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Essays on inequality and business cycles

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FACULDADE DE CIÊNCIAS ECONÔMICAS
CENTRO DE DESENVOLVIMENTO E PLANEJAMENTO REGIONAL

PAULO ROBERTO SANTOS CASACA

ESSAYS ON INEQUALITY AND BUSINESS CYCLES

Belo Horizonte
2019

PAULO ROBERTO SANTOS CASACA

ESSAYS ON INEQUALITY AND BUSINESS CYCLES

Tese apresentada ao curso de Doutorado em Economia do Centro de Desenvolvimento e Planejamento Regional da Faculdade de Ciências Econômicas da Universidade Federal de Minas Gerais, como requisito parcial à obtenção do Título de Doutor em Economia.

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Belo Horizonte
2019

Curso de Pós-Graduação em Economia da Faculdade de Ciências Econômicas

ATA DE DEFESA DE TESE DE **PAULO ROBERTO SANTOS CASACA** Nº. REGISTRO 2014657658. Às treze horas do dia oito do mês de abril de dois mil e dezenove, reuniu-se *na **Faculdade de Ciências Econômicas*** da Universidade Federal de Minas Gerais a Comissão Examinadora de TESE, indicada *"ad referendum"* pelo Colegiado do Curso em 20/03/2019, para julgar, em exame final, o trabalho final intitulado **"Essays on Inequality and Business Cycles",** requisito final para a obtenção do Grau de *Doutor em Economia,* área de concentração em Economia. Abrindo a sessão, o Presidente da Comissão, Prof. Mauro Sayar Ferreira após dar a conhecer aos presentes o teor das Normas Regulamentares do Trabalho Final, passou a palavra ao candidato, para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores, com a respectiva defesa do candidato. Logo após, a Comissão composta pelos professores, Mauro Sayar Ferreira (Orientador), Jose Heleno Faro (Coorientador), Rodrigo Jardim Raad, Rodrigo Menon Simões Moita, Jean-Marc Bonnisseau, Brice Magdalou (Participação por videoconferência) e Alain Jacques Chateauneuf (Participação por videoconferência) se reuniu, sem a presença do candidato e do público, para julgamento e expedição do resultado final. A Comissão APROVOU o candidato por unanimidade. O resultado final foi comunicado publicamente ao candidato pelo Presidente da Comissão. Nada mais havendo a tratar o Presidente encerrou a reunião e lavrou a presente ATA, que será assinada por todos os membros participantes da Comissão Examinadora. Belo Horizonte, 08 de abril de 2019.

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also for thousands of students before me. We cannot deny this opportunity to the new generations. So, I hope that the Brazilian state can keep providing it because without public education, our inglorious high inequality will never decrease and our nation will never reach prosperity.

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Resumo

Esta tese consiste em três capítulos na qual os dois primeiros estudam o papel da desigualdade em índices de desenvolvimento humano e o último capítulo analisa o efeito da ambiguidade Knightiana em um modelo Novo Keynesiano com choque de política fiscal.

No primeiro capítulo nós axiomatizamos, para o caso multidimensional, uma função de avaliação social que acomoda o princípio Pigou-Dalton e majoração da correlação crescente. Essa função de avaliação social é construída a partir da subclasse de funções inframodulares sobre risco propostas por Müller e Scarsini.

O segundo capítulo axiomatiza uma classe função de avaliação social multidimensional da qual o Índice de Desenvolvimento Humano ajustado à Desigualdade é um caso especial. Ademais, nós mostramos que essa classe de função de avaliação social concorda com o princípio de Pigou-Dalton e com a subclasse de funções inframodulares. Por fim, nós analisamos o contexto onde o planejador social não tem convicção sobre como ponderar os atributos da função e suas aplicações em relação à abordagem de aversão à incerteza.

O último capítulo analisa o efeito de um choque de confidência sobre a política fiscal através de um modelo Novo Keynesiano com agentes avessos à incerteza e rigidezes nominais. Nós também modelamos a preferência das famílias por ativos livres de riscos e comparamos os efeitos de um choque de prêmio de risco na atividade econômica. Nós encontramos que o choque de confidência na política fiscal pode gerar movimentos conjuntos entre produto, consumo, investimento e horas trabalhadas. O choque de prêmio de risco apresenta resultados similares sem que a política fiscal seja expansiva.

Abstract

This thesis comprises three chapters in which the first two study the role of inequality on human development indexes and the last chapter analyses the effect of Knightian ambiguity on a New Keynesian model with fiscal policy shock.

In the first chapter we axiomatize, in the multidimensional case, a social evaluation function that can accommodate a natural Pigou-Dalton principle and correlation increasing majorization. This is performed by building upon a simple subclass of inframodular functions proposed by Müller and Scarsini under risk.

The second chapter axiomatizes a class of multidimensional social evaluation function in which Inequality-adjusted Human Development Index is a special case. We, furthermore, show that this class of social evaluation function accommodates Pigou-Dalton principle and agrees with the subclass of inframodular functions. Finally, we analyze the context where the social planner is unsure on how to weigh different attributes and their implications to the inequality aversion approach.

The last chapter analyzes the effect of confidence shocks on fiscal policy through a New Keynesian model with ambiguity averse agents and nominal rigidities. We also model the household's preference for holding risk-free assets and compare the effects of risk premium shocks on economic activity. We find that confidence shocks on fiscal policy may generate business cycles comovement among output, consumption, investment and hours worked. Risk premium shock has similar results without expansive fiscal policy.

Chapter 1

Introduction (Version française)

Ce chapitre présente une introduction élargie aux trois chapitres suivants et contient les principaux résultats obtenus dans les essais sur l'inégalité. La section 2, présente, quant à lui, les principaux résultats de l'essai sur les cycles d'affaires.

1.1 Inégalité

Le sujet principal des deux premiers chapitres traite du le développement humain et l'inégalité.

1.1.1 Transferts multidimensionnels de Pigou-Dalton

Introduction

Les fonctions d'évaluation sociale multidimensionnelles ont gagné en pertinence au cours des dernières décennies, surtout en raison des nouvelles techniques abordées dans le cadre multidimensionnel des travaux pionniers de Atkinson (1970; 1987), Kolm (1976a;b; 1977) et Sen (1976). En particulier, Tsui (1995; 1999) et Gajdos and Weymark (2005) ont proposé des approches axiomatiques pour concevoir des mesures d'inégalité des revenus dans un contexte multi-attributs.

Dans ce chapitre, nous suivons l'approche additive traditionnelle et nous nous limitons adopter un transfert Pigou-Dalton spécifique, ce qui nous avons jugé pertinent. De plus, notre approche est cohérente avec la propriété de Majoration de Corrélation Croissante.

Tout d'abord, nous étudions une classe de fonctions d'évaluation sociale utilitaristes qui s'avérera inframodulaire. Plus précisément, ce chapitre vise à caractériser une classe de fonctions inframodulaires initialement proposée dans la littérature sur la décision sous risque.

Nous montrons également que les fonctions inframodulaires sont intimement liées à une généralisation naturelle au cas multidimensionnel du principe classique des transferts unidimensionnels de Pigou-Dalton. Ce point nous mènera à un simple axiome de "principe de transfert". De même, les fonctions inframodulaires étant sous-modulaires, il s'avère que notre fonction d'évaluation tiendra également compte de la propriété de Majoration de Corrélation Croissante.

Enfin, nous spécifions notre fonction d'évaluation sociale pour la comparer ensuite au célèbre IDH en utilisant des données réelles.

Notation, motivation et axiomes

Nous considérons n individus $1, \ldots, j, \ldots, n$ et $A_j \in \mathbb{R}^m$ est le vecteur colonne de m attributs de cet individu $A_j = (a_{1j}, \ldots, a_{ij}, \ldots, a_{mj})^t$; effectivement, les mêmes m attributs sont pris en compte pour chaque individu.

Désormais, pour n individus donnés, $A = (A_1, \ldots, A_j, \ldots, A_n)$ est la matrice $m \times n$ résumant la population considérée. Notez que $\mathcal{A}$ dénotera l'ensemble de ces matrices réelles A avec m lignes et n colonnes.

Dans le cas unidimensionnel, quand on considère n individus $1, \ldots, j, \ldots, n$ avec des revenus $x_1 \leq x_2 \leq \ldots \leq x_j \leq \ldots \leq x_n$, il est généralement supposé que si $x_j < x_{j+1}$, un transfert $\varepsilon > 0$ de l'individu $j+1$, vers l'individu j tel que $x_j + \varepsilon \leq x_{j+1} - \varepsilon$ aura comme effet de réduire l'inégalité. Ce transfert est ce que nous appelons transfert Pigou-Dalton.

Dans le cas m-dimensionnel avec $m \geq 1$, ce qui est le cas dans le chapitre, chaque individu j a un vecteur colonne $A_j \in \mathbb{R}^m$ de m attributs, la matrice $m \times n$, $A = (A_1, \ldots, A_j, \ldots, A_n)$ résume les données.

Notre objectif est d'axiomatiser des fonctions d'évaluation sociale additives, c'est-à-dire des fonctions d'évaluation sociale $I : \mathcal{A} \longrightarrow \mathbb{R}$ tel que $I(A) = \sum_{j=1}^n u(A_j)$, pour tout $A \in \mathcal{A}$, où $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ qui respecte une évaluation sociale décroissante dans le cas d'un tel transfert régressif de Pigou-Dalton comme observé ci-dessus. Donc, pour tout $(X, Y) \in \mathbb{R}^m \times \mathbb{R}^m$ tel que $X \leq Y$ et pour tout $\varepsilon \in \mathbb{R}^m_+$, u devrait satisfaire:

$$u(X) - u(X - \varepsilon) \geq u(Y + \varepsilon) - u(Y) \tag{1.1}$$

C'est la propriété habituelle de la concavité dans le cas *uni*-dimensionnel, au moins lorsque u est continu. De fait, nous montrons que les fonctions inframodulaires, satisfont la propriété désirée (1.1) et qu'elle sont cohérentes avec la Majoration de Corrélation Croissante. Rappelons:

Définition 1 *Une fonction $u : \mathbb{R}^m \to \mathbb{R}$ est dit inframodularie si:*

$$u(X + \varepsilon) - u(X) \geq u(Y + \varepsilon) - u(Y)$$

pour tous $X, Y \in \mathbb{R}^m$ avec $X \leq Y$ et $\varepsilon \in \mathbb{R}^m_+$.

Nous passons maintenant aux axiomes qui seront considérés afin de modéliser les préférences $\succsim$ du "planificateur social" (ou "observateur éthique") pour le bien-être global en tenant compte du fait que les inégalités ont un impact négatif sur le bien-être, mais aussi que tous les attributs sont "positifs", c'est-à-dire que toute augmentation de certains attributs a un effet positif sur le bien-être.

Ainsi, $\succsim$ est une relation de préférence sur $\mathcal{A}$ (si $\mathcal{A}_+$ ou $\mathcal{A}_{++}$ est considéré - ceci est spécifié dans le théorèmes). En effet, pour $A, B \in \mathcal{A}$, $A \succsim B$ signifie que A est faiblement préféré à B, $A \succ B$ signifie que A est strictement préféré à B, $A \sim B$ signifie que A et B sont considérés comme équivalents par le planificateur social. Notez que ces définitions sont cohérentes puisque $\succsim$ suppose un préordre total.

A.1 $\succsim$ est un préordre total; i.e., $\succsim$ est une transitive, complète et aussi une relation binaire réflexive sur $\mathcal{A}$.

A.2 Continuité: Soit $B \in \mathcal{A}$, donc $\{A \in \mathcal{A} | A \succsim B\}$ et $\{A \in \mathcal{A} | B \succsim A\}$ est fermé dans la topologie habituelle de $\mathbb{R}^{m \times n}$.

A.3 Monotonie: Pour tous $A, B \in \mathcal{A}$, $a_{ij} \geq b_{ij}$ for all i, j, implique $A \succsim B$. Si de plus $A \neq B$, puis $A \succ B$.

A.4 Indépendance: Pour tout j et (A, B); $(A_j, A_{-j}) \succsim (A_j^{'}, A_{-j}) \iff (A_j, B_{-j}) \succsim (A_j^{'}, B_{-j})$.

A.5 Anonymat: Pour toute matrice de permutation Π et pour tout $A \in \mathcal{A}$, on a $A \sim A\Pi$; i.e., $(A_1, \ldots, A_j, \ldots, A_n) \sim \left(A_{\sigma(1)}, \ldots, A_{\sigma(j)}, \ldots, A_{\sigma(n)}\right)$ où $\sigma : [1, n] \to [1, n]$ est une bijection.

A.6 Additivité: Pour tous A, A_j, B_j, C_j; $(A_j, A_{-j}) \sim (B_j, A_{-j}) \implies (A_j + C_j, A_{-j}) \sim (B_j + C_j, A_{-j})$.

A.7 Principe Pigou-Dalton: Soit $A = (A_1, \ldots, A_j, \ldots, A_n)$ tel que pour certains j_1, j_2 on a $A_{j_1} \leq A_{j_2}$ et soit $\varepsilon \in \mathbb{R}_+^m$ alors:

$A \succsim \left(A_1, \ldots, A_{j_1-1}, A_{j_1} - \varepsilon, A_{j_1+1}, \ldots, A_{j_2-1}, A_{j_2} + \varepsilon, A_{j_2+1}, \ldots, A_n\right) = A_\varepsilon$; en outre, $A \succ A_\varepsilon$ si $\varepsilon \in \mathbb{R}_+^m$ et $\varepsilon \neq 0$.

Fonctions d'évaluation sociale multidimensionnelle

Le théorème 1 propose une axiomatisation de la fonction d'évaluation sociale additive.

Théorème 1 *Une relation de préférence $\succsim$ sur $\mathcal{A}$ satisfait A.1, A.2, A.3, A.4 et A.5 si et seulement s'il existe: $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ croissante et continue, telle que:*

$$\textit{Pour tous } A, B \in \mathcal{A},\ A \succsim B \Longleftrightarrow \sum_{j=1}^{n} u(A_j) \geq \sum_{j=1}^{n} u(B_j)$$

où u est définie jusqu'à une transformation affine positive.

Nous arrivons maintenant au résultat principal de ce chapitre dans lequel nous caractérisons des fonctions d'évaluation sociale construites sur le type spécial de fonctions inframodulaires.

Théorème 2 *Une relation de préférence sur $\mathcal{A}$ satisfait A.1, A.2, A.3, A.4, A.5, A.6 et A.7 si et seulement s'il existe $\alpha_i > 0$, $i = 1, \ldots, m$, tel que $\sum_{i=1}^m \alpha_i = 1$ et il existe $\psi : \mathbb{R} \to \mathbb{R}$ strictement croissante, strictement concave et continue telle que:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) \geq \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot b_{ij}).$$

En outre, ces α_i's sont uniques et ψ est définie jusqu'à une transformation affine croissante.

Spécification de ψ et un indice d'inégalité relative

Spécifier ψ, nous introduisons deux axiomes:

A.8 Invariance absolue: Pour tous $A, B \in \mathcal{A}$ et pour tout $\lambda \in \mathbb{R}$, $A \sim B \Longleftrightarrow A + \lambda\mathbb{1} \sim B + \lambda\mathbb{1}$ où $\mathbb{1}$ est la matrice $m \times n$ avec 1 partout.

A.9 Invariance relative: Pour tous $A, B \in \mathcal{A}_{++}$ et pour tout $\lambda > 0$, $A \sim B \Longleftrightarrow \lambda A \sim \lambda B$.

Théorème 3 *Supposons que la relation de préférence $\succsim$ satisfasse A.1 à A.7, alors:*

· jusqu'à une transformation affine croissante $\psi(t) = -e^{-at}$ avec $a > 0$ si et seulement si A.8 est satisfait quand $\succsim$ est définie sur $\mathcal{A}$.

· jusqu'à une transformation affine croissante soit $\psi(t) = \ln(t)$, pour tout $t > 0$ ou $\psi(t) = t^a$, pour tout $t > 0$ où $a \neq 0$, $a < 1$ si et seulement si A.9 est satisfait quand $\succsim$ est définie sur $\mathcal{A}_{++}$.

Remarque Notons que dans les cas où tous les attributs sont strictement positifs, et si nous adoptons l'axiome *A.9* puis pour tout $A \in \mathcal{A}_{++}$ on pourrait adopter la fonction d'évaluation sociale $J(A) = \prod_{j=1}^{n}(\sum_{i=1}^{m} \alpha_i \cdot a_{ij})$.

En effet, dans un tel cas:

$$I(A) = \sum_{j=1}^{n} \ln(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) = \ln(\prod_{j=1}^{n}(\sum_{i=1}^{m} \alpha_i \cdot a_{ij})).$$

Nous nous concentrons sur l'indice relatif d'inégalité, qui est lié au choix de $\psi(\cdot) = \ln(\cdot)$. Cet indice semble être l'un des plus faciles à gérer et à prendre en compte dans notre cadre.

Corollaire du Théorème 3

L'indice d'inégalité correspondant lié à la fonction d'évaluation sociale définie sur $\mathcal{A}_{++}$, l'ensemble des $m \times n$ matrices avec des éléments positifs, satisfaisant *A.1* à *A.7* et *A.9* avec $\psi(t) = \ln(t)$, avec $t > 0$ est relatif et se trouve sous la forme,

$$1 - \left(\prod_{j=1}^{n} \frac{\sum\limits_{i=1}^{m} \alpha_i \cdot a_{ij}}{\sum\limits_{i=1}^{m} \alpha_i \cdot \mu_i} \right)^{\frac{1}{n}}$$

où μ_i, est la moyenne de i_{th} attribut dans lequel $i = 1, \ldots, m$.

En accord avec la Majoration de Corrélation Croissante.

La Majoration de Corrélation Croissante (MCC) est un concept créé par Boland and Proschan (1988) et introduit dans la littérature sur les inégalités par Tsui (1999).

Nous allons ici présenter ses définitions (Boland and Proschan, 1988).

Le Concept de Transfert de Corrélation Croissante (TCC)

Soit $A, B \in \mathcal{A}$, puis B est obtenu de A par un TCC s'il existe j_1, j_2 où $j_1 \neq j_2$ tel que $B_{j_1} = A_{j_1} \wedge A_{j_2}$ et $B_{j_2} = A_{j_1} \vee A_{j_2}$.

Un TCC est stricte chaque fois que ni $A_{j_1} \leq A_{j_2}$ ni $A_{j_2} \leq A_{j_1}$ produisent.

Le Concept de Majoration de Corrélation Croissante (MCC)

Soit $A, B \in \mathcal{A}$, puis $A >_c B$, i.e., A est strictement moins inégal pour le MCC si B peut être dérivé de A par une permutation de colonnes et une séquence finie de transferts de corrélation croissants, avec au moins un élément étant strict.

Nous pouvons maintenant affirmer et prouver que notre évaluation sociale est fonctionnelle du Théorème 2 ainsi que de tout respect social inframodulaire strict de la fonction *MCC.*

Notez que c'est le cas pour la fonction inframodulaire dans le Théorème 2.

Théorème 4 *Toute évaluation fonctionnelle inframodulaire stricte respecte la MCC.*

Analyse Empirique

Sur la base de *Théorème 3*, nous spécialisons $\psi(\cdot)$ comme $\ln(\cdot)$, ainsi, en considérant les cas dans lesquels tous les attributs sont strictement positifs, nous adoptons, pour tout $A \in \mathcal{A}_{++}$, $J(A)$ comme une fonction 'moyenne' d'évaluation sociale.

Nous visons à évaluer la pertinence de cette fonction inframodulaire à l'aide de données. Afin de le tester, nous avons décidé de faire une comparaison avec une autre fonction non inframodulaire, à savoir le célèbre Indice de Développement Humain (IDH).

Notre indice a besoin d'informations de niveau individuel pour être construit. Ainsi, nous décidons d'effectuer cette comparaison avec l'examen national brésilien pour les lycéens, appelé ENEM (Exame Nacional do Ensino Médio). Pour postuler $H(\cdot)$ avec ces données, nous avons concentré notre analyse sur trois attributs: sciences naturelles (a^s), langages (a^l) et mathématiques (a^m).

La population (n) est le nombre d'étudiants dans chaque ville. En suivant les règles de l'IDH, nous donnons ici le même poids aux attributs. La fonction $J(A)$ dans ce cas peut s'écrire,

$$J(A) = \frac{1}{n} \sum_{j=1}^{n} \ln \left(\frac{a_j^s + a_j^l + a_j^m}{3} \right).$$

L'IDH fournissant un résultat compris entre 0 et 1, nous avons décidé d'extraire le 'l'équivalent de certitude' de $J(A)$, i.e., $I(A) = \exp^{J(A)}$. Nous avons maintenant les deux indices fournissant des résultats dans l'intervalle $[0, 1]$.

Il est largement reconnu que la formule classique de l'IDH ne prend pas en compte le niveau d'inégalité dans un pays. Cependant, nous sommes intéressés par le contraste avec l'IDH classique pour détecter dans quelle mesure $I(A)$ est influencé par les 'intra' inégalités.

Premièrement, la différence entre les résultats $I(A)$ et $H(A)$ est relativement faible. Leur coefficient de corrélation est de $0,999$. La similarité des résultats convient, car elle montre que cette fonction fournit des résultats compatibles avec ceux de l'IDH.

En d'autres termes, nous voulons voir si l'inégalité est positivement corrélée avec $H(A) - I(A)$. Pour mesurer l'inégalité, nous avons résumé les valeurs des attributs pour chaque élève et extrait l'écart type de cette variable transformée. Nous voulons analyser la corrélation entre ces deux variables afin d'affirmer si $I(A)$ prend en compte ou non l'inégalité.

Nous avons donc calculé une régression linéaire comme un exercice hypothétique. Le $R^2 = 0,9592$ atteste que l'écart type explique fortement le comportement de $H(A) - I(A)$. La relation positive entre les variables est assez importante.

Par conséquent, sur la base de ces résultats, nous suggérons que cette fonction peut constituer une bonne alternative à l'Indice de Développement Humain ajusté aux Inégalités (IDHI).

1.1.2 Fonction Rawlsienne d'évaluation sociale ajustée aux inégalités

Introduction

L'inégalité sociale est-elle importante pour mesurer l'évaluation sociale? Si oui, quel devrait être le poids de l'inégalité dans une fonction d'évaluation sociale? Pour répondre à ces questions, il est essentiel de comprendre le rôle de l'inégalité dans le développement humain. Les politiques publiques doivent-elles induire une objectif de réduction des inégalités afin d'améliorer le bien-être global?

La littérature connexe présente un large éventail de possibilités dans lesquelles les inégalités pourraient entraver le développement humain. Cependant, certains résultats théoriques suggèrent que les inégalités peuvent aussi être positives pour la croissance économique.

Par conséquent, les inégalités *per se* sont traitées ici comme un élément important ayant un impact négatif sur le développement humain.

Les inégalités sociales et le développement humain ont souvent été mesurés essentiellement en termes de richesse ou de revenu. Ce type d'indicateur devrait être complété par d'autres attributs tels que la santé et l'alphabétisation, par exemple. Comme le souligne Tsui (1999), une croissance économique rapide peut ne pas nécessairement s'accompagner d'améliorations dans les domaines de la santé et de l'éducation.

En conséquence, afin d'évaluer le niveau de vie d'une manière plus large, l'Indice de Développement Humain (IDH) a été proposé. Cependant, l'IDH ne tient pas compte des inégalités au sein de la population. Pour prendre en compte les inégalités dans le développement humain, a été créé l'Indice de Développement Humain ajusté aux Inégalités (IDHI), où le niveau d'inégalité est mesuré pour "pénaliser" les résultats de l'IDH.

Dans ce chapitre, nous cherchons à caractériser une classe de fonctions d'évaluation sociale multidimensionnelle qui traitent des inégalités, tant entre les individus qu'entre les attributs. Une axiomatisation de cette fonction d'évaluation sociale est fournie. Nous montrons également que l'IDHI est un cas particulier de la classe de la fonction d'évaluation sociale que nous axiomatisons.

De plus, nous proposons un indice de bien-être qui conduit à une situation dans laquelle le planificateur social n'est pas sûr de la façon d'attribuer des pondération aux attributs lorsqu'ils sont fortement complémentaires.

Cadre et axiomes

Considérons une population finie $\mathcal{J} = \{1, \ldots, j, \ldots, n\}$ d'individus et un ensemble fini d'attributs $\mathcal{I} = \{1, \ldots, i, \ldots, m\}$. Chaque individu $j \in \mathcal{J}$ est doté de m attributs représentés par un vecteur (rangées) $a_i \in \mathbb{R}^m$.

Une matrice réele A est un $m \times n$ tableau de nombres réels. Soit $\mathbb{M}_{m\times n}$ l'espace vectoriel de toutes les m-par-n matrices réelles. Également, $\mathbb{M}^{+}_{m\times n}$ et $\mathbb{M}^{++}_{m\times n}$ sont des sous-ensembles de $\mathbb{M}_{m\times n}$ de matrices avec des éléments non négatifs et positifs, respectivement. Nous notons qu'une population de n individus dotés de m attributs peut être représentée par une matrice $A \in \mathbb{M}_{m\times n}$ avec m lignes et n colonnes.

Considérant une matrice $A \in \mathbb{R}^{m\times n} (\equiv \mathbb{M}_{m\times n})$, nous définissons $A_j \in \mathbb{R}^m (\equiv \mathbb{M}_{m\times 1})$ comme vecteur (matrice) de tous les attributs de l'individu $j \in \mathcal{J}$, représenté par la $j$ème colonne de A. Également, $A_i \in \mathbb{R}^n (\equiv \mathbb{M}_{1\times n})$ est défini comme le vecteur (matrice) de l'attribut $i \in \mathcal{I}$ à tous les individus, représentés par la $i$ème ligne de A. A_j peut être interprété comme un individu ou un sous-groupe de la société. De même, A_i peut être interprété comme un attribut ou un sous-groupe d'attributs. Pourtant, la matrice $(A_j^{'}, A_{-j})$ est la matrice A oú la colonne A_j est remplacée par la colonne $A_j^{'}$.

Nous présentons ensuite les axiomes qui caractérisent la préférence de ce planificateur social pour le développement humain.

Considérons un planificateur social qui gouverne une société (un pays ou une ville, par exemple) et dont l'objectif est d'améliorer le développement humain de cette société. Ce planificateur social a ses propres préférences sur la façon de mener des politiques sociales afin de faire face aux inégalités et d'améliorer le développement humain.

La préférence d'un planificateur social est donnée par une relation binaire $\succsim$ sur $\mathbb{M}_{m\times n}$. Pour $A, B \in \mathbb{M}_{m\times n}$, $A \succsim B$ signifie que A est faiblement préféré à B, $A \succ B$ signifie A est strictement préféré à B, $A \sim B$ signifie A and B sont considérés comme indifférents par le planificateur social.

A.1 Préordre Total: $\succsim$ est une complète, transitive relation binaire sur $\mathbb{M}_{m\times n}$. À savoir, pour tous $A, B, C \in \mathbb{M}_{m\times n}$, $A \succsim B$ or $B \succsim A$ (complétude). De plus, si $A \succsim B$ and $B \succsim C$ alors $A \succsim C$ (transitivité).

A.2 Continuité: Soit $B \in \mathbb{M}_{m\times n}$, $\{A \in \mathbb{M}_{m\times n} | A \succsim B\}$ et $\{A \in \mathbb{M}_{m\times n} | B \succsim A\}$ sont fermé dans $\mathbb{M}_{m\times n}$.

A.3 Monotonie: Pour tous $A, B \in \mathbb{M}_{m\times n}$, et pour tous i, j, si $a_{ij} \geq b_{ij}$ alors $A \succsim B$. Si de plus $A \neq B$, alors $A \succ B$.

A.4 Indépendance: Pour tous $A, B \in \mathbb{M}_{m\times n}$ et en prenant $j \in \{1, \ldots, n\}$ fixé avec $A'_j \in \mathbb{M}_{m\times 1}$; $(A_j, A_{-j}) \succsim (A'_j, A_{-j}) \iff (A_j, B_{-j}) \succsim (A'_j, B_{-j})$.

A.5 Anonymat: pour tout $A \in \mathbb{M}_{m\times n}$ et pour toute bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, on a $(A_1, \ldots, A_j, \ldots, A_n) \sim \big(A_{\sigma(1)}, \ldots, A_{\sigma(j)}, \ldots, A_{\sigma(n)}\big)$.

A.6 Additivité: Pour tous A, A_j, B_j, C_j; $(A_j, A_{-j}) \sim (B_j, A_{-j}) \implies (A_j + C_j, A_{-j}) \sim (B_j + C_j, A_{-j})$.

A.6 Fortement homothétique: Pour tout $A \in \mathbb{M}^+_{m\times n}$ et pour tous $B_j, C_j \in \mathbb{M}^+_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (C_j \otimes A_j, A_{-j}) \succsim (C_j \otimes B_j, A_{-j})$.

A.7 Principe Pigou-Dalton: Soit $A = (A_1, \ldots, A_j, \ldots, A_n)$ tel que pour certains j_1, j_2 on a $A_{j_1} \leq A_{j_2}$ et soit $\varepsilon \in \mathbb{R}^m_+$ alors:

$A \succsim \big(A_1, \ldots, A_{j_{1-1}}, A_{j_1} - \varepsilon, A_{j_{1+1}}, \ldots, A_{j_{2-1}}, A_{j_2} + \varepsilon, A_{j_{2+1}}, \ldots, A_n\big) = A_\varepsilon$; en outre, $A \succ A_\varepsilon$ si $\varepsilon \in \mathbb{R}^m_+$ et $\varepsilon \neq 0$.

Fonction d'évaluation sociale multidimensionnelle

Cette section présente le premier résultat.

Théorème 5 *Une relation de préférence $\succsim$ on $\mathbb{M}_{m\times n}$ satisfait A.1 à A.7 si, et seulement si, il existe $\alpha \in \Delta^{m-1}_{++}$ et il existe $\psi : \mathbb{R} \to \mathbb{R}$ strictement croissante, strictement concave et continue telle que:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) \geq \sum_{j=1}^{n} \psi(\prod_{i=1}^{m} b_{ij}^{\alpha_i}).$$

En outre, tels α_i sont uniques et ψ est défini jusqu'à une transformation affine croissante.

Le planificateur social attribue d'abord un poids pour chaque attribut et les agrège par une règle Cobb-Douglas. Puis il applique une fonction ψ et ajoute tous les individus.

Nous introduisons ensuite un autre axiome afin de formuler une proposition qui nous aide à démontrer la relation du Théorème 5 avec la formulation de l'IDHI.

A.8 Invariance Scalaire: Pour tous $A, B \in \mathbb{M}^+_{m\times n}$ et $k \in \mathbb{R}_{++}$; $A \succsim B \implies k \cdot A \succsim k \cdot B$.

Proposition 1 *Une relation de préférence $\succsim$ sur $\mathbb{M}^+_{m\times n}$ satisfait A.1 à A.8 si, et seulement si, jusqu'à une transformation affine croissante, soit $\psi(t) = \ln(t)$ ou $\psi(t) = t^\beta$ pour tout $t > 0$, où $\beta \in (0, 1)$*

$$I(A) = \left(\prod_{j=1}^{n} \prod_{i=1}^{m} a_{ij}^{\alpha_j} \right)^{\frac{1}{n}}$$

De même, en prenant $\psi(\cdot) = \ln(\cdot)$ et considérant que tous les attributs sont strictement positifs. Supposons également que tous les attributs ont le même poids. De plus, puisque la somme des logs est le log du produit, nous obtenons,

$$I(A) = \left[\prod_{j=1}^{n} \left(\prod_{i=1}^{m} a_{ij} \right)^{\frac{1}{m}} \right]^{\frac{1}{n}} \tag{1.2}$$

où $I(A) \in [0, 1]$. Prenant $m = 3$ nous constatons que la condition (1.2) est égal à la formule IHDI.

Fonction d'évaluation sociale: Une nouvelle proposition

Que se passe-t-il si le planificateur social n'est pas sûr de l'importance des attributs choisis pour évaluer le bien-être d'une société? Nous caractérisons ce contexte dans l'exemple A, montrant que l'axiome fortement homothétique peut être violé.

Cela signifie que l'index formulé dans le théorème 5 ne capture pas ce genre de comportement. Une façon de caractériser ce comportement est d'affaiblir l'axiome fortement homothétique et nous proposons donc une nouvelle fonction d'évaluation sociale.

Fonction Rawlsienne d'évaluation sociale ajustée aux inégalités

Comme l'axiome de l'homothéticité Forte est violé, nous remplaçons cet axiome par trois autres axiomes: Homothéticité *(A.9)*, Log-Convexité *(A.10)* et Invariance de Puissance *(A.11)*:

A.9 Homothéticité: Pour tout $A \in \mathbb{M}^+_{m\times n}$, $k \in \mathbb{R}_{++}$ et pour tous $A_j, B_j \in \mathbb{M}^+_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (k \cdot A_j, A_{-j}) \succsim (k \cdot B_j, A_{-j})$.

A.10 Log-Convexité: Si pour tout $A \in \mathbb{M}^+_{m\times n}$, $\lambda \in (0, 1)$ et pour tous $A_j, B_j \in \mathbb{M}^+_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (A_j^\lambda \otimes B_j^{1-\lambda}, A_{-j}) \succsim (B_j, A_{-j})$.

A.11 Invariance de Puissance: Si pour tout $A \in \mathbb{M}^+_{m\times n}$, $k \in \mathbb{R}_{++}$ et pour tous $A_j, B_j \in \mathbb{M}^+_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (A_j^k, A_{-j}) \succsim \left(B_j^k, A_{-j}\right)$.

Théorème 6 *Une relation de préférence $\succsim$ sur $\mathbb{M}_{m\times n}$ satisfait A.1 à A.5, A.7 et A.9 à A.11 si, et seulement si, il existe un ensemble unique, non vide, fermé et convexe $C \subset \Delta$ des mesures de probabilité tel que $\alpha \in \Delta^{m-1}_{++}$ et il existe $\psi : \mathbb{R} \to \mathbb{R}$ strictement croissante, strictement concave et continue telle que:*

$$A \succsim B \iff \sum_{j=1}^{n} \psi \left(\min_{\alpha \in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i} \right) \geq \sum_{j=1}^{n} \psi \left(\min_{\alpha \in C} \prod_{i=1}^{m} b_{ij}^{\alpha_i} \right).$$

De plus, ψ est définie jusqu'à une transformation affine croissante.

En outre, un axiome d'homothéticité forte dans le théorème 6 impliquerait que l'ensemble C possède une mesure de probabilité unique.

Afin de pouvoir calculer notre fonction d'évaluation sociale, nous spécifions maintenant $\psi(\cdot) = \ln(\cdot)$. Il s'avère que de simples calculs comme pour la proposition 1 conduisent au corollaire 1:

Corollaire 1 *Une relation de préférence* $\succsim$ *sur* $\mathbb{M}^{+}_{m\times n}$ *satisfait A.1 à A.5, A.7 et A.9 à A.11 si, et seulement si, il existe un ensemble unique, fermé et convexe* $C \subset \Delta$ *des mesures de probabilité telles que* $\alpha \in \Delta^{m-1}_{++}$ *et* $A \succsim B$ *si, et seulement si,* $I(A) \geq I(B)$ *où pour tous* A, B *avec des attributs strictement positifs:*

$$I(A) = \left(\prod_{j=1}^{n} \min_{\alpha \in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i} \right)^{\frac{1}{n}}$$

et une formule similaire pour $I(B)$.

Example B

Supposons que le planificateur social ne soit plus sûr de la meilleure façon d'attribuer des poids aux sous-composants de l'éducation. Il ne doute pas que les attributs de l'éducation devraient représenter conjointement 50% de la mesure. Il croit que leur poids est compris entre $[1/6, 1/3]$. De plus, il suit la fonction d'évaluation sociale fournie dans Corollaire 1, c'est-à-dire, définissant $a_j^{'} := a_{hj}^{\frac{1}{4}} a_{wj}^{\frac{1}{4}} \left(a_{e_1 j}^{\frac{1}{6}} a_{e_2 j}^{\frac{1}{3}} \right)$ et $a_j^{''} := a_{hj}^{\frac{1}{4}} a_{wj}^{\frac{1}{4}} \left(a_{e_1 j}^{\frac{1}{3}} a_{e_2 j}^{\frac{1}{6}} \right)$. La réécriture de l'indice donne:

$$I(A) = \left[\prod_{j=1}^{n} \min \left(a_j^{'}, a_j^{''} \right) \right]^{\frac{1}{n}}.$$

Plus le doute de l'agent est élevé pour attribuer des poids aux attributs, plus la gamme de possibilités de poids est élevée.

Nous complétons cette section en proposant un résultat final qui nous permet de traiter des cas simples et significatifs, qui semblent donc calculables pour les applications.

Théorème 7 *Supposons que le planificateur social est juste capable de donner une limite supérieure et inférieure, respectivement* a_i *and* b_i *pour le poids* α_i *de chaque attribut* $i = 1, \ldots, m$, *où* $0 < \underline{\delta_i} \leq \overline{\delta_i} < 1$. *Ensuite, le planificateur social est cohérent, c'est-à-dire, qu'il existe* $\alpha_i \geq 0$ *tel que* $\underline{\delta_i} \leq \alpha_i \leq \overline{\delta_i}$ *pour tout* i *et* $\sum_{i=1}^{m} \alpha_i = 1$ *si, et seulement si,*

$$\sum_{i=1}^{m} \underline{\delta_i} \leq 1 \leq \sum_{i=1}^{m} \overline{\delta_i} \tag{1.3}$$

En outre, la fonction d'évaluation sociale $I(A)$ *de toute* A *avec des attributs strictement positifs, comme proposé dans le Corollaire 1 est actuellement donné par la formule suivante:*

$$I(A) = \prod_{j=1}^{n} \exp \int_{\{1,\ldots,m\}} A_j \, dv \tag{1.4}$$

où $\int (\cdot)\, dv$ *est l'intégrale de Choquet en ce qui concerne* $v : E \in 2^{\{1,\ldots,m\}} \longrightarrow v(E) = \max \left\{ \sum_{i\in E} \underline{\delta_i}, 1 - \sum_{i \in E^c} \overline{\delta_i} \right\}$.

Corollaire 2 *Il semble que les planificateurs sociaux préfèrent peut-être envisager des limites symétriques quant à* $1/n$ *i.e. du type* $\underline{\delta_i} = 1/m - \varepsilon_i$, $\overline{\delta_i} = 1/m + \varepsilon_i$ *où* $\varepsilon_i \geq 0$ *pour tout* $i = 1, \ldots, m$. *Dans un tel cas, de simples comparaisons montrent que le Théorème 7 est valide si, et seulement si,* $0 \leq \varepsilon_i < 1 - 1/m$.

Rawls *versus* Harsanyi

La nouvelle proposition introduite dans le présent document peut être associée non seulement à l'idée d'aversion pour l'inégalité, mais exige également une approche normative sur la manière d'évaluer le développement humain du point de vue du planificateur social. Ce contexte implique invariablement une notion de justice et d'équité.

Afin d'éclairer cette discussion normative, nous revenons sur une discussion très intéressante qui a eu lieu dans les années 1970 entre John Rawls et John Harsanyi. Ils ont des points de vue assez différents sur la justice et, par conséquent, sur la manière de traiter l'inégalité, et ce différend, comme décrit par Moehler (2018), a approfondi et enrichi les recherches ultérieures depuis lors.

Pour Rawls, il s'agit d'une critique sévère de l'approche utilitariste, la principale préoccupation d'une société équitable devant être la fourniture de produits primaires à tous. Pour cette raison, les politiques publiques devraient accorder plus d'attention à la personne la plus défavorisée qu'au reste de la société.

D'un autre coté, Harsanyi était un fervent défenseur de la théorie de la justice des Rawls. Son approche suppose que les individus sont bayésiens et font leurs choix en fonction du risque ou de l'incertitude, attribuant des probabilités aux résultats et maximisant leur utilité attendue.

Notre exemple est utile pour décrire comment le théorème 6 peut représenter le point de vue de Harsanyi et Rawls dans une fonction d'évaluation sociale multidimensionnelle.

Notre fonction d'évaluation sociale traite les individus de la même façon que la théorie de Harsanyi, donnant le même poids à tous les individus que dans son modèle d'équiprobabilité. De plus, notre fonction d'évaluation sociale ouvre la possibilité de traiter les attributs comme dans la théorie de Rawls, c'est-à-dire que le planificateur social n'est pas sûr de la façon de peser les attributs, il donne plus d'importance aux résultats les plus défavorables.

1.2 Cycles d'affaires

1.2.1 Une politique budgétaire ambiguë

Introduction

La crise de 2008 a soulevé des questions sur la meilleure façon de promouvoir le développement durable à travers les politiques publiques. Les turbulences sur les marchés financiers créées par cette crise n'ont pas abouti à un consensus sur la manière de gérer la politique budgétaire. Ce contexte a élargi l'éventail des scénarios possibles et, par conséquent, accru l'incertitude quant à la politique budgétaire à mener.

À mesure que les risques augmentent, les agents deviennent plus attentifs aux décisions prises par les autorités. Il devient également plus important de prévoir ce que les gouvernements feront dans un proche avenir. Les ménages et les investisseurs craignent la possibilité d'une politique publique nuisant à leurs plans de consommation. En effet, de mauvaises politiques peuvent affecter la richesse future et les agents veulent anticiper ce type de scénario, empêchant les pertes futures.

En ce sens, l'incertitude gagne en pertinence pour comprendre les fluctuations de l'activité économique. Comment un ménage opposé à l'ambiguïté réagirait-il face à des informations intangibles contradictoires sur la future politique budgétaire? Cet chapitre vise à caractériser ce contexte dans un modèle New Keynesian, mettant en œuvre deux stratégies principales: 1) ajouter un choc d'ambiguïté *a la* Ilut and Schneider (2014) au choc de politique budgétaire et 2) ajouter un choc de prime de risque pour le risque sans liaisons comme proposé par Smets and Wouters (2007) et Fisher (2015).

Je suis Ilut and Schneider (2014) en modélisant les changements d'ambiguïté par des chocs de confiance. Les informations contradictoires qui soulèvent la perception d'ambiguïté sont perçues par les agents comme un choc de confiance. Toutefois, une baisse de la confiance des agents ne signifie pas que l'écart budgétaire se produira. Afin de comparer le choc de confiance suivi ou non d'un écart budgétaire, j'élargis l'analyse en ajoutant un autre choc qui capte la confiance des agents, de manière différente, indépendamment des dépenses publiques. Pour cela, j'introduis le choc de prime de risque, ou choc de liquidité comme dans Smets and Wouters (2007). Un choc positif de primes de risque fait augmenter la demande des agents en obligations sans risque. Ce mouvement peut être interprété comme un comportement de précaution des agents dont la décision d'exiger davantage d'obligations sans risque est perçue comme un moyen de réduire le risque de pertes futures.

L'hypothèse est que les agents opposés à l'ambiguïté ont des croyances multiples et agissent dans le pire des cas. Je définis ici la croyance la plus défavorable en considérant que les dépenses budgétaires non budgétisées de la part des décideurs politiques peuvent produire un rendement social plus faible que prévu, endommageant les comptes publics.

Un choc positif sur les primes de risque a un effet similaire lorsque les agents craignent que le gouvernement prenne une mauvaise décision à l'avenir. Une expansion de la demande d'obligations sans risque entraînera immédiatement une baisse correspondante de la consommation et des investissements, entraînant ainsi une baisse de la production. Ce comportement est également cohérent avec l'hypothèse ricardienne, c'est-à-dire que lorsque les agents craignent une future augmentation des impôts, ils visent à augmenter l'épargne actuelle pour lisser la consommation intertemporelle. La différence est que le gouvernement a introduit un motif préalable pour augmenter les impôts: les dépenses budgétaires non budgétisées.

Modèle

Je développe dans cet chapitre un nouveau modèle keynésien lancé par Christiano et al. (2005) et Smets and Wouters (2007) avec des prix et salaires échelonnés, des coûts d'ajustement des investissements et de nombreux éléments introduits dans la littérature depuis lors.

Les ménages du modèle sont représentés par une fonction d'utilité à vie avec un vecteur de variables d'état exogènes. Pour chaque période, les ménages projettent leur plan de consommation en choisissant le montant de la consommation, les heures travaillées et le montant des obligations d'État.

La formulation récursive est dynamiquement cohérente sur la base de l'approche introduite par Epstein and Schneider (2003). De plus, l'espérance rationnelle standard est obtenue comme un cas spécial si l'ensemble des probabilités conditionnelles ne contient qu'une seule croyance.

Le choc des primes de risque dénote une préférence stochastique pour la détention d'obligations sans risque émises par le gouvernement. La fonction de demande d'obligations représente la quantité réelle d'obligations d'État à une période achetées par les ménages. En outre, les ménages sont propriétaires du stock de capital installé, soumis aux coûts d'ajustement introduits par Christiano et al. (2005) et leur contrainte budgétaire compense les dépenses (consommation, investissement, taxes et obligations achetées) avec les revenus (obligations vendues, bénéfices en capital et salaires).

L'économie compte deux types de denrées périssables. Des entreprises parfaitement compétitives produisent des biens finaux qui sont fabriqués par la combinaison d'un continuum de biens intermédiaires. Les biens intermédiaires sont fabriqués par le capital, la main-d'œuvre et les biens finaux par des entreprises concurrentielles en situation de monopole utilisant une technologie d'agrégation Dixit-Stiglitz. L'entreprise de biens intermédiaires produit selon une fonction de production Cobb-Douglas avec une part de capital.

Je suis Calvo (1983) afin d'établir qu'un groupe aléatoire de bons producteurs intermédiaires réoptimise le prix à chaque période. Chaque fois que l'entreprise est en mesure de ré-optimiser son prix, elle maximise la valeur actualisée actuelle attendue des bénéfices futurs.

L'économie compte également deux types de main-d'œuvre: homogène et spécialisée. La première est proposée par les ≪agences pour l'emploi≫ et la seconde est demandée par ces agences. Comme dans Erceg et al. (2000), la main-d'œuvre spécialisée est fournie par un continuum de ménages dans un marché monopolistiquement concurrentiel et c'est la stratégie proposée par Smets and Wouters (2007) qui introduit la fixation de salaires échelonnés *a la* Calvo (1983).

Le secteur public de l'économie est composé de la Banque centrale et du gouvernement. La Banque centrale fixe le taux d'intérêt nominal concernant les écarts d'inflation par rapport à l'objectif et les écarts de production à son potentiel. La contrainte budgétaire à une période des pouvoirs publics compense les dépenses (dépenses publiques et remboursement des obligations vendues au cours de la période précédente) aux recettes (impôts perçus et obligations émises au cours de la période actuelle).

La règle budgétaire utilisée dans ce modèle stipule que la taxe forfaitaire est définie par la somme des obligations émises et des dépenses publiques, toutes deux réglementées par leurs réponses sensibles. Ces paramètres régissent la réponse de l'impôt aux émissions d'obligations et aux dépenses publiques, respectivement.

Enfin, la contrainte générale des ressources indique que la production est égale à la somme de la consommation privée, des investissements et des dépenses publiques.

Mécanisme du choc de confiance

Ce modèle permet aux agents d'avoir des croyances multiples sur les dépenses futures du gouvernement. Supposons donc que l'agent dispose d'informations contradictoires sur la question de savoir si le gouvernement poursuivra strictement le budget approuvé ou dépensera plus, au cours du trimestre suivant. Ces informations tronquées incitent les agents à considérer les dépenses publiques du trimestre suivant comme ambiguë. Une montée d'ambiguïté représente une augmentation du manque de confiance, ce qui explique le choc. Être opposé à l'ambiguïté signifie que les agents agissent comme si le pire des scénarios se produisait, c'est-à-dire que dans ce cas, le gouvernement dépenserait au-delà du budget.

Le terme μ_t^* mesure le manque de confiance. Par conséquent, la loi du mouvement sur les dépenses publiques, du point de vue des agents, est donnée par

$$\log\left(\frac{g_{t+1}}{\bar{g}}\right) = \rho_g \log\left(\frac{g_t}{\bar{g}}\right) + \mu_t^* + \upsilon_u \eta_{t+1}^u$$

où $\eta_t^u \sim \mathcal{N}(0,1)$.

La distribution de probabilité de l'ensemble de croyances exprimée par μ_t^* n'est pas connue, ce qui suggère pourquoi il est si difficile à prédire. Cet ensemble est désigné par,

$$\mu_t^* \in [a_t - 2|a_t|,\ a_t] \tag{1.5}$$

où a_t est défini comme l'ambiguïté variant dans le temps. Comme indiqué par Ilut and Schneider (2014), a_t représente un indicateur des informations intangibles disponibles à la date t sur les dépenses publiques à $t+1$.

Cet ensemble est symétrique, centré autour de zéro. Chaque élément de μ_t dans l'intervalle (1.5) est lié à une croyance contenue dans l'ensemble des probabilités conditionnelles de la fonction d'utilité des ménages. Lorsque cet ensemble n'a qu'une seule croyance, l'intervalle ne contiendra donc qu'un seul élément, caractérisant le paramètre d'attente rationnelle standard. Pourtant, plus les informations recueillies par les agents sont contradictoires, plus l'intervalle est long. L'ambiguïté variable dans le temps suit la loi du mouvement ci-dessous,

$$a_{t+1} - \bar{a} = \rho_a(a_t - \bar{a}) + \upsilon_a \eta^a_{t+1}$$

où $\eta^a_t \sim \mathcal{N}(0,1)$ et $\bar{a}$ est le niveau d'ambiguïté en régime permanent.

Le pire état stable des dépenses publiques en termes de trajectoire de production est,

$$g = \bar{g} + \frac{\bar{a}}{1-\rho_g}$$

Autrement dit, en l'absence d'ambiguïté, $g = \bar{g}$. Par exemple, supposons que $\bar{g} = 0,2$ (la part de la production du gouvernement représente 20% de la production), c'est-à-dire lorsque ces dépenses ne s'écartent pas du budget. Le pire état stable des dépenses publiques est alors la somme du paramètre non ambigu ($\bar{g}$) et du terme ambigu ($\bar{a}/(1-\rho_g)$). Plus $\bar{a}$ est élevé, plus la différence $g-\bar{g}$ est élevée. En d'autres termes, une grande ambiguïté de l'état d'équilibre signifie que les agents s'attendent à des dépenses publiques plus élevées que d'habitude.

Résultats

Il présente ici une comparaison du choc de confiance dans le pire des cas et du choc traditionnel de la politique budgétaire. Je compare également le choc de confiance avec le choc de la prime de risque. Enfin, est présenté l'effet de l'aversion au risque obligataire sur les principaux agrégats.

Du point de vue du pire des agents, la loi de mouvement de la politique budgétaire contient une composante ambiguë. L'objectif principal est de simuler un contexte dans lequel les agents perdent la confiance de la politique budgétaire au trimestre suivant. Les augmentations inattendues des dépenses publiques sont généralement considérées par les agents comme une mauvaise décision politique. Plus précisément, le pire scénario se produit lorsque $\mu^*_t = a_t$, c'est-à-dire l'écart maximal du budget public attendu par l'agent. Cet effet augmente les dépenses publiques, dans la pire des hypothèses des agents, et affecte le cycle économique.

Un choc de politique budgétaire augmente les dépenses publiques depuis le premier trimestre, générant un mouvement similaire de la production. La règle fiscale induit une augmentation respective de l'impôt et des obligations, ce qui réduit la consommation du ménage. Une offre plus importante d'obligations sans risque induit une hausse de leur taux d'intérêt, diminuant l'écart de taux d'intérêt. L'inflation et le coût marginal augmentent également, en opposition au choc de confiance.

La plupart des variables présentent une réponse inverse à une politique budgétaire positive et à un choc de confiance. En raison d'une augmentation de l'ambiguïté, la production, les heures et les taux d'intérêt (obligations et capital) commencent en dessous de l'état stationnaire.

D'un autre côté, l'investissement, la consommation et les salaires ont une réaction similaire aux deux chocs. L'investissement diminue en raison de l'effet d'éviction et la consommation suit la production et les heures diminuent avec une augmentation ultérieure des impôts.

Une autre comparaison intéressante est le choc de confiance et de prime de risque. Un choc positif de la prime de risque, entre autres possibilités, peut représenter une baisse de confiance des agents.

Une augmentation inattendue du choc budgétaire se traduit d'abord par une augmentation proportionnelle des dépenses publiques. De la contrainte des ressources, une augmentation des dépenses publiques génère une réduction actuelle de la consommation et de la production des ménages. L'investissement réagit à la perte de confiance, en baisse au cours des quelques premiers trimestres.

Le choc de confiance provoque au départ une déflation due à la baisse de la production. Puis, avec l'expansion ultérieure des dépenses publiques, l'inflation prend de l'ampleur, dépassant son état d'équilibre avant de converger vers l'équilibre.

En somme, une augmentation de l'ambiguïté a l'impact le plus négatif sur l'investissement et le capital parmi toutes les variables, indiquant que la confiance est un élément très important pour que les entrepreneurs décident d'investir ou non.

Considérons maintenant un effet similaire sur la confiance des agents, mais sans effet d'ambiguïté sur la réponse des dépenses publiques. Les gouvernements de mauvaise réputation appellent les agents à avoir une réaction de précaution pour atténuer le risque de perte. Cependant, ce comportement n'est pas toujours suivi d'une décision indésirable du gouvernement. Néanmoins, l'attitude adoptée par les agents est suffisante pour affecter le cycle économique. Ce choc positif sur la demande d'actifs sûrs et liquides peut être motivé par plusieurs caractéristiques. L'incertitude quant à la future politique budgétaire pourrait être considérée comme l'une des sources.

En comparant à la fois l'ambiguïté et les chocs de primes de risque, il est intéressant de voir que les deux génèrent une baisse de la production, de l'investissement et de la consommation. Malgré cela, ces chocs présentent certaines différences mécaniques.

Une augmentation de la demande d'obligations sans risque implique nécessairement une baisse de la consommation *via* la contrainte de ressources des ménages et également une baisse de la production par la contrainte de ressources. Du fait d'une baisse de la consommation, des investissements et de la production, l'inflation diminue également. Étant donné que les dépenses publiques augmentent l'inflation au-dessus de son état d'équilibre, les salaires réels baissent et mettent plus de temps à se rétablir en raison de l'adhésivité de Calvo.

En outre, comme l'a souligné Fisher (2015), le taux d'intérêt des obligations diminue à mesure que la demande d'obligations augmente. Le taux d'intérêt actuel sur le capital ne baisse qu'au début, comme l'a souligné Smets and Wouters (2007), et prend plus de temps à se rétablir par rapport au choc de confiance.

Étant donné que le choc des primes de risque, tel qu'introduit ici, est un élément récent de la littérature, j'analyse également l'effet du coefficient d'aversion au risque relatif sur les principaux agrégats et les variables de prix.

En bref, une augmentation de l'aversion au risque traduit une plus grande aversion au risque des agents, compte tenu du fait qu'ils sont moins disposés à prendre des risques compte tenu de la même richesse et du même contexte économique. Des valeurs plus élevées de l'aversion pour le risque peuvent avoir des effets négatifs sur la consommation, l'investissement, les heures travaillées et la production. Le fossé entre les taux d'intérêt augmente également, car l'augmentation de la demande d'obligations sans risque entraîne une baisse de leurs taux d'intérêt plus forte que le taux d'intérêt du capital pour les périodes suivantes.

Comme l'a souligné Fernández-Villaverde et al. (2015), la plupart des décisions économiques sont soumises à une incertitude omniprésente, en particulier à l'incertitude concernant la future politique budgétaire. Ce document apporte des preuves supplémentaires qu'une augmentation de l'incertitude de la politique budgétaire accroît l'effet négatif sur l'activité économique. En fait, il est important d'estimer les principaux paramètres pour comparer les résultats théoriques avec les données réelles.

Le présent document n'a pas l'intention de conclure que l'ambiguïté est la principale source de fluctuations du cycle économique, mais ces résultats mettent en évidence l'influence de l'ambiguïté sur l'activité économique.

Chapter 2

Introduction (English version)

This chapter aims at giving an extended introduction to the three subsequent chapters. The next section presents the main results obtained in the essays about inequality and section 2 the main results of the business cycle essay.

2.1 Inequality

The main subject of the two first chapters is human development and inequality.

2.1.1 Multidimensional Pigou-Dalton Transfers

Introduction

Multidimensional social evaluation functions has been gaining relevance during the last decades mainly due to new techniques that extend in the multidimensional setting the pioneering works by Atkinson (1970; 1987), Kolm (1976a;b; 1977) and Sen (1976). In particular, Tsui (1995; 1999) and Gajdos and Weymark (2005) have offered axiomatic approaches to designing income inequality measures in a multi-attribute context.

In this chapter, we follow the traditional additive approach and we confine ourselves to accommodating a particular Pigou-Dalton transfer, which we believe is relevant. Furthermore, our approach is consistent with the property of Correlation Increasing Majorization.

In actual fact, we investigate a class of utilitarian social evaluation functions that will prove to be inframodular. More precisely, this chapter aims at characterizing a class of inframodular functions initially proposed in the literature on decision under risk.

We also show that inframodular functions are intimately linked with a natural generalization to the multidimensional case of the classical unidimensional Pigou-Dalton transfers principle. This point will lead us to a key simple "transfers principle" axiom. Likewise, since inframodular functions are submodular, it turns out that our evaluation function will also accommodate the property of Correlation Increasing Majorization.

Finally, we specify our social evaluation function in order to compare it with the famous HDI by using actual data.

Notation, motivation and axioms

We consider n individuals $1, \ldots, j, \ldots, n$ and $A_j \in \mathbb{R}^m$ is the column-vector of the m attributes of this individual $A_j = (a_{1j}, \ldots, a_{ij}, \ldots, a_{mj})^t$; indeed, the same m attributes are considered for each individual.

Henceforth, for n given individuals, $A = (A_1, \ldots, A_j, \ldots, A_n)$ is the $m \times n$ matrix summarizing the considered population. Note that $\mathcal{A}$ will denote the set of such real matrices A with m rows and n columns.

In the unidimensional case, when considering n individuals $1, \ldots, j, \ldots, n$ with incomes $x_1 \leq x_2 \leq \ldots \leq x_j \leq \ldots \leq x_n$, it is usually assumed that if $x_j < x_{j+1}$ then a transfer $\varepsilon > 0$ from individual $j+1$, to individual j such that $x_j + \varepsilon \leq x_{j+1} - \varepsilon$ reduces inequality. This transfer is a Pigou-Dalton transfer.

Thus turning to the m-dimensional case where each individual j has a column-vector $A_j \in \mathbb{R}^m$ of m attributes, we get that the $m \times n$ matrix $A = (A_1, \ldots, A_j, \ldots, A_n)$ summarizes the data.

Our goal is to axiomatize additive social evaluation functions, i.e., social evaluation functions $I : \mathcal{A} \longrightarrow \mathbb{R}$ such that $I(A) = \sum_{j=1}^{n} u(A_j)$, for all $A \in \mathcal{A}$, where $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ agrees with a diminishing social evaluation in the case of such a Pigou-Dalton regressive transfer as above. It is immediate that for all $(X, Y) \in \mathbb{R}^m \times \mathbb{R}^m$ such that $X \leq Y$ and for all $\varepsilon \in \mathbb{R}^m_+$, u should satisfy:

$$u(X) - u(X - \varepsilon) \geq u(Y + \varepsilon) - u(Y) \tag{2.1}$$

This is the usual property of concavity in the *one*-dimensional case, at least when u is continuous. Actually, we show that *inframodular functions* satisfy the desired property (2.1), and are consistent with Correlation Increasing Majorization. Let us recall:

Definition 1 *A function $u : \mathbb{R}^m \to \mathbb{R}$ is said to be inframodular if:*

$$u(X + \varepsilon) - u(X) \geq u(Y + \varepsilon) - u(Y)$$

for all $X, Y \in \mathbb{R}^m$ with $X \leq Y$ and $\varepsilon \in \mathbb{R}^m_+$.

We now switch to the axioms which will be considered to model the preferences $\succsim$ of the "social planner" (or "ethical observer") for global welfare taking into account that inequalities have a negative impact on welfare, but also that all attributes are "positive" that is, an increase in some attribute has a positive effect on welfare.

Thus, $\succsim$ is a preference relation on $\mathcal{A}$ (if $\mathcal{A}_+$ or $\mathcal{A}_{++}$ is considered — this is specified in the theorems). Indeed, for $A, B \in \mathcal{A}$, $A \succsim B$ means that A is weakly preferred to B, $A \succ B$ means A is strictly preferred to B, $A \sim B$ means A and B are considered as equivalent by the social planner. Note that these definitions are consistent since $\succsim$ will be assumed to be a weak order.

A.1 $\succsim$ is a Weak Order; i.e., $\succsim$ is a transitive, complete hence also a reflexive binary relation on $\mathcal{A}$.

A.2 Continuity: Let $B \in \mathcal{A}$ be given, then $\{A \in \mathcal{A} | A \succsim B\}$ and $\{A \in \mathcal{A} | B \succsim A\}$ is closed in the usual topology of $\mathbb{R}^{m \times n}$.

A.3 Monotonicity: For all $A, B \in \mathcal{A}$, $a_{ij} \geq b_{ij}$ for all i, j, implies $A \succsim B$. If furthermore $A \neq B$, then $A \succ B$.

A.4 Independence: For all j and (A, B); $(A_j, A_{-j}) \succsim (A_j^{'}, A_{-j}) \iff (A_j, B_{-j}) \succsim (A_j^{'}, B_{-j})$.

A.5 Anonymity: For any permutation matrix Π and for all $A \in \mathcal{A}$, one has $A \sim A\Pi$; i.e., $(A_1, \ldots, A_j, \ldots, A_n) \sim \left(A_{\sigma(1)}, \ldots, A_{\sigma(j)}, \ldots, A_{\sigma(n)}\right)$ where $\sigma : [1, n] \to [1, n]$ is a bijection.

A.6 Additivity: For all A, A_j, B_j, C_j; $(A_j, A_{-j}) \sim (B_j, A_{-j}) \implies (A_j + C_j, A_{-j}) \sim (B_j + C_j, A_{-j})$.

A.7 Pigou-Dalton Principle: Let $A = (A_1, \ldots, A_j, \ldots, A_n)$ such that for some j_1, j_2 one has $A_{j_1} \leq A_{j_2}$ and let $\varepsilon \in \mathbb{R}_+^m$ then:

$A \succsim \big(A_1, \ldots, A_{j_{1-1}}, A_{j_1} - \varepsilon, A_{j_{1+1}}, \ldots, A_{j_{2-1}}, A_{j_2} + \varepsilon, A_{j_{2+1}}, \ldots, A_n\big) = A_\varepsilon$; furthermore, $A \succ A_\varepsilon$ if $\varepsilon \in \mathbb{R}_+^m$ and $\varepsilon \neq 0$.

Multidimensional social evaluation functions

Theorem 1 offers an axiomatization of the additive social evaluation function.

Theorem 1 *A preference relation $\succsim$ on $\mathcal{A}$ satisfies A.1, A.2, A.3, A.4 and A.5 if and only if there exists: $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ increasing and continuous, such that:*

$$\text{For all } A, B \in \mathcal{A},\ A \succsim B \iff \sum_{j=1}^{n} u(A_j) \geq \sum_{j=1}^{n} u(B_j)$$

where u is defined up to a positive affine transformation.

We come now to the main result of this chapter in which we characterize social evaluation functions built upon the special type of inframodular functions.

Theorem 2 *A preference relation on $\mathcal{A}$ satisfies A.1, A.2, A.3, A.4, A.5, A.6 and A.7 if and only if there exists $\alpha_i > 0$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} \alpha_i = 1$ and there exists $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous such that:*

$$A \succsim B \iff \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) \geq \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot b_{ij}).$$

Furthermore, such α_i's are unique and ψ is defined up to an increasing affine transformation.

Specification of ψ and a relative inequality index

To specify ψ, we introduce two axioms:

A.8 Absolute Invariance: For all $A, B \in \mathcal{A}$ and for all $\lambda \in \mathbb{R}$, $A \sim B \iff A + \lambda \mathbb{1} \sim B + \lambda \mathbb{1}$ where $\mathbb{1}$ is the matrix $m \times n$ with 1 everywhere.

A.9 Relative Invariance: For all $A, B \in \mathcal{A}_{++}$ and for all $\lambda > 0$, $A \sim B \iff \lambda A \sim \lambda B$.

Theorem 3 *Assume that the preference relation $\succsim$ satisfies A.1 to A.7, then:*

· up to an increasing affine transformation $\psi(t) = -e^{-at}$ with $a > 0$ if and only if A.8 is satisfied when $\succsim$ is defined on $\mathcal{A}$.

· up to an increasing affine transformation either $\psi(t) = \ln(t)$, for all $t > 0$ or $\psi(t) = t^a$, for all $t > 0$ where $a \neq 0$, $a < 1$ if and only if A.9 is satisfied when $\succsim$ is defined on $\mathcal{A}_{++}$.

Remark Notice that in cases in which all attributes are strictly positive, and if we adopt axiom *A.9* then for all $A \in \mathcal{A}_{++}$ one could adopt the social evaluation function $J(A) = \prod_{j=1}^{n}(\sum_{i=1}^{m} \alpha_i \cdot a_{ij})$.

Indeed, in such a case:

$$I(A) = \sum_{j=1}^{n} \ln(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) = \ln(\prod_{j=1}^{n}(\sum_{i=1}^{m} \alpha_i \cdot a_{ij})).$$

We focus on the relative inequality index, which is linked with the choice of $\psi(\cdot) = \ln(\cdot)$. This index appears to be one of the most tractable and relevant in our framework.

Corollary of Theorem 3

The corresponding inequality index related to the social evaluation function defined on $\mathcal{A}_{++}$, the set of $m \times n$ matrices with positive elements, satisfying *A.1* to *A.7* and *A.9* with $\psi(t) = \ln(t)$, with $t > 0$ is relative and has the form

$$1 - \left(\prod_{j=1}^{n} \frac{\sum_{i=1}^{m} \alpha_i \cdot a_{ij}}{\sum_{i=1}^{m} \alpha_i \cdot \mu_i} \right)^{\frac{1}{n}}$$

where μ_i, is the mean of i_{th} attribute in which $i = 1, \dots, m$.

Agreeing with correlation increasing majorization.

Correlation Increasing Majorization (*CIM*) is a concept due to Boland and Proschan (1988) and introduced into the inequality literature by Tsui (1999).

Let us introduce some definitions (see Boland and Proschan, 1988).

Concept of Correlation Increasing Transfer (*CIT*)

Let $A, B \in \mathcal{A}$, then B is obtained from A by a *CIT* if there exists j_1, j_2 where $j_1 \neq j_2$ such that $B_{j_1} = A_{j_1} \wedge A_{j_2}$ and $B_{j_2} = A_{j_1} \vee A_{j_2}$.

A *CIT* is strict whenever neither $A_{j_1} \leq A_{j_2}$ nor $A_{j_2} \leq A_{j_1}$ happen.

Concept of Correlation Increasing Majorization (*CIM*)

Let $A, B \in \mathcal{A}$, then $A >_c B$, i.e., A is strictly less unequal for the *CIM* if B may be derived from A by a permutation of columns and a finite sequence of Correlation Increasing Transfers with at least one being strict.

We can now state and prove that our social evaluation functional of Theorem 2 as well as any strict inframodular social functional respects *CIM*.

Note that this is the case for the inframodular function in the Theorem 2.

Theorem 4 *Any strict inframodular social evaluation functional respects CIM.*

Empirical Analysis

Based on *Theorem 3*, we specialize $\psi(\cdot)$ as $\ln(\cdot)$, thus, considering cases in which all of the attributes are strictly positive, we adopt, for all $A \in \mathcal{A}_{++}$, $J(A)$ as a 'mean' social evaluation function.

We aim at evaluating the pertinence of this inframodular function using data. In order to test it, we decided to make a comparison with another function that is not inframodular, namely the famous Human Development Index (HDI).

Our index needs the individual-level information to be built. Thereby, we decide to perform this comparison with the Brazilian national exam for high school students, called ENEM (Exame Nacional do Ensino Médio). To apply $H(\cdot)$ with these data, we focused our analysis on three attributes: natural sciences (a^s), languages (a^l) and mathematics (a^m).

The population (n) is the number of students in each town. Following HDI rules, here we also give the same weight to the attributes. The function $J(A)$ in this case, can be written as,

$$J(A) = \frac{1}{n}\sum_{j=1}^{n} \ln\left(\frac{a_j^s + a_j^l + a_j^m}{3}\right).$$

Since HDI provides an outcome between 0 and 1, we decided to extract the 'certainty equivalent' of $J(A)$, i.e., $I(A) = \exp^{J(A)}$. Then, now we have both indexes delivering results in the range $[0, 1]$.

It is widely known that classical HDI formula does not consider in its calculation the level of inequality within a country. However, we are interested in contrast with the classical HDI to detect to what extent $I(A)$ is influenced by the 'intra' inequalities.

Firstly, the difference between the $I(A)$ and $H(A)$ outcomes is relatively small. Their correlation coefficient is $0,999$. The similarity of the outcomes is suitable because it shows that this function provides the outcomes in a similar sense as HDI usually does.

In other words, we want to see whether inequality is positively correlated with $H(A) - I(A)$. To measure the inequality in this case, we summed the values of the attributes for each student and extracted the standard deviation of this transformed variable. We want to analyze the correlation between these two variables to assert whether $I(A)$ takes inequality into account or not.

We thus computed a linear regression as a hypothetical exercise. The $R^2 = 0.9592$ attests that standard deviation explains strongly $H(A) - I(A)$ behavior. The positive relationship between the variables is quite substantial.

Therefore, based on these results we suggest that this function could be a good alternative to IHDI.

2.1.2 Rawlsian Inequality-Adjusted Social Evaluation Function

Introduction

Is social inequality important to measure social evaluation? If yes, what should be the weight of inequality in a social evaluation function? To answer these questions, it is essential to understand the role of inequality in human development. Should public policies induce a reduction of inequality to improve global welfare?

The related literature presents a wide range of possibilities in which inequality might hamper human development. In the opposite sense, some theoretical results suggest that inequality can also be positive for economic growth.

Therefore, inequality *per se* is treated here as an important element of a negative impact on human development.

Social inequality and human development have often been measured essentially in terms of wealth or income. This kind of indicator should be supplemented by other attributes as health and literacy, for example. As Tsui (1999) points out, fast economic growth may not necessarily be accompanied by improvements in health and education.

Accordingly, to evaluate living standards more broadly, the Human Development Index (HDI) was proposed. However, HDI does not capture inequality in the population. To consider inequality in human development, the Inequality-adjusted Human Development Index (IHDI) was created, where the inequality level is measured to "penalize" HDI outcome.

In this chapter, we aim at characterizing a class of multidimensional social evaluation function which contemplates inequality, both among individuals and among attributes. It is provided an axiomatization to this social evaluation function. We also show that IHDI is a special case of the class of the social evaluation function that we axiomatize.

Moreover, we propose a well-being index that leads to a situation in which the social planner is unsure about how to assign weight to attributes when they are strongly complementary.

Framework and axioms

Let us consider a finite population $\mathcal{J} = \{1, \ldots, j, \ldots, n\}$ of individuals and a finite set of attributes $\mathcal{I} = \{1, \ldots, i, \ldots, m\}$. Each individual $j \in \mathcal{J}$ is endowed with m attributes represented by a (row) vector $a_i \in \mathbb{R}^m$.

A real matrix A is an $m \times n$ array of real numbers. Let $\mathbb{M}_{m\times n}$ be the vector space of all m-by-n real matrices. Also, $\mathbb{M}^{+}_{m\times n}$ and $\mathbb{M}^{++}_{m\times n}$ are subsets of $\mathbb{M}_{m\times n}$ of matrices with non-negative and positive elements, respectively. We note that a population of n individuals endowed with m attributes can be represented by a matrix $A \in \mathbb{M}_{m\times n}$ with m rows and n columns.

Given a matrix $A \in \mathbb{R}^{m\times n}(\equiv \mathbb{M}_{m\times n})$, we define $A_j \in \mathbb{R}^m(\equiv \mathbb{M}_{m\times 1})$ as the vector (matrix) of all attributes of the individual $j \in \mathcal{J}$, represented by the jth column of A. Also, $A_i \in \mathbb{R}^n(\equiv \mathbb{M}_{1\times n})$ is defined as the vector (matrix) of the attribute $i \in \mathcal{I}$ across all individuals, represented by the ith row of A. A_j may be interpreted as an individual, or a subgroup of society. Similarly, A_i may be interpreted as an attribute, or a subgroup of attributes. Yet, denote the matrix $(A^{'}_j, A_{-j})$, in which column A_j is replaced by column $A^{'}_j$ in the matrix A.

We next present the axioms that characterize this social planner's preference for human development.

Consider a social planner who governs a society (a country or a city, for example) and her goal is to improve the human development of this society. This social planner has her preferences about the way to carry out social policies to deal with inequality and improve human development.

A social planner's preference is given by a binary relation $\succsim$ on $\mathbb{M}_{m\times n}$. For $A, B \in \mathbb{M}_{m\times n}$, $A \succsim B$ means that A is weakly preferred to B, $A \succ B$ means A is strictly preferred to B, $A \sim B$ means A and B are considered as indifferent by the social planner.

A.1 Weak Order: $\succsim$ is a complete and transitive binary relation on $\mathbb{M}_{m\times n}$. That is, for all $A, B, C \in \mathbb{M}_{m\times n}$, $A \succsim B$ or $B \succsim A$ (completeness). Moreover, if $A \succsim B$ and $B \succsim C$ then $A \succsim C$ (transitivity).

A.2 Continuity: Given $B \in \mathbb{M}_{m\times n}$, $\{A \in \mathbb{M}_{m\times n} | A \succsim B\}$ and $\{A \in \mathbb{M}_{m\times n} | B \succsim A\}$ are closed in $\mathbb{M}_{m\times n}$.

A.3 Monotonicity: For all $A, B \in \mathbb{M}_{m\times n}$, and for all i, j, if $a_{ij} \geq b_{ij}$ then $A \succsim B$. If furthermore $A \neq B$, then $A \succ B$.

A.4 Independence: For all $A, B \in \mathbb{M}_{m\times n}$ and taking $j \in \{1, \ldots, n\}$ fixed with $A'_j \in \mathbb{M}_{m\times 1}$; $(A_j, A_{-j}) \succsim (A^{'}_j, A_{-j}) \iff (A_j, B_{-j}) \succsim (A^{'}_j, B_{-j})$.

A.5 Anonymity: For all $A \in \mathbb{M}_{m\times n}$ and for all bijection $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$, one has $(A_1,\ldots,A_j,\ldots,A_n) \sim \big(A_{\sigma(1)},\ldots,A_{\sigma(j)},\ldots,A_{\sigma(n)}\big)$.

A.6 Strongly Homothetic: For all $A \in \mathbb{M}^+_{m\times n}$ and for all $B_j, C_j \in \mathbb{M}^+_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (C_j \otimes A_j, A_{-j}) \succsim (C_j \otimes B_j, A_{-j})$.

A.7 Pigou-Dalton Principle: Let $A = (A_1,\ldots,A_j,\ldots,A_n)$ such that for some $j_1, j_2 \in \{1,\ldots,n\}$ one has $A_{j_1} \leq A_{j_2}$ and let $\varepsilon \in \mathbb{R}^m_+$ then:

$A \succsim \big(A_1,\ldots,A_{j_{1-1}}, A_{j_1}-\varepsilon, A_{j_{1+1}},\ldots,A_{j_{2-1}}, A_{j_2}+\varepsilon, A_{j_{2+1}},\ldots,A_n\big) =: A_\varepsilon$. Furthermore, $A \succ A_\varepsilon$ if $\varepsilon \in \mathbb{R}^m_+$ and $\varepsilon \neq 0$.

Multidimensional social evaluation function

This section presents the first result.

Theorem 5 *A preference relation $\succsim$ on $\mathbb{M}_{m\times n}$ satisfies A.1 to A.7 if, and only if, there exists $\alpha \in \Delta^{m-1}_{++}$ and there exists $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous such that:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi\Big(\prod_{i=1}^{m} a_{ij}^{\alpha_i}\Big) \geq \sum_{j=1}^{n} \psi\Big(\prod_{i=1}^{m} b_{ij}^{\alpha_i}\Big).$$

Furthermore, such α_i's are unique and ψ is defined up to an increasing affine transformation.

The social planner first assigns a weight for each attribute and aggregates them through a Cobb-Douglas rule. Then she applies a function ψ and adds all the individuals.

We next introduce another axiom to formulate a proposition that helps us to demonstrate the relation of Theorem 5 with the IHDI formulation.

A.8 Scale Invariance: For all $A, B \in \mathbb{M}^+_{m\times n}$ and $k \in \mathbb{R}_{++}$; $A \succsim B \implies k\cdot A \succsim k\cdot B$.

Proposition 2 *A preference relation $\succsim$ on $\mathbb{M}^+_{m\times n}$ satisfies A.1 to A.8 if, and only if, up to an increasing affine transformation, either $\psi(t) = \ln(t)$ or $\psi(t) = t^\beta$ for all $t > 0$, where $\beta \in (0,1)$*

$$I(A) = \left(\prod_{j=1}^{n}\prod_{i=1}^{m} a_{ij}^{\alpha_j}\right)^{\frac{1}{n}}$$

Likewise, taking $\psi(\cdot) = \ln(\cdot)$ and considering that all the attributes are strictly positive. Suppose also that all the attributes have the same weight. Moreover, since the sum of logs is the log of product, we get,

$$I(A) = \left[\prod_{j=1}^{n}\left(\prod_{i=1}^{m} a_{ij}\right)^{\frac{1}{m}}\right]^{\frac{1}{n}} \tag{2.2}$$

where $I(A) \in [0,1]$. Taking $m = 3$ we find that the condition (2.2) is equal to IHDI formula.

Social Evaluation Function: A new proposal

What happens if the social planner is not sure about the importance of the attributes chosen to assess a society's welfare? We characterize this context in Example A showing that the Strongly Homothetic axiom might be violated.

It means that the index formulated in Theorem 5 does not capture this kind of behavior. One way to characterize this behavior is to weaken the strongly Homothetic axiom and we thus propose a new social evaluation function.

Rawlsian Inequality-Adjusted Social Evaluation Function

Since Strong Homotheticity is violated, we replace this axiom by another three axioms: Homotheticity *(A.9)*, Log-Convexity *(A.10)* and Power Invariance *(A.11)*:

A.9 Homotheticity: For all $A \in \mathbb{M}^{+}_{m\times n}$, $k \in \mathbb{R}_{++}$ and for all $A_j, B_j \in \mathbb{M}^{+}_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (k \cdot A_j, A_{-j}) \succsim (k \cdot B_j, A_{-j})$.

A.10 Log-Convexity: If for all $A \in \mathbb{M}^{+}_{m\times n}$, $\lambda \in (0,1)$ and for all $A_j, B_j \in \mathbb{M}^{+}_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (A_j^{\lambda} \otimes B_j^{1-\lambda}, A_{-j}) \succsim (B_j, A_{-j})$.

A.11 Power Invariance: If for all $A \in \mathbb{M}^{+}_{m\times n}$, $k \in \mathbb{R}_{++}$ and for all $A_j, B_j \in \mathbb{M}^{+}_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (A_j^{k}, A_{-j}) \succsim \left(B_j^{k}, A_{-j}\right)$.

Theorem 6 *A preference relation $\succsim$ on $\mathbb{M}_{m\times n}$ satisfies A.1 to A.5, A.7 and A.9 to A.11 if, and only if, there exists a unique, non-empty, closed and convex set $C \subset \Delta$ of probability measures such that $\alpha \in \Delta^{m-1}_{++}$ and there exists $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous such that:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi\left(\min_{\alpha\in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i}\right) \geq \sum_{j=1}^{n} \psi\left(\min_{\alpha\in C} \prod_{i=1}^{m} b_{ij}^{\alpha_i}\right).$$

Furthermore, ψ is defined up to an increasing affine transformation.

In addition, Strong Homotheticity axiom in Theorem 6 would imply that the set C has a unique probability measure.

In order to be able to compute our social evaluation function, we now specify $\psi(\cdot) = \ln(\cdot)$. It turns out that simple computations as for Proposition 2 lead to Corollary 7:

Corollary 7 *A preference relation $\succsim$ on $\mathbb{M}^{+}_{m\times n}$ satisfies A.1 to A.5, A.7 and A.9 to A.11 if, and only if, there exists a unique, closed and convex set $C \subset \Delta$ of probability measures such that $\alpha \in \Delta^{m-1}_{++}$ and $A \succsim B$ if, and only if, $I(A) \geq I(B)$ where for all A, B with strictly positive attributes:*

$$I(A) = \left(\prod_{j=1}^{n} \min_{\alpha\in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i}\right)^{\frac{1}{n}}$$

and similar formula for $I(B)$.

Example B

Suppose the social planner is no longer sure about the best way to assign weights to education sub-components. She has no doubt that the education attributes should jointly represent 50% of the measurement. she believes that their weights are between $[1/6, 1/3]$. Furthermore, she follows the social evaluation function provided in Corollary 7, i.e. let us define $a_j^{'} := a_{hj}^{\frac{1}{4}} a_{wj}^{\frac{1}{4}} \left(a_{e_1 j}^{\frac{1}{6}} a_{e_2 j}^{\frac{1}{3}} \right)$ and $a_j^{''} := a_{hj}^{\frac{1}{4}} a_{wj}^{\frac{1}{4}} \left(a_{e_1 j}^{\frac{1}{3}} a_{e_2 j}^{\frac{1}{6}} \right)$. Rewriting the index,

$$I(A) = \left[\prod_{j=1}^{n} \min \left(a_j^{'}, a_j^{''} \right) \right]^{\frac{1}{n}}.$$

The higher is her dubiety to assign weights to the attributes, the higher is the range of weight possibilities.

We complete this section by proposing a final result that allows us to deal with simple meaningful cases, which furthermore appears to be tractably computable for applications.

Theorem 8 *Assume that the social planner is only able to give an upper and a lower bound respectively a_i and b_i for the weight α_i of each attribute $i = 1, \ldots, m$, where $0 < \underline{\delta_i} \leq \overline{\delta_i} < 1$. Then the social planner is consistent, i.e. there exists $\alpha_i \geq 0$ such that $\underline{\delta_i} \leq \alpha_i \leq \overline{\delta_i}$ for all i and $\sum_{i=1}^{m} \alpha_i = 1$ if, and only if,*

$$\sum_{i=1}^{m} \underline{\delta_i} \leq 1 \leq \sum_{i=1}^{m} \overline{\delta_i} \tag{2.3}$$

Furthermore, the social evaluation function $I(A)$ of any A with strictly positive attributes, as proposed in Corollary 7 is now given by the following formula:

$$I(A) = \prod_{j=1}^{n} \exp \int_{\{1,\ldots,m\}} A_j \; dv \tag{2.4}$$

where $\int (\cdot) \; dv$ is the Choquet integral with respect to $v : E \in 2^{\{1,\ldots,m\}} \longrightarrow v(E) = \max \left\{ \sum_{i \in E} \underline{\delta_i}, 1 - \sum_{i \in E^c} \overline{\delta_i} \right\}$.

Corollary 9 *It appears that social planners might prefer to envision bounds symmetrical with respect to $1/n$ i.e. of the type $\underline{\delta_i} = 1/m - \varepsilon_i$, $\overline{\delta_i} = 1/m + \varepsilon_i$ where $\varepsilon_i \geq 0$ for all $i = 1, \ldots, m$. In such a case, simple comparisons show that Theorem 8 is valid if, and only if, $0 \leq \varepsilon_i < 1 - 1/m$.*

Rawls *versus* Harsanyi

The new proposal introduced in this chapter may be associated not only with the idea of inequality aversion, but also requires a normative approach on how to assess human development from the social planner's perspective. This context involves a notion of justice and fairness, invariably.

To shed some light on this normative discussion, we revisit a very interesting discussion that occurred in the 1970's between John Rawls and John Harsanyi. They have quite different views about justice and, consequently, on how to deal with inequality, and this dispute, as described by Moehler (2018), has deepened and enriched subsequent researches since then.

For Rawls, a sharp critic of utilitarianism approach, the main concern of a fair society should be the provision of primary goods for all. For that reason, public policies should give more attention to the worst off individual than to the rest of society.

On the other hand, Harsanyi was a harsh critic of Rawls' theory of justice. His approach assumes that individuals are Bayesian and make their choices based on risk or uncertainty, assigning probabilities to outcomes and maximizing their expected utility.

Our example is useful to describe how theorem 2 may represent Harsanyi and Rawls' point of view in a multidimensional social evaluation function.

Our social evaluation function treats individuals similarly to Harsanyi's theory, giving the same weight to all individuals as in his equiprobability model. Additionally, our social evaluation function opens up the possibility to treat attributes as in Rawls's theory, i.e. since the social planner is unsure about how to weigh attributes, she gives more importance to the worst off outcome.

2.2 Business Cycles

2.2.1 Ambiguous Fiscal Policy

Introduction

The 2008 crisis has raised questions about the best way to promote sustainable development through public policies. The turmoil in financial markets created by this crisis did not lead to a consensus on how to manage fiscal policy. This context expanded the range of possible scenarios and, consequently, increased the uncertainty about the fiscal policy to be carried out.

As risks increase, agents become more alert about decisions taken by authorities. It also becomes more important to foresee what governments will do in the near future. Households and investors fear the possibility of a public policy harming their consumption plans. Indeed, bad policies may affect future wealth and agents want to anticipate this kind of scenario, preventing future losses.

In this sense, uncertainty has been gaining relevance to understanding fluctuations in economic activity. How would an ambiguity averse household react when facing conflicting intangible information about future fiscal policy? This chapter aims at characterizing this context in a New Keynesian model, implementing two main strategies: 1) adding a shock of ambiguity *a la* Ilut and Schneider (2014) into fiscal policy shock and 2) adding a risk premium shock for risk-free bonds as proposed by Smets and Wouters (2007) and Fisher (2015).

I follow Ilut and Schneider (2014) modeling changes in ambiguity through shocks to confidence. Conflicting information that raises the perception of ambiguity is perceived by agents as a shock of confidence. However, a drop in agents' confidence does not mean that the budget deviation will occur. In order to compare the shock of confidence followed or not by a budget deviation, I expand the analysis adding another shock that captures agents' confidence, in a different way, independently of the government expenditure. For that, I introduce the risk premium shock, or liquidity shock as in Smets and Wouters (2007). A positive shock of risk premium raise agents' demand for risk-free bonds. This movement can be interpreted as a precautionary behavior of the agents whose decision to demand more risk-free bonds is perceived as a way to reduce the risk of future losses.

The hypothesis is that ambiguity averse agents have multiple beliefs and act under the worst-case belief. I define the worst-case belief here by considering that non-budgeted fiscal expenditures on the part of policymakers may produce a lower than expected return to society, damaging public accounts.

A positive risk premium shock has a similar effect when agents fear that the government will make a bad decision in the future. An expansion of risk-free bonds demand will immediately generate a corresponding fall in consumption and investment, consequently dropping output. This behavior is also coherent with the Ricardian hypothesis, i.e. when agents fear a future increase in taxes they aim to increase current savings to smooth intertemporal consumption. The difference is that a prior motive is being inputted for the government to increase taxes: non-budgeted fiscal spending.

Model

I develop in this chapter a New Keynesian model as pioneered by Christiano et al. (2005) and Smets and Wouters (2007) with staggered prices and wages, investment adjustment cost and many elements introduced in the literature since then.

Households in the model are represented by a lifetime utility function with a vector of exogenous state variables. For every period, households project their consumption plan by choosing the amount of consumption, hours worked and the amount of government bonds.

The recursive formulation is dynamically consistent based on the approach introduced by Epstein and Schneider (2003). Moreover, the standard rational expectation is obtained as a special case if the set of conditional probabilities contains only one belief.

The risk premium shock denotes a stochastic preference for holding risk-free bonds issued by the government. The demand function for bonds represents the real quantity of one-period government bonds purchased by the households. Besides, households own the installed capital stock, subject to adjustment costs as introduced by Christiano et al. (2005) and their budget constraint offsets spending (consumption, investment, taxes, and bonds purchased) with revenues (bonds sold, capital profits, and wages).

The economy has two types of perishable goods. Perfectly competitive firms produce final goods that are made by the combination of a continuum of intermediate goods. Intermediate goods are made through the capital, labor, and final goods by monopolistically competitive firms using a Dixit-Stiglitz aggregation technology. Intermediate goods firm produces according to a Cobb-Douglas production function with capital share.

I follow Calvo (1983) to establish that a random group of intermediate good producers re-optimizes the price at every period. Whenever the firm is able to re-optimize its price, it maximizes the expected present discounted value of future profits.

The economy has also two types of labor: homogeneous and specialized. The first is offered by 'employment agencies' and the second is demanded by those agencies. As in Erceg et al. (2000), specialized labor is supplied by a continuum of households in a monopolistically competitive market and it is adopted here the strategy proposed by Smets and Wouters (2007) introducing staggered wages setting *a la* Calvo (1983).

The economy's public sector is composed of the Central Bank and the Government. The Central Bank sets the nominal interest rate concerning deviations of inflation from the target and output deviations from its potential. Government one-period budget constraint offsets spending (government expenditures and repayment of bonds sold in the previous period) with revenues (taxes collected and bonds issued in the current period).

The fiscal rule used in this model states that lump-sum tax is defined by the sum of bonds issued and government spending, both regulated by their sensitive responses. These parameters govern the response of tax to bonds issuance and government spending, respectively.

Finally, the general resource constraint states that output is equal to the sum of private consumption, investments, and government expenditures.

Mechanism of the confidence shock

This model allows agents to have multiple beliefs about future government expenditures. Suppose thus the agent has conflicting information about whether the government will strictly pursue the approved budget or spend more, in the following quarter. This truncated information induces agents to treat the following quarter's government expenditure as ambiguous. A raise of ambiguity represents an increase in the lack of confidence, which explains the shock. Being ambiguity averse means that agents act as if the worst-case scenario will happen, i.e., in this case, the government will spend beyond the budget.

The term μ_t^* measures the lack of confidence. Hence, government spending law of motion, from the perspective of the agents, is given by

$$\log\left(\frac{g_{t+1}}{\bar{g}}\right) = \rho_g \log\left(\frac{g_t}{\bar{g}}\right) + \mu_t^* + \upsilon_u \eta_{t+1}^u$$

where $\eta_t^u \sim \mathcal{N}(0,1)$.

It is not known the probability distribution of the belief set expressed by μ_t^*, suggesting why it is so hard to predict. This set is denoted by,

$$\mu_t^* \in [a_t - 2|a_t|,\ a_t] \tag{2.5}$$

where a_t is defined as the time-varying ambiguity. As pointed out by Ilut and Schneider (2014), a_t represents an indicator of intangible information available at date t about government expenditure at $t+1$.

This set is symmetric, centered around zero. Each element of μ_t in the interval (2.5) is related to one belief contained in the set of conditional probabilities in the households' utility function. When this set has only one belief, the interval will thus contain also only one element, characterizing the standard rational expectation setting. Yet, the more conflicting is the information gathered by agents, the greater the interval is. The time-varying ambiguity follows the law of motion below,

$$a_{t+1} - \bar{a} = \rho_a(a_t - \bar{a}) + \upsilon_a \eta_{t+1}^a$$

where $\eta_t^a \sim \mathcal{N}(0,1)$ and $\bar{a}$ is the steady state ambiguity level.

The worst-case steady state of government spending in terms of output path is,

$$g = \bar{g} + \frac{\bar{a}}{1-\rho_g}$$

That is, in the absence of ambiguity, $g = \bar{g}$. For example, suppose that $\bar{g} = 0.2$ (government's output share represents 20% of output), i.e. when such spending does not deviate from budget. Then the worst-case steady state of government spending is the sum of the unambiguous parameter ($\bar{g}$) and the ambiguous term ($\bar{a}/(1-\rho_g)$). The higher $\bar{a}$ is, the higher is the difference $g - \bar{g}$. In other words, a large steady state ambiguity means that agents expect government expenditures greater than usual.

Results

It is presented here a comparison of the confidence shock in the worst-case setup and traditional fiscal policy shock. I also compare the confidence shock with the risk premium shock. Finally, is presented the effect of the bonds risk aversion on the main aggregates.

From the agents' worst-case perspective, the law of motion of fiscal policy contains an ambiguous component. The main goal is to simulate a context in which agents lose the confidence of the fiscal policy in the following quarter. Unexpected increases in government expenditures are usually considered by agents as a bad political decision. More specifically, the worst scenario happens when $\mu_t^* = a_t$, i.e. the maximum deviation of the public budget expected by the agent. This effect increases government expenditures, in the agents' worst-case perspective, and affects the business cycle.

A fiscal policy shock increases government expenditure since the first quarter, generating a similar movement of output. The fiscal rule induces a respective increase in tax and bonds, which reduces the household's consumption. A larger supply of risk-free bonds, induces a rise in their interest rate, decreasing the interest rate spread. Inflation and marginal cost also increase, in opposition to the confidence shock.

Most of the variables present an inverse response to a positive fiscal policy and a confidence shock. Because of an increase of ambiguity, output, hours and the interest rates (bonds and capital) starts below the steady state.

On the other hand, investment, consumption, and wages have a similar reaction to both shocks. Investment drops as a result of crowding out effect and consumption follows output and hours decrease with a subsequent rise of taxes.

Another interesting comparison is the confidence and risk premium shock. A positive shock of risk premium, among other possibilities, may represent a fall of agents' confidence.

An unexpected increase of the fiscal shock is firstly reflected in a proportional increase in government expenditure. From the resource constraint, an increase in government expenditure generates a current reduction of household consumption and output. Investment reacts to the loss of confidence, falling during the few first quarters.

Confidence shock causes deflation at the beginning due to the drop in output. Then with the subsequent expansion of government spending, inflation gains momentum, surpassing its steady state before converging to equilibrium.

In sum, an increase of ambiguity has the most negative impact on investment and capital among all variables, denoting that confidence is a quite important element for entrepreneurs to decide whether to invest or not.

Consider now a similar effect on agents' confidence, but without ambiguity effect on government spending response. Governments with bad reputation call agents to have a precautional reaction to mitigate the risk of loss. However, this behavior is not always followed by a government undesirable decision. Nevertheless, the attitude taken by agents is enough to affect the business cycle. This positive shock to demand for safe and liquidity assets may be motivated by several features. Uncertainty about future fiscal policy might be considered as one of the sources.

In comparing both ambiguity and risk premium shocks, is interesting to see that both generate a comovement fall in output, investment, and consumption. Despite that, such shocks have some mechanic differences.

An increase in demand for risk-free bonds implies necessarily a fall of consumption *via* household's resource constraint and also a fall of output through resource constraint. As a result of a decrease in consumption, investment, and output, inflation also reduces. Since government spending increases inflation above its steady state, real wages decrease and takes longer to recover because of Calvo stickiness.

Furthermore, as pointed out by Fisher (2015), the interest rate of bonds decreases as demand for bonds increases. The current capital interest rate drops only in the beginning, as pointed out by Smets and Wouters (2007), and takes longer to recover compared with the confidence shock.

Since the risk premium shock, as introduced here, is a recent element in the literature, I also analyze the effect of the relative risk aversion coefficient on the main aggregates and price variables.

Briefly, an increase of the risk aversion reflects a greater aversion to risk by the agents, taking into account that they are less willing to take risks considering the same wealth and the same economic context. Larger values of the risk aversion potentialize negative effects on consumption, investment, hours worked and output. The wedge between the interest rates also increases, since the increment in the demand for risk-free bonds causes their interest rates to fall sharper than the capital interest rate for the subsequent periods.

As Fernández-Villaverde et al. (2015) pointed out, most of the economic decision-making is subject to pervasive uncertainty, in particular, to uncertainty about future fiscal policy. This chapter contributes with further evidence that an increase in fiscal policy uncertainty increases the negative effect on economic activity. In fact, it is important to estimate the main parameters to compare the theoretical results with actual data.

This chapter does not intend to conclude that ambiguity is the main source of business cycle fluctuations, but these findings highlight the influence of ambiguity on economic activity.

Chapter 3

Multidimensional Pigou-Dalton tranfers and social evaluation functions

This chapter is a result of the article "Multidimensional Pigou-Dalton transfers and social evaluation functions", made in collaboration with Alain Chateauneuf, Marcelo Basili, and Maurizio Franzini.

Abstract

We axiomatize, in the multidimensional case, a social evaluation function that can accommodate a natural Pigou-Dalton principle and correlation increasing majorization. This is performed by building upon a simple class of inframodular functions proposed by Müller and Scarsini under risk.

Keywords: Multidimensional inequality, Pigou-Dalton transfer, Inframodular functions, Increasing majorization, Human Development Index.

3.1 Introduction

There has been a resurgence of interest in multidimensional social evaluation functions mainly due to new techniques that extend in the multidimensional setting the pioneering works by Atkinson (1970; 1987), Kolm (1976a;b; 1977) and Sen (1976). In particular, Tsui (1995; 1999) and Gajdos and Weymark (2005) have offered axiomatic approaches to designing income inequality measures in a multiattribute context.

Although Tsui mainly used the additive approach, Gajdos and Weymark built upon the generalized Gini social function. These two different approaches are not at all innocuous. The former aggregates the attributes of each individual and then additively aggregates the resulting values; the latter evaluates the different attributes through a specific aggregation and then simply aggregates the values.

Note also that although Tsui's (1999) approach is a 'traditional' additive evaluation, Gajdos and Weymark (2005) adopted a non-additive approach, which was introduced by Weymark in his seminal paper in 1981.

In this chapter, we follow the traditional additive approach, but instead of imposing the majorization theory of the m-dimensional case as in Tsui (1995), we confine ourselves to accommodating a particular Pigou-Dalton transfer, which we believe is relevant. Furthermore, our approach is consistent with the meaningful property of Correlation Increasing Majorization (e.g. Tsui, 1999).

In actual fact, we investigate a class of utilitarian social evaluation functions (that consist in aggregating the individual utilities), where the utility function, the same for all the individuals and which depends on several attributes, will prove to be inframodular (a natural multivariate generalization of the notion of concavity). More precisely, the chapter aims at characterizing a class of inframodular functions initially proposed in the literature on decision under risk, by Müller and Scarsini (2012).

Letting $x = (x_1, \dots, x_m)$ be the list of endowments of an individual in each of the m attributes, the chapter investigates the class of inframodular utility functions that can be written as $u(x) = \psi\left(\sum_{i=1}^{m} \alpha_i x_i\right)$ where ψ is increasing and concave, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Notice that such a utility function requires the commensurability of different dimensions. Additionally, from Theorem 1 in Section 3.3, the fact that u is defined up to a positive affine transformation suggests that the underlying attributes are cardinally measurable. Although the same criticisms can be raised against the standard Human Development Index (HDI) approach, we have the feeling that such limitations allow however to treat most of the interesting situations.

Moreover, we show that inframodular functions are intimately linked with a natural generalization to the multidimensional case of the classical unidimensional Pigou-Dalton transfers principle, which allows one to consider situations where some of the welfare attributes are not transferable (to be compared with Bosmans et al. (2009)). This point will lead us to a key simple 'transfers principle' axiom. Likewise, since inframodular functions are submodular, it turns out that our evaluation function will also accommodate the property of Correlation Increasing Majorization.

Finally, we specify our social evaluation function to compare it with the famous HDI by using actual data.

The chapter is organized as follows. Section 3.2 presents the notation and also introduces the chapter's motivation and axioms. Section 3.3 offers our main theorem, specifically Theorem 2 which characterizes our social evaluation function. Section 3.4 aims at reducing the number of parameters, namely, specifying some fundamental ψ function, for our purpose. Thus, we deliver a tractable relative inequality index in the corollary of Theorem 3. Section 3.5 shows that our evaluation function can accommodate Correlation Increasing Majorization. Section 3.6 illustrates using Brazilian data in what way the current methodology differs from the popular HDI. Finally, Section 3.7 concludes and some proofs can be found in Appendix A and Appendix B.

3.2 Notation, motivation and axioms

We consider n individuals $1, \ldots, j, \ldots, n$ and $A_j \in \mathbb{R}^m$ is the column-vector of the m attributes of this individual $A_j = \begin{pmatrix} a_{1j} \\ . \\ a_{ij} \\ . \\ a_{mj} \end{pmatrix}$; indeed, the same m attributes are considered for each individual[1].

Henceforth, for n given individuals, $A = (A_1, \ldots, A_j, \ldots, A_n)$ is the $m \times n$ matrix summarizing the considered population. Note that $\mathcal{A}$ will denote the set of such real matrices A with m rows and n columns. $\mathcal{A}_+$ and $\mathcal{A}_{++}$ will denote the subset of $\mathcal{A}$ consisting of matrices with non negative elements and positive elements, respectively.

In the unidimensional case, when considering n individuals $1, \ldots, j, \ldots, n$ with incomes $x_1 \leq x_2 \leq \ldots \leq x_j \leq \ldots \leq x_n$, it is usually assumed that if $x_j < x_{j+1}$ then a transfer $\varepsilon > 0$ from individual $j+1$, to individual j such that $x_j + \varepsilon \leq x_{j+1} - \varepsilon$ reduces inequality. This transfer is a Pigou-Dalton transfer. Note that this is equivalent to assuming that if $x_j \leq x_{j+1}$, then modifying x_j into $x_j - \varepsilon$ and x_{j+1} into $x_{j+1} + \varepsilon$ increases inequality.

Thus turning to the m-dimensional case with $m \geq 1$, which is the topic of this chapter, where each individual j has a column-vector $A_j \in \mathbb{R}^m$ of m attributes, we get that the $m \times n$ matrix $A = (A_1, \ldots, A_j, \ldots, A_n)$ summarizes the data. Imagine that for two individuals their respective column attributes are X and Y, with $X \leq Y$ (i.e. $x_i \leq y_i$, for all $i = 1, \ldots, m$) and let $\varepsilon \in \mathbb{R}^m_+$ with $\varepsilon \neq 0$. In such a situation, the transfer of ε from X to Y will be a regressive Pigou-Dalton transfer because it would increase inequalities[2].

Our goal is to axiomatize additive social evaluation functions, i.e., social evaluation functions $I : \mathcal{A} \longrightarrow \mathbb{R}$ such that $I(A) = \sum_{j=1}^{n} u(A_j)$, for all $A \in \mathcal{A}$, where $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ agrees with a diminishing social evaluation in the case of such a Pigou-Dalton regressive transfer as above. It is immediate that for all $(X, Y) \in \mathbb{R}^m \times \mathbb{R}^m$ such that $X \leq Y$ and for all $\varepsilon \in \mathbb{R}^m_+$, u should satisfy:

$$u(X) - u(X - \varepsilon) \geq u(Y + \varepsilon) - u(Y) \tag{3.1}$$

This is the usual property of concavity in the *one*-dimensional case, at least when u is continuous. Actually, we show (see Lemma 1 in Appendix B) that *inframodular functions* (see Definition 2 below), extensively studied by Marinacci and Montrucchio (2005) and proposed by Müller and Scarsini (2012) as a meaningful representation of risk aversion in the multidimensional case, satisfy

[1]Note that throughout the chapter we assume $n \geq 3$. Indeed, to obtain an additive representation in a parsimonious way through the classical independence axiom as in Debreu's theorem (1960), it is known that $n > 2$ is required, which in any case appears to make sense for applications.

[2]Since this chapter has been performed, we are aware of a similar Pigou-Dalton principle introduced by Bosmans et al. (2009). Nevertheless main differences persist between the two chapters: our definition is model-free and our main motivation is to link this principle to inframodularity.

the desired property (3.1), and are consistent with the relevant property of Correlation Increasing Majorization[3]. Let us recall:

Definition 2 *A function $u : \mathbb{R}^m \to \mathbb{R}$ is said to be inframodular if:*

$$u(X+\varepsilon) - u(X) \geq u(Y+\varepsilon) - u(Y)$$

for all $X, Y \in \mathbb{R}^m$ with $X \leq Y$ and $\varepsilon \in \mathbb{R}^m_+$.

Inframodular functions may not be concave (e.g. Marinacci and Montrucchio, 2005); therefore, as observed by Müller and Scarsini (2001; 2012), inframodular functions do not match the property of risk aversion, which states that adding a random vector E with mean 0 to a constant multivariate vector is always unfavorable. Note also that multidimensional concave functions may not be inframodular. A typical example is the three-dimensional HDI function[4] $v : (x_1, x_2, x_3) \in [0,1]^3 \longrightarrow v(x_1, x_2, x_3) = x_1^{1/3} x_2^{1/3} x_3^{1/3}$. Thereby, an additive social evaluation function based on HDI might not respect the natural multidimensional Pigou-Dalton principle evoked above. As an example let us consider two individuals with row attributes respectively $X = (0.8, 0.64, 0.729)$ and $Y = (0.9, 0.81, 0.729)$ and let us transfer $\varepsilon = (0, 0, 0.1)$ from X to Y. A simple computation delivers $v(X-\varepsilon) + v(Y+\varepsilon) > v(X) + v(Y)$.

In actual fact the standard additive version of HDI would rank equally (X, Y) and $(X-\varepsilon, Y+\varepsilon)$ since:

$$HDI(X,Y) = v\left(\frac{X+Y}{2}\right) = v\left(\frac{X-\varepsilon+Y+\varepsilon}{2}\right) = HDI(X-\varepsilon, Y+\varepsilon)$$

As a consequence, this index would not take into account what appears as a clear deterioration of the social function with respect to inequality, when modifying (X, Y) into $(X-\varepsilon, Y+\varepsilon)$.

This contrasts with the additive social evaluation function that we propose in this chapter (e.g. Theorem 3 and Section 3.6), namely $I(X,Y) = u(X) + u(Y)$ where $u(x_1, x_2, x_3) = \ln \dfrac{x_1 + x_2 + x_3}{3}$ which delivers,

$$I(X,Y) = \ln(0.587) > I(X-\varepsilon, Y+\varepsilon) = \ln(0.583)$$

hence a ranking in accordance with the intuition.

As pointed out in the introduction, we intend to test the pertinence of inframodular social evaluations when compared to HDI. This explains why we focus on particularly tractable and meaningful inframodular functions u as proposed by Müller and Scarsini (2001; 2012) through:

$$u(x_1, \ldots, x_i, \ldots, x_m) = \psi\left(\sum_{i=1}^{m} \alpha_i x_i\right) \tag{3.2}$$

where ψ is increasing, concave and $\alpha_1, \ldots, \alpha_m \geq 0$.

Observe that such an u is a valuable function for our purpose, since, first, it makes sense to weight the different attributes in accordance with their importance $\alpha_i \geq 0$, and second, with such a ψ concave, the resulting u is *inframodular*, so it agrees with our definition of increasing inequality.

[3] Let $(X, Y) \in \mathbb{R}^m \times \mathbb{R}^m$, $X \wedge Y = (\ldots, \min(x_i, y_i), \ldots)$, $X \vee Y = (\ldots, \max(x_i, y_i), \ldots)$. Correlation Increasing Majorization stipulates the meaningful requirement that replacing two individuals endowed initially with X and Y by individuals endowed with $X \wedge Y$ and $X \vee Y$ increases inequality. Since an inframodular function u is submodular i.e. $u(X) + u(Y) \geq u(X \wedge Y) + u(X \vee Y)$, one gets this property.

[4] In 2010 the HDI functional has changed its additive form to a multiplicative form as introduced above. Section 3.6 includes a discussion about this 'new' HDI. Details are in Zambrano (2014).

Accordingly, in this chapter, we mainly axiomatize the social evaluation function of the type given by (3.2), present the natural usual axioms aiming to specify ψ, and propose a simple relative inequality index. Moreover, as mentioned in the Introduction, our social evaluation function is proved to satisfy the condition of Correlation Increasing Majorization.

We now switch to the axioms which will be considered to model the preferences $\succsim$ of the "social planner" (or "ethical observer") for global welfare taking into account that inequalities have a negative impact on welfare, but also that all attributes are "positive" that is, an increase in some attribute has a positive effect on welfare.

Thus, $\succsim$ is a preference relation on $\mathcal{A}$ (if $\mathcal{A}_+$ or $\mathcal{A}_{++}$ is considered — this is specified in the theorems). Indeed, for $A, B \in \mathcal{A}$, $A \succsim B$ means that A is weakly preferred to B, $A \succ B$ means A is strictly preferred to B, $A \sim B$ means A and B are considered as equivalent by the social planner. Note that these definitions are consistent since $\succsim$ will be assumed to be a weak order.

The first three axioms are standard; therefore, they do not require a particular explanation:

A.1 $\succsim$ is a Weak Order; i.e., $\succsim$ is a transitive, complete hence also a reflexive binary relation on $\mathcal{A}$.

A.2 Continuity: Let $B \in \mathcal{A}$ be given, then $\{A \in \mathcal{A} | A \succsim B\}$ and $\{A \in \mathcal{A} | B \succsim A\}$ is closed in the usual topology of $\mathbb{R}^{m \times n}$.

A.3 Monotonicity: For all $A, B \in \mathcal{A}$, $a_{ij} \geq b_{ij}$ for all i, j, implies $A \succsim B$. If furthermore $A \neq B$, then $A \succ B$.

For $A \in \mathcal{A}$ and $A_j^{'}$ a column of $\mathbb{R}^m$, $(A_j^{'}, A_{-j})$ denotes the matrix A where column A_j has been replaced by column $A_j^{'}$. Thus, the classical independence axiom states that the impact for the ranking of replacing a given individual by another one is the same if all the other individuals remain unchanged.

A.4 Independence: For all j and (A, B); $(A_j, A_{-j}) \succsim (A_j^{'}, A_{-j}) \iff (A_j, B_{-j}) \succsim (A_j^{'}, B_{-j})$.

Below is the usual anonymity axiom which states that the value of a distribution does not depend on the identity; only the value of the attributes matters.

A.5 Anonymity: For any permutation matrix Π and for all $A \in \mathcal{A}$, one has $A \sim A\Pi$; i.e., $(A_1, \ldots, A_j, \ldots, A_n) \sim \left(A_{\sigma(1)}, \ldots, A_{\sigma(j)}, \ldots, A_{\sigma(n)}\right)$ where $\sigma : [1, n] \to [1, n]$ is a bijection.

The additivity axiom $A.6$ allows through the independence axiom $A.4$ to show that the common preferences of individuals over $\mathbb{R}^m$ are additive in the sense of axiom[5] $A^*.6$, which is similar to the one used in Theorem $A2.1$ of (Wakker, 1989; p.161). Thus we obtain what Peter Wakker quoted as "de Finetti Theorem".

Roughly speaking, the additivity axiom states that if the social planner considers that, for a given individual, two given vectors of attributes are similarly valuable (this property being consistently defined through the independence axiom *A.4*) then they will remain similarly valuable if the same vector of attributes is added to both of them.

A.6 Additivity: For all A, A_j, B_j, C_j; $(A_j, A_{-j}) \sim (B_j, A_{-j}) \implies (A_j + C_j, A_{-j}) \sim (B_j + C_j, A_{-j})$.

Finally, axiom $A.7$ is crucial for our purpose. The following Pigou-Dalton principle is the direct translation of the fact that if for an individual j_1 all the attributes are smaller than for another one j_2, then, for any i, transferring a value $\varepsilon_i \geq 0$ of the attribute i from j_1 to j_2, clearly should increase the inequality (strictly increase if some $\varepsilon_i > 0$), leading to a worse social or welfare situation.

Indeed, the global progressive transfer $\varepsilon \in \mathbb{R}_+^m$ from individual j_1 to individual j_2 is "rank-preserving" in the sense that the individual j_2 who is initially richer than the individual j_1 in any

[5]See the Proof of Theorem 2, in Section 3.3

attribute remains richer when compared to j_1 after the transfer.

A.7 Pigou-Dalton Principle: Let $A = (A_1, \ldots, A_j, \ldots, A_n)$ such that for some j_1, j_2 one has $A_{j_1} \leq A_{j_2}$ and let $\varepsilon \in \mathbb{R}^m_+$ then:

$A \succsim \left(A_1, \ldots, A_{j_{1-1}}, A_{j_1} - \varepsilon, A_{j_{1+1}}, \ldots, A_{j_{2-1}}, A_{j_2} + \varepsilon, A_{j_{2+1}}, \ldots, A_n\right) = A_\varepsilon$; furthermore, $A \succ A_\varepsilon$ if $\varepsilon \in \mathbb{R}^m_+$ and $\varepsilon \neq 0$.

3.3 Multidimensional social evaluation functions

In this Section, multidimensional social evaluation functions are defined and characterized. Theorem 1 offers an axiomatization of the additive social evaluation function.

Theorem 1 *A preference relation $\succsim$ on $\mathcal{A}$ satisfies A.1, A.2, A.3, A.4 and A.5 if and only if there exists: $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ increasing and continuous, such that:*

$$\text{For all } A, B \in \mathcal{A},\ A \succsim B \Longleftrightarrow \sum_{j=1}^{n} u(A_j) \geq \sum_{j=1}^{n} u(B_j)$$

where u is defined up to a positive affine transformation (the proof is in Appendix A).

We come now to the main result of this chapter in which we characterize social evaluation functions built upon the special type of inframodular functions satisfying (3.2) as proposed in a different framework by Müller and Scarsini. Such a social function agrees with our Pigou-Dalton principle *A.7* and with the property of Correlation Increasing Majorization as shown respectively in Section 3.5 and Appendix B.

In this chapter, we consider situations where the relevant attributes are commensurable and cardinally measurable. Therefore it might make sense to first summarize the list of attributes as a weighted sum of these attributes, with positive weights summing to 1. Thus allowing each weight to value the relative importance of the corresponding attribute, this explains why in Theorem 2 below $\alpha_i > 0$ for all i and $\sum_{i=1}^{m} \alpha_i = 1$.

Theorem 2 *A preference relation on $\mathcal{A}$ satisfies A.1, A.2, A.3, A.4, A.5, A.6 and A.7 if and only if there exists $\alpha_i > 0$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} \alpha_i = 1$ and there exists $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous such that:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) \geq \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot b_{ij}).$$

Furthermore, such α_i's are unique and ψ is defined up to an increasing affine transformation.

Proof. The necessary part of the proof is straightforward since inframodular functions satisfy *A.7* (see Lemma 1 in Appendix B); thus we confine ourselves to provide the sufficiency part of the proof. From Theorem 1, we already know that the "social evaluation" of each individual j, by the social planner, is identically given by u up to a positive affine transformation, henceforth the corresponding induced preference relations $\succsim_j$ on $\mathbb{R}^m$ of every individual j are identical.

Let us denote $\overset{*}{\succsim}$ this common preference relation and let us show that there exists $\alpha_i > 0$, for all $i = 1, \ldots, m$, with $\sum_{i=1}^m \alpha_i = 1$ such that for all $(X, Y) \in \mathbb{R}^m \times \mathbb{R}^m$,

$$X \overset{*}{\succsim} Y \Longleftrightarrow \sum_{i=1}^{m} \alpha_i x_i \geq \sum_{i=1}^{m} \alpha_i y_i.$$

Note that, *A.1, A.2, A.3,* and *A.6* imply that $\overset{*}{\succsim}$ satisfies:

A.1*: $\overset{*}{\succsim}$ is a weak order

A.2*: Continuity: $X^{(p)}, X, Y \in \mathbb{R}^m$ then[6]

A.2.1*: $X^{(p)} \overset{*}{\succsim} Y$, for all p, $X^{(p)} \downarrow X \Longrightarrow X \overset{*}{\succsim} Y$

A.2.2*: as $Y \overset{*}{\succsim} X^{(p)}$, for all p, $X^{(p)} \uparrow X \Longrightarrow Y \overset{*}{\succsim} X$

A.3*: Monotonicity: $X, Y \in \mathbb{R}^m$, $X \geq Y \Longrightarrow X \overset{*}{\succsim} Y$, furthermore if $X \neq Y \Longrightarrow X \overset{*}{\succ} Y$

A.6*: Additivity: For all $X, Y, Z \in \mathbb{R}^m$, $X \overset{*}{\sim} Y \Longrightarrow X + Z \overset{*}{\sim} Y + Z$

For $X \in \mathbb{R}^m$ denote $K(X) :\equiv Inf \left\{ x \in \mathbb{R} \mid x \cdot \mathbb{1} \overset{*}{\succsim} X \right\}$, it is easy to see that $K(X)$ exists in $\mathbb{R}$, that $X \overset{*}{\succsim} Y$ *if and only if* $K(X) \geq K(Y)$ and that $K(x \cdot \mathbb{1}) = x$,

for all $x \in \mathbb{R}$, where indeed $\mathbb{1} = \begin{pmatrix} 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \in \mathbb{R}^m$

Let e_i be the i_{th} vector of the canonical basis of $\mathbb{R}^m$, i.e., $e_i = \begin{pmatrix} 0 \\ . \\ 1 \\ . \\ 0 \end{pmatrix}$ the i_{th} column, and let $\alpha_i :\equiv K(e_i)$, from *A*.3*, since $K(\mathbf{0}) = 0$, one gets $\alpha_i > 0$.

We now intend to show that

$$K(X) = \sum_{i=1}^{m} \alpha_i x_i \tag{3.3}$$

Note that since $K(\mathbb{1}) = 1$ this will entail $\sum_{i=1}^m \alpha_i = 1$.

To prove (3.3), let us show first that: for all $Y, Z \in \mathbb{R}^m$ one has

$$K(Y + Z) = K(Y) + K(Z) \tag{3.4}$$

Since $Y \overset{*}{\sim} K(Y) \cdot \mathbb{1}$ and $Z \overset{*}{\sim} K(Z) \cdot \mathbb{1}$, then *A*.6* implies (3.4), since $Y + Z \overset{*}{\sim} K(Y) \cdot \mathbb{1} + Z$ and $K(Y) \cdot \mathbb{1} + Z \overset{*}{\sim} (K(Y) + K(Z)) \cdot \mathbb{1}$, gives $Y + Z \overset{*}{\sim} (K(Y) + K(Z)) \cdot \mathbb{1}$

It turns out that $K(X) = \sum_{i=1}^m K(x_i \cdot e_i)$. It remains to show that $K(x_i \cdot e_i) = x_i \cdot K(e_i)$.

[6]Indeed, $X^{(p)} \downarrow X$ and $X^{(p)} \uparrow X$ means that the sequences $X^{(p)}$ are respectively decreasing (increasing) with respect to the point-wise order in $\mathbb{R}^m$ while converging towards X.

It is enough to prove that[7]:

$$K(x \cdot X) = x \cdot K(X)$$

for all $x \in \mathbb{R}$ and for all $X \in \mathbb{R}^m$.

This has been already proved for $x = 0$. So let us assume $x \in \mathbb{R}^*$.

Assume first $x \in \mathbb{Q}^*$ i.e. $x = {}^p/_q$, $p \in \mathbb{N}^*$, $q \in \mathbb{Z}^*$.

From (3.4) we have $K({}^p/_q \cdot X) = K(p \cdot {}^X/_q) = p \cdot K({}^X/_q)$ but $K(X) = K(q \cdot {}^X/_q) = q \cdot K({}^X/_q)$.

Therefore, $K({}^p/_q \cdot X) = {}^p/_q \cdot K(X)$.

From *A*.2* it is simple to see that $X_n \downarrow X \Longrightarrow K(X_n) \downarrow K(X)$ and $X_n \uparrow X \Longrightarrow K(X_n) \uparrow K(X)$, and that K is monotone i.e. $X \geq Y \Longrightarrow K(X) \geq K(Y)$.

Hence, let us consider now that $x \in \mathbb{R}$ and $x_n \in \mathbb{Q}$, such that $x_n \downarrow x$, and $y_n \in \mathbb{Q}$, such that $y_n \uparrow x$.

We know that from *A*.3* we can write $x_n \cdot X \geq x \cdot X \geq y_n \cdot X$ and it implies that $x_n \cdot X \overset{*}{\succsim} x \cdot X \overset{*}{\succsim} y_n \cdot X$. Then $K(x_n \cdot X) \geq K(x \cdot X) \geq K(y_n \cdot X)$.

Therefore, $x_n \cdot K(X) \geq K(x \cdot X) \geq y_n \cdot K(X)$ for all n, so letting $n \longrightarrow +\infty$, we get $K(x \cdot X) = x \cdot K(X)$, which completes the fact that:

$$X \overset{*}{\succsim} Y \textit{ if and only if } \sum_{i=1}^{m} \alpha_i x_i \geq \sum_{i=1}^{m} \alpha_i y_i.$$

We end the proof by showing that up to an increasing affine transformation there exists a unique $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave, and continuous such that for all $A, B \in \mathcal{A}$:

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) \geq \sum_{j=1}^{n} \psi(\sum_{i=1}^{m} \alpha_i \cdot b_{ij}).$$

From Theorem 1 there exists - up to a positive affine transformation - a unique u increasing and continuous such that, for all $X, Y \in \mathbb{R}^m$,

$$X \overset{*}{\succsim} Y \Longleftrightarrow u(X) \geq u(Y).$$

Since $K : \mathbb{R}^m \to \mathbb{R}$ is also a strictly increasing and continuous representation of $\overset{*}{\succsim}$, there exists up to a positive affine transformation a strictly increasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that $u = \psi \circ K$.

It remains to be proved that ψ is strictly concave.

It is enough to show that for all $(a, b) \in \mathbb{R}^2$, $a \leq b$ and for all $\varepsilon > 0$ one has $\psi(a) - \psi(a - \varepsilon) > \psi(b + \varepsilon) - \psi(b)$.

It is immediate to find $X, Y \in \mathbb{R}^m$ such that $X \leq Y$, $a = \sum_{i=1}^{m} \alpha_i x_i$ and $b = \sum_{i=1}^{m} \alpha_i y_i$, then from *A.7*, we get

$$u(X) - u(X - \varepsilon \mathbb{1}) > u(Y + \varepsilon \mathbb{1}) - u(Y).$$

Therefore, $\psi(a) - \psi(a - \varepsilon) > \psi(b + \varepsilon) - \psi(b)$, which completes the proof of Theorem 2 ■

[7] Indeed, below $\mathbb{Z}, \mathbb{N}$ and $\mathbb{Q}$ (respectively $\mathbb{Z}^*, \mathbb{N}^*$ and $\mathbb{Q}^*$) denote as usually the set of integers, non-negative integers, rational numbers (respectively non-null elements in $\mathbb{Z}, \mathbb{N}$ and $\mathbb{Q}$).

3.4 Specification of ψ and a relative inequality index

To specify ψ, we introduce two axioms that have a long tradition in the literature; see, for instance, Kolm (1976a;b) and, more recently, Gajdos and Weymark (2005).

A.8 Absolute Invariance: For all $A, B \in \mathcal{A}$ and for all $\lambda \in \mathbb{R}$, $A \sim B \Longleftrightarrow A + \lambda \mathbb{1} \sim B + \lambda \mathbb{1}$ where $\mathbb{1}$ is the matrix $m \times n$ with 1 everywhere.

This axiom usually called "absolute invariance" expresses that the inequalities remain unchanged if the same amount is added to all attributes and all individuals.

The following axiom usually called "relative invariance", which applies only if all the attributes are strictly positive, i.e., $A \in \mathcal{A}_{++}$ expresses that inequalities remain unchanged if all attributes are multiplied by the same positive number $\lambda > 0$ for all individuals.

A.9 Relative Invariance: For all $A, B \in \mathcal{A}_{++}$ and for all $\lambda > 0$, $A \sim B \Longleftrightarrow \lambda A \sim \lambda B$.

Theorem 3 *Assume that the preference relation $\succsim$ satisfies A.1 to A.7, then:*

· up to an increasing affine transformation $\psi(t) = -e^{-at}$ with $a > 0$ if and only if A.8 is satisfied when $\succsim$ is defined on $\mathcal{A}$.

· up to an increasing affine transformation either $\psi(t) = \ln(t)$, for all $t > 0$ or $\psi(t) = t^a$, for all $t > 0$ where $a \neq 0$, $a < 1$ if and only if A.9 is satisfied when $\succsim$ is defined on $\mathcal{A}_{++}$.

Proof. We prove only the *if* part; the *only if* part is straightforward.

Assume *A.8* is satisfied. It is easy to see that if $\sum_{j=1}^n \psi(x_j) = \sum_{j=1}^n \psi(y_j)$ where $x_j, y_j \in \mathbb{R}$, we must have $\sum_{j=1}^n \psi(x_j + k) = \sum_{j=1}^n \psi(y_j + k)$ for all $k \in \mathbb{R}$.

Therefore we can apply the results of the classical *one*-dimensional social welfare theory (see e.g. Kolm (1976a) or else Aczél (1966)) to get the desired result.

Now, assume *A.9* is satisfied and $\succsim$ is defined on $\mathcal{A}_{++}$. It is easy to see that $\sum_{j=1}^n \psi(x_j) = \sum_{j=1}^n \psi(y_j)$ where $x_j > 0$ and $y_j > 0$, for all j implies $\sum_{j=1}^n \psi(\lambda x_j) = \sum_{j=1}^n \psi(\lambda y_j)$ for all $\lambda > 0$. Then we can apply the results of the classical *one*-dimensional social welfare theory[8] ∎

Remark Notice that in cases in which all attributes are strictly positive, and if we adopt axiom *A.9* then for all $A \in \mathcal{A}_{++}$ one could adopt the social evaluation function $J(A) = \prod_{j=1}^n (\sum_{i=1}^m \alpha_i \cdot a_{ij})$.

Indeed, in such a case:

$$I(A) = \sum_{j=1}^{n} \ln(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) = \ln(\prod_{j=1}^{n} (\sum_{i=1}^{m} \alpha_i \cdot a_{ij})).$$

We focus on the relative inequality index, which is linked with the choice of $\psi(\cdot) = \ln(\cdot)$. This index appears to be one of the most tractable and relevant in our framework.

3.4.1 Corollary of Theorem 3

The corresponding inequality index related to the social evaluation function defined on $\mathcal{A}_{++}$, the set of $m \times n$ matrices with positive elements, satisfying *A.1* to *A.7* and *A.9* with $\psi(t) = \ln(t)$, with

[8] See e.g. Kolm (1976a;b), Atkinson (1970) or else Aczél (1966) to get this result.

$t > 0$ is relative and has the form

$$1 - \left(\prod_{j=1}^{n} \frac{\sum\limits_{i=1}^{m} \alpha_i \cdot a_{ij}}{\sum\limits_{i=1}^{m} \alpha_i \cdot \mu_i} \right)^{\frac{1}{n}}$$

where μ_i, is the mean of i_{th} attribute in which $i = 1, \ldots, m$.

Proof. Following Tsui (1995) and Kolm (1977), let us define the multidimensional inequality index $I_R(A)$ for $A \in \mathcal{A}_{++}$ as $I_R(A) = 1 - \delta(A)$ where $\delta(A) \in [0,1]$ is defined by $I(A) = I\left(\delta(A) \cdot A_\mu\right)$

where, A_μ is the $m \times n$ matrix where each column writes $\begin{pmatrix} \mu_1 \\ . \\ \mu_i \\ . \\ \mu_m \end{pmatrix}$.

From $I(A) = \sum_{j=1}^{n} \ln(\sum_{i=1}^{m} \alpha_i \cdot a_{ij}) = \ln \prod_{j=1}^{n} (\sum_{i=1}^{m} \alpha_i \cdot a_{ij})$ and $I(\delta(A) \cdot A_\mu) = \ln\left(\delta(A)^n \cdot (\sum_{i=1}^{m} \alpha_i \cdot \mu_i)^n\right)$.

Based on that, one gets the desired result, namely,

$$I_R(A) = 1 - \left(\prod_{j=1}^{n} \frac{\sum\limits_{i=1}^{m} \alpha_i \cdot a_{ij}}{\sum\limits_{i=1}^{m} \alpha_i \cdot \mu_i} \right)^{\frac{1}{n}}$$

■

3.5 Agreeing with correlation increasing majorization.

Correlation Increasing Majorization (*CIM*) is a concept due to Boland and Proschan (1988) and introduced into the inequality literature by Tsui (1999) who pointed out this type of majorization is known as an ordering of dependence in statistics (e.g. Shaked, 1982) and in economics of risks as ‘pairwise more risk’ (Richard, 1975). Note that *CIM* or *the majorization axiom* corresponds to Atkinson-Bourguignon ordering (Atkinson and Bourguignon, 1982), but Gajdos and Weymark (2005) observed that Bourguignon and Chakravarty (2003) raised reservations about this axiom, because *CIM* does not take into account individual preferences. Since the point of view of our social evaluation is to consider a social planner who aims to consider each individual in the same way, we do not concur with the previous reservation and agree with the motivating examples given by Tsui in 1999.

As an illustration, let A, B, C be the following three matrices summarizing the distributions of attributes,[9]

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

In accordance with Tsui (1999), we agree that intuitively the distribution of attributes summarized by C is most unequal followed by B and then by A.

[9]Each column represents an individual.

Note that column B_1 and B_3 are nothing else than $B_1 = A_1 \wedge A_3$ and $B_3 = A_1 \vee A_3$, where $\wedge$ and $\vee$ are the classical operators min and max.

Furthermore, $C_2 = B_2 \wedge B_3$ and $C_3 = B_2 \vee B_3$.

After this illustration, it is time to define formally *Correlation Increasing Majorization.*

First, let us introduce some definitions (see Boland and Proschan, 1988).

Concept of Correlation Increasing Transfer (*CIT*)

Let $A, B \in \mathcal{A}$, then B is obtained from A by a *CIT* if there exists j_1, j_2 where $j_1 \neq j_2$ such that $B_{j_1} = A_{j_1} \wedge A_{j_2}$ and $B_{j_2} = A_{j_1} \vee A_{j_2}$.

A *CIT* is strict whenever neither $A_{j_1} \leq A_{j_2}$ nor $A_{j_2} \leq A_{j_1}$ happen.

Concept of Correlation Increasing Majorization (*CIM*)

Let $A, B \in \mathcal{A}$, then $A >_c B$, i.e., A is strictly less unequal for the *CIM* if B may be derived from A by a permutation of columns and a finite sequence of Correlation Increasing Transfers with at least one being strict.

We can now state and prove that our social evaluation functional of Theorem 2 as well as any strict inframodular social functional respects *CIM.*

We say that an inframodular function u is strict if:

For all $(X, Y) \in \mathbb{R}^m \times \mathbb{R}^m$, $X < Y$, i.e., $X \leq Y$, $X \neq Y$ and $\varepsilon \in \mathbb{R}^m$, $\varepsilon \neq 0$.

Then one has $u(X + \varepsilon) - u(X) > u(Y + \varepsilon) - u(Y)$.

Note that this is the case for the inframodular function in the Theorem 2.

Theorem 4 *Any strict inframodular social evaluation functional respects CIM.*

Proof. It is enough to prove that if A_1 and A_2 are two columns in $\mathbb{R}^m$, and neither $A_1 \leq A_2$ nor $A_2 \leq A_1$, then the inframodular function u satisfies $u(A_1) + u(A_2) > u(A_1 \wedge A_2) + u(A_1 \vee A_2)$.

The proof of this point is given in Appendix B, Lemma 2 ■

3.6 Empirical Analysis

Based on *Theorem 3*, we specialize $\psi(\cdot)$ as $\ln(\cdot)$, thus, considering cases in which all of the attributes are strictly positive, we adopt, for all $A \in \mathcal{A}_{++}$, $J(A)$ as a 'mean' social evaluation function,

$$J(A) = \frac{1}{n} \sum_{j=1}^{n} \ln \sum_{i=1}^{m} \alpha_i a_{ij}$$

We aim at evaluating the pertinence of this inframodular function using data. To test it, we decided to make a comparison with another function that is not inframodular, namely the famous Human Development Index (HDI) launched by the United Nations in 1990.[10] The classical version of this index works with three variables (life expectancy (h), education (e) and income (w)) and provides a value, which allows us to obtain ranking among countries. Recently, in 2010, this index has been updated[11] and now its form (similar that presented in Section 3.2) is,

$$H(h, e, w) = h^{\frac{1}{3}} e^{\frac{1}{3}} w^{\frac{1}{3}}$$

[10] For more details, see UNDP (1990).

[11] Zambrano (2014) discusses this new index, its computation and axiomatization. See also Herrero et al. (2010).

where each variable is an index between 0 and 1 and consists, basically, in the population mean, for example, $h = \frac{1}{n}\sum_{j=1}^{n} h_j$.

Besides the famous role of HDI, we can simply consider this index as a way to aggregate different attributes, as well. As $H(h,e,w)$ provides an outcome between 0 and 1, we decided to extract the 'certainty equivalent' of $J(A)$, i.e., $I(A) = \exp^{J(A)}$. Then, now we have both indexes delivering results in the range $[0,1]$.

Unfortunately, we can not make a comparison between the indexes using the regular HDI database. In actual fact, only the means for each attribute by country are available. Our index needs the individual-level information to be built. Thereby, we decide to perform this comparison with another database that gives us the information level that we need. The database we have chosen is the Brazilian national exam for high school students.[12] The final notes in this exam are split up in five categories: natural sciences, human sciences, languages, mathematics and essay writing. To apply $H(\cdot)$ with these data, we focused our analysis on only three attributes, namely, natural sciences (a^s), languages (a^l) and mathematics (a^m). Then, our empirical analysis, while not performing a direct comparison with the classical HDI, is nevertheless indeed multidimensional, since we consider three attributes.

The population (n) is the number of students in each town. Following HDI rules, here we also give the same weight to the attributes. The function $J(A)$ in this case, can be written as,

$$J(A) = \frac{1}{n}\sum_{j=1}^{n} \ln\left(\frac{a_j^s + a_j^l + a_j^m}{3}\right).$$

It is widely known that classical HDI formula does not consider in its calculation the level of inequality within a country. For this, a specific index is available, called IHDI.[13] However, we are interested in contrast with the classical HDI to detect to what extent $I(A)$ is influenced by the 'intra' inequalities. In other words, we want to see whether this function delivers a worse result for towns that have more inequality among their students. In this case, as the classical HDI neglects inequality characteristics, this comparison could be a good option to test the effectiveness of this function concerning inequality.

Since Brazil has 5570 towns, we confined our analysis to Minas Gerais state. Below the descriptive statistics are presented.

Firstly, the difference between the $I(A)$ and $H(A)$ outcomes is relatively small. Their correlation coefficient is $0,999$. The similarity of the outcomes is suitable because it shows that this function provides the outcomes in a similar sense as HDI usually does. Nevertheless, we may see through the descriptive statistics table that there are some differences between the two functions' results and we are interested in them.

For example, despite the strong closeness among the outcomes, we found that $H(A)$ is always bigger than $I(A)$ for every town, and this difference varies. Thus, since HDI does not consider inequality in its computation, we would like to know if the size of the difference between the functions is related to the inequality level of the towns. In other words, we want to see whether

[12] This exam is called ENEM (Exame Nacional do Ensino Médio – National high school exam). This exam is non-mandatory and has been used both as an admission test for enrollment in federal universities and educational institutes, as well as for certification for a high school degree.

[13] Kovacevic (2010) offers a good review and discussion about the importance of the inequality to evaluate human development.

Table 3.1 – Descriptive Statistics

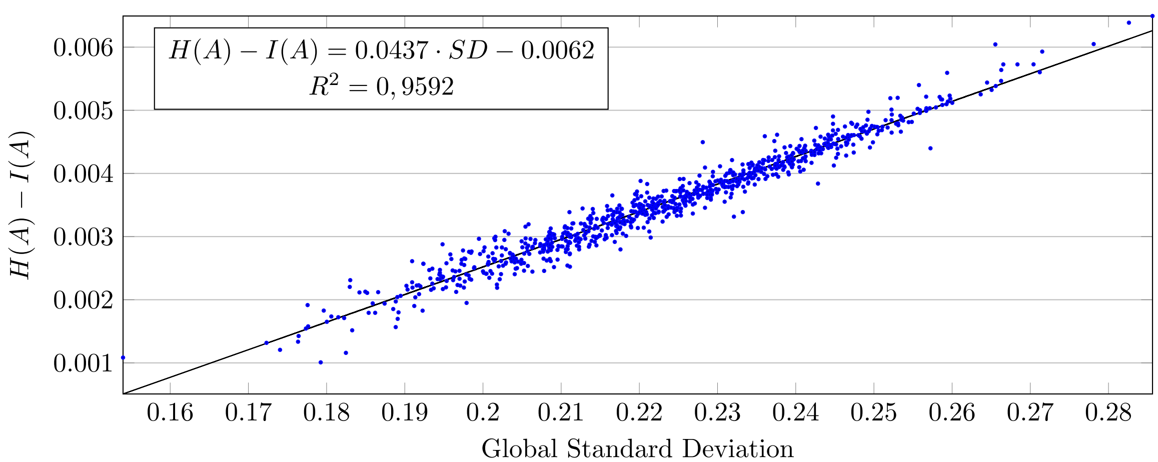
Variable	**Mean**	**Standard Deviation**	**Minimum**	**Maximum**
Science	0.49611	0.07456	0.34200	0.87640
Language	0.51850	0.06725	0.30620	0.79440
Math	0.49559	0.11077	0.31850	0.97360
Students		677.127		
		Results by Town		
Function	**Mean**	**Standard Deviation**	**Minimum**	**Maximum**
$\boldsymbol{I(A)}$	0.48412	0.01851	0.43577	0.53025
$\boldsymbol{H(A)}$	0.48765	0.01913	0.43748	0.53450
Towns		853		

Source: ENEM 2014.

inequality is positively correlated with $H(A) - I(A)$. To measure the inequality, in this case, we summed the values of the attributes for each student and extracted the standard deviation of this transformed variable. We want to analyze the correlation between these two variables to assert whether $I(A)$ takes inequality into account or not.

To answer this question, we need to evaluate these variables jointly. Below in Figure 3.1, one will find the dispersion graph of these two variables.

Figure 3.1 – Dispersion graph between $H(A) - I(A)$ and the global standard deviation of the attributes



We, also, computed a linear regression as a hypothetical exercise. The equation is written in the graph and depicted by the black line. The value of R^2 attests that standard deviation explains almost 96% of the $H(A) - I(A)$ behavior. The positive relationship between the variables is quite substantial.

Therefore, based on these results we suggest that this function could be a good alternative to IHDI. In short, we provide an inframodular function which can be used to aggregate several attributes (with different weights, if necessary), and takes into account the inequality inside the analyzed population.

3.7 Concluding remarks

This chapter aimed at characterizing a simple 'additive' social evaluation function based on a particular type of inframodular function proposed by Müller and Scarsini. In the multidimensional case, it allows us to respect what can be considered as a natural Pigou-Dalton principle. Furthermore, if the social planner treats every individual equally, which might be fair, our social evaluation functions agree with the property of Correlation Increasing Majorization, already suggested by Tsui.

Building upon a long tradition, we specify our functions to obtain a simple tractable relative inequality index. Finally, we propose an empirical analysis evaluating the pertinence of a specific inframodular evaluation function *à la* Müller and Scarsini when compared to the famous HDI functional.

We split up the Appendix in two parts: Appendix A includes the proof of Theorem 1. Appendix B gives the proofs of Lemma 1 and Lemma 2.

Appendix A

Theorem 1
Proof. We give only the sufficiency part since the necessary proof is immediate.

From *A.1, A.2, A.3, A.4* (*weak order, continuity, monotonicity and independence*) and $n \geq 3$, *Theorem 3* in Debreu (1960) implies that there exist n increasing[14] and continuous functions $u_j : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} u_j(A_j) \geq \sum_{j=1}^{n} u_j(B_j)$$

where the u_j are unique up to affine transformation $\alpha u_j + \beta_j$ *with* $\alpha > 0$ *and* $\beta_j \in \mathbb{R}$. Thus, we can assume that for all j, $u_j(0) = 0$.

From *A.5* (*Anonymity*), let us see that we can assume that there exists $u : \mathbb{R}^m \longrightarrow \mathbb{R}$, increasing and continuous such that:

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} u(A_j) \geq \sum_{j=1}^{n} u(B_j) \tag{A.1}$$

Then, fix u_j such that $u_j(0) = 0$, for all j. By symmetry, we only need to prove that $u_1 = u_2$. Take any $A_1 \in \mathbb{R}^m$ and consider $(A_1, 0, A_3, \ldots, A_n)$ and $(0, A_1, A_3, \ldots, A_n)$. Through *A.5*: $u_1(A_1) + u_2(0) + \sum_{j=3}^{n} u_j(A_j) = u_1(0) + u_2(A_1) + \sum_{j=3}^{n} u_j(A_j)$, this entails straightforwardly $u_1(A_1) = u_2(A_1)$; thus, $u_1 = u_2 = \ldots = u_n = u$. Therefore, there exists $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ increasing continuous (satisfying $u_j(0) = 0$) such that (A.1) holds. Clearly, u is defined up to a positive affine transformation ■

[14] Note that u_j increasing comes from our monotonicity axiom *A.3*.

Appendix B

Marinacci and Montrucchio (2005) provided a thorough analysis of 'Ultramodular Functions', thus (by reversing the inequality in the definition) of what Müller and Scarsini (2012) called 'Inframodular Functions' defined in Section 3.2.

We intend now to prove that inframodular functions agree with our Pigou-Dalton regressive transfers (see Introduction).

Lemma 1 if $u : \mathbb{R}^n \to \mathbb{R}$ is inframodular then u satisfies the property (3.1) quoted in Section 3.2.

Proof. Let $x, y \in \mathbb{R}^n$, $x \leq y$ and $\varepsilon \geq 0$. Set $x' = x - \varepsilon$ and $y' = y$, $\varepsilon' = \varepsilon$, so $x' \leq y'$ and $\varepsilon' \geq 0$. Then from the Definition 2 in Section 3.2 of u inframodular,

$$u(x' + \varepsilon) - u(x') \geq u(y' + \varepsilon) - u(y')$$

i.e.,

$$u(x) - u(x - \varepsilon) \geq u(y + \varepsilon) - u(y)$$

■

Lemma 2 **(**Proof of *Theorem 4*)

First, it is known that if u is *inframodular,* then u is *submodular,* i.e., for all $a, b \in \mathbb{R}^m$, $u(a) + u(b) \geq u(a \wedge b) + u(a \vee b)$ (see e.g. Marinacci and Montrucchio, 2005).

Let us show it again for the sake of completeness.

Let $x = a \wedge b$, so $a = a \wedge b + \varepsilon$ with $\varepsilon \geq 0$.

Let $y = b$, one has $x \leq y$ and $\varepsilon \geq 0$ then u inframodular implies $u(x+\varepsilon) - u(x) \geq u(y+\varepsilon) - u(y)$, i.e. $u(a) - u(a \wedge b) \geq u(b + a - a \wedge b) - u(b)$. But $b + a - a \wedge b = a \vee b$, hence, the result:

$u(a) + u(b) \geq u(a \wedge b) + u(a \vee b)$.

Thus one has $u(A_1) + u(A_2) \geq u(A_1 \vee A_2) + u(A_1 \wedge A_2)$.

Since by hypothesis neither $A_1 \leq A_2$ nor $A_2 \leq A_1$, u strict inframodular implies $u(A_1) + u(A_2) > u(A_1 \vee A_2) + u(A_1 \wedge A_2)$.

Actually since not $A_2 \leq A_1$, we get $A_1 \wedge A_2 < A_2$ then letting $x = A_1 \wedge A_2$, $y = A_2$, $\varepsilon = A_1 - A_1 \wedge A_2$, we get:

$u(x + \varepsilon) + u(y) > u(y + \varepsilon) + u(x)$, i.e. $u(A_1) + u(A_2) > u(A_1 \vee A_2) + u(A_1 \wedge A_2)$ ■

Chapter 4

Rawlsian inequality-adjusted social evaluation function

This chapter is a result of the working paper "Rawlsian inequality-adjusted social evaluation function", made in collaboration with Alain Chateauneuf and Jose Heleno Faro.

Abstract

A new methodology with a multiplicative structure for the calculations of the *Human Development Index* was unveiled in 2010 by the United Nations Development Programme. This chapter aims at axiomatizing a class of multidimensional social evaluation function in which *Inequality-adjusted Human Development Index* is a special case. We, furthermore, show that this class of social evaluation function accommodates the Pigou-Dalton principle and agrees with the subclass of inframodular functions. Finally, we analyze the context where the social planner is unsure of how to weigh different attributes and their implications to the inequality aversion approach.

Keywords: Multidimensional inequality, Pigou-Dalton transfer, Inframodular functions, Inequality-adjusted Human Development Index.

4.1 Introduction

Is social inequality important to measure social evaluation? If yes, what should be the weight of inequality in a social evaluation function?

To answer these questions, it is essential to understand the role of inequality in human development.[1] For instance, whether regions with less inequality are more developed. Should public policies induce a reduction of inequality to improve global welfare?

OECD (2015), for example, analyzed member countries over 30 years and found that income inequality has a sizeable and statistically significant negative impact on growth. One suggestion in this study is that inequality may affect economic performance by lowering investment opportunities of poorer population segments, especially in education. Indeed, this discussion is hardly new. Kravis (1960), using data from the 1950's, and Lydall (1968) found out that income is more equally distributed in richer countries.

The related literature presents a wide range of possibilities in which inequality might hamper human development. Alesina and Rodrik (1994) show that high inequality in income and land distribution are negatively associated with subsequent growth. Galor and Zeira (1993) demonstrate that wealth and income distribution affect output and investment in the short and long term. Thus, high levels of inequality result in under-investment in human capital mainly by the poorer portion of society and also curb social mobility and allocation of talents across occupations (Banerjee and Newman (1993), Owen and Weil (1998) and Checchi et al. (1999))

In the opposite sense, some theoretical results suggest that inequality can also be positive for economic growth. Kaldor (1955) advocated that the rich have a larger marginal propensity to save than the poor. Hence, if GDP growth is positively correlated with savings (in proportion of GDP), then unequal societies would grow faster. Stiglitz (1969) formalized this hypothesis by showing that in linear savings function aggregate capital accumulation behavior is independent of wealth distribution. As an extension, Bourguignon (1981) showed further that when savings is a convex function of income, different degrees of inequality generate multiple steady states. The main result is that output is larger in cases with greater inequality for both individual and aggregate levels.

Aghion et al. (1999) raised another aspect. Investment projects often involve large sunk costs. In the absence of an efficient market for shares (crowdfunding shares, for instance), big investments need a sufficient concentration of wealth.

In this sense, inequality cannot be viewed strictly as a negative aspect of human development or economic growth. Still, the relationship between inequality and human development is far from being well understood. It is thus possible to consider inequality to assess human development, but the inequality level may not be the only (or even the most) important feature to be considered.

Therefore, as discussed above, inequality *per se* is treated here as an important element of a negative impact on human development; a fully egalitarian society is not a benchmark to be sought here.

Social inequality and human development have often been measured essentially in terms of wealth or income.[2] However, many specialists have argued that confine the construction of a welfare indication in only income is insufficient and inadequate. This kind of indicator should be supplemented by other attributes as health and literacy, for example. As Tsui (1999) points out, fast

[1] We refer *human development* in a most broader way, like UNDP (2016) i.e. *"Human development implies that people must influence the processes that shape their lives. In all this, economic growth is an important means for human development, but not the end."* (p. 2)

[2] See, for instance, Atkinson (1970), Kolm (1976a;b), Foster and Sen (1997).

economic growth may not necessarily be accompanied by improvements in health and education.

Amartya Sen has raised several critiques of utilizing income as a single measure of human development (Sen, 1985; 1987; 1992). His capabilities approach emphasizes the importance of considering the "end" of development and not only the "means".[3] Sen also affirms that the human being is the "end" of development, the real objective of all activities. "Means" are the standard method to evaluate development, for example, GDP growth, GDP per capita and so on. Anand and Sen (1994) argue that these indicators are important but they do not measure properly the standard of living itself.

Accordingly, the Human Development Index (HDI) was proposed to evaluate living standards more broadly. That is, instead of assessing economic indicators to measure development, HDI gathers information related to health, education and wealth.

However, HDI does not capture inequality in the population. To consider inequality in human development, the Inequality-adjusted Human Development Index (IHDI) was created, where the inequality level is measured to "penalize" HDI outcomes, i.e. countries with higher inequality have an IHDI outcome smaller to their HDI.

In sum, we treat human development in the sense used by the Human Development Report (UNDP, 1990). We follow Sen's critique by working with a multidimensional social evaluation function.

In this chapter, we aim at characterizing a class of multidimensional social evaluation function which contemplates inequality, both among individuals and among attributes. It is provided an axiomatization to this social evaluation function. We also show that IHDI is a special case of the class of the social evaluation function that we axiomatize.

Moreover, we propose a well-being index that leads to a situation in which the social planner is unsure about how to assign weight to attributes when they are strongly complementary.

Lack of certainty in assigning a weight to attributes is treated here as the aversion of the social planner to inequality and for this reason, she gives more importance to the worst attribute. We thus relate this social planner's behavior with the approach proposed by Rawls (1971).

We restrict our focus in this chapter in analyzing possibilities on how to aggregate attributes to build a human development assessment concerning inequality. Idiosyncratic criteria to choose attributes and how to measure them are not treated here, although they might be considered as relevant as the aggregation of the social evaluation function.

This chapter is organized as follows: Section 4.2 includes the framework and the list of axioms; Section 4.3 presents the Theorem 1; Section 4.4 introduces an application of Theorem 1, methodology of IHDI and its relation with the main theorem; Section 4.5 proposes a new index by means of Theorem 2 and Section 4.6 is the conclusion.

4.2 Framework

Let us consider a finite population $\mathcal{J} = \{1, \ldots, j, \ldots, n\}$ of individuals and a finite set of attributes $\mathcal{I} = \{1, \ldots, i, \ldots, m\}$. Each individual $j \in \mathcal{J}$ is endowed with m attributes represented by a (row) vector $a_i \in \mathbb{R}^m$.

A real matrix A is an $m \times n$ array of real numbers. Let $\mathbb{M}_{m\times n}$ be the vector space of all m-by-n real matrices. Also, $\mathbb{M}^{+}_{m\times n}$ and $\mathbb{M}^{++}_{m\times n}$ are subsets of $\mathbb{M}_{m\times n}$ of matrices with non-negative and positive elements, respectively. We note that a population of n individuals endowed with m

[3]For more details about the capabilities approach, see Sen (1985).

attributes can be represented by a matrix $A \in \mathbb{M}_{m\times n}$ with m rows and n columns:

$$A \equiv (a_{ij})_{m\times n} := \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}.$$

Given a matrix $A \in \mathbb{R}^{m\times n}(\equiv \mathbb{M}_{m\times n})$, we define $A_j \in \mathbb{R}^m(\equiv \mathbb{M}_{m\times 1})$ as the vector (matrix) of all attributes of the individual $j \in \mathcal{J}$, represented by the jth column of A.[4] Also, $A_i \in \mathbb{R}^n(\equiv \mathbb{M}_{1\times n})$ is defined as the vector (matrix) of the attribute $i \in \mathcal{I}$ across all individuals, represented by the ith row of A. A_j may be interpreted as an individual, or a subgroup of society with different endowments on attributes. Similarly, A_i may be interpreted as an attribute, or a subgroup of attributes with different endowments to each individual. Yet, denote the matrix $(A_j^{'}, A_{-j})$, in which column A_j is replaced by column $A_j^{'}$ in the matrix A.

We say then that $I : \mathbb{M}_{m\times n} \to \mathbb{R}$ is an additive social evaluation function if there exists a utility index $u : \mathbb{R}^m \to \mathbb{R}$ over attributes such that for all $A \in \mathbb{M}_{m\times n}$,

$$I(A) = \sum_{j=1}^{n} u(A_j).$$

Given $A \in \mathbb{M}^+_{m\times n}$, $A_j, C_j, D_j \in \mathbb{M}^+_{m\times 1}$ and $\lambda, \beta > 0$ with $j \in \{1, \ldots, n\}$, we define $C_j^\lambda \otimes A_j^\beta := D_j$, where for any $i \in \{1, \ldots, m\}$:

$$d_{ij} = c_{ij}^\lambda \cdot a_{ij}^\beta$$

We also define the set $\Delta^{m-1}_{++} = \{\alpha \in \Delta^{m-1}_{+} : \sum_{k=1}^m \alpha_k = 1\}$.

Consider $\Sigma := 2^{\{1,\ldots,m\}}$, a capacity v on Σ is a mapping $v : \Sigma \longrightarrow [0, 1]$ such that:

(i) $v(\varnothing) = 0$ and $v(\{1, \ldots, m\}) = 1$;

(ii) for all $E, F \in \Sigma$, $E \subseteq F \Rightarrow v(E) \leq v(F)$.

Moreover, v is convex if for all $E, F \in \Sigma$,

$$v(E \cup F) \geq v(E) + v(F) - v(E \cap F).$$

Given $A_j = (a_{1j} \ldots a_{mj})^t$, if $a_{ij} \neq a_{i'j}$ for all $i, i' \in \{1, \ldots, m\}$ take a permutation $v : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$ such that $a_{\sigma(1)j} > a_{\sigma(2)j} > \ldots > a_{\sigma(m)j}$.

A_j can be viewed as a mapping $A_j : \{1, \ldots, m\} \longrightarrow \mathbb{R}$ where for all $i \in \{1, \ldots, m\}$, $A_j(i) := a_{ij}$. Denote $E_i := A_j^{-1}(a_{\sigma(i)j})$. The Choquet integral of A_j with respect to v is defined by:

$$\int_{\{1,\ldots,m\}} A_j \; dv := \sum_{i=1}^{m} (a_{\sigma(i)j} - a_{\sigma(i+1)j}) v \left(\bigcup_{k=1}^{i} E_k \right).$$

The case where there exist $i, i' \in \{1, \ldots, m\}$ such that $a_{ij} = a_{i'j}$ is analogous.

[4] So, A_j can be identified with the vector $(a_{1j} \; \ldots \; a_{mj}) \in \mathbb{R}^m$ or the matrix $(a_{1j} \; \ldots \; a_{mj})^t \in \mathbb{M}_{m\times 1}$.

4.2.1 Axioms

Consider a social planner who governs a society (a country or a city, for example) and her goal is to improve the human development of this society.[5] This social planner has her preferences about the way to carry out social policies to deal with inequality and improve human development.

We next present the axioms that characterize this social planner's preference for human development.

A social planner's preference is given by a binary relation $\succsim$ on $\mathbb{M}_{m\times n}$. For $A, B \in \mathbb{M}_{m\times n}$, $A \succsim B$ means that A is weakly preferred to B, $A \succ B$ means A is strictly preferred to B, $A \sim B$ means A and B are considered as indifferent by the social planner. Note that these definitions are consistent since $\succsim$ is assumed to be a weak order, i.e. social planner's preference is complete and transitive.

A.1 Weak Order: $\succsim$ is a complete and transitive binary relation on $\mathbb{M}_{m\times n}$. That is, for all $A, B, C \in \mathbb{M}_{m\times n}$, $A \succsim B$ or $B \succsim A$ (completeness). Moreover, if $A \succsim B$ and $B \succsim C$ then $A \succsim C$ (transitivity).

The next axiom guarantees that the social planner's preferences do not change suddenly.

A.2 Continuity: Given $B \in \mathbb{M}_{m\times n}$, $\{A \in \mathbb{M}_{m\times n} | A \succsim B\}$ and $\{A \in \mathbb{M}_{m\times n} | B \succsim A\}$ are closed in $\mathbb{M}_{m\times n}$.

The monotonicity axiom states that if all elements of the matrix are equal or greater than another matrix, then the social planner prefers the first one.

A.3 Monotonicity: For all $A, B \in \mathbb{M}_{m\times n}$, and for all i, j, if $a_{ij} \geq b_{ij}$ then $A \succsim B$. If furthermore $A \neq B$, then $A \succ B$.

The independence axiom states that if the social planner prefers A to $(A_j^{'}, A_{-j})$, then the ranking remains the same even if the rest of the population is modified (from A_{-j} to B_{-j}, for example).

A.4 Independence: For all $A, B \in \mathbb{M}_{m\times n}$ and taking $j \in \{1, \ldots, n\}$ fixed with $A'_j \in \mathbb{M}_{m\times 1}$; $(A_j, A_{-j}) \succsim (A_j^{'}, A_{-j}) \iff (A_j, B_{-j}) \succsim (A_j^{'}, B_{-j})$.

The anonymity axiom states that the value of a distribution does not depend on the individual's identity; only the values of the attributes matter.

A.5 Anonymity: For all $A \in \mathbb{M}_{m\times n}$ and for all bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, one has $(A_1, \ldots, A_j, \ldots, A_n) \sim \big(A_{\sigma(1)}, \ldots, A_{\sigma(j)}, \ldots, A_{\sigma(n)}\big)$.

The next axiom is based on Trockel (1989) and Faro (2013). It states that the social planner's preference regarding populations remains the same even after switch individuals, or groups of individuals, multiplying both by the same attributes' vector.

A.6 Strongly Homothetic: For all $A \in \mathbb{M}^+_{m\times n}$ and for all $B_j, C_j \in \mathbb{M}^+_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (C_j \otimes A_j, A_{-j}) \succsim (C_j \otimes B_j, A_{-j})$.

The following Pigou-Dalton principle states: if all attributes of individual j_1 are smaller than the respective attributes of j_2, then, for any attribute i, the transfer $\varepsilon_i \geq 0$ from j_1 to j_2, increases inequality,[6] resulting in a worse social situation.

Therefore, the global progressive transfer $\varepsilon \in \mathbb{M}^+_{m\times 1}$ from individual j_1 to individual j_2 is "rank-preserving" i.e., individual j_2 maintains himself with greater values for any attributes, than individual j_1 after the transfer.

A.7 Pigou-Dalton Principle: Let $A = (A_1, \ldots, A_j, \ldots, A_n)$ such that for some $j_1, j_2 \in \{1, \ldots, n\}$ one has $A_{j_1} \leq A_{j_2}$ and let $\varepsilon \in \mathbb{R}^m_+$ then:

[5] This social planner also might be considered as a group of authorities, for instance. The crucial point here is that this person or group is in charge of the human development improvement of this society.

[6] Strictly increase when $\varepsilon_i > 0$.

$A \succsim \big(A_1, \ldots, A_{j_1-1}, A_{j_1} - \varepsilon, A_{j_1+1}, \ldots, A_{j_2-1}, A_{j_2} + \varepsilon, A_{j_2+1}, \ldots, A_n\big) =: A_\varepsilon$. Furthermore, $A \succ A_\varepsilon$ if $\varepsilon \in \mathbb{R}^m_+$ and $\varepsilon \neq 0$.

4.3 Multidimensional social evaluation function

This section presents the first result, in which a social evaluation function agrees with the Pigou-Dalton principle in a multidimensional context. Attributes are aggregated through a weighted geometric mean and individuals through sum.

Theorem 1 *A preference relation $\succsim$ on $\mathbb{M}_{m\times n}$ satisfies A.1 to A.7 if, and only if, there exists $\alpha \in \Delta^{m-1}_{++}$ and there exists $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous such that:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi\Big(\prod_{i=1}^{m} a_{ij}^{\alpha_i}\Big) \geq \sum_{j=1}^{n} \psi\Big(\prod_{i=1}^{m} b_{ij}^{\alpha_i}\Big).$$

Furthermore, such α_i's are unique and ψ is defined up to an increasing affine transformation.

See Appendix for the proof.

This representation rules the way that the social planner aggregates both individuals and attributes to reach an outcome to assess the level of human development of this society.

She first assigns a weight for each attribute and aggregates them through a Cobb-Douglas rule. Then she applies a function ψ and adds all the individuals. This function ψ has an important role in the next section relating Theorem 1 with the IHDI formulation.

4.4 Special Case: IHDI

In this section, we present how IDHI is calculated and introduce the proposition demonstrating how IHDI is a special case of the function in Theorem 1.

4.4.1 Origins of IHDI

This index was proposed by Alkire and Foster (2010) aiming at adjusting HDI to reflect the distribution of human development achievements across the population and attributes. IHDI has been analyzed by the United Nation Development Program (UNDP) since 2010.

The idea of the index was conceived by Atkinson (1970) who created a family of inequality measures, which, in turn, was based on mean generalization. Hence, for the one-dimensional case, the generalized mean $\mu_q(A)$ is,

$$\mu_q(A) = \begin{cases} \left[\dfrac{(a_1^q + a_2^q + \ldots + a_n^q)}{n}\right]^{\frac{1}{q}} & \text{for } q \neq 0 \\ (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} & \text{for } q = 0 \end{cases}$$

Note that $\mu_1(a)$ is an arithmetic mean, $\mu_0(a)$ a geometric mean and $\mu_{-1}(a)$ an harmonic mean. Moreover, it is easy to see that μ_q is strictly increasing in q when at least two observations into the distribution are different.[7] The greater q is, the greater is the weight of the upper part of the a's

[7] When all a's values are equal, μ_q is constant in q.

distribution. On the other hand, as q decreases, more weight is attached to the lower end of the distribution. For instance, at the extreme case, that $q \to -\infty$, μ_q tends to the "Rawlsian" social welfare function (or *maximin rule*), where $\mu_q(a) = \min[a_1, a_2, \ldots, a_n]$, in which only the lowest part of the distribution is relevant.[8]

Therefore, the class of inequality measures proposed by Atkinson (1970) is given by,

$$I_q(a) = 1 - \left[\frac{\mu_q(a)}{\mu(a)}\right] \text{ for } q < 1. \tag{4.1}$$

As observed by Foster et al. (2005), Atkinson's idea was to compare a "bottom-sensitive" general mean with the "neutral" arithmetic mean. Taking $q < 1$, we always have the ratio $(\mu_q(a)/\mu(a))$ between 0 and 1. The smaller q is, the smaller will be this ratio and larger will be $I_q(a)$. It means that q reflects the sensibility level given to the lower part of the distribution, i.e. q might be considered as an "inequality aversion" parameter.

Consider, for instance, the case of maximum aversion. Then $\mu_q(a)$ will have the Rawlsian form and $I_q(a)$ reaches the largest value.

Though, in the IHDI case, we have $q = 0$. It means that the ratio in (4.1) is a geometric mean over an arithmetic mean. In this sense, $I_0(a)$ is used to adjust HDI for inequality.

Define HDI by $H_0 = (a_1 a_2 a_3)^{1/3}$, where each attribute a_i where $i = \{1, 2, 3\}$, is aggregated by a normalized arithmetic mean.[9] Then,

$$a_i^* = (1 - I_0)a_i$$

where a_i^* is an HDI attribute that is adjusted by its inequality level (I_0).

Then IHDI is the geometric mean of three adjusted attributes,

$$H_I = (a_1^* a_2^* a_3^*)^{\frac{1}{3}}$$

Note that $H_I \leq H_0$. When $I_0 = 0$, then $H_I = H_0$. This explains why HDI can be considered as *potential* Human Development, i.e., the IHDI is a special case when there is no inequality throughout the population.[10]

Yet, considering that $\mu(a)$ in (4.1) is obtained as a_i, i.e. a normalized arithmetic mean, then $\mu(a) = a_i$, and it is easy to see that $a_i^* = \mu_0(a)$.

It means that the attribute adjusted for inequality (a_i^*) is exactly the geometric mean of the attribute. Hence, for each attribute i,

$$a_i^* = \left[\prod_{j=1}^{n} a_{ij}\right]^{\frac{1}{n}} \tag{4.2}$$

From that, we can conclude that the IHDI is a geometric mean of geometric means. In fact, Alkire and Foster (2010) highlighted this aspect:

[8] See Rawls (1971; 2001) and Foster and Sen (1997).

[9] Each one of the HDI attributes has one type of normalization. This mean is normalized between the minimum and the maximum value found in each attributes' distribution. For life expectancy, for instance, a minimum value of 20 years is used instead of 0 years, since there is no country with a smaller life expectancy in the 20th century. For more details, see Zambrano (2014).

[10] For more details, see Kovacevic (2010), Section 5.

IHDI is the geometric mean of the geometric means of income, education and health, and each of the latter can either be calculated directly from the data or constructed from the arithmetic means and inequality levels. (p. 8).

Therefore, another way to write H_I is,

$$H_I = \left[\prod_{j=1}^{n} (a_{j1} a_{j2} a_{j3})^{\frac{1}{n}} \right]^{\frac{1}{3}} . \tag{4.3}$$

that can be viewed as a geometric mean of geometric means.

4.4.2 Theorem 1 and IHDI

We next introduce another axiom to formulate a proposition that helps us to demonstrate the relation of Theorem 1 with the IHDI formulation in (4.3).

The Scale Invariance axiom states that the social planner's preference remains the same when all the individuals' attributes are multiplied by the same scalar k.

A.8 Scale Invariance: For all $A, B \in \mathbb{M}^{+}_{m\times n}$ and $k \in \mathbb{R}_{++}$; $A \succsim B \implies k \cdot A \succsim k \cdot B$.

The scalar k can be viewed as a constant attribute vector.

Proposition 3 *A preference relation $\succsim$ on $\mathbb{M}^{+}_{m\times n}$ satisfies A.1 to A.8 if, and only if, up to an increasing affine transformation, either $\psi(t) = \ln(t)$ or $\psi(t) = t^{\beta}$ for all $t > 0$, where $\beta \in (0,1)$*

$$I(A) = \left(\prod_{j=1}^{n} \prod_{i=1}^{m} a_{ij}^{\alpha_j} \right)^{\frac{1}{n}}$$

See Appendix for the proof.

Likewise, taking $\psi(\cdot) = \ln(\cdot)$ and considering that all the attributes are strictly positive, let $G(A)$ be a "mean" social evaluation function,

$$G(A) = \frac{1}{n} \sum_{j=1}^{n} \ln \prod_{i=1}^{m} a_{ij}^{\alpha_j} . \tag{4.4}$$

Now, taking $a_{ij} \in (0,1]$,[11] we know that $G(A) \in (-\infty, 0]$. To avoid an index with this wide range of results, let $I(A) = \exp^{G(A)}$. Suppose also that all the attributes have the same weight, i.e., for m attributes each one of them is weighted by $1/m$. Moreover, since the sum of logs is the log of product, we get,

$$I(A) = \left[\prod_{j=1}^{n} \left(\prod_{i=1}^{m} a_{ij} \right)^{\frac{1}{m}} \right]^{\frac{1}{n}} \tag{4.5}$$

where $I(A) \in [0,1]$. Taking $m = 3$ we find that the condition (4.5) is equal to (4.3). This proves that IHDI is a special case of the social evaluation function proposed in Theorem 1.

[11] Although it is possible to have null values, this information is omitted here, since we are working with geometric mean.

4.5 Social Evaluation Function: A new proposal

What happens if the social planner is not sure about the importance of the attributes chosen to assess a society's welfare? In this section, we present in Example A a characterization of this context and propose a social evaluation function regarding this context in Theorem 2. We then relate this social evaluation function with Rawls and Harsanyi's inequality approaches.

4.5.1 Example A

Suppose that the social planner has to decide on which attribute to concentrate investments to improve the IHDI outcome of a country. The social planner is concerned with four attributes, namely: health (h), wealth (w), mean years of schooling (e_1) and expected years of schooling (e_2).[12]

The social planner does not assign the same importance to each attribute. She considers education more important than either health or wealth, but she does not know precisely how to weigh education attributes separately. Her main concern is to increase both attributes together because of their strong complementarity. More specifically, her lack of certainty on how to assess the education attributes separately does not mean that she gives the same importance to each of them. What she knows precisely is: improving education attributes together is more important compared to improve either health or wealth separately.

Moreover, the country where the social planner acts is ideally divided into three regions, namely: r_1, r_2, r_3. She got funding from an international institution to improve human development in the poorest region, which is r_1. For the sake of simplicity, the amount invested is 9. The situation in which the region r_1 receives investment in health is represented by the matrix A and the investment in mean years of schooling is B, where,

$$\begin{array}{cc} & \mathbf{A} \\ & \begin{array}{ccc} \mathbf{r_1} & \mathbf{r_2} & \mathbf{r_3} \end{array} \\ \begin{array}{c} \mathbf{h} \\ \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{w} \end{array} & \begin{bmatrix} 9 & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 1 & a_{32} & a_{33} \\ 1 & a_{42} & a_{43} \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} \mathbf{r_1} & \mathbf{r_2} & \mathbf{r_3} \end{array} \\ \begin{array}{c} \mathbf{h} \\ \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{w} \end{array} & \begin{bmatrix} 1 & a_{12} & a_{13} \\ 9 & a_{22} & a_{23} \\ 1 & a_{32} & a_{33} \\ 1 & a_{42} & a_{43} \end{bmatrix} \end{array}$$

Comparing both situations, she strictly prefers to invest in health than in mean years of schooling, i.e. $A \succ B$.

Now, suppose that she received extra funding to invest in the same region, but this time necessarily in the attribute (e_2), but she still needs to choose investing either in health or in mean years of schooling, i.e.

$$\begin{array}{cc} & \mathbf{\hat{A}} \\ & \begin{array}{ccc} \mathbf{r_1} & \mathbf{r_2} & \mathbf{r_3} \end{array} \\ \begin{array}{c} \mathbf{h} \\ \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{w} \end{array} & \begin{bmatrix} 9 & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 9 & a_{32} & a_{33} \\ 1 & a_{42} & a_{43} \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \mathbf{\hat{B}} \\ & \begin{array}{ccc} \mathbf{r_1} & \mathbf{r_2} & \mathbf{r_3} \end{array} \\ \begin{array}{c} \mathbf{h} \\ \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{w} \end{array} & \begin{bmatrix} 1 & a_{12} & a_{13} \\ 9 & a_{22} & a_{23} \\ 9 & a_{32} & a_{33} \\ 1 & a_{42} & a_{43} \end{bmatrix} \end{array}$$

[12]The way these investments will be applied and their effectiveness is not of concern here.

Since she gives more importance to invest in both education attributes, she inverts her preference, i.e. let $C_1 = (1, 1, 9, 1)^t \in \mathbb{M}_{4\times 1}$, $\hat{A} = (C_1 \otimes A_1, A_{-1})$ and $\hat{B} = (C_1 \otimes B_1, B_{-1})$.[13] Then $\hat{B} \succ \hat{A}$, which constitutes a violation of the Strongly Homothetic axiom.

It means that the index formulated in Theorem 1 does not capture this kind of behavior. The context in which the social planner has difficulties to weigh attributes is similar to Ellsberg paradox.[14] One way to characterize this behavior is to weaken the strongly Homothetic axiom *à la* Gilboa and Schmeidler (1989). In other words, we no longer obtain the ordering $(A_j, A_{-j}) \succsim (B_j, A_{-j})$ after multiplying the individual j by a non-constant attribute vector C_j above. Because of that, we follow Faro (2013) and substitute Strongly Homothetic axiom by Homotheticity *(A.9)*, Log-Convexity *(A.10)* and Power Invariance *(A.11)*.

4.5.2 Rawlsian Inequality-Adjusted Social Evaluation Function

Since Strong Homotheticity is violated by the example above, we weaken the conditions as Faro (2013), replacing this axiom by another three axioms: Homotheticity *(A.9)*, Log-Convexity *(A.10)* and Power Invariance *(A.11)*.

Homotheticity axiom means that the social planner's preference remains the same after multiplying all the attributes of individuals A_j and B_j by a scalar k.

A.9 Homotheticity: For all $A \in \mathbb{M}^{+}_{m\times n}$, $k \in \mathbb{R}_{++}$ and for all $A_j, B_j \in \mathbb{M}^{+}_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (k \cdot A_j, A_{-j}) \succsim (k \cdot B_j, A_{-j})$.

The scalar k may be considered as a constant attribute vector where $k \in \mathbb{M}^{++}_{m\times 1}$.

The next axiom states that if the social planner prefers (A_j, A_{-j}) over (B_j, A_{-j}) then she will prefer a Cobb-Douglas mixture $A_j^{\lambda} \otimes B_j^{1-\lambda}$ over B_j.

A.10 Log-Convexity: If for all $A \in \mathbb{M}^{+}_{m\times n}$, $\lambda \in (0, 1)$ and for all $A_j, B_j \in \mathbb{M}^{+}_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (A_j^{\lambda} \otimes B_j^{1-\lambda}, A_{-j}) \succsim (B_j, A_{-j})$.

Finally, the Power Invariance axiom states that the social planner's preference keeps the same preference with individuals' attributes A_j and B_j changed by the power of any positive scalar k.

A.11 Power Invariance: If for all $A \in \mathbb{M}^{+}_{m\times n}$, $k \in \mathbb{R}_{++}$ and for all $A_j, B_j \in \mathbb{M}^{+}_{m\times 1}$; $(A_j, A_{-j}) \succsim (B_j, A_{-j}) \implies (A_j^{k}, A_{-j}) \succsim \left(B_j^{k}, A_{-j}\right)$.

Theorem 2 *A preference relation $\succsim$ on $\mathbb{M}_{m\times n}$ satisfies A.1 to A.5, A.7 and A.9 to A.11 if, and only if, there exists a unique, non-empty, closed and convex set $C \subset \Delta$ of probability measures such that $\alpha \in \Delta^{m-1}_{++}$ and there exists $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous such that:*

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} \psi \left(\min_{\alpha \in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i} \right) \geq \sum_{j=1}^{n} \psi \left(\min_{\alpha \in C} \prod_{i=1}^{m} b_{ij}^{\alpha_i} \right).$$

Furthermore, ψ is defined up to an increasing affine transformation.

See Appendix for the proof.

Also, Strong Homotheticity axiom in Theorem 2 would imply that the set C has a unique probability measure.

[13]As we can see, $A_{-1} = B_{-1}$.

[14]See Ellsberg (1961) for more details.

In order to be able to compute our social evaluation function, we now specify ψ in Theorem 2 by requiring also axiom *A.8*, and accordingly choosing $\psi(\cdot) = \ln(\cdot)$. It turns out that simple computations as for Proposition 3 lead to Corollary 3:

Corollary 3 *A preference relation $\succsim$ on $\mathbb{M}^+_{m\times n}$ satisfies A.1 to A.5, A.7 and A.9 to A.11 if, and only if, there exists a unique, closed and convex set $C \subset \Delta$ of probability measures such that $\alpha \in \Delta^{m-1}_{++}$ and $A \succsim B$ if, and only if, $I(A) \geq I(B)$ where for all A, B with strictly positive attributes:*

$$I(A) = \left(\prod_{j=1}^{n} \min_{\alpha\in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i}\right)^{\frac{1}{n}}$$

and similar formula for $I(B)$.

4.5.3 Example B

In this section, we extend the idea given in Theorem 2, and more precisely in Corollary 3, for the case when the social planner is not able to weigh attributes properly. This section gives an insight into how to characterize this situation.

To be as clear as possible, let us take IHDI (H_I) as a point of departure.

$$H_I(A) = \left[\prod_{j=1}^{n} (a_{hj}a_{ej}a_{wj})^{\frac{1}{3}}\right]^{\frac{1}{n}}$$

The education attribute is built through the geometric mean of two sub-components: mean years of schooling (a_{e_1j}) and expected years of schooling (a_{e_2j}). Then,

$$H_I(A) = \left[\prod_{j=1}^{n} \left(a_{hj}^{\frac{1}{3}}a_{wj}^{\frac{1}{3}}a_{e_1j}^{\frac{1}{6}}a_{e_2j}^{\frac{1}{6}}\right)\right]^{\frac{1}{n}}.$$

Now, the social planner, who is in charge of assessing human development, is no longer sure about the best way to assign weights to education sub-components. She has no doubt that the education attributes should jointly represent 50% of the measurement, i.e. let $\underline{\alpha}$ be the probability of a_{e_1j} and $\overline{\alpha}$ the probability of a_{e_2j}. Then, $\underline{\alpha}+\overline{\alpha} = 1/2$. However, instead of assigning equal weights to each of them ($1/4$), she believes that their weights are between $[1/6, 1/3]$. Furthermore, she follows the social evaluation function provided in Corollary 3, i.e.

$$I(A) = \left[\prod_{j=1}^{n} a_{hj}^{\frac{1}{4}}a_{wj}^{\frac{1}{4}} \min\left(a_{e_1j}^{\frac{1}{6}}a_{e_2j}^{\frac{1}{3}}, a_{e_1j}^{\frac{1}{3}}a_{e_2j}^{\frac{1}{6}}\right)\right]^{\frac{1}{n}}.$$

Similarly, let us define $a_j^{'} := a_{hj}^{\frac{1}{4}}a_{wj}^{\frac{1}{4}}\left(a_{e_1j}^{\frac{1}{6}}a_{e_2j}^{\frac{1}{3}}\right)$ and $a_j^{''} := a_{hj}^{\frac{1}{4}}a_{wj}^{\frac{1}{4}}\left(a_{e_1j}^{\frac{1}{3}}a_{e_2j}^{\frac{1}{6}}\right)$. Rewriting the index,

$$I(A) = \left[\prod_{j=1}^{n} \min\left(a_j^{'}, a_j^{''}\right)\right]^{\frac{1}{n}}.$$

The higher is her dubiety to assign weights to the attributes, the higher is the range of weight possibilities. In the example above, the interval $[1/6, 1/3]$ was chosen, but in the extreme case, this range would be $[0, 1/2]$, i.e. the social planner would take only the worst attribute to compose the index.

Let us recall Example A in Section 4.5.1. Theorem 1 does not capture this context, since $A \succ B$ might imply $(C_1 \otimes B_1, A_{-1}) \succ (C_1 \otimes A_1, A_{-1})$ which violates Strongly Homothetic axiom. The specification proposed in Theorem 2 is consistent with Example A.

To demonstrate that, consider the probabilities presented in this example. It is easy to see that $A \succ B$, since A has a weight of $1/4$ and B's weight is $\min(1/6, 1/3) = 1/6$. Moreover, $(C_1 \otimes B_1, A_{-1})$ has a weight of $1/2$ and $(C_1 \otimes A_1, A_{-1})$ has a weight of $\min(1/4, 1/2) = 1/4$, which gives us that $(C_1 \otimes B_1, A_{-1}) \succ (C_1 \otimes A_1, A_{-1})$. This proves that the specification in Theorem 2 agrees with Example A.

We complete this section by proposing a final result that allows us to deal with simple meaningful cases, which furthermore appears to be tractably computable for applications.

Theorem 4 *Assume that the social planner is only able to give an upper and a lower bound respectively a_i and b_i for the weight α_i of each attribute $i = 1, \ldots, m$, where $0 < \underline{\delta_i} \leq \overline{\delta_i} < 1$. Then the social planner is consistent, i.e. there exists*[15] *$\alpha_i \geq 0$ such that $\underline{\delta_i} \leq \alpha_i \leq \overline{\delta_i}$ for all i and $\sum_{i=1}^{m} \alpha_i = 1$ if, and only if,*

$$\sum_{i=1}^{m} \underline{\delta_i} \leq 1 \leq \sum_{i=1}^{m} \overline{\delta_i} \tag{4.6}$$

Furthermore, the social evaluation function $I(A)$ of any A with strictly positive attributes, as proposed in Corollary 3 is now given by the following formula:

$$I(A) = \prod_{j=1}^{n} \exp \int_{\{1,\ldots,m\}} A_j \; dv \tag{4.7}$$

where $\int(\cdot)\,dv$ is the Choquet integral with respect to $v : E \in 2^{\{1,\ldots,m\}} \longrightarrow v(E) = \max\left\{\sum_{i\in E} \underline{\delta_i}, 1 - \sum_{i\in E^c} \overline{\delta_i}\right\}$.

Remark 1

Before giving the proof of Theorem 4, let us mention that in the case of general set C the computation of $I(A)$ contained in Corollary 3 is not so easy. As we will see, in the reasonable particular case of Theorem 4 the *min* condition *a la* Gilboa and Schmeidler (1989) turns out to be a *min* condition *a la* Schmeidler (1989) hence allowing to obtain an alternative formula through a Choquet integral.

Corollary 5 *It appears that social planners might prefer to envision bounds symmetrical with respect to $1/n$ i.e. of the type $\underline{\delta_i} = 1/m - \varepsilon_i$, $\overline{\delta_i} = 1/m + \varepsilon_i$ where $\varepsilon_i \geq 0$ for all $i = 1, \ldots, m$. In such a case, simple comparisons show that Theorem 4 is valid if, and only if, $0 \leq \varepsilon_i < 1 - 1/m$.*[16]

[15] Note that indeed consistency requires $\alpha_i > 0$ for any i.

[16] Remember that $m \geq 2$

4.5.4 Rawls *versus* Harsanyi

The new proposal introduced in this chapter may be associated not only with the idea of inequality aversion discussed in Section 4.4.1 but also requires a normative approach on how to assess human development from the social planner's perspective. This context involves a notion of justice and fairness, invariably.

To shed some light on this normative discussion, we revisit a very interesting discussion that occurred in the 1970's between John Rawls and John Harsanyi. They have quite different views about justice and, consequently, on how to deal with inequality, and this dispute, as described by Moehler (2018), have deepened and enriched subsequent researches since then.

Also in Section 4.4.1, we presented the social evaluation function proposed by Rawls, called *maximin rule*, whose goal is to maximize the welfare of the worst-off individual. This approach represents an extreme case of inequality aversion, as already discussed in that section. For Rawls, the main concern of a fair society should be the provision of primary goods for all. For that reason, public policies should give more attention to the worst off individual than to the rest of society.

Rawls develops his justice theory in two books (Rawls, 1971; 2001) and his principle of justice attempts to build an approach opposing utilitarianism. As pointed out by Fleurbaey et al. (2008), for Rawls utilitarianism fails to satisfy one of the main ideas of Immanuel Kant that individuals should be treated as ends in and of themselves, not just as means for promoting the social good.

Rawls' core moral ideal is the *original position* that justifies his principles of justice. The concept of original position means that individuals express a desire for certain social primary goods which give them a positive satisfaction (more is better than less). Indeed, such goods are not just tangible ones. The original position of individuals embodies Rawls' conception of citizenship, including the freedom to form and pursue a pleasant life and a notion of justice. Under the concept of the original position, individuals are seen as independent entities. Moreover, they are not aware of the specific preference profiles and circumstances of society. This idea constitutes the concept of *thick veil of ignorance*.

Furthermore, according to Rawls, the concept of the original position leads to a non-utilitarian conclusion about justice, which makes possible the application of the maximin rule.

For the application of this rule, Rawls imposes two main conditions: 1) individuals are not endowed with a justifiable basis for assigning specific probabilities to outcomes. This lack results from the concept of the thick veil of ignorance; 2) Individuals care essentially for the minimum level of goods that they receive. This condition is related to the sense of individuals as independent entities and to the principle of difference in which an ideal society follows a fair system of cooperation among free and equal citizens.

On the other hand, Harsanyi was a harsh critic of Rawls' theory of justice. His approach assumes that individuals are Bayesian and make their choices based on risk or uncertainty, assigning probabilities to outcomes and maximizing their expected utility. Harsanyi develops his point of view in several articles (Harsanyi, 1955; 1975a;b; 1978) where he mainly argues that individuals would choose an average utility principle to attain the highest degree of fairness and to fulfill the concept of justice.

In Harsanyi's theory, individuals know the specific circumstances of the society, but they do not know the existing preference profiles and, therefore, they cannot favor themselves or anyone else in their choices, i.e. individuals know the circumstances but do not know the names attached to the respective positions in society. This idea is at the core of the concept of the *thin veil of ignorance*, which is the opposite of Rawls' approach.

The thin veil of ignorance is a consequence of the moral ideal of impartiality and impersonality which is a key feature in Harsanyi's argument.

Under these concepts, individuals not only know their social position but they also believe to have an equal chance of occupying any other social position existing in their society. Considering this context, Harsanyi (1978) builds a hypothetical decision situation for rational individuals that he calls *equiprobability model*.

Yet, the equiprobability model does not require the assumption that all individuals are seen as separate entities, as thus the original position, and states that for the sake of justice the individuals' utility functions have the same weight. In this sense, it is rational to assign equal probabilities to all possible outcomes.

Despite sharing some similarities, Rawls and Harsanyi reach different conclusions. Their liberal point of view implies that the idea of justice must pass through the concept of impartiality. Both approaches reinforce the concept of impartiality from different perspectives. For Rawls, the thick veil of ignorance renders individuals unaware of the specific social and economic circumstances of the society, thus placing individuals' preferences under the concept of original position. For Harsanyi, the thin veil of ignorance ensures that besides knowing the circumstances of society, individuals do not know personal profiles, which prevents any possibility of partiality.

In sum, the Harsanyi-Rawls dispute takes into account the circumstances that the social planner may face with. To bring these theories to our context, we have to adapt some features. First, Harsanyi and Rawls are essentially dealing with the individual-level. Our context deals with an additional dimension, attribute-level. That is, instead of discussing probability assessments only about individuals, we also discuss assessments about attributes. Second, their discussion is wider than our context. In our case, the social planner has to decide the rule to measure human development taking into account the approaches presented above. Third, as already mentioned, Rawls' theory is deeply critical to the utilitarianism approach, which is a cornerstone of this chapter. Despite all this, our main inspiration from Rawls is the priority given to the worst-off individuals as embraced by Rawls in the maximin principle.

Rawls' condition to apply the maximin rule is that individuals are not able to assign specifics probabilities to outcomes. This condition is closely related to a situation where the social planner does not assess precisely the weight of the education attributes.

Our example is useful to describe how theorem 2 may represent Harsanyi and Rawls' point of view in a multidimensional social evaluation function where instead of considering the population perspective we are interested in how to treat misconceptions concerning weighing different attributes.

In sum, Harsanyi prefers to give the same weight to all individuals, while Rawls believes it is fairer to give more importance to the worst off individual. In this sense, our social evaluation function treats individuals similarly to Harsanyi's theory, giving the same weight to all individuals as in his equiprobability model. Additionally, our social evaluation function opens up the possibility to treat attributes as in Rawls's theory, i.e. since the social planner is unsure about how to weigh attributes, she gives more importance to the worst off outcome.

Incidentally, when the social planner has no doubt about the weight of the attributes and weighs them equally, then she would be following the IHDI format. Therefore, since IHDI treats both individuals and attributes according to Harsanyi, IHDI can be considered a special case of our social evaluation function.

4.6 Conclusion

The main goal of this chapter is to analyze a class of multidimensional social evaluation function of which IDHI is a special case. This analysis aimed at providing an axiomatization in a similar framework as the one in Basili et al. (2017). We also discuss the origin of IHDI and its inequality aversion degree, which allowed us to introduce a context where the strong homotheticity axiom is violated. To treat this violation, we provide another axiomatization, weakening the strong homotheticity axiom, resulting in another social evaluation function. We contextualize it, providing an example that accommodates this new social evaluation function where the social planner acts following Harsanyi and Rawls' approaches for the population and attributes dimension, respectively. This new proposal seems to be more inequality averse than IHDI.

Appendix

In the appendix, we provide the proof of our results.

Let us denote $\succsim^*$ as a preference relation induced from $\succsim$. The following lemma gives us some useful properties which are important for the proofs of the Theorems 1 and 2.

Lemma 1 *The axioms A.1, A.2, A.3, A.6, A.9, and A.10 imply that:*

A.1:* $\succsim^*$ *is weak order;*

A.2:* $\succsim^*$ *is continuous if given any sequence* $\{(a_p, b_p)\}_{p\in\mathbb{N}}$ *such that* $a_p \succsim^* b_p$ *for all* $n \geq 1$, *if* $a_p \overset{||\cdot||_\infty}{\longrightarrow} a \in \mathbb{R}^m_+$ *and* $b_p \overset{||\cdot||_\infty}{\longrightarrow} b \in \mathbb{R}^m_+$, *then* $a \succsim^* b$;

A.3:* $\succsim^*$ *is monotone if for all* $a, b \in \mathbb{R}^m_+$, $a \geq b \Longrightarrow a \succsim^* b$, *furthermore if* $a \neq b \Longrightarrow a \succ^* b$;

A.6:* $\succsim^*$ *is strongly homothetic if for all* $a, b, c \in \mathbb{R}^m_+$, $a \succsim^* b \Longrightarrow a \otimes c \succsim^* b \otimes c$;

A.8:* $\succsim^*$ *is homothetic if for all* $a, b \in \mathbb{M}^+_{m\times 1}$ *and* $k \in \mathbb{R}_{++}$, $a \succsim^* b \implies ka \succsim^* kb$;

A.9:* $\succsim^*$ *is log-convex if for all* $a, b \in \mathbb{M}^+_{m\times 1}$ *and* $\lambda \in (0,1)$, $a \succsim^* b \implies a^\lambda \otimes b^{1-\lambda} \succsim^* b$;

A.10:* $\succsim^*$ *is power invariant if for all* $a, b \in \mathbb{M}^+_{m\times 1}$ *and* $k \in \mathbb{R}_{++}$, $a \succsim^* b \implies a^k \succsim^* b^k$.

Proof. Let us define $r \in \mathbb{M}^+_{m\times 1}$ and $A^{(r)} \in \mathbb{M}^+_{m\times n}$ such that,

$$A^{(r)} := \begin{pmatrix} r_1 & 1 & \cdots & 1 \\ r_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ r_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n}$$

A.1*: (Completeness) It is easy to see that $A^{(a)} \succsim A^{(b)}$ or $A^{(b)} \succsim A^{(a)}$ or both. It implies $a \succsim^* b$ or $b \succsim^* a$ or both.

(Transitivity) If $A^{(a)} \succsim A^{(b)}$ and $A^{(b)} \succsim A^{(c)}$, then $A^{(a)} \succsim A^{(c)}$. It also implies that if $a \succsim^* b$ and $b \succsim^* c$, then $a \succsim^* c$.

A.2*: Suppose any sequence $\{(a_p, b_p)\}_{p\in\mathbb{N}}$ such that $a_p \succsim^* b_p$. Now, taking $A^{(a_p)}$ and $A^{(b_p)}$ where, from *A.2*,

$$A^{(a_p)} = \begin{pmatrix} a_{p1} & 1 & \cdots & 1 \\ a_{p2} & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{pm} & 1 & \cdots & 1 \end{pmatrix}_{m\times n} \succsim A^{(b_p)} = \begin{pmatrix} b_{p1} & 1 & \cdots & 1 \\ b_{p2} & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ b_{pm} & 1 & \cdots & 1 \end{pmatrix}_{m\times n}$$

If $A^{(a_p)} \overset{||\cdot||_\infty}{\longrightarrow} A^{(a)} \in \mathbb{M}^+_{m\times n}$ and $A^{(b_p)} \overset{||\cdot||_\infty}{\longrightarrow} A^{(b)} \in \mathbb{M}^+_{m\times n}$, then $A^{(a)} \succsim A^{(b)}$. It implies that $a_p \succsim^* b_p$ and, consequently, $a \succsim^* b$.

A.3*: Let $a, b \in \mathbb{M}^+_{m\times 1}$, such that $a \geq b$. Hence, from *A.3*, for all $i = 1, \cdots, m$, $a_i \geq b_i \implies A^{(a)} \geq A^{(b)} \implies A^{(a)} \succsim A^{(b)}$. Then $a \succsim^* b$.

A.6*: Let $a, b, c \in \mathbb{M}^+_{m\times 1}$ such that $a \succsim^* b$. Hence, $A^{(a)} \succsim A^{(b)}$, where,

$$A^{(a)} := \begin{pmatrix} a_1 & 1 & \cdots & 1 \\ a_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n} \quad \text{and } A^{(a)} := \begin{pmatrix} a_1 & 1 & \cdots & 1 \\ a_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n}$$

A.8*: (Homotheticity) Let $a, b \in \mathbb{M}^+_{m\times 1}$ such that $a \succsim^* b$. Then, if $A^{(a)} \succsim A^{(b)}$, by taking $k \in \mathbb{R}_{++}$, we can say that $A^{(a)} \succsim A^{(b)} \implies (kA_1^{(a)}, A_{-1}^{(a)}) \succsim (kA_1^{(b)}, A_{-1}^{(b)})$, which implies that $ka \succsim^* kb$.

A.9*: (Log-convexity) Let $a, b \in \mathbb{M}^+_{m\times 1}$ such that $a \succsim^* b$. Hence $A^{(a)} \succsim A^{(b)}$. Taking $\lambda \in [0, 1]$ if $A^{(a)} \succsim A^{(b)} \implies \Big((A_1^{(a)})^\lambda \otimes (A_1^{(b)})^{1-\lambda}, A_{-1}^{(a)}\Big) \succsim A^{(b)}$ where,

$$\Big((A_1^{(a)})^\lambda \otimes (A_1^{(b)})^{1-\lambda}, A_{-1}^{(a)}\Big) = \begin{pmatrix} a_1^\lambda b_1^{1-\lambda} & 1 & \cdots & 1 \\ a_2^\lambda b_2^{1-\lambda} & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_m^\lambda b_m^{1-\lambda} & 1 & \cdots & 1 \end{pmatrix}_{m\times n}$$

which implies that $a^\lambda \otimes b^{1-\lambda} \succsim^* b$.

A.10*: (Power Invariance) Let $a, b \in \mathbb{M}^+_{m\times 1}$ such that $a \succsim^* b$. Then, if $A^{(a)} \succsim A^{(b)}$, by taking $k \in \mathbb{R}_{++}$, we can say that $A^{(a)} \succsim A^{(b)} \implies \Big((A_1^{(a)})^k, A_{-1}^{(a)}\Big) \succsim \Big((A_1^{(b)})^k, A_{-1}^{(b)}\Big)$, where

$$\Big((A_1^{(a)})^k, A_{-1}^{(a)}\Big) = \begin{pmatrix} a_1^k & 1 & \cdots & 1 \\ a_2^k & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_m^k & 1 & \cdots & 1 \end{pmatrix}_{m\times n}$$

which implies that $a^k \succsim^* b^k$

■

Theorem 1

Proof.

Most of the *only if* part is straightforward. Our proof is limited to showing that the preference relation $\succsim$ satisfies independence, strong homotheticity, and Pigou-Dalton principle.

- *Independence:* We know that

  $\sum\limits_{j=1}^{n} \psi(\prod\limits_{i=1}^{m} a_{ij}^{\alpha_i}) \geq \sum\limits_{j\neq j_1} \psi(\prod\limits_{i=1}^{m} a_{ij}^{\alpha_i}) + \psi(\prod\limits_{i=1}^{m} ({a'_{ij_1}}^{\alpha_i})$

  $\iff \sum\limits_{j\neq j_1} \psi(\prod\limits_{i=1}^{m} b_{ij}^{\alpha_i}) + \psi(\prod\limits_{i=1}^{m} a_{ij_1}^{\alpha_i}) \geq \sum\limits_{j\neq j_1} \psi(\prod\limits_{i=1}^{m} b_{ij}^{\alpha_i}) + \psi(\prod\limits_{i=1}^{m} ({a'_{ij_1}}^{\alpha_i})$

  It turns out that,

$$\psi(\prod_{i=1}^{m} a_i^{\alpha_i}) \geq \psi(\prod_{i=1}^{m} ({a'_i}^{\alpha_i}) \iff \psi(\prod_{i=1}^{m} a_i^{\alpha_i}) \geq \psi(\prod_{i=1}^{m} ({a'_i}^{\alpha_i}).$$

- *Strong Homotheticity:* From the axiom, $\sum_{j=1}^{n} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) \geq \sum_{j \neq j_1} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) + \psi(\prod_{i=1}^{m} b_{ij_1}^{\alpha_i}) \Longrightarrow$ $\sum_{j \neq j_1} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) + \psi(\prod_{i=1}^{m} (c_i a_{ij_1})^{\alpha_i}) \geq \sum_{j \neq j_1} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) + \psi(\prod_{i=1}^{m} (c_i b_{ij_1})^{\alpha_i})$
  which gives us,

$$\psi(\prod_{i=1}^{m} a_i^{\alpha_i}) \geq \psi(\prod_{i=1}^{m} b_i^{\alpha_i}) \Longrightarrow \psi(\prod_{i=1}^{m} (c_i a_i)^{\alpha_i}) \geq \psi(\prod_{i=1}^{m} (c_i b_i)^{\alpha_i}).$$

- *Pigou-Dalton Principle:* Let $J : \mathbb{M}_{m \times n} \to \mathbb{R}$ where $J(A)$ and $J(A_\varepsilon)$ is:

$$J(A) = \sum_{j \neq j_1, j_2} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) + \psi(\prod_{i=1}^{m} a_{ij_1}^{\alpha_i}) + \psi(\prod_{i=1}^{m} a_{ij_2}^{\alpha_i})$$
$$J(A_\varepsilon) = \sum_{j \neq j_1, j_2} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) + \psi(\prod_{i=1}^{m} (a_{ij_1} - \varepsilon_i)^{\alpha_i}) + \psi(\prod_{i=1}^{m} (a_{ij_2} + \varepsilon_i)^{\alpha_i}).$$

  We must show that $J(A) \geq J(A_\varepsilon)$ where A_ε is the population in which the poorer individual j_1 transfers $\varepsilon \geq 0$ to the richer individual j_2.

  Since ψ is strictly concave, we can write that

$$\psi(\prod_{i=1}^{m} a_{ij_1}^{\alpha_i}) + \psi(\prod_{i=1}^{m} a_{ij_2}^{\alpha_i}) \geq \psi(\prod_{i=1}^{m} (a_{ij_1} - \varepsilon_i)^{\alpha_i}) + \psi(\prod_{i=1}^{m} (a_{ij_2} + \varepsilon_i)^{\alpha_i})$$

  Adding $\left(\sum_{j \neq j_1, j_2} \psi(\prod_{i=1}^{m} a_{ij}^{\alpha_i}) \right)$ to both sides of the equation, we conclude that $J(A) \geq J(A_\varepsilon)$.

This proves the *only if* part. Now, let us show the *if* part.

Under the axioms *A.1, A.2, A.3 and A.4* using Theorem 3 in Debreu (1960), there exist n continuous and increasing functions $u_j : \mathbb{R}_+^m \longrightarrow \mathbb{R}$, such that,[17] for all $A, B \in \mathbb{M}_{m \times n}^+$,

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} u_j(A_j) \geq \sum_{j=1}^{n} u_j(B_j).$$

where $A = (A_1 \ldots A_j \ldots A_n)$ and $B = (B_1 \ldots B_j \ldots B_n)$, with $A_j, B_j \in \mathbb{M}_{m \times 1}^+$. Yet, u_j are unique up to affine transformation $\alpha u_j + \beta_j$ with $\alpha > 0$ and $\beta_j \in \mathbb{R}$.

By the Axiom *A.5*, we obtain that $u_j = u_{j'}$, for all $j, j' \in \{1, \ldots, n\}$. In this sense, there exists $u : \mathbb{R}_+^n \longrightarrow \mathbb{R}$, such that

$$A \succsim B \Longleftrightarrow \sum_{j=1}^{n} u(A_j) \geq \sum_{j=1}^{n} u(B_j). \tag{8}$$

[17] For $n \geq 3$

Given $x, y \in \mathbb{R}^m_+$ we define $\succsim^*$ on $\mathbb{R}^m_+$ by

$$x \succsim^* y \overset{def}{\iff} \begin{pmatrix} x_1 & 1 & \cdots & 1 \\ x_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n} \succsim \begin{pmatrix} y_1 & 1 & \cdots & 1 \\ y_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ y_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n}.$$

where $\succsim^*$ is represented by u and satisfies the axioms *A.1, A.2, A.3 and A.6* in Theorem 1 of Faro (2013) where there exists $v : \mathbb{R}^m_+ \to \mathbb{R}$ such that $v(x) = \prod_{i=1}^m x_i^{\alpha_i}$ with $\alpha \in \Delta^{m-1}_{++}$. Then,

$$x \succsim^* y \iff v(x) \geq v(y)$$

By *A.6*, taking $C_1 := \{z_1, z_2, \ldots, z_m\}$,
$A^{(x)} \succsim A^{(y)} \implies (C_1 \otimes A_1^{(x)}, A_{-1}^{(x)}) \succsim (C_1 \otimes A_1^{(y)}, A_{-1}^{(y)})$, which implies that

$$\begin{pmatrix} z_1x_1 & 1 & \cdots & 1 \\ z_2x_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_mx_m & 1 & \cdots & 1 \end{pmatrix} \succsim \begin{pmatrix} z_1y_1 & 1 & \cdots & 1 \\ z_2y_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_my_m & 1 & \cdots & 1 \end{pmatrix} \implies x \otimes z \succsim^* y \otimes z$$

Once $\succsim^*$ satisfies *A*.1, A*.2, A*.3* and *A*.6*, then from Faro (2013) (Theorem 1, Remark 3), we obtain the representation below:

$$x \succsim^* y \iff v(x) \geq v(y).$$

where $v(x) = \prod_{i=1}^m x_i^{\alpha_i}$ and $v(y) = \prod_{i=1}^m y_i^{\alpha_i}$.

Since $v, u : \mathbb{M}_{m\times n} \to \mathbb{R}$ are representations of $\succsim^*$, there exists $\psi : \mathbb{R} \to \mathbb{R}$ such that $u = \psi \circ v$.

Now, let us show that ψ is strictly concave.

Suppose that for all $a, b \in \mathbb{R}^2$, $b \geq a$ and for all $\varepsilon > 0$, one has

$$\psi(a) - \psi(a - \varepsilon) > \psi(b + \varepsilon) - \psi(b) \tag{9}$$

From *A.7* we know that, for all $x, y, \mathbb{1} \in \mathbb{M}^+_{m\times 1}$, such that $x > y$ we get[18]

$$u(x) - u(x - \varepsilon\mathbb{1}) > u(y + \varepsilon\mathbb{1}) - u(y).$$

Since $a = \prod_{i=1}^m x_i^{\alpha_i}$ and $b = \prod_{i=1}^m y_i^{\alpha_i}$ the condition 9 holds. Then, up to an increasing affine transformation, ther exists a unique $\psi : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave and continuous, such that for all $A, B \in \mathbb{M}^+_{m\times n}$,

$$A \succsim B \iff \sum_{j=1}^n \psi\left(\prod_{i=1}^m a_{ij}^{\alpha_i}\right) \geq \sum_{j=1}^n \psi\left(\prod_{i=1}^m b_{ij}^{\alpha_i}\right)$$

■

Proposition 3
Proof. We prove only the *if* part. The *only if* part is straightforward.

[18] $\mathbb{1}$ is a column vector with 1 everywhere.

Assume the axiom *A.8* is satisfied and $\succsim$ is defined on $\mathbb{M}^{++}_{m\times n}$. For all $j \in \{1, \ldots, n\}$, it is easy to see that $\sum_{j=1}^{n} \psi(a_j) = \sum_{j=1}^{n} \psi(b_j)$ implies $\sum_{j=1}^{n} \psi(ka_j) = \sum_{j=1}^{n} \psi(kb_j)$, for all $a_j, b_j \in \mathbb{M}^{++}_{1\times n}$ and $k \in \mathbb{R}_{++}$. We can thus apply the results of the classical *one*-dimensional social welfare theory for our purpose.[19] ■

Theorem 2

Proof.

To prove the *only if* part, it remains to show that the preference relation $\succsim$ is homothetic, log-convex and power invariant.

Firstly, define $\Gamma : \mathbb{M}^{+}_{m\times n-1} \to \mathbb{R}$ and $\gamma : \mathbb{M}^{++}_{m\times 1} \to \mathbb{R}$ where:

$$\Gamma(a) := \sum_{j\neq j_1}^{n-1} \psi\left(\min_{\alpha\in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i}\right)$$
$$\gamma(a) := \psi\left(\min_{\alpha\in C} \prod_{i=1}^{m} a_{ij_1}^{\alpha_i}\right)$$

- *Homotheticity*: We know that, $\Gamma(a) + \gamma(a) \geq \Gamma(a) + \gamma(b)$. Taking $k \in \mathbb{R}_{++}$, is easy to see that $\Gamma(a) + k\gamma(a) \geq \Gamma(a) + k\gamma(b)$ implies that $\succsim$ satisfies *A.9*.
- *Log-convexity*: From *A.10*, $\Gamma(a) + \gamma(a) \geq \Gamma(a) + \gamma(b)$ must imply $\Gamma(a) + \gamma(a)^{\lambda}\gamma(b)^{1-\lambda} \geq \Gamma(a) + \gamma(b)$. It turns out that $\gamma(a)^{\lambda} \geq \gamma(b)^{\lambda}$ and, consequently, $\gamma(a) \geq \gamma(b)$. This proves that $\succsim$ satisfies both *A.10* and *A.11*.

Now the *if* part. The proof of Theorem 1 is rather similar. Given $x, y \in \mathbb{R}^m_+$ we define $\succsim^*$ on $\mathbb{R}^m_+$ by

$$x \succsim^* y \overset{def}{\Longleftrightarrow} \begin{pmatrix} x_1 & 1 & \cdots & 1 \\ x_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n} \succsim \begin{pmatrix} y_1 & 1 & \cdots & 1 \\ y_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ y_m & 1 & \cdots & 1 \end{pmatrix}_{m\times n}$$

where $\succsim^*$ is represented by u and satisfies the axioms *A.1, A.2, A.3, A.9, A.10* and *A.11* in Theorem 3 of Faro (2013) where there exists $v : \mathbb{R}^m_+ \to \mathbb{R}$ such that $\Gamma(x) = \prod_{i=1}^{m} x_i^{\alpha_i}$ with $\alpha \in \Delta^{m-1}_{++}$. Then,

$$x \succsim^* y \Longleftrightarrow \Gamma(x) \geq \Gamma(y) \tag{10}$$

Once $\succsim^*$ satisfies *A*.1, A*.2, A*.3, A*.8, A*.9* and *A*.10*, then from Faro (2013) Theorem 3, the condition (10) holds, i.e.

$$x \succsim^* y \iff v(x) \geq v(y).$$

where $v(x) = \min_{\alpha\in C} \prod_{i=1}^{m} x_i^{\alpha_i}$ and $v(y) = \min_{\alpha\in C} \prod_{i=1}^{m} y_i^{\alpha_i}$.

Since $v, u : \mathbb{M}_{m\times n} \to \mathbb{R}$ are representations of $\succsim^*$, there exists a concave $\psi : \mathbb{R} \to \mathbb{R}$ such that $u = \psi \circ v$, which concludes the proof.

[19] For more details, see Aczél (1966), Atkinson (1970) and Kolm (1976a;b).

■

Theorem 4

Proof. From Corollary 3 one needs to compute $\min_{\alpha \in C} \prod_{i=1}^{m} a_{ij}^{\alpha_i}$ where $C = \{\alpha \in \Delta_{++}^{m-1},\ \underline{\delta_i} \leq \alpha_i \leq \overline{\delta_i},\ i = 1, \ldots, m\}$, indeed the computation will be performed as soon as $\min_{\alpha \in C} \sum_{i=1}^{m} \alpha_i \ln(a_{ij})$ will be obtained for any j.

For a fixed j: denoting i and C as above, it turns out that in the language of uncertainty this is a situation of regular uncertainty, the terminology of Jean Yves Jaffray who coined it,[20] that is under condition (4.6), $C = core(v)$ where $v(E) = \max \alpha(E)$ for all $E \in 2^{\{1,\ldots,m\}}$ and furthermore here the value of $v(E)$ is given in (4.7). This result was obtained by De Campos et al. (1994) and also states that v is a convex capacity.

As a consequence building upon Schmeidler (1986) comes that:

$$\min_{\alpha \in core(v)} \sum_{i=1}^{m} \alpha_i \ln(a_{ij}) = \int_{\{1,\ldots,m\}} \ln a_j \ dv$$

hence the formula (4.7) of Theorem 4, which completes the proof

■

[20] See Jaffray (1988) and Chateauneuf and Jaffray (1989).

Chapter 5

Business Cycles with Ambiguous Fiscal Policy

This chapter is a result of the working paper "Business Cycles with Ambiguous Fiscal Policy".

Abstract

This chapter analyzes the effect of confidence shocks on fiscal policy through a New Keynesian model with ambiguity averse agents and nominal rigidities. I also model the household's preference for holding risk-free assets and compare the effects of risk premium shocks in economic activity. I find that ambiguity shocks on the fiscal policy may generate business cycle comovement among output, consumption, investment and hours worked.

Keywords: Ambiguity aversion, business cycles, fiscal policy, liquidity preference.

5.1 Introduction

The 2008 crisis has raised questions about the best way to promote sustainable development through public policies. The turmoil in financial markets created by this crisis did not lead to a consensus on how to manage fiscal policy. This context expanded the range of possible scenarios and, consequently, increased the uncertainty about the fiscal policy to be carried out.

As risks increase, agents become more alert about decisions taken by authorities. It also becomes more important to foresee what governments will do in the near future. Households and investors fear the possibility of a public policy harming their consumption plans. Indeed, bad policies may affect future wealth and agents want to anticipate this kind of scenario, preventing future losses.

In this sense, uncertainty has been gaining relevance to understanding fluctuations on economic activity. How would an ambiguity averse household react when facing conflicting intangible information about future fiscal policy? This chapter aims at characterizing this context in a New Keynesian model, implementing two main strategies: 1) adding a shock of ambiguity *a la* Ilut and Schneider (2014) into fiscal policy shock and 2) adding a risk premium shock for risk-free bonds as proposed by Smets and Wouters (2007) and Fisher (2015).

I follow Ilut and Schneider (2014) modeling changes in ambiguity through shocks to confidence. Conflicting information that raises the perception of ambiguity is perceived by agents as a shock of confidence. However, a drop in agents' confidence does not mean that the budget deviation will occur. In order to compare the shock of confidence followed or not by a budget deviation, I expand the analysis adding another shock that captures agents' confidence, in a different way, independently of the government expenditure. For that, I introduce the risk premium shock, or liquidity shock as in Smets and Wouters (2007). A positive shock of risk premium raise agents' demand for risk-free bonds. This movement can be interpreted as a precautionary behavior of the agents whose decision to demand more risk-free bonds is perceived as a way to reduce the risk of future losses.

The risk premium shock generates a wedge between the central bank interest rate for risk-free bonds and the return on assets held by households. Smets and Wouters (2007) point out that a positive shock on this wedge expands the return required by households on assets and reduces consumption. It also increases the cost of capital and reduces the value of investment.

The context analyzed in this chapter is straightforward. Countries with strong and democratic institutions generally have public budgets approved by policymakers (public authorities) during the previous year before budget execution. However, policymakers usually have some degree of freedom to execute the budget throughout the year. Developed democracies have also an efficient system of checks and balances that limits policymakers' ability to deviate from the approved budget. Nevertheless, policymakers usually have some "space" to take sudden decisions that bypass the budget, in case of emergency events for instance. Because of this autonomy, agents might fear an improper use of such budget deviation, that would harm economic activity and, ultimately, their wealth. Unexpected budget deviations are perceived by agents as ambiguous when conflicting information prevents them to foresee this scenario correctly. The baseline model aims to capture this kind of agents' behavior, i.e. through ambiguity shock on the next period's government spending.

Some studies associate an increase in uncertainty with declines in aggregates such as output, consumption and hours worked (Bloom, 2009; Bloom et al., 2018; Ilut and Schneider, 2014; Alexopoulos and Cohen, 2015; Fernández-Villaverde et al., 2015). Other studies find none or less impact of uncertainty on economic activity (Bachmann and Bayer, 2013; Bachmann et al., 2013; Chugh, 2016; Bekaert et al., 2013; Born and Pfeifer, 2014).

More specifically, two main studies analyzed the effect of uncertainty on fiscal policy. Exploring the time-varying volatility in tax structure and government spending, Fernández-Villaverde et al. (2015) found a considerable impact on output, consumption, investment, hours worked and prices. Moreover, they show that when the economy is at zero lower bound the effects are even greater. On the other hand, Born and Pfeifer (2014) concluded that the role of policy risk to explain business cycle through uncertainty shocks is small. A two-standard deviation shock to policy risk decreases output by only 0.065 percent. Both studies considered uncertainty as time-varying volatility.

Ilut and Schneider (2014) proposed a new approach that deals with ambiguity in a business cycle model. Instead of considering uncertainty as time-varying volatility, they introduced the concept of ambiguity (Knightian uncertainty) in a New Keynesian model, which is defined through averse ambiguity agents, who assess intangible information to predict future productivity. For example, a raise in conflicting information, puzzling agents' beliefs, is perceived as an increase in ambiguity. Ilut and Schneider found that shocks of productivity and ambiguity jointly explain about 2/3 of business cycle frequency in the US.

Based on a similar methodology, Masolo and Monti (2017) also investigate the wedge among long-run inflation expectations, trend inflation, and inflation target. In their model, the policymaker is also ambiguity averse. The results imply that in a situation of a high degree of ambiguity, as in the early 1980s, the policymaker should be more *hawkish* in comparison with the same context but without ambiguity.

Fisher (2015) micro-founded the risk premium shock deepening its analysis into a New Keynesian model. He argues that this shock is an important source of fluctuations to explain comovement among output, consumption, investment and hours worked. Some recent studies as Barsky et al. (2014) and Christiano et al. (2015) suggest that risk premium shock is specifically important to interpret post-2008 crisis dynamics.

The hypothesis is that ambiguity averse agents have multiple beliefs and act under the worst-case belief. I define the worst-case belief here by considering that non-budgeted fiscal expenditures on the part of policymakers may produce a lower than expected return to society, damaging public accounts. Since households are Ricardian agents, a sudden increase in fiscal spending generates in the future a tax increase or a fiscal contraction affecting economic activity.

A positive risk premium shock has a similar effect when agents fear that the government will make a bad decision in the future. An expansion of risk-free bonds demand will immediately generate a corresponding fall in consumption and investment, consequently dropping output. This behavior is also coherent with the Ricardian hypothesis, i.e. when agents fear a future increase in taxes they aim to increase current savings to smooth intertemporal consumption. The difference is that a prior motive is being inputted for the government to increase taxes: non-budgeted fiscal spending.

The outline of this chapter is as follows: Section 5.2 presents the model. Section 5.3 analyzes the impulse response functions (IRFs) and section 5.4 contains the conclusion.

5.2 Model

This section presents a New Keynesian model pioneered by Christiano et al. (2005) and Smets and Wouters (2007) with staggered prices and wages, investment adjustment cost and many elements introduced in the literature since then.

5.2.1 Households

Households in the model are represented by a lifetime utility function with a vector of exogenous state variables $s_t \in$ S, where $s^t = (s_1, \ldots, s_t)$ that denotes the history up to date t. For every history s^t households project their consumption plan C by choosing the amount of consumption C_t, hours worked h_t and the amount of government bonds B_t. Utility is recursively defined as,

$$U(C; s^t) = \log(C_t) + \frac{h_t^{1+\sigma_l}}{1+\sigma_l} + \varepsilon_t^b V(B_t) + \beta \min_{p \in \mathcal{P}_t(s^t)} \mathbb{E}^p[U_{t+1}(C; s^t, s_{t+1})] \tag{5.1}$$

where $\mathcal{P}_t(s^t)$ is the set of conditional probabilities about state of the economy in the next period (s_{t+1}). This recursive formulation is dynamically consistent. I follow the approach introduced by Epstein and Schneider (2003) where the update rule is based on rectangular sets of priors. The standard rational expectation is obtained as a special case if the set of conditional probabilities $\mathcal{P}_t(s^t)$ contains only one belief. The parameter $\sigma_l \geq 0$ is the Frisch elasticity of labor supply, and β is the discount factor.

The risk premium shock ε_t^b denotes a stochastic preference for holding risk-free bonds issued by the government. The law of motion follows an AR(1):

$$\log \varepsilon_t^b = \rho_b \log \varepsilon_{t-1}^b + \upsilon_b \eta_t^b \tag{5.2}$$

where $\eta_t^b \sim \mathcal{N}(0,1)$.

The function $V(B_t)$, represents the real quantity of one-period government bonds purchased by the household at date t.

As stated by Fisher (2015), $V(B_t)$ must be positive, increasing and concave. To evaluate the role of risk-free bonds demand on household utility, the risk-free bond demand is given as,

$$V(B_t) = \begin{cases} \frac{B_t^{1-\sigma_b}}{1-\sigma_b} & \text{if } \sigma_b \neq 1 \\ \log(B_t) & \text{if } \sigma_b = 1 \end{cases}$$

where $\sigma_b \geq 0$ is the Frisch elasticity for bonds, i.e. the higher the σ_b, the more willing are agents to demand risk-free bonds if price increases. Moreover, σ_b may be interpreted as a degree of agents' risk aversion. In section 5.3 is analyzed the effect of different values of σ_b on the main aggregates.

Households own the installed capital stock K_t, subject to adjustment costs as introduced by Christiano et al. (2005).

$$K_{t+1} = (1-\delta) K_t + \left[1 - \frac{1}{2}\kappa \left(\gamma - \frac{I_t}{I_{t-1}}\right)^2\right] I_t, \tag{5.3}$$

where I_t denotes gross investment and $\kappa > 0$ is a parameter that regulates capital adjustment. The parameter γ represents the labor productivity.

The household's budget constraint is given by,

$$P_t C_t + P_t I_t + B_t = B_{t-1} R_{t-1} + P_t r_t^k K_t + W_t h_t - T_t P_t \tag{5.4}$$

where P_t is the index of intermediate good prices, R_t is the gross nominal interest rate, r_t^k is the nominal interest rate that remunerates capital, W_t is the wage rate for homogeneous labor services and T_t denotes net lump-sum tax.

5.2.2 Firms

The economy has two types of perishable goods. Perfectly competitive firms produce final goods Y_t which are made by the combination of a continuum of intermediate goods $Y_{j,t}$. Intermediate goods are made through capital K_t, labor H_t and final goods by monopolistically competitive firms using a Dixit-Stiglitz aggregation technology,

$$Y_t = \left[\int_0^1 Y_{j,t}^{\frac{1}{\lambda_f}}\, dj\right]^{\lambda_f}, \tag{5.5}$$

The parameter λ_f regulates the substitution elasticity among intermediate goods, given by $\lambda_f/(\lambda_f - 1)$. Intermediate goods firm i produces $Y_{j,t}$ according to a Cobb-Douglas production function with capital share α,

$$Y_{j,t} = K_{j,t}^{\alpha}\left(\gamma^t H_{j,t}\right)^{1-\alpha} - \Phi\gamma^t \tag{5.6}$$

The fixed cost of production Φ grows with labor productivity, ensuring that profits are zero in the steady state.

Demand for intermediate good j is,

$$Y_{j,t} = Y_t\left(\frac{P_t}{P_{j,t}}\right)^{\frac{\lambda_f}{\lambda_f - 1}} \tag{5.7}$$

The index of intermediate prices P_t is defined as a continuum of intermediate goods prices as,

$$P_t = \left[\int_0^1 P_{j,t}^{\frac{1}{1-\lambda_f}}\, dj\right]^{(1-\lambda_f)} \tag{5.8}$$

I follow Calvo (1983) in order to establish that a random group of intermediate good producers $(1 - \xi_p)$ re-optimizes the price at every period following the rule $P_{it} = \bar{\Pi}P_{i,t-1}$, where $\bar{\Pi}$ is the steady state inflation.

Whenever the firm is able to re-optimize its price, it maximizes the expected present discounted value of future profits,

$$E_t\sum_{s=0}^{\infty}(\xi_p)^s M_{t,t+s}\left[P_{j,t+s}Y_{j,t+s} - W_{t+s}H_{j,t+s} - P_{t+s}r_{t+s}^k K_{j,t+s}\right] \tag{5.9}$$

where $M_{t,t+s}$ denotes the stochastic discount factor of households owning firms. Both the rental rate on capital services $P_t r_t^k$ and the wage rate for homogeneous labor services W_t are purchased in competitive factor markets.

5.2.3 Labor market

The economy has also two types of labor: homogeneous and specialized. The first is offered by 'employment agencies'. The second is demanded by those agencies following the rule,

$$h_{i,t} = H_t\left(\frac{W_t}{W_{i,t}}\right)^{\frac{\lambda_w}{\lambda_w - 1}} \tag{5.10}$$

where λ_w measures substitutability level among specialized labor and $W_{i,t}$ is the wage rate for the ith type of specialized labor. As in Erceg et al. (2000), specialized labor is supplied by a continuum of households in a monopolistically competitive market represented by,

$$H_t = \left[\int_0^1 (h_{i,t})^{\frac{1}{\lambda_w}} \right]^{\lambda_w} \tag{5.11}$$

It is adopted here the strategy proposed by Smets and Wouters (2007) introducing staggered wages setting *a la* Calvo (1983). Households optimize their wages with probability of $(1 - \xi_w)$. With probability of ξ_w wages are set by the rule $W_{i,t} = \bar{\Pi}\gamma W_{i,t-1}$, where $0 < \xi_w < 1$.

5.2.4 Central Bank and Government

The economy's public sector is composed of the Central Bank and the Government. The Central Bank sets the nominal interest rate R_t concerning deviations of inflation from the target $\bar{\Pi}$ and output deviations from its potential $\bar{Y}$. Nominal interest rate is set as,

$$\frac{R_t}{\bar{R}} = \left(\frac{R_{t-1}}{\bar{R}} \right)^{\rho_R} \left[\left(\frac{\Pi_t}{\bar{\Pi}} \right)^{\alpha_\pi} \left(\frac{Y_t}{\bar{Y}\gamma^{(1-\alpha)t}} \right)^{\alpha_y} \right]^{1-\rho_R} \tag{5.12}$$

where the nominal interest rate in steady state is given by $\bar{R} = \bar{\Pi}\gamma/\beta$. The parameter $\alpha_\pi > 1$ measures the effect of inflation on the interest rate and ρ_R sets the weight of the lagged interest rate deviation.

Government one-period budget constraint is

$$G_t + B_{t-1}R_{t-1} = T_t + B_t \tag{5.13}$$

where G_t denotes government expenditures, set as $G_t = g_t\gamma^t\bar{Y}$, where g_t is a stationary stochastic process in terms of steady state output path. That is, the government budget constraint states that its spending plus repayment of bonds must be equal to tax plus bond income.

The government issues bonds B_t, collects lump-sum tax T_t and spends G_t. From the perspective of the fiscal policy authority (who has no ambiguous view about future government expenditure) the law of motion for g_t is,

$$\log\left(\frac{g_{t+1}}{\bar{g}}\right) = \rho_g \log\left(\frac{g_t}{\bar{g}}\right) + \upsilon_g \eta^g_{t+1} \tag{5.14}$$

where $\eta^g_t \sim \mathcal{N}(0,1)$ and $\bar{g}$ is the steady state government spending in terms of output.

The fiscal rule was taken from Blanchard and Perotti (2002) and Galí et al. (2007).

$$T_t = \phi_b B_t + \phi_g G_t \tag{5.15}$$

This condition states that lump-sum tax is defined by the sum of bonds issued and government spending, both regulated by sensitive responses ϕ_b and ϕ_g. These parameters govern the response of tax to bonds issuance and government spending, respectively. For example, a high tax response to bonds (high ϕ_b), means that an increase in tax implies a larger bond issuance equal to government expenditure. The reasoning is the same for ϕ_g. For example, Galí et al. (2007) show that a positive comovement of consumption and output in response to government spending shocks requires a sufficiently high response of ϕ_b and sufficiently low response of ϕ_g.

Finally, the general resource constraint is,

$$Y_t = C_t + I_t + G_t \tag{5.16}$$

5.2.5 Mechanism of the confidence shock

Following Gilboa and Schmeidler (1989), Epstein and Schneider (2003) and Ilut and Schneider (2014), this model allows agents to have multiple beliefs about future government expenditures. Suppose thus the agent has conflicting information about whether the government will strictly pursue the approved budget or spend more, in the following quarter. This truncated information induces agents to treat the following quarter's government expenditure as ambiguous. A raise of ambiguity represents an increase in the lack of confidence, which explains the shock. This lack of confidence obstruct agents in inferring trustworthy predictions. Besides having multiple beliefs ambiguity is not desirable by agents, which explains the aversion. Being ambiguity averse, in the Gilboa and Schmeidler sense, means that agents act as if the worst-case scenario will happen, i.e., in this case, the government will spend beyond the budget.

When different intangible information is diffuse, indicating opposite scenarios, agents might be less confident about their forecast. This lack of confidence is not necessarily random and cannot be captured by the stochastic term η_t^g. The term μ_t^* measures the lack of confidence. Hence, government spending law of motion, from the perspective of the agents, is given by

$$\log\left(\frac{g_{t+1}}{\bar{g}}\right) = \rho_g \log\left(\frac{g_t}{\bar{g}}\right) + \mu_t^* + \upsilon_u \eta_{t+1}^u \tag{5.17}$$

where $\eta_t^u \sim \mathcal{N}(0,1)$.

It is not known the probability distribution of the belief set expressed by μ_t^*, suggesting why it is so hard to predict. This set is denoted by,

$$\mu_t^* \in [a_t - 2|a_t|,\ a_t] \tag{5.18}$$

where a_t is defined as the time-varying ambiguity. As pointed out by Ilut and Schneider (2014), a_t represents an indicator of intangible information available at date t about government expenditure at $t+1$.

This set is symmetric, centered around zero. Each element of μ_t in the interval (5.18) is related to one belief s_{t+1} contained in the set of conditional probabilities $\mathcal{P}_t(s^t)$ in the households' utility function (5.1). When $\mathcal{P}_t(s^t)$ has only one belief, the interval will thus contain also only one element, characterizing the standard rational expectation setting. Yet, the more conflicting is the information gathered by agents, the greater the interval is. The time-varying ambiguity follows the law of motion below,

$$a_{t+1} - \bar{a} = \rho_a(a_t - \bar{a}) + \upsilon_a \eta_{t+1}^a \tag{5.19}$$

where $\eta_t^a \sim \mathcal{N}(0,1)$ and $\bar{a}$ is the steady state ambiguity level.

Worst-case steady state

One of the main innovations of Ilut and Schneider (2014) was to characterize the context of worst-case in the model's dynamic state as well as in its steady state structure.

Following the same strategy, the worst-case steady state of government spending in terms of output path is,

$$g = \bar{g} + \frac{\bar{a}}{1 - \rho_g} \tag{5.20}$$

That is, in the absence of ambiguity, $g = \bar{g}$. For example, suppose that $\bar{g} = 0.2$ (government's output share represents 20% of output), i.e. when such spending does not deviate from budget. Then the worst-case steady state of government spending is the sum of the unambiguous parameter ($\bar{g}$) and the ambiguous term ($\bar{a}/(1-\rho_g)$). The higher $\bar{a}$ is, the higher is the difference $g - \bar{g}$. In other words, a large steady state ambiguity means that agents expect government expenditures greater than usual.

5.2.6 Parameters

Most of the parameters used in the model are common in the literature. The parameters' values are set quarterly. The demand elasticity of prices and wages is based in Christiano et al. (2005). Ambiguity is measured in the sense of Ilut and Schneider (2014). I assume that μ_t^* "looks like" a normal *iid* process and it is independent from η_t^g. I also assume that the empirical second moment ($1/T \sum_{t=1}^{T} \mu_t^* \eta_t^g$) converges to zero. Taken together, these two assumptions assure that the empirical mean of the innovation to g_t converges to zero and the empirical variance converges to $\upsilon_g^2 > \upsilon_u^2$.

The condition $\mu_t^* = a$ means that the best forecast in the set $[-a, a]$ is the supremum a. In this case, a is the closest forecast to the true conditional mean μ_t^*. Hence, since μ_t^* looks like an *iid* normal process with mean zero and variance $\upsilon_g^2 - \upsilon_u^2$, the confidence interval where the supremum is expresses the best forecast at least 5% of the time, then $a \leq 2\sqrt{\upsilon_g^2 - \upsilon_u^2}$. In this case, one should expect that the mean of the ambiguity $\bar{a}$ is smaller than υ_g. Larger values of $\bar{a}$ imply that the ambiguity must be excessively high in order to explain intangible information about government expenditure shocks. In order to control it, define $n \in (0, 1)$ where $\bar{a} = n\upsilon_g$ and $\upsilon_a = \upsilon_n \upsilon_g$. The values assumed in the model are on Table 5.1.

The values of the parameters ϕ_g and ϕ_b are taken from Galí et al. (2007) and those of Calvo price/wage from Smets and Wouters (2007).

5.3 Results

In this section is presented a comparison of the confidence shock in the worst-case setup and traditional fiscal policy shock. I also compare the confidence shock with the risk premium shock. Finally, is presented the effect of σ_b on the main aggregates.

5.3.1 Worst-case *versus* fiscal policy shock

The first question that raises about the model presented in Section 5.2 is the difference between a confidence shock and a fiscal policy shock. Considering the confidence shock under the worst-case scenario from the agents' perspective. To conceive this comparison, I have replicated the parameters' value of the law of motion g_t (dashed green) in a_t (solid blue) to simulate a similar dynamic behavior.

Table 5.1 – Structural Parameters

Parameter	Description	Value
α	Capital share	0.30
β	Discount factor	0.25
δ	Depreciation rate of capital	0.025
ξ_p	Calvo price	0.375
ξ_w	Calvo wage	0.375
γ	Natural growth rate	0.4
$\bar{\pi}$	Inflation target	0.6
κ	Investment adj. cost	4
α_π	Inflation response	1.75
α_y	Output Response	0.2
ρ_R	Interest Smoothing	0.5
n	Ambiguity Level	0.5
ρ_a	Ambiguity Persistence	0.5
ρ_g	Gov. spending Persistence	0.9
υ_n	Ambiguity	0.005
λ_f	Demand Elasticity Goods	1.2
λ_w	Demand Elasticity Wages	1.05
σ_l	Labor Supply Elasticity	1
Φ	Fixed Cost	0.0434
$\bar{g}$	Government Spending Share	0.2
υ_a	Ambiguity Standard Deviation	0.0005

A third comparison is included to represent the context when the ambiguity persistence (ρ_a) is zero. It means that the ambiguity shock on fiscal policy is confined to the stochastic component ($\upsilon_a \eta_t^a$). The absence of temporal persistence on ambiguity emphasizes the difference of the stochastic shock in a_t (solid red) and g_t (dashed green).

From the agents' worst-case perspective, the law of motion of fiscal policy contains an ambiguous component, following the strategy introduced by Ilut and Schneider (2014). The main goal is to simulate a context in which agents lose the confidence of the fiscal policy in the following quarter. Unexpected increases in government expenditures are usually considered by agents as a bad political decision.[1]

It is important to emphasize that this chapter does not assess whether an expansive fiscal policy is positive or negative for economic activity. The main point is the unexpected spending that surprises agents and harms the public budget. When the government suddenly applies public resources in areas not previously discussed with the society or even spends more than expected in ongoing projects, these decisions may negatively affect the economic activity. Usually, this kind of unexpected expenditure is perceived by agents as a loss of confidence, which increases future ambiguity.

More specifically, the worst scenario happens when $\mu_t^* = a_t$, i.e. the maximum deviation of the

[1] I do not analyze in this chapter the occurrence of exceptional events that produce a heavy negative impact on economy aggregates as, for instance, natural disasters or great economic depressions. The effect of unexpected expenditure in those cases may have a different dynamic compared with this model.

public budget expected by the agent. This effect increases government expenditures, in the agents' worst-case perspective, and affects the business cycle.

A fiscal policy shock increases government expenditure since the first quarter, generating a similar movement of output. The fiscal rule induces a respective increase in tax and bonds, which reduces the household's consumption.

A larger supply of risk-free bonds, induces a raise of their interest rate, decreasing the spread $(\mathbb{E}_t r^k_{t+1} - R_t)$. Inflation and marginal cost also increase, in opposition to the confidence shock.

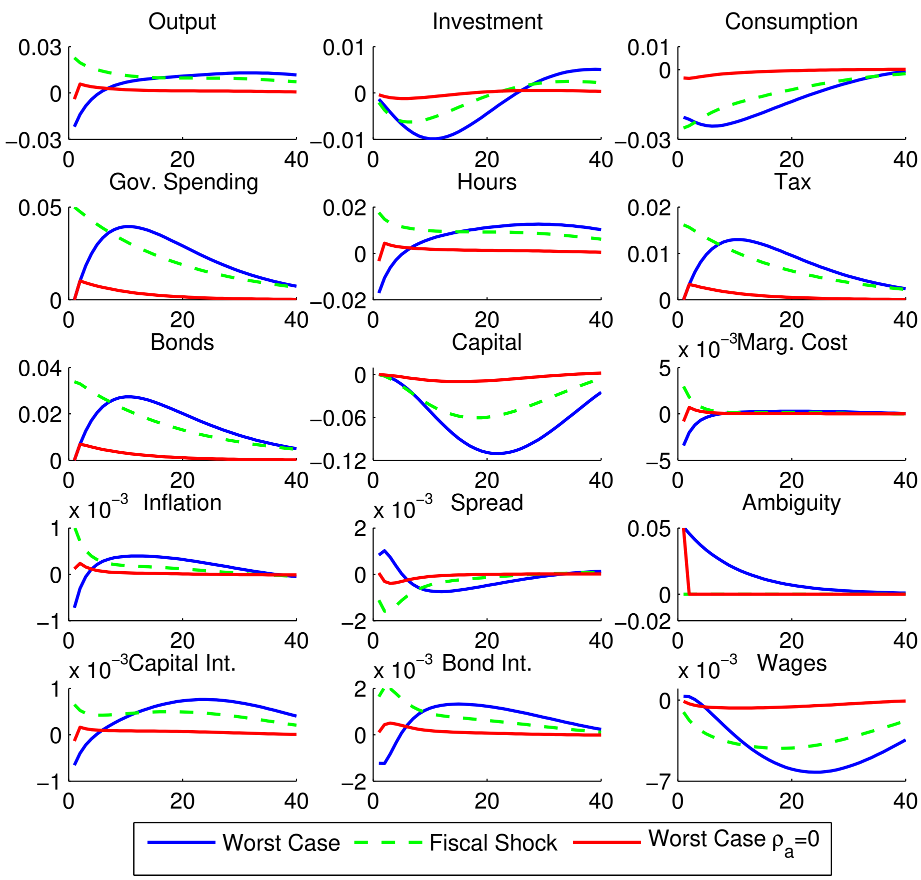


Figure 5.1 – Impulse Response of a positive confidence shock under the worst-case belief and government expenditure shock

Most of the variables in Figure 5.1 present an inverse response to a positive fiscal policy and a confidence shock. Because of an increase of ambiguity, output, hours and the interest rates (bonds and capital) starts below the steady state. Thereafter, with the rise of government spending, those variables recover smoothly as long as the confidence shock dissipates.

On the other hand, investment, consumption, and wages have a similar reaction to both shocks (blue and green lines). Real Wages decrease because of inflation and hours response to the fiscal policy. In the meantime, confidence shock makes real wages decrease because of the fall of inflation and hours.

Investment drops as a result of crowding out effect and consumption follows output and hours decrease with a subsequent rise of taxes.

Moreover, despite the consistency of the theoretical results, it is important also to evaluate these results by estimating some parameters with actual data. Another interesting exercise is to analyze whether the public debt is below or above its steady state to evaluate the impact of an unexpected expansion of government spending on output. Last, it is important to analyze the implications at the zero lower bound (ZLB) scenario.

5.3.2 Worst-case *versus* risk premium shock

Another interesting comparison is the confidence and risk premium shock. Once again, the former characterizes the perspective of ambiguity averse agents experiencing a negative shock on their confidence about future fiscal policy. The latter represents a positive shock on risk-free assets demand. A positive shock of risk premium, among other possibilities, may represent a fall of agents' confidence.

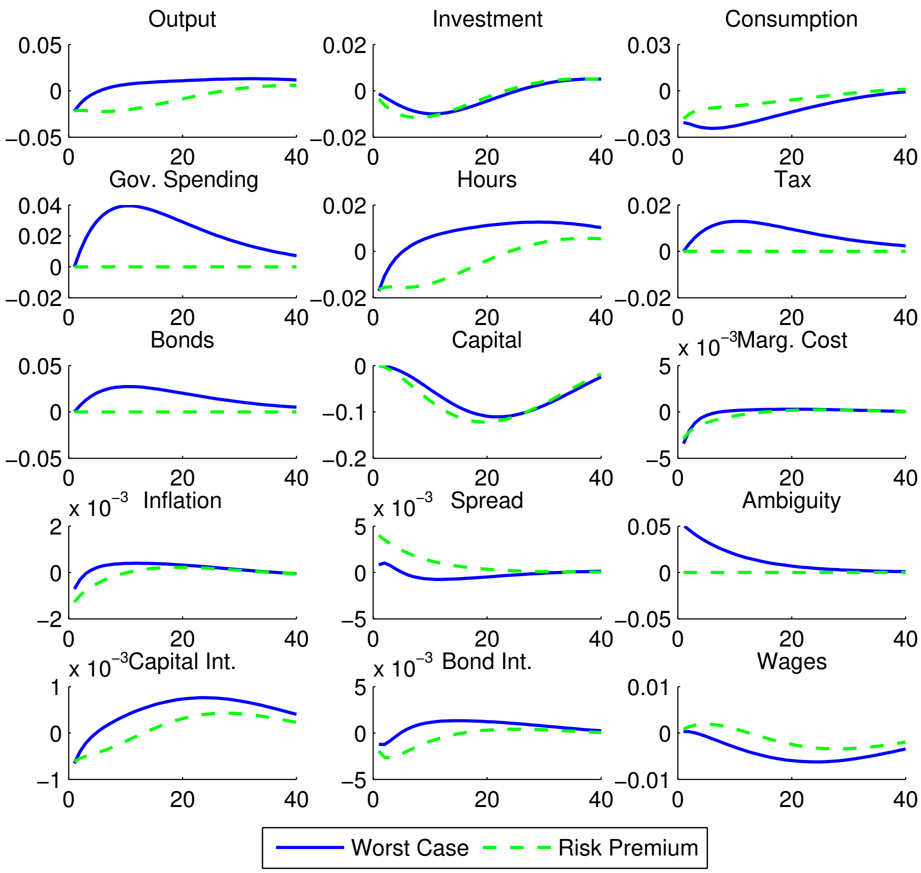


Figure 5.2 – Impulse Response of a positive confidence shock under worst-case belief and risk premium shock

The confidence shock under agents' worst-case perspective has an analogous effect as presented in Figure 5.1 except for the parameters of the law of motion a_t, which assumes the values in Table 5.1.

Figure 5.2 shows that an unexpected increase of the fiscal shock is firstly reflected in a proportional increase in government expenditure. From the resource constraint (5.16), an increase in government expenditure generates a current reduction of household consumption and output. Investment reacts to the loss of confidence, falling during the few first quarters. With adjustment costs, investment takes longer than consumption to converge to its steady state.

Output initially decreases because of the ambiguity positive shock, but then begins to recover due to the effective increase of government spending.

From the Fiscal rule (5.15), tax is regulated by government spending and bonds issue: in a context of expansive fiscal policy, the issuance of bonds must necessarily increase, generating a similar effect on tax.

Confidence shock causes deflation at the beginning due to the drop in output. Then with the subsequent expansion of government spending, inflation gains momentum, surpassing its steady state before converging to equilibrium.

Hours worked follow the same fluctuation, decreasing in the beginning because of ambiguity. Real wage drops due to deflation and takes longer to reach the steady state because of the rigidity implied by Calvo's rule. In sum, an increase of ambiguity has the most negative impact on investment and capital among all variables, denoting that confidence is a quite important element for entrepreneurs to decide whether to invest or not.

Consider now a similar effect on agents' confidence, but without ambiguity effect on government spending response. Governments with bad reputation call agents to have a precautional reaction to mitigate the risk of loss. However, this behavior is not always followed by a government undesirable decision. Nevertheless, the attitude taken by agents is enough to affect the business cycle. This background is interpreted here through a positive shock on risk-free bonds demand.

This positive shock to demand for safe and liquidity assets may be motivated by several features. Uncertainty about future fiscal policy might be considered as one of the sources. I follow here the same strategy used by Ilut and Schneider (2014) when they compare the confidence shock about future productivity with the disutility of work.

Christiano et al. (2015) refer to risk premium shock as a consumption wedge, given the negative effect on consumption caused by a larger demand for safe and liquid assets. As in their results, This consumption wedge leads to a deterioration in labor markets condition, with a persistent drop of hours worked and real wages.

In comparing both ambiguity and risk premium shocks, is interesting to see that both generate a comovement fall in output, investment, and consumption. Despite that, such shocks have some mechanic differences.

An increase in demand for risk-free bonds implies necessarily a fall of consumption *via* household's resource constraint (5.4) and also a fall of output through resource constraint (5.16). As a result of a decrease in consumption, investment, and output, inflation also reduces.

Since government spending increases inflation above its steady state, real wages decrease with the confidence shock and takes longer to recover because of Calvo stickiness. But with the risk premium shock, hours worked take longer to recover, expanding real wages during the first few quarters. However, real wages fall as inflation increases with the recovery of aggregate demand.

Furthermore, as pointed out by Fisher (2015), the interest rate of bonds decreases as demand for bonds increases.[2] The current capital interest rate drops only in the beginning, as pointed out by Smets and Wouters (2007), and takes longer to recover compared with the confidence shock.

The risk-free interest rate decreases less than the capital interest rate, exhibiting a larger spread ($\mathbb{E}_t r^k_{t+1} - R_t$), as the main results in the literature. This wedge takes more than 20 quarters to disappear.

According to Smets and Wouters (2007), this wedge increases the expected return on assets and reduces current consumption, as already mentioned. In addition, it also increases the cost of capital (in comparison to risk-free bonds) and reduces the value of investment and capital, as we can see in Figure 5.1.

[2]The supply of bond decreases jointly with the drop of its interest rate on the same scale. This fall is not captured by the graph in Figure 5.2 because the worst-case result has a bigger scale.

5.3.3 Risk aversion effect

Since the risk premium shock, as introduced here, is a recent element in the literature, I also analyze the effect of the relative risk aversion coefficient (σ_b) on the main aggregates and price variables.

Briefly, an increase of σ_b reflects a greater aversion to risk by the agents, taking into account that they are less willing to take risks considering the same wealth and the same economic context.

Figure 5.3 shows that larger values of σ_b potentialize negative effects on consumption, investment, hours worked and output. A decrease in output makes tax and bonds decrease as well, due to the fiscal rule and government budget constraint. The real wages increase in the first few quarters because of the fall of inflation and hours worked and decrease with the subsequent recovery of both variables.

Moreover, the wedge between capital interest rate at $t+1$ and risk-free bonds interest rates at t also increases, since the increment in the demand for risk-free bonds causes their interest rates to fall sharper than the capital interest rate for the subsequent periods.

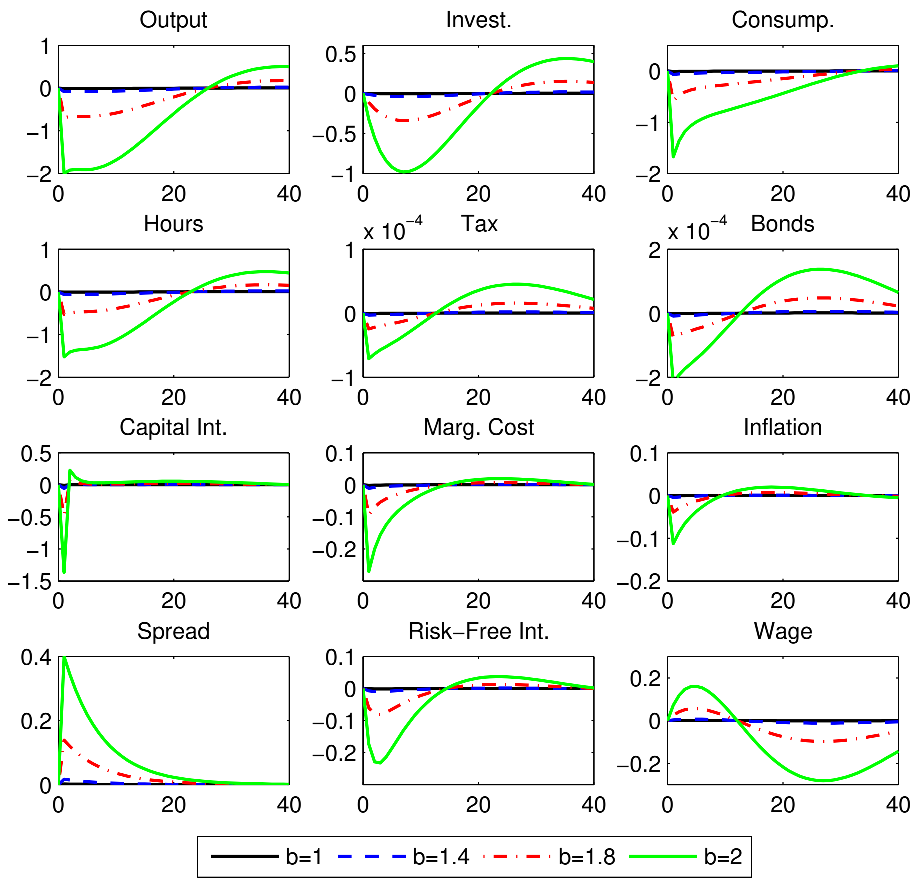


Figure 5.3 – Impulse Response Function to a increase of ε_t^b

5.4 Conclusion

Departing from a different uncertainty approach than the one used by Fernández-Villaverde et al. (2015), I found that the role of uncertainty in fiscal policy is, in the same sense, important to explain fluctuations of the main aggregates in the model.

As Fernández-Villaverde et al. (2015) pointed out, most of the economic decision-making is subject to pervasive uncertainty, in particular, to uncertainty about future fiscal policy. This chapter contributes with further evidence that an increase in fiscal policy uncertainty increases the negative effect on economic activity. In fact, it is important to estimate the main parameters to compare the theoretical results with actual data.

One natural extension of this chapter is the analysis of the effect of ambiguity on the business cycle through taxation on consumption, labor, and capital. It might be interesting to evaluate alternative features of New Keynesian models as well as alternative approaches to ambiguity.

This chapter does not intend to conclude that ambiguity is the main source of business cycle fluctuations, but these findings highlight the influence of ambiguity on economic activity.

The business cycles literature is just starting to explore the Knightian uncertainty approach and there still are plenty of features to take into account in future researches. It would be interesting to explore some other features in the business cycle through the ambiguity approach: open small economies, financial frictions and confidence shocks in prices of assets. Finally, the source of the confidence shock, which I treat here as exogenous, is an interesting element to be analyzed in a joint effort by economic and political science.

Appendix

It is presented in this appendix the dynamic equilibrium and First Order Condition (FOC) of the model. To induce stationarity, the variables are scaled to solve the model:

$$c_t = \frac{C_t}{\gamma^t};\; y_t = \frac{Y_t}{\gamma^t};\; k_t = \frac{K_t}{\gamma^t};\; i_t = \frac{I_t}{\gamma^t};\; \lambda_{z,t} = \lambda_t P_t \gamma^t$$

where λ_t is the Lagrange multiplier of the household budget constraint. The term ξ_t is the Lagrange multiplier of capital accumulation in consumption units,

$$Q_{K,t} = \frac{\xi_t}{\lambda_t}$$

Prices are determined as,

$$q_t = \frac{Q_{K,t}}{P_t};\; \bar{w}_t = \frac{W_t}{\gamma^t P_t}$$

and the variables controlled by the government are scaled by the output,

$$g_t = \frac{G_t}{Y_t};\; t_t = \frac{T_t}{Y_t};\; b_t = \frac{B_t}{P_t Y_t}$$

Moreover, Calvo prices and wages are scaled as,

$$p_t^* = (P_t)^{-1} \left(\int_0^1 P_{j,t}^{\frac{\lambda_f}{1-\lambda_f}} dj \right)^{\frac{1-\lambda_f}{\lambda_f}};\; w_t^* = (W_t)^{-1} \left(\int_0^1 W_{i,t}^{\frac{\lambda_w}{1-\lambda_w}} \right)^{\frac{1-\lambda_w}{\lambda_w}}$$

where households are referred to as i and firms as j.

Household specialized labor $h_{i,t}$ is aggregated as h_t, which allow us to write homogeneous labor l_t in terms of h_t,

$$l_t := \int_0^1 H_{j,t}\; dj = (w_t^*)^{\frac{\lambda_w}{\lambda_w - 1}} h_t; \quad \text{where } h_t := \int_0^1 h_{i,t}\; di$$

From the scaled form above, below is presented the nonlinear equilibrium conditions of the model. Conditional moments under worst-case belief $\mu_t^* = a_t$ is denoted by asterisk.

Households

Households maximize their utility (5.1) subject to budget constraint (5.4), capital accumulation (5.3) and labor demand (5.10). Marginal utility of consumption is given by,

$$\lambda_{z,t} = \frac{1}{c_t}$$

Capital accumulation decision (k_{t+1})

$$\lambda_{z,t} = E_t^* \frac{\beta}{\pi_{t+1}\gamma} \lambda_{z,t+1} R_{t+1}^k$$

Capital return is defined as,

$$R_t^k = \frac{\left(r^k + (1-\delta)\, q_t\right) \pi_t}{q_{t-1}}$$

Capital accumulates as,

$$k_{t+1} = \frac{(1-\delta)\, k_t}{\gamma} + \left[1 - S\left(\frac{i_t \gamma}{i_{t-1}}\right)\right] i_t$$

where the investment adjust is $S\left(\frac{i_t\gamma}{i_{t-1}}\right) = \frac{1}{2}\kappa\left(\gamma - \frac{i_t\gamma}{I_{i-1}}\right)^2$.

Investment decision (i_t),

$$\lambda_{z,t} = \lambda_{z,t} q_t \left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right) - S'\left(\frac{i_t\gamma}{i_{t-1}}\right)\frac{i_t\gamma}{i_{t-1}}\right] + \beta E_t^* \frac{\lambda_{z,t+1}}{\gamma} q_{t+1} S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2$$

Bond decision is,

$$\lambda_{z,t} = \frac{\varepsilon_t^b}{b_t^{\sigma_b}} + E_t^* \frac{\beta}{\pi_{t+1}\gamma} \lambda_{z,t+1} R_t$$

Firms

Firms minimize costs subject to the renting of labor and capital for each unit of output:

$$m_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\left(r_t^k\right)^{\alpha} \tilde{w}_t^{1-\alpha}$$

Moreover, firms also must equalize marginal costs with the cost to rent one unit of capital divided by the marginal capital productivity:

$$m_t = \frac{r_t^k}{\alpha\left(\frac{\gamma l_t}{k_t}\right)^{1-\alpha}},$$

where the capital-labor ratio is the same for all firms.

Production function is,

$$y_t = \left(p_t^*\right)^{\frac{\lambda_f}{\lambda_f - 1}}\left[\left(\frac{k_t}{\gamma}\right)^{\alpha} l_t^{1-\alpha} - \Phi\right]$$

Calvo Prices

Following Calvo's method, the price p_t^* is given as,

$$p_t^* = \left[(1-\xi_p)\left(\frac{\Psi_{p,t}}{F_{p,t}}\right)^{\frac{\lambda_f}{1-\lambda_f}} + \xi_p\left(\frac{\bar{\pi}}{\pi_t} p_{t-1}^*\right)^{\frac{\lambda_f}{1-\lambda_f}}\right]^{\frac{1-\lambda_f}{\lambda_f}}$$

where the terms $\Psi_{p,t}$ e $F_{p,t}$ are,

$$\Psi_{p,t} = \lambda_f \lambda_{z,t} y_t m_t + \beta \xi_p E_t^* \left(\frac{\bar{\pi}}{\pi_{t+1}} \right)^{\frac{\lambda_f}{1-\lambda_f}} \Psi_{p,t+1}$$

$$F_{p,t} = \lambda_{z,t} y_t + \beta \xi_p E_t^* \left(\frac{\bar{\pi}}{\pi_{t+1}} \right)^{\frac{1}{1-\lambda_f}} F_{p,t+1}$$

$$\Psi_{p,t} = F_{p,t} \left[(1-\xi_p)^{-1} \left(1 - \xi_p \left(\frac{\bar{\pi}}{\pi_t} \right)^{\frac{1}{1-\lambda_f}} \right) \right]^{1-\lambda_f}$$

Calvo wages
The wage (w_t^*) according to Calvo method is,

$$w_t^* = \left[(1-\xi_w) \left(\frac{\Psi_{w,t}}{\tilde{w}_t F_{w,t}} \right)^{\frac{\lambda_w}{1-\lambda_w(1+\sigma_l)}} + \xi_w \left(\frac{\bar{\pi}\gamma}{\pi_{w,t}} w_{t-1}^* \right)^{\frac{\lambda_w}{1-\lambda_w}} \right]^{\frac{1-\lambda_w}{\lambda_w}}$$

where the wage inflation is the following,

$$\pi_{w,t} = \pi_t \gamma \frac{\tilde{w}_t}{\tilde{w}_{t-1}}$$

and the terms $\Psi_{w,t}$ e $F_{w,t}$ are,

$$\Psi_{w,t} = l_t^{1+\sigma_l} + \beta \xi_w E_t^* \left(\frac{\bar{\pi}\gamma}{\pi_{w,t+1}} \right)^{\frac{\lambda_w}{1-\lambda_w}(1+\sigma_l)} \Psi_{w,t+1}$$

$$F_{w,t} = l_t \frac{\lambda_{z,t}}{\lambda_w} + \beta \xi_w \gamma^{\frac{1}{1-\lambda_w}} E_t^* \left(\frac{1}{\pi_{w,t+1}} \right)^{\frac{\lambda_w}{1-\lambda_w}} \frac{\bar{\pi}^{\frac{1}{1-\lambda_w}}}{\gamma \pi_{t+1}} F_{w,t+1}$$

$$\Psi_{w,t} = \tilde{w}_t F_{w,t} \left[(1-\xi_w)^{-1} \left(1 - \xi_w \left(\frac{\bar{\pi}\gamma}{\pi_{w,t}} \right)^{\frac{1}{1-\lambda_w}} \right) \right]^{1-\lambda_w(1+\sigma_l)}$$

Government
Government adjusts interest rate based on the Taylor rule,

$$\frac{R_t}{\bar{R}} = \left(\frac{R_{t-1}}{\bar{R}} \right)^{\rho_R} \left[\left(\frac{\pi_t}{\bar{\pi}} \right)^{\alpha_\pi} \left(\frac{y_t}{y} \right)^{\alpha_y} \right]^{1-\rho_R}$$

Government budget constraint is given by

$$b_t = b_{t-1} R_{t-1} + g_t - t_t$$

The fiscal rule is,

$$t_t = \phi_b b_t + \phi_g g_t \tag{21}$$

Production function,

$$y_t = (p^*)^{\frac{\lambda_f}{\lambda_f - 1}} \left[\left(\frac{k_t}{\gamma} \right)^{\alpha} l_t^{1-\alpha} - \Phi \right]$$

Finally, the aggregated resource constraint is,

$$y_t = c_t + i_t + g_t$$

which concludes the dynamic model with 22 dynamic equations and 22 endogenous variables, namely:

$c_t, i_t, y_t, l_t, b_t, t_t, k_{t+1}, \lambda_{z,t}, q_t, r_t^k, R_t^k, m_t, R_t, p_t^*, \pi_t, F_{p,t}, \Psi_{p,t}, w_t^*, F_{w,t}, \Psi_{w,t},$
$\tilde{w}_t, \pi_{w,t}$.

Bibliography

Aczél, J. (1966). *Lectures on functional equations and their applications*. New York: Academic Press.

Aghion, P., Caroli, E., and Garcia-Penalosa, C. (1999). Inequality and economic growth: the perspective of the new growth theories. *Journal of Economic literature*, 37(4):1615–1660.

Alesina, A. and Rodrik, D. (1994). Distributive politics and economic growth. *The Quarterly Journal of Economics*, 109(2):465–490.

Alexopoulos, M. and Cohen, J. (2015). The power of print: Uncertainty shocks, markets, and the economy. *International Review of Economics & Finance*, 40:8–28.

Alkire, S. and Foster, J. (2010). Designing the inequality-adjusted human development index. Technical report, Human Development Report Office (HDRO), United Nations Development Programme (UNDP).

Anand, S. and Sen, A. (1994). Human development index: Methodology and measurement. Technical report, Human Development Report Office (HDRO), United Nations Development Programme (UNDP).

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory*, 2(3):244–263.

Atkinson, A. B. (1987). On the measurement of poverty. *Econometrica*, 55(4):749–764.

Atkinson, A. B. and Bourguignon, F. (1982). The comparison of multi-dimensioned distributions of economic status. *The Review of Economic Studies*, 49(2):183–201.

Bachmann, R. and Bayer, C. (2013). 'Wait-and-See' business cycles? *Journal of Monetary Economics*, 60(6):704–719.

Bachmann, R., Elstner, S., and Sims, E. R. (2013). Uncertainty and economic activity: Evidence from business survey data. *American Economic Journal: Macroeconomics*, 5(2):217–49.

Banerjee, A. V. and Newman, A. F. (1993). Occupational choice and the process of development. *Journal of Political Economy*, 101(2):274–298.

Barsky, R., Justiniano, A., and Melosi, L. (2014). The natural rate of interest and its usefulness for monetary policy. *American Economic Review*, 104(5):37–43.

Basili, M., Casaca, P., Chateauneuf, A., and Franzini, M. (2017). Multidimensional Pigou–Dalton transfers and social evaluation functions. *Theory and Decision*, 83(4):573–590.

Bekaert, G., Hoerova, M., and Duca, M. L. (2013). Risk, uncertainty and monetary policy. *Journal of Monetary Economics*, 60(7):771–788.

Blanchard, O. and Perotti, R. (2002). An empirical characterization of the dynamic effects of changes in government spending and taxes on output. *The Quarterly Journal of Economics*, 117(4):1329–1368.

Bloom, N. (2009). The impact of uncertainty shocks. *Econometrica*, 77(3):623–685.

Bloom, N., Floetotto, M., Jaimovich, N., Saporta-Eksten, I., and Terry, S. J. (2018). Really uncertain business cycles. *Econometrica*, 86(3):1031–1065.

Boland, P. J. and Proschan, F. (1988). Multivariate arrangement increasing functions with applications in probability and statistics. *Journal of Multivariate Analysis*, 25(2):286–298.

Born, B. and Pfeifer, J. (2014). Policy risk and the business cycle. *Journal of Monetary Economics*, 68:68–85.

Bosmans, K., Lauwers, L., and Ooghe, E. (2009). A consistent multidimensional Pigou–Dalton transfer principle. *Journal of Economic Theory*, 144(3):1358–1371.

Bourguignon, F. (1981). Pareto superiority of unegalitarian equilibria in Stiglitz' model of wealth distribution with convex saving function. *Econometrica*, 49(6):1469–1475.

Bourguignon, F. and Chakravarty, S. R. (2003). The measurement of multidimensional poverty. *The Journal of Economic Inequality*, 1(1):25–49.

Calvo, G. A. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics*, 12(3):383–398.

Chateauneuf, A. and Jaffray, J.-Y. (1989). Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17(3):263–283.

Checchi, D., Ichino, A., and Rustichini, A. (1999). More equal but less mobile? Education financing and intergenerational mobility in Italy and in the US. *Journal of Public Economics*, 74(3):351–393.

Christiano, L. J., Eichenbaum, M., and Evans, C. L. (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy*, 113(1):1–45.

Christiano, L. J., Eichenbaum, M. S., and Trabandt, M. (2015). Understanding the great recession. *American Economic Journal: Macroeconomics*, 7(1):110–67.

Chugh, S. K. (2016). Firm risk and leverage-based business cycles. *Review of Economic Dynamics*, 20:111–131.

De Campos, L. M., Huete, J. F., and Moral, S. (1994). Probability intervals: a tool for uncertain reasoning. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2(02):167–196.

Debreu, G. (1960). Topological methods in cardinal utility theory. In In K. J. Arrow, S. K. and Suppes, P., editors, *Mathematical Methods in the Social Sciences.*, pages 16–26. Stanford University Press, Stanford. CA.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, 75:643–669.

Epstein, L. G. and Schneider, M. (2003). Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31.

Erceg, C. J., Henderson, D. W., and Levin, A. T. (2000). Optimal monetary policy with staggered wage and price contracts. *Journal of Monetary Economics*, 46(2):281–313.

Faro, J. H. (2013). Cobb-Douglas preferences under uncertainty. *Economic Theory*, 54(2):273–285.

Fernández-Villaverde, J., Guerrón-Quintana, P., Kuester, K., and Rubio-Ramírez, J. (2015). Fiscal volatility shocks and economic activity. *American Economic Review*, 105(11):3352–84.

Fisher, J. D. (2015). On the structural interpretation of the Smets–Wouters "risk premium" shock. *Journal of Money, Credit and Banking*, 47(2-3):511–516.

Fleurbaey, M., Salles, M., and Weymark, J. A. (2008). *Justice, political liberalism, and utilitarianism: themes from Harsanyi and Rawls.* Cambridge University Press Cambridge.

Foster, J. E., Lopez-Calva, L. F., and Szekely, M. (2005). Measuring the distribution of human development: methodology and an application to Mexico. *Journal of Human Development*, 6(1):5–25.

Foster, J. E. and Sen, A. (1997). *On economic inequality.* Oxford University Press.

Gajdos, T. and Weymark, J. A. (2005). Multidimensional generalized Gini indices. *Economic Theory*, 26(3):471–496.

Galí, J., López-Salido, J. D., and Vallés, J. (2007). Understanding the effects of government spending on consumption. *Journal of the European Economic Association*, 5(1):227–270.

Galor, O. and Zeira, J. (1993). Income distribution and macroeconomics. *The Review of Economic Studies*, 60(1):35–52.

Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153.

Harsanyi, J. C. (1955). Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of Political Economy*, 63(4):309–321.

Harsanyi, J. C. (1975a). Can the maximin principle serve as a basis for morality? a critique of John Rawls's theory. *American Political Science Review*, 69(2):594–606.

Harsanyi, J. C. (1975b). Nonlinear social welfare functions. *Theory and Decision*, 6(3):311–332.

Harsanyi, J. C. (1978). Bayesian decision theory and utilitarian ethics. *American Economic Review*, 68(2):223–228.

Herrero, C., Martínez, R., and Villar, A. (2010). Multidimensional social evaluation: an application to the measurement of human development. *Review of Income and Wealth*, 56(3):483–497.

Ilut, C. L. and Schneider, M. (2014). Ambiguous business cycles. *American Economic Review*, 104(8):2368–2399.

Jaffray, J.-Y. (1988). Application of linear utility theory to belief functions. In *International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 1–8. Springer.

Kaldor, N. (1955). Alternative theories of distribution. *The Review of Economic Studies*, 23(2):83–100.

Kolm, S.-C. (1976a). Unequal inequalities. I. *Journal of Economic Theory*, 12(3):416–442.

Kolm, S.-C. (1976b). Unequal inequalities. II. *Journal of Economic Theory*, 13(1):82–111.

Kolm, S.-C. (1977). Multidimensional egalitarianisms. *The Quarterly Journal of Economics*, 91(1):1–13.

Kovacevic, M. (2010). Measurement of inequality in human development – A review. *Measurement*, 35.

Kravis, I. B. (1960). International differences in the distribution of income. *The Review of Economics and Statistics*, 42:408–416.

Lydall, H. (1968). *The structure of earnings.* Oxford Press.

Marinacci, M. and Montrucchio, L. (2005). Ultramodular functions. *Mathematics of Operations Research*, 30(2):311–332.

Masolo, R. M. and Monti, F. (2017). Ambiguity, monetary policy and trend inflation. Technical report, London School of Economics and Political Science, LSE Library.

Moehler, M. (2018). The Rawls–Harsanyi dispute: A moral point of view. *Pacific Philosophical Quarterly*, 99(1):82–99.

Müller, A. and Scarsini, M. (2001). Stochastic comparison of random vectors with a common copula. *Mathematics of Operations Research*, 26(4):723–740.

Müller, A. and Scarsini, M. (2012). Fear of loss, inframodularity, and transfers. *Journal of Economic Theory*, 147(4):1490–1500.

OECD (2015). *In It Together: Why Less Inequality Benefits All.* OECD Publishing.

Owen, A. L. and Weil, D. N. (1998). Intergenerational earnings mobility, inequality and growth. *Journal of Monetary Economics*, 41(1):71–104.

Rawls, J. (1971). *A theory of justice.* Harvard University Press, Cambridge, MA.

Rawls, J. (2001). *Justice as fairness: A restatement.* Harvard University Press.

Richard, S. F. (1975). Multivariate risk aversion, utility independence and separable utility functions. *Management Science*, 22(1):12–21.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society*, 97(2):255–261.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, 57(3):571–587.

Sen, A. (1976). Poverty: An ordinal approach to measurement. *Econometrica*, 44(2):219–231.

Sen, A. (1985). Commodities and capabilities. lectures in economics: Theory, institutions. *Policy*, 7.

Sen, A. (1987). *The Standard of Living*. Cambridge University Press.

Sen, A. (1992). *Inequality reexamined*. Clarendon Press.

Shaked, M. (1982). A general theory of some positive dependence notions. *Journal of Multivariate Analysis*, 12(2):199–218.

Smets, F. and Wouters, R. (2007). Shocks and frictions in us business cycles: A bayesian dsge approach. *American Economic Review*, 97(3):586–606.

Stiglitz, J. (1969). Distribution of income and wealth among individuals. *Econometrica*, 37(3):382–397.

Trockel, W. (1989). Classification of budget-invariant monotonic preferences. *Economics Letters*, 30(1):7–10.

Tsui, K.-Y. (1995). Multidimensional generalizations of the relative and absolute inequality indices: the Atkinson-Kolm-Sen approach. *Journal of Economic Theory*, 67(1):251–265.

Tsui, K.-Y. (1999). Multidimensional inequality and multidimensional generalized entropy measures: An axiomatic derivation. *Social Choice and Welfare*, 16(1):145–157.

UNDP (1990). Human development report. *United Nations*.

UNDP (2016). Human development report. *United Nations*.

Wakker, P. P. (1989). *Additive representations of preferences: A new foundation of decision analysis*. Springer.

Weymark, J. A. (1981). Generalized Gini inequality indices. *Mathematical Social Sciences*, 1(4):409–430.

Zambrano, E. (2014). An axiomatization of the human development index. *Social Choice and Welfare*, 42(4):853–872.