

UNIVERSIDADE FEDERAL DE MINAS GERAIS
Engineering School
Graduate Program in Mechanical Engineering

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Torsional Analysis of Geared Turbomachinery Trains

Belo Horizonte
2025

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Torsional Analysis of Geared Turbomachinery Trains

Master's thesis submitted to the Graduate Program in Mechanical Engineering of Universidade Federal de Minas Gerais, in partial fulfilment of the requirements for obtaining the Master's degree in Mechanical Engineering.

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Belo Horizonte
2025

M637t	Milagres, João Carlos Ferreira. Torsional analysis of geared turbomachinery trains [recurso eletrônico] / João Carlos Ferreira Milagres. – 2025. 1 recurso online (113 f. : il., color.) : pdf. Orientador: Marco Túlio Corrêa de Faria. Dissertação (mestrado) – Universidade Federal de Minas Gerais, Escola de Engenharia. Inclui bibliografia. 1. Engenharia mecânica – Teses. 2. Vibrações mecânicas – Teses. 3. Trens de ferro – Teses. 4. Turbomáquinas – Teses. 5. Método dos elementos finitos – Teses. I. Faria, Marco Túlio Corrêa de. II. Universidade Federal de Minas Gerais. Escola de Engenharia. III. Título. CDU: 621(043)
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UNIVERSIDADE FEDERAL DE MINAS GERAIS
ESCOLA DE ENGENHARIA
PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA MECÂNICA

FOLHA DE APROVAÇÃO

"TORSIONAL ANALYSIS OF GEARED TURBOMACHINERY TRAINS"

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Dissertação submetida à Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em Engenharia Mecânica da Universidade Federal de Minas Gerais, constituída pelos Professores: Dr. Marco Túlio Correa de Faria (Orientador-Departamento de Engenharia Mecânica-UFMG), Dra. Maria Lucia Machado Duarte (Departamento de Engenharia Mecânica-UFMG) e Dr David Crombez (TotalEnergies), como parte dos requisitos necessários à obtenção do título de **"Mestre em Engenharia Mecânica"**, na área de concentração de **" Projeto e Sistemas"**.

Dissertação aprovada no dia 10 de dezembro de 2025.

Por:



Documento assinado eletronicamente por **Marco Tulio Correa de Faria, Professor do Magistério Superior**, em 15/12/2025, às 10:05, conforme horário oficial de Brasília, com fundamento no art. 5º do [Decreto nº 10.543, de 13 de novembro de 2020](#).



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ACKNOWLEDGMENTS

To my wife, Viviane, and my daughter, Maria Clara, for their love, support, and understanding during the necessary absences to fulfill my dream of obtaining a master's degree in mechanical engineering.

To my parents, Temístocles and Solange, for their encouragement, support, and prayers for the success of this stage of my career.

To my advisor, Prof. Marco Túlio Corrêa de Faria, for his efforts in keeping alive the important area of Rotor Dynamics at UFMG, and for sharing valuable knowledge about rotating machines acquired over years of dedication to research and teaching.

To Petrobras, for being a leading technology company in our country, which allows its employees to have contact with the technological frontier in various areas of knowledge, including turbomachinery, an area in which I have decided to delve deeper through this postgraduate program.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

RESUMO

Este trabalho apresenta a análise da vibração torcional de trens de turbomáquinas com engrenamento utilizando procedimento computacional baseado nos métodos de elementos finitos e de parâmetros concentrados. A análise é realizada para sistemas acionados por motores elétricos de rotação constante ou por motores com conversores de frequência (VFD), que devido à larga faixa operacional combinada com excitações torcionais adicionais, tornam a tarefa de manter margens de separação quase impossível. O procedimento computacional permite efetuar a análise torcional de trens de turbomáquinas com uma ou múltiplas derivações, baseada nas diretrizes do Instituto Americano do Petróleo (API). Considerações sobre a modelagem dos principais componentes dos trens são aplicadas para obter modelo de elementos finitos com inércias concentradas nos nós. O procedimento computacional consiste em obter o modelo de velocidade única equivalente, aplicar o método dos elementos finitos para obter as matrizes de massa, rigidez e amortecimento do trem e a equação de movimento do sistema. A partir da equação de movimento, obtém-se o problema de autovalor para a estimativa das frequências naturais torcionais e dos modos de vibração e constroem-se os diagramas de Campbell. Para os pontos de interferência identificados no diagrama de Campbell, são estimados na análise forçada em regime permanente os deslocamentos angulares nodais, funções de resposta em frequência, ângulos de torção dos elementos e torques e tensões de cisalhamento em cada elemento, seja a matriz de amortecimento proporcional (pode ser desacoplada pelos autovetores) ou não. As respostas transitórias devidas a curto-circuito de 2 e 3 fases são obtidas por integração numérica através de método de passo adaptativo da família Runge-Kutta, levando em conta não-linearidade associada com o torque resistente da turbomáquina. Casos de validação do procedimento são apresentados, demonstrando a concordância dos resultados com a literatura. Por fim, o procedimento é aplicado ao estudo de caso de um compressor acionado por motor elétrico de velocidade constante e sua versão modificada pela adição de um VFD, o que leva ao aparecimento de pontos de interferência na faixa de operação, exigindo análise de resposta forçada e fadiga para demonstrar a integridade do trem em regime permanente. A resposta do trem a excitações de curto-circuito é estimada para diferentes casos e ressonância com o 1º modo foi identificada para curto bifásico com utilização do VFD.

Palavras-chave: vibração torcional; trens de turbomáquinas com engrenagens; excitação torcional causada por VFD; método de elementos finitos

ABSTRACT

This work presents the torsional vibration analysis of geared turbomachinery trains using a computational procedure based on the finite element and lumped parameters methods. The analysis is performed for systems driven by constant speed electric motors or by variable frequency driver (VFD) motors, that induce additional torsional excitations and have wide operational speed range, which makes the task of keeping the separation margins almost impossible. The computational procedure allows the torsional analysis of geared turbomachinery trains to be performed for single or multiple branched systems, based on American Petroleum Institute (API) guidelines. Modeling considerations of the main train components are applied to obtain the finite element model with lumped inertias at the nodes. The computational procedure consists of obtaining the single speed equivalent model from the original geared model and applying the finite element method to obtain mass, stiffness and damping matrices of the train and the system equations of motion. From the equations of motion, the eigenvalue problem is derived to estimate the torsional natural frequencies and mode shapes and to plot the Campbell diagrams. For the interference points identified in the Campbell diagram, steady state forces response analysis is applied to estimate nodal angular displacements, frequency response functions, elements twisting angles and associated torques and shear stresses in each element, being the damping matrix proportional (can be decoupled by the eigenvectors) or not. Transient responses due to 2-phase and 3-phase short circuits are estimated by numerical integration using an adaptative stepsize Runge-Kutta method, considering the non-linearity associated with the turbomachinery resistant torque. Validation cases are presented and results agreement with literature are demonstrated. Finally, the procedure is applied to a case study of a constant speed electric motor driven compressor and its modified version by the inclusion of a VFD, which causes interference points within operational speed range, requiring forced response and fatigue analyses to demonstrate train integrity in steady state regime. Train response to short circuits torques is predicted for different cases and resonance with first torsional mode is identified for the 2-phase short circuit.

Keywords: torsional vibration; geared turbomachinery trains; VFD torsional excitation; finite element method

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LIST OF ABBREVIATIONS

AC	Alternating Current
API	American Petroleum Institute
CSI	Current Source Inverter
DC	Direct Current
DOF	Degree of Freedom
DOL	Direct-on-line
DTC	Direct Torque Control
E/M	Electro-mechanical
FAT	Factory Acceptance Test
FEM	Finite Element Method
FRF	Frequency Response Function
GHG	Greenhouse gases
GMF	Gear Meshing Frequency
IOGP	International Association of Oil & Gas Producers
LCI	Load Commutated Inverter
MDOF	Multiple Degree of Freedom
O&G	Oil & Gas
ODE	Ordinary Differential Equation
PF	Penetration Factor
PWM	Pulse Width Modulation
TNF	Torsional Natural Frequency
VFD	Variable Frequency Drive
VSI	Voltage Source Inverter

LIST OF SYMBOLS

a_i	Absolute error tolerance in the step i
$[A], [B], [C_{ss}], [D]$	State space model matrices: state, input-to-state, state-to-output and feedthrough matrices, respectively
c	Element Damping
$[C]$	Damping matrix
$[C_M]$	Modal damping matrix
e_i	Local error tolerance in the integration step i
$f_{AC_{motor}}$	AC motor excitations frequencies (Hz)
$f_{harmonic}$	Harmonic excitation frequencies due to the VFD
$f_{interharm}$	Inter- harmonic excitation frequencies
f_{switch}	VFD switching frequency
f_{vfd}	VFD output frequency (Hz)
G	Element shear modulus
$[H(s)]$	Frequency response function matrix
$[I]$	Identity matrix
J	Polar mass moment of inertia
J_A	Polar moment of inertia (second polar moment of area)
$[J]$	Consistent Polar mass moment of inertia matrix
$[J_L]$	Lumped polar mass moment of inertia matrix
k_a, k_b, k_c, k_d, k_e	Marin's equation modifying factors
k_t	Element torsional stiffness
KE	Kinetic Energy
$[K]$	Torsional stiffness matrix
$[\tilde{K}]$	Inertia-normalized stiffness matrix
l	Element length
$[L]$	Cholesky decomposition of $[J]$
n	Speed ratio relative to the reference shaft
N	Number of nodes
N_p	Number of poles of the motor
p_i	Number of pulses in the inverter
PE	Potential Energy
$[P]$	Matrix of eigenvectors in the transformed coordinate system
q_i	Generalized coordinates of the i-th lumped inertia
Q_i	Generalized forces acting at the i-th lumped inertia
R	Rayleigh dissipation function
r_e	Relative error tolerance
r_o	Element outer radius
$\{r\}$	Angular displacement in modal coordinate system
$\{\dot{r}\}$	Angular velocity in modal coordinate system

$\{\ddot{r}\}$	Angular acceleration in modal coordinate system
s	Laplace frequency variable
SF	Safety factor
SM	Separation margin
S_{se}	Shear Endurance Limit
S_{eq}	Equivalent Fully Reversing Shear Stress
S_u	Ultimate tensile strength
$[S]$	Matrix of mode shapes
t	Time
T	Torque applied to a node
T_1, T_2, a_1, a_2	Constants of the short circuit torque equation
$\{T(t)\}$	Torques vector
$U_i(\%)$	Relative strain energy of the i th element to the total of a mode
$\{v\}$	Eigenvector of $[\tilde{K}]$ in the transformed coordinate system
x	Axial coordinate along the element
$\{X(t)\}$	States vector
$ X_i $	Solution of the numerical integration in step i
α, β	Constants of the torque-speed relation in a turbomachinery
$\Delta\theta$	Element twisting angle
θ	Node angular displacement
$\{\theta\}$	Angular displacement vector
$\{\dot{\theta}\}$	Angular velocity vector
$\{\ddot{\theta}\}$	Angular acceleration vector
ξ	Modal damping ratio
ρ	Specific weight of the element
τ	Shear stress
$\emptyset(t)$	Transformed coordinate system
ω	Excitation Frequency (rad/s)
ω_n	Torsional natural frequency (rad/s)
ω_{feed}	Electrical frequency applied to the motor (rad/s)

Subscripts

a	Relative to the actual model
Cyclic	Cyclic value, without concentration factors
e	Relative to element
eff	Effective
eq	Relative to the equivalent single-speed model
i	Relative to element index i
Max	Maximum value
Mean	Mean value

Rated	Rated value
SC	Related to short circuit

Superscripts

T	Transpose of a matrix
-1	Inverse of a matrix

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1. INTRODUCTION

In different industries, there is a growing tendency towards the application of more efficient process control methods aiming to save energy and reduce greenhouse gas emissions. In the oil and gas (O&G) industry context, plant electrification has been recommended as one strategy to reduce its greenhouse gases (GHG) emissions. For example, in offshore O&G facilities, the electrification normally is achieved by replacing the gas turbines that drive large rotating equipment by electric motors, that can be fed by the grid supplied with renewable sources (e.g. power from shore) or by a more efficient centralized power generation facility, that could adopt combined cycle power plant (IOGP, 2022).

When the process equipment is a turbomachinery such as a pump or compressor and the driver is a motor, the use of Variable Frequency Drive (VFD) enables effective and efficient operation over a wide speed range, which is not the case when the machine is driven by a constant speed motor, since its capacity is controlled by recycling, or variable inlet guide vanes or throttling, which are effective but inefficient. Therefore, VFD motors have recently gained relevance as drivers for large turbomachinery (Venkataraman et al., 2019; Rotondo et al., 2009). A typical motor driven geared turbomachinery compressor train scheme is shown in figure 1:

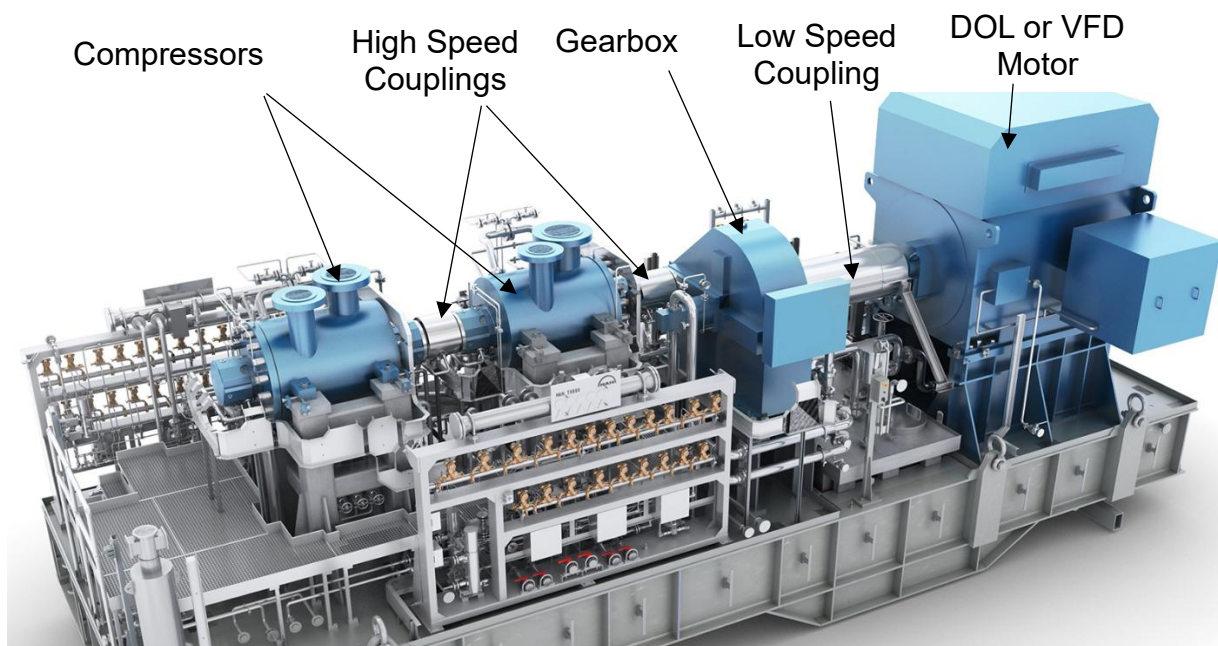


Figure 1: Typical motor driven geared turbomachinery train (Everllence, 2025)

Besides the aforementioned benefits, the use of VFDs in turbomachinery trains can pose a threat to the mechanical integrity of the train: VFDs use AC-DC-AC conversions

of the supplied voltage and current electric waves, and the resulting VFD output is not perfectly sinusoidal, but contains harmonic distortion, which leads to air-gap torque pulsation, with multiple harmonics of the output frequency being generated (Kita, 2016; Wachel; Szenasi, 1993), causing torsional vibration along the train rotors. VFDs also generate inter-harmonic or sidebands (non-integer harmonics) distortion, which are excitations associated with the difference between line frequency and VFD output frequency and sidebands of the Pulse Width Modulation (PWM) switching frequency (Huetten et al., 2013; API, 2019).

According to Vance, Zeidan and Murphy (2010), torsional vibration is the oscillatory twisting of the shafts that is superimposed to the running speed, being evidenced as an oscillatory variation of speed. In general, individual rotors are stiff enough in torsion to put their torsional natural frequencies (TNF) above the range of most torsional excitations, but when they are connected by shaft couplings, the torsional stiffness of coupling and connecting shafts are low enough to bring system TNF to the excitation frequencies range. Corcoran, Kocur and Mitsingas (2010) add that multibody and geared trains have increased likelihood of suffering torsional problems due to increased number of TNF. As noted by Wachel and Szenasi (1993) and Huetten et al. (2013), when a VFD motor is used, the wide operational range allowed, that can vary from 20-100% of the motor design speed, combined with the extra VFD excitation sources (multiple excitation harmonics of the output frequency and inter-harmonics), it is almost impossible to keep the required separation margins between excitation and natural frequencies.

All rotating machinery experience torsional vibration to some degree during regime and/or transient conditions. Historically, torsional vibration problems have been more commonly associated with diesel engines (reciprocating combustion engines) and marine propellers (Vance; Zeidan; Murphy, 2010). Torsional vibration failures on constant speed induction motor driven turbomachinery trains are not reported frequently in literature. On the other hand, with the growing application of VFDs in turbomachinery trains, numerous torsional vibration issues and failures have been associated with the VFDs, Bruha (2016), Corcoran, Kocur and Mitsingas (2010), Feese and Maxfield (2008) and Mishra et al. (2020) are only a sample of these cases.

Contrary to lateral vibrations that are easily detected through standard instrumentation such as proximity probes and accelerometers, rotating machinery trains are not

commonly instrumented to measure torsional vibration and the mechanical factory acceptance test (FAT) does not cover entire train testing due to the costs and project schedule impact (Corcoran; Kocur; Mitsingas, 2010). Furthermore, while excessive lateral vibration can be detected by rubbing of interstage seals and blades/impellers, torsional vibration seldom produces noise or transmit vibration to the foundation, so it is frequent that the first the first sign of its deleterious presence is a broken coupling, shaft, gear tooth, shrink fit slippage, etc. (Vance; Zeidan; Murphy, 2010). Figure 2 illustrates a coupling failure caused by torsional vibration:



Figure 2: Disc coupling fatigue failure caused by torsional vibration in a VFD motor driven compressor train (Corcoran; Kocur; Mitsingas, 2010)

Other relevant differences between torsional and lateral vibration are (Corbo; Melanoski, 1996; Vance; Zeidan; Murphy, 2010):

- Natural frequencies of lateral vibration are speed-dependent (gyroscopic effects and fluid-film bearings coefficients) while torsional vibration natural frequencies are independent of the shaft rotating speed.
- While the most common excitation of lateral vibration is the synchronous rotor imbalance, torsional vibration is not influenced by rotor imbalance, except in machines with gears where the lateral vibration produces dynamic torque. For geared trains, in many instances the torsional vibration can be identified as lateral vibration in the pinion and gear bearings.
- In lateral vibration, self-excited vibration (instability) is a concern while in torsional vibration rarely instability occurs, the exception being when the control system has speed control feedback, which is the case of many VFD controlled motors.
- In torsional analysis, all connected rotors of the train must be considered while lateral analysis is usually performed separately on each body in the train.

In turbomachinery trains, since the mechanical damping is usually very small, the response to a small continuous cyclic excitation torque at a frequency equal to any TNF can be a large alternating torque, that will cause fatigue and could lead to failure (Corcoran; Kocur; Mitsingas, 2010). In the typical turbomachinery geared train presented in figure 1, steady state excitation at TNFs can come from different mechanical sources, such as imbalance, misalignment, aerodynamic effects, gearing errors, etc. as well as from electrical sources, such as the VFD harmonics and inter-harmonics (Corbo; Melanoski, 1996). Transient torsional excitation caused by electric faults, such as short circuits, can result in high transient torques which can potentially excite torsional modes of the drive train and cause catastrophic failures (Aho; Nerg; Baum, 2017).

Torsional behavior prediction of turbomachinery trains in the design phase is essential to ensure train integrity with high availability and reliability, especially in the O&G industry, where most of the components are tailor-made and downtime cost is very high. To this end, American Petroleum Institute (API) provides in its Rotordynamic Tutorial (API, 2019), guidelines regarding modeling and analysis of turbomachinery trains under torsional vibration. Other works, such as Wachel and Szenasi (1993) and Corbo and Melanoski (1996) also provide guidelines regarding torsional vibration analysis. Whenever the train is driven by a VFD motor, due to the relevant number of failures associated with torsional vibration in such trains, the criticality of performing a thorough torsional analysis in the design phase is substantially increased (API, 2019).

Torsional analysis of turbomachinery trains involves mathematical modeling of all the train components to obtain equations of motion that characterizes the dynamics of the system. These equations are solved for the free vibration case (natural frequencies and mode shapes determination) and forced response, to determine the stresses in all elements. The torsional analysis of a complex turbomachinery train is not feasible to be performed accurately by hand calculation, usually a computational code is required. A conventional torsional vibration analysis normally comprises (API, 2019):

- a) Undamped torsional natural frequency analysis, which is the determination of the natural frequencies and mode shapes. The TNFs are then plotted together with the excitation sources to determine if there is any interference point within operating speed range or, if not, what are the separation margins.

- b) Steady state forced response analysis, which is the determination of cyclic stress for each model component and expected fatigue life, required if it is not possible to comply with separation margins.
- c) Transient response analysis, involving the computation of dynamic torques and gear tooth loads under transient conditions, e.g., electric motor/generator short circuit or start-up.

In some special cases, other analysis might be performed, although not mandatory as per API standards:

- d) Closed loop coupled electromechanical torsional analysis must be performed to analytically evaluate torsional stability of VFD driven trains which use speed control feedback or when the VFD is used to provide torsional damping to the system.
- e) Lateral-torsional coupled analysis should be considered for some geared turbomachinery trains, although the conventional approach is to assume torsional and lateral vibrations are independent of each other.

1.1.Objectives

This work objective is to perform the torsional analysis of the dynamic response of geared turbomachinery trains using the Finite Element Method (FEM) and the Lumped Mass Modeling to apply it to the execution of a torsional analysis according to the guidelines of API. More specifically, the objectives are:

- Develop a computational procedure that can be applied to different rotating equipment trains, single or multiple branched geared, to:
 - Assembly the equivalent single speed system matrices of inertia, damping and stiffness.
 - Estimate the natural frequencies, mode shapes and relative strain energy distribution.
 - Generate the Campbell Diagram of the train and calculate the separation margins of the interference points.
 - Estimate the steady state response of the complete train for each interference point, with the determination of elements twisting angles, transmitted cyclic torques and shear stresses.

- Estimate the transient response of the complete train due to short circuit excitations, with the determination of maximum cyclic torques and shear stresses.
- Apply the procedure to model different rotating equipment trains presented in the literature to validate the developed code and demonstrate its functionalities.
- Perform a conventional torsional analysis based on API with the use of the developed code through a case study of a geared turbomachinery train driven by a constant speed electric motor driven compressor and its modified variable speed version by the inclusion of a VFD.

2. LITERATURE REVIEW

Some papers, technical reports, and design tutorials about the historical advancements of the modeling of torsional vibration problems, the modeling of the components of geared turbomachinery trains, the description of torsional excitations sources, some examples of successful design and troubleshooting of geared turbomachinery trains are described in this chapter. Since variable frequency drives (VFDs) add complexity into the torsional analysis of geared turbomachinery trains, special attention is given to the description of VFD-induced torsional vibration.

2.1. Torsional vibration in geared turbomachinery trains

According to Vance, Zeidan and Murphy (2010), torsional vibration is the oscillatory twisting of the shafts that is superimposed to the running speed, being evidenced as an oscillatory variation of speed. All rotating machinery experience torsional vibration to some degree during regime and/or transient conditions. Contrary to lateral vibration analysis, which is often performed on component-by-component basis, torsional models normally include all coupled rotors, since, in general, individual rotors are stiff enough in torsion to put their TNF above the range of most torsional excitations. When rotors are connected via shaft and couplings, each individual rotor acts as a single massive inertia. The torsional stiffness values of coupling and connecting shafts are relatively low, resulting in lower system TNFs, that is often in the excitation frequencies range. Corcoran, Kocur and Mitsingas (2010) add that multibody and geared trains have increased likelihood of suffering torsional problems due to increased number of TNFs. The gears also convert lateral vibration on gear shafts from different sources, such as unbalance, misalignment, etc. into torsional vibration and vice versa (Wachel; Szenasi, 1993).

While lateral vibration can be detected by proximity probes and accelerometers, torsional vibration is typically not monitored in turbomachinery trains, which combined with the usual low damping level, can lead to severe damage without previous warning, even with relatively low excitations (Corcoran; Kocur; Mitsingas, 2010). Therefore, predicting torsional behavior accurately is essential to ensure train integrity, especially when dealing with critical turbomachinery applications.

Historically, torsional vibration problems were more commonly associated with reciprocating combustion engines and marine propellers (Vance; Zeidan; Murphy,

2010). In the early 1900s, the classical Holzer method was developed to address failures in ship propulsion reciprocating steam engines. This iterative method was originally developed as a hand calculation procedure for the determination of system torsional natural frequencies and modes, but it can also be conveniently implemented in computers (Wachel; Szenasi, 1993). The method is applicable to lumped-parameter models. It was the first algorithm devised to compute natural frequencies of large-order torsional systems. Before, the available energy methods and empirical rules could provide good estimates of the fundamental mode only.

According to Corbo and Melanoski (1996), the basis of the Holzer method is that free vibration can occur with no external torques acting on the system only if the vibration frequency is a natural frequency of the system, therefore, a guess value of frequency is selected and the angular twist of each component is calculated sequentially, assuming that the first disk vibrates with arbitrary amplitude of 1 radian, the process being repeated until boundary conditions are satisfied (i.e., torque should be zero at the end disks). Besides the classical Holzer method, there are other procedures that can also be classified as transfer matrix methods, being considered as Holzer method variants, since they consist in multiplying individual element transfer matrices to obtain expressions for torque and displacement of the last disk in terms of those at the first. There are Holzer variants that can be applied to damped forced vibration analysis.

In recent years, the torsional analysis by the Holzer method and other transfer matrix method variants have been replaced by FEM for single branch and multiple branches systems (Wu; Chen, 2001). With the advance of computational capabilities, the FEM gradually overtook the tabular methods, especially for the modelling and analysis of more complex, multiple branches systems, becoming a norm today for industry practice (Rao, 2011; API, 2019). The natural frequencies and mode shapes are typically calculated by Matrix-Eigenvalue methods while steady state forced vibration response is obtained by analytical methods such as Modal Analysis and Matrix-Eigenvalue or by numerical integration methods such as Newmark or Runge-Kutta (Corbo; Melanoski, 1996).

Wu and Chen (2001) analyzed torsional vibration of two complex gear-branched systems. Their approach consisted of simplifying the original system by the elimination of the dependent angle at each gear pair, resulting in an equivalent single speed model, then applying the conventional FEM. The authors claimed that this procedure was

much simpler than previous FEM procedures applicable to gear-branched systems that required element-by-element or overall transformations to deal with the fact that the total independent DOF was less than the number of rotating inertias. Undamped TNFs (eigenvalues) and mode shapes (eigenvectors) were calculated using the Jacobi method and the estimated TNFs were compared to other authors results using different FEM procedures and Holzer method. Steady state forced vibration response was obtained with Newmark direct integration method.

In geared systems, lateral-torsional vibration interaction can occur, and, in some instances, torsional vibration can lead to severe lateral vibration (Bruha, 2016; Corcoran; Kocur; Matsingas, 2010; Kita, 2016). The finite element modeling is the standard choice for coupled lateral-torsional vibration analysis of geared trains, since it is required to model flexible rotor-bearing systems, including lateral and torsional degrees of freedom. Considerations about this topic are included in section 2.2.

The relevance of predicting torsional behavior accurately of geared turbomachinery trains is reflected in different tutorials devoted to the torsional analysis, such as Wachel and Szenasi (1993), Corbo and Melanoski (1996) and API (2019). These tutorials describe procedures and methods for the analysis of torsional vibration in rotating machinery, highlighting the special attention required by VFD motor driven trains, covering: modeling considerations for the main train components; undamped TNF analysis, consisting of calculation of the torsional natural frequencies and mode shapes and development of interference (Campbell) diagram; steady state forced response analysis, consisting of calculation of dynamic torsional oscillations of all masses and the dynamic torsional stresses of all components for the resonance points identified; transient simulation of start-up and fault conditions of motors.

In the early 1980s, the evolution of semiconductors and microelectronics technologies led to the increase in VFD industrial use. Torsional vibration problem induced by VFD in the cement industry was discussed by Mayer (1981). It described fundamentals of torsional vibration in mechanical trains, using lumped parameter models. The trains were divided into electrical supply system, drive motors, mechanical system and load system and excitation from each part was described. Several cement industry trains were presented and design guidelines based on experience discussed.

Hudson (1996) reported the successful design and field testing procedure of a train composed by a VFD, 7.7 MW synchronous motor, gearbox, compressor, elastomeric

low speed coupling and disc coupling. The train model was finite element based, containing 14 nodes where lumped inertias were included. Since the elastomeric coupling stiffness is non-linear, the TNFs calculation were repeated for different operational (transmitted torque) conditions. The damping of the elastomeric coupling was modelled as discrete dampers. The VFD torsional excitations were composed of 20 torque harmonics and it was not possible to detune all of them from the TNFs, so steady state response was determined. Start-up simulation was also carried out, considering a combined torque input of the 8 more relevant torque harmonics. Field test measurements using strain gages located at the low speed coupling confirmed the design acceptability, as predicted in the analysis.

Hudson (1996) performed an open loop analysis, which means that there was no feedback control from the mechanical system to the VFD. This is the traditional approach to torsional analysis (Bruha, 2016). However, many modern compressors driven by VFDs have feedback control, which provides good controllability for speed and torque but the interaction between the complex control loop and the entire train might lead to failures if the control system is not properly tuned (Kita, 2016). If the VFD control system uses measured signals, such as motor speed, stator current, stator voltages, etc. to adjust the speed and/or torque provided by the motor and these signals contain oscillations due to torsional vibration, they might adversely affect the drive behavior (Bruha, 2016). To analyze such systems, a coupled electro-mechanical model is required. This subject is further discussed in section 2.3.

2.2. Modeling the typical turbomachinery components

In turbomachinery trains, the torsional system is composed of different components, such as a driver (motor or turbine), shafts, bearings, couplings, gearbox, driven equipment (generator, compressor or pump). Here, modeling guidelines are presented for the elements contained in a turbomachinery train composed of a compressor driven by an electric motor through a gearbox.

2.2.1. Electric motor

Many motor suppliers provide mass-elastic data of the equipment as a one-station model consisting of single inertia and single stiffness. However, more detailed modeling is recommended, using multiple stations connected by shaft elements to represent the inertia distribution along axial length, especially if the analyst is

concerned with accurate determination of higher order natural frequencies of the system. In systems composed of motor, gear and a compressor, the third system torsional mode is mainly governed by stiffness and inertia of motor core such that the first mode of motor subsystem is nearly the same as the system third mode (API, 2019).

2.2.2. Shafts

Frequently, turbomachinery shafts have low inertia compared to the driver and driven equipment and gears inertia, therefore, lumped parameter approach usually provides good results. If the inertia of shafts are comparable to the major inertias of the train components, then a more accurate procedure might be required, such as using more lumped stations or formulating the elements with a finite element approach (Corbo; Melanoski, 1996).

If the shaft is stepped (different diameters), the smaller shaft penetrates the larger by a distance defined as “penetration factor” (PF), which causes the length of the smaller shaft to effectively be increased and the larger to be reduced by the amount PF (API, 2019).

Most rotating machines have shrunk-on components (sleeves, collars, etc.), that may contribute to the shaft torsional stiffness. If the component has a fit with the ratio of length to diameter greater than 1, the shaft is assumed to be free to twist up to a point referred as “point of rigidity” and beyond this point, it is assumed that there is no deflection (Corbo; Melanoski, 1996). One common assumption of coupling vendors is the rule $1/3 - 2/3$, where the shaft is assumed to twist freely over $1/3$ of the length of the hub interference fit (Wachel; Szenasi, 1993).

2.2.3. Couplings

Different types of flexible couplings are used in turbomachinery to connect the train elements: gear, metallic flexible (disc or diaphragm type) or elastomeric, being the metallic flexible the most common (API, 2019). The flexible disk packs are composed of thin sheet metal shims stacked together, presented as black rectangles in figure 3:

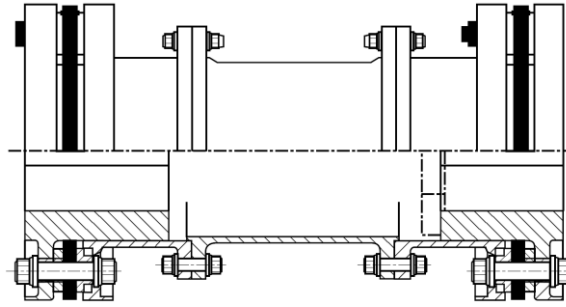


Figure 3: Flexible disk pack coupling (API, 2019)

Elastic couplings are normally modelled as torsional inertias for each hub connected by an equivalent coupling stiffness (Vance; Zeidan; Murphy, 2010). The coupling can be seen as an assembly of torsional springs in series. The coupling manufacturer shall provide the mass-elastic data for the train torsional analysis (API, 2019).

In motor/gear/compressor systems couplings have direct influence on the first two modes and there are many cases where a torsional resonance is eliminated by simple modifying the coupling design (API, 2019).

According to API (2019), in most couplings that do not contain elastomers (elastomeric type), the torsional stiffness can be assumed as constant, independent of the torque. Special consideration is required if there is a hydraulic coupling in the system, which transfers the torque through the kinetic energy of the working fluid. In this case, the system can be treated by 2 modelling approaches: assume that the coupling has zero stiffness and analyze the two connected subsystems separately or include an equivalent stiffness that is dependent on the output speed and transmitted power (non-linear stiffness).

2.2.4. Driven turbomachinery (Compressor or pumps)

In general, compressors and pump impellers are modeled as discrete lumped inertias. For pumps, the fluid inertia shall be accounted for in the inertia of the impeller (wet inertia) (Corbo; Melanoski, 1996).

In a motor-driven turbomachinery train, the energy provided by the motor is used to compress the fluid. The torque-speed relation of an impeller is (Corbo; Melanoski, 1996):

$$T = \alpha \omega^\beta \quad (1)$$

where α and β are constants, being β between 2 and 3. For many practical impellers, the torque is proportional to the square of the speed ($\beta = 2$). For example, Hudson (1996) uses equation 1 with $\beta = 2$ to calculate the motor transmitted torque for an operating speed lower than the rated.

2.2.5. Gears

The rotational inertia of the gear and pinion is normally provided by the manufacturer. The gear mesh can be represented by a spring and damper along the pressure line (tangent to base circles). Typically, the gear mesh stiffness is high compared to the shafts and coupling stiffnesses and can be considered torsionally rigid, having minor influence on the lower torsional natural frequencies (the ones that are normally of concern in most of torsional problems) and in general only being relevant in the calculation of very high natural frequencies or when there are multiple gear meshes in the system (Wachel; Szenasi, 1993; Corbo; Melanoski, 1996).

Gear backlash is not a concern under normal circumstances, where the teeth are continuously loaded in one direction. However, if there is a vibratory torque where the magnitude exceeds the average transmitted torque, this could cause torque reversal, which cause teeth to pound against each other, causing wear and eventually fatigue fracture (Corbo; Melanoski, 1996).

There are some geared systems where coupling between torsional and lateral motion occurs, although the standard practice is to evaluate lateral and torsional vibration separately (most of the references). Moreover, the relevant API standards for compressors or pumps do not require coupled lateral-torsional analysis. There are many works in the literature related to mathematical models used in gear dynamics, which include teeth stiffness (constant or time-varying), backlash and damping, causing the system to present nonlinear behavior. A detailed review about these models is presented by Shin and Palazollo (2020). Factors that influence the decision of performing a lateral-torsional analysis are presented as reference (API, 2019):

- Normally, the higher the lateral flexibility of the gear shafts and their supports, greater the likelihood of problems. This is usually related to the excessive flexibility of the gear shaft bearings, especially the one closest to the gear mesh.
- Since gearshaft bearing stiffnesses increases with the load, the potential for problems is higher at light loads.

- The torsional modes that are capable to couple with a lateral mode are those with considerable motion at gear mesh.
- If a torsional natural frequency is close to one lateral natural frequency, the chances for coupled mode instabilities are greater.
- If there is a torsional mode close to 50% of the operating speed of one of the gearshafts, the chance for a coupled mode instability increases.

Bruha (2016), Corcoran, Kocur and Mitsingas (2010) and Kita (2016) present cases of troubleshooting where excessive torsional vibration could be identified by abnormal lateral vibration on gear bearings at the 1st TNF of the system.

2.2.6. Damping

Most of torsional analysis are performed with a generic modal damping ratio applied to all elements since actual magnitudes of damping due to individual sources within a train are usually small and almost impossible to determine analytically. According to API (2019), the following values are generally conservative:

- For trains employing rigid couplings (e.g. turbine-generator trains): 0.5% of critical damping
- Ung geared trains employing couplings – 1.0% of critical damping
- Geared trains having gear shafts supported by fluid-film bearings: 1.5%-2.0% of critical damping

In a literature survey by Corbo and Melanoski (1996), 1% damping ratio is the minimum value found and became the baseline for ungeared systems. Geared systems damping is much higher than ungeared systems due to the squeeze-film effect that occurs in fluid-film bearings supporting the gears when torsional vibration occurs (API, 2019). For a typical O&G train compressor, 0.6-1.7% of the critical damping is frequently assumed (Corcoran; Kocur; Mitsingas, 2010).

Sometimes, it is desirable to model each discrete damper of the system, which is a more complex task. These dampers can be classified as external, connecting one model node and ground, being only dependent on the absolute angular speed of that node, for example, damping provided by bearings or impellers (load damping), or internal, acting between two adjacent inertias of the system, being dependent on the difference in the angular speed of the two nodes, for example, elastomeric or hydraulic couplings (Corbo; Melanoski, 1996). When performing a lateral-torsional analysis, this

more complex modeling procedure is required to account for the damping provided by the bearings.

2.3. Steady state torsional excitations in turbomachinery geared trains

For turbomachinery trains composed of an induction motor, gearbox and a centrifugal compressor or pump, the main torsional excitation sources that should be considered in a torsional analysis are summarized below. Since the trains under consideration are geared, the typical lateral vibration excitation sources are also present as torsional excitations.

2.3.1. Compressors and pumps

In a typical torsional analysis, it is common practice to include excitations at the frequencies of one per revolution (1X) and two per revolution (2X) originated in generic sources such as unbalance, eccentricity, misalignment and ellipticity. It is recommended that an excitation with 1.0% amplitude (zero-peak) of the mean operating torque is applied for synchronous (1X) frequency excitations and 0.5% for nonsynchronous frequencies (Wachel; Szenasi, 1993).

Other excitation that can occur is related to impellers, where torque variations occur at blades passing frequency due to pressure disturbances resulting from vanes passing a stationary object such as a volute or diffuser entrance (Corbo; Melanoski, 1996). If the casing contains diffuser vanes, they also can produce excitations. The excitation frequency is equal to the number of impellers or vanes multiplied by the shaft speed. Based on experience, an excitation level of $1/(\text{number of blades})$ percent zero-peak of steady-steady torque is used when blade passing frequency coincides with torsional natural frequencies (Wachel; Szenasi, 1993).

2.3.2. Gearing

According to Wachel and Szenasi (1993), the magnitude of torsional excitations arising in high quality gearing is quite low. Gear unbalance, misalignment and lateral vibrations of the gear shaft can cause torsional vibration, typically less than 1.0% (zero-peak) of transmitted torque. It should be noted that badly worn gearing can cause excitations many times the magnitudes defined for high quality gearing. The excitation frequencies occur at 1X and 2X, of both the gear and pinion, and gear mesh frequency (GMF). The GMF can be defined as the number of teeth on that shaft's gear multiplied by shaft

speed and a conservative assumption for the excitation magnitude is 1.0% (zero-peak) of the transmitted torque (Corbo; Melanoski, 1996).

2.3.3. Constant speed induction motors

Excitations from directly-on-line (DOL) connected induction motors at operating speed are relatively low. The primary excitations occur at 1x line frequency and 2X line frequency. According to Wachel and Szenasi (1993), although, in theory, there should be no net dynamic excitation for a three-phase motor, pulsating torques at 1X line frequency were documented in torsional tests. 2X line frequency excitations can be generated by unsymmetric voltage supply or rotor eccentricity. Therefore, a conservative approach is to assume at least 1.0% (zero-peak) of output torque for 1x line frequency excitations and 0.5% (zero-peak) for 2X line frequency (Wachel; Szenasi, 1993; API, 2019). In general, the manufacturer should be consulted.

2.3.4. Induction motor controlled by VFDs

The VFD system allows the motor speed variation by varying the input frequency to the motor. This is made possible by AC-DC-AC conversion achieved by electrical switching action, which results in the generation of ripple, distortion or pulsation on the AC output electrical current, which result in pulsating torques in motor air-gap (Kita, 2016). Figure 4 illustrates the typical VFD scheme:

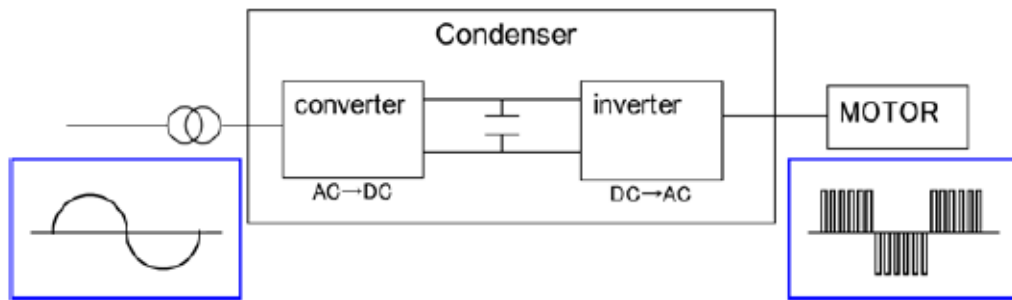


Figure 4: Typical VFD schematic (Kita, 2016)

The relation between motor synchronous speed (N_{rpm}) and electrical frequency supplied to it by the VFD (f_{vfd}) is given by (Corbo; Melanoski, 1996):

$$N_{rpm} = 60 * f_{vfd} * \frac{2}{N_p} \quad (2)$$

where N_p is the number of poles. The VFDs generate excitations by themselves and alter the excitation generated by the motor. Instead of the 1X and 2X line frequency excitations, the motor excitations become speed-dependent, appearing as positively

sloped lines on the Campbell Diagram. Ignoring the effect of the slip, the AC motor excitations frequencies ($f_{AC_{motor}}$) are (Corbo; Melanoski, 1996):

$$f_{AC_{motor}} = k * \frac{2}{N_p} * f_{vfd} \quad (3)$$

For $k=1$ and $k=2$.

In general, the VFDs can be classified among 3 categories (API, 2019):

- 1- Current Source Inverter (CSI)
- 2- Load Commutated Inverter (LCI)
- 3- Voltage Source Inverter (VSI)

In turbomachinery industry, the most common types are VSIs and LCIs, with pros and cons for each one, the selection being based on power and voltage range and complexity. In general, VSI are used for power ratings up to 25 MW and LCI is preferred for the range 15-120 MW and is only applicable to synchronous motors (Huetten et al., 2013).

The excitation magnitudes and frequencies caused by VFDs varies depending on the type and design, therefore it is highly recommended to obtain the actual data for a torsional analysis with the VFD vendors, close collaboration among parties is key to success (Hudson, 1996; Huetten et al., 2013). Nevertheless, the literature contains some guidelines for a preliminary assessment independently of the manufacturer.

For a Voltage Source Inverter (VSI), which is the most common VFD type in the range up to 25 MW, the following torsional excitations sources should be considered:

- a- Harmonic
- b- Inter-Harmonic (Sideband)
- c- Broadband
- d- Closed-loop electro-mechanical interaction between the VFD and the turbomachinery

A brief description of each one is provided:

- a- Harmonic excitations

The harmonic excitation frequencies ($f_{harmonic}$) are given by (Kita, 2016):

$$f_{harmonic} = p_i * f_{vfd} * 1, 2, 3 \dots \quad (4)$$

where p_i is the number of pulses in the inverter, the most common being equal to 6. Most VSIs use Pulse Width Modulation (PWM) to control the voltage being fed to the motor. The VFD output is composed by pulses, since the inverter switches have to be fully on or off. The modulation scheme uses the spacing and duration of pulses to maximize the fundamental and minimizing the harmonics, but the switching frequency still appears (API, 2019). These voltage harmonics will drive current that will produce harmonics in the torque.

The usual amplitude differs in literature, Bruha (2016) claimed that modern VSIs have typical torque ripple in the range 2-10% of motor rated torque. In Corbo and Melanoski (1996), the typical torque informed for VSIs with PWM lies in the range 5-10% for the 6X excitation, which normally is larger than the others. API (2019) reports one case of estimated VSI torsional excitation of 1.3% of motor rated torque based on measured lateral vibration and estimated torsional amplification factor and states that the VSI air-gap torque sub-synchronous components are smaller than 1.0% of the motor rated torque. The harmonics magnitude often varies for each harmonic order (6X, 12X, 18X, etc.) of the VFD output frequency (Hudson, 1996; Bruha, 2016). Besides, the load-dependence is smaller than in CSI and LCI types, therefore it may have higher excitations at no load condition and lower excitations at rated torque.

b- Inter-harmonic (sideband) excitations

In VSIs, inter-harmonics can be caused by 1) frequency differences between line frequency and VFD output frequency and 2) sidebands of the PWM switching frequency. In VSIs, normally only the second is relevant and the frequencies ($f_{interharm}$) are given by the following equation (Kita, 2016):

$$f_{interharm} = m * f_{switch} \pm (q * f_{vfd}) \quad (5)$$

where f_{switch} is the VFD switching frequency and m, q are specific positive integers. The rules for m, q values depend on VSI design. There are VFDs that do not even obey equations 4 and 5.

According to API (2019), the magnitude of the inter-harmonic excitations for VSIs with PWM are less than 1% but since they have the potential to excite the first mode at speed close to maximum continuous operating speed, even this small excitation can be dangerous. For example, figure 5 shows the difference between harmonic (left) and inter-harmonic (right) excitations acting on the same machine: the inter-harmonic

excitation, which can have negatively sloped lines, causes 2 interferences with the 1st TNF within operational speed range:

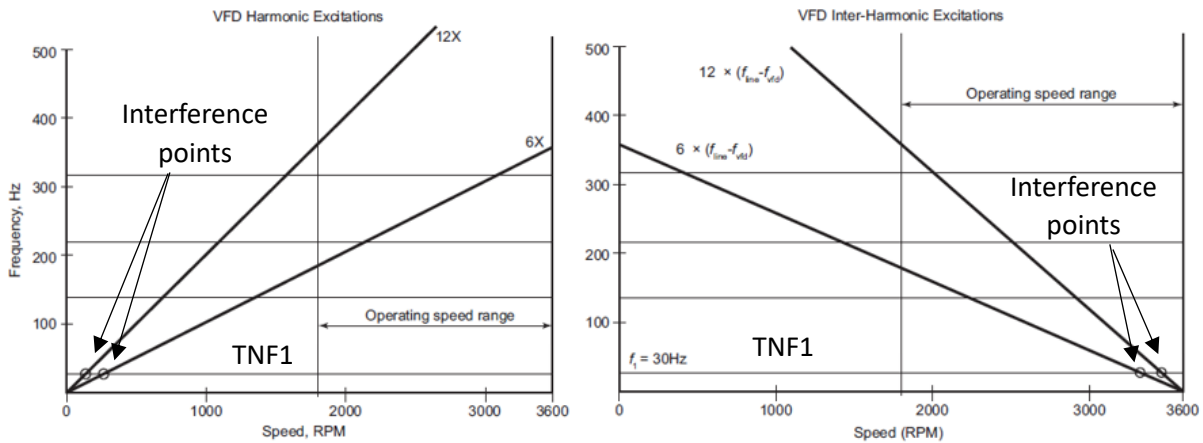


Figure 5: Harmonic (left) and inter-harmonic (right) excitations (API, 2019)

c- Broadband Excitations

When PWM is based on hysteresis control, the excitations are no longer discrete frequencies, since hysteresis type control of motor current (or torque) is random, therefore, there is no repeating pattern, the excitation is a continuous function similar to an acoustic noise spectrum. If the noise spectrum has a constant value, it is called white noise. In Rotondo et al. (2008), a field case was described where the measured VFD excitation was broadband, the drive employed Direct Torque Control (DTC) modulation, which is a hysteretic control, so the excitation had no specific frequency (excitation spectrum is a continuous function, similar to an acoustic noise spectrum). According to API (2019), the torques at broadband spectrum (white noise) are usually of very low level to be a cause of concern regarding excitation of TNFs.

d- Closed-loop electro-mechanical (E/M) interaction

The torsional analysis of VFD controlled turbomachinery trains performed using the torque harmonic data (torque ripple produced by the frequency converter due to discrete switching of power semiconductors), as described in “a” and “b”, derived independently of the actual mechanical system coupled with the VFD and motor may not be sufficient for proper design (Mishra et al., 2020; Bruha, 2016).

Traditionally, torsional analysis is an open loop analysis, the excitations are defined and torque responses calculated, the reaction of the drive control is ignored. However, if the drive uses feedback control (e.g., motor speed, torque, stator current, stator

voltage, etc.), the torsional analysis might predict wrong torsional response since these signals (measured or estimated from other measured parameters) might contain oscillations due to torsional vibration caused by different sources, which will affect drive behavior (Mishra et al., 2020; Bruha, 2016). Figure 6 shows a schematic of a closed-loop coupled electrical-mechanical system:

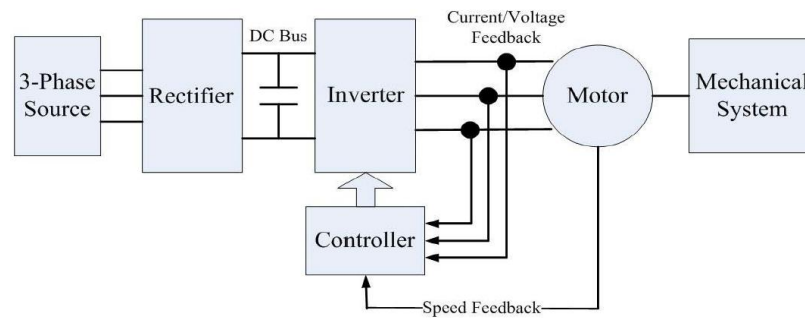


Figure 6: Closed-loop electro-mechanical system (Mishra et al., 2020)

In a closed loop system, the reaction of the VFD control shall be such that torsional vibration is damped or at least not further excited, resulting in control instability. If the control damping is positive (proper setting of the control parameters), the VFD behavior will help to reduce torsional vibration severity, providing more damping to the system, which reduces the torsional vibration. If control damping provided to the system is negative, the overall system damping will be reduced, which might destabilize the system, with the initial oscillation being amplified until the weakest link of the system, typically the coupling, fails (Bruha, 2016).

There are several cases in literature describing positive and negative outcomes associated with proper or improper VFD control parametrization.

Mishra et al. (2020) describes a compressor controlled by VFD with an open-loop voltage/frequency (v/f) control, in which the harmonic spectrum of the PWM-inverted current/voltage contained integers of the electrical fundamental frequency and the compressor was experiencing severe torsional vibration at the 1st TNF when operating at certain speed, that was reduced due to the implementation of closed-loop control.

Corcoran, Kocur and Mitsingas (2010) presented a case of a compressor train in which the coupling failed due to vibration induced by the VFD. In this case, the excessive torsional vibration was identified by abnormal lateral vibration at the gears, coincident with 1st TNF. The VFD vendor was unable to tune the parameters of the closed loop

control algorithm, so the final solution was the implementation of an open-loop Volts/Hz algorithm, without speed feedback.

Rotondo et al. (2009), Kita (2016), Huetten et al. (2013) are other authors that consider E/M interaction in turbomachinery systems controlled by VFDs.

In closed-loop electrical-mechanical systems, a coupled electro-mechanical (E/M) simulation is recommended, although not mandatory as per API standards. In general, the decision to run E/M simulation should be left to the end user and should be based on the complexity of the train and level of “novelties” in the proposed VFD, as compared to similar trains (API, 2019).

2.3.5. Steady state excitations summary

Table 1 presents a summary of the typical excitation sources, frequencies and torque magnitudes as percentage of the operating torque.

Table 1: Summary of the typical excitation sources, frequencies and magnitude as percentage of operating torque

Source	Frequency	Amplitude 0-pk (related to operating torque)
Generic 1X (unbalance, eccentricity, misalignment, etc.)	1 x speed of each connected rotor	1%
Generic 2X (misalignment, ellipticity,etc.)	2 x speed of each connected rotor	0.5%
Gearing (pinion/gear: Np/Ng teeth)	1 x speed of each connected rotor	1.0%
	2 x speed of each connected rotor	0.5%
	GMF: Np * pinion speed and Ng * gear speed	1.0%
Blade/ vanes passing frequency (Impeller with B blades and diffuser with V vanes)	B x speed	1/B %
	V x speed	1/V%
Induction AC motor with Np poles (constant speed)	1 x Line Frequency	1%
	2 x Line Frequency	0.5%
	Np x speed	0.5%
Induction AC motor with Np poles (variable speed)	0,5 x Np x speed	1%
	Np x speed	0.5%
Variable Frequency Drive (VFD) See notes 1 and 2	Harmonics	
	pi x VFD output frequency x 1,2,3...	2-10%
	Inter-harmonics (see note 3)	
	Ressonance with first mode at MCOS.	1%

Note 1: Highly dependent on VFD type and design features, VFD vendor shall always be consulted.

Note 2: The provided information is related to VSI type

Note 3: There is no defined typical excitation frequencies. Worst case scenario is assumed (API, 2019)

2.4. Transient torsional excitations

Electric motor driven turbomachinery trains are subject to different torsional excitations that are transient in nature, such as motor start-up, breaker reclosure after power interruption or electrical faults (e.g., short circuits) torques. Breaker reclosure can be prevented by time delay relays and other problems by fault detection relays in the switchgear. Direct-On-Line (DOL) start-ups of induction motors can generate large torques at line frequency that decrease rapidly but usually are not analyzed, unless the train has a mode with significant modal amplitude at the motor core, at or near line frequency (API, 2019). Therefore, the transient torsional analyses normally performed are Direct-On-Line (DOL) start-ups of synchronous motors and fault conditions of motors, such as motor short circuit, that can produce braking torques several times the rated torque of the motor (Wachel; Szenasi, 1993). Nowadays, most of the motors used in industrial turbomachinery trains are induction (asynchronous) motors. Therefore, for induction motor driven turbomachinery, the most relevant transient analysis to be performed is the short circuit analysis (Aho; Nerg; Baum, 2017).

2.4.1. Short circuit simulation

Even though short circuit in induction motors can lead to catastrophic failures, frequently short circuit analysis is only performed when the train is driven by a DOL motor, based on the assumption that VFD motors are protected by the fast reaction of the VFD and its protection functions (Aho; Nerg; Baum, 2017). In both DOL or VFD motors, the short circuit current and the resulting electromagnetic torque are limited only by the motor short circuit impedance, it is not limited by the frequency converter unless additional impedance is applied to the output (Aho; Nerg; Baum, 2017).

Short circuits current and torque characteristics are normally provided by the motor manufacturers. These torques are considerable, 3-15 times the rated full load-torque (Aho; Nerg; Baum, 2017), which can potentially excite torsional modes of the train and create high loads on couplings, gears and shafts. The short circuit torque equations generally have the following form (Wachel; Szenasi, 1993; Aho; Nerg; Baum, 2017):

$$T_{sc}(t) = -T_{rated} \left(T_1 e^{-\frac{t}{a1}} \sin(\omega_{feed} t) - T_2 e^{-\frac{t}{a2}} \sin(2\omega_{feed} t) \right) \quad (6)$$

In equation 6, ω_{feed} is the electrical frequency fed to the motor. The parameters T_1 , T_2 , $a1$ and $a2$ are constants dependent on the equivalent circuit parameters of the motor,

such as voltage, number of poles, stator and rotor inductance, stator reactance, stator and rotor resistance and angular speed of magnetic field. The negative sign accounts for the fact that during a short circuit, energy stored in the motor magnetic fields causes it to act briefly as an induction generator. Therefore the torque is in the opposite direction of motor's normal operating torque, acting as a braking torque (Aho; Nerg; Baum, 2017).

Three-phase short circuit phase implies that all three phases of the power system are simultaneously short circuited to each other through a direct connection. The second term of equation 6, which oscillates with two times the electrical frequency fed to the motor, only exists for 2-phase short circuits (line-to-line short between two phases of the phase circuits) (Wachel; Szenasi, 1993; Aho; Nerg; Baum, 2017).

Since short circuits generate excitations only at 1X line and 2X line frequency for a constant speed motor or 1x and 2x VFD output frequency for VFD motors, the analysis is only needed if there are torsional modes that could be excited by these frequencies (API, 2019). When the train is driven by a VFD motor, the excitations occur at 1x and 2X electrical fed frequency (variable) and since the operational speed range is usually wide, the likelihood of an interference point in the Campbell Diagram, i.e. resonance condition, is greater than for constant speed motors (API, 2019).

Since short circuits are rare events, the acceptability criteria is simply survival: the maximum stresses in all shaft elements shall be below their shear yield strength, the torques at couplings and gears shall not exceed vendor's peak (momentary) torque rating and peak torques must be lower than the minimum torque required to cause relative motion at all train interference fits (API, 2019).

In transient simulation of machine start-ups or short circuits, non-linear behavior of components shall be considered, for example, if the coupling is elastomeric, the non-linear torque x stiffness behavior shall be considered, it is not possible to assume constant stiffness (Hudson, 1996). To obtain the transient response, the most robust solution procedures are based on numerical integration methods, such as Newmark Direct Integration method, which allows to directly solve 2nd order differential equations, or Runge-Kutta methods, which requires the representation of the equations of motion as 1st order Ordinary Differential Equations (ODEs). The numerical integration methods can handle nonlinearities (Corbo; Melanoski, 1996).

3. METHODOLOGY

In this chapter, the methodology adopted in the implementation of the computational procedure to obtain the torsional dynamic response and to analyze geared turbomachinery trains is presented. Initially, it is presented how the equivalent single speed model of a geared turbomachinery train is obtained based on the elimination of the degree of freedom (DOF) associated with the output gear (pinion) of each driven rotor and substitution of variables to represent the driven rotors dynamic behavior in the same speed as the driver shaft, that is the reference adopted for the single speed model. After that, the finite element procedure applied to derive the system equations of motion of the equivalent system is presented. Then, the undamped torsional natural frequency analysis, steady state forced response analysis and short circuit response analysis procedures are covered. Finally, the computational procedure workflow and the validation cases are described. The computational procedure was implemented in MATLAB.

The computational procedure was developed to model and analyze different geared trains configurations, single branch, i.e., having one driver shaft and one driven shaft (for example, see figure 1) or multiple branches, i.e., one driver shaft and multiple driven shafts (for example, see figure 21). Here, a simplified model of the motor driven compressors train of figure 1 is presented:

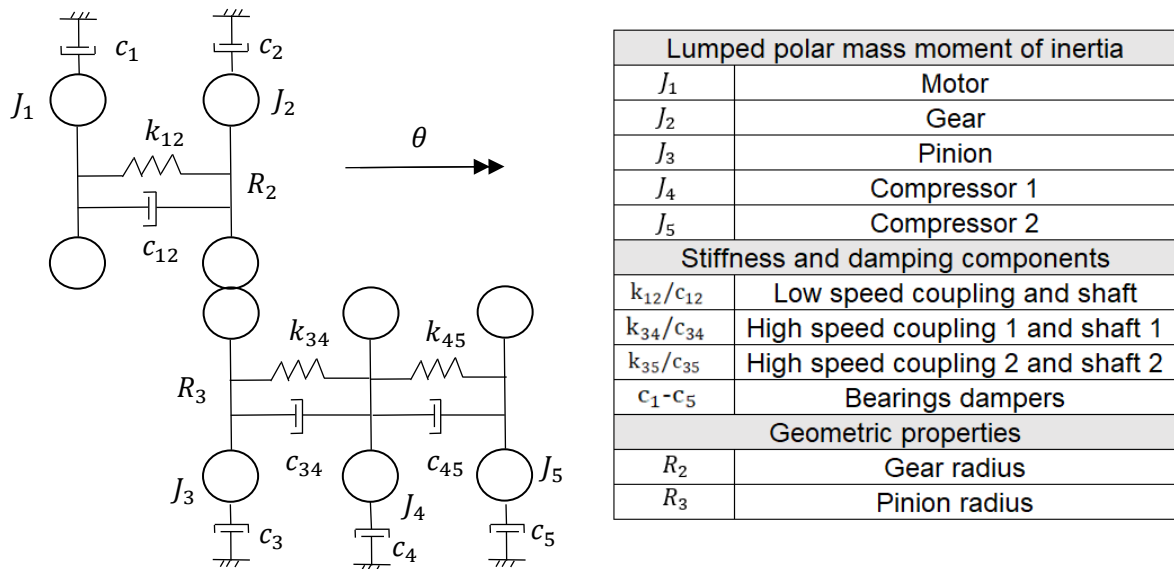


Figure 7: Geared motor driven turbomachinery train model

The geared turbomachinery train of figure 1 is modeled for torsional vibration analysis as a series of lumped inertias connected by stiffness and damping elements. Typically, the modeling of such a train would require dozens of lumped inertias and connecting elements to represent the system with high accuracy, if the guidelines reviewed on section 2.2 are followed, but a simplified model as shown in figure 7 is sufficient for preliminary design assessment, especially for the prediction of the coupling modes associated with the lower TNFs (the most critical modes), although might not be enough to predict higher order natural frequencies associated with the components correctly. For example, in a motor-gear-compressor train, the 3rd mode is almost exclusively governed by the stiffness and inertia characteristics of motor core (API, 2019).

In the model of figure 7, the motor, compressors and gears are represented as one lumped inertia (J_i) each, while couplings and shafts are combined and represented as stiffness (k_{ij}) and damping (c_{ij}) elements. Shafts and coupling hubs inertias are lumped together with adjacent major components. Absolute velocity-dependent torques (e.g., bearings, fluid impellers, etc.) are represented by “ground” dampers (c_i). External torques can be applied to any lumped inertia, typically, electrical excitation is applied to the motor centerline while mechanical excitations are applied at gear mesh.

The computational procedure for the modeling and analysis of geared turbomachinery trains adopts the following simplifying assumptions:

- The continuous torsional system can be modeled using discrete elements.
- The model is purely torsional, coupling between torsional vibration and lateral vibration is not considered.
- The shaft elements are modeled as distributed parameter models by default, adopting a linear angle of twist between the two element nodes. However, the analyst can choose to use lumped parameter model if required.
- Gear teeth are assumed rigid when compared to shaft elements, backlash and damping in gears are neglected.
- Damping can be modeled as modal damping ratio or by discrete dampers, that shall be linear.

- If a VFD is used, the control method is open loop, there is no feedback signal from the mechanical system to the VFD control system, i.e., no electromechanical (E/M) interaction.
- For the determination of natural frequencies and mode shapes of turbomachinery trains, damping is negligible.
- For forced response calculations, forcing torques are harmonic.
- For steady state forced response analysis, the torsional system is linear, a consequence of small angular deflections.
- For short circuit simulation, the speed-dependent torque associated with the driven equipment (e.g. compressor) is considered (non-linear behavior).

3.1. Discrete equations of motion of the torsional vibration of geared systems

The computational procedure derives the discrete equations of motion using the FEM to obtain the mass polar moment of inertia, stiffness and damping matrices of an equivalent single speed system. Since the computational procedure was developed to model and analyze different geared trains configurations, here the procedure is presented using the model of the turbomachinery train shown in figure 7 as reference.

In section 3.1.1, the procedure to obtain the equivalent single speed model is introduced and the discrete equations of motion are derived using the Lagrange method. In section 3.1.2, the procedure adopted to obtain the system matrices based on the FEM is presented. Section 3.1.3 describes the procedure to obtain the state space model of the system.

3.1.1. Equivalent single speed model and its equations of motion

The discrete equations of motion can be obtained by the principle that the net external torque acting on the system is equal to the rate of change of its angular momentum or by the Lagrange method, which being an energy-based approach is convenient for geared systems since there are shafts rotating with different speeds due to the increasing gearbox and it is desirable to represent the system as an equivalent single speed model, with same kinetic and strain energies of the actual model.

For a system with viscous damping, Lagrange's equations can be written as (Inman, 2014):

$$\frac{d}{dt} \left(\frac{\partial KE}{\partial \dot{q}_i} \right) - \frac{\partial KE}{\partial q_i} + \frac{\partial PE}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i \quad (7)$$

where KE is the kinetic energy, PE is the potential energy, R is the Rayleigh dissipation function, q_i are the generalized coordinates ($\theta_i(t)$ of the i -th inertia) and Q_i are the generalized forces ($T_i(t)$ acting at the i -th inertia). For the geared turbomachinery train model of figure 7, the following relations are obtained:

$$KE = \frac{1}{2} [J_1 \dot{\theta}_1^2 + J_2 \dot{\theta}_2^2 + J_3 \dot{\theta}_3^2 + J_4 \dot{\theta}_4^2 + J_5 \dot{\theta}_5^2] \quad (8)$$

$$PE = \frac{1}{2} [k_{12}(\theta_2 - \theta_1)^2 + k_{34}(\theta_4 - \theta_3)^2 + k_{45}(\theta_5 - \theta_4)^2] \quad (9)$$

$$R = \frac{1}{2} [c_1 \dot{\theta}_1^2 + c_2 \dot{\theta}_2^2 + c_3 \dot{\theta}_3^2 + c_4 \dot{\theta}_4^2 + c_5 \dot{\theta}_5^2] + \frac{1}{2} [c_{12}(\dot{\theta}_2 - \dot{\theta}_1)^2 + c_{34}(\dot{\theta}_4 - \dot{\theta}_3)^2 + c_{45}(\dot{\theta}_5 - \dot{\theta}_4)^2] \quad (10)$$

Equivalent single speed model

The herein procedure is described in a similar fashion by Wachel and Szenasi (1993), Corbo and Melanoski (1996), Wu and Chen (2001) and Vance, Zeidan and Murphy (2010).

Defining the motor (driver) shaft as the reference speed shaft and n as the speed of the driven shaft relative to the reference (driver) shaft, it can be written for the mating gears:

$$n = -\frac{R_2}{R_3} = \frac{\theta_3}{\theta_2} = \frac{\dot{\theta}_3}{\dot{\theta}_2} \quad (11)$$

where the magnitude of n is equal to the gear ratio. The signal shall be considered positive, if the mating gears rotate in the same direction, or negative, if the gears rotate in opposite directions (typical configuration).

Equation 11 allows the description of the angular motion of the pinion (J_3) in terms of the gear (J_2) motion, therefore eliminating the DOF associated with the pinion. Besides, the angular motion of compressors 1 (J_4) and 2 (J_5) can be described in the same reference speed of the motor (driver) by the application of the following substitutions:

$$\theta_4 = n\hat{\theta}_4 ; \theta_5 = n\hat{\theta}_5 \quad (12)$$

where $\hat{\theta}_4$ and $\hat{\theta}_5$ are the coordinates representing the angular motion of compressors 1 (J_4) and 2 (J_5) in the equivalent single speed coordinate system.

In more general terms, for any inertia, when converting the angular displacement of a branch of a geared system from its actual speed to the equivalent single speed reference, the following relation applies:

$$\theta_a = n\theta_{eq} \quad (13)$$

Furthermore, the external torques applied at the actual system inertias shall be converted to equivalent torques at the equivalent single speed model, so that they cause the same twisting effect in the system as the actual torque applied at the original geared system:

$$T_{eq} = nT_a \quad (14)$$

Substituting equations 11 and 12 into equations 8-10, the system kinetic and potential energy and Rayleigh dissipation function become:

$$KE = \frac{1}{2} \left[J_1 \dot{\theta}_1^2 + (J_2 + n^2 J_3) \dot{\theta}_2^2 + n^2 J_4 \dot{\hat{\theta}}_4^2 + n^2 J_5 \dot{\hat{\theta}}_5^2 \right] \quad (15)$$

$$PE = \frac{1}{2} \left[k_{12} (\theta_2 - \theta_1)^2 + n^2 k_{34} (\hat{\theta}_4 - \theta_2)^2 + n^2 k_{45} (\hat{\theta}_5 - \hat{\theta}_4)^2 \right] \quad (16)$$

$$R = \frac{1}{2} \left[c_1 \dot{\theta}_1^2 + (c_2 + n^2 c_3) \dot{\theta}_2^2 + n^2 c_4 \dot{\hat{\theta}}_4^2 + n^2 c_5 \dot{\hat{\theta}}_5^2 \right] + \frac{1}{2} \left[c_{12} (\dot{\theta}_2 - \dot{\theta}_1)^2 + n^2 c_{34} (\dot{\hat{\theta}}_4 - \dot{\theta}_2)^2 + n^2 c_{45} (\dot{\hat{\theta}}_5 - \dot{\hat{\theta}}_4)^2 \right] \quad (17)$$

From equations 15-17, the equivalent single speed model can be represented as:

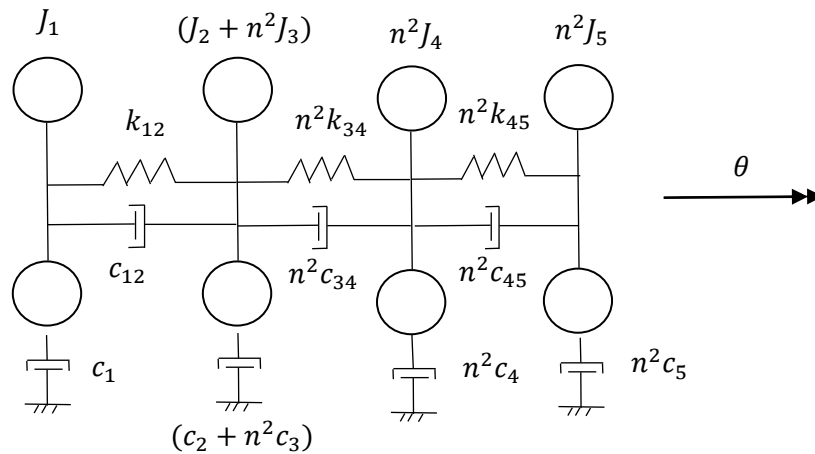


Figure 8: Equivalent single speed model for the geared turbomachinery train

In more general terms, when converting the polar mass moment of inertia, stiffness and damping of a branch of a geared system from its actual speed to the equivalent single speed reference, the following relations applies:

$$J_{eq} = n^2 J_a \quad (18)$$

$$k_{eq} = n^2 k_a \quad (19)$$

$$c_{eq} = n^2 c_a \quad (20)$$

Replacing equations 14-17 into equation 7 and taking the partial derivatives relative to each generalized coordinate, the equations of motion are obtained:

For $q_i = \theta_1$:

$$J_1 \ddot{\theta}_1 + c_1 \dot{\theta}_1 + c_{12}(\dot{\theta}_1 - \dot{\theta}_2) + k_{12}(\theta_1 - \theta_2) = T_1 \quad (21)$$

For $q_i = \theta_2$:

$$(J_2 + n^2 J_3) \ddot{\theta}_2 + (c_2 + n^2 c_3) \dot{\theta}_2 + c_{12}(\dot{\theta}_2 - \dot{\theta}_1) + n^2 c_{34}(\dot{\theta}_2 - \dot{\hat{\theta}}_4) + k_{12}(\theta_2 - \theta_1) + n^2 k_{34}(\theta_2 - \hat{\theta}_4) = T_2 + nT_3 \quad (22)$$

For $q_i = \hat{\theta}_4$:

$$n^2 J_4 \ddot{\hat{\theta}}_4 + n^2 c_4 \dot{\hat{\theta}}_4 + n^2 c_{34}(\dot{\hat{\theta}}_4 - \dot{\theta}_2) + n^2 c_{45}(\dot{\hat{\theta}}_4 - \dot{\hat{\theta}}_5) + n^2 k_{34}(\hat{\theta}_4 - \theta_2) + n^2 k_{45}(\hat{\theta}_4 - \hat{\theta}_5) = nT_4 \quad (23)$$

For $q_i = \hat{\theta}_5$:

$$n^2 J_5 \ddot{\hat{\theta}}_5 + n^2 c_5 \dot{\hat{\theta}}_5 + n^2 c_{45}(\dot{\hat{\theta}}_5 - \dot{\hat{\theta}}_4) + n^2 k_{45}(\hat{\theta}_5 - \hat{\theta}_4) = nT_5 \quad (24)$$

Equations 21-24 can be written in the matrix form:

$$[J]\{\ddot{\theta}\} + [C]\{\dot{\theta}\} + [K]\{\theta\} = \{T(t)\} \quad (25)$$

where:

$$[J] = \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 + n^2 J_3 & 0 & 0 \\ 0 & 0 & n^2 J_4 & 0 \\ 0 & 0 & 0 & n^2 J_5 \end{bmatrix} \quad (26)$$

$$[C] = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 + n^2 c_3 & 0 & 0 \\ 0 & 0 & n^2 c_4 & 0 \\ 0 & 0 & 0 & n^2 c_5 \end{bmatrix} + \begin{bmatrix} c_{12} & -c_{12} & 0 & 0 \\ -c_{12} & c_{12} + n^2 c_{34} & -n^2 c_{34} & 0 \\ 0 & -n^2 c_{34} & n^2(c_{34} + c_{45}) & -n^2 c_{45} \\ 0 & 0 & -n^2 c_{45} & n^2 c_{45} \end{bmatrix} \quad (27)$$

$$[K] = \begin{bmatrix} k_{12} & -k_{12} & 0 & 0 \\ -k_{12} & k_{12} + n^2 k_{34} & -n^2 k_{34} & 0 \\ 0 & -n^2 k_{34} & n^2 (k_{34} + k_{45}) & -n^2 k_{45} \\ 0 & 0 & -n^2 k_{45} & n^2 k_{45} \end{bmatrix} \quad (28)$$

$$\{T(t)\} = \{T_1 \quad T_2 + nT_3 \quad nT_4 \quad nT_5\}^T \quad (29)$$

$$\{\theta(t)\} = \{\theta_1 \quad \theta_2 \quad \hat{\theta}_4 \quad \hat{\theta}_5\}^T \quad (30)$$

Equation 25 of the equivalent single speed model is solved in terms of the equivalent single speed coordinates and equation 13 is used to obtain the angular response of each node in the actual coordinate system.

3.1.2. Equations of motion using the FEM

In the previous section, it was demonstrated how the discrete equations of motion for a geared turbomachinery train can be obtained based on the Lagrange method: a simplified model with 5 lumped inertias was considered and shafts and coupling hubs polar mass moment of inertia were lumped together with adjacent major components. However, when modeling a turbomachinery train according to the guidelines reviewed in section 2.2, dozens of lumped inertias and connecting elements might be required to properly represent the system and shafts inertia should be considered. In this case, it is convenient to use the FEM to obtain the global polar mass moment of inertia, stiffness and damping matrices.

An equivalent single speed turbomachinery model under torsion can be represented as a finite element model, as shown in figure 9, which contains N nodes and $N-1$ torsional elements connecting the nodes. Each node has one DOF represented by an angular displacement $\theta_i(t)$. All rotary inertia such as impellers, gears, couplings, etc. are lumped into discrete disks located at the nodes (J_i), which contain all inertia but no flexibility, while the shaft elements (El_i) contain flexibility and inertia. The shaft dissipation of vibratory energy is represented by dampers connecting two nodes (relative twisting motion of shafts or couplings) or connecting a node to the “ground” (C_i). Torque loads can be applied to any node.

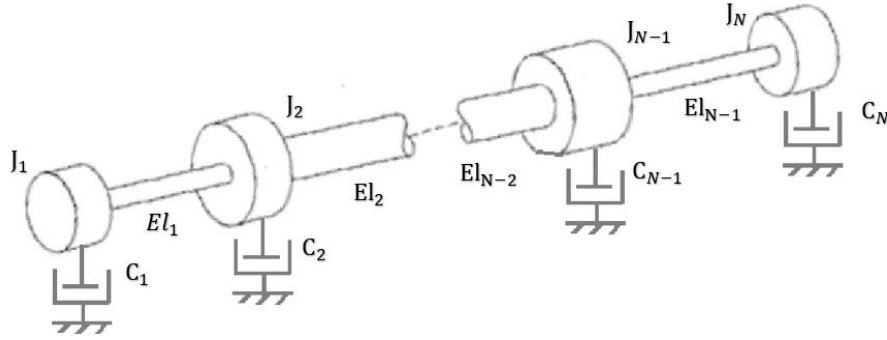


Figure 9: Generic equivalent single speed turbomachinery train model

The FEM consists of describing the equations of motion for each element separately and then combining all by the superposition principle of the Direct Stiffness Method (Walker, 2004). The procedure is presented in sections 3.1.2.1 to 3.1.2.3.

3.1.2.1. Finite element modeling of torsional elements

A torsional linear finite element is represented in figure 10:

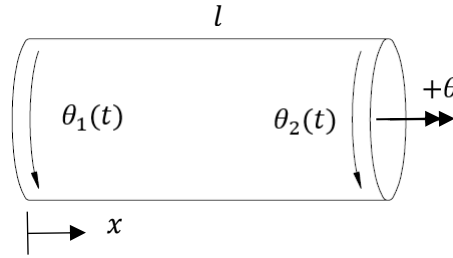


Figure 10: Typical torsional element

Assuming a linear angle of twist, the linear displacement θ at any point along the shaft element can be written as (Walker, 2004):

$$\theta(x, t) = \left(1 - \frac{x}{l}\right) \theta_1(t) + \left(\frac{x}{l}\right) \theta_2(t) \quad (31)$$

Applying Lagrange's method to the torsional element shown in figure 10 and using equation 31, it can be demonstrated that the consistent element inertia matrix and the stiffness element matrix are expressed by (Walker, 2004):

$$[J]_e = \frac{J}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (32)$$

$$[K]_e = k_t \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (33)$$

If a discrete damper of magnitude “c” connects each end node of the element, the element damping matrix is expressed by:

$$[C]_e = c \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (34)$$

In geared systems, where the rotors rotate at different speeds and directions, it is necessary to obtain an equivalent model of the system at the same reference speed, accounting for the differences in kinetic and strain energy associated with the different speeds, therefore the element matrices equations 32-34 shall be multiplied by n^2 , where n is the speed of the element relative to the reference shaft, typically the driver shaft (Walker, 2004; Corbo; Melanoski, 1996):

$$[J]_e = \frac{n^2 J}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (35)$$

$$[K]_e = n^2 k_t \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (36)$$

$$[C]_e = n^2 c \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (37)$$

It shall be remarked that n might be positive, if the mating gears rotate in the same direction, or negative, if the gears rotate in opposite direction.

The applied torques in the actual model shall be converted to the equivalent torque at the reference speed by equation 14. The angular displacements estimated by the solution of equation 25 in the equivalent single speed reference shall be related to the actual angular displacements by equation 13.

If required by the analyst, the computational procedure can also compute shaft inertia using the lumped parameter approach. In this case there is no interpolation function between nodes 1 and 2 of the element and the element inertia is equally divided between nodes, which results for the torsional element and for the equivalent single speed model element:

$$[J_L]_e = \frac{J}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (38)$$

$$[J_L]_e = \frac{n^2 J}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (39)$$

In this case, the global inertia matrix will be diagonal, differently from the finite element approach where the matrix was only symmetric, but not diagonal. Improved accuracy in analysis results can be achieved by formulating energy-distributed inertia matrices since they better approximate the kinetic energy of the system (Walker, 2004). The

lumped vs. energy-distributed inertia matrices influence on shaft natural frequencies is discussed in Appendix A.

3.1.2.2. Global matrices assembly for the equivalent single speed system

Here, the FEM procedure is demonstrated for the model of figure 7, with its equivalent single speed model represented in figure 8, obtained by the application of equations 11 and 12. The first step is to define the element matrices:

Element 1 (low speed coupling and shaft):

$$\begin{array}{cc} \theta_1 & \theta_2 \\ \dot{\theta}_1 & \dot{\theta}_2 \\ \ddot{\theta}_1 & \ddot{\theta}_2 \end{array} \quad [k]_1 = {}^1k_{12} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; [c]_1 = {}^1c_{12} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; [J]_1 = \frac{{}^1J_{12}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (40)$$

Element 2 (high speed coupling 1 + shaft 1):

$$\begin{array}{cc} \theta_2 & \hat{\theta}_4 \\ \dot{\theta}_2 & \dot{\hat{\theta}}_4 \\ \ddot{\theta}_2 & \ddot{\hat{\theta}}_4 \end{array} \quad [k]_2 = n^2 k_{34} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; [c]_2 = n^2 c_{34} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; [J]_2 = \frac{n^2 J_{34}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (41)$$

Element 3 (high speed coupling 2 + shaft 2):

$$\begin{array}{cc} \hat{\theta}_4 & \hat{\theta}_5 \\ \dot{\hat{\theta}}_4 & \dot{\hat{\theta}}_5 \\ \ddot{\hat{\theta}}_4 & \ddot{\hat{\theta}}_5 \end{array} \quad [k]_3 = n^2 k_{45} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; [c]_3 = n^2 c_{45} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; [J]_3 = \frac{n^2 J_{45}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (42)$$

After the definition of all element matrices, they are combined by the superposition principle of the Direct Stiffness Method (Walker, 2004). The components of the element matrices are added to the global matrices in the row and column that correspond to the two global end nodes of each element:

$$[K] = \begin{bmatrix} \theta_1 & \theta_2 & \hat{\theta}_4 & \hat{\theta}_5 \\ k_{12} & -k_{12} & 0 & 0 \\ -k_{12} & k_{12} + n^2 k_{34} & -n^2 k_{34} & 0 \\ 0 & -n^2 k_{34} & n^2 k_{34} + n^2 k_{45} & -n^2 k_{45} \\ 0 & 0 & -n^2 k_{45} & n^2 k_{45} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \hat{\theta}_4 \\ \hat{\theta}_5 \end{bmatrix} \quad (43)$$

$$[C]^{(1)} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_4 & \dot{\theta}_5 \\ c_{12} & -c_{12} & 0 & 0 \\ -c_{12} & c_{12} + n^2 c_{34} & -n^2 c_{34} & 0 \\ 0 & -n^2 c_{34} & n^2 c_{34} + n^2 c_{45} & -n^2 c_{45} \\ 0 & 0 & -n^2 c_{45} & n^2 c_{45} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix} \quad (44)$$

$$[J]^{(1)} = \frac{1}{6} * \begin{bmatrix} \ddot{\theta}_1 & \ddot{\theta}_2 & \ddot{\theta}_4 & \ddot{\theta}_5 \\ 2J_{12} & J_{12} & 0 & 0 \\ J_{12} & 2J_{12} + 2n^2 J_{34} & n^2 J_{34} & 0 \\ 0 & n^2 J_{34} & 2n^2 J_{34} + 2n^2 J_{45} & n^2 J_{45} \\ 0 & 0 & n^2 J_{45} & 2n^2 J_{45} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{bmatrix} \quad (45)$$

Additionally, lumped inertias and “ground” dampers are added to the main diagonal of the polar mass moment of inertia matrix and damping matrix, respectively, in the row and column that correspond to the node where they are located. All lumped inertias and ground dampers are converted to the reference speed using equations 18 and 20:

$$[J]^{(2)} = \begin{bmatrix} \ddot{\theta}_1 & \ddot{\theta}_2 & \ddot{\theta}_4 & \ddot{\theta}_5 \\ J_1 & 0 & 0 & 0 \\ 0 & J_2 + n^2 J_3 & 0 & 0 \\ 0 & 0 & n^2 J_4 & 0 \\ 0 & 0 & 0 & n^2 J_5 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{bmatrix} \quad (46)$$

$$[C]^{(2)} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_4 & \dot{\theta}_5 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 + n^2 c_3 & 0 & 0 \\ 0 & 0 & n^2 c_4 & 0 \\ 0 & 0 & 0 & n^2 c_5 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix} \quad (47)$$

Torque vector, which accounts for the torque at each node, can be easily obtained by direct application of equation 14. The total polar mass moment of inertia matrix is $[J] = [J]^{(1)} + [J]^{(2)}$ and the total damping matrix is $[C] = [C]^{(1)} + [C]^{(2)}$. Comparing equations 43-47 to equations 26-28, both methods agree (exception is the equation 45, that was only computed using FEM), the advantage of the FEM being the ease to implement for larger systems and to account for the distributed inertia of the shafts.

3.1.3. State-space representation

The equations of motion (equation 25), which are second order differential equations, can be represented in the state-space form, so that N second order differential equations are replaced by $2N$ first order differential equations (Inman, 2014), that allows the time response to be obtained by numerical integration using Euler, 4th order Runge-Kutta or other methods. For damped systems, the method also allows to solve the eigenvalue problem, to obtain the N natural frequencies and modal damping (Inman, 2014).

The equations of motion are:

$$[J]\{\ddot{\theta}\} + [C]\{\dot{\theta}\} + [K]\{\theta\} = \{T(t)\} \quad (25)$$

The most general state-space representation of a dynamic system is given by:

$$\begin{cases} \{\dot{X}(t)\} = [A]\{X(t)\} + [B]\{T(t)\} \\ \{\theta(t)\} = [C_{ss}]\{X(t)\} + [D]\{T(t)\} \end{cases} \quad (48)$$

where $\{X(t)\}$ is the state vector and $[A]$, $[B]$, $[C_{ss}]$ and $[D]$ are the state, input-to-state, state-to-output and feedthrough matrices, respectively. The first equation is the state equation and the second is the output or observation equation.

The state vector can be defined as:

$$\{X(t)\} = \begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} = \begin{Bmatrix} \{\theta(t)\} \\ \{\dot{\theta}(t)\} \end{Bmatrix} \quad (49)$$

In this case, it can be written:

$$\{\dot{X}_1\} = \{\dot{\theta}(t)\} = \{X_2\} \quad (50)$$

Substituting equations 49 and 50 into equation 25, and rearranging, the equations of state are obtained in the form of equation 48 (Inman, 2014):

$$\{\dot{X}(t)\} = \begin{Bmatrix} \{\dot{X}_1\} \\ \{\dot{X}_2\} \end{Bmatrix} = [A]\{X(t)\} + [B]\{T(t)\} = \begin{bmatrix} [0] & [I] \\ -[J]^{-1}[K] & -[J]^{-1}[C] \end{bmatrix} \begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} + \begin{bmatrix} [0] \\ [J]^{-1} \end{bmatrix} \{T(t)\} \quad (51)$$

To calculate the twisting in each element, it is required to determine the angular displacements $\{\theta(t)\}$, therefore the desired output is obtained from equation 48 as:

$$\{\theta(t)\} = [C_{ss}]\{X(t)\} + [D]\{T(t)\} = \begin{bmatrix} [I] & [0] \end{bmatrix} \begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} + [0]\{T(t)\} \quad (52)$$

Since the state-space model requires the damping matrix $[C]$ in physical coordinate system, not the modal damping matrix $[C_M]$, but in many cases the analyst only know the modal damping ξ_i , a reverse transformation to $diag(2\xi_i\omega_{ni})$ can be applied to obtain $[C]$:

$$[C_M] = diag(2\xi_i\omega_{ni}) = [S]^T[C][S] \quad (53)$$

Hence,

$$[C] = [S]^{-T}[C_M][S]^{-1} \quad (54)$$

3.2. Undamped torsional natural frequency analysis (eigenvalue analysis)

In most turbomachinery trains, the damping is small and the undamped natural frequencies are nearly equal to the damped ones (Corbo; Melanoski, 1996), therefore, the turbomachinery standards such as API only require the determination of undamped TNFs.

In the undamped torsional natural frequency analysis (eigenvalue analysis), the purpose is to obtain the natural frequencies and mode shapes of vibration. The procedure described hereafter is presented in Inman (2014). It can be accomplished by solving the homogeneous form of the equation of motion of the system:

$$[J]\{\ddot{\theta}\} + [K]\{\theta\} = 0 \quad (55)$$

Considering the physical nature of $[J]$ and $[K]$ is that they are symmetrical, the vibration problem can be connected to a symmetric eigenvalue problem using the Cholesky decomposition (equation 56) and by a change in the coordinate system (equation 57) to obtain a symmetric mass-normalized matrix $\tilde{K}$ (equation 58) (Inman, 2014):

$$[J] = [L][L]^T \quad (56)$$

$$\{\theta(t)\} = ([L]^T)^{-1} \{\phi(t)\} \quad (57)$$

where $[L]$ is a lower triangular matrix. Substituting equations 56 and 57 into equation 55 and pre-multiplying by $[L]^{-1}$, it is obtained:

$$[I]\{\ddot{\phi}\} + [\tilde{K}]\{\phi\} = 0 \quad (58)$$

where $[\tilde{K}] = [L]^{-1}[K]([L]^T)^{-1}$. Assuming a harmonic solution in the form $\vec{\phi}(t) = \vec{v}e^{j\omega t}$, where $\vec{v}$ is a vector of constants that cannot be all zeros to result in movement, result in the algebraic symmetric eigenvalue problem:

$$[\tilde{K}]\{v\} = \omega^2\{v\} \quad (59)$$

where ω^2 are the eigenvalues and the vectors $\vec{v}$ are the corresponding eigenvectors, which have the desirable feature of being orthogonal to each other. For a system with N equations, the N eigenvectors can be grouped in an orthogonal matrix:

$$[P] = [\vec{v}_1 \vec{v}_2 \dots \vec{v}_n] \quad (60)$$

since the eigenvectors $\vec{v}_i$ are obtained in coordinate system $\emptyset$, they are not the mode shapes of the original system described by equation 55. The eigenvectors $\vec{v}_i$ are related to the mode shapes by equation 57. Therefore, the matrix of mode shapes (modal matrix) is defined as:

$$[S] = ([L]^T)^{-1}[P] \quad (61)$$

The mode shape for each natural frequency (columns of $[S]$) can be plotted as a dimensionless relative angular displacement versus axial distance along the train, for the equivalent single speed model. Since the mode shapes only represent the relative angular displacements of one node in relation to the others (the absolute values are meaningless), these plots are typically normalized to unity based on the location of maximum angular displacement and provide information about which components experience most of the twisting (API, 2019), an example is presented in figure 11.

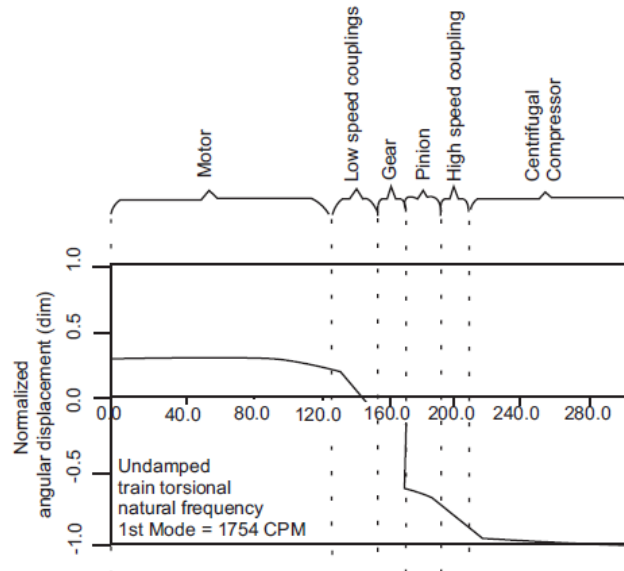


Figure 11: Example of a mode shape plot for the 1st natural frequency of a train (API, 2019)

The modes can be divided into two types: Coupling Modes, when most of the twist occurs at the couplings and Component Body Modes, when most of twist occurs at the train components such as motors, compressors, gearbox, etc. Usually, Coupling

Modes are more dangerous than the others and for a train with N couplings, the first N modes will be coupling modes (API, 2019).

Relative strain energy distribution

From mode shapes, the relative strain energy distribution, defined as the percentage of strain energy associated with one element relative to the total strain energy of the system can be obtained for each system mode. The relative strain energy distribution helps the analyst in the determination of which model elements are most influential in modifying the natural frequency of a mode (common task during design phase) and how the modelling accuracy of one element might influence the results (Mayer, 1981; API, 2019). The relative strain energy ($U_i(\%)$) for the i-th element of a system with N nodes (N-1 elements) for a given mode is given by:

$$U_i(\%) = \frac{k_i(\Delta\theta_i)^2}{\sum_{j=1}^{j=N-1} k_j(\Delta\theta_j)^2} * 100 \quad (62)$$

where k_i and $\Delta\theta_i$ are the stiffness and twisting of the i-th element.

3.2.1. Campbell (Interference) diagrams

Once the natural frequencies are determined, it is usual to plot the Campbell Diagram (aka interference diagram). Corbo and Malanoski (1996) discussed different forms of plotting the Campbell Diagram for geared systems. The natural frequencies appear as horizontal lines, the operation range is represented by vertical lines and the excitations that are harmonics of the rotating speed appear as inclined lines. If an excitation frequency does not vary with the rotating frequency of the machine, for example, those associated with the electrical line frequency in constant speed motors, it appear as a horizontal line and should be represented differently from the natural frequency lines to avoid confusion (e.g. dotted lines). Figure 12 is an example of a Campbell diagram:

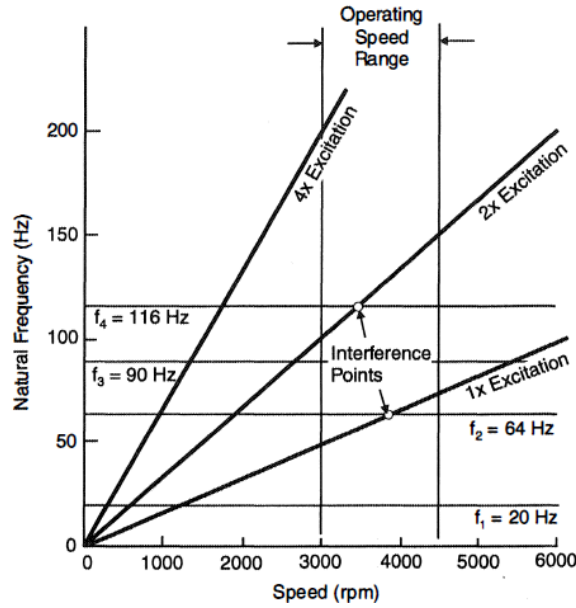


Figure 12: Example of a Campbell Diagram (Corbo; Melanoski, 1996)

In geared turbomachinery, all excitations related to each shaft can be conveniently represented in a single Campbell Diagram, with all excitations referenced to the reference speed (motor speed). To do that, all excitations related to the driven shaft shall have their order multiplied by the gear ratio (Corbo; Melanoski, 1996). In this work, this representation is adopted.

Separation Margin

To account for the uncertainties regarding the determination of the natural frequencies, the separation margin is defined as (API, 2019):

$$SM = \frac{\Delta N}{N_{oplimit}} * 100 \quad (63)$$

Where $N_{oplimit}$ is the top or bottom limit of speed range and ΔN is the difference between the speed at intersection point and $N_{oplimit}$. Analogously, $N_{oplimit}$ could be defined as the excitation frequency at operating speed and ΔN as the difference between the excitation frequency and the natural frequency.

According to API (2019), for pumps and compressors, the minimum separation margin shall be 10% between any excitation frequency and intersection point frequency (torsional natural frequency) to ensure there is no interference in the operational range.

3.3. Steady state forced response analysis

The system equations of motion (equation 25) of the forced damped vibration are composed of second order differential equations. Depending on the matrix $[C]$, different procedures may be applied to solve the equations. If $[C]$ is proportional to $[J]$ and $[K]$, that is, $[C]$ can be written as a linear combination of $[J]$ and $[K]$, then the eigenmodes can be applied to decouple the equations. The eigenvectors cannot in general be used to decouple equations of motion unless $[C][J]^{-1}[K] = [K][J]^{-1}[C]$ holds (Inman, 2014).

If $[C]$ is not proportional, the equations of motion cannot be decoupled. One method to obtain the forced steady-state response of a multiple degree of freedom (MDOF) system is to use the state-space representation combined with a Laplace transform to obtain the FRFs (Frequency Response Functions), which are steady-state functions in the frequency domain (Inman, 2014). For a MDOF system of N outputs (nodes displacements) and M inputs (torque loads), there will be $N \times M$ FRFs. If the system is linear, the outputs at each node due to each input torque can be superimposed (phasor summation) to obtain the combined output.

The torsional steady state forced response analysis consists of identifying all the potential interference points in the Campbell Diagram, and for each point, calculate transmitted torques and stresses associated with the respective excitation torque related to that interference point for all train components, that shall be evaluated for infinite life (API, 2019).

3.3.1. Modal analysis procedure

If the system equations of motion (equation 25) can be decoupled using the Modal Analysis procedure, i.e., $[C]$ is proportional, the principle of orthogonality of modes can be applied to transform matrices $[J]$, $[K]$ and $[C]$ to diagonal matrices using the modal matrix $[S]$, so that in a coordinate system r the equations are uncoupled (Inman, 2014). Equation 64 establishes the relation between the coordinate system θ and r that are applied to equation 25, which is then pre-multiplied by $[S]^T$ to obtain equation 65:

$$\{\theta(t)\} = [S]\{r(t)\} \quad (64)$$

$$[I]\{\ddot{r}\} + [S]^T[C][S]\{\dot{r}\} + [S]^T[K][S]\{r\} = [S]^T\{T(t)\} \quad (65)$$

Equation 65 can be rewritten to highlight the diagonal nature of the transformed damping and stiffness matrices:

$$[I]\{\ddot{r}\} + \text{diag}(2\xi_i\omega_{ni})\{\dot{r}\} + \text{diag}(\omega_i^2)\{r\} = [S]^T\{T(t)\} \quad (66)$$

If there is a harmonic forcing torque of the form $T(t) = T_0 \cos(\omega t)$ acting on the system, equation 66 is a set of single degree of freedom forced vibration equations of the form:

$$\ddot{r} + 2\xi_i\omega_{ni}\dot{r} + \omega_{ni}^2 r = T_{0s} \cos(\omega t) \quad (67)$$

The steady-state solution of equation 67 is (Inman, 2014):

$$r(t) = A_s \cos \omega t + B_s \sin \omega t = X \cos(\omega t - \gamma) \quad (68)$$

$$A_s = \frac{T_{0s}(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2} \quad (69)$$

$$B_s = \frac{T_{0s}2\xi\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2} \quad (70)$$

$$X = \sqrt{A_s^2 + B_s^2} \quad (71)$$

$$\gamma = \tan^{-1} \frac{B_s}{A_s} \quad (72)$$

If in the system there are different torque harmonics applied to the same inertia, i.e., $T(t) = T_1 \cos(\omega_1 t) + T_2 \cos(\omega_2 t) + \dots + T_m \cos(\omega_m t)$, then the computational routine needs to be ran for each harmonic and the combined response for each node (sum of each response considering magnitude and phase) can be obtained by the principle of linear superposition (Inman, 2014).

Equation 68 is transformed back to the original (physical) coordinate system θ of the equivalent single speed model by the application of equation 64 and the actual angular displacement of each node is calculated by equation 13.

With the calculated actual torsion angles for each node, the twisting angles $\Delta\theta = (\theta_{i+1} - \theta_i)$ are calculated. For train components such as couplings, the transmitted torque T and twisting angle $\Delta\theta$ are related by the effective stiffness K_{eff} by (Budynas; Nisbett, 2011):

$$T = K_{eff} \Delta\theta \quad (73)$$

For shaft elements, the maximum nominal cyclic shear stress (τ_{max}) occurs at its outer radius (r_o) and is given by (Budynas; Nisbett, 2011):

$$\tau_{max} = \frac{Tr_o}{J_A} = \frac{Gr_o\Delta\theta}{l} \quad (74)$$

where l is the length and G is the shear modulus of the element. Fatigue analysis is required to determine if the design is acceptable (API, 2019). For the train components such as couplings and gears, the assessment is performed with the support of the manufacturer, based on the estimated cyclic torque. For shaft elements, the assessment is performed as described in section 3.3.3.

3.3.2. Laplace transformation + matrix inversion method

The FRFs are steady-state functions that describe how a system responds to sinusoidal input of frequency ω (Inman, 2014). Applying Laplace transform to the state-space representation of the system (equation 48), a set of first order algebraic equations can be obtained instead of first order differential equations:

$$\begin{cases} (s[I] - [A])\{X(s)\} = [B]\{T(s)\} \\ \{\theta(s)\} = [C_{ss}]\{X(s)\} + [D]\{T(s)\} \end{cases} \quad (75)$$

Isolating $\{X(s)\}$ in the first expression of equation 75 and substituting it in the second expression, the angular displacement can be obtained in the frequency domain:

$$\{\theta(s)\} = [[C_{ss}](s[I] - [A])^{-1}[B] + [D]]\{T(s)\} = [H(s)]\{T(s)\} \quad (76)$$

where $[H(s)]$ is the frequency response function matrix. The FRFs for each node due to each torque input can be obtained by making $s = j\omega$, where ω is the excitation frequency.

The calculation of output (angular displacement) at i^{th} node due to a torque applied at node j^{th} (with all other torques equal to zero) for the k^{th} frequency are provided by the elements $H_{i,j,k}$ of the FRF matrix. Since $H_{i,j,k}$ is a complex number, it can be expressed by a magnitude $M_{i,j,k}$ and phase $\phi_{i,j,k}$ (phasor representation) and a FRF can be generated.

The resultant magnitude and phase at each node are obtained by superimposing the individual responses (phasors summation) to obtain the resultant angular displacement and phase for each node. Knowing the nodes angular displacement, the twisting angle for each element $\Delta\theta = (\theta_{i+1} - \theta_i)$ is estimated.

For train components such as couplings, the transmitted torque is determined by equation 73 and for shaft elements, the maximum nominal cyclic shear stress is determined by equation 74.

Fatigue analysis is required to determine if the design is acceptable. For train components such as couplings, the assessment is performed with the support of the manufacturer, based on the estimated cyclic torque. For shaft elements, the assessment is performed as described in section 3.3.3.

3.3.3. High cycle fatigue analysis for shaft elements

For each interference point in the Campbell Diagram, after the determination of the maximum nominal cyclic shear stress on each shaft element, a fatigue analysis shall be performed. The mean stress should be calculated from the steady-state torque at the operation conditions consistent with the interference point being evaluated. The cyclic and mean stresses shall be multiplied by concentration factors for each system element and finally, superimposes them on a Goodman Diagram to determine if the design is acceptable, i.e., infinite fatigue life. (API, 2019)

The fatigue analysis procedure for elements under torsion consists in determining the safety factor (SF), that is the ratio between the Shear Endurance Limit (S_{se}) and the Equivalent Fully Reversing Shear Stress (S_{eq}) (equation 77). In practice, the design is acceptable if the Safety Factor (SF) is equal or higher than 2.0 (API, 2019).

$$SF = \frac{S_{se}}{S_{eq}} \quad (77)$$

Since endurance limit data is often not available, the Shear Endurance Limit can be calculated by Marin Equation (Budynas; Nisbett, 2011):

$$S_{se} = 0.5S_u k_a k_b k_c k_d k_e \quad (78)$$

In equation 78, S_u is the material ultimate tensile strength. For pure torsion, the loading factor (k_c) predicted by the Distortion Energy Theory is 0.577. Expressions for the surface finishing factor (k_a), the size modification factor (k_b), temperature factor (k_d) and reliability factor (k_e) are found in Budynas and Nisbett (2011).

The nominal cyclic and mean shear stresses shall be multiplied by the concentration factors associated with shaft geometry and loading type (torsion) to obtain the maximum cyclic (τ_{cyclic}) and mean stresses (τ_{mean}). The Equivalent Fully Reversing

Shear Stress based on the Goodman criteria is presented in equation 79 (Corbo; Melanoski, 1996):

$$S_{eq} = \tau_{cyclic} S_{su} / (S_{su} - \tau_{mean}) \quad (79)$$

In equation 79, S_{su} stands for Ultimate Shear Stress, which is equivalent to 0.577 times the Ultimate Tensile Strength (S_u), based on Distortion Energy Theory. In Budynas and Nisbett (2011), the Goodman Diagram is built with the intersection with the x axis at S_u . Corbo and Melanoski (1996) and API (2019) adopt the intersection at S_{su} , which is more conservative, the resulting Goodman Diagram, built according to equation 79 is:

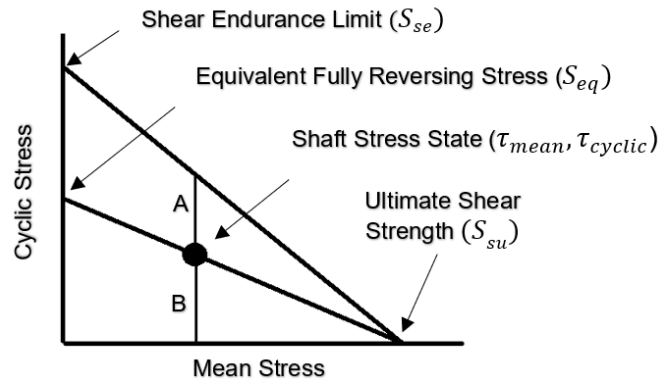


Figure 13: Typical Goodman Diagram

The safety factor can be graphically interpreted as $SF = (A + B)/B$ or the stress state can be converted to an equivalent state with mean stress equal to zero, as defined by equation 77.

3.4. Transient simulation of short circuits

In this work, the transient response is obtained by the numerical integration the system of first order differential equations (equation 51) using the *ode45* MATLAB function, with the initial conditions for the angular displacements and angular speeds of each node defined as $\vec{X}_1(0) = \vec{\theta}_0$ and $\vec{X}_2(0) = \vec{\dot{\theta}}_0$.

$$\{\dot{X}(t)\} = \begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{Bmatrix} = [A]\{X(t)\} + [B]\{T(t)\} = \begin{bmatrix} [0] & [I] \\ -[J]^{-1}[K] & -[J]^{-1}[C] \end{bmatrix} \begin{Bmatrix} \{X_1\} \\ \{X_2\} \end{Bmatrix} + \begin{bmatrix} [0] \\ [J]^{-1} \end{bmatrix} \{T(t)\} \quad (51)$$

For the numerical simulation, if the motor torque defined by equation 6 is applied at node 1 and the load torque (compressor or pump) defined by equation 1 is applied at the last node N, equation 51 can be written as:

$$\{\dot{X}(t)\} = [A]\{X(t)\} + \begin{bmatrix} \{0\} \\ \{f_1\} \end{bmatrix} e^{-\frac{t}{a1}} \sin(\omega_{feed} t) + \begin{bmatrix} \{0\} \\ \{f_2\} \end{bmatrix} e^{-\frac{t}{a2}} \sin(2\omega_{feed} t) + \begin{bmatrix} \{0\} \\ \{f_3\} \end{bmatrix} \frac{X_N^2(t)}{\omega_{rated}^2} \quad (79)$$

where X_N is the state variable which represents the angular speed at the last node N (compressor location) and the vectors $\vec{f}_1$, $\vec{f}_2$ and $\vec{f}_3$ are:

$$f_1 = [J]^{-1} * \begin{Bmatrix} -T_{rated} * T_1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}; \quad f_2 = [J]^{-1} * \begin{Bmatrix} +T_{rated} * T_2 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}; \quad f_3 = [J]^{-1} * \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ -T_{rated} \end{Bmatrix} \quad (80)$$

Since the system of equations 79 has a term with the node speed ($X_N(t) = \dot{\theta}_N(t)$) squared, it is a nonlinear equation.

3.4.1. Numerical integration of 1st order ODE in MATLAB

In MATLAB, there are many options of 1st order ODE solvers, all of them with adaptative step size, the best one for each problem depending on the ODE equation/system type (if it is nonstiff or stiff) and the required accuracy. For most ODE problems, the recommended solver as first choice is *ode45*, especially for nonstiff problems with medium accuracy requirements (The Mathworks Inc, 2025).

There is no precise definition about the stiffness of an ODE, in general, ODE stiffness occurs when there is a difference in the scaling of two solution components somewhere in the problem domain, with the components varying in drastically different time scales. In this case, a nonstiff solver is unable to solve the problem or are extremely slow (The Mathworks Inc, 2025).

In this work, *ode45* was adopted to numerically integrate the equations of motion to obtain the time response of the train due to a short circuit torque. The suitability of the solver was confirmed through Validation Case 4.

According to The Mathworks Inc. (2025), *ode45* solver is an explicit Runge-Kutta method based on the Dormand-Prince pair, basically at each step it calculates the response using two Runge-Kutta predictions, one of 5th order and other of 4th order, and subtract them to obtain the local truncation error, that shall be lower than the predefined error tolerance. If the calculated error is higher than the limit, then the step size is reduced and if the error is too small the step is increased. In the *ode45* solver, the local error is defined as:

$$|e_i| \leq \max(r_e * |X_i|, a_i) \quad (81)$$

where r_e is the relative and a_i the absolute tolerances, with default values of $r_e = 10^{-3}$ and $a_i = 10^{-6}$. If the solution $|X|$ is smaller than the absolute tolerance, the solver does

not need to obtain any correct digits in $|X|$. Because of that, the definition of the value of the absolute tolerance shall consider the scale of the solution components. The relative tolerance measures the error relative to each solution component, therefore, it controls the number of correct digits in all solution components, except those smaller than the absolute tolerance.

3.5. Torsional vibration computational procedure flowchart

The computational procedure flowchart is presented in the figure 14. The procedure outputs are highlighted in grey.

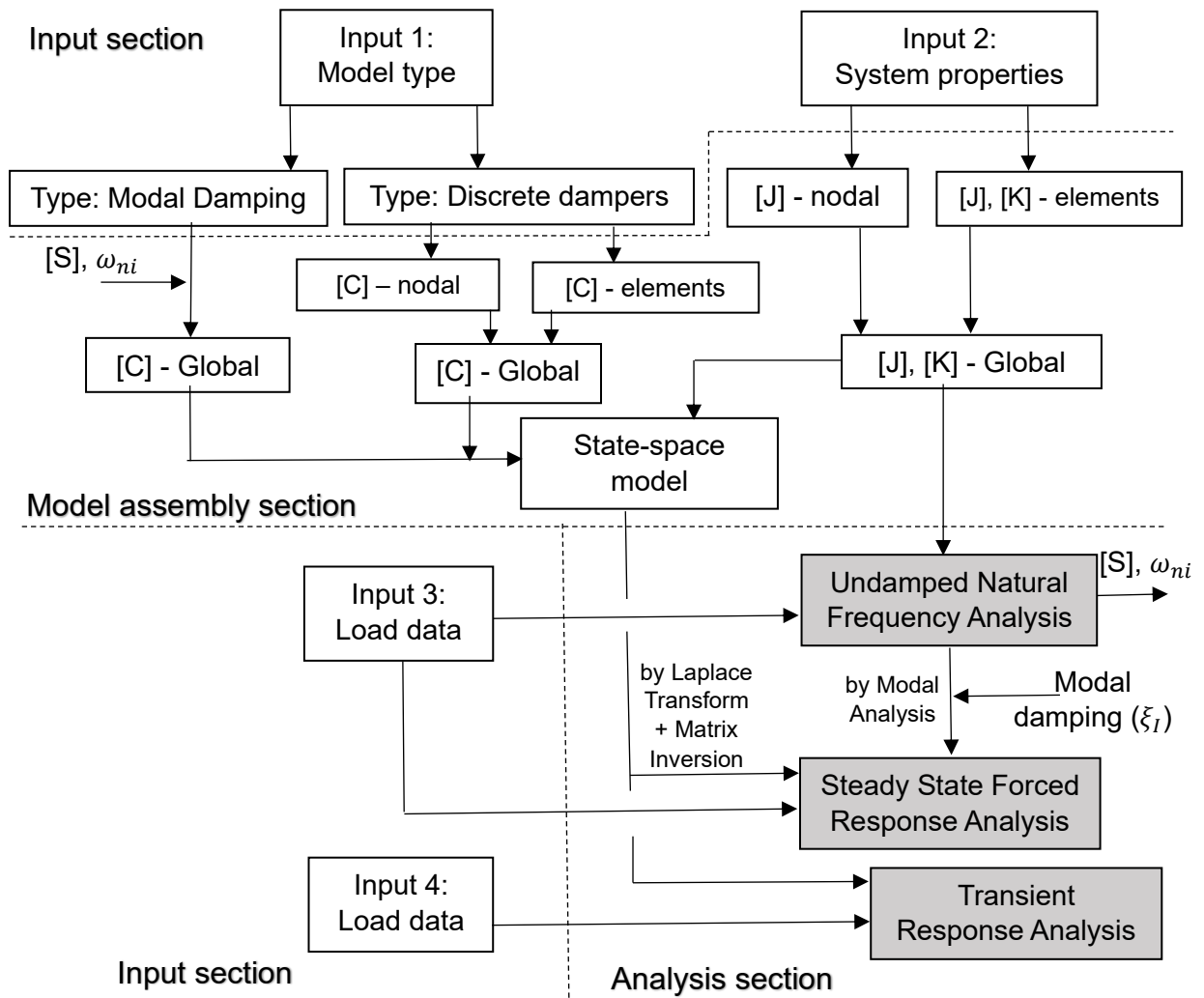


Figure 14: Procedure flowchart for Torsional Analysis of Geared Turbomachinery Trains

For convenience, the procedure flowchart is divided in input, model assembly and analysis sections.

The required inputs are grouped in 4 sets:

- Input 1 – Model options: user needs to inform if shaft polar mass moment of inertia will be considered or not; if the element inertia matrices will be based in consistent formulation (FEM) or lumped; and if the damping model will consider modal damping or if discrete dampers will be used.
- Input 2 – Train Configuration and Properties: user to inform all lumped inertias (placed at the nodes), element properties connecting each node (diameter, length, specific weight, shear modulus) for shafts or torsional stiffness for couplings, speed ratio of each node and element (depends on the gear ratio of each gearbox in the model), operating speed and speed range, driver type (constant speed or variable), damping properties (modal damping for each mode or discrete damper between each node and ground and between nodes).
- Input 3 –Harmonic Excitation orders and steady state excitation: user to inform electrical line frequency, all excitation orders (1x, 2x, 6x, etc. referred to the reference speed) to be plotted in the Campbell Diagram; torque magnitude, phase and frequency applied to each node for the Steady State Forced Response Analysis. Shaft material and geometric concentration factors data for the fatigue analysis.
- Input 4 – Transient simulation data: user to inform the parameters for the two-phase and three-phase short circuit torque x time behavior; motor rated torque; motor electrical fed frequency before short circuit; initial angular position and speed of each node; duration of the simulation; absolute and relative errors for the *ode45* integration function.

The following outputs are generated by the procedure:

- Undamped Natural Frequency Analysis: Torsional Natural Frequencies, mode shapes vector, mode shapes plot, relative strain energy distribution for each mode, Campbell Diagram plot and separation margins.
- Steady State Forced Response Analysis: Node angular displacements and element twisting angles (magnitude and phase) considering the gear ratios to obtain the actual values (not the equivalent single speed), FRFs (magnitude and phase) plots of nodes angular displacements, transmitted torques for couplings and cyclic shear stresses and fatigue safety factors for shaft elements.

- Transient Response Analysis: Time response of the node angular displacements and elements twisting angles, transmitted torques for couplings and shear stresses for each shaft due to 2-phase and 3-phase short circuits.

The computational procedure description is as follows:

First, input sets 1 and 2 shall be provided by the analyst, taking into account the modeling guidelines described in section 2.2. The global equivalent single speed matrices ($[J]$, $[K]$ and $[C]$) and state-space model matrices are obtained according to the procedure described in section 3.1.

The TNFs, mode shapes and relative strain energy distribution are computed according to section 3.2. With the harmonic excitation orders (1x, 2x, 3x, 6x, etc.) provided in the input set 3, Campbell Diagram and separation margins are calculated according to section 3.2.1. The interference points within the operational range are identified.

For each interference point identified in the Campbell Diagram, the analyst needs to input the excitation frequency and applied torques magnitude and phase at each node of the model (input set 3) and a Steady State Forced Response Analysis as described in section 3.3 is performed, using the Modal Analysis Procedure (section 3.3.1) when the damping is input as modal damping ratio or by the Laplace Transform + Matrix Inversion procedure (section 3.3.2) for the more general case where $[C]$ is not proportional to $[J]$ and $[K]$ and also when the damping is input as modal damping ratio. The elements twisting angles calculated are converted to the actual coordinates from the equivalent speed coordinate system and the transmitted torques are estimated for couplings and cyclic shear stresses for shafts. Fatigue assessment is performed for the shaft elements.

Input set 4 is applied to the state space model and 2-phase and 3-phase short circuit simulation is performed as described in section 3.4. Time responses are obtained.

3.6. Validation of the computational procedure

The computational procedure is validated by comparison with other results in the literature. Five validation cases are presented in section 4.1, each one applicable to the validation of different code features. A brief description of each validation case is provided below:

- Validation case 1: Validation of the procedure of reduction of a single gear system to an equivalent single speed model (equivalent inertia, damping and stiffness matrices were obtained), neglecting the shafts inertias. Calculation of natural frequencies and mode shapes. Forced response obtained by Modal Analysis Procedure (modal damping applied to the model) as well as by FRFs, considering one single excitation torque applied to the model. Reference: Brockett and Barrett (1994)
- Validation case 2: same model of case 1, except for the damping and forcing torques: the damping was provided by one discrete damper connected to ground and other connected between two inertias of the model. Besides, two torques were applied to the model, so the angular response of each node due to each input needed to be superimposed (magnitude and phase). Reference: Brockett and Barrett (1994)
- Validation case 3: Validation of the procedure of reduction of a multiple branch geared system to an equivalent single speed model (equivalent inertia, damping and stiffness matrices were obtained using FEM), considering shafts inertia influence. Calculation of natural frequencies, mode shapes and forced response for the system, which had 3 discrete dampers connected to ground. Reference: Wu and Chen (2001).
- Validation case 4: Time response simulation by numerical integration using an adaptative step Runge-Kutta Method (*ode45*). Definition of required absolute and relative error tolerances for torsional vibration response simulation.
- Validation case 5: mesh sensitivity is assessed for the turbomachinery train used for the case study.

4. RESULTS

This chapter is divided into two sections: validation of the torsional vibration procedure (section 4.1) and application of the validated procedure to a case study of a geared turbomachinery train (section 4.2).

4.1. Validation of the torsional vibration procedure

In this section the results generated with the computational procedure are presented. The results are compared to literature references and to results produced by commercial codes such as TWIST2, developed by The Rotating Machinery and Controls (ROMAC) lab at the University of Virginia (Brockett; Barrett, 1994).

4.1.1. Validation case 1

This example is presented in the ROMAC manual of code TWIST2 by Brockett and Barrett (1994). The model used for the code validation is presented in figure 15 and table 2.

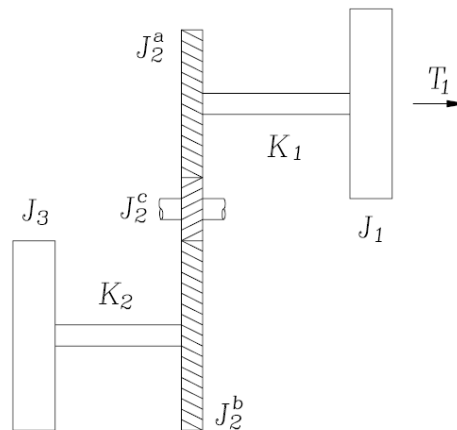


Figure 15: Turbomachinery train model (Brockett; Barrett,1994)

Table 2: Validation case 1 - model data

Inertias id	Polar mass moment of inertia (lbf-in-s ²)	Length (in)	Radius (in)	Speed ratio
J_1	10	-	-	1
J_{2a}	2	-	-	1
J_{2b}	6	-	-	0.5
J_{2c}	neglected	-	-	-3
J_3	24	-	-	0.5
Shafts id	-			
K_1	neglected	40	1.5	1
K_2	neglected	30	2	0.5

Constitutive parameters: $\rho = 0,2830 \text{ lbf/in}^3$ and $G = 0,12 * 10^8 \text{ lbf/in}^2$.

The equivalent single speed model built for the analysis is presented in Figure 16, with the reference speed being defined as the inertia J_1 speed. Gears inertias J_{2a} , J_{2b} and J_{2c} are combined as J_{2eq} and inertia J_3 is represented in the single speed model as J_{3eq} . Shaft torsional stiffness K_2 of the actual model is represented as K_{2eq} in the equivalent single speed model.

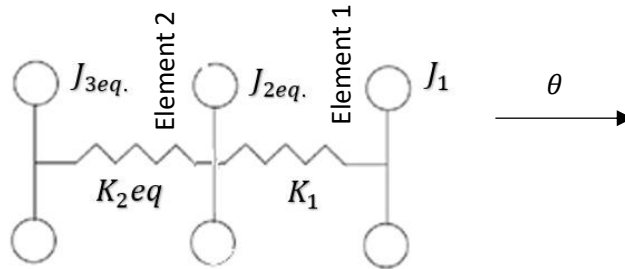


Figure 16: Validation case 1 - Equivalent single speed model

For the steady state torsional analysis, a forcing torque of 28.8 lbf-in is applied at inertia J_1 , with an excitation frequency of 1200 rpm and a modal damping ratio of 2% is assumed for all modes.

The results obtained using TWIST2 code (Annex A) and the present one are the same and they are summarized in tables 3 and 4.

Table 3: Results of the Undamped Torsional Natural Frequency calculations

Undamped Torsional Natural Frequency Analysis			
	Eigenvalues (rad/s):		
	ω_{n1}	ω_{n2}	ω_{n3}
	0	141.557	329.443
Associated eigenvectors			
Equivalent single speed model			
Node	Mode 1	Mode 2	Mode 3
1	1.0000	-0.6821	-0.1593
2	1.0000	0.2346	1.0000
3	1.0000	1.0000	-0.3179
Relative strain energy distribution (%)			
Element	Mode 1	Mode 2	Mode 3
1	0	57.66	42.34
2	0	42.34	57.66

Table 4: Validation Case 1 - Steady State Forced Response results

$$(\theta_i = Mag * \exp(\omega t - Phase))$$

Steady state forced response analysis		
Excitation Frequency: 1200 rpm		
Angular response (°)		
Node	Amplitude	Phase
θ_1	0.0113	13.931
θ_{2a}	0.0116	-175.332
θ_{2b}	0.0058	-175.332
θ_{2c}	-0.0348	-175.332
θ_3	0.0292	-172.167
Max shear cyclic stress (psi)		
Element	Amplitude	Phase
1	89.688	-170.768
2	61.438	-170.077

Frequency response functions of the angular response and phase of nodes 1, 2 and 3 for the excitation range of 1200 to 3492 rpm are presented in Figure 17 to Figure 19. The results are also consistent with TWIST2 code (Annex A) results, for all the frequency range.

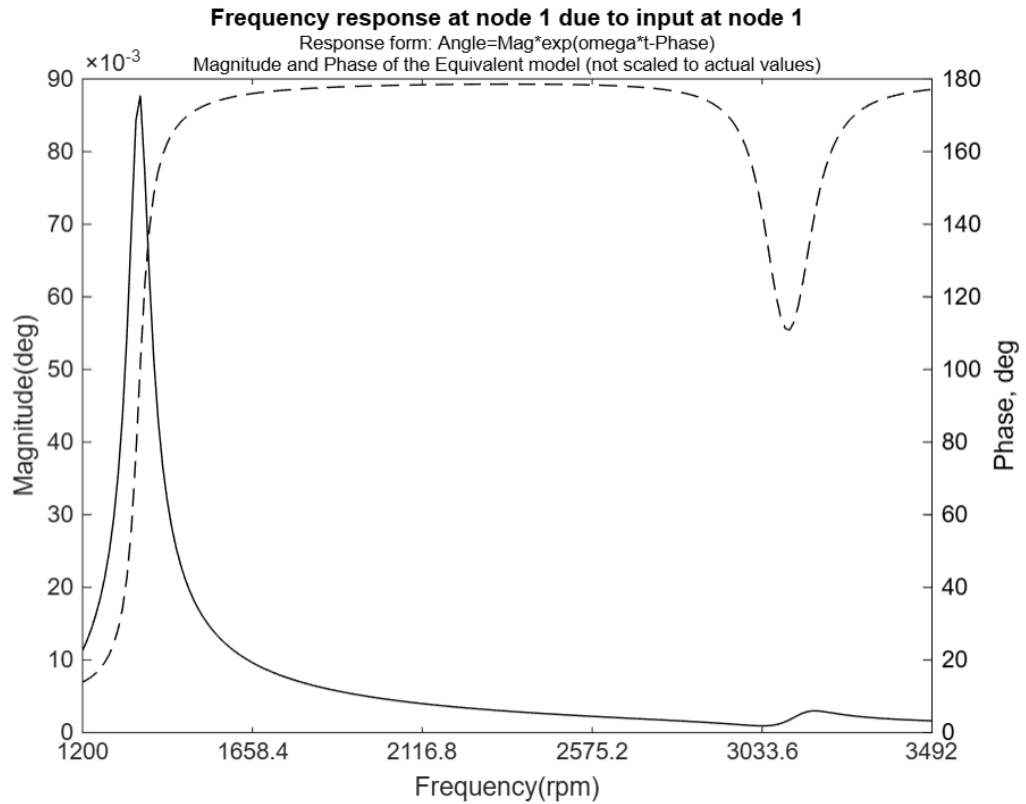


Figure 17: Validation case 1 - FRF of node 1 due to torque input at node 1

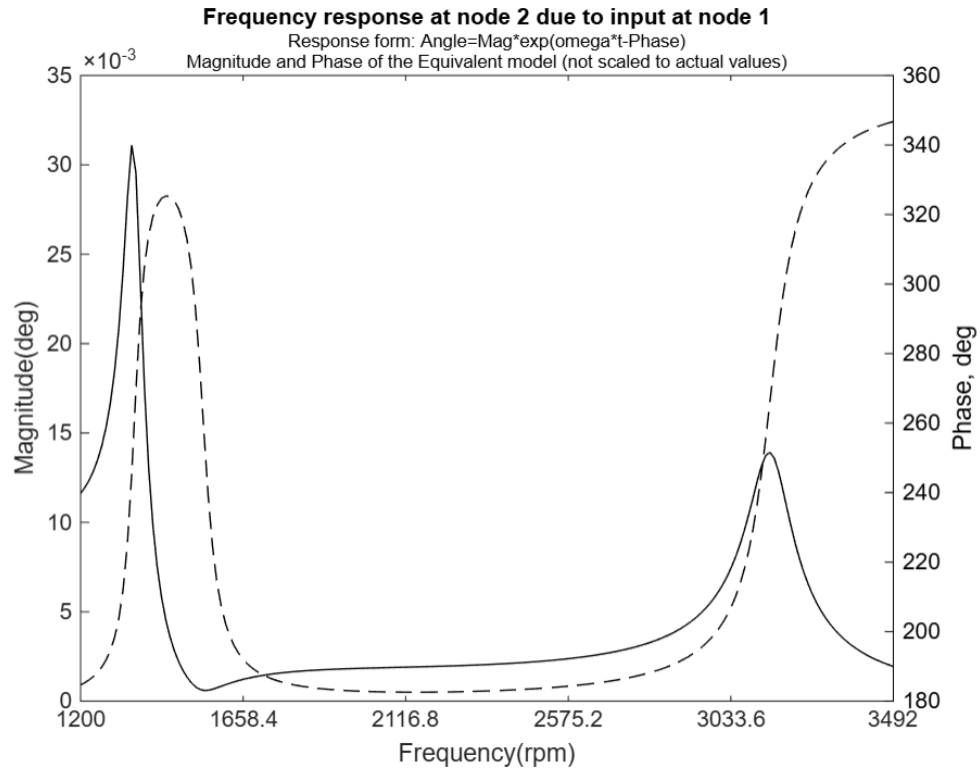


Figure 18: Validation case 1 - FRF of node 2 due to torque input at node 1

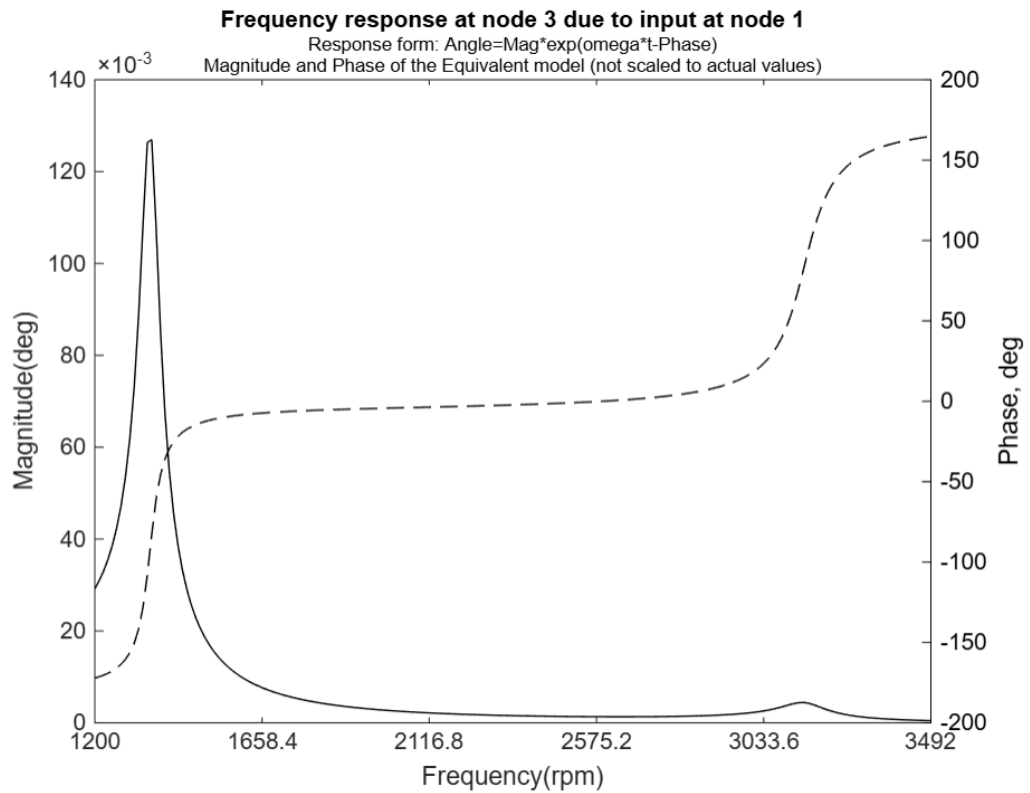


Figure 19: Validation case 1 - FRF of node 3 due to torque input at node 1

With validation case 1, the following features of the code were validated, with results being in full agreement with results presented in TWIST2 (Annex A):

- Procedure of reduction of a single gear system to an equivalent single speed model (equivalent polar mass moment of inertia, damping and stiffness matrices were obtained), neglecting the shafts inertias.
- Undamped Natural Frequency Analysis: Calculation of natural frequencies, mode shapes and relative strain energy distribution.
- Steady State Forced Response Analysis: Angular response (amplitude and phase) of each node as well as cyclic shear stresses (amplitude and phase) of each element were obtained using the Modal Analysis Procedure, considering one single excitation torque applied to the model.
- Steady State Forced Response Analysis: FRFs of the angular response (amplitude and phase) of each node were obtained, for a fixed amplitude and phase excitation torque with variable excitation frequency over a defined range.

4.1.2. Validation case 2

This example is presented in the ROMAC manual of the TWIST2 code by Brockett and Barrett (1994). Here it is adapted so that the amplitude and phase of the torques are only considered for excitation frequency of 1200 rpm (in Brockett and Barrett (1994) the amplitudes and phases are linearly varied with the excitation frequency). It is a similar model of validation case 1, but the viscous dampers are directly modeled instead of the use of the conventional modal damping ratio approach. The $[C]$ matrix can no longer be diagonalized, so Laplace transform was applied to the state space equations of motion and matrix inversion procedure was applied to simultaneously solve the equations of motion and obtain the nodal displacements and element twisting angles. Besides, because there are two forcing torques, the magnitude and phase of angular displacement for each node due to each torque at each frequency were combined to obtain the resultant magnitude and phase of nodal angular displacement. The model is presented in Figure 20.

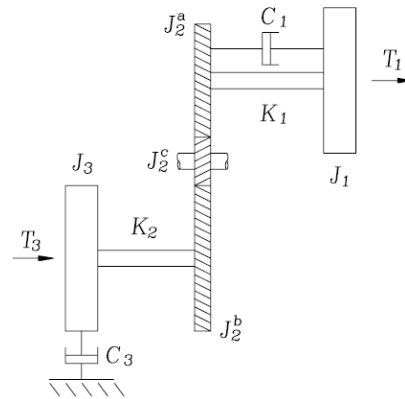


Figure 20: Modified turbomachinery train model (Brockett; Barrett, 1994)

Model data: Mass-elastic data and constitutive parameters according to case 1.

Dampers: $C_1 = 16 \text{ lbf} \cdot \text{in} \cdot \text{s}$ and $C_3 = 10 \text{ lbf} \cdot \text{in} \cdot \text{s}$

Torques: $T_1 = 2,40 \text{ lbf} \cdot \text{ft}$, $\text{Phase} = 0^\circ$ and $T_3 = 0,50 \text{ lbf} \cdot \text{ft}$, $\text{Phase} = 265^\circ$ both at 1200 rpm

The results obtained using TWIST2 code (Annex A) and the present one were the same. The results are presented in table 5.

Table 5: Validation Case 2 - Results for the Steady State Forced Response

$$(\theta_i = \text{Mag} * \exp(\omega t - \text{Phase}))$$

Steady state forced response analysis		
Excitation Frequency: 1200 rpm		
Angular response (°)		
Node	Amplitude	Phase
θ_1	0.0122	18.651
θ_{2a}	0.0117	-177.339
θ_{2b}	0.012	-177.339
θ_{2c}	0.0234	-177.339
θ_3	0.0300	-171.820
Max shear cyclic stress (psi)		
Element	Amplitude	Phase
1	92.840	-169.185
2	64.115	-168.307

With validation case 2, the following features of the code are validated, in addition to the features validated by case 1, with results being in fully agreement with results presented in TWIST2 (Annex A):

- Steady State Forced Response Analysis: Angular responses (amplitude and phase) of each node as well as cyclic shear stresses (amplitude and phase) of each element were obtained for a model containing two different excitation

torques of same frequency applied to different inertias and with damping matrix $[C]$ being not proportional to $[J]$ and $[K]$, so equations of motion could not be decoupled. The angular responses of each node due to each input torque were superimposed (magnitude and phase).

4.1.3. Validation case 3

Wu and Chen (2001) performed eigenvalue and forced response analyses for a multiple branch geared system under torsion. Forced response was estimated using Newmark Direct Integration Method to directly solve 2nd order differential equations. The original machine scheme and the equivalent single speed model are shown in Figure 21.

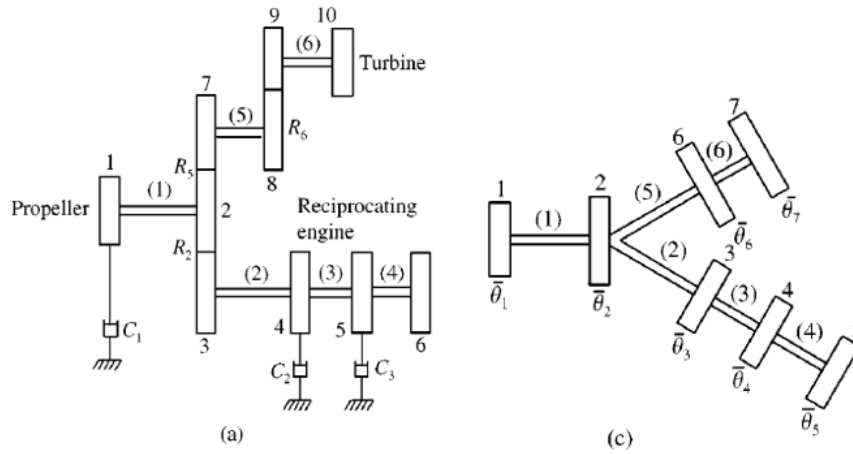


Figure 21: Actual machine model (left) and single speed equivalent model (right)

Machine mass-elastic data, gears ratio and shafts constitutive parameters are provided in the paper. For the forced response, a sinusoidal forcing torque is applied at disk 1 for 3 different values of damping, with $C_1 = C_2 = C_3$ in each of the 3 cases.

In this example, the $[C]$ matrix cannot be diagonalized, so state variable modeling and Laplace Transform are used to obtain the nodal angular displacements. Wu and Chen (2001) did not provide the element stresses for the forced response, only the maximum angular displacement at disk 1.

The eigenvalue analysis comparison for the undamped system is presented in table 6 for 5 modes (excluding rigid body mode). The influence of considering or not the shafts inertias for this machine was also assessed and it is negligible (maximum difference is 1,1% for 3rd mode). The differences between the results of Wu and Chen (2001) and the computational procedure presented in this work are lower than 0,1%.

Table 6: Comparison between natural frequencies obtained by Wu and Chen (2001) and author's code

Mode	Natural Frequencies (rad/s)			
	Shaft mass neglected		Shaft mass considered	
	Wu and Chen	Present work	Wu and Chen	Present work
1	23.0899	23.0900	23.0870	23.0867
2	41.3228	41.3231	41.2902	41.2896
3	217.6013	217.6012	215.2224	215.2024
4	376.9155	376.9228	376.6524	376.6640
5	712.6985	712.6906	710.4786	710.8485

For the forced response, the maximum angular displacement amplitude at disk 1 is compared. Results consistent with those obtained by Wu and Chen (2001) and are presented in table 7.

Table 7: Validation case 3 - Comparison of node 1 torsional amplitude response

Maximum amplitude at node 1 (rad)		
C	Wu and Chen	Present work
lbf.in.s/rad		
1200	0.077	0.0764
3600	0.052	0.0514
6000	0.037	0.0365

With validation case 3, the following features of the code were validated:

- Procedure of reduction of a multiple branch gear system to an equivalent single speed model was applied and equivalent polar mass moment of inertia, damping and stiffness matrices were obtained using FEM. Shafts inertia influence was considered.
- Calculation of natural frequencies, mode shapes and forced response for the system were performed, with the results obtained by the developed procedure being consistent with those published by Wu and Chen (2001). Since the system had 3 discrete dampers connected to ground, the damping matrix was not proportional, so Laplace transform and matrix inversion procedure was applied.

4.1.4. Validation case 4

The torsional vibration procedure developed in this work applies the MATLAB ODE solver “*ode45*” to numerically integrate the equations of motion in the form of equation 79 to obtain the transient response for the short circuit simulation. Since *ode45* is an adaptative step-size algorithm, the method accuracy depends on the absolute and relative errors set for the determination of a step acceptability (equation 81).

In this example, the model of the case study (section 4.2) was excited with a forcing function with the maximum frequency that is present in a short circuit of a conventional induction motor (2x line frequency) and the steady-state analytical and numerical results are compared for element 1 (motor shaft). The forcing function applied to node 1 and steady-state twisting angle of element 1 obtained by the application of modal analysis procedure are:

$$\tau_{twist} = 500 * \cos(628,32 * t) \text{ Nm} \quad (82)$$

$$\theta_{twist} = 0,0063 * \cos(628,32 * t + 179,7)^\circ \quad (83)$$

First, the ode45 function was applied with the default error tolerances (absolute error: $1 * 10^{-6}$ and relative error: $1 * 10^{-3}$) and all initial conditions (angular displacement and speed of each node) equal to zero. The numerical solution is plotted together with the analytical solution in figure 22.

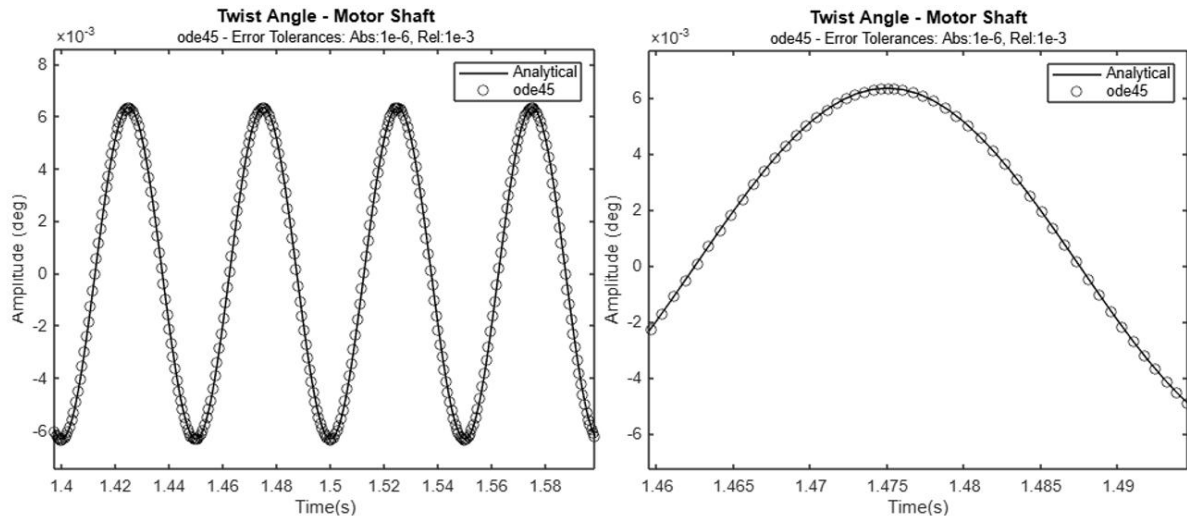


Figure 22: Steady-state twisting angle response of shaft motor – analytical and numerical solution by ode45 with default error tolerances, initial speed=0 rad/s

The accuracy and stability of the numerical solution for the maximum excitation frequency required for short circuit simulation was demonstrated with default (error tolerances (figure 22), with the numerical solution converging to the steady-state analytical solution after few oscillation cycles. However, short circuit simulation might require initial conditions different of zero for position and speed. In the case study, all model elements are initially twisted and rotating with speed equal to the train nominal speed of 2955 rpm (309,4 rad/s). As can be seen in equation 83, the twisting angle amplitude is usually small for torsional vibration, therefore, the nodes displacements shall be calculated with enough number of correct digits so that the twisting angle,

which is the angular displacement difference between two adjacent nodes, is correctly calculated. In *ode45*, the number of correct digits in the solution components is controlled by the relative error parameter. Figure 23 shows the motor shaft twisting angle obtained by numerical integration using *ode45* with default relative error tolerances (1×10^{-3}), for the initial speed of 309,4 rad/s for all nodes.

Comparing figures 22 and 23, it is clear the steady-state solution for the problem with an initial speed of 309,4 rad/s (figure 23) presents abnormal oscillatory behavior of very high frequency with the analytical solution as a mean value and not related to the physics of the problem. This happens because the standard relative error tolerance is not enough to ensure appropriate number of correct digits in the state variables (angular displacement and speed) at each integration step due to the difference in the order of magnitude of the initial speed $O(10^2)$ and the twisting angles amplitude $O(10^{-4})$. Improving the relative error tolerance, the abnormal oscillation in the response is eliminated in figure 24.

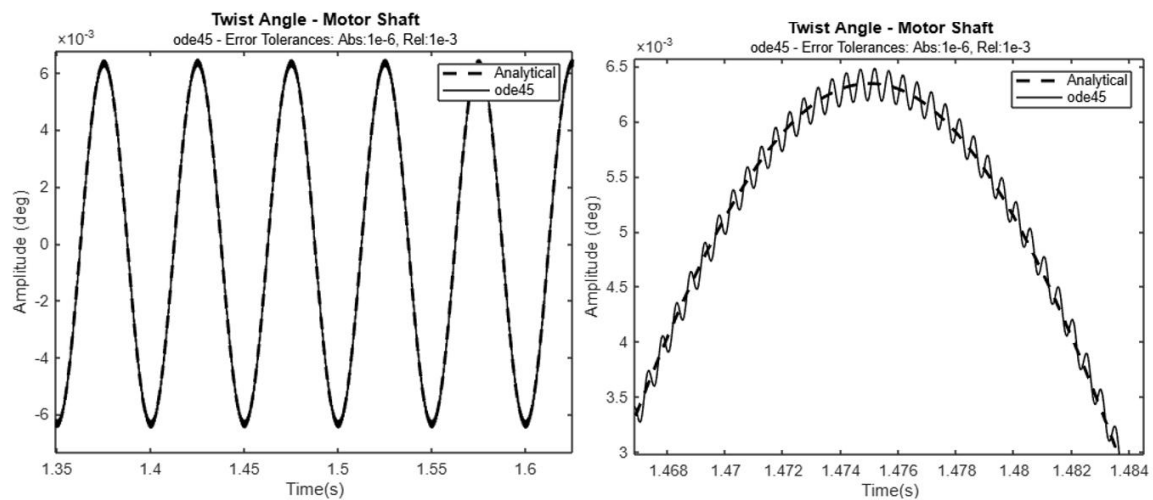


Figure 23: Steady-state twisting angle response of shaft motor – analytical and numerical solution by *ode45* with default error tolerances, initial speed=309,4 rad/s

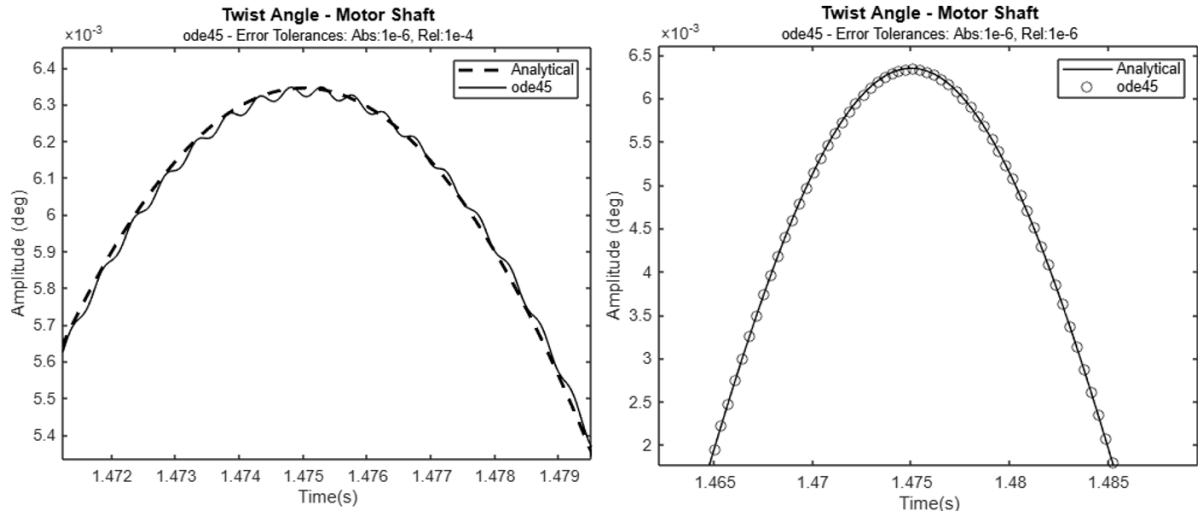


Figure 24: Steady-state twisting angle response of shaft motor – analytical and numerical solution by *ode45* with modified relative error tolerances (left: $1 \cdot 10^{-4}$, right: $1 \cdot 10^{-6}$)

In figure 24, it can be noted that reducing the tolerance for the relative error to $1 \cdot 10^{-6}$ eliminates the abnormal oscillation, which demonstrates that default MATLAB tolerances for *ode45* function is not enough to accurately calculate the twisting angles response for the short circuit response, which might be started with considerable initial speeds.

In the present work, the average time step of *ode45* for the improved tolerances case (absolute error: $1 \cdot 10^{-6}$ and relative error: $1 \cdot 10^{-6}$, figure 24 right) was $3,62 \cdot 10^{-5} \text{ s}$, same order of magnitude that what was applied in Aho, Nerg and Baum (2017).

With validation case 4, the following features of the code were validated:

- Numerical solution by integration of the equations of motion using an adaptative step Runge-Kutta Method (*ode45*) was demonstrated for excitation frequencies
- The influence of the initial conditions, especially initial speed, in the definition of required absolute and relative error tolerances for torsional vibration response caused by short circuits in a typical turbomachinery train was discussed and minimum values required for consistent solution were determined.

4.1.5. Validation case 5

Here the mesh sensitivity is assessed for the turbomachinery train used for the case study (section 4.2). Mesh sensitivity for torsional vibration of a uniform bar with both ends free and with discs (inertias) at each end is presented in appendix A.

The influence of mesh refinement is assessed for the turbomachinery geared train of the case study. The first 4 non-zero system natural frequencies are calculated for different cases: not considering the motor and compressor shafts inertia and considering the shafts inertia and modelling each of them with 1, 2 and 5 elements, using consistent element matrix formulation. The relative errors are calculated assuming the TNFs calculated by the discretization with 5 elements as the reference value. The results are presented in table 8.

Table 8: Influence of shafts inertia and their mesh refinement on the torsional natural frequencies of the case study system

TNF	Estimated natural frequencies (Hz)				Relative Error (%)		
	No shaft inertia	With shaft inertias - elements per shaft			No shaft inertia	1 element	2 elements
		1	2	5			
1st	69.5	69.4	69.2	69.3	0.35	0.19	-0.08
2nd	223.3	223.4	223.4	223.1	0.10	0.15	0.16
3rd	786.8	781.0	771.2	770.2	2.16	1.41	0.14
4th	2024.4	1991.9	1997.8	1993.5	1.55	-0.08	0.22

Based on the results shown in table 8, motor and compressor shafts inertia of the case study model are of minor importance to the accuracy of the torsional analysis, especially for the first 2 modes. The relative error for the 1st and 2nd natural frequencies is lower than 0.5% and the maximum error is just 2,16% for the 3rd natural frequency. The separation margin required between any interference point and a TNF is 10% (API, 2019), which covers modelling simplifications, especially in the preliminary design phase, when detailed train information is not available and the main concern is focused on the prediction of the coupling modes associated with the lower TNFs (the most critical modes). The conclusion is that shaft elements do not need to be subdivided based on length/diameter ratio, they should start/end at diameter changes along the rotors and on lumped inertia locations.

4.2. Case study of VFD induced torsional vibration of a geared turbomachinery train

The validated computational procedure is applied to a case study of a constant speed turbomachinery train (section 4.2.1) and its modified version by the introduction of a VFD (section 4.2.2), that widened the operational range and generated harmonic torsional excitation (6x, 12x, etc.) and inter-harmonic excitation. Short circuit response simulation is presented in section 4.2.3.

4.2.1. Compressor train in the original condition (constant speed motor)

The model used in this case study was adapted from Vance, Zeidan and Murphy (2010):

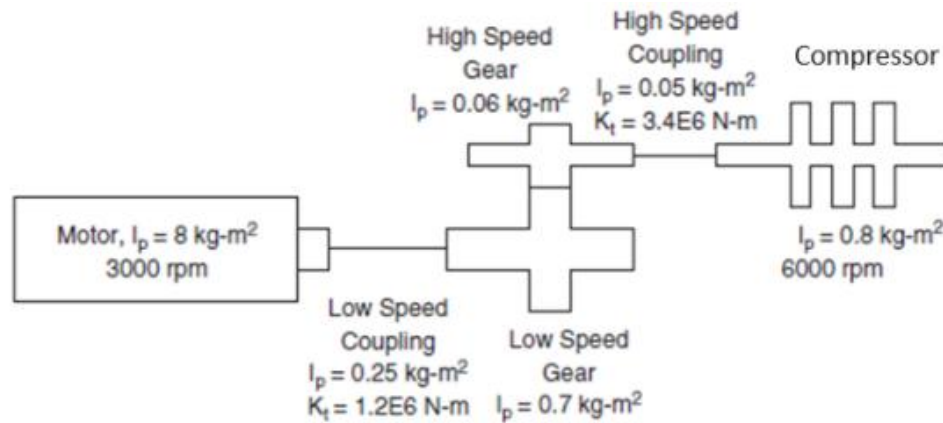


Figure 25: Turbomachinery train model used for the case study (Vance; Zeidan; Murphy, 2010)

Besides, the components presented in the figure, the motor output shaft and compressor input shaft are in series with the couplings and are considered in the torsional model. Their geometric properties are informed in table 9:

Table 9: Motor output shaft and compressor input shaft properties

Shaft data	Motor output	Compressor input
Length (mm)	180	270
Diameter (mm)	80	60
Material	AISI 4140 (Quenched and Tempered)	
Shear modulus (GPa)	80	
Ultimate Tensile Strength (MPa)	1000	
Yield Strength (MPa)	830	

Since the mass-elastic data available is not as detailed as recommended by the guidelines reviewed in section 2.2, the resulting model is simplified and suitable for preliminary design assessment, especially for the prediction of the coupling modes associated with the lower TNFs (the most critical modes), although it might not be enough to accurately predict higher order natural frequencies associated with the components (motor, gearbox and compressor).

The full torsional model is presented in figure 26 and model parameters are shown in table 10:

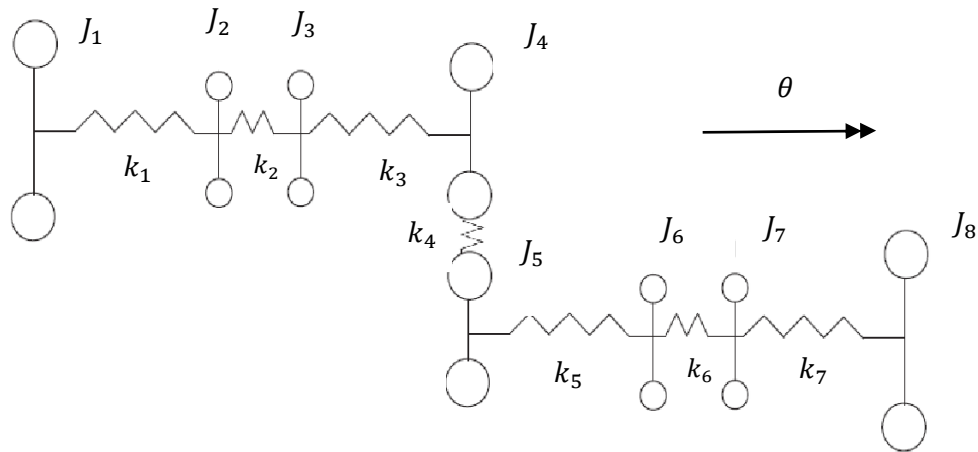


Figure 26: Full torsional model schematic

Table 10: Model parameters – system physical properties

Id	Component	Polar mass moment of inertia (kg*m ²)	Length (m)	Diameter (m)	Torsional stiffness (N-m)	Speed ratio
J_1	Motor	8	-	-	-	1
J_2	Half LS Coupling	0.125	-	-	-	1
J_3	Half LS Coupling	0.125	-	-	-	1
J_4	Gear	0.7	-	-	-	1
J_5	Pinion	0.06	-	-	-	-2
J_6	Half HS Coupling	0.025	-	-	-	-2
J_7	Half HS Coupling	0.025	-	-	-	-2
J_8	Compressor	0.8	-	-	-	-2
k_1	Motor output shaft	0.006	0.18	0.08	1,787,000	1
k_2	LS Coupling	-	-	-	1,200,000	1
k_3	Coupling-gear connection	-	-	-	rigid	1
k_4	Gear tooth flexibility	-	-	-	rigid	-
k_5	Coupling-pinion connection	-	-	-	rigid	-2
k_6	HS Coupling	-	-	-	3,400,000	-2
k_7	Compressor input shaft	0.003	0.27	0.06	339,300	-2

In this case study, the motor and compressor shafts are modeled with a single element each, with minor impact in the TNFs estimation, as discussed in section 4.1.5.

The low and high speed couplings are flexible disk pack couplings with the properties given in table 10. Fatigue strength was not provided.

Since the gear teeth are rigid (k_4) and the stiffnesses of the connection between gearbox and low and high speed couplings are also rigid (k_3 and k_5 , respectively), the equivalent single speed model, correcting the stiffnesses and polar mass moment of inertias to the reference speed is obtained. The model is represented by inertias connected by stiffnesses elements (figure 27) with model parameters shown in table 11:

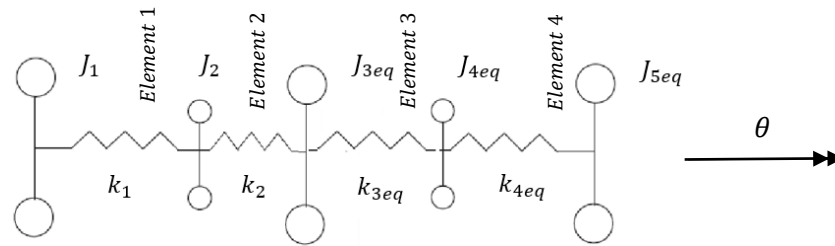


Figure 27: Equivalent single speed torsional model

Table 11: Equivalent single speed model parameters

Id	Component	Polar mass moment of inertia (kg*m ²)	Torsional stiffness (N-m)
J_1	Motor	8	-
J_2	Half LS Coupling	0.125	-
J_{3eq}	Gearbox+half couplings (LS+HS)	1.165	-
J_{4eq}	Half HS Coupling	0.100	-
J_{5eq}	Compressor	3.2	-
k_1	Motor output shaft	0.006	1,787,000
k_2	LS Coupling	-	1,200,000
k_{3eq}	HS Coupling	-	13,600,000
k_{4eq}	Compressor input shaft	0.011	1,508,000

The system motor and gearbox information are presented in table 12:

Table 12: Drive motor and gear ratio data

Motor type	Induction
Motor power (kW)	1550
Number of Poles	2
Line Frequency (Hz)	50
Motor Speed (rpm)	2970
Motor rated Torque (kNm)	5,0
Gear ratio (output/input)	2
Gear teeth number	30
Pinion teeth number	15
Compressor speed (rpm)	5940

Undamped torsional natural frequency analysis (eigenvalue analysis)

The calculation procedure described in Section 3.2 is applied to compute the system natural frequencies (eigenvalues) and mode shapes (eigenvectors) to the equivalent single speed system. The results are summarized in table 13:

Table 13: Undamped natural frequencies and mode shapes for the equivalent model

	Undamped Torsional Natural Frequency Analysis				
	Eigenvalues (Hz):				
	ω_{n0}	ω_{n1}	ω_{n2}	ω_{n3}	ω_{n4}
	0	69.5	223.3	786.8	2024.4
	Associated eigenvectors - equivalent single speed model				
	Mode 0	Mode 1	Mode 2	Mode 3	Mode 4
θ_1	1.0000	-0.4863	-0.0517	-0.0092	0.0000
θ_2	1.0000	-0.0709	0.4041	1.0000	0.0055
θ_{3eq}	1.0000	0.5493	1.0000	-0.0425	-0.0783
θ_{4eq}	1.0000	0.5950	0.8839	-0.0455	1.0000
θ_{5eq}	1.0000	1.0000	-0.2782	0.0009	-0.0029

The mode shapes are presented in figure 28 (rigid body mode omitted):

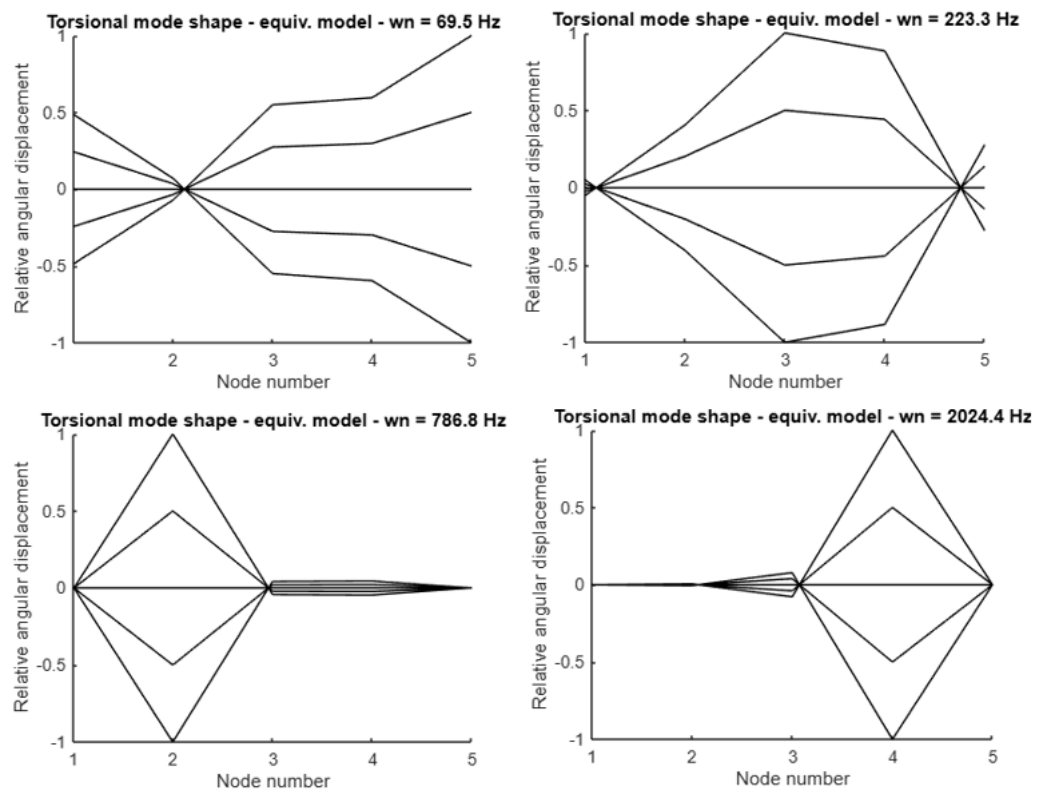


Figure 28: Mode shapes: 1 (top-left), 2 (top-right), 3 (bottom-left) and 4 (bottom-right)

The calculated relative strain energy distribution of each element relative to the total strain energy of the system is presented in table 14:

Table 14: Relative strain energy distribution for modes 1 to 4

Relative Strain Energy Distribution (%)				
Element	Mode 1	Mode 2	Mode 3	Mode 4
1	29.5	12.3	58.2	0.0
2	44.1	14.1	41.7	0.0
3	2.7	6.1	0.0	91.2
4	23.7	67.5	0.0	8.7

From the analysis of the mode shapes (table 13 and figure 28) and relative strain energy distribution of each mode (table 14), it is clear that modes 3 and 4 are not true system phenomenon, but are related only to specific parts of the train: mode 3 is related to the low speed rotor (motor shaft) while mode 4 is related to the high speed rotor (compressor shaft). These subsystem natural frequencies are uncoupled from the remainder of the system. This fact was confirmed by modeling each rotor separately, nearly the same natural frequencies and mode shapes were obtained for the separate sub-systems. Despite of that, these frequencies still represent possible relevant vibration modes and shall be considered in the torsional analysis.

Regarding modes 1 and 2, the strain energy is distributed along the train, both low and highspeed rotors are twisted at each mode, therefore they are global modes, involving the twisting of the whole train. For mode 1, 73,6% of the strain energy is related to the low speed shaft and most of the twisting occurs at the coupling, therefore it can be classified as a “Coupling Mode” while in mode 2, 73,6% of the strain energy is related to the high speed shaft and not at the coupling, what can be explained in this case by the fact that the coupling is much more stiff then the compressor input shaft.

Campbell (Interference) Diagram

For the constant speed case, the following excitations are considered for each shaft:

Motor shaft:

Generic mechanical: 1x and 2x (unbalance, misalignment, eccentricity, ellipticity, etc.)

Gearing: 1x and 2x; GMF: Gear teeth number x speed = 30x

Electrical (unsymmetric voltage supply or rotor eccentricity): 1x and 2x line frequency

Compressor shaft:

Generic mechanical: 1x and 2x

Gearing: 1x and 2x; GMF: Pinion teeth number x speed = 15x

Blade passing frequency: 6 blades x speed = 6x

All excitations related to each shaft can be conveniently represented in a single Campbell Diagram, with all excitations referenced to the motor speed. To do that, all excitations related to the compressor shaft shall have their order multiplied by the gear ratio. Therefore, the following excitations are represented in the diagram:

Generic mechanical: 1x, 2x and 4x

Gearing: 1x, 2x, 4x and 30x

Line Frequency: 1x and 2x line frequency

Blade passing frequency: 12x

The combined Campbell Diagram is presented in figure 29 in the full version (all excitation sources and natural frequencies) and a zoomed version in figure 30 (only 1st and 2nd TNF and 1x, 2x, 4x and line frequency excitations):

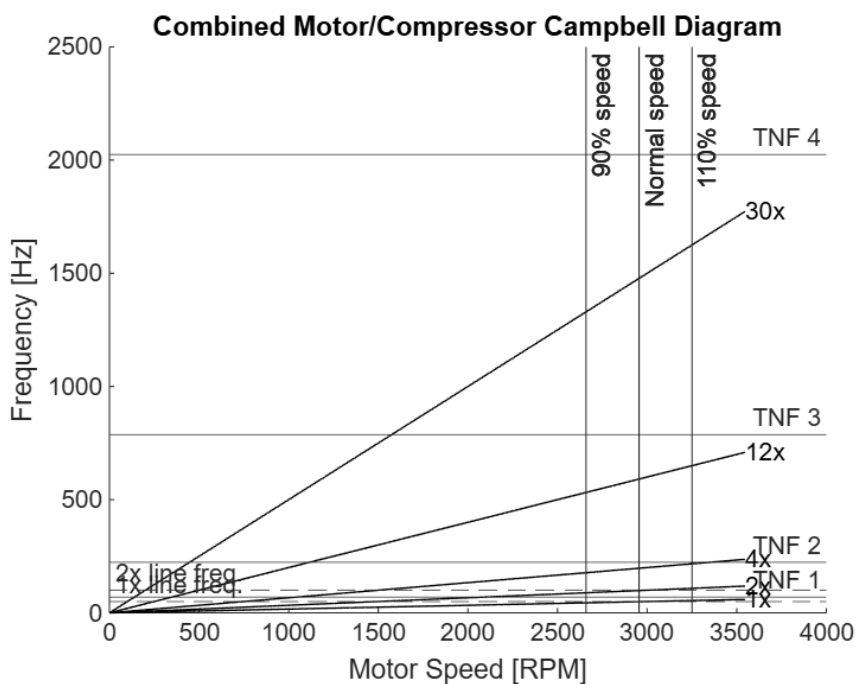


Figure 29: Full Campbell Diagram – all excitations and natural frequencies

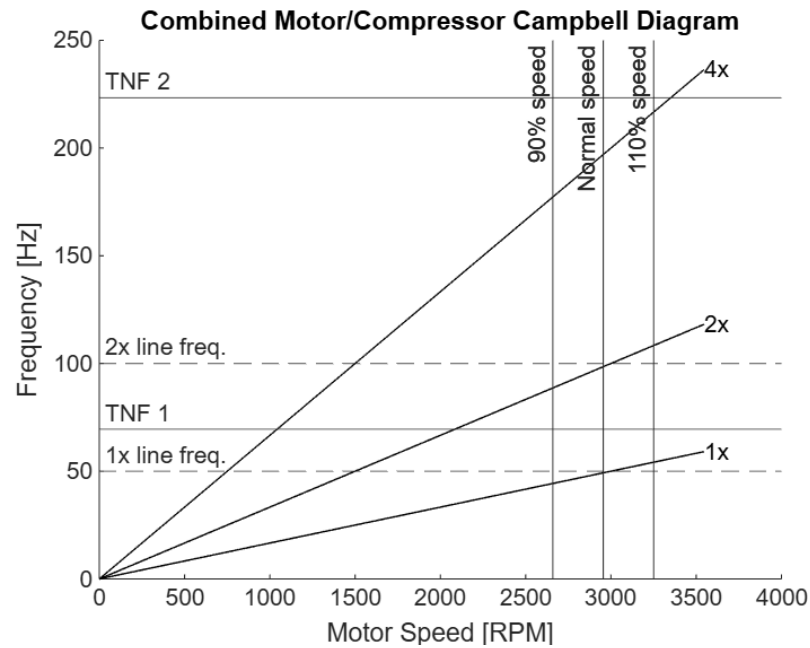


Figure 30: Zoomed Campbell Diagram – 1st and 2nd TNF and 1x, 2x, 4x and line frequency excitations

In figures 29 and 30, the operating speed of 2955 rpm and the 10% separation margin required according to API Standards are represented as vertical lines. The TNFs are represented as continuous horizontal lines. The 1x and 2x line frequencies are represented as horizontal dashed lines and the other excitations are represented as inclined continuous lines. From figure 29, it is clear there are no interference points in the operational speed and in the region from 90-110% of operational speed (minimum separation margin of +/-10% required by API) for the excitations 12x and 30x. From figure 30, the conclusion is the same regarding excitations 1x, 2x, 4x. Separation margins of the excitations and their closest TNF are presented in table 15:

Table 15: Excitation frequencies and separation margins

Excitation frequencies			Separation margin	
Order	rpm	Hz	%	Reference
1x line	3000	50	39.0	to 1st TNF (69.5 Hz)
2x line	6000	100	30.5	to 1st TNF (69.5 Hz)
1x	2955	49.25	41.1	to 1st TNF (69.5 Hz)
2x	5910	98.5	29.4	to 1st TNF (69.5 Hz)
4x	11820	197	13.4	to 2nd TNF (223.3 Hz)
12x	35460	591	33.1	to 3rd TNF (786.8 Hz)
30x	88650	1477.5	37.0	to 4th TNF (2024.4 Hz)

Since there are no interference points in the operational range, i.e., the minimum required separation margins are achieved, steady state forced response analysis is not required.

4.2.2. Modified compressor train with the introduction of a VFD

In this section, turbomachinery train modification to improve energy efficiency is considered. The modification consists of installing a VSI type VFD to control motor speed, that now varies from 70 to 110% of the nominal speed (required 10% separation margins included).

The following excitations are considered for each shaft:

Motor shaft:

Generic mechanical: 1x and 2x (unbalance, misalignment, eccentricity, ellipticity, etc.)

Gearing: 1x and 2x; GMF: Gear teeth number x speed = 30x

Electrical (unsymmetric voltage supply or rotor eccentricity): 1x, 2x motor fed frequency

VFD (VSI type) harmonics: 6x, 12x, 18x and 24x

Compressor shaft:

Generic mechanical: 1x and 2x

Gear: 1x and 2x; GMF: Pinion teeth number x speed = 15x

Blade passing frequency: 6 blades x speed = 6x

All excitations related to each shaft can be conveniently represented in a single Campbell Diagram, with all excitations referenced to the motor speed. To do that, all excitations related to the compressor shaft shall have their order multiplied by the gear ratio. Therefore, the following excitations are represented in the diagram:

Generic mechanical: 1x, 2x and 4x

Gearing: 1x, 2x, 4x and 30x

Blade passing frequency: 12x

Electrical: 1x and 2x

VFD (VSI type): 6x, 12x, 18x and 24x

The combined Campbell Diagram is presented in figure 31 in the full version (all excitation sources and natural frequencies) and a zoomed version in figure 32 (only 1st and 2nd TNF and 1x, 2x, 4x and 6x excitations):

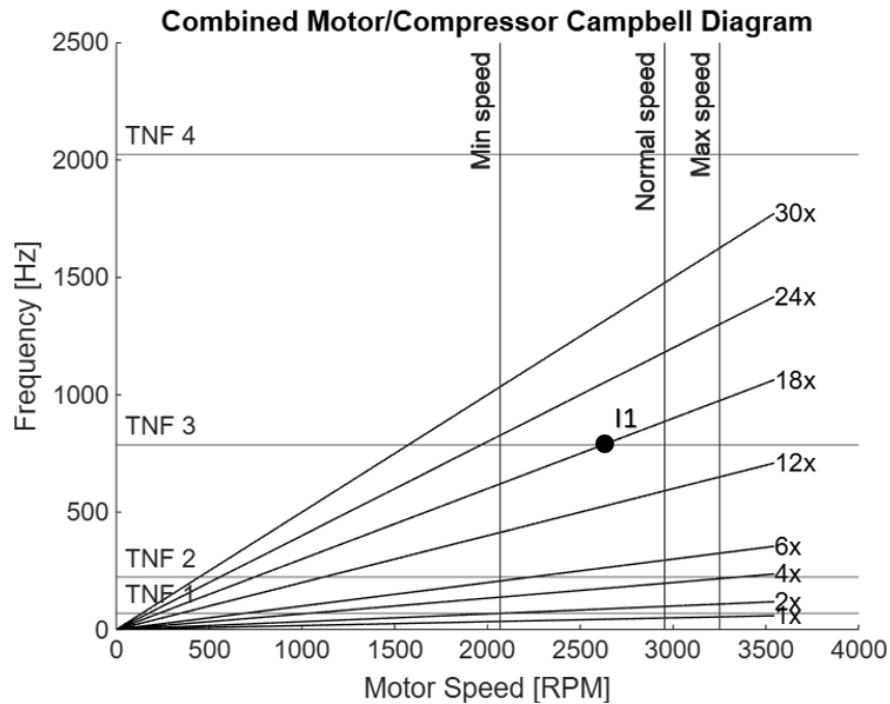


Figure 31: Full Campbell Diagram – all excitations and natural frequencies

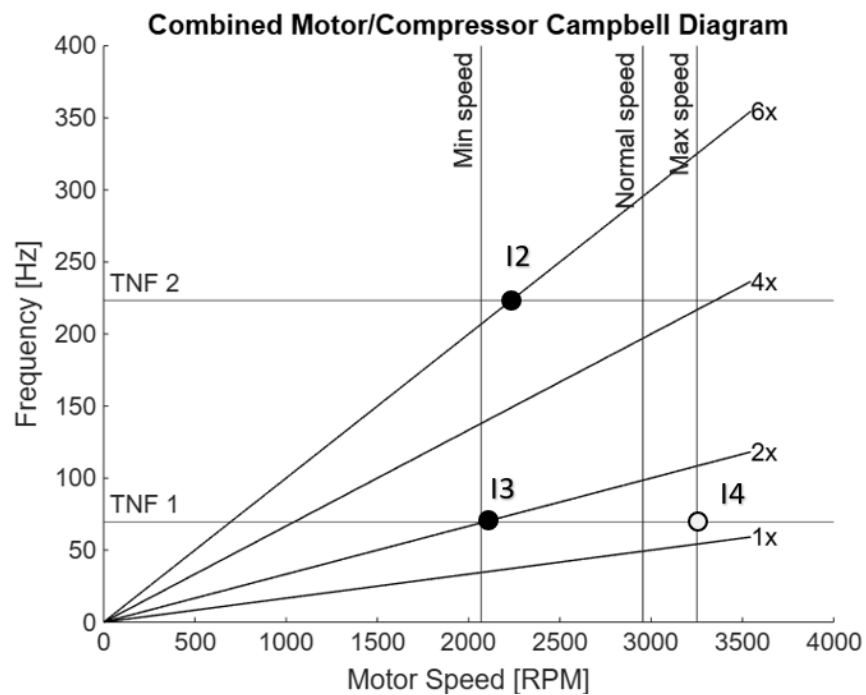


Figure 32: Campbell Diagram – 1st and 2nd TNF and 1x, 2x, 4x and 6x excitations

In this variable speed case, the additional excitation sources as well as the widened operational range causes interference points I1, I2 and I3 in the Campbell Diagrams, which is a typical situation for VFD controlled turbomachinery trains. Additionally, an I4 interference point due to inter-harmonic excitation at the first torsional natural frequency and maximum continuous operating speed with 1% of the rated torque is considered (conservative assumption, as recommended by API (2019)). To evaluate if

the stress levels associated with each interference point are acceptable, a steady-state response analysis is performed for each point.

It is important to highlight that, according to API (2019), if there is an interference between 1x excitation and the Coupling Modes (1st and 2nd modes), the steady-state forced response analysis cannot be applied and the train design needs to be modified. This is the situation with interference point I3, since it is caused by a 2x excitation order, that in this example coincides with 1x compressor shaft frequency since the gear ratio is 2. In this example, the interference is not eliminated, however if this was a real case, the modification would be highly recommended.

The interference points with the respective excitations considered in the steady-state response analysis are presented in table 16:

Table 16: Interference points and associated torsional loads applied to the model

Interference point	Excitation order	Source	Excitation applied to model	Location	Excitation frequency
I1	18x	VFD harmonic	10% of Motor rated torque	Motor	3rd TNF (786.8 Hz)
I2	6x	VFD harmonic	10% of Motor rated torque	Motor	2nd TNF (223.3 Hz)
I3	2x	Generic mechanical (2x at motor speed and 1x at compressor speed)	0.5% for motor shaft and 1% for compressor shaft of the Operating torque	Gear/Pinion	1st TNF (69.5 Hz)
		Gearing (2x at motor shaft speed and 1x of compressor shaft)	0.5% for motor shaft (at gear) or 1% for compressor shaft (at pinion) of the Operating torque	Gear/Pinion	
		Electrical	0.5% of Motor rated torque	Motor	
I4	not applicable	VFD inter-harmonic	1% of Motor rated torque	Motor	1st TNF (69.5 Hz)

Since the VFD type is VSI, which produces torsional excitations that have a weak load-dependence, i.e., the operating point of the machine (speed) does not affect the excitation torques magnitude, the excitations applied to the model are a percentage of motor rated torque. For the mechanical excitation sources, the torque magnitude is a percentage of the mean operating torque for each shaft.

The steady-state response analysis is carried out with a conservative modal damping of 1.0% for all modes. For geared trains supported by fluid film bearings, which is the case of this train, 1.5% of modal damping would be more realistic. The following results are obtained for each interference point:

Table 17: Actual angular displacements and phase angles

Node	Interference 1		Interference 2		Interference 3		Interference 4	
	Amplitude(°)	Phase(°)	Amplitude(°)	Phase(°)	Amplitude(°)	Phase(°)	Amplitude(°)	Phase(°)
θ_1	0.0002	165.1	0.0022	145.3	0.0162	89.7	0.0324	89.7
θ_2	0.0042	-91.0	0.0100	-98.1	0.0024	102.3	0.0048	102.3
θ_{3gear}	0.0002	87.7	0.0246	-91.2	0.0183	-94.4	0.0366	-94.4
$\theta_{3pinion}$	-0.0004	87.7	-0.0492	-91.2	-0.0366	-94.4	-0.0732	-94.4
θ_4	-0.0004	87.9	-0.0434	-91.1	-0.0397	-94.2	-0.0794	-94.2
θ_5	0.0000	-93.1	-0.0137	87.4	-0.0666	-93.4	-0.1333	-93.4

Table 18: Actual twisting angles, cyclic torques and cyclic shear stresses

Element		Interference 1			Interference 2		
		Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)	Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)
Id	Name						
1	Motor shaft	0.0043	133.0	1.3	0.0112	349.5	3.5
2	LS Coupling	0.0044	92.3	NA	0.0147	307.2	NA
3	HS Coupling	0.0000	-1.5	NA	-0.0057	-338.3	NA
4	Compressor shaft	0.0004	-2.6	-0.1	-0.0571	-375.6	-8.9
Element		Interference 3			Interference 4		
		Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)	Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)
Id	Name						
1	Motor shaft	0.0138	431.4	4.3	0.0277	862.9	8.6
2	LS Coupling	0.0206	432.4	NA	0.0413	864.8	NA
3	HS Coupling	-0.0030	-180.7	NA	-0.0061	-361.4	NA
4	Compressor shaft	-0.0270	-177.4	-4.2	-0.0539	-354.8	-8.4

Tables 17 and 18 present the results for the single speed model of figure 27, with node displacements and twisting angles corrected to the actual train configuration, which considers the gear ratio of the gearbox. Twisting angle of each element is calculated by the subtraction of the adjacent nodes displacement, considering magnitude and phase. Cyclic torques and shear stresses for the elements are presented in table 18, except for the shear stress of the coupling, that due to its complex geometry (flexible elements, bolted connections, spacers, etc.), needs to be calculated by the coupling manufacturer. Usually, coupling manufacturers provide the maximum cyclic torque that the coupling can withstand for infinite fatigue life.

Interference point 1 is related to the excitation of system 3rd mode, which was demonstrated to be a subsystem TNF (see table 14), therefore, it only causes relevant nodes displacement and shear stresses at the low-speed rotor (motor shaft). Since the stresses magnitudes are much lower than the other cases, it is not necessary to perform fatigue analysis for this point. This can be explained by the fact that the 3rd mode has no relevant vibratory activity at the motor (see table 13 and figure 28), therefore the excitation applied at the motor node does not excite the mode effectively.

Interference points 2 and 4 are related to harmonic and inter-harmonic VFD excitations of the 2nd and 1st torsional modes, respectively. Despite the fact the torsional load applied for inter-harmonic excitation (point 4) is only 1% of motor rated torque, compared to 10% for the harmonic excitation (point 2), the shear stresses magnitudes are much higher for the motor shaft and of similar magnitude for the compressor shaft, because inter-harmonic excitation was applied at 1st mode TNF while the harmonic excitation was applied at 2nd mode TNF. According to the mode shapes, table 13 and figure 28, both modes involve overall train motion, however the 2nd mode has no relevant vibratory activity at the motor, therefore the excitation applied at the motor node does not excite the mode effectively. On the other hand, the 1st mode involves relevant vibratory activity at the motor, therefore the system is effectively excited, resulting in relevant transmitted torques and shear stresses along the train.

Interference point 3, associated with 2x excitation, could be generated by electrical or by mechanical sources (table 16). A sensitivity analysis was performed applying the electrical excitation at the motor and the mechanical excitations at the gears and the most critical situation identified is related to the electrical excitation.

Fatigue analysis is performed for all points, with nominal mean stresses obtained from the operating torque (speed-dependent) at each shaft and with nominal alternate stresses presented in table 18. The acceptability criteria is infinite fatigue life for all the train components, with a minimum safety factor of 2.0. Detailed fatigue calculation is presented in Appendix B. Table 19 summarizes the main results, the calculated safety factors are all higher than 2, therefore infinite life is predicted for the motor and compressor shafts.

Table 19: Fatigue Analysis summary for interference points 2, 3 and 4

Motor shaft	Interference point		
	2	3	4
Nominal alternating stress (MPa)	3.5	4.3	8.6
Nominal mean stress (MPa)	28	25	60
Equivalent fully reversing stress (MPa)	12.3	14.8	37.6
Shear Endurance limit (MPa)	119.2		
Calculated Safety factor	9.7	8.0	3.2
Compressor shaft	Interference point		
	2	3	4
Nominal alternating stress (MPa)	8.9	4.2	8.4
Nominal mean stress (MPa)	34	29	71
Equivalent fully reversing stress (MPa)	32.4	14.9	40.1
Shear Endurance limit (MPa)	124.7		
Calculated Safety factor	3.9	8.4	3.1

In some instances, the combination of different harmonic torques that could be exciting the system at same time might be required. API (2019) requires this approach only for reciprocating machines, where the combined response due to multiple harmonics might be larger than the response due to a single excitation frequency. In the present case, it is worth noting that the intersections between electrical (1x, 2x) / VFD excitation frequencies (6x, 12x, 18x, 24x) and TNFs occur far from each other in terms of motor speed for each mode, therefore the train is not analyzed with a combined excitation torque (considering magnitude and phase of each harmonic) containing all VFD harmonics. The convenience of performing such analysis for VFD motor driven trains should be evaluated on a case-by-case basis and with the direct involvement of the VFD manufacturer, that should provide all the required VFD harmonics data.

4.2.3. Transient Response Analysis

In this section, it is presented the transient response analysis of short circuit of the induction motor, which is the transient analysis normally performed for an induction motor driven turbomachinery train.

For the constant speed motor (section 4.2.1), separation margins are 39,0% for 1x line frequency and 30,5% for 2x line frequency (table 15), which means that they exceed the minimum separation margin requirement of 10%. Nonetheless, simulating the short circuit and performing and mechanical integrity assessment is a recommended design practice.

For the variable speed train (section 4.2.2), the motor fed frequency is variable and the electrical excitation frequencies occur at 1x and 2x the VFD output frequency. As per figure 32, there is an interference in the Campbell diagram related to the 2x excitation, which means that a short circuit analysis is mandatory for the running speed that is associated with the resonance (interference) point.

The following simulations are performed in this work:

- I. Three-phase short circuit for the constant speed case, where the motor was running at the nominal speed (2955 rpm) before the event (motor fed frequency of 50 Hz);
- II. Two-phase short circuit for the constant speed case, where the motor was running at the nominal speed (2955 rpm) before the event (motor fed frequency of 50 Hz);

- III. Three-phase short circuit for the variable speed case, where the motor was running at 2085 rpm before the event (motor feed frequency of 34.75 Hz), which corresponds to the interference point of 2x motor feed frequency with the 1st TNF (69.5 Hz);
- IV. Two-phase short circuit for the variable speed case, where the motor was running at 2085 rpm before the event (motor feed frequency of 34.75 Hz), which corresponds to the interference point of 2x motor feed frequency with the 1st TNF (69.5 Hz);

The equivalent single speed model represented in figure 27 is used in the simulations.

Initial conditions

There is a set of initial conditions for cases I and II and another for III and IV. For cases I and II, the mean transmitted torque before the short circuit is the motor rated torque, i.e, 5000 Nm. For cases III and IV, since the operational speed is 2085 rpm, the mean transmitted torque is $5000 \cdot (2085/2955)^2 = 2490$ Nm (from equation 1). With these applied torques, the initial angular displacements are calculated, assuming the angular displacement of inertia 1 equal to 0, since the system is free (there is a rigid body mode):

$$\theta_{0_{Case-I-II}} = \{0 \quad 0.0028 \quad 0.0070 \quad 0.0073 \quad 0.0107\} rad$$

$$\theta_{0_{Case-III-IV}} = \{0 \quad 0.0014 \quad 0.0035 \quad 0.0037 \quad 0.0053\} rad$$

The initial speed for the simulation is the running speed, that is:

$$\dot{\theta}_{0_{Case-I-II}} = \{309.5 \quad 309.5 \quad 309.5 \quad 309.5 \quad 309.5\} rad/s$$

$$\dot{\theta}_{0_{Case-III-IV}} = \{218.3 \quad 218.3 \quad 218.3 \quad 218.3 \quad 218.3\} rad/s$$

Applied torques

The 3-phase and 2-phase short circuit torques have the form of equations 84 and 85:

$$T_{sc3ph} = -T_{rated} \left(T_1 e^{-\frac{t}{a1}} \sin(\omega_{feed} t) \right) \quad (84)$$

$$T_{sc2ph} = -T_{rated} \left(T_2 e^{-\frac{t}{a2}} \sin(\omega_{feed} t) - T_3 e^{-\frac{t}{a3}} \sin(2\omega_{feed} t) \right) \quad (85)$$

Most of the parameters of equations 84 and 85 are dependent on the motor design and shall be supplied by the manufacturer. For this case study, the following parameters are considered:

$$T_{rated} = 5.0 \text{ kNm}, T_1 = T_2 = 6.0, T_3 = 3.0, a_1 = 0.030, a_2 = a_3 = 0.026$$

The short circuit excitation torques, normalized by the rated torque (per unit, P.U.) are presented for cases I and II (figure 33) and III and IV (figure 34):

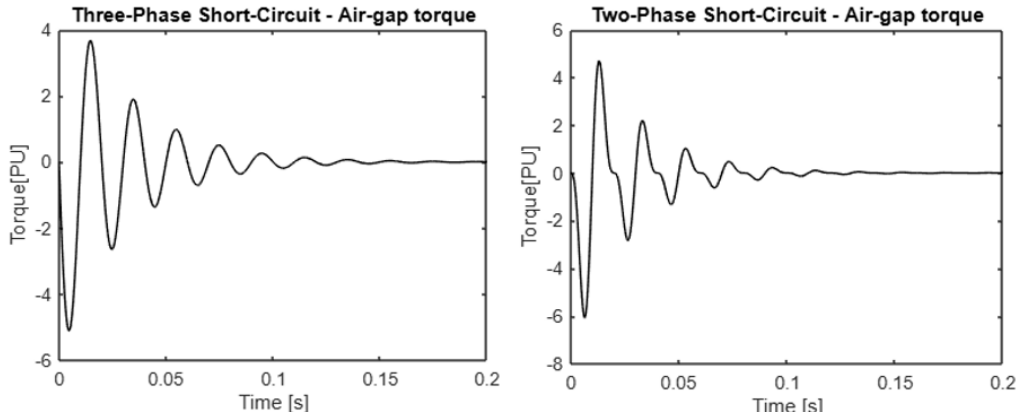


Figure 33: Short circuit excitation torques for cases I and II

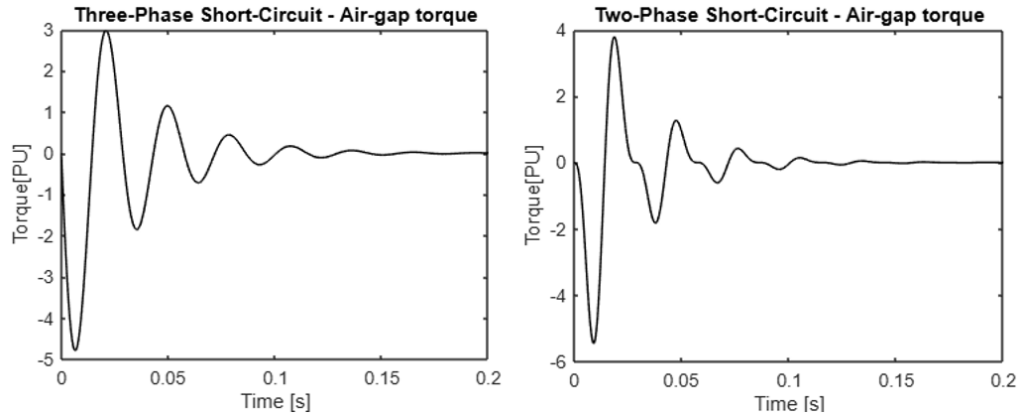


Figure 34: Short circuit excitation torques for cases III and IV

The load torque due to the compressor is considered in transient analysis. The load torque is derived from equation 1 and is defined as:

$$T_{load}(t) = -\left(\frac{T_{rated}}{\omega_{rated}^2}\right) * \omega_{op}^2(t) \quad (86)$$

Since the load torque is dependent on the compressor operational speed squared, that varies with time during the transient regime, the compressor equation of motion is non-linear. During the numerical integration of the system equations of motion, the load torque is updated at each step based on the load-torque equation.

Short circuit simulation results

With the initial conditions and applied loads defined, the numerical integration of equation 79 was performed in MATLAB using *ode45* function, with the absolute and relative tolerances of $1 * 10^{-6}$ (improved relative tolerance, as discussed in section 4.1.4). Each node displacement at each time step is calculated and the twisting angle of each model element is calculated by the difference of adjacent nodes displacement at each time. The actual twisting angles (corrected by the gear ratio) and the respective transmitted torques and shear stresses at each element are obtained. The torques at each element for cases I and IV (cases II and III results are not plotted) are presented in figures 35 and 36.

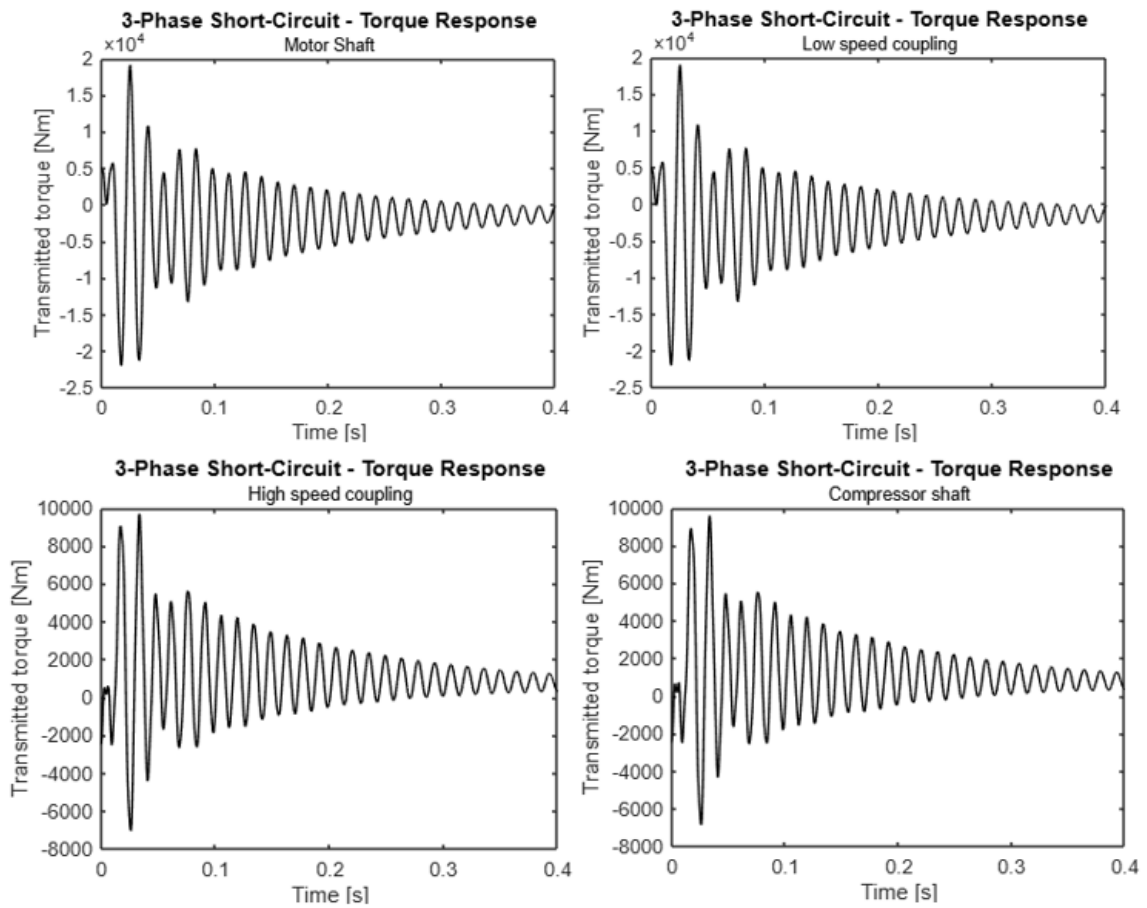


Figure 35: Torque at each model element – Case I

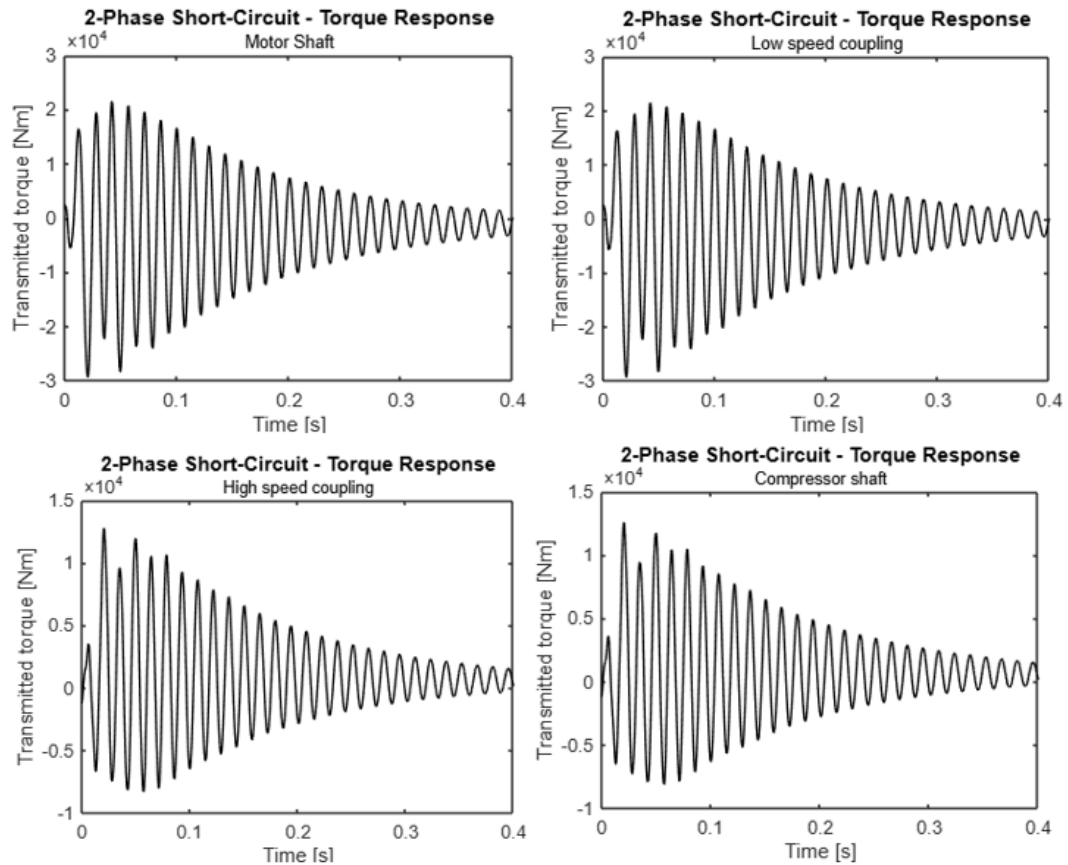


Figure 36: Torque at each model element – Case IV

Table 20 summarizes, for cases I to IV, the maximum twisting angles and transmitted torques for all train elements and shaft shear stresses:

Table 20: Maximum twisting angle, torques and shaft shear stresses on the train elements during short circuit simulation

Element		Case I			Case II		
		Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)	Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)
Id	Name						
1	Motor shaft	0.7074	22067	219.5	0.9338	29127	289.7
2	LS Coupling	1.0500	21994	NA	1.3925	29167	NA
3	HS Coupling	0.1641	9736	NA	0.2017	11971	NA
4	Compressor shaft	1.4648	9638	227.3	1.7846	11742	276.9
Element		Case III			Case IV		
		Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)	Twisting angle (°)	Transmitted Torque (Nm)	Shear stress (MPa)
Id	Name						
1	Motor shaft	0.4753	14827	147.5	0.8774	27369	272.2
2	LS Coupling	0.7035	14736	NA	1.3072	27379	NA
3	HS Coupling	0.1054	6256	NA	0.1933	11473	NA
4	Compressor shaft	0.9366	6162	145.3	1.7146	11282	266.0

According to the results summarized in table 20, the 2-phase short circuit is more severe for the constant speed motor (case II) as well as for the variable speed motor (case IV) when compared to the 3-phase short circuit. Even though the operating (mean) torque for cases III and IV before short circuit was nearly half the operating

torque of cases I and II, the maximum transmitted torques of the 2-phase short circuit of the variable speed motor (case IV) are similar to those of case II (-6.0% for the motor shaft and -3.9% for the compressor shaft), which can be explained by the excitation of the first TNF by the 2X component of the short circuit torque. The same is not observed for the 3-phase short circuit (cases I and III), where there is no resonance of the short circuit torque with the 1st torsional mode in both cases.

The short circuit excitation torque and response torque on the motor shaft, normalized by motor rated torque (per unit, P.U.), are presented in figures 37 (cases I and II) and 38 (cases III and IV).

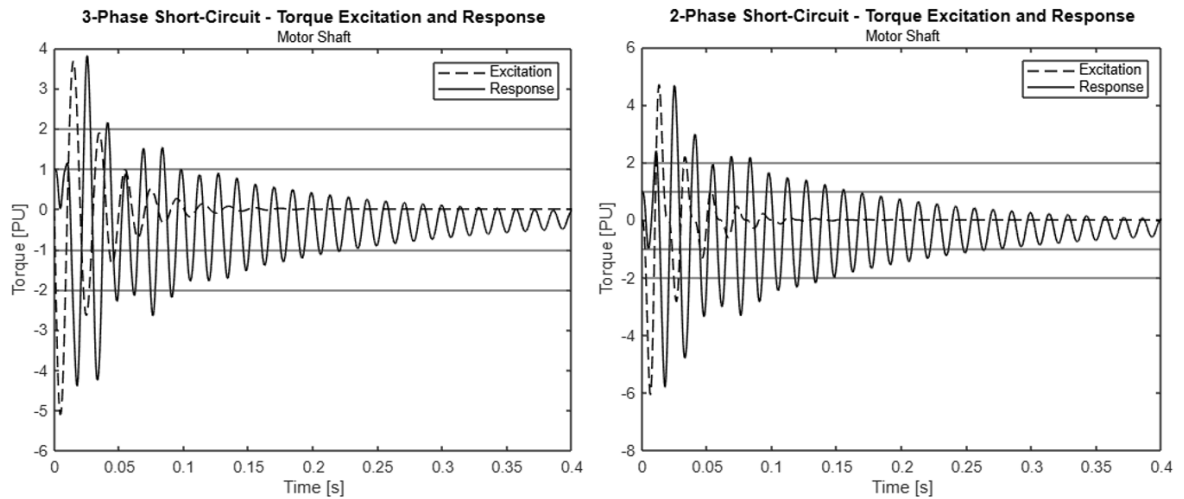


Figure 37: Excitation and response torque of motor shaft for cases I (left) and II (right)

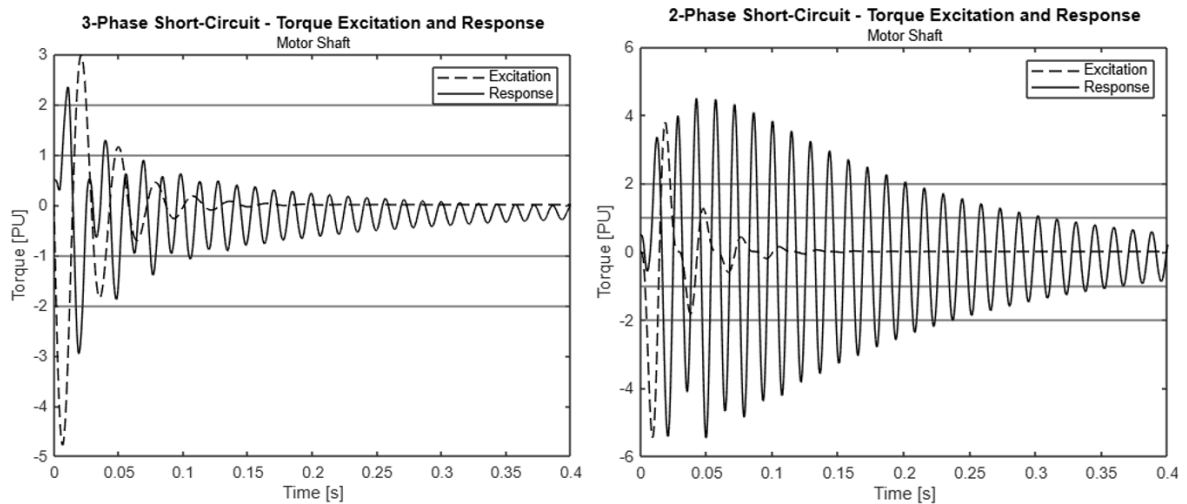


Figure 38: Excitation and response torque of motor shaft for cases III (left) and IV (right)

The short circuit torque vanishes in 0.15s, as shown in figures 33 and 34, but the oscillatory behavior on the train remains longer since the damping is small (modal damping ratio is 0.01). The mean torque in each train element asymptotically goes to zero, since the breaking torque imposed by the load (compressor) decreases with the

square of the rotational speed. Besides the maximum transmitted torques/stresses at each train component, the number of cycles may differ depending on the excitation frequency: in figure 37, since there is a relevant separation margin between the 1x/2x line frequency (50/100 Hz) and the first torsional mode (69.5 Hz), the twisting in the train elements induced by the short circuit excitation decreases rapidly. The same is true for the 3-phase short circuit of the variable speed drive case (case III, shown in figure 38), since it only has 1x excitation component, therefore no interference with 1st TNF. Different situation occur in the 2-phase short circuit of the variable speed drive case (case IV, shown in figure 38), because the 2x excitation frequency coincides with the first TNF and resonance occurs, the twisting of the train elements is much more severe, there are multiple cycles with the transmitted torque exceeding two times motor rated torque.

Design assessment

As per recommended by API (2019), fault conditions such as short circuits should be assumed as a onetime event. In synchronous motors startups, that induce relevant torques and are expected to happen hundreds of times during the machine life, low cycle fatigue assessment is required to calculate the number of startups that are possible without achieving the onset of fatigue failure. For short circuit transient analysis, since it is a rare event to happen, the acceptability criteria is simply survival: the maximum stresses in all shaft elements shall be below their shear yield strength, the torques at couplings and gears shall not exceed vendor's peak torque rating and peak torques must be lower than the minimum torque required to cause relative motion at all train interference fits.

For the compressor and motor shafts, the properties of table 9 are considered. The shaft material is ductile, therefore the main theories to predict the yielding are: the "Maximum Shear-Stress Theory", which predicts $S_{sy} = 0.5S_y$ or the "Distortion-Energy Theory", which predicts $S_{sy} = 0.577S_y$ (Budynas; Nisbett, 2011). Using the second relation, the shear yield stress is 478.9 MPa. Comparing this value to the maximum nominal shear stresses of table 20, the safety factor for the motor and compressor shafts are 1.65 and 1.73, respectively. Considering that the shafts have stress concentrators (keyways, shoulder fillets, etc.), local yielding could occur if the stress concentration factors exceeded the calculated safety factors.

Finally, it is important to mention that the most conservative design practice would be to perform a low cycle fatigue assessment, especially for case IV, to ensure that the cumulated damage would not be enough to achieve onset fatigue failure limit. The most suitable method to assess low cycle fatigue is strain-life method, which involves more detailed analysis of the plastic deformation at localized regions. The traditional stress-life method, that was applied for the high cycle fatigue analysis of the steady-state regime, is not accurate for low cycle applications (Budynas; Nisbett, 2011). Life prediction using the strain-life method is beyond the scope of the present work. Moreover, it is not required by API (2019) for short circuit analysis.

5. CONCLUSIONS

This work shows that the computational procedure implemented to analyze the torsional vibration of geared turbomachinery trains can render reliable results about the dynamic response of different system configurations.

The developed computational procedure consists of obtaining the single speed equivalent model from the original geared system model accounting for the differences in kinetic and strain energies associated with different shaft speeds; applying the finite element method to obtain the mass, stiffness and damping element matrices and assembling the global matrices of the system, which characterizes the train torsional dynamics, by the superposition principle of the direct stiffness method; solving the equations of motion to obtain torsional natural frequencies, mode shapes, relative strain energy distribution, Campbell diagrams, separation margins, elements twisting angles and associated torques / shear stresses for steady state and transient conditions.

The validation of the computational procedure was performed by comparison with results from other authors (validation cases 1 to 3), considering different trains configurations that encompass most of turbomachinery found in real applications: single or multiple branched geared systems, system damping inputted as modal damping ratios (equations of motion can be decoupled by Modal Analysis) or discrete dampers (equations of motion to be solved simultaneously, cannot be decoupled), single or multiple torques acting on any model node with different magnitudes and phases.

In validation case 4, it was observed that, despite the numerical integration method using an adaptative step Runge-Kutta Method, *ode45*, the default relative error tolerance ($1 * 10^{-3}$), which controls the number of correct digits in the solution components, was not enough to ensure accurate and stable numerical solution (i.e., node angular displacements and twisting angles, as consequence) when the typical running speed of a turbomachinery train is used as the initial condition. This is because the twisting angles in each model element are typically small $O(10^{-4})$ while the speed is high $O(10^2)$, requiring the solution (node angular displacement) to be calculated with an improved relative error tolerance ($1 * 10^{-6}$ produced acceptable results).

In validation case 5, mesh sensitivity for systems under torsional vibration was assessed. It was observed that the refinement of the mesh of the shaft elements does not improve the results accuracy in turbomachinery systems where the shafts have low inertia compared to the driver and driven equipment and gears inertia, which is the case of most real turbomachinery trains. Normally, lumped parameter approach produces similar results to the finite element approach, and, in some systems, the shaft inertia effect is negligible, especially if the analyst concern is related to the lower vibration modes, that causes vibration in the whole train (global modes). Shaft sections do not need to be divided based on length/diameter ratio, but elements should start/end at diameter changes, major inertia locations and gears centerline.

Besides mesh sensitivity, this work reviewed aspects to be considered to produce an accurate model for torsional vibration analysis of turbomachinery. One key aspect is that the trains have components from different manufacturers, that should provide the required data for the analysis, for example: couplings have complex geometry and direct influence the first two torsional modes of motor/gear/compressor systems; the electric motor modeling using multiple stations is preferable over simplified single station model typically provided by manufacturers, consisting of one inertia and stiffness; in the gearbox, the rotational inertia of gear and pinions shall be informed by the manufacturer, as well as the gear tooth stiffness, that typically can be assumed rigid, having minor influence on the lower TNFs; shafts are normally stepped, penetration factors shall be considered to properly represent shaft torsional stiffness as well as shrunk-on components (sleeves, collars, etc.). Typically, the detailed modeling of turbomachinery trains requires dozens of lumped inertias and connecting elements to represent the system with high accuracy, but simplified models can be useful for preliminary design assessment, especially for the prediction of the coupling modes associated with the lower TNFs (the most critical modes).

Geared turbomachinery trains are largely used in industry, especially in the O&G segment. In the case study presented in this work, API Rotordynamics Tutorial (API RP 684), which provides modeling and analysis guidelines on torsional vibration of rotating equipment, was applied along with complementary design guidelines from different authors. The computational procedure was applied to perform all the calculations for the analysis of a constant speed electric motor driven compressor and its modified variable speed version by the inclusion of a VFD. The mass-elastic data

used was simplified data from the literature, suitable for preliminary design assessment, especially for the prediction of the coupling modes associated with the lower TNFs (the most critical modes). Moreover, torsional excitations caused by VFDs were not provided by a manufacturer, which is highly recommended since the excitation orders and magnitudes can vary substantially between different VFD models. Despite these limitations, this preliminary design approach applied is often required due to overall project time constraint, since it might not be possible to specify, purchase and obtain the actual design data of critical and tailor-made items, such as VFDs, from the vendors in the required project time. When adopting such approach, the analyst should be diligent to promptly update the model once the detailed vendor data is provided. The computational procedure is suitable for assessing most turbomachinery trains with more detailed mass-elastic data and with inputs of the different components manufacturers.

For the constant speed case, TNFs, mode shapes, relative strain energy distribution were calculated, and Campbell Diagrams were plotted, considering excitations associated with mechanical sources as well as electrical line frequency (due to the gears, lateral excitations can generate torsional excitations) and no interference point (resonance) was identified, so steady state forced response analysis was not required. This is a typical situation for constant speed trains, where often the designer can modify the system, such as coupling stiffness, to ensure the minimum separation margins required.

For the variable speed version, the VFD enabled a wider operation range and generated harmonics and inter-harmonic or sidebands (non-integer harmonics) of the VFD output frequency, creating torque oscillation that caused 4 interference points in the operating region. In this case, a steady state forced response analysis was performed and the node angular displacements (magnitude and phase), maximum element twisting angles, transmitted torques and cyclic stresses were calculated for each point and used for the high cycle fatigue analysis to ensure train mechanical integrity. It is worth noting that inter-harmonic excitation of the 1st TNF (point I4) caused much higher shear stresses at the motor shaft and of similar magnitude for the compressor shaft than the excitation of the 2nd TNF (point I2), despite the fact that excitation amplitude applied to the 2nd mode being 10x higher than the 1st mode and modal damping ratios being equal in all modes. This is explained by the mode shapes

difference, since the 1st mode has relevant vibratory activity at the motor region, which is not the case of the 2nd mode, therefore, the electrical excitation applied at motor region excites much more effectively the 1st mode than the 2nd mode, leading to larger vibration amplitudes and cyclic shear stresses.

Finally, transient response caused by 2-phase and 3-phase motor short circuits were predicted in the case study, for constant speed and VFD motor driven trains. For the constant speed case, short circuit simulation was not required since there was no interference between 1x and 2x electrical line frequencies (constant in this case) and TNFs, but it was performed for comparison with the VFD motor case, where motor electrical fed frequency became speed-dependent due to the VFD, causing the 2x electrical component to interfere with the TNF1 (point I3) when the motor is at 2085 rpm ($1/2 \cdot \text{TNF1}$). The state variable equations of motion, containing the non-linear speed-dependent load term, were numerically integrated using *ode45* and the time response was determined (nodes angular response, element twisting angles and transmitted torques). For the VFD motor case, the maximum 3-phase short circuit response torque for the shaft motor response achieved 2.96 p.u. (abs) and decreased to less than 1.0 p.u (abs) in few cycles, while 2-phase short circuit achieved 5.46 p.u. (abs) and took multiple cycles to achieve 1.0 p.u (abs), due to the resonance.

The case study demonstrated that VFD motor driven trains require careful torsional vibration analysis due to the wider operation range, harmonics and inter-harmonics excitations and an increased likelihood of a short circuit triggering a resonance with first mode. The current trend towards the application of more efficient process control methods and electrification of critical process turbomachinery trains aiming to save energy and reduce GHG emissions is leading to an increase in the VFD application. In this context, the importance of this work lies in the presentation of an overview of torsional vibrations on geared turbomachinery trains, the practical application of the torsional vibration analysis procedure recommended by API (2019) and the development of a computational procedure that can be used by designers and end-users in the analysis of torsional vibration in geared turbomachinery trains.

5.1. Future works

The computational procedure presented in this work performs purely torsional open loop analysis of the mechanical system, the excitations are defined and torque responses calculated, the reaction of the VFD control system is ignored. Since many VFDs use feedback control (e.g., motor speed, torque, etc.), closed loop coupled electromechanical torsional analysis is required to evaluate torsional stability of VFD driven trains or when the VFD is used to provide torsional damping to the system to reduce torsional vibration amplitude. There are several cases in literature describing positive and negative outcomes associated with VFD control parametrization.

Other relevant aspect is that this work assumed that torsional and lateral vibrations are independent of each other, which is the standard assumption in most analysis performed in the industry. However, there are reported cases of geared systems where coupling between torsional and lateral motion occurred during operation. Coupled torsional-lateral vibration in geared turbomachinery is a phenomenon not fully understood, its analysis is not even required by the relevant equipment standards, therefore the implementation of a computational procedure to the analysis of this kind of the phenomenon and definition of clear design guidelines to manufacturers and end-users would be very useful.

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APPENDIX A – Mesh sensitivity for FEM and lumped parameter shaft models

Here a comparison between lumped parameter and finite element modeling of torsional elements is presented. The purpose is to check how the inertia distribution of shaft elements influences their natural frequencies and if the computational procedure was providing the expected results.

The analytical expression for the natural frequencies of a continuous uniform bar under torsion, with both ends free is Inman (2014):

$$\omega_n = \frac{z\pi}{L} \sqrt{\frac{G}{\rho}}, \quad \text{with } z = 0, 1, 2, \dots \quad (87)$$

The first 4 non-zero natural frequencies were calculated for a consistent inertia matrix formulation of the elements and lumped inertia matrix formulation, for the same bar modeled with different element quantities. The results are summarized in tables 21 and 22.

Bar properties: $L = 1,0m$, $r = 0,05m$, $G = 80 GPa$, $\rho = 7800 kg/m^3$

Table 21: Torsional natural frequencies (rad/s) for a free-free uniform bar

Mode	Analytical (rad/s)	1 element		2 elements		5 elements		20 elements		40 elements	
		Consistent	Lumped	Consistent	Lumped	Consistent	Lumped	Consistent	Lumped	Consistent	Lumped
0	0	0	0	0	0	0	0	0	0	0	0
1	10061,1	10843	6405	11025	9058	10204	9896,5	10065	10051	10061	10059
2	20122,3	-	-	22370	12810	21408	18824	20193	20040	20137	20102
3	30183,4	-	-	-	-	34357	25909	30446	29905	30244	30114
4	40244,6	-	-	-	-	47868	30458	40886	39586	40398	40079

Table 22: Relative errors (%) for the free-free uniform bar

Mode	1 element		2 elements		5 elements		20 elements		40 elements	
	Consistent	Lumped	Consistent	Lumped	Consistent	Lumped	Consistent	Lumped	Consistent	Lumped
1	7,8	-36,3	9,6	-10,0	1,4	-1,6	0,0	-0,1	0,0	0,0
2	-	-	11,2	-36,3	6,4	-6,5	0,4	-0,4	0,1	-0,1
3	-	-	-	-	13,8	-14,2	0,9	-0,9	0,2	-0,2
4	-	-	-	-	18,9	-24,3	1,6	-1,6	0,4	-0,4

From tables 21 and 22, the conclusions are: 1) the computational procedure is working properly in terms of global inertia and stiffness matrices assembly, for both consistent and lumped element inertia formulations, the convergence to the analytical solution improves with mesh refinement, as expected; 2) Consistent formulation of the element inertias matrix provides better results than lumped formulation and should be always preferred, especially if in the rotor the shaft inertia is comparable to the lumped inertias.

APPENDIX B – Case Study: Fatigue Analysis based on Goodman criteria

Interference Point 2

P	1550	kW	Power
N_nom	2955	rpm	Nominal speed
Gr	2	-	Gear Ratio
Tmrared	5,0	kNm	Motor rated torque
TNF	223,3	Hz	2nd TNF (interence point I2)
Nop	2233	rpm	Operating speed related to the interference point I2
Tmop	2,860	kNm	Operating torque at critical speed related to I2 - motor shaft
Tmop	1,43	kNm	Operating torque at critical speed related to I2 - compressor shaft
Ta	501	Nm	I2 - Alternate torque amplitude applied to the model (10% at motor)

Material properties

Material	AISI 4140 Q&T		
Sut	1000	MPa	Ultimate tensile strength
Sy	830	MPa	Yield strength
Ssut	577	MPa	Ultimate shear strength

Equivalent fully reversing stress calculation

Shaft:	Motor	Compressor		
τ_{a0}	3,5	8,9	MPa	Alternate cyclic stress - nominal (from steady-state torsional analysis)
D	0,08	0,06	m	Shaft diameter (compressor)
J	4,02E-06	1,27E-06	m ⁴	Area polar moment of inertia
τ_{m0}	28	34	MPa	Mean stress - nominal (associated with motor rated torque)
kt	3	3	-	Theoretical stress concentration factor
q	1	1	-	Notch sensivity factor
Kf	3	3		Stress concentration factor
τ_{m_max}	85,4	101,2	MPa	Mean stress - maximum
τ_{a_max}	10,5	26,7	MPa	Alternate cyclic stress - maximum
Seq	12,3	32,4	MPa	Equivalent fully reversing stress

Shear Endurance Limit Calculation

Se'	500	500	MPa	Endurance limit of rotating-beam specimen
ka	0,723	0,723	-	Surface finish factor
kb	0,759	0,794	-	Size factor
kc	0,577	0,577	-	Load Factor (shear load = 0,577)
kd	1	1	-	Temperature factor
ke	0,753	0,753	-	Reliability factor
kf	1	1	-	Miscellaneous factor
Se	119,2	124,7	MPa	Shear Endurance Limit

Safety Factor Calculation

SF_req	2,0	2,0	-	Required safety factor
SF	9,67	3,85	-	Calculated safety factor

Interference Point 3

P	1550	kW	Power
N_nom	2955	rpm	Nominal speed
Gr	2	-	Gear Ratio
Tmrared	5.0	kNm	Motor rated torque
TNF	69.5	Hz	First TNF (interence point I3)
Nop	2085	rpm	Operating speed related to the interference point I3
Tmop	2.49	kNm	Operating torque at critical speed related to I3 - motor shaft
Tmop	1.25	kNm	Operating torque at critical speed related to I3 - compressor shaft
Ta	25.0	Nm	Alternate torque amplitude applied to the model (0.5% at motor)

Material properties

Material	AISI 4140 Q&T		
Sut	1000	MPa	Ultimate tensile strength
Sy	830	MPa	Yield strength
Ssut	577	MPa	Ultimate shear strength

Equivalent fully reversing stress calculation

Shaft:	Motor	Compressor		
ta0	4.3	4.2	MPa	Alternate cyclic stress - nominal (from steady-state torsional analysis)
D	0.08	0.06	m	Shaft diameter (compressor)
J	4.02E-06	1.27E-06	m^4	Area polar moment of inertia
tm0	25	29	MPa	Mean stress - nominal (associated with operating torque)
kt	3	3	-	Theoretical stress concentration factor
q	1	1	-	Notch sensivity factor
Kf	3	3		Stress concentration factor
tm_max	74.4	88.2	MPa	Mean stress - maximum
ta_max	12.9	12.6	MPa	Alternate cyclic stress - maximum
Seq	14.8	14.9	MPa	Equivalent fully reversing stress

Shear Endurance Limit Calculation

Se'	500	500	MPa	Endurance limit of rotating-beam specimen
ka	0.723	0.723	-	Surface finish factor
kb	0.759	0.794	-	Size factor
kc	0.577	0.577	-	Load Factor (shear load = 0,577)
kd	1	1	-	Temperature factor
ke	0.753	0.753	-	Reliability factor
kf	1	1	-	Miscellaneous factor
Se	119.2	124.7	MPa	Shear Endurance Limit

Safety Factor Calculation

SF_req	2.0	2.0	-	Required safety factor
SF	8.0	8.4	-	Calculated safety factor

Interference Point 4

P	1550	kW	Power
N_nom	2955	rpm	Nominal speed
Gr	2	-	Gear Ratio
Tmrared	5,0	kNm	Motor rated torque
TNF	69,5	Hz	2nd TNF (interence point I2)
Nop	3251	rpm	Operating speed related to the interference point I4
Tmop	6,06	kNm	Operating torque at critical speed related to I4 - motor shaft
Tmop	3,03	kNm	Operating torque at critical speed related to I4 - compressor shaft
Ta	50	Nm	I4 - Alternate torque amplitude applied to the model (1% at motor)

Material properties

Material	AISI 4140 Q&T		
Sut	1000	MPa	Ultimate tensile strength
Sy	830	MPa	Yield strength
Ssut	577	MPa	Ultimate shear strength

Equivalent fully reversing stress calculation

Shaft:	Motor	Compressor		
τ_{a0}	8,6	8,4	MPa	Alternate cyclic stress - nominal (from steady-state torsional analysis)
D	0,08	0,06	m	Shaft diameter (compressor)
J	4,02E-06	1,27E-06	m ⁴	Area polar moment of inertia
τ_{m0}	60	71	MPa	Mean stress - nominal (associated with motor rated torque)
kt	3	3	-	Theoretical stress concentration factor
q	1	1	-	Notch sensivity factor
Kf	3	3		Stress concentration factor
τ_{m_max}	180,9	214,4	MPa	Mean stress - maximum
τ_{a_max}	25,8	25,2	MPa	Alternate cyclic stress - maximum
Seq	37,6	40,1	MPa	Equivalent fully reversing stress

Shear Endurance Limit Calculation

Se'	500	500	MPa	Endurance limit of rotating-beam specimen
ka	0,723	0,723	-	Surface finish factor
kb	0,759	0,794	-	Size factor
kc	0,577	0,577	-	Load Factor (shear load = 0,577)
kd	1	1	-	Temperature factor
ke	0,753	0,753	-	Reliability factor - 99,9%
kf	1	1	-	Miscellaneous factor
Se	119,2	124,7	MPa	Shear Endurance Limit

Safety Factor Calculation

SF_req	2,0	2,0	-	Required safety factor
SF	3,17	3,11	-	Calculated safety factor

ANNEX A – Validation Cases 1 and 2 – Results by TWIST2 code

For validation cases 1 and 2, the results generated by TWIST2 Code are presented below. These cases were originally designated as “Example 3” and “Example 4” in the TWIST2 Manual and were used to demonstrate the TWIST2 capabilities.

A.1 Validation Case 1

Input data

TWIST2 - TORSIONAL CRITICAL SPEED & FORCED RESPONSE - VER 3.00 - NOV 01, 1993

THREE MASS TORSIONAL MODEL OF THOMSON TEXTBOOK PROBLEM. PROBLEM 12.18.
FORCING TORQUE AT NODE 1 WITH A VALUE OF 2.4 FT-LBF AND PHASE OF ZERO.
PLOTS OF EIGENVECTORS AND FORCED RESPONSE PRODUCED. (EXAMPLE 3 IN MANUAL)
DATA READ FROM FILE: tw3dat

ELEMENT PROPERTIES:

ELM NO	REF ELM	OUTER RAD (IN)	INNER RAD (IN)	LENGTH (IN)	SPECIFIC WT (LBF/IN ³)	SHEAR MOD (LBF/IN ²)	STIFF FRAC	SPEED RATIO
1	-	.7500	.0000	40.00	.2830	.1200E+08	1.000	1.0000
2	-	1.0000	.0000	30.00	.2830	.1200E+08	1.000	.5000

GLOBAL STIFFNESS MULTIPLYING FACTOR = 1.0000

'REF ELM' ENTRY INDICATES 45.0 DEGREE RULE USED IN DETERMINING STIFFNESS.

ELEMENT-NODE CONNECTIVITY DATA: ELEMENT INERTIA AND SCALED STIFFNESS:
(BEFORE DISK MODEL CHANGES TO STIFFNESS)

ELEM NO.	GLOBAL NODE 1	GLOBAL NODE 2	ELEM NO.	ELEM INERTIA (LBF-S ² -IN)	ELEM STIFF (LBF-IN)
1	1	2	1	.145604E-01	.149103E+06
2	2	3	2	.862840E-02	.157080E+06

EXTERNAL LUMPED INERTIAS AND DISK MODELS:

NODE NO.	INERTIA (LBF-S ² -IN)	----- DISK MODEL PROPERTIES -----					
		OUTER RAD (IN)	LENGTH (IN)	SPECIFIC WT (LBF/IN ³)	SHEAR MOD (LBF/IN ²)	STIFF FRAC	SPEED RATIO
1	10.000	-	-	-	-	-	1.0000
2	2.000	-	-	-	-	-	1.0000
2	6.000	-	-	-	-	-	.5000
3	24.000	-	-	-	-	-	.5000

TOTAL SHAFT INERTIA RELATIVE TO THE REFERENCE SHAFT (LBF-S²-IN): .19500E+02
ELEMENT INERTIAS NOT INCLUDED IN MASS MATRIX AND TOTAL INERTIA VALUE.

CONSTANT TORQUE MAGNITUDE AND PHASE OVER

RPM RANGE: 1200.0 TO 3500.0 WITH STEP 12.0:

NODE	MAGNITUDE (FT-LBF)	PHASE (DEG)	SPEED RATIO
1	2.4000	.0	1.0000

DAMPING RATIO FOR ALL MODES = .02000

Results

MODE 1 NATURAL FREQ = .00 RAD/SEC = .00 RPM

NODE (NOT SPEED NORMALIZED)	EIGENVECTOR	ELEMENT (SPEED NORMALIZED)	STRAIN ENERGY
1	1.000000	1	.000%
2	1.000000	2	.000%
3	1.000000		

MODE 2 NATURAL FREQ = 141.56 RAD/SEC = 1351.77 RPM

NODE (NOT SPEED NORMALIZED)	EIGENVECTOR	ELEMENT (SPEED NORMALIZED)	STRAIN ENERGY
1	-.682108	1	57.655%
2	.234593	2	42.345%
3	1.000000		

MODE 3 NATURAL FREQ = 329.44 RAD/SEC = 3145.95 RPM

NODE (NOT SPEED NORMALIZED)	EIGENVECTOR	ELEMENT (SPEED NORMALIZED)	STRAIN ENERGY
1	-.159260	1	42.345%
2	1.000000	2	57.655%
3	-.317900		

FORCED RESPONSE INFORMATION:

MINIMUM NUMBER OF EIGENVECTORS THAT SHOULD
BE USED IN FORCED RESPONSE ANALYSIS IS 3
NUMBER OF EIGENVECTORS ACTUALLY USED IS 3
MAGNITUDE AND PHASE RESPONSE ARE IN THE FORM: MAGNITUDE*EXP(OMEGA*T-PHASE)
SHEAR STRESS AND PHASE RESPONSE ARE IN THE FORM: STRESS*EXP(OMEGA*T-PHASE)
ANGULAR RESPONSES ARE NOT SPEED RATIO NORMALIZED
SHEAR STRESSES ARE SPEED RATIO NORMALIZED
SHEAR STRESS VALUES LISTED (0-PEAK) ARE MAXIMUM VALUES FOR EACH ELEMENT
SHEAR STRESSES DO NOT INCLUDE STRESS CONCENTRATION FACTORS
SHEAR STRESSES REFLECT THE EFFECT OF DISK MODELS ON THE SHAFT ELEMENTS

ANGULAR RESPONSE AT RPM = 1200.0			ELEMENT STRESSES AT RPM = 1200.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.11290E-01	13.931	1	.89688E+02	-170.768
2	.11623E-01	-175.332	2	.61438E+02	-170.077
3	.29194E-01	-172.167			

ANGULAR RESPONSE AT RPM = 1212.0			ELEMENT STRESSES AT RPM = 1212.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.12690E-01	14.559	1	.96455E+02	-169.922
2	.11952E-01	-174.680	2	.66455E+02	-169.217
3	.30956E-01	-171.323			

ANGULAR RESPONSE AT RPM = 1224.0			ELEMENT STRESSES AT RPM = 1224.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.14318E-01	15.363	1	.10442E+03	-168.922
2	.12359E-01	-173.888	2	.72365E+02	-168.202
3	.33052E-01	-170.325			

ANGULAR RESPONSE AT RPM = 3468.0			ELEMENT STRESSES AT RPM = 3468.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.16712E-02	176.829	1	.14804E+02	-9.217
2	.21152E-02	-13.991	2	.92035E+01	165.602
3	.52167E-03	163.948			

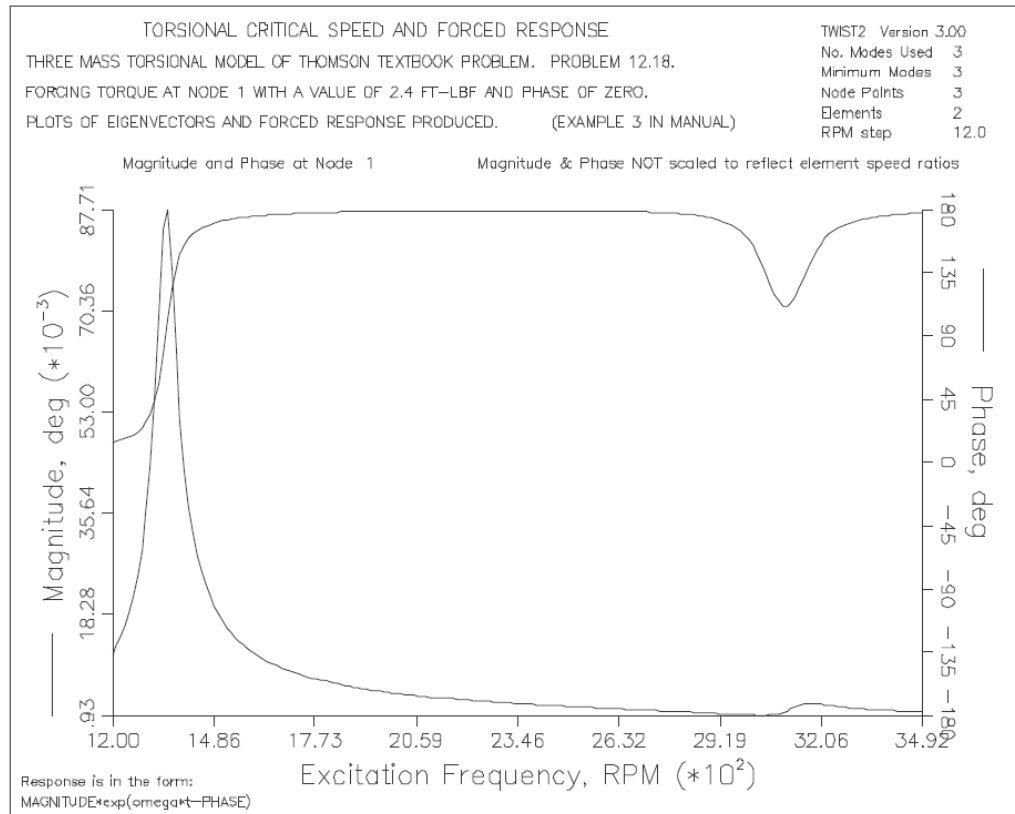


Figure A1: Magnitude and phase as a function of excitation frequency for node 1

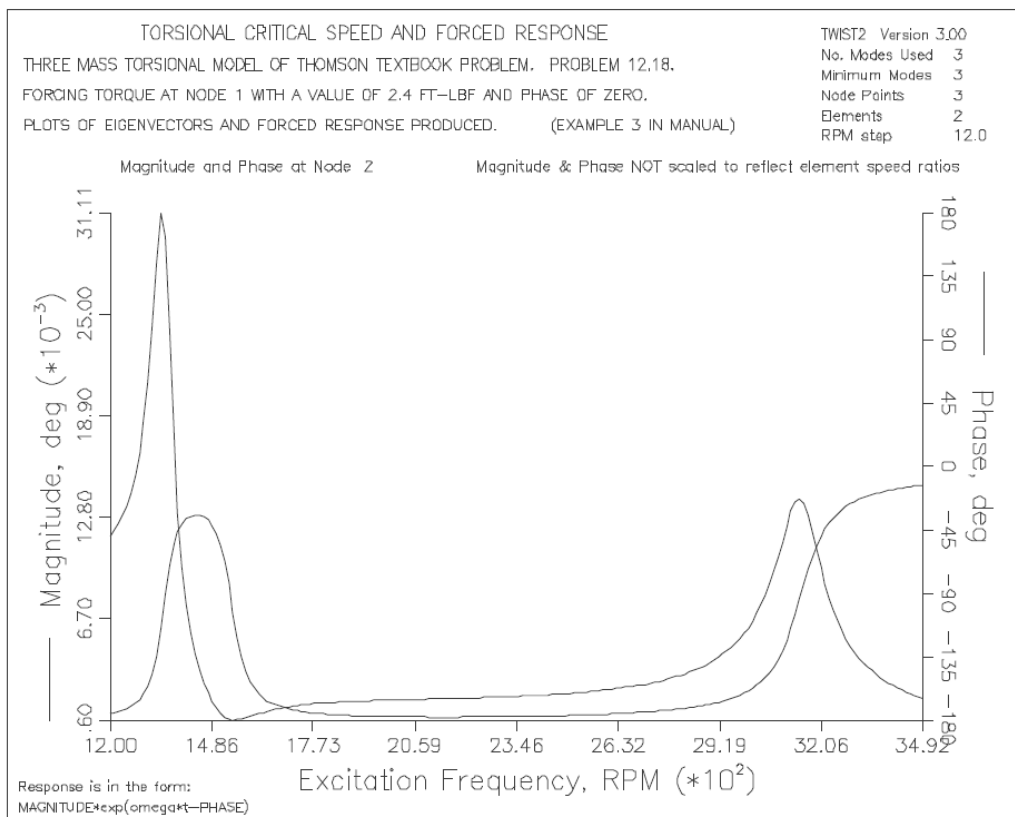


Figure A2: Magnitude and phase as a function of excitation frequency for node 2

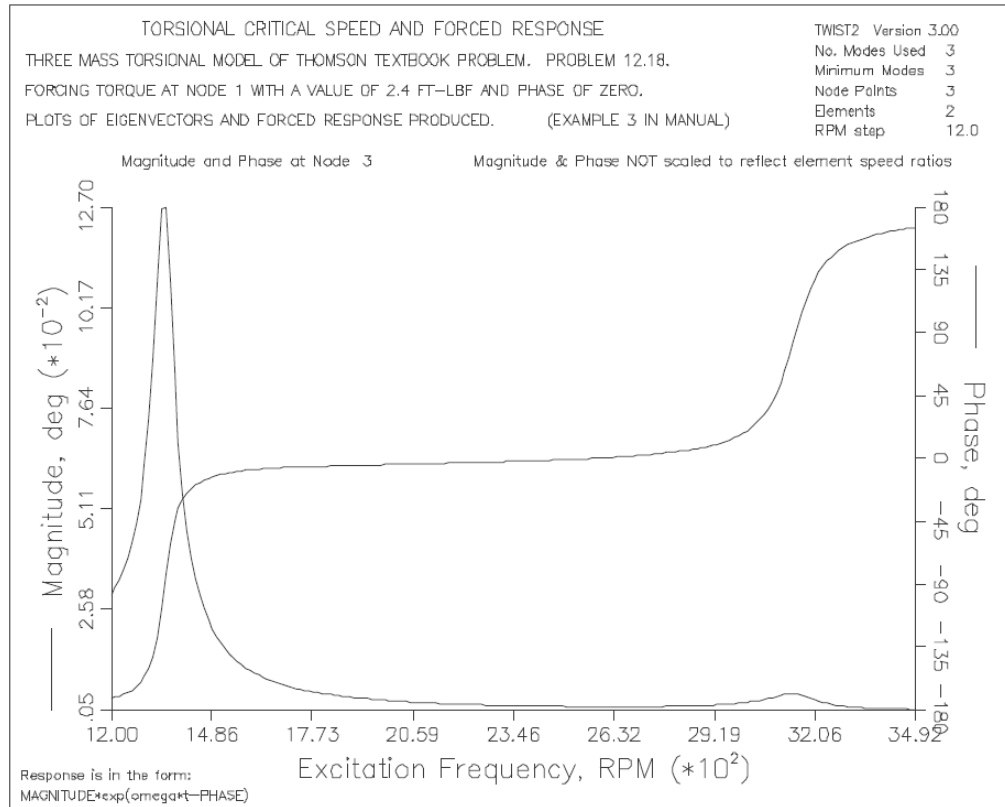


Figure A3: Magnitude and phase as a function of excitation frequency for node 3

A.2 Validation Case 2

Input data

TWIST2 - TORSIONAL CRITICAL SPEED & FORCED RESPONSE - VER 3.00 - NOV 01, 1993

THREE MASS TORSIONAL MODEL OF THOMSON TEXTBOOK PROBLEM. PROBLEM 12.18.
FORCING TORQUE AT NODES 1 AND 3. PLOTS OF EIGENVECTORS AND FORCED RESPONSE
OUTPUT AT ALL THREE NODES PRODUCED. (EXAMPLE 4 IN MANUAL)
DATA READ FROM FILE: tw4dat

ELEMENT PROPERTIES:

ELM NO	REF ELM	OUTER RAD (IN)	INNER RAD (IN)	LENGTH (IN)	SPECIFIC WT (LBF/IN ³)	SHEAR MOD (LBF/IN ²)	STIFF FRAC	SPEED RATIO
1	-	.7500	.0000	40.00	.2830	.1200E+08	1.000	1.0000
2	-	1.0000	.0000	30.00	.2830	.1200E+08	1.000	.5000

GLOBAL STIFFNESS MULTIPLYING FACTOR = 1.0000

'REF ELM' ENTRY INDICATES 45.0 DEGREE RULE USED IN DETERMINING STIFFNESS.

ELEMENT-NODE CONNECTIVITY DATA: ELEMENT INERTIA AND SCALED STIFFNESS: (BEFORE DISK MODEL CHANGES TO STIFFNESS)

ELEM NO.	GLOBAL NODE 1	GLOBAL NODE 2	ELEM NO.	ELEM INERTIA (LBF-S ² -IN)	ELEM STIFF (LBF-IN)
1	1	2	1	.145604E-01	.149103E+06
2	2	3	2	.862840E-02	.157080E+06

EXTERNAL LUMPED INERTIAS AND DISK MODELS:

NODE NO.	INERTIA (LBF-S ² -IN)	DISK MODEL PROPERTIES					
		OUTER RAD (IN)	LENGTH (IN)	SPECIFIC WT (LBF/IN ³)	SHEAR MOD (LBF/IN ²)	STIFF FRAC	SPEED RATIO
1	10.000	-	-	-	-	-	1.0000
2	2.000	-	-	-	-	-	1.0000
2	6.000	-	-	-	-	-	.5000
3	24.000	-	-	-	-	-	.5000

TOTAL SHAFT INERTIA RELATIVE TO THE REFERENCE SHAFT (LBF-S²-IN): .19500E+02
ELEMENT INERTIAS NOT INCLUDED IN MASS MATRIX AND TOTAL INERTIA VALUE.

VARIABLE TORQUE MAGNITUDE AND PHASE OVER
RPM RANGE 1200.0 TO 2690.0 WITH STEP 10.0:

NODE	RPM	MAGNITUDE (FT-LBF)	PHASE (DEG)	SPEED RATIO
1	1200.0	2.4000	.0	1.0000
3	1200.0	.5000	265.0	.5000
1	1250.0	2.6000	10.0	1.0000
3	1250.0	.7000	245.0	.5000

DAMPERS ADDED TO SYSTEM:

NODE(S)	VALUE (LBF-IN-S)	SPEED RATIO OF NODE(S)
3	10.000	.5000
1, 2	16.000	1.0000

Results

MODE 1 NATURAL FREQ = .00 RAD/SEC = .00 RPM

NODE (NOT SPEED NORMALIZED)	EIGENVECTOR	ELEMENT (SPEED NORMALIZED)	STRAIN ENERGY
1	1.000000	1	.000%
2	1.000000	2	.000%
3	1.000000		

MODE 2 NATURAL FREQ = 141.56 RAD/SEC = 1351.77 RPM

NODE (NOT SPEED NORMALIZED)	EIGENVECTOR	ELEMENT (SPEED NORMALIZED)	STRAIN ENERGY
1	-.682108	1	57.655%
2	.234593	2	42.345%
3	1.000000		

MODE 3 NATURAL FREQ = 329.44 RAD/SEC = 3145.95 RPM

NODE (NOT SPEED NORMALIZED)	EIGENVECTOR	ELEMENT (SPEED NORMALIZED)	STRAIN ENERGY
1	-.159260	1	42.345%
2	1.000000	2	57.655%
3	-.317900		

FORCED RESPONSE INFORMATION:

MINIMUM NUMBER OF EIGENVECTORS THAT SHOULD
BE USED IN FORCED RESPONSE ANALYSIS IS 3
NUMBER OF EIGENVECTORS ACTUALLY USED IS 3
MAGNITUDE AND PHASE RESPONSE ARE IN THE FORM: MAGNITUDE*EXP(OMEGA*T-PHASE)
SHEAR STRESS AND PHASE RESPONSE ARE IN THE FORM: STRESS*EXP(OMEGA*T-PHASE)
ANGULAR RESPONSES ARE NOT SPEED RATIO NORMALIZED
SHEAR STRESSES ARE SPEED RATIO NORMALIZED
SHEAR STRESS VALUES LISTED (O-PEAK) ARE MAXIMUM VALUES FOR EACH ELEMENT
SHEAR STRESSES DO NOT INCLUDE STRESS CONCENTRATION FACTORS
SHEAR STRESSES REFLECT THE EFFECT OF DISK MODELS ON THE SHAFT ELEMENTS

ANGULAR RESPONSE AT RPM = 1200.0			ELEMENT STRESSES AT RPM = 1200.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.12172E-01	18.651	1	.92840E+02	-169.185
2	.11701E-01	-177.339	2	.64115E+02	-168.307
3	.29980E-01	-171.820			

ANGULAR RESPONSE AT RPM = 1210.0			ELEMENT STRESSES AT RPM = 1210.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.14012E-01	20.306	1	.10214E+03	-166.723
2	.12223E-01	-174.788	2	.71072E+02	-165.942
3	.32493E-01	-169.258			

ANGULAR RESPONSE AT RPM = 1220.0			ELEMENT STRESSES AT RPM = 1220.0		
NODE	MAGNITUDE (DEG)	PHASE (DEG)	ELEMENT	MAX STRESS (PSI)	PHASE (DEG)
1	.16147E-01	21.834	1	.11300E+03	-164.371
2	.12840E-01	-172.184	2	.79196E+02	-163.714
3	.35439E-01	-166.773			