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Novel tests for stability analysis of time-delayed systems: a linear matrix inequality-based approach

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NOVEL TESTS FOR STABILITY ANALYSIS OF TIME-DELAYED SYSTEMS: A LINEAR MATRIX INEQUALITY-BASED APPROACH

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Resumo

A análise de estabilidade de sistemas com retardo no tempo tem sido um campo de pesquisa muito ativo nas últimas décadas. O interesse se baseia, em parte, no fato de o atraso ser um fenômeno inerente a uma ampla classe de sistemas encontrados nas mais diversas áreas, como engenharia, biologia e economia. Além disso, os sistemas com atraso são representados por equações diferenciais infinito-dimensionais, o que torna a determinação de sua estabilidade um problema particularmente mais complexo. Apesar dos diversos avanços obtidos na área nos últimos anos, os métodos existentes para lidar com esse problema ainda possuem as suas limitações. Levando em consideração este cenário, nesta tese são propostos novos métodos para analisar a estabilidade de sistemas lineares invariantes no tempo (LTI) com atraso constante e variante no tempo. Primeiramente, são propostas novas condições necessárias e suficientes para análise de estabilidade independente do retardo de sistemas sujeitos a retardo constante no tempo. O método proposto é baseado no uso da desigualdade de Lyapunov, definida por meio de polinômios matriciais dependentes da frequência, e formulado em termos de desigualdades matriciais lineares (LMIs), resultado obtido graças ao Lema de Kalman-Yakubovich-Popov (KYP). Como segunda contribuição, são propostas novas condições suficientes para a estabilidade dependente do atraso de sistemas com atraso variante no tempo. Tais condições são obtidas a partir do uso de um novo funcional de Lyapunov-Krasovskii e dependente de parâmetros. Este critério de estabilidade é especificado como uma condição de negatividade de uma função quadrática parametrizada pelo atraso. Neste trabalho também é proposto um método para converter essa condição em um problema de otimização convexa de dimensão finita que pode ser verificado de maneira exata em termos de condições LMIs. Finalmente, as condições de estabilidade dependente do atraso são estendidas para o caso de sistemas com múltiplos atrasos variantes no tempo. Exemplos numéricos mostram que os métodos propostos podem levar a resultados menos conservadores quando comparados aos resultados fornecidos por abordagens similares encontradas na literatura.

Palavras-chave: Sistemas com retardo no tempo. Análise de estabilidade. Desigualdades Matriciais Lineares. Teoria de Lyapunov-Krasovskii. Lema de Kalman-Yakubovich-Popov.

Abstract

Stability analysis of time-delayed systems has been a very active research field in control theory over the last decades. The interest relies on the fact that the time-delay is an inherent phenomenon of a broad class of systems in different fields, like engineering, biology, and economics. Moreover, time-delay systems belong to the class of infinite-dimensional differential equations, making their stability analysis a very complex problem. Although the improvements over the last years, the existing methods to deal with this problem still have limitations. In this thesis, improved methods are proposed to assess the stability of linear time-invariant (LTI) systems with constant and time-varying delay. A first contribution is an alternative method to check exactly the (*strong*) delay-independent stability of systems with constant time-delay. The proposed approach is to use a frequency-dependent Lyapunov stability inequality, with matrices of polynomial type, to indirectly determine the existence of crossing roots on the imaginary axis. The resulting stability criterion is formulated as a linear matrix inequality condition using the Kalman-Yakubovich-Popov (KYP) lemma. As a second contribution, new sufficient conditions are proposed for delay-dependent stability of systems with time-varying delay based on a new augmented affine parameter-dependent Lyapunov-Krasovskii functional (LKF). This stability criterion is specified as a negativity condition for a quadratic function parameterized by the delay. It is also presented a method to translate such a condition into a finite-dimensional convex optimization problem that can be checked exactly in terms of LMI conditions. Finally, the delay-dependent stability conditions are extended to the case of systems with multiple time-varying delays. Numerical examples taken from the literature show that the proposed methods can improve the related existing results.

Keywords: Time-delay systems. Stability analysis. Linear matrix inequalities. Lyapunov-Krasovskii theory. Kalman-Yakubovich-Popov lemma.

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Abbreviations

FWM	Free-Weighting Matrix
IQC	Integral Quadratic Constraints
KYP	Kalman-Yakubovich-Popov
LKF	Lyapunov-Krasovskii Functional
LMI	Linear Matrix Inequality
LTI	Linear Time Invariant
SISO	Single-Input Single-Output
TDS	Time-Delay System(s)

Notation

$\mathbb{R}^{n\times m}$	Set of $n \times m$ real matrices
$\mathbb{C}^{n\times m}$	Set of $n \times m$ complex matrices
$\mathbb{F}^{n\times m}$	Set of $n \times m$ real or complex matrices
$\mathbb{S}^{n}$	Set of $n \times n$ symmetric real matrices
$\mathbb{S}^{n}_{+}$	Set of $n \times n$ real, positive definite matrices
$\mathbb{C}_{+}$	Closed right half plane of the complex plane
$\mathbb{D}$	Closed unit disc
M^T	Transpose of the matrix M
M^*	Conjugate transpose of M
M^{-1}	Inverse of the matrix M
$\det(M)$	Determinant of the matrix M
$\mathrm{adj}(M)$	Adjugate of the matrix M
$\mathrm{vec}(M)$	Vector formed by stacking columns of M
$\mathrm{diag}(M_1, ..., M_n)$	Block-diagonal matrix whose elements are M_1,..., M_n
$\mathrm{col}(x, y)$	Column vector with x stacked on top of y
$\mathrm{sym}(M)$	Denotes $M + M^T$
$\rho(M)$	Spectral radius of M
$\alpha_{\max}(M)$	Spectral abscissa of M
$\lambda_{\min(\max)}(M)$	Minimum (maximum) real part of eigenvalues of M
$M \succ 0$ ($M \prec 0$)	Positive (negative) definite matrix
$\otimes$	Kronecker product
$\oplus$	Kronecker sum
$\mathrm{sign}(\alpha)$	The sign function of a real number α defined to be $+1$ for $\alpha > 0$, -1 for $\alpha < 0$, or 0 otherwise.
I_n	Identity matrix of size $n \times n$
$0_{n\times m}$	Zero matrix of size $n \times m$

Contents

Chapter 1

Introduction

Time-delay is a common phenomenon in a variety of control processes. The delay naturally appears in processes in which the transportation of mass, energy, and information is present, as well as it can be introduced by sensors, actuators, and field networks. Chemical processes, as stirred-tank reactors and flow temperature-composition control (Sipahi et al., 2011); metal cutting processes (Gu et al., 2003), and biological systems (Djema et al., 2017; Niculescu et al., 2010; MacDonald, 2008) are some examples in which the delay is the result of an inherent phenomenon of the system. The internal combustion engine (Gu et al., 2003), on the other hand, is an example in which the delay results from feedback control action, whereas in multi-agent systems (dos Santos Junior et al., 2018; Savino et al., 2017), such as teleoperated systems (de Lima et al., 2020) and vehicular platoons (Souza et al., 2019; Alves Neto et al., 2019), the delay is mainly due to the exchange of information between the agents. Furthermore, time-delays are frequently used to simplify models of high orders (Normey-Rico and Camacho, 2007; Richard, 2003).

It is well-known that delays may have different effects on system stability. Typically, the presence of delay in the control loop is regarded as a source of performance degradation and instability. For single-input single-output (SISO) systems, this effect can be explained in the frequency domain by the extra decrease in the system phase introduced by the delay, which may cause instability (Normey-Rico and Camacho, 2007; Niculescu, 2001). However, for some systems, the delay may have a stabilizing effect. The following example is presented to illustrate it.

Example 1.1 (Gu et al., 2003, page 173)**.** *Consider the system given by:*

$$\ddot{y}(t) - 0.1\dot{y}(t) + y(t) = u(t). \tag{1.1}$$

System (1.1) *is unstable in the absence of control input, i.e.,* $u(t) = 0$*, but it can be stabilizable using a derivative feedback* $u(t) = -k\dot{y}(t)$*, for* $k > 0.1$*. If* $\dot{y}(t)$ *is not available for feedback, one can use the finite difference*

$$u(t) = y(t-\tau) - y(t) = -\tau\frac{y(t) - y(t-\tau)}{\tau}.$$

to approximate $-\tau\dot{y}(t)$, with τ being a sufficiently small number. In this case, the delay τ has a stabilizing effect.

The delayed resonator and distillation columns are examples of practical applications where the delay is intentionally introduced to stabilize or enhance the system performance (Gomes da Silva Jr. and Leite, 2007, (in Portuguese)).

Because of the frequent occurrence of delay and its intriguing characteristics, stability analysis and stabilization of time-delay systems (TDS) have been a very active research field in the past few decades and have been a topic of discussion in many surveys and monographs. For an introduction in basic concepts of linear time-invariant (LTI) systems with time-delay, see Gomes da Silva Jr. and Leite (2007) (in Portuguese). A survey of general TDS can be found, for instance, in Fridman (2014a); Kharitonov (2013); Sipahi et al. (2011); Wu et al. (2010); Gu et al. (2003); Niculescu (2001); Kharitonov (1999), and the references therein. Despite the significant advances on this subject, it is noteworthy that there are still some open problems related to the stability analysis and stabilization of time-delay systems.

One of the most challenging aspects of the time-delay system is its infinite-dimensional nature, which highly complicates the analysis of stability and the design of controllers. A possible approach to overcome this issue is to use rational approximations for the time-delay, allowing the use of tools for finite dimensional systems. A well-known example of such kind of method is the Padé approximation. Although this approach may be useful, controllers that stabilize a system obtained by the Padé approximation may be destabilizing for the original system, as shown in the following example.

Example 1.2 (Adapted from Silva et al., 2005, page 84)**.** *Consider a system represented by the transcendental transfer function*

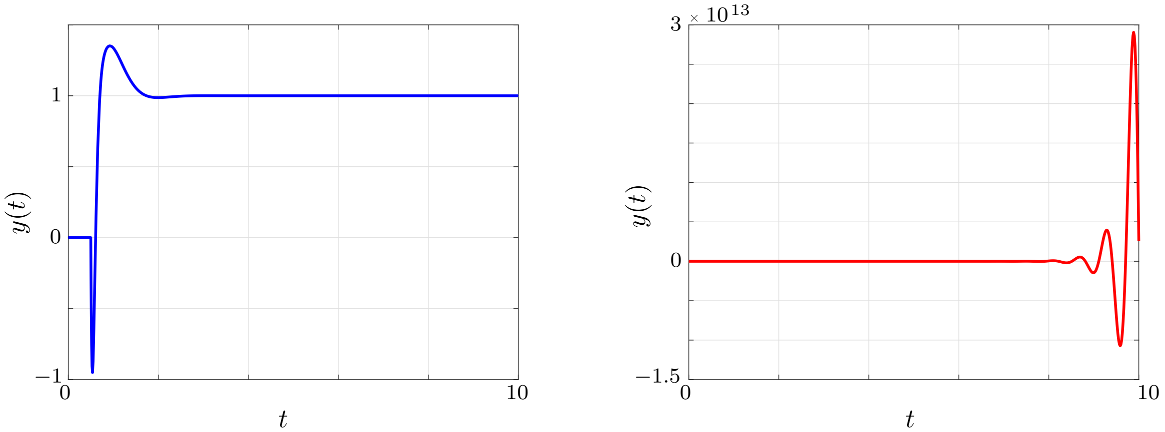
$$G(s) = \frac{1.6667}{2.9036s + 1} e^{-0.2475s}. \tag{1.2}$$

Using a first-order Padé approximation, the following rational transfer function for $G(s)$ is obtained:

$$G_m(s) = \frac{1.6667}{(2.9036s + 1)} \frac{(-0.1238s + 1)}{(0.1238s + 1)}. \tag{1.3}$$

Considering the approximate model (1.3), the stabilizing PID controller with gains $k_p = 10$, $k_i = 20$ and $k_d = 1$ is designed. Figure 1.2 depicts the step response of the transcendental transfer function (1.2) and of the rational transfer function (1.3) in closed-loop with the proposed PID controller. Notice that this controller leads to an unstable closed-loop system with the original model.

The previous example illustrates one of the main drawbacks of the use of rational approximations for time-delay, which motivates and justifies the use of more specialized and,



(a) Rational transfer function (1.3).

(b) Transcendental transfer function (1.2).

Figure 1.1: Step response of transfer functions (1.2) and (1.3) in closed-loop with the designed PID controller.

unfortunately, more complicated techniques for the study of time-delayed systems.

The methods to assess the stability of time-delay systems are often classified into two categories: delay-independent or delay-dependent ones. A system is said to be delay-independent stable if its stability is maintained for all admissible delay values. Otherwise, the system is said to be delay-dependent stable. Methods for determining delay-independent stability are of great interest because the delay-independent stability automatically implies robustness with respect to the time-delay. This is an important property, since for some systems, it may be hard to estimate delays values in practice. On the other hand, delay-dependent stability methods are suitable with a larger range of systems, since most systems are stable only for certain values of time-delay.

Both delay-independent and delay-dependent stability have been studied in frequency and time domains. The earlier frequency-domain methods attempt to determine whether all roots of the characteristic equation lie in the left half-plane. Despite providing necessary and sufficient conditions for stability analysis, this approach is only suitable for systems with constant time-delay (Wu et al., 2010). To deal with time-varying delay and/or uncertainties in the frequency-domain, an interesting approach consists in modeling the time-delay system as a closed-loop between a delay free system and the delay operator. This representation enables to use tools from the robust control theory, as the Small Gain Theorem (Fridman, 2014a; Gu et al., 2003) and the Integral Quadratic Constraints (IQC) framework (Seiler, 2015; Veenman and Scherer, 2013; Kao and Rantzer, 2007). However, to obtain design conditions under the IQC framework is not trivial.

The time-domain methods for TDS are based basically on two remarkable results known as Razumikhin (Razumikhin, 1956) and Lyapunov-Krasovskii (Krasovskii, 1963) theorems, which are a direct generalization of the second Lyapunov method to time-delay systems. The

Lyapunov-Krasovskii method usually leads to less conservative results than the Razumikhin method, and it is more frequently used. An advantage of such approaches is the ease of handling with time-varying uncertainties, nonlinearities and to cope with problems of control and state observer of a large class of systems (Silva et al., 2020b,a; Oliveira et al., 2017; Parada et al., 2016; Savino et al., 2016; Fenili et al., 2014; Gomes et al., 2014). Allied with these benefits, the Razumikhin and Lyapunov-Krasovskii formalisms also provide constructive stability conditions which can be formulated as linear matrix inequalities (LMI).

The idea of formulating control-related problems as linear matrix inequalities comes from the theory of Lyapunov at the end of the 19th century (Boyd et al., 1994; Palhares and Gonçalves, 2007, (in Portuguese)). However, the LMI formulation became a powerful tool to solve such problems only in the late 1980s, with the development of efficient interior-point methods to solve LMIs (Nesterov and Nemirovsky, 1988). An LMI is a set of convex constraints, which means that it can be solved exactly and in polynomial time by convex optimization algorithms. There are currently several optimization packages available to solve LMIs, as the LMILAB (Gahinet et al., 1995), SeDuMi (Sturm, 1999), MOSEK (Andersen and Andersen, 2000), SDPT3 (Toh et al., 1999). Therefore, formulating a problem as an LMI is an effective way to solve it. Several problems of stability and performance analysis can readily be expressed in terms of LMIs. On the other hand, to derive LMI stability conditions for time-delay systems within the Lyapunov-Krasovskii theory framework, it is necessary to use bounding methods to transform some expressions into suitable LMI forms, which unavoidably introduces conservatism in the resulting stability conditions. Since this step is crucial in deriving LMI conditions, much effort has been made to obtain methods that provide thinner bounds (Gu, 2000; Fridman and Shaked, 2003; Wu et al., 2004; Park et al., 2011; Seuret and Gouaisbaut, 2013; Park et al., 2015; Seuret and Gouaisbaut, 2015).

It is important to mention that infinite-dimensional frequency-dependent conditions, as the ones obtained by the IQC method, can also be formulated as finite-dimensional frequency-independent linear matrix inequalities conditions by the use of the Kalman-Yakubovich-Popov (KYP) lemma (Rantzer, 1996). For a complete overview of LMI methods in system and control theory, see Boyd et al. (1994).

This thesis investigates both the stability independent and dependent of delay under the LMI framework. The study is restricted to the linear time-invariant systems with constant and time-varying delay. The next two sections briefly describe some remarkable contributions that have directly influenced the direction of this research.

1.1 Delay-independent stability methods

Delay-independent stability have been studied in the literature using different types of techniques. The earlier methods were based on the analysis of a polynomial with two variables associated with the system characteristic equation (Hertz et al., 1984; Kamen, 1983, 1982, 1980). Another important frequency-domain criterion was proposed in Chen and Latchman (1995), which led to necessary and sufficient conditions similar to small gain conditions in robust stability analysis. This approach is based on frequency sweeping tests and, therefore, may lack accuracy in certain cases and also does not generalize to the problem of controller design.

Regarding the LMI framework, most of conditions for asymptotic stability independent of the delay has been obtained using Lyapunov-Krasovskii functional method (Fridman, 2014a; He et al., 2005; Fridman, 2001; Kolmanovskii, 1999; Boyd et al., 1994). As mentioned before, an advantage of LMI based methods is that they can be usually generalized to cope with problems of filters and controller design. However, typically these conditions are only sufficient and, therefore, conservative.

We can also find in the literature necessary and sufficient delay-independent conditions formulated as LMIs, but in a smaller number of works. In Bliman (2002) interesting and elegant results for verifying *strong* delay-independent stability (a slightly stricter notion of delay-independent stability, which will be discussed later, *in Chapter 2*) have been obtained. The method is based on a family of LMIs, which becomes an exact stability test as the LMIs dimension increases. Li et al. (2016) have proposed a method based on discretization of the frequency domain into several subintervals. In this case, the method becomes an exact stability test as the number of frequency subintervals increases. The main drawback of these two results is the undecidability on the instability, since it is not known a priori the dimension of the LMIs in Bliman (2002), or the number of frequency subintervals in Li et al. (2016), that invalidates stability. Alternatively, in recent papers (Souza et al., 2018; Souza, 2018) an exact new LMI condition with a priori known dimension for *strong* delay-independent stability was proposed. This condition, however, has two main disadvantages: i) it is not easy to be extended to the synthesis of filters and stabilizing controllers, and ii) it lies in a high dimensional LMI constraint, such that it becomes impractical for high-dimensional systems. Therefore, an important problem to be investigated is how to obtain LMI conditions that lead to an exact characterization of the delay-independent stability and are possibly suitable to the design of filters and controllers.

1.2 Delay-dependent stability methods

Among the methodologies to deal with the problem of delay-dependent stability, the Lyapunov-Krasovskii method allied with the LMI framework have been widely utilized. For the case of linear TDS, it is known a general structure for the Lyapunov-Krasovskii functional that is necessary and sufficient for the stability (Kharitonov and Zhabko, 2003). However, it is very difficult to construct such functional, mainly in the case of systems with time-varying delays. For this reason, alternative Lyapunov-Krasovskii functionals are continuously proposed in the literature.

In general, the methods for reducing the conservativeness of LMI conditions based on the Lyapunov-Krasovskii method rely in two main approaches: constructing functionals with more information on the delay and delayed states (Fridman et al., 2009; Yue et al., 2009; Gu et al., 2003; Fridman and Shaked, 2002; Sun et al., 2010; Lee and Park, 2017; Zhang et al., 2017b), and/or using improved integral inequalities to obtain thinner bounds for some quadratic integral terms appearing in the functional derivative (Gu et al., 2003; Seuret and Gouaisbaut, 2013, 2015; Lee et al., 2017). Several papers have shown through numerical examples that, when these integral inequalities are used with LKFs appropriately chosen, they can contribute significantly to conservatism reduction. See, for example, Zhang et al. (2020b); Lee (2020); Chen et al. (2019); Zhang et al. (2019); Seuret and Gouaisbaut (2018); Park et al. (2018); Lee and Park (2018); Chen et al. (2018); Zhang et al. (2017a,b); Liu et al. (2017); Kim (2016), to cite a few of most recent contributions.

Recently, by employing improved integral inequalities and new functional choices, less conservative stability criteria were presented as negativity conditions for quadratic functions within a finite interval of the form

$$\mathcal{F}(\tau(t)) = \xi^T(t)\left(\tau^2(t)\mathcal{M}_2 + \tau(t)\mathcal{M}_1 + \mathcal{M}_0\right)\xi(t) \prec 0, \quad \forall \tau(t) \in [0, \tau_M],$$

where $\xi(t)$ is the vector of states and $\mathcal{M}_2$, $\mathcal{M}_1$, and $\mathcal{M}_0$ are real matrices of appropriate dimensions, and $\tau(t)$ is the time-delay. However, since the pointwise evaluation of $\mathcal{F}(\tau(t))$ within the finite interval is not practical, some sufficient conditions have been proposed to realize such evaluation via a practical test. See, for example, Zhang et al. (2020b); Lee (2020); Chen et al. (2019); Kim (2016). Thus, there is still room to improve these methods.

1.3 Objectives and methodology

This work aims to take a step towards filling some gaps in the stability analysis of TDS described in the previous sections. Basically, the main objective is to derive improved criteria for assessing delay-independent and delay-dependent stability. More specifically, this thesis aims to:

- provide new necessary and sufficient delay-independent stability conditions for LTI systems with constant time-delay;

- develop an exact method for evaluating stability conditions formulated as quadratic functions with respect to the time-varying delay;
- achieve less conservative conditions for delay-dependent stability analysis of LTI systems with time-varying delay by using the Lyapunov-Krasovskii method.

The problem of delay-independent stability is investigated using a frequency-domain methodology, whereas the delay-dependent stability is studied using the time-domain Lyapunov-Krasovskii approach. In both cases, the problems are translated into convex finite-dimensional ones within the LMI framework. It is noteworthy that reducing the problem of stability analysis to an LMI formulation is an effective way of solving it, since LMI conditions can be solved exactly and in polynomial time by algorithms of convex optimization.

1.4 Outline of the chapters

This manuscript is divided into four chapters, and the remaining chapters are organized as follows. The problem of delay-independent stability of systems with constant time-delay is addressed in *Chapter 2*. Firstly, it is provided some concepts and definitions related to this problem in the frequency-domain. Then, the first contribution of this thesis is presented: a new necessary and sufficient LMI condition for assessing the *strong* delay-independent stability of LTI systems subject to a single constant delay. The proposed method follows from the use of matrix polynomials constraints and the Kalman-Yakubovich-Popov lemma to translate the infinite-dimensional frequency-dependent conditions into finite-dimensional ones.

Chapter 3 is dedicated to delay-dependent stability analysis of systems with time-varying delay based on the Lyapunov-Krasovskii method. The contribution presented in that chapter is twofold. It is proposed an augmented affine parameter-dependent Lyapunov-Krasovskii functional, which provides a less conservative stability analysis criterion in terms of a negativity condition over a quadratic function within a finite interval. It is also shown how to attest such negativity condition via a finite-dimensional convex optimization problem that can be checked exactly in terms of LMIs. Some extensions to systems with multiple time-varying delays are also provided.

Finally, *Chapter 4* presents the conclusions and the future research directions of this study. The publications that resulted from this doctoral research are also presented.

Chapter 2

Delay-independent stability of linear systems with constant time-delay

This chapter investigates the delay-independent stability of linear systems with constant time-delay, i.e., we are interested in verifying if stability is maintained for all possible delay values. Based on the use of a frequency-dependent matrix polynomial constraint, new necessary and sufficient conditions are proposed to check the *strong* delay-independent stability. The resulting infinite-dimensional frequency-dependent conditions are formulated as finite-dimensional frequency-independent LMIs conditions by the use of the Kalman-Yakubovich-Popov lemma.

2.1 Problem statement and preliminaries concepts

Consider the linear state-delayed system of the form

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) \\ x(t) = \varphi(\vartheta), \quad \vartheta : [-\tau,\ 0] \to \mathbb{R}^n \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; A_0, $A_1 \in \mathbb{R}^{n\times n}$ are given system matrices; $\tau \geq 0$ is the constant delay parameter, and $\varphi(\vartheta)$ is an initial condition.

The aim of the delay-independent stability is to characterize if the origin of the system (2.1) is an asymptotically stable equilibrium point for all possible values of τ.

In the same spirit as for systems free of delays, the time-delay system (2.1) is asymptotically stable for a given τ if and only if all roots of the characteristic function

$$\Delta(s, e^{-s\tau}) := \det(sI - A_0 - A_1 e^{-s\tau}) \tag{2.2}$$

are located in the open left half of the complex plane.

One can note that (2.2) is a transcendental function and, therefore, has an infinite number of solutions, which apparently makes impossible the problem of determining the system stability. However, the solutions of the characteristic equation $\Delta(s, e^{-s\tau}) = 0$ have some interesting properties: the number of roots to the right of any vertical line of the complex plane is always finite, and the roots behave continuously with respect to variations of the delay (Gu et al., 2003, Theorem 1.5). This continuity property will be instrumental in deriving some results in this chapter.

An alternative approach to characterize the delay-independent stability of the system (2.1) is to consider the characteristic function (2.2) as a bivariate polynomial. For this purpose, let the complex variable $z = e^{-s\tau}$. Since we are interested in stability for all values of τ, the variables s and z can be interpreted as two independent variables, except at the origin of the complex plane where z is uniquely determined by s. In this context, one can often find two different notions of delay-independent stability in terms of the following polynomial in two variables (Niculescu, 2001, Remark 4.3):

$$\Delta(s, z) := \det(sI - A_0 - A_1 z).$$

The first one is delay-independent stability, as stated next:

Definition 2.1. *System (2.1) is delay-independent stable if* $\Delta(s, z) \neq 0$ *with* $z = e^{-s\tau}$, $\forall s \in \mathbb{C}_+$, *and* $\forall \tau \geq 0$.

The *strong* delay-independent stability is a slightly more restrictive condition, and can be defined as follows:

Definition 2.2. *System (2.1) is* strongly *delay-independent stable if* $\Delta(s, z) \neq 0$, $\forall s \in \mathbb{C}_+$ *and* $z \in \mathbb{D}$.

According to Definitions 2.1 and 2.2, we note that in the latter s and z are regarded as completely independent variables, whereas in the former z is uniquely determined by $s\tau$ when $s\tau = 0$. That is, the only difference between the two stability notions is at the origin of the complex plane, i.e., $s = 0$. Therefore the notion of *strong* delay-independent stability is slightly more restrictive than the former. Another important feature is that the *strong* delay-independent stability is robust against perturbations in the system matrices A_0 and A_1, while the delay-independent stability is not (Bliman, 2001, 2002). The following frequency-sweeping test, proposed in Chen and Latchman (1995), helps enlighten these issues:

Lemma 2.1 (Chen and Latchman 1995, Theorem 3.1)**.** *System (2.1) is delay-independent stable if and only if the following conditions hold*

$i)$ A_0 *is Hurwitz;*

$ii)$ $\rho((j\omega I - A_0)^{-1} A_1) < 1, \forall \omega > 0,$

$iii)$ *either*

- $a)$ $\rho(A_0^{-1}A_1) < 1$ *or*
- $b)$ $\rho(A_0^{-1}A_1) = 1$ *and* $\det(A_0 + A_1) \neq 0$.

Moreover, if condition in $ii)$ is also verified for $\omega = 0$, then system (2.1) is strongly delay-independent stable. ◇

The infinite-dimensional condition in the above lemma shows that the notion of *strong* delay-independent stability is stricter only because the spectral radius condition should also be verified for $\omega = 0$. Moreover, the conditions in Lemma 2.1 also have a robust control interpretation, which shows why the *strong* delay-independent stability is robust with respect to system matrices uncertainties. Recall that the following equivalence holds for any structured uncertainty $\Delta = \{\delta I_n : \delta \in \mathbb{C}\}$ (Zhou et al., 1996, page 269)

$$\rho((j\omega I - A_0)^{-1}A_1) = \mu_\Delta((j\omega I - A_0)^{-1}A_1), \forall\omega, \tag{2.3}$$

where μ_Δ denotes the structured singular value associated with the uncertainty Δ. The equivalence in (2.3) implies that solving the problem of *strong* delay-independent stability is equivalent to solving the robust stability problem. That is, the stability is ensured for all uncertain systems obtained when replacing the delay by any proper stable perturbation such that $||\Delta||_\infty < 1$.

In this chapter, we will focus on the *strong* delay-independent stability. The goal is to translate the infinite-dimensional condition in Definition 2.2 into a finite-dimensional convex optimization problem. Despite being a particular case of delay-independent stability, the *strong* delay-independent stability is a well-studied problem in the literature and it may be sufficiently general from a practical point of view, as will be shown in Example 2.3.

2.2 A Novel LMI condition for *strong* delay-independent stability

In order to construct an LMI based test for the *strong* delay-independent stability of system (2.1), firstly, it is presented a technical result that provides alternative frequency-domain conditions to check the condition in Definition 2.2.

Due to the continuity property of the roots of the characteristic equation $\Delta(s, e^{-s\tau}) = 0$, we can formulate the following result in Lemma 2.2. Basically, it states that solving the problem of *strong* delay-independent stability is equivalent to determine if a frequency-dependent matrix, defined as $F(e^{-j\theta}) = A_0 + A_1e^{-j\theta}$, is Hurwitz for all values of $\theta \in [0,\ 2\pi]$. Moreover, it also provides exact conditions to check if $F(e^{-j\theta})$ is a Hurwitz matrix for all admissible values of the frequency-dependent parameter θ.

Lemma 2.2. *Let $F(e^{-j\theta}) = A_0 + A_1 e^{-j\theta}$. The following statements are equivalent.*

i) Time-delay system (2.1) is strongly delay-independent stable;

ii) $F(e^{-j\theta})$ is Hurwitz $\forall\theta \in [0,\ 2\pi]$;

iii) $\exists$ a Hermitian matrix $P(e^{-j\theta}) \succ 0$ of appropriate dimension such that

$$F^*(e^{-j\theta})P(e^{-j\theta}) + P(e^{-j\theta})F(e^{-j\theta}) \prec 0 \tag{2.4}$$

holds for all $\theta \in [0,\ 2\pi]$;

iv) $P(e^{-j\theta}) \succ 0$ and (2.4) hold for all $\theta \in [0,\ 2\pi]$, where

$$P(e^{-j\theta}) = \sum_{\ell=-k}^{k} P_\ell e^{j\theta\ell} \ \ \textit{with} \ \ P_{-\ell} = P_\ell^T \in \mathbb{R}^{n\times n} \tag{2.5}$$

and for any integer k such that $0 \le k^\star \le k \le \bar{k} := n^2 - 1$, in which $k^\star$ is an appropriate integer;

v) $A_0 + A_1$ is Hurwitz and (2.4) with $P(e^{-j\theta})$ given in (2.5) holds for all $\theta \in [0,\ 2\pi]$ and for any integer k such that $0 \le k^\star \le k \le \bar{k} := n^2 - 1$.

◇

Proof. The equivalence between statements $i)$ and $ii)$ follows immediately from the relationship $\Delta(j\omega, e^{-j\theta}) = \det\big(j\omega - F(e^{-j\theta})\big)$ (Kamen, 1982; Li et al., 2016).

The statements $ii)$ and $iii)$ are equivalents due to the Lyapunov stability criterion for matrices over the field of complex numbers (Hogben, 2014, page 26-4).

To show the equivalence between items $iii)$ and $iv)$, i.e., $P(e^{-j\theta})$ in (2.4) admits the polynomial expansion (2.5) without conservatism, recall that the solvability of (2.4) is equivalent to the solvability of

$$F^*(e^{-j\theta})P(e^{-j\theta}) + P(e^{-j\theta})F(e^{-j\theta}) = -I_n, \ \ \forall\theta \in [0,\ 2\pi]. \tag{2.6}$$

The solution $P(e^{-j\theta})$ to this equation, when $ii)$ holds, is given explicitly as

$$P(e^{-j\theta}) = \int_0^{+\infty} e^{F^*(e^{-j\theta})t} e^{F(e^{-j\theta})t} dt, \tag{2.7}$$

which is analytic w.r.t. the parameter $e^{-j\theta}$ and its conjugate $e^{j\theta}$. Thus, $P(e^{-j\theta})$ can be expressed in terms of an infinite sum of powers of $e^{-j\theta}$ and $e^{j\theta}$. As shown in Bliman (2004), this latter expansion can be truncated due to uniform convergence of the integral at $t = +\infty$ with respect to $e^{-j\theta}$. Therefore, (2.4) admits a solution of polynomial type

$$P(e^{-j\theta}) = \sum_{\ell=-k}^{k} P_\ell e^{j\theta\ell} \text{ with } P_{-\ell} = P_\ell^T \in \mathbb{R}^{n\times n} \tag{2.8}$$

for a sufficiently large k.

Although proving that $P(e^{-j\theta})$ admits a polynomial expansion without conservatism, in Bliman (2004) it is not provided a priori bound on the degree of the truncated polynomial. In the sequel, it is determined an upper bound value on the degree for such expansion, i.e., $\bar{k} = n^2 - 1$, in item $iv)$ following similar steps as in Zhang et al. (2003). Suppose that item $ii)$ holds. Therefore, there exists a unique $P(e^{-j\theta}) \succ 0$ which satisfies the Lyapunov equation

$$F^*(e^{-j\theta})P(e^{-j\theta}) + P(e^{-j\theta})F(e^{-j\theta}) + Q(e^{-j\theta}) = 0,$$

for any $Q(e^{-j\theta}) \succ 0$. Solving this equation for $P(e^{-j\theta})$ (Proposition 1.1, Appendix A) one has

$$[F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})]\text{vec}(P(e^{-j\theta})) = -\text{vec}(Q(e^{-j\theta})). \tag{2.9}$$

Let us consider

$$Q(e^{-j\theta}) = \left|\det\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right)\right| I_n, \tag{2.10}$$

which constitutes a suitable choice, used for the sake of demonstration simplicity. Since $ii)$ holds, i.e., $F(e^{-j\theta})$ is Hurwitz, then $F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})$ is also Hurwitz. Thus,

$$\det\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right) \neq 0.$$

Consequently, $Q(e^{-j\theta})$ in (2.10) is positive definite for all $\theta \in [0, 2\pi]$. Using (2.10) in (2.9) yields

$$[F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})]\text{vec}(P(e^{-j\theta})) = -\left|\det\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right)\right|\text{vec}(I_n).$$

Therefore,

$$\begin{aligned}\text{vec}(P(e^{-j\theta})) &= -\left|\det\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right)\right| \frac{\text{adj}(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta}))}{\det\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right)}\text{vec}(I_n)\\ &= \sigma(e^{-j\theta})\,\text{adj}\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right)\text{vec}(I_n),\end{aligned}$$

where $\sigma(e^{-j\theta}) = -\text{sign}\Big(\det\left(F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta})\right)\Big)$. The last equality holds by Lemma 1.12

(Appendix A). Moreover, note that

$$\begin{aligned} F^T(e^{-j\theta}) \oplus F^*(e^{-j\theta}) &= (A_0^T + e^{-j\theta}A_1^T) \oplus (A_0 + e^{-j\theta}A_1)^* \\ &= \mathcal{A}_0 + e^{-j\theta}\mathcal{A}_{-1} + e^{j\theta}\mathcal{A}_1, \end{aligned}$$

with $\mathcal{A}_0 = A_0^T \oplus A_0^T \in \mathbb{R}^{n^2 \times n^2}$, $\mathcal{A}_{-1} = A_1^T \otimes I_n \in \mathbb{R}^{n^2 \times n^2}$, and $\mathcal{A}_1 = I_n \otimes A_1^T \in \mathbb{R}^{n^2 \times n^2}$. Thus, according to Lemma 1.11, there exists constant matrices N_ℓ such that

$$\text{adj}(\mathcal{A}_0 + e^{-j\theta}\mathcal{A}_{-1} + e^{j\theta}\mathcal{A}_1) = \sum_{\ell=-k}^{k} e^{-j\theta\ell}N_\ell$$

where $k^\star \leq k \leq \bar{k} = n^2 - 1$, and N_ℓ can be calculated from $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_{-1}$. Since the inverse of $\text{vec}(.)$ exists, $P(e^{-j\theta})$ admits the solution

$$P(e^{-j\theta}) = \sum_{\ell=-k}^{k} e^{-j\theta\ell}P_\ell.$$

with $P_\ell = \sigma(e^{-j\theta})\text{vec}^{-1}(N_\ell \text{vec}(I_n)) \in \mathbb{R}^{n \times n}$. In addition, due to the specific structure of matrices $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_{-1}$, we have that $P_{-\ell} = P_\ell^T$.

Finally, it is shown that the items $iv)$ and $v)$ are equivalent, which means that we can simply check if $A_0 + A_1$ is Hurwitz instead of checking if the constraint $P(e^{-j\theta}) \succ 0$ on the Lyapunov function is satisfied for all $\theta \in [0,\ 2\pi]$. The necessity part of condition $v)$ is evident by item $iv)$ of Lemma 2.2. Let us prove the sufficiency. If $F(e^{-j\theta})$ is Hurwitz for $\theta = 0$, that is $A_0 + A_1$ is Hurwitz, then (2.4) ensures that $P(e^{-j\theta}) \succ 0$ for $\theta = 0$. Moreover, because of the continuity of the eigenvalues of $P(e^{-j\theta})$ with respect to θ and the condition (2.4), the matrix $P(e^{-j\theta})$ remains positive defined for all $\theta \in (0,\ 2\pi]$. This is true since for the real part of an eigenvalue of $P(e^{-j\theta})$ to become negative it has first to become zero, which implies that the condition (2.4) does not hold. Therefore, the items $iv)$ and $v)$ are equivalent, which concludes the proof. □

In the above lemma, $k^\star$ represents the optimum value of k, i.e., the smaller degree for the expansion of the matrix $P(e^{-j\theta})$ that ensures necessity in items $iv)$ and $v)$. Therefore, for any $k < k^\star$, the statements $vi)$ and $v)$ are only sufficient. Moreover, although the value of $k^\star$ is unknown a priori, one can simply set $k = \bar{k}$ for guaranteeing exactness.

Notwithstanding the previous lemma presents alternative conditions to check the *strong* delay-independent stability, it is still dependent on θ. Therefore, in the following, it is shown how to replace the dependence on θ by a matrix-valued decision variable. The main technical result used then is the well-known Kalman-Yakubovich-Popov (KYP) Lemma, which converts an infinite-dimensional frequency domain inequality into a finite-dimensional LMI that can be efficiently solved numerically using polynomial-time algorithms (Gahinet et al., 1995).

Lemma 2.3 (KYP lemma (Rantzer, 1996)). *Given matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ of compatible dimensions, the infinite dimensional Frequency Domain Inequality*

$$\begin{bmatrix} (e^{j\theta}I - \mathcal{A})^{-1}\mathcal{B} \\ I \end{bmatrix}^* \mathcal{M} \begin{bmatrix} (e^{j\theta}I - \mathcal{A})^{-1}\mathcal{B} \\ I \end{bmatrix} \prec 0$$

holds for all $\theta \in \mathbb{R}$, if and only if the following inequality holds for some symmetric matrix $\mathcal{Q}$

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & -\mathcal{Q} \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ I & 0 \end{bmatrix} + \mathcal{M} \prec 0.$$

◇

In light of Lemma 2.2 and Lemma 2.3, we can obtain a new LMI condition for testing the *strong* delay-independent stability. In order to obtain such a result, we first note that the Lyapunov matrix $P(e^{j\theta})$ given in (2.5) can be rewritten as

$$P(e^{-j\theta}) = (\Theta[k] \otimes I_n) P_\#[k] (\Theta[k] \otimes I_n)^* \tag{2.11}$$

where $\Theta[k] = \begin{bmatrix} e^{j\theta k} & e^{j\theta(k-1)} & \dots & e^{j\theta} & 1 \end{bmatrix}$ and

$$P_\#[k] = \begin{bmatrix} P_0 & P_1 & \dots & P_k \\ P_1^T & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_k^T & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(k+1)n \times (k+1)n}$$

and, similarly, the condition in (2.4) with $P(e^{-j\theta})$ given in (2.5) can be rewritten as

$$(\Theta[k+1] \otimes I_n) \big(G_\#[k] + G_\#^T[k] \big) (\Theta[k+1] \otimes I_n)^* \prec 0 \tag{2.12}$$

where $G_\# \in \mathbb{R}^{(k+2)n \times (k+2)n}$ is given by

$$G_\#[k] = \left(\begin{bmatrix} I_{(k+1)n} \\ 0_{n \times (k+1)n} \end{bmatrix} P_\#[k] \begin{bmatrix} I_{(k+1)n} \\ 0_{n \times (k+1)n} \end{bmatrix}^T \right) \big(I_{k+2} \otimes A_0 \big)$$
$$+ \begin{bmatrix} P_1 & P_2 & \dots & P_k & 0 & 0 \\ P_0 & 0 & \dots & 0 & 0 & 0 \\ P_1^T & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ P_k^T & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \big(I_{k+2} \otimes A_1 \big). \tag{2.13}$$

Moreover, to find a nonconservative way of checking the frequency-dependent inequalities in (2.11) and (2.12), the following lemma is presented, which is a direct consequence of Lemma 2.3.

Lemma 2.4. *: Let* $\Xi \in \mathbb{R}^{(k+1)n \times (k+1)n}$*. The frequency matrix inequality*

$$(\Theta[k] \otimes I_n)\Xi(\Theta[k] \otimes I_n)^* \prec 0$$

where $\Theta[k] = \begin{bmatrix} e^{j\theta k} & e^{j\theta(k-1)} & \dots & e^{j\theta} & 1 \end{bmatrix}$ *holds for all* $\theta \in \mathbb{R}$ *if and only if*

$$\begin{bmatrix} A_\#[k] & B_\#[k] \\ I & 0 \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \begin{bmatrix} A_\#[k] & B_\#[k] \\ I & 0 \end{bmatrix} + \Xi \prec 0$$

where

$$\begin{aligned} A_\#[k] &= \begin{bmatrix} 0 & I_{k-1} \\ 0 & 0 \end{bmatrix} \otimes I_n \; \in \mathbb{R}^{kn \times kn} \text{ and} \\ B_\#[k] &= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^T \otimes I_n \; \in \mathbb{R}^{kn \times n}, \end{aligned}$$

holds for some symmetric matrix $Q \in \mathbb{R}^{kn \times kn}$*.* ◇

Proof. The equivalence holds because

$$\begin{bmatrix} (e^{j\theta} I_{kn} - A_\#[k])^{-1} B_\#[k] \\ I_n \end{bmatrix} = (\Theta[k] \otimes I_n)^*$$

with $A_\#[k]$, $B_\#[k]$, and $\Theta[k]$ given in the lemma. Then an application of Lemma 2.3 (KYP lemma) concludes the proof. □

Thus, we are now in a position to present the main contribution of this chapter.

Theorem 2.1. *Let* $A_\#[k]$ *and* $B_\#[k]$ *be given in Lemma 2.4 and* $G_\#[k]$ *in (2.13). The time-delay system in (2.1) is strongly delay-independent stable if and only if the following two conditions hold*

i) $A_0 + A_1$ *is Hurwitz;*

ii) $\exists\, P_\ell \in \mathbb{R}^{n \times n}$*, for* $\ell = 1, \dots, k$*,* $P_0 \in \mathbb{S}^n$ *and* $Q \in \mathbb{S}^{(k+1)n}$*, such that*

$$\begin{bmatrix} A_\#^T[k+1] & I \\ B_\#^T[k+1] & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} \begin{bmatrix} A_\#^T[k+1] & I \\ B_\#^T[k+1] & 0 \end{bmatrix}^T + G_\#[k] + G_\#^T[k] \prec 0, \qquad (2.14)$$

for $k = \bar{k} := n^2 - 1$, *or even for any integer* k *such that* $0 \leq k^\star \leq k \leq \bar{k}$ *in which* $k^\star$ *is an (a priori unknown) appropriate integer.*

Proof. The proof follows from the combination of item $v)$ in Lemma 2.2, the frequency-dependent inequality in (2.12), and Lemma 2.4. □

The previous theorem establishes that for an appropriate k, the LMI condition (2.14) is necessary and sufficient to attest the *strong* delay-independent stability of system (2.1). Theorem 2.1 does not provide the value for $k^\star$, as a result inherited from Lemma 2.2. However, it does provide an upper bound for $k^\star$, namely $\bar{k}$, which guarantees the exactness of the proposed result. The downside of setting $k = \bar{k}$ is the resulting dimension of the LMI condition. On the other hand, for any value of k, the proposed LMI condition solvability ensures *strong* delay-independent stability. Thus for high-dimensional systems, k can be set as a small value. As illustrated in the next section, usually $k = 1$ provides exact results, which shows that the upper bound provided for $k^\star$, i.e., $\bar{k}$, is usually conservative.

Besides, it is noteworthy that the existence of $\bar{k}$ allows us to obtain decidability on the instability case. It is the main advantage of the proposed method compared to the previous results in Bliman (2002) and Li et al. (2016), see Example 2.3. Another advantage of the proposed LMI condition in Theorem 2.1 is that the system matrices appear linearly. For this reason, apparently, the proposed method is more suitable to be extended to the synthesis of filters and controllers.

Remark 1. *For unidimensional systems we have* $\bar{k} = k^\star = 0$, *which means that* $P(e^{-j\theta})$ *in* (2.4) *can be chosen as a constant matrix without conservatism. In this case, LMI* (2.14) *in Theorem 2.1 reduces to*

$$\begin{bmatrix} A_0^T P_0 + P_0 A_0 - Q & A_1^T P_0 \\ * & Q \end{bmatrix} \prec 0,$$

which is the same LMI obtained under the Lyapunov-Krasovskii framework, when the Lyapunov-Krasovskii functional candidate is chosen as

$$V(x_t) = x^T(t)\mathcal{P}x(t) + \int_{t-\tau}^{t} x^T(s)\mathcal{Q}x(s)ds,$$

with $\mathcal{P} = P_0 \succ 0$ *and* $\mathcal{Q} = -Q$, $Q \prec 0$.

Indeed, in Bliman (2000) a more general result is proved: the Lyapunov-Krasovskii approach is nonconservative for strong delay-independent stability analysis when matrices A_0 *and* A_1 *commute.*

2.3 Numerical Examples

This section presents two numerical examples to illustrate the application and effectiveness of Theorem 2.1. Example 2.3 illustrates that Theorem 2.1 can eliminate the undecidability of the instability and shows that the *strong* delay-independent stability property may be sufficiently general from a practical point of view. In Example 2.4, it is analyzed the computational complexity of the proposed condition and the related methods in the literature.

The LMI conditions were solved by using the LMI parser YALMIP and the SDP solver MOSEK. The results are compared to those from Souza et al. (2018), Li et al. (2016) and Bliman (2002).

Example 2.3. *Consider the system* (2.1) *with*

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \quad \textit{and} \quad A_1 = \beta \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \tag{2.15}$$

where $\beta \in \mathbb{R}$.

Firstly, assume that $\beta = 1$*. Applying the proposed condition in Theorem 2.1, with* $\bar{k} = 3$*, one can find that the LMI in* (2.14) *is infeasible. Therefore, the system is not strongly delay-independent stable, yielding the same conclusion in Souza et al. (2018, Example 2). On the other hand, the method in Bliman (2002) does not provide an LMI dimension, and the method in Li et al. (2016), a number of frequency subintervals, that guarantees the exact solution. Therefore, for this example, the methods in Bliman (2002); Li et al. (2016) (by themselves) do not provide a conclusion, because one can always believe that the LMI dimension considered in Bliman (2002) or the number of frequency subintervals tested in Li et al. (2016) was not high enough to verify the strong delay-independent stability.*

In the following, it is analyzed how restrictive is the notion of strong delay-independent stability. An application of Theorem 2.1 yields that the maximum value of β *for which the system is strongly delay-independent stable is* $\beta = 0.89999$*. This result can be checked by Lemma 2.1 using a fine frequency grid. Moreover, by Lemma 2.1, the system is delay-independent stable for* $\beta = 0.9$*, but not strongly delay-independent stable, because* $\rho\big((j\omega I - A_0)^{-1}A_1\big) \nless 1$ *only at* $\omega = 0$*,* $\rho(A_0^{-1}A_1) = 1$*, and* $\det(A_0 + A_1) \neq 0$*. Figure 2.1 illustrates the spectral radius* $\rho\big((j\omega I - A_0)^{-1}A_1\big)$ *with* $\beta = 0.9$ *graphically by plotting its values on a fine frequency grid. For this example, it was used steps of* $\omega = 0.001$*. Also, one can easily check that for* $\beta > 0.9$*, we have that* $\rho\big((j\omega I - A_0)^{-1}A_1\big) \nless 1$ *for some* $\omega > 0$*. Therefore, the system is not delay-independent stable. Thus, as the previous analysis illustrates, the strong delay-independent stability may be sufficiently general from a practical point of view.*

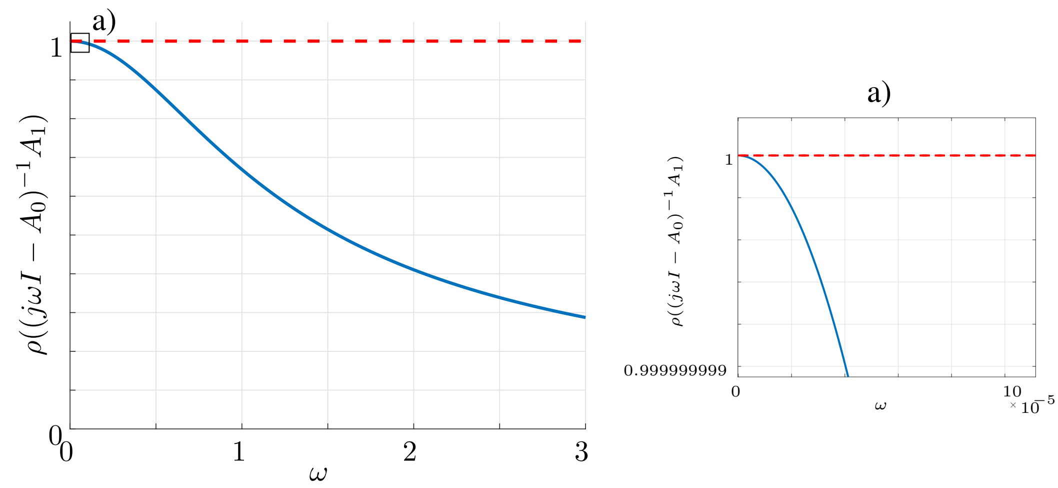


Figure 2.1: Spectral radius $\rho((j\omega I - A_0)^{-1}A_1)$, with A_0 and A_1 given in (2.15) and $\beta = 0.9$, evaluated on a fine grid. Example 2.3.

Example 2.4. *Consider the time-delay system* (2.1) *with* A_0 *and* A_1 *given by*

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix}, \quad A_1 = \beta \begin{bmatrix} -0.05 & 0.005 & 0.25 & 0 \\ 0.005 & 0.005 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -0.5 & 0 \end{bmatrix}, \text{ and } \beta \in \mathbb{R}. \tag{2.16}$$

It is known that the maximum value of β *to maintain delay-independent stability is* $\bar{\beta}^\star = 1.21955$. *See, for instance, Gu et al. (2003, Example 3.3) and Li et al. (2016, Example 1).*

The results in Table 2.1 are obtained by applying the proposed Theorem 2.1 and some similar ones to test the strong delay-independent stability of this system. The table summarizes the maximum value of β *achieved (denoted by* $\bar{\beta}$*), and the number of real scalar variables (NoV) needed per each method applied. One can see that all methods applied can verify the exact delay-independent stability margin of the system, i.e.,* $\bar{\beta} = \bar{\beta}^\star$*. However, it does not come at the same cost of computational complexity, as indicated by the value of NoV.*

Table 2.1: Maximum value of β ($\bar{\beta}$) and number of decision variables (NoV) needed for different methods, with A_0 and A_1 given in (2.16). Example 2.4.

Method	$\bar{\beta}$	NoV
Souza et al. (2018, Theorem 1)	1.21955	528
Li et al. (2016, Theorem 4 ($\eta = 10$))	1.21955	480
Bliman (2002, Theorem 1 ($k = 2$))	1.21955	72
Li et al. (2016, Theorem 2 ($\kappa = 2$))	1.21955	58
Theorem 2.1 ($k = 1$)	1.21955	62

In Table 2.1, we see that the proposed Theorem 2.1 for $k = 1$ *provides the exact an-*

swer. It, in turn, means that the Lyapunov matrix $P(e^{-j\theta})$ given in (2.5) with $k = 1$, i.e. $P(e^{-j\theta}) = e^{-j\theta}P_1^T + P_0 + e^{j\theta}P_1$, satisfies the conditions in item $ii)$ of Lemma 2.2. For an illustration of the solution found, Figure 2.2 shows the minimum real part of the eigenvalues of the matrix $P(e^{-j\theta})$ and the maximum real part of the eigenvalues of $Z(e^{-j\theta}) := F^*(e^{-j\theta})P(e^{-j\theta}) + P(e^{-j\theta})F^*(e^{-j\theta})$ evaluated on a fine grid in θ. On the other hand, following the formula in Theorem 2.1, we compute $\bar{k} = 15$, which illustrates how that the upper bound of k provided by Theorem 2.1 may be conservative.

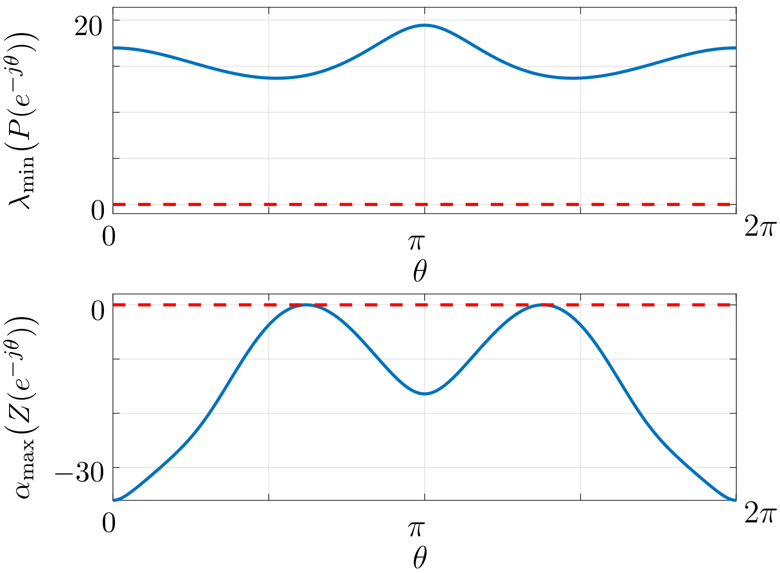


Figure 2.2: Minimum real part of the eigenvalues of $P(e^{-j\theta})$ and spectral abscissa of $Z(e^{-j\theta}) := F^*(e^{-j\theta})P(e^{-j\theta}) + P(e^{-j\theta})F^*(e^{-j\theta})$ evaluated on a fine grid in θ with $k = 1$, A_0 and A_1 in (2.16), $\beta = 1.21955$, and P_0 and P_1 are the solution to the LMI feasibility test in Theorem 2.1. Example 2.4.

The results in Table 2.1 show that the method in Li et al. (2016, Theorem 2 ($\kappa = 2$)) is computationally more efficient than the others listed, but this depends on the specific system data considered. For instance, as in Li et al. (2016, Remark 7), consider now that the third row of the matrix A_1 is replaced by $\beta[0.7\ \ -0.2\ \ 0.5\ \ -0.18]$. Then, Table 2.2 is assembled, which shows that all the methods reveal the same strong delay-independent stability margin. However, the proposed one needs less NoV. Figure 2.3 illustrates the accuracy of the results from the methods in Table 2.2 verifying the condition $i)$ of Lemma 2.2 using a fine grid in θ. Moreover, for illustration of the solution found, the conditions in item $ii)$ of Lemma 2.2 are illustrated in Figure 2.4.

Table 2.2: Maximum value of β ($\bar{\beta}$) and the number of decision variables (NoV) for different methods, with A_0 and A_1 given in (2.16) and replacing the third row of matrix A_1 in (2.16) by $\beta[0.7\ \ -0.2\ \ 0.5\ \ -0.18]$. Example 2.4.

Method	$\bar{\beta}$	NoV
Li et al. (2016, Theorem 4 ($\eta = 301$))	1.45307	14448
Souza et al. (2018, Theorem 1)	1.45307	528
Li et al. (2016, Theorem 2 ($\kappa = 5$))	1.45307	240
Bliman (2002, Theorem 1 ($k = 2$))	1.45307	72
Theorem 2.1 ($k = 1$)	1.45307	62

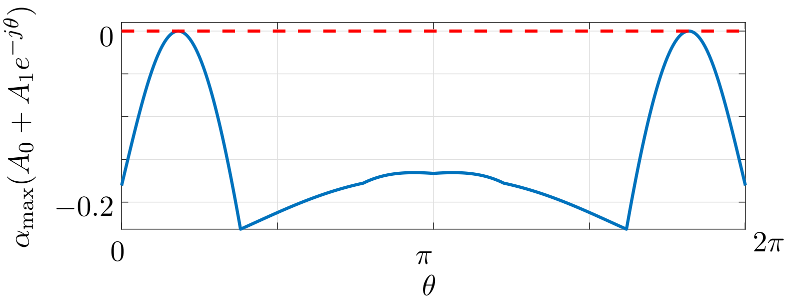


Figure 2.3: Spectral abscissa of $A_0 + A_1 e^{-j\theta}$ for Example 2.4, with the third row of the matrix A_1 in (2.16) replaced by $\beta[0.7\ \ -0.2\ \ 0.5\ \ -0.18]$ and $\beta = 1.45307$.

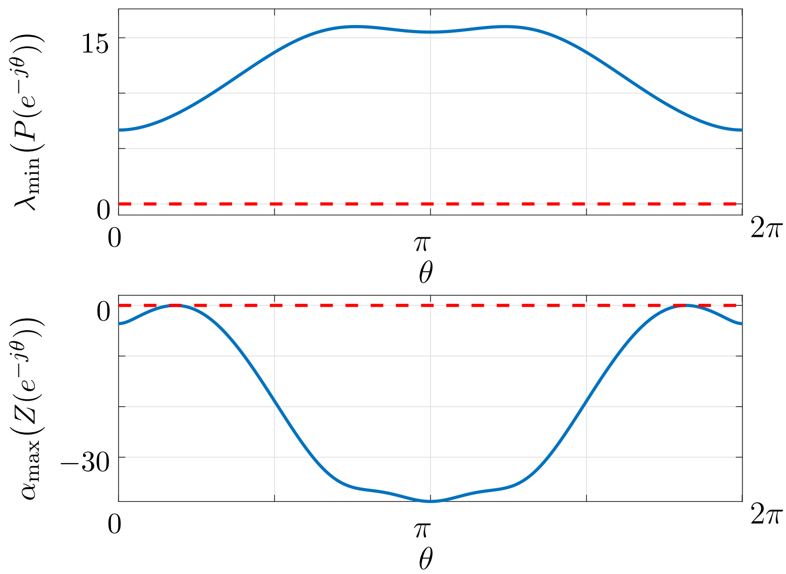


Figure 2.4: Minimum real part of the eigenvalues of $P(e^{-j\theta})$ and spectral abscissa of $Z(e^{-j\theta}) := F^*(e^{-j\theta})P(e^{-j\theta}) + P(e^{-j\theta})F^*(e^{-j\theta})$ evaluated on a fine grid in θ with $k = 1$, A_0 and A_1 in (2.16), where the third row of A_1 is replaced by $\beta[0.7\ \ -0.2\ \ 0.5\ \ -0.18]$, and P_0 and P_1 are the solution to the LMI feasibility test in Theorem 2.1. Example 2.4.

Finally, Table 2.3 summarizes the number of decision variables, as a function of system dimension, required by each method analyzed. Notice that, for $k = 1$, the proposed theorem needs fewer decision variables than Theorems 2 and 4 in Li et al. (2016), and Theorem 1 in Bliman (2002) when the a priori unknown parameters of those methods (κ, η, and k, respectively) are greater than 1. Similarly, for $k < 3$, Theorem 2.1 is computationally more efficient than Theorem 1 in Souza et al. (2018). It is noteworthy that for high-dimensional systems, Theorem 2.1 requires fewer decision variables than the method in Souza et al. (2018) even for higher values of k.

Table 2.3: Number of decision variables as a function of the system dimension.

Method	NoV
Bliman (2002, Theorem 1 for a given k)	$k^2n^2 + kn$
Li et al. (2016, Theorem 2 for a given κ)	$(3\kappa + 0.5)n^2 + 0.5n$
Li et al. (2016, Theorem 4 for a given η)	$3\eta n^2$
Souza et al. (2018, Theorem 1)	$2n^4 + n^2$
Theorem 2.1 for a given k	$(0.5k^2 + 2k + 1)n^2 + (0.5k + 1)n$

2.4 Publication

The results presented in this chapter have been published in:

- de Oliveira, F. S. S. and Souza, F. O. (2019). Strong delay-independent stability of linear delay systems. *Journal of the Franklin Institute*, 356(10):5421–5433.

Chapter 3

Delay-dependent stability of linear systems with time-varying delay

In practical applications, the assumptions of a system with a constant time-delay and its delay-independent stability property, made in the previous chapter, might be quite restrictive. In this chapter, it is investigated the delay-dependent stability of a broader class of time-delay systems, in which the delay can be represented by a time-varying function. More specifically, the main goal of this chapter is to provide less conservative conditions to characterize whether the origin is an asymptotically stable equilibrium point of the linear state-delayed system given by

$$\begin{cases} \dot{x}(t) = & A_0 x(t) + A_1 x(t - \tau(t)) \\ x(\vartheta) = & \varphi(\vartheta), \quad \vartheta : [-\tau_M,\, 0] \to \mathbb{R}^n \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_0,\ A_1 \in \mathbb{R}^{n\times n}$ are constant system matrices, $\varphi(\vartheta)$ is an initial condition, and $\tau(t)$ is a continuous and differentiable function used to describe the time-varying delay. It is also assumed that $\tau(t)$ satisfies the conditions

$$0 \le \tau(t) \le \tau_M < \infty, \quad \mu_1 \le \dot{\tau}(t) \le \mu_2 < 1, \tag{3.2}$$

with τ_M, μ_1, and μ_2 being known constant scalars.

The constraint $\dot{\tau}(t) < 1$ on delay derivative is more than a technical condition; it guarantees causality and regularity of the solutions. Notice that the condition (3.2) imposes that the function $t - \tau(t)$, which represents the evolution of the delayed information with respect to time, is strictly increasing. From a practical point of view, it means that the delayed information affects the system following a chronological order (Seuret, 2017). This type of constraint naturally appears, for example, in the variable transport delay of fluid flow control systems. On the other hand, there are some practical applications, as networked control systems, in which such a constraint may be violated. For a detailed discussion on the constraints on the delay derivative, the readers are referred to Verriest (2010); Michiels and Verriest (2011); Verriest (2011).

To derive less conservative conditions to assess the delay-dependent stability of system (3.1) under constraints (3.2), the Lyapunov-Krasovskii method is employed and the resulting stability conditions are expressed as linear matrix inequalities.

3.1 Preliminaries

3.1.1 Overview of Lyapunov-Krasovskii methods

The Lyapunov-Krasovskii method is a generalization of the second Lyapunov method to time-delay systems. Unlike for a delay-free system, a function $x_t = x(t+\vartheta), \forall\vartheta \in [-\tau_M,\ 0]$ is required to completely specify the evolution of a system with delay. Therefore, the LK method requires the construction of a proper functional $V(t, x_t)$, instead of Lyapunov function that depend on the instant state $x(t)$, which is positive definite and strictly decreasing along the system trajectories.

Initially, the Lyapunov-Krasovskii method was formulated considering functionals that depend only on x_t, but in some cases, functionals that depend on the state derivatives are useful, and the method has been extended to this case (Fridman, 2014b; Zhang and Han, 2012). When restricting the Lyapunov functional to be a quadratic one and depending on state derivatives, the Lyapunov-Krasovskii theorem for linear TDS can be restricted to the following formulation[1].

Theorem 3.1. *The time-delay system* (3.1) *is asymptotically stable if there exist a quadratic Lyapunov-Krasovskii functional (LKF)* $V(t, x_t, \dot{x}_t)$ *such that for some* ϵ_i $(i = 1, 2, 3)$

$$\epsilon_1 \parallel x(t) \parallel^2 \leq V(t, x_t, \dot{x}_t) \leq \epsilon_2 \parallel x_t \parallel_W^2, \tag{3.3}$$

and its derivative along the solution of (3.1) *satisfies*

$$\dot{V}(t, x_t, \dot{x}_t) \leq -\epsilon_3 \parallel x(t) \parallel^2, \tag{3.4}$$

where $\parallel x_t \parallel_W^2 = \sup_{\vartheta\in[-\tau_M,0]} \parallel x_t(\vartheta) \parallel^2 + \int_{-\tau_M}^{0} \parallel \dot{x}_t(\vartheta) \parallel^2 d\vartheta.$ ◇

The main issue within the application of the above theorem is the choice of an appropriate LKF. For linear systems with a constant delay, it has been proved that the existence of a complete quadratic Lyapunov-Krasovskii functional of the type

$$\begin{aligned} V(x_t) = \; & x^T(t)Px(t) + 2x^T(t)\int_{-\tau_M}^{0} Q(\xi)x(\xi)d\xi \\ & + \int_{-\tau_M}^{0}\int_{-\tau_M}^{0} x^T(\xi)R(\xi,\eta)x(\eta)d\eta d\xi + \int_{-\tau_M}^{0} x^T(\xi)S(\xi)x(\xi)d\xi, \end{aligned} \tag{3.5}$$

[1]The complete version of the Lyapunov-Krasovskii theorem can be found, for instance, in Gu et al. (2003).

is indeed a necessary and sufficient condition for stability (Kharitonov and Zhabko, 2003; Gu, 2013). The complete quadratic LKF is defined in terms of matrix functions, which highly complicates the formulation of stability conditions as LMIs conditions. The first attempt to provide tractable LMI conditions based on functional (3.5) is due to Gu (1997). The method is based on the piecewise linear discretization of the matrix functions and, as the number of discretization increases, the test becomes nonconservative. An improvement of this method was proposed in Gu (2001, 2003), and the convergence was later proven in Gu (2013). An alternative approach that also provides tractable stability conditions was presented in Peet et al. (2006, 2009). It consists in restricting the matrix functions in (3.5) to be a class of polynomial matrices, and then, the system stability is addressed via the Sum of Squares (SoS) framework. In Peet and Bliman (2011), it is proved that this method is nonconservative. The main drawback of the approaches mentioned above is their complexity in terms of implementation. Moreover, for the case of systems with time-varying delay, it is still challenging to construct an exact Lyapunov-Krasovskii functional. That is the reason why alternative LKF has been considered in the literature.

To reduce the inherent conservatism of not considering a complete-type LKF, several strategies have been proposed. The idea behind them is to enrich the functional to exploit more information on the delayed states. The most common approaches are summarized next:

- *Augmented functionals*: in this approach, the LKF is constructed by augmenting the non-integral and/or integral terms using more state information, such as the derivative of the state vector, delayed state and its derivative, and (multiple) integral of the state vector. By doing that, some new matrix variables are introduced in the problem, which possibly provides more freedom for checking the feasibility of the LMI conditions in the stability criteria. For some noteworthy results, see He et al. (2005); Seuret and Gouaisbaut (2013); Park et al. (2015); Zhang et al. (2017b); Seuret and Gouaisbaut (2018); Park et al. (2018); Zhang et al. (2019).

- *Multiple integrals terms*: an important improvement on Lyapunov-Krasovskii functionals was proposed in Sun et al. (2009), by introducing a triple integral term of the form

$$\int_{-\tau_M}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\lambda d\theta,$$

  which inserts new integral terms to the functional derivative. Since then, many extensions to more general multiple integral functional have been proposed in the literature (Kim, 2011; Fang and Park, 2013; Gyurkovics and Takács, 2016).

- *Delay-partitioning method*: this method consists of the discretization of the delay in subintervals, of the same or different lengths, and augmenting the state variables introducing intermediate values of the delayed state. Numerical experiments have shown that the method reduces conservatism when taking thinner discretizations. See, for instance, Gouaisbaut and Peaucelle (2006, 2007); Fridman et al. (2009); Han (2009); Feng et al. (2013) for some results in this direction.

- *Delay-dependent matrices*: basically, this approach contributes to reducing the conservatism of the criteria by exploiting the information of the bounds of the delay and its derivative. Examples of usage of this approach can be found in Fridman et al. (2009); Xu et al. (2017); Lee and Park (2017); Kwon et al. (2018); Zhang et al. (2020c) and references therein.

3.1.2 Integral inequalities

Another crucial issue in the Lyapunov-Krasovskii method allied with LMI framework is the treatment with quadratic integral terms, usually of the form

$$\int_{t-\tau}^{t} \dot{x}^T(t) R \dot{x}(s) ds, \tag{3.6}$$

appearing in the derivative of the chosen LKF.

The first delay-dependent conditions were derived using some model transformations. See, for example, Kolmanovskii and Richard (1999). The idea is to transform the system with discrete delays into a distributed delay system through the Leibniz-Newton formula. This transformation introduces an integral cross term in the LKF derivative that can be bounded by well-known inequalities, as the Young's inequality, for example. Then, the bounding of the cross term can be used to compensate (3.6). The main drawback of this approach is that the models are not equivalent. The stability of the transformed system implies the stability of the original one, but the reverse is not always true (Gu and Niculescu, 2001). An equivalent model transformation, known as the descriptor method, was proposed in Fridman (2001); Fridman and Shaked (2003). Since an equivalent model transformation is used, this latter approach is less conservative than the former.

Another useful approach to deal with the integral terms is the Free-Weighting Matrix (FWM) approach (Wu et al., 2004). It consists of using free weighting matrices to indicate the relationship between the terms in the Leibniz-Newton formula, without any system transformation nor bounding techniques for cross terms. Numerical examples through the literature have shown that the FWM method can leads to less conservative results than the ones based on model transformations (He et al., 2007).

Despite the improvements of the FWM method, most of the recent LMI-based results are obtained using another approach, the application of integral inequalities. This approach is regarded as more convenient because it directly provides a bound for the quadratic integral term and usually requires fewer variables. The first integral inequality, known as Jensen inequality (Gu, 2000), has been widely used in the literature. The Jensen inequality was then improved by the Wirtinger (Seuret and Gouaisbaut, 2013) and the Auxiliary Function-based (Park et al., 2015) integral inequalities. Another remarkable integral inequality, called Bessel-Legendre integral inequality, was proposed by Seuret and Gouaisbaut (2015). It generalizes all the previous ones, providing a more accurate bound for the integral term (3.6). The Bessel-Legendre inequality is given in the next lemma.

Lemma 3.5 (Bessel-Legendre integral inequality (Seuret and Gouaisbaut, 2015; Zhang et al., 2019))**.** *For an integer $d \geq 0$, any $\mathcal{R} \in \mathbb{S}^n_+$, any differentiable function x in $[a,b] \to \mathbb{R}^n$, the following inequality holds*

$$\int_a^b \dot{x}^T(u)\mathcal{R}\dot{x}(u)du \geq \frac{1}{b-a}\Omega_d^T \tilde{\Theta}_d^T \tilde{\mathcal{R}}_d \tilde{\Theta}_d \Omega_d,$$

where

$$\Omega_d = \mathrm{col}\{x(b), x(a), \lambda_1, ..., \lambda_d\}, \textit{ with } \lambda_i := \int_a^b \frac{(b-s)^{i-1}}{(b-a)^i} x(s)ds, \quad i = 1, 2, ..., d.$$

$$\tilde{\Theta}_d = \Theta_d \Lambda_d, \textit{ with}$$

$$\Lambda_d = \begin{bmatrix} I & -I & 0 & 0 & \dots & 0 \\ 0 & -I & I & 0 & \dots & 0 \\ I & -I & 0 & 2I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -I & 0 & 0 & \dots & dI \end{bmatrix},$$

$$\Theta_d = \begin{bmatrix} I & 0 & \dots & 0 \\ & (-1)^1\binom{1}{1}\binom{1+1}{1}I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ & (-1)^1\binom{d}{1}\binom{d+1}{1}I & \dots & (-1)^d\binom{d}{d}\binom{d+d}{1}I \end{bmatrix},$$

and $\tilde{\mathcal{R}}_d = \mathrm{diag}(\mathcal{R}, 3\mathcal{R}, ..., (2d+1)\mathcal{R})$. ◇

For $d = 0$, $d = 1$, and $d = 2$, the Bessel-Legendre inequality reduces to the Jensen, Wirtinger, and Auxiliary Function-based integral inequalities, respectively. As the parameter d increases, a more accurate bound for (3.6) is obtained. However, the stability criterion based on higher-order Bessel-Legendre inequality may be of the same conservatism as the one using a lower order if an appropriate augmented LKF is not used. A comprehensive study on the effect of bounding inequalities and augmented LKF on the conservatism of criteria can be found in

Zhang et al. (2017a).

In the case of systems with a time-varying delay, the aforementioned integral inequalities lead to non-convex terms with respect to the time-varying delay. To translate it into a convex one, a possible approach is using the delay-dependent reciprocally convex lemma, stated below. This lemma, however, provides only a sufficient condition for bounding the non-convex terms.

Lemma 3.6 (Delay-dependent reciprocally convex lemma (Zhang et al., 2017b))**.** *Let $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{S}_+^m$; $\sigma_1, \sigma_2 \in \mathbb{R}^m$ and a scalar $\alpha \in (0,1)$. If there exist matrices $X_1, X_2 \in \mathbb{S}^m$ and $Y_1, Y_2 \in \mathbb{R}^{m\times m}$ such that*

$$\begin{bmatrix} \mathcal{R}_1 - X_1 & Y_1 \\ * & \mathcal{R}_2 \end{bmatrix} \succeq 0, \begin{bmatrix} \mathcal{R}_1 & Y_2 \\ * & \mathcal{R}_2 - X_2 \end{bmatrix} \succeq 0, \tag{3.7}$$

then the following inequality holds

$$\begin{aligned} \frac{1}{\alpha}\sigma_1^T\mathcal{R}_1\sigma_1 + \frac{1}{1-\alpha}\sigma_2^T\mathcal{R}_2\sigma_2 \ \succeq \ & \sigma_1^T[\mathcal{R}_1 + (1-\alpha)X_1]\sigma_1 \\ & + \sigma_2^T(\mathcal{R}_2 + \alpha X_2)\sigma_2 + 2\sigma_1^T[\alpha Y_1 + (1-\alpha)Y_2]\sigma_2. \end{aligned}$$

◇

Lemma 3.6 is a generalization of the reciprocally convex combination lemma from Park et al. (2011). It introduces more slack variables than the one in Park et al. (2011), which results in an additional degree of freedom. By setting $X_1 = X_2 = 0$ and $Y_1 = Y_2 = Y$, the result of Park et al. (2011) is recovered.

Remark 2. *The number of decision variables in Lemma 3.6 can be reduced by setting $X_1 = R_1 - Y_1R_2^{-1}Y_1^T$ and $X_2 = R_2 - Y_2^TR_1^{-1}Y_2$. In Zhang et al. (2017b) it is demonstrated that this choice for X_1 and X_2 is nonconservative. Moreover, this choice automatically satisfies the condition* (3.7).

3.1.3 Stability conditions as quadratic functions w.r.t. the time-delay

To take full advantage of the use of improved integral inequalities and/or delay-dependent matrices in the Lyapunov-Krasovskii functionals, some stability criteria have been expressed as negativity condition for a quadratic function of the time-varying delay $\tau(t)$ within a finite interval of the form

$$\mathcal{F}(\tau(t)) = \xi^T(t)\left(\tau^2(t)\mathcal{M}_2 + \tau(t)\mathcal{M}_1 + \mathcal{M}_0\right)\xi(t) \prec 0, \quad \forall \tau(t) \in [0, \tau_M], \tag{3.8}$$

where $\xi(t)$ is an augmented vector of states and $\mathcal{M}_2$, $\mathcal{M}_1$, and $\mathcal{M}_0$ are real matrices of appropriate dimensions. Handling this type of condition is difficult because $\mathcal{F}(\tau(t))$ is a non-convex function with respect to $\tau(t)$ on $[0, \tau_M]$ if $\mathcal{M}_2 \prec 0$. Since the pointwise evaluation of (3.8) is not practical, some methods have been proposed to overcome this problem.

To overcome a condition expressed as the one in (3.8), an alternative approach is to use an augmented vector $\bar{\xi}(t)$, composed of $\xi(t)$ and some linear combinations of its elements, for example, $\bar{\xi}(t) = \mathrm{col}\{\xi(t), \tau(t)\nu_1(t), ..., \tau(t)\nu_N(t)\}$ where $\nu_1(t), ..., \nu_N(t)$ are elements of $\xi(t)$. The purpose of this approach is to transform (3.8) into a linear condition

$$\bar{\mathcal{F}}(\tau(t)) = \bar{\xi}^T(t)\left(\tau(t)\bar{\mathcal{M}}_1 + \bar{\mathcal{M}}_0\right)\bar{\xi}(t) \prec 0, \quad \forall \tau(t) \in [0, \tau_M], \tag{3.9}$$

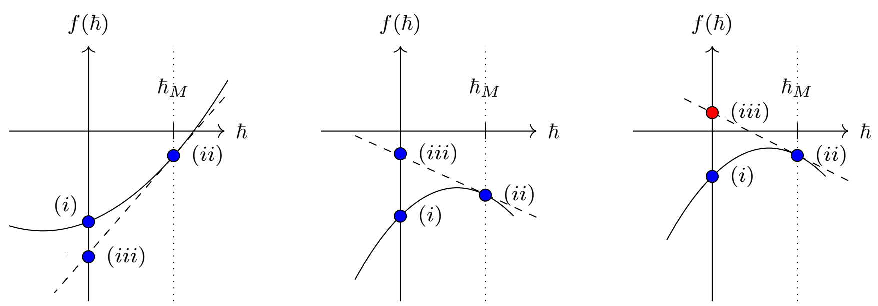
which can be easily handled. On the other hand, (3.9) cannot be directly guaranteed because of the existence of zero-valued diagonal terms therein caused by the augmentation of $\xi(t)$. Hence, it is necessary to introduce some zero-valued terms based on the linear relationships among the elements of $\bar{\xi}(t)$. An advantage of this approach is that it can be employed to deal with any criterion formulated as a polynomial function. On the other hand, it provides only a sufficient condition to evaluate such function. Hereafter, this approach will be referred as matrix injection-based method.

A simple method to directly evaluate (3.8) was proposed by Kim (2016), based on geometric properties of quadratic functions. As well as the method mentioned above, the conditions in Kim (2016), presented in the next lemma, are only sufficient.

Lemma 3.7 (Kim 2016)**.** *Let* $f(\hbar) = a_2\hbar^2 + a_1\hbar + a_0$*, where* $a_2, a_1, a_0 \in \mathbb{R}$*. If*

$$(i)\ f(0) < 0, \quad (ii)\ f(\hbar_M) < 0, \quad (iii)\ f(0) - \hbar_M^2 a_2 < 0,$$

then $f(\hbar) < 0, \forall \hbar \in [0, \hbar_M]$. ◇



a) Convex function b) Concave function c) Concave function

Figure 3.1: Graphical interpretation of Lemma 3.7. The highlighted points correspond to the points tested by conditions (i), (ii), and (iii).

Figure 3.1 presents a graphical interpretation of Lemma 3.7. The highlighted points in the graphs correspond to the points tested by conditions (i), (ii), and (iii). One can note that if $a_2 > 0$, $f(\hbar)$ is a convex function and the conditions (i) and (ii) are necessary and sufficient to ensure $f(\hbar) < 0$ within the interval $[0, \hbar_M]$ (see Fig. 3.1a). On the other hand, if $f(\hbar)$ is concave the conditions (i) and (ii) are no longer sufficient, see graphs b) and c) in Fig. 3.1. This explains the need of condition (iii), which, however, is not necessary. Condition (iii) implies that the tangent line to the curve $f(\hbar)$ at the point $\hbar_M$ should be negative at $\hbar = 0$. Thus it may be quite conservative, especially when the tangent line has a slope that is far from zero. The graphs b) and c) in Fig. 3.1 illustrate the conservativeness of condition (iii). Particularly, for the function depicted in Fig. 3.1c, the condition (iii) does not hold and the Lemma 3.7 fails, which shows how the conservatism of condition (iii) may lead to erroneous conclusion.

Very recently, in Chen et al. (2019) it was proposed an improved version of Lemma 3.7 to reduce its conservativeness. In this method, the delay interval $[0, \tau_M]$ is divided into multiple subintervals of equal length and, for each subinterval, the conditions (ii) and (iii) of Lemma 3.7 are tested. As the number of subintervals increases, the method becomes less conservative. Fig. 3.2 depicts the points tested by this method, marked with the symbol "□", and the points tested by Lemma 3.7 (marked with "○"). Note that for the depicted function, the method of Chen et al. (2019) can correctly assert its negativity while Lemma 3.7 fails. However, this approach also provides only a sufficient condition to evaluate (3.8). Other methods were reported recently in Zhang et al. (2020b) and Lee (2020), but none of them provide conditions to evaluate exactly the function (3.8).

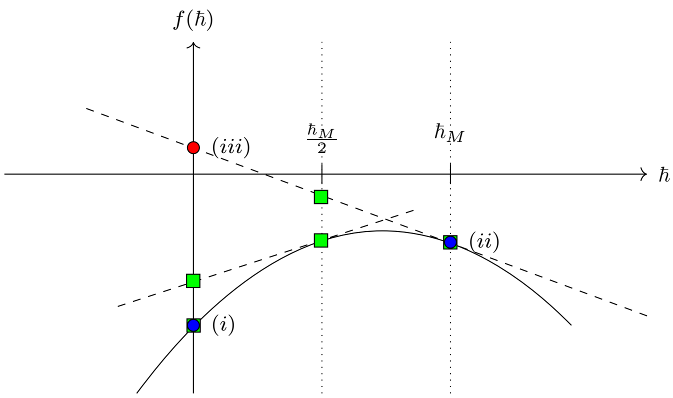


Figure 3.2: Graphical interpretation of an improved version of Lemma 3.7. The points marked with ○ and □ correspond to the points tested by Lemma 3.7 and its improved version, respectively.

Based on the previous discussion, one can see the relevance of obtaining an exact test for negativity verification of a quadratic function within a finite interval. Therefore, a contribution of this thesis is to present how the negativity of a quadratic function within a closed interval can be tested non conservatively in terms of LMIs. To this end, consider the following result borrowed from the robust control literature.

Lemma 3.8 (Zhang et al. 2003, 2010)**.** *Let* $\Theta \in \mathbb{S}^m$*,* $J, C \in \mathbb{R}^{k \times m}$*. The following statements are equivalent.*

(i) *$\zeta^T\Theta\zeta < 0$, $\forall\zeta \neq 0 \in \mathbb{R}^m$ which satisfy $(J - \delta C)\zeta = 0$, for some real scalar δ such that $|\delta| \leq 1$.*

(ii) *There exist $D \in \mathbb{S}^k_+$ and skew-symmetric matrix $G \in \mathbb{R}^{k\times k}$ such that*

$$\Theta \prec \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}.$$

◇

Notice that the previous lemma establishes the equivalence between a parameter-dependent inequality, for all parameter-dependent vectors, and a linear matrix inequality. A similar result can be obtained to evaluate condition (3.8) in a practical manner. The key issue is to rewrite (3.8) as item $i)$ of Lemma 3.8. The proposed Lemmas 3.9 and 3.10, given in the sequel, show how it can be done.

Lemma 3.9. *Given $J = \begin{bmatrix}(\hbar_M/2)I_p & -I_p\end{bmatrix} \in \mathbb{R}^{p\times 2p}$ and $C = \begin{bmatrix}(\hbar_M/2)I_p & 0_p\end{bmatrix} \in \mathbb{R}^{p\times 2p}$, with $\hbar_M$ being a known scalar, the sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ defined next are equal.*

$$\begin{aligned} \mathcal{Z}_1 &:= \left\{\zeta \in \mathbb{R}^{2p} : (J - \delta C)\zeta = 0, \textit{some } \delta \in [-1, 1]\right\}, \\ \mathcal{Z}_2 &:= \left\{\zeta \in \mathbb{R}^{2p} : \zeta = [I_p \;\; \hbar I_p]^T\xi, \hbar \in [0, \hbar_M], \xi \in \mathbb{R}^p\right\}. \end{aligned}$$

◇

Proof. $\mathcal{Z}_2 \subseteq \mathcal{Z}_1$: Let $\zeta \in \mathcal{Z}_2$ and note that $\delta \in [-1, 1]$ can be written as a function of $\hbar \in [0, \hbar_M]$ as $\delta = 1 - 2\hbar/\hbar_M$ then

$$\begin{aligned} [J - (1 - 2\hbar/\hbar_M)C]\zeta &= \left(\begin{bmatrix}(\hbar_M/2)I_p & -I_p\end{bmatrix} - (1 - 2\hbar/\hbar_M)\begin{bmatrix}(\hbar_M/2)I_p & 0_p\end{bmatrix}\right)\zeta \\ &= \left(\begin{bmatrix}0 & -I_p\end{bmatrix} + 2\hbar/\hbar_M\begin{bmatrix}(\hbar_M/2)I_p & 0_p\end{bmatrix}\right)\zeta \\ &= \begin{bmatrix}\hbar I_p & -I_p\end{bmatrix}\begin{bmatrix}I_p & \hbar I_p\end{bmatrix}^T\xi = 0. \end{aligned}$$

$\mathcal{Z}_1 \subseteq \mathcal{Z}_2$: Let $\zeta \in \mathcal{Z}_1$ and rewrite $\zeta = \text{col}\{\xi, \nu\}$, with $\xi, \nu \in \mathbb{R}^p$. Thus

$$\left(\begin{bmatrix}(\hbar_M/2)I_p & -I_p\end{bmatrix} - \delta\begin{bmatrix}(\hbar_M/2)I_p & 0_p\end{bmatrix}\right)\begin{bmatrix}\xi \\ \nu\end{bmatrix} = 0, \;\; \text{some } \delta \in [-1, 1],$$

implies that

$$\zeta = \begin{bmatrix} I_p \\ (\hbar_M/2)(1 - \delta)\, I_p \end{bmatrix}\xi.$$

Then the claim holds because the image of the function $f(\delta) = (\hbar_M/2)(1 - \delta)$ for all $\delta \in [-1,\; 1]$ lies in the domain of $\hbar$. □

In light of Lemma 3.9, we are in position to present a necessary and sufficient LMI condition to evaluate stability conditions formulated as quadratic functions. This result is stated next.

Lemma 3.10. *Let $\Lambda_2, \Lambda_1, \Lambda_0 \in \mathbb{S}^p$ and $\xi \in \mathbb{R}^p$. Then the inequality*

$$\xi^T \left(\hbar^2 \Lambda_2 + \hbar \Lambda_1 + \Lambda_0\right) \xi < 0 \tag{3.10}$$

holds for all $\hbar \in [0, \hbar_M]$ if and only if there exist $D \in \mathbb{S}^p_+$ and skew-symmetric matrix $G \in \mathbb{R}^{p\times p}$ such that

$$\begin{bmatrix} \Lambda_0 & \frac{1}{2}\Lambda_1 \\ \frac{1}{2}\Lambda_1 & \Lambda_2 \end{bmatrix} \prec \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}, \tag{3.11}$$

where

$$C = \begin{bmatrix} (\hbar_M/2) I_p & 0_p \end{bmatrix} \ \text{and} \ J = \begin{bmatrix} (\hbar_M/2) I_p & -I_p \end{bmatrix}. \tag{3.12}$$

◇

Proof. Let $\zeta = \begin{bmatrix} I_p & \hbar I_p \end{bmatrix}^T \xi$ and rewrite (3.10) as

$$\xi^T \begin{bmatrix} I_p \\ \hbar I_p \end{bmatrix}^T \begin{bmatrix} \Lambda_0 & \frac{1}{2}\Lambda_1 \\ \frac{1}{2}\Lambda_1 & \Lambda_2 \end{bmatrix} \begin{bmatrix} I_p \\ \hbar I_p \end{bmatrix} \xi < 0.$$

Further, from Lemma 3.9 it follows that all $\zeta \neq 0$ that satisfies $(J - \delta C)\zeta = 0$, for some $\delta \in [-1,\ 1]$, is of the form $\zeta = \begin{bmatrix} I_p & \hbar I_p \end{bmatrix}^T \xi$, with C and J defined in (3.12). Then an application of Lemma 3.8 concludes the proof. □

To the best author's knowledge, Lemma 3.10 is the unique exact method available in the literature so far. Since the recent publication of this result in de Oliveira and Souza (2020a), Lemma 3.10 has already been used by other authors in Zhang et al. (2020c) and Zhang et al. (2020a), which may indicate the importance of this result.

3.2 New LMI conditions for delay-dependent stability

In this section, new LMI conditions for assessing the stability of systems with time-varying delay given by (3.1) are provided. In a similar manner as in the recent literature (see Kim (2016); Zhang et al. (2017b); Chen et al. (2019)), the proposed stability criterion in its primary form is expressed in terms of the negativity of a quadratic function parameterized by the delay. Then, to evaluate the proposed criterion, three different strategies are employed: Lemma 3.7, Lemma 3.10, and the matrix injection-based approach. As will be shown in Section 3.2.1, the proposed criterion evaluated using Lemma 3.10 can lead to less conservative results.

To assess the stability of system (3.1), it is proposed the use of the following augmented affine parameter-dependent Lyapunov-Krasovskii functional candidate inspired by the papers of Lee and Park (2017) and Zhang et al. (2017b):

$$V(x_t, \dot{x}_t) = V_1(x_t) + V_2(x_t) + \tau_M V_3(\dot{x}_t), \tag{3.13}$$

where $x_t = x(t+\theta),\ \theta \in [-\tau_M,\ 0]$ and

$$\begin{aligned}
V_1(x_t) &= \eta_1^T(t)\mathcal{P}_1(t)\eta_1(t) + \eta_2^T(t)\mathcal{P}_2(t)\eta_2(t),\\
V_2(x_t) &= \int_{t-\tau(t)}^{t} \eta_3^T(t,s)Q_1\eta_3(t,s)ds + \int_{t-\tau_M}^{t-\tau(t)} \eta_4^T(t,s)Q_2\eta_4(t,s)ds,\\
V_3(\dot{x}_t) &= \int_{t-\tau_M}^{t-\tau(t)} (\tau_M - t + s)\dot{x}^T(s)R_1\dot{x}(s)ds + \int_{t-\tau(t)}^{t} (\tau_M - t + s)\dot{x}^T(s)R_2\dot{x}(s)ds,
\end{aligned}$$

with $\mathcal{P}_1(t) = \tau(t)P_{11} + P_{12}$, $\mathcal{P}_2(t) = (\tau_M - \tau(t))P_{21} + P_{22}$,

$$\eta_0^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & x^T(t-\tau_M) \end{bmatrix},$$

$$\eta_1^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & \int_{t-\tau(t)}^{t} x^T(s)ds \end{bmatrix}, \tag{3.14}$$

$$\eta_2^T(t) = \begin{bmatrix} x^T(t-\tau(t)) & x^T(t-\tau_M) & \int_{t-\tau_M}^{t-\tau(t)} x^T(s)ds \end{bmatrix}, \tag{3.15}$$

$$\eta_3^T(t,s) = \begin{bmatrix} \dot{x}^T(s) & x^T(s) & \eta_0^T(t) & \int_{s}^{t} x^T(\theta)d\theta \end{bmatrix}, \tag{3.16}$$

$$\eta_4^T(t,s) = \begin{bmatrix} \dot{x}^T(s) & x^T(s) & \eta_0^T(t) & \int_{s}^{t-\tau(t)} x^T(\theta)d\theta \end{bmatrix}. \tag{3.17}$$

An important feature of the proposed LKF candidate is to take advantage of the use of the second-order Bessel-Legendre integral inequality. Moreover, there are three main differences between functional candidate (3.13) and the one in Zhang et al. (2017b), namely:

i) the inclusion of the parameter-dependent term $V_1(x_t)$ to exploit the information on the bounds of the delay and its derivative;

ii) the change on the limits of integration of the integral terms in $\eta_3(t,s)$ and $\eta_4(t,s)$;

iii) the integration over $[0,\ \tau_M]$ in the term $V_3(\dot{x}_t)$ is split into two integrals, and for each integral term a different matrix variable is associated. The idea behind this change is to derive conditions more suitable with the convex combination in Lemma 3.6.

As will be illustrated in Section 3.2.1, these changes are important in reducing conservativeness.

In light of the Lyapunov-Krasovskii functional candidate in (3.13), a novel stability criterion for system (3.1) is stated in the next theorem.

Theorem 3.2. *Let be given the delay upper bound τ_M and the lower and upper bounds of the delay derivative, μ_1 and μ_2, respectively. System (3.1) is asymptotically stable if there exist matrices $P_{11} \in \mathbb{S}^{3n}$, $P_{12} \in \mathbb{S}_+^{3n}$, $P_{21} \in \mathbb{S}^{3n}$, $P_{22} \in \mathbb{S}_+^{3n}$, $Q_i \in \mathbb{S}_+^{6n}$, $R_i \in \mathbb{S}_+^{n}$, $X_i \in \mathbb{S}^{3n}$, $Y_i \in \mathbb{R}^{3n\times 3n}$, $i = 1, 2$, such that the following conditions hold for $\mu_{j=1,2}$, and $\forall\, \tau(t) \in [0,\ \tau_M]$.*

$$\tau_M P_{11} + P_{12} \succ 0,\ \tau_M P_{21} + P_{22} \succ 0, \tag{3.18}$$

$$\begin{bmatrix} \tilde{R}_1 - X_1 & Y_1 \\ * & \tilde{R}_2 \end{bmatrix} \succeq 0,\ \begin{bmatrix} \tilde{R}_1 & Y_2 \\ * & \tilde{R}_2 - X_2 \end{bmatrix} \succeq 0, \tag{3.19}$$

$$\tau^2(t)\Upsilon_2(\mu_j) + \tau(t)\Upsilon_1(\mu_j) + \Upsilon_0(\mu_j) \prec 0, \tag{3.20}$$

where $\tilde{R}_i = \mathrm{diag}\{R_i, 3R_i, 5R_i\}$ for $i = 1, 2$ and

$$\Upsilon_0(\dot{\tau}(t)) = \Psi_0(\dot{\tau}(t)) - \Gamma_2^T X_2 \Gamma_2 \tag{3.21}$$

$$\begin{aligned} \Psi_0(\dot{\tau}(t)) = {} & \mathrm{sym}\{\mathcal{D}_1^T P_{12}\mathcal{C}_{11}\} + \dot{h}(t) C_{11}^T P_{11} C_{11} + \mathrm{sym}\{\mathcal{D}_2^T(\tau_M P_{21} + P_{22})\mathcal{C}_{21}\} \\ & - \dot{\tau}(t) C_{21}^T P_{21} C_{21} + \mathcal{C}_3^T Q_1 \mathcal{C}_3 - (1 - \dot{h}(t))\mathcal{C}_{41}^T Q_1 \mathcal{C}_{41} + \mathrm{sym}\{\mathcal{C}_{50}^T Q_1 \mathcal{D}_3\} \\ & + (1 - \dot{\tau}(t))\mathcal{C}_6^T Q_2 \mathcal{C}_6 - \mathcal{C}_{71}^T Q_2 \mathcal{C}_{71} + \mathrm{sym}\{(\mathcal{C}_{80} + \tau_M \mathcal{C}_{81} + \tau_M^2 \mathcal{C}_{82})^T Q_2 \mathcal{D}_4\} \\ & + \tau_M^2 \mathcal{C}_0^T R_2 \mathcal{C}_0 + \tau_M^2 (1 - \dot{h}(t)) e_8^T (R_1 - R_2) e_8 - \Gamma_1^T \tilde{R}_1 \Gamma_1 - \Gamma_2^T \tilde{R}_2 \Gamma_2 - \mathrm{sym}\{\Gamma_1^T Y_1 \Gamma_2\}, \end{aligned} \tag{3.22}$$

$$\Upsilon_1(\dot{\tau}(t)) = \Psi_1(\dot{\tau}(t)) - (1/\tau_M)\Gamma_1^T X_1 \Gamma_1 + (1/\tau_M)\Gamma_2^T X_2 \Gamma_2, \tag{3.23}$$

$$\begin{aligned} \Psi_1(\dot{\tau}(t)) = {} & \mathrm{sym}\{\mathcal{D}_1^T P_{12}\mathcal{C}_{12} + \mathcal{D}_2^T(\tau_M P_{21} + P_{22})\mathcal{C}_{22} + \dot{h}(t)\mathcal{C}_{11}^T P_{11}\mathcal{C}_{12} \\ & - \mathcal{D}_2^T P_{21}\mathcal{C}_{21} - \dot{\tau}(t)\mathcal{C}_{21}^T P_{21}\mathcal{C}_{22} + \mathcal{D}_1^T P_{11}\mathcal{C}_{11} + \mathcal{C}_{51}^T Q_1 \mathcal{D}_3 - (1 - \dot{\tau}(t))\mathcal{C}_{41}^T Q_1 \mathcal{C}_{42} \\ & - \mathcal{C}_{71}^T Q_2 \mathcal{C}_{72} - (\mathcal{C}_{81}^T + 2\tau_M \mathcal{C}_{82}) Q_2 \mathcal{D}_4\} - (1 - \dot{\tau}(t))\tau_M e_8^T (R_1 - R_2) e_8 \\ & + \mathrm{sym}\{(1/\tau_M)\Gamma_1^T (Y_1 - Y_2)\Gamma_2\}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \Upsilon_2(\dot{\tau}(t)) = {} & \mathrm{sym}\{\mathcal{D}_1^T P_{11}\mathcal{C}_{12} - \mathcal{D}_2^T P_{21}\mathcal{C}_{22} + \mathcal{D}_3^T Q_1 \mathcal{C}_{52} + \mathcal{D}_4^T Q_2 \mathcal{C}_{82}\} \\ & + \dot{\tau}(t)\mathcal{C}_{12}^T P_{11}\mathcal{C}_{12} - \dot{\tau}(t)\mathcal{C}_{22}^T P_{21}\mathcal{C}_{22} - (1 - \dot{\tau}(t)) C_{42}^T Q_1 \mathcal{C}_{42} - \mathcal{C}_{72}^T Q_2 \mathcal{C}_{72}, \end{aligned} \tag{3.25}$$

with,

$$\mathcal{D}_1 = \mathrm{col}\{\mathcal{C}_0, (1-\dot{\tau}(t))e_8, e_1-(1-\dot{\tau}(t))e_2\},$$
$$\mathcal{D}_2 = \mathrm{col}\{(1-\dot{\tau}(t))e_8, e_9, (1-\dot{\tau}(t))e_2-e_3\},$$
$$\mathcal{D}_3 = \mathrm{col}\{e_0, e_0, \mathcal{C}_0, (1-\dot{\tau}(t))e_8, e_9, e_1\},$$
$$\mathcal{D}_4 = \mathrm{col}\{e_0, e_0, \mathcal{C}_0, (1-\dot{\tau}(t))e_8, e_9, (1-\dot{\tau}(t))e_2\},$$

$$\begin{aligned}
&\mathcal{C}_0 = A_0e_1 + A_1e_2, && \mathcal{C}_{11} = \mathrm{col}\{e_1, e_2, e_0\},\\
&\mathcal{C}_{12} = \mathrm{col}\{e_0, e_0, e_6\}, && \mathcal{C}_{21} = \mathrm{col}\{e_2, e_3, \tau_M e_4\},\\
&\mathcal{C}_{22} = \mathrm{col}\{e_0, e_0, -e_4\}, && \mathcal{C}_{3} = \mathrm{col}\{\mathcal{C}_0, e_1, e_1, e_2, e_3, e_0\},\\
&\mathcal{C}_{41} = \mathrm{col}\{e_8, e_2, e_1, e_2, e_3, e_0\}, && \mathcal{C}_{42} = \mathrm{col}\{e_0, e_0, e_0, e_0, e_0, e_6\},\\
&\mathcal{C}_{50} = \mathrm{col}\{e_1-e_2, e_0, e_0, e_0, e_0, e_0\}, && \mathcal{C}_{51} = \mathrm{col}\{e_0, e_6, e_1, e_2, e_3, e_0\},\\
&\mathcal{C}_{52} = \mathrm{col}\{e_0, e_0, e_0, e_0, e_0, e_6-e_7\}, && \mathcal{C}_{71} = \mathrm{col}\{e_9, e_3, e_1, e_2, e_3, \tau_M e_4\},\\
&\mathcal{C}_{72} = \mathrm{col}\{e_0, e_0, e_0, e_0, e_0, -e_4\}, && \mathcal{C}_{80} = \mathrm{col}\{e_2-e_3, e_0, e_0, e_0, e_0, e_0\}\\
&\mathcal{C}_{81} = \mathrm{col}\{e_0, e_4, e_1, e_2, e_3, e_0\}, && \mathcal{C}_{82} = \mathrm{col}\{e_0, e_0, e_0, e_0, e_0, e_4-e_5\},
\end{aligned}$$

$$\Gamma_1 = \mathrm{col}\{e_2-e_3, e_2+e_3-2e_4, e_2-e_3-6e_4+12e_5\},$$
$$\Gamma_2 = \mathrm{col}\{e_1-e_2, e_2+e_1-2e_6, e_1-e_2-6e_6+12e_7\}, \tag{3.26}$$

and $e_i = [0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(9-i)n}]$, *for* $i = 1, 2, ..., 9$, *and* $e_0 = 0_{n\times 9n}$. ◇

Proof. The Lyapunov-Krasovskii functional condition is guaranteed by imposing $\mathcal{P}_i(t) \succ 0$, $Q_i \succ 0$, and $R_i \succ 0$ for $i = 1, 2$. Note that $\mathcal{P}_i(t)$ is an affine function in $\tau(t) \in [0,\ \tau_M]$, then $\mathcal{P}_i(t) \succ 0$ is ensured by imposing

$$P_{i2} \succ 0, \tau_M P_{i1} + P_{i2} \succ 0, \;\; \text{for} \;\; i = 1, 2.$$

Hereafter, it is shown that the Lyapunov-Krasovskii derivative condition $\dot{V}(x_t, \dot{x}_t) < 0$ is satisfied if the inequalities in (3.19) and (3.20) hold. For simplicity of notation, it is defined the augmented vector

$$\xi(t) = \mathrm{col}\{x(t), x(t-\tau(t)), x(t-\tau_M), \rho_1(t), \rho_2(t), \rho_3(t), \rho_4(t), \dot{x}(t-\tau(t)), \dot{x}(t-\tau_M)\}, \tag{3.27}$$

with

$$\rho_1(t) = \int_{t-\tau_M}^{t-\tau(t)} \frac{x(s)}{\tau_M - \tau(t)} ds, \;\; \rho_2(t) = \int_{t-\tau_M}^{t-\tau(t)} \frac{(t-\tau(t)-s)x(s)}{(\tau_M-\tau(t))^2} ds,$$
$$\rho_3(t) = \int_{t-\tau(t)}^{t} \frac{x(s)}{\tau(t)} ds, \;\; \text{and} \;\; \rho_4(t) = \int_{t-\tau(t)}^{t} \frac{(t-s)x(s)}{\tau^2(t)} ds. \tag{3.28}$$

Taking the time derivative of (3.13) yields

$$
\begin{aligned}
\dot{V}(x_t,\dot{x}_t) &= 2\dot{\eta}_1^T(t)\mathcal{P}_1(t)\eta_1(t) + \eta_1^T(t)\dot{\mathcal{P}}_1(t)\eta_1(t) + 2\dot{\eta}_2^T(t)\mathcal{P}_2(t)\eta_2(t) \\
&+ \eta_2^T(t)\dot{\mathcal{P}}_2(t)\eta_2(t) - (1-\dot{\tau}(t))\eta_3^T(t,t-\tau(t))Q_1\eta_3(t,t-\tau(t)) \\
&+ \eta_3^T(t,t)Q_1\eta_3(t,t) + 2\int_{t-\tau(t)}^{t}\eta_3^T(t,s)dsQ_1\frac{\partial}{\partial t}\eta_3(t,s) \\
&+ (1-\dot{\tau}(t))\eta_4^T(t,t-\tau(t))Q_2\eta_4(t,t-\tau(t)) \\
&- \eta_4^T(t,t-\tau_M)Q_2\eta_4(t,t-\tau_M) + 2\int_{t-\tau_M}^{t-\tau(t)}\eta_4^T(t,s)dsQ_2\frac{\partial}{\partial t}\eta_4(t,s) \\
&+ \tau_M^2\dot{x}^T(t)R_2\dot{x}(t) + (1-\dot{\tau}(t))(h_M^2-\tau(t)\tau_M)\dot{x}^T(t-\tau(t))(R_1-R_2)\dot{x}(t-\tau(t)) \\
&- \tau_M\left(\int_{t-\tau_M}^{t-\tau(t)}\dot{x}^T(s)R_1\dot{x}(s)ds + \int_{t-\tau(t)}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds\right),
\end{aligned}
\tag{3.29}
$$

where $\dot{\eta}_1(t)=\xi^T(t)\mathcal{D}_1$, $\dot{\eta}_2(t)=\xi^T(t)\mathcal{D}_2$, $\eta_3(t,t)=\mathcal{C}_3\xi(t)$, $\eta_3(t,t-\tau(t))=(\mathcal{C}_{41}+\tau(t)\mathcal{C}_{42})\xi(t)$, $\eta_4(t,t-\tau(t))=\mathcal{C}_6\xi(t)$, $\eta_4(t,t-\tau_M)=(\mathcal{C}_{71}+\tau(t)\mathcal{C}_{72})\xi(t)$, and

$$
\begin{aligned}
&\int_{t-\tau(t)}^{t}\eta_3^T(t,s)dsQ_1\frac{\partial}{\partial t}\eta_3(t,s) = \\
&\qquad \xi^T(t)(\mathcal{C}_{50}+\tau(t)\mathcal{C}_{51}+h^2(t)\mathcal{C}_{52})Q_1\mathcal{D}_3\xi(t), \\
&\int_{t-\tau_M}^{t-\tau(t)}\eta_4^T(t,s)dsQ_2\frac{\partial}{\partial t}\eta_4(t,s) = \\
&\qquad \xi^T(t)(\mathcal{C}_{80}+\alpha\tau_M\mathcal{C}_{81}+(\tau_M-\tau(t))^2\mathcal{C}_{82})Q_2\mathcal{D}_4\xi(t),
\end{aligned}
$$

with $\alpha=(\tau_M-\tau(t))/\tau_M$.

Under the assumption that $R_1\succ 0$ and $R_2\succ 0$, an upper bound for the last two integral terms in (3.29) can be obtained by applying Lemma 3.5 with $d=2$, which leads to

$$
\begin{aligned}
&\tau_M\left(\int_{t-\tau_M}^{t-\tau(t)}\dot{x}^T(s)R_1\dot{x}(s)ds + \int_{t-\tau(t)}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds\right) \\
&\succeq \frac{1}{\alpha}\xi^T(t)\Gamma_1^T\tilde{R}_1\Gamma_1\xi(t) + \frac{1}{1-\alpha}\xi^T(t)\Gamma_2^T\tilde{R}_2\Gamma_2\xi(t),
\end{aligned}
$$

where $\tilde{R}_i=\mathrm{diag}\{R_i,3R_i,5R_i\}$ for $i=1,2$ and $\Gamma_1,\ \Gamma_2$ are given in Theorem 3.2. Hence, we can apply Lemma 3.6 to obtain

$$
\begin{aligned}
&\frac{1}{\alpha}\xi^T(t)\Gamma_1^T\tilde{R}_1\Gamma_1\xi(t) + \frac{1}{1-\alpha}\xi^T(t)\Gamma_2^T\tilde{R}_2\Gamma_2\xi(t) \\
&\succeq \xi^T(t)\{\Gamma_1^T[\tilde{R}_1+(1-\alpha)X_1]\Gamma_1 \\
&+ 2\Gamma_1^T[\alpha Y_1+(1-\alpha)Y_2]\Gamma_2 + \Gamma_2^T(\tilde{R}_2+\alpha X_2)\Gamma_2\}\xi(t).
\end{aligned}
$$

Thus based on the previous two inequalities we have that

$$\dot{V}(x_t, \dot{x}_t) \leq \xi^T(t)[\tau^2(t)\Upsilon_2(\dot{\tau}(t)) + \tau(t)\Upsilon_1(\dot{\tau}(t)) + \Upsilon_0(\dot{\tau}(t))]\xi(t),$$

with $\Upsilon_0(\dot{\tau}(t))$, $\Upsilon_1(\dot{\tau}(t))$, and $\Upsilon_2(\dot{\tau}(t))$ given in (3.21), (3.23), and (3.25), respectively.

Therefore, under the constraint (3.19), the Lyapunov-Krasovskii derivative condition is satisfied if (3.20) holds, concluding this proof. □

The previous theorem provides new conditions for analyzing the stability of system (3.1), however evaluating the condition in (3.20) within the finite interval for all $\tau(t) \in [0,\ \tau_M]$ is not practical. To translate the Theorem 3.2 into a finite-dimensional convex optimization problem, three different techniques are considered: Lemmas 3.7 and 3.10, and the matrix injection-based approach. It is emphasized that Lemma 3.10 leads to an exact LMI characterization of the quadratic condition in (3.20).

Accordingly, the following three corollaries are provided. The first one is the combination of Theorem 3.2 with Lemma 3.7, setting $a_2 = \xi^T(t)\Upsilon_2(\dot{\tau}(t))\xi(t)$, $a_1 = \xi^T(t)\Upsilon_1(\dot{\tau}(t))\xi(t)$, and $a_0 = \xi^T(t)\Upsilon_0(\dot{\tau}(t))\xi(t)$. It is also considered that $X_1 = \tilde{R}_1 - Y_1\tilde{R}_2^{-1}Y_1^T$ and $X_2 = \tilde{R}_2 - Y_2^T\tilde{R}_1^{-1}Y_2$ to reduce the number of decision variables. Then, the result in Corollary 1 follows from applying the Schur complement.

Corollary 1. *Let be given the delay upper bound τ_M and the lower and upper bounds of the delay derivative, μ_1 and μ_2, respectively. System (3.1) is asymptotically stable if there exist matrices $P_{11} \in \mathbb{S}^{3n}$, $P_{12} \in \mathbb{S}^{3n}_+$, $P_{21} \in \mathbb{S}^{3n}$, $P_{22} \in \mathbb{S}^{3n}_+$, $Q_i \in \mathbb{S}^{6n}_+$, $R_i \in \mathbb{S}^{n}_+$, $Y_i \in \mathbb{R}^{3n\times 3n}$, for $i = 1, 2$, such that the following LMI conditions hold for $j = 1, 2$.*

$$\tau_M P_{11} + P_{12} \succ 0,\ \tau_M P_{21} + P_{22} \succ 0,$$

$$\begin{bmatrix} \Psi_0(\mu_j) - \Gamma_2^T\tilde{R}_2\Gamma_2 & \Gamma_2^T Y_2^T \\ * & -\tilde{R}_1 \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \tau_M^2\Upsilon_2(\mu_j) + \tau_M\Psi_1(\mu_j) + \Psi_0(\mu_j) - \Gamma_1^T\tilde{R}_1\Gamma_1 & \Gamma_1^T Y_1 \\ * & -\tilde{R}_2 \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} -\tau_M^2\Upsilon_2(\mu_j) + \Psi_0(\mu_j) - \Gamma_2^T\tilde{R}_2\Gamma_2 & \Gamma_2^T Y_2^T \\ * & -\tilde{R}_1 \end{bmatrix} \prec 0,$$

where $\Psi_0(\dot{\tau}(t))$, $\Psi_1(\dot{\tau}(t))$, *and* $\Upsilon_2(\dot{\tau}(t))$ *are given in Theorem* 3.2. ◇

Similarly to the previous corollary the next one is obtained using Theorem 3.2, but now invoking Lemma 3.10. In this case, we cannot reduce the number of decision variables by setting $X_1 = \tilde{R}_1 - Y_1\tilde{R}_2^{-1}Y_1^T$ and $X_2 = \tilde{R}_2 - Y_2^T\tilde{R}_1^{-1}Y_2$, because it is not possible to apply Schur complement due to the structure of the resulting stability condition.

Corollary 2. *Let be given the delay upper bound* τ_M *and the lower and upper bounds of the delay derivative,* μ_1 *and* μ_2*, respectively. System* (3.1) *is asymptotically stable if there exist matrices* $P_{11} \in \mathbb{S}^{3n}$, $P_{12} \in \mathbb{S}^{3n}_+$, $P_{21} \in \mathbb{S}^{3n}$, $P_{22} \in \mathbb{S}^{3n}_+$, $Q_i \in \mathbb{S}^{6n}_+$, $R_i \in \mathbb{S}^{n}_+$, $D_i \in \mathbb{S}^{9n}_+$, $X_i \in \mathbb{S}^{3n}$, $Y_i \in \mathbb{R}^{3n\times 3n}$ *and skew-symmetric matrices* $G_i \in \mathbb{R}^{9n\times 9n}$*, for* $i = 1,2$*, such that the following conditions hold for* $j = 1,2$.

$$\tau_M P_{11} + P_{12} \succ 0,\ \tau_M P_{21} + P_{22} \succ 0,$$

$$\begin{bmatrix} \tilde{R}_1 - X_1 & Y_1 \\ * & \tilde{R}_2 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R}_1 & Y_2 \\ * & \tilde{R}_2 - X_2 \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} \Upsilon_0(\mu_j) & \frac{1}{2}\Upsilon_1(\mu_j) \\ \frac{1}{2}\Upsilon_1(\mu_j) & \Upsilon_2(\mu_j) \end{bmatrix} - \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D_j & G_j \\ * & D_j \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} \prec 0, \tag{3.30}$$

where $C = \begin{bmatrix} (\tau_M/2)I_{9n} & 0_{9n} \end{bmatrix}$, $J = \begin{bmatrix} (\tau_M/2)I_{9n} & -I_{9n} \end{bmatrix}$, *and* $\Upsilon_0(\dot{\tau}(t))$, $\Upsilon_1(\dot{\tau}(t))$, *and* $\Upsilon_2(\dot{\tau}(t))$ *given in* (3.21), (3.23), *and* (3.25), *respectively.* ◇

Finally, Corollary 3 is obtained using the matrix injection-based method, that is, by augmenting the vector (3.27) with linearly dependent terms and introducing an appropriate zero-valued term.

Corollary 3. *Let be given the delay upper bound* τ_M *and the lower and upper bounds of the delay derivative,* μ_1 *and* μ_2*, respectively. System* (3.1) *is asymptotically stable if there exist matrices* $P_{i1} \in \mathbb{S}^{3n}$, $P_{i2} \in \mathbb{S}^{3n}_+$, $Q_i \in \mathbb{S}^{6n}_+$, $R_i \in \mathbb{S}^{n}_+$, $Y_i \in \mathbb{R}^{3n\times 3n}$*, for* $i = 1,2$*, and* $M \in \mathbb{R}^{13n\times 4n}$*, such that the following conditions hold for* $j = 1,2$.

$$\tau_M P_{11} + P_{12} \succ 0,\ \tau_M P_{21} + P_{22} \succ 0,$$

$$\begin{bmatrix} \tilde{\Upsilon}_0(\mu_j) - \tilde{\Gamma}_2^T\tilde{R}_2\tilde{\Gamma}_2 & \tilde{\Gamma}_2^T Y_2^T \\ * & -\tilde{R}_1 \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} \tau_M\tilde{\Upsilon}_1(\mu_j) + \tilde{\Upsilon}_0(\mu_j) - \Gamma_1^T\tilde{R}_1\tilde{\Gamma}_1 & \tilde{\Gamma}_1^T Y_1 \\ * & -\tilde{R}_2 \end{bmatrix} \prec 0,$$

where

$$\begin{aligned}\tilde{\Upsilon}_0(\dot{\tau}(t)) = \ & \mathrm{sym}\{\tilde{\mathcal{D}}_1^T P_{12}\tilde{\mathcal{C}}_1\} + \dot{\tau}(t)\tilde{\mathcal{C}}_1^T P_{11}\tilde{\mathcal{C}}_1 + \mathrm{sym}\{\tilde{\mathcal{D}}_2^T(\tau_M P_{21} + P_{22})\tilde{\mathcal{C}}_{21}\} - \dot{\tau}(t)\tilde{\mathcal{C}}_2^T P_{21}\tilde{\mathcal{C}}_2 \\ & + \tilde{\mathcal{C}}_3^T Q_1\tilde{\mathcal{C}}_3 - (1-\dot{h}(t))\tilde{\mathcal{C}}_4^T Q_1\tilde{\mathcal{C}}_4 + \mathrm{sym}\{\tilde{\mathcal{C}}_{50}^T Q_1\tilde{\mathcal{D}}_3\} + (1-\dot{\tau}(t))\tilde{\mathcal{C}}_6^T Q_2\tilde{\mathcal{C}}_6 \\ & - \tilde{\mathcal{C}}_7^T Q_2\tilde{\mathcal{C}}_7 + \mathrm{sym}\tilde{\mathcal{C}}_{80}^T Q_2\tilde{\mathcal{D}}_4\} + \tau_M^2\tilde{\mathcal{C}}_0^T R_2\tilde{\mathcal{C}}_0 - \tilde{\Gamma}_1^T\tilde{R}_1\tilde{\Gamma}_1 - \tilde{\Gamma}_2^T\tilde{R}_2\tilde{\Gamma}_2 - \mathrm{sym}\{\tilde{\Gamma}_1^T Y_1\tilde{\Gamma}_2\} \\ & + \tau_M^2(1-\dot{h}(t))\tilde{e}_8^T(R_1 - R_2)\tilde{e}_8 + \mathrm{sym}\{M\tilde{\mathcal{C}}_{90}\},\end{aligned}$$

$$\begin{aligned}\tilde{\Upsilon}_1(\dot{\tau}(t)) = \ & \mathrm{sym}\{\tilde{\mathcal{D}}_1^T P_{11}\tilde{\mathcal{C}}_1 - \tilde{\mathcal{D}}_2^T P_{21}\tilde{\mathcal{C}}_2 + \tilde{\mathcal{C}}_{51}^T Q_1\tilde{\mathcal{D}}_3 + \tilde{\mathcal{C}}_{81}^T Q_2\tilde{\mathcal{D}}_4 + M\tilde{\mathcal{C}}_{91}\} \\ & - (1-\dot{\tau}(t))\tau_M\tilde{e}_8^T(R_1 - R_2)\tilde{e}_8 + \mathrm{sym}\{(1/\tau_M)\tilde{\Gamma}_1^T(Y_1 - Y_2)\tilde{\Gamma}_2\},\end{aligned}$$

with,

$$\begin{aligned}
&\tilde{\mathcal{C}}_0 = A_0\tilde{e}_1 + A_1\tilde{e}_2, && \tilde{\mathcal{C}}_1 = \mathrm{col}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_{12}\},\\
&\tilde{\mathcal{C}}_2 = \mathrm{col}\{\tilde{e}_2, \tilde{e}_3, \tilde{e}_{10}\}, && \tilde{\mathcal{C}}_3 = \mathrm{col}\{\tilde{\mathcal{C}}_0, \tilde{e}_1, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_0\},\\
&\tilde{\mathcal{C}}_4 = \mathrm{col}\{\tilde{e}_8, \tilde{e}_2, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_{12}\}, && \tilde{\mathcal{C}}_{50} = \mathrm{col}\{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_{12}, \tilde{e}_0, \tilde{e}_0, \tilde{e}_0, \tilde{e}_0\},\\
&\tilde{\mathcal{C}}_{51} = \mathrm{col}\{\tilde{e}_0, \tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_{12} - \tilde{e}_{13}\}, && \tilde{\mathcal{C}}_6 = \mathrm{col}\{\tilde{e}_8, \tilde{e}_2, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_0\},\\
&\tilde{\mathcal{C}}_7 = \mathrm{col}\{\tilde{e}_9, \tilde{e}_3, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_{10}\}, && \tilde{\mathcal{C}}_{91} = \mathrm{col}\{-\tilde{e}_4, -\tilde{e}_5, \tilde{e}_6, \tilde{e}_7\},\\
&\tilde{\mathcal{C}}_{80} = \mathrm{col}\{\tilde{e}_2 - \tilde{e}_3, \tilde{e}_{10}, \tau_M\tilde{e}_1, \tau_M\tilde{e}_2, \tau_M\tilde{e}_3, \tau_M(\tilde{e}_{10} - \tilde{e}_{11})\}\\
&\tilde{\mathcal{C}}_{81} = \mathrm{col}\{\tilde{e}_0, \tilde{e}_0, -\tilde{e}_1, -\tilde{e}_2, -\tilde{e}_3, \tilde{e}_{11} - \tilde{e}_{10}\},\\
&\tilde{\mathcal{C}}_{90} = \mathrm{col}\{\tau_M\tilde{e}_4 - \tilde{e}_{10}, \tau_M\tilde{e}_5 - \tilde{e}_{11}, -\tilde{e}_{12}, -\tilde{e}_{13}\},
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&\tilde{\mathcal{D}}_1 = \mathrm{col}\{\tilde{\mathcal{C}}_0, (1-\dot{\tau}(t))\tilde{e}_8, \tilde{e}_1 - (1-\dot{\tau}(t))\tilde{e}_2\},\\
&\tilde{\mathcal{D}}_2 = \mathrm{col}\{(1-\dot{\tau}(t))\tilde{e}_8, \tilde{e}_9, (1-\dot{\tau}(t))\tilde{e}_2 - \tilde{e}_3\},\\
&\tilde{\mathcal{D}}_3 = \mathrm{col}\{\tilde{e}_0, \tilde{e}_0, \tilde{\mathcal{C}}_0, (1-\dot{\tau}(t))\tilde{e}_8, \tilde{e}_9, \tilde{e}_1\},\\
&\tilde{\mathcal{D}}_4 = \mathrm{col}\{\tilde{e}_0, \tilde{e}_0, \tilde{\mathcal{C}}_0, (1-\dot{\tau}(t))\tilde{e}_8, \tilde{e}_9, (1-\dot{\tau}(t))\tilde{e}_2\},
\end{aligned}$$

$$\begin{aligned}
&\tilde{\Gamma}_1 = \mathrm{col}\{\tilde{e}_2 - \tilde{e}_3, \tilde{e}_2 + \tilde{e}_3 - 2\tilde{e}_4, \tilde{e}_2 - \tilde{e}_3 - 6\tilde{e}_4 + 12\tilde{e}_5\},\\
&\tilde{\Gamma}_2 = \mathrm{col}\{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_2 + \tilde{e}_1 - 2\tilde{e}_6, \tilde{e}_1 - \tilde{e}_2 - 6\tilde{e}_6 + 12\tilde{e}_7\},
\end{aligned}$$

and $\tilde{e}_i = [0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(13-i)n}]$*, for* $i = 1, 2, ..., 13$*, and* $\tilde{e}_0 = 0_{n\times 13n}$. ◇

Proof. It follows the same lines of the proof of Theorem 3.2, by defining a new augmented vector

$$\tilde{\xi}(t) = \mathrm{col}\{\xi(t), (\tau_M - \tau(t))\rho_1(t), (\tau_M - \tau(t))\rho_2(t), \tau(t)\rho_3(t), \tau(t)\rho_4(t)\},$$

with $\xi(t)$ and $\rho_i(t), i = 1, ..., 4$, given in (3.27) and (3.28), and adding the following zero-valued term to the functional derivative

$$2\tilde{\xi}(t)^T M(\tilde{\mathcal{C}}_{90} + \tau(t)\tilde{\mathcal{C}}_{91})\tilde{\xi}(t) = 0,$$

with M being a $13n \times 4n$ free matrix, and $\tilde{\mathcal{C}}_{90},\ \tilde{\mathcal{C}}_{91}$ given in (3.31). □

As previously mentioned, the proposed corollaries present numerical tractable LMI conditions to attest to the quadratic condition in (3.20). As a result of Lemma 3.10, Corollary 2 is less conservative, but it requires the determination of four new additional matrices D_i and G_i, for $i = 1, 2$. To summing up, the numerical complexity due to the application of Lemma 3.10 is significantly increased with the addition of $162n^2$ scalar decision variables. On the other hand, Corollary 3 requires the determination of just one additional matrix $M \in \mathbb{R}^{13n \times 4n}$, i. e., $52n^2$ scalar decision variables. Therefore, the improvement in Corollary 2 over Corollaries 1 and 3 came with a considerable increase in the numerical complexity, especially for high-dimensional systems.

Remark 3. *The LKF (3.13) can be simplified to obtain stability analysis conditions for time-delays systems when just part or no information on the bounds of the delay derivative is available. However, in those cases, there are no improvements over the existing results in the literature since we can no longer take advantage of convexity properties involving the bounds of the delay derivative.*

3.2.1 Numerical Examples

In this section, the stability of three benchmark systems drawn from the literature is investigated. The computations are carried out in MATLAB using the LMI parser YALMIP and the SDP solver MOSEK.

Example 3.5. *Consider the systems listed in Table 3.1. Given bounds for the delay derivative $\dot{\tau}(t) \in [\mu_1,\ \mu_2]$, setting $\mu_1 = -\mu_2$, we search for the maximum admissible delay upper bound τ_M such that the LMI conditions in Corollaries 1, 2, and 3 are feasible.*

The results are listed in Tables 3.4, 3.5 and 3.6, and are compared with those from some recent LMI methods presented in the literature. For ease of comparison, the main features of such methods are listed in Table 3.2. The number of decision variables, as a function of the system dimension, required by each method analyzed is shown in Table 3.3.

Table 3.1: System matrices.

System 1: $A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$

System 2: $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$

System 3: $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$

Table 3.2: Main features of the methods considered for comparison purpose.

Method	Integral inequality	Approach to deal with the quadratic function
Zhang et al. (2017b, Proposition 1)	2^{nd} order Bessel-Legendre	Lemma 3.7
Chen et al. (2018, Theorem 1)	3^{rd} order Bessel-Legendre	Lemma 3.7
Park et al. (2018, Theorem 3)	Auxiliary function	Matrix injection
Zhang et al. (2019, Proposition 2)	4^{th} order Bessel-Legendre	Matrix injection
Chen et al. (2019, Theorem 1)	2^{nd} order Bessel-Legendre	Improved version of Lemma 3.7
Zhang et al. (2020a, Theorem 1)	2^{nd} order Bessel-Legendre and improved triple-integral inequality	Lemma 3.10
Zhang et al. (2020c, Proposition 1)	4^{th} order Bessel-Legendre	-[2]

Table 3.3: Number of decision variables (NoV) as a function of the system dimension.

Method	NoV
Zhang et al. (2017b, Proposition 1)	$54.5n^2 + 6.5n$
Chen et al. (2018, Theorem 1)	$108n^2 + 12n$
Zhang et al. (2019, Proposition 2, N=4)	$216n^2 + 11n$
Chen et al. (2019, Theorem 1 (C3))	$104n^2 + 15n$
Park et al. (2018, Theorem 3)	$221.5n^2 + 12.5n$
Corollary 1	$73n^2 + 13n$
Corollary 3	$152n^2 + 13n$
Corollary 2	$235n^2 + 34n$
Zhang et al. (2020a, Theorem 1)	$249n^2 + 15.5n$
Zhang et al. (2020c, Proposition 1)	$367n^2 + 35n + 6$

[2]The stability criterion is formulated as a cubic function w.r.t. the time-varying delay. To evaluate such a function, the authors have generalized the proposed Lemma 3.10 to deal with any polynomial function.

Table 3.4: Maximum admissible upper bound τ_M of the delay $\tau(t)$ for given $\mu_1 = -\mu_2$ and number of decision variables (NoV). System 1.

Method	μ_2 0.1	0.5	0.8	NoV
Zhang et al. (2017b, Proposition 1)	4.910	3.233	2.789	231
Chen et al. (2018, Theorem 1)	4.942	3.309	2.882	456
Zhang et al. (2019, Proposition 2, N=4)	4.929	3.252	2.823	886
Chen et al. (2019, Theorem 1 (C3))	4.939	3.298	2.869	446
Park et al. (2018, Theorem 3)	4.944	3.305	2.850	911
Corollary 1	4.938	3.281	2.854	318
Corollary 3	4.938	3.302	2.875	634
Corollary 2	5.044	3.443	2.983	1008
Zhang et al. (2020a, Theorem 1)	5.084	3.482	3.005	1027
Zhang et al. (2020c, Proposition 1)	5.147	3.673	3.243	1544

Table 3.5: Maximum admissible upper bound τ_M of the delay $\tau(t)$ for given $\mu_1 = -\mu_2$. System 2.

Method	μ_2 0.1	0.2	0.5	0.8
Zhang et al. (2017b, Proposition 1)	7.230	4.556	2.509	1.940
Chen et al. (2018, Theorem 1)	7.400	4.795	2.717	2.089
Chen et al. (2019, Theorem 1 (C3))	7.401	4.765	2.709	2.091
Park et al. (2018, Theorem 3)	7.550	4.902	2.714	2.054
Corollary 1	7.307	4.655	2.612	2.023
Corollary 3	7.438	4.863	2.715	2.068
Corollary 2	7.685	4.969	2.774	2.117
Zhang et al. (2020a, Theorem 1)	7.714	5.003	2.809	2.149

Table 3.6: Maximum admissible upper bound τ_M of the delay $\tau(t)$ for given $\mu_1 = -\mu_2$. System 3.

Method	μ_2 0.05	0.1	0.5
Zhang et al. (2017b, Proposition 1)	2.637	2.474	2.042
Chen et al. (2018, Theorem 1)	2.657	2.511	2.116
Park et al. (2018, Theorem 3)	2.658	2.509	2.105
Corollary 1	2.641	2.488	2.090
Corollary 3	2.648	2.499	2.113
Corollary 2	2.675	2.536	2.137

Firstly, the results from the proposed three corollaries are compared. One can see that Corollary 2 leads to the least conservative results for all systems. This benefit, however, comes at the cost of increased computational complexity. Moreover, note that Corollary 3 can produce less conservative results than Corollary 1. This improvement is due to the additional matrix inserted in the problem. Comparing the results from Corollaries 1, 2, and 3, to those from recent literature, we note that the proposed conditions presented in Corollary 1 outperform some of the results, which reveals the conservatism reduction merit of the proposed functional.

Corollary 3 and Park et al. (2018, Theorem 3) use the same approach to deal with the quadratic function, but the latter achieved less conservative results in all tests. The improvement of Park et al. (2018, Theorem 3) is due to the LKF proposed in that paper.

On the other hand, the results of Corollary 2 are less conservative than all of the methods listed in Table 3.2 published before 2020. It is noteworthy that evaluating exactly a criterion formulated as a quadratic function and based on the second-order Bessel-Legendre inequality may leads to less conservative results than using a criterion based on the Bessel-Legendre inequality of higher orders, as in Chen et al. (2018); Zhang et al. (2019).

Finally, it is noticeable that results of Zhang et al. (2020a, Theorem 1) and Zhang et al. (2020c, Proposition 1) outperform those of Corollary 2. Such newer methods reveal the added scientific value of ours results presented in de Oliveira and Souza (2020a), since the latter recent papers have used some of our ideas presented on it. In Zhang et al. (2020a), the proposed Lemma 3.10 is used to evaluate a stability criterion formulated as a quadratic function w.r.t. the delay, whereas in Zhang et al. (2020c), the authors have extended the result of Lemma 3.10 to deal with any polynomial function and have used it to evaluate a stability criterion formulated as a cubic function w.r.t. delay.

3.3 Systems with multiple time-varying delays

The numerical examples presented in the previous section have shown that stability criteria obtained using improved integral inequalities and formulated as quadratic/cubic functions w.r.t. the time-varying delay can provide less conservative results when the function is evaluated using an exact method. One of the main drawbacks of this approach is that the stability conditions cannot be easily extend to the case of systems with multiple time-varying delays, since it is difficult to evaluate non-conservatively the sum of several polynomial functions. On the other hand, stability conditions formulated as an affine function w.r.t. the delay, as in Corollary 3, are suitable to such extension. In this section, it is presented new stability criteria for systems with multiple time-varying delays inspired by the conditions in Corollary 3.

3.3.1 Preliminaries concepts and problem description

Most of the existing stability conditions for systems with a single delay can be extended to the case of systems with multiple time-delays. However, a direct extension of these conditions may leads to a conservative stability criterion. The works on stability of system with multiple time-delays try to reduce the conservatism by exploiting all possible information for the relationship among the time-delays. The first attempt in this direction is due to He et al. (2006). The authors have proposed a new Lyapunov-Krasovskii functional with a double integral term of the form

$$\sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \int_{-\tau_{M,j}}^{-\tau_{M,i}} \int_{t+\theta}^{t} \dot{x}^T(s) R_{i,j} \dot{x}(s) ds d\theta, \tag{3.32}$$

instead of the usual term

$$\sum_{i=1}^{N} \int_{-\tau_{M,i}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R_i \dot{x}(s) ds d\theta, \tag{3.33}$$

where N is the number of delays of the system. The idea behind the term (3.32) is to take the relationship between each pair of constant delays into account, which results in a less conservative stability condition. By using the Jensen integral inequality, the result in He et al. (2006) was extended for the case of multiple time-varying delays in Liao et al. (2014). The authors also considered the relationship between the current delayed states and the states subject to the maximum allowable bound of each time-varying delay. Recently, based on an augmented Lyapunov-Krasovskii functional and the Wirtinger-based integral inequality, Ko et al. (2018) have proposed a partitioning approach in the derivative of the double integral term (3.33) to exploit all possible information of the relationships among the current states, the current delayed states, and the states subject to the maximum allowable delay bounds. As a result, a less conservative stability criterion was obtained. However, from the work of Ko et al. (2018) it is unclear what has been more important in reducing the conservatism, the use of an augmented functional with an improved integral inequality (Wirtinger inequality) or the partitioning approach.

We have noted that some of the relationships among the time-delays, introduced by the methods mentioned above, can appear implicitly in the LKF derivative if an appropriate augmented LKF is used. Considering this, the result in Corollary 3 is adapted to deal with systems with multiple time-varying delays by appropriate changing the functional (3.13). It is important to mention that the results presented in the next sections are not an exact extension of Corollary 3. Some terms of functional (3.13) have been removed to reduce the computational complexity of the resulting LMI conditions. A discussion about the advantages and drawbacks of using such an approach is also provided.

To begin with, consider the following system with multiple time-varying delays:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t - \tau_i(t)), \\ x(\vartheta) = \varphi(\vartheta), \;\; \vartheta : [-\tau_M, \; 0] \to \mathbb{R}^n \end{cases} \tag{3.34}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector; $A_i \in \mathbb{R}^{n \times n}, i = 0, 1, .., N$, are constant matrices; and $\varphi(\vartheta)$ is an initial condition, with $\tau_M = \max_{i=1,\ldots,N}(\tau_{M,i})$. Such kind of delay differential equation has been used to model numerous systems, as vehicle platoons in Souza et al. (2019).

Hereafter, it will be assumed that the time-varying delays $\tau_i(t)$ satisfy one of the following constraints:

Case i)**:** $\tau_i(t)$ is continuous and the lower and upper bounds of $\dot{\tau}_i(t)$ are known

$$0 \leq \tau_i(t) \leq \tau_{M,i}, \;\; \mu_{m,i} \leq \dot{\tau}_i(t) \leq \mu_{M,i} < 1, \tag{3.35}$$

Case ii)**:** $\tau_i(t)$ is continuous and only the upper bound of $\dot{\tau}_i(t)$ is known

$$0 \leq \tau_i(t) \leq \tau_{M,i}, \;\; \dot{\tau}_i(t) \leq \mu_{M,i}, \tag{3.36}$$

with $\tau_{M,i}, \; \mu_{m,i}$, and $\mu_{M,i}$ being known real constants for $i = 1, 2, ..., N$.

Note that in **Case** i) the delay functions satisfy

$$(\tau_1(t), ..., \tau_N(t), \dot{\tau}_1(t), ..., \dot{\tau}_N(t)) \in \mathcal{H} \subset \mathbb{R}^{2N}, \quad \forall t > 0, \tag{3.37}$$

where $\mathcal{H}$ is the polytope $[0, \; \tau_{M,1}] \times ... \times [0, \; \tau_{M,N}] \times [\mu_{m,1}, \; \mu_{M,1}] \times ... \times [\mu_{m,N}, \; \mu_{M,N}]$.

It is important to mention that Liao et al. (2014) and Ko et al. (2018) consider only systems under conditions in **Case** ii).

3.3.2 Stability criterion under Case i)

The stability of the time-delay system (3.34), in which all $\tau_i(t)$ satisfy the conditions in (3.35), is studied by choosing the following quadratic Lyapunov-Krasovskii functional:

$$\bar{V}_{i)}(t, x_t, \dot{x}_t) = \bar{V}_1(t, x_t) + \bar{V}_2(t, x_t) + \bar{V}_3(t, \dot{x}_t), \tag{3.38}$$

where $x_t = x(t + \vartheta), \; \vartheta \in [-\tau_M, \; 0]$, and

$$\bar{V}_1(t, x_t) = \bar{\eta}_1^T(t) P \bar{\eta}_1(t), \tag{3.39}$$

$$\bar{V}_2(t,x_t) = \sum_{i=1}^{N} \left(\int_{t-\tau_i(t)}^{t} \bar{\eta}_2^T(t,s) Q_{1,i} \bar{\eta}_2(t,s) ds + \int_{t-\tau_{M,i}}^{t-\tau_i(t)} \bar{\eta}_2^T(t,s) Q_{2,i} \bar{\eta}_2(t,s) ds \right), \tag{3.40}$$

$$\begin{aligned} \bar{V}_3(t,\dot{x}_t) = \sum_{i=1}^{N} \tau_{M,i} \Bigg(& \int_{t-\tau_{M,i}}^{t-\tau_i(t)} (\tau_{M,i} - t + s) \dot{x}^T(s) R_{1,i} \dot{x}(s) ds \\ & + \int_{t-\tau_i(t)}^{t} (\tau_{M,i} - t + s) \dot{x}^T(s) R_{2,i} \dot{x}(s) ds \Bigg), \end{aligned} \tag{3.41}$$

with

$$\begin{aligned} \bar{\eta}_1(t) = \text{col}\{ & x(t), x(t-\tau_1(t)), ..., x(t-\tau_N(t)), x(t-\tau_{M,1}), ..., x(t-\tau_{M,N}), \\ & \int_{t-\tau_1(t)}^{t} x(s)ds, ..., \int_{t-\tau_N(t)}^{t} x(s)ds, \int_{t-\tau_{M,1}}^{t-\tau_1(t)} x(s)ds, ..., \int_{t-\tau_{M,N}}^{t-\tau_N(t)} x(s)ds, \\ & \int_{t-\tau_{M,1}}^{t} (t-s)x(s)ds, ..., \int_{t-\tau_{M,N}}^{t} (t-s)x(s)ds \}, \end{aligned}$$

$$\bar{\eta}_2(t,s) = \text{col}\{\dot{x}(s), x(s), x(t), \int_s^t x(\theta)d\theta\}.$$

As mentioned before, (3.38) is not an exact extension of functional (3.13) to the case of multiple time-varying delays. Some changes are introduced in the first and second terms of (3.13) to reduce the computational complexity of the conditions. More specifically, the terms $x(t-\tau(t))$ and $x(t-\tau_M)$ were removed from the augmented vector $\eta_3(t,s)$ in (3.16) and it was set $\eta_3(t,s) = \eta_4(t,s)$. It was also set $\mathcal{P}(t) = P$ and merged the augmented vectors $\eta_1(t)$ and $\eta_2(t)$ into a unique vector. And finally, to partially compensate the choice $\eta_3(t,s) = \eta_4(t,s)$, the term $\int_{t-\tau_M}^{t}(t-s)x(s)ds$ was included in the new augmented vector. The inclusion of $\int_{t-\tau_M}^{t}(t-s)x(s)ds$ leads to criterion formulated as a cubic function w.r.t. to the time-varying delay $\tau(t)$, but it can be avoided employing the same approach used in Corollary 3 to deal with the quadratic function. The influence of these changes in terms of the stability analysis of systems with a single time-varying delay is shown in Example 3.6.

Considering the proposed functional candidate (3.38) a stability condition for system (3.34) is given in the next theorem.

Theorem 3.3. *Let $\tau_{M,i}$, $\mu_{m,i}$, and $\mu_{M,i}$ be given for $i = 1,2,...,N$. The time-delay system (3.34), under constraints in (3.35), is asymptotically stable if there exist matrices $P \in \mathbb{S}_+^{(1+5N)n}$, $Q_{1,i}$, $Q_{2,i} \in \mathbb{S}_+^{4n}$, $\mathbb{R}_{1,i}$, $R_{2,i} \in \mathbb{S}_+^{n}$, $M \in \mathbb{R}^{(12N+1)n \times 4Nn}$, $X_{1,i}$, $X_{2,i} \in \mathbb{S}^{3n}$, and $Y_{1,i}$, $Y_{2,i} \in \mathbb{R}^{3n\times 3n}$, for $i = 1,2,...,N$, such that the following conditions hold for $i = 1,2,...,N$*

$$\begin{bmatrix} \tilde{R}_{1,i} - X_{1,i} & Y_{1,i} \\ * & \tilde{R}_{2,i} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R}_{1,i} & Y_{2,i} \\ * & \tilde{R}_{2,i} - X_{2,i} \end{bmatrix} \succeq 0, \tag{3.42}$$

and the following condition holds at the vertices of $\mathcal{H}$, i.e., for all the combinations of $\tau_i(t) = 0,\ \tau_{M,i}$ *and* $\dot{\tau}_i(t) = \mu_{m,i},\ \mu_{M,i}\ (i = 1, ..., N)$*:*

$$\bar{\Upsilon}(\tau_1(t), ..., \tau_N(t), \dot{\tau}_1(t), ..., \dot{\tau}_N(t)) \prec 0, \tag{3.43}$$

in which $\tilde{R}_{1,i} = \mathrm{diag}(R_{1,i}, 3R_{1,i}, 5R_{1,i})$, $\tilde{R}_{2,i} = \mathrm{diag}(R_{2,i}, 3R_{2,i}, 5R_{2,i})$, *for* $i = 1, 2, ..., N$, *and*

$$\begin{aligned}
&\bar{\Upsilon}(\tau_1(t), ..., \tau_N(t), \dot{\tau}_1(t), ..., \dot{\tau}_N(t)) = \mathrm{sym}\{\bar{\mathcal{D}}(\dot{\tau}_i(t))^T P(\bar{\mathcal{C}}_{10} + \sum_{i=1}^{N} \tau_i(t)\bar{\mathcal{C}}_{11,i})\} + \\
&\quad + \sum_{i=1}^{N} \big[\bar{\mathcal{C}}_{2,i}^T Q_{1,i}\bar{\mathcal{C}}_{2,i} + (1 - \dot{\tau}_i(t))\bar{\mathcal{C}}_{3,i}^T (Q_{2,i} - Q_{1,i})\bar{\mathcal{C}}_{3,i} - \bar{\mathcal{C}}_{6,i}^T Q_{2,i}\bar{\mathcal{C}}_{6,i} \\
&\quad + \mathrm{sym}\{(\bar{\mathcal{C}}_{40,i} + \tau_i(t)\bar{\mathcal{C}}_{41,i})^T Q_{1,i}\bar{\mathcal{C}}_5\} + \mathrm{sym}\{(\bar{\mathcal{C}}_{70,i} + \tau_i(t)\bar{\mathcal{C}}_{71,i})^T Q_{2,i}\bar{\mathcal{C}}_5\} \\
&\quad + \tau_{M,i}^2 \bar{\mathcal{C}}_0^T R_{2,i}\bar{\mathcal{C}}_0 + \tau_{M,i}(1 - \dot{\tau}_i(t))(\tau_{M,i} - \tau_i(t))\bar{e}_{6N+1+i}^T (R_{1,i} - R_{2,i})\bar{e}_{6N+1+i} \\
&\quad - \bar{\Gamma}_{1,i}^T[\tilde{R}_{1,i} + (1 - \alpha_i)X_{1,i}]\bar{\Gamma}_{1,i} - \bar{\Gamma}_{2,i}^T(\tilde{R}_{2,i} + \alpha_i X_{2,i})\bar{\Gamma}_{2,i} \\
&\quad - \mathrm{sym}\{\bar{\Gamma}_{1,i}^T[\alpha_i Y_{1,i} + (1 - \alpha_i)Y_{2,i}]\bar{\Gamma}_{2,i}\} + M(\bar{\mathcal{C}}_{80} + \sum_{i=1}^{N} \tau_i(t)\bar{\mathcal{C}}_{81,i})\ \big],
\end{aligned} \tag{3.44}$$

with

$$\alpha_i = (\tau_{M,i} - \tau_i(t))/\tau_{M,i}, \quad \bar{\mathcal{C}}_0 = \sum_{i=1}^{N+1} A_{i-1}\bar{e}_i, \tag{3.45}$$

$$\begin{aligned}
&\bar{\mathcal{C}}_{10} = \mathrm{col}\{\bar{e}_1, \bar{e}_2, ..., \bar{e}_{N+1}, \bar{e}_{N+2}, ..., \bar{e}_{2N+1}, \bar{e}_{10N+2}, ..., \bar{e}_{11N+1}, \bar{e}_{8N+2}, ..., \bar{e}_{9N+1}, \\
&\qquad \tau_{M,1}\bar{e}_{9N+2}, ..., \tau_{M,N}\bar{e}_{10N+1}\}, \\
&\bar{\mathcal{C}}_{11,i} = \mathrm{col}\{\underbrace{\bar{e}_0, .., \bar{e}_0,}_{4N+i} \bar{e}_{11N+1+i} - \bar{e}_{9N+1+i} + \bar{e}_{8N+1+i}, \underbrace{\bar{e}_0, .., \bar{e}_0,}_{N-i}\}, \\
&\bar{\mathcal{C}}_{2,i} = \mathrm{col}\{\bar{\mathcal{C}}_0, \bar{e}_1, \bar{e}_1, \bar{e}_0\}, \qquad \bar{\mathcal{C}}_{3,i} = \mathrm{col}\{\bar{e}_{6N+1+i}, \bar{e}_{1+i}, \bar{e}_1, \bar{e}_{10N+1+i}\}, \\
&\bar{\mathcal{C}}_{40,i} = \mathrm{col}\{\bar{e}_1 - \bar{e}_{1+i}, \bar{e}_{10N+1+i}, \bar{e}_0, \bar{e}_0\}, \quad \bar{\mathcal{C}}_{41,i} = \mathrm{col}\{\bar{e}_0, \bar{e}_0, \bar{e}_1, \bar{e}_{10N+1+i} - \bar{e}_{11N+1+i}\}, \\
&\bar{\mathcal{C}}_5 = \mathrm{col}\{\bar{e}_0, \bar{e}_0, \bar{\mathcal{C}}_0, \bar{e}_1\}, \qquad \bar{\mathcal{C}}_{6,i} = \mathrm{col}\{\bar{e}_{7N+1+i}, \bar{e}_{N+1+i}, \bar{e}_1, \bar{e}_{8N+1+i} + \bar{e}_{10N+1+i}\}, \\
&\bar{\mathcal{C}}_{70,i} = \mathrm{col}\{\bar{e}_{1+i} - \bar{e}_{N+1+i}, \bar{e}_{8N+1+i}, \tau_{M,i}\bar{e}_1, \tau_{M,i}(\bar{e}_{8N+1+i} - \bar{e}_{9N+1+i} + \bar{e}_{10N+1+i})\}, \\
&\bar{\mathcal{C}}_{71,i} = \mathrm{col}\{\bar{e}_0, \bar{e}_0, -\bar{e}_1, -\bar{e}_{8N+1+i} + \bar{e}_{9N+1+i} - \bar{e}_{10N+1+i}\}, \\
&\bar{\mathcal{C}}_{80} = \mathrm{col}\{\tau_{M,1}\bar{e}_{2N+2} - \bar{e}_{8N+2}, ..., \tau_{M,N}\bar{e}_{3N+1} - \bar{e}_{9N+1}, \tau_{M,1}\bar{e}_{3N+2} - \bar{e}_{9N+2}, ..., \\
&\qquad \tau_{M,N}\bar{e}_{4N+1} - \bar{e}_{10N+1}, -\bar{e}_{10N+2}, ..., -\bar{e}_{11N+1}, -\bar{e}_{11N+2}, ..., -\bar{e}_{12N+1}, \}, \\
&\bar{\mathcal{C}}_{81,i} = \mathrm{col}\{\underbrace{\bar{e}_0, .., \bar{e}_0,}_{i-1} -\bar{e}_{2N+1+i}, \underbrace{\bar{e}_0, .., \bar{e}_0,}_{N-i} \underbrace{\bar{e}_0, .., \bar{e}_0,}_{i-1} -\bar{e}_{3N+1+i} \underbrace{\bar{e}_0, .., \bar{e}_0,}_{N-i} \underbrace{\bar{e}_0, .., \bar{e}_0,}_{i-1} \bar{e}_{4N+1+i}, \underbrace{\bar{e}_0, .., \bar{e}_0}_{N-i} \\
&\qquad \underbrace{\bar{e}_0, .., \bar{e}_0,}_{i-1} \bar{e}_{5N+1+i}, \underbrace{\bar{e}_0, .., \bar{e}_0,}_{N-i}\},
\end{aligned} \tag{3.46}$$

$$\begin{aligned}\bar{\mathcal{D}}(\dot{\tau}_i(t)) = \text{col}\{&\bar{\mathcal{C}}_0, (1-\dot{\tau}_1(t))\bar{e}_{6N+2}, ..., (1-\dot{\tau}_N(t))\bar{e}_{7N+1}, \bar{e}_{7N+2}, ..., \bar{e}_{8N+1}, \bar{e}_1 - (1-\dot{\tau}_1(t))\bar{e}_2,\\ &..., \bar{e}_1 - (1-\dot{\tau}_N(t))\bar{e}_{N+1}, (1-\dot{\tau}_1(t))\bar{e}_2 - \bar{e}_{N+2}, ..., (1-\dot{\tau}_N(t))\bar{e}_{N+1} - \bar{e}_{2N+1},\\ &-\tau_{M,1}\bar{e}_{N+2} + \bar{e}_{10N+2} + \bar{e}_{8N+2}, ..., -\tau_{M,N}\bar{e}_{2N+1} + \bar{e}_{11N+1} + \bar{e}_{9N+1}\},\\ \bar{\Gamma}_{1,i} = \text{col}\{&\bar{e}_{1+i} - \bar{e}_{N+1+i}, \bar{e}_{1+i} + \bar{e}_{N+1+i} - 2\bar{e}_{2N+1+i}, \bar{e}_{1+i} - \bar{e}_{N+1+i} - 6\bar{e}_{2N+1+i} + 12\bar{e}_{3N+1+i}\},\\ \bar{\Gamma}_{2,i} = \text{col}\{&\bar{e}_1 - \bar{e}_{1+i}, \bar{e}_1 + \bar{e}_{1+i} - 2\bar{e}_{4N+1+i}, \bar{e}_1 - \bar{e}_{1+i} - 6\bar{e}_{4N+1+i} + 12\bar{e}_{5N+1+i}\},\end{aligned}$$

and $\bar{e}_i = [0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(12N+1-i)n}]$, *for* $i = 1, 2, ..., 12N+1$. $\diamond$

Proof. The proof is similar to the one of Theorem 3.2 and Corollary 3, and it will be only sketched. Firstly, the following augmented vector is introduced

$$\begin{aligned}\bar{\xi}(t) = \text{col}\{&x(t), x(t-\tau_1(t)), ..., x(t-\tau_N(t)), x(t-\tau_{M,1}), ..., x(t-\tau_{M,N}), \rho_{1,1}(t), ..., \rho_{1,N}(t),\\ &\rho_{2,1}(t), ..., \rho_{2,N}(t), \rho_{3,1}(t), ..., \rho_{3,N}(t), \rho_{4,1}(t), ..., \rho_{4,N}(t), \dot{x}(t-\tau_1(t)), ..., \dot{x}(t-\tau_N(t)),\\ &\dot{x}(t-\tau_{M,1}), ..., \dot{x}(t-\tau_{M,N}), (\tau_{M,1}-\tau_1(t))\rho_{1,1}(t), ..., (\tau_{M,N}-\tau_N(t))\rho_{1,N}(t),\\ &(\tau_{M,1}-\tau_1(t))\rho_{2,1}(t), ..., (\tau_{M,N}-\tau_N(t))\rho_{2,N}(t), \tau_1(t)\rho_{3,1}(t), ..., \tau_N(t)\rho_{3,N}(t),\\ &\tau_1(t)\rho_{4,1}(t), ..., \tau_N(t)\rho_{4,N}(t)\}\end{aligned} \tag{3.47}$$

with

$$\begin{aligned}\rho_{1,i}(t) &= \int_{t-\tau_{M,i}}^{t-\tau_i(t)} \frac{x(s)}{\tau_{M,i}-\tau_i(t)} ds, \quad \rho_{2,i}(t) = \int_{t-\tau_{M,i}}^{t-\tau_i(t)} \frac{(t-\tau_i(t)-s)x(s)}{(\tau_{M,i}-\tau_i(t))^2} ds,\\ \rho_{3,i}(t) &= \int_{t-\tau_i(t)}^{t} \frac{x(s)}{\tau_i(t)} ds, \quad \rho_{4,i}(t) = \int_{t-\tau_i(t)}^{t} \frac{(t-s)x(s)}{\tau_i(t)^2} ds.\end{aligned} \tag{3.48}$$

The positiveness of the proposed Lyapunov-Krasovskii functional (3.38) is guaranteed if:

$$\begin{gathered}P \succ 0,\\ Q_{1,i} \succ 0, \;\; Q_{2,i} \succ 0, \;\; R_{1,i} \succ 0, \text{ and } R_{2,i} \succ 0, \;\; \text{for } \; i = 1, 2, ..., N.\end{gathered}$$

After applying Lemma 3.5 with $d = 2$, Lemma 3.6, and considering the zero-valued term

$$\bar{\xi}^T(t)\left[M(\bar{\mathcal{C}}_{80} + \sum_{i=1}^{N}\tau_i(t)\bar{\mathcal{C}}_{81,i})\right]\bar{\xi}(t) = 0$$

for any matrix $M \in \mathbb{R}^{(12N+1)n\times 4Nn}$, the time-derivative of the functional candidate 3.38, along the system trajectories, can be bounded by

$$\dot{V}_{i)}(t, x_t, \dot{x}_t) \leq \bar{\xi}^T(t)\bar{\Upsilon}(\tau_1(t), ..., \tau_N(t), \dot{\tau}_1(t), ..., \dot{\tau}_N(t))\bar{\xi}(t),$$

with $\bar{\Upsilon}(\tau_i(t), \dot{\tau}_i(t))$ given (3.44). Thus, the Lyapunov-Krasovskii derivative condition is satisfied if

$$\bar{\Upsilon}(\tau_1(t), ..., \tau_N(t), \dot{\tau}_1(t), ..., \dot{\tau}_N(t)) \prec 0. \tag{3.49}$$

Note that $\bar{\Upsilon}(\tau_i(t), \dot{\tau}_i(t))$ is a multi-affine function on $\tau_i(t)$ and $\dot{\tau}_i(t)$ and, therefore, convex in these parameters (Belta and Habets, 2006). Hence, if LMI (3.43) is satisfied at the vertices of $\mathcal{H}$, then (3.49) also holds for all $(\tau_1(t), ..., \tau_N(t), \dot{\tau}_i(t), ..., \dot{\tau}_N(t)) \in \mathcal{H}$, which concludes the proof. □

3.3.3 Stability criterion under Case $ii)$

When the information on the lower bound of $\dot{\tau}_i(t)$ is not known, a similar result to Theorem 3.3 can be established by properly modifying some terms of the functional (3.38). In this case, however, we cannot take advantage of convexity properties of the reciprocally convex inequality in Lemma 3.6, since the lower bound of $\dot{\tau}_i(t)$ is not available. To obtain a stability criterion for time-delay systems under conditions in (3.36), the following LKF candidate is proposed:

$$V_{ii)}(t, x_t, \dot{x}_t) = \hat{V}_1(t, x_t) + \hat{V}_2(t, x_t) + \hat{V}_3(t, \dot{x}_t), \tag{3.50}$$

in which

$$\hat{V}_1(t, x_t) = \hat{\eta}_1^T(t) P \hat{\eta}_1(t), \tag{3.51}$$

$$\hat{V}_2(t, x_t) = \sum_{i=1}^{N} \left(\int_{t-\tau_i(t)}^{t} \hat{\eta}_2^T(t,s) Q_{1,i} \hat{\eta}_2(t,s) ds + \int_{t-\tau_{M,i}}^{t-\tau_i(t)} \hat{\eta}_2^T(t,s) Q_{2,i} \hat{\eta}_2(t,s) ds \right), \tag{3.52}$$

$$\hat{V}_3(t, \dot{x}_t) = \sum_{i=1}^{N} \tau_{M,i} \left(\int_{-\tau_{M,i}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R_i \dot{x}(s) ds d\theta \right), \tag{3.53}$$

with

$$\hat{\eta}_1(t) = \text{col}\{\int_{t-\tau_{M,1}}^{t} x(s)ds, ..., \int_{t-\tau_{M,N}}^{t} x(s)ds, \int_{t-\tau_{M,1}}^{t} (t-s)x(s)ds, ..., \int_{t-\tau_{M,N}}^{t} (t-s)x(s)ds\}.$$

$$\hat{\eta}_2(t) = \text{col}\{x(s), x(t), \int_s^t x(\theta) d\theta\}.$$

In light of the functional in (3.50), the result stated below is obtained.

Theorem 3.4. *Let $\tau_{M,i}$ and $\mu_{M,i}$ be given for $i = 1, 2, ..., N$. The time-delay system (3.34), under constraints in (3.36), is asymptotically stable if there exist matrices $P \in \mathbb{S}_+^{2Nn}$, $Q_{1,i}$, $Q_{2,i} \in \mathbb{S}_+^{3n}$, $R_i \in \mathbb{S}_+^{n}$, $M \in \mathbb{R}^{(10N+1)n \times 4Nn}$, $X_{1,i}$, $X_{2,i} \in \mathbb{S}^{3n}$, and $Y_{1,i}$, $Y_{2,i} \in \mathbb{R}^{3n \times 3n}$, for $i = 1, 2, ..., N$, such that the following conditions hold for $i = 1, 2, ..., N$*

$$Q_{1,i} \succ Q_{2,i} \tag{3.54}$$

$$\begin{bmatrix} \tilde{R}_i - X_{1,i} & Y_{1,i} \\ * & \tilde{R}_i \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R}_i & Y_{2,i} \\ * & \tilde{R}_i - X_{2,i} \end{bmatrix} \succeq 0, \tag{3.55}$$

and the following condition holds for all the combinations $\tau_i(t) = 0$, $\tau_i(t) = \tau_{M,i}$ *(*$i = 1, 2..., N$*)*

$$\hat{\Upsilon}(\tau_1(t), ..., \tau_N(t)) \prec 0, \tag{3.56}$$

in which $\tilde{R}_i = \mathrm{diag}(R_i, 3R_i, 5R_i)$, *for* $i = 1, 2, ..., N$, *and*

$$\begin{aligned}
\hat{\Upsilon}(\tau_1(t), ..., \tau_N(t)) &= \mathrm{sym}\{\hat{\mathcal{C}}_1^T P(\hat{\mathcal{C}}_{20} + \sum_{i=1}^{N} \tau_i(t)\hat{\mathcal{C}}_{21,i})\} + \mathrm{sym}\{M(\hat{\mathcal{C}}_{90} + \sum_{i=1}^{N} \tau_i(t)\hat{\mathcal{C}}_{91,i})\} \\
&+ \sum_{i=1}^{N} \Big[\hat{\mathcal{C}}_3^T Q_{1,i}\hat{\mathcal{C}}_3 + (1-\mu_{M,i})\hat{\mathcal{C}}_{4,i}^T(Q_{1,i} - Q_{2,i})\hat{\mathcal{C}}_{4,i} + \mathrm{sym}\{(\hat{\mathcal{C}}_{50,i} + \tau_i(t)\hat{\mathcal{C}}_{51,i})^T Q_{1,i}\hat{\mathcal{C}}_6\} \\
&- \hat{\mathcal{C}}_{7,i}^T Q_{2,i}\hat{\mathcal{C}}_{7,i} + \mathrm{sym}\{(\hat{\mathcal{C}}_{80,i} + \tau_i(t)\hat{\mathcal{C}}_{81,i})^T Q_{2,i}\hat{\mathcal{C}}_6\} + \tau_{M,i}^2\hat{\mathcal{C}}_0^T R_i\hat{\mathcal{C}}_0 - \hat{\Gamma}_{2,i}^T(\tilde{R}_i + \alpha_i X_{2,i})\hat{\Gamma}_{2,i} \\
&- \hat{\Gamma}_{1,i}^T[\tilde{R}_i + (1-\alpha_i)X_{1,i}]\hat{\Gamma}_{1,i} - \mathrm{sym}\{\hat{\Gamma}_{1,i}^T[\alpha_i Y_{1,i} + (1-\alpha_i)Y_{2,i}]\hat{\Gamma}_{2,i}\}\ \Big],
\end{aligned}$$

with α_i *given in* (3.45), $\hat{\mathcal{C}}_0 = \sum_{i=1}^{N+1} A_{i-1}\hat{e}_i$, *and*

$$\begin{aligned}
\hat{\mathcal{C}}_1 &= \mathrm{col}\{\hat{\mathcal{C}}_0, \hat{e}_1 - \hat{e}_{N+2}, ..., \hat{e}_1 - \hat{e}_{2N+1}, -\tau_{M,1}\hat{e}_{N+2} + \hat{e}_{8N+2} + \hat{e}_{6N+2}, ..., \\
&\qquad - \tau_{M,N}\hat{e}_{2N+1} + \hat{e}_{9N+1} + \hat{e}_{7N+1}\}, \\
\hat{\mathcal{C}}_{20} &= \mathrm{col}\{\hat{e}_1, \hat{e}_{8N+2} + \hat{e}_{6N+2}, ..., \hat{e}_{9N+1} + \hat{e}_{7N+1}, \tau_{M,1}\hat{e}_{7N+2}, ..., \tau_{M,N}\hat{e}_{8N+1}\}, \\
\hat{\mathcal{C}}_{21,i} &= \mathrm{col}\{\underbrace{\hat{e}_0, .., \hat{e}_0,}_{i-1} \hat{e}_{9N+1+i} - \hat{e}_{7N+1+i} + \hat{e}_{6N+1+i}, \underbrace{\hat{e}_0, .., \hat{e}_0}_{N-i}\}, \\
\hat{\mathcal{C}}_3 &= \mathrm{col}\{\hat{e}_1, \hat{e}_1, \hat{e}_0\}, \qquad \hat{\mathcal{C}}_{4,i} = \mathrm{col}\{\hat{e}_{1+i}, \hat{e}_1, \hat{e}_{8N+1+i}\}, \\
\hat{\mathcal{C}}_{50,i} &= \mathrm{col}\{\hat{e}_{8N+1+i}, \hat{e}_0, \hat{e}_0\}, \qquad \hat{\mathcal{C}}_{51,i} = \mathrm{col}\{\hat{e}_0, \hat{e}_1, \hat{e}_{8N+1+i} - \hat{e}_{9N+1+i}\}, \\
\hat{\mathcal{C}}_6 &= \mathrm{col}\{\hat{e}_0, \hat{\mathcal{C}}_0, \hat{e}_1\}, \qquad \hat{\mathcal{C}}_{7,i} = \mathrm{col}\{\hat{e}_{N+1+i}, \hat{e}_1, \hat{e}_{8N+1+i} + \hat{e}_{6N+1+i}\}, \\
\hat{\mathcal{C}}_{80,i} &= \mathrm{col}\{\hat{e}_{6N+1+i}, \tau_{M,i}\hat{e}_1, \tau_{M,i}(\hat{e}_{8N+1+i} - \hat{e}_{7N+1+i} + \hat{e}_{6N+1+i})\}, \\
\hat{\mathcal{C}}_{81,i} &= \mathrm{col}\{\hat{e}_0, -\hat{e}_1, -\hat{e}_{8N+1+i} + \hat{e}_{7N+1+i} - \hat{e}_{6N+1+i}\}, \\
\hat{\mathcal{C}}_{90} &= \mathrm{col}\{\tau_{M,1}\hat{e}_{2N+2} - \hat{e}_{6N+2}, ..., \tau_{M,N}\hat{e}_{3N+1} - \hat{e}_{7N+1}, \tau_{M,1}\hat{e}_{3N+2} - \hat{e}_{7N+2}, ..., \\
&\qquad \tau_{M,N}\hat{e}_{4N+1} - \hat{e}_{8N+1}, -\hat{e}_{8N+2}, ..., -\hat{e}_{9N+1}, -\hat{e}_{9N+2}, ..., -\hat{e}_{10N+1}\}, \\
\hat{\mathcal{C}}_{91,i} &= \mathrm{col}\{\underbrace{\hat{e}_0, .., \hat{e}_0,}_{i-1} -\hat{e}_{2N+1+i}, \underbrace{\hat{e}_0, .., \hat{e}_0,}_{N-i} \underbrace{\hat{e}_0, .., \hat{e}_0,}_{i-1} -\hat{e}_{3N+1+i}, \underbrace{\hat{e}_0, .., \hat{e}_0,}_{N-i} \underbrace{\hat{e}_0, .., \hat{e}_0,}_{i-1} \hat{e}_{5N+1+i}, \underbrace{\hat{e}_0, .., \hat{e}_0,}_{N-i} \\
&\qquad \underbrace{\hat{e}_0, .., \hat{e}_0,}_{i-1} \hat{e}_{6N+1+i} \underbrace{\hat{e}_0, .., \hat{e}_0}_{N-i}\}, \\
\hat{\Gamma}_{1,i} &= \mathrm{col}\{\hat{e}_{1+i} - \hat{e}_{N+1+i}, \hat{e}_{1+i} + \hat{e}_{N+1+i} - 2\hat{e}_{2N+1+i}, \hat{e}_{1+i} - \hat{e}_{N+1+i} - 6\hat{e}_{2N+1+i} + 12\hat{e}_{3N+1+i}\}, \\
\hat{\Gamma}_{2,i} &= \mathrm{col}\{\hat{e}_1 - \hat{e}_{1+i}, \hat{e}_1 + \hat{e}_{1+i} - 2\hat{e}_{4N+1+i}, \hat{e}_1 - \hat{e}_{1+i} - 6\hat{e}_{4N+1+i} + 12\hat{e}_{5N+1+i}\},
\end{aligned} \tag{3.57}$$

and $\hat{e}_i = [0_{n\times(i-1)n} \ \ I_n \ \ 0_{n\times(10N+1-i)n}]$, *for* $i = 1, 2, ..., 10N+1$, *and* $\hat{e}_0 = 0_{n\times(10N+1)n}$. ⋄

Proof. The proof follows the same lines of the proof of Theorem 3.3. The positiveness of the LKF candidate (3.50) is guaranteed if:

$$P \succ 0,$$
$$Q_{1,i} \succ 0, \;\; Q_{2,i} \succ 0, \;\; \text{and } R_i \succ 0, \;\; \text{for } i = 1, 2, ..., N.$$

By defining the augmented vector

$$\begin{aligned}\hat{\xi}(t) = \mathrm{col}\{&x(t), x(t-h_1(t)), ..., x(t-\tau_N(t)), x(t-\tau_{M,1}), ..., x(t-\tau_{M,N}), \rho_{1,1}(t), ..., \rho_{1,N}(t),\\ &\rho_{2,1}(t), ..., \rho_{2,N}(t), \rho_{3,1}(t), ..., \rho_{3,N}(t), \rho_{4,1}(t), ..., \rho_{4,N}(t), (\tau_{M,1}-\tau_1(t))\rho_{1,1}(t), ...,\\ &(\tau_{M,N}-\tau_N(t))\rho_{1,N}(t), (\tau_{M,1}-\tau_1(t))\rho_{2,1}(t), ..., (\tau_{M,N}-\tau_N(t))\rho_{2,N}(t),\\ &\tau_1(t)\rho_{3,1}(t), ..., \tau_N(t)\rho_{3,N}(t), \tau_1(t)\rho_{4,1}(t), ..., \tau_N(t)\rho_{4,N}(t)\},\end{aligned}$$

with $\rho_{1,i}(t)$, $\rho_{2,i}(t)$, $\rho_{3,i}(t)$ and $\rho_{4,i}(t)$ defined in (3.48), the functional derivative, along the system trajectories, can be rewritten as

$$\begin{aligned}\dot{V}_{ii)}(t, x_t, \dot{x}_t) = \hat{\xi}^T(t)\Bigg[&\mathrm{sym}\{\hat{\mathcal{C}}_1^T P(\hat{\mathcal{C}}_{20} + \sum_{i=1}^{N}\tau_i(t)\hat{\mathcal{C}}_{21,i})\} + \sum_{i=1}^{N}\Big[\hat{\mathcal{C}}_3^T Q_{1,i}\hat{\mathcal{C}}_3 - (1-\dot{\tau}_i(t))\hat{\mathcal{C}}_{4,i}^T \times\\ &(Q_{1,i} - Q_{2,i})\hat{\mathcal{C}}_{4,i} + \mathrm{sym}\{(\hat{\mathcal{C}}_{50,i} + \tau_i(t)\hat{\mathcal{C}}_{51,i})^T Q_{1,i}\hat{\mathcal{C}}_6\} - \hat{\mathcal{C}}_{7,i}^T Q_{2,i}\hat{\mathcal{C}}_{7,i}\\ &+ \mathrm{sym}\{(\hat{\mathcal{C}}_{80,i} + \tau_i(t)\hat{\mathcal{C}}_{81,i})^T Q_{2,i}\hat{\mathcal{C}}_6\} + \tau_{M,i}^2\hat{\mathcal{C}}_0^T R_i\hat{\mathcal{C}}_0\Big]\Bigg]\hat{\xi}(t)\\ &- \sum_{i=1}^{N}\left(\int_{t-\tau_i(t)}^{t}\dot{x}^T(s)R_i\dot{x}(s)ds + \int_{t-\tau_{M,i}}^{t-\tau_i(t)}\dot{x}^T(s)R_i\dot{x}(s)ds\right).\end{aligned}$$

Suppose that $Q_{1,i} \succ Q_{2,i}$ for $i = 1, ..., N$. Then, $\dot{V}_{ii)}(t, x_t, \dot{x}_t)$ can be bounded as

$$\begin{aligned}\dot{V}_{ii)}(t, x_t, \dot{x}_t) \le \hat{\xi}^T(t)\Bigg[&\mathrm{sym}\{\hat{\mathcal{C}}_1^T P(\hat{\mathcal{C}}_{20} + \sum_{i=1}^{N}\tau_i(t)\hat{\mathcal{C}}_{21,i})\} + \sum_{i=1}^{N}\Big[\hat{\mathcal{C}}_3^T Q_{1,i}\hat{\mathcal{C}}_3 - (1-\mu_{M,i})\hat{\mathcal{C}}_{4,i}^T \times\\ &(Q_{1,i} - Q_{2,i})\hat{\mathcal{C}}_{4,i} + \mathrm{sym}\{(\hat{\mathcal{C}}_{50,i} + \tau_i(t)\hat{\mathcal{C}}_{51,i})^T Q_{1,i}\hat{\mathcal{C}}_6\} - \hat{\mathcal{C}}_{7,i}^T Q_{2,i}\hat{\mathcal{C}}_{7,i}\\ &+ \mathrm{sym}\{(\hat{\mathcal{C}}_{80,i} + \tau_i(t)\hat{\mathcal{C}}_{81,i})^T Q_{2,i}\hat{\mathcal{C}}_6\} + \tau_{M,i}^2\hat{\mathcal{C}}_0^T R_i\hat{\mathcal{C}}_0\Big]\Bigg]\hat{\xi}(t)\\ &- \sum_{i=1}^{N}\left(\int_{t-\tau_i(t)}^{t}\dot{x}^T(s)R_i\dot{x}(s)ds + \int_{t-\tau_{M,i}}^{t-\tau_i(t)}\dot{x}^T(s)R_i\dot{x}(s)ds\right).\end{aligned}$$

The proof is then completed by invoking Lemma 3.5 with $d = 2$, Lemma 3.6, and adding the zero-valued term

$$\hat{\xi}^T(t)\left[M(\hat{\mathcal{C}}_{90} + \sum_{i=1}^{N}\tau_i(t)\hat{\mathcal{C}}_{91,i})\right]\hat{\xi}(t) = 0 \tag{3.58}$$

to the LKF derivative. □

3.3.4 Numerical Examples

In this section, two numerical examples are presented to illustrate the effectiveness of the proposed Theorem 3.3 and Theorem 3.4. In Example 3.6, it is analyzed the stability of systems with a single time-varying delay considered previously in Example 3.5. The aim is to show the influence of the changes proposed in functional (3.38) considering $N = 1$. Example 3.7 illustrates the performance of Theorem 3.3 and Theorem 3.4 in the case of systems with multiple time-varying delays. The results are compared to those of Liao et al. (2014) and Ko et al. (2018).

Example 3.6. *Consider system (3.34) with $N = 1$ and matrices A_0 and A_1 given in Table 3.1. As in Example 3.5, it is considered that $\mu_{M,1} = -\mu_{m,1} \triangleq \mu$.*

The same tests presented in Example 3.5 have been accomplished using Theorem 3.3 with $N = 1$. Namely, it was sought the maximum $\tau_{M,1}$ of $\tau_1(t)$ that the system remains stable for different values of μ. The results are listed in Tables 3.7, 3.8, and 3.9, and reveal that Theorem 3.3 outperforms only the results of Zhang et al. (2017b, Proposition 1) for all the systems analyzed. For System 2 with $\mu \leq 0.5$, Theorem 3.3 also leads to less conservative results than those from proposed Corollary 1. On the other hand, Theorem 3.3 requires the optimization of $114n^2 + 11n$ scalar decision variables, which is the lowest computational complexity among the matrix injection-based methods considered in Example 3.5.

Table 3.7: Maximum admissible upper bound $\tau_{M,1}$ of the delay $\tau_1(t)$ for given $\mu_{M,1} = -\mu_{m,1} \triangleq \mu$, and number of decision variables (NoV). Example 3.6 (System 1).

Method	μ 0.1	0.5	0.8	NoV
Zhang et al. (2017b, Proposition 1)	4.910	3.233	2.789	231
Chen et al. (2018, Theorem 1)	4.942	3.309	2.882	456
Zhang et al. (2019, Proposition 2, N=4)	4.929	3.252	2.823	886
Chen et al. (2019, Theorem 1 (C3))	4.939	3.298	2.869	446
Park et al. (2018, Theorem 3)	4.944	3.305	2.850	911
Corollary 1	4.938	3.281	2.854	318
Corollary 3	4.938	3.302	2.875	634
Corollary 2	5.044	3.443	2.983	1008
Zhang et al. (2020a, Theorem 1)	5.084	3.482	3.005	1027
Zhang et al. (2020c, Proposition 1)	5.147	3.673	3.258	1544
Theorem 3.3	4.930	3.243	2.820	478

Table 3.8: Maximum admissible upper bound $\tau_{M,1}$ of the delay $\tau_1(t)$ for given $\mu_{M,1} = -\mu_{m,1} \triangleq \mu$. Example 3.6 (System 2).

Method	μ			
	0.1	0.2	0.5	0.8
Zhang et al. (2017b, Proposition 1)	7.230	4.556	2.509	1.940
Chen et al. (2018, Theorem 1)	7.400	4.795	2.717	2.089
Chen et al. (2019, Theorem 1 (C3))	7.401	4.765	2.709	2.091
Park et al. (2018, Theorem 3)	7.550	4.902	2.714	2.054
Corollary 1	7.307	4.655	2.612	2.023
Corollary 3	7.438	4.863	2.715	2.068
Corollary 2	7.685	4.969	2.774	2.117
Zhang et al. (2020a, Theorem 1)	7.714	5.003	2.809	2.149
Theorem 3.3	7.310	4.687	2.636	2.010

Table 3.9: Maximum admissible upper bound $\tau_{M,1}$ of the delay $\tau_1(t)$ for given $\mu_{M,1} = -\mu_{m,1} \triangleq \mu$. Example 3.6 (System 3).

Method	μ_2		
	0.05	0.1	0.5
Zhang et al. (2017b, Proposition 1)	2.637	2.474	2.042
Chen et al. (2018, Theorem 1)	2.657	2.511	2.116
Park et al. (2018, Theorem 3)	2.658	2.509	2.105
Corollary 1	2.641	2.488	2.090
Corollary 3	2.648	2.499	2.113
Corollary 2	2.675	2.536	2.137
Theorem 3.3	2.643	2.486	2.058

Example 3.7. *Consider system* (3.34) *with $N = 2$ and matrices A_0, A_1, and A_2 given in Table 3.10, and assume that $\mu_{M,i=1,2} \triangleq \mu$ and $\mu_{m,i=1,2} = -\mu$.*

Table 3.10: System matrices for Example 3.7.

System 1:	$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 0.6 \\ -0.4 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -0.6 \\ -0.6 & 0 \end{bmatrix}$,
System 2:	$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Firstly, it is performed some tests for System 1 and System 2 subject to constant delays, i.e., $\tau_1(t) = \tau_1$ and $\tau_2(t) = \tau_2$. Figure 3.3 shows the stability ranges obtained with the method of Liao et al. (2014) and Theorems 3.3 and 3.4. It is clear that the proposed method significantly enlarges the stability domains of τ_1 and τ_2. For System 2, the proposed theorems also

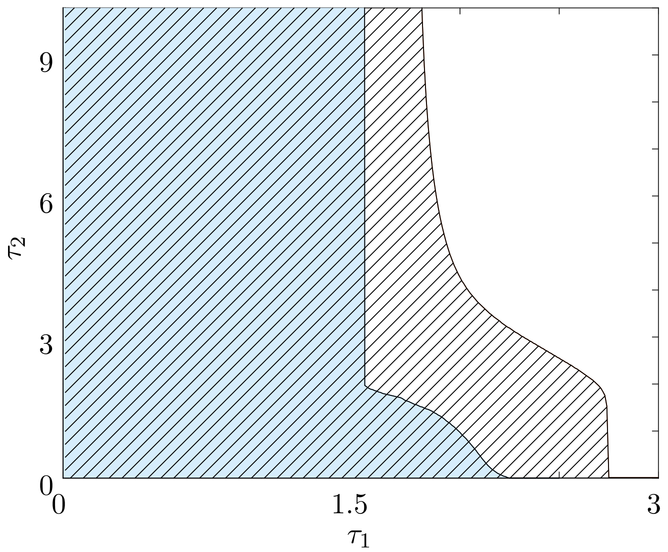


Figure 3.3: Comparison of stability regions for System 1 obtained with Theorem 1 of Liao et al. (2014) (blue area) and Theorems 3.3 and 3.4 (hatched area).

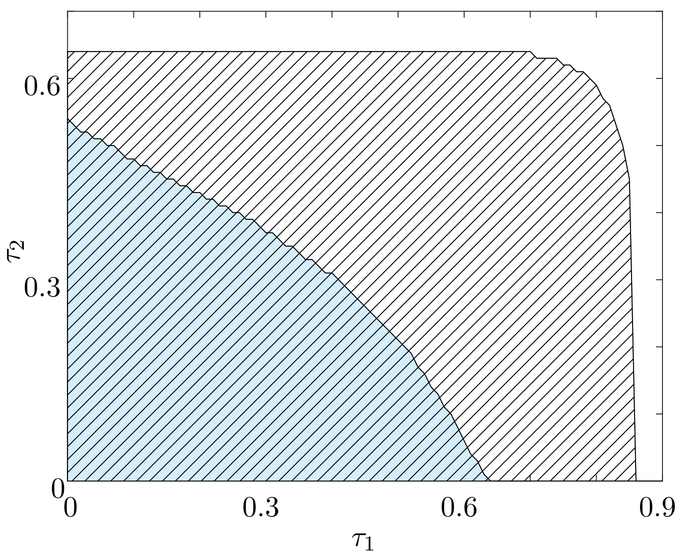


Figure 3.4: Comparison of stability regions for System 2 obtained with Theorem 1 of Liao et al. (2014) (blue area) and Theorems 3.3 and 3.4 (hatched area).

outperform the ones in Liao et al. (2014), as illustrated in Figure 3.4. It is noteworthy that there is no improvement of Theorem 3.3 over Theorem 3.4 in case of constant delays.

To illustrate the effectiveness of the proposed theorems in the case of multiple time-varying delays, it was sought the maximum allowable delay bound of $\tau_2(t)$, i.e., $\tau_{M,2}$, given $\tau_{M,1}$ and μ such that the system remains asymptotically stable. The results of the tests performed are listed in Tables 3.11 and 3.12. Comparing the results of Theorems 3.3 and 3.4, one can see that Theorem 3.3 achieves larger upper bounds of $\tau_{M,2}$. This improvement is due to the extra information on the derivative lower bound exploited in that criterion.

Once again, the results of the proposed theorems overcome the ones obtained by Liao et al. (2014) and Ko et al. (2018). It is worth pointing out that whereas Theorem 3.4 asserts the stability of the systems in all tests performed, Liao et al. (2014, Theorem 1) and Ko et al. (2018, Theorem 3) did not verify the stability of the systems in some tests.

Table 3.11: Maximum upper bound $\tau_{M,2}$ for $\tau_2(t)$ given $\tau_{M,1}$ and $\mu_{M,i} = \mu$, for $i = 1, 2$, and number of decision variables (NoV). Example 3.7 (System 1).

Method		$\mu = 0.01$					NoV
	$\tau_{M,1}$	1.6	1.7	1.8	1.9	2.0	
Liao et al. (2014, Theorem 1)	$\tau_{M,2}$	1.78	1.66	1.48	1.24	0.85	27
Ko et al. (2018, Theorem 3)	$\tau_{M,2}$	2.35	2.21	2.12	2.05	2.02	235
Theorem 3.4	$\tau_{M,2}$	> 100	> 100	8.57	5.03	4.02	1437
Theorem 3.3	$\tau_{M,2}$	> 100	> 100	> 100	> 100	> 100	1026
		$\mu = 0.9$					
	$\tau_{M,1}$	0.5	0.8	1.0	1.2	1.3	
Liao et al. (2014, Theorem 1)	$\tau_{M,2}$	1.32	0.38	–	–	–	27
Ko et al. (2018, Theorem 3)	$\tau_{M,2}$	2.6	2.03	1.69	1.41	1.31	235
Theorem 3.4	$\tau_{M,2}$	4.89	2.79	2.16	1.61	1.37	1437
Theorem 3.3	$\tau_{M,2}$	> 100	94	56	56	3.56	1026

Table 3.12: Maximum upper bound $\tau_{M,2}$ for $\tau_2(t)$ given $\tau_{M,1}$ and $\mu_{M,i} = \mu$, for $i = 1, 2$, and number of decision variables (NoV). Example 3.7 (System 2).

Method		$\mu = 0.1$					NoV
	$\tau_{M,1}$	0.1	0.15	0.2	0.25	0.3	
Liao et al. (2014, Theorem 1)	$\tau_{M,2}$	0.4	0.37	0.34	0.31	0.27	27
Ko et al. (2018, Theorem 3)	$\tau_{M,2}$	0.49	0.49	0.49	0.48	0.47	235
Theorem 3.4	$\tau_{M,2}$	0.57	0.56	0.56	0.56	0.55	1437
Theorem 3.3	$\tau_{M,2}$	0.60	0.60	0.60	0.60	0.59	1026
		$\mu = 0.5$					
Liao et al. (2014, Theorem 1)	$\tau_{M,2}$	0.22	0.19	0.08	–	–	27
Ko et al. (2018, Theorem 3)	$\tau_{M,2}$	0.39	0.34	0.30	0.26	–	235
Theorem 3.4	$\tau_{M,2}$	0.46	0.41	0.36	0.32	0.27	1437
Theorem 3.3	$\tau_{M,2}$	0.52	0.48	0.44	0.40	0.37	1026

Table 3.13 summarizes the number of decision variables, as a function of the system dimension, of each method analyzed. As in the case of single delays, the improvement of the proposed Theorem 3.3 and Theorem 3.4 came with a considerable increased computational cost. Note that most of the decision variables in Theorem 3.3 and Theorem 3.4 come from the matrix M in (3.58), introduced to avoid a stability criterion expressed in terms of cubic functions on time-delays $\tau_i(t)$.

The numerical results presented above suggest that introducing appropriate augmented terms in the functional candidate and using improved integral inequalities is more effective in *reducing the conservativeness than exploiting all the possible relationships between the time-varying delays, as in Ko et al. (2018, Theorem 3). This advantage, however, comes at a cost in increasing computational complexity.*

Table 3.13: Number of decision variables (NoV) required by the methods used in Example 3.7.

Method	NoV
Liao et al. (2014, Theorem 1)	$N[(0.5N+1)n^2+(0.5N+1)n]+0.5(n^2+n)$
Ko et al. (2018, Theorem 3)	$N[(12.5N+2.5)n^2+2n]+1.5n^2+0.5n$
Theorem 3.3	$N[(60.5N+53)n^2+10.5n]+0.5(n^2+n)$
Theorem 3.4	$N[(42N+40.5)n^2+7.5n]$

3.4 Publications

The results described in this chapter have been published in de Oliveira and Souza (2020a) and de Oliveira and Souza (2020b) for systems with single and multiple time-varying delays, respectively. The results in de Oliveira and Souza (2020a) generalize the preliminary results presented in de Oliveira et al. (2018).

Chapter 4

Conclusion

This thesis has investigated the problem of stability of linear time-invariant systems with time-delay and new methods have been proposed to improve some results existing in the literature. The first contribution, presented in *Chapter 2*, is an alternative method for testing the *strong* delay-independent stability of systems with a single constant delay. Our approach is to use a frequency-dependent Lyapunov stability inequality with matrices of polynomial type. A new LMI condition of increasing dimension was obtained using the Kalman–Yakubovich–Popov (KYP) lemma. A merit of the proposed condition is to overcome the main drawback of conditions in Li et al. (2016); Bliman (2002), i.e., eliminating the undecidability on the instability since it provides priori known upper bound dimension for which the LMI condition is necessary and sufficient for stability. Moreover, unlike the method in Souza et al. (2018), in the proposed LMI condition, the system matrices appear linearly, which may perhaps be extended to obtain linear synthesis conditions. Finally, numerical experiments have shown that our method usually has a reduced computational complexity compared to Souza et al. (2018), making it attractive for handling high-dimensional systems.

Chapter 3 has presented some contributions to the delay-dependent stability of systems with time-varying delay. As discussed in that chapter, some recent stability conditions in the literature have been formulated as negativity for a quadratic function parameterized by the delay. Then, as a first contribution of the chapter, a method was presented in Lemma 3.10 to translate such a condition into a finite-dimensional convex optimization problem that can be checked exactly in terms of LMI conditions. Although this result has been presented in the context of continuous-time systems, it can also be applied to conditions for discrete-time systems. Soon after the publication of this result in de Oliveira and Souza (2020a), Lemma 3.10 has been used in Zhang et al. (2020a) and extended to deal with any polynomial function by other authors in Zhang et al. (2020c), revealing the interest of this type of result in the literature.

The second contribution presented in *Chapter 3* is a new stability criterion specified as a negativity condition for a quadratic function parameterized by the delay. This result is obtained from an augmented affine parameter-dependent Lyapunov-Krasovskii functional adequately chosen. Numerical experiments on three benchmark systems have shown that the proposed stability criterion, when evaluated using Lemma 3.10, can lead to less conservative results than some new methods in the literature. The main drawback of the proposed approach is the large number of decision variables required by Lemma 3.10, which may be a problem in handling high dimensional systems. As a final contribution, it was shown how to extend the stability conditions for systems with a single time-varying delay to systems with multiple time-varying delays. Once again, the proposed stability criterion has outperformed the related methods, but the computational complexity may be an obstacle in using this method in practical applications that consider several delays, as the control of vehicular platoon described in Souza et al. (2019).

4.1 Future research

Some suggestions for future research are outlined below.

Deriving state feedback control design conditions from the stability analysis conditions proposed in Chapter 2. The stability analysis results presented in Chapter 2 apparently are extendable to synthesis, since the system matrices appear linearly in the proposed LMI conditions. Indeed, this extension is trivial for unidimensional systems because there is just one Lyapunov matrix variable multiplying the system matrices. So, the problem can be linearized by performing a congruence transformation and a simple change of variables. However, for systems with dimension $n \geq 2$, this extension is still challenging. Basically, it seems there are two ways to do that. The first one is to consider a priori known structure for the controller and replace the closed-loop system matrices in the LMI condition of Theorem 2.1. The main problem within this approach is linearizing the resulting conditions since several Lyapunov matrices multiply the controller gain(s). In the literature, one can find some strategies to solve this problem, but they are usually conservative.

The second approach is to consider an unknown structure for the controller and try to linearize the frequency domain condition (2.4) in Lemma 2.2 and then apply the KYP Lemma to obtain a finite-dimensional LMI condition. To make it clearer, let the closed-loop system given by

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t) \\ u(t) = K(x(t), x(t-\tau)) \end{cases}$$

where $K(x(t), x(t-\tau))$ is a controller to be determined. Then, $F(e^{-j\theta})$ in Lemma 2.2 is given by $F(e^{-j\theta}) = A_0 + A_1 e^{-j\theta} + BK(e^{-j\theta})$ and condition $iii)$ becomes

$$\bar{P}(e^{-j\theta}) \succ 0,$$
$$\bar{P}(e^{-j\theta})(A_0 + A_1 e^{-j\theta})^* + (A_0 + A_1 e^{-j\theta})\bar{P}(e^{-j\theta}) + BL(e^{-j\theta}) + L^*(e^{-j\theta})B^T \prec 0,$$

after applying a congruence transformation, pre and post multiplying (2.4) by $P^{-1}(e^{-j\theta})$ and

$P^{-1}(e^{-j\theta})$, and the change of variables $\bar{P}(e^{-j\theta}) = P^{-1}(e^{-j\theta})$ and $L(e^{-j\theta}) = K(e^{-j\theta})Q(e^{-j\theta})$. Hence, if the above conditions hold, a stabilizing controller is given by

$$K(e^{-j\theta}) = L(e^{-j\theta})\bar{P}^{-1}(e^{-j\theta}). \tag{4.1}$$

Note that the inverse of $\bar{P}(e^{-j\theta})$ contain terms in both $e^{-j\theta}$ and $e^{j\theta}$, which means that the expression for $K(e^{-j\theta})$ in the time-domain has non-causal terms. Therefore, the controller (4.1) is not physically realizable. So, the problem within this approach is finding a proper realization for the controller.

The problem mentioned above was the subject of study during the doctoral sandwich internship at the University of California, San Diego, under the supervision of Professor Mauricio de Oliveira. This work, however, is still ongoing.

Reducing the computational complexity of the stability analysis conditions proposed in Chapter 3. As shown in *Chapter 3*, Lemma 3.10 is instrumental in reducing the conservativeness of stability conditions formulated as quadratic functions. However, it may require more decision variables than those in the chosen Lyapunov-Krasovskii functional. Likewise, the matrix injection-based method and the delay-dependent reciprocally convex lemma also introduce several decision variables in the LMI conditions. A problem of practical interest is how to reduce the computational complexity without inserting conservatism in the criterion. This is particularly important for high-dimensional and multiple time-varying delay systems.

Deriving control design conditions based on the delay-dependent stability analysis conditions proposed in Chapter 3. The numerical experiments in this manuscript have shown that the proposed delay-dependent stability analysis criteria can yield nice results. It is expected that control design conditions obtained from the proposed analysis conditions also lead to less conservative results. However, how to obtain the LMI synthesis conditions in a nonconservative way is a challenging problem.

4.2 Publications

The publications related to the contributions of this thesis are listed below:

- de Oliveira, F. S. S. and Souza, F. O. (2020b). Improved delay-dependent stability criteria for linear systems with multiple time-varying delays. *International Journal of Control*, pages 1–9.

- de Oliveira, F. S. S. and Souza, F. O. (2020a). Further refinements in stability conditions for time-varying delay systems. *Applied Mathematics and Computation*, 369:124866.

- de Oliveira, F. S. S. and Souza, F. O. (2019). Strong delay-independent stability of linear delay systems. *Journal of the Franklin Institute*, 356(10):5421–5433.

- de Oliveira, F. S. S., Lima, M. V., and Souza, F. O. (2018). Further improvements on stability analysis for uncertain time-delayed linear systems. In *XXII Congresso Brasileiro de Automática - CBA2018*, pages 1–6, João Pessoa - PB, Brasil.

Bibliography

Alves Neto, A., Mozelli, L. A., and Souza, F. O. (2019). Control of air-ground convoy subject to communication time delay. *Computers & Electrical Engineering*, 76:213–224.

Andersen, E. D. and Andersen, K. D. (2000). The MOSEK interior point optimizer for linear programming: An implementation of the homogeneous algorithm. In *Applied Optimization*, pages 197–232. Springer US.

Belta, C. and Habets, L. C. G. J. M. (2006). Controlling a class of nonlinear systems on rectangles. *IEEE Transactions on Automatic Control*, 51(11):1749–1759.

Bernstein, D. S. (2009). *Matrix mathematics: theory, facts, and formulas*. Princeton University Press, New Jersey.

Bliman, P.-A. (2000). Lyapunov-Krasovskii method and strong delay-independent stability of linear delay systems. *IFAC Proceedings Volumes*, 33(23):33–37.

Bliman, P.-A. (2001). LMI characterization of the strong delay-independent stability of linear delay systems via quadratic Lyapunov–Krasovskii functionals. *Systems & Control Letters*, 43(4):263 – 274.

Bliman, P.-A. (2002). Lyapunov equation for the stability of linear delay systems of retarded and neutral type. *IEEE Transactions on Automatic Control*, 47(2):327–335.

Bliman, P.-A. (2004). A convex approach to robust stability for linear systems with uncertain scalar parameters. *SIAM Journal on Control and Optimization*, 42(6):2016–2042.

Boyd, S., Ghaoui, L. E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics.

Chen, J. and Latchman, H. (1995). Frequency sweeping tests for stability independent of delay. *IEEE Transactions on Automatic Control*, 40(9):1640–1645.

Chen, J., Park, J., and Xu, S. (2018). Stability analysis of continuous-time systems with time-varying delay using new Lyapunov-Krasovskii functionals. *Journal of the Franklin Institute*, 355:5957–5967.

Chen, J., Park, J. H., and Xu, S. (2019). Stability analysis of systems with time-varying delay: a quadratic-partitioning method. *IET Control Theory & Applications*, 13(18):3184–3189.

de Lima, M. V., Mozelli, L. A., Neto, A. A., and Souza, F. O. (2020). A simple algebraic criterion for stability of bilateral teleoperation systems under time-varying delays. *Mechanical Systems and Signal Processing*, 137:106217.

de Oliveira, F. S. S., Lima, M. V., and Souza, F. O. (2018). Further improvements on stability analysis for uncertain time-delayed linear systems. In *XXII Congresso Brasileiro de Automática - CBA2018*, pages 1–6, João Pessoa - PB, Brasil.

de Oliveira, F. S. S. and Souza, F. O. (2019). Strong delay-independent stability of linear delay systems. *Journal of the Franklin Institute*, 356(10):5421–5433.

de Oliveira, F. S. S. and Souza, F. O. (2020a). Further refinements in stability conditions for time-varying delay systems. *Applied Mathematics and Computation*, 369:124866.

de Oliveira, F. S. S. and Souza, F. O. (2020b). Improved delay-dependent stability criteria for linear systems with multiple time-varying delays. *International Journal of Control*, pages 1–9.

de Oliveira, F. S. S., Souza, F. O., and Palhares, R. M. (2017). PID tuning for time-varying delay systems based on modified Smith predictor. *IFAC-PapersOnLine*, 50(1):1269–1274.

Djema, W., Mazenc, F., and Bonnet, C. (2017). Stability analysis and robustness results for a nonlinear system with distributed delays describing hematopoiesis. *Systems & Control Letters*, 102:93–101.

dos Santos Junior, C. R. P., Carvalho, J. R. H., Souza, F. O., and Savino, H. J. (2018). Exponential consensus with decay rate estimation for heterogeneous multi-agent systems. *Journal of Intelligent & Robotic Systems*, 95(2):543–553.

Fang, M. and Park, J. H. (2013). A multiple integral approach to stability of neutral time-delay systems. *Applied Mathematics and Computation*, 224:714–718.

Feng, Z., Lam, J., and Yang, G.-H. (2013). Optimal partitioning method for stability analysis of continuous/discrete delay systems. *International Journal of Robust and Nonlinear Control*, 25(4):559–574.

Fenili, E. P., Souza, F. O., and Mozelli, L. A. (2014). Sintonia de PID via LMIs: imposição de tempo de acomodação em sistemas com retardo incerto [*PID tunning via LMIs: imposition of settling time in systems with uncertain time-delay]*. In *XX Congresso Brasileiro de Automática - CBA2014*, pages 1127–1134, Belo Horizoznte - MG, Brasil. (In Portuguese).

Fridman, E. (2001). New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43:309–319.

Fridman, E. (2014a). *Introduction to time-delay systems: Analysis and Control*. Systems & Control: Foundations & Applications. Birkhäuser Basel, 1 edition.

Fridman, E. (2014b). Tutorial on Lyapunov-based methods for time-delay systems. *European Journal of Control*, 20:217–283.

Fridman, E. and Shaked, U. (2002). A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Trans. on Automatic Control*, 47:253–270.

Fridman, E. and Shaked, U. (2003). Delay-dependent stability and H_∞ control: constant and time-varying delays. *International Journal of Control*, 76(1):48–60.

Fridman, E., Shaked, U., and Liu, K. (2009). New conditions for delay-derivative-dependent stability. *Automatica*, 45:2723–2727.

Gahinet, P., Nemirovski, A., Laub, A. J., and Chilali, M. (1995). *LMI control Toolbox user's guide*. The Math Works Inc.

Gomes, R. H. A., Souza, F. O., and Mozelli, L. A. (2014). Síntese de controladores TS com taxa de convergência pré-especificada para sistemas com retardo [*Synthesis of TS controllers with pre-specified convergence rate for time-delay systems*]. In *XX Congresso Brasileiro de Automática - CBA2014*, pages 1808–1815, Belo Horizoznte - MG, Brasil. (In Portuguese).

Gomes da Silva Jr., J. M. and Leite, V. J. S. (2007). Sistemas lineares com atraso de tempo [*Linear systems with time-delay*]. In Aguirre, L. A., editor, *Enciclopédia de Automática: Controle e Automação*, volume 2, pages 108–124. Blucher. (In Portuguese).

Gouaisbaut, F. and Peaucelle, D. (2006). Delay-dependent robust stability of time delay systems. In *Preprints of the 5th IFAC Symposium on Robust Control Design*, France, Toulouse.

Gouaisbaut, F. and Peaucelle, D. (2007). Robust stability of time-delay systems with interval delays. In *2007 46th IEEE Conference on Decision and Control*. IEEE.

Gu, K. (1997). Discretized LMI set in the stability problem of linear uncertain time-delay systems. *International Journal of Control*, 68(4):923–934.

Gu, K. (2000). An integral inequality in the stability problem of time-delay systems. In *Proceeding of the 39th IEEE Conference on Decision and Control*, pages 2805–2810, Sydney, Australia.

Gu, K. (2001). A further refinement of discretized Lyapunov functional method for the stability of time-delay systems. *International Journal of Control*, 74(10):967–976.

Gu, K. (2003). Refined discretized Lyapunov functional method for systems with multiple delays. *International Journal of Robust and Nonlinear Control*, 13(11):1017–1033.

Gu, K. (2013). *Complete quadratic Lyapunov-Krasovskii functional: limitations, computational efficiency, and convergence*, pages 1–19. World Scientific, Singapore.

Gu, K., Kharitonov, V., and Chen, J. (2003). *Stability of time-delay systems*. Birkhäuser, Boston, MA.

Gu, K. and Niculescu, S.-I. (2001). Further remarks on additional dynamics in various model transformations of linear delay systems. *IEEE Transactions on Automatic Control*, 46(3):497–500.

Gyurkovics, É. and Takács, T. (2016). Multiple integral inequalities and stability analysis of time delay systems. *Systems & Control Letters*, 96:72–80.

Han, Q.-L. (2009). A discrete delay decomposition approach to stability of linear retarded and neutral systems. *Automatica*, 45(2):517–524.

He, Y., Wang, Q. G., Lin, C., and Wu, M. (2005). Augmented Lyapunov functional and delay-dependent stability criteria for neutral systems. *International Journal of Robust and Nonlinear Control*, 15:923–933.

He, Y., Wang, Q.-G., Xie, L., and Lin, C. (2007). Further improvement of free-weighting matrices technique for systems with time-varying delay. *IEEE Transactions on Automatic Control*, 52(2):293–299.

He, Y., Wu, M., and She, J. (2006). Delay-dependent stability criteria for linear systems with multiple time delays. *IEEE Proceedings - Control Theory and Applications*, 153:447–452.

Hertz, D., Jury, E., and Zeheb, E. (1984). Stability independent and dependent of delay for delay differential systems. *Journal of the Franklin Institute*, 318(3):143–150.

Hogben, L., editor (2014). *Handbook of Linear Algebra*. CRC Press, 2nd edition.

Kamen, E. (1982). Linear systems with commensurate time delays: stability and stabilization independent of delay. *IEEE Transactions on Automatic Control*, 27(2):367–375.

Kamen, E. (1983). Correction to "Linear systems with commensurate time delays: Stability and stabilization independent of delay". *IEEE Transactions on Automatic Control*, 28(2):248–249.

Kamen, E. W. (1980). On the relationship between zero criteria for two-variable polynomials and asymptotic stability of delay differential-equations. *IEEE Transactions on Automatic Control*, 25(5):983–984.

Kao, C.-Y. and Rantzer, A. (2007). Stability analysis of systems with uncertain time-varying delays. *Automatica*, 43:959–970.

Kharitonov, V. L. (1999). Robust stability analysis of time delay systems: a survey. *Annual Reviews in Control*, 23(1):185–196.

Kharitonov, V. L. (2013). *Time-delay systems: Lyapunov functionals and matrices*. Control engineering (Birkhäuser). Birkhauser.

Kharitonov, V. L. and Zhabko, A. P. (2003). Lyapunov-Krasovskii approach to robust stability analysis of time-delay systems. *Automatica*, 39:15–20.

Kim, J.-H. (2011). Note on stability of linear systems with time-varying delay. *Automatica*, 47:2118–2121.

Kim, J. H. (2016). Further improvement of Jensen inequality and application to stability of time-delayed systems. *Automatica*, 64:121–125.

Ko, K., Lee, W., Park, P., and Sung, D. (2018). Delays-dependent region partitioning approach for stability criterion of linear systems with multiple time-varying delays. *Automatica*, 87:389–394.

Kolmanovskii, V. and Richard, J.-P. (1999). Stability of some linear systems with delays. *IEEE Transactions on Automatic Control*, 44(5):984–989.

Kolmanovskii, V. B. (1999). On the Liapunov-Krasovskii functionals for stability analysis of linear delay systems. *International Journal of Control*, 72:374–384.

Krasovskii, N. N. (1963). *Stability of Motion*. Stanford University Press, Stanford, CA. (In Russian).

Kwon, W., Koo, B., and Lee, S. (2018). Novel Lyapunov–Krasovskii functional with delay-dependent matrix for stability of time-varying delay systems. *Applied Mathematics and Computation*, 320:149–157.

Lee, S. Y., Lee, W. I., and Park, P. G. (2017). Polynomials-based integral inequality for stability analysis of linear systems with time-varying delays. *Journal of the Franklin Institute*, 354:2053–2067.

Lee, T. H. (2020). Geometry-based conditions for a quadratic function: Application to stability of time-varying delay systems. *IEEE Access*, pages 1–1.

Lee, T. H. and Park, H. (2017). A novel Lyapunov functional for stability of time-varying delay systems via matrix-refined-function. *Automatica*, 80:239–242.

Lee, T. H. and Park, J. H. (2018). Improved stability conditions of time-varying delay systems based on new Lyapunov functionals. *Journal of the Franklin Institute*, 355:1176–1191.

Li, X., Gao, H., and Gu, K. (2016). Delay-independent stability analysis of linear time-delay systems based on frequency discretization. *Automatica*, 70:288 – 294.

Liao, F., Wu, J., and Tomizuka, M. (2014). An improved delay-dependent stability criterion for linear uncertain systems with multiple time-varying delays. *International Journal of Control*, 87:861–873.

Liu, K., Seuret, A., and Xia, Y. (2017). Stability analysis of systems with time-varying delays via the second-order Bessel-Legendre inequality. *Automatica*, 76:138–142.

MacDonald, N. (2008). *Biological delay systems: linear stability theory*. Number 9 in Cambridge Studies in Mathematical Biology. Cambridge University Press, Cambridge.

Michiels, W. and Verriest, E. I. (2011). A systems theoretic analysis of fast varying and state dependent delays. In *IEEE Conference on Decision and Control and European Control Conference*. IEEE.

Nesterov, Y. and Nemirovsky, A. (1988). A general approach to polynomial-time algorithms design for convex programming. Technical report, Central Economical and Mathematical Institute. USSR Academy of Sciences, Moscow.

Niculescu, S.-I. (2001). *Delay effects on stability: a robust control approach*, volume 269 of *Lecture notes in control and information sciences*. Springer, Berlin, Germany.

Niculescu, S.-I., Kim, P. S., Gu, K., Lee, P. P., and Levy, D. (2010). Stability crossing boundaries of delay systems modeling immune dynamics in leukemia. *Discrete & Continuous Dynamical Systems - B*, 13(1):129–156.

Normey-Rico, J. E. and Camacho, E. F. (2007). *Control of Dead-time Processes*. Springer-Verlag London.

Oliveira, T. G., Palhares, R. M., Campos, V. C. S., Queiroz, P. S., and Gonçalves, E. N. (2017). Improved Takagi-Sugeno fuzzy output tracking control for nonlinear networked control systems. *Journal of the Franklin Institute*, 354(16):7280–7305.

Palhares, R. M. and Gonçalves, E. N. (2007). Desigualdades matriciais lineares em controle [*Linear Matrix Inequalities in control]*. In Aguirre, L. A., editor, *Enciclopédia de Automática – Controle & Automação*, pages 155–195. Blucher. (In Portuguese).

Parada, M., Sbarbaro, D., Borges, R. A., and Peres, P. L. D. (2016). Robust PI and PID design for first- and second-order processes with zeros, time-delay and structured uncertainties. *International Journal of Systems Science*, 48(1):95–106.

Park, M., Kwon, O., and Ryu, J. (2018). Advanced stability criteria for linear systems with time-varying delays. *Journal of the Franklin Institute*, 355(1):520–543.

Park, P., Ko, J. W., and Jeong, C. (2011). Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47(1):235 – 238.

Park, P., Lee, W. I., and Lee, S. Y. (2015). Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. *Journal of the Franklin Institute*, 352:1378–1396.

Peet, M., Papachristodoulou, A., and Lall, S. (2006). Positive forms and stability of linear time-delay systems. In *Proceedings of the 45th IEEE Conference on Decision and Control*. IEEE.

Peet, M. M. and Bliman, P.-A. (2011). On the conservatism of the Sum-of-Squares method for analysis of time-delayed systems. *Automatica*, 47(11):2406–2411.

Peet, M. M., Papachristodoulou, A., and Lall, S. (2009). Positive forms and stability of linear time-delay systems. *SIAM Journal on Control and Optimization*, 47(6):3237–3258.

Rantzer, A. (1996). On the Kalman-Yakubovich-Popov lemma. *Systems & Control Letters*, 28(1):7 – 10.

Razumikhin, B. S. (1956). On the stability of systems with a delay. *Prikl. Mat. Meh*, 20:500–512. (In Russian).

Richard, J. P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1694.

Sarachik, P. E. (1997). *Principles of Linear Systems*. Cambridge University Press, Cambridge, 1 edition.

Savino, H. J., Souza, F. O., and Pimenta, L. C. A. (2016). Consensus with guaranteed convergence rate of high-order integrator agents in the presence of time-varying delays. *International Journal of Systems Science*, 47(10):2475–2486.

Savino, H. J., Souza, F. O., and Pimenta, L. C. A. (2017). Consensus on intervals of communication delay. *International Journal of Automation and Computing*, 15(1):13–24.

Seiler, P. (2015). Stability analysis with dissipation inequalities and Integral Quadratic Constraints. *IEEE Transactions on Automatic Control*, 60:1704–1709.

Seuret, A. (2017). Contributions to the stability analysis and control of networked systems. Technical report, Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier).

Seuret, A. and Gouaisbaut, F. (2013). Wirtinger-based integral inequality: Application to time-delay systems. *Automatica*, 49:2860–2866.

Seuret, A. and Gouaisbaut, F. (2015). Hierarchy of LMI conditions for the stability of time-delay systems. *Systems & Control Letters*, 81:1–7.

Seuret, A. and Gouaisbaut, F. (2018). Stability of linear systems with time-varying delays using Bessel-Legendre inequalities. *IEEE Transactions on Automatic Control*, 63:225–232.

Silva, G. J., Datta, A., and Bhattachaiyya, S. P. (2005). *PID Controllers for Time-Delay Systems*. Birkhäuser Boston.

Silva, L. F., Leite, V. J., Castelan, E. B., Klug, M., and Guelton, K. (2020a). Local stabilization of nonlinear discrete-time systems with time-varying delay in the states and saturating actuators. *Information Sciences*, 518:272–285.

Silva, T. C., Souza, F. O., and Pimenta, L. C. A. (2020b). Distributed formation-containment control with Euler-Lagrange systems subject to input saturation and communication delays. *International Journal of Robust and Nonlinear Control*, 30(7):2999–3022.

Sipahi, R., Niculescu, S. I., Abdallah, C. T., Michiels, W., and Gu, K. (2011). Stability and stabilization of systems with time delay - limitations and opportunities. *IEEE Control Systems Magazine*, 31:38–65.

Souza, F. O. (2018). An exact LMI condition for the strong delay-independent stability analysis of neutral delay systems. *International Journal of Robust and Nonlinear Control*, 28(17):5375–5385.

Souza, F. O., de Oliveira, M. C., and Palhares, R. M. (2018). A simple necessary and sufficient LMI condition for the strong delay-independent stability of LTI systems with single delay. *Automatica*, 89:407–410.

Souza, F. O., Torres, L. A. B., Mozelli, L. A., and Neto, A. A. (2019). Stability and formation error of homogeneous vehicular platoons with communication time delays. *IEEE Transactions on Intelligent Transportation Systems*, pages 1–12.

Sturm, J. F. (1999). Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625–653.

Sun, J., Liu, G. P., and Chen, J. (2009). Delay-dependent stability and stabilization of neutral time-delay systems. *International Journal of Robust and Nonlinear Control*, 19(12):1364–1375.

Sun, J., Liu, G. P., Chen, J., and Rees, D. (2010). Delay-dependent stability and stabilization of neutral time-delay systems. *Automatica*, 46:466–470.

Toh, K. C., Todd, M. J., and Tütüncü, R. H. (1999). SDPT3 – a Matlab software package for semidefinite programming, Version 1.3. *Optimization Methods and Software*, 11(1-4):545–581.

Veenman, J. and Scherer, C. W. (2013). Stability analysis with integral quadratic constraints: A dissipativity based proof. In *Proceedings of the 52nd IEEE Conference on Decision and Control*, pages 3770–3775, Florence, Italy.

Verriest, E. I. (2010). Well-posedness of problems involving time-varying delays. In *Proceedings of the 18th International Symposium on Mathematical Theory of Networks and Systems*, pages 1985–1988, Budapest, Hungary: Eötvös Loráand University.

Verriest, E. I. (2011). Inconsistencies in systems with time-varying delays and their resolution. *IMA Journal of Mathematical Control and Information*, 28(2):147–162.

Wu, M., He, Y., and She, J.-H. (2010). *Stability analysis and robust control of time-delay systems*. Springer.

Wu, M., He, Y., She, J.-H., and Liu, G.-P. (2004). Delay-dependent criteria for robust stability of time-varying delay system. *Automatica*, 40:1435–1439.

Xu, H.-T., Zhang, C.-K., Jiang, L., and Smith, J. (2017). Stability analysis of linear systems with two additive time-varying delays via delay-product-type Lyapunov functional. *Applied Mathematical Modelling*, 45:955–964.

Yue, D., Tian, E., and Zhang, Y. (2009). A piecewise analysis method to stability analysis of linear continuous/discrete systems with time-varying delay. *International Journal of Nonlinear and Robust Control*, 19:1493–1518.

Zhang, B.-L., Cheng, L., Pan, K., and Zhang, X.-M. (2020a). Reducing conservatism of stability criteria for linear systems with time-varying delay using an improved triple-integral inequality. *Applied Mathematics and Computation*, 380:125254.

Zhang, C.-K., He, Y., Jiang, L., and Wu, M. (2017a). Notes on stability of time-delay systems: Bounding inequalities and augmented Lyapunov-Krasovskii functionals. *IEEE Transactions on Automatic Control*, 62(10):5331–5336.

Zhang, C.-K., Long, F., He, Y., Yao, W., Jiang, L., and Wu, M. (2020b). A relaxed quadratic function negative-determination lemma and its application to time-delay systems. *Automatica*, 113:108764.

Zhang, X., Tsiotras, P., and Iwasaki, T. (2003). Parameter-dependent Lyapunov functions for exact stability analysis of single-parameter dependent LTI systems. In *42nd IEEE Conference on Decision and Control*, pages 5168–5173, Maui, Hawaii USA. Paper FrM01-6.

Zhang, X., Tsiotras, P., and Iwasaki, T. (2010). Lyapunov-based exact stability analysis and synthesis for linear single-parameter dependent systems. *International Journal of Control*, 83(9):1823–1838.

Zhang, X.-M., Han, G.-L., Seuret, A., and Gouaisbaut, F. (2017b). An improved reciprocally convex inequality and an augmented Lyapunov-Krasovskii functional for stability of linear systems with time-varying delays. *Automatica*, 76:138–142.

Zhang, X.-M. and Han, Q.-L. (2012). Novel delay-derivative-dependent stability criteria using new bounding techniques. *International Journal of Robust and Nonlinear Control*, 23(13):1419–1432.

Zhang, X.-M., Han, Q.-L., and Ge, X. (2020c). Novel stability criteria for linear time-delay systems using Lyapunov-Krasovskii functionals with a cubic polynomial on time-varying delay. *IEEE/CAA Journal of Automatica Sinica*, pages 1–9.

Zhang, X.-M., Han, Q.-L., Seuret, A., Gouaisbaut, F., and He, Y. (2019). An overview of recent advances in stability of linear systems with time-varying delays. *IET Control Theory & Applications*, 13:1–16.

Zhou, K., Doyle, J. C., and Glover, K. (1996). *Robust and Optiomal Control*. Prentice Hall, Inc, Englewood Cliffs, NJ.

Appendix A

Some technical lemmas

This appendix presents some useful matrix lemmas used to construct the main results in the Chapter 2.

Proposition 1.1. *(Bernstein, 2009, Proposition 7.2.4) Let $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{m\times m}$, and $Q \in \mathbb{F}^{n\times m}$. Then $P \in \mathbb{F}^{n\times m}$ satisfies the equation*

$$AP + PB + Q = 0 \tag{A.1}$$

if and only if P satisfies

$$(B^T \oplus A)\text{vec}(P) + \text{vec}(Q) = 0.$$

Therefore, $(B^T \oplus A)$ is nonsingular if and only if there exists a unique matrix P satisfying (A.1), which is given by:

$$P = \text{vec}^{-1}\left[(B^T \oplus A)^{-1}\text{vec}(Q)\right].$$

Lemma 1.11. *Let $A, B, C \in \mathbb{R}^{n\times n}$ and $z \in \mathbb{C}$. Then*

$$\text{adj}(A + zB + z^{-1}C) = \sum_{\ell=-p^\star}^{p^\star} z^\ell N_\ell,$$

for an appropriate positive integer $p^\star \le \bar{p} := n-1$ and non-null matrices $N_\ell \in \mathbb{R}^{n\times n}$. ⋄

Proof. The entry associated with the i-th row and j-th column of the adjugate of a matrix M in $\mathbb{C}^{n\times n}$ is defined as

$$[\text{adj}(M)]_{ij} := (-1)^{i+j}\det([M]_{[ji]}),$$

where $[M]_{[ji]}$ is the $(n-1)\times(n-1)$ submatrix of M obtained by deleting the i-th row and j-th column of M. Let $M(z) = A + zB + \bar{z}C$, with $\bar{z} = z^{-1}$. Since $M(z)$ is a polynomial in $(z, \bar{z})$, and each entry of $\text{adj}\big(M(z)\big)$ is obtained from the determinant of an $(n-1)\times(n-1)$ matrix, then all $\big[\text{adj}\big(M(z)\big)\big]_{ij}$ is a polynomial in $(z, \bar{z})$ of degree at most $(n-1)$ (Sarachik, 1997), which yields the results. □

Lemma 1.12. *Let* $F \in \mathbb{C}^{n \times n}$. *Then* $\det(F^T \oplus F^*) \in \mathbb{R}$. $\diamond$

Proof. Since the eigenvalues of $F^T \oplus F^*$ are all possible pairwise sums of the eigenvalues of F^T and F^* (Bernstein, 2009, Proposition 7.2.3), the eigenvalues of $F^T \oplus F^*$ will always occur in complex conjugate pairs. Hence, the result of the lemma follows because the determinant of a matrix is the product of its eigenvalues. $\square$