhood $\mathcal{W}$ of x_0 in Σ such that $t \mapsto \theta(x,t)^{\frac{1}{n-1}}$ does not have critical points for all $x \in \mathcal{W}$. From (3.15) and the hypothesis, we have $\frac{\partial}{\partial t}\theta(x,t)^{\frac{1}{n-1}} < 0$ for all $t \geq 0$ so that $t \mapsto \theta(x,t)^{\frac{1}{n-1}}$ is (strictly) decreasing along $[0,\infty)$ and therefore so is $t \mapsto \hat{\theta}(x,t)$, for all $x \in \mathcal{W}$, since we have seen that equality in (1.4) implies $\hat{\theta} = \theta$. Hence, for $0 < R' < R$, a simple analysis of the asymptotic behavior of $vol(\mathcal{T}_\Omega(R))/|\mathbb{B}^n|R^n$, where the numerator satisfies a chain of inequalities of the type

$$\begin{aligned}
vol(\mathcal{T}_\Omega(R)) &= |\Omega| + \int_{\mathcal{W}}\int_0^R \det J_{\xi(x)}(t)dtd\sigma(x) + \int_{\Sigma\setminus\mathcal{W}}\int_0^R \det J_{\xi(x)}(t)dtd\sigma(x)\\
&= |\Omega| + \int_{\mathcal{W}}\int_{R'}^R \theta(x,t)X(x,t)^{n-1}dtd\sigma(x) + \int_{\mathcal{W}}\int_0^{R'} \det J_{\xi(x)}(t)dtd\sigma(x) +\\
&\quad \int_{\Sigma\setminus\mathcal{W}}\int_0^R \det J_{\xi(x)}(t)dtd\sigma(x)\\
&< |\Omega| + \int_{\mathcal{W}}\theta(x,R')\int_{R'}^R X(x,t)^{n-1}dtd\sigma(x) + \int_{\mathcal{W}}\int_0^{R'} \det J_{\xi(x)}(t)dtd\sigma(x) +\\
&\quad \int_{\Sigma\setminus\mathcal{W}}\int_0^R \det J_{\xi(x)}(t)dtd\sigma(x)
\end{aligned}$$

yields

$$AVR(g) < \frac{e^{(n-1)b_0}}{|\mathbb{S}^{n-1}|}\int_\Sigma \left(\left|\frac{\mathrm{H}}{n-1}\right|(1+b_0)+b_1\right)^{n-1} d\sigma,$$

contradicting equality in (1.4). This proves the first assertion of the proposition. To prove *(D .1)* suppose $\frac{\partial}{\partial t}\theta(x_0,r)^{\frac{1}{n-1}} = 0$ for all $r \geq \tau$ for some positive τ and some $x_0 \in \Sigma$. Then, $\theta(x_0,t)^{\frac{1}{n-1}} = y_{x_0}(t)/\mathbb{X}_{x_0}(t) = \mathrm{c} \in \mathbb{R}$ for $t \geq \tau$ (it is easy to see that $\mathrm{c} = 1$, but this has no role here). Then, we must have $y_{x_0}(t) = \mathrm{c}\mathbb{X}_{x_0}(t)$, for all $t \geq \tau$. Therefore,

$$0 = y''_{x_0}(t) = \lambda(t)y_{x_0}(t)$$

for all $t \geq \tau$ and because $y_{x_0} > 0$ this proves that λ has compact support. To see *(D.2)* notice that $t \mapsto \theta(x_0,t)^{\frac{1}{n-1}}$ is eventually increasing, by hypothesis. Since $\theta = \hat{\theta}$ and $\sup\{\hat{\theta}(x,t) : t \in [R',\infty)\} = 1$, we conclude that the limit of $\theta(x,r)$ for $r \to \infty$ is equal to 1 which completes this case. □

It is relevant to point out that the first assertion of Proposition 3.2.3 is equivalent to:

Suppose that $\lambda \neq 0$. If there exists $x_0 \in \Sigma$ and an open neighborhood $\mathcal{W}$ of x_0 in Σ such that $t \mapsto \theta(x,t)^{\frac{1}{n-1}}$ does not have critical points on $\mathcal{W}$ then the inequality (1.4) is strict.

3.2.2 The Ricci curvature of the foliation

In this section we investigate some properties of the level sets of the function u defined in (3.20). Thus, we are assuming equality (1.4) in the main theorem holds so that $M\backslash\Omega$ splits isometrically as a warped product $[r_0,\infty)\times_\varrho \Sigma$ and the level sets $u^{-1}(c)$ are just the slices $\{s\}\times\Sigma$, where $\int_0^s \varphi = c$. Let $S = u^{-1}(c)$, with $c > 0$ (a regular value), be any level set of u. The unit and normal vector field along S is the gradient $\mathrm{grad}_g r$ of the distance (2.3), which is the same as the vector field ∂_r, so that its shape operator is given by

$$\langle A_{\partial_r}X, Y\rangle = -\langle \nabla_X \partial_r, Y\rangle = -D^2 r(X,Y) = -\frac{\Delta r}{n-1}\langle X, Y\rangle, \quad X, Y \in \mathfrak{X}(S).$$

To understand the relation between the Ricci curvature of the ambient to the Ricci curvature of these level sets, we need one more element: the sectional curvature of planes spanned by pairs of vectors $\{\partial_r, \mathbf{e}\}$ where $\mathbf{e}$ is any unit vector in TS, say $\mathbf{e} \in T_qS$. Denote by E any local vertical extension of $\mathbf{e}$ (that is, a vector field E which is everywhere tangent to the fibers of the Riemannian submersion $r : M\setminus\bar{\Omega} \to \mathbb{R}$). We have

$$\begin{aligned}
\sec_M(\mathbf{e},\partial_r) &= -\langle \nabla_{\partial_r}\nabla_E \partial_r, E\rangle - \langle \nabla_{[E,\partial_r]}\partial_r, E\rangle \\
&= -\frac{\partial}{\partial r} D^2 r(E,E) + \langle \nabla_E \partial_r, \nabla_{\partial_r} E\rangle - D^2 r([E,\partial_r],E) \\
&= -\frac{\partial}{\partial r}\left(\frac{\Delta r}{n-1}\right) - \left(\frac{\Delta r}{n-1}\right)^2
\end{aligned}$$

We observe that this is in accordance with *(E2)* since

$$\begin{aligned}
\mathrm{Ric}_M(\partial_r,\partial_r) &= \sum_{k=1}^{n-1}\sec_M(e_k,\partial_r) = -(n-1)\left\{\frac{\partial}{\partial r}\frac{\Delta r}{n-1} + \left(\frac{\Delta r}{n-1}\right)^2\right\} \\
&= -(n-1)\left\{\left(\frac{\varphi'}{\varphi}\right)' + \left(\frac{\varphi'}{\varphi}\right)^2\right\} = -(n-1)\big(\lambda\circ \mathrm{dist}_{p_0}\big),
\end{aligned}$$

by (3.19), and in terms of an orthonormal basis $e_1,\dots,e_{n-1}$ of T_qS. We compute the intrinsic Ricci curvature of S

$$\begin{aligned}\mathrm{Ric}_S(\mathbf{e},\mathbf{e}) &= \sum_{k=1}^{n-2}\sec_S(\mathbf{e},e_k)\\ &= \mathrm{Ric}_M(\mathbf{e},\mathbf{e}) - \sec_M(\mathbf{e},\partial_r) + \sum_k^{n-2}\langle A_{\partial_r}\mathbf{e},\mathbf{e}\rangle\langle A_{\partial_r}e_k,e_k\rangle\\ &= \mathrm{Ric}_M(\mathbf{e},\mathbf{e}) + \frac{\partial}{\partial r}\frac{H}{n-1} + (n-1)\left(\frac{H}{n-1}\right)^2\end{aligned}$$

where $e_1,...,e_{n-2}$ is an orthonormal basis of $\{\mathbf{e}\}^\perp$ in T_qS. Using the hypothesis on the Ricci curvature of g, that is, $\mathrm{Ric}_M \geq -(n-1)(\lambda\circ \mathrm{dist}_{p_0})g$ and (3.19) again, we get

$$\mathrm{Ric}_S(\mathbf{e},\mathbf{e}) \geq -(n-2)\frac{\partial}{\partial r}\frac{H}{n-1}.$$

Trivially, $M\setminus\Omega$ cannot be Einstein (from the beginning, clearly, M cannot be Einstein — unless it is Ricci flat). Moreover, there is no restriction about Σ being Einstein (cf. Corollary 9.107 in [7]).

3.3 A sharp geometric inequality, 2^{nd} case

In this section we sketch the proof of Theorem 1.3.2. We first digress about its preliminary conditions.

Digression 3.3.1. [*The notion of asymptotic* volume ratio*]
Assume (M,g) has asymptotically* nonnegative Ricci curvature relatively to Γ_0, where Γ_0 is a connected and closed hypersurface. Assume additionally that Γ_0 separates the ambient M, that is, $M\setminus\Gamma_0 = M^+\cup M^-$ where both M^+ and M^- are connected subsets of M, at least one of these components is unbounded and the other is possibly empty. Assume M^+ to be the unbounded component. Take a unit and normal vector field $\nu:\Gamma_0\to\nu\Gamma_0$ that points into M^+. Our main interest relies on the analysis of the asymptotic behavior of

$$vol(\mathcal{T}^+_{\Gamma_0}(R)) = \int_{\Gamma_0}\int_0^{R\wedge\tau_c(x)} \det J_{\nu(x)}(t)dtd\sigma_0(x),$$

where $d\sigma_0$ is the induced volume element on Γ_0. We observe now that all comparisons from Section 3.1 go through if we replace the distance dist_{p_0} by dist_{Γ_0}. Indeed, one defines the Jacobian $\mathscr{J}$ just as in formula (3.1) and notices that the only difference between the comparisons there and the case at hand is that we are now considering geodesics γ_{ν_x} in M with $\dot\gamma_{\nu_x}(0) = \nu_x$ which implies:

$$\mathrm{dist}_{\Gamma_0}(\gamma_{\nu_x}(t)) = t \;\Rightarrow\; \lambda(\mathrm{dist}_{\Gamma_0}(\gamma_{\nu_x}(t))) = \lambda(t),\;\; 0\leq t\leq \tau_c(\nu_x).$$

In particular, if the mean curvature H of $\Gamma_{\mathbf{0}}$ is constant along $\Gamma_{\mathbf{0}}$ then the solution y described in Lemma 3.1.2 and the function $\mathbb{X} = \mathbb{X}(x,t)$ do not depend on x (that will be the case, for instance, for slices in a class of warped product spaces that we will analyse in a moment). However, for the comparison (1.4) to be accomplished with $\Gamma_{\mathbf{0}}$ in place of Σ we must further assume that (M, g) has a well defined asymptotic volume ratio. Based on this, we assume that:

$(\mathcal{VR})$ the function $(0,\infty) \ni R \mapsto \Theta^{*}(R) \doteq \frac{vol(\mathcal{T}^{+}_{\Gamma_{\mathbf{0}}}(R))}{\omega_n R^n}$ is (eventually) nonincreasing.

Then, define the asymptotic* volume ratio of g as

$$AVR^{*}(g) \doteq \lim_{R\to\infty} \Theta^{*}(R). \tag{3.21}$$

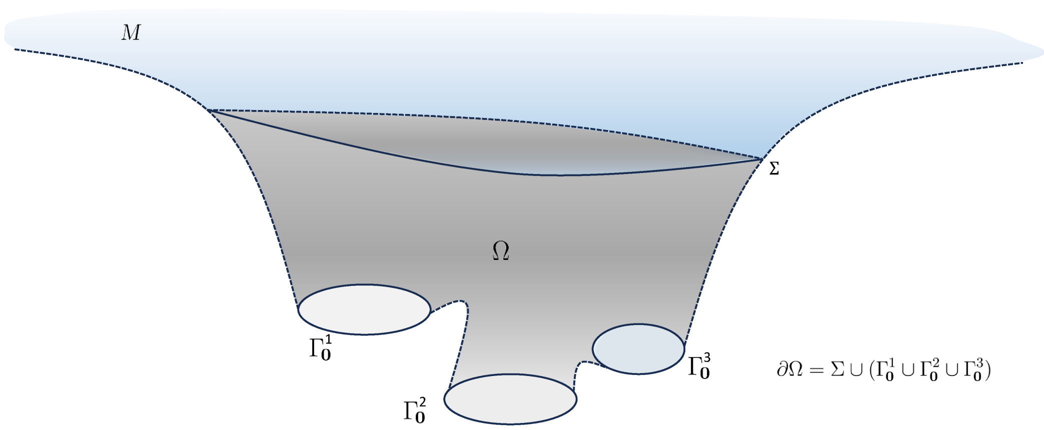
We should think of $vol(\mathcal{T}^{+}_{\Gamma_{\mathbf{0}}}(R))$ as the volume of an oriented tubular neighborhood, where the orientation was chosen relatively to some unbounded component, namely M^{+}. Observe that if $\Gamma_{\mathbf{0}}$ is null-homologous, so that $\partial\Omega = \Gamma_{\mathbf{0}}$ for some open and bounded set Ω then $M \backslash \Gamma_{\mathbf{0}} = (M \backslash \bar{\Omega}) \cup \Omega$, with $M \setminus \bar{\Omega}$ occupying the role of the unbounded component M^{+} and

$$\lim_{R\to\infty} \frac{vol(\mathcal{T}^{+}_{\Gamma_{\mathbf{0}}}(R))}{\omega_n R^n} = \lim_{R\to\infty} \frac{vol(\mathcal{T}_{\Omega}(R))}{\omega_n R^n}$$

In consequence, the definition of $AVR^{*}(g)$ is natural, although not through geodesic balls. Proceeding exactly as in Proof of inequality (4), we easily obtain the inequality (1.8) for $\Gamma_{\mathbf{0}}$ instead of Σ under these conditions. In particular, this exposition implies our inequality (1.14). ☐

Remark 3.3.2. In some cases, the hypothesis on the connectedness of the base hypersurface $\Gamma_{\mathbf{0}}$ may be dropped. In fact, a priori, the only requirement we need to impose to obtain inequality (1.8) is that $M^{+} \setminus \Gamma_{\mathbf{0}}$ is connected (see Figure 3.3, where $\Gamma_{\mathbf{0}}$ has three components).

Figure 3.3: A noncompact Riemannian manifold M with boundary $\Gamma_{\mathbf{0}} = \bigcup_{k=1}^{3} \Gamma_{\mathbf{0}}^{k}$



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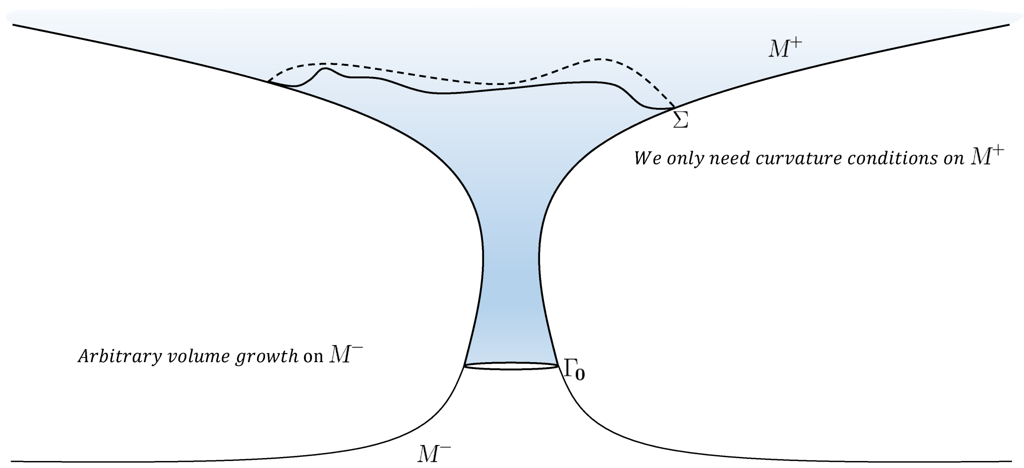
Remark 3.3.3. Since our arguments deals only with oriented tubular neighborhoods $\mathcal{T}_{\Sigma}^{+}(R)$ of hypersurfaces $\Sigma \subset M^{+}$ we may dispense curvature hypothesis on the whole ambient, when $M^{-} \neq \emptyset$ (see Figure 3.4).

Figure 3.4: A noncompact Riemannian manifold with two unbounded components



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Next, we show how the proof of Theorem 1.3.1 may be adapted to the class of asymptotic* nonnegatively curved spaces under the circumstances described above.

Proof of Theorem 1.3.2. Here we use both the distances $w \mapsto \mathrm{dist}_{\Gamma_0}(w)$ and $r^{\bullet}(w) = \mathrm{dist}_{\Sigma}(w)$, where $w \in M^{+}$, and the proof follows virtually the same lines of Theorem 1.3.1 by switching the distance dist_{p_0} by dist_{Γ_0}, so we skip minor details. We must analyse the asymptotics of

$$\frac{vol(\mathcal{T}_{\Sigma}^{+}(R))}{|\mathbb{B}^n(R)|} = \frac{n}{R^n|\mathbb{S}^{n-1}|}\int_{\Sigma}\int_0^{R\wedge\tau_c(x)} \det J_{\xi(x)}(t)dtd\sigma(x),$$

where $\tau_c(x)$ refers to the cut time of the geodesics with initial velocity ξ_x. For $x \in \Sigma$, define the function $\mathscr{J}_x^{*} : [0, t_f(\xi_x)) \to \mathbb{R}$ by

$$\mathscr{J}_x^{*}(t) \doteq \left(\det J_{\xi(x)}(t)\right)^{1/(n-1)}. \tag{3.22}$$

Since $\mathrm{Ric}_w \geq -(n-1)(\lambda \circ \mathrm{dist}_{\Gamma_0})(w)g_w$, and $\lambda \neq 0$ proceeding as in Section 3.1 we readily obtain

- $(\mathscr{J}_x^{*})''(t) - \left(\lambda \circ \mathrm{dist}_{\Gamma_0} \circ \gamma_{\xi_x}\right)(t)\mathscr{J}_x^{*}(t) \leq 0$, and
- $\mathscr{J}_x^{*}(t) \leq y_x^{*}(t) \leq \mathbb{X}_x(t)$,

for all $x \in \Sigma$ and all $t \in [0, t_f(\xi_x))$, where the function $\mathbb{X}_x$ is defined in (3.12) and y_x^{*} is the unique solution to the following IVP

$$\begin{cases} (y_x^*)''(t) - \lambda(\mathrm{dist}_{\Gamma_0}(\gamma_{\xi_x}(t)))y_x^*(t) = 0, \\ y_x^*(0) = 1, \;\; (y_x^*)'(0) = |h|(x), \end{cases}$$

where $h = \mathrm{H}/(n-1)$, with H being the mean curvature of Σ. By hypothesis, there exists an open set Ω with finite volume such that $\partial\Omega = \Gamma_0 \cup \Sigma$. Then $vol(\mathcal{T}^+_{\Gamma_0}(R)) \leq |\Omega| + vol(\mathcal{T}^+_\Sigma(R))$ so that, proceeding exactly as in Proof of inequality (4) and letting $R \to \infty$ in a chain of inequalities of the type

$$\begin{aligned} \frac{vol(\mathcal{T}^+_{\Gamma_0}(R))}{\omega_n R^n} &\leq \frac{|\Omega|}{|\mathbb{B}^n(R)|} + \frac{1}{|\mathbb{B}^n(R)|}\int_\Sigma \int_0^{R\wedge\tau_c(x)} \mathscr{J}^*(x,t)^{n-1} dt d\sigma(x) \\ &\leq \frac{|\Omega|}{|\mathbb{B}^n(R)|} + \frac{n}{R^n|\mathbb{S}^{n-1}|}\int_\Sigma \int_0^R \mathbb{X}(x,t)^{n-1} dt d\sigma(x) \end{aligned}$$

grants us inequality (1.8).

Equality discussion. Suppose we have the equality

$$e^{(n-1)b_0}\int_\Sigma \left(\left| \frac{\mathrm{H}}{n-1} \right| (1+b_0) + b_1 \right)^{n-1} d\sigma = AVR^*(g)|\mathbb{S}^{n-1}|. \tag{3.23}$$

First observe $\tau_c \equiv \infty$ over Σ along the directions given by ξ. Analogously to what was done in Digression 3.2.1 we define the functions $\hat{\theta}^*, \theta^* : \Sigma \times [0,\infty) \to \mathbb{R}$ setting

$$\hat{\theta}^*(x,t)^{\frac{1}{n-1}} = \left(\frac{\mathscr{J}^*(x,t)}{\mathbb{X}(x,t)} \right) \text{ and } \theta^*(x,t)^{\frac{1}{n-1}} = \left(\frac{y^*(x,t)}{\mathbb{X}(x,t)} \right).$$

Now, the fact that $\frac{\mathscr{J}^*(x,t)}{\mathbb{X}(x,t)} \leq \frac{\mathscr{J}^*(x,t)}{y^*(x,t)}$, $x \in \Sigma$, $t \geq 0$ together with the monotonicity of $t \mapsto \frac{\mathscr{J}^*_x(t)}{y^*_x(t)}$ and equality (3.23) gives $\sup\{\hat{\theta}^*(x,t) : t \in [R',\infty)\} = 1$, for almost every $x \in \Sigma$ and for all $R' > 0$ which yields $\mathscr{J}^*(x,t) \equiv y^*(x,t)$ on $\Sigma \times [0,\infty)$ so that the mean curvature H of Σ is nonnegative and

(E1)* $\mathcal{H}_{r^\bullet} = \frac{\Delta r^\bullet}{n-1}\mathcal{P}_{\partial_{r^\bullet}}$;

(E2)* $\mathrm{Ric}\left(\frac{\partial}{\partial r^\bullet}, \frac{\partial}{\partial r^\bullet}\right) = -(n-1)(\lambda \circ \mathrm{dist}_{\Gamma_0})$,

both hold on $\Phi(\Sigma\times[0,\infty))$, where Φ is defined by (2.5). Repeating the argument in the **Equality discussion** in Section 3.2 we deduce that Σ is a totally umbilic hypersurface with locally constant mean curvature. In addition to that, if Σ is connected then *(E1*)* and *(E2*)* together with the nonexistence of critical points for $r^\bullet$, the fact $\tau_c \equiv \infty$ over Σ and elementary Morse theory yield that the referred distance induces a smooth, codimension one umbilical foliation $\mathcal{F}^*(X)$ whose leaves $\Sigma^*_t = \Phi\big(\Sigma \times \{t\}\big)$ are connected hypersurfaces with constant mean curvature $\mathrm{H}[t]$, whence $M^+ \setminus \Omega$ has only one end. The aforementioned foliation is also induced by the closed and conformal vector field given by

$$X\big|_w = \varphi(r^\bullet(w))\partial_{r^\bullet}\big|_w, \quad \text{and defined on } M^+ \setminus \Omega \ \big(\text{observe } C(\Sigma) = \emptyset\big)$$

where φ is the solution to $\varphi'(t) - \frac{\mathrm{H}[t]}{n-1}\varphi(t) = 0$, with initial condition $\varphi(0) = 1$. Following the ideas of Montiel (see Proposition 3 in [37]) one shows that $M^+ \setminus \Omega$ is locally a warped product. Following the ideas in our **Equality discussion** in the preceding section, one shows that

$$M^+ \setminus \Omega \ \text{ is isometric to } \ \big([r_0, \infty) \times \Sigma, dr \otimes dr + \varrho(r)^2 g_\Sigma\big), \ \text{ where } \ r_0 = \mathrm{dist}_g(\Gamma_0, \Sigma)$$

and $\varrho(r) = \varphi(r - r_0)$ is the solution to $\varrho''(t) - \lambda(t)\varrho(t) = 0$, with the required initial conditions, so that Γ_0 and Σ are equidistant hypersurfaces (see Remark 3.2.2) with r_0 being the distance between them. □

3.4 Application to warped product spaces

In this section, we show that Corollary 1.3.4 is a direct consequence of Theorem 1.3.2 and set the stage for the proof of the converse of our Willmore-type inequality in asymptotic and asymptotic* nonnegatively curved spaces. Moreover, we evaluate explicitly the constants $b_0(\lambda)$, $b_1(\lambda)$ and $AVR^*(g_S)$ for the 3-dimensional Schwarzschild manifold.

Lemma 3.4.1. $\big[$*Basic properties of* (M, g) $\big]$
Let (M, g) *be the warped product described in the introduction, that is*

$$M \doteq [0, \infty) \times N, \ \ g \doteq dr \otimes dr + h(r)^2 g_N,$$

and satisfying the conditions $(\boldsymbol{\Lambda 1})$—$(\boldsymbol{\Lambda 3})$. *Then,* (M, g) *has asymptotically* nonnegative Ricci curvature relatively to* Γ_0 *and associated function* λ *with finite* $AVR^*(g)$*, as defined in Digression 3.3.1. In addition to that, if the condition* $(\boldsymbol{\Lambda 4})$ *holds then* $AVR^*(g) > 0$.

Proof. Let $\{E_1 \doteq \partial_r, E_2 ..., E_n\}$ be a local orthonormal frame for M so that $g(E_l, E_k) = \delta_{lk}$ and set $e_k = hE_k$, for $k = 2, ..., n$. It follows from elementary properties of warped product spaces (see e.g. [25], pp. 59 – 60) that the Ricci curvature of M in the radial direction is given by

$$\mathrm{Ric}_M(\partial_r, \partial_r) = -(n-1)\frac{h''(r)}{h(r)}.$$

Whereas in spacial directions

$$\begin{aligned}\mathrm{Ric}_M(E_k,E_k) &= \mathrm{Ric}_N(E_k,E_k) - \left(\frac{h''(r)}{h(r)} + (n-2)\frac{h'(r)^2}{h(r)^2}\right) \\ &\geq -(n-1)\left[\frac{1}{(n-1)}\frac{h''(r)}{h(r)} - \left(\frac{n-2}{n-1}\right)\frac{\rho - h'(r)^2}{h(r)^2}\right].\end{aligned}$$

Let λ be the function from hypothesis ($\mathbf{\Lambda 2}$) so as to λ is nonincreasing and b_0 is well defined. The computations above show that

$$\mathrm{Ric}_M \geq -(n-1)\big(\lambda \circ \mathrm{dist}_{\Gamma_0}\big) g$$

so that the first claim is verified. If ($\mathbf{\Lambda 3}$) holds then $t \mapsto \frac{h(t)^{n-1}}{t^{n-1}}$ is eventually nonincreasing. By Lemma 2.2 in [56] the function

$$(0,\infty) \ni R \ \mapsto \frac{1}{\displaystyle\int_0^R t^{n-1}dt}\int_0^R h(t)^{n-1}dt$$

is also eventually nonincreasing. Since the (open) geodesic tube around Γ_0 of radius $R<\infty$ is equal to $\mathcal{T}_{\Gamma_0}(R) = [0,R)\times N$ we have

$$\frac{vol(\mathcal{T}_{\Gamma_0}(R))}{|\mathbb{B}^n(R)|} = \frac{|N|}{|\mathbb{S}^{n-1}|}\frac{1}{\displaystyle\int_0^R t^{n-1}dt}\int_0^R h(t)^{n-1}dt. \tag{3.24}$$

Hence, for big values of radius R, the function $R \mapsto vol(\mathcal{T}_{\Gamma_0}(R))/|\mathbb{B}^n(R)|$ is nonincreasing and we have a finite asymptotic* volume ratio. If ($\mathbf{\Lambda 4}$) holds then it is easy to see that $AVR^*(g) > 0$. □

We now reveal how Corollary 1.3.4 is obtained from Theorem 1.3.2. Choose the open set M^+ from Theorem 1.3.2 to be equal to $M\setminus\Gamma_0$ and $M^- = \emptyset$. Since Γ_0 and Σ are homologous, there exists an open set Ω with $|\Omega|<\infty$ such that $\partial\Omega = \Sigma\cup\Gamma_0$. The second inequality in the corollary follows from here and the first was observed in Digression 3.3.1. With respect to the rigidity statement, if equality in (1.15) holds then by Theorem 1.3.2 the manifold $M\setminus(\Omega\cup\Gamma_0)$ is isometric to the model (1.9) (here we use the connectedness assumption on Σ), the hypersurfaces Γ_0 and Σ are equidistant and $\therefore$ Σ is a slice $\{r_0\}\times N$, where r_0 is the distance between Γ_0 and Σ. Clearly, $h \equiv \varrho$ over $[r_0,\infty)$ so that all the necessary conditions are met. The proof of the sufficiency along with the converse of Theorems 1.3.1 and 1.3.2 will be shown in the Subsection 3.4.1.

At this point, considering the warped product from Lemma 3.4.1 it is interesting to look at the behavior of the function $\mathscr{F}:[0,\infty)\to\mathbb{R}$ given by

$$\mathscr{F}(t) = \int_N \left(\left|\frac{h'(t)}{h(t)}\right|(1+b_0)+b_1\right)^{n-1} dvol_N.$$

The slices $\{r\} \times N$ for which equality in (1.15) occurs represent (global) minima of $\mathscr{F}$. If (**Λ4**) holds so that $h'(t) \geq 0$ for $t \geq \tau_0$ then

$$\mathscr{F}'(t) = \frac{(n-1)|N|}{(1+b_0)^{-1}} \left(\frac{h'(t)}{h(t)}(1+b_0) + b_1 \right)^{n-2} \left(\frac{h''(t)}{h(t)} - \frac{h'(t)^2}{h(t)^2} \right), \;\; t \geq \tau_0.$$

If equality in (1.15) holds for the slice $\Sigma = \{\tau_0\} \times N$ then $\mathscr{F}'(t) \geq 0$ on a neighborhood of τ_0 and

$$\lim_{t \to \infty} \mathscr{F}'(t) = 0.$$

3.4.1 The converse of Theorems 1.3.1 and 1.3.2

We are now ready to prove the converse of Theorems 1.3.1 and 1.3.2. If Ω is null-homologous (resp. homologous to Γ_0) then $M \setminus \Omega$ (resp. $M^+ \setminus \Omega$) is isometric to a warped product of the type

$$\left([r_0, \infty) \times \Sigma, dr \otimes dr + \varrho(r)^2 g_\Sigma \right)$$

where Σ is a connected constant mean curvature hypersurface in M and ϱ is the solution to the Jacobi equation $\varrho'' - \lambda\varrho = 0$ over $[r_0, \infty)$ with the given initial conditions, i.e., $\varrho(r_0) = 1$ and $\varrho'(r_0) = H/(n-1)$. We divide this part in two cases, the first being that case where Ω is null-homologous.

1st case. — Here r_0 is the distance between p_0 and Σ, Ω is a geodesic ball of radius r_0 and ϱ satisfies the limit (1.6). A note about the asymptotic volume ration is in order. By definition, $AVR(g) = \lim_{r\to\infty} \frac{vol(B_{p_0}(r))}{\omega_n r^n}$. In our case, we may also compute this quantity by

$$AVR(g) = \lim_{R \to \infty} \frac{vol(\mathcal{T}_\Sigma^+(R))}{\omega_n R^n} \tag{3.25}$$

by an easy examination of the asymptotics of the inequality $vol(B_{p_0}(R)) \leq vol(\mathcal{T}_\Omega^+(R))$ when divided by $\omega_n R^n$ and the fact that equality in (1.4) holds. Moreover, for $R > 0$, we have $\mathcal{T}_\Sigma^+(R) = [r_0, r_0 + R) \times \Sigma$ and according to formulas (3.24) and (3.25), we have

$$AVR(g) = \lim_{R \to \infty} \left(\frac{|\Sigma|}{|\mathbb{S}^{n-1}|} \left(\int_0^R t^{n-1} dt \right)^{-1} \int_{r_0}^{r_0+R} \varrho(t)^{n-1} dt \right)$$

thereby, alongside an application of L'Hôpital's rule, we obtain

$$\begin{aligned}
\frac{AVR(g)|\mathbb{S}^{n-1}|}{|\Sigma|} &= \lim_{R\to\infty}\left(\frac{\varrho(r_0+R)}{R}\right)^{n-1} \\
&= \left(\lim_{R\to\infty}\frac{\varrho(r_0+R)}{R}\right)^{n-1} \\
&= \left[e^{b_0}\left(\frac{\mathrm{H}}{(n-1)}(1+b_0)+b_1\right)\right]^{n-1}.
\end{aligned}$$

This implies that we have equality in (1.4).

2nd case. — Here r_0 is the distance between the equidistant hypersurfaces Γ_0 and Σ and ϱ satisfies the limit (1.10). Now, for $R > r_0$ we have $vol(\mathcal{T}^+_{\Gamma_0}(R)) = |\Omega| + vol(\mathcal{T}^+_\Sigma(R-r_0))$ and also $\mathcal{T}^+_\Sigma(R-r_0) = [r_0, R-r_0) \times \Sigma$. Again, by formula (3.24) and the definition of the asymptotic* volume ratio (see (3.21)), we have

$$AVR^*(g) = \lim_{R\to\infty}\frac{vol(\mathcal{T}^+_\Sigma(R-r_0))}{\omega_n R^n} = \lim_{R\to\infty}\frac{|\Sigma|}{|\mathbb{S}^{n-1}|}\frac{1}{\displaystyle\int_0^R t^{n-1}dt}\int_{r_0}^{R-r_0}\varrho(t)^{n-1}dt.$$

Clearly we may proceed as before to achieve equality in this case. □

3.4.2 Application to the Schwarzschild and Reissner-Nordstrom manifolds

A particular set of examples of the above construction comes from considering Riemannian warped products (M, g) of the following type

$$M = (\underline{s}, \infty) \times N, \quad g = \frac{1}{\omega(s)}ds \otimes ds + s^2 g_N, \tag{3.26}$$

where $\underline{s} > 0$ and $\omega(s)$ is a smooth function on $[\underline{s}, \infty)$. As described in [8], to see the metric g into the form (1.13), one considers the change of variables $F : [\underline{s}, \infty) \to \mathbb{R}$ defined by $F'(s) = 1/\sqrt{\omega(s)}$ and $F(\underline{s}) = 0$. Then, the substitution $r = F(s)$ brings (3.26) into the desired form and if $h : [0, \infty) \to [\underline{s}, \infty)$ denotes the inverse of F an easy computation gives

$$h'(r) = \sqrt{\omega(s)} \text{ and } h''(r) = \frac{1}{2}\omega'(s),$$

where $s = h(r)$. With respect to the change of variables $r = F(s)$, the functions λ_1 and λ_2 become

$$\lambda_1(s) = \frac{\omega'(s)}{2s} \text{ and } \lambda_2(s) = \frac{\omega'(s)}{2(n-1)s} - \left(\frac{n-2}{n-1}\right)\frac{\rho - \omega(s)}{s^2},$$

the condition (**Λ1**) remains unchanged while (**Λ2**) — (**Λ4**) are equivalent to the following pair of conditions

- there exists a continuous nonnegative nonincreasing function $\lambda : [\underline{s}, \infty) \to \mathbb{R}$ s.t. $\lambda \geq \max\{\lambda_1, \lambda_2\}$ and

$$b_0 = \int_{\underline{s}}^{\infty} F(s)\lambda(s)F'(s)ds < \infty \tag{3.27}$$

- the function $(\underline{s}, \infty) \ni s \mapsto \frac{s}{F(s)}$ is eventually monotone decreasing.

It is straightforward to check that, after the change described above, the Schwarzschild and Reissner-Nordstrom spaces introduced in (1.16) and (1.17) respectively, fulfils the conditions $(\mathbf{\Lambda 1})$—$(\mathbf{\Lambda 4})$. In particular, since these spaces have smallest Ricci curvature in the radial direction, there is an evident choice for the function λ.

Example 3.4.2. [*The constants $b_0(\lambda)$, $b_1(\lambda)$ and $AVR^{*}(g_S)$ for the 3-d Schwarzschild space*] The (3-dimensional) model of the Schwarzschild space we are adopting in this example is

$$M = [0, \infty) \times \mathbb{S}^2, \quad g = dr \otimes dr + h(r)^2 g_{\mathbb{S}^2}, \tag{3.28}$$

where h has $h'(r) = \sqrt{1 - ms(r)^{-1}}$ so that it meets the initial condition $h(0) = m$. It is straightforward to check that M has asymptotically* nonnegative Ricci curvature relatively to $\Gamma_{\mathbf{0}} = \{0\} \times \mathbb{S}^2$:

$$\mathrm{Ric}_M \geq -(n-1)\frac{h''(r)}{h(r)} g.$$

In this case we have the clear choice $\lambda(r) = \frac{h''(r)}{h(r)}$ for the associated function. Also, in this example, Σ is taken to be equal to $\Gamma_{\mathbf{0}}$. We begin by computing the constant b_1. Using the substitution $r = F(s)$, the definition of b_1 and the change of variables $u = \omega(s)$, where $\omega(s) = 1 - ms^{-1}$, we get

$$b_1 = \int_0^{\infty} \lambda(t)dt = \frac{m}{2} \int_{\underline{s}}^{\infty} \frac{s^{-3}}{\sqrt{\omega(s)}} ds = \frac{2}{3}\frac{1}{m}$$

We now evaluate $AVR^{*}(g)$. Using formula (3.24), using the substitution $r = F(s)$ and the change of variables $u = \omega(s)$ again, we have

$$vol(\mathcal{T}_\Sigma(R)) = \frac{m^3|\mathbb{S}^2|}{48}\left\{\frac{2\sqrt{\omega}}{m^3 s(R)^{-3}}\left(15\omega^2 - 40\omega + 33\right) - 15\log\left(\frac{ms(R)^{-1}}{(1+\sqrt{w})^2}\right)\right\}, \quad \omega = \omega(s(R)).$$

Therefore,

$$
\begin{aligned}
AVR^{*}(g) &= \lim_{R\to\infty}\frac{vol(\mathcal{T}_{\Sigma}(R))}{|\mathbb{B}^3(R)|}\\
&= \lim_{R\to\infty}\frac{m^3}{8R^3}\left\{\frac{\sqrt{\omega}}{m^3 s(R)^{-3}}\left(15\omega^2-40\omega+33\right)-\frac{15}{2}\log\left(\frac{ms(R)^{-1}}{(1+\sqrt{w})^2}\right)\right\}=1.
\end{aligned}
$$

To compute $b_0(\lambda)$ note that $F(s)=(1/2)\big(2s\sqrt{\omega}+m\log(1+\sqrt{\omega})-m\log(1-\sqrt{\omega})\big)$. Then,

$$
b_0=\frac{m}{2}\int_{\underline{s}}^{\infty}\frac{F(s)}{s^3\sqrt{\omega(s)}}ds=\frac{1}{3}\big(1+\log 4\big). \quad \square \tag{3.29}
$$

Observe that it is natural to make an attempt to extract a model for the geometry (1.9) from the Schwarzschild space by changing the warping function. We now describe one such experiment.

Let κ be a positive real number and define $m \doteq \frac{2}{3\kappa}e^{b_0}$, where b_0 is given by (3.29). As in the previous example, choose the warping function h with initial condition $h(0)=m$. Then define $\varrho:[0,\infty)\to\mathbb{R}$ by $\varrho(r)\doteq m^{-1}h(r)$, so that $\varrho(0)=1$ and $\varrho'(0)=0$ and consider the Riemannian manifold

$$
M=[0,\infty)\times\mathbb{S}^2,\;\; g=dr\otimes dr+\varrho(r)^2 g_{\mathbb{S}^2}.
$$

It is straightforward to check that the Ricci curvature of g is smallest in the radial direction provided $m\geq 1$. Therefore, $\mathrm{Ric}_M\geq-(n-1)(\lambda\circ\mathrm{dist}_{\Gamma_0})g$, where the associated function λ is equal to $\varrho''/\varrho=h''/h$ so that it satisfies the Jacobi equation $\varrho''-\lambda\varrho=0$. As a result, the numbers b_1 and b_0 remain the same as those calculated in the previous example. However,

$$
\lim_{r\to\infty}\frac{\varrho(r)}{(e^{b_0}b_1)r}=\lim_{r\to\infty}\frac{3}{2e^{b_0}}\frac{h(r)}{r}=\frac{3}{2e^{b_0}}\neq 1,
$$

and we do not achieve equality in (1.14).

3.4.3 The Schwarzschild space fulfils $(\Lambda 1)$—$(\Lambda 4)$

We now demonstrate that the Schwarzschild manifold fulfils the hypothesis described above. It is easy to see that $s\mapsto\frac{s}{F(s)}$ is monotone decreasing. In fact, differentiate the function $g(s)=\frac{F(s)}{s}$ to obtain $g'(s)=\frac{F'(s)s-F(s)}{s^2}$. The numerator of $g'(s)$ satisfies

$$
(F'(s)s-F(s))'=F''(s)s=-\frac{m(n-2)}{2\omega(s)^{3/2}}s^{-n}<0.
$$

Since the numerator of g' is decreasing and $\lim_{s\to\infty} g'(s) = 0$ we have $g' \geq 0$ and $\therefore g$ is increasing. This proves that (M, g) has a well defined asymptotic volume ratio. The other property that needs to be checked is $b_0(\lambda) < \infty$. For this, we subdivide the interval $(\underline{s}, \infty)$ into $(m, 2m]$ and $[2m, \infty)$ to write

$$b_0 = \frac{m(n-2)}{2}\left(\int_{\underline{s}}^{2\underline{s}} F(s)s^{-n}F'(s)ds + \int_{2\underline{s}}^{\infty} F(s)s^{-n}F'(s)ds\right)$$

In the interval $m < s \leq 2m$ we have $1/(2\underline{s})^n \leq s^{-n} \leq 1/\underline{s}^n$ so that

$$\int_{\underline{s}}^{2\underline{s}} \frac{F(s)F'(s)}{s^n}ds \leq \frac{1}{\underline{s}^n}\int_{\underline{s}}^{2\underline{s}} F(s)F'(s)ds = \frac{1}{\underline{s}^n}\left(F(2\underline{s})^2 - \int_{\underline{s}}^{2\underline{s}} F(s)F'(s)ds\right).$$

In the interval $2m \leq s < \infty$, we have $1/\sqrt{\omega(s)} \leq 1/\sqrt{\omega(2m)}$ so that

$$\int_{2m}^{\infty} F(s)s^{-n}F'(s)ds \leq \frac{1}{\sqrt{\omega(2m)}}\left\{\frac{F(2m)(2m)^{1-n}}{n-1} + \frac{2}{m(n-1)(n-2)}\left(1-\sqrt{\omega(2m)}\right)\right\}.$$

3.5 Building examples employing conformal approach

Let $\bar{g}$ be the standard Euclidean metric on $\mathbb{R}^n$. We want to study the possibility to find a metric g, conformal to the Euclidean metric $\bar{g}$, say $g = e^{2f}\bar{g}$ such that $(\mathbb{R}^n, g)$ has asymptotically nonnegative Ricci curvature with base point the origin $\mathbf{0}$, and such that f is a radial function, i.e.

$$\mathrm{Ric}_g\big|_w \geq -(n-1)(\lambda \circ \mathrm{dist}_g(\mathbf{0}, w))g\big|_w \text{ and } f(w) = f(|w|),\ \ w \in \mathbb{R}^n, \tag{$\star\star$}$$

for some associated function λ and some $f \in C^\infty(\mathbb{R}^n)$, where $|w|$ refers to the usual Euclidean distance from the origin. Set $s(w) = |w|$, $w \in \mathbb{R}^n$, in order for, strictly speaking, $f = \varphi(s)$, for some $\varphi \in C^\infty$, and in terms of the natural $\bar{g}$-orthonormal basis $\{e_i\}$ of $\mathbb{R}^n$ write $w = w^k e_k$. We begin this study examining what properties the conformal factor f must satisfy. It is widely known (see e.g. [43], pp. 90 – 91) that the Ricci curvature of g is given by

$$\mathrm{Ric}_g = -(n-2)\bar{D}^2 f + (n-2)df \otimes df - \Delta_{\bar{g}} f\bar{g} - (n-2)|df|^2_{\bar{g}}\bar{g}$$

where the quantities on the RHS of the above equation are considered with respect to the metric $\bar{g}$. Since f is a radial function, denoting by $\{dw^i\}$ the dual coframe to $\{e_i\}$ so that $dw^l(e_k) = \delta_{lk}$, we have

$$\bar{D}^2 f\big|_w = \left(f''(s) - \frac{f'(s)}{s}\right)\frac{w^j w^k}{s^2} dw^j \otimes dw^k\big|_w + \frac{f'(s)}{s}\bar{g}\big|_w, \text{ for } w \in \mathbb{R}^n$$

so that

$$\begin{aligned} \mathrm{Ric}_g\big|_w = &-(n-2)\left(f''(s) - \frac{f'(s)}{s} - f'(s)^2\right)\frac{w^j w^k}{s^2} dw^j \otimes dw^k\big|_w + \\ &- \left(f''(s) + (2n-3)\frac{f'(s)}{s} + (n-2)f'(s)^2\right)\bar{g}\big|_w. \end{aligned}$$

Now, the inequality (⋆⋆) is realized for some associated function λ and some smooth radial function f iff

$$\begin{aligned} &-(n-2)\left(f''(s) - \frac{f'(s)}{s} - f'(s)^2\right)\frac{w^j w^k}{s^2} dw^j \otimes dw^k\big|_w + \\ &- \left(f''(s) + (2n-3)\frac{f'(s)}{s} + (n-2)f'(s)^2 - (n-1)\big(\lambda(r(w))\big)e^{2f(s)}\right)\bar{g}\big|_w \geq 0, \end{aligned}$$

for all $w \in \mathbb{R}^n$, where $r(w) = \mathrm{dist}_g(\mathbf{0}, w)$ is the distance induced by g, as defined in (2.3).

CHAPTER 4

SPACES WITH NONNEGATIVE BAKRY-ÉMERY RICCI CURVATURE

In this chapter we consider smooth metric-measure spaces to give a simple approach to weighted volume measures on a Riemannian manifold and achieve a Willmore-type inequality based on the techniques of asymptotic volume ratio, as developed on the previous chapter, and the knowledge of volume growth of geodesic balls in manifolds with lower bounds on the Bakry-Émery Ricci curvature. This chapter is ordered as follows. In Section 4.1 we prove some elementary inequalities for the Bakry-Émery Ricci tensor which are the stepping-stones for volume comparison. Next, in Section 4.2, we introduce the corresponding notion of asymptotic volume ratio for spaces with nonnegative Bakry-Émery Ricci curvature and, finally, in the last section we prove a Willmore-type inequality for these spaces.

4.1 Mean curvature and volume comparison

An approach to comparison geometry based on Ricci curvature is given by the ***Bochner formula***:

$$\frac{1}{2}\Delta|\nabla u|^2 = |D^2u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\mathrm{grad}_g u, \mathrm{grad}_g u), \quad u \in C^\infty(M). \tag{4.1}$$

Relatively to the measure $e^{-f}dvol_g$, where f is any smooth function on M, the intrinsic self-adjoint ***f-Laplacian*** (or weighted Laplacian) is

$$\Delta_f u = \Delta u - \langle \mathrm{grad}_g u, \mathrm{grad}_g f\rangle, \quad u \in C^\infty(M). \tag{4.2}$$

We now recall the corresponding Bochner formula for this Laplacian and the Bakry-Émery Ricci tensor. For any $u \in C^\infty(M)$ (see e.g. [52]) it is straightforward to get

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |D^2u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f^N(\mathrm{grad}_g u, \mathrm{grad}_g u) + \frac{1}{N}(df\,\mathrm{grad}_g u)^2.$$

On the one hand, using the inequality

$$\frac{(\Delta u)^2}{n} + \frac{1}{N}\langle \nabla u, \nabla f\rangle^2 \geq \frac{(\Delta_f u)^2}{n+N}$$

we get the following Bochner formula for the N-Bakry-Émery Ricci tensor,

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f u)^2}{n+N} + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f^N(\mathrm{grad}_g u, \mathrm{grad}_g u). \tag{4.3}$$

On the other hand, for the ∞-Bakry-Émery Ricci tensor we have

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f(\mathrm{grad}_g u, \mathrm{grad}_g u).$$

We now turn our attention to mean curvature comparison. Recall that the mean curvature measures the relative rate of change of the volume element of the geodesic spheres (when considering distance to a point) and the volume element of the level sets (when considering distance to a hypersurface). We now restrict our attention to the distance (2.3). With respect to the measure $e^{-f}dvol_g$, the weighted mean curvature is $m_f = m - \frac{\partial}{\partial r} f$ where, as in Chapter 2, $m = \mathrm{H}$ is the mean curvature of the level sets with respect to $-\mathrm{grad}_g r$.

Recall that we denote by ξ the unit, outward-pointing and normal vector field along $\partial\Omega$. The corresponding quantity in the model space we are comparing m_f with is defined by $m_0^k \doteq \frac{(k-1)\mathrm{H}_f}{k-1+(\mathrm{H}_f)r}$, where $\mathrm{H}_f = \mathrm{H} - \langle \xi, \mathrm{grad}_g f\rangle$ and k is understood to be the dimension of the model. For instance, when dealing with the N-Bakry-Émery Ricci tensor then the underlying dimension is $k = n+N$. In general, we will omit the superscript k and the dimension is supposed to be known by context. Observe, finally, that $m_f(0) = \mathrm{H}_f$.

Lemma 4.1.1. [*Mean curvature comparison for N-Bakry-Émery*]
Let (M, g) *be a Riemannian manifold and* Σ *a closed hypersurface. Assume* $\mathrm{Ric}_f^N(\partial_r, \partial_r) \geq 0$. *Then*

$$m_f(r) \leq m_0(r), \tag{4.4}$$

holds up to the first occurrence of a cut-point.

Proof. Applying the (modified Bochner) inequality (4.3) to the distance (2.3), we get

$$0 = \frac{1}{2}\Delta_f |\nabla r|^2 \geq \frac{m_f^2}{n+N-1} + \frac{\partial}{\partial r} m_f.$$

Since $m_0' = -\frac{1}{n+N-1} m_0^2$ we immediately get (4.4), by Sturm-Liouville comparison arguments. □

Lemma 4.1.2. [*Mean curvature comparison for ∞-Bakry-Émery*]
Let (M, g) be a Riemannian manifold and $\Sigma \subset M$ a closed hypersurface, realized as the boundary of an open and bounded set Ω. Assume $\mathrm{Ric}_f(\partial_r, \partial_r) \geq 0$ along the integral curves of ∂_r, where r is the oriented distance (2.3). If $\partial_r f \geq -a$, where $a \in \mathbb{R}$ is ≥ 0, and $\mathrm{H}_f \geq 0$ then

$$m_f(r) \leq a + m_0(r), \tag{4.5}$$

holds up to the first occurrence of a cut-point.

Proof. By the usual Bochner formula applied to the distance (2.3), we have $0 \geq m^2/(n-1) + m' + \mathrm{Ric}(\mathrm{grad}_g r, \mathrm{grad}_g r)$. Summing and subtracting $D^2 f$ applied to $\mathrm{grad}_g r$ and using the curvature condition $\mathrm{Ric}_f(\partial_r, \partial_r) \geq 0$, we obtain:

$$\begin{aligned}(m - m_0)' &\leq -\frac{m^2}{n-1} - \mathrm{Ric}(\mathrm{grad}_g r, \mathrm{grad}_g r) + \frac{m_0^2}{n-1} \\ &\leq -\frac{m^2 - m_0^2}{n-1} + D^2 f(\mathrm{grad}_g r, \mathrm{grad}_g r).\end{aligned}$$

Defining the auxiliary function $h(r) = (n-1) + (\mathrm{H}_f) r$ so that $(h^2)' = \frac{2}{n-1} h^2 m_0$ we henceforth obtain

$$\begin{aligned}\left[h^2(m - m_0)\right]' &\leq h^2 \left[-\frac{(m - m_0)^2}{n-1} + D^2 f(\mathrm{grad}_g r, \mathrm{grad}_g r)\right] \\ &\leq h^2 D^2 f(\mathrm{grad}_g r, \mathrm{grad}_g r).\end{aligned}$$

Integrating from zero to r and integrating by parts we get

$$\begin{aligned}h^2(r)\big(m(r) - m_0(r)\big) - h^2(0)\big(m(0) - m_0(0)\big) &\leq \int_0^r h^2(t) \frac{\partial^2 f}{\partial t^2}(t) dt \\ &= h^2(r)\partial_t f(r) - h^2(0)\partial_t f(0) - \int_0^r \frac{\partial h(t)^2}{\partial t} \frac{\partial f}{\partial t}(t) dt\end{aligned}$$

which, since $\big(m(0) - m_0(0) - \partial_t f(0)\big) = 0$, is the same as

$$h^2(r)(m_f(r) - m_0(r)) \leq -\int_0^r (h(t)^2)' \partial_t f(t) dt.$$

Now, if $\partial_r f \geq -a$ and $\mathrm{H}_f \geq 0$ then

$$h^2(r)(m_f(r) - m_0(r)) \leq a(h(r)^2 - h(0)^2) \leq a h(r)^2,$$

so that

$$m_f(r) - m_0(r) \leq a. \quad \square$$

In the previous result, it is important to observe that the auxiliary function satisfies $h \geq n-1$ (and it is nondecreasing) so that we may only possibly recover sharpness if $a = 0$. For this reason, in what follows, we will consider this case.

It is now straightforward to get comparisons for the volume element based on lower bounds for the Bakry-Émery Ricci tensors. Indeed, firstly consider the coefficient function of the measure $e^{-f}dvol_g$ induced by the coordinates (2.5), which we denote here by $e^{-f}\det J_\xi$, i.e.,

$$\Phi^*\left(e^{-f}dvol_g\right) = e^{-(f\circ\Phi)}\det J_\xi d\sigma \wedge dr,$$

then

$$\frac{\partial}{\partial t}\log\left(e^{-f}\det J_\xi\right) = \left(-\partial_t f + \operatorname{tr} U\right) = m_f.$$

Secondly, define the function $\theta : \Sigma \times [0,\infty) \to \mathbb{R}$ by

$$\theta(t,x) = \frac{e^{-f(x)}\det J_{\xi_x}}{\left(1 + \frac{\mathrm{H}_f(x)}{L}t\right)^L}, \quad \text{where } L \in \mathbb{Z},\ L \geq 1\ \ (f = f\circ\Phi).$$

It is immediate that

$$\frac{\partial}{\partial t}\theta = \theta\left\{m_f - \frac{\mathrm{H}_f}{1 + \frac{\mathrm{H}_f}{L}t}\right\}.$$

Thirdly, and finally, on the one hand, if we have a nonnegative N-Bakry-Émery Ricci curvature $\mathrm{Ric}_f^N \geq 0$ then take $L = n + N - 1$ so that by Lemma 4.1.1, we get

$$\frac{\partial}{\partial t}\theta = \theta\left\{m_f - m_0\right\} \leq 0.$$

On the other hand, if we have a nonnegative ∞-Bakry-Émery Ricci tensor $\mathrm{Ric}_f \geq 0$ and, moreover, if $\mathrm{H}_f \geq 0$ and $\partial_r f \geq 0$, then take $L = n - 1$ and conclude, by Lemma 4.1.2, that

$$\frac{\partial}{\partial t}\theta = \theta\left\{m_f - m_0\right\} \leq 0.$$

In each case the function θ is nonincreasing for each $x \in \Sigma$ so that $\theta \leq \theta(0) = e^{-f(x)}$.

4.2 The notion of f-asymptotic volume ratio

In this section we establish the corresponding notion of **a**symptotic **v**olume **r**atio (AVR) when we have nonnegativity of the Bakry-Émery Ricci curvature.

Suppose (M, g) is a noncompact, complete, n-dimensional Riemannian manifold and let $p_0 \in M$ be an arbitrary (base) point. Let $r^\bullet(w) = \mathrm{dist}_g(w, p_0)$ be the distance to the point p_0.

- If $\mathrm{Ric}_f^N \geq 0$ for some $N \geq 1$ then it is well known (see e.g. [44], [5], [35]) that

  (f-$\mathcal{VR}$) the function $(0, \infty) \ni R \mapsto \Theta^f(R) \doteq \frac{vol_f(B_{p_0}(R))}{\omega_{n+N} R^{n+N}}$ is nonincreasing.

- If $\mathrm{Ric}_f \geq 0$ and, in addition to that, $\partial_{r^\bullet} f \geq 0$ along minimal geodesic segments from p_0 then it is known (see e.g. [52]) that

  (f-$\mathcal{VR}$) the function $(0, \infty) \ni R \mapsto \Theta^f(R) \doteq \frac{vol_f(B_{p_0}(R))}{\omega_n R^n}$ is nonincreasing.

Now, regardless of these two curvature assumptions, we have exhibited a monotonous quantity, which is denoted by the same symbol, namely Θ^f. Henceforth, we convention that the quantity we are referring to depend on which curvature assumption is in effect. Based on this, we may introduce the ***f*-Asymptotic Volume Ratio** by

$$f\text{-}AVR(g) = \lim_{R \to \infty} \Theta^f(R). \tag{4.6}$$

Example 4.2.1. [*Gaussian soliton*]
Let $M \doteq \mathbb{R}^n$ with the usual Euclidean metric $\bar{g}$ and let $f(x) = (\lambda/2)|x|^2$, so that $\bar{D}^2 f = \lambda \bar{g}$ and $\mathrm{Ric}_f = \lambda \bar{g}$. This example shows that, although the underlying ambient is noncompact, we may have $\mathrm{Ric}_f \geq \lambda \bar{g}$ and $\lambda > 0$. Moreover, still in the case $\lambda > 0$, we have

$$\lim_{R \to \infty} vol_f(B_{\mathbf{0}}(R)) = \int_{\mathbb{R}^n} e^{-\frac{\lambda}{2}|x|^2} dx < \infty$$

and $\partial_{r^\bullet} f = \langle \bar{\nabla} f, \bar{\nabla} r \rangle = \lambda |x| > 0$, where $r^\bullet(x) = |x|$, so that

$$f\text{-}AVR(\bar{g}) = 0. \quad \square$$

The previous example is a particular case of a more general scenery, as is shown in the next result (see [52], pp. 390 or [38]).

Theorem 4.2.2. [*Volume finiteness I*]
Let (M, g) be complete Riemannian manifold and $f \in C^\infty(M)$ such that $\mathrm{Ric}_f \geq \lambda > 0$. Then $vol_f(M)$ is finite and M has finite fundamental group.

For the $N < \infty$ case, we can reach stronger conclusions and draw comparisons with the classical Myers' Theorem ([51], [39]).

Theorem 4.2.3. [*Volume finiteness II*]
Let (M, g) be a complete Riemannian manifold and let $f \in C^\infty(M)$ be a function such that $\mathrm{Ric}_f^N \geq (n-1)H > 0$, with $N < \infty$. Then, M is compact and $diam_g(M) \leq \sqrt{\frac{n+N-1}{n-1}}\frac{\pi}{\sqrt{H}}$.

Needless to say, under the circumstances described in the theorem above, we have f-$AVR(g) = 0$. At this point, it is clear that, in the case $\mathrm{Ric}_f \geq 0$, we have made two assumptions about the (potential) function f. The first is that, to accomplish mean curvature comparison, we needed to assume that the time derivatives of f along minimizing geodesics from the boundary are nonnegative. The second is that, in order to have a well defined asymptotic volume ratio, we needed that the time derivative of f along radial geodesics starting at a given base point, are nonnegative. For the purpose of obtaining a Willmore-type inequality, we need these two conditions to be satisfied at once.

For example, these two conditions are fulfilled if we let our $\Omega \subset M$ be a connected, open and bounded set with smooth boundary Σ and assume that there is a point $p_0 \in \Omega$ from which all radial geodesics starting at p_0 meet Σ perpendicularly. Then, take $f \in C^\infty(M)$ having nonnegative partial derivatives along radial geodesics emanating from p_0.

4.3 Willmore-type inequality for nonnegative Bakry-Émery Ricci curvature

In this section we establish the Willmore-type inequality for the Bakry-Émery Ricci tensor. For convenience of the reader, we enunciate it here. We start with the $N < \infty$ case.

Theorem 4.3.1. [*Willmore-type inequality for nonnegative N-Bakry-Émery*]
Let (M, g) be a noncompact, complete n-dimensional Riemannian manifold with nonnegative N-Bakry-Émery Ricci curvature. Let Ω be an open and bounded set with smooth boundary $\Sigma = \partial\Omega$, whose mean curvature is H. Then

$$\int_\Sigma \left|\frac{\mathrm{H}_f(x)}{n+N-1}\right|^{n+N-1} e^{-f(x)} d\sigma(x) \geq f\text{-}AVR(g)|\mathbb{S}^{n+N-1}| \tag{4.7}$$

where f-$AVR(g)$ is the f-asymptotic volume ratio of g and $\mathrm{H}_f = \mathrm{H} - \langle \xi, grad_g f\rangle$.

Proof. As in the previous chapter, denote by $\mathcal{T}_\Omega(R)$ the geodesic tube of radius R about Ω. Then, compute the volume of this tube with respect to the measure $e^{-f} dvol_g$ to get

$$
\begin{aligned}
vol_f(\mathcal{T}_\Omega(R)) &= vol_f(\Omega) + \int_\Sigma \int_0^{R\wedge\tau_c(x)} e^{-f(x)} \det J_{\xi(x)}(t)\,dt\,d\sigma(x) \\
&\leq vol_f(\Omega) + \int_\Sigma \int_0^{R\wedge\tau_c(x)} e^{-f(x)} \left(1 + \frac{\mathrm{H}_f(x)}{n+N-1} r\right)^{n+N-1} dt\,d\sigma(x) \\
&\leq vol_f(\Omega) + \int_\Sigma \int_0^{R\wedge\tau_c(x)} e^{-f(x)} \left(1 + \frac{\mathrm{H}_f^+(x)}{n+N-1} r\right)^{n+N-1} dt\,d\sigma(x) \\
&\leq vol_f(\Omega) + \int_\Sigma \int_0^{R} e^{-f(x)} \left(1 + \frac{\mathrm{H}_f^+(x)}{n+N-1} r\right)^{n+N-1} dt\,d\sigma(x) \\
&= vol_f(\Omega) + \frac{R^{n+N}}{n+N} \int_\Sigma e^{-f(x)} \left(\frac{\mathrm{H}_f^+(x)}{n+N-1}\right)^{n+N-1} d\sigma(x) + \mathcal{O}(R^{n+N-1}),
\end{aligned}
$$

Dividing both sides of the inequality above by $\omega_{n+N} R^{n+N} = |\mathbb{S}^{n+N-1}| R^{n+N}/(n+N)$ and letting $R \to \infty$ we obtain

$$
f\text{-}AVR(g) \leq \frac{1}{|\mathbb{S}^{n+N-1}|} \int_\Sigma \left(\frac{\mathrm{H}_f^+(x)}{n+N-1}\right)^{n+N-1} e^{-f(x)} d\sigma(x),
$$

which clearly implies (4.7). □

Corollary 4.3.2. $\big[$*Non-existence of closed, f-minimal hypersurfaces I* $\big]$
If (M, g) is complete and noncompact with $\mathrm{Ric}_f^N \geq 0$ then there is no f-minimal hypersurface Σ realized as the boundary of an open and bounded set, provided f-$AVR(g) > 0$.

Next, we establish the $N = \infty$ case.

Theorem 4.3.3. $\big[$*Willmore-type inequality for nonnegative ∞-Bakry-Émery* $\big]$
Let (M, g) be a noncompact, complete n-dimensional Riemannian manifold with nonnegative ∞-Bakry-Émery Ricci curvature. Let Ω be an open and bounded set with smooth boundary $\Sigma = \partial\Omega$, whose mean curvature is H. If $\partial_r f \geq 0$ and $\mathrm{H}_f \geq 0$ then

$$
\int_\Sigma \left|\frac{\mathrm{H}_f(x)}{n-1}\right|^{n-1} e^{-f(x)} d\sigma(x) \geq f\text{-}AVR(g)|\mathbb{S}^{n-1}| \tag{4.8}
$$

where f-$AVR(g)$ is the f-asymptotic volume ratio of g and $\mathrm{H}_f = \mathrm{H} - \langle \xi, grad_g f\rangle$.

Proof. We have

$$
\begin{aligned}
vol_f(\mathcal{T}_\Omega(R)) &= vol_f(\Omega) + \int_\Sigma \int_0^{R\wedge\tau_c(x)} e^{-f(x)} \det J_{\xi(x)}(t)\,dt\,d\sigma(x) \\
&\leq vol_f(\Omega) + \int_\Sigma \int_0^{R} e^{-f(x)} \left(1 + \frac{\mathrm{H}_f(x)}{n-1} r\right)^{n-1} dt\,d\sigma(x) \\
&= vol_f(\Omega) + \frac{R^n}{n} \int_\Sigma e^{-f(x)} \left(\frac{\mathrm{H}_f(x)}{n-1}\right)^{n-1} d\sigma(x) + \mathcal{O}(R^{n-1}),
\end{aligned}
$$

Dividing both sides of the inequality above by $\omega_n R^n = |\mathbb{S}^{n-1}|R^n/n$ and letting $R \to \infty$ we obtain (4.8). $\square$

Corollary 4.3.4. $\big[$*Non-existence of closed, f-minimal hypersurfaces II* $\big]$
Under the same conditions of Theorem 4.3.3, if f-$AVR(g) > 0$ then there is no f-minimal hypersurface in M realized as the boundary of an open and bounded set.

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APPENDIX A

REMAINING CONSIDERATIONS

In this final chapter we discuss in more depth some topics left along the text. Firstly, regarding the Willmore-type inequality in spaces of nonnegative Ricci curvature, we provide sufficient conditions on the open set Ω that guarantees that the whole space is isometric to the Euclidean space. Secondly, we show how to obtain a weaker version of the Willmore-type inequality using Sobolev inequalities, as discussed in the introduction.

A.1 The geometry of Ω

Here we consider the equality case in equation (1.7), i.e., suppose

$$\int_{\Sigma} \left| \frac{\mathrm{H}}{n-1} \right|^{n-1} d\sigma = AVR(g)|\mathbb{S}^{n-1}| > 0.$$

Then, as described in [3], (M, g) has Euclidean Volume Growth and $M \setminus \Omega$ splits isometrically as the warped product $[r_0, \infty) \times_h \Sigma$ (which is $\therefore$ endowed with metric $dr \otimes dr + h(r)^2 g_{\Sigma}$, where $h(r) = r/r_0$) and Σ is a connected totally umbilic hypersurface with constant mean curvature H. The nonnegativity of the Ricci tensor together with elementary properties of warped product spaces (cf. Corollary 2.2.2. in [25]) gives

$$\mathrm{Ric}_{\Sigma} \geq (n-2)\frac{1}{r_0^2} g_{\Sigma}.$$

It then becomes noteworthy to ask what geometric properties of Ω, or its boundary, prevent the whole space to be isometric with $\mathbb{R}^n$ with standard Euclidean metric. It is straightforward to check that if the diameter or the area of Σ fulfills, respectively, $\mathrm{diam}_g(\Sigma) \geq \pi r_0$ or $|\Sigma| \geq r_0^{n-1}|\mathbb{S}^{n-1}|$ then, in fact, (M, g) is isometric to the Euclidean space. An open question is if $vol(\Omega) = \omega_n$ yields the same conclusion.

In a different direction since $\bar{\Omega}$ is compact with MCB we may use a sharp result for $\bar{\Omega}$ based only on its intrinsic distance to yield global conclusions. More specifically, by a theorem of Li [34], since Σ has constant mean curvature, say $\mathrm{H} = (n-1)k > 0$, we have

$$\sup_{x \in \Omega} \operatorname{dist}_g(x, \Sigma) \leq \frac{1}{k}. \tag{A.1}$$

Furthermore, equality in (A.1) holds if and only if $\bar{\Omega}$ is isometric to a closed Euclidean ball of radius $1/k$, in which case $AVR(g) = 1$ so that (M, g) is isometric to the Euclidean space.

A.2 A weaker Willmore-type inequality

We now describe how to obtain a weaker version of the Willmore-type inequality using Sobolev inequalities in spaces of nonnegative sectional curvature. Let $\Sigma^{n-1} \subset M$ be any closed hypersurface in a complete noncompact Riemannian manifold (M, g) with nonnegative sectional curvature. Assume $\Sigma = \partial\Omega$, where Ω is open and bounded in M. Then, take $f = 1$ over Σ and, via the identification $M \cong M \times \{0\}$, realize Σ as a codimension 2 submanifold in $M \times \mathbb{R}$, which is endowed with the product metric $\tilde{g} = g + dr \otimes dr$, apply Corollary 1.5 from [10] to get

$$\int_\Sigma |\vec{H}| d\sigma \geq (n-1)|\mathbb{B}^{n-1}|^{\frac{1}{n-1}} |AVR(\tilde{g})^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}}, \tag{A.2}$$

where $\mathbb{B}^k$ is the euclidean unit ball in $\mathbb{R}^k$. A simple application of Hölder's inequality with $p = n-1$ and $q = (n-1)/(n-2)$ on the left hand side of the above yields

$$\left(\int_\Sigma |\vec{H}|^{n-1} d\sigma \right)^{\frac{1}{n-1}} \geq (n-1)|\mathbb{B}^{n-1}|^{\frac{1}{n-1}} AVR(g)^{\frac{1}{n-1}},$$

where we have used $AVR(g) = AVR(\tilde{g})$. As a result,

$$\int_\Sigma \left| \frac{\vec{H}}{n-1} \right|^{n-1} d\sigma \geq \frac{|\mathbb{S}^{n-2}|}{n-1} AVR(g).$$

A.3 A generalization of Bishop-Gromov

The next theorem is from [56] and is stated here for convenience of the reader.

Theorem A.3.1. [*Volume Comparison for asymptotic nonnegatively curved spaces*] *Let (M, g) be a complete open n-dimensional Riemannian manifold with*

$$\mathrm{Ric}_w \geq -(n-1)(\lambda \circ dist_{p_0})(w),$$

for some associated function λ satisfying (1.1). Then,

$$\frac{vol(B_{p_0}(R))}{vol(B_{p_0}(r))} \leq e^{(n-1)b_0} \left(\frac{R}{r} \right)^n, \quad R \geq r > 0.$$

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