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$*$-Superalgebras and exponential growth

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ABSTRACT

In this paper, we study the exponential growth of $*$-graded identities of a finite dimensional $*$-superalgebra A over a field F of characteristic zero. If a $*$-superalgebra A satisfies a non-trivial identity, then the sequence $\{c_n^{\text{gri}}(A)\}_{n\geq 1}$ of $*$-graded codimensions of A is exponentially bounded and so we study the $*$-graded exponent $\exp^{\text{gri}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{\text{gri}}(A)}$ of A. We show that $\exp^{\text{gri}}(A) = \dim_F(A)$ if and only if A is a simple $*$-superalgebra and F is the symmetric even center of A. Also, we characterize the finite dimensional $*$-superalgebras such that $\exp^{\text{gri}}(A) \leq 1$ by the exclusion of four $*$-superalgebras from $\text{var}^{\text{gri}}(A)$ and construct eleven $*$-superalgebras $E_i, i = 1, \ldots, 11$, with the following property: $\exp^{\text{gri}}(A) > 2$ if and only if $E_i \in \text{var}^{\text{gri}}(A)$, for some $i \in \{1, \ldots, 11\}$. As a consequence, we characterize the finite dimensional $*$-superalgebras A such that $\exp^{\text{gri}}(A) = 2$.

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1. Introduction

Let A be an associative algebra over a field F of characteristic zero. It is well known that if A is a PI-algebra, i.e. if A satisfies a non-trivial ordinary identity, then the sequence $\{c_n(A)\}_{n\geq 1}$ of codimensions of A is exponentially bounded, i.e. there exist constants a, k such that $c_n(A) \leq ak^n$, for all $n \geq 1$ [10]. In the 1980's, Amitsur conjectured that the limit $\exp(A) := \lim_{n\to\infty} \sqrt[n]{c_n(A)}$, called exponent of A, exists and is a non-negative integer. In 1999, Giambruno and Zaicev [4,5] confirmed this conjecture and provided an explicit formula to compute the exponent of any PI-algebra A.

It is natural to ask if a similar phenomenon is true in the setting of algebras with some additional structure, e.g. algebras with involution and G-graded algebras, where G is a finite group. In [6] Giambruno and Zaicev confirmed the analog of Amitsur's conjecture for finite dimensional algebras with involution and in [1] Aljadeff and Giambruno confirmed the analog of Amitsur's conjecture for G-graded PI-algebras, where G is a finite group.

Later, in [8] Gordienko considered the so-called algebras with a generalized H-action. Algebras with involution and G-graded algebras are particular cases of algebras with a generalized H-action. In this paper he confirmed the analog of Amitsur's conjecture in case A is a finite dimensional algebra with a generalized H-action.

In [3], Giambruno, dos Santos and Vieira studied a new class of algebras: $*$-superalgebras. A $*$-superalgebra is a superalgebra $A = A^{(0)} \oplus A^{(1)}$ endowed with an involution which preserves the homogeneous components $A^{(0)}$ and $A^{(1)}$. By following the analogous procedure of the ordinary case, they studied the behavior of $*$-graded codimensions $c_n^{\mathrm{gri}}(A)$ of a $*$-superalgebra A and proved that if A is a PI-algebra, then $c_n^{\mathrm{gri}}(A)$ is exponentially bounded. Also, they classified the finite dimensional $*$-superalgebras of polynomial growth, i.e. $*$-superalgebras A such that $c_n^{\mathrm{gri}}(A) \leq an^k$, for some constants a, k, for all $n \geq 1$, by excluding four $*$-superalgebras from $\mathrm{var}^{\mathrm{gri}}(A)$, the $*$-supervariety generated by A. Since a $*$-superalgebra A can be viewed as an algebra with a generalized FG-action, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on A by automorphisms and antiautomorphisms, the analog of Amitsur's conjecture was confirmed for $*$-superalgebras in [8].

This paper is devoted to the study of the exponential growth of $*$-graded codimensions and the $*$-graded exponent $\exp^{\mathrm{gri}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$ of a finite dimensional $*$-superalgebra A over a field F of characteristic zero. First, we characterize the finite dimensional simple $*$-superalgebras as those $*$-superalgebras such that $\exp^{\mathrm{gri}}(A) = \dim_F(A)$ and the field F coincides with the symmetric even center of A. After, as a consequence of the results proved in [3], we characterize the finite dimensional $*$-superalgebras such that $\exp^{\mathrm{gri}}(A) \leq 1$. In the last section, we give a characterization of finite dimensional $*$-superalgebras such that $\exp^{\mathrm{gri}}(A) > 2$ by excluding a finite number of $*$-superalgebras from $\mathrm{var}^{\mathrm{gri}}(A)$. As a consequence, we characterize the finite dimensional $*$-superalgebras such that $\exp^{\mathrm{gri}}(A) = 2$.

It is worth mentioning that the results presented here have already been proved for algebras, superalgebras and algebras with involution in [4,7,6,2,9], respectively.

2. Superalgebras and graded involutions

Throughout this paper we will denote by F a field of characteristic zero and by A an associative algebra over F.

An algebra A is a *superalgebra* (*or a* $\mathbb{Z}_2$*-graded algebra*) if A has a decomposition $A = A^{(0)} \oplus A^{(1)}$ where $A^{(0)}$ and $A^{(1)}$ are subspaces such that $A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)}$ and $A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}$. The subspaces $A^{(0)}$ and $A^{(1)}$ are called *homogeneous components of* A. We shall denote by $A = (A^{(0)}, A^{(1)})$ the superalgebra A, where $A^{(0)}$ and $A^{(1)}$ are the homogeneous components of A. We say that a subspace B of A is a *graded subspace* if $B = B^{(0)} \oplus B^{(1)}$, where $B^{(0)} = B \cap A^{(0)}$ and $B^{(1)} = B \cap A^{(1)}$.

We remind the reader that if $A = (A^{(0)}, A^{(1)})$ is a superalgebra, then $\varphi \in \mathrm{Aut}(A)$ defined by $\varphi(a^{(0)} + a^{(1)}) = a^{(0)} - a^{(1)}$, where $a^{(0)} \in A^{(0)}$, $a^{(1)} \in A^{(1)}$, is an automorphism of order at most 2. Conversely, any automorphism $\varphi \in \mathrm{Aut}(A)$ of order at most 2 determines a $\mathbb{Z}_2$-grading on A by setting $A^{(0)} = \{a + \varphi(a) : a \in A\}$ and $A^{(1)} = \{a - \varphi(a) : a \in A\}$.

An involution on the algebra A is just an antiautomorphism of order at most 2 on A which we shall denote by $*$. In this case, we write $A^+ = \{a \in A : a^* = a\}$ and $A^- = \{a \in A : a^* = -a\}$ for the sets of symmetric and skew elements of A, respectively. Clearly $A = A^+ \oplus A^-$, since $\mathrm{char}(F) \neq 2$.

An involution $*$ on a superalgebra that preserves the homogeneous components $A^{(0)}$ and $A^{(1)}$, i.e. $(A^{(0)})^* = A^{(0)}$ and $(A^{(1)})^* = A^{(1)}$, is called *graded involution*. A superalgebra A endowed with a graded involution $*$ is called $*$*-superalgebra*. In this case, we have that $\varphi \circ * = * \circ \varphi$, where φ is the automorphism of order at most 2 determined by the $\mathbb{Z}_2$-grading, and the subspaces A^+ and A^- are graded, i.e.

$$A = (A^{(0)})^+ \oplus (A^{(1)})^+ \oplus (A^{(0)})^- \oplus (A^{(1)})^-.$$

We start by giving some examples of $*$-superalgebras. We let D_* be the algebra $D = F \oplus F$ with trivial grading and exchange involution $(a, b)^* = (b, a)$. We also consider D^{gr} to be the algebra $D = F \oplus F$ with grading $D = (F(1, 1), F(1, -1))$ and trivial involution.

Next, we define M to be the following subalgebra of the algebra $UT_4(F)$ of 4×4 upper triangular matrices:

$$M = \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

We let M_* be the algebra M with trivial grading and reflection involution, i.e. the involution obtained by flipping the matrix along its secondary diagonal

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

On the other hand, we can define the following grading on the algebra M:

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

The algebra M endowed with this grading and with the reflection involution will be denoted by M^{gri}. It is not difficult to see that the algebras M_*, D_*, D^{gr} and M^{gri} are $*$-superalgebras.

Let X be a countable set of non-commuting variables. We write the set X as the disjoint union of four countable sets $X = Y_0 \cup Y_1 \cup Z_0 \cup Z_1$, where $Y_0 = \{y_{1,0}, y_{2,0}, \ldots\}$, $Y_1 = \{y_{1,1}, y_{2,1}, \ldots\}$, $Z_0 = \{z_{1,0}, z_{2,0}, \ldots\}$ and $Z_1 = \{z_{1,1}, z_{2,1}, \ldots\}$. We can define the free $*$-superalgebra $\mathcal{F} = F\langle X|\mathbb{Z}_2, *\rangle$ of countable rank on X by giving a superstructure on $\mathcal{F}$ where we require that the variables of $Y_0 \cup Z_0$ are homogeneous of degree 0 and those of $Y_1 \cup Z_1$ are homogeneous of degree 1. We also define an involution on $\mathcal{F}$ by requiring that the variables of $Y_0 \cup Y_1$ are symmetric and those of $Z_0 \cup Z_1$ are skew.

It is easily seen that $\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ has a structure of $*$-superalgebra, where $\mathcal{F}^{(0)}$ is the span of all monomials in the variables of X which have an even number of variables of degree 1, $\mathcal{F}^{(1)}$ is the span of all monomials in the variables of X which have an odd number of variables of degree 1, $(\mathcal{F}^{(0)})^* = \mathcal{F}^{(0)}$ and $(\mathcal{F}^{(1)})^* = \mathcal{F}^{(1)}$. The elements of $\mathcal{F}$ are called $(\mathbb{Z}_2, *)$-*polynomials*.

We say that

$$f = f(y_{1,0}, \ldots, y_{m,0}, y_{1,1}, \ldots, y_{n,1}, z_{1,0}, \ldots, z_{p,0}, z_{1,1}, \ldots, z_{q,1}) \in \mathcal{F}$$

is a $(\mathbb{Z}_2, *)$-*identity* for the $*$-superalgebra A, and we write $f \equiv 0$ on A, if

$$f(a^+_{1,0}, \ldots, a^+_{m,0}, a^+_{1,1}, \ldots, a^+_{n,1}, a^-_{1,0}, \ldots, a^-_{p,0}, a^-_{1,1}, \ldots, a^-_{q,1}) = 0,$$

for all $a^+_{1,0}, \ldots, a^+_{m,0} \in (A^{(0)})^+$, $a^+_{1,1}, \ldots, a^+_{n,1} \in (A^{(1)})^+$, $a^-_{1,0}, \ldots, a^-_{p,0} \in (A^{(0)})^-$ and $a^-_{1,1}, \ldots, a^-_{q,1} \in (A^{(1)})^-$.

The set

$$\mathrm{Id}^{\mathrm{gri}}(A) := \{f \in \mathcal{F} : f \equiv 0 \text{ on } A\}$$

is an ideal of $\mathcal{F}$, called *ideal of* $(\mathbb{Z}_2, *)$*-identities of* A. We notice that $\mathrm{Id}^{\mathrm{gri}}(A)$ is a T_2^*-ideal of $\mathcal{F}$, i.e. an ideal invariant under all endomorphisms of $\mathcal{F}$ that preserve the superstructure and commute with the involution. Since $\mathrm{char}(F) = 0$, $\mathrm{Id}^{\mathrm{gri}}(A)$ is determined by its multilinear polynomials and so we define

$$P_n^{\mathrm{gri}} := \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} : \sigma \in S_n, w_i = y_{i,g_i} \text{ or } w_i = z_{i,g_i}, g_i = 0, 1\},$$

the space of multilinear polynomials in the first n variables.

The dimension of the quotient space

$$P_n^{\mathrm{gri}}(A) := \frac{P_n^{\mathrm{gri}}}{P_n^{\mathrm{gri}} \cap \mathrm{Id}^{\mathrm{gri}}(A)}$$

is called *the n-th $*$-graded codimension of A* and it is denoted by $c_n^{\mathrm{gri}}(A)$.

Let A be a $*$-superalgebra. In [3], the authors studied the behavior of the sequence of $*$-graded codimensions of A. We start with the following lemma.

Lemma 1 *([3], Lemma 3.1). Let A be a $*$-superalgebra. Then for any $n \geq 1$, we have $c_n(A) \leq c_n^{\mathrm{gri}}(A) \leq 4^n c_n(A)$.*

It is well known that an algebra A is a PI-algebra, i.e. A satisfies a non-trivial ordinary identity, if and only if $c_n(A)$ is exponentially bounded [10]. Thus, as an immediate consequence of the previous lemma, we have the following corollary.

Corollary 2 *([3], Corollary 3.2). Let A be a $*$-superalgebra. Then A is a PI-algebra if and only if its sequence of $*$-graded codimensions $\{c_n^{\mathrm{gri}}(A)\}_{n\geq 1}$ is exponentially bounded.*

Since any finite dimensional algebra A is a PI-algebra, we have the following corollary.

Corollary 3. *Let A be a finite dimensional $*$-superalgebra. Then the sequence of $*$-graded codimensions $\{c_n^{\mathrm{gri}}(A)\}_{n\geq 1}$ is exponentially bounded.*

Given a $*$-superalgebra A, we shall denote by $\mathrm{var}^{\mathrm{gri}}(A)$ the variety of $*$-superalgebras generated by A, that is, $\mathrm{var}^{\mathrm{gri}}(A)$ is the class of all $*$-superalgebras B such that $\mathrm{Id}^{\mathrm{gri}}(A) \subseteq \mathrm{Id}^{\mathrm{gri}}(B)$. Consequently $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B)$ if and only if $\mathrm{Id}^{\mathrm{gri}}(A) = \mathrm{Id}^{\mathrm{gri}}(B)$.

In [3], the authors characterized varieties $\mathcal{V} = \mathrm{var}^{\mathrm{gri}}(A)$ which are generated by finite dimensional $*$-superalgebras of polynomial growth, i.e. $*$-superalgebras A such that $c_n^{\mathrm{gri}}(A) \leq an^k$, for all $n \geq 1$, where a, k are constants, by the exclusion of four $*$-superalgebras from $\mathcal{V}$.

The main theorem in [3] is the following.

Theorem 4 *([3], Theorem 8.6). Let A be a finite dimensional $*$-superalgebra over a field of characteristic zero. Then $c_n^{\mathrm{gri}}(A)$ is polynomially bounded if and only if M_*, D_*, D^{gr}, $M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$.*

Next, we present some results on the structure of $*$-superalgebras which were given in [3] and will be important in the development of the next sections.

Let A be a $*$-superalgebra, φ the automorphism of order at most 2 determined by the $\mathbb{Z}_2$-grading and I an ideal (subalgebra) of A. We say that I is a *$*$-graded ideal (subalgebra)* if $I^\varphi = I$ and $I^* = I$. A $*$-superalgebra A is a *simple $*$-superalgebra* if $A^2 \neq \{0\}$ and A has no non-zero proper $*$-graded ideals. The next theorem is a generalization of Wedderburn–Malcev theorem.

Theorem 5 *([3], Theorem 7.3). Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero and let $J(A)$ denote the Jacobson radical of A. Then:*

1. *$J(A)$ is a $*$-graded ideal;*
2. *If A is a simple $*$-superalgebra, then either A is simple or A is $*$-simple or $A = B \oplus B^\varphi$ for some $*$-simple ideal B;*
3. *If A is semisimple, then A is a finite direct sum of simple $*$-superalgebras;*
4. *If F is algebraically closed, then $A = A_1 \oplus \cdots \oplus A_m + J(A)$, where each algebra $A_i, i = 1, \ldots, m$, is a simple $*$-superalgebra.*

In [3] the authors gave a classification of the finite dimensional simple $*$-superalgebras over an algebraically closed field F of characteristic zero.

Theorem 6 *([3], Theorem 7.6). Let A be a finite dimensional simple $*$-superalgebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following $*$-superalgebras:*

1. *$M_{k,l}(F)$, with $k \geq 1, k \geq l \geq 0$, with transpose or symplectic involution (the symplectic involution can occur only when $k = l$);*
2. *$M_{k,l}(F) \oplus M_{k,l}(F)^{op}$, with $k \geq 1$, $k \geq l \geq 0$, with induced grading and exchange involution;*
3. *$M_n(F) + cM_n(F)$, $c^2 = 1$, with involution given by $(a + cb)^\dagger = a^* - cb^*$, where $*$ denotes the transpose or symplectic involution;*
4. *$M_n(F) + cM_n(F)$, $c^2 = 1$, with involution given by $(a + cb)^\dagger = a^* + cb^*$, where $*$ denotes the transpose or symplectic involution;*
5. *$(M_n(F) + cM_n(F)) \oplus (M_n(F) + cM_n(F))^{op}$, $c^2 = 1$, with grading*

$$(M_n(F) \oplus M_n(F)^{op}, c(M_n(F) \oplus M_n(F)^{op}))$$

and exchange involution.

3. The $*$-graded exponent

In this section, we study the $*$-graded exponent of a $*$-superalgebra A. We use the $*$-graded exponent to characterize simple $*$-superalgebras and $*$-superalgebras having polynomial growth of the $*$-graded codimensions.

Let A be a finite dimensional $*$-superalgebra. We define by

$$\underline{\exp}^{\mathrm{gri}}(A) := \liminf_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$$

the *lower $*$-graded exponent* of A and by

$$\overline{\exp}^{\mathrm{gri}}(A) := \limsup_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$$

the *upper $*$-graded exponent* of A. If $\underline{\exp}^{\mathrm{gri}}(A) = \overline{\exp}^{\mathrm{gri}}(A)$, the limit

$$\exp^{\mathrm{gri}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$$

is called the *$*$-graded exponent* of A.

A natural question is if the Amitsur's conjecture holds on $*$-superalgebras, i.e. if the $*$-graded exponent of a finite dimensional $*$-superalgebra exists and is a non-negative integer. In [8], Gordienko gave a positive answer for this question and we will discuss this below.

Let H be an associative algebra with 1. We say that an associative algebra A is an algebra with a generalized H-action if A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ and for every $h \in H$, there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ such that, for all $a, b \in A$,

$$h(ab) = \sum_i (h'_i(a))(h''_i(b)) + (h'''_i(b))(h''''_i(a))).$$

Let A be a $*$-superalgebra and φ the automorphism of order at most 2 determined by the $\mathbb{Z}_2$-grading. It is easily seen that the group $G = \{1, \varphi\} \times \{1, *\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on A by automorphisms and antiautomorphisms and so, by [8, Example 6], A is an algebra with a generalized FG-action.

In our case, we can translate the main theorem of [8] as follows.

Theorem 7 *([8], Theorem 5). Let A be a finite dimensional $*$-superalgebra over an algebraically closed field F of characteristic zero. Then there exist an integer $d \geq 0$ and constants C_1, C_2, r_1, r_2 such that $C_1 > 0$ and*

$$C_1 n^{r_1} d^n \leq c_n^{\mathrm{gri}}(A) \leq C_2 n^{r_2} d^n.$$

As a consequence, $\exp^{\mathrm{gri}}(A)$ *exists and is a non-negative integer. Furthermore, if* B *is a maximal semisimple* $*$*-graded subalgebra of* A *and* $J = J(A)$ *is the Jacobson radical of* A*, then*

$$\exp^{\mathrm{gri}}(A) = \max_i \dim_F(C_1^{(i)} + \cdots + C_k^{(i)}),$$

where $C_1^{(i)}, \ldots, C_k^{(i)}$ *are distinct simple* $*$*-graded subalgebras of* B *and*

$$C_1^{(i)} J C_2^{(i)} J \cdots J C_{k-1}^{(i)} J C_k^{(i)} \neq \{0\}.$$

We remark that, by Theorem 5, if F is an algebraically closed field, then it is always possible to write $A = B + J(A)$, where B is a maximal semisimple $*$-graded subalgebra of A and $J(A)$ is a $*$-graded ideal of A.

The next remark will be useful in the proofs of the results of this paper.

Remark 1. Let A be a $*$-superalgebra over a field F of characteristic zero and $\bar{F}$ the algebraic closure of F. Then $\bar{A} = A \otimes_F \bar{F}$ is a $*$-superalgebra with involution $(a \otimes \alpha)^* = a^* \otimes \alpha$ and grading $\bar{A} = (A^{(0)} \otimes_F \bar{F}, A^{(1)} \otimes_F \bar{F})$. Moreover, $\dim_F(A) = \dim_{\bar{F}}(\bar{A})$, $\mathrm{Id}^{\mathrm{gri}}(A) = \mathrm{Id}^{\mathrm{gri}}(\bar{A})$, viewed as $*$-superalgebras over F, and $c_n^{\mathrm{gri}}(A) = c_n^{\mathrm{gri}}(\bar{A})$. As a consequence, $\exp^{\mathrm{gri}}(A) = \exp^{\mathrm{gri}}(\bar{A})$. Thus, if necessary, to questions about the $*$-graded exponent, we may assume that F is an algebraically closed field.

As an immediate consequence of the previous remark and Theorem 7, we have the following corollary.

Corollary 8. *Let* A *be a finite dimensional* $*$*-superalgebra over a field* F *of characteristic zero. Then* $\exp^{\mathrm{gri}}(A)$ *exists, is a integer and* $\exp^{\mathrm{gri}}(A) \leq \dim_F(A)$*.*

Now, we start the characterization of simple $*$-superalgebras through the $*$-graded exponent.

Lemma 9. *Let* A *be a* $*$*-superalgebra and let* $\mathcal{Z} = \mathcal{Z}(A)$ *be the center of* A*. Then:*

1. $\mathcal{Z}$ *is a* $*$*-graded subalgebra of* A*. As a consequence,*

$$\mathcal{Z} = (\mathcal{Z}^{(0)})^+ \oplus (\mathcal{Z}^{(1)})^+ \oplus (\mathcal{Z}^{(0)})^- \oplus (\mathcal{Z}^{(1)})^-;$$

2. *If* A *is a finite dimensional* $*$*-superalgebra, simple as an algebra, then* $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta)$*, where* $\alpha \in (\mathcal{Z}^{(0)})^-, \beta \in \mathcal{Z}^{(1)}$ *and* $\mathfrak{Z} = (\mathcal{Z}^{(0)})^+$*. As a consequence,* $[\mathcal{Z} : \mathfrak{Z}] \leq 4$*.*

Proof.

1. Let φ be the automorphism of order at most 2 determined by the $\mathbb{Z}_2$-grading and let $a \in \mathcal{Z}$. If $b \in A$, then there exists $c \in A$ such that $c^\varphi = b$ and

$$a^\varphi b = a^\varphi c^\varphi = (ac)^\varphi = (ca)^\varphi = ba^\varphi.$$

   Hence, $\mathcal{Z}^\varphi = \mathcal{Z}$. Analogously, $\mathcal{Z}^* = \mathcal{Z}$ and hence $\mathcal{Z}$ is a $*$-graded subalgebra of A.
2. Let A be a finite dimensional $*$-superalgebra, simple as an algebra. Then $\mathcal{Z}$ is a field and $F \subseteq \mathfrak{Z} \subseteq \mathcal{Z}^{(0)} \subseteq \mathcal{Z}$ are fields extensions, where $\mathfrak{Z} = (\mathcal{Z}^{(0)})^+$. Let $\alpha \in (\mathcal{Z}^{(0)})^-$. Then α and $-\alpha$ are the roots of $f(x) = x^2 - \alpha^2 \in \mathfrak{Z}[x]$ and so $f(x)$ is irreducible in $\mathfrak{Z}[x]$. Hence $[\mathcal{Z}^{(0)} : \mathfrak{Z}] = 2$ and $\mathcal{Z}^{(0)} = \mathfrak{Z}(\alpha)$. Analogously, $\mathcal{Z} = \mathcal{Z}^{(0)}(\beta)$, where $\beta \in \mathcal{Z}^{(1)}$ and $[\mathcal{Z} : \mathcal{Z}^{(0)}] = 2$. Hence, $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta)$ and $[\mathcal{Z} : \mathfrak{Z}] \leq 4$.

We are in condition to prove the main theorem of this section.

Theorem 10. *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero and $\mathfrak{Z} = (\mathcal{Z}(A)^{(0)})^+$.*

1. *If A is a simple $*$-superalgebra, then $\exp^{\mathrm{gri}}(A) = \dim_{\mathfrak{Z}}(A)$;*
2. *If A is a semisimple $*$-superalgebra and $A = A_1 \oplus \cdots \oplus A_m$ is a decomposition of A into simple $*$-superalgebras, then $\exp^{\mathrm{gri}}(A) = \max_{1 \leq i \leq m} \dim_{\mathfrak{Z}_i}(A_i)$, where $\mathfrak{Z}_i = (\mathcal{Z}(A_i)^{(0)})^+$;*
3. *$\exp^{\mathrm{gri}}(A) = \dim_F(A)$ if and only if A is a simple $*$-superalgebra and $F = \mathfrak{Z}$.*

Proof.

1. Let A be a simple $*$-superalgebra over a field F of characteristic zero and let $\bar{F}$ be the algebraic closure of F. By Theorem 5, we have that either A is simple or A is $*$-simple or $A = B \oplus B^\varphi$, where B is a simple $*$-ideal of A. First, suppose that A is simple. Then, by Lemma 9, $\mathcal{Z}$ is a field and either $\mathcal{Z} = \mathfrak{Z}$ or $\mathcal{Z} = \mathfrak{Z}(\alpha), \alpha \in (\mathcal{Z}^{(0)})^-$, or $\mathcal{Z} = \mathfrak{Z}(\beta), \beta \in \mathcal{Z}^{(1)}$, or $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta), \alpha \in (\mathcal{Z}^{(0)})^-, \beta \in \mathcal{Z}^{(1)}$. If $\mathcal{Z} = \mathfrak{Z}$, then

$$\begin{aligned}\mathfrak{Z} \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} F_i \otimes_F \bar{F} \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i,\end{aligned}$$

   where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$. Therefore

$$\begin{aligned} A \otimes_F \bar{F} &\cong A \otimes_{\mathfrak{Z}} \mathfrak{Z} \otimes_F \bar{F} \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathfrak{Z}} \bar{F}_i), \end{aligned}$$

where each summand $A \otimes_{\mathfrak{Z}} \bar{F}_i$ is a central simple algebra over $\bar{F}$ with induced structure of $*$-superalgebra. Moreover,

$$[\mathfrak{Z} : F] \dim_{\mathfrak{Z}}(A) = \dim_F(A) = \dim_{\bar{F}}(A \otimes_F \bar{F}) = [\mathfrak{Z} : F] \dim_{\bar{F}}(A \otimes_{\mathfrak{Z}} \bar{F}_i).$$

Thus, $\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}(A \otimes_{\mathfrak{Z}} \bar{F}_i)$ and, by Theorem 7, we have that

$$\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}(A \otimes_{\mathfrak{Z}} \bar{F}_i) = \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) = \exp^{\mathrm{gri}}(A).$$

Now, suppose that $\mathcal{Z} = \mathfrak{Z}(\alpha), \alpha \in (\mathcal{Z}^{(0)})^-$. Then

$$\begin{aligned} \mathcal{Z} \otimes_F \bar{F} &\cong \mathcal{Z} \otimes_{\mathfrak{Z}} \mathfrak{Z} \otimes_F \bar{F} \\ &\cong \mathcal{Z} \otimes_{\mathfrak{Z}} \left(\bigoplus_{i=1}^{[\mathfrak{Z}:F]} F_i \otimes_F \bar{F} \right) \\ &\cong \mathcal{Z} \otimes_{\mathfrak{Z}} \left(\bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \right) \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (\mathfrak{Z} \oplus \mathfrak{Z}) \otimes_{\mathfrak{Z}} \bar{F}_i \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \oplus \bar{F}_i, \end{aligned}$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$. Therefore

$$\begin{aligned} A \otimes_F \bar{F} &\cong A \otimes_{\mathcal{Z}} \mathcal{Z} \otimes_F \bar{F} \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} A \otimes_{\mathcal{Z}} (\bar{F}_i \oplus \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i). \end{aligned}$$

On each summand $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$, φ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^\varphi = a_1^\varphi \otimes f_1 + a_2^\varphi \otimes f_2$ and $*$ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^* = a_1^* \otimes f_2 + a_2^* \otimes f_1$. Hence, $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and, as in the previous case, it follows that

$$\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}((A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)) = \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) = \exp^{\mathrm{gri}}(A).$$

If $\mathcal{Z} = \mathfrak{Z}(\beta), \beta \in \mathcal{Z}^{(1)}$, then, as in the previous case,

$$\mathcal{Z} \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \oplus \bar{F}_i$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$, and

$$A \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i).$$

On each summand $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$, φ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^{\varphi} = a_1^{\varphi} \otimes f_2 + a_2^{\varphi} \otimes f_1$ and $*$ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^* = a_1^* \otimes f_1 + a_2^* \otimes f_2$, if $\beta \in (\mathcal{Z}^{(1)})^+$, and as $(a_1 \otimes f_1 + a_2 \otimes f_2)^* = a_1^* \otimes f_2 + a_2^* \otimes f_1$, if $\beta \in (\mathcal{Z}^{(1)})^-$. In any case, $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and, as in the previous case, it follows that

$$\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}((A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)) = \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) = \exp^{\mathrm{gri}}(A).$$

Finally, if $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta), \alpha \in (\mathcal{Z}^{(0)})^-, \beta \in (\mathcal{Z}^{(1)})$ then, as before,

$$\mathcal{Z} \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \oplus \bar{F}_i \oplus \bar{F}_i \oplus \bar{F}_i$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$, and

$$A \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i).$$

On each summand $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$, φ acts as

$$(a_1 \otimes f_1 + a_2 \otimes f_2 + a_3 \otimes f_3 + a_4 \otimes f_4)^{\varphi} = a_1^{\varphi} \otimes f_2 + a_2^{\varphi} \otimes f_1 + a_3^{\varphi} \otimes f_4 + a_4^{\varphi} \otimes f_3$$

and $*$ acts as

$$(a_1 \otimes f_1 + a_2 \otimes f_2 + a_3 \otimes f_3 + a_4 \otimes f_4)^* = a_1^* \otimes f_3 + a_2^* \otimes f_4 + a_4^* \otimes f_1 + a_4^* \otimes f_2,$$

if $\beta \in (\mathcal{Z}^{(1)})^+$ and as

$$(a_1 \otimes f_1 + a_2 \otimes f_2 + a_3 \otimes f_3 + a_4 \otimes f_4)^* = a_1^* \otimes f_4 + a_2^* \otimes f_3 + a_4^* \otimes f_2 + a_4^* \otimes f_1,$$

if $\beta \in (\mathcal{Z}^{(1)})^-$. In any case, $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and hence

$$\begin{aligned}\dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}((A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A).\end{aligned}$$

This proves (1) in case A is a simple algebra.

Now, suppose that A is $*$-simple but not simple. Then $A \cong C \oplus C^*$ where C is a simple algebra. Notice that the map $\psi : C \oplus C^* \to C \oplus C^{op}$, where C^{op} denotes de opposite algebra of C, defined by $\psi(a, b^*) = (a, b)$ is an isomorphism of algebras with involution, where $C \oplus C^{op}$ is endowed with the exchange involution. If $C^\varphi = C$, then $(C^{op})^\varphi \cong C^{op}$ and $\mathfrak{Z} \cong \mathcal{Z}(C)^{(0)}$. If $\mathcal{Z}(C)^{(0)} = \mathcal{Z}(C)$, then

$$\mathcal{Z}(C) \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} \bar{F}_i,$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathcal{Z}(C) : F]$. Therefore,

$$\begin{aligned}A \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (A \otimes_{\mathcal{Z}(C)} \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)\end{aligned}$$

and $(C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$. Thus, as before, we get

$$\begin{aligned}\dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}((C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A).\end{aligned}$$

If $\mathcal{Z}(C)^{(0)} \neq \mathcal{Z}(C)$, then $\mathcal{Z}(C) \cong \mathfrak{Z}(\gamma)$, where $\gamma \in \mathcal{Z}(C)^{(1)}$. We have that

$$\mathcal{Z}(C) \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} \bar{F}_i \oplus \bar{F}_i,$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathcal{Z}(C) : F]$ and

$$\begin{aligned}A \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} A \otimes_{\mathcal{Z}(C)} (\bar{F}_i \oplus \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i).\end{aligned}$$

Each summand $((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and

$$\begin{aligned} \dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}(((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A). \end{aligned}$$

If $C^\varphi \neq C$, then $C^\varphi \cong C^{op}$ and $\mathfrak{Z} \cong \mathcal{Z}(C)$. Thus

$$\mathcal{Z}(C) \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} \bar{F}_i,$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathcal{Z}(C) : F]$. Therefore,

$$\begin{aligned} A \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (A \otimes_{\mathcal{Z}(C)} \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i) \end{aligned}$$

and $(C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$. Thus, as before, we get

$$\begin{aligned} \dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}((C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A). \end{aligned}$$

Finally, suppose that $A = B \oplus B^\varphi$, where B is a simple $*$-algebra. In this case, $\mathfrak{Z} \cong \mathcal{Z}(B)^+$. If B is a simple algebra, then, as in the previous case, we get that $\dim_{\mathfrak{Z}}(A) = \exp^{\mathrm{gri}}(A)$. If B is not simple, then $B = C \oplus C^*$, where C is a simple algebra, $\mathfrak{Z} \cong \mathcal{Z}(B)^+ \cong \mathcal{Z}(C)$ and, as before, $\dim_{\mathfrak{Z}}(A) = \exp^{\mathrm{gri}}(A)$. This proves (1).

2. Suppose that A is a semisimple $*$-superalgebra. Then $A = A_1 \oplus \cdots \oplus A_m$, where each A_i, $i = 1, \ldots, m$, is a simple $*$-superalgebra. Thus,

$$A \otimes_F \bar{F} \cong \bigoplus_{i=1}^{m} A_i \otimes_F \bar{F}.$$

Now, by part (1), for each $i = 1, \ldots, m$,

$$A_i \otimes_F \bar{F} \cong B_{i1} \oplus \cdots \oplus B_{it_i},$$

where $t_i = [\mathfrak{Z}_i : F]$, $\bar{F}_j \cong \bar{F}$, $j = 1, \ldots, t_i$, $B_{i1} \cong \cdots \cong B_{it_i}$, each B_{ij} is a simple $*$-superalgebra over $\bar{F}$ and $\dim_{\bar{F}}(B_{ij}) = \dim_{\mathfrak{Z}_i}(A_i)$, $j = 1, \ldots, t_i$. Hence, by

Theorem 7,

$$\begin{aligned}\exp^{\mathrm{gri}}(A) &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \max_{1\le i\le m} \dim_{\bar{F}}(B_{i1}) \\ &= \max_{1\le i\le m} \dim_{\mathfrak{Z}_i}(A_i).\end{aligned}$$

3. In order to prove (3), by part (1), we only need to show that $\exp^{\mathrm{gri}}(A) = \dim_F(A)$ implies that A is a simple $*$-superalgebra and $F = \mathfrak{Z}$. Let $\bar{A} = A \otimes_F \bar{F}$. Then $\dim_F(A) = \dim_{\bar{F}}(\bar{A}) = \exp^{\mathrm{gri}}(\bar{A}) = \exp^{\mathrm{gri}}(A)$. If $\bar{A}$ is nilpotent, then $\exp^{\mathrm{gri}}(\bar{A}) = 0$, a contradiction. Thus $\bar{A}$ contains a maximal semisimple $*$-superalgebra $B = B_1 \oplus \cdots \oplus B_m$ and $\dim_{\bar{F}}(\bar{A}) = \exp^{\mathrm{gri}}(\bar{A}) = \dim_{\bar{F}}(C)$, where C is a suitable $*$-graded subalgebra of B. Hence, A is a semisimple $*$-superalgebra and, by part (2), $\dim_{\bar{F}}(\bar{A}) = \dim_{\bar{F}} B_i$, for some $i \in \{1, \ldots, m\}$, and so A is a simple $*$-superalgebra. Hence, by part (1), $\dim_F(A) = \exp^{\mathrm{gri}}(A) = \dim_{\mathfrak{Z}}(A)$ implies that $F = \mathfrak{Z}$. This complete the proof of the theorem.

In the next theorem the authors characterized the finite dimensional $*$-superalgebras over an algebraically closed field of characteristic zero whose sequence of $*$-graded codimensions is polynomially bounded by considering the behavior of the sequence of ordinary codimensions.

Theorem 11 *([3], Theorem 8.3). Let A be a finite dimensional $*$-superalgebra over an algebraically closed field F and let $J(A)$ denote its Jacobson radical. Then $c_n^{\mathrm{gri}}(A)$ is polynomially bounded if and only if*

1. *$c_n(A)$ is polynomially bounded;*
2. *$A = B + J(A)$, where B is a maximal semisimple subalgebra of A with trivial induced $\mathbb{Z}_2$-grading and trivial induced involution.*

The next theorem is a characterization of finite dimensional $*$-superalgebras having polynomial growth of the $*$-graded codimensions.

Theorem 12. *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. The following conditions are equivalent:*

1. *$\exp^{\mathrm{gri}}(A) \le 1$;*
2. *$c_n^{\mathrm{gri}}(A)$ is polynomially bounded;*
3. *M_*, D_*, D^{gr}, $M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$.*

Proof. Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. The equivalence (2) $\Leftrightarrow$ (3) is the Theorem 4 and it is clear that (2) $\Rightarrow$ (1).

Therefore, we only need to show that (1) $\Rightarrow$ (2). Let $\bar{F}$ be the algebraic closure of F and consider $\bar{A} = A \otimes_F \bar{F}$. By Theorem 5, we can write $\bar{A} = \bar{B} + J(\bar{A})$, where $\bar{B}$ is maximal semisimple $*$-graded subalgebra of $\bar{A}$ and $J(\bar{A})$ is the Jacobson radical of $\bar{A}$. By Remark 1, we have that $\exp^{\mathrm{gri}}(\bar{A}) = \exp^{\mathrm{gri}}(A) \leq 1$. Hence, by Theorem 7, $\bar{B}$ has trivial induced $\mathbb{Z}_2$-grading and trivial induced involution. Also, by Lemma 1, we have that $\exp(\bar{A}) \leq \exp^{\mathrm{gri}}(\bar{A}) \leq 1$. Thus, $c_n(\bar{A})$ is polynomially bounded and, by Theorem 11, $c_n^{\mathrm{gri}}(\bar{A})$ is polynomially bounded. Since, by Remark 1, $c_n^{\mathrm{gri}}(\bar{A}) = c_n^{\mathrm{gri}}(A)$, we have that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded and the theorem is proved.

4. $*$-Supervarieties with $\exp^{\mathrm{gri}}(\mathcal{V}) \geq 2$

In the previous section, we characterized finite dimensional $*$-superalgebras A such that $\exp^{\mathrm{gri}}(A) \leq 1$. In this section, we characterize finite dimensional $*$-superalgebras A such that $\exp^{\mathrm{gri}}(A) \geq 2$.

Recall that the algebra $UT_n(F)$ of upper triangular matrices of order n can be endowed with the involution $(a_{ij})^* = a_{n+1-j,n+1-i}$, called reflection involution. This involution is obtained by flipping the matrix along its secondary diagonal. Any subalgebra of $UT_n(F)$, for some $n \geq 1$, appearing in this section will be endowed with this involution.

Consider the following $*$-superalgebras:

1. $E_1 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 & 0 \\ 0 & b & f & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & g & h \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a,b,c,d,e,f,g,h,i \in F \right\}$ with trivial grading and reflection involution;
2. $E_2 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 & 0 \\ 0 & b & f & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & g & h \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a,b,c,d,e,f,g,h,i \in F \right\}$ with grading

$$\left(\begin{pmatrix} a & d & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

and reflection involution;

3. $E_3 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 & 0 \\ 0 & b & f & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & g & h \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e, f, g, h, i \in F \right\}$ with grading

$$\left(\begin{pmatrix} a & 0 & e & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & h \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

and reflection involution;

4. $E_4 = \left\{ \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & e \\ 0 & 0 & 0 & c \end{pmatrix} : a, b, c, d, e \in F \right\}$ with trivial grading and reflection involution;

5. $E_5 = \left\{ \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & e \\ 0 & 0 & 0 & c \end{pmatrix} : a, b, c, d, e \in F \right\}$ with grading

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

and reflection involution;

6. $E_6 = \left\{ \begin{pmatrix} a + \alpha b & e + \alpha f & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g + \alpha h \\ 0 & 0 & 0 & a + \alpha b \end{pmatrix} : a, b, d, e, f, g \in F, \alpha^2 = 1 \right\}$ with grading

$$\left(\begin{pmatrix} a & e & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} \alpha b & \alpha f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha h \\ 0 & 0 & 0 & \alpha b \end{pmatrix} \right)$$

and reflection involution;

7. $E_7 = M_2(F)$ with trivial grading and transpose involution;
8. $E_8 = M_2(F)$ with trivial grading and symplectic involution;
9. $E_9 = M_{1,1}(F)$ with transpose involution;

10. $E_{10} = M_{1,1}(F)$ with symplectic involution;
11. $E_{11} = (F + cF) \oplus (F + cF)$ with grading $(F + F, c(F + F))$ and exchange involution.

In the next 4 lemmas, by Remark 1, we assume that F is an algebraically closed field.

Lemma 13. $\exp^{\text{gri}}(E_i) = 3, i = 1, 2, 3$.

Proof. We have that the Wedderburn–Malcev decompositions as $*$-superalgebras of $E_i, i = 1, 2, 3$, are the same: $E_i = A_1 \oplus A_2 \oplus A_3 + J(E_i), i = 1, 2, 3$, where $A_1 = F(e_{11} + e_{66}), A_2 = F(e_{22} + e_{55}), A_3 = F(e_{33} + e_{44})$ and

$$J(E_i) = Fe_{12} \oplus Fe_{13} \oplus Fe_{23} \oplus Fe_{45} \oplus Fe_{46} \oplus Fe_{56}.$$

We get that $A_1 J(E_i) A_2 J(E_i) A_3 \neq \{0\}$ and, by Theorem 7, it follows that $\exp^{\text{gri}}(E_i) = 3, i = 1, 2, 3$.

Lemma 14. $\exp^{\text{gri}}(E_i) = 3, i = 4, 5$.

Proof. We have that the Wedderburn–Malcev decompositions as $*$-superalgebras of $E_i, i = 4, 5$, are the same: $E_i = A_1 \oplus A_2 + J(E_i), i = 4, 5$, where $A_1 = Fe_{11} \oplus F_{44}$, $A_2 = F(e_{22} + e_{33})$ and $J(E_i) = Fe_{12} \oplus Fe_{34}$. We get that $A_1 J(E_i) A_2 \neq \{0\}$ and, by Theorem 7, it follows that $\exp^{\text{gri}}(E_i) = 3, i = 4, 5$.

Lemma 15. $\exp^{\text{gri}}(E_6) = 3$.

Proof. We have that the Wedderburn–Malcev decomposition as $*$-superalgebra of E_6 is $E_6 = A_1 \oplus A_2 + J(E_6)$, where $A_1 = F\mathbb{Z}_2(e_{11} + e_{44}), A_2 = F(e_{22} + e_{33})$ and $J(E_6) = F\mathbb{Z}_2(e_{12}) \oplus F\mathbb{Z}_2(e_{34})$. We get that $A_1 J(E_6) A_2 \neq \{0\}$ and, by Theorem 7, it follows that $\exp^{\text{gri}}(E_6) = 3$.

Lemma 16. $\exp^{\text{gri}}(E_i) = 4, i = 7, \ldots, 11$.

Proof. The result follows from Theorems 6 and 10.

We remind the reader that we denote by D_* the algebra $D = F \oplus F$ with trivial grading and exchange involution and by D^{gr} the algebra $D = F \oplus F$ with grading $(F(1, 1), F(1, -1)) \cong F + cF, c^2 = 1$, and trivial involution.

From now on F will be a field of characteristic zero and A a finite dimensional $*$-superalgebra over F. By Theorem 5, if F is an algebraically closed field, we can write $A = A_1 \oplus \cdots \oplus A_m + J$, where each algebra A_i, $i = 1, \ldots, m$, is a simple $*$-superalgebra and $J = J(A)$ is the Jacobson radical of A.

Lemma 17. *Suppose that F is algebraically closed and $\exp^{\text{gri}}(A) > 2$. If there exist three distinct $*$-graded simple components $A_i \cong A_k \cong A_l \cong F$ such that $A_i J A_k J A_l \neq \{0\}$, then $E_i \in \text{var}^{\text{gri}}(A)$ for some $i \in \{1, 2, 3\}$.*

Proof. Let e_1, e_2, e_3 be the unit elements of A_i, A_k and A_l, respectively. Then $e_n^2 = e_n$, $e_n \in A_n^{(0)}$, $e_n^* = e_n$ and $e_r e_s = \delta_{rs} e_r$ for $r, s = 1, 2, 3$ and $n \in \{i, k, l\}$.

Since $e_1 J e_2 J e_3 \neq \{0\}$, let $m \geq 1$ be the greatest integer such that $J^m \neq \{0\}$ and $e_a J e_b J e_c \subseteq J^m$, for all permutations (a, b, c) of $(1, 2, 3)$. Let $\bar{A} = A/J^{m+1}$. Then $\bar{A}$ is a $*$-superalgebra and $\bar{A} \in \mathrm{var}^{\mathrm{gri}}(A)$. Let $\bar{e}_i = e_i + J^{m+1}, i = 1, 2, 3$. Then $\bar{e}_i, i = 1, 2, 3$, are orthogonal idempotents of $\bar{A}$ such that, by eventually renaming the idempotents, $\bar{e}_1 \bar{J} \bar{e}_2 \bar{J} \bar{e}_3 \neq \{0\}$, where $\bar{J} = J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Also, $\bar{e}_a \bar{J} \bar{e}_b \bar{J} \bar{e}_c \bar{J} = \bar{J} \bar{e}_a \bar{J} \bar{e}_b \bar{J} \bar{e}_c = \{0\}$, for all permutations (a, b, c) of $(1, 2, 3)$. Hence, we may assume that in A we have $e_1 J e_2 J e_3 \neq \{0\}$ and $J e_a J e_b J e_c = e_a J e_b J e_c J = \{0\}$ for all permutations (a, b, c) of $(1, 2, 3)$.

Let I be the ideal of A generated by $\{e_n J e_m J e_n : m, n \in \{1, 2, 3\}, m \neq n\}$. Since the idempotents $e_i, i = 1, 2, 3$, are symmetric and have homogeneous degree 0, we get that I is a $*$-graded ideal of A, $e_1 J e_2 J e_3 \not\subset I$ and $A/I \in \mathrm{var}^{\mathrm{gri}}(A)$. Hence, we may assume that in A we have $e_1 J e_2 J e_3 \neq \{0\}$ and $e_m J e_n J e_m = \{0\}$, $m, n \in \{1, 2, 3\}, m \neq n$.

Since $e_1 J e_2 J e_3 \neq \{0\}$, there exist $j_1 = j_1^{(0)} + j_1^{(1)}$, $j_2 = j_2^{(0)} + j_2^{(1)} \in J$, with $j_1^{(0)}, j_2^{(0)} \in J^{(0)}$, $j_1^{(1)}, j_2^{(1)} \in J^{(1)}$ such that

$$\begin{aligned} e_1 j_1 e_2 j_2 e_3 &= e_1 (j_1^{(0)} + j_1^{(1)}) e_2 (j_2^{(0)} + j_2^{(1)}) e_3 \\ &= e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 + e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 + e_1 j_1^{(1)} e_2 j_2^{(0)} e_3 + e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \\ &\neq 0. \end{aligned}$$

Therefore, one of the following inequalities must hold:

1. $e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 \neq 0$;
2. $e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 \neq 0$;
3. $e_1 j_1^{(1)} e_2 j_2^{(0)} e_3 \neq 0$;
4. $e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \neq 0$.

Suppose that (1) holds. Then $e_1 j_1^{(0)} e_2 \neq 0$ and $e_2 j_2^{(0)} e_3 \neq 0$. Let U_1 be the $*$-superalgebra linearly generated by the elements e_1, e_2, e_3, $e_1 j_1^{(0)} e_2$, $e_2 (j_1^{(0)})^* e_1$, $e_2 j_2^{(0)} e_3$, $e_3 (j_2^{(0)})^* e_2$, $e_1 j_1^{(0)} e_2 j_2^{(0)} e_3$, $e_3 (j_2^{(0)})^* e_2 (j_1^{(0)})^* e_1$. Notice that U_1 has trivial induced $\mathbb{Z}_2$-grading. Then, the map $\psi_1 : U_1 \to E_1$ defined by

$$\begin{aligned} &e_1 \mapsto e_{11} + e_{66}, && e_2 \mapsto e_{22} + e_{55}, \\ &e_3 \mapsto e_{33} + e_{44}, && e_1 j_1^{(0)} e_2 \mapsto e_{12}, \\ &e_2 (j_1^{(0)})^* e_1 \mapsto e_{56}, && e_2 j_2^{(0)} e_3 \mapsto e_{23}, \\ &e_3 (j_2^{(0)})^* e_2 \mapsto e_{45}, && e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 \mapsto e_{13}, \\ &e_3 (j_2^{(0)})^* e_2 (j_1^{(0)})^* e_1 \mapsto e_{46} \end{aligned}$$

is an isomorphism of $*$-superalgebras. Hence, $E_1 \in \mathrm{var}^{\mathrm{gri}}(A)$.

Now, suppose that (2) holds. Then $e_1 j_1^{(0)} e_2 \neq 0$ and $e_2 j_2^{(1)} e_3 \neq 0$. Let U_2 be the $*$-superalgebra linearly generated by the elements e_1, e_2, e_3, $e_1 j_1^{(0)} e_2$, $e_2 (j_1^{(0)})^* e_1$, $e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2$, $e_1 j_1^{(0)} e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2 (j_1^{(0)})^* e_1$. Notice that U_2 has induced $\mathbb{Z}_2$-grading $U_2 = (U_2^{(0)}, U_2^{(1)})$ where

$$U_2^{(0)} = \mathrm{span}_F\{e_1, e_2, e_3, e_1 j_1^{(0)} e_2, e_2 (j_1^{(0)})^* e_1\}$$

and

$$U_2^{(1)} = \mathrm{span}_F\{e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2, e_1 j_1^{(0)} e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2 (j_1^{(0)})^* e_1\}.$$

Then, the map $\psi_2 : U_2 \to E_2$ defined by

$$\begin{aligned} &e_1 \mapsto e_{11} + e_{66}, && e_2 \mapsto e_{22} + e_{55}, \\ &e_3 \mapsto e_{33} + e_{44}, && e_1 j_1^{(0)} e_2 \mapsto e_{12}, \\ &e_2 (j_1^{(0)})^* e_1 \mapsto e_{56}, && e_2 j_2^{(1)} e_3 \mapsto e_{23}, \\ &e_3 (j_2^{(1)})^* e_2 \mapsto e_{45}, && e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 \mapsto e_{13}, \\ &e_3 (j_2^{(1)})^* e_2 (j_1^{(0)})^* e_1 \mapsto e_{46} \end{aligned}$$

is an isomorphism of $*$-superalgebras. Hence, $E_2 \in \mathrm{var}^{\mathrm{gri}}(A)$. Analogously, if (3) holds, then $E_2 \in \mathrm{var}^{\mathrm{gri}}(A)$.

Finally, suppose that (4) holds. Then $e_1 j_1^{(1)} e_2 \neq 0$ and $e_2 j_2^{(1)} e_3 \neq 0$. Let U_3 be the $*$-superalgebra linearly generated by the elements e_1, e_2, e_3, $e_1 j_1^{(1)} e_2$, $e_2 (j_1^{(1)})^* e_1$, $e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2$, $e_1 j_1^{(1)} e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2 (j_1^{(1)})^* e_1$. Notice that U_3 has induced $\mathbb{Z}_2$-grading $U_3 = (U_3^{(0)}, U_3^{(1)})$ where

$$U_3^{(0)} = \mathrm{span}_F\{e_1, e_2, e_3, e_1 j_1^{(1)} e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2 (j_1^{(1)})^* e_1\}$$

and

$$U_3^{(1)} = \mathrm{span}_F\{e_1 j_1^{(1)} e_2, e_2 (j_1^{(1)})^* e_1, e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2\}.$$

Then, the map $\psi_3 : U_3 \to E_3$ defined by

$$\begin{aligned} &e_1 \mapsto e_{11} + e_{66}, && e_2 \mapsto e_{22} + e_{55}, \\ &e_3 \mapsto e_{33} + e_{44}, && e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \mapsto e_{13}, \\ &e_3 (j_2^{(1)})^* e_2 (j_1^{(1)})^* e_1 \mapsto e_{46}, && e_1 j_1^{(1)} e_2 \mapsto e_{12}, \\ &e_2 (j_1^{(1)})^* e_1 \mapsto e_{56}, && e_2 j_2^{(1)} e_3 \mapsto e_{23}, \\ &e_3 (j_2^{(1)})^* e_2 \mapsto e_{45} \end{aligned}$$

is an isomorphism of $*$-superalgebras. Hence, $E_3 \in \mathrm{var}^{\mathrm{gri}}(A)$. This completes the proof.

Lemma 18. *Suppose that F is algebraically closed and* $\exp^{\mathrm{gri}}(A) > 2$. *If there exist two $*$-graded simple components $A_i \cong F$ and $A_k \cong D_*$ such that either $A_iJA_k \neq \{0\}$ or $A_kJA_i \neq \{0\}$ then E_4 or $E_5 \in \mathrm{var}^{\mathrm{gri}}(A)$.*

Proof. Suppose first that $A_iJA_k \neq \{0\}$. Let e_i and e_k be the unit elements of A_i and A_k, respectively. Then $e_n^2 = e_n = e_n^*$, $e_n \in A_n^{(0)}$, $e_re_s = \delta_{rs}e_r$ for $r, s, n \in \{i, k\}$.

Since $e_iJe_k \neq \{0\}$, let $m \geq 1$ be the greatest integer such that $J^m \neq \{0\}$ and $e_aJe_b \subseteq J^m$, $a, b \in \{i, k\}$. Let $\bar{A} = A/J^{m+1}$. Then $\bar{A}$ is a $*$-superalgebra and $\bar{A} \in \mathrm{var}^{\mathrm{gri}}(A)$. Let $\bar{e}_n = e_n + J^{m+1}, n = i, k$. Then $\bar{e}_n, n = i, k$, are orthogonal idempotents of $\bar{A}$ such that, by eventually renaming the idempotents, $\bar{e}_i\bar{J}\bar{e}_k \neq \{0\}$, where $\bar{J} = J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Also, $\bar{e}_a\bar{J}\bar{e}_b\bar{J} = \bar{J}\bar{e}_a\bar{J}\bar{e}_b = \{0\}$, $a, b \in \{i, k\}$. Hence, we may assume that in A we have $e_iJe_k \neq \{0\}$ and $Je_aJe_b = e_aJe_bJ = \{0\}$, $a, b \in \{i, k\}$. Writing $e_i = e_1$ and $e_k = e_2 + e_3$, we have that $e_1^* = e_1$ and $e_2^* = e_3$.

Since $A_iJA_k \neq \{0\}$, there exists $j = j^{(0)} + j^{(1)} \in J$, $j^{(0)} \in J^{(0)}$, $j^{(1)} \in J^{(1)}$ such that

$$e_1(j^{(0)} + j^{(1)})(e_2 + e_3) = e_1j^{(0)}e_2 + e_1j^{(0)}e_3 + e_1j^{(1)}e_2 + e_1j^{(1)}e_3 \neq 0.$$

Therefore, one of the following inequalities must hold:

1. $e_1j^{(0)}e_2 \neq 0$;
2. $e_1j^{(0)}e_3 \neq 0$;
3. $e_1j^{(1)}e_2 \neq 0$;
4. $e_1j^{(1)}e_3 \neq 0$.

Suppose that (1) holds. Let H_1 be the $*$-superalgebra linearly generated by the elements $e_1, e_2, e_3, e_1j^{(0)}e_2, e_3(j^{(0)})^*e_1$. Notice that H_1 has trivial induced $\mathbb{Z}_2$-grading. Then, the map $\psi_1 : H_1 \to E_4$ defined by

$$\begin{aligned} &e_1 \mapsto e_{22} + e_{33}, \qquad && e_2 \mapsto e_{44} \\ &e_3 \mapsto e_{11} && e_1j^{(0)}e_2 \mapsto e_{34} \\ &e_3(j^{(0)})^*e_1 \mapsto e_{12} && \end{aligned}$$

is an isomorphism of $*$-superalgebras. Hence, $E_4 \in \mathrm{var}^{\mathrm{gri}}(A)$. Analogously, if (2) holds, then $E_4 \in \mathrm{var}^{\mathrm{gri}}(A)$.

Suppose that (3) holds. Let H_2 be the $*$-superalgebra linearly generated by the elements $e_1, e_2, e_3, e_1j^{(1)}e_2, e_3(j^{(1)})^*e_1$. Notice that H_2 has induced $\mathbb{Z}_2$-grading $H_2 = (H_2^{(0)}, H_2^{(1)})$ where

$$H_2^{(0)} = \mathrm{span}_F\{e_1, e_2, e_3\}$$

and

$$H_2^{(1)} = \mathrm{span}_F\{e_1j^{(1)}e_2, e_3(j^{(1)})^*e_1\}.$$

Then, the map $\psi_2 : H_2 \to E_5$ defined by

$$\begin{array}{ll} e_1 \mapsto e_{22} + e_{33}, & e_2 \mapsto e_{44} \\ e_3 \mapsto e_{11} & e_1 j^{(1)} e_2 \mapsto e_{34} \\ e_3 (j^{(1)})^* e_1 \mapsto e_{12} & \end{array}$$

is an isomorphism of $*$-superalgebras. Hence, $E_5 \in \mathrm{var}^{\mathrm{gri}}(A)$. Analogously, if (4) holds, then $E_5 \in \mathrm{var}^{\mathrm{gri}}(A)$.

The case $A_k J A_i \neq \{0\}$ is analogous.

Lemma 19. *Suppose that F is algebraically closed and $\exp^{\mathrm{gri}}(A) > 2$. If there exist two $*$-graded simple components $A_i \cong F$ and $A_k \cong D^{\mathrm{gr}}$ such that either $A_i J A_k \neq \{0\}$ or $A_k J A_i \neq \{0\}$ then $E_6 \in \mathrm{var}^{\mathrm{gri}}(A)$.*

Proof. Let e_1 and e_2 be the unit elements of A_i and A_k, respectively. Then $e_n^2 = e_n$, $e_n \in A_n^{(0)}$, $e_n^* = e_n$ and $e_r e_s = \delta_{rs} e_r$ for $r, s = 1, 2$ and $n \in \{i, k\}$.

Since $e_1 J e_2 \neq \{0\}$, let $m \geq 1$ be the greatest integer such that $J^m \neq \{0\}$ and $e_a J e_b \subseteq J^m$, $a, b \in \{1, 2\}$. Let $\bar{A} = A/J^{m+1}$. Then $\bar{A}$ is a $*$-superalgebra and $\bar{A} \in \mathrm{var}^{\mathrm{gri}}(A)$. Let $\bar{e}_i = e_i + J^{m+1}, i = 1, 2$. Then $\bar{e}_i, i = 1, 2$, are orthogonal idempotents of $\bar{A}$ such that, by eventually renaming the idempotents, $\bar{e}_1 \bar{J} \bar{e}_2 \neq \{0\}$, where $\bar{J} = J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Also, $\bar{e}_a \bar{J} \bar{e}_b \bar{J} = \bar{J} \bar{e}_a \bar{J} \bar{e}_b = \{0\}$, $a, b \in \{1, 2\}$. Hence, we may assume that in A we have $e_1 J e_2 \neq \{0\}$ and $J e_a J e_b = e_a J e_b J = \{0\}$, $a, b \in \{1, 2\}$.

Since $e_1 J e_2 \neq \{0\}$, there exists $j = j^{(0)} + j^{(1)} \in J$, $j^{(0)} \in J^{(0)}, j^{(1)} \in J^{(1)}$ such that

$$e_1 (j^{(0)} + j^{(1)}) e_2 = e_1 j^{(0)} e_2 + e_1 j^{(1)} e_2 \neq 0.$$

Thus, we must have either $e_1 j^{(0)} e_2 \neq 0$ or $e_1 j^{(1)} e_2 \neq 0$. If $e_1 j^{(1)} e_2 \neq 0$, by multiplying by c on the right, we may assume that $e_1 j^{(0)} e_2 \neq 0$, for some $j^{(0)} \in J^{(0)}$.

Let H be the $*$-superalgebra linearly generated by the elements e_1, e_2, ce_2, $e_1 j^{(0)} e_2$, $ce_1 j^{(0)} e_2$, $e_2 (j^{(0)})^* e_1$, $ce_2 (j^{(0)})^* e_1$. Notice that H has induced $\mathbb{Z}_2$-grading $H = (H^{(0)}, H^{(1)})$ where

$$H^{(0)} = \mathrm{span}_F \{e_1, e_2, e_1 j^{(0)} e_2, e_2 (j^{(0)})^* e_1\}$$

and

$$H^{(1)} = \mathrm{span}_F \{ce_2, ce_1 j^{(0)} e_2, ce_2 (j^{(0)})^* e_1\}.$$

Then, the map $\psi : H \to E_6$ defined by

$$\begin{array}{ll} e_1 \mapsto e_{22} + e_{33} & e_2 \mapsto e_{11} + e_{44} \\ ce_2 \mapsto \alpha(e_{11} + e_{44}) & e_1 j^{(0)} e_2 \mapsto e_{34} \end{array}$$

$$ce_1 j^{(0)} e_2 \mapsto \alpha e_{34} \qquad e_2(j^{(0)})^* e_1 \mapsto e_{12}$$
$$ce_2(j^{(0)})^* e_1 \mapsto \alpha e_{12}$$

is an isomorphism of $*$-superalgebras. Hence $E_6 \in \mathrm{var}^{\mathrm{gri}}(A)$.

The case $A_k J A_i \neq \{0\}$ is analogous.

The next remark will be useful in the proof of the main theorem.

Remark 2.

1. If $M_{k,l}(F)$, with $k+l \geq 2, l \geq 0$, with transpose or symplectic involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then either $M_2(F)$, with trivial grading, or $M_{1,1}(F)$, with transpose or symplectic involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$;
2. If $M_{k,l}(F) \oplus M_{k,l}(F)^{op}$, with $k+l \geq 2, l \geq 0$, with induced grading and exchange involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then either $M_2(F)$ with trivial grading or $M_{1,1}(F)$, with transpose involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$;
3. If $M_n(F) + cM_n(F)$, $n \geq 2$, with involution given by $(a+cb)^\dagger = a^* \pm cb^*$, where $*$ denotes the transpose or symplectic involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then $M_n(F)$, with trivial grading and transpose or symplectic involution, and $(F+cF)\oplus(F+cF)$, with grading $(F+F, c(F+F))$ and exchange involution, lie in $\mathrm{var}^{\mathrm{gri}}(A)$. Hence, $M_2(F)$, with trivial grading and transpose or symplectic involution, and $(F+cF)\oplus(F+cF)$, with grading $(F+F, c(F+F))$ and exchange involution, lie in $\mathrm{var}^{\mathrm{gri}}(A)$.
4. If $(M_n(F) + cM_n(F)) \oplus (M_n(F) + cM_n(F))^{op}$, $n \geq 2$, with grading

$$(M_n(F) \oplus M_n(F)^{op}, c(M_n(F) \oplus M_n(F)^{op}))$$

   and exchange involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then $M_n(F) + cM_n(F)$, with involution given by $(a+cb)^\dagger = a^* \pm cb^*$, where $*$ denotes the transpose or symplectic involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$. Hence, $M_2(F)$, with trivial grading and transpose or symplectic involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$.

Now we are in condition to proof the main theorem of this paper.

Theorem 20. *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. Then $\exp^{\mathrm{gri}}(A) > 2$ if and only if $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$, for some $i \in \{1, \ldots, 11\}$.*

Proof. By Remark 1, we may assume that F is an algebraically closed field. If, for some $i \in \{1, \ldots, 11\}$, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$ then, by Lemmas 13, 14, 15 and 16, $\exp^{\mathrm{gri}}(A) > 2$.

Conversely, suppose that $\exp^{\mathrm{gri}}(A) > 2$. By Theorem 5, we can write $A = A_1 \oplus \cdots \oplus A_m + J$, where each algebra A_i, $i = 1, \ldots, m$, is a simple $*$-superalgebra and $J = J(A)$ is the Jacobson radical of A. If, for some $i \in \{1, \ldots, m\}$, A_i is isomorphic to one of the simple $*$-superalgebras given in Theorem 6 with $\dim_F(A_i) \geq 4$, then, by Remark 2, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$ for some $i \in \{7, 8, 9, 10, 11\}$.

Since $\exp^{\mathrm{gri}}(A) > 2$, by Theorem 7, there exist distinct $*$-graded simple components $A_{i_1}, \dots, A_{i_n}$ such that $A_{i_1} J \cdots J A_{i_n} \neq \{0\}$ and $\dim_F(A_{i_1} + \cdots + A_{i_n}) > 2$. By the above, we may assume that one of the following possibilities occurs:

1. there exist distinct A_i, A_k, A_l such that $A_i J A_k J A_l \neq \{0\}$ and $A_i \cong A_k \cong A_l \cong F$;
2. for some $i \neq k$, $A_i J A_k \neq \{0\}$ where $A_i \cong F$ and $A_k \cong D_*$;
3. for some $i \neq k$, $A_i J A_k \neq \{0\}$ where $A_i \cong F$ and $A_k \cong D^{\mathrm{gr}}$.

If (1) holds, then, by Lemma 17, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$, for some $i \in \{1, 2, 3\}$. If (2) holds, then, by Lemma 18, either E_4 or $E_5 \in \mathrm{var}^{\mathrm{gri}}(A)$. Finally, if (3) holds, then, by Lemma 19, $E_6 \in \mathrm{var}^{\mathrm{gri}}(A)$. The proof is complete.

We can notice that the above list of $*$-superalgebras cannot be reduced. In fact, we have the following proposition.

Proposition 21. *For all $i, j \in \{1, \dots, 11\}, i \neq j, \mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$.*

Proof. We shall prove the proposition in several steps by utilizing different arguments.

- it is clear that if $\mathrm{Id}^{\mathrm{gri}}(E_i) \subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, then $\exp^{\mathrm{gri}}(E_j) \leq \exp^{\mathrm{gri}}(E_i)$. Hence, by Lemmas 13, 14, 15 and 16, $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$ for $i \in \{1, 2, 3, 4, 5, 6\}$ and $j \in \{7, 8, 9, 10, 11\}$;
- the $*$-superalgebras $E_i, i \in \{1, 4, 7, 8\}$ have trivial $\mathbb{Z}_2$-grading. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{1, 4, 7, 8\}$, $j \in \{2, 3, 5, 6, 9, 10, 11\}$;
- $z_{1,0}^2 \in \mathrm{Id}^{\mathrm{gri}}(E_i), i \in \{2, 3, 6\}$ and $z_{1,0}^2 \notin \mathrm{Id}^{\mathrm{gri}}(E_j), i \in \{1, 4, 5\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{2, 3, 6\}$, $j \in \{1, 4, 5\}$;
- $z_{1,0}^3 \in \mathrm{Id}^{\mathrm{gri}}(E_1)$ and $z_{1,0}^3 \notin \mathrm{Id}^{\mathrm{gri}}(E_4)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_1) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_4)$;
- $y_{1,1}^2 \in \mathrm{Id}^{\mathrm{gri}}(E_2)$ and $y_{1,1}^2 \notin \mathrm{Id}^{\mathrm{gri}}(E_j), j \in \{3, 6\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_2) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{3, 6\}$;
- $z_{1,0} y_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(E_3)$ and $z_{1,0} y_{1,1} \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2, 6\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_3) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2, 6\}$;
- $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}] \in \mathrm{Id}^{\mathrm{gri}}(E_4)$ and $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}] \notin \mathrm{Id}^{\mathrm{gri}}(E_1)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_4) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_1)$;
- $[y_{1,0}, y_{2,0}] \in \mathrm{Id}^{\mathrm{gri}}(E_i), i \in \{5, 8, 10\}$ and $[y_{1,0}, y_{2,0}] \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1, 2, 3, 4, 6, 7\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{5, 8, 10\}$, $j \in \{1, 2, 3, 4, 6, 7\}$;
- $z_{1,0} z_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(E_6)$ and $z_{1,0} z_{1,1} \notin \mathrm{Id}^{\mathrm{gri}}(E_2)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_6) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_2)$;
- $z_{1,1}^2 \in \mathrm{Id}^{\mathrm{gri}}(E_6)$ and $z_{1,1}^2 \notin \mathrm{Id}^{\mathrm{gri}}(E_3)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_6) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_3)$;
- $[z_{1,0}, z_{2,0}] \in \mathrm{Id}^{\mathrm{gri}}(E_i)$, $i \in \{7, 10\}$ and $[z_{1,0}, z_{2,0}] \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1, 4, 8\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{7, 10\}$, $j \in \{1, 4, 8\}$;
- $z_{1,0} \in \mathrm{Id}^{\mathrm{gri}}(E_9)$ and $z_{1,0} \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_9) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}$;

- $y_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(E_{10})$ and $y_{1,1} \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2,3,5,6,9,11\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_{10}) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2,3,5,6,9,11\}$;
- the $*$-superalgebra E_{11} is commutative and the $*$-superalgebras E_j are not, $j \in \{1,\dots,10\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_{11}) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1,\dots,10\}$.

These facts prove the proposition.

As a consequence of Theorems 12 and 20, we have the following characterization of finite dimensional $*$-superalgebras A such that $\exp^{\mathrm{gri}}(A) = 2$.

Corollary 22. *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. Then $\exp^{\mathrm{gri}}(A) = 2$ if and only if $E_i \notin \mathrm{var}^{\mathrm{gri}}(A)$, for every $i \in \{1,\dots,11\}$, and either $D_*, D^{\mathrm{gr}}, M_*$ or $M^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(A)$.*

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