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Typical properties of contractive operators on separable Hilbert spaces

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RESUMO

Este é um trabalho focado em Análise Funcional, mais precisamente na compreensão das propriedades típicas de operadores contrativos em diferentes topologias. Estudamos o espaço de contrações $C(H)$ em um espaço de Hilbert separável H e nosso principal objetivo é discutir a existência de uma órbita unitária típica de um operador e suas propriedades espectrais típicas. Como as propriedades típicas dependem da topologia que escolhemos para $C(H)$ discutimos os problemas mencionados anteriormente em cinco topologias diferentes: a topologia fraca, a topologia polinomial fraca, a topologia forte, a topologia forte-estrela e a topologia da norma. Vimos que essas topologias não necessariamente possuem as mesmas propriedades; na topologia fraca, na topologia polinomial fraca e na topologia forte-estrela, encontramos que duas contrações típicas não são equivalentes unitariamente. No entanto, na topologia forte, todas as contrações típicas são equivalentes unitariamente a um operador shift. Na topologia da norma, investigamos quão difícil é obter resultados semelhantes aos anteriores.

Palavras-chave: análise funcional; propriedades espectrais; contrações; propriedades típicas

ABSTRACT

This work is focused in Functional Analysis, more precisely in understanding the typical properties of contractive operators in different topologies. We study the space of contractions $C(H)$ in a separable Hilbert space H and our main goal is to discuss the existence of a typical unitary orbit of an operator and its typical spectral properties. Since typical properties depend on which topology we put on $C(H)$, we discuss the previously stated problems in five different topologies, the weak topology, the polynomial weak topology, the strong topology, the strong-star topology and the norm topology. We find that these topologies non-necessarily have the same properties, in weak topology, polynomial weak topology and strong-star topology we find that two typical contractions are not unitarly equivalent. However, in the strong topology all of the typical contractions are unitarly equivalent to a shift. In the norm topology we investigate how difficult is to obtain similar results in this topology.

Keywords: functional analysis; spectral properties; contractions; typical properties

Symbols

$\mathrm{Cl}\, A$	closure of A
$\mathrm{Cl}_\tau\, A$	closure of A relative to topology τ
$\mathrm{int}\, A$	interior of A
$\mathrm{int}_\tau\, A$	interior of A relative to topology τ
$\langle \cdot, \cdot \rangle$	inner product
$\mathrm{span}\{U\}$	linear subspace generated by U
$U^\perp$	$\{x \in H : \langle x, u \rangle = 0\ \forall u \in U\}$
$U \perp V$	$\langle u, v \rangle = 0$ for all $u \in U, v \in V$
$V \leqslant H$	V is a linear subspace of H
$H \cong K$	H and K are isomorphic
$H \cong_U K$	H and K are unitarily equivalent
$\bigoplus_{j \in J} H_j$	external direct sum of Hilbert Spaces $(H_j)_{j \in J}$
$\bigoplus_{j \in J}^{\mathrm{inn}} H_j$	internal direct sum of Hilbert Spaces $(H_j)_{j \in J}$
P_V	orthogonal projection onto V
B_V	$\{v \in V : \|\|v\|\| \leqslant 1\}$
S_V	$\{v \in V : \|\|v\|\| = 1\}$
$B(H)$	set of bounded linear operators $H \to H$
$C(H)$	set of contractive linear operators $H \to H$
$U(H)$	set of contractive unitary linear operators $H \to H$
$P(H)$	set of contractive positive linear operators $H \to H$
T^*	adjoint of T
$\mathcal{O}(T)$	$\{UTU^{-1} : U \in U(H)\}$
$\sigma(T)$	spectrum of T
$\sigma_p(T)$	point spectrum of T
$\sigma_c(T)$	continuous spectrum of T
$\sigma_r(T)$	residual spectrum of T
$\rho(T)$	resolvent of T
$\mathrm{supp}(\mu)$	support of a measure μ
$\mu \sim \nu$	μ and ν are equivalent
$\mu \perp \nu$	μ and ν are mutually singular

Contents

INTRODUCTION

This work is focused in Functional Analysis, more precisely in understanding the typical properties of contractive operators in different topologies. Many studies that focused on the typical behavior have been done over the years, the main one studied in this text, i.e., [6], the classical result which states that in $(C[0,1], ||\cdot||_\infty)$ the continuous nowhere differentiable functions are typical (see [17]), the Wonderland Theorem proved by Barry Simon (see [19]), which states that in every regular metric space of operators X an typical operator has only singular continuous spectrum, are examples of results in the so-called "soft-analysis".

As mentioned before, here we study in details the main results of [6] and when necessary we use other articles in order to prove the necessary results.

We study the space of contractions $C(H)$ in a separable Hilbert space H, i.e., the space of operators $H \to H$ with norm less or equal 1, and our main goal is discuss the existence of a typical unitary orbit of an operator and its typical spectral properties. One careful person could note that typical properties depend on which topology we put on $C(H)$, so study these problems in different topologies is a natural thing. In our studies, we discuss the previously stated problems in five different topologies.

We expect that the reader is familiar with the standard theory of Linear Algebra (a good reference is [3]), Topology (a good reference is [13]) and Functional Analysis (good references are [2] and [17]). Another important theory for understanding some proofs is the theory of spectral measures and Borel Functional Calculus (a good reference is [16]), however it is possible to read and understand mostly of the theorems without this knowledge.

Chapter 0 recalls some definitions and results that are needed along the text. If you feel comfortable with meager sets, G_δ sets, analytic sets, Polish Spaces, the Baire property, Baire Category Theorem and the Spectral Theorem, you can proceed to Chapter 1.

In Chapter 1 we define the main objects that will be studied here and summarize some of the general properties of the topologies. We also prove some results that involve more than one topology at same time.

In Chapter 2 we investigate the main properties of the weak topology and after that we study the typical properties of contractions with this topology. Some of the main results include: every weak typical contraction is unitary; the continuous spectrum of a weak typical contraction is all $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and, both, point spectrum and residual spectrum of a typical contraction is empty; two weak typical contractions are not unitarily equivalent.

In Chapter 3 we investigate the polynomial weak topology and after that we study the typical properties of contractions with this topology. Some of the main results include: $C(H)$ with the polynomial weak topology is a Polish space; the theory of polynomial weak typical contractions are exactly the theory of weak typical contractions.

In Chapter 4 we investigate the strong topology and after that we study the typical properties of contractions with this topology. Some of the main results include: all of the strong typical contractions are unitarily equivalent to a shift; in consequence the point spectrum of strong typical operators is the set $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$; the continuous spectrum is $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and the residual spectrum is empty.

In Chapter 5 we investigate the strong-star topology and after that we study the typical properties of contractions with this topology. Some of the main results include: strong-star typical contractions are the product of a unitary operator and a positive contractive operator with dense range; two strong-star typical contractions are not unitarily equivalent.

In Chapter 6 we investigate the norm topology and how difficult is to obtain similar results in this topology.

About the reading order, readers interested in a specific topology can go directly to the chapter about it after reading Chapter 1. However, we recommend the reader to read the main results of Chapter 2 before a deep dive into Chapter 3.

Regarding the proofs, a curious reader will note that most of the them are labelled with letters. These letters are just to make the reading more pleasurable, a point to a little pause to drink a coffee if you want to.

0. PRELIMINARIES

This chapter is a tool box which summarizes some definitions and theorems that are used along the text. Basically, here we recall some important definitions and theorems indicating a reference for non-familiar readers.

Definition 0.1. *A subset of a topological space X is called*

1. *meager in X if it can be written as a countable union of nowhere dense sets, i.e., a countable union of sets that it closure has empty interior.*
2. *non-meager in X if it is not meager in X.*
3. *co-meager in X if it is the complement of a meager set in X.*

Theorem 0.2. (*see* [20]) *If $U \subset A$ is meager and $V \subset B$, then $U \times V \subset A \times B$ is meager.*

Theorem 0.3. (*see* [13]) *If $A \subset Y \subset X$ and A is meager in Y, then A is meager in X.*

It is a well-know result that if Y is dense or open, the converse of Theorem 0.3 follows.

Definition 0.4. *A Baire space is any topological space satisfying any of the following statements (all of them are equivalent):*

1. *Every countable intersection of dense open sets is dense.*
2. *Every countable union of closed sets with empty interior has empty interior.*
3. *Every meager set has empty interior.*
4. *Every nonempty open set is non-meager.*
5. *Every co-meager set is dense.*
6. *Whenever a countable union of closed sets has an interior point, at least one of the closed sets has an interior point.*

An interesting fact about Baire spaces is that the notion of co-meager and meager sets are mutually excludent, i.e., a set cannot be meager and co-meager at the same time. Namely, if some set A is meager and co-meager in a Baire space X, we have that A^C is also meager, and so $A \cup A^C = X$ is meager. Therefore X has empty interior which is an absurd. With this in mind, we can understand meager sets as "small sets" and co-meagers sets as "large sets".

Remember that the Baire Theorem guarantees that any complete metric space is a Baire space (see Chapter 6 of [17]). Since our focus here is on Hilbert spaces which are complete metric spaces, we are in fact investigating some properties over Baire spaces. We point out here that we will say that a metric is complete if the space endowed with this metric is complete.

Definition 0.5. *A Polish space is a separable completely metrizable space.*

Naturally Polish spaces are Baire spaces too.

The next result characterizes metric spaces out of topological spaces:

Urysohn Metrization Theorem. (*see* [13]) *Let X be a second countable topological space. Then X is metrizable if and only if is T_1 and regular.*

Recall that:

Definition 0.6. *A topological space X is called*

- *second countable if there exists a countable collection $\mathcal{U}$ of open subsets of X such that any open subset of X can be written as the union of elements of some subfamily of $\mathcal{U}$. $\mathcal{U}$ is called a countable base of X;*
- *T_1 if for every pair of distinct points, each one has a neighborhood not containing the other point;*
- *regular if given any closed set F and any point x that does not belong to F, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint.*

In the appendix B we also present a characterization of Polish spaces.

Definition 0.7. *A subset $A \subset X$ of a topological space X is said to have the Baire property (or BP) if there exist an open set $U \subset X$ such that $A \triangle U$ is a meager subset of X. Here, $\triangle$ denotes the symmetric difference. (Sometimes, sets with the Baire property are called almost open sets).*

Another important class of sets are the analytic sets.

Definition 0.8. *Let A be a subset of X a Polish space. A is called an analytic set if there exists a Polish space Y and a continuous function $f : Y \to X$ such that $f(Y) = A$.*

There exists a useful result, usually attributed to Luzin and Sierpiński, which guarantees that every analytic set has the BP. We will write it down because of its usefulness, however, even if it isn't a standard result, we will not provide its proof. It involves a lot of descriptive set theory, which isn't the scope of this work (see [12] for the details).

Theorem 0.9. *Every analytic set has the BP.*

Definition 0.10. *A set in a Topological Space is called G_δ if it is an countable intersection of open sets.*

As was said before, in Appendix B we present a characterization of Polish spaces. However, there exists a simple and useful characterization of Polish subspaces of Polish spaces.

Theorem 0.11. (*see* [12]) *A subspace of a Polish space is Polish if and only if it is a G_δ.*

We also have a useful tool to characterize co-meager sets after G_δ sets. Since we didn't find any reference with a proof of such characterization we present one below.

Theorem 0.12. *Let X be a topological space. If Y is a dense G_δ subset of X then Y is co-meager.*

Proof. Write $Y = \bigcap_{n\in\mathbb{N}} U_n$, where each U_n is a open set. In order to prove that Y is co-meager it suffices show that each U_n^C is a nowhere dense set, i.e., that $\operatorname{int}\big(\operatorname{Cl}(U_n^C)\big) = \varnothing$.

Note that $\operatorname{Cl}(U_n^C) = U_n^C$, since U_n is open. Moreover, since U_n is dense it follows that $\operatorname{int}\big(\operatorname{Cl}(U_n^C)\big) = \varnothing$.

■

An important notion to keep in mind is the following:

Theorem 0.13. *Let X be a topological space and C be the class of all pairs of the form (x, y) where x is a net in X and y is an element of X. For any subset U with $y \in U$, we say that a net x converges to y with respect to U if x is eventually in U. We denote this by $x \to_U y$. Let*

$$\mathcal{T} = \{U \subset X \mid (x, y) \in C \text{ and } y \in U \text{ imply } x \to_U y\}.$$

Then $\mathcal{T}$ is a topology on X and its called the topology induced by this notion of convergence.

Proof. Obviously $x \to_X y$ for any pair $(x, y) \in C$ and $x \to_\varnothing y$ is vacuously true. Then $\varnothing, X \in \mathcal{T}$.

Let $U, V \in \mathcal{T}$, we will show that $U \cap V \in \mathcal{T}$. Let $y \in U \cap V$ and $(x, y) \in C$, since x is eventually in U and V there are i, j such that $x_r \in U$ and $x_s \in V$ for all $r \geqslant i$ and $s \geqslant j$. Since x is defined in a directed set there is a k such that $k \geqslant i, j$. It is clear that $x_t \in U \cap V$ for every $t \geqslant k$. We conclude that $U \cap V \in \mathcal{T}$.

Let $U_\alpha \in \mathcal{T}$ for every $\alpha \in I$, where I is an index set. We want to show that $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}$, so we pick $y \in \bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}$ and any $(x, y) \in C$. Using definition there exist an α such that $y \in U_\alpha$, x is eventually in $U_\alpha \subset \bigcup_{\alpha\in I} U_\alpha$. We conclude that x is eventually in $\bigcup_{\alpha\in I} U_\alpha$ and so $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology.

■

Definition 0.14. *Let $\mathcal{A}$ be the Borel σ-algebra in $\mathbb{R}$, H be a Hilbert space and let $\operatorname{Proj}(H)$ be the set of orthogonal projection operators on H. A (spectral) resolution of the identity on H is a map*

$$P : \mathcal{A} \to \operatorname{Proj}(H)$$

so that

1. *$P(\mathbb{R}) = I$, and*
2. *If $\Lambda = \bigcup_{j\in\mathbb{N}} \Lambda_j$, with $\Lambda_j \in \mathcal{A}$ and are pairwise disjoint for each $j \in \mathbb{N}$, then one has the strong limit*

$$P(\Lambda) = s\text{-}\lim \sum_{j=1}^{n} P(\Lambda_j)$$

Using this notion of resolution of the identity we can define a notion of integral which results in a linear operator (as discussed in [16]). The principal result of this theory to our studies is the following.

Theorem 0.15. (*see* [16]) *To each self-adjoint operator* $T : \operatorname{dom} T \subset H \to H$ *corresponds a unique resolution of the identity* P^T *on* H*, so that*

$$T = \int t dP^T(t)$$

.

1. GENERAL DEFINITIONS AND RESULTS

In what follows always let H be a Hilbert separable space. We know that, under this assumption, the space admits an orthonormal basis.

We will use the standard notation $\langle \cdot, \cdot \rangle$ for the inner product in H. If $U \subset H$, span$\{U\}$ will denotes the linear subspace of H generated by U, and $U^{\perp} := \{x \in H : \langle x, u \rangle = 0 \ \forall u \in U\}$. If $U, V \subset H$, then $U \perp V$ indicates that $\langle u, v \rangle = 0$ for each $u \in U, v \in V$. $V \leqslant H$ denotes that V is a linear subspace of H.

Let X be a Baire space and let Φ be a property on the points of X. Φ is called a typical property on X, or that a typical element of X satisfies Φ, if $\{x \in X \mid x \text{ satisfies } \Phi\}$ is a co-meager subset of X. Note that if $\{\Phi_n\}_{n \in \mathbb{N}}$ is a collection of typical properties on X, then a typical element of X satisfies all of the Φ_n properties simultaneously.

If $V \leqslant H$, we denote the orthogonal projection onto V by P_V. Furthermore, let $B_V := \{v \in V : ||v|| \leqslant 1\}$ and $S_V := \{v \in V : ||v|| = 1\}$.

Set for each $T \in B(H)$ it unitary orbit $\mathcal{O}(T) := \{UTU^{-1} : U \in U(H)\}$ and denote the adjoint of T by T^*. The spectrum of T is denoted by $\sigma(T)$, its point spectrum by $\sigma_p(T)$, its continuous spectrum by $\sigma_c(T)$ and its residual spectrum by $\sigma_r(T)$. The resolvent of T is denoted by $\rho(T)$. (See [17] for details).

Definition 1.1. *Let $B(H)$, $C(H)$, $U(H)$ and $P(H)$ denote the sets of bounded, contractive, unitary and contractive positive self-adjoint linear $H \to H$ operators respectively.*

Definition 1.2. *Let $T, T_\alpha \in B(H)$ with $\alpha \in \Lambda$, where Λ is any directed set.*

1. *We say that $\{T_\alpha\}_{\alpha \in \Lambda}$ converges to T weakly if for every $x, y \in H$, $\lim_\alpha \langle T_\alpha x, y \rangle = \langle Tx, y \rangle$. We denote this by $T = w\text{-}\lim_\alpha T_\alpha$, and the topology induced by this notion of convergence is called the weak topology. Topological notions referring to this topology are preceded by w-.*

2. *We say that $\{T_\alpha\}_{\alpha \in \Lambda}$ converges to T weakly polynomially if for every $k \in \mathbb{N}$, $w\text{-}\lim_\alpha T_\alpha^k = T^k$. We denote this by $T = pw\text{-}\lim_\alpha T_\alpha$, and the topology induced by this notion of convergence is called the polynomial weak topology. Topological notions referring to this topology are preceded by pw-.*

3. *We say that $\{T_\alpha\}_{\alpha \in \Lambda}$ converges to T strongly if for every $x \in H$, $\lim_\alpha T_\alpha x = Tx$. We denote this by $T = s\text{-}\lim_\alpha T_\alpha$, and the topology induced by this notion of convergence is called the strong topology. Topological notions referring to this topology are preceded by s-.*

4. *We say that $\{T_\alpha\}_{\alpha \in \Lambda}$ converges to T in the strong-star sense, if $T = s\text{-}\lim_\alpha T_\alpha$ and $T^* = s\text{-}\lim_\alpha T_\alpha^*$. We denote this by $T = s^*\text{-}\lim_\alpha T_\alpha$, and the topology induced by this notion of convergence is called the strong-star topology. Topological notions referring to this topology are preceded by s^*-.*

The obvious observation is that $w \subset pw \subset s \subset s^*$.

The next result helps us to understand some properties of the topologies defined above. The proof of each statement will be presented in the Chapter of the respetive topology.

Theorem 1.3. *Let $\{e_i : i \in \mathbb{N} \setminus \{0\}\} \subset B_H$ be a dense subset.*

1. *For every $A, B \in B(H)$, set*

$$d_w(A, B) = \sum_{i,j \in \mathbb{N} \setminus \{0\}} 2^{-i-j} |\langle Ae_i, e_j \rangle - \langle Be_i, e_j \rangle|$$

Then, d_w is a complete separable metric on $C(H)$ which generates the weak topology.

2. *For every $A, B \in B(H)$, set*

$$d_{pw}(A, B) = \sum_{i,j,n \in \mathbb{N} \setminus \{0\}} 2^{-i-j-n} |\langle A^n e_i, e_j \rangle - \langle B^n e_i, e_j \rangle|$$

Then, d_{pw} is is a complete separable metric on $C(H)$ which generates the polynomial weak topology.

3. *For every $A, B \in B(H)$, set*

$$d_s(A, B) = \sum_{i \in \mathbb{N} \setminus \{0\}} 2^{-i} ||Ae_i - Be_i||$$

Then, d_s is a complete separable metric on $C(H)$ which generates the strong topology.

4. *The strong-star topology in $C(H)$ is generated by the metric $d_{s^*}(A, B) = d_s(A, B) + d_s(A^*, B^*)$, and this metric is complete.*

We observe that Theorem 1.3 presents a characterization of these topologies by sequences.

Theorem 1.4. *The weak, polynomial weak, strong, and strong-star topologies coincide over $U(H)$. Moreover, $U(H)$ is a Polish space when endowed with such topologies.*

Proof. As noted before one has $w \subset pw \subset s \subset s^*$ over $B(H)$. We will show that $s \subset w$ and $s^* \subset s$.

Let us start proving that the strong-star limit of a net $(U_\alpha)_{\alpha \in \Lambda} \subset U(H)$, say U, is also unitary. To see this, note that

$$\begin{aligned} ||Ux|| &= ||\lim U_\alpha x|| = \lim ||U_\alpha x|| = \lim ||x|| = ||x|| \\ ||U^* x|| &= ||\lim U^*_\alpha x|| = \lim ||U^*_\alpha x|| = \lim ||x|| = ||x|| \end{aligned}$$

given that for each $\alpha \in \Lambda$ and each $x \in H$ we have $||U_\alpha x|| = ||U^*_\alpha x|| = ||x||$. So, both U, U^* are isometries, from which follows that U is unitary.

Let $(U_\alpha)_{\alpha \in \Lambda}$ be a net such that $\text{s-}\lim U_\alpha = U$ for some $U \in U(H)$. We will show that $\text{s*-}\lim U_\alpha = U$. Thus, fix $x \in H$, $U \in U(H)$ and note that the only thing we need to prove is that $\text{s-}\lim U_\alpha^* = U^*$. So,

$$\begin{aligned}||U_\alpha^* x - U^* x||^2 &= ||U_\alpha^* x||^2 + ||U^* x||^2 - 2\,\mathrm{Re}\langle U_\alpha^* x, U^* x\rangle \\ &= 2||x||^2 - 2\,\mathrm{Re}\langle U_\alpha^* x, U^* x\rangle\end{aligned}$$

Then, since $\text{w-}\lim U_\alpha = U$, we get $\langle U_\alpha^* x, U^* x\rangle = \langle x, U_\alpha U^* x\rangle \to \langle x, UU^* x\rangle = \langle x, x\rangle = ||x||^2$ and so

$$||U_\alpha^* x - U^* x||^2 \to 2||x||^2 - 2\,\mathrm{Re}\,||x||^2 = 0$$

Therefore, $s^* \subset s$.

Now, let $(U_\alpha)_{\alpha\in\Lambda}$ be a net such that $\text{w-}\lim U_\alpha = U$ for some $U \in U(H)$. We will show that $\text{s-}\lim U_\alpha = U$. Fix $x \in H$, $U \in U(H)$, and note that

$$\begin{aligned}||U_\alpha x - Ux||^2 &= \langle U_\alpha x, U_\alpha x\rangle + \langle Ux, Ux\rangle - 2\,\mathrm{Re}\langle U_\alpha x, Ux\rangle \\ &= 2\langle x, x\rangle - 2\,\mathrm{Re}\langle U_\alpha x, Ux\rangle\end{aligned}$$

Then, since $\text{w-}\lim U_\alpha = U$ we have $\lim\langle U_\alpha x, Ux\rangle = \langle Ux, Ux\rangle = \langle x, x\rangle$ and so

$$\begin{aligned}\lim ||U_\alpha x - Ux||^2 &= \lim(2\langle x, x\rangle - 2\,\mathrm{Re}\langle U_\alpha x, Ux\rangle) \\ &= 2\langle x, x\rangle - 2\,\mathrm{Re}\langle x, x\rangle = 0\end{aligned}$$

This shows that $\text{s-}\lim U_\alpha = U$, and so $s \subset w$. We conclude now that $w = s = s^*$, and since $w \subset pw \subset s \subset s^*$ it follows that $w = pw = s = s^*$ in $U(H)$.

It remains to prove that $U(H)$ is Polish. By Theorem 1.2 $(C(H), s^*)$ is a complete metric space and so, by the fact that $U(H)$ is s*-closed in $C(H)$ we conclude that $U(H)$ is a complete metric space. Using again Theorem 1.2, $(C(H), s)$ is a separable metric space, so $U(H) \subset C(H)$ is separable. Hence, $U(H)$ is a separable complete metric space, i.e., a Polish space.

■

An interesting phenomenon is that unitary equivalence of Hilbert spaces preserves the convergences with respect to each one of these topologies:

Theorem 1.5. *Suppose that $H \cong_U K$ (i.e., H and K unitarily equivalent), let $\psi : H \to K$ be a unitary operator, let $(A_n)_n \subset C(H)$ and $A \in C(H)$. Then*

1. *if $\text{w-}\lim A_n = A$ then $\text{w-}\lim \psi A_n \psi^{-1} = \psi A \psi^{-1}$;*
2. *if $\text{pw-}\lim A_n = A$ then $\text{pw-}\lim \psi A_n \psi^{-1} = \psi A \psi^{-1}$;*
3. *if $\text{s-}\lim A_n = A$ then $\text{s-}\lim \psi A_n \psi^{-1} = \psi A \psi^{-1}$;*
4. *if $\text{s}^*\text{-}\lim A_n = A$ then $\text{s}^*\text{-}\lim \psi A_n \psi^{-1} = \psi A \psi^{-1}$.*

Proof. (1) Fix $x, y \in K$, then

$$\langle \psi(A_n - A)\psi^{-1}x, y\rangle_K = \langle (A_n - A)\psi^{-1}x, \psi^{-1}y\rangle_H \xrightarrow{n\to\infty} 0$$

and so $\text{w-}\lim \psi A_n \psi^{-1} = \psi A \psi^{-1}$.

(2) If $\text{pw-}\lim A_n = A$, then $\text{w-}\lim A_n^k = A^k$ for every $k \in \mathbb{N}$. By using item (1), we obtain $\text{w-}\lim \psi A_n^k \psi^{-1} = \psi A^k \psi^{-1}$ for every $k \in \mathbb{N}$. And for each $k \in \mathbb{N}$ we also note that

$$(\psi A_n \psi^{-1})^k = \psi A_n \psi^{-1} \psi A_n \psi^{-1} \cdots \psi A_n \psi^{-1} = \psi A_n^k \psi^{-1}$$

Analogously we have $(\psi A\psi^{-1})^k = \psi A^k\psi^{-1}$ for each $k \in \mathbb{N}$. Therefore $\text{pw-}\lim \psi A_n\psi^{-1} = \psi A\psi^{-1}$.

(3) Fix $x \in K$. Then

$$||\psi(A_n - A)\psi^{-1}x||_K = ||(A_n - A)\psi^{-1}x||_H \xrightarrow{n\to\infty} 0$$

and so $\text{s-}\lim \psi A_n\psi^{-1} = \psi A\psi^{-1}$.

(4) If $\text{s}^*\text{-}\lim A_n = A$ then $\text{s-}\lim A_n = A$ and $\text{s-}\lim A_n^* = A^*$. By using item (3), we obtain $\text{s-}\lim \psi A_n\psi^{-1} = \psi A\psi^{-1}$ and $\text{s-}\lim \psi A_n^*\psi^{-1} = \psi A^*\psi^{-1}$. Moreover, we note that

$$(\psi A\psi^{-1})^* = (\psi^{-1})^* A^* \psi^* = \psi A^* \psi^{-1}$$

and so $(\psi A_n\psi^{-1})^* = \psi A_n^*\psi^{-1}$ by the same argument. Thus, $\text{s}^*\text{-}\lim \psi A_n\psi^{-1} = \psi A\psi^{-1}$.

■

2. THE WEAK TOPOLOGY

2.1 Understanding the topology

Recall that we have defined the weak topology by using nets, i.e., we have defined the topology induced by the convergence of nets. This is natural by considering the results discussed here, but in order to write some of the following proofs we will need a description in terms of open sets.

Definition 2.1. *Define the WOT topology (Weak Operator Topology) in $B(H)$ by the coarsest topology for which the map $f_{x,y} : T \to \langle Tx, y\rangle$ is continuous for every $x, y \in H$. In other words, the WOT topology is the topology generated by the sub-basis consisting of the elements*

$$f_{x,y}^{-1}(B(Tx, \varepsilon)) = \{A \in B(H) \ : \ |\langle (A - T)x, y\rangle| < \varepsilon\}$$

where $x, y \in H$, $\varepsilon > 0$ and $T \in B(H)$.

The next result shows that this topology coincides with the weak topology previously defined.

Theorem 2.2. *A net $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ converges to a $T \in B(H)$ in the WOT topology if, and only if, $w\text{-}\lim T_\alpha = T$*

Proof. (a) Let $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ be a net converging to $T \in B(H)$ in the WOT topology. Fix $x, y \in H$ and note that for every $\varepsilon > 0$, the set $V_\varepsilon = \{A \in B(H) \ : \ |\langle (A-T)x, y\rangle| < \varepsilon\}$ is a neighborhood of T in the WOT topology. By using the convergence of $(T_\alpha)_\alpha$, we conclude that there exists $\beta \in \Lambda$ such that for every $\alpha > \beta$, $T_\alpha \in V_\varepsilon$. With this, if $\alpha > \beta$, then

$$|\langle (T_\alpha - T)x, y\rangle| < \varepsilon$$

and since $x, y \in H$ are arbitrary we conclude that $\text{w-}\lim T_\alpha = T$.

(b) Let $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ be a net such that $\text{w-}\lim T_\alpha = T$. Pick a neighborhood V of T in the WOT topology. We can assume that

$$V = \bigcap_{i=1}^{k} \{A \in B(H) \ : \ |\langle (A - T)x_i, y_i\rangle| < \varepsilon\}$$

for some $k \in \mathbb{N}$, and $x_i, y_i \in H$ for all $i \leqslant k$.

Since $\text{w-}\lim T_\alpha = T$ for all $i \leqslant k$, there exists $\Gamma_i \in \Lambda$ such that for all $\alpha > \Gamma_i$

$$|\langle (T_\alpha - T)x_i, y_i\rangle| < \varepsilon$$

Let $\Gamma \in \Lambda$ be such that $\Gamma > \Gamma_i$ for every $i \leqslant k$, so if $\alpha > \Gamma$ for all $i \leqslant k$, then $|\langle (T_\alpha - T)x_i, y_i\rangle| < \varepsilon$. In particular, $T_\alpha \in \bigcap_{i=1}^{k}\{A \in B(H) \ : \ |\langle (A - T)x_i, y_i\rangle| < \varepsilon\}$. This concludes the proof.

■

Now we are able to prove the metrizability of this topology in $C(H)$, as stated in Theorem 1.3.

Theorem 2.3. *Let $\{e_i : i \geqslant 1\} \subset H$ be a dense subset of B_H. Then,*

$$d_w(A,B) = \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Ae_i, e_j\rangle - \langle Be_i, e_j\rangle|$$

is a complete separable metric on $C(H)$ which generates the weak topology.

Proof. (a) We start proving that $d_w(A,B)$ is indeed a metric. Define

$$||A||_w := \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Ae_i, e_j\rangle|$$

and note that $||A-B||_w = d_w(A,B)$, so it is sufficient to show that $||\cdot||_w$ is a norm.

- Note that

$$\begin{aligned} ||A||_w &= \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Ae_i, e_j\rangle| \leqslant \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}||A||\cdot||e_i||\cdot||e_j|| \\ &= ||A|| \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j} = ||A|| \end{aligned}$$

  so the sum is well defined.
- Fix $\alpha \in \mathbb{C}$. Then,

$$||\alpha A||_w = \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle \alpha Ae_i, e_j\rangle| = |\alpha| \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Ae_i, e_j\rangle| = |\alpha|\cdot||A||_w$$

  .
- Fix $A, B \in C(H)$. Then

$$\begin{aligned} ||A+B||_w &= \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle (A+B)e_i, e_j\rangle| = \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Ae_i, e_j\rangle + \langle Be_i, e_j\rangle| \\ &\leqslant \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Ae_i, e_j\rangle| + \sum_{i,j\in\mathbb{N}\setminus\{0\}} 2^{-i-j}|\langle Be_i, e_j\rangle| = ||A||_w + ||B||_w \end{aligned}$$

- Since $2^{-i-j}|\langle Ae_i, e_j\rangle| \geqslant 0$ for all $i, j \in \mathbb{N}$ we have that $||A||_w \geqslant 0$.
- If $A = 0$, it is obvious that $||A||_w = 0$.

  On the other hand if $||A||_w = 0$ we obtain that $|\langle Ae_i, e_j\rangle| = 0$ for every $i, j \in \mathbb{N}$. Fix $x_0, y_0 \in H \setminus \{0\}$ and note that $x = x_0/||x_0||, y = y_0/||y_0|| \in S_H$, so there exists subsequences $(e_{n_k})_k, (e_{m_k})_k$ such that $\lim_k e_{n_k} = x$ and $\lim_k e_{m_k} = y$. Thus,

$$\begin{aligned}
|\langle Ax, y\rangle| &= |\langle A(x-e_{n_k}), y\rangle + \langle Ae_{n_k}, y\rangle| \\
&= |\langle A(x-e_{n_k}), y-e_{m_k}\rangle + \langle Ae_{n_k}, y-e_{m_k}\rangle + \langle A(x-e_{n_k}), e_{m_k}\rangle \\
&\quad + \langle Ae_{n_k}, e_{m_k}\rangle| \\
&\leqslant |\langle A(x-e_{n_k}), y-e_{m_k}\rangle| + |\langle Ae_{n_k}, y-e_{m_k}\rangle| + |\langle A(x-e_{n_k}), e_{m_k}\rangle| \\
&\quad + |\langle Ae_{n_k}, e_{m_k}\rangle| \\
&\leqslant ||A||\cdot||x-e_{n_k}||\cdot||y-e_{m_k}|| + ||Ae_{n_k}||\cdot||y-e_{m_k}|| \\
&\quad + ||A||\cdot||x-e_{n_k}||\cdot||e_{m_k}|| \xrightarrow{k\to\infty} 0
\end{aligned}$$

and so, $\langle Ax, y\rangle = 0$. By making some calculation we obtain that $\langle Ax_0, y_0\rangle = 0$, and since $x_0, y_0 \in H\setminus\{0\}$ are arbitrary, it follows that $A = 0$.

Hence, $||\cdot||_w$ is a norm in $C(H)$, and it follows that $d_s(A,B)$ is a metric in $C(H)$. Now, we want to see that this metric generates exactly the weak topology as previously defined.

(b) We begin by fixing V a w-open set in $C(H)$, by Theorem 2.2 we can assume that there exists $k \in \mathbb{N}$, $T \in C(H)$, $\varepsilon > 0$ and $x_i, y_i \in B_H$ for all $i \leqslant k$ such that

$$V = \bigcap_{i=1}^{k}\{A \in B(H) \ : \ |\langle (A-T)x_i, y_i\rangle| < \varepsilon\}$$

We want to obtain a $\delta > 0$ such that the ball

$$B_w(T,\delta) = \{A \in C(H) : d_w(A,T) < \delta\}$$

is a subset of V. Note that if $d_w(A,T) < \delta$, then $|\langle Ae_i, e_j\rangle - \langle Te_i, e_j\rangle| < 2^{i+j}\delta$ for each $i, j \in \mathbb{N}\setminus\{0\}$.

Now we use the density of $\{e_i : i \geqslant 1\} \subset H$ to obtain subsequences $(\psi_n^i)_{n\in\mathbb{N}}$ and $(\xi_n^i)_{n\in\mathbb{N}}$ of $\{e_i : i \geqslant 1\}$ such that $\lim_n \psi_n^i = y_i$ and $\lim_n \xi_n^i = x_i$ for every $i \leqslant k$.

Let $C > 0$ be such that $C^2 < \varepsilon/8$ and $C < \varepsilon/8$. We can obtain, for every $i \leqslant k$, a $N_i \in \mathbb{N}$ such that $n \geqslant N_i$ implies $||x_i - \xi_n^i|| < C$ and $||y_i - \psi_n^i|| < C$. Fix $N = \max_{i\leqslant k} N_i$, and now for this N write $\psi_N^i = e_{p(i)}$ and $\xi_N^i = e_{q(i)}$. If $M = \max_{i\leqslant k} p(i) + q(i)$, choose $\delta > 0$ such that $2^M\delta < \varepsilon/4$. Finally, if $A \in B_w(T,\delta)$, it follows that for every $i \leqslant k$:

$$\begin{aligned}
|\langle (T-A)x_i, y_i\rangle| &= |\langle (T-A)(x_i-\xi_N^i), y_i\rangle + \langle (T-A)\xi_N^i, y_i\rangle| \\
&\leqslant |\langle (T-A)(x_i-\xi_N^i), y_i-\psi_N^i\rangle| + |\langle (T-A)\xi_N^i, y_i-\psi_N^i\rangle| \\
&\quad + |\langle (T-A)(x_i-\xi_N^i), \psi_N^i\rangle| + |\langle (T-A)\xi_N^i, \psi_N^i\rangle| \\
&\leqslant ||T-A||\cdot||x_i-\xi_N^i||\cdot||y_i-\psi_N^i|| + ||T-A||\cdot||\xi_N^i||\cdot||y_i-\psi_N^i|| \\
&\quad + ||T-A||\cdot||x_i-\xi_N^i||\cdot||\psi_N^i|| + |\langle (T-A)e_{q(i)}, e_{p(i)}\rangle| \\
&\leqslant 2C^2 + 2C + 2C + 2^{p(i)+q(i)}\delta \leqslant 2\cdot\frac{\varepsilon}{8} + 2\cdot\frac{\varepsilon}{8} + 2\cdot\frac{\varepsilon}{8} + 2^M\delta \\
&< 3\cdot\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon
\end{aligned}$$

therefore, $A \in \bigcap_{i=1}^{k}\{A \in B(H) \ : \ |\langle (A-T)x_i, y_i\rangle| < \varepsilon\}$.

(c) Let $T \in C(H)$, $\varepsilon > 0$ and set $B_w(T,\varepsilon) = \{A \in C(H) : d_w(A,T) < \varepsilon\}$. We have to obtain $\delta > 0$, $k \in \mathbb{N}$ and $x_i, y_i \in B_H$ with $i \leqslant p$ such that the set

$$V = \bigcap_{i=1}^{p} \{A \in B(H) \ : \ |\langle (A-T)x_i, y_i \rangle| < \delta\}$$

is contained in $B_w(T, \varepsilon)$. Let $k \in \mathbb{N}$ be such that $2^{2(-k+1)} < \varepsilon/4$, and for every $j \leqslant k$, let

$$V_j = \bigcap_{i=1}^{k} \{A \in B(H) \ : \ |\langle (A-T)e_i, e_j \rangle| < \delta\}$$

and

$$V = \bigcap_{j=1}^{k} V_j \, .$$

If $\delta = \varepsilon/4$, then $A \in V$ implies

$$\begin{aligned} d_w(A,T) &= \sum_{i,j \in \mathbb{N} \setminus \{0\}} 2^{-i-j} |\langle (A-T)e_i, e_j \rangle| \\ &= \sum_{1 \leqslant i,j \leqslant k} 2^{-i-j} |\langle (A-T)e_i, e_j \rangle| + \sum_{i,j > k} 2^{-i-j} |\langle (A-T)e_i, e_j \rangle| \\ &\leqslant \frac{\varepsilon}{4} \Big(\sum_{1 \leqslant i,j \leqslant k} 2^{-i-j} \Big) + \sum_{i,j>k} 2^{-i-j} (||A|| + ||T||) ||e_i|| \cdot ||e_j|| \\ &\leqslant \frac{\varepsilon}{4} + 2 \sum_{i=k}^{\infty} \sum_{j=k}^{\infty} 2^{-i-j} = \frac{\varepsilon}{4} + 2 \cdot 2^{2(-k+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \, , \end{aligned}$$

from which follows that $A \in B_w(T, \varepsilon)$.

(d) Since every compact metric space is complete, so it is sufficient to show that $C(H)$ is a compact space. We follow the arguments presented in [5].

For each $g \in B_H$ let B_g be a copy of B_H equipped with the weak topology. Since H is reflexive, we use Kakutani's Theorem to conclude that every B_g is compact. In particular, $X = \prod_{g \in B_H} B_g$ is compact due to Tychonoff's Theorem.

Let $\phi : (C(H), w) \to X$ be defined by $\phi(T) = (Tg)_{g \in B_H}$. We claim that ϕ is continuous and injective.

Namely, let $(T_\alpha)_{\alpha \in \Lambda} \subset C(H)$ be a net such that $\text{w-}\lim T_\alpha = T$ for some $T \in C(H)$. We know that for each fixed $g \in B_H$ we have that $\lim \langle T_\alpha g, y \rangle = \langle Tg, y \rangle$ for every $y \in H$. So, $\text{w-}\lim T_\alpha g = Tg$, and in particular, $\lim (T_\alpha g)_{g \in B_H} = (Tg)_{g \in B_H}$. Hence, $\lim \phi(T_\alpha) = \phi(T)$ and we conclude that ϕ is continuous.

If $\phi(T) = \phi(S)$, it follows that $Tx = Sx$ for every $x \in B_H$, and so $T = S$. Therefore, ϕ is injective.

It remains to prove that $\phi(C(H))$ is closed in X (which is compact), and so compact. Let $(T_\alpha)_{\alpha \in \Lambda} \subset C(H)$ be such that $\lim \phi(T_\alpha) = F \in X$. If π_g denotes the projection into the g-th coordinate, we have that $\text{w-}\lim T_\alpha g = \pi_g(F)$, so we need to construct a linear map $T : H \to H$ such that $Tg = \pi_g(F)$ for every $g \in B_H$.

Define, for each $g \in B_H$, the map $Tg = \pi_g(F)$ and extend it to all H by linearity.

Fix $\beta \in \mathbb{C}$ and $x, y \in H$. If $\beta x + y \notin B_H$, the linearity follows directly from the definition, but if $\beta x + y \in B_H$, then

$$\begin{aligned} T(\beta x + y) &= \pi_{(\beta x+y)}(F) = \text{w-}\lim T_\alpha(\beta x + y) \\ &= \beta \text{w-}\lim T_\alpha x + \text{w-}\lim T_\alpha y = \\ &= \beta \pi_x(F) + \pi_y(F) = \beta T x + T y\,. \end{aligned}$$

So, T is a linear map such that $Tg = \pi_g(F)$ for every $g \in B_H$. Note that T is also contractive: if $g \in B_H$ we have that $||T_\alpha g|| \leqslant 1$ for every $\alpha \in \Lambda$, i.e., $T_\alpha g \in B_H$ for every $\alpha \in \Lambda$. Recall that B_H is a closed convex set, so it is weakly closed (see [2]). We conclude that $Tg = \pi_g(F) \in B_H$, i.e., $||Tg|| \leqslant 1$. Taking the supremum we conclude that $||T|| \leqslant 1$, then $T \in C(H)$, and obviously $F = \phi(T)$. Hence, $\phi(C(H))$ is closed in H.

Since every B_g is Hausdorff, X is also Hausdorff. And so, by using that ϕ is continuous we conclude that $C(H)$ is compact.

(e) In Theorem 4.2 we have shown that the space $(C(H), s)$ is separable. Since $w \subset s$, it follows that the topology $(C(H), w)$ is also separable.

■

2.2 Studying the properties of w-typical contractions

A w-typical contraction in $C(H)$ is a unitary operator, i.e., $U(H)$ is w-co-meager in $C(H)$. This was proved by Eisner in [7] and we will show in details what she did.

Lemma 2.4. *The weak closure of $U(H)$ is $C(H)$*

Proof. It follows from Theorem 3.5 that $\mathrm{Cl}_{pw}\, U(H) = C(H)$. Now by $w \subset pw$ we obtain $C(H) = \mathrm{Cl}_{pw}\, U(H) \subset \mathrm{Cl}_w\, U(H) \subset C(H)$, i.e., $\mathrm{Cl}_w\, U(H) = C(H)$.

■

Lemma 2.5. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of linear operators in H such that $w\text{-}\lim T_n = T$. If $||T_n x|| \leqslant ||Tx||$ for all $x \in H$, then $s\text{-}\lim T_n = T$.*

Proof. If $x \in H$, then

$$\begin{aligned} ||T_n x - Tx||^2 &= ||T_n x||^2 + ||Tx||^2 - 2\,\mathrm{Re}\langle T_n x, Tx\rangle \\ &\leqslant 2||Tx||^2 - 2\,\mathrm{Re}\langle T_n x, Tx\rangle\,. \end{aligned}$$

Note that $2||Tx||^2$ is a real number, so we can write the last part of the equation as $2\,\mathrm{Re}\langle Tx - T_n x, Tx\rangle$, which converges to zero since $\text{w-}\lim T_n = T$. With that, $||T_n x - Tx||^2$ converges to zero and we conclude that $\text{s-}\lim T_n = T$.

■

Denote by $I(H)$ the set of all isometries $H \to H$ and recall that $U(H) \subset I(H) \subset C(H)$. We now state and prove the main result of this section

Theorem 2.6. *$U(H)$ is w-co-meager in $C(H)$*

Proof. This proof is divided into two steps. In the first step we show that $I(H)\backslash U(H)$ is a w-meager set, in the second one that $C(H)\backslash I(H)$ is a w-meager set. By combining both results it is easy to show that $C(H)\backslash U(H)$ is w-meager. Before we proceed, fix $\{x_j\}_{j\in\mathbb{N}}$ a dense subset of $H\backslash\{0\}$.

(a) First of all, note that $I(H)\backslash U(H)$ is exactly the set of non-invertible isometries. Now, let $T \in I(H)\backslash U(H)$.

Note that $\operatorname{Im} T$ is closed. Indeed, if $y \in \operatorname{Cl}(\operatorname{Im} T)$, then there exists $(z_n)_{n\in\mathbb{N}} \subset H$ such that $Tz_n \to y$. The sequence $(Tz_n)_n$ converges, so it is a Cauchy sequence. By using that T is a isometry we conclude that (z_n) is also a Cauchy sequence. Since H is Hilbert, there exists $z \in H$ such that $z_n \to z$, and then $y = \lim_n Tz_n = Tz$. So, $y \in \operatorname{Im} T$.

Since T is not invertible, $\operatorname{Im}(T) \neq H$, so there exists $j \in \mathbb{N}$ such that $d(x_j, \operatorname{Im}(T)) > 0$. By setting $M_{j,k} = \{T \in I(H) : d(x_j, \operatorname{Im} T) > 1/k\}$, we can write

$$I(H)\backslash U(H) = \bigcup_{k,j=1}^{\infty} M_{j,k}\,.$$

Note that in order to prove that for each $j, k \in \mathbb{N}$, $\operatorname{int}_w\big(\operatorname{Cl}_w(M_{j,k})\big) = \varnothing$, it is sufficient to prove that $U(H) \cap \operatorname{Cl}_w(M_{j,k}) = \varnothing$. This is because if we pick $x \in \operatorname{int}_w\big(\operatorname{Cl}_w(M_{j,k})\big)$ then there exists a w-open subset $V \subset C(H)$ such that $x \in V \subset \operatorname{Cl}_w(M_{j,k})$. By Lemma 2.4 $U(H)$ is a dense subset, so there exists $p \in V \cap U(H)$, and so $p \in U(H) \cap \operatorname{Cl}_w(M_{j,k})$.

Fix $j, k \in \mathbb{N}$ and suppose that $U \in U(H) \cap \operatorname{Cl}_w(M_{j,k})$. Thus, there exists $\{T_n\}_{n\in\mathbb{N}} \subset M_{j,k}$ such that w-$\lim T_n = U$. But $||T_n x|| = ||x|| = ||Ux||$ for all $n \in \mathbb{N}$, given that each T_n is an isometry and U is unitary. By Lemma 2.5 s-$\lim T_n = U$.

If $y = U^{-1}x_j$, then $\lim T_n y = Uy = x_j$. Hence, $\lim_n d(x_j, \operatorname{Im} T_n) = 0$,which is absurd, because $T_n \in M_{j,k}$ for all $n \in \mathbb{N}$, i.e., $d(x_j, \operatorname{Im} T_n) > 1/k$ for all $n \in \mathbb{N}$. Therefore, for each $j, k \in \mathbb{N}$, $U(H) \cap \operatorname{Cl}_w(M_{j,k}) = \varnothing$, and by the previous discussion, $\operatorname{int}_w\big(\operatorname{Cl}_w(M_{j,k})\big) = \varnothing$, for all $j, k \in \mathbb{N}$. This concludes the proof that $I(H)\backslash U(H)$ is a w-meager set.

(b) Now we show that $C(H)\backslash I(H)$ is w-meager. Set, for each $j, k \in \mathbb{N}$

$$N_{j,k} := \left\{T \in C(H) : \frac{||Tx_j||}{||x_j||} < 1 - \frac{1}{k}\right\}$$

We claim that

$$C(H)\backslash I(H) = \bigcup_{k,j=1}^{\infty} N_{j,k}\,.$$

Namely, suppose that $T \in N_{j,k}$, for some $j, k \in \mathbb{N}$, and that $T \in I(H)$. Then, $||Tx_j||/||x_j|| = ||x_j||/||x_j|| = 1$ and $||Tx_j||/||x_j|| < 1 - 1/k < 1$, an absurd. So, if $T \in \bigcup_{k,j=1}^{\infty} N_{j,k}$, then $T \in C(H) \setminus I(H)$.

On the other hand, if $T \in C(H)\backslash I(H)$, then there exists x_j such that $||Tx_j|| < ||x_j||$. Suppose, on the contrary, that for every $j \in \mathbb{N}$, $||Tx_j|| \geqslant ||x_j||$. By picking an arbitrary $y \in H$ we obtain a subsequence $\{\zeta_m\}_{m\in\mathbb{N}}$ of $\{x_j\}_{j\in\mathbb{N}}$ which converges to y. With that, $||Ty|| = ||\lim_m T\zeta_m|| = \lim_m ||T\zeta_m|| \geqslant \lim_m ||\zeta_m|| = ||y||$. On the another hand, by using the fact that T is a contraction, we get $||Ty|| \leqslant ||T|| \cdot ||y|| \leqslant ||y||$. Now, by combining both results we conclude that for each $y \in H$, $||y|| \leqslant ||Ty|| \leqslant ||y||$, i.e., then $||Ty|| = ||y||$, and so T is an isometry. Hence, there exists x_j such that $||Tx_j|| < ||x_j||$, and so there exists $k \in \mathbb{N}$ such that $T \in N_{j,k}$.

Again, for the same reason, it is enough to show that $U(H) \cap \operatorname{Cl}_w(N_{j,k}) = \varnothing$, for every $j, k \in \mathbb{N}$.

Let $U \in U(H)$ and choose a sequence $(U_n) \subset N_{j,k}$ such that $\text{w-}\lim U_n = U$. Since $||U_n|| \leqslant 1$, then $||U_n x|| \leqslant ||U_n|| \cdot ||x|| \leqslant ||x|| = ||Ux||$. So, by using the Lemma 2.5, $\text{s-}\lim U_n = U$.

Note that for each $x_j \in H$, $\lim_n ||U_n x_j|| = ||U x_j|| = ||x_j||$, so

$$1 - \frac{1}{k} > \frac{||U_n x_j||}{||x_j||} \xrightarrow{n\to\infty} 1\,,$$

which is absurd. Hence, by the previous discussion, $C(H)\backslash I(H)$ is also w-meager.

(c) Note that

$$\begin{aligned} C(H)\backslash U(H) &= \big((C(H)\backslash I(H)) \cup I(H)\big)\backslash U(H) \\ &= \big((C(H)\backslash I(H))\backslash U(H)\big) \cup (I(H)\backslash U(H)) \\ &= (C(H)\backslash I(H)) \cup (I(H)\backslash U(H)) \end{aligned}$$

Then, $C(H)\backslash U(H)$ is the union of w-meager sets, and so $C(H)\backslash U(H)$ is also w-meager. This proves that $U(H)$ is w-co-meager in $C(H)$.

■

This is an interesting result. By Theorem 1.4, the weak topology coincides with the strong-star topology in $U(H)$, so we understand $U(H)$ with the weak topology really well. Moreover, we have the following result:

Theorem 2.7. *A is w-co-meager in $C(H)$ if, and only if, $A \cap U(H)$ is w-co-meager in $U(H)$.*

Proof. (a) Suppose that A is w-co-meager in $C(H)$, so $C(H) \setminus A$ is w-meager in $C(H)$. In particular, $U(H) \cap \big(C(H) \setminus A\big)$ is w-meager in $C(H)$. Since $U(H)$ is w-dense by Lemma 2.4, $U(H) \cap \big(C(H) \setminus A\big)$ is w-meager in $U(H)$. Thus, the set $U(H) \setminus \Big(U(H) \cap \big(C(H) \setminus A\big)\Big)$ is w-co-meager in $U(H)$, but

$$U(H) \setminus \Big(U(H) \cap \big(C(H) \setminus A\big)\Big) = A \cap U(H)$$

This shows that $A \cap U(H)$ is w-meager in $U(H)$.

(b) If $A \cap U(H)$ is w-co-meager in $U(H)$, then $U(H) \setminus \big(A \cap U(H)\big)$ is w-meager in $U(H)$. By Theorem 0.3, $U(H) \setminus \big(A \cap U(H)\big)$ is w-meager in $C(H)$, hence $C(H) \setminus \Big(U(H) \setminus \big(A \cap U(H)\big)\Big)$ is w-co-meager in $C(H)$. Now

$$C(H) \setminus \Big(U(H) \setminus \big(A \cap U(H)\big)\Big) = A \cup (C(H) \setminus U(H))\,,$$

and so $A \cup (C(H) \setminus U(H))$ is w-co-meager in $C(H)$. Since $(C(H) \setminus U(H))$ is w-meager in $C(H)$, we conclude that A is w-co-meager in $C(H)$.

■

With this result, we conclude that w-typical properties on $C(H)$ are w-typical properties on $U(H)$. Recall that $U(H)$ is a Baire Space so it makes sense to ask about typical properties on $U(H)$.

2.3 Spectral properties of w-typical contractions

A necessary tool for the characterization of the spectrum of a w-typical contraction is measure theory (for unfamiliar readers a good reference is [1]). We will state many results below, however we will only prove some of them. Before proceed some new notation is required.

If $(X, \mathcal{A})$ is a measurable space and μ, ν are two measures on that space, we say that μ and ν are equivalent ($\mu \sim \nu$) if $\mu(A) = 0$ if, and only if, $\nu(A) = 0$. We say that μ and ν are mutually singular ($\mu \perp \nu$) if there exists disjoint sets $A, B \in \mathcal{A}$ such that $\mu(U) = 0$ for all U measurable subset of A, and $\nu(U) = 0$ for all U measurable subset of B. We say that x is an atom of a measure μ if $\mu(\{x\}) > 0$.

If (X, τ) is a topological space, we let $B(\tau)$ be the Borel σ-algebra generated by τ and make $(X, B(\tau))$ a measurable space. If μ is a measure on $(X, B(\tau))$, we define $\operatorname{supp} \mu = \{x \in X : \forall\ N_x \text{ neighborhood of } x \text{ we have } \mu(N_x) > 0\}$.

Definition 2.8. *Denote by $\mathcal{P}$ the collection of probability measures over the Borel sets of $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. We say that $\mu_n \in \mathcal{P}$ converges weakly to μ on $\mathcal{P}$ if*

$$\int_{S^1} f d\mu_n \to \int_{S^1} f d\mu$$

for all continuous functions defined on S^1.

Definition 2.9. *Let $U \in U(H)$, fix $(x_n)_{n\geqslant 1}$ an orthonormal basis of H and let F_U denote the resolution of identity of U. We define the maximal spectral type of U by the probability measure*

$$\mu_U(A) = \sum_{n=1}^{\infty} 2^{-n} \langle F_U(A)x_n, x_n \rangle$$

for every $A \in B(S^1)$.

The first thing to show is that the choice of the orthonormal basis doesn't change the class of μ_U.

Lemma 2.10. *If $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ are both orthonormal basis of H, then*

$$\sum_{n=1}^{\infty} 2^{-n} \langle F_U(A)x_n, x_n \rangle \sim \sum_{n=1}^{\infty} 2^{-n} \langle F_U(A)y_n, y_n \rangle .$$

Proof. Denote by $\mu_U^x(A) = \sum_{n=1}^{\infty} 2^{-n} \langle F_U(A)x_n, x_n \rangle$ and by $\mu_U^y(A) = \sum_{n=1}^{\infty} 2^{-n} \langle F_U(A)y_n, y_n \rangle$.

(a) Suppose that $\mu_U^x(A) = 0$. Then, for every $n \in \mathbb{N}$, $\langle F_U(A)x_n, x_n \rangle = 0$. Fix $m \in \mathbb{N}$ and let $\psi = y_m$. We have that $\psi = \lim_k x_{n_k}$ for some subsequence $(x_{n_k})_k$, so

$$\begin{aligned} |\langle F_U(A)\psi, \psi \rangle| &= |\langle F_U(A)(\psi - x_{n_k}), \psi \rangle + \langle F_U(A)x_{n_k}, \psi \rangle| \leqslant |\langle F_U(A)(\psi - x_{n_k}), \psi - x_{n_k} \rangle| \\ &\quad + |\langle F_U(A)x_{n_k}, \psi - x_{n_k} \rangle| + |\langle F_U(A)(\psi - x_{n_k}), x_{n_k} \rangle| + |\langle F_U(A)x_{n_k}, x_{n_k} \rangle| \\ &\leqslant ||\psi - x_{n_k}||^2 + ||F_U(A)x_{n_k}|| \cdot ||\psi - x_{n_k}|| + ||F_U(A)|| \cdot ||\psi - x_{n_k}|| \cdot ||x_{n_k}|| \\ &\xrightarrow{k\to\infty} 0 \end{aligned}$$

Then, $\langle F_U(A)y_m, y_m \rangle = 0$, and since m is arbitrary it follows that $\mu_U^y(A) = 0$.

(b) In item (a) we shown that $\mu_U^x(A) = 0$ implies $\mu_U^y(A) = 0$ for each $A \in B(S^1)$. Thus, item (b) follows from item (a) swapping x and y.

■

We define a function $f : (U(H), w) \to \mathcal{P}$ by the law $f(U) = \mu_U$. Such function has some good properties:

Theorem 2.11. (*see* [14]) *The function $f : (U(H), w) \to \mathcal{P}$ given by the law $f(U) = \mu_U$ satisfies the following conditions:*

1. *f is a continuous function;*
2. *If $C \subset \mathcal{P}$ is measure class invariant (i.e., if $\mu \in C$ and $\nu \sim \mu$, then $\nu \in C$), then $f^{-1}(C)$ is unitary invariant (i.e., if $U \in f^{-1}(C)$ and $W \in U(H)$, then $WUW^{-1} \in f^{-1}(C)$).*

Theorem 2.12. (*see* [14]) *The following sets are w-co-meager in $U(H)$:*

1. *$\{U \in U(H) : \mu_U$ is atomless$\}$;*
2. *$\{U \in U(H) : \operatorname{supp} \mu_U = S^1\}$;*
3. *$\{V \in U(H) : \mu_V \perp \nu\}$ for any ν measure on S^1.*

Theorem 2.12 is really useful in the study of the spectral properties of a w-typical contraction.

Theorem 2.13. *If $U \in U(H)$ the following statements holds:*

1. *$\sigma(U) = \operatorname{supp} \mu_U$;*
2. *$\sigma_p(U) = \{\lambda \in \mathbb{C} : \lambda$ is an atom of $\mu_U\}$.*

Proof. (1) Recall that $\sigma(U) = \operatorname{supp} F_U$, where

$$\operatorname{supp} F_U := \{\lambda \in \mathbb{C} : F_U(N) \neq 0\,, \forall \text{ neighborhood of } \lambda\}\,.$$

Firstly, we show that if $\lambda \in \sigma(U)$ then $\lambda \in \operatorname{supp} F_U$. Namely, let V be a neighborhood of λ and denote by E the closed subspace such that $F_U(V)$ projects. Since E is closed in a separable metric space, E is separable itself. Fix $\{z_n\}_{n\geqslant 1}$ an orthonormal basis of E, and by using the same argument, let $\{w_n\}_{n\geqslant 1}$ be an orthonormal basis of $E^\perp$. Hence, $\{t_n\}_{n\geqslant 1}$, defined by $t_{2n} = z_n$ and $t_{2n-1} = w_n$, is an orthonormal basis of H. Let

$$\nu_U(A) = \sum_{n=1}^{\infty} 2^{-n} \langle F_U(A) t_n, t_n \rangle\,.$$

Suppose that $\mu_U(V) = 0$. Then, $\nu_U(V) = 0$, since $\mu_U \sim \nu_U$. Note that when $t_{2n-1} = w_n$, we have

$$\langle F_U(V) t_{2n-1}, t_{2n-1} \rangle = \langle F_U(V) w_n, w_n \rangle = 0\,,$$

since $F_U(V) w_n \in E$ and $w_n \in E^\perp$. So,

$$0 = \sum_{n=1}^{\infty} 2^{-2n} \langle F_U(V) z_n, z_n \rangle\,,$$

and we conclude that $0 = \langle F_U(V)z_n, z_n\rangle = ||z_n||^2$. This is absurd, since $\{z_n\}_{n\geqslant 1}$ is an orthonormal basis of E. Thus $\mu_U(V) \neq 0$, and since V is an arbitrary neighborhood of λ, we conclude that $\lambda \in \operatorname{supp}\mu_U$.

On the other hand, if $\lambda \in \operatorname{supp}\mu_U$, let V be a neighborhood of λ, it follows that

$$0 < \sum_{n=1}^{\infty} 2^{-n}\langle F_U(V)x_n, x_n\rangle\,.$$

Thus, there exists $n \in \mathbb{N}$ such that $0 < \langle F_U(V)x_n, x_n\rangle$, and then

$$0 < \langle F_U(V)x_n, x_n\rangle = \langle F_U(V)^2 x_n, x_n\rangle = \langle F_U(V)x_n, F_U(V)x_n\rangle = ||F_U(V)x_n||^2$$

We conclude that there exists $n \in \mathbb{N}$ such that $F_U(V)x_n \neq 0$, and so $F_U(V) \neq 0$. Since V is an arbitrary neighborhood of λ, we conclude that $\lambda \in \operatorname{supp} F_U$, so $\lambda \in \sigma(U)$.

(2) Recall that $\lambda \in \sigma_p(U)$ if, and only if, $F_U(\{\lambda\}) \neq 0$.

Suppose that $\lambda \in \sigma_p(U)$, so $F_U(\{\lambda\}) \neq 0$. Therefore, there exists $x \in S_H$ such that $F_U(\{\lambda\})x \neq 0$. Let $\{t_n\}_{n\geqslant 1}$ be an orthonormal basis of H such that $t_1 = x$ and let

$$\nu_U(A) = \sum_{n=1}^{\infty} 2^{-n}\langle F_U(A)t_n, t_n\rangle$$

By using that $\langle F_U(\{\lambda\})t_1, t_1\rangle > 0$, we get $\nu_U(\{\lambda\}) > 0$. Since $\mu_U \sim \nu_U$, we have that $\mu_U(\{\lambda\}) > 0$, and so λ is an atom of μ_U.

On the other hand, if λ is an atom of μ_U, then $\mu_U(\{\lambda\}) > 0$, and so, there exists $n \geqslant 1$ such that $\langle F_U(A)x_n, x_n\rangle > 0$. Thus,

$$0 < \langle F_U(\{\lambda\})x_n, x_n\rangle = \langle F_U(\{\lambda\})^2 x_n, x_n\rangle = \langle F_U(\{\lambda\})x_n, F_U(\{\lambda\})x_n\rangle = ||F_U(\{\lambda\})x_n||^2\,,$$

from which follows that $F_U(\{\lambda\}) \neq 0$. Therefore, $\lambda \in \sigma_p(U)$.

■

Now we can describe the spectrum of a w-typical contraction.

Theorem 2.14. *A w-typical contraction U satisfies $\sigma_c(U) = S^1$*

Proof. By Theorem 2.6, a w-typical contraction U is unitary. By Theorem 2.12 and Theorem 2.13 we have that

$$\{U \in U(H) : \sigma_p(U) = \varnothing\} = \{U \in U(H) : \mu_U \text{ is atomless}\}$$

and

$$\{U \in U(H) : \sigma(U) = S^1\} = \{U \in U(H) : \operatorname{supp}\mu_U = S^1\}$$

are both w-co-meager in $U(H)$. Since $\sigma_r(U) = \varnothing$ for every unitary operator (see [15]), so we conclude that $\sigma_c(U) = S^1$.

■

2.4 Unitary equivalence of w-typical contractions

Before we investigate the existence of w-co-meager unitary orbits in $C(H)$, we need one result info about the maximal spectral type.

Theorem 2.15. *Let $U, V \in U(H)$. Then*

$$\mu_U \sim \mu_{VUV^{-1}} .$$

Proof. Again, the proof follows from a well-know result about spectral measures. Namely, it follows that for every $x \in H$, $\langle F_U(A)x, x\rangle \sim \langle F_{VUV^{-1}}(A)x, x\rangle$.

If $\mu_U(A) = 0$, then $\langle F_U(A)x_n, x_n\rangle = 0$ for all $n \in \mathbb{N}$, and so $\langle F_{VUV^{-1}}(A)x_n, x_n\rangle = 0$ for all $n \in \mathbb{N}$. Thus, $\nu_{VUV^{-1}}(A) = 0$.

On the other hand, if $\mu_{VUV^{-1}}(A) = 0$, then $\langle F_{VUV^{-1}}(A)x_n, x_n\rangle = 0$ for all $n \in \mathbb{N}$. Thus, $\langle F_U(A)x_n, x_n\rangle = 0$, and so $\mu_U(A) = 0$.

■

Lemma 2.16. *Let $(A_n)_n, (S_n)_n \subset C(H)$ and $A, S \in C(H)$. If $w\text{-}\lim A_n = A$ and $s\text{-}\lim S_n = S$, then $w\text{-}\lim A_nS_n = AS$. In particular, if $(S_n)_n \subset U(H)$ and $S \in U(H)$, then $w\text{-}\lim S_nA_nS_n^{-1} = SAS^{-1}$.*

Proof. (a) Assume that $\text{w-}\lim A_n = A$ and $\text{s-}\lim S_n = S$. Let $x, y \in H$, then

$$\begin{aligned}|\langle A_nS_nx - ASx, y\rangle| &\leqslant |\langle A_nS_nx - A_nSx, y\rangle| + |\langle A_nSx - ASx, y\rangle| \\ &\leqslant ||A_n|| \cdot ||S_nx - Sx|| \cdot ||y|| + |\langle A_nSx - ASx, y\rangle| \\ &\leqslant ||S_nx - Sx|| \cdot ||y|| + |\langle A_nSx - ASx, y\rangle| \to 0\,,\end{aligned}$$

from which follows that $\text{w-}\lim A_nS_n = AS$.

(b) Note that if $\text{w-}\lim A_n = A$ then $\text{w-}\lim A_n^* = A^*$. Let $x, y \in H$, then

$$\begin{aligned}|\langle (A_n^* - A^*)x, y\rangle| &= |\langle x, (A_n - A)y\rangle| \\ &= |\overline{\langle (A_n - A)y, x\rangle}| \\ &= |\langle (A_n - A)y, x\rangle| \to 0\,.\end{aligned}$$

By Theorem 1.4, one has $(U(H), w) = (U(H), s^*)$, so $\text{s-}\lim S_n^{-1} = S^{-1}$. It follows from (a) that $\text{w-}\lim A_n^*S_n^{-1} = AS^{-1}$, and by using the same argument as before $\text{w-}\lim S_nA_n = SA$. Using (a) again, one concludes that $\text{w-}\lim S_nA_nS_n^{-1} = SAS^{-1}$.

■

Theorem 2.17. *For every $U \in C(H)$, $\mathcal{O}(U)$ is an w-meager subset of $C(H)$. In particular, w-typical contractions are not unitarly equivalent.*

Proof. (a) Let $U \in C(H)$. If $U \notin U(H)$, since $U(H)$ is unitary invariant we have $\mathcal{O}(U) \subset C(H) \setminus U(H)$. Since $U(H)$ is w-co-meager, it follows that $\mathcal{O}(U)$ is w-meager.

Assume now that $U \in U(H)$. In this case it follows from Theorem 2.12 that for any ν measure on S^1, the set $\mathcal{P}(\nu) = \{V \in U(H) : \mu_V \perp \nu\}$ is w-co-meager.

Now, by Theorem 2.15, the maximal spectral type is invariant under conjugacy in $U(H)$, i.e., for every $V \in U(H)$, $\mu_U \sim \mu_{VUV^{-1}}$. In particular, it is impossible to have

$\mu_U \perp \mu_{VUV^{-1}}$, i.e., $\mathcal{O}(U) \subset U(H) \setminus \mathcal{P}(\mu_U)$. Since $U(H) \setminus \mathcal{P}(\mu_U)$ is w-meager, we conclude that $\mathcal{O}(U)$ is w-meager.

(b) Set

$$E := \{(U_1, U_2) \in C(H) \times C(H) : U_1 \text{ and } U_2 \text{ are unitarily equivalent}\}$$

Since $(C(H), w)$ and $(U(H), w)$ are Polish spaces we have that $(C(H) \times C(H), w \times w)$ and $(C(H) \times U(H), w \times w)$ are also Polish spaces. We can define $f : C(H) \times U(H) \to C(H) \times C(H)$ by the law $f(A, U) = (A, UAU^{-1})$, and immediately conclude that $f(C(H) \times U(H)) = E$.

We claim that f is continuous. Namely, let $(A_n, U_n)_n \subset C(H) \times U(H)$ and $(A, U) \in C(H) \times U(H)$ be such that $(A_n, U_n) \xrightarrow{w \times w} (A, U)$. Note that $\text{w-}\lim A_n = A$ is immediate. By using the Lemma 2.16 we have that $\text{w-}\lim U_n A U_n^{-1} = UAU^{-1}$, and so $(\text{w} \times \text{w}) - \lim f(A_n, U_n) = f(A, U)$.

Since E is the image of a continuous function between Polish spaces, E is analytic and, by using Theorem 0.9, has the BP. We will use Kuratowski-Ulam Theorem to show that E is meager in $(C(H) \times C(H), w \times w)$ (see Appendix A for definitions and a proof). First of all, let $T \in C(H)$ and note that

$$\begin{aligned} E_T &= \{S \in C(H) : (T, S) \in E\} = \{S \in C(H) : S \text{ and } T \text{ are unitarily equivalent}\} \\ &= \{UTU^{-1} : U \in U(H)\} = \mathcal{O}(T) \end{aligned}$$

By item (a), we have that E_T is w-meager in $C(H)$ for all $T \in C(H)$, so we conclude that

$$\{T \in C(H) : E_T \text{ is w-meager}\} = C(H)\,,$$

and so $\{T \in C(H) : E_T \text{ is w-meager}\}$ is w-co-meager. By Kuratowski-Ulam Theorem E is meager in $(C(H) \times C(H), w \times w)$, and then we conclude that two w-typical contractions are not unitary equivalent.

■

3. THE POLYNOMIAL WEAK TOPOLOGY

3.1 Understanding the topology

The method used to understand the polynomial weak topology is a bit different from what we do in the others sections. In the first theorem of the section, we prove that the pw-topology is metrizable, and so second countable, T_1 and regular. After that, we prove that the of pw-topology is Polish, and to do that we use as discussed in [6]: the Strong Choquet characterization of this type of space (see Appendix B for details).

Theorem 3.1. *Let $\{e_i : i \geqslant 1\} \subset H$ be an dense subset of B_H. Then, the pw-topology is generated by the metric*

$$d_{pw}(A, B) = \sum_{i,j,n \in \mathbb{N} \setminus \{0\}} 2^{-i-j-n} |\langle A^n e_i, e_j \rangle - \langle B^n e_i, e_j \rangle|$$

Proof. (a) The first thing to do is to show that the series is absolutely convergent. Note that

$$\begin{aligned} d_{pw}(A, B) &= \sum_{i,j,n \in \mathbb{N} \setminus \{0\}} 2^{-i-j-n} |\langle (A^n - B^n) e_i, e_j \rangle| \\ &\leqslant \sum_{i,j,n \in \mathbb{N} \setminus \{0\}} 2^{-i-j-n} (||A_n|| + ||B_n||) ||e_i|| \cdot ||e_j|| \\ &\leqslant 2 \sum_{i,j,n \in \mathbb{N} \setminus \{0\}} 2^{-i-j-n} < \infty \end{aligned}$$

In particular, it is commutatively convergent. Thus, we can write

$$d_{pw}(A, B) = \sum_{n=1}^{\infty} 2^{-n} \Big(\sum_{i,j \in \mathbb{N} \setminus \{0\}} 2^{-i-j} |\langle A^n e_i, e_j \rangle - \langle B^n e_i, e_j \rangle| \Big) = \sum_{n=1}^{\infty} 2^{-n} d_w(A^n, B^n)$$

Now, we can use this in order to show that d_{pw} is a metric. Namely:

- if $A = B$, then $A^n = B^n$ for every $n \in \mathbb{N}$, so $d_w(A^n, B^n) = 0$, and then $d_{pw}(A, B) = 0$;
- if $A \neq B$, then $d_w(A, B) > 0$, and since $d_w(A^n, B^n) \geqslant 0$ for every $n \in \mathbb{N}$, it follows that $d_{pw}(A, B) > 0$;
- we know that $d_w(A, B) = d_w(B, A)$, so

$$d_{pw}(A, B) = \sum_{n=1}^{\infty} 2^{-n} d_w(A^n, B^n) = \sum_{n=1}^{\infty} 2^{-n} d_w(B^n, A^n) = d_{pw}(B, A)\,;$$

- note that

$$\begin{aligned} d_{pw}(A,C) &= \sum_{n=1}^{\infty} 2^{-n} d_w(A^n, C^n) \leqslant \sum_{n=1}^{\infty} 2^{-n}(d_w(A^n, B^n) + d_w(B^n, C^n)) = \\ &= \sum_{n=1}^{\infty} 2^{-n} d_w(A^n, B^n) + \sum_{n=1}^{\infty} 2^{-n} d_w(B^n, C^n) \\ &= d_{pw}(A,B) + d_{pw}(B,C)\,. \end{aligned}$$

(b) Now, we show that the topology generated by this metric is exactly the pw-topology. Let $A_\alpha \xrightarrow{d} A$ denote the fact that the net $(A_\alpha)_{\alpha\in\Lambda}$ converges to A with respect to this metric d_{pw}

Let $(A_\alpha)_{\alpha\in\Lambda} \subset C(H)$ and $A \in C(H)$ be such that $A_\alpha \xrightarrow{d} A$. Thus, for every $\varepsilon > 0$, there exists $\beta \in \Lambda$ such that for every $\alpha > \beta$, $d_{pw}(A_\alpha, A) < \varepsilon$. In particular, $d_w(A^n_\alpha, A^n) < 2^n\varepsilon$ for every $n \in \mathbb{N}$.

For each fixed $n \in \mathbb{N}$ and $\varepsilon > 0$. There exists $\beta \in \Lambda$ such that for every $\alpha > \beta$, $d_{pw}(A_\alpha, A) < 2^{-n}\varepsilon$. In particular, $d_w(A^n_\alpha, A^n) < \varepsilon$. So, $\text{w-}\lim A^n_\alpha = A^n$, for each $n \in \mathbb{N}$, from which follows that $\text{pw-}\lim A_\alpha = A$.

(c) Let $(A_\alpha)_{\alpha\in\Lambda} \subset C(H)$ and $A \in C(H)$ be such that $\text{pw-}\lim A_\alpha = A$. Let $\varepsilon > 0$ and choose $M \in \mathbb{N}$ such that $\sum_{n=M}^{\infty} 2^{-n+1} < \varepsilon/2$.

It follows from $\text{pw-}\lim A_\alpha = A$ that $\text{w-}\lim A^n_\alpha = A^n$ for every $n \in \mathbb{N}$. Hence, fro each $n \in \mathbb{N}$ there exists $\beta_k \in \Lambda$ such that for every $\alpha > \beta_k$ we have $d_w(A^n, B^n) < 2^{n-1}\varepsilon/(M-1)$. Choose $\beta > \beta_k$ for every $n < M$, and note that if $\alpha > \beta$, then

$$\begin{aligned} d_{pw}(A_\alpha, A) &= \sum_{n=1}^{\infty} 2^{-n} d_w(A^n_\alpha, A^n) = \sum_{n=1}^{M-1} 2^{-n} d_w(A^n_\alpha, A^n) + \sum_{n=M}^{\infty} 2^{-n} d_w(A^n_\alpha, A^n) \\ &< \sum_{n=1}^{M-1} \left(2^{-n} 2^{n-1} \frac{\varepsilon}{M-1}\right) + \sum_{n=M}^{\infty} 2^{-n} ||A^n_\alpha - A^n|| \\ &\leqslant \frac{\varepsilon(M-1)}{2(M-1)} + \sum_{n=M}^{\infty} 2^{-n+1} < \varepsilon\,. \end{aligned}$$

So, for every $\varepsilon > 0$, there exists $\beta \in \Lambda$ such that for every $\alpha > \beta$, $d_{pw}(A_\alpha, A) < \varepsilon$, i.e., $A_\alpha \xrightarrow{d} A$. This shows that the topology generated by d_{pw} is exactly the pw-topology.

■

Before we proceed, some lemmas are required.

Lemma 3.2. *Let $(A_n)_n \subset B(H)$ and $A \in B(H)$ be such that $\text{w-}\lim A_n = A$. If for every $x \in S_H$, $||Ax|| \geqslant \limsup_{n\in\mathbb{N}} ||A_n x||$, then $A = \text{s-}\lim A_n$.*

Proof. Let $x \in S_H$. Then,

$$||Ax - A_n x||^2 = ||Ax||^2 + ||A_n x||^2 - 2\,\mathrm{Re}\langle Ax, A_n x\rangle\,.$$

By taking the limsup of both sides, we have

$$
\begin{aligned}
0 &\leqslant \limsup_{n\in\mathbb{N}} ||Ax - A_n x||^2 = ||Ax||^2 + \limsup_{n\in\mathbb{N}} ||A_n x||^2 - 2\lim_n \operatorname{Re}\langle Ax, A_n x\rangle \\
&\leqslant 2||Ax||^2 - 2\operatorname{Re}\langle Ax, Ax\rangle = 0\,.
\end{aligned}
$$

Since $x \in S_H$ is arbitrary, it follows that $A = \text{s-}\lim A_n$.

■

Lemma 3.3. *Let $n > 0$, $\{x_i : i < n\} \subset S_H$ and $B \in C(H)$. Then, for every $\varepsilon > 0$, there exists a w-open set $W \subset C(H)$ such that $B \in W$, and for every $A \in W$, we have $||Ax_i|| \geqslant ||Bx_i|| - \varepsilon$, for every $i < n$.*

Proof. (a) We start proving the singleton case, i.e., when $\{x_i : i < n\} = \{x_0\}$. If $Bx_0 = 0$, the result holds for every W w-open set containing B. So, we can assume $Bx_0 \neq 0$.

Fix $\varepsilon > 0$ and consider the set

$$W = \{A \in C(H) : |\langle Ax_0, Bx_0\rangle| > ||Bx_0||^2 - ||Bx_0||\varepsilon\}\,,$$

which obviously contains B. We claim that W is w-open.

We will show that $W^C = \{A \in C(H) : |\langle Ax_0, Bx_0\rangle| \leqslant ||Bx_0||^2 - ||Bx_0||\varepsilon\}$ is w-closed. Let $(T_n)_n \subset W^C$ and $T \in C(H)$ be such that $\text{w-}\lim T_n = T$. In particular, $\lim\langle T_n x_0, Bx_0\rangle = \langle Tx_0, Bx_0\rangle$, and then

$$|\langle T_n x_0, Bx_0\rangle| \leqslant ||Bx_0||^2 - ||Bx_0||\varepsilon \;\Rightarrow\; |\langle Tx_0, Bx_0\rangle| \leqslant ||Bx_0||^2 - ||Bx_0||\varepsilon\,,$$

therefore, $T \in W^C$.

Now let $A \in W$. By using Cauchy-Schwarz inequality we get

$$||Ax_0|| \geqslant \frac{|\langle Ax_0, Bx_0\rangle|}{||Bx_0||} > \frac{||Bx_0||^2 - ||Bx_0||\varepsilon}{||Bx_0||} = ||Bx_0|| - \varepsilon\,,$$

concluding the proof in this case.

(b) For the general case, let $\{x_i : i < n\} \subset S_H$ and $\varepsilon > 0$. For every $i < n$, we can use item (a) in order to obtain a w-open set $W_i \subset C(H)$ such that $B \in W_i$ and such that for every $A \in W_i$, $||Ax_i|| \geqslant ||Bx_i|| - \varepsilon$.

If we set $W = \bigcap_{i<n} W_i$, then W is w-open and $B \in W$. Moreover, if $A \in W$, then for every $i < n$ $A \in W_i$ and so $||Ax_i|| \geqslant ||Bx_i|| - \varepsilon$.

■

Theorem 3.4. *The set $C(H)$ endowed with the polynomial weak topology is a Polish Space.*

Proof. (a) By using Urysohn Metrization Theorem and the fact that $(C(H), pw)$ is a separable metric space, we conclude that $(C(H), pw)$ is second countable, T_1 and regular. It remains to show that it is a strong Choquet space. (see Definition B.3 and Corollary B.8 in Appendix B).

Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of S_H and suppose that the n-th move of player I is (A_n, U_n), where $A_n \in U_n \subset C(H)$ and U_n is a pw-open set.

(b) We claim that there exists a w-open set W_n, with the following properties:

1. $A_n \in W_n \subset \mathrm{Cl}_w(W_n) \subset \{A \in C(H) : d_w(A, A_n) < 1/(n+1)\}$;
2. $\mathrm{Cl}_w(W_n) \subset W_{n-1}$;
3. for each $A \in W_n$ and each $i \leqslant n$, $||Ax_i|| \geqslant ||A_n x_i|| \geqslant ||A_n x_i|| - 1/(n+1)$.

We obtain the set W_n by applying the Lemma 3.3 with $\varepsilon = 1/(n+1)$ and $B = A_n$. Namely, by Lemma 3.3 there exists a w-open set $Z_n \subset C(H)$ such that $A_n \in Z_n$, and for every $A \in Z_n$ and each $i \leqslant n$, $||Ax_i|| \geqslant ||A_n x_i|| - 1/(n + +1)$.

Note that since W_{n-1} is a non-empty w-open set, there exists $\delta > 0$ such that $B_w(A_n, \delta) \subset W_{n-1}$ and $\mathrm{Cl}_w \left(B_w(A_n, \delta)\right) \subset W_{n-1}$. Let $r = \min\{\delta, 1/(2n+2)\}$ and set $W_n := Z_n \cap B_w(A_n, r)$.

Since $A_n \in Z_n$ and $A_n \in B_w(A_n, r)$, $W_n \neq \varnothing$. Also, since Z_n and $B_w(A_n, r)$ are both w-open, W_n is w-open too.

Naturally, $\mathrm{Cl}_w(W_n) \subset \mathrm{Cl}_w \left(B_w(A_n, r)\right)$. Since $B_w(A_n, r) \subset B_w(A_n, \delta)$ and $B_w(A_n, r) \subset B_w(A_n, 1/(2n+2))$, we have

$$\underset{w}{\mathrm{Cl}}(W_n) \subset \underset{w}{\mathrm{Cl}} \left(B_w(A_n, r)\right) \subset \underset{w}{\mathrm{Cl}} \left(B_w(A_n, \delta)\right) \subset W_{n-1}$$

and

$$\begin{aligned} \underset{w}{\mathrm{Cl}}(W_n) &\subset \underset{w}{\mathrm{Cl}} \left(B_w(A_n, r)\right) \\ &\subset \underset{w}{\mathrm{Cl}} \left(B_w(A_n, 1/(2n+2))\right) \\ &\subset B_w(A_n, 1/(n+1)) = \{A \in C(H) : d_w(A, A_n) < 1/(n+1)\}, \end{aligned}$$

and so, W_n is the desired set.

(c) Let player II respond to the move of player I by playing a pw-open set $V_n \subset U_n$ such that

$$A_n \in V_n \subset \underset{pw}{\mathrm{Cl}}(V_n) \subset U_n \cap W_n$$

Such V_n exists because $U_n \cap W_n$ is a w-open set, in particular pw-open, and by using that pw-topology is metrizable, we can find $\gamma > 0$ such that $B_{pw}(A_n, \gamma) \subset U_n \cap W_n$ and $\mathrm{Cl}_{pw} \left(B_{pw}(A_n, \gamma)\right) \subset U_n \cap W_n$.

(d) In this step we show that the strategy defined in the previous items is winning for player II (see Definition B.3). Let $\{(A_n, U_n), V_n : n \in \mathbb{N}\}$ be a run in the game in which player II follows the above strategy.

Note that

$$V_{n+1} \subset U_{n+1} \cap W_{n+1} \subset W_{n+1} \subset W_n\,,$$

and since $A_{n+1} \in V_{n+1}$, we have that $A_{n+1} \in W_n$. Then $d_w(A_{n+1}, A_n) < 1/(n+1)$ and hence A_n is weakly convergent.

Let $A = \text{w-}\lim A_n$. By using that

$$W_0 \supset W_1 \supset W_2 \supset \cdots$$

and $A_n \in W_n$, we get $A_m \in W_n$ for every $m \geqslant n$. Since $\mathrm{Cl}_w(W_n) \subset W_{n-1}$ it follows that $A \in W_n$ for every $n \in \mathbb{N}$. So, it follows that for each $i, n \in \mathbb{N}$, $||Ax_i|| \geqslant ||A_n x_i|| - 1/(n+1)$, and so $||Ax_i|| \geqslant \limsup_{n \in \mathbb{N}} ||A_n x_i||$ for every $i \in \mathbb{N}$.

If we fix $x \in S_H$ there exists a sequence $(x_{n_k})_k \subset \{x_n : n \in \mathbb{N}\}$ such that $\lim_k x_{n_k} =$

x. For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n_k \geqslant N$ implies $||x - x_{n_k}|| < \varepsilon$. Then,

$$||A_n x|| \leqslant ||A_n x - A_n x_{n_k}|| + ||A_n x_{n_k}|| \leqslant ||A_n|| \cdot ||x - x_{n_k}|| + ||A_n x_{n_k}|| \leqslant \varepsilon + ||A_n x_{n_k}||\,,$$

from which follows that

$$\limsup_n ||A_n x|| \leqslant \varepsilon + \limsup_n ||A_n x_{n_k}|| \leqslant \varepsilon + ||A x_{n_k}|| \xrightarrow{k\to\infty} \varepsilon + ||Ax||\,.$$

Then, $\limsup_n ||A_n x|| \leqslant \varepsilon + ||Ax||$ and since ε is arbitrary, we conclude that $\limsup_n ||A_n x|| \leqslant ||Ax||$. It follows from Lemma 3.2 that $A = \text{s-}\lim A_n$, and in particular, $A = \text{pw-}\lim A_n$.

We know that $A_n \in V_n$, and then we have

$$V_n \subset \underset{pw}{\mathrm{Cl}}(V_n) \subset U_n \cap W_n \subset U_n \subset V_{n-1}\,,$$

from which follows that $A_m \in V_n$ for every $m \geqslant n$. Therefore, $A \in \mathrm{Cl}_{pw} V_n$ for every $n \in \mathbb{N}$. Since $\mathrm{Cl}_{pw}(V_n) \subset V_{n-1}$, we conclude that $A \in V_n$ for every $n \in \mathbb{N}$ and so $A \in \bigcap_{n\in\mathbb{N}} V_n$. This shows that the strategy is winning for player II.

■

Note that, by the same argument as in the proof of Theorem 3.4, $C(H)$ is a Polish space for any metrizable topology on $C(H)$ which is finer than the weak topology and coarser than the strong topology.

3.2 Studying the properties of pw-typical contractions

In this section, we prove that the theory of typical pw-contractions is exactly the theory of w-contractions, i.e., we will prove that a property Φ is pw-typical in $C(H)$ if, and only if, Φ is w-typical in $C(H)$. Before we proceed, we need a result which states that $\mathrm{Cl}_{pw}(U(H)) = C(H)$.

Theorem 3.5. *(proved by Peller in [18]) $U(H)$ is pw-dense in $C(H)$.*

Proof. (a) Let $(K, (\cdot|\cdot))$ be a separable Hilbert space and let R be a contraction in K. Define $\tilde{R} : \bigoplus_{n=1}^{\infty} K \to \bigoplus_{n=1}^{\infty} K$ by the law $\tilde{R}(x_0, x_1, \dots) = (Rx_0, 0, 0, 0, \dots)$ and $\mathscr{D}_R : K \to K$ by the law $\mathscr{D}_R = (I - R^*R)^{1/2}$. We note that $I - R^*R$ is self-adjoint, and so the function $\mathscr{D}_R$ is well defined. Moreover,

$$\begin{aligned} ||\mathscr{D}_R x||^2 + ||Rx||^2 &= ((I - R^*R)^{1/2}x|(I - R^*R)^{1/2}x) + (Rx|Rx) \\ &= ((I - R^*R)x|x) + (R^*Rx|x) = (x|x) = ||x||^2\,. \end{aligned}$$

In particular, $\mathscr{D}_R$ is a contraction, because

$$||\mathscr{D}_R x||^2 = ||x||^2 - ||Rx||^2 \leqslant ||x||^2$$

Let $A_n : \bigoplus_{n=1}^{\infty} K \to \bigoplus_{n=1}^{\infty} K$ given by

$$A_n(x_0, x_1, \dots) = (Rx_0, \underbrace{0, \dots, 0}_{n}, \mathscr{D}_R x_0, x_1, \dots)$$

Note that A_n is an isometric operator:

$$\begin{aligned}||A_n(x_0,x_1,\dots)||_{\oplus}^2 &= ||Rx_0||^2 + ||\mathscr{D}_R x_0||^2 + \sum_{i=1}^{\infty}||x_i||^2\\ &= ||x_0||^2 + \sum_{i=1}^{\infty}||x_i||^2 = \sum_{i=0}^{\infty}||x_i||^2\\ &= ||(x_0,x_1,\dots)||_{\oplus}^2.\end{aligned}$$

We claim that $\text{pw-}\lim A_n = \tilde{R}$. In order to prove this we denote the inner product of $\bigoplus_{n=1}^{\infty} K$ by $[\cdot,\cdot]$ and fix $f=(f_0,\dots), g=(g_0,\dots) \in \bigoplus_{n=1}^{\infty} K$. Note that for every $k \in \mathbb{N}$, we have

$$A_n^k f = (R^k f_0, \underbrace{0,\dots,0}_{n}, \mathscr{D}_R R^{k-1} f_0, \underbrace{0,\dots,0}_{n}, \mathscr{D}_R R^{k-2} f_0, \dots, \mathscr{D}_R f_0, f_1, f_2, \dots)$$

Fix $n,k \in \mathbb{N}$. We can rewrite $f \in \bigoplus_{n=1}^{\infty} K$ as $f = P(f) + R(f)$, where

$$P(f) = (f_0, f_1, \dots, f_{kn+k}, 0, 0, \dots)$$

and $R(f) = f - P(f)$. Now, note that

$$\begin{aligned}\left|[A_n^k f, g] - [\tilde{R}^k f, g]\right| &= \big|[P(A_n^k f - \tilde{R}^k f), P(g)] + [Q(A_n^k f - \tilde{R}^k f), P(g)]\\ &\quad + [P(A_n^k f - \tilde{R}^k f), Q(g)] + [Q(A_n^k f - \tilde{R}^k f), Q(g)]\big|\\ &= \left|[P(A_n^k f - \tilde{R}^k f), P(g)] + [Q(A_n^k f - \tilde{R}^k f), Q(g)]\right|\\ &\leqslant \left|[P(A_n^k f - \tilde{R}^k f), P(g)]\right| + \left|[Q(A_n^k f - \tilde{R}^k f), Q(g)]\right|\\ &\leqslant \left|\sum_{i=1}^{k}(\mathscr{D}_R R^{k-i} f_0 | g_{in+i})\right| + ||Q(A_n^k f - \tilde{R}^k f)||_{\oplus}||Q(g)||_{\oplus}\\ &\leqslant \sum_{i=1}^{k}||f_0||\cdot||g_{in+i}|| + \Big(\sum_{i=1}^{\infty}||f_i||^2\Big)^{1/2}\Big(\sum_{i=kn+k+1}^{\infty}||g_i||^2\Big)^{1/2}\\ &= ||f_0||\sum_{i=1}^{k}||g_{in+i}|| + ||f||_{\oplus}\Big(\sum_{i=kn+k+1}^{\infty}||g_i||^2\Big)^{1/2}\end{aligned}$$

Given that $g \in \bigoplus_{n=1}^{\infty} K$, we know that the series $\sum_{i=1}^{\infty}||g_i||^2$ converges. In particular, we have that $\sum_{i=kn+k+1}^{\infty}||g_i||^2 \xrightarrow{n\to\infty} 0$ and $||g_{in+i}|| \xrightarrow{n\to\infty} 0$. So, we conclude that

$$\left|[A_n^k f, g] - [\tilde{R}^k f, g]\right| \leqslant ||f_0||\sum_{i=1}^{k}||g_{in+i}|| + ||f||_{\oplus}\Big(\sum_{i=kn+k+1}^{\infty}||g_i||^2\Big)^{1/2} \xrightarrow{n\to\infty} 0$$

Hence, $\text{pw-}\lim A_n = \tilde{R}$, and since A_n is an isometry, we conclude that $\tilde{R}$ can be approximated by isometries.

(b) Let T be a contraction in H. Firstly, we prove that the operator T can be approximated by isometric operators.

Fix $\mathcal{B} = \{x_n\}_{n=1}^{\infty}$ an orthonormal basis of H. Let $K_1 = \text{Cl}\left(\text{span}\{x_{2n}\}_{n=1}^{\infty}\right)$ and $K_m = \text{Cl}\left(\text{span}\left(\{x_{2n}\}_{n=1}^{\infty} \cup \{x_{2n-1}\}_{n=1}^{m-1}\right)\right)$ for every $m > 1$.

Note that for every $n \in \mathbb{N}$, $K_n \subset K_{n+1}$. Moreover, for every $n \in \mathbb{N}$, $K_n \cong K_{n+1}$ since the Hilbert dimensions of both spaces are equal.

Let P_n be the orthogonal projection over K_n, and note that $\text{s-}\lim P_n = I$: for each $x \in H$, we can find $\lambda_j \in \mathbb{C}$, with $j \in \mathbb{N} \setminus \{0\}$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n x_n = \Big(\sum_{n=1}^{\infty} \lambda_{2n} x_{2n} + \sum_{n=1}^{m-1} \lambda_{2n-1} x_{2n-1} \Big) + \sum_{n=m}^{\infty} \lambda_{2n-1} x_{2n-1} = P_m x + \sum_{n=m}^{\infty} \lambda_{2n-1} x_{2n-1} \,,$$

and so

$$||P_m x - x|| = \Big|\Big| \sum_{n=m}^{\infty} \lambda_{2n-1} x_{2n-1} \Big|\Big| \xrightarrow{m\to\infty} 0 \,,$$

thus, $(P_n)_n$ is a sequence of operators in H such that $P_n(H) \subset P_{n+1}(H)$ and $\text{s-}\lim P_n = I$.

Let $n \in \mathbb{N}$ and note that H is equal to $\bigoplus_{t\in\mathbb{N}}^{\text{inn}} K_n^{(t)}$, the inner direct sum of the orthogonal sets $(K_n^{(t)})_{t\in\mathbb{N}}$, where we define these sets by induction: let $K_n^{(0)} = K_n$, and for a given $K_n^{(t)}$ we use the above construction to obtain an infinite dimensional and infinite co-dimensional closed subspace $K_n^{(t+1)} \perp \bigoplus_{t=1}^{m} K_n^{(t)}$. In particular, every $K_n^{(t)}$ is unitarily equivalent to K_n, and so

$$H = \bigoplus_{t\in\mathbb{N}}^{\text{inn}} K_n^{(t)} \cong_U \bigoplus_{t\in\mathbb{N}} K_n^{(t)} \cong_U \bigoplus_{t\in\mathbb{N}} K_n$$

Let $\psi_1 : \bigoplus_{t\in\mathbb{N}}^{\text{inn}} K_n^{(t)} \to \bigoplus_{t\in\mathbb{N}} K_n^{(t)}$ the standard unitary operator (i.e., ψ_1 is the inverse of the function f given by the rule $f(k_0, k_1, \dots) = \sum_{t\in\mathbb{N}} k_t$) and let $\psi_2 : \bigoplus_{t\in\mathbb{N}} K_n^{(t)} \to \bigoplus_{t\in\mathbb{N}} K_n$ be a unitary operator such that its restriction to $K_n^{(0)} \oplus \{0\} \oplus \{0\} \oplus \cdots$ is the identity operator.

Let $T_n = P_n T\big|_{K_n}$ and $\tilde{T}_n : \bigoplus_{t\in\mathbb{N}} K_n \to \bigoplus_{t\in\mathbb{N}} K_n$ be given by $\tilde{T}_n(z_0, z_1, \dots) = (T_n z_0, 0, \dots)$. In particular, $\psi_2^{-1} \tilde{T}_n \psi_2(z_0, z_1, \dots) = (T_n z_0, 0, \dots)$. Moreover, $T_n z_0 = \psi_1^{-1} \psi_2^{-1} \tilde{T}_n \psi_2 \psi_1 (\sum_{n=1}^{\infty} z_i)$.

If $f \in K_m$, then for every $n \geqslant m$ we have $||T_n f - Tf|| = ||P_n Tf - Tf|| \to 0$, since $\text{s-}\lim P_n = I$. Moreover, the sequence $(T_n)_n$ converges strongly to T, given that $\bigcup_{n=1}^{\infty} K_n = H$. In particular, $\psi_1^{-1} \psi_2^{-1} \tilde{T}_n \psi_2 \psi_1$ converges strongly to T. Then ,$\tilde{T}_n$ converges strongly to $\psi_2 \psi_1 T \psi_1^{-1} \psi_2^{-1}$, and in particular, it converges weakly.

By item (a) there exists a sequence of isometries $(A_m^{(n)})_{m\in\mathbb{N}}$ such that $\text{pw-}\lim_m A_m^{(n)} = \tilde{T}_n$, i.e., we can approximated $\tilde{T}_n$ in the pw-topology by isometries, and since ψ_1 and ψ_2 are unitary operators, we conclude that T_n can be approximated by isometries in the pw-topology. Since $\text{pw-}\lim T_n = T$ and the pw-topology is metrizable we conclude that T is also approximated by isometries.

(c) The last step consist in showing that any isometric operator V can be approximated by unitary operators. According to Wold's decomposition (see Appendix C), there exist V-invariant subspaces $H_1, H_2 \leqslant H$ such that $\mathcal{U} = V\big|_{H_1}$ is unitary and $V_0 = V\big|_{H_2}$ is a right shift in H_2, i.e., there exists a Hilbert space K and a unitary operator $\psi : H_2 \to \bigoplus_{n=1}^{\infty} K$ such that $\psi^{-1} V \psi(k_0, k_1, \dots) = (0, k_0, k_1, \dots)$.

We define V_n in H_2 so that

$$\psi^{-1} V_n \psi(k_0, k_1, \dots) = (k_n, k_0, k_1, \dots, k_{n-1}, k_{n+1}, \dots) \,.$$

The operator $\psi^{-1}V_n\psi$ is obviously a bijection. Moreover

$$\begin{aligned}||\psi^{-1}V_n\psi(k_0,k_1,\dots)||_{\oplus} &= ||(k_n,k_0,k_1,\dots,k_{n-1},k_{n+1},\dots)||_{\oplus}\\ &= \sum_{i=0}^{\infty}||k_i||_K^2 = ||(k_0,k_1,\dots)||_{\oplus}\,.\end{aligned}$$

Therefore, $\psi^{-1}V_n\psi$ is unitary, and so V_n. We claim that $\operatorname{s-lim} V_n = V$ in H_2. Namely, let $(k_0,k_1,\dots) \in \bigoplus_{n=1}^{\infty} K$, and note that

$$\begin{aligned}||\psi^{-1}(V_n - V_0)\psi(k_0,k_1,\dots)||_{\oplus} &= ||(k_n,0,\dots,0,k_{n+1}-k_n,k_{n+2}-k_{n+1},\dots)||_{\oplus}\\ &\leqslant ||k_n||_K + ||(0,0,\dots,0,k_{n+1},k_{n+2},\dots)||_{\oplus}\\ &\quad + ||(0,0,\dots,0,k_n,k_{n+1},\dots)||_{\oplus}\\ &= ||k_n||_K + \Big(\sum_{i\geqslant 1}||k_{n+i}||_K^2\Big)^{1/2} + \Big(\sum_{i\geqslant 0}||k_{n+i}||_K^2\Big)^{1/2}\\ &\leqslant ||k_n||_K + 2\Big(\sum_{i\geqslant 0}||k_{n+i}||_K^2\Big)^{1/2}\\ &= ||k_n||_K + 2\Big(\sum_{i\geqslant n}||k_i||_K^2\Big)^{1/2} \xrightarrow{n\to\infty} 0\end{aligned}$$

Then, $\operatorname{s-lim}\psi^{-1}V_n\psi = \psi^{-1}V\psi$ in $\bigoplus_{n=1}^{\infty} K$. In particular, $\operatorname{s-lim} V_n = V$ in H_2.

Let $\mathcal{U}_n = \mathcal{U} \oplus V_n$, which is unitary since V_n and $\mathcal{U}$ are unitary, and if we write $x = P_{H_1}x + P_{H_2}x$, then

$$||(\mathcal{U}_n - V)x|| = ||(\mathcal{U}\oplus V_n - \mathcal{U}\oplus V_0)(P_{H_1}x + P_{H_2}x)|| = ||V_nP_{H_2}x - V_0P_{H_2}x|| \xrightarrow{n\to\infty} 0\,.$$

This shows that $\operatorname{s-lim}\mathcal{U}_n = V$, and since every $\mathcal{U}_n$ is unitary we conclude that V can be approximated by unitary operators.

■

A directly consequence of Theorem 3.5 is the following result.

Corollary 3.6. *The set $U(H)$ is pw-co-meager pw-G_δ in $C(H)$.*

Proof. It follows from Theorem 3.5 that $U(H)$ is pw-dense in $C(H)$. We know that $(U(H), w)$ is Polish, by Theorem 1.4, and that $(C(H), w)$ is also Polish, so $U(H)$ is w-G_δ. In particular, since every w-open set is pw-open, we have that $U(H)$ is pw-G_δ. Therefore, $U(H)$ is pw-co-meager in $C(H)$.

■

The last result in this chapter shows the previously stated claim that the theory of typical pw-contractions is exactly the theory of typical w-contractions.

Theorem 3.7. *A set $\mathcal{C} \subset C(H)$ is pw-co-meager in $C(H)$ if, and only if, $\mathcal{C}$ is w-co-meager in $C(H)$. In particular, a property Φ of contractions is pw-typical if, and only if, Φ is w-typical.*

Proof. By Theorem 3.5 and Corollary 3.6, $U(H)$ is a pw-co-meager subset of the Polish space $C(H)$. By Theorem 1.4 the weak and polynomial weak topologies coincide on $U(H)$, and in these topologies, $U(H)$ is a Polish space, and so a Baire space. So, we can discuss in $U(H)$ typical properties related to Baire spaces.

Since $U(H) \subset C(H)$ is pw-co-meager, it is in particular pw-dense. Hence, a set $\mathcal{M} \subset U(H)$ is pw-meager in $U(H)$ if, and only if, it is pw-meager in $C(H)$.

Since $U(H)$ is pw-co-meager, a set $\mathcal{C} \subset C(H)$ is pw-co-meager in $C(H)$ if, and only if, $\mathcal{C} \cap U(H) \subset C(H)$ is pw-co-meager in $C(H)$. By the discussion above, it follows that this is equivalent to $\mathcal{C} \cap U(H) \subset U(H)$ be pw-co-meager in $U(H)$, which is the same as saying that $\mathcal{C} \cap U(H) \subset U(H)$ is w-co-meager in $U(H)$.

We conclude that $\mathcal{C} \subset C(H)$ is pw-co-meager in $C(H)$ if, and only if, $\mathcal{C} \cap U(H) \subset U(H)$ is w-co-meager in $U(H)$. By Theorem 2.7, this is exactly the same as saying that $\mathcal{C} \subset C(H)$ is w-co-meager in $C(H)$.

■

4. THE STRONG TOPOLOGY

4.1 Understanding the topology

Recall that we have defined the Strong Topology by using nets, i.e., we have defined the topology saying how the convergence of nets works. However, for some of the results, it is better to define its open sets.

Definition 4.1. *Define the SOT topology (Strong Operator Topology) in $B(H)$ as the coarsest topology for which the evaluation map $f_x : T \to Tx$ is continuous for every $x \in H$. In another words, the SOT topology is the topology generated by the sub-basis consisting of the elements*

$$f_x^{-1}(B_\varepsilon(Tx)) = \{T_0 \in B(H) \ : \ ||T_0x - Tx|| < \varepsilon\},$$

where $x \in H$, $\varepsilon > 0$ and $T \in B(H)$.

Theorem 4.2. *A net $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ converges to $T \in B(H)$ in the SOT topology if, and only if, s-$\lim T_\alpha = T$.*

Proof. First of all, let $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ be a net that converges to $T \in B(H)$ in the SOT topology, and let $x \in H$. For a fixed $\varepsilon > 0$, let $V_\varepsilon = \{T_0 \in B(X) \ : \ ||T_0x - Tx|| < \varepsilon\}$ be a SOT-neighborhood of T. Then, there exists $\beta \in \Lambda$ such that for each $\alpha > \beta$ it is true that $T_\alpha \in V_\varepsilon$.

Then, for each $\alpha > \beta$, $||T_\alpha x - Tx|| < \varepsilon$, from which follows that s-$\lim T_\alpha = T$.

On the other hand, let $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ be a net such that s-$\lim T_\alpha = T$. Let V be a SOT-neighborhood of T, by taking a subset of V it is possible to assume that

$$V = \{T_0 \in B(H) \ : \ \text{for all } i \in I \ ||T_0x_i - Tx_i|| < \varepsilon\},$$

where I is some fixed finite index set.

Since s-$\lim T_\alpha = T$, it follow that for each $i \in I$, $\lim T_\alpha x_i = Tx_i$. Or, in other words, for each $\delta > 0$ there exists $\beta_i \in \Lambda$ such that $||T_\alpha x_i - Tx_i|| < \delta$, for all $\alpha > \beta$. Then, let $\delta = \varepsilon$ and let $\beta > \beta_i$ for all $i \in I$ (which is possible since I is finite). Now, note that if $\alpha > \beta \geqslant \beta_i$, then $||T_\alpha x_i - Tx_i|| < \varepsilon$ for all $i \in I$. So, for all $\alpha > \beta$, we conclude that $T_\alpha \in V$, then $(T_\alpha)_{\alpha\in\Lambda} \subset B(H)$ converges to $T \in B(H)$ in the SOT topology.

∎

Now we are able to prove the metrizability of this topology in $C(H)$, as stated in Theorem 1.3.

Theorem 4.3. *Let $\{e_i : i \geqslant 1\} \subset H$ be a dense subset of B_H. Then,*

$$d_s(A,B) = \sum_{i=1}^{\infty} 2^{-i} ||Ae_i - Be_i||$$

is a complete separable metric on $C(H)$ which generates the strong topology.

Proof. (a) First of all, define $||A||_s := \sum_{i=1}^{\infty} 2^{-i}||Ae_i||$ for every $A \in C(H)$, and note that this is a norm:

- Note that
$$||A||_s = \sum_{i=1}^{\infty} 2^{-i}||Ae_i|| \leqslant \sum_{i=1}^{\infty} 2^{-i}||A|| = ||A||$$
so $||A||_s$ is well defined.

- Obviously, $||A||_s \geqslant 0$.

- If $||A||_s = 0$, then for every $i \in \mathbb{N}$ we have $||Ae_i|| = 0$. If $y \in B_H$, there exists a sequence $(e_{n_k})_k$ such that $\lim_k e_{n_k} = y$, and then $\lim_k Ae_{n_k} = Ay$. But $Ae_{n_k} = 0$, so $Ay = 0$. since the result follows for every $y \in B_H$, one concludes that $A = 0$. If $A = 0$, obviously $||A||_s = 0$.

- If $\alpha \in \mathbb{C}$, then
$$||\alpha A||_s = \sum_{i=1}^{\infty} 2^{-i}||\alpha Ae_i|| = \sum_{i=1}^{\infty} |\alpha| 2^{-i}||Ae_i|| = |\alpha| \sum_{i=1}^{\infty} 2^{-i}||Ae_i|| = |\alpha| \cdot ||A||_s\,.$$

- If $A, B \in C(H)$, then
$$||A+B||_s = \sum_{i=1}^{\infty} 2^{-i}||Ae_i + Be_i|| \leqslant \sum_{i=1}^{\infty} 2^{-i}(||Ae_i|| + ||Be_i||) = ||A||_s + ||B||_s\,.$$

Hence, $|| \cdot ||_s$ is a norm in $C(H)$, and since $d_s(A,B) = ||A-B||_s$, it follows that $d_s(A,B)$ is a metric in $C(H)$. Now we want to show that this metric generates the strong topology as defined.

(b) Let $T \in C(H)$ and let V be an s-neighborhood of T. We want to obtain $r > 0$ such that
$$U = \{A \in C(H) : d_s(A,T) < r\} \subset V\,.$$

We can assume that for each $\varepsilon > 0$ and $k \in \mathbb{N}$,

$$V = \bigcap_{i=1}^{k} \{A \in C(H) : ||Ay_i - Ty_i|| < \varepsilon\}\,.$$

Without loss of generality, we can assume that $||y_i|| \leqslant 1$ for every $i = 1, \ldots, k$. Then, for each $i \in \{1, \ldots, k\}$, we can use that $\{e_j : j \geqslant 1\} \subset H$ is a dense subset of B_H in order to obtain $n_i \in \mathbb{N}$ such that

$$||y_i - e_{n_i}|| < \varepsilon/4\,.$$

We can also choose $r > 0$ such that

$$2^{n_i} r < \varepsilon/2 \,.$$

We state that $U \subset V$. Namely, if $d_s(A,T) < r$, then $2^{-i}||Ae_i - Te_i|| < r$ for every $i \geqslant 1$. In particular, $2^{-n_i}||Ae_{n_i} - Te_{n_i}|| < r$ for every $i \in \{1, \ldots, k\}$.

So, if $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned} ||Ay_i - Ty_i|| &= ||(A-T)(y_i - e_{n_i}) + (A-T)e_{n_i}|| \\ &\leqslant ||(A-T)(y_i - e_{n_i})|| + ||(A-T)e_{n_i}|| \\ &< ||(A-T)|| \cdot ||(y_i - e_{n_i})|| + 2^{n_i} r \\ &\leqslant (||A|| + ||T||)\frac{\varepsilon}{4} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

We conclude that if $A \in U$, then $A \in V$.

(c) Let $T \in C(H)$. For each $r > 0$ we have to obtain a s-neighborhood $V \subset U = \{A \in C(H) : d_s(A,T) < r\}$. So, we need to obtain $\varepsilon > 0$, $k \in \mathbb{N}$ and $y_1, \ldots, y_k$ such that

$$\bigcap_{i=1}^{k} \{A \in C(H) : ||Ay_i - Ty_i|| < \varepsilon\} \subset U$$

We claim that it is enough to choose $\varepsilon = r/2$, k such that $2^{-k+1} < r/2$ and $y_i = e_i$ for every $i \in \{1, \ldots, k\}$. To see this, pick $A \in V$ and see that

$$\begin{aligned} d_s(A,T) &= \sum_{i=1}^{k} 2^{-i}||Ae_i - Te_i|| + \sum_{i=k+1}^{\infty} 2^{-i}||Ae_i - Te_i|| \\ &< \varepsilon \sum_{i=1}^{k} 2^{-i} + \sum_{i=k+1}^{\infty} 2^{-i}(||A|| + ||T||) \\ &\leqslant \varepsilon + 2 \sum_{i=k+1}^{\infty} 2^{-i} \leqslant \varepsilon + 2\Big(\frac{2^{-(k+1)}}{1 - \frac{1}{2}}\Big) \\ &\leqslant \varepsilon + 2^{-k+1} < \frac{r}{2} + \frac{r}{2} = r \end{aligned}$$

Hence, $V \subset U$. Concluding the proof of that metric d_s generates the strong topology. Now, we have to shown that this metric is complete and separable.

(d) Let's show that the metric is complete. Let $(A_n)_n \subset C(H)$ be a Cauchy sequence with respect to d_s metric. For each $\varepsilon > 0$ and each $i \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that if $n, m > N$, then $d_s(A_n, A_m) < 2^{-i}\varepsilon$. So, it follows that for every $i \in \mathbb{N} \setminus \{0\}$, $||A_n e_i - A_m e_i|| < \varepsilon$. Hence, $(A_n e_i)_n$ is a Cauchy sequence in H for every $i \in \mathbb{N} \setminus \{0\}$.

We can now define $Ae_i = \lim A_n e_i$ and by using the density of $(e_i)_i$ in B_H we can extend it for every $x \in B_H$, and then we can extend it again by linearly for every $x \in H$. We claim that $A \in C(H)$. Namely, for each $x \in B_H$, let $(e_{k_i})_i$ be such that $\lim e_{k_i} = x$. Now,

$$||Ax|| = \lim_i ||Ae_{k_i}|| = \lim_i \lim_n ||A_n e_{k_i}|| \leqslant \lim_i \lim_n ||A_n|| \cdot ||e_{k_i}|| \leqslant 1 \,,$$

and we are done.

One way to see that $\lim d_s(A, A_n) = 0$ is to check if $\text{s-}\lim A_n = A$, after all, we proved that the strong topology is generated by this metric. Fix some $\delta > 0$, and again let $x \in B_H$ and let $(e_{k_i})_i$ be such that $\lim e_{k_i} = x$. Then let e_{k_i}be such that

$$||x - e_{k_i}|| < \frac{\delta}{4},$$

and for such e_{k_i}, there exists $N \in \mathbb{N}$ such that for every $n > N$,

$$||(A_n - A)e_{k_i}|| < \frac{\delta}{2}.$$

So,

$$\begin{aligned} ||A_n x - Ax|| &\leqslant ||(A_n - A)(x - e_{k_i})|| + ||(A_n - A)e_{k_i}|| \leqslant \\ &\leqslant ||(A_n - A)|| \cdot ||(x - e_{k_i})|| + ||(A_n - A)e_{k_i}|| \\ &< 2\frac{\delta}{4} + \frac{\delta}{2} = \delta, \end{aligned}$$

which shows that $\lim A_n x = Ax$ for all $x \in B_H$. If $y \in H \setminus \{0\}$, then $y/||y|| \in B_H$, and so

$$\lim A_n\Big(\frac{y}{||y||}\Big) = A\Big(\frac{y}{||y||}\Big) \;\Rightarrow\; \frac{1}{||y||}\lim A_n y = \frac{1}{||y||}Ay \;\Rightarrow\; \lim A_n y = Ay,$$

and finally, that $\text{s-}\lim A_n = A$. This shows that d_s is complete.

(e) In order to prove that $C(H)$ is separable when endowed with the strong topology let $(x_k)_{k\geqslant 1}$ be a dense subset of H and set $A = \{(Tx_k)_{k\geqslant 1} : T \in C(H)\} \subset H^{\mathbb{N}}$. Since H is separable and metrizable, the same is true for $H^{\mathbb{N}}$, so A is separable. Let $\{T_n\}_{n\geqslant 1}$ be the set of operators corresponding to a countable dense subset of A.

We claim that $(T_n)_{n\geqslant 1}$ is s-dense in $C(H)$. Namely, let $T \in C(H)$ and since $(Tx_k)_{k\geqslant 1} \in A$ there exists $(T_{n_j})_{j\geqslant 1}$ such that $(T_{n_j}x_k)_{j\geqslant 1} \to (Tx_k)_{k\geqslant 1}$ when $j \to \infty$. In particular, $\lim_j T_{n_j}x_k = Tx_k$ for all $k \geqslant 1$.

Fix $y \in H$ and $\varepsilon > 0$. By using the density of $(x_k)_{k\geqslant 1}$, there exists $k \in \mathbb{N}$ such that

$$||y - x_k|| < \frac{\varepsilon}{4}$$

and, for such fixed k, let N be such that $n_j > N$, then

$$||(T_{n_j} - T)x_k|| < \frac{\varepsilon}{2}.$$

Now, if $n_j > N$, then

$$\begin{aligned} ||T_{n_j}y - Ty|| &\leqslant ||(T_{n_j} - T)(y - x_k)|| + ||(T_{n_j} - T)x_k|| \\ &\leqslant 2||y - x_k|| + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So, $\text{s-}\lim T_{n_j} = T$ and we conclude that strong topology is separable.

■

4.2 Some general results and lemmas

The theory of s-typical contractive operators depends on the following fact: in some sense all of the s-typical contractions are unitarily equivalent to a shift. With that we can understand all the theory understanding this shift. In particular, if we want to study spectral properties of typical operator it is sufficient to study spectral properties of this shift.

The problem is: every beautiful painting demands a hard work of the artist... We will need a great amount of lemmas and another theorems to create this piece of art. Let us begin.

Lemma 4.4. *Let $x, y \in S_H$ be such that $x \neq -y$, and set $\alpha = (2 + 2\operatorname{Re}\langle x, y\rangle)^{-1/2}$. Then $||\alpha(x + y)|| = 1$.*

Proof. Note that

$$\begin{aligned} ||\alpha(x+y)||^2 &= \alpha^2\langle x+y, x+y\rangle = \alpha^2(||x||^2 + ||y||^2 + 2\operatorname{Re}\langle x, y\rangle) \\ &= \alpha^2(2 + 2\operatorname{Re}\langle x, y\rangle) = \alpha^2\alpha^{-2} = 1\,, \end{aligned}$$

and we are done.

■

Lemma 4.5. *Let $A \in C(H)$, $(b_n)_{n\in\mathbb{N}} \subset S_H$, and $z \in S_H$ be such that* $\lim Ab_n = z$*. Then, $(b_n)_{n\in\mathbb{N}}$ is convergent.*

Proof. We will prove that the sequence $(b_n)_{n\in\mathbb{N}}$ is Cauchy. Let $\varepsilon > 0$ and note that for n and m sufficiently large, $b_n \neq -b_m$. Namely, if for all $N \in \mathbb{N}$ there exist $n(N), m(N) > N$ such that $b_{n(N)} = -b_{m(N)}$, then there exists a subsequence $(c_k)_k$ of $(b_n)_n$ such that for all $k \in \mathbb{N}$, $c_k = -c_{k+1}$. Since $\lim Ab_n = z$, we have

$$2||Ac_{2k}|| = ||Ac_{2k} - Ac_{2k+1}|| \xrightarrow{k\to\infty} 0\,.$$

On the other hand $\lim ||Ac_{2k}|| = z$. By the uniqueness of the limit, it follows that $z = 0$, an absurd, since $z \in S_H$. This shows that for n and m sufficiently large, $b_n \neq -b_m$.

Thus, by n and m sufficiently large $\alpha_{n,m} = (2 + 2\operatorname{Re}\langle b_n, b_m\rangle)^{-1/2}$ is well defined, and then, by Lemma 4.4 it follows that $||\alpha_{n,m}(b_n + b_m)|| = 1$. Now, note that

$$||2z|| \leqslant ||2z - (Ab_n - z) - (Ab_m - z)|| + ||(Ab_n - z)|| + ||(Ab_m - z)||\,.$$

And since $||z|| = 1$, we obtain

$$2 - ||(Ab_n - z)|| - ||(Ab_m - z)|| \leqslant ||Ab_n + Ab_m||\,.$$

So,

$$\begin{aligned} \alpha_{n,m}(2 - ||(Ab_n - z)|| - ||(Ab_m - z)||) &\leqslant \alpha_{n,m}||Ab_n + Ab_m|| \\ &\leqslant ||A|| \cdot ||\alpha_{n,m}(b_n + b_m)|| \\ &= ||A|| \leqslant 1\,. \end{aligned}$$

Now, since $\lim Ab_n = z$, there exists $N \in \mathbb{N}$ such that $||Ab_n - z|| < \varepsilon/8$ for all $n > N$. Thus, if $n, m > N$, then that $2 - ||Ab_n - z|| - ||Ab_m - z|| \geqslant 2 - \varepsilon/4$ then

$$\alpha_{n,m} \leqslant \frac{1}{2 - ||Ab_n - z|| - ||Ab_m - z||} \leqslant \frac{1}{2 - \varepsilon/4}$$

Thus, $(2 - \varepsilon/4)^2 \leqslant \alpha_{n,m}^{-2} = 2 + 2\,\mathrm{Re}\langle b_n, b_m\rangle$, from which follows that

$$\begin{aligned} ||b_n - b_m||^2 = ||b_n||^2 + ||b_m||^2 - 2\,\mathrm{Re}\langle b_n, b_m\rangle = 2 - 2\,\mathrm{Re}\langle b_n, b_m\rangle \\ \leqslant 2 + 2 - (2 - \varepsilon/4)^2 = \varepsilon - \varepsilon^2/16 \leqslant \varepsilon\,, \end{aligned}$$

proving that the sequence $(b_n)_{n\in\mathbb{N}}$ is Cauchy.

■

Lemma 4.6. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $\{e_i : i < n\}$ be an orthonormal family. Let $\{f_i : i < n\} \subset H$ be such that $||f_i - e_i|| < 1/n$ for all $i < n$. Then, $\{f_i : i < n\}$ is linearly independent.*

Proof. Suppose that for each $i < n$, there exists α_i such that $\sum_{i<n} |\alpha_i| > 0$ and $\sum_{i<n} \alpha_i f_i = 0$. Then, we have

$$\begin{aligned} \Big|\Big|\sum_{i<n} \alpha_i e_i\Big|\Big|^2 = \Big|\Big|\sum_{i<n} \alpha_i e_i - \sum_{i<n} \alpha_i f_i\Big|\Big|^2 \leqslant \Big(\sum_{i<n} |\alpha_i| \cdot ||e_i - f_i||\Big)^2 \\ < \Big(\frac{1}{n}\sum_{i<n} |\alpha_i|\Big)^2 \leqslant \frac{1}{n}\sum_{i<n} |\alpha_i|^2 = \frac{1}{n}\Big|\Big|\sum_{i<n} \alpha_i e_i\Big|\Big|^2 \end{aligned}$$

So, $\Big|\Big|\sum_{i<n} \alpha_i e_i\Big|\Big|^2 < \frac{1}{n}\Big|\Big|\sum_{i<n} \alpha_i e_i\Big|\Big|^2$, but since $\sum_{i<n} |\alpha_i| > 0$, we conclude that $\sum_{i<n} \alpha_i e_i \neq 0$ and then $1 < 1/n$, an absurd.

Hence, $\sum_{i<n} |\alpha_i| = 0$ or $\sum_{i<n} \alpha_i f_i \neq 0$. In each case, we conclude that the set $\{f_i\}_i$ is linearly independent.

■

Lemma 4.7. *Let $V_0, V_1 \leqslant H$ be subspaces satisfying $V_0 \perp V_1$. Let $A_i : V_i \to H$, with $i = 1, 2$, be contractive linear operators such that $A_0(V_0) \perp A_1(V_1)$. Then, $A : \mathrm{span}\{V_0, V_1\} \to H$, given by*

$$A(\alpha v_0 + \beta v_1) = \alpha A_0(v_0) + \beta A_1(v_1)\,,$$

is also contractive.

Proof. Let $v \in \mathrm{span}\{V_0, V_1\}$ be such that $||v|| = 1$. We can write $v = \xi_0 + \xi_1$, where $\xi_i \in V_i$.

In case $\xi_0 = 0$ or $\xi_1 = 0$ we use the fact that A_i is contractive and the fact that $||\xi_i|| \leqslant ||v||$ to obtain

$$||Av|| = ||A_i\xi_i|| \leqslant ||A_i|| \cdot ||\xi_i|| \leqslant 1\,.$$

In case both are different from zero, we write $v = \alpha_0 v_0 + \alpha_1 v_1$, where $\alpha_i = ||\xi_i||$ and $v_i = \xi_i/||\xi_i||$. With these definitions, we obtain $|\alpha_0|^2 + |\alpha_1|^2 = 1$ and $v_i \in S_{V_i}$. Now, note that

$$||Av||^2 \leqslant |\alpha_0|^2 \cdot ||A_0 v_0||^2 + |\alpha_1|^2 \cdot ||A_1 v_1||^2 \leqslant |\alpha_0|^2 + |\alpha_1|^2 = 1\,,$$

concluding the proof.

■

4.3 Strongly stable contractions

In this section we begin the study of s-typical properties in $C(H)$. We will define a special type of contraction, and after that we will prove that this special contraction is s-typical.

Definition 4.8. *We say that a contraction A is strongly stable if* $s\text{-}\lim A^n = 0$. *We denote the set of all strongly stable contractions by $\mathcal{S}$.*

Remark 4.9. *We note that a contraction is strongly stable if, and only if for each $x \in S_H$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $||A^N x|| < \varepsilon$. Indeed, for each $\varepsilon > 0$ and $y \in H\backslash\{0\}$, set $x = y/||y|| \in S_H$, pick N satisfying the property above for $\varepsilon/||y||$ and conclude that if $n \geqslant N$, $||A^n x|| \leqslant ||A^{n-N}|| \cdot ||A^N x|| < \varepsilon/||y||$, showing that $||A^n y|| < \varepsilon$. The other implication is obvious.*

Theorem 4.10. *$\mathcal{S}$ is an s-co-meager, s-G_δ subset of $C(H)$.*

Proof. (a) The first thing to note is that $\mathfrak{C}(H) := \{A \in C(H) : ||A|| < 1\}$ is norm-dense in $C(H)$. To see this, we use the fact that $A = \lim(1 - 2^{-n})A$, and so, for $n \geqslant 0$,

$$||(1 - 2^{-n})A|| = |1 - 2^{-n}| \cdot ||A|| \leqslant |1 - 2^{-n}| < 1$$

Then, since $\mathfrak{C}(H)$ is norm-dense, it is s-dense in $C(H)$.

The second thing is that every operator in $\mathfrak{C}(H)$ is strongly stable. Indeed, by taking $x \in S_H$, it follows that

$$||A^n x|| \leqslant ||A||^n \cdot ||x|| = ||A||^n \xrightarrow{n\to\infty} 0\,.$$

So, since $\mathfrak{C}(H) \subset \mathcal{S} \subset C(H)$ and $\mathfrak{C}(H)$ is s-dense in $C(H)$, we conclude that $\mathcal{S}$ is s-dense in $C(H)$.

(b) It remains to prove that $\mathcal{S}$ is s-G_δ. For that, let $\{x_i : i \in \mathbb{N}\}$ be a dense subset of S_H. Now, we claim that

$$\mathcal{S} = \bigcap_{i,j\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \{A \in C(H) : ||A^n x_i|| < 2^{-j}\}\,.$$

Let us show that this equality holds and afterwards that $\{A \in C(H) : ||A^n x_i|| < 2^{-j}\}$ is s-open for all $i, j, n \in \mathbb{N}$.

If $A \in \mathcal{S}$, it follows from Remark 4.9 that for each $x \in S_H$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $||A^N x|| < \varepsilon$. We conclude that for every $i, j \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $||A^N x_i|| < 2^{-j}$, in other words, $A \in \bigcap_{i,j} \bigcup_n \{A \in C(H) : ||A^n x_i|| < 2^{-j}\}$.

On the other hand, let $A \in \mathcal{A} := \bigcap_{i,j} \bigcup_n \{A \in C(H) : ||A^n x_i|| < 2^{-j}\}$, $x \in S_H$ and $\varepsilon > 0$. Since $\{x_i : i \in \mathbb{N}\}$ is dense in S_H there exists $k \in \mathbb{N}$ satisfying $||x_k - x|| < \varepsilon/2$. Let $j \in \mathbb{N}$ be such that $2^{-j} < \varepsilon/2$, and by using the properties of $\mathcal{A}$, exists $N \in \mathbb{N}$ such that $||A^N x_k|| < 2^{-j}$, so

$$||A^N x|| \leqslant ||A^N(x - x_k)|| + ||A^N x_k|| \leqslant ||A||^N \cdot ||x - x_k|| + ||A^N x_k|| < \varepsilon/2 + 2^{-j} < \varepsilon\,.$$

Hence, $A \in \mathcal{S}$.

(c) To see that $\{A \in C(H) : ||A^n x_i|| < 2^{-j}\}$ is s-open for all $i, j, n \in \mathbb{N}$ it suffices to show that its complement is s-closed. For each $i, j, n \in \mathbb{N}$, let $(T_m)_{m\in\mathbb{N}} \subset \{A \in C(H) : ||A^n x_i|| \geqslant 2^{-j}\}$ and $T \in C(H)$ be such that s-$\lim T_m = T$. We know that for every $k \in \mathbb{N}$, s-$\lim T_m^k = T^k$. In particular, s-$\lim T_m^n = T^n$, and then

$$||T^n x_i|| = ||\lim T_m^n x_i|| = \lim ||T_m^n x_i|| \geqslant \lim 2^{-j} = 2^{-j}\,.$$

Hence, $T \in \{A \in C(H) : ||A^n x_i|| \geqslant 2^{-j}\}$, proving that this set is closed.

■

4.4 Studying the properties of s-typical contractions

The main goal of this section is prove the following result.

Theorem 4.11. *Let $\mathcal{G}$ denote the set of contractive operators A satisfying the following properties:*

1. *$S_H \subset A(S_H)$, i.e., for every $y \in S_H$ there exists $x \in S_H$ such that $Ax = y$;*
2. $\dim(\ker A) = \infty$

Then $\mathcal{G}$ is an s-co-meager subset of $C(H)$.

In order to prove this theorem we will study the two properties separately, denoting by $\mathcal{G}_1$ and $\mathcal{G}_2$ the contractions satisfying only the Property 1 and Property 2 respectively. Given that, $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$, we conclude that it is enough prove that $\mathcal{G}_1$ and $\mathcal{G}_2$ are both s-co-meager in $C(H)$.

The strategy of the proof that $\mathcal{G}_1$ is an s-co-meager subset involves a result from Linear Algebra called Singular Value Decomposition, or simply SVD (see [10] and [11] for details).

Singular Value Decomposition. *Let A be an $m \times n$ complex matrix, $q = \min\{m, n\}$ and $r = \operatorname{rank} A$. Then, there exists is an $m \times n$ matrix $\Sigma = [\sigma_{ij}]$, with $\sigma_{ij} = 0$ for all $i \neq j$ and $\sigma_{11} \geqslant \sigma_{22} \geqslant \cdots \geqslant \sigma_{rr} > 0 = \sigma_{(r+1)(r+1)} = \cdots = \sigma_{qq}$, and there exist a unitary $m \times m$ matrix V and a unitary $n \times n$ matrix W such that $A = V\Sigma W^*$.*

In particular, if $A : V \to W$ is a linear map with $\dim V = n$ and $\dim W = m$, one can find an orthonormal basis $\{v_i\}_{i<n}$ of V and $\{w_i\}_{i<m}$ of W such that A maps the i-th basis vector of V to a non-negative multiple of the i-th basis vector of W, for every $i < \dim A(V)$, and it maps the other basis vectors to zero.

With SVD, we can prove the following result:

Lemma 4.12. *Let $V \leqslant H$ be a finite dimensional subspace and let $A : V \to H$ be a contractive linear operator. Let $W = A(V)$ and let $Y \leqslant H$ be an arbitrary subspace satisfying $Y \perp W$. Then, for all $X \leqslant H$ satisfying $X \perp V$ and $\dim X = \dim W + \dim Y$, there exists a contractive linear operator $\tilde{A} : \operatorname{span}\{V, X\} \to H$ such that $\tilde{A}\big|_V = A$ and $\tilde{A}(B_{\operatorname{span}\{V,X\}}) = B_{\operatorname{span}\{W,Y\}}$.*

Proof. (a) Case $Y = \{0\}$.

Let $\dim V = n$ and $\dim W = m$. Obviously, $m \leqslant n$. The SVD states that there exists an orthonormal base $\{v_i : i < n\} \subset V$ such that $\{Av_i : i < n\}$ are pairwise orthogonal. Moreover, $||Av_i|| > 0$ if, and only if, $i < m$. Set $w_i = Av_i/||Av_i||$ for all $i < m$.

Let $X \leqslant H$ be such that $X \perp V$ and $\dim X = \dim W + \dim Y = m$. Fix an orthonormal basis $\{x_i : i < m\} \subset X$ and define $\tilde{A} : \mathrm{span}\{V, X\} \to H$ such that $\tilde{A}\big|_V = A$ and $\tilde{A}x_i = \sqrt{1 - ||Av_i||^2} \cdot w_i$ for all $i < m$. We claim that $\tilde{A}$ satisfies the properties we want.

We begin by checking that $\tilde{A}$ is a contraction. Let $u \in S_{\mathrm{span}\{V,X\}}$, i.e., $u \in \mathrm{span}\{V, X\}$ and $||u|| = 1$. We can write $u = \sum_{i<n} \alpha_i v_i + \sum_{i<m} \beta_i x_i$ for some $\alpha_i, \beta_i \in \mathbb{C}$, with $1 = ||u||^2 = \sum_{i<n} |\alpha_i|^2 + \sum_{i<m} |\beta_i|^2$. Then,

$$
\begin{aligned}
||\tilde{A}u||^2 &= \Big|\Big| \sum_{i<n} \alpha_i A v_i + \sum_{i<m} \beta_i \sqrt{1 - ||Av_i||^2} \cdot w_i \Big|\Big|^2 \\
&= \Big|\Big| \sum_{i<m} \alpha_i ||Av_i|| \cdot w_i + \sum_{i<m} \beta_i \sqrt{1 - ||Av_i||^2} \cdot w_i \Big|\Big|^2 \\
&= \sum_{i<m} \Big|\Big| (\alpha_i ||Av_i|| + \beta_i \sqrt{1 - ||Av_i||^2}) \cdot w_i \Big|\Big|^2 \\
&\leqslant \sum_{i<m} (|\alpha_i| \cdot ||Av_i|| + |\beta_i| \sqrt{1 - ||Av_i||^2})^2 .
\end{aligned}
$$

If $0 \leqslant p, q, r \leqslant 1$, we use the Cauchy-Schwarz inequality in $\mathbb{R}^2$ in order to obtain

$$
\begin{aligned}
(pr + q\sqrt{1 - r^2})^2 &= |\langle (p, q), (r, \sqrt{1 - r^2}) \rangle_{\mathbb{R}^2}|^2 \\
&\leqslant ||(p, q)||^2_{\mathbb{R}^2} ||(r, \sqrt{1 - r^2})||^2_{\mathbb{R}^2} \\
&= (p^2 + q^2)(r^2 + 1 - r^2) = p^2 + q^2 .
\end{aligned}
$$

Then, $(pr + q\sqrt{1 - r^2})^2 \leqslant p^2 + q^2$, and by using this in previous relation we obtain

$$
\begin{aligned}
||\tilde{A}u||^2 &\leqslant \sum_{i<m} (|\alpha_i| \cdot ||Av_i|| + |\beta_i| \sqrt{1 - ||Av_i||^2})^2 \\
&\leqslant \sum_{i<m} |\alpha_i|^2 + |\beta_i|^2 \leqslant \sum_{i<n} |\alpha_i|^2 + \sum_{i<m} |\beta_i|^2 = 1 .
\end{aligned}
$$

Next, we show that for each $y \in B_W$, there exists $x \in B_{\mathrm{span}\{V,X\}}$ such that $\tilde{A}x = y$. Let $y = \sum_{i<m} \beta_i w_i$, where $\beta_i \in \mathbb{C}$ are such that $||y||^2 = \sum_{i<m} |\beta_i|^2 \leqslant 1$. We define

$$
x = \sum_{i<m} \beta_i (||Av_i|| v_i + \sqrt{1 - ||Av_i||^2} \cdot x_i) ,
$$

and claim that $x \in B_{\mathrm{span}\{V,X\}}$, with $\tilde{A}x = y$. First, note that

$$
||x||^2 = \sum_{i<m} |\beta_i|^2 ||Av_i||^2 + |\beta_i|^2 (1 - ||Av_i||^2) = \sum_{i<m} |\beta_i|^2 \leqslant 1 ,
$$

and so $x \in B_{\text{span}\{V,X\}}$. Moreover,

$$\begin{aligned}\tilde{A}x &= \sum_{i<m} \beta_i(||Av_i||\tilde{A}v_i + \sqrt{1-||Av_i||^2}\tilde{A}x_i) \\ &= \sum_{i<m} \beta_i\big(||Av_i||Av_i + (1-||Av_i||^2)w_i\big) = \\ &= \sum_{i<m} \beta_i\big(||Av_i||^2 w_i + (1-||Av_i||^2)w_i\big) \\ &= \sum_{i<m} \beta_i w_i = y\end{aligned}$$

Then, $\tilde{A}$ is a contractive linear operator such that $\tilde{A}\big|_V = A$ and $\tilde{A}(B_{\text{span}\{V,X\}}) = B_{\text{span}\{W,Y\}}$. This concludes the proof of the case $Y = \{0\}$.

(b) General case.

Since $\dim X = \dim W + \dim Y$ write $X = X_0 \oplus X_1$ where $X_0 \perp X_1$, $\dim X_0 = \dim W$ and $\dim X_1 = \dim Y$. By the Case (a) there exists a contraction $\tilde{A} : \text{span}\{V, X_0\} \to H$ such that $\tilde{A}\big|_V = A$ and $\tilde{A}(B_{\text{span}\{V,X_0\}}) = B_W$. We extend such $\tilde{A}$ by setting $\tilde{A}\big|_{X_1} : X_1 \to Y$ as any isometric isomorphism (which exists since $\dim X_1 = \dim Y$). By Lemma 4.7 we conclude that $\tilde{A}$ is also contractive.

It remains to prove that $\tilde{A}(B_{\text{span}\{V,X\}}) = B_{\text{span}\{W,Y\}}$. Since $\tilde{A}$ is contractive, we just have to show that $B_{\text{span}\{W,Y\}} \subset \tilde{A}(B_{\text{span}\{V,X\}})$. Let $u \in B_{\text{span}\{W,Y\}}$. We can write $u = w+y$, with $w \in W$ and $y \in Y$. Since $\tilde{A}\big|_{X_1}$ is an isometric isomorphism, there exists $x_1 \in X_1$ such that $\tilde{A}x_1 = y$ and $||x_1|| = ||y||$.

For a vector space U and an $\alpha \geqslant 0$ we denote $B_U^\alpha = \alpha B_U$. It is easy to conclude that $\tilde{A}(B^\alpha_{\text{span}\{V,X_0\}}) = B_W^\alpha$ for every $\alpha \geqslant 0$. Fix $\alpha = \sqrt{||u||^2 - ||y||^2} = ||w||$, it follows that if $w \in B_W^\alpha$, then there exists $\xi \in B^\alpha_{\text{span}\{V,X_0\}}$ such that $\tilde{A}\xi = w$.

We know that $X_1 \perp X_0$, and since $X_0 \oplus X_1 = X$ and $X \perp V$, we have that $\xi \perp x_1$, and so

$$||\xi + x_1||^2 = ||\xi||^2 + ||x_1||^2 \leqslant \alpha^2 + ||x_1||^2 = ||w||^2 + ||y||^2 = ||u||^2 \leqslant 1\,.$$

Then, $\xi + x_1 \in B_{\text{span}\{V,X\}}$ and $\tilde{A}(\xi + x_1) = w + y = u$, proving that $\tilde{A}(B_{\text{span}\{V,X\}}) = B_{\text{span}\{W,Y\}}$.

■

Theorem 4.13. *$\mathcal{G}_1$ is a s-co-meager subset of $C(H)$.*

Proof. (a) Let

$$\mathcal{M} = \{A \in C(H) : \forall\, \varepsilon > 0 \text{ and } y \in S_H\ \exists\, x \in S_H \text{ such that } ||y - Ax|| < \varepsilon\}$$

Firstly, we show that $\mathcal{M}$ is an s-dense subset s-G_δ of $C(H)$. Let $y \in S_H$, $\varepsilon > 0$, and define the set

$$C(y,\varepsilon) = \{A \in C(H) : \exists\, x \in S_H \text{ such that } ||y - Ax|| < \varepsilon\}\,,$$

which is s-open.

To see this, we show that $C(y,\varepsilon)^C$ is s-closed, i.e., that the set $\{A \in C(H) : \forall\, x \in S_H, ||y - Ax|| \geqslant \varepsilon\}$ is s-closed. Let $(T_n)_{n\in\mathbb{N}} \subset C(y,\varepsilon)^C$ and let $T \in C(H)$ be such that s-$\lim T_n = T$. Then,

$$||y - Tx|| = \lim ||y - T_n x|| \geqslant \varepsilon$$

which shows what we need.

Now, we prove that $C(y, \varepsilon)$ is s-dense in $C(H)$. Let $U \subset C(H)$, and by taking a subset if necessary, we can assume that $U = \bigcap_{i \in I}\{A \in C(H) : ||y_i - Ax_i|| < \varepsilon_i\}$, where I is a finite index set, $x_i, y_i \in H$ and $\varepsilon_i > 0$ for all $i \in I$.

Let $V = \text{span}\{x_i : i \in I\}$ and let $A \in U$. Note that $AP_V \in U$, so we can assume that $A\big|_{V^\perp} = 0$. Set $W = A(V)$ and note that since V is finite dimensional W is also, so we can obtain $Y \leqslant H$, with $\dim(Y) = 1$, such that $Y \perp W$ and $y \in \text{span}\{W, Y\}$. Explicitly, if $y \in W$ let Y any one dimensional subspace such that $Y \perp W$. Otherwise, let $Y = \text{span}\{P_{W^\perp}(y)\}$.

Let now $X \leqslant H$ be a $(\dim W + \dim Y)$-dimensional subspace such that $X \perp V$, which obviously exists since $\dim(V) < \infty$ and $\dim(H) = \infty$. By Lemma 4.12 there exists a contraction $\tilde{A} : \text{span}\{V, X\} \to H$ such that $\tilde{A}\big|_V = A\big|_V$ and $\tilde{A}(B_{\text{span}\{V,X\}}) = B_{\text{span}\{W,Y\}}$. In particular, there exists a $x \in B_{\text{span}\{V,X\}}$ such that $\tilde{A}x = y$. Since $\tilde{A}$ is contractive, $||y|| = 1$ and $||x|| \leqslant 1$, it follows that

$$1 = ||y|| = ||\tilde{A}x|| \leqslant ||\tilde{A}|| \cdot ||x|| = ||x|| \leqslant 1$$

Hence, $x \in S_H$.

We extend $\tilde{A}$ by setting $\tilde{A}\big|_{\text{span}\{V,X\}^\perp} = 0$, so $\tilde{A} \in U \cap C(y, \varepsilon)$ (which is true because $\tilde{A}$ extends A and $A \in U$). On the other hand, $||\tilde{A}x - y|| = 0 < \varepsilon$, so $\tilde{A} \in C(y, \varepsilon)$. Then, $C(y, \varepsilon)$ is s-dense.

(b) We will prove that

$$\mathcal{M} = \bigcap_{\substack{n \in \mathbb{N} \\ y \in D}} C(y, 2^{-n}) \tag{4.1}$$

where $D \subset H$ is a countable dense subset. The inclusion $\mathcal{M} \subset \bigcap C(y, 2^{-n})$ is obvious. In order to prove the other one, let $A \in \bigcap_{n \in \mathbb{N}, y \in D} C(y, 2^{-n})$. Let $\varepsilon > 0$ and $y_0 \in S_H$. Since D is dense, there exists $y \in D$ such that $||y_0 - y|| < \varepsilon/2$. Let $n \in \mathbb{N}$ be such that $2^{-n} < \varepsilon/2$, and since $A \in C(y, 2^{-n})$, there exists $x \in S_H$ satisfying $||y - Ax|| < 2^{-n}$. Hence,

$$||y_0 - Ax|| \leqslant ||y - Ax|| + ||y_0 - y|| < \varepsilon/2 + 2^{-n} < \varepsilon\,.$$

Then, relation (4.1) holds and by Baire Category Theorem, it follows that $\mathcal{M}$ is an s-dense s-G_δ.

(c) Now, we claim that $\mathcal{M} = \mathcal{G}_1$. Namely, for $A \in \mathcal{G}_1$, $y \in S_H$, there exists $x \in S_H$ such that $Ax = y$. Then, $||Ax - y|| = 0 < \varepsilon$ for every $\varepsilon > 0$, and so $A \in \mathcal{M}$.

On the other hand, if $A \in \mathcal{M}$ and $z \in S_H$ there exist $(b_n)_{n \in \mathbb{N}} \subset S_H$ such that $||Ab_n - z|| < 2^{-n}$, so $\lim Ab_n = z$. By Lemma 4.5, $(b_n)_{n \in \mathbb{N}}$ converges to some $b \in S_H$. So,

$$z = \lim Ab_n = A(\lim b_n) = Ab\,,$$

from which follows that $A \in \mathcal{G}_1$.

∎

Now, we show that the set $\mathcal{G}_2$ is s-co-meager.

Lemma 4.14. *The set of contractive operators A such that, for each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $Z \leqslant H$ such that $\dim Z \geqslant n$ and $||A|_Z|| < \varepsilon$, is an s-dense s-G_δ subset of $C(H)$.*

Proof. (a) For each $n \in \mathbb{N}$ and $\varepsilon > 0$, we define the set

$$C(n,\varepsilon) = \{A \in C(H) : \exists\, Z \leqslant H \text{ such that } \dim Z \geqslant n \text{ and } ||A|_Z|| < \varepsilon\}.$$

We claim that $C(n,\varepsilon)$ is s-open for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Namely, for each $A \in C(n,\varepsilon)$ there exists $Z \leqslant H$ such that $\dim Z \geqslant n$ and $||A|_Z|| < \varepsilon$. We can assume without loss of generality that Z finite dimensional (this because if there exists an infinite dimensional space W satisfying these properties we can pick any finite dimensional subset Z with $\dim Z \geqslant n$, and then $||A|_Z|| \leqslant ||A|_W||$). Thus, we can assume that $\dim Z = k < \infty$. Let $\{x_i : i < k\}$ be an orthonormal base of Z.

Since every norms in Z are equivalent, there exists $K > 0$ such that for each $z \in Z$, $||z||_1 \leqslant K||z||$, where $||\sum_{i<k} \alpha_i x_i||_1 = \sum_{i<k} |\alpha_i|$. Let $\eta = \varepsilon - ||A|_Z||$ and define

$$U = \bigcap_{i<k} \{T \in C(H) : ||(T-A)x_i|| < \eta/2K\},$$

which is an s-neighborhood of A, by Theorem 4.2.

We want to show that $U \subset C(n,\varepsilon)$. Thus, let $T \in U$ and note that

$$||T|_Z|| \leqslant ||(T-A)|_Z|| + ||A|_Z||. \tag{4.2}$$

We already know how to control $||A|_Z||$, so we just need to control the other norm. Let $\sum_{i<k} \alpha_i x_i \in Z$ be such that $||\sum_{i<k} \alpha_i x_i|| = 1$, so

$$\begin{aligned}
\Big\|(T-A)\Big(\sum_{i<k} \alpha_i x_i\Big)\Big\| &= \Big\|\sum_{i<k} \alpha_i (T-A)x_i\Big\| \\
&\leqslant \sum_{i<k} (|\alpha_i| \cdot ||(T-A)x_i||) \\
&< \sum_{i<k} |\alpha_i| \frac{\eta}{2K} = \frac{\eta}{2K} \Big\|\sum_{i<k} \alpha_i x_i\Big\|_1 \\
&\leqslant \frac{\eta}{2K} \cdot K \Big\|\sum_{i<k} \alpha_i x_i\Big\| = \frac{\eta}{2}.
\end{aligned}$$

Hence, $||(T-A)|_Z|| \leqslant \eta/2 < \eta$, and by relation (4.2), we conclude that

$$||T|_Z|| < \eta + ||A|_Z|| = \varepsilon.$$

Thus $T \in C(n,\varepsilon)$, and we conclude that $C(n,\varepsilon)$ is s-open.

(b) The next step is show that $C(n,\varepsilon)$ is s-dense. Let $U \subset C(H)$ be any non-empty s-open set, and again, by taking a subset if needed we can assume that

$$U = \bigcap_{i \in I} \{A \in C(H) : ||y_i - Av_i|| < \varepsilon_i\}$$

where I is finite, $v_i, y_i \in H$ and $\varepsilon_i > 0$.

Let $A \in U$, $V = \operatorname{span}\{v_i : i \in I\}$ and define $B \in C(H)$ by $B\big|_V = A\big|_V$ and $B\big|_{V^\perp} = 0$. Obviously $B \in U$, and since $\dim V^\perp = \infty$ is such that $||B\big|_{V^\perp}|| = 0 < \varepsilon$, we have $B \in C(n, \varepsilon)$. So, $B \in C(n, \varepsilon) \cap U$ and we conclude that $C(n, \varepsilon)$ is s-dense.

(c) Now we claim that the set of contractive operators A such that, for each $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $Z \leqslant H$ such that $\dim Z \leqslant n$ and $||A\big|_Z|| < \varepsilon$, is exactly the set

$$\bigcap_{n,m\in\mathbb{N}} C(n, 2^{-m})$$

This is straightforward, because if A is a contractive operator such that, for each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $Z \leqslant H$ such that $\dim Z \leqslant n$ and $||A\big|_Z|| < \varepsilon$ we let $\varepsilon = 2^{-m}$ and then conclude that $A \in C(n, 2^{-m})$ for all $n, m \in \mathbb{N}$. On the other hand, for each $\varepsilon > 0$ and each $A \in \bigcap_{n,m\in\mathbb{N}} C(n, 2^{-m})$ let $m \in \mathbb{N}$ be such that $2^{-m} < \varepsilon$. Then, there exists $Z \leqslant H$ such that $\dim Z \geqslant n$ and $||A\big|_Z|| < 2^{-m} < \varepsilon$.

By Baire Category Theorem it follows that $\bigcap_{n,m\in\mathbb{N}} C(n, 2^{-m})$ is s-dense and s-G_δ.

■

Theorem 4.15. *$\mathcal{G}_2$ is an s-co-meager subset of $C(H)$.*

Proof. By Theorem 4.13 and Lemma 4.14, the set of contractive operators A satisfying

1. for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $Z \leqslant H$ such that $\dim Z \geqslant n$ and $||A\big|_Z|| < \varepsilon/n$
2. for every $y \in S_H$, there exists $x \in S_H$ such that $Ax = y$

is an s-co-meager subset of $C(H)$. We will show that every element of this set satisfies $\dim(\ker A) = \infty$.

Pick A satisfying the Properties 1 and 2, let $n \in \mathbb{N}$ and choose $\varepsilon = 1$. By Property 1 there exists $Z \leqslant H$ such that $\dim Z \geqslant n$ and $||A\big|_Z|| < 1/n$. Let $\{e_i\}_{i=0}^{n-1} \subset Z$ be an orthonormal set. For each $i < n$, there exist $f_i \in H$ such that $||f_i - e_i|| < 1/n$ and $Af_i = 0$.

Fix $i < n$: if $Ae_i = 0$, set $f_i = e_i$. Otherwise, by Property 2 there exists $x_i \in S_H$ such that $Ax_i = Ae_i/||Ae_i||$. Set in this case, $f_i = e_i - ||Ae_i||x_i$. With that, we get

$$Af_i = Ae_i - ||Ae_i||\frac{Ae_i}{||Ae_i||} = 0$$

Note that $f_i \neq 0$. Indeed, if $f_i = 0$, then $e_i = ||Ae_i||x_i$, and by taking the norm in both sides we get $1 = ||Ae_i||$, which is absurd since $e_i \in Z$ and $||A\big|_Z|| < 1/n$.

In both cases, $||f_i - e_i|| = 0 < 1/n$ or $||f_i - e_i|| = ||Ae_i|| < 1/n$ concluding the construction.

It follows from Lemma 4.6 that $\{f_i : i < n\}$ is linearly independent, but since $\{f_i : i < n\} \subset \ker A$, then $\dim\ker A \geqslant n$. Hence, $\dim\ker A = \infty$.

■

Remark 4.16. *Since a finite intersection of co-meager is co-meager in any topology, Theorem 4.11 is a direct consequence of Theorem 4.13 and Theorem 4.15.*

Before we proceed to the proof of the main result of this chapter, let us study some of properties that the operators in $\mathcal{G}$ satisfy.

Theorem 4.17. *Let $A \in \mathcal{G}$. Then $AA^* = I$ and $A^*A = P_{\operatorname{Ran} A^*}$, $\operatorname{Ran} A^*$ is infinite-dimensional and infinite-co-dimensional subspace of H. In particular, A^* is an isometry (hence A is a co-isometry) and, A is an isometry on $(\ker A)^{\perp}$.*

Proof. (a) Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis of H. By the definition of $\mathcal{G}$, for every $i \in \mathbb{N}$ there exists $a_i \in S_H$ such that $Aa_i = e_i$. We claim that $A^*e_i = a_i$. First, note that

$$0 = \langle Aa_i - e_i, Aa_i - e_i\rangle = \langle Aa_i, Aa_i\rangle - 2\operatorname{Re}\langle Aa_i, e_i\rangle + \langle e_i, e_i\rangle = 2 - 2\operatorname{Re}\langle Aa_i, e_i\rangle\,,$$

and since $\langle A^*e_i, a_i\rangle = \overline{\langle Aa_i, e_i\rangle}$, we conclude that $\operatorname{Re}\langle A^*e_i, a_i\rangle = 1$. We also have that

$$1 = |\langle e_i, e_i\rangle| = |\langle e_i, Aa_i\rangle| = |\langle A^*e_i, a_i\rangle| \leqslant ||A^*e_i|| \leqslant ||A^*|| \cdot ||e_i|| \leqslant 1\,.$$

Then, $||A^*e_i|| = 1$, and we conclude that

$$\langle A^*e_i - a_i, A^*e_i - a_i\rangle = \langle A^*e_i, A^*e_i\rangle - 2\operatorname{Re}\langle A^*e_i, a_i\rangle + \langle a_i, a_i\rangle = 1 - 2 + 1 = 0$$

So, $A^*e_i = a_i$ and with this we have $AA^*e_i = Aa_i = e_i$, from which follows that $AA^* = I$.

(b) Note that

$$(A^*A)^2 = A^*AA^*A = A^*(AA^*)A = A^*IA = A^*A\,.$$

Moreover, $(A^*A)^* = A^*A$. Then, A^*A is a projection onto $Z = \{x \in H : A^*Ax = x\}$. Obviously, $Z \subset \operatorname{Ran} A^*$; on the other hand, if $x \in \operatorname{Ran} A^*$, there exists $y \in H$ such that $A^*y = x$. Then,

$$A^*Ax = A^*AA^*y = A^*y = x\,,$$

So $x \in Z$, and we conclude that $A^*A = P_{\operatorname{Ran} A^*}$.

(c) Since $(A^*)^*A^* = AA^* = I$, it follows that A^* is an isometry. Now, if $x \in \operatorname{Ran} A^*$, there exists $y \in H$ such that $x = A^*y$ and

$$\langle Ax, Ax\rangle = \langle AA^*y, AA^*y\rangle = \langle AA^*y, y\rangle = \langle A^*y, A^*y\rangle = \langle x, x\rangle\,.$$

Then, A is a isometry in $\operatorname{Ran} A^*$.

(d) Since $\{e_i : i \in \mathbb{N}\}$ is an orthonormal set and A^* is an isometry, it follows that $\{a_i : i \in \mathbb{N}\}$ is also orthonormal. Given that $\operatorname{Ran} A^* = \operatorname{Cl}\big(\operatorname{span}\{A^*e_i : i \in \mathbb{N}\}\big) = \operatorname{Cl}\big(\operatorname{span}\{a_i : i \in \mathbb{N}\}\big)$ the Hilbert dimension of H is infinite, and so its the algebraic dimension is also infinite.

(e) Since A^* is a isometry, we have that $\operatorname{Ran} A^*$ is a closed subspace and so $(\ker A)^{\perp} = \operatorname{Cl}(\operatorname{Ran} A^*) = \operatorname{Ran} A^*$. We conclude that $\operatorname{codim}(\ker A)^{\perp} = \operatorname{codim} \operatorname{Ran} A^*$. Note that

$$H\big/(\ker A)^{\perp} \cong \ker A\,,$$

then, $\operatorname{codim} \operatorname{Ran} A^* = \dim\ker A = \infty$, since $A \in \mathcal{G}$. Then $\operatorname{Ran} A^*$ is infinite co-dimensional. ■

4.5 Unitary equivalence of s-typical contractions

Since all infinite-dimensional complex separable Hilbert spaces are isometrically isomorphic, we can restrict ourselves to study a particular one.

Definition 4.18. *Let* $H = \ell^2(\mathbb{N} \times \mathbb{N})$ *and denote the canonical orthonormal basis of* H *by* $\{e_i(n) : i, n \in \mathbb{N}\}$*, i.e.,*

$$e_i(n)(j,k) = \begin{cases} 1 & j = i \text{ and } k = n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We define the infinite-dimensional left shift operator S *by the law* $Se_0(n) = 0$ *and* $Se_{i+1}(n) = e_i(n)$ *for all* $i, n \in \mathbb{N}$*.*

The first thing to note is that $S \in C(H)$. Since $||Se_2(0)|| = ||e_1(0)|| = 1$, we have that $||S|| \geqslant 1$. On the other hand, for each $x \in H$ there exists $\{\lambda_{in}\}_{i,n\in\mathbb{N}}$ such that $x = \sum_{i,n\in\mathbb{N}} \lambda_{in} e_i(n)$, and then

$$\begin{aligned} ||Sx||^2 &= \Big|\Big| \sum_{i,n\in\mathbb{N}} \lambda_{in} S e_i(n) \Big|\Big|^2 = \Big|\Big| \sum_{\substack{i\in\mathbb{N}\setminus\{0\}\\ n\in\mathbb{N}}} \lambda_{in} e_{i-1}(n) \Big|\Big|^2 \\ &= \sum_{\substack{i\in\mathbb{N}\setminus\{0\}\\ n\in\mathbb{N}}} |\lambda_{in}|^2 \leqslant \sum_{i,n\in\mathbb{N}} |\lambda_{in}|^2 = ||x||^2\,. \end{aligned}$$

This shows that $||S|| \leqslant 1$, proving that $||S|| = 1$.

Theorem 4.19. *The set* $\mathcal{O}(S) = \{USU^{-1} : U \in U(H)\}$ *is an s-co-meager subset of* $C(H)$*.*

Proof. (a) By Theorems 4.10 and 4.11, it is enough to show that every $A \in \mathcal{S} \cap \mathcal{G}$ is unitary equivalent to the operator S. By Theorem 4.16 A^* is an isometry, so by Wold Decomposition (see Appendix C) we have $H = H_U \oplus H_S$, where $A^*\big|_{H_U}$ is unitary and $A^*\big|_{H_S}$ is a right shift.

(b) We claim that $H_U = \{0\}$. Recall that $H_U = \bigcap_{i\in\mathbb{N}} H_i$, where $H_i = (A^*)^i(H)$. Let $v \in H_U$. For every $i \in \mathbb{N}$, there exists $v_i \in H$ such that $(A^*)^i v_i = v$. Using that A^* is an isometry, we obtain that $A^i v = v_i$, but since $A \in \mathcal{S}$, A is strongly stable, and so $0 = \lim_i A^i v = \lim_i v_i$. Now, note that

$$0 \leqslant ||v|| = ||(A^*)^i v_i|| \leqslant ||(A^*)^i|| \cdot ||v_i|| \leqslant ||v_i|| \xrightarrow{i\to\infty} 0$$

Thus, $v = 0$ and so $H_U = \{0\}$.

(c) We claim that $A\big|_{H_S}$ is unitary equivalent to S. Recall that $H_S \cong_U \bigoplus_{i\in\mathbb{N}} M_0$, where $M_0 = H \ominus (A^*)(H) = (\operatorname{Ran} A^*)^\perp = \ker A$. Since A^* is a right shift in $\bigoplus_{i\in\mathbb{N}} \ker A$, it follows that its adjoint A is a left shift.

We claim that $\ell^2(\mathbb{N} \times \mathbb{N}) \cong_U \bigoplus_{i\in\mathbb{N}} \ker A$. To see this let $f = (f_0, f_1, \dots) \in \bigoplus_{i\in\mathbb{N}} \ker A$, where $f_i \in \ker A$ for every $i \in \mathbb{N}$, and $\{e_i\}_{i\in\mathbb{N}}$ be a basis for $\ker A$. We can write $f_j = \sum_{n=0}^{\infty} \lambda_n^j e_n$ for some $\lambda_n^j \in \mathbb{C}$, and then define a function $F : \bigoplus_{i\in\mathbb{N}} \ker A \to \ell^2(\mathbb{N} \times \mathbb{N})$ by the law $(Ff)(n,m) = \lambda_m^n$ for every $f \in \bigoplus_{i\in\mathbb{N}} \ker A$.

In the next steps we will prove that F is an unitary operator and that $A = F^{-1}SF$.

(d) We begin by showing that F is well defined, i.e., that Ff really lies in $\ell^2(\mathbb{N}\times\mathbb{N})$. Let $f \in \bigoplus_{i\in\mathbb{N}} \ker A$ be such that $f_j = \sum_{n=0}^{\infty} \lambda_n^j e_n$. Then,

$$||f||_\oplus^2 = \sum_{n\in\mathbb{N}} ||f_n||^2 = \sum_{n\in\mathbb{N}} \Big|\Big| \sum_{m\in\mathbb{N}} \lambda_m^n e_m \Big|\Big|^2 = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |\lambda_m^n|^2$$

On the other hand,

$$||Ff||_{\ell^2}^2 = \sum_{n,m\in\mathbb{N}} |\lambda_m^n|^2 \,,$$

from which follows that $||Ff||_{\ell^2} = ||f||_{\oplus}$. This shows that F is well defined and also that is an isometry.

(e) We now show that F is indeed a linear map. Let $f, g \in \bigoplus_{i\in\mathbb{N}} \ker A$ be such that $f_j = \sum_{n=0}^{\infty} \lambda_n^j e_n$ and $g_j = \sum_{n=0}^{\infty} \mu_n^j e_n$, and let $\zeta \in \mathbb{C}$ too. We have that

$$\zeta f + g = (\zeta f_0 + g_0, \zeta f_1 + g_1, \dots) = \Big(\sum_{n=0}^{\infty} (\zeta\lambda_n^0 + \mu_n^0) e_n, \sum_{n=0}^{\infty} (\zeta\lambda_n^1 + \mu_n^1) e_n, \dots \Big) .$$

Then,

$$F(\zeta f + g)(n,m) = \zeta\lambda_m^n + \mu_m^n = \zeta Ff + Fg \,,$$

and so f is a linear map.

(f) To see that F is surjective, let $\sum_{n,m\in\mathbb{N}} \lambda_m^n e_n(m) \in \ell^2(\mathbb{N}\times\mathbb{N})$. Let $f \in \bigoplus_{i\in\mathbb{N}} \ker A$ be such that $f_n = \sum_{m\in\mathbb{N}} \lambda_m^n e_n$. First of all, $f \in \bigoplus_{i\in\mathbb{N}} \ker A$, since

$$\sum_{n\in\mathbb{N}} ||f_n||^2 = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} |\lambda_m^n|^2 < \infty$$

(here we use that $\sum_{n,m\in\mathbb{N}} \lambda_m^n e_n(m) \in \ell^2(\mathbb{N}\times\mathbb{N})$).

Now, we claim that $Ff = \sum_{n,m\in\mathbb{N}} \lambda_m^n e_n(m)$. Note that for each $j, k \in \mathbb{N}$, $(Ff)(j,k) = \lambda_k^j$, and on the other hand

$$\Big(\sum_{n,m\in\mathbb{N}} \lambda_m^n e_n(m) \Big)(j,k) = \sum_{n,m\in\mathbb{N}} \lambda_m^n (e_n(m)(j,k)) = \lambda_k^j \,.$$

Since j, k are arbitrary, it follows that $Ff = \sum_{n,m\in\mathbb{N}} \lambda_m^n e_n(m)$, and so F is surjective.

(g) Since F is unitary, it also is invertible, so if $FAf = SFf$ for every $f \in \bigoplus_{i\in\mathbb{N}} \ker A$, then $FA = SF$ and $A = F^{-1}SF$.

Fix $f \in \bigoplus_{i\in\mathbb{N}} \ker A$ with $f_j = \sum_{n=0}^{\infty} \lambda_n^j e_n$. We have that

$$Af = (f_1, f_2, \dots)$$

Then, $(Af)_j = f_{j+1}$. We can write $(Af)_j = \sum_{n=0}^{\infty} \eta_n^j e_n$, so $\eta_n^j = \lambda_n^{j+1}$. Therefore, we have $(FAf)(j,k) = \eta_k^j = \lambda_k^{j+1}$.

On the other hand, we have $(Ff)(j,k) = \lambda_k^j$. We can write

$$Ff = \sum_{n,m\in\mathbb{N}} \lambda_m^n e_n(m) \,,$$

and then,

$$SFf = \sum_{n,m\in\mathbb{N}} \lambda_m^n S e_n(m) = \sum_{\substack{n\in\mathbb{N}\setminus\{0\} \\ m\in\mathbb{N}}} \lambda_m^n e_{n-1}(m) = \sum_{n,m\in\mathbb{N}} \lambda_m^{n+1} e_n(m) \,,$$

from which follows that $(SFf)(j,k) = \sum_{n,m\in\mathbb{N}} \lambda_m^{n+1}(e_n(m)(j,k)) = \lambda_k^{j+1}$, so $(SFf)(j,k) = (FAf)(j,k)$ for every $j,k \in \mathbb{N}$. Now, by the previous discussion, it follows that $A = F^{-1}SF$. We conclude that $A \in \mathcal{O}(S)$, and we are done.

■

This shows us that the study of s-typical properties of operators reduces to the study of the properties of S. In particular, the spectral properties of s-typical operators can be obtained by studying the spectrum of S. Moreover, in the proof of Theorem 4.19 we showed that A is basically a left shift (since $H_U = \{0\}$ and A^* is a right shift in $\bigoplus_{i\in\mathbb{N}} \ker A$), thus, such spectral properties are the properties of a left shift.:

Corollary 4.20. *An s-typical contraction A satisfies*

1. $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$;
2. $\sigma_c(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$;
3. $\sigma_r(A) = \varnothing$.

5. THE STRONG-STAR TOPOLOGY

5.1 Understanding the topology

After studying a really interesting topology where all the theory of typical operators reduces to the study of only one operator, we return to a topology where typical operators are not unitary equivalent. The reason of this change is the fact that now we have a topology where we control the operator and its adjoint together, or in other words, the function $A \mapsto A^*$ is continuous in strong-star topology.

Definition 5.1. *We define the $*$-SOT topology in $B(H)$ as the coarsest topology that makes the map $\rho_x : T \to ||Tx|| + ||T^*x||$ continuous. In other words, $*$-SOT is the topology generated by the sub-basis consisting of the sets*

$$\{A \in B(H) : \rho_x(T - A) < \varepsilon\}$$

for each $x \in H$, $T \in B(H)$ and $\varepsilon > 0$.

Theorem 5.2. *The s^*-topology coincides with the $*$-SOT topology.*

Proof. (a) Let $(T_\alpha)_{\alpha\in\Lambda}$ and $T \in B(H)$ be such that $\text{s}^*\text{-}\lim T_\alpha = T$, i.e., $\text{s-}\lim T_\alpha = T$ and $\text{s-}\lim T_\alpha^* = T^*$. Let V be a $*$-SOT-neighborhood of T, and by taking a subset of V if necessary we can assume that there exists $n \in \mathbb{N}$ and, for each $i \leqslant n$, $x_i \in H$ such that

$$V = \bigcap_{i=0}^{n} \{A \in B(H) : ||Tx_i - Ax_i|| + ||T^*x_i - A^*x_i|| < \varepsilon\}\,.$$

It follows, from $\text{s}^*\text{-}\lim T_\alpha = T$, that for each $i \leqslant n$ there exists $\beta_i, \beta_i^* \in \Lambda$ such that:

- if $\alpha > \beta_i$ then $||(T_\alpha - T)x_i|| < \varepsilon/2$;
- if $\alpha > \beta_i^*$ then $||(T_\alpha^* - T^*)x_i|| < \varepsilon/2$.

Let $\beta \in \Lambda$ be such that $\beta > \beta_i$ and $\beta > \beta_i^*$ for every $i < n$. Then, if $\alpha > \beta$, it follows that $\alpha > \beta_i, \beta_i^*$ for every $i < n$, and so

$$||(T_\alpha^* - T^*)x_i|| + ||(T_\alpha^* - T^*)x_i|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

This shows that $T_\alpha \in V$ for every $\alpha > \beta$, and the limit in the $*$-SOT topology of this net coincides with its limit in the strong-star topology.

(b) On the other hand, let $(T_\alpha)_{\alpha\in\Lambda}$ be a net that converges to T in the $*$-SOT topology. Let $x \in H$ and note that $V = \{A \in B(H) : ||Tx - Ax|| + ||T^*x - A^*x|| < \varepsilon\}$ is a $*$-SOT-neighborhood of T. So, there exists $\beta \in \Lambda$ such that for each $\alpha > \beta$, $T_\alpha \in V$. We conclude that for each $\alpha > \beta$,

$$||(T_\alpha - T)x|| \leqslant ||(T_\alpha - T)x|| + ||(T_\alpha^* - T^*)x|| < \varepsilon \text{ and}$$
$$||(T_\alpha^* - T^*)x|| \leqslant ||(T_\alpha^* - T^*)x|| + ||(T_\alpha - T)x|| < \varepsilon\,.$$

Now, given that x is arbitrary, it follows that $\text{s-}\lim T_\alpha = T$ and $\text{s-}\lim T_\alpha^* = T^*$, so $\text{s}^*\text{-}\lim T_\alpha = T$.

■

Theorem 5.3. *The strong-star topology in $C(H)$ is generated by the metric $d_{s^*}(A,B) = d_s(A,B) + d_s(A^*,B^*)$, and this metric is complete.*

Proof. The first thing we note is that $d_{s^*}(A,B)$ is obviously a metric:

- $d_{s^*}(A,A) = d_s(A,A) + d_s(A^*,A^*) = 0$
- If $A \neq B$, then $d_s(A,B) > 0$ and $d_s(A^*,B^*) > 0$, so, $d_{s^*}(A,B) > 0$
- $d_{s^*}(A,B) = d_s(A,B) + d_s(A^*,B^*) = d_s(B,A) + d_s(B^*,A^*) = d_{s^*}(B,A)$, for all $A, B \in C(H)$
- $d_{s^*}(A,C) = d_s(A,C) + d_s(A^*,C^*) \leqslant d_s(A,B) + d_s(B,C) + d_s(A^*,B^*) + d_s(B^*,C^*) = d_{s^*}(A,B) + d_{s^*}(B,C)$, for all $A, B, C \in C(H)$

Now we show that this metric induces the strong-star topology.

(a) Let $T \in C(H)$ and let V be an s*-neighborhood of T. We need to obtain $r > 0$ such that

$$U = \{A \in C(H) : d_{s^*}(A,T) < r\} \subset V$$

We can assume that

$$V = \bigcap_{i=1}^{k} \{A \in C(H) : ||Ay_i - Ty_i|| + ||A^*y_i - T^*y_i|| < \varepsilon\}$$

for some $\varepsilon > 0$ and $k \in \mathbb{N}$. Without loss of generality, we can assume that $||y_i|| \leqslant 1$ for every $i = 1, \ldots, k$. For each fixed $i \in \{1, \ldots, k\}$, by using the fact that $\{e_i : i \geqslant 1\} \subset H$ is a dense subset of B_H, there exists $n_i \in \mathbb{N}$ such that

$$||y_i - e_{n_i}|| < \varepsilon/8$$

So, let $r > 0$ be such that for every $i \leqslant k$

$$2^{n_i} r < \varepsilon/2\,.$$

We state that $U \subset V$. Namely, if $d_{s^*}(A,T) < r$, then $2^{-i}(||Ae_i - Te_i|| + ||A^*e_i - T^*e_i||) < r$ for every $i \geqslant 1$; in particular, $2^{-n_i}(||Ae_{n_i} - Te_{n_i}|| + ||A^*e_{n_i} - T^*e_{n_i}||) < r$ for every $i \in \{1, \ldots, k\}$.

So, if $i \in \{1, \dots, k\}$, then

$$\begin{aligned}
||Ay_i - Ty_i|| + ||A^*y_i - T^*y_i|| &= ||(A-T)(y_i - e_{n_i}) + (A-T)e_{n_i}|| \\
&\quad + ||(A^* - T^*)(y_i - e_{n_i}) + (A^* - T^*)e_{n_i}|| \\
&\leqslant ||(A-T)(y_i - e_{n_i})|| + ||(A-T)e_{n_i}|| \\
&\quad + ||(A^* - T^*)(y_i - e_{n_i})|| + ||(A^* - T^*)e_{n_i}|| \\
&< ||A - T|| \cdot ||y_i - e_{n_i}|| + ||A^* - T^*|| \cdot ||y_i - e_{n_i}|| + 2^{n_i} r \\
&\leqslant 4\frac{\varepsilon}{8} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

This shows that if $A \in U$ then $A \in V$.

(b) Let $T \in C(H)$. For each $r > 0$, we have to obtain an s^*-neighborhood $V \subset U = \{A \in C(H) : d_{s^*}(A,T) < r\}$, that is, we have to obtain $\varepsilon > 0$, $k \in \mathbb{N}$ and $y_1, \dots, y_k$ such that

$$\bigcap_{i=1}^{k} \{A \in C(H) : ||Ay_i - Ty_i|| + ||A^*y_i - T^*y_i|| < \varepsilon\} \subset U$$

Let $\varepsilon = r/2$, $k \in \mathbb{N}$ be such that $2^{-k+2} < r/2$ and $y_i = e_i$ for every $i \in \{1, \dots, k\}$. Now, we show that $V \subset U$. Indeed, if $A \in V$, then

$$\begin{aligned}
d_{s^*}(A,T) &= \sum_{i=1}^{\infty} 2^{-i}(||Ae_i - Te_i|| + ||A^*y_i - T^*y_i||) \\
&= \sum_{i=1}^{k} 2^{-i}(||Ae_i - Te_i|| + ||A^*y_i - T^*y_i||) \\
&\quad + \sum_{i=k+1}^{\infty} 2^{-i}(||Ae_i - Be_i|| + ||A^*y_i - T^*y_i||) \\
&< \varepsilon \sum_{i=1}^{k} 2^{-i} + \sum_{i=k+1}^{\infty} 2^{-i}(||A|| + ||T|| + ||A^*|| + ||T^*||) \\
&\leqslant \varepsilon + 2^2 \sum_{i=k+1}^{\infty} 2^{-i} \\
&\leqslant \varepsilon + 2^2 \Big(\frac{2^{-(k+1)}}{1 - \frac{1}{2}}\Big) \leqslant \varepsilon + 2^{-k+2} < \frac{r}{2} + \frac{r}{2} = r\,,
\end{aligned}$$

proving that $V \subset U$.

(c) Now, we show that this metric is complete. Let $(A_n)_n$ be a Cauchy sequence with respect to d_{s^*} metric and let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that if $n, m > N$, then $d_{s^*}(A_n, A_m) < \varepsilon$. Since $d_{s^*}(A,B) = d_s(A,B) + d_s(A^*, B^*)$, it follows that if $n, m > N$, then $d_s(A_n, A_m) < \varepsilon$ and $d_s(A_n^*, A_m^*) < \varepsilon$. So, $(A_n)_n$ and $(A_n^*)_n$ are both Cauchy sequences in d_s.

Now, since d_s is a complete metric, there exists $A, B \in C(H)$ such that $0 = \lim d_s(A_n, A) = \lim d_s(A_n^*, B)$. And now, we note that to every $x, y \in H$,

$$\langle Ax, y\rangle = \lim\langle A_n x, y\rangle = \lim\langle x, A_n^* y\rangle = \langle x, By\rangle\,,$$

and so $A^* = B$. Hence, $\lim d_{s^*}(A, A_n) = \lim d_s(A, A_n) + \lim d_s(A^*, A_n^*) = 0$. ■

As it was said before, the map $A \mapsto A^*$ is s*-continuous. Moreover, it is an s*-homeomorphism. In the next two theorems, we prove this and write down some useful consequences.

Theorem 5.4. *The map $\zeta : (C(H), s^*) \to (C(H), s^*)$ given by $\zeta(A) = A^*$ is an s*-homeomorphism.*

Proof. The first thing to note is that $\zeta^2 = I$, so $\zeta^{-1} = \zeta$. It remains to show that ζ is s*-continuous.

Let $(T_n)_n \subset C(H)$ and $T \in C(H)$ be such that $\text{s}^*\text{-}\lim T_n = T$. We want to show that $\text{s}^*\text{-}\lim \zeta(T_n) = \zeta(T)$, i.e., that $\text{s}^*\text{-}\lim T_n^* = T^*$. This is obvious, since $\text{s}^*\text{-}\lim T_n = T$ implies $\text{s-}\lim T_n = T$ and $\text{s-}\lim T_n^* = T^*$, and so $\text{s}^*\text{-}\lim T_n^* = T^*$.

■

Corollary 5.5. *Let $\mathcal{U} \subset C(H)$ and set $\mathcal{U}^* := \{A^* : A \in \mathcal{V}\} = \zeta(\mathcal{U})$. Then,*

1. *$\mathcal{U}$ is s*-open if, and only if, $\mathcal{U}^*$ is s*-open;*
2. *$\mathcal{U}$ is s*-closed if, and only if, $\mathcal{U}^*$ is s*-closed;*
3. *$\mathcal{U}$ is s*-G_δ if, and only if, $\mathcal{U}^*$ is s*-G_δ;*
4. *$\mathcal{U}$ is s*-co-meager if, and only if, $\mathcal{U}^*$ is s*-co-meager.*

5.2 Studying the properties of s*-typical contractions

Since we are dealing with a more rigid topology, one may expect a more complicated structure of the theory of typical operators. Unfortunately this is true, but on the other hand we have an interesting theory to investigate. The main result of this section reduces the theory of typical operators to the theory of typical properties of unitary and positive self-adjoint operators in the better understood strong topology.

Some preparations are required.

Lemma 5.6. *Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis in H and define $D \in C(H)$ by the law $De_0 = 0$, $De_{i+1} = e_i$ for each $i \in \mathbb{N}$. Then, for every $\lambda \in \mathbb{C}$, $\operatorname{Ran}(D - \lambda I)$ is dense in H.*

Proof. Essentially D is a left shift, so the proof use this fact. It is easy to check that D^* is given by $D^*(e_i) = e_{i+1}$, so we can use the set identity $\operatorname{Cl}\big(\operatorname{Ran}(D - \lambda I)\big) = \big(\ker(D^* - \overline{\lambda} I)\big)^\perp$.

Note that $\ker(D^* - \overline{\lambda} I) = \{0\}$ for every $\lambda \in \mathbb{C}$. Namely, let $x = \sum_{n=0}^\infty \eta_i e_i \in \ker(D^* - \overline{\lambda} I)$, so

$$\sum_{n=0}^\infty \eta_i e_{i+1} = \overline{\lambda} \sum_{n=0}^\infty \eta_i e_i \,,$$

from which follows that $\eta_0 = 0$ and $\eta_i = \overline{\lambda} \eta_{i+1}$ $(i \in \mathbb{N})$. Thus, $\eta_i = 0$ for every $i \in \mathbb{N}$, and then $x = 0$. This shows that $\operatorname{Cl}\big(\operatorname{Ran}(D - \lambda I)\big) = \big(\ker(D^* - \overline{\lambda} I)\big)^\perp = H$.

■

Theorem 5.7. *The set $\mathcal{T} = \{A \in C(H) : \forall\ \lambda \in \mathbb{C}\ \operatorname{Ran}(A - \lambda I)$ is dense in $H\}$ is s*-co-meager and s*-G_δ in $C(H)$.*

Proof. (a) Firstly, we show that $\mathcal{T}$ is s*-dense in $C(H)$. Let $U \subset C(H)$ be a non-empty s*-open set. Then, there exist $\{x_i : i < n\} \subset S_H$, $0 < \varepsilon < 1$ and $A \in U$ such that, for each $B \in C(H)$ satisfying $||Bx_i - Ax_i|| \leqslant \varepsilon$ and $||B^*x_i - A^*x_i|| \leqslant 2\varepsilon$ for all $i < n$, we have that $B \in U$.

Namely, there exist $\{x_i : i < n\} \subset S_H$, $\delta > 0$ and $A \in U$ such that for each $B \in C(H)$ such that $(||Bx_i - Ax_i||^2 + ||B^*x_i - A^*x_i||^2)^{1/2} < \delta$, we have $B \in U$. By fixing $\varepsilon = \min\{\delta/(2\sqrt{5}), 1\}$ and $B \in C(H)$ satisfying $||Bx_i - Ax_i|| \leqslant \varepsilon$, $||B^*x_i - A^*x_i|| \leqslant 2\varepsilon$, we get

$$(||Bx_i - Ax_i||^2 + ||B^*x_i - A^*x_i||^2)^{1/2} \leqslant \left(\frac{\delta^2}{20} + \frac{\delta^2}{5}\right)^{1/2} = \left(\frac{\delta^2}{4}\right)^{1/2} = \frac{\delta}{2} < \delta$$

So, $B \in U$.

(b) What we want now is to obtain $B \in U \cap \mathcal{T}$, i.e., $B \in U$ such that for each $\lambda \in \mathbb{C}$, $\operatorname{Ran}(B - \lambda I)$ is dense in H.

Set $V = \operatorname{span}\{x_i, Ax_i, A^*x_i : i < n\}$ and define $Q : V \to V$ by $Qv = (1-\varepsilon)P_V Av$. Let $\dim V = m$ and let $\{x_i(0) : i < m\}$ be an orthonormal basis in V. Since $\dim V^{\perp} = \infty$, we can pick an orthonormal basis $\{x_i(j) : i < m, j \in \mathbb{N} \setminus \{0\}\}$ in $V^{\perp}$ and we can define $T : V^{\perp} \to H$ by $Tx_i(j) = x_i(j-1)$, for every $i < m, j \in \mathbb{N} \setminus \{0\}$. Now, set $B = Q \oplus \varepsilon T$. We need to show that $B \in U \cap \mathcal{T}$.

The first thing to show is that B is contractive. Note that

$$||Q|| = (1-\varepsilon)\big|\big|P_V A\big|_A\big|\big| \leqslant (1-\varepsilon)||P_v|| \cdot ||A|| \leqslant 1 - \varepsilon\,.$$

Now, let $x = \sum_{i,j} \lambda_{i,j} x_i(j) \in V^{\perp}$. So,

$$||Tx||^2 = \Big|\Big| \sum_{\substack{j \in \mathbb{N}\setminus\{0\} \\ i<m}} \lambda_{i,j} T x_i(j) \Big|\Big|^2 = \Big|\Big| \sum_{\substack{j \in \mathbb{N}\setminus\{0\} \\ i<m}} \lambda_{i,j} x_i(j-1) \Big|\Big|^2 = \sum_{\substack{j \in \mathbb{N}\setminus\{0\} \\ i<m}} |\lambda_{i,j}|^2 = ||x||^2\,.$$

We conclude that $||Tx||/||x|| = 1$ and $||T|| = 1$. Thus,

$$||B|| \leqslant ||Q|| + \varepsilon||T|| \leqslant 1 - \varepsilon + \varepsilon = 1\,,$$

which proves that B is a contractive operator.

(c) The next step is to prove that for each $\lambda \in \mathbb{C}$, $\operatorname{Ran}(B - \lambda I)$ dense in H. For each $\lambda \in \mathbb{C}$, we claim that

$$\operatorname{Ran}\left(T - \frac{\lambda}{\varepsilon} I\big|_{V^{\perp}}\right) \subset \operatorname{Ran}\left(B\big|_{V^{\perp}} - \lambda I\big|_{V^{\perp}}\right) \subset \operatorname{Ran}(B - \lambda I)$$

The second inclusion is quite obvious. In order to prove the first one, pick $x \in \operatorname{Ran}\big(T - \frac{\lambda}{\varepsilon} I\big|_{V^{\perp}}\big)$, so there exists $y \in V^{\perp}$ such that $Ty - \frac{\lambda}{\varepsilon} y = x$. Then, $\varepsilon T(y/\varepsilon) - \lambda(y/\varepsilon) = x$. So, $x = \big(B\big|_{V^{\perp}} - \lambda I\big|_{V^{\perp}}\big)(y/\varepsilon)$, which implies $x \in \operatorname{Ran}\big(B\big|_{V^{\perp}} - \lambda I\big|_{V^{\perp}}\big)$.

Thus, it suffices to prove that $\operatorname{Ran}\big(T - \frac{\lambda}{\varepsilon} I\big|_{V^{\perp}}\big)$ is dense in H. Let $\eta = \lambda/\varepsilon$ and set, for each $i < m$, the sets $V_i := \overline{\operatorname{span}\{x_i(j) : j \in \mathbb{N}\}}$, $W_i := \overline{\operatorname{span}\{x_i(j) : j \in \mathbb{N} \setminus \{0\}\}}$, and define the map $D_i : V_i \to V_i$ by the law $D_i(x_i(0)) = 0$ and $D_i(x_i(j)) = x_i(j-1)$ for each $j \in \mathbb{N} \setminus \{0\}$.

The first two things we point out here is that $V^{\perp} = \bigoplus_{i=0}^{m-1} W_i$ and $H = \bigoplus_{i=0}^{m-1} V_i$. Let us prove it.

Fix $x \in V^{\perp}$. Since $\{x_i(j) : i < m, j \in \mathbb{N} \setminus \{0\}\}$ is a basis of $V^{\perp}$, it follows that

$$x = \sum_{\substack{j \in \mathbb{N}\setminus\{0\} \\ i<m}} \alpha_{i,j} x_i(j) \tag{5.1}$$

for some $\alpha_{i,j} \in \mathbb{C}$. Notes that for each $i < m$, $\sum_{j\in\mathbb{N}\setminus\{0\}} \alpha_{i,j} x_i(j) \in W_i$, so $x \in \bigoplus_{i=0}^{m-1} W_i$. On the other hand, if $x \in \bigoplus_{i=0}^{m-1} W_i$, then $x = \sum_{i=0}^{m-1} \sum_{j\in\mathbb{N}\setminus\{0\}} \alpha_{i,j} x_i(j)$, and so (5.1) holds. This proves that $x \in V^{\perp}$.

Now, if $x \in H = V \oplus V^{\perp}$ it follows that $P_V x = \sum_{i=0}^{m-1} \alpha_{i,0} x_i(0)$ for some $\alpha_{i,0} \in \mathbb{C}$ ($\{x_i(0) : i < m\}$ is a base of V), and $P_{V^{\perp}} x = \sum_{j\in\mathbb{N}\setminus\{0\}, i<m} \alpha_{i,j} x_i(j)$ for some $\alpha_{i,j} \in \mathbb{C}$. Then,

$$x = P_V x + P_{V^{\perp}} x = \sum_{i=0}^{m-1} \alpha_{i,0} x_i(0) + \sum_{\substack{j \in \mathbb{N}\setminus\{0\} \\ i<m}} \alpha_{i,j} x_i(j) = \sum_{\substack{j \in \mathbb{N} \\ i<m}} \alpha_{i,j} x_i(j) = \sum_{i<m} \sum_{j \in \mathbb{N}} \alpha_{i,j} x_i(j) \,,$$

proving that $x \in \bigoplus_{i=0}^{m-1} V_i$. The other inclusion is obvious. Hence, $V^{\perp} = \bigoplus_{i=0}^{m-1} W_i$ and $H = \bigoplus_{i=0}^{m-1} V_i$.

By Lemma 5.6 Ran $\left(D_i - \eta I\big|_{V_i}\right)$ is dense in V_i for every $i < m$. Our next step is to show that Ran $\left(D_i - \eta I\big|_{V_i}\right) =$ Ran $\left(D_i\big|_{W_i} - \eta I\big|_{W_i}\right)$, and with that we can prove that Ran $\left(D_i\big|_{W_i} - \eta I\big|_{W_i}\right)$ is dense in V_i.

Since $W_i \subset V_i$, one of the inclusions is obvious. In order to prove the other one, let $\sum_{j=0}^{\infty} \alpha_j x_i(j) \in V_i$ and note that

$$(D_i - \eta I)\Big(\sum_{j=0}^{\infty} \alpha_j x_i(j)\Big) = \sum_{j=1}^{\infty} \alpha_j x_i(j-1) - \sum_{j=0}^{\infty} \eta \alpha_j x_i(j) = \sum_{j=0}^{\infty} (\alpha_{j+1} - \eta\alpha_j) x_i(j) \,.$$

Now, note that $(D_i - \eta I)\big(\sum_{j=1}^{\infty} \omega_j x_i(j)\big) = (D_i - \eta I)\big(\sum_{j=0}^{\infty} \alpha_j x_i(j)\big)$, with $\omega_j = \alpha_j - \eta^j \alpha_0$. Namely,

$$\begin{aligned}(D_i - \eta I)\Big(\sum_{j=1}^{\infty} \omega_j x_i(j)\Big) &= \sum_{j=1}^{\infty} \omega_j x_i(j-1) - \sum_{j=1}^{\infty} \eta\omega_j x_i(j) \\ &= \omega_1 x_i(0) + \sum_{j=1}^{\infty} (\omega_{j+1} - \eta\omega_j) x_i(j) \,.\end{aligned}$$

Since $\omega_1 = \alpha_1 - \eta\alpha_0$ and

$$\omega_{j+1} - \eta\omega_j = \alpha_{j+1} - \eta^{j+1}\alpha_0 - \eta\alpha_j + \eta^{j+1}\alpha_0 = \alpha_{j+1} - \eta\alpha_j \,,$$

it follows that

$$(D_i - \eta I)\Big(\sum_{j=1}^{\infty} \omega_j x_i(j)\Big) = \sum_{j=0}^{\infty} (\alpha_{j+1} - \eta\alpha_j) x_i(j) = (D_i - \eta I)\Big(\sum_{j=0}^{\infty} \alpha_j x_i(j)\Big) \,.$$

This shows that Ran $\left(D_i - \eta I\big|_{V_i}\right) =$ Ran $\left(D_i\big|_{W_i} - \eta I\big|_{W_i}\right)$, and by the previous discussion, we conclude that Ran $\left(D_i\big|_{W_i} - \eta I\big|_{W_i}\right)$ is dense in V_i.

It remains to prove that $\operatorname{Ran}(T-\eta I|_{V^\perp}) = \bigoplus_{i=0}^{m-1} \operatorname{Ran}\big(D_i|_{W_i} - \eta I|_{W_i}\big)$. Indeed,

$$\begin{aligned}
(T-\eta I)\Big(\sum_{\substack{j\in\mathbb{N}\setminus\{0\}\\ i<m}} \alpha_{i,j} x_i(j)\Big) &= \sum_{\substack{j\in\mathbb{N}\setminus\{0\}\\ i<m}} \alpha_{i,j} x_i(j-1) - \sum_{\substack{j\in\mathbb{N}\setminus\{0\}\\ i<m}} \eta\alpha_{i,j} x_i(j) \\
&= \sum_{\substack{j\in\mathbb{N}\setminus\{0\}\\ i<m}} \alpha_{i,j}(x_i(j-1) - \eta x_i(j)) \\
&= \sum_{i=0}^{m-1} \sum_{j\in\mathbb{N}\setminus\{0\}} (D_i - \eta I)(\alpha_{i,j} x_i(j)) \\
&= \sum_{i=0}^{m-1} \Big[(D_i - \eta I)\Big(\sum_{j\in\mathbb{N}\setminus\{0\}} \alpha_{i,j} x_i(j)\Big)\Big].
\end{aligned}$$

By putting all the pieces together, we conclude that

$$\overline{\operatorname{Ran}(T-\eta I|_{V^\perp})} = \overline{\bigoplus_{i=0}^{m-1} \operatorname{Ran}\big(D_i|_{W_i} - \eta I|_{W_i}\big)} = \bigoplus_{i=0}^{m-1} \overline{\operatorname{Ran}\big(D_i|_{W_i} - \eta I|_{W_i}\big)} = \bigoplus_{i=0}^{m-1} V_i = H\,.$$

With this, it follows that $\operatorname{Ran}(B-\lambda I)$ is dense in H, and so $B \in \mathcal{T}$.

(d) Now we need to show that $B \in U$. What we will do is ensure the conditions we stated before, i.e., we will prove that $||Bx_i - Ax_i|| \leqslant \varepsilon$, $||B^*x_i - A^*x_i|| \leqslant 2\varepsilon$ for all $i < n$.

Recall that $V = \operatorname{span}\{x_i, Ax_i, A^*x_i : i < n\}$ and $B = Q \oplus \varepsilon T$, where Q acts in V and T in $V^\perp$, so

$$||Bx_i - Ax_i|| = ||Qx_i - Ax_i|| = ||(1-\varepsilon)P_V Ax_i - P_V Ax_i|| = ||\varepsilon P_V Ax_i|| \leqslant \varepsilon\,.$$

Another form to write B is $B = QP_V + \varepsilon T P_{V^\perp}$, so $B^* = P_V Q^* + \varepsilon P_{V^\perp} T^*$ and

$$\begin{aligned}
||B^*x_i - A^*x_i|| &= ||P_V Q^* x_i + \varepsilon P_{V^\perp} T^* x_i - A^* x_i|| \\
&\leqslant \varepsilon ||P_{V^\perp} T^* x_i|| + ||P_V Q^* x_i - A^* x_i|| \\
&\leqslant \varepsilon ||P_{V^\perp}|| \cdot ||T^*|| \cdot ||x_i|| + ||P_V|| \cdot ||Q^* x_i - A^* x_i|| \\
&\leqslant \varepsilon + ||(1-\varepsilon) P_V A^* x_i - P_V A^* x_i|| \\
&= \varepsilon + ||\varepsilon P_V A^* x_i|| \leqslant 2\varepsilon\,.
\end{aligned}$$

Thus, $B \in U \cap \mathcal{T}$, and since U is an arbitrary s*-open set, we conclude that $\mathcal{T}$ is s*-dense in $C(H)$.

(e) It remains to prove that $\mathcal{T}$ is s*-G_δ in $C(H)$. We start by defining, for each $y \in S_H$, $\delta > 0$ and $L \geqslant 0$, the set

$$R(y,\delta,L) = \{A \in C(H) : \exists\, \lambda \in \mathbb{C} \text{ with } |\lambda| \leqslant L \text{ such that } \operatorname{dist}\big(\operatorname{Ran}(A-\lambda I), y\big) \geqslant \delta\}$$

Let us show that for each $y \in S_H$, $\delta > 0$, $R(y,\delta,1)$ is s-closed. For that, let $(A_n)_{n\in\mathbb{N}} \subset R(y,\delta,1)$ and $A \in C(H)$ be such that $\text{s-}\lim A_n = A$. For each $n \in \mathbb{N}$, we know that $A_n \in R(y,\delta,1)$, so there exists $\lambda_n \in \mathbb{C}$ such that $|\lambda_n| \leqslant 1$ and $\operatorname{dist}\big(\operatorname{Ran}(A_n - \lambda_n I), y\big) \geqslant \delta$. Since $|\lambda_n| \leqslant 1$ for every $n \in \mathbb{N}$, there exist a subsequence $(\lambda_{n_k})_{n\in\mathbb{N}}$ of $(\lambda_n)_n$ and $\lambda \in \mathbb{C}$ such that $\lambda_{n_k} \to \lambda$. We will show that $\operatorname{dist}\big(\operatorname{Ran}(A-\lambda I), y\big) \geqslant \delta$.

Suppose that this is false, i.e., that there exists $x \in H$ such that $||Ax - \lambda x - y|| < \delta$. Since $\lim A_{n_k} x = Ax$ and $\lim \lambda_{n_k} = \lambda$, it follows that for n_k sufficiently large, $||A_{n_k} x - \lambda_{n_k} x - y|| < \delta$. This happens because

$$\begin{aligned} ||A_{n_k} x - \lambda_{n_k} x - y|| &\leqslant ||(A_{n_k} - \lambda_{n_k} I)x - (A - \lambda I)x|| + ||Ax - \lambda x - y|| \\ &< ||(A_{n_k} - \lambda_{n_k} I)x - (A - \lambda I)x|| + \delta \xrightarrow{k\to\infty} \delta\,. \end{aligned}$$

But this contradicts the fact that $\operatorname{dist}(\operatorname{Ran}(A_n - \lambda_n I), y) \geqslant \delta$ for every $n \in \mathbb{N}$. Hence, $R(y, \delta, 1)$ is s-closed, and consequently s*-closed (in fact, if $(A_n)_{n\in\mathbb{N}} \subset R(y, \delta, 1)$ and s*- $\lim A_n = A$, then s- $\lim A_n = A$, and since $R(y, \delta, 1)$ is strongly closed, it follows that $A \in R(y, \delta, 1)$).

(f) We claim that if $Y \subset S_H$ is a countable dense subset, then

$$\mathcal{T} = C(H) \backslash \Big(\bigcup_{\substack{y\in Y \\ n\in\mathbb{N}}} R(y, 2^{-n}, 1) \Big)\,.$$

Namely, let $A \in \mathcal{T}$. Then, for each $\lambda \in \mathbb{C}$, $\operatorname{Cl}\big(\operatorname{Ran}(A - \lambda I)\big) = H$. In particular, for each $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$ and for each $y \in Y$, $\operatorname{dist}(\operatorname{Ran}(A - \lambda I), y) < 2^{-n}$.

On the other hand, let $A \in C(H)$ be such that for each $y \in Y$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$, $\operatorname{dist}(\operatorname{Ran}(A - \lambda I), y) < 2^{-n}$. Let $z \in S_H$. Since Y is dense, there exists a sequence $(z_m)_{m\in\mathbb{N}} \subset Y$ such that $\lim z_m = z$. Now, for each $m \in \mathbb{N}$ there exists $h_m \in \operatorname{Ran}(A - \lambda I)$ such that $||h_m - z_m|| < 2^{-m}$. Thus,

$$||h_m - z|| \leqslant ||h_m - z_m|| + ||z_m - z|| < 2^{-m} + ||z_m - z|| \xrightarrow{m\to\infty} 0$$

and we conclude that $\operatorname{Ran}(A - \lambda I)$ is dense. Hence, $T \in \mathcal{T}$, and so $\mathcal{T}$ is a s*-G_δ subset of $C(H)$.

■

For technical reasons we state the following corollary of Theorem 5,7:

Corollary 5.8. *The set*

$$\mathcal{E} = \{A \in C(H) : \operatorname{Ran}(A) \textit{ is dense in } H\}$$

is s-co-meager and s*-G_δ in $C(H)$.*

Proof. (a) The set $\mathcal{T}$ defined in Theorem 5.7 is s*-co-meager and $\mathcal{T} \subset \mathcal{E}$, so $\mathcal{E}$ is s*-co-meager.

(b) By using the same terminology presented in the proof of Theorem 5.7, we claim that

$$\mathcal{E} = C(H) \setminus \bigcup_{y\in Y, n\in\mathbb{N}} R(y, 2^{-n}, 0) \tag{5.2}$$

and that $R(y, 2^{-n}, 0)$ is s*-closed. In order to show that for each $y \in Y, n \in \mathbb{N}$, $R(y, 2^{-n}, 0)$

is s*-closed, we note that

$$\begin{aligned} R(y, 2^{-n}, 0) &= \{A \in C(H) : \exists\, \lambda \in \mathbb{C} \text{ with } |\lambda| \leqslant 0 \text{ such that } \operatorname{dist}\big(\operatorname{Ran}(A - \lambda I), y\big) \geqslant 2^{-n}\} \\ &= \{A \in C(H) : \operatorname{dist}\big(\operatorname{Ran}(A), y\big) \geqslant 2^{-n}\}\,. \end{aligned}$$

If $(A_m)_{m\in\mathbb{N}} \subset R(y, 2^{-n}, 0)$ is such that $\text{s*-}\lim A_m = A$, the first thing we note is that $\operatorname{dist}\big(\operatorname{Ran}(A_m), y\big) \geqslant 2^{-n}$. Hence, for every $x \in H$ and $m \in \mathbb{N}$, $||A_m x - y|| \geqslant 2^{-n}$, then $2^{-n} \leqslant \lim_m ||A_m x - y|| = ||Ax - y||$. We conclude that $A \in R(y, 2^{-n}, 0)$.

(c) Let us prove relation (5.2). Let us $A \in \mathcal{E}$, then $\operatorname{Ran}(A)$ is dense, so for each $n \in \mathbb{N}$ and $y \in Y$, $\operatorname{dist}(\operatorname{Ran}(A), y) < 2^{-n}$. We conclude that for each $n \in \mathbb{N}$ and $y \in Y$, $A \notin R(y, 2^{-n}, 0)$, so $A \in C(H) \setminus \bigcup_{y\in Y, n\in\mathbb{N}} R(y, 2^{-n}, 0)$.

On the other hand, if $A \in C(H)\setminus\bigcup_{y\in Y,n\in\mathbb{N}} R(y, 2^{-n}, 0)$, then $A \in C(H)\setminus R(y, 2^{-n}, 0)$ for each $n \in \mathbb{N}$ and $y \in Y$. So for each $n \in \mathbb{N}$ and $y \in Y$, $\operatorname{dist}(\operatorname{Ran}(A), y) < 2^{-n}$, i.e., for each $n \in \mathbb{N}$ and $y \in Y$, there exists $x_n^y \in H$ such that $||Ax_n^y - y|| < 2^{-n}$. Hence, the sequence $(x_n^y)_{n\in\mathbb{N}}$ satisfies $\lim_n Ax_n^y = y$.

Let $h \in H \setminus \{0\}$. Then, $z := h/||h|| \in S_H$. By the density of Y, there exists $y \in Y$ such that $||z - y|| < \varepsilon/(2||h||)$. For this y there exists $m \in \mathbb{N}$ such that $||Ax_m^y - y|| < 2^{-m} < \varepsilon/(2||h||)$. Then,

$$||z - Ax_m^y|| \leqslant ||z - y|| + ||y - Ax_m^y|| < \frac{\varepsilon}{||h||}\,,$$

and we conclude that $||h - A(||h||x_m^y)|| < \varepsilon$. Since h is arbitrary, $\operatorname{Ran}(A)$ is dense and so $A \in \mathcal{E}$.

■

We also need a similar result for positive self-adjoint operators.

Theorem 5.9. *The set*

$$\mathcal{P} = \{P \in P(H) : \operatorname{Ran}(P) \textit{ is dense in } H\}$$

is s-co-meager and s-G_δ in $P(H)$.

Proof. (a) We begin showing that $\mathcal{P}$ is s-dense in $P(H)$. Let $U \subset P(H)$ be an s-open non-empty subset. Then, there exist $n \in \mathbb{N}$, $\{x_i : i < n\} \subset S_H$, $\varepsilon > 0$ and $A \in U$ such that for each $B \in P(H)$ such that $||Bx_i - Ax_i|| \leqslant \varepsilon$ for each $i < n$ it follows that $B \in U$. Thus, there exists $B \in U$ such that $\operatorname{Ran} B$ is dense in H. We will, during the proof, assume that $\varepsilon \leqslant 1$

(b) Set $V = \operatorname{span}\{x_i, Ax_i : i < n\}$. Let $Q : V \to V$ be any invertible contractive self-adjoint operator such that $||Q - P_V A\big|_V|| \leqslant \varepsilon$, one possibility is

$$Q = \frac{\varepsilon}{2} P_V A\big|_V + \frac{\varepsilon}{2} I\big|_V$$

- To see that Q is invertible, it is enough to show that Q is injective. This is because V is finite dimensional and V is the domain and the counter-domain of Q. Let $v \in V$ be such that $Qv = 0$, so

$$P_V Av + v = 0 \Rightarrow P_V Av = -v\,,$$

and then $-1 \in \sigma(P_V A\big|_V)$. On the other hand, we have for each $v \in V$ that

$$\langle P_V Av, v\rangle = \langle Av, P_V v\rangle = \langle Av, v\rangle \geqslant 0\,.$$

So, $P_V A\big|_V$ is a positive operator, and then $\sigma(P_V A\big|_V) \subset [0, +\infty)$, an absurd. Hence, Q is invertible.

- Note that

$$||Q|| = \left|\left|\frac{\varepsilon}{2}P_V A\big|_V + \frac{\varepsilon}{2}I\big|_V\right|\right| \leqslant \frac{\varepsilon}{2}||P_V A\big|_V|| + \frac{\varepsilon}{2}||I\big|_V|| \leqslant \varepsilon \leqslant 1\,,$$

hence, Q is contractive.

- Let $v \in V$, so

$$\begin{aligned}\langle Qv, v\rangle &= \langle(\varepsilon/2)P_V Av + (\varepsilon/2)Iv, v\rangle\\ &= \frac{\varepsilon}{2}\langle P_V Av, v\rangle + \frac{\varepsilon}{2}\langle v, v\rangle\\ &= \frac{\varepsilon}{2}\langle Av, v\rangle + \frac{\varepsilon}{2}\langle v, v\rangle \geqslant 0\,.\end{aligned}$$

Then, Q is positive and in particular self-adjoint.

- Note that

$$||Q - P_V A\big|_V|| = ||(\varepsilon/2)I\big|_V - (\varepsilon/2)P_V A\big|_V|| \leqslant \frac{\varepsilon}{2}||I\big|_V|| + \frac{\varepsilon}{2}||P_V A\big|_V|| \leqslant \varepsilon\,.$$

(c) Set $B = Q \oplus I\big|_{V^\perp}$. We claim that B is a positive self-adjoint operator, $||B\big|_V|| \leqslant 1$, $||B\big|_{V^\perp}|| = 1$ and B is invertible.

- Let $x \in H$. Write $x = P_V x + P_{V^\perp}x$, so

$$\begin{aligned}\langle Bx, x\rangle &= \langle QP_V x + P_{V^\perp}x, x\rangle = \langle QP_V x, x\rangle + \langle P_{V^\perp}x, x\rangle\\ &= \langle QP_V x, P_V x\rangle + \langle QP_V x, P_{V^\perp}x\rangle + \langle P_{V^\perp}x, P_{V^\perp}x\rangle + \langle P_{V^\perp}x, P_V x\rangle\\ &= \langle QP_V x, P_V x\rangle + \langle P_{V^\perp}x, P_{V^\perp}x\rangle \geqslant 0\,,\end{aligned}$$

in the last step, we have used that Q is positive. This shows that B is a positive operator, and in particular self-adjoint.

- $||B\big|_V|| \leqslant 1$, $||B\big|_{V^\perp}|| = 1$ follow directly from $||Q|| \leqslant 1$ and $||I|| = 1$. In particular, $||B|| \leqslant 1$.

- To see that B is invertible, recall that Q is invertible, so the operator $D = Q^{-1} \oplus I\big|_{V^\perp}$ satisfies $BD = DB = I$.

Then, $B \in P(H)$ and since B is invertible, $\mathrm{Ran}(B)$ is dense H, and so $B \in \mathcal{P}$.

(d) Now we prove that $B \in U$. namely, let $i < n$, and note that

$$||Bx_i - Ax_i|| = ||Qx_i - Ax_i|| = ||Qx_i - P_V A\big|_V x_i|| \leqslant ||Q - P_V A\big|_V|| \cdot ||x_i|| \leqslant \varepsilon\,.$$

Hence, $B \in U$. Since U is an arbitrary s-open set and $B \in U \cap \mathcal{P}$, we conclude that $\mathcal{P}$ is an s-dense set.

(e) It remains to prove that $\mathcal{P}$ is s-G_δ. For $y \in S_H$ and $\delta > 0$ define

$$R(y, \delta) = \{A \in P(H) : \text{dist}(\text{Ran}(A), y) \geqslant \delta\}$$

In order to prove that this set is s-closed, let $(A_m)_{m \in \mathbb{N}} \subset R(y, \delta)$ be such that s-$\lim A_m = A$. The first thing to note is that $\text{dist}\big(\text{Ran}(A_m), y\big) \geqslant \delta$. Hence, for every $x \in H$ and $m \in \mathbb{N}$, $||A_m x - y|| \geqslant \delta$, then $\delta \leqslant \lim_m ||A_m x - y|| = ||Ax - y||$. We conclude that $A \in R(y, \delta)$.

(f) Let $Y \subset S_H$ be a dense countable subset. We claim that

$$\mathcal{P} = P(H) \setminus \bigcup_{y \in Y, n \in \mathbb{N}} R(y, 2^{-n})$$

The proof of this equality uses the same arguments presented in the proof of relation (5.2). Using this the proof ends.

■

We know that in $P(H)$ it is possible to define the square root of any operator (see [16] for details). The next result investigates some continuity properties of the square root on $P(H)$. After that, we will gather all the tools needed to prove the main result of the chapter.

Lemma 5.10. *The function* $\cdot^{1/2} : P(H) \to P(H)$, $A \mapsto A^{1/2}$ *is s-continuous. In particular,* $\cdot^{1/2}$ *is* s^**-continuous.*

Proof. If $T \in P(H)$, then $\sigma(T) \subset [0, +\infty)$. Moreover, $||T|| \leqslant 1$ and so the spectral radius of T is less or equal to 1. Hence, $\sigma(A) \subset [0, 1]$.

We know that the polynomial functions are dense in $(C([0,1], \mathbb{R}), ||\cdot||_\infty)$ (see Stone-Weierstrass Theorem in [9]), so for each $\varepsilon > 0$, there exists a polynomial $p : [0,1] \to \mathbb{R}$ such that

$$\max_{x \in [0,1]} |p(x) - x^{1/2}| = ||p(x) - x^{1/2}||_\infty < \varepsilon$$

Set $f(x) = p(x) - x^{1/2}$ and let $(A_n)_n \subset P(H)$, $A \in P(H)$ be such that s-$\lim A_n = A$. Then, by the functional calculus, $||f(A)|| \leqslant ||f||_\infty < \varepsilon$ and $||f(A_n)|| \leqslant ||f||_\infty < \varepsilon$ (see [16]). Therefore,

$$\begin{aligned} ||A^{1/2}x - A_n^{1/2}x|| &\leqslant ||A^{1/2}x - p(A)x|| + ||p(A)x - p(A_n)x|| + ||p(A_n)x - A_n^{1/2}x|| \\ &\leqslant ||f(A)|| \cdot ||x|| + ||p(A)x - p(A_n)x|| + ||f(A_n)|| \cdot ||x|| \\ &\leqslant 2\varepsilon + ||p(A)x - p(A_n)x||\,. \end{aligned}$$

We note that s-$\lim A_n^k = A_n^k$ for every $k \in \mathbb{N}$, and so s-$\lim p(A_n) = p(A)$. Thus,

$$0 \leqslant \limsup ||A^{1/2}x - A_n^{1/2}x|| \leqslant 2\varepsilon\,,$$

from which follows that $\limsup ||A^{1/2}x - A_n^{1/2}x|| = 0$. Similarly we prove that $\liminf ||A^{1/2}x - A_n^{1/2}x|| = 0$ and so $\lim ||A^{1/2}x - A_n^{1/2}x|| = 0$. Hence, s-$\lim A_n^{1/2} = A^{1/2}$

Note that in $P(H)$ the strong and the strong-star topologies coincide given that for each $A \in P(H)$, $A^* = A$. Hence, $\cdot^{1/2}$ is s*-continuous.

■

The main result of the section characterizes the structure of s*-typical contractions:

Theorem 5.11. *There exists an* s^**-co-meager and* s^**-*G_δ *set,* $\mathcal{H} \subset C(H)$*, and an s-co-meager s-*G_δ*,* $\mathcal{P} \subset P(H)$*, such that the function* $\Psi : U(H) \times \mathcal{P} \to \mathcal{H}$ *given by* $\Psi(U, P) = UP$ *is a homeomorphism, where* $U(H)$ *and* $\mathcal{P}$ *are endowed with the strong topology and* $\mathcal{H}$ *is endowed with the strong-star topology.*

Moreover, if $(\psi_0, \psi_1) : \mathcal{H} \to U(H) \times \mathcal{P}$ *denotes the inverse of* Ψ*, then for each* $A \in \mathcal{H}$ *and* $U \in U(H)$*,* $UAU^{-1} \in \mathcal{H}$ *and* $\psi_i(UAU^{-1}) = U\psi_i(A)U^{-1}$ $(i = 0, 1)$*.*

Proof. (a) With the notation of Corollary 5.8 set

$$\mathcal{H} = \mathcal{E} \cap \mathcal{E}^* = \{A \in C(H) : \mathrm{Ran}(A) \text{ and } \mathrm{Ran}(A^*) \text{ are dense in } H\}.$$

It follows from Corollaries 5.5 and 5.8 that $\mathcal{H}$ is s*-G_δ and s*-co-meager.

(b) Define $\psi_1 : \mathcal{H} \to P(H)$ by $\psi_1(A) = (A^*A)^{1/2}$. Let us show that this function is well defined. The first thing to note is that $(A^*A)^{1/2}$ is positive and $||(A^*A)^{1/2}|| = ||(A^*A)||^{1/2} \leqslant 1$, hence $(A^*A)^{1/2} \in P(H)$.

Using the notation as in Theorem 5.9, we claim that ψ_1 maps to $\mathcal{P}$. Note that $\mathrm{Ran}(A^*A)$ is dense in H: namely, for $h \in H$ and $\varepsilon > 0$, it follows from $\mathrm{Ran}(A^*)$ and $\mathrm{Ran}(A)$ both being dense in H that there exists $y \in H$ such that $||h - A^*y|| < \varepsilon/2$ and $z \in H$ such that $||y - Az|| < \varepsilon/2$. Then,

$$||h - A^*Az|| \leqslant ||h - A^*y|| + ||A^*y - A^*Az|| < \frac{\varepsilon}{2} + ||A^*|| \cdot ||y - Az|| < \varepsilon,$$

and we conclude that $\mathrm{Ran}(A^*A)$ is dense. Since $\mathrm{Ran}(A^*A) \subset \mathrm{Ran}(A^*A)^{1/2}$, $\mathrm{Ran}(A^*A)^{1/2}$ is also dense.

(c) We need to show that ψ_1 is an s*-continuous function. Define the map $F : \mathcal{H} \to P(H)$ by the law $F(A) = A^*A$ and note that $\psi_1 = F^{1/2}$. If F is s*-continuous, then ψ_1 is s*-continuous, given that the map $\cdot^{1/2} : P(H) \to P(H)$ is s*-continuous by Lemma 5.10.

Let $(A_n)_n \subset C(H)$ and $A \in C(H)$ be such that $\text{s*-}\lim A_n = A$, and so $\text{s-}\lim A_n = A$ and $\text{s-}\lim A_n^* = A^*$. Therefore, $\text{s-}\lim A_n^*A_n = A^*A$, and since $(T^*T)^* = T^*T$ for every $T \in B(H)$, we conclude that $\text{s-}\lim(A_n^*A_n)^* = (A^*A)^*$, and so $\text{s*-}\lim A_n^*A_n = A^*A$. Thus, F is s*-continuous, from which follows that $\psi_1 : \mathcal{H} \to \mathcal{P}$ is also s*-continuous.

(d) Now we need to define ψ_0. Recall that $\psi_1(A)$ is a positive self-adjoint operator with dense range for each $A \in \mathcal{H}$. So, since $H = \mathrm{Cl}(\mathrm{Ran}(\psi(A))) = (\ker(\psi_1(A)))^\perp$, we conclude that $\ker(\psi_1(A)) = \{0\}$ and hence $\psi_1(A)$ is injective. Thus, we can define the operator $\psi_1(A)^{-1} : \mathrm{Ran}\,\psi_1(A) \to H$, which is closed and positive. Namely,

- we use the positivity of $\psi_1(A)$ to show that $\psi_1^{-1}(A)$ is positive. Let $\psi_1(A)y = x \in \mathrm{Ran}\,\psi_1(A)$,so

$$\langle \psi_1^{-1}(A)x, x\rangle = \langle \psi_1^{-1}(A)\psi_1(A)y, \psi_1(A)y\rangle = \langle y, \psi_1(A)y\rangle \geqslant 0;$$

- Let $(x_n)_n$ be such that $x_n \to x$ and $\psi_1^{-1}(A)x_n \to y$. Since $x_n \in \mathrm{Ran}\,\psi_1(A)$, there exists $y_n \in H$ such that $\psi_1(A)y_n = x_n$, so $y_n = \psi_1(A)^{-1}\psi_1(A)y_n = \psi_1(A)^{-1}x_n \to y$. By the continuity of $\psi_1(A)$ it follows that $x_n = \psi_1(A)y_n \to \psi_1(A)y$, and by the uniqueness of the limit, one concludes that $\psi_1(A)y = x$. Moreover, $y = \psi_1(A)^{-1}x$.

We can extend densely $\psi_1(A)^{-1}$ to a function defined in all H. Now, we consider the operator $A\psi_1(A)^{-1}$, which has a dense range and it is an isometry. Namely,

- since $\operatorname{Ran}\psi_1(A)^{-1} = H$ and $\operatorname{Cl}(A) = H$ (given that $A \in \mathcal{H}$), it follows that $\operatorname{Cl}\operatorname{Ran}(A\psi_1(A)^{-1}) = H$;
- For each $x \in \operatorname{Ran}\psi_1(A)^{-1}$, we get

$$\begin{aligned}\langle A\psi_1(A)^{-1}x, A\psi_1(A)^{-1}x\rangle &= \langle \psi_1(A)^{-1}A^*A\psi_1(A)^{-1}x, x\rangle \\ &= \langle \psi_1(A)^{-1}\psi_1(A)^2\psi_1(A)^{-1}x, x\rangle \\ &= \langle x, x\rangle,\end{aligned}$$

  and so $A\psi_1(A)^{-1}$ is an isometry.

Hence, $A\psi_1(A)^{-1} : \operatorname{Ran}\psi_1(A) \to \operatorname{Ran} A$ has a dense range and is a isometry, so we can extend it to an operator $\psi_0(A) : H \to H$, which is unitary. Indeed,

- for each $y \in H$, given that $\operatorname{Cl}(\operatorname{Ran} A) = H$, there exists $(y_n)_n \subset \operatorname{Ran} A$ such that $\lim y_n = y$. For each $n \in \mathbb{N}$, there exists $z_n \in \operatorname{Ran}\psi_1^{-1}(A)$ such that $A\psi_1(A)^{-1}z_n = y_n$, so

$$||y_n - y_m|| = ||A\psi_1(A)^{-1}(z_n - z_m)|| = ||z_n - z_m||$$

  Since $(y_n)_n$ is a Cauchy sequence, the sequence $(z_n)_n$ is also Cauchy, so there exists $z \in H$ such that $z_n \to z$. By continuity, it follows that $A\psi_1(A)^{-1}z = y$, proving that ψ_0 is surjective.
- Given that $A\psi_1(A)^{-1}$ is an isometry it follows that ψ_0 is also an isometry. Now, let $(y_n)_n \subset \operatorname{Ran}\psi_1(A)$ and $y \in H$ be such that $y = \lim y_n$. Thus,

$$||A\psi_1(A)^{-1}y|| = \lim ||A\psi_1(A)^{-1}y_n|| = \lim ||y_n|| = ||y||$$

and $\psi_0(A)$ is a unitary operator.

(e) We claim now that every $A \in \mathcal{H}$ can be written as $A = \psi_0(A)\psi_1(A)$, where $\psi_0(A) \in U(H)$ and $\psi_1(A) \in \mathcal{P}$. This decomposition is unique and unitary invariant, i.e., for each $A \in \mathcal{H}$ and $U \in U(H)$, $\psi_i(UAU^{-1}) = U\psi_i(A)U^{-1}$ $(i = 0, 1)$.

First of all, if $x \in H$, then $\psi_1(A)x \in \operatorname{Ran}\psi_1(A)$ and $\psi_0(A)\psi_1(A)x = A\psi_1(A)^{-1}\psi_1(A)x = Ax$, hence $\psi_0(A)\psi_1(A) = A$.

In order to prove the uniqueness, let $U, V \in U(H)$ and $P, Q \in \mathcal{P}$ be such that $UP = VQ$. Then,

$$P^2 = PU^*UP = (UP)^*UP = (VQ)^*VQ = QV^*VQ = Q^2,$$

and by the uniqueness of the square root of positive operators, we conclude that $P = Q$. So $UP = VP$, and since $\operatorname{Cl}(\operatorname{Ran} P) = H$ we conclude that $U = V$.

If $A \in \mathcal{H}$ and $U \in U(H)$, then $UAU^{-1} \in \mathcal{H}$ and

$$UAU^{-1} = U\psi_0(A)\psi_1(A)U^{-1} = U\psi_0(A)U \cdot U^{-1}\psi_1(A)U^{-1}.$$

By the uniqueness of the decomposition, we conclude that $\psi_i(UAU^{-1}) = U\psi_i(A)U^{-1}$ $(i = 0, 1)$.

(f) Recall that $\Psi : U(H)\times\mathcal{P} \to \mathcal{H}$ is given by $\Psi(U, P) = UP$ where $U(H)$ and $\mathcal{P}$ are endowed with the strong topology and $\mathcal{H}$ is endowed with the strong-star topology. This map in continuous. Namely, let $(U_n)_n \subset U(H)$, $(P_n)_n \subset \mathcal{P}$ and $U \in U(H)$, $P \in \mathcal{P}$ be such that s-$\lim U_n = U$ and s-$\lim P_n = P$. Hence, s*-$\lim U_n = U$ and s*-$\lim P_n = P$ (since $P_n^* = P_n$, $P^* = P$, $U_n^* = U_n^{-1}$ and $U^* = U^{-1}$), from which follows that s-$\lim U_n P_n = UP$ and s-$\lim P_n^* U_n^* = P^* U^*$. Hence, s*-$\lim \Psi(U_n, P_n) = \Psi(U, P)$.

It follows from item (e) that Ψ is injective, and it is obviously that (ψ_0, ψ_1) is the inverse of Ψ. It remains to prove that ψ_0 is s*-continuous.

(g) Let $(A_n)_n \subset \mathcal{H}$ and $A \in \mathcal{H}$ be such that s*-$\lim A_n = A$. For each $y \in \operatorname{Ran}(\psi_1(A))$ there exists $x \in H$ such that $y = \psi_1(A)x$. So,

$$\begin{aligned}
||\psi_0(A_n)y - \psi_0(A)y|| &= ||\psi_0(A_n)\psi_1(A)x - \psi_0(A)\psi_1(A)x|| \\
&\leqslant ||\psi_0(A_n)\psi_1(A)x - \psi_0(A_n)\psi_1(A_n)x|| \\
&\quad + ||\psi_0(A_n)\psi_1(A_n)x - \psi_0(A)\psi_1(A)x|| \\
&\leqslant ||\psi_0(A_n)|| \cdot ||\psi_1(A)x - \psi_1(A_n)x|| + ||A_n x - Ax|| \\
&= ||\psi_1(A)x - \psi_1(A_n)x|| + ||A_n x - Ax|| .
\end{aligned}$$

Now, since ψ_1 is s*-continuous it follows that $||\psi_0(A_n)y - \psi_0(A)y|| \to 0$. By the density of $\operatorname{Ran}\psi_1(A)$ we conclude that s-$\lim \psi_0(A_n) = \psi_0(A)$, and since $\psi_0(A) \in U(H)$. It follows that s*-$\lim \psi_0(A_n) = \psi_0(A)$. This proves that ψ_0 is s*-continuous.

■

5.3 Unitary equivalence of s*-typical contractions

Corollary 5.12. *For every $A \in C(H)$, $\mathcal{O}(A)$ is an s*-meager subset of $C(H)$. In particular, s*-typical contractions are not unitarily equivalent.*

Proof. Let $A \in C(H)$ be arbitrary. If $A \notin \mathcal{H}$, since $\mathcal{H}$ is unitary invariant we have $\mathcal{O}(A) \subset C(H) \setminus \mathcal{H}$. Since $\mathcal{H}$ is s*-co-meager, it follows that $\mathcal{O}(A)$ is s*-meager. So, we can assume that $A \in \mathcal{H}$, in this case we have

$$\begin{aligned}
\Psi^{-1}(\mathcal{O}(A)) &= \{(\psi_0(UAU^{-1}), \psi_1(UAU^{-1})) : U \in U(H)\} \\
&= \{(U\psi_0(A)U^{-1}, U\psi_1(A)U^{-1}) : U \in U(H)\} \\
&\subset \{U\psi_0(A)U^{-1} : U \in U(H)\} \times \{U\psi_1(A)U^{-1} : U \in U(H)\} \\
&= \mathcal{O}(\psi_0(A)) \times \mathcal{O}(\psi_1(A))
\end{aligned}$$

It follows from Theorem 1.4 that $(U(H), s) = (U(H), w)$ and by combining this result with Theorem 2.17, we conclude that $\mathcal{O}(\psi_0(A))$ is s-meager. By Theorem 0.2 we conclude that $\Psi^{-1}(\mathcal{O}(A))$ is s×s-meager in $U(H) \times \mathcal{P}$. Since Ψ is a homeomorphism, it follows that $\mathcal{O}(A)$ is s*-meager in $\mathcal{H}$, and consequentially in $C(H)$.

The second part of the proof follows the same arguments presented in the proof of Theorem 2.17.

■

5.4 Spectral properties of s*-typical contractions

The main result in this section states that the point spectrum and the residual spectrum are empty. Moreover, we will prove that typically the continuous spectrum is equal to $\{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

Lemma 5.13. *The set*

$$\mathcal{S} = \{A \in C(H) : \forall\, \lambda \in \mathbb{C}, |\lambda| \leqslant 1 \text{ we have } \lambda \in \sigma(A)\}$$

is s^*-co-meager and s^*-G_δ in $C(H)$.

Proof. (a) We begin proving that $\mathcal{S}$ is s*-dense. Let $U \in C(H)$ be an s*-open set. Then, there exist $n \in \mathbb{N}$, $\{x_i : i < n\} \subset S_H$, $\varepsilon > 0$ and $A \in U$ such that for each $B \in C(H)$ satisfying $||Bx_i - Ax_i|| < \varepsilon$ and $||B^*x_i - A^*x_i|| < \varepsilon$ for every $i < n$, we have $B \in U$.

Set $V = \text{span}\{x_i, Ax_i, A^*x_i : i < n\}$ and let $Q : V \to V$ given by $Q = P_V A\big|_V$. Let $T \in C(V^\perp)$ be such that $\lambda \in \sigma(T)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$ (for instance, let T be the left shift, $T\xi_0 = 0$ and $T\xi_i = \xi_{i-1}$ for $(\xi_n)_{n\in\mathbb{N}}$ an orthonormal basis of $V^\perp$). Set $B = Q \oplus T$ and since $\sigma(T) \subset \sigma(B)$, we have $\lambda \in \sigma(B)$, so $B \in \mathcal{S}$.

On the other hand, for each $i < n$ we have

$$||Bx_i - Ax_i|| = ||Qx_i - Ax_i|| = ||P_V A\big|_V x_i - Ax_i|| = 0 < \varepsilon \text{ and}$$
$$||B^*x_i - A^*x_i|| = ||Q^*x_i - A^*x_i|| = ||P_V A^*\big|_V x_i - A^*x_i|| = 0 < \varepsilon\,,$$

and then $B \in U$. Since U is an arbitrary s*-open set, we conclude that $\mathcal{S}$ is s*-dense.

(b) It remains to prove that $\mathcal{S}$ is an s*-G_δ set. In order to do that, we define, for every $\delta > 0$, the set

$$S(\delta) = \{A \in C(H) : \exists \lambda \in \mathbb{C} \text{ such that } |\lambda| \leqslant 1,\ ||(A - \lambda I)x|| \geqslant \delta$$
$$\text{and } ||(A^* - \overline{\lambda} I)x|| \geqslant \delta\ \forall x \in S_H\}$$

Note that $S(\delta)$ is s*-closed: if $(A_n)_n \in S(\delta)$ and $A \in C(H)$ are such that s*-$\lim A_n = A$, we have for each $x \in S_H$

$$\delta \leqslant ||(A_n - \lambda I)x|| \Rightarrow \delta \leqslant \lim ||(A_n - \lambda I)x|| = ||(A - \lambda I)x|| \text{ and}$$
$$\delta \leqslant ||(A_n^* - \overline{\lambda} I)x|| \Rightarrow \delta \leqslant \lim ||(A_n^* - \overline{\lambda} I)x|| = ||(A^* - \overline{\lambda} I)x||\,.$$

So, $A \in S(\delta)$. Now, we claim that

$$\mathcal{S} = C(H) \setminus \bigcup_n S(2^{-n})$$

Actually, we will show that $C(H) \setminus \mathcal{S} = \bigcup_n S(2^{-n})$.

Let $A \in C(H) \setminus \mathcal{S}$. Then, there exists $\lambda \in \mathbb{C}$, $|\lambda| \leqslant 1$ such that $\lambda \in \rho(A)$, and $(A - \lambda I)^{-1}, (A^* - \overline{\lambda} I)^{-1} \in B(H)$. By denoting $A_\lambda = A - \lambda I$ and $A^*_{\overline{\lambda}} = A^* - \overline{\lambda} I$, we have that

$$1 = ||x|| = ||A_\lambda^{-1} A_\lambda x|| \leqslant ||A_\lambda^{-1}|| \cdot ||A_\lambda x|| \text{ and}$$
$$1 = ||x|| = ||(A^*_{\overline{\lambda}})^{-1} A^*_{\overline{\lambda}} x|| \leqslant ||(A^*_{\overline{\lambda}})^{-1}|| \cdot ||A^*_{\overline{\lambda}} x||$$

Then, $||A_\lambda x|| \geqslant 1/||A_\lambda^{-1}||$ and $||A^*_{\overline{\lambda}} x|| \geqslant 1/||(A^*_{\overline{\lambda}})^{-1}||$. Let $\delta = \min\{1/||A_\lambda^{-1}||, 1/||(A^*_{\overline{\lambda}})^{-1}||\}$

and $n \in \mathbb{N}$ such that $2^{-n} \leqslant \delta$. Then,

$$2^{-n} \leqslant \delta \leqslant ||A_\lambda x|| = ||(A - \lambda I)x|| \text{ and}$$
$$2^{-n} \leqslant \delta \leqslant ||A^*_{\overline{\lambda}} x|| = ||(A^* - \overline{\lambda} I)x||\,.$$

With this, we conclude that $A \in S(2^{-n})$, and so $A \in \bigcup_n S(2^{-n})$.

On the other hand, if $A \in \bigcup_n S(2^{-n})$ then there exists $n \in \mathbb{N}$ such that $A \in S(2^{-n})$. Hence, there exists $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$, such that $||(A-\lambda I)x|| \geqslant 2^{-n}$ and $||(A^* - \overline{\lambda} I)x|| \geqslant 2^{-n}$ for every $x \in S_H$. In particular, $||(A - \lambda I)x|| > 0$ and $||(A^* - \overline{\lambda} I)x|| > 0$ for every $x \in S_H$, so $\lambda \notin \sigma_p(A)$ and $\overline{\lambda} \notin \sigma_p(A^*)$.

Note that $(\ker(A^* - \overline{\lambda} I))^\perp = \mathrm{Cl}(\mathrm{Ran}(A - \lambda I))$, so it follows from $\overline{\lambda} \notin \sigma_p(A^*)$ that $\mathrm{Cl}(\mathrm{Ran}(A - \lambda I)) = \{0\}^\perp = H$. Hence, $\mathrm{Ran}(A - \lambda I)$ is dense in H, and then $\lambda \notin \sigma_r(A)$.

The last step is prove that $\lambda \notin \sigma_c(A)$. Let $\lambda \in \sigma_c(A)$ and then $\mathrm{Cl}(\mathrm{Ran}(A-\lambda I)) = H$, but $\mathrm{Ran}(A-\lambda I) \neq H$. We claim that $\mathrm{Ran}(A-\lambda I)$ is a closed set. Let $\xi \in \mathrm{Cl}(\mathrm{Ran}(A-\lambda I))$, so there exists $(\eta_j)_{j\in\mathbb{N}} \subset H$ such that $\lim_j (A - \lambda I)\eta_j = \xi$. By using that $A \in S(2^{-n})$, we get

$$||\eta_j - \eta_k|| 2^{-n} \leqslant ||(A - \lambda I)(\eta_j - \eta_k)|| \xrightarrow{j,k\to\infty} 0\,.$$

So, $(\eta_j)_{j\in\mathbb{N}}$ is a Cauchy sequence that converges to a $\eta \in H$. By the continuity of $A - \lambda I$, we conclude that $(A - \lambda I)\eta = \xi$. So, $\mathrm{Ran}(A - \lambda I)$ is a closed set, and since $\mathrm{Cl}(\mathrm{Ran}(A - \lambda I)) = H$, it follows that $\mathrm{Ran}(A - \lambda I) = H$, an absurd. This shows that $\lambda \notin \sigma_c(A)$.

Since $\lambda \notin \sigma_c(A) \cup \sigma_r(A) \cup \sigma_p(A) = \sigma(A)$ and $\lambda \in \mathbb{C}$, $|\lambda| \leqslant 1$ we conclude that $A \notin \mathcal{S}$, and we are done.

∎

Theorem 5.14. *An s*-typical contraction satisfies* $\sigma_c(A) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

Proof. By Theorem 5.7, an s*-typical contraction belongs to $\mathcal{T} \cap \mathcal{T}^*$, i.e., an s*-typical contraction A is such that $\mathrm{Cl}(\mathrm{Ran}(A - \lambda I)) = \mathrm{Cl}(\mathrm{Ran}(A^* - \lambda I)) = H$ for each $\lambda \in \mathbb{C}$. Hence, $\sigma_r(A) = \sigma_r(A^*) = \varnothing$.

Since $(\ker(A-\lambda I))^\perp = \mathrm{Cl}(\mathrm{Ran}(A^* - \overline{\lambda} I)) = H$ and $(\ker(A^* - \lambda I))^\perp = \mathrm{Cl}(\mathrm{Ran}(A - \overline{\lambda} I)) = H$, it follows that $\ker(A^* - \lambda I) = \{0\}$. Hence $\sigma_p(A) = \sigma_p(A^*) = \varnothing$.

We recall that the spectral radius of a contraction A is limited by $||A|| \leqslant 1$, so $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$. It follows from Theorem 5.13 that $\{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\} \subset \sigma(A)$, hence $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$. By the preceding discussion conclude that $\sigma_c(A) = \{A \in \mathbb{C} : |\lambda| \leqslant 1\}$.

∎

6. THE NORM TOPOLOGY

Before anything we note that in this chapter, all the topological notions will refer to the norm topology.

In the norm topology, unfortunately, it is not possible to give a simple description of the spectral properties of typical operators. One intuitive explanation for this fact is that the space $B(H)$ is not separable. To see this, we recall a classic example

Example 6.1. *Let* $\ell^\infty = \{x \in \mathbb{C}^{\mathbb{N}} : \sup_{1\leqslant j<\infty} |x_j| < \infty\}$ *be endowed with the norm* $||x||_\infty = \sup_{1\leqslant j<\infty} |x_j| < \infty$*. This space is not separable. Indeed, if we pick some sequence* $(y^n)_{n\in\mathbb{N}} \subset \ell^\infty$*, where* $y^n = (y^n_j)_{j=1}^\infty$*, we can define* $x = (x_j)_{j=1}^\infty$ *with* $x_j = 0$ *if* $|y^j_j| \geqslant 1$ *and* $x_j = y^j_j + 1$ *otherwise.*

With that, $||x||_\infty \leqslant 2$ *and* $||x - y^n||_\infty \geqslant 1$ *for each* $n \in \mathbb{N}$*. Therefore, there doesn't exist a dense sequence in* ℓ^∞*.*

Theorem 6.2. *The space* $B(H)$ *is not separable.*

Proof. Since H is separable, it has a countable orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Now for every $\lambda \in \ell^\infty$, define the operator $T_\lambda : H \to H$ by the law $T_\lambda(e_n) = \lambda_n e_n$ over the elements of the basis, and then extend it linearly and densely over H.

We define now the map $i : \ell^\infty \to B(H)$ by $i(\lambda) = T_\lambda$. We claim that i is an isometry. Namely,

- let $\lambda, \gamma \in \ell^\infty$ and $\alpha \in \mathbb{C}$. Naturally, for each $j \in \mathbb{N}$ we have $(\alpha\lambda + \gamma)_j = \alpha\lambda_j + \gamma_j$, so

$$T_{\alpha\lambda+\gamma}(e_n) = (\alpha\lambda + \gamma)_n e_n = \alpha\lambda_n e_n + \gamma_n e_n = \alpha T_\lambda e_n + T_\gamma e_n$$

  It follow that $i(\alpha\lambda + \gamma) = T_{\alpha\lambda+\gamma} = \alpha T_\lambda + T_\gamma = \alpha i(\lambda) + i(\gamma)$.

- Let $x \in H$, and let $(\eta_n)_n \subset \mathbb{C}$ be such that $x = \sum_{n\in\mathbb{N}} \eta_n e_n$. Then,

$$||T_\lambda x||^2 = \left|\left| \sum_{n\in\mathbb{N}} \eta_n \lambda_n e_n \right|\right|^2 = \sum_{n\in\mathbb{N}} |\eta_n|^2 |\lambda_n|^2 ||e_n||^2 \leqslant ||\lambda||_\infty^2 ||x||^2 ,$$

  from which follows that $||T_\lambda|| \leqslant ||\lambda||_\infty$. Now, let $y = \sum_{n\in\mathbb{N}} \eta_n e_n$, where $\eta_j = \frac{||\lambda||_\infty}{|\lambda_j|}$, to j the first index where $|\lambda_j| \neq 0$, and $\eta_k = 0$ when $k \neq j$ (The case $\lambda = 0$ is obvious). With that

$$||T_\lambda y|| = \left|\left| \frac{||\lambda||_\infty}{|\lambda_j|} \lambda_j e_j \right|\right| = ||\lambda||_\infty$$

  By combining both results we conclude that $||T_\lambda|| = ||\lambda||_\infty$.

Thus, ℓ^∞ is a subspace of $B(H)$. Since $B(H)$ is metric, if it is separable then ℓ^∞ wold be separable too, an absurd. So, $B(H)$ is not separable.

■

6.1 Stability of the spectrum

We will also see some of the properties of $B(H)$ that make the theory of typical spectral properties really complicated. Before this, we will need some terminology and results.

Definition 6.3. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$. We say that $\lambda \in \sigma(A)$ is stable if there exists $\varepsilon > 0$ such that for every $D \in B(H)$ with $||D|| < \varepsilon$, $\lambda \in \sigma(A+D)$. Similarly, we say that $\lambda \in \sigma_p(A)$ is p-stable if there exists $\varepsilon > 0$ such that for every $D \in B(H)$ with $||D|| < \varepsilon$, $\lambda \in \sigma_p(A+D)$.*

Theorem 6.4. (*see* [8]) *An operator A on a separable Hilbert space H is not in the closure of the set $\mathcal{G} = \{T \in B(H) : T \text{ is invertible}\}$ if, and only if, there exists a constant $k > 0$ such that $||Ax|| \geqslant k||x||$ for every $x \in (\ker A)^{\perp}$, and $\dim(\ker A) \neq \dim(\mathrm{Ran}(A))^{\perp}$.*

The proof of this result goes beyond the scope of this work, so we will only cite it and use it. We need it in order to prove the following result.

Theorem 6.5. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$. Then, the following statements are equivalent:*

1. *$\lambda \in \sigma(A)$ is stable;*
2. *$\dim(\ker(A - \lambda I)) \neq \dim(\mathrm{Ran}(A - \lambda I))^{\perp}$ and there exists $\varepsilon > 0$ such that for every $x \in (\ker(A - \lambda I))^{\perp}$ we have $||(A - \lambda I)x|| \geqslant \varepsilon||x||$.*

Before we present the proof of Theorem 6.5 another result is required.

Lemma 6.6. *$\lambda \in \sigma(A)$ is not stable if, and only if, $A - \lambda I \in \mathrm{Cl}(\mathcal{G})$*

Proof. ($\Rightarrow$) If $\lambda \in \sigma(A)$ is not stable, then for every $\varepsilon > 0$ there exists $D \in B(H)$ such that $||D|| < \varepsilon$ and $\varepsilon \in \rho(A + D)$. So, under these conditions, $A - \lambda I + D$ is invertible. Let, for each $n \in \mathbb{N}$, D_n be given as above by letting $\varepsilon = 1/n$. Set, for each $n \in \mathbb{N}$, $T_n = A - \lambda I + D_n$.

Then, $T_n \in \mathcal{G}$ and

$$||T_n - A + \lambda I|| = ||A - \lambda I + D_n - A + \lambda I|| = ||D_n|| < \frac{1}{n} \xrightarrow{n\to\infty} 0\,,$$

which shows that $T_n \to A - \lambda I$, and so $A - \lambda I \in \mathrm{Cl}(\mathcal{G})$.

($\Leftarrow$) Suppose that $A - \lambda I = \lim T_n$, where every T_n is invertible. For each $\varepsilon > 0$, let $n \in \mathbb{N}$ be such that $||T_n - A + \lambda I|| < \varepsilon$. Set $D_n = T_n - A + \lambda I$, so $||D_n|| < \varepsilon$. We claim that $\lambda \in \rho(A + D_n)$. Namely, since

$$A + D_n - \lambda I = A + T_n - A + \lambda I - \lambda I = T_n\,,$$

it follows that $A + D_n - \lambda I$ invertible. So, $\lambda \in \rho(A + D_n)$.

∎

Proof of Theorem 6.5. It follows from Lemma 6.6 that

$$\lambda \in \sigma(A) \text{ is stable if, and only if, } A - \lambda I \notin \mathrm{Cl}(\mathcal{G}) \tag{6.1}$$

(1) implies (2). If $\lambda \in \sigma(A)$ is stable, then $A - \lambda I \notin \mathrm{Cl}(\mathcal{G})$. It follows from Theorem 6.4 that $\dim(\ker(A - \lambda I)) \neq \dim(\mathrm{Ran}(A - \lambda I))^{\perp}$, and there exists $\varepsilon > 0$ such that for every $x \in (\ker(A - \lambda I))^{\perp}$ we have $||(A - \lambda I)x|| \geqslant \varepsilon||x||$.

(2) implies (1). If $\dim(\ker(A - \lambda I)) \neq \dim(\mathrm{Ran}(A - \lambda I))^{\perp}$ and if there exists $\varepsilon > 0$ such that for every $x \in (\ker(A - \lambda I))^{\perp}$, we have $||(A - \lambda I)x|| \geqslant \varepsilon||x||$, then it follows from Theorem 6.4 that $A - \lambda I \notin \mathrm{Cl}(\mathcal{G})$. By relation (6.1), $\lambda \in \sigma(A)$ is stable.

■

We prove a similar result for the point spectrum.

Theorem 6.7. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$. Then the following are equivalent:*

1. *$\lambda \in \sigma_p(A)$ is p-stable;*
2. *$\dim(\ker(A - \lambda I)) > \dim(\mathrm{Ran}(A - \lambda I))^{\perp}$ and there exists $\varepsilon > 0$ such that for every $x \in (\ker(A - \lambda I))^{\perp}$, $||(A - \lambda I)x|| \geqslant \varepsilon||x||$.*

Proof. We prove first the case $\lambda = 0$.

(1) implies (2). If $0 \in \sigma_p(A)$ is p-stable, then $0 \in \sigma(A)$ and it is stable. So, it follows from Theorem 6.5 that $\dim(\ker(A)) \neq \dim(\mathrm{Ran}(A))^{\perp}$ and there exists $\varepsilon > 0$ such that for every $x \in (\ker(A))^{\perp}$, $||Ax|| \geqslant \varepsilon||x||$.

Suppose that $\dim(\ker(A)) < \dim(\mathrm{Ran}(A))^{\perp}$. We will obtain for each $\varepsilon > 0$ an operator $D \in B(H)$ such that $||D|| < \varepsilon$, $D\big|_{(\ker A)^{\perp}} = 0$ and that for every $x \in \ker(A) \setminus \{0\}$, $Dx \in \mathrm{Ran}(A)^{\perp} \setminus \{0\}$. By assuming that such D exists we conclude that:

- if $x \in \ker(A) \setminus \{0\}$, then $Dx \neq 0$ and $Ax = 0$;
- if $x \in (\ker(A))^{\perp} \setminus \{0\}$, then $Dx = 0$ and $Ax \neq 0$.

So, if $x \in H \setminus \{0\}$, we can write $x = x_0 + x_1$, with $x_0 \in \ker(A)$ and $x_1 \in (\ker(A))^{\perp}$, and then

$$(A + D)x = (A + D)(x_0 + x_1) = Dx_0 + Ax_1 .$$

Since for every $x \in \ker(A)\setminus\{0\}$, $Dx \in \mathrm{Ran}(A)^{\perp}\setminus\{0\}$, we conclude that $Dx_0 + Ax_1 \neq 0$. So, for each $x \in H \setminus \{0\}$ we get $(A + D)x \neq 0$, and then $0 \notin \sigma_p(A + D)$, which is absurd, since 0 is p-stable. Therefore, $\dim(\ker(A)) > \dim(\mathrm{Ran}(A))^{\perp}$.

Obtaining of D: For each $\varepsilon > 0$, let $(k_n)_{n\in K}$ be an orthonormal basis for $\ker A$ and let $(r_n)_{n\in R}$ be an orthonormal basis for $(\mathrm{Ran}(A))^{\perp}$. We are assuming that $|K| < |R|$, so there exists an injective function $i : K \to R$. Now, we define D by $Dk_n = \frac{\varepsilon}{2} r_{i(n)}$ for each $n \in K$, and $Dx = 0$ for each $x \in (\ker(A))^{\perp}$. We extend D linearly and densely over H.

Of course, $D\big|_{(\ker A)^{\perp}} = 0$ and for each $x \in \ker(A) \setminus \{0\}$ we have $Dx \in \mathrm{Ran}(A)^{\perp} \setminus \{0\}$. If $x \in H \setminus \{0\}$, we write $x = \sum_{n\in K} \eta_n k_n + x_1$, where $\sum_{n\in K} \eta_n k_n \in \ker(A)$ and $x_1 \in (\ker A)^{\perp}$. Hence,

$$\begin{aligned} ||Dx||^2 &= \Big|\Big| \sum_{n\in K} \eta_n D k_n + Dx_1 \Big|\Big|^2 = \Big|\Big| \sum_{n\in K} \eta_n \frac{\varepsilon}{2} r_{i(n)} \Big|\Big|^2 \\ &= \sum_{n\in K} |\eta_n|^2 \frac{\varepsilon^2}{4} \leqslant \frac{\varepsilon^2}{4} \Big(\sum_{n\in K} |\eta_n|^2 + |x_1|^2 \Big) \\ &= \Big(\frac{\varepsilon}{2}\Big)^2 ||x||^2 \end{aligned}$$

So, $||D|| < \varepsilon$, by the previous discussion we conclude the proof of this part of the statement.

(2) implies (1). Since $\dim\ker(A) > \dim(\mathrm{Ran}(A))^{\perp}$, it follows that $\dim\ker(A) > 0$, and so $0 \in \sigma_p(A)$.

(a) Note that $A : (\ker(A))^{\perp} \to \mathrm{Ran}(A)$ is a bounded invertible operator. Naturally, A is bounded. In order to prove that A is injective, let $x \in (\ker(A))^{\perp}$ with $Ax = 0$; then, $x = 0$.

Now, if $y \in \mathrm{Ran}(A)$ there exists $x \in H$ such that $Ax = y$. Write $x = x_0 + x_1$, with $x_0 \in \ker(A)$ and $x_1 \in (\ker(A))^{\perp}$, so $y = Ax = Ax_0 + Ax_1 = Ax_1$. Hence, $x_1 \in (\ker(A))^{\perp}$ and $Ax_1 = y$, which proves that A is surjective.

Another thing to prove is that $\mathrm{Ran}(A)$ is a closed co-finite-dimensional subspace of H. Let $(x_n)_n \subset (\ker(A))^{\perp}$ be such that $\lim Ax_n = y$. By our hypothesis, we have that

$$||x_n - x_m|| \leqslant \frac{1}{\varepsilon}|||Ax_n - Ax_m||$$

Since $(Ax_n)_n$ converges, it is Cauchy, and so by the previous relation it follows that $(x_n)_n$ is also Cauchy. So, there exists $x \in (\ker(A))^{\perp}$ such that $\lim x_n = x$, from which follows that $Ax = y$ and $y \in \mathrm{Ran}(A)$. Hence, $\mathrm{Ran}(A)$ is closed.

In order to prove the statement regarding the co-finite-dimensional subspace, just consider the relation

$$\begin{aligned}\mathrm{codim}(\mathrm{Ran}(A)) &= \dim\Big(H/\,\mathrm{Ran}(A)\Big) = \dim(\mathrm{Ran}(A))^{\perp}\\ &< \dim\ker(A) \leqslant \dim(H) = \aleph_0\,.\end{aligned}$$

With that, we let $k = 1 + \dim(\mathrm{Ran}(A))^{\perp}$.

(b) Let $D \in B(H)$ be such that $||D|| < \frac{\varepsilon}{2k^2}$.

For every $x \in \ker(A)$, consider the following sequence:

- $x_0 = x$;
- If x_n is defined, write $Dx_n = u_n + v_n$ with $u_n \in \mathrm{Ran}(A)$ and $v_n \in (\mathrm{Ran}(A))^{\perp}$. Let $x_{n+1} = A^{-1}u_n$.

  We set

$$\xi(x) = \sum_{n\in\mathbb{N}}(-1)^n x_n \qquad \& \qquad \rho(x) = \sum_{n\in\mathbb{N}}(-1)^n v_n$$

First of all, we need to show that these functions are well defined. Note that, by the hypothesis there exists $\varepsilon > 0$ such that for every $x \in (\ker(A))^{\perp}$, $||Ax|| \geqslant \varepsilon||x||$. If we write $y = Ax \in \mathrm{Ran}(A)$, it follows that

$$||Ax|| \geqslant \varepsilon||x|| \Rightarrow ||y|| \geqslant \varepsilon||A^{-1}y|| \Rightarrow \frac{1}{\varepsilon} \geqslant ||A^{-1}||\,,$$

and then,

$$||x_{n+1}|| \leqslant \frac{||u_n||}{\varepsilon} \leqslant \frac{||Dx_n||}{\varepsilon} \leqslant ||D||\frac{||x_n||}{\varepsilon} < \frac{||x_n||}{2k^2}\,. \tag{6.2}$$

By continuing the process we show that $||x_n|| < \frac{||x_0||}{(2k^2)^n} = \frac{||x||}{(2k^2)^n}$, and then

$$||\xi(x)|| \leqslant ||x|| \sum_{n\in\mathbb{N}} \left|\frac{(-1)^n}{(2k^2)^n}\right| = ||x|| \sum_{n\in\mathbb{N}} \frac{1}{(2k^2)^n} < \infty\,,$$

proving that ξ is well defined.

In order to prove that $\rho(x)$ is well defined we use equation (6.2) to show that

$$||v_n|| \leqslant ||Dx_n|| \leqslant \varepsilon \frac{||x_n||}{2k^2} \leqslant \frac{\varepsilon ||x||}{(2k^2)^{n+1}}\,;$$

then,

$$||\rho(x)|| \leqslant \varepsilon ||x|| \sum_{n\in\mathbb{N}} \frac{1}{(2k^2)^{n+1}} < \infty\,.$$

Moreover, these functions are linear and $\rho(x) \in \mathrm{Ran}(A)^\perp$ (The last assertion follows from the fact that $v_n \in \mathrm{Ran}(A)^\perp$ and that $\mathrm{Ran}(A)^\perp$ is closed). Namely,

- $(\alpha x + y)_0 = \alpha x + y = \alpha x_0 + y_0$;
- suppose now that $(\alpha x + y)_n = \alpha x_n + y_n$, so

$$\begin{aligned}(\alpha x + y)_{n+1} &= A^{-1}(P_{\mathrm{Ran}(A)} D(\alpha x + y)_n) = A^{-1} P_{\mathrm{Ran}(A)} D(\alpha x_n + y_n) \\ &= \alpha A^{-1} P_{\mathrm{Ran}(A)} D x_n + A^{-1} P_{\mathrm{Ran}(A)} D y_n = \alpha x_{n+1} + y_{n+1}\,;\end{aligned}$$

- note that $v_n = P_{\mathrm{Ran}(A)^\perp} D x_n$. By using the previous two items we conclude that $P_{\mathrm{Ran}(A)^\perp} D(\alpha x + y)_n = \alpha P_{\mathrm{Ran}(A)^\perp} D x_n + P_{\mathrm{Ran}(A)^\perp} D y_n$.

By using that the linearity of $\xi(x)$ and $\rho(x)$ follows.

(c) We claim that for each $x \in B_{\ker(A)}$, $||\xi(x) - x|| < 1/k$. Before we proceed, we need to consider study the polynomial function $P(x) = 2x^2 - x - 1$. This polynomial has roots $x = 1$ and $x = -1/2$, and so we can conclude that $P(x) > 0$ if $x > 1$. We recall that $k = 1 + \dim(\mathrm{Ran}(A))^\perp$, and it follows that $P(k) \geqslant 0$, i.e.,

$$0 \leqslant 2k^2 - k - 1 \Rightarrow k \leqslant 2k^2 - 1 \Rightarrow \frac{1}{2k^2 - 1} \leqslant \frac{1}{k}$$

Therefore, $\frac{1}{2k^2-1} \leqslant \frac{1}{k}$. Now, let $x \in B_{\ker(A)}$. So,

$$\begin{aligned}||\xi(x) - x|| &= \left|\left| \sum_{n\in\mathbb{N}\setminus\{0\}} (-1)^n x_n \right|\right| \leqslant \sum_{n\in\mathbb{N}\setminus\{0\}} ||x_n|| < ||x|| \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{(2k^2)^n} \\ &\leqslant \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{(2k^2)^n} = \frac{1}{1 - \frac{1}{(2k^2)}} - 1 = \frac{2k^2 - 2k^2 + 1}{2k^2 - 1} \leqslant \frac{1}{k}\,.\end{aligned}$$

Hence, $||\xi(x) - x|| < 1/k$. By Lemma 4.6, if $\{x(i) : i < k\}$ is an orthonormal family in $\ker(A)$, then $\{\xi(x(i)) : i < k\}$ are linearly independent.

(d) Recall that $x_0 = x \in \ker(A)$, and so

$$\begin{aligned}(A+D)\xi(x) &= (A+D)\Big(\sum_{n\in\mathbb{N}}(-1)^n x_n\Big) = \sum_{n\in\mathbb{N}}(-1)^n(Ax_n + Dx_n)\\ &= Ax_0 + \sum_{n\in\mathbb{N}\setminus\{0\}}(-1)^n Ax_n + \sum_{n\in\mathbb{N}}(-1)^n(u_n+v_n)\\ &= \sum_{n\in\mathbb{N}\setminus\{0\}}(-1)^n Ax_n + \sum_{n\in\mathbb{N}}(-1)^n Ax_{n+1} + \sum_{n\in\mathbb{N}}(-1)^n v_n\\ &= \sum_{n\in\mathbb{N}}(-1)^n v_n = \rho(x)\,.\end{aligned}$$

It follows from $\dim\ker(A) \geqslant k$ that there exists an orthonormal family $\{x(i) : i < k\}$ of $\ker(A)$. Recall now that $\dim(\mathrm{Ran}(A))^\perp < k$ and $\rho(x) \in \mathrm{Ran}(A)^\perp$, so there exists $x \in \mathrm{span}\{x(i) : i < k\} \setminus \{0\}$ such that $\rho(x) = 0$. Then, $(A+D)\xi(x) = 0$ and $0 \in \sigma_p(0)$.

This concludes the proof of this part of the statement, after all, we have shown that there exists $\delta > 0$, where $\delta = \frac{\varepsilon}{2k^2}$, such that for every $D \in B(H)$ with $||D|| < \delta$, $0 \in \sigma_p(A+D)$.

(e) In order to prove the general case, we apply what we have proved to $T = A - \lambda I$. We only need to show that

$$0 \in \sigma_p(T) \text{ is p-stable if, and only if, } \lambda \in \sigma_p(A) \text{ is p-stable.}$$

If $0 \in \sigma_p(T)$, then there exists $x \in H \setminus \{0\}$ such that $Tx = 0$, which by its turn is true if, and only if, there exists $x \in H \setminus \{0\}$ such that $Ax = \lambda x$, i.e., $\lambda \in \sigma_p(A)$.

Now note that if $0 \in \sigma_p(T)$, then there exists $\varepsilon > 0$ such that for every $D \in B(H)$ with $||D|| < \varepsilon$, $0 \in \sigma_p(T+D)$. We note that $\sigma_p(T+D) = \sigma_p(A+D-\lambda I)$, so it follows from the previous discussion that $0 \in \sigma_p(T+D)$ if, and only if, $\lambda \in \sigma_p(A+D)$. Then, $0 \in \sigma_p(T)$ is p-stable if, and only if, there exists $\varepsilon > 0$ such that for every $D \in B(H)$ with $||D|| < \varepsilon$, $\lambda \in \sigma_p(A+D)$, i.e., $\lambda \in \sigma_p(A)$ is p-stable. This concludes the proof.

■

6.2 Topological properties of some sets

As it was said before, some properties of $B(H)$ make the theory of typical spectral properties more complicated than what we have seen before for the others topologies in $C(H)$. In order to prove the main results of this section, we need to recall some general facts.

Definition 6.8. *Let $A \in B(H)$. The approximate point spectrum of A, $\sigma_{ap}(A)$, is defined as*

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : \exists (x_n)_n \subset H \text{ such that } ||x_n|| = 1\, \forall n \in \mathbb{N} \text{ and } ||(A-\lambda I)x_n|| \to 0\}$$

A sequence $(x_n)_n \subset H$ satisfying the above conditions are usually called a Weyl sequence for A at $\lambda \in \mathbb{C}$.

The first thing to note is that the nomenclature approximate point spectrum makes sense, that is $\sigma_{ap}(A) \subset \sigma(A)$. Suppose that $\lambda \in \sigma_{ap}(A)$, but $A - \lambda I$ invertible. It follows from Definition 6.8 that there exists $(x_n)_n \subset H$ such that $||x_n|| = 1$ for every $n \in \mathbb{N}$

and $||(A-\lambda I)x_n|| \to 0$. On the other hand, $1 = ||x_n|| = ||(A-\lambda I)^{-1}(A-\lambda I)x_n|| \leqslant ||(A-\lambda I)^{-1}|| \cdot ||(A-\lambda I)x_n|| \to 0$, an absurd. Hence, $\lambda \in \sigma(A)$.

Another thing to note is that $\sigma_p(A) \subset \sigma_{ap}(A)$, although the point spectrum may be empty. Our main goal now is to show that $\sigma_{ap}(A)$ is never empty. In the process, we prove some topological properties of some sets.

Lemma 6.9. *If $A \in B(H)$ is such that $||A - I|| < 1$, then A is invertible.*

Proof. Let $B = I - A$, so $||B|| = r < 1$. Then,

$$Z = \sum_{n=0}^{\infty} B^n$$

converges in $B(H)$. If we write $Z_n = I + B + \cdots + B^n$, we obtain

$$Z^n(I-B) = I + B + \cdots + B^n - B - B^2 - \cdots - B^{n+1} = I - B^{n+1}$$

But then $||B^{n+1}|| \leqslant r^{n+1} \to 0$, so $Z(I-B) = I$. Similarly, $(I-B)Z = I$. So, $I - B$ is invertible, with $(I-B)^{-1} = \sum_{n=0}^{\infty} B^n$. We conclude the proof by noting that $I - B = A$. ∎

Theorem 6.10. *Let $G = \{A \in B(H) : A \text{ is invertible}\}$, $G_l = \{A \in B(H) : A \text{ is left invertible}\}$ and $G_r = \{A \in B(H) : A \text{ is right invertible}\}$. Then, G_l, G_r and G are open subsets of $B(H)$.*

Proof. (a) Let $A \in G_l$ and $B \in B(H)$ be such that $BA = I$. We claim that $B(A; ||B||^{-1}) \subset G_l$.

If $||T - A|| < ||B||^{-1}$, then $||BT - I|| = ||B(T-A)|| < 1$. It follows from Lemma 6.9 that $Z = BT$ is invertible. Set $X = Z^{-1}B$, so

$$XT = Z^{-1}BT = Z^{-1}Z = I$$

This proves that T is left invertible. Hence $B(A; ||B||^{-1}) \subset G_l$ and we conclude that G_l is open.

(b) Similarly to G_r, we show that $B(A; ||B||^{-1}) \subset G_r$ for $A \in G_r$, where $B \in B(H)$ is such that $AB = 1$.

If $||T - A|| < ||B||^{-1}$, then $||TB - I|| = ||(T-A)B|| < 1$. It follows from Lemma 6.9 that $Z = TB$ is invertible. Set $X = BZ^{-1}$, so

$$TX = TBZ^{-1} = ZZ^{-1} = I$$

This proves that T is right invertible. Hence $B(A; ||B||^{-1}) \subset G_r$ and we conclude that G_r is open.

(c) Since $G = G_l \cup G_r$, it follows that G is also open. ∎

A direct consequence of the arguments presented in the proof of Theorem 6.10 is the following result

Corollary 6.11. *Let $A, B \in B(H)$, then*

1. *If $BA = I$ and $||T - A|| \leqslant ||B||^{-1}$, then T is left invertible;*
2. *If $AB = I$ and $||T - A|| \leqslant ||B||^{-1}$, then T is right invertible.*

One could look at these results and want to generalize then in some sense. A good try is to replace $B(H)$ with some general Banach algebra with identity. Indeed, Conway proves in [4] Lemma 6.9, Theorem 6.10 and Corollary 6.11 in a Banach algebra with identity.

Now we prove that the approximate point spectrum of a bounded operator is always nonempty. After that we will be able to prove the main results of the Chapter.

Theorem 6.12. *If $A \in B(H)$, $\partial\sigma(A) \subset \sigma_{ap}(A)$. In particular, $\sigma_{ap}(A) \neq \varnothing$*

Proof. Let $\lambda \in \partial\sigma(A)$ and $\{\lambda_n\} \subset \rho(A)$ be such that $\lambda_n \to \lambda$. We claim that $||(A - \lambda_n I)^{-1}|| \to \infty$.

Suppose that this is false. Then, by taking a subsequence if necessary, there exists $M > 0$ such that $||(A - \lambda_n I)^{-1}|| < M$ for each $n \in \mathbb{N}$. Since $\lambda_n \to \lambda$, let n be sufficiently large so that $|\lambda_n - \lambda| < M^{-1}$. Then,

$$||(A - \lambda I) - (A - \lambda_n I)|| \leqslant |\lambda_n - \lambda| < M^{-1} \leqslant ||(A - \lambda_n I)^{-1}||^{-1} .$$

It follows from $\{\lambda_n\} \subset \rho(A)$ that $A - \lambda_n I$ is invertible, and then by Corollary 6.11, $A - \lambda I$ is invertible. Therefore, $\lambda \in \rho(A)$, which is absurd since $\partial\sigma(A) \subset \mathrm{Cl}(\sigma(A)) = \sigma(A)$. Hence, $||(A - \lambda_n I)^{-1}|| \to \infty$.

Let $(x_n)_n \subset H$ be such that $||x_n|| = 1$ and $\alpha_n := ||(A - \lambda_n I)^{-1}x_n|| > ||(A - \lambda_n I)^{-1}|| - 1/n$. It follows from previous claim that $\alpha_n \to \infty$. Set $y_n = \alpha_n^{-1}(A - \lambda_n I)^{-1}x_n$, so $||y_n|| = |\alpha_n^{-1}||\alpha_n| = 1$. Now,

$$\begin{aligned}(A - \lambda I)y_n &= (A - \lambda_n I)y_n + (\lambda_n I - \lambda I)y_n = \alpha_n^{-1}x_n + (\lambda_n - \lambda)y_n \\ &\Rightarrow ||(A - \lambda I)y_n|| \leqslant |\alpha_n^{-1}| + |\lambda_n - \lambda| \to 0 ,\end{aligned}$$

from which follows that $\lambda \in \sigma_{ap}(A)$.

We know that the spectrum $\sigma(A)$ is bounded by $\mathrm{Cl}(B(0, ||A||))$. Moreover, $\mathbb{C}$ is a connected set, so $\partial\sigma(A) \neq \varnothing$, proving that $\sigma_{ap}(A) \neq \varnothing$.

■

We are finally able to state and prove the main results of this chapter. The first one shows that some important sets have non-empty interior, and that others are nowhere dense. We note that such results will destroy our hopes of finding a simple description of typical spectral properties.

Theorem 6.13. *Let $\lambda \in \mathbb{C}$. Then the following sets of operators have non-empty interior:*

1. $\{A \in B(H) : \lambda \notin \sigma(A)\}$;
2. $\{A \in B(H) : \lambda \in \sigma_r(A)\}$;
3. $\{A \in B(H) : \lambda \in \sigma_p(A)\}$.

In particular, the following sets have non-empty interior:

4. $\{A \in B(H) : \mathrm{Ran}(A - \lambda I) = H\}$;
5. $\{A \in B(H) : \mathrm{Ran}(A - \lambda I)$ *is not dense in* $H\}$.

On the other hand, the following sets of operators are nowhere dense:

6. $\{A \in B(H) : \lambda \in \sigma_c(A)\}$;
7. $\{A \in B(H) : \mathrm{Ran}(A - \lambda I)$ *is dense in* H *but not equal to* $H\}$.

Proof. Let $S_n(\lambda)$ be the set in item n ($n = 1, \dots, 7$) related with $\lambda \in \mathbb{C}$. For instance $S_3(1) = \{A \in B(H) : 1 \in \sigma_p(A)\}$.

First of all, we prove the results in case $\lambda = 0$, and after that we study the general case.

(a) $S_1(0)$ e $S_4(0)$: Note that $\{A \in B(H) : 0 \notin \sigma(A)\} = \{A \in B(H) : A \text{ is invertible}\}$, which is an open set by Theorem 6.10. Now, since every invertible operator satisfies $\mathrm{Ran}(A) = H$, it follows that $S_1(0) \subset S_4(0)$, so $\mathrm{int}\, S_4(0) \neq \varnothing$.

(b) $S_3(0)$: Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of H, and let D be the defined left shift in this basis, i.e., $De_0 = 0$ and $De_{i+1} = e_i$ for every $i > 0$. Since $De_0 = 0$ it follows that $D \in S_3(0)$.

It is obvious that $\mathrm{Ran}(D) = H$ and $\ker(D) = \mathrm{span}\{e_0\}$, so it follows that

$$\dim(\mathrm{Ran}(D))^{\perp} = 0 < 1 = \dim\ker(D)$$

. Moreover, if $x \in (\ker(D))^{\perp} = \mathrm{Cl}(\mathrm{span}\{e_i\}_{i=1}^{\infty})$ we write $x = \sum_{i=1}^{\infty} \eta_i e_i$ and so

$$||Dx||^2 = \left|\left|\sum_{i=1}^{\infty} \eta_i e_{i-1}\right|\right|^2 = \sum_{i=1}^{\infty} |\eta_i|^2 = ||x||^2$$

Hence, $||Dx|| \geqslant ||x||$, and by Theorem 6.7, we conclude that 0 is p-stable. Then there exists $\varepsilon > 0$ such that for each $S \in B(H)$ with $||S|| < \varepsilon$, $0 \in \sigma_p(S + D)$.

We claim that $B(D, \varepsilon) \subset S_3(0)$. Namely, let $A \in B(D, \varepsilon)$, so $||A - D|| < \varepsilon$. It follows from the p-stability of 0 that $0 \in \sigma_p(A - D + D) = \sigma_p(A)$, so $A \in S_3(0)$.

(c) $S_6(0)$ and $S_7(0)$: It is obvious that $S_6(0) \subset S_7(0)$, so it is sufficient to study $S_7(0)$.

Let $A \in B(H)$ be such that $\mathrm{Cl}(\mathrm{Ran}(A)) = H$ but $\mathrm{Ran}(A) \neq H$. Then, A is not invertible, which means that $0 \in \sigma(A)$.

Now, note that if there exists $\varepsilon > 0$ such that $||Ax|| \geqslant \varepsilon ||x||$ for every $x \in (\ker(A))^{\perp}$, then $\mathrm{Ran}(A)$ closed. This is because if $(Ax_n)_n$ converges to some y in H, then

$$||x_n - x_m|| \leqslant \frac{1}{\varepsilon}||Ax_n - Ax_m|| \xrightarrow{n,m\to\infty} 0\,.$$

Hence, $(x_n)_n$ is a Cauchy sequence in H, and there exists $x \in H$ such that $x = \lim x_n$ and $Ax = y$. Thus, $y \in \mathrm{Ran}(A)$ showing that $\mathrm{Ran}(A)$ is closed. This contradicts the fact that $A \in S_7(0)$. With that, 0 do not satisfy the second condition in Theorem 6.5, and 0 is not stable.

By Lemma 6.6, $A \in \mathrm{Cl}(\mathcal{G})$, where $\mathcal{G} = \{T \in B(H) : T \text{ is invertible}\}$. Since A is not invertible, we get $A \in \partial\mathcal{G}$, and so $S_7(0) \subset \partial\mathcal{G}$.

We use that to obtain $\mathrm{Cl}(S_7(0)) \subset \mathrm{Cl}(\partial\mathcal{G}) = \partial\mathcal{G}$, and so $\mathrm{int}\,\mathrm{Cl}(S_7(0)) \subset \mathrm{int}\,\partial\mathcal{G} = \varnothing$. Hence, both $S_6(0)$ and $S_7(0)$ are nowhere dense.

(d) $S_2(0)$ and $S_5(0)$: Let D^* be the right shift over an orthonormal basis $(e_i)_{i\in\mathbb{N}}$

(see the proof of item (b)). Note that $0 \notin \sigma_p(D^*)$, since if $x = \sum_{i=0}^{\infty} \eta_i e_i$, then

$$0 = D^* x = \sum_{i=0}^{\infty} \eta_i e_i = \sum_{i=0}^{\infty} \eta_i e_{i+1}$$

and so $\eta_i = 0$ for each $i \in \mathbb{N}$. Another property of D^* is that it is not surjective (namely, $\text{span}\{e_0\} \subset (\text{Ran}(D^*))^C$). Hence, $0 \in \sigma(D^*) \setminus \sigma_p(D^*)$. It follows that $D^* \in S_2(0)$, since $\text{Ran}(D^*) = \text{Cl}(\text{span}\{e_i\}_{i=1}^{\infty})$, and so, $\text{Cl}(\text{Ran}(D^*)) \neq H$.

We know that $||D^*x|| = ||x||$ for each $x \in H$. Moreover, $\dim\ker(A) = 0 \neq 1 = \dim(\text{Ran}(A))^{\perp}$, so, 0 is stable. Then, there exists $\varepsilon > 0$ such that for each $S \in B(H)$ such that $||S|| < \varepsilon$, $0 \in \sigma(S + D^*)$.

Now, we will show that $0 \notin \sigma_p(D^*)$ is also stable, i.e, that there exists $\varepsilon' > 0$ such that for every $S \in B(H)$ with $||S|| < \varepsilon'$, $0 \in \sigma_p(D^* + S)$. Fix $\varepsilon' = \frac{1}{2}\min\{\varepsilon, 1\}$ and let $S \in B(H)$ be such that $||S|| < \delta$. If $0 \in \sigma_p(S + D^*)$, then there exists $x \in H$ such that $D^*x = -Sx$, so $||Sx|| = ||D^*x|| = ||x||$. Therefore, $1 \leqslant ||S|| < \delta \leqslant 1$, an absurd.

By combining the previous results, we conclude that there exists $\delta > 0$ (namely, $\delta = \min\{\varepsilon, \varepsilon'\}$) such that for each $S \in B(H)$ satisfying $||S|| < \delta$, $0 \in \sigma(S+D^*)\setminus\sigma_p(S+D^*)$. Hence,

$$B(D^*, \delta) \subset \{A \in B(H) : 0 \in \sigma(A) \setminus \sigma_p(A)\} = S_2(0) \cup S_6(0) \subset S_2(0) \cup \partial\mathcal{G}\,.$$

Let $V = B(D^*, \delta) \setminus \partial\mathcal{G}$. Note that $V \neq \varnothing$ (namely, $\partial\mathcal{G}$ has empty interior, so it is not possible that $B(D^*, \delta) \subset \partial\mathcal{G}$), and V is open ($B(D^*, \delta)$ and $\partial\mathcal{G}$ is closed). Hence, $V \subset S_2(0)$ and so $S_2(0)$ has non-empty interior.

Since $S_2(0) \subset S_5(0)$, it follows that $S_5(0)$ has non-empty interior.

(e) The general case. Let $\lambda \in \{C\}$ and define $f_\lambda(T) = T + \lambda I$. We claim that f_λ is a homeomorphism. Namely,

- if $f_\lambda(T) = f_\lambda(U)$, then $T + \lambda I = U + \lambda I$. So, $T = U$ and we conclude that f_λ is injective;
- if $T \in B(H)$, we let $S = T - \lambda I \in B(H)$. So, $f_\lambda(S) = S + \lambda I = T$ and f_λ is surjective;
- let $V \subset B(H)$ be an open set and let $T + \lambda I \in f_\lambda(V)$. Thus, $T \in V$, and there exists $\varepsilon > 0$ such that $B(T, \varepsilon) \subset V$. We claim that $B(f_\lambda(T), \varepsilon) \subset f_\lambda(V)$.

  Let $U + \lambda I \in B(f_\lambda(T), \varepsilon)$. Note that $||U - T|| = ||U + \lambda I - T - \lambda I|| < \varepsilon$, so $U \in B(T, \varepsilon) \subset V$. Hence, $U + \lambda I \in f_\lambda(V)$, proving that $f_\lambda(V)$ is open. We conclude that f_λ is an open map;
- Note that $f_\lambda^{-1} = f_{-\lambda}$. Namely, $f_\lambda f_{-\lambda}(T) = T - \lambda I + \lambda I = T$ and $f_{-\lambda} f_\lambda(T) = T + \lambda I - \lambda I = T$. By the previous discussion f_λ^{-1} is an open map too. We conclude that f_λ is continuous and then is a homeomorphism.

(f) Note that $S_1(\lambda) = \{A \in B(H) : \lambda \notin \sigma(A)\} = \{A \in B(H) : A - \lambda I \text{ is invertible}\} = \{T + \lambda I \in B(H) : T \text{ is invertible}\} = \{T + \lambda I \in B(H) : 0 \notin \sigma(A)\} = f_\lambda(S_1(0))$. Since $S_1(0)$ has non-empty interior and f_λ is a homeomorphism, we conclude that $S_1(\lambda)$ have non-empty interior.

Like we did before, $S_1(\lambda) \subset S_4(\lambda)$, so $S_4(\lambda)$ has non-empty interior.

(g) Note that

$$\begin{aligned}S_2(\lambda) &= \{A \in B(H) : \lambda \in \sigma_r(A)\}\\ &= \{A \in B(H) : A - \lambda I \text{ is injective and } \operatorname{Cl}(\operatorname{Ran}(A - \lambda I) \neq H\}\\ &= \{T + \lambda I : T \text{ is injective and } \operatorname{Cl}(\operatorname{Ran}(T)) \neq H\}\\ &= f_\lambda(S_2(0))\end{aligned}$$

Since $S_2(0)$ has non-empty interior and f_λ is a homeomorphism, we conclude that $S_2(\lambda)$ has non-empty interior.

Moreover, $S_2(\lambda) \subset S_5(\lambda)$, so $S_5(\lambda)$ has non-empty interior.

(h) $S_3(\lambda) = \{A \in B(H) : \lambda \in \sigma_p(A)\} = \{A \in B(H) : A - \lambda I \text{ is not injective}\} = \{T + \lambda I \in B(H) : T \text{ is not injective}\} = \{T + \lambda I : 0 \in \sigma_p(T)\} = f_\lambda(S_3(0))$. Since $S_3(0)$ has non-empty interior and f_λ is a homeomorphism, we conclude that $S_3(\lambda)$ has non-empty interior.

(i) $S_7(\lambda) = \{A \in B(H) : \operatorname{Ran}(A - \lambda I) \text{ is dense in } H \text{ but not equal to } H\} = \{T + \lambda I \in B(H) : \operatorname{Ran}(T) \text{ is dense in } H \text{ but not equal to } H\} = f_\lambda(S_7(0))$. Since $S_7(0)$ is nowhere dense and f_λ is a homeomorphism, we conclude that $S_7(\lambda)$ is nowhere dense.

Clearly $S_6(\lambda) \subset S_7(\lambda)$, hence $S_6(\lambda)$ is nowhere dense. This ends the proof of the theorem.

■

As we have said, these results show that it is hard to present a simple description of typical spectral properties in the norm topology. For example, fix $\lambda \in \mathbb{C}$; then, $\{A \in B(H) : \lambda \notin \sigma(A)\}$ has non-empty interior. One could imagine that this set is co-meager, but it follows from Baire Category Theorem that a meager set, i.e., the complement of a co-meager set, has empty interior.

On the other hand $\{A \in B(H) : \lambda \in \sigma_r(A)\}$ is a subset of the complement of $\{A \in B(H) : \lambda \notin \sigma(A)\}$, and by Theorem 6.13 we know that it has non-empty interior, an absurd. We can prove similar results for any of the sets in Theorem 6.13.

Theorem 6.14. *The set* $\{A \in B(H) : \sigma_p(A) \neq \varnothing\}$ *is dense in* $B(H)$.

Proof. Let $B \in B(H)$. Since $\sigma_{ap}(B) \neq \varnothing$, there exist $\lambda \in \sigma(B)$ and a sequence $(x_n)_n \subset S_H$ such that $\lim ||Bx_n - \lambda x_n|| = 0$.

For every $n \in \mathbb{N}$, let P_n be the orthonormal projection onto $\operatorname{span}\{x_n\}$, and set $A_n = B(I - P_n) + \lambda P_n$. Note that

$$A_n x_n = Bx_n - Bx_n + \lambda x_n = \lambda x_n \,.$$

So, $\lambda \in \sigma_p(A_n)$, and $\sigma_p(A_n) \neq \varnothing$.

We claim that $||A_n - B|| = ||\lambda x_n - Bx_n||$. On one hand, if $x \in H$ is such that $||x|| = 1$, we can write $x = P_n x + Rx$, where $Rx \in (\operatorname{span}\{x_n\})^\perp$. We can see that $||P_n x|| \leqslant ||x|| = 1$, and if we write $P_n x = \eta_n x_n$ for some $\eta_n \in \mathbb{C}$, it follows that $|\eta_n| = ||\eta_n x_n|| = ||P_n x|| \leqslant 1$, so

$$||(A_n - B)x|| = ||\lambda P_n x - BP_n x|| = |\eta_n| \cdot ||\lambda x_n - Bx_n|| \leqslant ||\lambda x_n - Bx_n|| \,.$$

This shows that $||A_n - B|| \leqslant ||\lambda x_n - Bx_n||$. On the other hand,

$$||(A_n - B)x_n|| = ||\lambda x_n - Bx_n||,$$

So $||A_n - B|| \geqslant ||\lambda x_n - Bx_n||$, and we conclude that $||A_n - B|| = ||\lambda x_n - Bx_n|| \to 0$. Hence, $\lim A_n = B$ and $A_n \in \{A \in B(H) : \sigma_p(A) \neq \varnothing\}$ for each $n \in \mathbb{N}$. This concludes the proof of the theorem.

■

BIBLIOGRAPHY

[1] BARTLE, R. G. **The Elements of Integration and Lebesgue Measure**. [s.l.]: John Wiley & Sons, 1995

[2] BREZIS, H. **Functional Analysis, Sobolev Spaces and Partial Differential Equations**. 1.ed. New York: Springer, 2010

[3] COELHO, F. U. & LOURENÇO, M. L. **Um Curso de Álgebra Linear**. 2.ed. São Paulo: EDUSP, 2018

[4] CONWAY, J. B. **A Course in Functional Analysis**. 2.ed. New York: Springer, 1990

[5] CONWAY, J. B. **A Course in Operator Theory**. 1.ed. [s.l.]: American Mathematical Society, 2000

[6] EISNER, T. & MÁTRAI, T. **On Typical Properties of Hilbert Space Operators**. Israel Journal of Mathematics **195**, p. 247-281, 2013

[7] EISNER, T. **A "typical" contraction is unitary**. Enseignement des Mathématiques (2) **56**, p. 403-410, 2010

[8] FELDMAN, J. & KADISON, R. V. **The closure of the regular operators in a ring of operators**. Proceedings of the American Mathematical Society 5, p. 909–916, 1954

[9] FOLLAND, G. B. **Real analysis: modern techniques and their applications**. 2.ed. [s.l.]: John Wiley & Sons, 1999

[10] HORN, R. A. & JOHNSON, C. R. **Matrix Analysis**. 2.ed. New York: Cambridge University Press, 1985

[11] HORN, R. A. & JOHNSON, C. R. **Topics in matrix analysis**. 1.ed. New York: Cambridge University Press, 1991

[12] KECHRIS, A. S. **Classical Descriptive Set Theory**. 1.ed. New York: Springer, 1995

[13] MUNKRES, J. R. **Topology**. 2.ed. [s.l.]: Pearson, 2000

[14] NADKARNI, M. G. **Spectral Theory of Dynamical Systems**. Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 1998

[15] NAYLOR, A.W. & SELL, G. R. **Linear Operator Theory in Engineering and Sciences**. 1.ed. New York: Springer, 2000

[16] OLIVEIRA, C. R. **Intermediate Spectral Theory and Quantum Dynamics**. 1.ed. Berlin: Birkhäuser Verlag AG, 2009

[17] OLIVEIRA, C. R. **Introdução à Análise Funcional**. 1.ed. Rio de Janeiro: IMPA, 2018

[18] PELLER, V. V. **Estimates of operator polynomials in the space Lp with respect to the multiplicative norm**. Journal of Mathematical Sciences **16**, p. 1139–1149, 1981

[19] SIMON, B. **Operators with singular continuous spectrum: I. general operators**. Ann. of Math. **141** p. 131-145, 1995

[20] SRIVASTAVA, S. M. **A Course on Borel Sets**. 1.ed. New York: Springer, 1998

[21] TAKESAKI, M. **Theory of Operator Algebras I**. 1.ed. New York: Springer, 1979

A. The Kuratowski-Ulam Theorem

The discussion presented here follows [20], and with some additions of [12].

For $E \subset X \times Y$, $x \in X$ and $y \in Y$, we set

$$E_x = \{y \in Y : (x, y) \in E\}$$

and

$$E^y = \{x \in X : (x, y) \in E\}$$

Our main goal is prove the following result:

The Kuratowski-Ulam Theorem. *Let X, Y be second countable Baire spaces and suppose that $A \subset X \times Y$ has the Baire property. Then, the following statements are equivalent.*

1. *A is meager (respec. co-meager);*
2. *$\{x \in X : A_x$ is meager (respec. co-meager)$\}$ is co-meager;*
3. *$\{y \in Y : A^y$ is meager (respec. co-meager)$\}$ is co-meager.*

In what follows, if $A, U \subset X$ we will say that A is meager in U if $A \cap U$ is meager in U. We begin proving the following result:

Lemma A.1. *Let X be a topological space and suppose that $A \subset X$ has the BP. Then, either A is meager or there exists a nonempty open set $U \subset X$ on which A is co-meager (i.e., $A \cap U$ is co-meager in U). If X is a Baire space, exactly one of these alternative holds.*

Proof. Since A has the BP, there exists an open set U such that $M := A \triangle U$ is meager. If A is non-meager then $U \neq \varnothing$ and A is co-meager in U (namely, since $U \setminus A \subset M$, we conclude that $U \setminus A$ is meager in U).

In particular, if both of the alternatives hold, then A is meager and co-meager in U. Since U is a non-empty open set and X is Baire, we conclude that U is Baire. Hence, it is impossible to $A \cap U$ be meager and co-meager at the same time.

■

Lemma A.2. *Let X, Y be topological spaces and $A \subset X \times Y$. Then, for any $x \in X$,*

1. *$(A_x)^C = (A^C)_x$;*
2. *if A_x is nowhere dense then A_x^C is dense;*
3. *if A open (respec. closed) then A_x is open (respec. closed);*
4. *if A is closed then A_x is nowhere dense if and only if A_x^C is dense.*

Note that by item 1 in Lemma A.2, it makes sense to write A_x^C without parenthesis in the following sentences. The order of the operations performed in the set A does not change the final obtained set.

Proof. (1) Note that $y \in (A_x)^C$ if, and only if, $(x,y) \notin A$, and this is equivalent to say that $(x,y) \in A^C$; on the other hand, this occurs if, and only if, $y \in (A^C)_x$. We conclude that $(A_x)^C = (A^C)_x$.

(2) Suppose that A_x is nowhere dense, so $\text{int}(\text{Cl}(A_x)) = \varnothing$, and then

$$\text{int}(A_x) \subset \text{int}(\text{Cl}(A_x)) = \varnothing\,;$$

hence, $\text{int}(A_x) = \varnothing$, and then $\text{Cl}(A_x^C) = Y$. So, A_x^C is dense.

(3) For each $x \in X$ let $f_x : Y \to X \times Y$ such that $f_x(y) = (x,y)$, and then $A_x = f_x^{-1}(A)$. We just have to prove that f_x is continuous.. Let $(y_\alpha)_\alpha \subset Y$ be a net that converges to $y \in Y$; then, $f_x(y_\alpha) = (x, y_\alpha)$ converges to $f_x(y) = (x,y)$, proving that f is continuous. So, if A open (respec. closed) implies A_x is open (respec. closed).

(4) One implication follows from item (2). Now, let A be a closed subset such that A_x^C is dense. Since A_x is closed, by item (3), it follows that

$$\text{int}(\text{Cl}\, A_x) = \text{int}\, A_x$$

Since $Y = \text{Cl}(A_x^C)$, it follows that $\text{int}\, A_x = \varnothing$, and A_x is nowhere dense.

■

Lemma A.3. *Let X be a Baire Space and let Y be second countable. Let $A \subset X \times Y$ be a closed, nowhere dense set. Then,*

$$\{x \in X : A_x \textit{ is nowhere dense}\}$$

is a dense G_δ set.

Proof. Let $A \subset X \times Y$ be closed and nowhere dense, let $(V_n)_n$ be a countable basis for Y and let $U = A^C$; then, U is open and dense. Set, for each $n \in \mathbb{N}$

$$W_n = \{x \in X : U_x \cap V_n \neq \varnothing\}\,,$$

and note that

$$\begin{aligned} W_n &= \{x \in X : \exists\, y \in V_n \text{ such that } (x,y) \in U\} = \\ &= \{x \in X : \exists\, y \in Y \text{ such that } (x,y) \in U \cap (X \times V_n)\} \\ &= \pi_X(U \cap (X \times V_n))\,, \end{aligned}$$

where π_X stands for the projection over X. Since projections are open maps, we conclude that W_n is open. We claim that W_n is also dense.

Suppose that this is not the case and note that $(X \setminus \text{Cl}(W_n)) \times V_n$ is a non-empty open set. Moreover, if $(x,y) \in U \cap \big((X \setminus \text{Cl}(W_n)) \times V_n\big)$ then $x \in X \setminus \text{Cl}(W_n)$, and so $x \notin W_n$, which implies $U_x \cap V_n = \varnothing$. Thus, there does not exist $z \in V_n$ such that $(x,z) \in U$, which is absurd, since $(x,y) \in U$, and we conclude that $U \cap \big((X \setminus \text{Cl}(W_n)) \times V_n\big) = \varnothing$. Then, $(X \setminus \text{Cl}(W_n)) \times V_n$ is a non-empty open set disjoint from U, contradicting the fact that U is dense. Hence, W_n is dense.

It follows from Lemma A.2

$$A_x \text{ is nowhere dense if, and only if, } U_x \text{ is dense}\,,$$

so

$$\{x \in X : A_x \text{ is nowhere dense}\} = \{x \in X : U_x \text{ is dense}\} = \bigcap_{n\in\mathbb{N}} W_n\,.$$

The result now follows from the fact that X is a Baire space.

■

Corollary A.4. *Let X be a Baire Space and let Y be second countable. Let $A \subset X \times Y$ be a nowhere dense set. Then,*

$$\{x \in X : A_x \textit{ is nowhere dense}\}$$

is a co-meager set.

Proof. If A is a nowhere dense set, then $\mathrm{Cl}(A)$ is also nowhere dense. It follows from Lemma A.3 that $\{x \in X : (\mathrm{Cl}\,A)_x$ is nowhere dense$\}$ is a dense G_δ set and in particular co-meager.

We claim that

$$\{x \in X : (\mathrm{Cl}\,A)_x \text{ is nowhere dense}\} \subset \{x \in X : A_x \text{ is nowhere dense}\}\,.$$

Let $x \in X$ be such that $(\mathrm{Cl}\,A)_x$ is nowhere dense, i.e., $\mathrm{int}\,\mathrm{Cl}(\mathrm{Cl}\,A)_x = \varnothing$. It follows from Lemma A.2 that $\mathrm{int}(\mathrm{Cl}\,A)_x = \varnothing$.

We want to prove that A_x is nowhere dense, i.e., $\mathrm{int}(\mathrm{Cl}(A_x)) = \varnothing$. It suffices to prove that $\mathrm{Cl}(A_x) \subset (\mathrm{Cl}\,A)_x$.

Let $y \in \mathrm{Cl}\,A_x$; then, there exists a net $(y_\alpha)_{\alpha\in\Lambda} \subset A_x$ such that $y_\alpha \to y$. We have that $((x, y_\alpha))_{\alpha\in\Lambda} \subset A$ and $(x, y_\alpha) \to (x, y)$, so $(x, y) \in \mathrm{Cl}\,A$. This prove that that $y \in (\mathrm{Cl}\,A)_x$, and therefore the inclusion $\mathrm{Cl}\,A_x \subset (\mathrm{Cl}\,A)_x$.

Hence, A_x is nowhere dense, and so

$$\{x \in X : (\mathrm{Cl}\,A)_x \text{ is nowhere dense}\} \subset \{x \in X : A_x \text{ is nowhere dense}\}\,.$$

Since $\{x \in X : (\mathrm{Cl}\,A)_x$ is nowhere dense$\}$ is co-meager, $\{x \in X : A_x$ is nowhere dense$\}$ is also co-meager.

■

Definition A.5. *Let X be a non-empty set, let Y be a topological space, let $A \subset X \times Y$ and let U be a non-empty open set in Y. We define*

$$A^{\triangle U} = \{x \in X : A_x \textit{ is non-meager in } U\}$$

and

$$A^{*U} = \{x \in X : A_x \textit{ is co-meager in } U\}\,.$$

Remark A.6. *If U is a Baire space it follows from Definition A.5 that $A^{*U} \subset A^{\triangle U}$.*

Lemma A.7. *Let X be a Baire Space, let Y be a second countable Baire space and let $A \subset X \times Y$ be such that it satisfies the Baire property. Then, the following statements are equivalent:*

1. *A is meager;*

2. $\{x \in X : A_x$ *is meager*$\}$ *is co-meager.*

Proof. (a) If A is meager, then we can write $A = \bigcup_n A_n$ where A_n is nowhere dense for every $n \in \mathbb{N}$. It follows from Corollary A.4 that $\{x \in X : (A_n)_x$ is nowhere dense$\}$ is a co-meager set.

Since nowhere dense implies meager, we have that

$$\{x \in X : (A_n)_x \text{ is nowhere dense}\} \subset \{x \in X : (A_n)_x \text{ is meager}\}$$

and then $\{x \in X : (A_n)_x$ is meager$\}$ is co-meager. In particular, the set $\bigcap_n \{x \in X : (A_n)_x$ is meager$\}$ is co-meager. We claim that

$$\bigcap_n \{x \in X : (A_n)_x \text{ is meager}\} \subset \{x \in X : A_x \text{ is meager}\}$$

Namely, let $x \in X$ be such that $(A_n)_x$ is meager for every $n \in \mathbb{N}$. Note that if $y \in A_x$, then $(x, y) \in A$ and there exists $m \in \mathbb{N}$ such that $(x, y) \in A_m$, so $y \in (A_m)_x$. Therefore, $A_x \subset \bigcup_n (A_n)_x$.

Since every $(A_n)_x$ is meager, we conclude that $\bigcup_n (A_n)_x$ is also meager, and then A_x is meager. We conclude that $\bigcap_n \{x \in X : (A_n)_x$ is meager$\} \subset \{x \in X : A_x$ is meager$\}$, and so $\{x \in X : A_x$ is meager$\}$ is co-meager.

(b) Suppose that A is non-meager. Since A has the BP, by Lemma A.1, there exists a non-empty open $U \subset X$ and a non-empty open $V \subset Y$ such that A is co-meager in $U \times V$. So, it follow from the previous discussion that $\{x \in U : A_x$ is meager in $V\}$ is co-meager in U. But

$$\{x \in U : A_x \text{ is meager in } V\} \subset \{x \in X : A_x \text{ is meager in } V\} = A^{*V},$$

and then A^{*V} is co-meager in U.

Note that U is non-meager: if U is meager, since X is a Baire space, it follows that U has empty interior, an absurd since U is a non-empty open set.

Since U is non-meager, it follows that A^{*V} is non-meager. Namely, if A^{*V} is a meager set, then A^{*V} is meager in U, an absurd since U is a Baire space.

We claim that $A^{\triangle Y}$ is non-meager. Note that V is a non-empty set in a Baire space, so V is also a Baire space, and then $A^{*V} \subset A^{\triangle V} \subset A^{\triangle Y}$. If $A^{\triangle Y}$ is a meager set, then A^{*V} is also meager, an absurd.

On the other hand, if $\{x \in X : A_x$ is meager$\}$ is co-meager, then its complement is meager in X. But the complement of this set is

$$\{x \in X : A_x \text{ is non-meager}\} = A^{\triangle Y},$$

contradicting the fact that $A^{\triangle Y}$ is co-meager in Y. Then, A is meager.

∎

We are finally ready to prove the Kuratowski-Ulam Theorem.

The Kuratowski-Ulam Theorem. *Let X, Y be second countable Baire spaces and suppose that $A \subset X \times Y$ has the Baire property. Then, the following statements are equivalent.*

1. A is meager (respec. co-meager);

2. $\{x \in X : A_x$ *is meager (respec. co-meager)*$\}$ *is co-meager;*

3. $\{y \in Y : A^y$ *is meager (respec. co-meager)*$\}$ *is co-meager.*

Proof. (a) Since X and Y are both second countable and Baire spaces, it follows from Lemma A.7 that the following statements are equivalent:

1. A is meager;
2. $\{x \in X : A_x$ is meager$\}$ is co-meager;
3. $\{y \in Y : A^y$ is meager$\}$ is co-meager.

(b) Now, if A is co-meager then A^C is meager and then the following statements are equivalent

1. A is co-meager;
2. $\{x \in X : A_x^C$ is meager$\}$ is co-meager;
3. $\{y \in Y : (A^y)^C$ is meager$\}$ is co-meager.

Now, it follows from Lemma A.2 that

$$\{x \in X : A_x^C \text{ is meager}\} = \{x \in X : A_x \text{ is co-meager}\}$$

and

$$\{y \in Y : (A^y)^C \text{ is meager}\} = \{y \in Y : A^y \text{ is co-meager}\},$$

which shows that the following statements are also equivalent

1. A is co-meager;
2. $\{x \in X : A_x$ is co-meager$\}$ is co-meager;
3. $\{y \in Y : A^y$ is co-meager$\}$ is co-meager.

■

B. The characterization of Polish spaces through Strong Choquet spaces

B.1 A world of definitions

We follow the strategy presented in [12].

Let A be a non-empty set and let $n \in \mathbb{N}$. We denote ny A^n the set of finite n-sequences $s = (s_0, s_1, \ldots, s_{n-1})$ of elements of A. When $n = 0$, $A^0 = \{\varnothing\}$, where $\varnothing$ denotes the empty sequence.

If $s \in A^n$ and $m \leqslant n$ we set $s|m = (s_0, \ldots, s_{m-1})$ (if $m = 0$, then $s|0 = \varnothing$). If s, t are finite sequences from A, we say that s is an initial segment of t or that t is an extension of s if there exists $m \leqslant \text{length}(t)$ such that $s = t|m$ (we write $s \subset t$).

Two finite sequences are compatible if one of them is an initial segment of the other; otherwise, we say that they are incompatible. We denote two incompatible sequences s, t by $s \perp t$.

Let $A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A_n$ and let $A^{\mathbb{N}}$ be the set of all infinite sequences of elements of A. The concatenation of two finite sequences $s = (s_i)_{i<n}$ and $t = (t_j)_{j<m}$ is $s \wedge t = (s_0, s_1, \ldots, s_{n-1}, t_0, t_1, \ldots, t_{m-1})$ (when we concatenate a sequence s with a 1-sequence (a), we usually write $s \wedge a$ instead of $s \wedge (a)$).

Definition B.1. *A tree on A is a subset $T \subset A^{<\mathbb{N}}$ closed under initial segments, i.e., if $t \in T$ and $s \subset t$, then $s \in T$. (in particular, $\varnothing \in T$ if T is non-empty).*

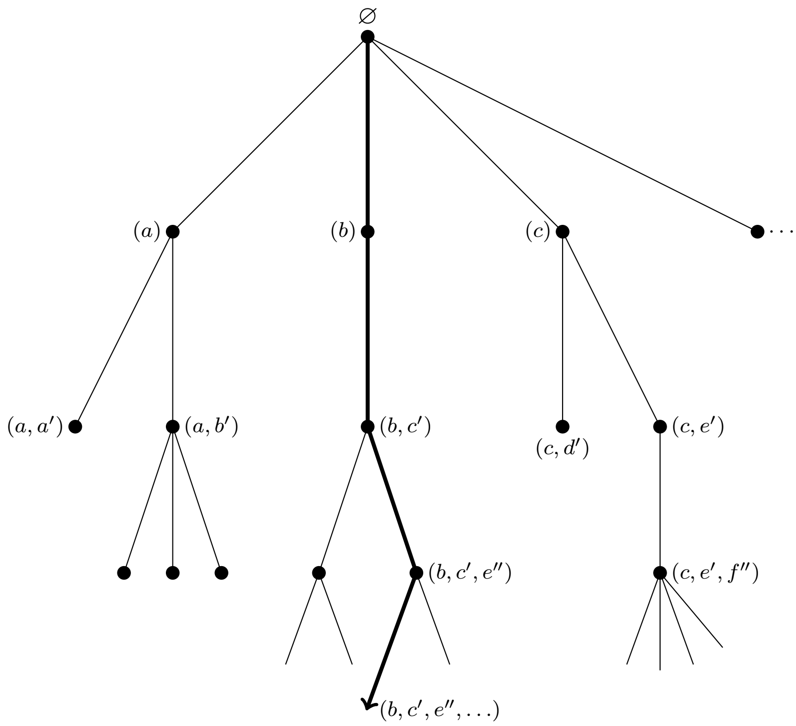
We call the elements of T the nodes of T. An infinite branch of T is a sequence $x \in A^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $x|n \in T$.

The body of T, denoted by $[T]$, is the set of all infinite branches of T, i.e.,

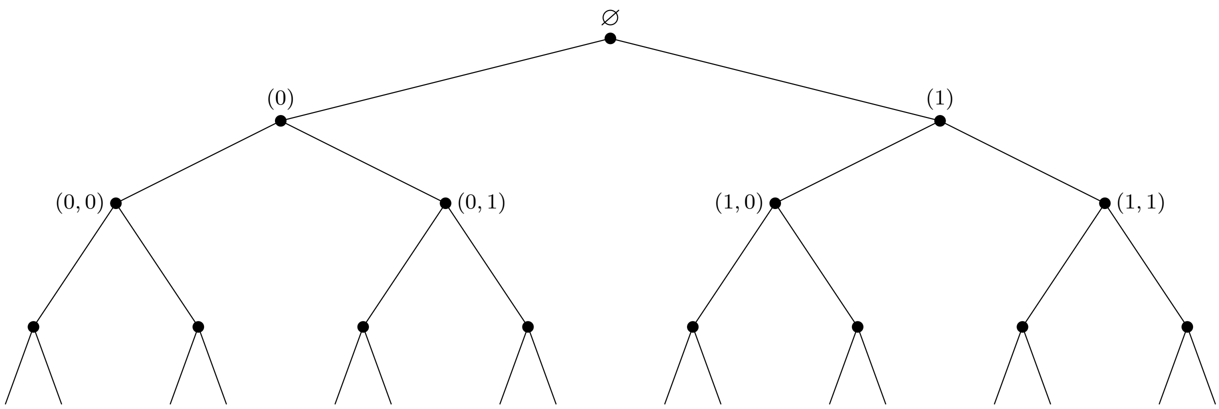
$$[T] = \{x \in A^{\mathbb{N}} : \forall\, n \in \mathbb{N}\; x|n \in T\}\,.$$

We call a tree T pruned if every $s \in T$ has a proper extension $t \supsetneq s$, $t \in T$.

Obviously, this notion of tree is different from the notion presented in Graph Theory; however, some intuition is also present here. Usually, we may visualize trees as follows:



The bold line represents an infinite branch $(b, c', e'', \dots) \in [T]$. The tree above is not pruned; a good example of a pruned tree is the full binary tree $\{0, 1\}^{<\mathbb{N}}$ pictured below:



B.2 König's Lemma

Definition B.2. *If T is a tree on A, we call T finite splitting if for every $s \in T$, there exist at most finitely many $a \in A$ such that $s \wedge a \in T$.*

König's Lemma. *Let T be a tree on A. If T is finite splitting then, $[T] \neq \varnothing$ if, and only if, T is infinite.*

Proof. (a) Suppose that $[T] \neq \varnothing$. Then, there exists an infinite branch in T, i.e., there exists $x \in A^{\mathbb{N}}$ such that $x|n \in T$ for every $n \in \mathbb{N}$. In particular, $\{x|n : n \in \mathbb{N}\} \subset T$, and so T is infinite.

(b) Suppose that $[T] = \varnothing$. Note that $\varnothing$ splits in a finite number of nodes, say $(a_1), \dots, (a_m) \in T$. Again, each one of these nodes (a_i) splits in a finite number of nodes;

thus, by induction, each (a_i) splits in a finite number of paths. Since the body of T is empty, we conclude that these paths will end in a finite time, so T is finite.

■

B.3 Strong Choquet games and spaces

Definition B.3. *Let X be a non-empty topological space. The strong Choquet game G_X^s is defined as follows:*

$$\begin{array}{llllll} I & x_0, U_0 & & x_1, U_1 & & \cdots \\ II & & V_0 & & V_1 & \cdots \end{array}$$

Players I and II take turns playing non-empty open sets of X with the requirement of I playing a set $U_n \subset V_{n-1}$ with a point $x_n \in U_n$, while II must play $V_n \subset U_n$ with $x_n \in V_n$. So, one has

$$U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots$$

with $x_n \in U_n, V_n$.

Player II wins this run of the game if $\bigcap_n V_n\big(= \bigcap_n U_n\big) \neq \varnothing$, and player I wins if $\bigcap_n V_n\big(= \bigcap_n U_n\big) = \varnothing$.

A strategy for a player in this game is a "rule" that tells him how to play. For example, a strategy to player II works as follows: given the player I previous moves $(x_0, U_0), \ldots, (x_n, U_n)$ tell how II have to choose V_n. Formally, this is defined as follows.

Definition B.4. *Let T be the tree of legal positions in the strong Choquet game G_X^s, i.e., T consists of all finite sequences $((x_0, U_0), V_0, (x_1, U_1), \ldots, V_n)$ and $((x_0, U_0), V_0, \ldots, (x_n, U_n))$ where U_i and V_i are non-empty open subsets of X, $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots \supset V_n$ and $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \cdots \supset U_n$ with $x_i \in U_i, V_i$ for every $i \in \mathbb{N}$. A strategy for II in G_X^s is a subtree $\sigma \subset T$ such that*

1. *σ is non-empty;*
2. *If $((x_0, U_0), V_0, (x_1, U_1), \ldots, V_n) \in \sigma$, then for each open non-empty $U_{n+1} \subset V_n$ and $x_{n+1} \in U_{n+1}$, $((x_0, U_0), V_0, \ldots, V_n, (x_{n+1}, U_{n+1})) \in \sigma$;*
3. *If $((x_0, U_0), V_0, \ldots, (x_n, U_n)) \in \sigma$, then for a unique $V_n \subset U_n$ with $x_n \in V_n$, $((x_0, U_0), V_0, \ldots, (x_n, U_n), V_n) \in \sigma$.*

We say that a position $p \in T$ is compatible with σ if $p \in \sigma$. A run of the game $((x_0, U_0), V_0, \ldots)$ is compatible with σ if $((x_0, U_0), V_0, \ldots) \in [\sigma]$. The strategy σ is a winning strategy for II if he wins every run compatible with σ.

A space X is called strong Choquet if player II has a winning strategy in G_X^s. The main goal of this appendix is to use strong Choquet spaces to characterize the Polish spaces. This approach of using topological games to characterize spaces with a given property is really interesting and have many results. For instance, we can characterize Baire spaces using another type of game, called Choquet game (for interested readers, we recommend [12]). We need some auxiliary results in what follows.

Lemma B.5. *A non-empty completely metrizable space X is strong Choquet.*

Proof. Denote the metric of X by d and suppose that the n-th move of player I is (x_n, U_n), with U_n open and $x_n \in U_n$. Since U_n is open, there exists $\varepsilon_n > 0$ such that $B(x_n, \varepsilon_n) \subset U_n$. Set $\delta_n = \min\{\varepsilon_n, 1/n\}$ and let the response of player II to be $V_n = B(x_n, \delta_n)$. Naturally, V_n is open, $x_n \in V_n$ and $V_n \subset U_n$.

Let us prove that with this strategy the player II always wins. Fix a run of the game $((x_0, U_0), V_0, \dots)$ with II using the above strategy. We claim that the sequence $(x_n)_n$ is Cauchy. This happens because for each $m, n \geqslant k$, $x_m, x_n \in V_k = B(x_k, \delta_k)$, and so

$$d(x_n, x_m) < 2\delta_k \leqslant \frac{2}{k} \xrightarrow{k\to\infty} 0$$

Since X is complete, there exists $x \in X$ such that $\lim x_n = x$. For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\operatorname{Cl} V_m \subset V_n$, so $x \in \operatorname{Cl}(V_m) \subset V_n$; moreover, $x \in V_k$ for every $k < n$. Since $n \in \mathbb{N}$ is arbitrary, it follows that $x \in V_k$ for every $k \in \mathbb{N}$.

We conclude that $x \in \bigcap_k V_k$, and so player II wins this run of the game. Hence, X is strong Choquet.

■

Lemma B.6. *Let (Y, d) be a separable metric space. Let $\mathcal{U}$ be a family of non-empty open sets in Y. Then, $\mathcal{U}$ has a point-finite refinement $\mathcal{V}$, i.e., $\mathcal{V}$ is a family of non-empty open sets with $\bigcup_{U\in\mathcal{U}} U = \bigcup_{V\in\mathcal{V}} V$, such that for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$ and for every $y \in Y$ the set $\{V \in \mathcal{V} : y \in V\}$ is finite. Moreover, given $\varepsilon > 0$ we can also assume that* $\operatorname{diam}(V) < \varepsilon$ *for every $V \in \mathcal{V}$.*

Proof. The first thing to note is that Y is second countable. Since it is separable, there exists a dense countable subset $\mathcal{B}$, and so the set $\{B(x, \varepsilon) : x \in \mathcal{B}, \varepsilon \in \mathbb{Q}\setminus\{0\}\}$ is a countable basis of Y.

By the previously paragraph, there exists a sequence $(U_n)_n$ of open sets such that $\bigcup_n U_n = \bigcup_{U\in\mathcal{U}} U$ and that for every $n \in \mathbb{N}$, there exists $U \in \mathcal{U}$ such that $U_n \subset U$. Moreover, given $\varepsilon > 0$ we can assume that $\operatorname{diam}(U_n) < \varepsilon$.

Let $U_n = \bigcup_{p\in\mathbb{N}} U_n^{(p)}$ with $U_n^{(p)}$ open, $U_n^{(p)} \subset U_n^{(p+1)}$ and $\operatorname{Cl} U_n^{(p)} \subset U_n$. One way to do this is to note that $U_n = B(x, \varepsilon)$ for some $x \in \mathcal{B}$ and some $\varepsilon \in \mathbb{Q} \setminus \{0\}$. Then, define $U_n^{(p)} = B(x, \varepsilon - \varepsilon/(p+1))$, so $U_n = \bigcup_{p\in\mathbb{N}} U_n^{(p)}$ and the properties follow.

Set $V_m = U_m \setminus \bigcup_{n<m}(\operatorname{Cl} U_n^{(m)})$ and note that since every finite union of closed sets is closed, it follows that V_m is open.

We claim that $\bigcup_n V_n = \bigcup_n U_n$. Namely, $V_m \subset U_m$, so $\bigcup_n V_n \subset \bigcup_n U_n$. On the other hand, let $x \in \bigcup_n U_n$ and let $m \in \mathbb{N}$ be the least index for which $x \in U_m$. We claim that $x \in V_m$. Of course, $x \in U_m$, suppose that $x \in \bigcup_{n<m}(\operatorname{Cl} U_n^{(m)})$. Then, there exists $j < m$ such that $x \in \operatorname{Cl}(U_j^{(m)}) \subset U_j$, which is absurd since m is the least index for which $x \in U_m$. We conclude that $x \in V_m$, and so $x \in \bigcup_n V_n$.

Let $\mathcal{V} = \{V_n : V_n \neq \varnothing\}$. It follow from the previous discussion that $\bigcup_{U\in\mathcal{U}} U = \bigcup_{V\in\mathcal{V}} V$. Moreover, $V_n \subset U_n$, and by construction there exists $U \in \mathcal{U}$ such that $U_n \subset U$, so $V_n \subset U$. It remains to prove that for every $y \in Y$, the set $\{V \in \mathcal{V} : y \in V\}$ is finite.

Let $y \in Y$. If $\{V \in \mathcal{V} : y \in V\} = \varnothing$ there is nothing to prove, so suppose that this set is not empty. Then, there exists $n \in \mathbb{N}$ such that $y \in V_n \subset U_n = \bigcup_{p\in\mathbb{N}} U_n^{(p)}$. We conclude that there exists $p \in \mathbb{N}$ such that $y \in U_n^{(p)}$, and since

$$U_n^{(p)} \subset U_n^{(p+1)} \subset U_n^{(p+2)} \subset \cdots,$$

we conclude that $y \in U_n^{(j)}$ for every $j \geqslant p$. Then, if $m > p$ and $m > n$,

$$y \in \bigcup_{j<m} (\operatorname{Cl} U_j^{(m)}),$$

and so $y \notin V_m$ for every $m > \max\{p, n\}$. We conclude that if $V_m \in \{V \in \mathcal{V} : y \in V\}$, then $m \leqslant \max\{p, n\}$, and hence the set $\{V \in \mathcal{V} : y \in V\}$ is finite.

■

Theorem B.7. *Let X be a non-empty separable metrizable space and let $\hat{X}$ be a Polish space in which X is dense. Then, the following statements are equivalent:*

1. *X is strong Choquet;*
2. *X is G_δ in $\hat{X}$;*
3. *X is Polish.*

Proof. We know that (2) implies (3) and by Lemma B.5, it follows that (3) implies (1). So, it remains to prove that (1) implies (2).

(a) Suppose that X is strong Choquet, fix a metric d on $\hat{X}$ and a winning strategy σ for player II in G_X^s. In this step, we will construct a tree S of sequences of the form $(x_0, (V_0, \hat{V}_0), x_1, \dots, x_n)$ or $(x_0, (V_0, \hat{V}_0), x_1, \dots, x_n, (V_n, \hat{V}_n))$, satisfying:

- V_i is open in X and $\hat{V}_i$ is open in $\hat{X}$;
- $x_i \in V_i$ and $x_i \in \hat{V}_{i-1} \cap X$ (with $\hat{V}_{-1} = \hat{X}$);
- $\hat{V}_i \cap X \subset V_i$;
- $\hat{V}_0 \supset \hat{V}_1 \supset \cdots$;
- If $U_n = \hat{V}_{n-1} \cap X$ then the infinite sequence $((x_0, U_0), V_0, (x_1, U_1) \dots)$ is compatible with σ.

Additionally, for each $p = (x_0, (V_0, \hat{V}_0), x_1, \dots, x_{n-1}, (V_{n-1}, \hat{V}_{n-1})) \in S$ (including the empty sequence), if $\hat{\mathcal{V}}_p = \{\hat{V}_n : (x_0, (V_0, \hat{V}_0), x_1, \dots, (V_{n-1}, \hat{V}_{n-1}), x_n, (V_n, \hat{V}_n)) \in S\}$, then:

- $X \cap \hat{V}_{n-1} \subset \bigcup_{V \in \hat{\mathcal{V}}_p} V$;
- for every $\hat{V}_n \in \hat{\mathcal{V}}_p$, $\operatorname{diam}(\hat{V}_n) < 2^{-n}$;
- for every $\hat{x} \in \hat{X}$, there exist at most finitely many $(x_n, (V_n, \hat{V}_n))$ such that

$$(x_0, (V_0, \hat{V}_0), x_1, \dots, (V_{n-1}, \hat{V}_{n-1}), x_n, (V_n, \hat{V}_n)) \in S$$

and $\hat{x} \in \hat{V}_n$.

The construction uses induction. We begin with putting $\varnothing \in S$.

Fix $p = (x_0, (V_0, \hat{V}_0), \dots, (V_{n-1}, \hat{V}_{n-1})) \in S$ and $x \in \hat{V}_{n-1} \cap X$. Choose $V_{p \wedge x}$ as the only answer of player II in σ in the run of the game $((x_0, X), V_0, (x_1, \hat{V}_0 \cap X) \dots, (x, \hat{V}_{n-1} \cap X))$.

Since $V_{p\wedge x}$ is open in X and X is a subspace of $\hat{X}$, there exists an open set $Z_{p\wedge x} \subset \hat{X}$ such that $Z_{p\wedge x} \cap X = V_{p\wedge x}$. Set $O_{p\wedge x} = Z_{p\wedge x} \cap \hat{V}_{n-1}$ and note that $O_{p\wedge x}$ is a non-empty open set in $\hat{X}$ (given that $x \in O_{p\wedge x}$) that satisfies

$$O_{p\wedge x} = Z_{p\wedge x} \cap \hat{V}_{n-1} \subset \hat{V}_{n-1}$$

and

$$O_{p\wedge x} \cap X = Z_{p\wedge x} \cap \hat{V}_{n-1} \cap X = V_{p\wedge x} \cap \hat{V}_{n-1} \subset V_{p\wedge x}\,.$$

So, $x, V_{p\wedge x}$ and $O_{p\wedge x}$ are candidates to be some of the x_n, V_n and $\hat{V}_n$, respectively. Unfortunately, we cannot guarantee the other properties we want for S with these choices. However, we can adjust this collection to be what we want. By Zorn's Lemma, we can construct the maximal collection $\mathcal{O}_{p\wedge x}$ of sets $\hat{O}$ satisfying:

- $\hat{O}$ is open in $\hat{X}$;
- $\hat{O} \cap X \subset V_{p\wedge x}$;
- $\hat{V}_{n-1} \supset \hat{O}$.

The previously constructed $O_{p\wedge x}$ are of course in $\mathcal{O}_{p\wedge x}$. Now, let

$$\mathcal{O}_p = \bigcup_{x\in\hat{V}_{n-1}\cap X} \mathcal{O}_{p\wedge x}$$

and use Lemma B.6 to refine $\mathcal{O}_p$ into a family $\mathcal{U}_p$ of non-empty open sets satisfying the following properties: $\bigcup_{O\in\mathcal{O}_p} O = \bigcup_{U\in\mathcal{U}_p} U$; for every $U \in \mathcal{U}_p$ there exists $O \in \mathcal{O}_p$ such that $U \subset O$; for every $\hat{x} \in \hat{X}$ the set $\{U \in \mathcal{U}_p : \hat{x} \in U\}$ is finite and $\operatorname{diam}(U) < 2^{-n}$ for every $U \in \mathcal{U}_p$.

Moreover, each $\hat{V} \in \mathcal{U}_p$ satisfyies the following properties:

- $\hat{V}$ is open in $\hat{X}$;
- since there exists $O \in \mathcal{O}_p$ such that $\hat{V} \subset O$ and for some $x \in \hat{V}_{n-1} \cap X$ we have that $O \cap X \subset V_{p\wedge x}$, we conclude that $\hat{V} \cap X \subset V_{p\wedge x}$;
- since there exists $O \in \mathcal{O}_p$ such that $\hat{V} \subset O$ and $O \subset \hat{V}_{n-1}$, it follows that $\hat{V} \subset \hat{V}_{n-1}$.

For every $U \in \mathcal{U}_p$, there exists $O \in \mathcal{O}_p$ such that $U \subset O$, and for this O there exists $x \in \hat{V}_{n-1} \cap X$ such that $O \in \mathcal{O}_{p\wedge x}$. Choose exactly one x satisfying $O \in \mathcal{O}_{p\wedge x}$ and put $p \wedge x$ and $p \wedge (x, (V_{p\wedge x}, U))$ in S. We claim that this definition of S satisfies all the previously stated properties.

By using the previous construction, it is obvious that

- V_i is open in X and $\hat{V}_i$ is open in $\hat{X}$;
- $x_i \in V_i$ and $x_i \in \hat{V}_{i-1} \cap X$ (with $\hat{V}_{-1} = \hat{X}$);
- $\hat{V}_i \cap X \subset V_i$;
- $\hat{V}_0 \supset \hat{V}_1 \supset \cdots$;
- If $U_n = \hat{V}_{n-1} \cap X$ then the infinite sequence $((x_0, U_0), V_0, (x_1, U_1) \dots)$ is compatible with σ.

In order to prove the other property, fix $p = (x_0, (V_0, \hat{V}_0), x_1, \dots, x_{n-1}, (V_{n-1}, \hat{V}_{n-1})) \in S$ and $\hat{\mathcal{V}}_p = \{\hat{V}_n : (x_0, (V_0, \hat{V}_0), x_1, \dots, (V_{n-1}, \hat{V}_{n-1}), x_n, (V_n, \hat{V}_n)) \in S\}$. The first thing to note is the obvious fact that $\hat{\mathcal{V}}_p = \mathcal{U}_p$.

We begin showing that $X \cap \hat{V}_{n-1} \subset \bigcup_{V\in\hat{\mathcal{V}}_p} V$. If $x \in X \cap \hat{V}_{n-1}$ then by the previous discussion let the set $O_{p\wedge x}$ be as in our construction. Thus, $x \in O_{p\wedge x}$ and $O_{p\wedge x} \in \mathcal{O}_{p\wedge x} \subset \mathcal{O}_p$, so

$$x \in \bigcup_{O\in\mathcal{O}_p} O = \bigcup_{U\in\mathcal{U}_p} U = \bigcup_{V\in\hat{\mathcal{V}}_p} V\,.$$

This shows that $x \in \bigcup_{V \in \hat{\mathcal{V}}_p} V$.

The fact that every $\hat{V}_n \in \hat{\mathcal{V}}_p$ satisfies that $\operatorname{diam}(\hat{V}_n) < 2^{-n}$ comes from $\hat{\mathcal{V}}_p = \mathcal{U}_p$ and the fact that every element of $\mathcal{U}_p$ has its diameter less than 2^{-n}.

If $\hat{x} \in \hat{X}$, it follows from our construction that the set $\{U \in \mathcal{U}_p : \hat{x} \in U\}$ is finite. And since for each U, there exists a unique $x \in \hat{V}_{n-1} \cap X$ such that $p \wedge (x, (V_{p \wedge x}, U)) \in S$, we conclude that there are at most finitely many $(x_n, (V_n, \hat{V}_n))$ with

$$(x_0, (V_0, \hat{V}_0), x_1, \ldots, (V_{n-1}, \hat{V}_{n-1}), x_n, (V_n, \hat{V}_n)) \in S$$

and $\hat{x} \in \hat{V}_n$.

(b) Let $Z_n = \{\hat{V}_n : (x_0, (V_0, \hat{V}_0), x_1, \ldots, x_n, (V_n, \hat{V}_n)) \in S\}$ and $W_n = \bigcup_{\hat{V}_n \in Z_n} \hat{V}_n$. Since each $\hat{V}_n$ is open in $\hat{X}$, W_n is also open in $\hat{X}$. We will prove by induction that $X \subset W_n$ for each $n \in \mathbb{N}$.

For the case $n = 0$, let $p = \varnothing$; then, $\hat{\mathcal{V}}_p = \{\hat{V}_0 : (x_0, (V_0, \hat{V}_0)) \in S\}$. It follows from item (a) that $X \cap \hat{V}_{-1} \subset \bigcup_{V \in \hat{\mathcal{V}}_p} V$, but then $X = X \cap \hat{V}_{-1}$, so

$$X \subset \bigcup_{V \in \hat{\mathcal{V}}_p} V = W_0$$

Now, assume that $X \in W_m$ for every $m \leqslant n$. Let $x \in X$ and note that since $X \subset W_n$, there exist $\hat{V}_n$ such that $x \in \hat{V}_n$, and a sequence $(x_0, \ldots, x_n, (V_n, \hat{V}_n)) \in S$, label it by p. By the construction of S, it follows that $X \cap \hat{V}_n \subset \bigcup_{V \in \hat{\mathcal{V}}_p} V \subset W_{n+1}$, and so $x \in W_{n+1}$. This proves that $X \subset W_{n+1}$.

So, $X \subset W_n$ for every $n \in \mathbb{N}$, that is, $X \subset \bigcap_n W_n$.

(c) We claim that $X = \bigcap_n W_n$. Namely, let $\hat{x} \in \bigcap_n W_n$ and let $S_{\hat{x}}$ be the subtree of S consisting of all initial segments of the sequences $(x_0, (V_0, \hat{V}_0), x_1, \ldots, x_n, (V_n, \hat{V}_n)) \in S$ for which $\hat{x} \in \hat{V}_n$. Since $\hat{x} \in W_n$ for every $n \in \mathbb{N}$, there exists $(x_0, (V_0, \hat{V}_0), x_1, \ldots, x_n, (V_n, \hat{V}_n)) \in S$ for which $\hat{x} \in \hat{V}_n$ for every $n \in \mathbb{N}$. Hence, $S_{\hat{x}}$ is infinite.

For each $p = (x_0, (V_0, \hat{V}_0), x_1, \ldots, x_{n-1}, (V_{n-1}, \hat{V}_{n-1})) \in S$ it follows from item (a) that there exist at most finitely many $(x_n, (V_n, \hat{V}_n))$ with $p \wedge (x_n, (V_n, \hat{V}_n)) \in S$ and $\hat{x} \in \hat{V}_n$. This proves that $S_{\hat{x}}$ is finite splitting, and so by König's Lemma, we conclude that $[S_{\hat{x}}] \neq \varnothing$.

Now, let $q = (x_0, (V_0, \hat{V}_0), \ldots) \in [S_{\hat{x}}]$. Item (a) implies that $((x_0, X), V_0, (x_1, \hat{V}_0 \cap X) \ldots)$ is compatible with σ. Since σ is a winning strategy for player II, we conclude that $\bigcap_n (\hat{V}_n \cap X) \neq \varnothing$.

On the other hand, $\hat{x} \in \hat{V}_n$ for every $n \in \mathbb{N}$ and $\operatorname{diam}(\hat{V}_n) < 2^{-n}$, so $\bigcap_n \hat{V}_n = \{\hat{x}\}$ and

$$X \cap \{\hat{x}\} = X \cap \bigcap_n \hat{V}_n = \bigcap_n (X \cap \hat{V}_n) \neq \varnothing\,.$$

This shows that $\hat{x} \in X$, and so $X = \bigcap_n W_n$. Since every W_n is an open subset of $\hat{X}$ it follows that X is G_δ subset of $\hat{X}$.

■

Corollary B.8. *A nonempty, second countable topological space is Polish if, and only if, it is T_1, regular and strong Choquet.*

Proof. (a) Suppose that Y is a second countable Polish space. In particular, Y is a second countable and metrizable space, so by Urysohn Metrization Theorem (see Chapter 0) it follows that Y is T_1 and regular. Another thing to note is that we can let Y be simultaneously X and $\hat{X}$ in the previously theorem. In particular, since Y is Polish, we conclude that Y is strong Choquet.

(b) Suppose that Y is a second countable, T_1, regular and strong Choquet space. It follows from Urysohn Metrization Theorem that Y is metrizable. We know that we can find a complete metrizable space $\hat{Y}$ in which Y is dense, and since Y is separable and dense, $\hat{Y}$ is also. Since Y is strong Choquet, it follows from Theorem B.7 that Y is Polish. ■

C. Wold's decomposition

Wold's Decomposition. *Let H be a Hilbert space and let $V : H \longrightarrow H$ be an isometry. Then, there exist K_1, K_2, V-invariant subspaces of H, such that*

$$H = K_1 \oplus K_2\,,$$

where $V\big|_{K_1}$ is unitary and $V\big|_{K_2}$ is the right shift in K_2 (i.e., there exists a Hilbert space K such that $K_2 \cong_U \bigoplus_{i\geqslant 0} K$ and V is a right shift on this space).

Proof. (a) Let H be a Hilbert Space and let V be an isometry in H. We first note that $V^m(H)$ is closed for every $m \in \mathbb{N}$. Namely, let $(V^m(x_n))_{n\in\mathbb{N}}$ and $y \in H$ be such that $\lim_n V^m(x_n) = y$. Since $(V^m(x_n))_{n\in\mathbb{N}}$ converges in H, it is a Cauchy sequence. Now, given that $||V^m(x_n - x_k)|| = ||x_n - x_k||$ we conclude that $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence; let x be its limit. It follows from the continuity of V that $V^m x = y$. Then $y \in V^m(H)$, and $V^m(H)$ is closed.

(b) Let, for each $i \in \mathbb{N}$, $H_i = V^i(H)$. Thus,

$$H = H_0 \supset H_1 \supset H_2 \supset \cdots$$

Now, we use the fact that H_i is closed in H, and consequently in H_{i-1}, to define $M_i = H_i \ominus H_{i+1}$ for each $i \in \mathbb{N}$.

Now we claim that

$$H = \Big(\bigcap_{i\geqslant 0} H_i\Big) \oplus \Big(\bigoplus_{i\geqslant 0}^{\text{inn}} M_i\Big) \tag{C.1}$$

from this, it suffices to set $K_1 = \bigcap_{i\geqslant 0} H_i$ and $K_2 = \bigoplus_{i\geqslant 0}^{\text{inn}} M_i$, and we are done.

First of all, let us show that these spaces are well defined. If $h_1, h_2 \in \bigcap_{i\geqslant 0} H_i$ and $\alpha \in \mathbb{C}$, then for each $i \in \mathbb{N}$ $h_1, h_2 \in H_i$. So, for each $i \in \mathbb{N}$, $\alpha h_1 + h_2 \in H_i$, and consequently $\alpha h_1 + h_2 \in \bigcap_{i\geqslant 0} H_i$. Then, this intersection is a vector subspace and in particular isn't empty.

In order to prove that $\bigoplus_{i\geqslant 0}^{\text{inn}} M_i$ is well defined we need to show that for each $i, j \in \mathbb{N}$ such that $i \neq j$, $M_i \perp M_j$. Without loss of generality, assume $j > i$. Since $M_j \subset H_j \subset H_{i+1}$ and $M_i = H_i \ominus H_{i+1}$ it follows that $M_i \perp M_j$ and we are done.

(c) Let us see prove relation (C.1). Let $x \in H$. Since $H = H_0 = H_1 \oplus M_0$ there exist $h_1 \in H_1$ and $m_0 \in M_0$ such that $x = h_1 + m_0$. If $h_1 \in \bigcap_{i\geqslant 0} H_i$ we are done, given that $M_j \subset \bigoplus_{i\geqslant 0}^{\text{inn}} M_i$ for every $j \in \mathbb{N}$. Otherwise, since $H_1 = H_2 \oplus M_1$, there exist $h_2 \in H_2$ and $m_1 \in M_1$ such that $h_1 = h_2 + m_1$ for some . Now, $x = h_2 + \sum_{i=0}^{1} m_i$. Again, if $h_2 \in \bigcap_{i\geqslant 0} H_i$ we are done, otherwise we continue.

Suppose that this process never ends. Then for each $n \in \mathbb{N}$ it follows that $x =$

$h_n + \sum_{i=0}^{n-1} m_i$, $h_n = h_{n+1} + m_n$, where $h_i \in H_i$ and $m_i \in M_i$ for each $i \in \mathbb{N}$. If we can prove that the sequence $(h_n)_{n\in\mathbb{N}}$ converges to some $h \in \bigcap_{i\geqslant 0} H_i$ we are done; namely, taking the limit in both sides of $x = h_n + \sum_{i=0}^{n-1} m_i$, we get

$$x = h + \sum_{i=0}^{\infty} m_i$$

The series $\sum_{i=0}^{\infty} m_i$ converges, given that for each $n \in \mathbb{N}$

$$\sum_{i=0}^{n} ||m_i||^2 \leqslant ||x||^2$$

(and so the sequence of partial sums is bounded and monotone). Moreover, it follows from the fact that $M_i \perp M_j$, for every $i \neq j$, that

$$\Big|\Big|\sum_{i=0}^{\infty} m_i\Big|\Big|^2 = \sum_{i=0}^{\infty} ||m_i||^2 < \infty\,.$$

So, let us prove that the sequence $(h_n)_n$ converges. If $m \geqslant n$ (without loss of generality), then

$$||h_m - h_n||^2 = \Big|\Big|h_m - \Big(h_m + \sum_{i=n}^{m-1} m_i\Big)\Big|\Big|^2 = \Big|\Big|\sum_{i=n}^{m-1} m_i\Big|\Big|^2 = \sum_{i=n}^{m-1} ||m_i||^2 \xrightarrow{n,m\to\infty} 0\,,$$

proving that this sequence is a Cauchy sequence and so it converges in H to some h. It remains to show that $h \in \bigcap H_i$. Let $i \in \mathbb{N}$, then for each $n \geqslant i$, $h_n \in H_n \subset H_i$, and since H_i is closed it follows that $\lim_n h_n = h \in H_i$. Hence, $h \in \bigcap_{i\geqslant 0} H_i$.

Let $x \in \left(\bigcap_{i\geqslant 0} H_i\right) \cap \left(\bigoplus_{i\geqslant 0}^{\text{inn}} M_i\right)$. We can write $x = \sum_{i\geqslant 0} m_i$, with $m_i \in M_i$ for every $i \in \mathbb{N}$. Note that $x \in H_i$ for every $I \in \mathbb{N}$. In particular, $\langle x, m_i\rangle = 0$ and then

$$||x||^2 = \langle x, \sum_{i\geqslant 0} m_i\rangle = \sum_{i\geqslant 0}\langle x, m_i\rangle = 0\,.$$

Then $x = 0$, and so $\left(\bigcap_{i\geqslant 0} H_i\right) \perp \left(\bigoplus_{i\geqslant 0}^{\text{inn}} M_i\right)$.

(d) Our next step consisting in showing that these spaces are both V-invariant. For each $h \in \bigcap_{i\geqslant 0} H_i$and each $n \in \mathbb{N}$ there exists $z_n \in H$ such that $V^n z_n = h$. In particular, for each $n \in \mathbb{N}$ we have that $V^{n+1} z_n = Vh$, and then for each $n \in \mathbb{N}$, it is true that $Vh \in H_n$. Hence, $Vh \in \bigcap_{i\geqslant 0} H_i$. This shows that $\bigcap_{i\geqslant 0} H_i$ is V-invariant.

Now, let $x = \sum_{n\in\mathbb{N}} m_i \in \bigoplus_{i\geqslant 0}^{\text{inn}} M_i$, with $m_i \in M_i$, so $Vx = \sum_{n\in\mathbb{N}} Vm_i$. We will show that $Vm_i \in M_{i+1}$. Since $m_i \in M_i = H_i \ominus H_{i-1}$, it follows that $Vm_i \in V(H_i) = H_{i+1}$, and so it remains to show that for every $y \in H_i$, $\langle Vm_i, y\rangle = 0$. Let $y \in H_i$, so there exists $y_0 \in H_{i-1}$ such that $y = Vy_0$,and then

$$\langle Vm_i, y\rangle = \langle Vm_i, Vy_0\rangle = \langle m_i, y_0\rangle = 0$$

(here we are using the fact that $m_i \in H_i \ominus H_{i-1}$). Then, $Vm_i \in M_{i+1}$ and we conclude that $Vx \in \bigoplus_{i\geqslant 0}^{\text{inn}} M_i$.

(e) We know that V is an isometry in H; in particular, it is a isometry when we restrict to $V : K_1 \to K_1$. In order to prove the surjectivity, let $x \in K_1$ and note that $x \in H_i$ for every $i \geqslant 0$. Then, for each $i \geqslant 0$ there exists $y_i \in H$ such that $V^i y_i = x$; in particular, $Vy_1 = x$. We will prove that $y_1 \in K_1$. Let $i \geqslant 0$ and note that

$$||y_1 - V^i y_{i+1}|| = ||Vy_1 - V^{i+1} y_{i+1}|| = ||x - x|| = 0\,;$$

then, $V^i y_{i+1} = y_1$, and since i is arbitrary, we conclude that $y_1 \in K_1$. This proves that V is unitary in K_1.

(f) Write $K_2 \cong_U \bigoplus_{i\geqslant 0} M_i$. We have proved that $V(M_i) \subset M_{i+1}$ and with this it follows that

$$V(m_0, m_1, m_2, \dots) = (0, Vm_0, Vm_1, Vm_2, \dots)$$

Note now that $M_i = V^i(H \ominus V(H))$. Namely, let $x \in M_i$, so $x \in H_i$ and $x \perp H_{i+1}$. Since $x \in H_i$ there exists $y \in H$ such that $x = V^i(y)$. We claim that $x \in V^i(H \ominus V(H))$. Let $Vh \in V(H)$, since V is an isometry, it follows that

$$\langle y, Vh\rangle = \langle Vy, V^2 h\rangle = \cdots = \langle V^i(y), V^{i+1}(h)\rangle = \langle x, V^{i+1}(h)\rangle = 0\,,$$

so $y \in H \ominus V(H)$ and then $x \in V^i(H \ominus V(H))$.

On the other hand, if $x \in V^i(H \ominus V(H))$, then there exists $y \in H \ominus V(H)$ such that $V^i(y) = x$. Obviously, $x \in H_i$ by definition. Moreover, if $V^{i+1}h \in H_{i+1}$, then

$$\langle x, V^{i+1}(h)\rangle = \langle V^i(y), V^{i+1}(h)\rangle = \cdots = \langle y, Vh\rangle = 0\,,$$

so $x \in H_i \ominus H_{i+1}$, from which that $M_i = V^i(H \ominus V(H))$.

Recall that V is an isometry, so for each $i \in \mathbb{N}$,

$$\dim M_0 = \dim H \ominus V(H) = \dim V^i(H \ominus V(H)) = \dim M_i\,.$$

Hence, $M_0 \cong_U M_i$ for every $i \in \mathbb{N}$. By setting $K = M_0$, we can write

$$K_2 \cong_U \bigoplus_{i\geqslant 0} M_i \cong_U \bigoplus_{i\geqslant 0} K$$

and V is the right shift over $\bigoplus_{i\geqslant 0} K$. This concludes the proof.

■