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Raquel Salgado Saraiva

Examples of discontinuity for the Lyapunov exponents

Belo Horizonte
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Raquel Salgado Saraiva

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Examples of discontinuity for the Lyapunov exponents

RAQUEL SALGADO SARAIVA

Tese defendida e aprovada pela banca examinadora constituída por:

Profa. Karina Daniela Marín
Orientadora - UFMG

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Prof. Pablo Daniel Carrasco Correa
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Belo Horizonte, 05 de fevereiro de 2026.

Av. Antônio Carlos, 6627 – Campus Pampulha - Caixa Postal: 702
CEP-31270-901 - Belo Horizonte – Minas Gerais - Fone (31) 3409-5963
e-mail: pgmat@mat.ufmg.br - home page: http://www.mat.ufmg.br/pgmat

Abstract

This work is concerned with the study of mechanisms producing discontinuity of the Lyapunov exponents viewed as functions of linear cocycles. More specifically, we work in the setting where the base dynamics is a shift on the space of bi-lateral sequences over a finite alphabet, endowed with the Bernoulli probability measure.

The examples we construct consist of families of non–fiber-bunched Hölder continuous linear cocycles taking values in $\mathrm{SL}(2,\mathbb{R})$. Under suitable conditions, we show that such cocycles can be approximated in the C^{α}-topology by cocycles with vanishing exponents. In particular, this answers a question posed by Butler on whether, in a specific setting, a cocycle may be approximated by cocycles with zero Lyapunov exponents.

Our results provide new examples and further evidence of the complexity of the discontinuity phenomenon for Lyapunov exponents.

Keywords: Dynamical systems; ergodic theory; linear cocycles; Lyapunov exponents; continuity and discontinuity.

Resumo

Este trabalho trata do estudo de mecanismos que produzem descontinuidade dos expoentes de Lyapunov quando vistos como funções de cociclos lineares. Mais especificamente, trabalhamos no contexto em que a dinâmica de base é o shift no espaço de sequências bilaterais sobre um alfabeto finito, munido da medida de probabilidade de Bernoulli.

Os exemplos que construímos consistem em famílias de cociclos lineares Hölder contínuos não fiber-bunched com valores em $\mathrm{SL}(2,\mathbb{R})$. Sob condições adequadas, mostramos que tais cociclos podem ser aproximados na topologia C^α por cociclos com expoentes de Lyapunov nulos. Em particular, este resultado responde a uma questão proposta por Butler sobre se, em um contexto específico, um cociclo pode ser aproximado por cociclos com expoentes de Lyapunov nulos.

Nossos resultados fornecem novos exemplos e evidências adicionais da complexidade do fenômeno de descontinuidade dos expoentes de Lyapunov.

Palavras-chave: Sistemas dinâmicos; teoria ergódica; cociclos lineares; expoentes de Lyapunov; continuidade e descontinuidade.

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Introduction

The definition of Lyapunov exponents was introduced in stability theory for differential equations by A. M. Lyapunov, in his thesis, in the late nineteenth century. In ergodic theory, the existence of Lyapunov exponents was later established by results of Furstenberg-Kesten [12] and Oseledets [21]. Since then, this topic has become a central area of dynamical systems, interacting with several areas of Mathematics, such as random matrix products in stochastic processes and smooth dynamical systems through Pesin's theory of non-uniform hyperbolicity [22]. Moreover, Lyapunov exponents are related with metric entropy of smooth dynamical systems and the geometry of measures, as explored by Ruelle [24], Pesin [23] and Ledrappier-Young [15], [16].

The Lyapunov exponents describe the asymptotic behavior of orbits under a given dynamical system. Focusing in the theory of Lyapunov exponents for linear cocycles, an interesting question is how the Lyapunov exponents vary as functions of the cocycle. This problem has motivated significant research over the past decades. The answer depends strongly on the topology considered and in the type of the constructed pertubation. For our discussion, we are concerned with the case of cocycles taking values in $\mathrm{SL}(2,\mathbb{R})$.

In this context, Bochi in [5] showed that if a continuous cocycle over a fixed ergodic, aperiodic, invertible map on a compact space is a C^0-continuity point for Lyapunov exponents, then it must be either uniformly hyperbolic or has zero Lyapunov exponents. Thus, continuity holds only at cocycles that are either uniformly hyperbolic or have trivial spectra. As a consequence, discontinuity of Lyapunov exponents is typical. A similar conclusion, previously stated by Mañé [19] in the 1980s, holds when restricted to the class of derivative cocycles of area-preserving C^1 diffeomorphisms on surfaces.

While discontinuities of Lyapunov exponents are common, there are notable situations in which continuity has been verified. For random products of 2×2 matrices, continuity was established in both the Bernoulli and Markov settings by Bocker and Viana [7] and by Malheiro and Viana [17] respectively. In higher dimensions the picture is more subtle, but recent work by Avila, Eskin, and Viana [2] announced continuity for i.i.d. random products of matrices, indicating that such stability is not restricted to the low-dimensional case.

For Hölder continuous cocycles, Backes, Brown and Butler [4] proved that, when restricted to the set of cocycles satisfying the quasi-conformality condition, known as *fiber-bunching*, over hyperbolic homeomorphism systems on compact metric space and with invariant probability measure with local product structure, the Lyapunov exponents vary continuously. Their result applies to cocycles admitting invariant holonomies, a property ensured by the fiber-bunching condition.

Within this context, a natural question is formulated: what can be said about the behavior of Hölder continuous cocycles that fail to be fiber-bunched? This problem, which remains open, guided the development of this thesis. In the following theorem, which was obtained in joint work with E. Mamani, the base dynamics is the Bernoulli shift on a space M of bi-lateral sequences over an alphabet of $l+1$ symbols, $l \geq 1$, endowed with the probability measure $\mu = \left(\sum_{i=0}^{l} p_i \delta_i\right)^{\mathbb{Z}}$, where p_i are positive numbers and $\sum_{i=0}^{l} p_i = 1$.

Theorem A. *For any $\alpha > 0$, let $A : M \to \mathrm{SL}(2,\mathbb{R})$ be a cocycle such that, for $i \in \{0, \cdots, l\}$,*

$$A(x) = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix} \quad \text{if } x \in [0; i], \tag{1}$$

where there exist distinct indices $j, r \in \{0, \cdots, l\}$ such that $a_j < 1 < a_r$, and positive weights $(p_0, \ldots, p_l)$ satisfying that $\lambda_+(A, \mu) \neq 0$. Define

$$\eta = \min\{a_j^{-1}, a_r\} \quad \text{and} \quad \sigma = \max\{a_j^{-1}, a_r\}.$$

If

$$2^{3\alpha} < \sigma^2$$

and

$$2^{\left(2 + \frac{\log \eta}{\log \sigma}\right)\alpha} \leq \eta^2$$

then A is a discontinuity point for Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$.

We also show that such a cocycle A can be approximated by perturbations with zero Lyapunov exponents. The proof of the vanishing of the Lyapunov exponents in Theorem A uses a new argument that relies on the explicit form of the perturbations. This provides an original approach to establishing the nullity of the Lyapunov exponents.

Bocker and Viana [7] gave examples of cocycles defined by two matrices as in (1), with the same expansion constant, $a_1 = \sigma = a_0^{-1}$, and proved the existence of approximations by cocycles with zero Lyapunov exponents via a contrapositive argument. More precisely, they constructed examples of locally constant cocycles with non-zero Lyapunov exponents, very far from being fiber-bunched that are discontinuity points for Lyapunov exponents in the Hölder topology. Improving the techniques used in this counter-example, Butler [6] obtained non-fiber-bunched cocycles arbitrarily close to being it and still approximable in the Hölder topology by cocyles with arbitrarily small Lyapunov exponents. However, his approach does not allow the cocycle in his example to be approximated by cocycles with zero Lyapunov exponents. In this sense, his result provides a form of flexibility for Lyapunov exponents that does not include the value zero. In that setting, Butler asked [6] whether it is possible to construct approximated cocycles with zero Lyapunov exponents. The next result, a direct corollary of our main theorem, gives a partial answer to this question.

Corollary 1. *For any $\alpha > 0$ and let $A : \{0,1\}^{\mathbb{Z}} \to \mathrm{SL}(2,\mathbb{R})$ be a locally constant cocycle defined as in (1) with $a_0 = \sigma^{-1}$ and $a_1 = \sigma$, for some $\sigma > 1$ that satisfies*

$$\sigma^2 \geq 2^{3\alpha}.$$

Then, there exist α-Hölder continuous cocycles $B : \{0,1\}^{\mathbb{Z}} \to \mathrm{SL}(2,\mathbb{R})$ with zero Lyapunov exponents that are arbitrarily close to A in the α-Hölder topology. In particular, A is a discontinuity point for the Lyapunov exponents in the space $C^\alpha\left(\{0,1\}^{\mathbb{Z}}, \mathrm{SL}(2,\mathbb{R})\right)$.

Continuing in the setting of locally constant cocycles over a left shift on $l+1$ symbols, we obtain another result on discontinuity points of the Lyapunov exponents. Here, the hypothesis is weaker than in Theorem A, as Theorem B allows more freedom in the choice of the horizontal contraction constant leading to discontinuity.

Theorem B. *In the same setting of Theorem A, let $(p_0, \cdots, p_l)$ be any set of weights such that*

$$2^\alpha < \prod_{i=0}^{l} a_i^{2p_i}$$

and there exists an indice $j \in \{0, \cdots, l\}$ such that

$$2^{2\alpha} \leq \left(a_j^{-1}\right)^2,$$

then for every value $\kappa \in (0, \lambda_+(A, \mu)]$ there exists an α-Hölder continuous cocycle B, arbitrarily close to A, such that $\lambda_+(B, \mu) = \kappa$. In particular, A is a discontinuity for the Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2, \mathbb{R}))$.

We also present examples of discontinuity points for the Lyapunov exponents in a topology weaker than the C^α-topology. This topology is induced by the following norm: for $0 < \delta < 1$, and $A \in C^0(M, \mathrm{SL}(2, \mathbb{R})$,

$$\|A\|_{\delta-\log} = \sup_{x \in M} \|A(x)\| + \sup_{x \neq y \in M} \left\{ \|A(x) - A(y)\| \left(\log \frac{1}{d(x,y)} \right)^\delta \right\}$$

where d is the distance on M.

Following the same principle used previously, that is, constructing perturbations that interchange the Oseledets subspaces, we carry out this procedure for a locally constant cocycle A generated by a hyperbolic matrix and the identity, that is, for $\sigma > 1$,

$$\begin{aligned} A : \{0,1\}^{\mathbb{Z}} &\to \mathrm{SL}(2, \mathbb{R}) \\ x &\mapsto A(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x_0 = 0, \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1. \end{cases} \end{aligned}$$

We obtain the following:

Theorem C. *A is a discontinuity point for the Lyapunov exponents in the $C_{\delta-\log}$-topology for every $0 < \delta < 1$. Furthermore, in this topology, A can be approximated by cocycles with vanishing Lyapunov exponents.*

The remainder of the thesis is organized as follows. In Chapter 1, we review some important definitions and classical results that will be used in the rest of the text, including the definitions of linear cocycles, Lyapunov exponents and induced cocycles. In Chapter 2, we revisit the example constructed by Bocker and Viana. Chapter 3 contains the construction of the cocycle A, the statement of Theorem A, its proof, and associated corollaries. In Chapter 4, we refine the hypotheses of Theorem A and still obtain discontinuity points in Theorem B, although we no longer guarantee the existence of perturbations with zero Lyapunov exponents. Chapter 5, investigates discontinuity phenomena for cocycles involving the identity matrix, introduces an intermediate topology between the Hölder and the C^0 topologies, and establishes Theorem C. Finally, in Chapter 6, we discuss several open problems related to this framework.

Chapter 1

Preliminaries

The purpose of this chapter is to present the fundamental notions and ideas that form the basis of our analysis in this thesis and on which our main results are formulated.

In Section 1.1, we introduce the central topic of study and provide illustrative examples, culminating in the definition of a special class of functions that play an important role in our work. In Section 1.2, we define the Lyapunov exponents, the key functions whose asymptotic behavior we aim to investigate. Finally, in Section 1.3, we examine a new dynamical system induced by the original object we are considering and establish some important properties of this derived system.

1.1 Linear cocycles

Linear cocycles are the basic object in our study. This framework naturally appears in the study of products of random matrices, differential equations with time-dependent coefficients, and Lyapunov exponents in dynamical systems.

Intuitively, a cocycle describes how a linear action evolves along the orbits of a dynamical system. In other words, while the base map governs the dynamics on the space, the cocycle records the way linear transformations are composed as we follow the trajectory of a point under the dynamic.

We give precisely their definition in this section and introduce some examples. Our focus is on the two dimensional case.

Let $(X, \mathcal{C}, \nu)$ be a probability space, where X is a compact set. Consider the product space $M = X^{\mathbb{Z}}$ endowed with the product σ-algebra $\mathcal{B} = \mathcal{C}^{\mathbb{Z}}$ and the product measure $\mu = \nu^{\mathbb{Z}}$. Let $f : M \to M$ be the shift map, which is a measure-preserving transformation, defined on M by

$$\begin{aligned} f : \qquad & M \to M \\ & (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}. \end{aligned}$$

Consider $A : M \to \mathrm{SL}(2, \mathbb{R})$ a measurable function taking values in the special linear group $\mathrm{SL}(2, \mathbb{R})$ of real 2×2 matrices with determinant 1. This will be the setting in which the thesis is developed.

The *linear cocycle* generated by A over the base f is

$$\begin{aligned} F_A : M \times \mathbb{R}^2 &\to M \times \mathbb{R}^2 \\ (x, v) &\mapsto (f(x), A(x)v). \end{aligned}$$

This map acts simultaneously on the base point x and on the vector v in the fiber $\mathbb{R}^2$, the base evolves according to f, and the fiber is transformed linearly by $A(x)$.

Note that, for every integer $n \geq 1$, the n-th iterate of F_A is given by

$$F_A^n(x, v) = (f^n(x), A^n(x)v),$$

where

$$A^n(x) = A(f^{n-1}(x)) \cdots A(x). \tag{1.1}$$

Thus, $A^n(x)$ represents the product of matrices along the first n iterates of x.

Since f is invertible, then F_A is invertible as well, and, in this case, we have

$$F_A^{-n}(x, v) = (f^{-n}(x), A^{-n}(x)v),$$

where

$$A^{-n}(x) = A(f^{-n}(x))^{-1} \cdots A(f^{-1}(x))^{-1} = A^n(f^{-n}(x))^{-1}.$$

Since the base map f is fixed, we will denote the cocycle F_A by A itself.

Example 1.1. *(Random products)*

A classical example of a linear cocycle arises from random products.

Define the map $A : M \to \mathrm{SL}(2, \mathbb{R})$ as the function that depends only on the zero-th coordinate. That is, for each point in $x = (x_i)_{i\in\mathbb{Z}} \in M$,

$$A((x_i)_{i\in\mathbb{Z}}) = B(x_0),$$

where $B : X \to \mathrm{SL}(2, \mathbb{R})$ is a given function.

The associated cocycle $F_A : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$ defined by A over f is:

$$F_A((x_i)_{i\in\mathbb{Z}}, v) = ((x_{i+1})_{i\in\mathbb{Z}}, B(x_0)v).$$

Observe that the n^{th} iterate of A is given by

$$A^n(x) = A(f^{n-1}(x)) \cdots A(x) = B(x_{n-1}) \cdots B(x_0),$$

which corresponds exactly to the product of n consecutive random matrices.

Such a cocycle is called *locally constant linear cocycle*, meaning that the generating function A depends only on a finite number of coordinates, in this case, only on the zero-th one. Locally constant cocycles are useful because they let us build discrete, symbolic models of more complicated dynamical systems. Even though they are simple, they can still show interesting behavior, like discontinuities in the Lyapunov exponents.

The next example is another locally constant linear cocycle, but one that plays a more specific role in our study. It will be analyzed in detail in Chapter 2 and in Chapter 5 as a counterexample to the continuity of Lyapunov exponents.

Example 1.2. *(A Symbolic Cocycle with Two Matrices)*

Let $M = \{0, 1\}^{\mathbb{Z}}$ the space of bi-infinite sequences of zeros and ones, equipped with the product measure $\mu_p = \nu_p^{\mathbb{Z}}$ where ν_p is the probability measure on $\{0, 1\}$ defined by $\nu_p = p\delta_1 + (1 - p)\delta_0$ and p is positive.

Given parameters $\sigma \geq \eta > 1$, define the measurable function $\mathrm{A}_{\sigma\eta} : M \to \mathrm{SL}(2, \mathbb{R})$ by

$$\mathrm{A}_{\sigma\eta}(x) = \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} \text{ if } x_0 = 0$$

and

$$\mathrm{A}_{\sigma\eta}(x) = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \text{ if } x_0 = 1.$$

That is, the cocycle multiplies the vector by one of two diagonal matrices, depending on the current symbol at the zero coordinate.

Since the matrices are diagonal, their products remain diagonal and depend only on how many times each symbol, 1 or 0, appears in the first n iterates of $x \in M$. Explicitly,

$$A^n_{\sigma\eta}(x) = \begin{pmatrix} \sigma^{S_n(x)}\eta^{-(n-S_n(x))} & 0 \\ 0 & \sigma^{-S_n(x)}\eta^{n-S_n(x)} \end{pmatrix}, \tag{1.2}$$

where $S_n(x)$ and $n - S_n(x)$ are, respectively, the quantity of how many times the symbols 1 and 0 appear in the point x up to iterate n, that is,

$$\begin{aligned} S_n(x) &= \#\{j = 0, ..., n-1; f^j(x) \in [0;1]\} \\ &= \sum_{j=0}^{n-1} \chi_{[0;1]} f^j(x), \end{aligned}$$

and

$$\begin{aligned} n - S_n(x) &= \#\{j = 0, ..., n-1; f^j(x) \in [0;0]\} \\ &= \sum_{j=0}^{n-1} \chi_{[0;0]} f^j(x). \end{aligned}$$

1.1.1 Hyperbolic Cocycles

We now introduce a class of cocycles whose behavior is well understood.

In order to work with continuous cocycles and to obtain uniform estimates, we endow the product space $M = X^{\mathbb{Z}}$ with a natural metric that makes it compact and metrizable. Thus, assume that (X, d_X) is a compact metric set and consider the standard product metric on M defined by

$$d(x, y) = \sup_{i \in \mathbb{Z}} \frac{d_X(x_i, y_i)}{2^{|i|}}, \tag{1.3}$$

where $x = (x_i)_{i\in\mathbb{Z}},\ y = (y_i)_{i\in\mathbb{Z}} \in M$. With this metric, (M, d) is a compact metric space. Note also that the shift map $f : M \to M$ is a homeomorphism.

We say that a continuous cocycle $A : M \to \mathrm{SL}(2, \mathbb{R})$ is (uniformly) *hyperbolic* if there exist constants $C_0 > 0$ and $0 < \zeta < 1$ such that, for every $x \in M$, the vector space $\mathbb{R}^2$ admits a splitting

$$\mathbb{R}^2 = E^s_x \oplus E^u_x$$

with the following properties:

i) **Invariance of the splitting:** $A(x)E^s_x = E^s_{f(x)}$ and $A(x)E^u_x = E^u_{f(x)}$;

ii) **Uniform contraction/expansion:** for every $v^s \in E^s_x$, $v^u \in E^u_x$ and $n \geq 1$

$$\|A^n(x)v^s\| \leq C_0\zeta^n\|v^s\| \text{ and } \|A^{-n}(x)v^u\| \leq C_0\zeta^n\|v^u\|.$$

($\|.\|$ is taken to be the standard Euclidean norm on $\mathbb{R}^2$).

Also, there is a useful and classical characterization of hyperbolicity, which states that: a cocycle A is hyperbolic if and only if there exist constants $C_1 > 0$ and $\vartheta > 1$ such that

$$\|A^n(x)\| \geq C_1\vartheta^n \quad \text{for every } x \in M \text{ and } n \geq 1. \tag{1.4}$$

This criterion expresses hyperbolicity through uniform exponential growth of the norms of the iterates of the cocycle. In particular, it immediately excludes from the definition of hyperbolicity all cocycles whose behavior oscillates or remains bounded along some orbit (see Chapter 2 of [26] for more details).

Let us revisit the cocycle introduced in Example 1.2 with parameter $\sigma = \eta$. From Equation (1.2), we have for every $x \in M$ and $n \geq 1$

$$\|A_\sigma^n(x)\| = \sigma^{|S_n(x)-(n-S_n(x))|}.$$

Thus, the norm depends only on how unbalanced the number of symbols 1 and 0 is along the orbit segment.

Then, consider the periodic point $\tilde{x} \in M$ whose symbolic sequence consists of n_0 consecutive symbols 1 followed by n_0 consecutive symbols 0, and this block is repeated periodically, for some $n_0 \geq 1$. The period of this point is $2n_0$.

For any $k \geq 1$,

$$\|A_\sigma^{k2n_0}(\tilde{x})\| = \sigma^{|kn_0-kn_0|} = 1.$$

Along this periodic orbit, the cocycle exhibits no growth at all on the subsequence of times $2kn_0$. Consequently, it is impossible to find constants $C_1 > 0$ and $\vartheta > 1$ satisfying the hyperbolicity condition (1.4) for every $k \geq 1$ and for this $\tilde{x} \in M$. This shows that the cocycle A_σ is not hyperbolic. When $\sigma \neq \eta$, the cocycle $\mathrm{A}_{\sigma\eta}$ is also non-hyperbolic, as we will discuss in the next section.

1.2 Lyapunov Exponents

The classical theorem of Furstenberg-Kesten [12] provides the existence of the extremal Lyapunov exponents under general assumptions. It states that, provided a suitable integrability condition, both the norm and the co-norm of the iterates of a linear cocycle admit well-defined exponential growth rates for almost every point.

We denote by $L^1(\mu)$ the space of μ-integrable functions on M.

Theorem 1.3 (Furstenberg-Kesten). *If $\log^+\|A^{\pm}\| \in L^1(\mu)$, then the numbers*

$$\lambda_+(A,\mu,x) = \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)\| \quad \textit{and} \quad \lambda_-(A,\mu,x) = \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)^{-1}\|^{-1}, \tag{1.5}$$

exist for μ-almost every $x \in M$. Moreover, the functions $\lambda_\pm$ are f-invariant and μ-integrable with

$$\begin{aligned}\int \lambda_+\,d\mu &= \lim_n \frac{1}{n}\int \log\|A^n(x)\|\,d\mu(x),\\ \int \lambda_-\,d\mu &= \lim_n \frac{1}{n}\int \log\|A^n(x)^{-1}\|^{-1}\,d\mu(x).\end{aligned}$$

We call these limits in (1.5), when they exist, as (extremal) *Lyapunov exponents* of the cocycle A at the point $x \in M$.

This theorem is a direct consequence of Kingman's Subadditive Ergodic Theorem [13]. Note that the theorem does not assume continuity or invertibility of f, nor any kind of hyperbolicity, the result is purely measurable.

Since the Bernoulli shift (f,μ) is ergodic, then the Lyapunov exponents $\lambda_\pm$ are constants in a full measure set because they are f-invariant. In this case, we simply write $\lambda_+(A,\mu)$ and $\lambda_-(A,\mu)$.

Since A is $\mathrm{SL}(2,\mathbb{R})$-valued, there is a fundamental relation connecting the two extremal Lyapunov exponents: for μ-almost every $x \in M$,

$$\lambda_+(A,\mu) + \lambda_-(A,\mu) = 0. \tag{1.6}$$

This follows directly from the definition of norm and elementary linear algebra. Let B be a matrix in $\mathrm{SL}(2,\mathbb{R})$,

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then,

$$B^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \quad B^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}; \quad (B^{-1})^T = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix},$$

where B^T denotes the transpose matrix of B.

The norm of B is the largest eigenvalue of the symmetric matrix B^TB, that is, the largest root of its characteristic polynomial:

$$\lambda^2 - \lambda \mathrm{Tr}(B^TB) + \det(B^TB) = 0$$

which becomes

$$\lambda^2 - \lambda(a^2+b^2+c^2+d^2) + ((a^2+c^2)(b^2+d^2) - (ab+cd)^2) = 0.$$

Performing the same computation for $(B^{-1})^TB^{-1}$, we obtain the polynomial

$$\gamma^2 - \gamma \mathrm{Tr}\left((B^{-1})^TB^{-1}\right) + \det\left((B^{-1})^TB^{-1}\right) = 0$$

and a direct calculation shows that this polynomial coincides with the one above. Thus the largest eigenvalues agree, and we conclude that $\|B\| = \|B^{-1}\|$.

Since for every $x \in M$ and for every $n \geq 1$, the product $A^n(x)$ also lies in $\mathrm{SL}(2,\mathbb{R})$, and the preceding observation gives

$$\|A^n(x)\| = \|(A^n(x))^{-1}\|.$$

Hence, for μ-almost every point in M,

$$\begin{aligned}
\lambda_+(A,\mu) + \lambda_-(A,\mu) &= \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| + \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1} \\
&= \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \|A^n(x)^{-1}\|^{-1} \\
&= \lim_{n\to\infty} \frac{1}{n} \log 1 \\
&= 0.
\end{aligned}$$

Example 1.4. If the cocycle A is hyperbolic, then $\lambda_+(A,\mu) > 0$ for μ-almost every $x \in M$. Indeed, by inequality (1.4), we have, for every x and every $n \geq 1$,

$$\|A^n(x)\| \geq C_1 \vartheta^n.$$

Thus, for μ-almost every $x \in M$,

$$\lambda_+(A,\mu) \geq \lim_n \frac{1}{n} \log(C_1 \vartheta^n) = \log \vartheta > 0,$$

since $\vartheta > 1$.

Example 1.5. We now compute the Lyapunov exponents of the cocycle from Example 1.2. First, note that $\log^+ \|A^{\pm}_{\sigma\eta}\| \in L^1(\mu_p)$. We previously obtained, that for every $x \in M$ and $n \geq 1$

$$\|A^n_{\sigma\eta}(x)\| = \max\left\{\eta^{-(n-S_n(x))}\sigma^{S_n(x)}, \eta^{n-S_n(x)}\sigma^{-S_n(x)}\right\}.$$

Therefore, for μ_p-almost every $x \in M$,

$$\lambda_+(\mathrm{A}_{\sigma\eta},\mu_p) = \max\left\{\lim_{n\to\infty} \frac{1}{n} \log(\eta^{-(n-S_n(x))}\sigma^{S_n(x)}), \lim_{n\to\infty} \frac{1}{n} \log(\eta^{n-S_n(x)}\sigma^{-S_n(x)})\right\}.$$

Since $S_n(x) = \sum_{j=0}^{n-1} \chi_{[0;1]} f^j(x)$ and $n - S_n(x) = \sum_{j=0}^{n-1} \chi_{[0;0]} f^j(x)$ and since (f, μ_p) is an ergodic system, by Birkhoff's Ergodic Theorem, we have for μ_p-almost every $x \in M$,

$$\lim_n \frac{1}{n} S_n(x) = \mu_p([0;1]) = p, \tag{1.7}$$

$$\lim_n \frac{1}{n}(n - S_n(x)) = \mu_p([0;0]) = 1 - p. \tag{1.8}$$

Combining these, we obtain

$$\lambda_+(\mathrm{A}_{\sigma\eta}, \mu_p) = |(1-p)\log\eta - p\log\sigma|.$$

Using (1.6)

$$\lambda_-(\mathrm{A}_{\sigma\eta}, \mu_p) = -|(1-p)\log\eta - p\log\sigma|.$$

Example 1.6. Using the two previous examples, we see that $\mathrm{A}_{\sigma\eta}$ is not uniformly hyperbolic. Indeed, if

$$p = \frac{\log\eta}{\log\sigma\eta},$$

then $\lambda_+(\mathrm{A}_{\sigma\eta}, \mu_p) = 0$, so the cocycle cannot exhibit uniform hyperbolicity.

As we have seen, the Furstenberg–Kesten theorem provides exponential growth rates for the norm and co-norm of the cocycle. The next result, known as the Oseledets' Multiplicative Ergodic Theorem, [21], offers a more refined description. It gives growth rates for every nonzero vector and establishes an invariant splitting whenever the Lyapunov exponents are nontrivial.

Theorem 1.7 (Oseledets). *Assume* $\log^+\|A^{\pm}\| \in L^1(\mu)$*,* μ *is an ergodic measure and* $f : M \to M$ *is invertible then, for* μ*-almost every* $x \in M$*, exactly one of the following occurs:*

i) $\lambda_-(A,\mu) = \lambda_+(A,\mu)$ *and for all nonzero* $v \in \mathbb{R}^2$*,*

$$\lim_{n\to\pm\infty} \frac{1}{n}\log\|A^n(x)v\| = \lambda_\pm(A,\mu).$$

ii) $\lambda_-(A,\mu) < \lambda_+(A,\mu)$ *and there is a direct sum decomposition* $\mathbb{R}^2 = E^s_x \oplus E^u_x$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\log\|A^n(x)v\| = \begin{cases} \lambda_-(A,\mu) & \text{if } v \in E^s_x \setminus \{0\} \\ \lambda_+(A,\mu) & \text{if } v \in \mathbb{R}^2 \setminus E^s_x, \end{cases}$$

$$\lim_{n\to-\infty} \frac{1}{n}\log\|A^n(x)v\| = \begin{cases} \lambda_+(A,\mu) & \text{if } v \in E^u_x \setminus \{0\} \\ \lambda_-(A,\mu) & \text{if } v \in \mathbb{R}^2 \setminus E^u_x. \end{cases}$$

Moreover, the subspaces, called Oseledets subspaces, are A-invariant, that is,

$$A(x)E^s_x = E^s_{f(x)} \text{ and } A(x)E^u_x = E^u_{f(x)},$$

and the angle between these two lines decreases subexponentially along orbits:

$$\lim_{n\to\pm\infty} \frac{1}{n}\log|\sin\angle(E^s_{f^n(x)}, E^u_{f^n(x)})| = 0.$$

Observe that since A is $\mathrm{SL}(2,\mathbb{R})$-valued, the relation $\lambda_+(A,\mu) + \lambda_-(A,\mu) = 0$ implies that Case (ii) holds precisely when $\lambda_+(A,\mu) > 0$. In particular, if $\lambda_+(A,\mu) = 0$, then necessarily $\lambda_+(A,\mu) = \lambda_-(A,\mu)$, and the splitting collapses.

1.3 Inducing maps

In this section, we introduce a construction that allows us to study an induced dynamical system derived from the original pair (f, μ), and to relate their respective Lyapunov exponents. This construction is classical and relies on Poincaré Recurrence theorem.

Recall that (M, d) is a compact metric space and the shift map $f : M \to M$ is a homeomorphism. Also, μ is an ergodic probability measure on M. Let $A : M \to \mathrm{SL}(2, \mathbb{R})$ be a continuous function.

Consider $Z \subset M$ a measurable subset with positive measure $\mu(Z) > 0$. We denote by μ_Z the normalized restriction of μ to Z, that is

$$\mu_Z(E) = \frac{\mu(E)}{\mu(Z)} \quad \text{for every measurable set } E \subset Z.$$

For each $x \in M$, let $\tau_Z(x)$ be the time of the first return of x to Z, that is, $\tau_Z(x) = \inf\{n \geq 1 : f^n(x) \in Z\}$. Define, then, $g : Z \to Z$ the first return map by

$$\begin{aligned} g : Z &\to Z \\ x &\mapsto f^{\tau_Z(x)}(x). \end{aligned}$$

By Poincaré Recurrence theorem, $\tau_Z(x)$ is finite for μ-almost every $x \in Z$, and hence g is defined on a full μ_Z-measure subset of Z. Moreover, since the system (f, μ) is ergodic, the induced system (g, μ_Z) is also ergodic; see, for instance, Theorem 1 in section 5 of [8].

We now define the induced linear cocycle over g. Let $G : Z \times \mathbb{R}^2 \to Z \times \mathbb{R}^2$

$$G(x, v) = (g(x), B(x)v),$$

where the function $B : Z \to \mathrm{GL}(2, \mathbb{R})$ defined by

$$B(x) = A^{\tau_Z(x)}(x).$$

is the product of the original matrices along the return orbit segment of x. Note that the induced map g and the cocycle G are only measurable functions.

The next result shows that the Lyapunov exponents of the induced cocycle G, which exist and are well defined by item 1. of the next proposition, differ from those of the original cocycle only by a multiplicative factor $c(x)$, which corresponds to the asymptotic average return time. This classical fact appears, for example, as Proposition 4.18 in [26]. This result will be used in the next chapter.

Proposition 1.8. *Since the shift map $f : M \to M$ is invertible and μ is ergodic, then:*

1. *The probability measure μ_Z is g-invariant and $\log^+ \left\| B^{\pm 1} \right\| \in L^1(\mu_Z)$.*
2. *The Oseledets decomposition of the induced cocycle G is the restriction of the Oseledets decomposition of the cocycle F.*
3. *For μ_Z-almost every $x \in Z$, the Lyapunov exponents satisfy*

$$\lambda_{\pm}(B, \mu_Z) = \frac{1}{\mu(Z)} \lambda_{\pm}(A, \mu). \tag{1.9}$$

Equation (1.9) follows from the relation below, which holds without assuming that μ is ergodic.

For μ_Z-almost every $x \in Z$, there exists $c(x) \geq 1$ such that the Lyapunov exponents satisfy

$$\lambda_{\pm}(G, \mu_Z, x) = c(x)\lambda_{\pm}(F, \mu, x), \tag{1.10}$$

where

$$c(x) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \tau_Z(g^j(x)),$$

is the average of the return map.

In order to justify this, first note that τ_Z is integrable relative to μ_Z. By Birkhoff's Ergodic Theorem and by Kac's Theorem,

$$\begin{aligned} \int_Z c(x)\, d\mu(x) &= \int_Z \tau_Z(x)\, d\mu(x) \\ &= \mu(M) - \mu(E_0^*), \end{aligned}$$

where $E_0^* = \{x \in M; f^k(x) \notin Z \ \forall k \geq 0\}$ is the set of points of M that never enter in Z.

In particular, the set E_0^* is f-invariant. Since the system (f, μ) is ergodic, it follows that $\mu(E_0^*) = 0$. Hence,

$$\int_Z \tau_Z(x)\, d\mu(x) = 1$$

Also, the Lyapunov exponents are constant in a full measure set,

$$\lambda_\pm(B, \mu_Z) = \lambda_\pm(B, \mu_Z, x) \text{ for } \mu_Z\text{-almost every } x,$$

and

$$\lambda_\pm(A, \mu) = \lambda_\pm(A, \mu, x) \text{ for } \mu\text{-almost every } x.$$

Using the relation from the proposition (1.10)

$$\begin{aligned} \lambda_\pm(B, \mu_Z)\mu(Z) = \int_Z \lambda_\pm(B, \mu_Z, x)\, d\mu(x) &= \int_Z c(x)\lambda_\pm(A, \mu, x)\, d\mu(x) \\ &= \lambda_\pm(A, \mu) \int_Z \tau_Z(x)\, d\mu(x). \end{aligned}$$

Therefore we obtain the desired relation (1.9), since $\int_Z \tau_Z(x)\, d\mu(x) = 1$.

Thus, the Lyapunov exponents of the induced system are simply the original Lyapunov exponents rescaled by the reciprocal of the measure of the inducing set Z.

Chapter 2

Bocker and Viana's Example

In this chapter, we investigate the continuity problem of Lyapunov exponents for linear cocycles when the measure remains fixed but the cocycle varies, a central topic in smooth ergodic theory.

We begin by recalling the characterization of continuity points in the C^0-topology, given by the classical results of Mañé and Bochi. We then turn our attention to the Hölder topology, where the situation is subtler.

The theorem of Backes, Brown, and Butler guarantees the continuity of Lyapunov exponents for fiber-bunched cocycles, providing a sufficient condition that ensures regular dependence on the base dynamics. However, when this condition fails, the continuity may break down. In fact, examples of discontinuous for Lyapunov exponents are known to appear even for simple cocycles over symbolic systems, as shown by Bocker–Viana and Butler.

2.1 The continuity problem in the C^0-topology

We remain in the same framework as in the previous chapter: $(M = X^{\mathbb{Z}}, d)$ is a compact metric product space with the metric d defined in (1.3), and (f, μ) denotes the Bernoulli shift.

In this setting, as we discussed previously, the Lyapunov exponents $\lambda_{\pm}$ are constant on a full measure set, that is, $\lambda_{\pm}(A, \mu) = \lambda_{\pm}(A, \mu, x)$ for μ-almost every $x \in M$. Therefore, since the measure μ is fixed, the Lyapunov exponents depend only on the linear cocycle A and not on the choice of the base point.

A natural and fundamental question in smooth ergodic theory is whether this dependence on A is continuous with respect to the chosen topology on the space of cocycles. The answer to this question turns out to depend strongly on both the regularity class of the cocycle and the topology considered.

As a direct consequence of Theorem (1.3), the function

$$A \mapsto \lambda_+(A, \mu),$$

is upper semi-continuous while the map

$$A \mapsto \lambda_-(A, \mu),$$

is lower semi-continuous. Thus, whenever

$$\lambda_+(A, \mu) = \lambda_-(A, \mu) = 0,$$

A is a continuity point for the Lyapunov exponents.

The full characterization of continuity points in the C^0-topology was obtained through the works of Mañé [19] and Bochi [5]. Their results completely solve the continuity question for continuous $\mathrm{SL}(2, \mathbb{R})$-cocycles over invertible base dynamics when the measure μ is *aperiodic*, that is, when the set of periodic points has zero μ-measure. Since the Bernoulli shift is generated by a homeomorphism and an aperiodic measure, the Mañé–Bochi theorem applies directly in our setting, yielding the following conclusion:

Theorem 2.1 (Mañé-Bochi)**.** *A continuous function $A : M \to \mathrm{SL}(2,\mathbb{R})$ is a continuity point for the Lyapunov exponents in $C^0(M, \mathrm{SL}(2,\mathbb{R}))$ if and only if the corresponding cocycle is uniformly hyperbolic over the support of μ or $\lambda_\pm(A,\mu) = 0$.*

This theorem shows that, in the continuous setting, the presence of nonzero Lyapunov exponents is precisely what causes discontinuity. In particular, continuity holds only in the two extremal cases: when all Lyapunov exponents vanish or when the cocycle is uniformly hyperbolic.

In order to prove this result, Bochi, based on an outline sketched by Mañé, completed the proof and demonstrated that every cocycle with non-vanishing Lyapunov exponents and that is not uniformly hyperbolic can be approximated in the C^0-topology by cocycles with zero Lyapunov exponents.

2.2 The continuity problem in the C^α-topology

The previous theorem provides a sufficient and necessary condition for identifying continuity points of the Lyapunov exponents in the space $C^0(M, \mathrm{SL}(2,\mathbb{R}))$. In contrast, in the space of Hölder continuous cocycles over a Bernoulli base dynamics, continuity holds in a substantially larger set: an additional assumption guarantees continuity for all cocycles satisfying it, thereby expanding the class of continuity points.

Fix $\alpha > 0$. A cocycle $A : M \to \mathrm{SL}(2,\mathbb{R})$ is *α-Hölder continuous* if there is a constant $C > 0$ such that

$$\|A(x) - A(y)\| \leq C d(x,y)^\alpha \quad \text{for every } \; x, y \in M.$$

We denote by $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$ the space of all α-Hölder continuous cocycles $A : M \to \mathrm{SL}(2,\mathbb{R})$ over f, endowed with the α-Hölder norm defined by

$$\|A\|_\alpha = \sup_{x\in M} \|A(x)\| + \sup_{x\neq y \in M} \frac{\|A(x) - A(y)\|}{d(x,y)^\alpha}. \tag{2.1}$$

Here, $\|.\|$ is taken to be the standard Euclidean norm on $\mathbb{R}^2$ along with the associated operator norm on $\mathrm{SL}(2,\mathbb{R})$.

From now on, we consider $M = \{0, 1, \cdots, l\}^{\mathbb{Z}}$, $l \geq 1$, the space of bi-infinite sequences with $l + 1$ symbols, endowed with the distance d, defined by

$$d(x,y) = 2^{-N(x,y)} \quad \text{for } x = (x_i)_{i\in\mathbb{Z}},\ y = (y_i)_{i\in\mathbb{Z}} \in \mathrm{M}, \tag{2.2}$$

where

$$N(x,y) = \sup\{N \geq 0 : x_i = y_i \text{ for every } \; |i| < N\}.$$

Recall that the dynamic is given by the shift map on M

$$\begin{array}{rccc} f: & M & \to & M \\ & (x_i)_{i\in\mathbb{Z}} & \mapsto & (x_{i+1})_{i\in\mathbb{Z}}. \end{array}$$

Also, μ is the product measure, $\mu = \nu^{\mathbb{Z}}$, where $p_i = \nu(\{i\}) > 0$, for every $i \in \{0, 1, \cdots, l\}$, and ν is the probability measure of $\{0, 1, \cdots, l\}$.

Definition 2.2. For $\alpha > 0$, a cocycle $A \in C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$ is α-fiber-bunched if for every $x \in M$

$$\|A(x)\|\|A(x)^{-1}\| < 2^\alpha.$$

The constant 2 in the definition above is related to the particular choice of the distance in (2.2) and reflects the expansion rate of the hyperbolic homeomorphism f. More generally, if we choose another constant $\theta \in (0,1)$ and define $d_\theta(x,y) = \theta^{N(x,y)}$, the resulting metric is Hölder equivalent to (2.2), and hence induces the same topology on M.

The next result, proved by Backes, Brown and Butler [4], establishes the continuity for the Lyapunov exponents when restricted to the class of fiber-bunched cocycles. In our setting, this yields the following:

Theorem 2.3 (Backes-Brown-Butler)**.** *Let A be a α-fiber-bunched cocycle, $\alpha > 0$, then A is a continuity point for the Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$.*

This theorem shows that, although Lyapunov exponents may be discontinuous in more general setting for Hölder cocycles, the fiber-bunching condition provides a sufficient form of domination ensuring their continuity. Geometrically, being fiber-bunched means that the action of the cocycle along the fibers does not distort vectors too much compared to the rate at which the base dynamics separates nearby points. In other words, the non-conformality of the cocycle is uniformly dominated by the contraction of the base map. This domination allows one to build invariant holonomies and control the variation of the Oseledets directions, which are essential ingredients for proving continuity of Lyapunov exponents.

2.3 Example of Bocker-Viana

The theorem of Backes, Brown, and Butler establishes that the Lyapunov exponents vary continuously on the subset of α-fiber-bunched cocycles in $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$. However, when the fiber-bunching condition fails, the situation becomes significantly more delicate. In contrast to the C^0-topology, where the Mañé–Bochi theorem completely characterizes the continuity points, continuity in the Hölder topology is far from being typical. Indeed, several works (see, for instance, Bocker–Viana [7] and Butler [6]) have shown that discontinuities of Lyapunov exponents naturally appear among Hölder continuous cocycles that are not fiber-bunched.

In this section, we revisit these examples, which consist of locally constant cocycles over Bernoulli shifts. For that, we consider the space M to the case $l = 1$, and, for convenience, we denote the corresponding space by $M_1 = \{0,1\}^{\mathbb{Z}}$.

For each parameter $\sigma > 1$, recall the cocycle introduced in Example 1.2, associated with the function $\mathrm{A}_\sigma : M_1 \to \mathrm{SL}(2,\mathbb{R})$ defined by

$$\begin{aligned} A_\sigma : M_1 &\to \mathrm{SL}(2,\mathbb{R}) \\ (x_i)_{i\in\mathbb{Z}} &\mapsto \begin{cases} \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} & \text{if } x_0 = 0, \\ \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1. \end{cases} \end{aligned} \tag{2.3}$$

Observe that, since the cocycle is locally constant, A_σ is α-Hölder continuous for any $\alpha > 0$.

Also, remember that $\mu_p = \nu_p^{\mathbb{Z}}$ is the Bernoulli measure, $\nu_p = p\delta_1 + (1-p)\delta_0$ with $p \in (0,1)$.

We concluded (more specifically in Example (1.5)) that the Lyapunov exponents are $\lambda_\pm(\mathrm{A}_\sigma, \mu_p) = \pm|(1-2p)\log\sigma|$. Therefore, since μ_p is an aperiodic measure, by Theorem 2.1, A_σ is a continuity point in $C^0(M_1, \mathrm{SL}(2,\mathbb{R}))$ if and only if $p = \frac{1}{2}$.

Thus, from now on, we consider $p \neq \frac{1}{2}$.

Observe that the cocycle A_σ satisfies the fiber bunching condition when $\sigma^2 < 2^\alpha$, making it a continuity point for the Lyapunov exponents in $C^\alpha(M_1, \mathrm{SL}(2,\mathbb{R}))$, by Theorem 2.3. However, in [7], provided that $\sigma^2 > 2^{4\alpha}$, Bocker and Viana show that A_σ is a discontinuity point for Lyapunov exponents in $C^\alpha(M_1, \mathrm{SL}(2,\mathbb{R}))$ with respect to μ_p with $p \notin \{0, 1/2, 1\}$. Notably, this cocycle is strongly non-fiber-bunched. In their construction, refining the techniques of Mañé-Bochi [5] and Knill [14], they showed how to approximate A_σ, in the α-Hölder topology, by cocycles whose Lyapunov exponents vanish identically.

Theorem 2.4 (Bocker-Viana)**.** *For any $\alpha > 0$ such that*

$$2^{4\alpha} < \sigma^2, \tag{2.4}$$

there exist α-Hölder continuous cocycles $B : M_1 \to \mathrm{SL}(2,\mathbb{R})$ with vanishing Lyapunov exponents which are arbitrarily close to A_σ in the α-Hölder norm.

In the same context, in [6], Butler constructed cocycles that are arbitrarily close to satisfying the fiber-bunching condition while they are still discontinuity points for the Lyapunov exponents in the α-Hölder topology. Precisely, he proved this for cocycles A_σ whose parameters satisfy $\sigma^{4p-2} \geq 2^\alpha$ and $p \in (1/2, 1)$. Observe that these discontinuity points approach the fiber-bunching condition as $p \to 1^-$ in the Bernoulli measure μ_p. In particular, the same result holds if

$$\sigma^2 \geq 2^{2\alpha} \text{ and } p \in (3/4, 1). \tag{2.5}$$

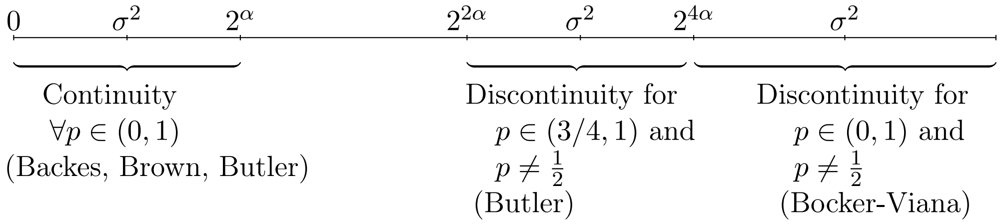
In order to prove this discontinuity result, he constructed cocycles arbitrarily close to A_σ with small Lyapunov exponents. However, his technique does not allow one to approximate cocycles with zero Lyapunov exponents.

Theorem 2.5 (Butler)**.** *For any $\alpha > 0$, let $p \in (1/2, 1)$ such that*

$$\sigma^{4p-2} \leq 2^\alpha.$$

Then, for each open neighborhood $\mathcal{U} \subset C^\alpha(M_1, \mathrm{SL}(2,\mathbb{R}))$ of A_σ and every $\kappa \in (0, \lambda_+(A_\sigma, \mu_p)]$, there is a cocycle $L \in \mathcal{U}$ such that $\lambda_+(L, \mu_p) = \kappa$.

In summary, we have the following:



Also, in [6], Butler raises an open question regarding the possibility of approximation by cocycles with zero Lyapunov exponents when $\sigma^2 \in (2^{2\alpha}, 2^{4\alpha})$. As a consequence of our central theorem, stated in the next chapter, we obtain a partial answer to this question, as will be discussed later.

Chapter 3

Statement and Proof of Theorem A

As studied in the previous chapter, even in relatively simple settings, such as that of locally constant cocycles, the Lyapunov exponents may exhibit points of discontinuity when considered with respect to the Hölder topology.

In this chapter, we work in the following setting: a bi-infinite sequence space over $l+1$ symbols, with $l \geq 1$, and cocycles that depend only on the first coordinate. The cocycles we consider are non-fiber-bunched and have nonzero Lyapunov exponents. Moreover, under suitable conditions, they also provide examples of discontinuity points for the Lyapunov exponents.

Within this framework, Section 3.1 describes the construction of the cocycle and specifies the restrictions it must satisfy. In the same section, we establish our discontinuity result and present an alternative proof of the vanishing of the Lyapunov exponent. Section 3.2 then focuses on a broader class of such cocycles, which are non-locally constant and non-fiber-bunched.

3.1 Locally Constant Case

In this section, we study the natural extension of the example presented in Example 1.2.

For that, we consider $M = \{0, \cdots, l\}^{\mathbb{Z}}$ with $l \geq 1$, let $f : M \to M$ be the shift map and let μ be the Bernoulli measure determined by a probability vector $(p_0, \cdots, p_l)$, where $p_i = \nu(\{i\}) > 0$, for every $i \in \{0, 1, \cdots, l\}$, and ν is the probability measure of $\{0, 1, \cdots, l\}$.

As mentioned above, we consider the cocycle associated with the function $A : M \to \mathrm{SL}(2, \mathbb{R})$ defined by

$$A|_{[0;i]} = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix}, \tag{3.1}$$

where $a_i > 0$ for every $i \in \{0, \cdots, l\}$, and we only require that exist two indices r and j such that $a_r > 1$ and $a_j < 1$.

Since A is a locally constant cocycle, its Lyapunov exponents can be computed explicity as

$$\lambda_{\pm}(A, \mu) = \pm \left| \sum_{a_k > 1} p_k \log a_k - \sum_{a_k < 1} p_k \log a_k^{-1} \right|. \tag{3.2}$$

We also require that the weights $(p_0, \ldots, p_l)$ are chosen such that $\lambda_+(A, \mu) \neq 0$.

Moreover, since the cocycle depends only on the first coordinate, the map A is α-Hölder continuous for every $\alpha > 0$.

For the sake of simplicity, we define

$$\eta := \min\{a_j^{-1}, a_r\} \quad \text{and} \quad \sigma := \max\{a_j^{-1}, a_r\}.$$

The next result provides another concrete and explicit example of discontinuity points in the Hölder topology for the Lyapunov exponents.

Theorem A. *For any $\alpha > 0$, with $a_j < 1 < a_r$ and positive weights $(p_0, \ldots, p_l)$ satisfying that $\lambda_+(A, \mu) \neq 0$ such that*

$$2^{3\alpha} < \sigma^2 \tag{3.3}$$

and

$$2^{\left(2+\frac{\log \eta}{\log \sigma}\right)\alpha} \leq \eta^2, \tag{3.4}$$

there exist α-Hölder continuous cocycles $B : M \to \mathrm{SL}(2, \mathbb{R})$ with $\lambda_\pm(B, \mu) = 0$ which are arbitrarily close to A in the α-Hölder norm. Consequently, A is a discontinuity point for Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2, \mathbb{R}))$.

Remark 3.1. If condition (3.3) does not hold, then there is no constant $1 < \eta < \sigma$ such that

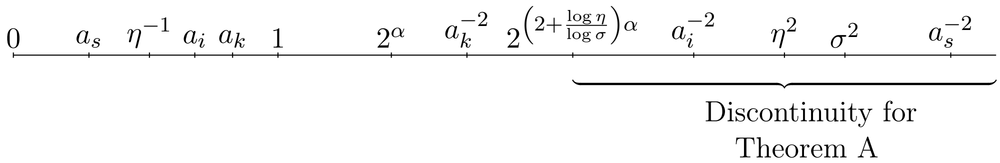
$$2^{\left(2+\frac{\log \eta}{\log \sigma}\right)\alpha} \leq \eta^2 < 2^{3\alpha}.$$

Consequently, the notion of asymmetry between the matrices in the cocycle does not improve the hypotheses of the theorem. For this reason, we assume condition (3.3).

Note that the parameters a_i appearing in the definition of A that are smaller than η do not require any additional assumptions. For instance, if $a_k < 1$ with $a_k^{-1} < \eta$ or $a_k > 1$ with $a_k < \eta$ there is no need to impose any relation between a_k and the quantity $2^{\left(2+\frac{\log \eta}{\log \sigma}\right)\alpha}$, the value of a_k may lie on either side of this bound without affecting the construction.

This observation highlights an important feature of our setting: we may choose the largest parameter, η, to play the decisive role in the argument, while the remaining parameters retain substantial freedom.

As an illustration of a possible configuration of these parameters, we may choose a number $a_k < 1$ such that $2^\alpha < a_k^{-2} < 2^{\left(2+\frac{\log \eta}{\log \sigma}\right)\alpha}$.



Observe that in Theorem A the condition depends on the smallest constant involved. This is a consequence of how the cylinder and the perturbation are constructed. The cylinder is always formed by first placing the symbols corresponding to the constant η, followed by those corresponding to the constant σ. What changes is the number of occurrences of each symbol. Since η is the smaller of the two chosen constants, the matrix associated with it expands the vertical direction (if $\eta = a_r$) or contracts the horizontal direction (if $\eta = a_j^{-1}$) less than the matrix associated with σ. As a consequence, more iterates are required to complete the first exchange than the second one. Accordingly, fewer iterates are needed for the second exchange, which forces the angle required to close this exchange to be smaller, precisely the origin of our condition.

Proof. In order to prove the theorem, we will construct suitable cocycles B arbitrarily close to A in the α- Hölder topology. This construction refines the technique developed by Bocker and Viana in [7], which, in turn, builds on methods from Mañé-Bochi, [5], and Knill, [14]. We then show that these cocycles have zero Lyapunov exponents.

For the remainder of the proof, we assume that $\sigma = a_r$ and $\eta = a_j^{-1}$. The other case can be treated analogously, however, the perturbation constructed is slightly different, being composed of a rotation matrix and two shear matrices. These shear matrices preserve the horizontal direction, while not preserving the vertical one. Nevertheless, it still produces the interchange of the Oseledets subspaces, and in this case we begin the analysis from the vertical direction. We will carry out the construction assuming the strict inequality

$$2^{\left(2+\frac{\log\eta}{\log\sigma}\right)\alpha} < \eta^2. \tag{3.5}$$

The reason is that the case of equality in Theorem A follows as a direct consequence of the theorem under the assumption of strict inequality. Indeed, suppose $2^{\left(2+\frac{\log\eta}{\log\sigma}\right)\alpha} = \eta^2$. Let $\mathcal{U}$ be an open neighborhood of A in $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$, choose $r > 0$ sufficiently small so that $\eta + r < \sigma$, and such that the cocycle defined by the matrix

$$\begin{pmatrix} (\eta+r)^{-1} & 0 \\ 0 & \eta+r \end{pmatrix},$$

on the set $x \in [0; j]$, denoted by A', belongs to the open set $\mathcal{U}$, that is, $A' \in \mathcal{U}$. Since $2^{\left(2+\frac{\log\eta}{\log\sigma}\right)\alpha} < (\eta+r)^2$, we may apply Theorem A to the cocycle A', which yields a cocycle $B' \in \mathcal{U}$ with zero Lyapunov exponents. This provides the desired cocycle B' arbitrarily close to A in the α-Hölder topology, even when $2^{\left(2+\frac{\log\eta}{\log\sigma}\right)\alpha} = \eta^2$.

Now, recall that Lyapunov exponents of A are nonzero. In this case, for μ-almost every $x \in M$, Oseledets' Theorem, theorem (1.7), provides a decomposition of $\mathbb{R}^2$ into f-invariant subspaces, $\mathbb{R}^2 = E^s_x \oplus E^u_x$. Since the cocycle A is defined by diagonal matrices, it follows that these Oseledets subspaces coincide μ-almost everywhere with the vertical and horizontal bundles:

$$\mathbb{R}^2 = H_x \oplus V_x = \mathbb{R}(1,0) \oplus \mathbb{R}(0,1). \tag{3.6}$$

The central strategy in Bocker-Viana's approach [7] is to construct a cocycle B, close to A in the Hölder topology, that exchanges the Oseledets subspaces of A. When this exchange property is implemented at points on a cylinder, it guarantees that the resulting cocycle B has zero Lyapunov exponents.

We apply the idea of exchanging the Oseledets subspaces and adjust the approximation procedure so that, under appropriate conditions, the cocycles A become discontinuity points of the Lyapunov exponents. In our construction, we crucially use the difference between the expansion and the contraction rates of the matrices associated with the cocycle A, since $\sigma \geq \eta$.

Now, we fix several key parameters. Let $\beta \in \left[\frac{\log\eta}{\log\sigma}, 1\right]$ be a rational number satisfying $2^{(2+\beta)\alpha} < \eta^2$, and let $\gamma \in [1, 2)$. For each $k \in \mathbb{N}$ such that βk is an integer, we define the cylinder

$$Z_k = [0; \underbrace{j \dots j}_{k \text{ times}}\ \underbrace{r \dots r}_{\beta k \text{ times}}] \subset M,$$

where the symbol j appears k times, while the symbol r appears βk times. Note that the cylinder Z_k is determined by fixing these $k + \beta k$ coordinates. Setting $n = k(\beta+1)$, we observe from the definition that the collection $\{f^i(Z_k)\}_{i=0}^{n-1}$ is pairwise disjoint. Writing $c = (1 + \eta^{2k(\gamma-2)})^{-1/2}$, this last property enables us to define a modified cocycle $R_k : M \to \mathrm{SL}(2,\mathbb{R})$ by

$$R_k(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \eta^{-\gamma k} & 1 \end{pmatrix} & \text{if } x \in Z_k, \\[2ex] c\begin{pmatrix} 1 & -\eta^{k(\gamma-2)} \\ \eta^{k(\gamma-2)} & 1 \end{pmatrix} & \text{if } x \in f^k(Z_k), \\[2ex] \begin{pmatrix} 1 & 0 \\ \eta^{k(2-\gamma)}\sigma^{-2\beta k} & 1 \end{pmatrix} & \text{if } x \in f^{n-1}(Z_k), \\[2ex] \mathrm{Id} & \text{otherwise.} \end{cases} \tag{3.7}$$

Finally, the perturbed cocycle of interest is defined by

$$B_k(x) = A(x)R_k(x). \tag{3.8}$$

Observe that cocycle R_k is constant on each cylinder, and consequently both R_k and B_k are α-Hölder continuous, $B_k \in C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$.

We emphasize that the main strategy of the proof lies in selecting a smaller parameter, specifically, the one representing the angle between the horizontal axis and the vector $R_k(x)e_1$, with $x \in Z_k$. As a result, we are able to perform the first exchange of Oseledets subspaces (from H_x to V_x) with a smaller Hölder distance between the perturbed and the original cocycle. This allows us to obtain a weaker hypothesis.

Another key argument of our mechanism is exploiting the asymmetry in the definition of the cocycle A. When $\sigma > \eta$, due to the difference in the hyperbolicity between the matrices, fewer quantity of symbols r are needed in the cylinder Z_k. Since σ induces stronger expansion, fewer iterations are required for the cocycle B_k, when restricted to e_2, to approach the horizontal axis, all while keeping the Hölder distance under control. Moreover, by increasing the length of the cylinder, the distance between points inside and outside the cylinder becomes larger, which allows for an improvement in the required condition.

An explicit calculation using Equations (3.7) and (3.8) shows that for every $x \in Z_k$,

$$B_k^n(x) = c \begin{pmatrix} 0 & -\eta^{(\gamma-1)k}\sigma^{\beta k} \\ \sigma^{-\beta k}(\eta^{(\gamma-3)k} + \eta^{(1-\gamma)k}) & 0 \end{pmatrix}. \tag{3.9}$$

From this formula, we immediately see that the cocycle B_k satisfies the *exchange property* on Z_k: for every $x \in Z_k$,

$$B_k^n(x)H_x = V_{f^n(x)} \quad \text{and} \quad B_k^n(x)V_x = H_{f^n(x)}. \tag{3.10}$$

This key property will be essential for computing the Lyapunov exponents of B_k.

It remains only to prove that cocycle B_k can be made arbitrarily close to A in the α-Hölder topology. This is established by the following lemma.

Lemma 3.2. *For sufficiently large $k \in \mathbb{N}$ with $\beta k \in \mathbb{N}$, the α-Hölder norm $\|A - B_k\|_\alpha$ becomes arbitrarily small.*

Proof. For the first term in formula (2.1), by examining all cases in formulas (3.1),(3.7) and (3.8), we directly obtain

$$\|A - B_k\|_\infty \leq \sup_{x \in M} \|A(x)\| \max\{\eta^{-\gamma k}, \eta^{-k(2-\gamma)}, \eta^{k(2-\gamma)}\sigma^{-2\beta k}\}.$$

Since $\eta^{2-\gamma} < \eta^2$ and $\frac{\log \eta}{\log \sigma} \leq \beta$, we have $\frac{\eta^{(2-\gamma)}}{\sigma^{2\beta}} < 1$. Therefore, from our parameter choices of $\gamma, \beta, \eta, \sigma$, it follows that $\|A - B_k\|_\infty \to 0$ as $k \to \infty$.

Now we analyze the second term of formula (2.1). First, observe that cocycle B_k modifies A only on three cylinders: Z_k, $f^k(Z_k)$, and $f^{n-1}(Z_k)$. We consider several cases based on these cylinders. Suppose first that x and y do not belong to any of these cylinders. From definitions (3.1), (3.7), and (3.8), it follows that $R_k(x) = R_k(y) = \mathrm{Id}$, and hence $B_k(x) = A(x)$ and $B_k(y) = A(y)$. Therefore, the second term vanishes, and the Hölder norm satisfies $\|A - B_k\|_\alpha = \|A - B_k\|_\infty \to 0$ as $k \to \infty$ by the argument in the previous paragraph.

Next, suppose x and y belong to the same cylinder (Z_k, $f^k(Z_k)$, or $f^{n-1}(Z_k)$). By definition, $A(x) - B_k(x) = A(y) - B_k(y)$, so again the second term vanishes, and the conclusion follows as in the first case.

It only remains to analyze the cases where x belongs to one of the cylinders (Z_k, $f^k(Z_k)$, or $f^{n-1}(Z_k)$) and y does not. In these cases, if $A(x) \neq A(y)$, then by Definition 3.1, we have $x_0 \neq y_0$, and thus $d(x, y) = 1$. Applying the triangle inequality, we obtain:

$$\frac{\|A(x) - B_k(x) - A(y) + B_k(y)\|}{d(x, y)^\alpha} \leq 2\|A - B_k\|_\infty.$$

Therefore, the Hölder norm satisfies $\|A - B_k\|_\alpha \leq 3\|A - B_k\|_\infty \to 0$ as $k \to \infty$. For the remaining cases, we assume $A(x) = A(y)$. Using this equation and (3.8), we can estimate the second term of (2.1) as follows:

$$\frac{\|A(x) - B_k(x) - A(y) + B_k(y)\|}{d(x, y)^\alpha} \leq \frac{\|A\|_\infty \|R_k(x) - R_k(y)\|}{d(x, y)^\alpha}. \tag{3.11}$$

For this purpose, we will consider the following cases, accounting for possibilities of R_k:

- Case 1: $x \in Z_k$ and $y \notin Z_k$.
  Here, we have $N(x, y) \leq k(\beta + 1)$ and $d(x, y)^{-\alpha} \leq 2^{k(1+\beta)\alpha}$, hence
  $$\frac{\|A(x)(R_k(x) - \mathrm{Id})\|}{d(x, y)^\alpha} \leq \eta \left(\frac{2^{(1+\beta)\alpha}}{\eta^\gamma}\right)^k.$$
- Case 2: $x \in f^{n-1}(Z_k)$ and $y \notin f^{n-1}(Z_k)$.
  Similarly, $N(x, y) \leq k(\beta + 1)$, which implies
  $$\frac{\|A(x)\|\|R_k(x) - \mathrm{Id}\|}{d(x, y)^\alpha} \leq \sigma \left(\frac{2^{(1+\beta)\alpha}\eta^{(2-\gamma)}}{\sigma^{2\beta}}\right)^k.$$
- Case 3: $x \in f^k(Z_k)$ and $y \notin f^k(Z_k)$.
  Now, by definition of the cylinder $f^k(Z_k)$, $N(x, y) \leq k$ and $d(x, y)^{-\alpha} \leq 2^{k\alpha}$, yielding
  $$\frac{\|A\|_\infty \|R_k(x) - R_k(y)\|}{d(x, y)^\alpha} \leq \sigma \left(\frac{2^\alpha}{\eta^{(2-\gamma)}}\right)^k.$$

From these estimates and equation (3.11), it is clear that Hölder norm $\|A - B_k\|_\alpha \to 0$ as $k \to \infty$ provided the following three inequalities hold:

$$2^{(1+\beta)\alpha} < \eta^\gamma, \tag{3.12}$$

$$2^{(1+\beta)\alpha} < \frac{\sigma^{2\beta}}{\eta^{(2-\gamma)}}, \tag{3.13}$$

$$2^\alpha < \eta^{(2-\gamma)}. \tag{3.14}$$

First, we observe that inequality (3.12) implies (3.13). Indeed, since $\beta \geq \frac{\log \eta}{\log \sigma}$, it follows that $\eta^\gamma \leq \frac{\sigma^{2\beta}}{\eta^{(2-\gamma)}}$, which establishes inequality (3.13). Consequently, we only need to satisfy inequalities (3.12) and (3.14). In order to optimize the parameter choice, note that $2 - \gamma$ is a decreasing function of γ while $\frac{\gamma}{1+\beta}$ is an increasing one. Therefore, the optimal parameter choice occurs when $\gamma = \frac{2(1+\beta)}{2+\beta}$. With this choice, the condition

$$2^{\alpha(2+\beta)} < \eta^2$$

implies both inequalities (3.12) and (3.14), which completes the proof. □

It remains to prove the nullity of the Lyapunov exponents for the perturbation B_k. Since the construction used in the proof depends only on the constants η and σ, and the remaining parameters have no influence on the nullity argument, as we will see shortly, we may, for simplicity, restrict our attention to the case $l = 1$, that is, $M = \{0,1\}^{\mathbb{Z}} =: M_1$. The construction in the case $l = 1$ then extends naturally to any $l > 1$.

In other words, we consider the cocycle associated with the function $\mathrm{A}_{\sigma\eta} : M_1 \to \mathrm{SL}(2,\mathbb{R})$ defined in Example 1.2 by

$$\begin{aligned} \mathrm{A}_{\sigma\eta} : M_1 &\to \mathrm{SL}(2,\mathbb{R}) \\ (x_i)_{i\in\mathbb{Z}} &\mapsto \begin{cases} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} & \text{if } x_0 = 0 \\ \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1 \end{cases} \end{aligned}$$

Recalling the notation introduced in Section 2.3, we now denote the Bernoulli measure by $\mu_p = \nu_p^{\mathbb{Z}}$, where $\nu_p = p\delta_1 + (1-p)\delta_0$ with $p \in (0,1)$.

Since μ_p is an aperiodic measure, Theorem 2.1 implies that $\mathrm{A}_{\sigma\eta}$ is a continuity point in $C^0(M, \mathrm{SL}(2,\mathbb{R}))$ if and only if $\lambda_+(\mathrm{A}_{\sigma\eta}, \mu_p) = 0$. This equality holds precisely when $p = \frac{\log\eta}{\log\eta\sigma}$, since $\lambda_+(\mathrm{A}_{\sigma\eta}, \mu_p) = |p\log\sigma - (1-p)\log\eta|$.

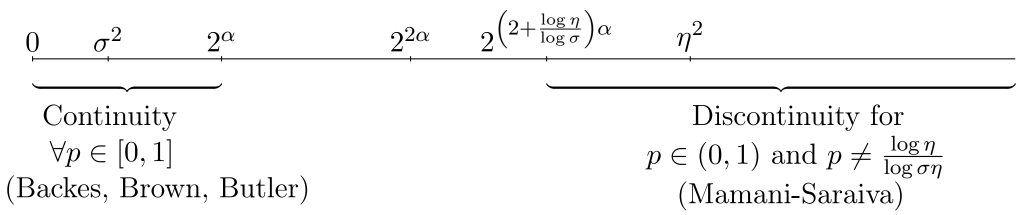
Hence, in what follows, we shall assume that $p \neq \frac{\log\eta}{\log\eta\sigma}$.

Observe that the cocycle $\mathrm{A}_{\sigma\eta}$ is α-fiber-bunched if and only if $\sigma^2 < 2^\alpha$. In this case, by Theorem 2.3, $\mathrm{A}_{\sigma\eta}$ is a continuity point for the Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$.

We now proceed to calculate the Lyapunov exponents of B_k.

In the present setting, our method, in contrast to what is typically done in discontinuity counterexamples, permits explicit computation of the Lyapunov exponents. This is possible because the cocycle $\mathrm{A}_{\sigma\eta}$ is constant in cylinders $[0; i]$, for $i = \{0,1\}$.

Hence, Theorem A presents this configuration concerning the points of discontinuity:



Proposition 3.3. *$\lambda_\pm(B_k, \mu_p) = 0$ for every k such that $\beta k \in \mathbb{Z}$.*

Proof. We fix $k \in \mathbb{N}$ sufficiently large with $\beta k \in \mathbb{N}$ and we consider the cylinder $Z_k = [0; 0...01...1]$ where the symbol 0 appears k times and the symbol 1, βk times.

Recall that for every $x \in Z_k$, we have

$$B_k^n(x) = c \begin{pmatrix} 0 & -\eta^{(\gamma-1)k}\sigma^{\beta k} \\ \sigma^{-\beta k}(\eta^{(\gamma-3)k} + \eta^{(1-\gamma)k}) & 0 \end{pmatrix} \tag{3.15}$$

The central idea for computing the Lyapunov exponents of the system (M_1, f, μ_p, B_k) is to restrict the problem to a subsystem based on the cylinder Z_k as we did before. To make this precise, let us recall some definitions and notation. Let $Z := Z_k$ and $\mu := \mu_p$, and recall that τ_Z denotes the first return time to Z under f. That is, for every $x \in Z$,

$$\tau_Z(x) = \inf\{m \geq 1 : f^m(x) \in Z\}.$$

Furthermore, for every $m \geq 1$, let $\tau_Z^{(m)}$ denote the m-th return time to Z under f. Specifically, for every $x \in Z$,

$$\tau_Z^{(m)}(x) = \sum_{i=0}^{m-1} \tau_Z\big(f^{\tau_Z^{(i)}}(x)\big), \tag{3.16}$$

where $\tau_Z^{(m)}(x)$ counts the total number of iterations under f required for x to return to Z exactly m times. Here, we use the convention that $\tau_Z^{(0)}(x) = 0$. Let $\pi\colon M_1 \to \{0,1\}$ be the projection onto the 0-th coordinate. For every $m \geq 1$ and every $x \in M_1$, the number of occurrences of the symbol 1 in the first m coordinates of x is given by

$$S_m(x) = \sum_{i=0}^{m-1} \pi \circ f^i(x) \tag{3.17}$$

Consequently, for every $x \in Z$, we can define

$$S_{\tau_Z}(x) = \sum_{i=0}^{\tau_Z(x)-1} \pi \circ f^i(x).$$

Let $\mu_Z = \frac{\mu|_Z}{\mu(Z)}$ be the normalized restriction of μ to Z, which is invariant under the first return map f^{τ_Z}. It is a well-known fact that the subsystem (Z, f^{τ_Z}, μ_Z) is ergodic. Moreover, recalling Definition 1.1, we have an induced cocycle $B_k^{\tau_Z}$ defined for $x \in Z$ by the product

$$B_k^{\tau_Z}(x) := B_k^{\tau_Z(x)}(x) = B_k(f^{\tau_Z(x)-1}(x)) \cdots B_k(f(x))B_k(x).$$

This gives the desired subsystem $(Z, f^{\tau_Z}, \mu_Z, B_k^{\tau_Z})$, as discussed in Section 1.3. The relevance of this subsystem stems from the following relation between the Lyapunov exponents:

$$\lambda_\pm(B_k^{\tau_Z}, \mu_Z) = \frac{\lambda_\pm(B_k, \mu)}{\mu(Z)}, \tag{3.18}$$

which is a consequence from Proposition 1.8. Thus, the problem reduces to showing the vanishing of Lyapunov exponents for the induced system, i.e., proving that $\lambda_\pm(B_k^{\tau_Z}, \mu_Z) = 0$.

Let us compute some iterates of the cocycle $B_k^{\tau_Z}$. Using Equations (3.7) and (3.8) along with the first return time τ_Z, for every $x \in Z$ we obtain

$$B_k^{\tau_Z}(x) = c \begin{pmatrix} 0 & -\sigma^{S_{\tau_Z}(x)}\eta^{-\tau_Z(x)+S_{\tau_Z}(x)+k\gamma} \\ \sigma^{-S_{\tau_Z}(x)}\eta^{\tau_Z(x)-S_{\tau_Z}(x)}(\eta^{-4k+k\gamma}+\eta^{-k\gamma}) & 0 \end{pmatrix}, \tag{3.19}$$

where $c = (1 + \eta^{2k(\gamma-2)})^{-1/2}$. Note that the normalization constant satisfies the identity $c^2\eta^{k\gamma}(\eta^{-4k+k\gamma} + \eta^{-k\gamma}) = 1$.

For the second return time $\tau_Z^{(2)}$, we have

$$B_k^{\tau_Z^{(2)}}(x) = \begin{pmatrix} -\sigma^{R(f^{\tau_Z}(x))-R(x)}\eta^{-S(f^{\tau_Z}(x))+S(x)} & 0 \\ 0 & -\sigma^{-R(f^{\tau_Z}(x))+R(x)}\eta^{S(f^{\tau_Z}(x))-S(x)} \end{pmatrix}, \tag{3.20}$$

where we define

$$R(x) = S_{\tau_Z}(x) \quad \text{and} \quad S(x) = \tau_Z(x) - S_{\tau_Z}(x). \tag{3.21}$$

By induction on $j \geq 1$, we can compute the cocycle iterates at even return times:

$$B_k^{\tau_Z^{(2j)}}(x) = \begin{pmatrix} (-1)^j\sigma^{c_j(x)}\eta^{-b_j(x)} & 0 \\ 0 & (-1)^j\sigma^{-c_j(x)}\eta^{b_j(x)} \end{pmatrix}, \tag{3.22}$$

with

$$c_j(x) = \sum_{i=1}^{j} \left(R(f^{\tau_Z^{(2i-1)}}(x)) - R(f^{\tau_Z^{(2i-2)}}(x)) \right) \text{ and } b_j(x) = \sum_{i=1}^{j} \left(S(f^{\tau_Z^{(2i-1)}}(x)) - S(f^{\tau_Z^{(2i-2)}}(x)) \right). \tag{3.23}$$

Observe that in the case $l > 1$, for instance $l = 2$, the induced cocycle at even return times is exactly the same as in (3.22), except that the diagonal entries include the additional factor $a_2^{d_j(x)}$, where the function d_j in the exponent is defined as in (3.23), but counts the symbol 2 instead of the symbols 0 and 1.

From this representation, denoting $m_j = \tau_Z^{(2j)}$, note that at the even return times the induced cocycle $B_k^{m_j}(x)$ takes a diagonal form reflecting expansion in one direction and contraction in the transverse direction, or vice versa, depending on the point x. Specifically, the first direction is multiplied by $\sigma^{c_j(x)}\eta^{-b_j(x)}$ and the second by its reciprocal $\sigma^{-c_j(x)}\eta^{b_j(x)}$. Thus, one direction is expanded while the other is contracted, and which one expands depends on the signs of $c_j(x)$ and $b_j(x)$.

Therefore, we obtain the norm estimate $\|B_k^{m_j}(x)\| \leq \sigma^{|c_j(x)|}\eta^{|b_j(x)|}$, which yields the following bound on the growth rate:

$$\lim_{j\to\infty} \frac{1}{m_j} \log \|B_k^{m_j}(x)\| \leq \left(\lim_{j\to\infty} \frac{|c_j(x)|}{m_j} \right) \log \sigma + \left(\lim_{j\to\infty} \frac{|b_j(x)|}{m_j} \right) \log \eta. \tag{3.24}$$

Since $B_k^{\tau_Z}$ is an $\mathrm{SL}(2,\mathbb{R})$-valued cocycle, the Furstenberg-Kesten Theorem [12] implies that for μ_Z-almost every $x \in Z$, the following limit exists and satisfies

$$\lambda_+(B_k^{\tau_Z}, \mu_Z) = \lim_{m\to\infty} \frac{1}{m} \log \left\| B_k^{\tau_Z^{(m)}}(x) \right\| \geq 0.$$

To complete the proof, it remains to establish the key estimate:

$$\lim_{j\to\infty} \frac{1}{m_j} \log \left\| B_k^{m_j}(x) \right\| \leq 0. \tag{3.25}$$

Since the limit exists, and $\{m_j\}_{j\in\mathbb{N}} = \{\tau_Z^{(2j)}\}_{j\in\mathbb{N}}$ is a subsequence of return times $\{\tau_Z^{(m)}\}_{m\in\mathbb{N}}$, the limit along this subsequence must satisfy $\lim_{j\to\infty} \frac{1}{m_j} \log \|B_k^{m_j}(x)\| = 0$. Consequently, the full limit must agree: $\lambda_+(B_k^{\tau_Z}, \mu_Z) = 0$. This establishes the vanishing of the Lyapunov exponents.

This result, philosofically, was already expected. By the definition of the induced cocycle at even return times, given in (3.22), the alternating behavior of expansion and contraction along invariant directions, governed by the coefficients $c_j(x)$ and $b_j(x)$ defined in (3.23), determines the asymptotic behavior of vectors under the cocycle action. Since these coefficients are sums of differences evaluated along the orbit at specific return times, they exhibit oscillations and cancellations that, for μ-almost every point, cause the average exponential growth rate, that is, the Lyapunov exponent, to vanish. In other words, despite the presence of local expansion and contraction, the global dynamics of the induced cocycle balance these effects, resulting in zero Lyapunov exponents.

In order to establish the inequality (3.25), we first require the following lemma.

Lemma 3.4. *The functions $R : Z \to \mathbb{R}$ and $S : Z \to \mathbb{R}$ defined in Equation* (3.21) *belong to $L^1(\mu_Z)$.*

Proof. We begin by analyzing the function R. For each $m > n$, define the function $T_m : Z \to \mathbb{R}$ as

$$T_m(x) := S_m(x) - S_n(x) = \sum_{i=n}^{m-1} \pi \circ f^i(x).$$

Using the first return time τ_Z, we observe that

$$R(x) = T_{\tau_Z}(x) + k\beta + 1. \tag{3.26}$$

We note that T_{τ_Z} is a sum of random variables $X_i = \pi \circ f^i$ for $i > n$, which are independent and identically distributed (i.i.d.) with respect to μ_Z. Furthermore, the sum T_{τ_Z} has a stopping time $\tau_Z - n$ with respect to the filtration of Borel σ-algebras on Z induced by the sequence $\{X_i\}_{i>n}$. The mean and variance of X_i are given by

$$\mathbb{E}[X_i] = \int_Z \pi \circ f^i \, d\mu_Z = \int_{[i;0]} \pi \circ f^i \, d\mu_Z + \int_{[i;1]} \pi \circ f^i \, d\mu_Z = p,$$

$$\mathrm{Var}(X_i) = \int_Z \left(\pi \circ f^i - p\right)^2 d\mu_Z = \int_{[i;0]} (0-p)^2 d\mu_Z + \int_{[i;1]} (1-p)^2 d\mu_Z = p(1-p).$$

These properties allow us to apply Wald's identity:

$$\mathbb{E}\left[\sum_{i=1}^{\tau} X_i\right] = \mathbb{E}[X_1] \cdot \mathbb{E}[\tau],$$

where $\{X_i\}$ are i.i.d. random variables and τ is a stopping time adapted to $\{X_i\}$. Applying this to T_{τ_Z}, we obtain

$$\int_Z T_{\tau_Z} \, d\mu_Z = \left(\int_Z \pi \circ f^n \, d\mu_Z\right)\left(\int_Z (\tau_Z - n) \, d\mu_Z\right) = p\left(\frac{1}{\mu(Z)} - n\right).$$

From (3.26), we conclude that $R \in L^1(\mu_Z)$, with

$$\int_Z R \, d\mu_Z = p\left(\frac{1}{\mu(Z)} - n\right) + (k\beta + 1).$$

Finally, the integrability of S follows immediately, since $S = \tau_Z - R$ is a sum of L^1-functions with respect to μ_Z. □

Lemma 3.5. *Let $m_j(x) = \tau_Z^{(2j)}(x)$ be the $2j$-th return time of x to Z. Then, for μ_Z-almost every $x \in Z$*

$$\limsup_{j\to\infty} \frac{1}{m_j} \log \|B_k^{m_j}(x)\| \leq 0.$$

Proof. We will show that both terms on the right side of Equation (3.24) vanish, from which the result will follow. Let $g = f^{\tau_Z^{(2)}}$ denote the second return time map. From Equation (3.23), we obtain

$$\lim_{j\to\infty} \frac{1}{m_j}|c_j(x)| = \lim_{j\to\infty} \frac{j}{m_j} \cdot \lim_{j\to\infty} \frac{1}{j}\left|\sum_{i=0}^{j-1} R(g^i(f^{\tau_Z}(x))) - \sum_{i=0}^{j-1} R(g^i(x))\right|. \tag{3.27}$$

By Birkhoff's Ergodic Theorem and Kac's Theorem, we have

$$\lim_{j\to\infty} \frac{m_j}{j} = \lim_{j\to\infty} \frac{1}{j}\sum_{i=0}^{2j} \tau_Z(f^{\tau_Z^{(i)}}(x)) = 2\int_Z \tau_Z \, d\mu_Z = \frac{2}{\mu(Z)}.$$

Since f^{τ_Z} is ergodic and Bernoulli, the subsystem (g, μ_Z) is also ergodic, see [11]. By Lemma 3.4, $R \in L^1(\mu_Z)$, and applying Birkhoff's Ergodic Theorem to (3.27) yields

$$\lim_{j\to\infty} \frac{1}{j}\sum_{i=0}^{j-1} R(g^i(f^{\tau_Z}(x))) = \int_Z R \, d\mu_Z = \lim_{j\to\infty} \frac{1}{j}\sum_{i=0}^{j-1} R(g^i(x)).$$

Combining these results, we conclude that

$$\lim_{j\to\infty}\frac{1}{m_j}|c_j(x)| = \frac{\mu(Z)}{2}\cdot\left|\int_Z R\,d\mu_Z - \int_Z R\,d\mu_Z\right| = 0. \tag{3.28}$$

Similarly, using Equation (3.23) to express the second term of (3.24) in terms of g, we obtain

$$\lim_{j\to\infty}\frac{1}{m_j}|b_j(x)| = \lim_{j\to\infty}\frac{j}{m_j}\cdot\lim_{j\to\infty}\frac{1}{j}\left|\sum_{i=0}^{j-1}S(g^i(f^{\tau_Z}(x))) - \sum_{i=0}^{j-1}S(g^i(x))\right|. \tag{3.29}$$

Since Lemma 3.4 also guarantees $S \in L^1(\mu_Z)$, the same argument applied to S shows that $\lim_j \frac{1}{m_j}|b_j(x)| = 0$.

For instance, when $l = 2$, we proceed in the same way as we did to prove that $\lim_j \frac{1}{m_j}|d_j(x)| = 0$. □

Thus, combining this lemma with the fact that $\lambda_+(B_k^{\tau_Z}, \mu_Z) \geq 0$, we deduce from relation (3.18) that $\lambda_+(B_k, \mu_p) = 0$ for every k such that $\beta k \in \mathbb{Z}$. □

Therefore, this completes the proof of the theorem. □

3.1.1 Bocker and Viana's Revised Example

In this section, we reconsider the example of Bocker-Viana, introduced in Chapter 2. This example can be interpreted as a particular case of the cocycle $\mathrm{A}_{\sigma\eta}$, defined in Example 1.2, when $\sigma = \eta$.

As a corollary of Theorem A, we provided an affirmative answer to the previously open question raised by Butler: when $\sigma^2 \in (2^{3\alpha}, 2^{4\alpha})$, the cocycle A_σ can be approximated by cocycles with vanishing Lyapunov exponents.

Theorem 3.6. *For any $\alpha > 0$ and $\sigma > 1$ satisfying*

$$\sigma^2 \geq 2^{3\alpha}$$

Then, there exist α-Hölder continuous cocycles $B : M_1 \to \mathrm{SL}(2,\mathbb{R})$ with zero Lyapunov exponents that are arbitrarily close to A_σ in the α-Hölder topology. In particular, A_σ is a discontinuity point for the Lyapunov exponents in the space $C^\alpha(M_1, \mathrm{SL}(2,\mathbb{R}))$.

It is important to note that no restrictions on the weights of the measure were imposed in order to obtain discontinuity. Moreover, our result is stronger than that obtained in Theorem (2.5), since we show that for every C^α-neighborhood of the cocycle A_σ there exists a cocycle with vanishing Lyapunov exponents.

3.2 Generalized Example

As discussed in the previous section, discontinuities arise even in simple settings, such as locally constant cocycles. This naturally leads us to ask what phenomena may occur in more general and less rigid systems.

In this section, we still consider the following framework: a bi-infinite sequence space over $l+1$ symbols, $l \geq 1$, and a Hölder continuous cocycle $A : M \to \mathrm{SL}(2,\mathbb{R})$ of the form

$$A(x) = \begin{pmatrix} a(x) & 0 \\ 0 & a(x)^{-1} \end{pmatrix}$$

where $a : M \to \mathbb{R}$ is a Hölder continuous function and $a(x) \neq 0$ for every $x \in M$.

Although this cocycle remains diagonal, the passage to this more general context requires us to impose a few restrictions. These restrictions, however, still allow considerably more flexibility than the locally constant case treated earlier.

We consider two cylinders, $\mathrm{C}_1^{\mathrm{q}_1}$ and $\mathrm{C}_2^{\mathrm{q}_2}$, where q_1 and q_2 denote, respectively, the number of symbols in each cylinder. Thus, we write $\mathrm{C}_1^{\mathrm{q}_1} = [0; x_0 \dots x_{q_1-1}]$ and $\mathrm{C}_2^{\mathrm{q}_2} = [0; y_0 \dots y_{q_2-1}]$, and assume that

$$f^j(\mathrm{C}_1^{\mathrm{q}_1}) \cap f^i(\mathrm{C}_2^{\mathrm{q}_2}) = \emptyset \qquad \forall j \in \{0, \cdots, q_1 - 1\} \quad \text{and} \quad \forall i \in \{0, \cdots, q_2 - 1\} \tag{3.30}$$

Example 3.7. The cylinders $C_1 = [0, 0]$ and $C_2 = [0; 1]$, where $q_1 = q_2 = 1$, satisfy the above condition. Another example is given by $C_1 = [0; 010]$ and $C_2 = [0; 111]$, which also satisfy (3.30), now with $q_1 = q_2 = 3$. Observe that it is not necessary to assume that the cylinders are composed of disjoint symbols.

Example 3.8. We may also consider $C_1 = [0; l]$ and $C_2 = [0; 0 \cdots l - 1]$. These cylinders satisfy condition (3.30), and in this case $q_1 = 1$ and $q_2 = l$. Note that we do not impose any restrictions on the values of q_1 and q_2 in the definition.

We fix two constants $\sigma \geq \eta > 1$.

For these parameters, we define the cocycle $A : M \to \mathrm{SL}(2, \mathbb{R})$ by:

$$\begin{aligned} A : M &\to \mathrm{SL}(2, \mathbb{R}) \\ x &\mapsto \begin{cases} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} & \text{if } x \in \bigcup_{i=0}^{q_1-1} f^i(\mathrm{C}_1^{\mathrm{q}_1}), \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x \in \bigcup_{i=0}^{q_2-1} f^i(\mathrm{C}_2^{\mathrm{q}_2}), \\ \begin{pmatrix} a(x) & 0 \\ 0 & a(x)^{-1} \end{pmatrix} & \text{otherwise} \end{cases} \end{aligned} \tag{3.31}$$

where the function $a : M \to \mathbb{R}$ is α-Hölder continuous and satisfies the following condition:

$$\int_{M \setminus C} \log a(x) d\mu(x) \geq 0 \tag{3.32}$$

with $C = (\bigcup_{i=0}^{q_1-1} f^i(\mathrm{C}_1^{\mathrm{q}_1}) \cup \bigcup_{i=0}^{q_2-1} f^i(\mathrm{C}_2^{\mathrm{q}_2}))$. The function $a(x)$ may be chosen to be identically 1 outside the set C, in which case condition (3.32) still holds.

Note that A is well defined, as the sets $f^j\,(\mathrm{C}_1^{\mathrm{q}_1})$ and $f^i(\mathrm{C}_2^{\mathrm{q}_2})$ are disjoint for all $0 \leq j \leq q_1 - 1$ and $0 \leq i \leq q_2 - 1$, by the choice of the cylinders.

Since A coincides with a outside a finite union of cylinder sets, it inherits the Hölder continuity of a, and is therefore an α-Hölder continuous cocycle.

Remark 3.9. The cocycle A generalizes the locally constant cocycle A defined in (3.1), in particular, $\mathrm{A}_{\sigma\eta}$ defined in Example 1.2 over the space M_1, considering the cylinders of Example 3.7, $\mathrm{C}_1^{\mathrm{q}_1} = [0; 0]$ and $\mathrm{C}_2^{\mathrm{q}_2} = [0; 1]$.

We also assume that the weights of the measure μ are chosen such that

$$-\log \eta\mu(\cup_{i=0}^{q_1-1} f^i(\mathrm{C}_1^{\mathrm{q}_1})) + \log \sigma\mu(\cup_{i=0}^{q_2-1} f^i(\mathrm{C}_2^{\mathrm{q}_2})) > 0. \tag{3.33}$$

With these choices, A has positive Lyapunov exponents, and we now proceed to prove this.

Suppose, by contradiction, that $\lambda_+(A, \mu) = 0$. Then, by Oseledets' theorem, for μ-almost every point $x \in M$, we have

$$\lim_n \frac{1}{n} \log \|A^n(x)v\| = 0 \quad \forall v \in \mathbb{R}^2$$

Given $x \in M$ in this full measure set, define

$$g_A(x) = \frac{\|A(x)v\|}{\|v\|}$$

where $v \in H_x$ denotes the horizontal direction in $\mathbb{R}^2$. The function g_A is well defined, meaning that it does not depend on the particular choice of the vector v, since the subspace H_x is one-dimensional. Moreover, because A is a diagonal cocycle, the horizontal subspace is A-invariant. Therefore, by Birkhoff Ergodic Theorem, the Lyapunov exponent can be computed as

$$\lambda_+(A, \mu) = \int \log g_A(x) d\mu(x)$$

Therefore, under the assumption that $\lambda_+(A, \mu) = 0$, we obtain that

$$0 = \lambda_+(A, \mu) = \int \log a(x) d\mu(x)$$

On the other hand, we now turn to the analysis of the integral, which is positive as a consequence of hypotheses (3.33) and (3.32):

$$\begin{aligned}
\int_M \log a d\mu &= \int_{\bigcup_{i=0}^{q_1-1} f^i(\mathrm{C}_1^{\mathrm{q}_1})} \log a d\mu + \int_{\bigcup_{i=0}^{q_2-1} f^i(\mathrm{C}_2^{\mathrm{q}_2})} \log a d\mu + \int_{M \setminus C} \log a d\mu \\
&= \int_{\bigcup_{i=0}^{q_1-1} f^i(\mathrm{C}_1^{\mathrm{q}_1})} \log \eta^{-1} d\mu + \int_{\bigcup_{i=0}^{q_2-1} f^i(\mathrm{C}_2^{\mathrm{q}_2})} \log \sigma d\mu + \int_{M \setminus C} \log a d\mu \\
&= \log \sigma \mu \left(\bigcup_{i=0}^{q_2-1} f^i(\mathrm{C}_2^{\mathrm{q}_2}) \right) - \log \eta \mu \left(\bigcup_{i=0}^{q_1-1} f^i(\mathrm{C}_1^{\mathrm{q}_1}) \right) + \int_{M \setminus C} \log a d\mu \\
&> 0
\end{aligned}$$

This contradiction implies that $\lambda_+(A, \mu) > 0$.

Moreover, the cocycle A is not uniform hyperbolic, which follows from the same argument used in Example 1.6. Indeed, we may choose a measure μ whose associated weights satisfy $\lambda_+(A, \mu) = 0$.

Therefore, since A has positive Lyapunov exponent, it is a C^0-discontinuity point for the Lyapunov exponents, by Mañé-Bochi's Theorem. Furthermore, it can be approximated by cocycles with zero Lyapunov exponents. We are thus interested in understanding when this phenomenon can occur in the C^α-topology.

This cocycle is fiber-bunched when $\|A(x)\|\|A(x)^{-1}\| < 2^\alpha$, that is, when $\max_{x \in M}\{a(x), a(x)^{-1}\}^2 < 2^\alpha$. According to Backes, Brown and Butler's Theorem, that is, Theorem 2.3, in this case, A is a C^α-continuity point.

While the Backes, Brown and Butler's Theorem guarantees continuity under the fiber-bunching condition, next corollary shows that once this domination fails, discontinuities can indeed occur, even for simple, non-locally constant diagonal cocycles over a Bernoulli shift.

Moreover, the theorem highlights that the occurrence of discontinuity can be determined solely from the local behavior of the cocycle on specific cylinders. In particular, it suffices to consider matrices that act in opposite ways on the same direction, one contracting and the other expanding the horizontal axis, to produce the mechanism leading to discontinuity.

However, this example still lies far from being fiber bunching, because, in contrast, the condition appearing in Corollary 3.10 is about the constants η and σ, which may be much smaller than the number $\max_{x \in M}\{a(x), a(x)^{-1}\}$.

Corollary 3.10. *For any $\alpha > 0$ and $\eta \leq \sigma$ such that*

$$2^{3\alpha} < \sigma^2 \tag{3.34}$$

and

$$2^{\left(2+\frac{\log\eta}{\log\sigma}\right)\alpha} \leq \eta^2 \tag{3.35}$$

and for any choice of positive weights $(p_0, \cdots, p_l)$ *satisfying* (3.33)*, there exist* α*-Hölder continuous cocycles* $B : M \to \mathrm{SL}(2,\mathbb{R})$ *with vanishing Lyapunov exponents which are arbitrarily close to* A *in the* α*-Hölder norm. Therefore,* A *is a discontinuity point for Lyapunov exponents in* $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$.

Proof. The perturbation that realizes the conclusion of the corollary is the same as the one constructed in Theorem A, with slight modifications. However, since the cocycle A is not locally constant, we cannot apply the proof of Proposition 3.3 directly. Instead, we follow the argument used in [7].

Recall that $\beta \in \left[\frac{\log\eta}{\log\sigma}, 1\right]$ is a sufficiently small rational number satisfying $2^{(2+\beta)\alpha} < \eta^2$, and $\gamma \in [1,2)$. The difference in the proofs lies in the choice of the cylinder on which the perturbation is defined. For each $r \in \mathbb{N}$ such that βr is integer, we denote by $k = rq_1q_2$ and define the cylinder

$$Z_k = [0; \underbrace{x_0 \dots x_{q_1-1}}_{rq_2 \text{ times}}\, \underbrace{y_0 \dots y_{q_2-1}}_{\beta r q_1 \text{ times}}] \subset M,$$

where the word $x_0 \cdots x_{q_1-1}$ appears rq_2 times and the word $y_0 \cdots y_{q_2-1}$ appears $\beta r q_1$ times. Note that the cylinder Z_k therefore contains $k + \beta k$ elements. Setting $n = k(\beta+1)$, we observe from the definition that the collection $\{f^i(Z_k)\}_{i=0}^{n-1}$ is pairwise disjoint, meaning $f^i(Z_k) \cap f^j(Z_k) = \emptyset$ for all $0 \leq i < j \leq n-1$. This disjointness is a consequence of the fact that the two cylinders and all their iterates are mutually disjoint.

The modified cocycle $R_k : M \to \mathrm{SL}(2,\mathbb{R})$ is defined as in (3.7), and the perturbed cocycle of interest is defined by $B_k(x) = A(x)R_k(x)$.

An explicit calculation again shows that for every $x \in Z_k$,

$$B_k^n(x) H_x = V_{f^n(x)} \quad \text{and} \quad B_k^n(x) V_x = H_{f^n(x)}. \tag{3.36}$$

The same conclusion as in Lemma 3.2 holds here: for sufficiently large $k \in \mathbb{N}$ with $\beta k \in \mathbb{N}$, the α-Hölder norm $\|A - B_k\|_\alpha$ becomes arbitrarily small. The proof follows the same lines.

Now, to conclude the proof of the Corollary, it remains to show that the Lyapunov exponents of the cocycle B_k vanish. The proof of the following result is the same as in ([26], Proposition 9.13) and we reproduce it here for completeness.

Proposition 3.11. $\lambda_\pm(B_k, \mu) = 0$ for every k multiple of q_1q_2 such that $\beta k \in \mathbb{Z}$.

Proof. Fix $k \geq 1$, and from this point onward, denote $Z_k =: Z$, $B_k =: B$. Recall that the Oseledets decomposition for the cocycle A is given by $\mathbb{R}^2 = H_x \oplus V_x$, since it is diagonal, and let μ_Z denote the normalized restriction of μ to Z. Furthermore, the map $f^{\tau_Z} : Z \to Z$ refers to the first return map, which is defined on a full measure subset of Z.

Assume, for the sake of contradiction, that $\lambda_\pm(B,\mu) \neq 0$. Then, there exists an Oseledets decomposition associated to the cocycle B, namely $\mathbb{R}^2 = E^s_{B,x} \oplus E^u_{B,x}$. We consider the induced linear cocycle over the return map f^{τ_Z}, as introduced in section 1.3

$$G : Z \times \mathbb{R}^2 \to Z \times \mathbb{R}^2, \quad G(x,v) = \left(f^{\tau_Z(x)}(x), B^{\tau_Z(x)}(x)v\right).$$

By construction,

$$B^{\tau_Z(x)}(x)\,(H_x) = V_{f^{\tau_Z}(x)} \quad \text{and} \quad B^{\tau_Z(x)}(x)\,(V_x) = H_{f^{\tau_Z}(x)} \quad \mu_Z\text{-almost everywhere}. \tag{3.37}$$

By Proposition 1.8, the Lyapunov exponents $\lambda_{\pm}(B^{\tau_Z}, \mu_Z)$ are different from zero, and the Oseledets decomposition of G is given by $\mathbb{R}^2 = E^s_{B,x} \oplus E^u_{B,x}$ restricted to the domain Z. Define a probably measure m on $Z \times \mathbb{PR}^2$ by

$$m(X) = \frac{1}{2}\mu_Z\left(\{x \in Z : (x, [H_x]) \in X\}\right) + \frac{1}{2}\mu_Z\left(\{x \in Z : (x, [V_x]) \in X\}\right).$$

Observe that m is constructed so that it projects down to μ_Z and its conditional measures on the fibers are

$$x \mapsto \frac{1}{2}\delta_{H_x} + \frac{1}{2}\delta_{V_x}.$$

From (3.37) and using that μ_Z is f^{τ_Z}-invariant, the measure m is invariant under $\mathbb{P}B^{\tau_Z}$, where $\mathbb{P}B^{\tau_Z}$ is the projective cocycle

$$\mathbb{P}B^{\tau_Z} : Z \times \mathbb{PR}^2 \to Z \times \mathbb{PR}^2, \qquad \mathbb{P}B^{\tau_Z}(x, [v]) = (f^{\tau_Z}(x), [B^{\tau_Z}(x)v]).$$

Also, m may be written as a linear convex combination $m = am^s + bm^u$, with $a, b \geq 0$ and $a + b = 1$ where m^s and m^u are probability measures defined on $Z \times \mathbb{PR}^2$ by

$$m^s(X) = \mu_Z\left(\left\{x \in Z : \left(x, [E^s_{B,x}]\right) \in X\right\}\right),$$
$$m^u(X) = \mu_Z\left(\left\{x \in Z : \left(x, [E^u_{B,x}]\right) \in X\right\}\right),$$

(see Lemma 5.24 in [26] for instance). Notice that m^s and m^u are $\mathbb{P}B^{\tau_Z}$-invariant since μ_Z is f^{τ_Z}-invariant and $B^{\tau_Z}(x)E^u_{B,x} = E^u_{B,f^{\tau_Z}(x)}$, $B^{\tau_Z}(x)E^s_{B,x} = E^s_{B,f^{\tau_Z}(x)}$.

The next step is to prove that m is ergodic. This fact implies that m must coincide with m^s or m^u. And it is a contradiction in any case, because the conditional probabilities of m are supported on exactly two points on each fiber, whereas the conditional probabilities of both m^u and m^s are supported on a single point. With this contradiction, we prove the proposition and consequently, the corollary.

Lemma 3.12. *The probability measure m is ergodic.*

Proof. Suppose that m is not ergodic, meaning that there exists an $\mathbb{P}B^{\tau_Z}$-invariant set $W \subset Z \times \mathbb{PR}^2$ with $0 < m(W) < 1$.

Let W_0 be the set of $x \in Z$ whose fiber $W \cap \left(\{x\} \times \mathbb{PR}^2\right)$ contains neither H_x nor V_x, that is,

$$W_0 = \{x \in Z; (x, [H_x]) \notin W \text{ and } (x, [V_x]) \notin W\}$$

In view of (3.37), W_0 is a (f^{τ_Z}, μ_Z)-invariant set and so its $\mu_Z(W_0) \in \{0, 1\}$. Since $m(W) > 0$, then $\mu_Z(\{x \in Z; [H_x] \in W\}) > 0$ and/or $\mu_Z(\{x \in Z; [V_x] \in W\}) > 0$. Therefore we have $\mu_Z(W_0) = 0$.

Also, consider W_2 the set of $x \in Z$ whose fiber contains both V_x and H_x, that is,

$$W_2 = \{x \in Z; (x, [V_x]) \in W \text{ and } (x, [H_x]) \in W\}$$

Similarly, $m(W) < 1$ implies that $\mu_Z(W_2) = 0$. Now let

$$W_H = \{x \in Z; (x, [H_x]) \in W \text{ but } (x, [V_x]) \notin W\}$$

and

$$W_V = \{x \in Z; (x, [V_x]) \in W \text{ but } (x, [H_x]) \notin W\}$$

The previous observations show that $\mu_Z(W_H \cup W_V) = 1$ and it follows from (3.37) that

$$f^{\tau_Z}(W_H) = W_V \text{ and } f^{\tau_Z}(W_V) = W_H$$

up to a measure set.

Thus, since μ_Z is f^{τ_Z}-invariant, $\mu_Z(W_H) = \frac{1}{2} = \mu_Z(W_V)$, $(f^{\tau_Z})^2(W_H) = W_H$ and $(f^{\tau_Z})^2(W_V) = W_V$. This implies that $(f^{\tau_Z})^2$ is not ergodic, contradicting the fact that f^{τ_Z} is conjugated with a Bernoulli shift (for more details, see Chapter 9 of [26]) and, in particular, the second return map of f must be ergodic. $\square$

□

□

Remark 3.13. If we now assume that $\sigma < \eta$, we obtain the same result under the same hypotheses. The difference arises in the construction of the perturbation. In this case, the contraction in the horizontal direction is stronger than in the vertical direction. Consequently, we consider a cylinder in which the word $x_0 \cdots x_{q_1-1}$ appears fewer times than the word $y_0 \cdots y_{q_2-1}$. More precisely, we define $Z_k = [0; x_0 \cdots x_{q_1-1}\, y_0 \cdots y_{q_2-1}]$, where the first word appears $rq_2\beta$ times and the second word appears rq_1 times, with $k = rq_1q_2$ and $\beta \in \left[\frac{\log \sigma}{\log \eta}, 1\right]$.

Chapter 4

Statement and Proof of Theorem B

In this chapter, we continue to explore the central theme of this thesis, the construction and analysis of cocycles exhibiting discontinuity of Lyapunov exponents. For this purpose, we retake to examine the example introduced in Section 3.1.

In the following, we prove a result that expands the range of cocycles for which discontinuity occurs, illustrating that the behavior observed in the initial example is not isolated but can in fact be replicated within a broader class. We then clarify in what sense our result constitutes an improvement and what restrictions remain.

For that, we revisit some definitions and notation introduced in the previous chapter. We then establish a refinement of the results previously obtained, specifically, an improvement over the conditions in (3.4) and (3.3) from Theorem A. This statement allows us to obtain a broader family of discontinuity points for the Lyapunov exponents.

However, the argument developed here leads to a different type of result. We only obtain a flexibility result for positive exponents. More precisely, within any neighborhood of the original cocycle, and for every positive number strictly smaller than the Lyapunov exponent of the unperturbed cocycle, we can construct a C^α-continuous linear cocycle whose Lyapunov exponent realizes the chosen value.

This phenomenon shows that, although we cannot force the Lyapunov exponent to vanish in this setting, we still retain information about the asymptotic behavior of infinitely many cocycles arbitrarily close to the original one. In particular, they all exhibit exponential growth, but at a rate strictly smaller than that of the original cocycle.

As mentioned above, for the remainder of this chapter we work with $M = \{0, \cdots, l\}$, associated with the shift map f as the dynamic base, the Bernoulli measure μ, and the cocycle $A : M \to \mathrm{SL}(2, \mathbb{R})$ defined by

$$A|_{[0;i]} = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix} \tag{4.1}$$

where $a_i \neq 0$ for every $i \in \{0, \cdots, l\}$.

In addition, we assume that there exist indices j and r such that $a_j < 1$ and $a_r > 1$. Note that it is also irrelevant which of them provides the stronger expansion or contraction.

Recall that the Lyapunov exponents of this cocycle are given by the expression (3.2). For convenience, set

$$\lambda := \sum_{a_i > 1} p_i \log a_i - \sum_{a_i < 1} p_i \log a_i^{-1}.$$

Without loss of generality, we assume that the weights $(p_0, \cdots, p_l)$ are chosen so that

$$\lambda > 0.$$

Theorem B. *For any $\alpha > 0$, let $a_0, \cdots, a_l$ be positive numbers such that there exist indices j and r with $a_j < 1 < a_r$. Let $(p_0, \cdots, p_l)$ be any set of weights such that*

$$2^\alpha < \prod_{i=0}^{l} a_i^{2p_i} \tag{4.2}$$

and

$$2^{2\alpha} \leq \left(a_j^{-1}\right)^2 \tag{4.3}$$

then for every value $\kappa \in (0, \lambda]$ there exists an α-Hölder continuous cocycle B, arbitrarily close to A, such that $\lambda_+(B, \mu) = \kappa$. Therefore, A is a discontinuity for the Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2, \mathbb{R}))$.

We would like to make a few remarks regarding the assumptions of the theorem.

Considering the symbolic space with only two symbols, 0 and 1, the locally constant cocycle A coincides with $\mathrm{A}_{\sigma\eta}$ defined in Example 1.2. By Thereom B, if

$$2^{2\alpha} \leq \eta^2, \quad \text{and} \quad 2^\alpha < \sigma^{2p}\eta^{2(p-1)}$$

for some $p \in \left(\frac{3\log\eta}{2(\log\sigma + \log\eta)}, 1\right)$, then the same conclusion applies. However, this hypothesis can be improved. Using that the space consists of only two symbols and that $\sigma > \eta$, where η is the rate of contraction in the horizontal direction, and following the same argument as in the proof of Theorem B, we obtain the following refinement:

Corollary 4.1. *For any $\alpha > 0$, let $p \in \left(\frac{3}{4}, 1\right)$ and $1 < \eta < \sigma$, such that*

$$2^\alpha < \sigma^{4p-2} \tag{4.4}$$

and

$$2^{2\alpha} \leq \eta^2 \tag{4.5}$$

then the conclusion of Theorem B holds for $\mathrm{A}_{\sigma\eta}$.

Observe that the assumption $\lambda > 0$ in Theorem B is equivalent, in Butler's setting, Theorem 2.5, to requiring $p \in (1/2, 1)$. Under this condition, we have $\lambda_+(A, \mu) = \lambda$.

A cocycle $A \in C^\alpha(M, \mathrm{SL}(2, \mathbb{R}))$ is said to be *non-uniformly fiber-bunched* if its extremal Lyapunov exponents satisfy

$$\lambda_+(A, \mu) - \lambda_-(A, \mu) = 2\lambda_+(A, \mu) \leq \alpha \log 2 \tag{4.6}$$

The non-uniformly fiber-bunched condition captures a form of domination that holds not at every point, but only along a full measure set of typical orbits. In this setting, the rate at which the cocycle separates nearby directions is sufficiently small to be controlled by the regularity of the base dynamics and the Hölder continuity of the matrices. In particular, every fiber-bunched cocycle is non-uniformly fiber-bunched, but the converse is not true. Indeed, consider the cocycle A_σ defined in (2.3), with $\sigma > 1$ satisfying $2^{3\alpha} \leq \sigma^2$. By Definition 2.2, this cocycle is not α-fiber-bunched. However, we can choose a weight $p = \frac{1}{2} - \varepsilon$ for a small positive ε, and then $\lambda_+(A_\sigma, \mu) = \log \sigma^{2\varepsilon}$. Hence, if $\varepsilon < \frac{\alpha \log 2}{2 \log \sigma}$, condition (4.6) is satisfied, that is, A_σ is non-uniformly fiber-bunched.

In Theorem B, condition (4.2) explicitly excludes this situation. In fact,

$$\alpha \log 2 < 2\sum_{i=0}^{l} p_i \log a_i = 2\lambda = 2\lambda_+(A, \mu).$$

which means that A is not a non-uniformly fiber-bunched cocycle.

In contrast, Theorem A allows for greater flexibility in the choice of both the cocycle and the measure. There, we may work with cocycles that are non-uniformly fiber-bunched, as long as the Lyapunov exponents are non-zero, a situation that is explicitly not allow in Theorem B.

On the other hand, Theorem B yields a broader class of discontinuity points. Indeed, it applies to any cocycle, provided that it is not non-uniformly fiber-bunched, possessing at least one constant $a_j < 1$ such that $2^{2\alpha} \leq (a_j^{-1})^2$. In comparison, the condition (3.4) in Theorem A is more restrictive.

Thus, while Theorem A allows greater freedom in the choice of both the cocycle and the measure and guarantees that the original cocycle can be approximated by cocycles with zero Lyapunov exponents, Theorem B contains a wider range of discontinuity phenomena when the two results are compared for a fixed measure. However, in the setting of Theorem B, one can only ensure approximation by cocycles with arbitrarily small, but not necessarily zero, Lyapunov exponents.

Proof of Theorem B. In order to prove the theorem, we follow the standard strategy, that is, we construct a perturbation that interchanges the horizontal and vertical axes and then show that this perturbed cocycle has Lyapunov exponents that are small, but nonzero.

To achieve this, we divide the proof into two parts. In the first part, we define a cocycle that sends the horizontal direction to the vertical one at certain points in the return times iterates. However, this cocycle does not yet realize both interchanges. So, after that we construct a new cocycle, derived from the one defined previously, which performs both interchanges while remaining close to the original cocycle in the α-Hölder topology.

Finally, in the second part, we compute the Lyapunov exponents of this cocycle, then concluding the discontinuity result.

The technique used in this proof is based on Butler's technique, in [6].

- *First Part:*

Assume that $2^{2\alpha} < a_j^{-2}$. The case of equality follows directly from the theorem under the hypothesis of strict inequality, as discussed in Chapter 3.

Let $N \in \mathbb{N}$ and consider the cylinder

$$Z = [0; rj \cdots j]$$

where the symbol j appears in N coordinates.

By the definition of the cylinder, we have that $f^i(Z) \cap f^s(Z) = \emptyset$ for $0 \leq i < s \leq N$.

Next, choose a small parameter $\varrho > 0$ such that the following inequality still holds:

$$a_j^{2(p_j + l\varrho)} \prod_{a_i > 1} a_i^{2(p_i - \varrho)} \prod_{a_i < 1,\ i \neq k} a_i^{2(p_i + \varrho)} > 2^{\alpha}. \tag{4.7}$$

Also, we fix some notation for the shear matrices that will be used in our construction. For any $\rho \in \mathbb{R}$, define

$$R_1^{\rho} = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_2^{\rho} = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} \tag{4.8}$$

Observe that each of these matrices leaves one of the coordinate directions invariant: specifically, the direction e_i is invariant under R_i^{ρ} for $i \in \{1, 2\}$.

Fix $\varepsilon > 0$ and consider the cocycle associated with the following function:

$$\begin{aligned} \tilde{B}_N : M &\to \mathrm{SL}(2, \mathbb{R}) \\ (x_i)_{i \in \mathbb{Z}} &\mapsto \tilde{B}_N(x) = \begin{cases} A(x) R_2^{\varepsilon 2^{-\alpha N}} & \text{if } x \in Z \\ R_1^{-\theta} A(x) & \text{if } x \in f^{N-1}(Z) \\ A(x) & \text{otherwise} \end{cases} \end{aligned} \tag{4.9}$$

where $\theta = \varepsilon^{-1} 2^{\alpha N} a_j^{2(N-1)} a_r^2$.

The function $\tilde{B}_N$ is well defined, since the domains on which it is defined are pairwise disjoint. Moreover, for $x \in Z$, we have

$$\begin{aligned}\tilde{B}_N^N(x) &= \tilde{B}_N(f^{N-1}(x)) \cdots \tilde{B}_N(x) \\ &= \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_j^{(N-1)} & 0 \\ 0 & a_j^{-(N-1)} \end{pmatrix} \begin{pmatrix} a_r & 0 \\ 0 & a_r^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon 2^{-\alpha N} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\theta a_j^{-(N-1)} a_r^{-1} \\ \varepsilon 2^{-\alpha N} a_j^{-(N-1)} a_r^{-1} & a_j^{-(N-1)} a_r^{-1} \end{pmatrix}\end{aligned}$$

Recall that we denote τ_Z the return time of the set Z. Then, for every $x \in Z$, the action of the cocycle after τ_Z iterates of the dynamics is given by:

$$\begin{aligned}\tilde{B}_N^{\tau_Z}(x) &= \tilde{B}_N(f^{\tau_Z - 1}(x)) \cdots \tilde{B}_N(f^N(x)) \tilde{B}_N^N(x) \\ &= \begin{pmatrix} d_{\tau_Z}(x) & 0 \\ 0 & d_{\tau_Z}(x)^{-1} \end{pmatrix} \begin{pmatrix} 0 & -\theta a_j^{-(N-1)} a_r^{-1} \\ \varepsilon 2^{-\alpha N} a_j^{-(N-1)} a_r^{-1} & a_j^{-(N-1)} a_r^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\theta a_j^{-(N-1)} a_r^{-1} d_{\tau_Z}(x) \\ \varepsilon 2^{-\alpha N} a_j^{-(N-1)} a_r^{-1} d_{\tau_Z}(x)^{-1} & a_j^{-(N-1)} a_r^{-1} d_{\tau_Z}(x)^{-1} \end{pmatrix}\end{aligned}$$

Here,

$$d_{\tau_Z}(x) = a_r^{D^r_{\tau_Z}(x)-1} a_j^{D^j_{\tau_Z}(x)-(N-1)} \prod_{i \neq \{k,j\}} a_i^{D^i_{\tau_Z}(x)}$$

where, for each $x \in Z$, the random variable $D^i_{\tau_Z}(x)$ denotes the number of occurrences of the symbol i in the orbit segment of x up to time $\tau_Z(x) - 1$. More explicitly,

$$\begin{aligned}D^i_{\tau_Z}(x) &= \#\{0 \le n \le \tau_Z(x); f^n(x) \in [0;i]\} \\ &= \sum_{n=0}^{\tau_Z(x)-1} \chi_{[0;i]} \circ f^n(x)\end{aligned}$$

Substituting this expression, we obtain

$$\tilde{B}_N^{\tau_Z}(x) = \begin{pmatrix} 0 & -\theta a_r^{D^r_{\tau_Z}(x)-2} a_j^{D^j_{\tau_Z}(x)-2(N-1)} \prod_{i \neq \{r,j\}} a_i^{D^i_{\tau_Z}(x)} \\ \prod_{i=0}^{l} a_i^{-D^i_{\tau_Z}(x)} \varepsilon 2^{-\alpha N} & \prod_{i=0}^{l} a_i^{-D^i_{\tau_Z}(x)} \end{pmatrix}$$

Observe, therefore, that for every point $x \in Z$,

$$\tilde{B}_N^{\tau_Z}(x) e_1 = \left(\prod_{i=0}^{l} a_i^{-D^i_{\tau_Z}(x)} \varepsilon 2^{-\alpha N} \right) e_2$$

Hence, this cocycle sends the horizontal direction e_1 to a multiple of the vertical direction e_2, and it remains close to A in the α-Hölder topology.

Claim: For N sufficiently large, we have $\|A - \tilde{B}_N\|_\alpha < (\text{Const.})\varepsilon$

Proof. As before, we analyze the two terms appearing in the definition of the norm in (2.1). For the first term, we obtain

$$\|A - \tilde{B}_N\| = \sup_{x \in M} \|A(x) - \tilde{B}_N(x)\| \leq \max_{0 \leq i \leq l} \{a_i, a_i^{-1}\} \max\{\theta, \varepsilon 2^{-\alpha N}\}$$

Recall that $\theta = \varepsilon^{-1} 2^{\alpha N} a_j^{2(N-1)} a_r^2$, and by hypothesis, $a_j^{-2} > 2^{2\alpha}$, that is, $\frac{2^{2\alpha}}{a_j^{-2}} < 1$. Then, for N sufficiently large,

$$\left(\frac{2^{2\alpha}}{a_j^{-2}}\right)^N < \left(\frac{\varepsilon}{a_j^{-1} a_r}\right)^2$$

It follows,

$$\theta = \varepsilon^{-1} 2^{\alpha N} a_j^{2(N-1)} a_r^2 < \varepsilon 2^{-\alpha N}$$

Consequently,

$$\|A - \tilde{B}_N\| \leq \max_{0 \leq i \leq l} \{a_i, a_i^{-1}\} \varepsilon 2^{-\alpha N}$$

For the second term in the norm, we must consider all possible positions of the points x and y. The only nontrivial case occurs when x and y belong to different cylinders in the domain of definition of $\tilde{B}_N$; in all other cases, the difference vanishes. Observe that, since the angle θ is bounded by $\varepsilon 2^{-\alpha N}$ and the diameter of the cylinders is greater than $2^{-\alpha N}$ by the definition of Z, we have in all relevant cases that

$$\frac{\|A(x) - \tilde{B}_N(x) - A(y) + \tilde{B}_N(y)\|}{d(x,y)^\alpha} \leq \text{const } \varepsilon$$

Hence, $\|A - \tilde{B}_N\|_\alpha < (\text{Const})\varepsilon$ for N sufficiently large. □

However, the cocycle $\tilde{B}_N$ is not yet the perturbation we aim to construct. To obtain the desired cocycle, we must define it on specific sets whose properties will later be essential for computing the Lyapunov exponents.

Let us consider a finite word $v = v_0 \cdots v_{m-1}$ with $m \geq N+1$, where $v_i \in \{0, \cdots, l\}$ satisfying the following properties:

- $v_0 = r$;
- $v_i = j$ for $1 \leq i \leq N$;
- the subword $v_{N+1} \cdots v_{m-1}$ does not contain the form $rj \cdots j$ consisting of one r followed by N consecutive j's.

Words satisfying these conditions will be called *Z-return blocks.* We denote by $|v| := m$ the length of the Z-return block v.

As an example, if $x \in Z$ is a point that returns to Z after τ_Z iterates, then the corresponding Z-return block associated with x is $[x]_0^{\tau_Z} = x_0 x_1 \cdots x_{\tau_Z - 1}$, which we denote by $v^1(x)$.

For a Z-return block v, we define the cylinder associated with v by

$$\mathcal{C}(v) = \{y \in M;\, y_i = v_i,\, 0 \leq i \leq |v| - 1,\, y_{|v|} = r,\, y_{|v|+i} = j,\, 1 \leq i \leq N\}$$

We also define the *good set* $G \subset Z$, consisting of points $x \in Z$ such that

$$G = \left\{x \in Z; N^2 \leq \tau_Z(x) \leq \frac{\omega}{\mu(Z)} \text{ and } |D^i_{\tau_Z}(x) - p_i \tau_Z(x)| \leq \varrho \tau_Z(x) \quad \forall i \neq j\right\}$$

where $\omega > 0$ is a constant to be determined in the second part of the proof.

Finally, we define the set $\mathcal{G}_1$ of Z-return blocks whose associated cylinders satisfy:

$$\mathcal{G}_1 = \{v; \mathcal{C}(v) \subset G\}$$

As mentioned earlier, our goal is to construct a perturbation that interchanges the horizontal and vertical directions after τ_Z iterates. This interchange will play a key role in the computation of the Lyapunov exponents. In the first perturbation we introduce, however, the construction allows for only a single change of direction. By definition of the cocycle, the axis e_2 is not invariant under $\tilde{B}_N$. Nevertheless, because of the dynamics, the behavior of the iterates depends sensitively on the symbolic sequence of the point. In particular, if a point's sequence contains a larger proportion of the symbol r (or any symbol whose corresponding coefficient a_i exceeds 1) up to time τ_Z, the corresponding vector tends to form a smaller angle with the horizontal direction. It is therefore essential to control this angle carefully in order to ensure that the perturbed cocycle remains α-Hölder continuous. This is precisely the motivation for introducing the good set G: it provides uniform bounds on the frequency of the symbols in the sequence and on the size of the return times. With this control in place, we can guarantee that the perturbation effectively interchanges the horizontal and vertical directions on a subset of Z with large measure, ensuring the desired geometric behavior of the cocycle.

Consider a Z-return block v in the set $\mathcal{G}_1$. This means that there exists a point $y \in G$ such that $v = v^1(y)$. For any point contained in the cylinder generated by v, that is $x \in \mathcal{C}(v)$, we define the following angle

$$\begin{aligned} \delta(v) &= \theta^{-1} d_{\tau_Z}(x)^{-2} \\ &= \theta^{-1} a_r^{-2D^r_{\tau_Z}(x)+2} a_j^{-2D^j_{\tau_Z}(x)+2(N-1)} \textstyle\prod_{i\neq\{r,j\}} a_i^{-2D^i_{\tau_Z}(x)} \end{aligned} \tag{4.10}$$

Using this angle $\delta(v)$, we obtain

$$\begin{aligned} R_2^{\delta(v)} \tilde{B}_N^{\tau_Z}(x) &= \begin{pmatrix} 1 & 0 \\ \delta(v) & 1 \end{pmatrix} \\ &\cdot \begin{pmatrix} 0 & -\theta a_r^{D^r_{\tau_Z}(x)-2} a_j^{D^j_{\tau_Z}(x)-2(N-1)} \prod_{i\neq\{r,j\}} a_i^{D^i_{\tau_Z}(x)} \\ \prod_{i=0}^l a_i^{-D^i_{\tau_Z}(x)} \varepsilon 2^{-\alpha N} & \prod_{i=0}^l a_i^{-D^i_{\tau_Z}(x)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\theta a_r^{D^r_{\tau_Z}(x)-2} a_j^{D^j_{\tau_Z}(x)-2(N-1)} \prod_{i\neq\{r,j\}} a_i^{D^i_{\tau_Z}(x)} \\ \prod_{i=0}^l a_i^{-D^i_{\tau_Z}(x)} \varepsilon 2^{-\alpha N} & 0 \end{pmatrix} \end{aligned}$$

We then define a new cocycle based on the previous one as follows:

$$\begin{aligned} B_N : M &\to \mathrm{SL}(2,\mathbb{R}) \\ (x_i)_{i\in\mathbb{Z}} &\mapsto \begin{cases} R_2^{\delta(v)} \tilde{B}_N(x) & \text{if } x \in \bigcup_{v\in Z\text{-return blocks}} f^{|v|-1}\mathcal{C}(v) \\ \tilde{B}_N(x) & \text{otherwise} \end{cases} \end{aligned} \tag{4.11}$$

It is well defined by construction of the sets where it is supported.

From the definition of the cocycle, it follows that, for every $x \in G$,

$$B_N^{\tau_Z}(x) e_1 = c_1 e_2 \text{ and } B_N^{\tau_Z}(x) e_2 = c_2 e_1.$$

We aim to estimate an upper bound for the angle $\delta(v)$ and to show that this bound decays exponentially, which will ensure that the perturbation is α-Hölder continuous.

Recall that

$$d_{\tau_Z}^{-2}(x) = \theta^{-1} a_r^{-2D^r_{\tau_Z}(x)+2} a_j^{-2D^j_{\tau_Z}(x)+2(N-1)} \prod_{i\neq\{r,j\}} a_i^{-2D^i_{\tau_Z}(x)}$$

By definition, for every point $x \in G$, and for every $i \neq j$, we have

$$(-\varrho + p_i)\tau_Z(x) \leq D^i_{\tau_Z}(x) \leq (\varrho + p_i)\tau_Z(x).$$

Using these inequalities, we obtain that

$$a_r^{-2D^r_{\tau_Z}(x)+2} \leq a_r^{-2(p_r-\varrho)\tau_Z(x)+2};$$

for $i \neq r$ such that $a_i > 1$,

$$a_i^{-2D^i_{\tau_Z}(x)} \leq a_i^{-2(p_i-\varrho)\tau_Z(x)}; \tag{4.12}$$

for $i \neq j$ such that $a_i < 1$,

$$a_i^{-2D^i_{\tau_Z}(x)} \leq a_i^{-2(p_i+\varrho)\tau_Z(x)};$$

and

$$a_j^{-2D^j_{\tau_Z}(x)+2(N-1)} \leq a_j^{-2\tau_Z(x)(p_j+l\varrho)+2(N-1)} \tag{4.13}$$

We now define the following parameter $\xi > 0$ as

$$\xi := 2^\alpha a_j^{-2(p_j+l\varrho)} \prod_{a_i>1} a_i^{-2(p_i-\varrho)} \prod_{a_i<1,\ i\neq j} a_i^{-2(p_i+\varrho)}.$$

By definition of ϱ in (4.7) we have $\xi < 1$.

Also, by definition of θ, there exists a constant C, independent of N, such that

$$C^{-1}\varepsilon 2^{-\alpha N} \leq \theta \leq C\varepsilon 2^{-\alpha N}$$

Hence,

$$C^{-1}\varepsilon^2 2^{-2\alpha N} a_j^{2N-2} a_r^2 \leq \varepsilon 2^{-\alpha N} a_j^{2N-2} a_r^2 \theta \leq C\varepsilon^2 2^{-2\alpha N} a_j^{2N-2} a_r^2$$

Since $x \in G$, we have $\tau_Z(x) \geq N^2$. Using the assumption $a_j^{-2} > 2^{2\alpha}$, it follows that for N sufficiently large,

$$\begin{aligned}
\left(2^\alpha a_j^{-2(p_j+l\varrho)} \prod_{a_i>1} a_i^{-2(p_i-\varrho)} \prod_{a_i<1,\ i\neq j} a_i^{-2(p_i+\varrho)}\right)^{\tau_Z(x)} &= \xi^{\tau_Z(x)} \\
&\leq \xi^{N^2} \\
&< C^{-1}\frac{a_j^{-2N}}{2^{2\alpha N}}\varepsilon^2 a_r^{-2} a_j^2 \\
&< \varepsilon 2^{-\alpha N} a_j^{-2N+2} a_r^{-2}\theta
\end{aligned}$$

Rearranging this inequality, we conclude that

$$\delta(v) = \theta^{-1} d_{\tau_Z}(x)^{-2} < \varepsilon 2^{-\alpha(\tau_Z(x)+N)} \tag{4.14}$$

With this, we conclude that B_N is an α-Hölder continuous cocycle. Moreover, since B_N differs from $\tilde{B}_N$ only on cylinders of diameter at least $2^{-(N+|v|)}$, where v is a Z-return block, we obtain that

$$\|\tilde{B}_N - B_N\|_\alpha < (\text{const.})\varepsilon.$$

Consequently, B_N is ε-close to A in the α-Hölder norm.

- *Second Part:*

In this part of the proof, we compute the Lyapunov exponents of B_N. For this specific cocycle, we will show that for any given number $0 < \kappa \leq \lambda$, one can choose N large enough such that $\lambda_+(B_N, \mu) < \kappa$. Thus, our goal now is to establish this bound. Once this is achieved, we proceed to prove the flexibility argument.

As we discussed in Proposition 1.8, there is a relation between $B_N^{\tau_Z}$ and B_N given by the formula

$$\lambda_+(B_N^{\tau_Z}, \mu_Z) = \frac{\lambda_+(B_N, \mu)}{\mu(Z)} \tag{4.15}$$

where μ_Z denotes the normalized measure on Z.

Recall that for $x \in Z$, the second return time of this point to Z under the dynamics of f is defined by

$$\tau_Z^{(2)}(x) = \tau_Z(x) + \tau_Z(f^{\tau_Z}(x))$$

By the Subadditive Ergodic Theorem, we then obtain the bound

$$\lambda_+(B_N^{\tau_Z}, \mu_Z) \leq \frac{1}{2}\int_Z \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z. \tag{4.16}$$

Therefore, given this bounded, we now compute the integral by splitting it over the sets $Z \backslash G$, $Z \backslash f^{-\tau_Z}(G)$ and $G \cap f^{-\tau_Z}(G)$.

First observe that since $f^i(Z) \cap Z = \emptyset$ for $1 \leq i \leq N$, we may apply a result of Abadi-Vergne [1], specifically equation (2) in their paper, which states that there is a constant $\varsigma > 0$ independent of N such that for all $a > 1$ and $N > 0$,

$$\mu_Z(\{x \in Z; \tau_Z(x) > a\mu(Z)^{-1}\} \leq \varsigma e^{-a} \tag{4.17}$$

We then fix a parameter $\omega > 1$ such that $\varsigma \log(\max_{0\leq i\leq l}\{a_i, a_i^{-1}\}^2)\frac{\omega e^{-\omega}}{(1-e^{-\omega})^2} < \frac{\kappa}{100}$. Such a choice is possible because the limit of $\frac{xe^{-x}}{(1-e^{-x})^2}$ is 0 as x goes to infinity. This is the value of ω used in the definition of the good set G.

Let us now consider the set $Z \backslash G$.

First, we consider the points in $Z \backslash G$ that lie in the set $Q_\omega = \{x \in Z : \tau_Z(x) > \omega\mu(Z)^{-1}\}$. Using the Abadi-Vergne estimate, we have $\mu_Z(Q_\omega) \leq \varsigma e^{-\omega} < \frac{\kappa}{100}$.

Since $Q_\omega = \cup_{n\geq 1}(Q_{n\omega} \backslash Q_{(n+1)\omega})$, we obtain

$$\begin{aligned}
\int_{Q_\omega} \log \|B_N^{\tau_Z}\| d\mu_Z &\leq \sum_{n\geq 1} \int_{Q_{n\omega}\backslash Q_{(n+1)\omega}} \log \|B_N\|^{\tau_Z} d\mu_Z \\
&\leq \log\left(\max_{0\leq i\leq l}\{a_i, a_i^{-1}\}^2\right) \sum_{n\geq 1} \int_{Q_{n\omega}\backslash Q_{(n+1)\omega}} \tau_Z \, d\mu_Z \\
&\leq \log\left(\max_{0\leq i\leq l}\left\{a_i, a_i^{-1}\right\}^2\right) \sum_{n\geq 1} \frac{(n+1)\omega\mu_Z(Q_{n\omega})}{\mu(Z)} \\
&\leq \frac{\varsigma \log(\max_{0\leq i\leq l}\{a_i, a_i^{-1}\}^2)}{\mu(Z)} \sum_{n\geq 1} e^{-n\omega}(n+1)\omega \\
&< \frac{\kappa}{100\mu(Z)}
\end{aligned}$$

Note that the good set can be written as a finite intersection of the following sets:

$$\tilde{G} = \left\{x \in Z; N^2 \leq \tau_Z(x) \leq \frac{\omega}{\mu(Z)}\right\}$$

and, for $i \neq j$,

$$G_i = \{x \in Z;\ |D^i_{\tau_Z}(x) - p_i\tau_Z(x)| \leq \varrho\tau_Z(x)\}$$

Thus, $G = \tilde{G} \cap \bigcap_{i\neq j} G_i$.

If $x \in Z\backslash(G \cup Q_\omega)$, then $\tau_Z(x) \leq \omega\mu(Z)^{-1}$, and there are l possible reasons why x does not belong to $G \cup Q_\omega$. They are:

1. $x \in \tilde{G}^c\backslash Q_\omega$.
2. $x \in G_i^c\backslash(\tilde{G}^c \cup Q_\omega)$ for some $i \neq j$.

Observe that in item 2 we must consider all indices $0 \leq i \neq j \leq l$. This is why we obtain l different possibilities.

In the case of item 1, we have $N + 2 \leq \tau_Z(x) \leq N^2$. Therefore, for each $N + 1 < m$,

$$\{x \in Z : \tau_Z(x) = n\} \subset \{x \in Z : f^n(x) \in Z\} \subset [n; rj \cdots j].$$

Since, for $x = (x_m)_{\{m\in\mathbb{Z}\}}$, the coordinates x_m with $m \geq N + 2$ or $m < 0$ are independent and identically distributed with respect to μ_Z, then for $m > N + 1$

$$\begin{aligned} \mu_Z(\{x \in Z : \tau_Z(x) = n\}) &\leq p_r \prod_{i=1}^{N} \mu_Z([n+i; j]) \\ &\leq p_r p_j^{N+1} \\ &= \mu(Z) \end{aligned}$$

Hence,

$$\begin{aligned} \mu_Z(\{x \in Z : \tau_Z(x) < n\}) &= \sum_{s=N+2}^{n-1} \mu_Z(\{x \in Z : \tau_Z(x) = s\}) \\ &\leq n\mu(Z) \end{aligned}$$

We conclude that

$$\mu_Z(\tilde{G}^c\backslash Q_\omega) \leq N^2\mu(Z) \tag{4.18}$$

For the second item, fix some $i \neq j$ and consider a point x in $G_i^c\backslash(\tilde{G}^c \cup Q_\omega)$. The remaining cases are analogous.

Since $x \notin \tilde{G}^c$, we have $\tau_Z(x) \geq N^2$, and therefore

$$\{x \in Z; \tau_Z(x) \geq N^2\} \subset \bigcup_{n=N^2}^{\infty} \{x \in Z; \tau_Z(x) \geq n\}$$

On the other hand, because $x \notin G_i$, we obtain $|D^i_{\tau_Z}(x) - p_i\tau_Z(x)| > \varrho\tau_Z(x)$. Combining both facts, we have

$$x \in \bigcup_{n=N^2}^{\infty} \{x \in Z; |D^i_n(x) - p_i n| > \varrho n\}$$

Resulting in the following estimate:

$$\mu_Z(G_i^c\backslash(\tilde{G}^c \cup Q_\omega)) \leq \sum_{n=N^2}^{\infty} \mu_Z(\{x \in Z; |D^i_n(x) - p_i n| > \varrho n\}) \tag{4.19}$$

Recall that for any $y \in M$,

$$D^i_n(y) = \sum_{m=0}^{n-1} \chi_{[0;i]} f^{-m}(y)$$

which is a sum of independent and identically distributed random variables taking values in $[0,1]$. Thus,

$$\begin{aligned}\mathbb{E}[D_n^i] &= \sum_{m=0}^{n-1} \mathbb{E}[\chi_{[0;i]} f^{-m}] \\ &= \sum_{m=0}^{n-1} \int \chi_{[0;i]} f^{-m} d\mu_Z \\ &= np_i\end{aligned}$$

Applying the Chernoff inequality, [25], we obtain

$$\mu_Z(\{x \in Z; |D_n^i(x) - p_i n| > \varrho n\}) \leq e^{-\frac{\varrho^2 n}{2}}$$

Substituting this bound into (4.19), we have

$$\begin{aligned}\mu_Z(G_i^c \backslash (\tilde{G}^c \cup Q_\omega)) &\leq \sum_{n=N^2}^{\infty} e^{-\frac{\varrho^2 n}{2}} \\ &\leq \frac{e^{-\frac{\varrho^2 N^2}{2}}}{1 - e^{-\frac{\varrho^2}{2}}}\end{aligned}$$

Hence,

$$\mu_Z(Z \backslash (G \cap Q_\omega)) < N^2 \mu(Z) + (l-1) \frac{e^{-\frac{\varrho^2 N^2}{2}}}{1 - e^{-\frac{\varrho^2}{2}}}$$

which converges to zero as $N \to \infty$. Thus, for N sufficiently large,

$$\mu_Z(Z \backslash (G \cap Q_\omega)) < \frac{\kappa}{100\omega \log(\max_{0 \leq i \leq l}\{a_i, a_i^{-1}\}^2)} \tag{4.20}$$

Therefore, since $\mu_Z(Q_\omega) < \frac{\kappa}{100\omega \log(\max_{0 \leq i \leq l}\{a_i, a_i^{-1}\}^2)}$,

$$\mu_Z(Z \backslash G) < \frac{\kappa}{50\omega \log(\max_{0 \leq i \leq l}\{a_i, a_i^{-1}\}^2)} \tag{4.21}$$

Proposition 4.2. $\frac{1}{2} \int_{Z \backslash G} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z < \frac{3\kappa}{100\mu(Z)}$

Proof. By the definition of a linear cocycle, we can bound the integral by the sum of two other integrals as follows:

$$\frac{1}{2} \int_{Z \backslash G} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z \leq \frac{1}{2} \int_{Z \backslash G} \log \|B_N^{\tau_Z}\| d\mu_Z + \frac{1}{2} \int_{Z \backslash G} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z$$

Using previous estimates, the first integral is bounded by:

$$\begin{aligned}\frac{1}{2} \int_{Z \backslash G} \log \|B_N^{\tau_Z}\| d\mu_Z &\leq \frac{1}{2} \int_{Z \backslash (G \cap Q_\omega)} \log \|B_N^{\tau_Z}\| d\mu_Z + \frac{1}{2} \int_{Q_\omega} \log \|B_N^{\tau_Z}\| d\mu_Z \\ &\leq \frac{1}{2\mu(Z)} \log \left(\max_{0 \leq i \leq l} \{a_i, a_i^{-1}\}^2 \right) \omega \mu_Z(Z \backslash (G \cap Q_\omega)) + \frac{\kappa}{200\mu(Z)} \\ &< \frac{\kappa}{100\mu(Z)}\end{aligned}$$

For the second one, we split it into two additional integrals:

$$\int_{G^c} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z = \int_{G^c \cap f^{-\tau_Z}(G)} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z + \int_{G^c \cap f^{-\tau_Z}(G)^c} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z$$

Repeating the same arguments used for the set $Z \backslash G$, but now computing the integrals over the sets $f^{-\tau_Z}(Q_\omega)$ and $Z \backslash (f^{-\tau_Z}(G) \cap f^{-\tau_Z}(Q_\omega))$, we obtain

$$\begin{aligned} \int_{G^c \cap f^{-\tau_Z}(G)^c} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z &\leq \int_{f^{-\tau_Z}(G)^c} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z \\ &< \frac{\kappa}{50\mu(Z)} \end{aligned}$$

For the remaining integral, note that if x in $G^c \cap f^{-\tau_Z}(G)$, then $f^{\tau_Z}(x) \in G$, and consequently $\tau_Z(f^{\tau_Z}(x)) \leq \frac{\omega}{\mu(Z)}$. Hence,

$$\begin{aligned} \int_{G^c \cap f^{-\tau_Z}(G)} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z &\leq \frac{\omega}{\mu(Z)} \log \left(\max_{0 \leq i \leq l} \{a_i, a_i^{-1}\}^2 \right) \mu_Z(G^c) \\ &< \frac{\kappa}{50\mu(Z)} \end{aligned}$$

Combining the two estimates, we conclude that

$$\frac{1}{2} \int_{Z \backslash G} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z < \frac{\kappa}{50\mu(Z)}$$

Thus,

$$\frac{1}{2} \int_{Z \backslash G} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z < \frac{3\kappa}{100\mu(Z)}$$

□

We have the same estimate on $Z \backslash f^{-\tau_Z}(G)$:

Proposition 4.3. $\frac{1}{2} \int_{Z \backslash f^{-\tau_Z}(G)} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z < \frac{3\kappa}{100\mu(Z)}$

Proof. By definition,

$$\frac{1}{2} \int_{Z \backslash f^{-\tau_Z}(G)} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z \leq \frac{1}{2} \int_{Z \backslash f^{-\tau_Z}(G)} \log \|B_N^{\tau_Z}\| d\mu_Z + \frac{1}{2} \int_{Z \backslash f^{-\tau_Z}(G)} \log \|B_N^{\tau_Z} \circ f^{\tau_Z}\| d\mu_Z$$

Applying once again the arguments employed in the preceding proposition, we obtain the desired estimate.

□

Therefore, combining both results we conclude that

$$\int_{Z \backslash (G \cap f^{-\tau_Z}(G))} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z < \frac{3\kappa}{50\mu(Z)}.$$

In order to conclude the bound (4.16), we need to analyze $\int_{G \cap f^{-\tau_Z}(G)} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z$.

Proposition 4.4. $\frac{1}{2} \int_{G \cap f^{-\tau_Z}(G)} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z < \frac{\log(a_r^{-2}\varepsilon)}{2} + N \frac{\log C}{2} + \frac{\kappa}{50\mu(Z)}$

Proof. Observe that if $x \in G \cap f^{-\tau_Z}(G)$, then $x \in G$ and $f^{\tau_Z}(x) \in G$, that is, the first and second return times of x are bounded by $\frac{\omega}{\mu(Z)}$.

By definition of B_N, we have that for every $x \in G$,

$$B_N^{\tau_Z}(x) = \begin{pmatrix} 0 & -\theta a_r^{D^r_{\tau_Z}(x)-2} a_j^{D^j_{\tau_Z}(x)-2(N-1)} \prod_{i\neq\{r,j\}} a_i^{D^i_{\tau_Z}(x)} \\ \prod_{i=0}^{l} a_i^{-D^i_{\tau_Z}(x)} \varepsilon 2^{-\alpha N} & 0 \end{pmatrix} \tag{4.22}$$

and for every $x \in G \cap f^{-\tau_Z}(G)$,

$$B_N^{\tau_Z^{(2)}}(x) = \begin{pmatrix} -\theta\varepsilon 2^{-\alpha N} a_r^{-2} a_j^{-2(N-1)} \prod_{i=0}^{l} a_i^{R_i(x)} & 0 \\ 0 & -\theta\varepsilon 2^{-\alpha N} a_r^{-2} a_j^{-2(N-1)} \prod_{i=0}^{l} a_i^{-R_i(x)} \end{pmatrix} \tag{4.23}$$

where $R_i(x) = D^i_{\tau_Z}(f^{\tau_Z}(x)) - D^i_{\tau_Z}(x)$ with $i \in \{0, \cdots, l\}$.

Using (4.23), we have that

$$\log \left\| B_N^{\tau_Z^{(2)}}(x) \right\| \leq \log(a_r^{-2}\varepsilon) + N \log C + \sum_{i;a_i>1} |R_i(x)| \log a_i^2 + \sum_{i;a_i<1} |R_i(x)| \log a_i^{-2} \tag{4.24}$$

where $C = \frac{a_j^{-2}}{2^{2\alpha}}$. Thus,

$$\begin{aligned} \int_{G\cap f^{-\tau_Z}(G)} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z \ &\leq \ \int_{G\cap f^{-\tau_Z}(G)} (\log(a_r^{-2}\varepsilon) + N\log C) d\mu_Z \\ &+ \sum_{i;a_i>1} \log a_i^2 \int_{G\cap f^{-\tau_Z}(G)} |R_i(x)| d\mu_Z \\ &+ \sum_{i;a_i<1} \log a_i^{-2} \int_{G\cap f^{-\tau_Z}(G)} |R_i(x)| d\mu_Z \end{aligned}$$

Define for each $i \in \{0, \cdots, l\}$,

$$T^i_m(x) := \sum_{n=N+2}^{m-1} \chi_{[0;i]} \circ f^n(x) \ \text{ for } x \in Z \ \text{ and } \ m > N+1.$$

We focus on the case $i = j$ since the remaining ones are analogous and for simplicity, we write $T_m = T^j_m$ from now on.

Notice that, $T_m(x) = D^j_m(x) - N$ is a sum of random variables $\chi_{[0;j]} \circ f^n$ with $n > N+1$. These variables are independent and identically distributed with respect to μ_Z.

Define, for $x \in Z$, $\varphi(x) := T_{\tau_Z}(x) - T_{\tau_Z}(f^{\tau_Z}(x))$. A direct computation gives

$$\varphi(x) = D^j_{\tau_Z}(x) - D^j_{\tau_Z}(f^{\tau_Z}(x))$$

and therefore

$$\int_{G\cap f^{-\tau_Z}(G)} |D^j_{\tau_Z} \circ f^{\tau_Z} - D^j_{\tau_Z}| d\mu_Z = \int_{G\cap f^{-\tau_Z}(G)} |\varphi| d\mu_Z$$

Observe that $T_{\tau_Z} \in L^1(\mu_Z)$. The proof of this fact follows exactly the same argument used in Lemma 3.4. Moreover,

$$\int_Z T_{\tau_Z} d\mu_Z = p_j \left(\frac{1}{\mu(Z)} - (N+1) \right)$$

Since T_{τ_Z} is integrable and μ_Z is f^{τ_Z}-invariant, it follows that

$$\int_Z \varphi d\mu_Z = 0 \tag{4.25}$$

Therefore, the first moment of φ is 0. Now, applying the second identity of Wald in T_{τ_Z}, that is,

$$\mathbb{E}\left[\left(\sum_{i=1}^{\tau} X_i - \mathbb{E}\left[\sum_{i=1}^{\tau} X_i\right]\right)^2\right] = \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] \cdot \mathbb{E}[\tau]$$

where X_i are random variables and τ is the stopping time with respect to the sequence of σ-algebras associated with this sequence $\{X_i\}_i$, we obtain

$$\begin{aligned}
\int_Z \left(T_{\tau_Z} - \int_Z T_{\tau_Z} d\mu_Z\right)^2 d\mu_Z &= \left(\int_Z \left(\chi_{[0;j]} \circ f^{N+1} - \int_Z \chi_{[0;j]} \circ f^{N+1} d\mu_Z\right)^2 d\mu_Z\right) \\
&\cdot \left(\int_Z (\tau_Z - (N+1) d\mu_Z\right) \\
&= p_j(1-p_j)\left(\frac{1}{\mu(Z)} - (N+1)\right)
\end{aligned}$$

Note that the random variables T_{τ_Z} and $T_{\tau_Z} \circ f^{\tau_Z}$ are independent and identically distributed. Then, denoting by $\mathrm{Var}(X)$ the variance of a random variable X,

$$\begin{aligned}
\int_Z \varphi^2 d\mu_Z &= 2\,\mathrm{Var}(T_{\tau_Z}) \\
&= 2\int_Z \left(T_{\tau_Z} - \int_Z T_{\tau_Z} d\mu_Z\right)^2 d\mu_Z \\
&= 2p_j(1-p_j)\left(\frac{1}{\mu(Z)} - (N+1)\right) \\
&\leq \frac{r_j}{\mu(Z)}
\end{aligned}$$

where $r_j \geq 2p_j(1-p_j)(1-(N+1)\mu(Z))$

Consider the sets

$$A = \{x \in G \cap f^{-\tau_Z}(G);\ |\varphi(x)| > \mu(Z)^{-\frac{3}{4}}\}$$

and

$$B = \{x \in G \cap f^{-\tau_Z}(G);\ |\varphi(x)| \leq \mu(Z)^{-\frac{3}{4}}\}$$

Applying the Chebyshev's inequality (here we use (4.25), that is, the first moment of φ vanish),

$$\begin{aligned}
\mu(A) &\leq \frac{r_j}{\mu(Z)}\mu(Z)^{\frac{3}{2}} \\
&= r_j \mu(Z)^{\frac{1}{2}}
\end{aligned}$$

Since $\tau_Z(x) \leq \frac{\omega}{\mu(Z)}$ and $\tau_Z(f^{\tau_Z}(x)) \leq \frac{\omega}{\mu(Z)}$ for every $x \in G \cap f^{-\tau_Z}(G)$

$$\begin{aligned}
|\varphi(x)| &\leq \left|\sum_{n=0}^{\tau_Z(x)-1} \chi_{[0;j]} \circ f^n(x)\right| + \left|\sum_{n=0}^{\tau_Z(f^{\tau_Z}(x))-1} \chi_{[0;j]} \circ f^n(f^{\tau_Z}(x))\right| \\
&\leq \frac{2\omega}{\mu(Z)}
\end{aligned}$$

Hence,

$$\int_A |\varphi| d\mu_Z \le \frac{2\omega r_j}{\mu(Z)^{\frac{1}{2}}}$$

and

$$\int_B |\varphi| d\mu_Z \le \mu(B)\mu(Z)^{-\frac{3}{4}}$$

Combining these results, we obtain that

$$\int_{G\cap f^{-\tau_Z}(G)} |\varphi| d\mu_Z \le c_{(N;j)}\mu(Z)^{-\frac{3}{4}}$$

where $c_{(N;j)} = \mu(B) + 2\omega r_j \mu(Z)^{\frac{1}{4}}$. Observe that for N large enough, $c_{(N;j)}$ is sufficiently small. Then, choose N large enough such that

$$\int_{G\cap f^{-\tau_Z}(G)} |\varphi| d\mu_Z < \frac{\kappa}{25l \log a_j^{-2} \mu(Z)}$$

Similarly, we reproduce the same idea for the other indices i and we find N large enough such that

$$\sum_{i;a_i>1} \log a_i^2 \int_{G\cap f^{-\tau_Z}(G)} |R_i(x)| d\mu_Z + \sum_{i;a_i<1} \log a_i^{-2} \int_{G\cap f^{-\tau_Z}(G)} |R_i(x)| d\mu_Z \le \frac{\kappa}{25\mu(Z)} \tag{4.26}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\int_{G\cap f^{-\tau_Z}(G)} \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z &\le \frac{1}{2}\int_{G\cap f^{-\tau_Z}(G)} (\log(a_r^{-2}\varepsilon) + N\log C) d\mu_Z + \frac{\kappa}{50\mu(Z)} \\ &< \frac{\log(a_r^{-2}\varepsilon)}{2} + N\frac{\log C}{2} + \frac{\kappa}{50\mu(Z)} \end{aligned}$$

□

Combining the estimates of these three last propositions we conclude that

$$\frac{1}{2}\int_Z \log \left\| B_N^{\tau_Z^{(2)}} \right\| d\mu_Z < \frac{2\kappa}{25\mu(Z)} + \frac{\log(a_r^{-2}\varepsilon)}{2} + N\frac{\log C}{2} \le \frac{\kappa}{\mu(Z)}$$

for N large enough.

Using this estimate together with relations (4.15) and (4.16), we obtain that

$$\lambda_+(B_N, \mu) < \kappa \tag{4.27}$$

At this stage, we have shown that the Lyapunov exponent of our perturbed cocycle can be made arbitrarily small. However, to complete the flexibility argument we must find a linear cocycle for which the largest Lyapunov exponent attains the value κ.

For each parameter $t \in [0, 1]$, we consider the family of linear cocycles

$$L_{t;N} = (1-t)A + tB_N$$

These cocycles vary continuously with t and satisfy that $L_{0;N} = A$ and $L_{1;N} = B_N$. Moreover, each cocycle $L_{t;N}$ is locally constant on the same family of cylinders as B_N. Since each cocycle is a convex combination of two α-Hölder continuous cocycles, it is itself α-Hölder continuous. By choosing ε sufficiently small, we ensure that the entire path $\{L_{t;N}\}_{t\in[0,1]}$ is contained in the neighborhood $\mathcal{U}$ of A.

In particular, the map $t \to \lambda_+(L_{t;N}, \mu)$ is continuous on $[0,1]$, as ensured by the Theorem 2.8 of [4], since the family of cocycles $\{L_{t;N}\}_{t\in[0,1]}$ are locally constant on cylinders with the same size. Also, $\lambda_+(L_{1;N}, \mu) < \kappa$ and $\lambda_+(L_{0;N}, \mu) = \lambda$, then the intermediate value property implies the existence of a parameter $t_0 \in [0,1]$ such that

$$\lambda_+(L_{t_0;N}, \mu) = \kappa$$

This completes the proof of the theorem. □

We would like to remark that the condition of Theorem A, $2^{\left(2+\frac{\log \eta}{\log \sigma}\right)\alpha} \leq \eta^2$, and the condition in Theorem B, $2^{2\alpha} \leq (a_j^{-1})^2$, depend only on the contraction rate considered of the cocycle. In our setting, this rate corresponds to the smallest parameter in the definition of the cocycle $\mathrm{A}_{\sigma\eta}$ and to the smallest parameter in the definition of the cocycle A.

Note that if we choose $\eta > 1$ or $a_j^{-1} > 1$ sufficiently close to 1 so that the above inequalities are satisfied, then $\mathrm{A}_{\sigma\eta}$ and A are discontinuity points for Lyapunov exponents with respect to μ_p and μ, respectively, in some Hölder topology. However, if we consider $\eta = 1$ or $a_j = 1$ there is no $\alpha > 0$ such that

$$2^{\left(2+\frac{\log \eta}{\log \sigma}\right)\alpha} \leq 1 \quad \text{or} \quad 2^{2\alpha} \leq 1.$$

In these cases, we do not know under which conditions $\mathrm{A}_{\sigma 1}$ or the cocycle A, defined solely by parameters $a_i \geq 1$, are discontinuity points for Lyapunov exponents in $C^\alpha(M, \mathrm{SL}(2,\mathbb{R}))$.

We discuss the case of the cocycle $\mathrm{A}_{\sigma 1}$ in Chapter 5, since the analysis of the other case is analogous.

Chapter 5

Intermediate Topologies

In this chapter, we investigate new examples of cocycles that lie on the boundary of uniformly hyperbolic cocycles. These examples illustrate the subtle transition between uniform and non-uniform hyperbolicity, providing insight into how Lyapunov exponents behave near the boundary of hyperbolic settings.

To advance this question and motivated by the goal of understanding the behavior of cocycles in broader settings beyond the Hölder framework, we introduce intermediate topologies, finer than the C^0 topology, which provide a suitable setting for analyzing the continuity problem of Lyapunov exponents in this particular example.

5.1 Topologies

In this section, we consider again the product space (M, d) with the distance defined in (1.3) and a continuous function $A : M \to \mathrm{SL}(2, \mathbb{R})$. We will only use the fact that M is compact. Our goal is to introduce several classes of regularity for such functions, starting from the usual Hölder continuity and moving gradually toward weaker forms of continuity that appear naturally in the study of linear cocycles with limited smoothness, [9].

Let us begin by recalling the classical notion. A function A is *α-Hölder continuous*, for some $\alpha > 0$, if it belongs to the space $C^\alpha(M, \mathrm{SL}(2, \mathbb{R}))$, that is, if $\|A\|_\alpha < \infty$, where

$$\|A\|_\alpha = \sup_{x \in M} \|A(x)\| + \sup_{x \neq y \in M} \frac{\|A(x) - A(y)\|}{d(x, y)^\alpha}$$

This is the standard regularity condition; it can be interpreted in terms of the Hölder constant of A, which quantifies how much A can vary between two points relative to a power of their distance. A smaller constant corresponds to a more regular A.

A weaker version of this condition can be defined by replacing the power of the distance with a logarithmic term.

We say that A is a *weak Hölder continuous* function if $A \in C_{weak}(M, \mathrm{SL}(2, \mathbb{R}))$, that is, if

$$\|A\|_{weak} = \sup_{x \in M} \|A(x)\| + \sup_{x \neq y \in M} \left\{ \|A(x) - A(y)\| \exp\left(\alpha \left(\log \frac{1}{d(x, y)} \right)^\theta \right) \right\} < \infty$$

for some $\alpha, \theta \leq 1$.

We can note that, due to logarithmic term, this definition allows for slower rates of continuity decay as points approach each other. Also, when $\theta = 1$, the estimate above coincides with the classical Hölder continuity, so the weak Hölder condition truly generalizes the standard case.

An even more relaxed form of regularity is obtained by introducing iterated logarithmic terms.

The function A is said to be (γ, κ)*-log-Hölder continuous* if $A \in C_{(\gamma,\kappa)}(M, \mathrm{SL}(2, \mathbb{R}))$, that is, if

$$\|A\|_{(\gamma,\kappa)} = \sup_{x\in M} \|A(x)\| + \sup_{x\neq y\in M} \left\{ \|A(x) - A(y)\| \exp\left(\kappa \left(\log\log \frac{1}{d(x,y)}\right)^{\gamma}\right)\right\} < \infty$$

where $\gamma, \kappa \geq 1$.

This class consists of functions whose regularity decays at a double-logarithmic rate, allowing for even slower convergence than in the weak Hölder case. When $\gamma = 1 = \kappa$, the definition reduces to the usual log-Hölder continuity, which we introduce next. Throughout, we will use the notation 1-log-Hölder continuity to refer to the standard log-Hölder case, since we will work with broader regularity classes that generalize this type of continuity.

Then, we say that A is a 1*-log-Hölder continuous* function if $A \in C_{1-\log}(M, \mathrm{SL}(2, \mathbb{R}))$, i.e, if

$$\|A\|_{1-\log} = \sup_{x\in M} \|A(x)\| + \sup_{x\neq y\in M} \left\{ \|A(x) - A(y)\| \left(\log \frac{1}{d(x,y)}\right)\right\} < \infty$$

Finally, for $0 < \delta < 1$, we define the δ*-log-Hölder continuity* by requiring that $A \in C_{\delta-\log}(M, \mathrm{SL}(2, \mathbb{R}))$, where

$$\|A\|_{\delta-\log} = \sup_{x\in M} \|A(x)\| + \sup_{x\neq y\in M} \left\{ \|A(x) - A(y)\| \left(\log \frac{1}{d(x,y)}\right)^{\delta}\right\} < \infty$$

This class provides a continuous transition between log-Hölder and merely continuous functions, reflecting very mild degree of regularity.

These definitions form a natural relation between the spaces describing decreasing levels of regularity. Indeed, one can verify the following inclusions

$$C^{\alpha} \subset C_{weak} \subset C_{(\gamma,\kappa)} \subset C_{1-\log} \subset C_{\delta-\log} \subset C^0 \tag{5.1}$$

showing how each new space properly extends the previous one.

Example 5.1. The cocycle $\mathrm{A}_{\sigma\eta}$ studied in the Chapters 1 and 3 is α-Hölder continuous for every $\alpha > 0$, as we have already seen. Consequently, by the inclusion (5.1), this cocycle belongs to all the spaces defined above. Moreover, for every choice of σ and η and, for all weights $p > 0$ such that $p \neq \frac{\log \eta}{\log \sigma\eta}$, the cocycle $\mathrm{A}_{\sigma\eta}$ is a discontinuity point for the Lyapunov exponents with respect to μ_p in each of these intermediate topologies.

In contrast with those topologies, the situation in the α-Hölder topology is more subtle: discontinuity is not the only possible behavior. Indeed, depending on the parameters, there exist regions where the Lyapunov exponents vary continuously, as well as regions of discontinuity, as discussed earlier.

The proof of this claim follows from the fact that the norms defining these topologies involve logarithmic terms. Considering the perturbation defined in (3.8) and using Lemma 3.2, which shows that $\|(B_k - \mathrm{A}_{\sigma\eta})(x) - (B_k - \mathrm{A}_{\sigma\eta})(y)\|$ decays exponentially for $x \neq y \in M_1$, we conclude that B_k is arbitrarily close to $\mathrm{A}_{\sigma\eta}$ in each of those norms, without imposing any restriction on σ and η.

This framework provides a flexible way to describe cocycles with different degrees of smoothness, which is often crucial in the analysis of dynamical systems, as in the case for the cocycle with the identity, introduced in the next section.

5.2 Cocycle with the Identity

We return, in this section, to the fixed notation in the previous chapter. Let M_1 denote the space of bi-infinite sequences of two symbols, 0 and 1, $M_1 = \{0, 1\}^{\mathbb{Z}}$ and $f : M_1 \to M_1$ be the shift map. We consider the Bernoulli measure μ_p on M_1 with weight $p \in (0, 1)$.

Here, we introduce the cocycle associated with a function taking values in diagonal matrices, one of which is the identity matrix. We recall some known results concerning this cocycle and discuss the structural restrictions that such a class of cocycle imposes.

Fix $\sigma > 1$. We consider the cocycle associated with the function $\mathrm{A}_{\sigma 1} : M_1 \to \mathrm{SL}(2, \mathbb{R})$ defined by

$$A_{\sigma 1}(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x_0 = 0 \\ \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1 \end{cases}$$

Observe that the cocycle $\mathrm{A}_{\sigma 1}$ is α-Hölder continuous for every $\alpha > 0$, since it is locally constant. Consequently, by the inclusions in (5.1), $\mathrm{A}_{\sigma 1}$ belongs to each of the regularity spaces defined in the previous section.

Moreover, the Lyapunov exponents associated with this cocycle are given by

$$\lambda_{\pm}(\mathrm{A}_{\sigma 1}, \mu_p) = \pm p \log \sigma$$

which are nonzero since p is positive.

Note that, according to the criterion (1.4), this cocycle is not uniformly hyperbolic. Indeed, if we consider the fixed point $\tilde{x} \in M_1$ consisting entirely of the symbol 0, then for every $n \geq 1$,

$$\|A^n_{\sigma 1}(\tilde{x})\| = 1$$

Hence, it is impossible to find constants satisfying inequality (1.4) for this point $\tilde{x}$ and for all $n \geq 1$.

Nevertheless, for every $\sigma > 1$, the cocycle $\mathrm{A}_{\sigma 1}$ lies on the boundary of the set of uniformly hyperbolic cocycles. Indeed, to see this, consider the family of cocycles $\mathrm{A}_{\sigma\eta}$. When $\eta > 1$ is sufficiently close to one, the cocycle $\mathrm{A}_{\sigma\eta}$ is not uniformly hyperbolic, as shown in Example 1.6, while remaining close to $\mathrm{A}_{\sigma 1}$ in the appropriate topology. On the other hand, when $\eta < 1$ is sufficiently close to 1, the same cocycle $\mathrm{A}_{\sigma\eta}$ is still close to $\mathrm{A}_{\sigma 1}$, but in this case it is uniformly hyperbolic, with hyperbolic constant equal to $\eta^{-1}\sigma$. Hence, the cocycle $\mathrm{A}_{\sigma 1}$ sits precisely at the interface between continuity and discontinuity of the Lyapunov exponents, depending on the level of regularity.

Therefore, by Theorem 2.1, $\mathrm{A}_{\sigma 1}$ is a discontinuity point for the Lyapunov exponents with respect to μ_p in the C^0- topology.

In constrast, when we consider the C^α-topology, Theorem 2.3 provides a sufficiently condition for continuity: $\mathrm{A}_{\sigma 1}$ is a continuity point

$$\|A_{\sigma 1}(x)\| \left\|A_{\sigma 1}(x)^{-1}\right\| < 2^\alpha \quad \forall x \in M_1$$

And this inequality holds precisely if and only if $\sigma^2 < 2^\alpha$. Hence, $\mathrm{A}_{\sigma 1}$ is a C^α-continuity point for the Lyapunov exponents with respect to μ_p whenever $\sigma^2 < 2^\alpha$.

Conversely, when $\sigma^2 \geq 2^\alpha$, it is unknown whether the corresponding cocycle is a continuity or discontinuity point in $C^\alpha(M_1, \mathrm{SL}(2, \mathbb{R}))$. Then, the cocycle presents the following configuration:

The techniques employed in the proof of Theorem A cannot be directly applied in this setting, since they rely crucially on the existence of some iterate of the cocycle that contracts the first coordinate in $\mathbb{R}^2$, a property that $\mathrm{A}_{\sigma 1}$ does not possess. The idea would be to construct a perturbation that produces the desired effect, that is, a contraction along the first coordinate on a certain subset. However, we cannot ensure that this new cocycle is Hölder continuous. Moreover, it is far from the original cocycle in the α-norm.

Rather than restrict ourselves to the Hölder framework, we investigate intermediate topologies between C^α and C^0, which provide a more flexible framework for analyzing the behavior of the Lyapunov exponents for cocycles near $\mathrm{A}_{\sigma 1}$. By enlarging the class of admissible cocycles, we are able to analyze how the discontinuity of the Lyapunov exponents behaves when weaker regularity assumptions are considered. This broader perspective allows us to understand whether the phenomenon persists or changes when additional elements are included in the topology.

5.3 Discontinuity Example in $C_{\delta-\log}$-topology

In this section, we state and prove the Theorem C, which establishes that the cocycle $\mathrm{A}_{\sigma 1}$ is a discontinuity point for the Lyapunov exponents in the $C_{\delta-\log}$ topology, for all parameters $\sigma > 1$ and $\delta < 1$. As previously discussed, this topology is finer than the C^0 topology.

The proof follows the same general strategy as that of Theorem A. We construct a perturbation of the original cocycle that interchanges the Oseledets subspaces. This property forces the Lyapunov exponents of the perturbed cocycle to vanish, yielding the desired discontinuity. In the present setting, however, the weaker regularity of the $C_{\delta-\log}$ topology allows for greater flexibility, and we are in fact able to construct two distinct perturbations with this property.

Theorem C. $\mathrm{A}_{\sigma 1}$ *is a discontinuity point for the Lyapunov exponents in the* $C_{\delta-\log}$*-topology. In fact, in this topology,* $A_{\sigma 1}$ *can be approximated by cocycles with vanishing Lyapunov exponents.*

Proof. First, recall that $V_x = \mathbb{R}(1,0)$ and $H_x = \mathbb{R}(0,1)$ denote, respectively, the vertical and horizontal line bundles. Since the cocycle $\mathrm{A}_{\sigma 1}$ is defined by diagonal matrices, both bundles are invariant under its action. Moreover, these subspaces coincide almost everywhere with the Oseledets subspaces associated with the cocycle.

Our goal is to construct two perturbations of $\mathrm{A}_{\sigma 1}$, arbitrarily close to it in the $C^{\delta-\log}$, that interchange the Oseledets subspaces. The existence of such perturbations implies, by the technique developed in the previous chapter, that the Lyapunov exponents of the perturbed cocycles vanish.

The general idea is as follows. In the first perturbation, we modify the original cocycle by introducing a sequence of small rotations along the orbit of points contained in a suitably chosen cylinder. These rotations will progressively alter the directions of the invariant subspaces until the vertical and horizontal ones are exchanged.

The second perturbation is based on a different geometric mechanism. We begin by applying a small shear that slightly tilts the vector e_1 from the horizontal direction. Then, along the orbit, we compose the dynamics with hyperbolic matrices that contract the horizontal direction while expanding the vertical one. After a sufficient number of iterations, we apply a small rotation to the image of e_1 aligning it with the vertical axis. Using the same rotation, we then move the vector e_2 from the vertical direction to one forming a small angle with the vertical axis. The action of the hyperbolic matrix of the cocycle then amplifies this change, bringing the vector closer to the horizontal direction. Finally, a second shear completes the interchange between the two invariant directions.

- *First Perturbation*:

Let $k \in \mathbb{N}$ and consider the cylinder $Z_k = [0; 0 \cdots 01]$ where the symbol 0 appears k times in Z_k. By construction, the colection of sets $\{f^i(Z_k)\}_{i=0}^{k-1}$ are pairwise disjoint, that is, $f^i(Z_k) \cap f^j(Z_k) = \emptyset$ for $0 \leq i < j \leq k-1$.

We now define a perturbation $B_k : M_1 \to \mathrm{SL}(2,\mathbb{R})$ by

$$B_k(x) = \begin{cases} \mathrm{A}_{\sigma 1}(x) R_{\theta_k} & \text{if } x \in \bigcup_{i=1}^{k-1} f^i(Z_k) \\ \\ \mathrm{A}_{\sigma 1}(x) & \text{otherwise} \end{cases} \tag{5.2}$$

where $\theta_k = \frac{\pi}{2k}$ and R_{θ_k} is the rotation matrix, $R_{\theta_k} = \begin{pmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos(\theta_k) \end{pmatrix}$.

Note that $B_k \in C_{\delta-\log}$, $\forall k \geq 1$, since B_k is constant on cylinders of diameter at least 2^{-k}. Furthermore, observe that if $x \in Z_k$, then

$$B_k^k(x) = R_{k\theta_k} = R_{\frac{\pi}{2}}$$

In other words, for every $x \in Z_k$, the k-th iterate of the cocycle B_k act as a rotation by an angle of $\frac{\pi}{2}$.

Therefore,

$$B_k^k(x) H_x = V_{f^k(x)} \text{ and } B_k^k(x) V_x = H_{f^k(x)} \quad \forall x \in Z_k$$

Using the definition of the perturbation (5.2), we now calculate some iterates of the induced cocycle $B_k^{\tau_{Z_k}}$:

$$\begin{aligned} B_k^{\tau_{Z_k}}(x) &= B_k(f^{\tau_{Z_k}-1}(x)) \cdots B_k(f^k(x)) B_k^k(x) &(5.3) \\ &= \begin{pmatrix} 0 & -\sigma^{S_{\tau_{Z_k}}(x)+1} \\ \sigma^{-S_{\tau_{Z_k}}(x)-1} & 0 \end{pmatrix} &(5.4) \end{aligned}$$

where the function $S_m : M_1 \to \mathbb{R}$, for $m \geq 1$, is defined in (3.17).

Repeating the same procedure for the second return time, we obtain:

$$B_k^{\tau_{Z_k}^{(2)}}(x) = \begin{pmatrix} -\sigma^{S_{\tau_{Z_k}}(f^{\tau_{Z_k}}(x)) - S_{\tau_{Z_k}}(x)} & 0 \\ 0 & -\sigma^{-S_{\tau_{Z_k}}(f^{\tau_{Z_k}}(x)) + S_{\tau_{Z_k}}(x)} \end{pmatrix} \tag{5.5}$$

Moreover, for iterates corresponding to even return times, an induction argument on $j \geq 1$ yields

$$B_k^{\tau_{Z_k}^{(2j)}}(x) = \begin{pmatrix} (-1)^j \sigma^{c_j(x)} & 0 \\ 0 & (-1)^j \sigma^{-c_j(x)} \end{pmatrix} \tag{5.6}$$

where the function c_j, defined in (3.23), is given by

$$c_j(x) = \sum_{i=1}^{j} \left(S_{\tau_{Z_k}}(f^{\tau_{Z_k}^{(2i-1)}}(x)) - S_{\tau_{Z_k}}(f^{\tau_{Z_k}^{(2i-2)}}(x)) \right) \tag{5.7}$$

Then, denoting again $\tau_{Z_k}^{(2j)} =: m_j$, we obtain the following estimate:

$$\lim_{j\to\infty} \frac{1}{m_j} \log \|B_k^{m_j}(x)\| \leq \left(\lim_{j\to\infty} \frac{|c_j(x)|}{m_j} \right) \log \sigma$$

Using (3.28), we see that this limit is bounded above by zero. Moreover, by Furstenberg-Kesten Theorem, it is also bounded below by zero. Hence, we may conclude that

$$\lambda_+(B_k, \mu_p) = 0 \quad \forall k \geq 1.$$

Finally, we show that B_k converges to $\mathrm{A}_{\sigma 1}$ in this topology. More precisely, we prove that for every $\varepsilon > 0$ there is $k_0 > 0$ such that for all $k \geq k_0$,

$$\|B_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} < \varepsilon.$$

To this end, fix $\varepsilon > 0$ and recall that

$$\|B_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} = \|B_k - \mathrm{A}_{\sigma 1}\|_0 + \sup_{x \neq y \in M} \left\{ \|B_k(x) - \mathrm{A}_{\sigma 1}(x) - B_k(y) + \mathrm{A}_{\sigma 1}(y)\| \left(\log \frac{1}{d(x,y)}\right)^{\delta} \right\}$$

We first observe that the first term of the above sum is bounded by $\sigma \frac{\pi}{2k}$, which clearly decays to zero as $k \to \infty$.

To handle the second term, we define, for all $x, y \in M_1$,

$$T_k(x,y) = \|B_k(x) - \mathrm{A}_{\sigma 1}(x) - B_k(y) + \mathrm{A}_{\sigma 1}(y)\| \left(\log \frac{1}{d(x,y)}\right)^{\delta}$$

and we analyze $T_k(x,y)$ by considering the possible relative positions of x and y.

If x and y belong to different cylinders $[0;a]$ with $a \in \{0,1\}$, then their distance satisfies $d(x,y) = 1$. Consequently, the logarithmic tem vanishes and we obtain $T_k(x,y) = 0$.

On the other hand, if x and y are in the same cylinder, $[0;0]$ or $[0;1]$, we have $\mathrm{A}_{\sigma 1}(x) = \mathrm{A}_{\sigma 1}(y)$. In this situation, two subcases may occur.

If one of the points belongs to $\bigcup_{i=1}^{k-1} Z_k$ and the other does not, we have the estimates $d(x,y)^{-1} \leq 2^k$ and $\|B_k(x) - B_k(y)\| = \frac{\pi}{2k}$. Hence,

$$T_k(x,y) \leq \frac{\pi}{2k} k^{\delta} (\log 2)^{\delta} \xrightarrow[k \to \infty]{} 0 \tag{5.8}$$

since $\delta < 1$.

In all remaining cases, the matrices B_k and $\mathrm{A}_{\sigma 1}$ coincide on both points, and thus $T_k(x,y) = 0$.

Therefore, for sufficiently large k, we conclude that $\|B_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} < \varepsilon$.

It is worth emphasizing that the assumption $\delta < 1$ plays a crucial role here: it ensures that term $\frac{\pi}{2k} k^{\delta} (\log 2)^{\delta}$ indeed tends to zero, which in turn guarantees convergence in the δ-logarithmic norm.

- *Second Perturbation*:

Now, let $k \in \mathbb{N}$ and consider a different cylinder, $W_k = [0; 0 \cdots 01 \cdots 1]$ where the number 0 appears $k+1$ times and the number 1 appears k times. In this construction, we intentionally choose the symbol 1 to appear more frequently than in the previous cylinder Z_k, since the hyperbolic matrix will be used to produce the interchange of subspaces.

As before, the first $2k$ iterates of the cylinder are pairwise disjoint, that is, $f^i(W_k) \cap f^j(W_k) = \emptyset$ for $0 \leq i < j \leq 2k$.

Fix $\beta > 0$ such that $1 > \beta > \delta$. And, finally, we define the cocycle associated to the function $L_k : M_1 \to \mathrm{SL}(2, \mathbb{R})$ by

$$L_k(x) = \begin{cases} \mathrm{A}_{\sigma 1}(x) \begin{pmatrix} 1 & 0 \\ k^{-\beta} & 1 \end{pmatrix} & \text{if } x \in W_k \\[2ex] \mathrm{A}_{\sigma 1}(x) \begin{pmatrix} (1 + k^{-\beta})^{-1} & 0 \\ 0 & (1 + k^{-\beta}) \end{pmatrix} & \text{if } x \in \bigcup_{i=1}^{k} f^i(W_k) \\[2ex] \mathrm{A}_{\sigma 1}(x) R_{\theta_k} & \text{if } x \in f^{k+1}(W_k) \\[2ex] \mathrm{A}_{\sigma 1}(x) \begin{pmatrix} 1 & 0 \\ \tilde{\gamma}(k) & 1 \end{pmatrix} & \text{if } x \in f^{2k}(W_k) \\[2ex] \mathrm{A}_{\sigma 1}(x) & \text{otherwise} \end{cases} \tag{5.9}$$

where θ_k is such that

$$\tan(\theta_k) = \frac{1}{k^{-\beta}(1+k^{-\beta})^{2k}}$$

and

$$\tilde{\gamma}(k) = \frac{k^{-\beta}(1+k^{-\beta})^{2k}}{\sigma^{2k}} = \frac{1}{\tan(\theta_k)\sigma^{2k}}. \tag{5.10}$$

Observe that, due to the disjointness of the iterates of W_k, the function L_k is well defined for all $k \geq 1$. In addition, L_k is $\delta - \log$ Hölder continuous for every $k \geq 1$, since it remains constant on cylinders of diameter at least 2^{-2k+1}.

For any $x \in W_k$, by the definition of L_k, we have

$$\begin{aligned} L_k^{2k+1}(x) &= L_k(f^{2k}(x)) \cdots L_k(x) \\ &= \begin{pmatrix} 0 & -\sin(\theta_k)\sigma^{k+1}(1+k^{-\beta})^k \\ \sigma^{-k-1}\frac{(1+k^{-\beta})^{-k}}{\sin(\theta_k)} & 0 \end{pmatrix} \end{aligned}$$

Hence, for every $x \in W_k$

$$L_k^{2k+1}(x)H_x = V_{f^{2k+1}(x)} \text{ and } L_k^{2k+1}(x)V_x = H_{f^{2k+1}(x)}$$

Once again, we can compute the first and second return times as we did previously, in order to determine the Lyapunov exponets of the perturbation using their definition. Thus, for μ_p-almost every point $x \in M_1$, we have

$$L_k^{\tau_{W_k}}(x) = \begin{pmatrix} 0 & \sin(\theta_k)\sigma^{S_{\tau_{W_k}}(x)}(1+k^{-\beta})^k \\ \sigma^{-S_{\tau_{W_k}}(x)}\frac{(1+k^{-\beta})^{-k}}{\sin(\theta_k)} & 0 \end{pmatrix} \tag{5.11}$$

Similarly, the second return time is given by

$$L_k^{\tau_{W_k}^{(2)}}(x) = \begin{pmatrix} \sigma^{c_1(x)} & 0 \\ 0 & \sigma^{-c_1(x)} \end{pmatrix}$$

More generally, by recursion, for every $j \geq 1$ we obtain

$$L_k^{\tau_{W_k}^{(2j)}}(x) = \begin{pmatrix} \sigma^{c_j(x)} & 0 \\ 0 & \sigma^{-c_j(x)} \end{pmatrix} \tag{5.12}$$

where the function c_j is the same as that defined in (5.7), but now the return times are taken with respect to the cylinder W_k.

Following the same reasoning as in the previous perturbation, and by the very definition of the perturbation, we conclude that the Lyapunov exponents of L_k vanish. In other words, for all $k \geq 1$,

$$\lambda_+(L_k, \mu_p) = 0$$

It remains to show that, for k sufficiently large, L_k is close to $\mathrm{A}_{\sigma 1}$ in the $\delta - \log$ norm. As in the case of the first perturbation, our goal is to prove that for each $\varepsilon > 0$ there is $k_0 > 0$ such that $\|L_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} < \varepsilon$ for all $k \geq k_0$.

For this purpose, we aim to estimate

$$\begin{aligned} \|L_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} &= \|L_k - \mathrm{A}_{\sigma 1}\|_0 \\ &+ \sup_{x\neq y\in M_1}\left\{\|(L_k - \mathrm{A}_{\sigma 1})(x) - (L_k + \mathrm{A}_{\sigma 1})(y)\| \left(\log\frac{1}{d(x,y)}\right)^{\delta}\right\} \end{aligned} \tag{5.13}$$

We begin by analyzing the first term of the sum, $\|L_k - \mathrm{A}_{\sigma 1}\|_0$. Recall that $\sigma > 1$, $\tan(\theta_k) = \frac{k^\beta}{(1+k^{-\beta})^{2k}}$ and $\tilde{\gamma}(k) = \frac{(1+k^{-\beta})^{2k}}{k^\beta \sigma^{2k}}$. From these definitions, we have

$$\lim_{k\to\infty} \tilde{\gamma}(k) = \lim_{k\to\infty} \frac{1}{k^\beta} \left(\frac{1+k^{-\beta}}{\sigma} \right)^{2k} = 0 \tag{5.14}$$

$$\lim_{k\to\infty} \tan(\theta_k) = \lim_{k\to\infty} \frac{k^\beta}{(1+k^{-\beta})^{2k}} = 0 \tag{5.15}$$

Since $\|L_k - \mathrm{A}_{\sigma 1}\|_0 \leq \sigma \max\{k^{-\beta},\ \tilde{\gamma}(k),\ \theta_k\}$, we conclude from (5.14) and (5.15) that the uniform distance between L_k and $\mathrm{A}_{\sigma 1}$ tends to zero. In particular, for every $\varepsilon > 0$ there exists $\tilde{k} > 0$ such that

$$\|L_k - \mathrm{A}_{\sigma 1}\|_0 < \varepsilon \quad \text{for all } k \geq \tilde{k}.$$

In order to conclude the proof, we now estimate the second term in the $\delta-\log$ norm of (5.13). Once again, we analyze all possible configurations of the points x and y in M_1.

Let us denote

$$\tilde{T_k}(x,y) = \|L_k(x) - \mathrm{A}_{\sigma 1}(x) - L_k(y) + \mathrm{A}_{\sigma 1}(y)\| \left(\log \frac{1}{d(x,y)} \right)^\delta$$

If x and y belong to different cylinders, then $d(x,y) = 1$ and, consequently, the logarithmic factor vanishes. In this case, we immediately have $\tilde{T_k}(x,y) = 0$.

If, on the other hand, x and y belong to the same cylinder, then $\mathrm{A}_{\sigma 1}(x) = \mathrm{A}_{\sigma 1}(y)$, so that only the variation of L_k contributes to the estimate of $\tilde{T_k}(x,y)$. We now analyze this variation by considering, separately, both cases about x and y:

- Case 1: $x, y \in [0; 0]$

  In this case, since both points share the same initial symbol 0, we will examine the possible positions of x and y relative to the sets W_k and its iterates $f^i(W_k)$ with $0 < i \leq k$.

  1. $x \in W_k$ and $y \notin \bigcup_{i=0}^k f^i(W_k)$.
     Notice that their distance satisfies $d(x,y)^{-1} \leq 2^{2k+1}$ and by the construction of L_k, the difference between its values at these points is small, namely $\|L_k(x) - L_k(y)\| \leq k^{-\beta}$. Combining these two estimates yields

     $$\tilde{T_k}(x,y) \leq \frac{(2k+1)^\delta (\log 2)^\delta}{k^\beta} = (\log 2)^\delta \left(\frac{2k+1}{k^{\frac{\beta}{\delta}}} \right)^\delta$$

     Since $\frac{\beta}{\delta} > 1$, the exponent of k in the denominator dominates, and it follows that

     $$\lim_{k\to\infty} \tilde{T_k}(x,y) = 0.$$

  2. $x \in \bigcup_{i=1}^k f^i(W_k)$ and $y \notin \bigcup_{i=0}^k f^i(W_k)$.
     In this configuration, x belong to one of the first k forward images of W_k, while y remains outside this union. The same reasoning applies, we have $d(x,y)^{-1} \leq 2^{2k}$ and $\|L_k(x) - L_k(y)\| \leq k^{-\beta}$. Hence,

     $$\tilde{T_k}(x,y) \leq \frac{k^\delta (2\log 2)^\delta}{k^\beta}$$

     Since $\beta > \delta$, the right-hand side tends to zero, so

     $$\lim_{k\to\infty} \tilde{T_k}(x,y) = 0.$$

3. $x \in W_k$ and $y \in \bigcup_{i=1}^{k} f^i(W_k)$.
Here, both points belong to the orbit of W_k, but under different iterates. From the size of the cylinder W_k, we have $d(x,y)^{-1} \leq 2^k$ and by the definition of the perturbation, it follows that $\|L_k(x) - L_k(y)\| \leq \frac{2}{k^\beta}$. Therefore,

$$\tilde{T}_k(x,y) \leq 2(\log 2)^\delta \frac{k^\delta}{k^\beta}$$

Using again $\beta > \delta$, the limit of this expression as $k \to \infty$ is zero:

$$\lim_{k\to\infty} \tilde{T}_k(x,y) \leq \lim_{k\to\infty} 2(\log 2)^\delta \frac{k^\delta}{k^\beta} = 0$$

4. All other configurations of $x, y \in [0;0]$
For any other configuration within this cylinder, the function L_k takes the same value at x and y, hence $\tilde{T}_k(x,y) = 0$.

- Case 2: $x, y \in [0;1]$.

We now consider the situation in which both points x and y begin with the symbol 1. As in the previous case, we examine how the function L_k varies on different subsets of the space and estimate $\tilde{T}_k(x,y)$ accordingly.

If x and y belong to the same image $f^i(W_k)$ for some $i \in \{k+1, \cdots, 2k\}$, then $L_k(x) = L_k(y)$ and consequently $\tilde{T}_k(x,y) = 0$. The same conclusion holds if neither x nor y belong to $f^{k+1}(W_k) \cup f^{2k}(W_k)$, since L_k takes constant values on the complement of these sets.

Hence, we only need to analyze the cases in which one (or both) of the points lies in $f^{k+1}(W_k)$ or $f^{2k}(W_k)$. So, the remaining possibilities are the following:

1. $x \in f^{k+1}(W_k)$ and $y \notin f^{k+1}(W_k) \cup f^{2k}(W_k)$.
Here, we have $d(x,y)^{-1} \leq 2^{k+1}$ and the variation of L_k in this region satisfies $\|L_k(x) - L_k(y)\| \leq \sigma \frac{k^\beta}{(1+k^{-\beta})^{2k}}$. Combining these two estimates, we obtain

$$\tilde{T}_k(x,y) \leq \frac{\sigma k^\beta (k+1)^\delta (\log 2)^\delta}{(1+k^{-\beta})^{2k}}$$

Since the denominator grows exponentially while the numerator grows only polynomially, this expression tends to zero as $k \to \infty$. Thus

$$\lim_{k\to\infty} \tilde{T}_k(x,y) = 0.$$

2. $x \in f^{2k}(W_k)$ and $y \notin f^{k+1}(W_k) \cup f^{2k}(W_k)$.
By construction, the symbolic distance between the points satisfies $d(x,y)^{-1} \leq 2^{2k+1}$ and the difference of L_k along these two points is controlled by $\|L_k(x) - L_k(y)\| \leq \sigma\tilde{\gamma}(k)$. Using this inequality and the definition of $\tilde{\gamma}(k)$ in (5.10), we obtain

$$\tilde{T}_k(x,y) \leq \sigma(\log 2)^\delta \left(\frac{2k+1}{k^{\frac{\beta}{\delta}}}\right)^\delta \left(\frac{1+k^{-\beta}}{\sigma}\right)^{2k}.$$

The first factor decays polynomially because $\beta > \delta$, and the second decays exponentially since $\sigma > 1$. Hence, both terms ensure that

$$\lim_{k\to\infty} \tilde{T}_k(x,y) = 0.$$

3. $x \in f^{k+1}(W_k)$ and $y \in f^{2k}(W_k)$.
   In this configuration, both points belong to images of W_k where the perturbation acts, but at different times.
   Since their symbolic sequences differs in the first coordinate, we have $N(x, y) = 1$ and thus, $d(x, y)^{-1} = 2$. Moreover, the difference of L_k at these points satisfies $\|L_k(x) - L_k(y)\| \leq \sigma \left(\tilde{\gamma} + \frac{k^\beta}{(1+k^{-\beta})^{2k}}\right)$. Therefore,

$$\tilde{T}_k(x, y) \leq \sigma(\log 2)^\delta \left(\tilde{\gamma} + \frac{k^\beta}{(1 + k^{-\beta})^{2k}}\right)$$

   and since both terms inside the parentheses vanish as $k \to \infty$, we conclude that

$$\lim_{k\to\infty} \tilde{T}_k(x, y) = 0$$

In summary, we have established that

$$\lim_{k\to\infty} \tilde{T}_k(x, y) = 0 \text{ for every } x, y \in M_1.$$

Combining this with the previous estimate on the uniform norm, we may choose $k_0 > \tilde{k}$ such that

$$\|L_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} < \varepsilon \quad \forall k \geq k_0.$$

Therefore, we have proved that for every $\varepsilon > 0$, one can construct a $\delta -\log$ Hölder continuous cocycle L_k such that $\lambda_+(L_k, \mu_p) = 0$ and is arbitrarily close to the locally constant cocycle $\mathrm{A}_{\sigma 1}$

$$\|L_k - \mathrm{A}_{\sigma 1}\|_{\delta-\log} < \varepsilon.$$

This completes the construction and the proof. □

Chapter 6

Open Problems

In this final chapter, we present and discuss some open problems concerning the continuity and discontinuity of Lyapunov exponents in the setting explored in this thesis. These problems indicate possible directions for further research and suggest how the techniques and phenomena discussed here may extend to more general contexts, such as the Anosov setting.

In the first section, we revisit the example presented in Chapter 3 and discuss some of its key features. We also identify the main question that remains open in that specific context.

In the following section, we analyze the cocycle constructed in Chapter 5 and highlight a particular behavior exhibited by this example. This leads us to consider the problem of continuity beyond the fiber-bunching condition within this framework.

In Section 6.3, we introduce another family of $\mathrm{SL}(2,\mathbb{R})$-valued cocycles that retains the essential dynamical features of the example presented in Chapter 3, namely, the presence of expansion and contraction in both horizontal and vertical directions. However, in this new construction, we relax one of the main assumptions of the previous model by allowing the defining matrices of the cocycle to be non-diagonal. We discuss some difficulties that prevented us from establishing a discontinuity result.

In the final two sections, we slightly change the setting in which we have been working. Our goal is to search for discontinuity phenomena in much more general systems, such as Anosov diffeomorphisms on compact manifolds. In each case, we remark open problems.

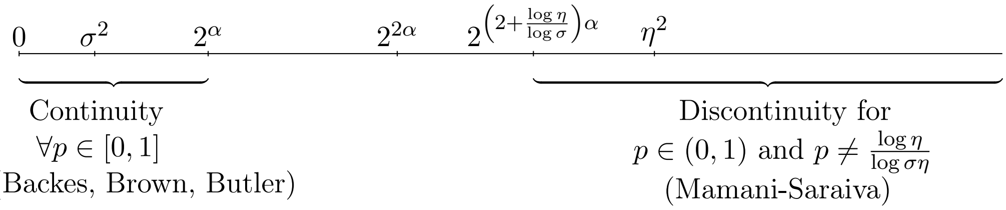
6.1 Characterizing Continuity for a Cocycle Generated by Two Hyperbolic Matrices

In this section, we revisit the Example 1.2, which was also discussed in Chapter 3. Our goal is to review what is currently understood about continuity and discontinuity of Lyapunov exponents in this family of cocycles, and to highlight what remains unknown.

Recall that the cocycle under consideration is defined by:

$$\begin{aligned} \mathrm{A}_{\sigma\eta} : M_1 &\to \mathrm{SL}(2,\mathbb{R}) \\ x &\mapsto \mathrm{A}_{\sigma\eta}(x) = \begin{cases} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} & \text{if } x_0 = 0 \\ \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1 \end{cases} \end{aligned}$$

For this family, the following configuration summarizes the known parameter ranges for which continuity or discontinuity of Lyapunov exponents occurs:



Observe that this cocycle is non-uniformly fiber-bunched whenever inequality (4.6) holds. Explicity, for $p \in (0,1)$,

$$\sigma^{2p}\eta^{2p-2} \leq 2^{\alpha} \tag{6.1}$$

However, the hypothesis of Corollary 4.1 do not contradict the non-uniformly fiber-bunched condition. Indeed, the assumptions

$$\eta^2 > 2^{2\alpha}, \quad \sigma^{4p-2} \geq 2^{\alpha} \text{ and } p \in (3/4, 1) \tag{6.2}$$

ensure that the cocycle $\mathrm{A}_{\sigma\eta}$ is a discontinuity point for the Lyapunov exponents with respect to the measure μ_p. Moreover, we may choose a constant η_0 that is still smaller than σ and p such that,

$$\sigma^{2p}\eta_0^{2p-2} \leq 2^{\alpha} \leq \sigma^{4p-2} \quad \text{and} \quad \eta_0^2 > 2^{2\alpha}$$

With this choice, the cocycle remains a discontinuity point for the Lyapunov exponents, while at the same time being non-uniformly fiber-bunched.

At the moment, the information above essentially constitutes all that is known about the continuity problem for this specific diagonal cocycle. In view of this, one is naturally led to the following question: *Does fiber-bunching characterize continuity for* $\mathrm{A}_{\sigma\eta}$*?*

More precisely: is $\mathrm{A}_{\sigma\eta}$ a continuity point for the Lyapunov exponents if and only if it is fiber-bunched? The available evidence suggests that the answer may well be affirmative, although a complete proof remains open.

6.2 Case of Hyperbolic and Identity Matrices

In this section, we return to the continuity problem for the cocycle involving the identity matrix, previously studied in Chapter 5. In that chapter, we proved a discontinuity result, though only with respect to a weaker topology.

Let us recall the cocyle under consideration. Define $\mathrm{A}_{\sigma 1} : M_1 \to \mathrm{SL}(2,\mathbb{R})$ by

$$A_{\sigma 1}(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x_0 = 0 \\ \\ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1 \end{cases}$$

where $\sigma > 1$. We consider the Bernoulli product measure $\mu_p = (p\delta_1 + (1-p)\delta_0)^{\mathbb{Z}}$ with $p > 0$.

The Lyapunov exponents of this cocycle are $\lambda_{\pm}(\mathrm{A}_{\sigma 1}, \mu_p) = \pm p \log \sigma$ which is non-zero for our choice of parameters.

The cocycle presents a structure similar of uniform hyperbolicity once we restrict it to an induced system. More precisely, consider the cylinder $Z_1 = [0; 1]$ and denote by τ_{Z_1} the first return time to Z_1. Then, for μ_p-almost every $x \in Z_1$, $f^{\tau_{Z_1}}(x)$ also belongs to Z_1. Consequently,

$$\begin{aligned} A_{\sigma 1}^{\tau_{Z_1}}(x) &= A_{\sigma 1}(f^{\tau_{Z_1}(x)-1}(x)) \cdots A_{\sigma 1}(x) \\ \\ &= \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \end{aligned}$$

Recursively, for every $m \geq 1$ and for almost every point in Z_1, the m-th iterate of the induced cocycle $A_{\sigma 1}^{\tau_{Z_1}}$ is

$$A_{\sigma 1}^{\tau_{Z_1}^{(m)}}(x) = \begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^{-m} \end{pmatrix}$$

This shows that the induced cocycle $A_{\sigma 1}^{\tau_{Z_1}}$ is uniformly hyperbolic, with expansion rate σ. Although the original cocycle is not hyperbolic and admits a trivial invariant splitting, this induced uniform hyperbolicity suggests that the system may exhibit regularity properties incompatible with discontinuity in the α-Hölder topology.

Motivated by this structure, we may ask: *Is there continuity beyond the fiber-bunching condition for* $\mathrm{A}_{\sigma 1}$*?*

Explicity, for any $\alpha > 0$, does there exist a constant $\sigma > 1$ satisfying $2^\alpha \leq \sigma^2$ such that $\mathrm{A}_{\sigma 1}$ is a continuity point for the Lyapunov exponents? We believe that the answer for this question is affirmative, and a natural direction to pursue is to investigate cocycles satisfying the non-uniformly fiber-bunched condition.

Although uniform domination between the fibers and the base dynamics fails in this setting, non-uniform fiber-bunching still guarantees domination along a full measure set of typical orbits. As a consequence, the cocycle admits non-uniform invariant holonomies, which are defined almost everywhere and vary continuously along Pesin stable and unstable manifolds for typical points. These holonomies retain many of the key structural features present in the uniform case, such as invariance under the cocycle, while relaxing the requirement of uniform control. Combined with the fact that the induced cocycle can be uniformly hyperbolic, this suggests that continuity results might extend beyond the uniformly fiber-bunched condition.

6.3 Case of Hyperbolic and Hyperbolic non-diagonal Matrices

In this section, we treat the case when one matrix of the cocycle is non-diagonal. Then, we introduce the cocycle under study and discuss its main properties, such as its Lyapunov exponents and Oseledets subspaces. We also discuss some difficulties that arise in the non-diagonal setting, which make the analysis more delicate than in the diagonal case.

As in the previous chapters, we denote $M_1 = \{0,1\}^{\mathbb{Z}}$, $f : M_1 \to M_1$ the shift map, and μ_p the Bernoulli measure associated with the parameters $p \in (0,1)$. And we fix two real constants satisfying $1 < \eta \leq \sigma$.

For each $b \in \mathbb{R}$, $b \neq 0$, we define the cocycle associated to the function $A_b : M_1 \to \mathrm{SL}(2,\mathbb{R})$, given by

$$\begin{aligned} A_b : M_1 &\to \mathrm{SL}(2,\mathbb{R}) \\ (x_i)_{i\in\mathbb{Z}} &\mapsto \begin{cases} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} & \text{if } x_0 = 0 \\[2ex] \begin{pmatrix} \sigma & 0 \\ b & \sigma^{-1} \end{pmatrix} & \text{if } x_0 = 1 \end{cases} \end{aligned}$$

Note that the parameter b does not affect the regularity of the cocycle. In particular, A_b remains locally constant, and hence it is α-Hölder continuous for every positive α.

When $b = 0$, the cocycle reduces to a diagonal one, coinciding with the example presented in Chapter 3. The introduction of the non-diagonal term b modify the dynamics in a significant way: the horizontal direction is no longer invariant under the action of the both matrices. In fact, only the vertical direction remains invariant when $b \neq 0$, and the lack of symmetry between horizontal and vertical directions produces a richer geometric structure in the projective action of the cocycle.

By the Furstenberg–Kesten's Theorem, we know that the Lyapunov exponents of this cocycle exist for μ_p-almost every point in M_1. Moreover, since every hyperbolic matrix is conjugate to a diagonal hyperbolic matrix and the Lyapunov exponents are invariant under conjugacy, we conclude that

$$\lambda_\pm(A_b, \mu_p) = \lambda_\pm(\mathrm{A}_{\sigma\eta}, \mu_p)$$

Hence, natural questions arise: *For which parameters $\sigma \geq \eta > 1$ is the cocycle A_b a discontinuity point for the Lyapunov exponents? Does the answer depend on b?*

When these exponents are nonzero, Oseledets' Theorem guarantees the existence of two measurable invariant subspaces (the Oseledets subspaces) corresponding to the stable and unstable directions. Unlike in the diagonal case, one of these subspaces does not coincide with the coordinate axes anymore. Instead, it depends nontrivially on the orbit of the base point and on the parameter b, reflecting the coupling introduced by the off-diagonal term. This dependence introduces a significant obstacle to applying the strategy of interchanging Oseledets subspaces in order to prove discontinuity. Indeed, for each point one would need to design a perturbation adapted to the behavior of the Oseledets subspaces along its orbit. In particular, if the subspaces associated with the orbit of a given point never approach the complementary Oseledets subspace, the construction would require a perturbation based solely on rotations. As discussed in Chapter 5, such a perturbation cannot be made Hölder continuous, which prevents the direct application of this strategy.

On the other hand, if b is sufficiently small, the Oseledets subspace that does not coincide with the vertical direction is close to the horizontal axis. In this case, we could apply the same method used in the other examples, that is, to interchange the subspaces using the definition of the cocycle as a composition of hyperbolic matrices. However, in this setting, the angles required to complete the interchange depend on the base point. Consequently, there is no guarantee that the resulting angle function is Hölder continuous, which constitutes a serious obstruction to the method, since the perturbations are required to be Hölder continuous. This difficulty will be addressed explicitly in the next section.

6.4 Discontinuity in the General Setting

In the first three sections of this chapter, we discuss open problems in a specific setting, the shift map on the space of bi-infinite sequences over two symbols, equipped with a Bernoulli measure and locally constant cocycles defined by two triangular matrices.

In this section, we still aim to obtain discontinuity results, but now in a more general setting. More precisely, within the context where Theorem 2.3 applies, we seek conditions on the norm and the co-norm of the cocycle that nevertheless ensure discontinuity of the Lyapunov exponents.

We now fix the general setting and notation for this section. Let M be a subshift of finite type on l symbols, let f be the left shift on M and let μ be a Markov measure.

We consider the α-Hölder continuous cocycle defined by the function $A : M \to \mathrm{SL}(2, \mathbb{R})$:

$$A(x) = \begin{pmatrix} a(x) & 0 \\ 0 & a(x)^{-1} \end{pmatrix}$$

where $a : M \to \mathbb{R}$ is a Hölder continuous function and $a(x) \neq 1$ for every $x \in M$.

Moreover, we assume that A is not uniformly hyperbolic and that it has positive upper Lyapunov exponent, $\lambda_+(A, \mu) > 0$.

Since the cocycle is diagonal, the Oseledets subspaces are the horizontal and vertical axes at μ-almost every point of M.

Our strategy to prove discontinuity of the Lyapunov exponents follows the classical argument used throughout the thesis: we seek to interchange the Oseledets subspaces. In order to obtain this property, we adapted the argument introduced by Mañé-Bochi, [5]. Since these subspaces coincide with only two fixed directions, such an interchange can be achieved either by small rotations or by using the hyperbolic matrices that appear in the definition of the cocycle. However, as shown in Chapter 5, the method based only on small rotations cannot be applied here, because we cannot guarantee that the modified cocycle remains α-close to A. Thus, our approach is to exploit the dynamics of the cocycle itself to implement this exchange. We would like to identify conditions on the function a that allow us to construct a perturbation cocycle that performs this interchange while remaining close to A in the α-norm.

To this end, let n be a large natural number, and consider the set of points whose first n iterates satisfy $a(f^k(x)) < 1$ for $0 \leq k \leq n-1$, and whose next n iterates satisfy $a(f^{n+k}(x)) > 1$ for $0 \leq k \leq n-1$.

Let x be a point in the set under consideration and let $\gamma \in (0,1)$ be the hyperbolicity constant of f. Choose $r > 0$ so that for every point y in the open ball $B(x,r)$, the defining property of the set holds, that is, $a(f^k(y)) < 1$ for $0 \leq k \leq n-1$, which is possible since the function a is continuous and n is fixed. We may also choose $r > 0$, possibly smaller, so that every point $y \in B(f^n(x), \gamma^n r)$ satisfies $a(f^{n+k}(y)) > 1$ for $0 \leq k \leq n-1$.

Then, to produce the first exchange after n iterates, that is, sending the horizontal direction to the vertical one, we use the usual type of perturbation supported in the open balls $B(x,r)$ and $B(f^n(x), \gamma^n r)$: we compose the dynamics with two rotations, of angles α_0 and α_1. More precisely, let A be the perturbed cocycle and at the point $x \in M$ the perturbation corresponds to replacing the action of $A^n(x)$ by

$$\hat{A}^n(x) = \begin{pmatrix} \cos(\alpha_1) & -\sin(\alpha_1) \\ \sin(\alpha_1) & \cos(\alpha_1) \end{pmatrix} \begin{pmatrix} a^n(x) & 0 \\ 0 & a^n(x)^{-1} \end{pmatrix} \begin{pmatrix} \cos(\alpha_0) & -\sin(\alpha_0) \\ \sin(\alpha_0) & \cos(\alpha_0) \end{pmatrix}$$

And the first entry of this product is

$$a^n(x)\cos(\alpha_0)\cos(\alpha_1) - \sin(\alpha_0)\sin(\alpha_1)a^n(x)^{-1}.$$

In order to obtain the desired exchange, we require this term to vanish. This forces α_1 to satisfy

$$\tan(\alpha_1) = \frac{\cos(\alpha_0)}{\sin(\alpha_0)} a^n(x)^2 \tag{6.3}$$

Thus, the angle α_1 is not fixed, depends explicitly on the point x and on the number of iterates n.

However, to ensure that the perturbation is α-close to A, the angles α_0 and α_1 must vary in an α-Hölder way. In particular, we require

$$\frac{\|\hat{A}(z) - A(z) - \hat{A}(y) + A(y)\|}{d(z,y)^\alpha} < \varepsilon \qquad \text{for } y \neq z.$$

for some fixed $\varepsilon > 0$.

Consider now a point $z \in B(x,r)$ and a point $y \notin B(x,r)$, so that $d(z,y) \geq r$. Then,

$$\frac{\alpha_0}{d(z,y)^\alpha} < \frac{\alpha_0}{r^\alpha}$$

and therefore, $\alpha_0 < \varepsilon r^\alpha$.

Applying the same reasoning in the second ball of perturbation, we obtain that

$$\frac{\alpha_1(z)}{d(f^n(z),y)^\alpha} < \frac{\alpha_1(z)}{\gamma^n r^\alpha}$$

so, for every point $z \in B(x,r)$, $\alpha_1(z) < \varepsilon\gamma^n r^\alpha$.

Thus, the angle α_1 must be exponentially small. However, by its definition in (6.3), α_1 depends on the point and on the iterate n. Making $\alpha_1(x)$ exponentially small requires increasing n, but increasing n in turn forces the perturbation to be supported on smaller sets. This produces a circular argument: to make the perturbation small we must take n large, but taking n large shrinks the region where the perturbation can act.

Also, since the angle depends on the point, there is no reason to expect that the map $x \to \alpha_1(x)$ satisfies a uniform Hölder bound. Consequently, we cannot guarantee that the perturbed cocycle $\hat{A}$ is α-Hölder continuous.

6.5 Anosov Case

Since in the previous section we encountered difficulties arising from the fact that the cocycle was not locally constant, and the perturbed cocycle therefore depended on the base point, we now attempt to construct an example of a discontinuity point for the Lyapunov exponents in a different setting. More precisely, we aim to build a cocycle that is constant on a suitable subset of the base, while allowing the base dynamics to be different from the full shift.

For this section, let M be a compact manifold and $f : M \to M$ an Anosov, transitive, volume preserving diffeomorphism with hyperbolicity constant $\gamma \in (0, 1)$. Let μ be an ergodic probability in the Lebesgue class. Also, consider an α-Hölder continuous cocycle $A : M \to \mathrm{SL}(2, \mathbb{R})$.

It is a well known fact that, under these assumptions, the manifold M admits a Markov partition for f, denoted by $\mathcal{R} = \{R_1, \cdots, R_s\}$. Associated with this partition, we define a transition matrix $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} 1 & \text{if } f(\mathring{R}_i) \cap \mathring{R}_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This matrix determines a subshift of finite type (Σ_B, σ_B), whose admissible sequences are determined by B.

There exists a natural semiconjugacy between these two systems, given by a surjective Hölder continuous map $\Pi : \Sigma_B \to M$.

Since we can construct cocycles over the subshift of finite type that are constant on suitable sets, as discussed in Chapter 3, and which can be approximated by cocycles with vanishing Lyapunov exponents, a natural first attempt to produce examples of discontinuity in the manifold setting would be to transfer such a cocycle from (Σ_B, σ_B) to (M, f) via the semiconjugacy Π. However, this approach encounters a fundamental difficulty: the map Π does not preserve distances. For instance, if two points in Σ_B differ in their zeroth coordinate, their distance is equal to 1. In contrast, on the manifold M, one can find sequences of points lying in different rectangles of the Markov partition whose mutual distances converge to zero. As a consequence, although one can define a perturbation on M induced by the perturbation constructed on Σ_B, the fact of that distance is not preserved prevents us from ensuring that this perturbation remains close to the original cocycle in the Hölder norm.

In this setting, when attempting to apply the strategy of interchanging Oseledets subspaces, the main difficulty lies not in defining an original cocycle with the desired properties, such as being non–fiber-bunched, non–uniformly hyperbolic, Hölder continuous, and with nonzero Lyapunov exponents, but rather in constructing a Hölder continuous perturbation of it that realizes the exchange of subspaces. Unlike the shift space, which enjoys a well-controlled metric structure, the manifold M does not have such uniform distance behavior. As a consequence, it becomes necessary to define suitable regions where the perturbation can be performed while preserving Hölder regularity. However, the introduction of these regions may interfere with the mechanism used to interchange the Oseledets subspaces, therefore creating an additional obstruction to the procedure.

With this difficult, which has not yet been overcome, the following question remains open:

In the Anosov case, do there exist Hölder continuous cocycles that are non–fiber-bunched and are discontinuity points for the Lyapunov exponents?

We expect that this question admits affirmative answer.

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