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**NON-CLASSICAL COMMON-CAUSE AND
DIRECT-CAUSE**

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NON-CLASSICAL COMMON-CAUSE AND DIRECT-CAUSE

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ATA DA SESSÃO DE ARGUIÇÃO DA 391ª TESE DO PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA, DEFENDIDA POR MARCELLO NERY GARCIA VIDAL DE BARROS orientado pelo professor Reinaldo Oliveira Vianna e coorientado pelo doutor Marco Túlio Coelho Quintino, para obtenção do grau de **DOUTOR EM CIÊNCIAS, área de concentração física**. Às 9:00 horas de vinte e um de janeiro de dois mil e vinte e dois reuniu-se, por videoconferência, a Comissão Examinadora, composta pelos professores **Reinaldo Oliveira Vianna** (Orientador - Departamento de Física/UFMG), **Carlos Henrique Monken** (Departamento de Física/UFMG), **Pablo Lima Saldanha** (Departamento de Física/UFMG), **Rafael Chaves** (IIP/UFRN), **Rafael Luiz da Silva Rabelo** (IFGW/UNICAMP) e pelo doutor **Marco Túlio Coelho Quintino** (Coorientador - IQOQI, Vienna), para dar cumprimento ao Artigo 37 do Regimento Geral da UFMG, submetendo o Mestre **MARCELLO NERY GARCIA VIDAL DE BARROS** à arguição de seu trabalho de Tese de Doutorado, que recebeu o título de "**Non-classical common-cause and direct-cause**". O candidato fez uma exposição oral de seu trabalho durante aproximadamente 50 minutos. Após esta, os membros da comissão prosseguiram com a sua arguição, e apresentaram seus pareceres individuais sobre o trabalho, concluindo pela aprovação do candidato.

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Errata sheet

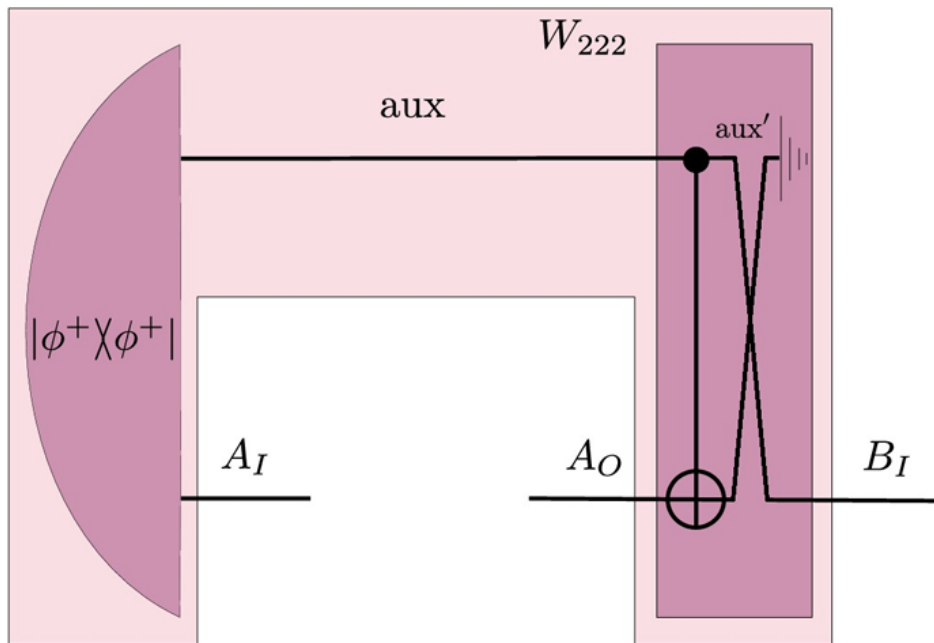
In section 5.2, the process W_{222} is not correctly defined. The correct process is obtained by changing the control qubit with the target qubit on the CNOT operation, which can be obtained either by inverting the CNOT or applying a SWAP gate just after the CNOT.

It can be written as

$$W_{222} := \text{tr}_{\text{aux}'} \left(|\phi^+\rangle\langle\phi^+|^{A_I \text{aux}} * |U_{\text{CNOT}}\rangle\langle U_{\text{CNOT}}|^{\text{aux}A_O/\text{aux}'B_I} * |U_{\text{SWAP}}\rangle\langle U_{\text{SWAP}}|^{\text{aux}A_O/B_I \text{aux}'} \right) \quad (1)$$

$$:= \text{tr}_{\text{aux}'} \left(|\phi^+\rangle\langle\phi^+|^{A_I \text{aux}} * |U_{\text{CNOT}}\rangle\langle U_{\text{CNOT}}|^{\text{aux}A_O/B_I \text{aux}'} \right). \quad (2)$$

The following figure represents the correct circuit:



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¹ Conjuntos com interseções não-vazias uns com os outros.

² Urubus.

"The greatest teacher, failure is."

Yoda — Star Wars Episode VIII:
The Last Jedi

Abstract

Guided by the intuition of coherent superposition of causal relations, recent works presented quantum processes without classical common-cause and direct-cause explanation, that is, processes which cannot be written as probabilistic mixtures of quantum common-cause and quantum direct-cause relations (CCDC). In this work, we analyse the minimum requirements for a quantum process to fail to admit a CCDC explanation and present “simple” processes, which we prove to be the most robust ones against general noise. These simple processes can be performed by preparing a maximally entangled state and applying the identity quantum channel, thus not requiring an explicit coherent mixture of common-cause and direct-cause, exploiting the possibility of a process to have both relations simultaneously. We then prove that, although all bipartite direct-cause processes are bipartite separable operators, there exist bipartite separable processes which are not direct-cause. This shows that the problem of deciding whether a process is direct-cause *is not* equivalent to entanglement certification, and points out the limitations of entanglement methods to detect non-classical CCDC processes. We also present a semidefinite programming hierarchy that can detect and quantify the non-classical CCDC robustnesses of every non-classical CCDC process. Among other results, our numerical methods allow us to show that the simple processes presented here are likely to be also the maximally robust against white noise. Finally, we explore the equivalence between bipartite direct-cause processes and bipartite processes without quantum memory, to present a separable process which cannot be performed as a process without quantum memory.

Keywords: non-classical causal relations; common-cause; direct-cause.

Resumo

Guiados pela intuição sobre superposições coerentes de relações de causalidade, trabalhos recentes apresentaram processos quânticos sem explicação clássica de causa comum e causa direta, ou seja, processos que não podem ser descritos como misturas probabilísticas de relações quânticas de causa comum e causa direta (CCDC). Neste trabalho, analisamos os requisitos mínimos para que um processo quântico falhe em admitir uma explicação CCDC e apresentamos processos “simples”, os quais provamos serem os mais robustos contra ruído generalizado. Esses processos simples podem ser realizados através da preparação de um estado maximamente emaranhado e a aplicação do canal quântico identidade, não requerendo, assim, uma mistura coerente explícita de causa comum e causa direta, explorando a possibilidade de um processo ter ambas as relações simultaneamente. Provamos então que, embora todos os processos bipartidos de causa direta sejam representados por operadores bipartidos separáveis, existem processos bipartidos separáveis que não são de causa direta. Isso mostra que o problema de determinar se um processo é de causa direta *não é* equivalente à certificação de emaranhamento, apontando as limitações dos métodos de emaranhamento para determinar não classicalidade em processos CCDC. Apresentamos também uma hierarquia de programação semidefinida que pode detectar e quantificar a robustez de CCDC não clássica de todo processo CCDC não clássico. Entre outros resultados, nossos métodos numéricos permitem mostrar que os processos simples aqui apresentados são provavelmente também os maximamente robustos a ruído branco. Finalmente, exploramos a equivalência entre processos bipartidos de causa direta e processos bipartidos sem memória quântica, para apresentar um processo separável que não pode ser realizado como um processo sem memória quântica.

Palavras-chave: causalidade não-clássica; causa comum; causa direta.

List of abbreviations and acronyms

CC	Common-cause
CCDC	Common-cause and direct-cause, also used to refer to processes with classical common-cause and direct-cause explanation
CP	Completely positive
DC	Direct-cause
POVM	Positive Operator-Valued Measure
PPT	Positive Partial Transposition
TP	Trace-preserving
SDP	Semi-definite Program

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Introduction

Common-cause and direct-cause relations are the building blocks of classical causal models, as a causal model of multiple variables consists of combinations of common-cause and direct-cause relations between them. Understanding the relationship between cause and effects is one of the fundamental goals of several physical theories and of statistical analysis [3]. Also, determining the causal relation behind a correlation of two objects is a fundamental problem in causal inference theory and plays a main role in topics such as hypothesis testing, social sciences, medicine, and machine learning [4, 5]. In order to analyse causality in quantum phenomena, recent works proposed new frameworks of theory of Bayesian inference [6] and causal modelling [7], where understanding common-cause and direct-cause relations in quantum mechanics plays a fundamental role.

Quantum causal processes consist of a sequence of quantum operations and may be analysed from different equivalent perspectives, such as sequential quantum operations via non-Markovian processes [8–11], fragments of quantum circuits via quantum combs [12, 13], quantum channels with memory [14], and quantum strategies for playing a n -turn game [15]. When referring to causal modelling, a process which can be written as a probabilistic mixture of common-cause (CC) and direct-cause (DC) processes is said to admit a classical common-cause or direct-cause (CCDC) explanation, since the causal relations in such process could be simulated classically, by sampling from a probability distribution and then implementing either a common-cause or direct-cause process.

In Ref. [16] the authors consider the different possibilities of combining CC and DC processes and, inspired by coherent mixture of quantum channels, the authors of Ref. [1] experimentally certify the existence of a quantum process with no classical CCDC explanation. Also, inspired by the *Quantum Switch* [17–19], Ref. [2] presents a coherent superposition of common-cause and direct-cause processes which cannot be explained by a classical CCDC.

In this work, we analyse quantum processes which cannot be decomposed as probabilistic mixtures of common-cause and direct-cause ones, hence admitting no classical CCDC explanation. We focus on the bipartite case, the simplest scenario where such processes can exist. In this simplest scenario, exploring the possibility of a process to have both CC and DC relations simultaneously, we present a process which can be simply realized by preparing a maximally entangled state and an identity channel, not needing an interpretation of it as a coherent mixture of causal relations. When

attempting to work with the minimum non-trivial dimensions, we propose another process, which requires a controlled-NOT operation. We prove that these processes are the most robust ones against their worst possible noise, known as generalized noise. We also develop a semidefinite programming (SDP) numerical approach to quantify the non-classical CCDC property of a process, based on its robustness against white and general noise. Our numerical methods allow us to show that these simple processes presented here are also maximally robust against white noise when considering qubits, and to quantify the non-classical CCDC property in any bipartite ordered process.

We also investigate the differences and similarities between quantum entanglement and processes without a direct-cause explanation, showing that there exist separable processes which are not direct-cause. This example points out the limitations of purely entanglement-based methods to detect non direct-cause processes and answers a conjecture first raised in Ref. [20].

Finally, we connect our results with a different related field by proving that for the bipartite case, processes without quantum memory [20] are equivalent to processes having a direct-cause decomposition. With that, we present a bipartite separable process which cannot be realized as a process without quantum memory and contribute to the understanding of the relation between entanglement and of quantum memory [20–24].

Part I

Mathematical Preliminaries

1 Linear operators and measurements, Maps and Supermaps

In this chapter, we present some definitions and properties of linear operators and maps which are most important for this work. We denote A and B finite dimensional complex vector spaces, and $\mathcal{L}(A, B)$ the set of linear operators mapping A to B . For simplicity, we denote $\mathcal{L}(A)$ the set of linear operators mapping A to itself and, throughout this entire work, we will always assume that the complex vector spaces involved have finite dimension. More details about this subject can be found in Ref. [25]

1.1 Linear operators

The following definitions correspond to the most important classes of linear operators considered in this work. Consider $\{|i\rangle\}_{i=0}^{d_A-1}$ the computational basis for a complex vector space A with dimension d_A . An operator $X \in \mathcal{L}(A)$ decomposed in the computational basis is written as $X = \sum_{i,j} \gamma_{ij} |i\rangle\langle j|$, where γ_{ij} is a complex number. The transposition of X , denoted as X^T , is $X^T := \sum_{i,j} \gamma_{ij} |j\rangle\langle i|$, and the complex conjugate of X is $X^* := \sum_{i,j} \overline{\gamma_{ij}} |i\rangle\langle j|$. The *adjoint* of X , denoted as X^\dagger is $X^\dagger := (X^T)^* = (X^*)^T$.

Definition 1.1.1 (Hermitian Operator). An operator $X \in \mathcal{L}(A)$ is Hermitian if $A = A^\dagger$. The set of Hermitian operators acting on A is denoted by $\text{Herm}(A)$.

Throughout this work, we adopt the convention of identifying the spaces where each operator acts by a superscript label. In the above example, the operator X is equivalently represented by X^A within this convention.

An important property of Hermitian operators is that their eigenvalues are necessarily real numbers. This makes Hermitian operators very important in Quantum Theory, as *observables*, *i.e.*, physical quantities that can be measured, can always be represented by Hermitian operators, with the possible values of such physical quantities being associated with the eigenvalues of the operators.

Definition 1.1.2 (Positive Semidefinite Operator). An operator $X \in \mathcal{L}(A)$ is positive semidefinite if it is Hermitian and every eigenvalue is non-negative.

The following items are equivalent definitions for positive semidefinite operators:

- (i) $\langle \psi | X | \psi \rangle$ is a non-negative real number for every $|\psi\rangle \in A$;

- (ii) For every two positive semidefinite operators $X, Y \in \mathcal{L}(A)$, $\text{tr}(XY) \geq 0$;
- (iii) $X = Z^\dagger Z$ for some complex vector space B and $Z : A \rightarrow B$.

In this work, we represent a positive semidefinite operator X by $X \succeq 0$.

Positive semidefinite operators can represent many physical objects in quantum mechanics, such as elements of a positive operator-valued measure (POVM), quantum states, and others.

Definition 1.1.3 (Projector). A linear operator P is an orthogonal projection, or simply called a projector, if P is Hermitian and $P^2 = P$.

Quantum states, *i.e.*, the mathematical object that provides probability distribution for measurements in quantum mechanics, are represented by *density operators*.

Definition 1.1.4 (Density operators). A linear operator ρ is a density operator if $\text{tr}(\rho) = 1$ and $\rho \succeq 0$.

From density operators, one can get the probability of obtaining an outcome of a measurement, generally represented by a positive operator-valued measure (POVM).

1.2 Measurements

Definition 1.2.1 (Positive-operator valued measure (POVM)). A POVM is a set of positive semidefinite operators $\{M_i\}$, $M_i \in \mathcal{L}(A)$ that satisfies

$$\sum_i M_i = \mathbb{1}^A. \quad (1.1)$$

The POVM element M_i is related to the measurement outcome i . The probability of obtaining the measurement outcome i from a state $\rho \in \mathcal{L}(A)$ is given by the Born's rule

$$p(i|\rho, \{M_i\}) = \text{tr}(\rho M_i). \quad (1.2)$$

A *projective measurement* is a particular case of POVM, when the elements M_i also satisfy

$$M_i M_j = \delta_{ij} M_i. \quad (1.3)$$

1.3 Maps

The mathematical description of dynamical evolutions of density matrices in quantum mechanics is made in terms of *linear maps*, which we simply refer to as *maps*.

A linear map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ takes an operator from $\mathcal{L}(A_I)$ to another in $\mathcal{L}(A_O)$, where A_I and A_O are complex finite dimensional vector spaces of input and output. We begin by defining some important properties of linear maps, before introducing one of the most important classes of linear maps in quantum theory, the *quantum channels*, which represent deterministic transformations of quantum states.

1.3.1 Properties of Linear Maps

Definition 1.3.1 (Trace-preserving (TP)). A linear map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ is trace-preserving if it satisfies

$$\text{tr}(\tilde{\Lambda}(\rho)) = \text{tr}(\rho), \quad (1.4)$$

for every operator $\rho \in \mathcal{L}(A_I)$.

Definition 1.3.2 (Positive). A linear map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ is positive if

$$\tilde{\Lambda}(\rho) \succeq 0, \quad (1.5)$$

for all $\rho \succeq 0, \rho \in \mathcal{L}(A_I)$.

Definition 1.3.3 (Completely Positive (CP)). Consider an auxiliary system represented by a finite dimensional complex space denoted by “aux”. The linear map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ is *completely positive* if it satisfies

$$\tilde{\Lambda} \otimes \tilde{\mathbb{I}}(\sigma) \succeq 0 \quad (1.6)$$

for all $\sigma \succeq 0$ acting on $A_I \otimes \text{aux}$, and for any dimension of aux, with $\tilde{\mathbb{I}} : \mathcal{L}(\text{aux}) \rightarrow \mathcal{L}(\text{aux})$ being an identity map.

1.3.2 Channels and Instruments

A quantum channel is a deterministic communication resource capable of transmitting quantum information, encoded in a quantum state.

Definition 1.3.4 (Quantum Channels). A linear map $\tilde{\Phi} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ is a *Quantum Channel* if it is completely positive and trace preserving (CPTP).

The property of complete-positivity of a map ensures that the dynamical evolution of a quantum state results in a positive semidefinite operator, independently of whether the operator represents the state of a single system, or a composite system of the evolved state and an auxiliary system. Together with the property of trace-preserving, these properties ensure that the result of the dynamical evolution is a quantum state, represented by a density operator.

While quantum channels represent deterministic transformations of quantum states, quantum mechanics allows probabilistic transformations, which are mathematically described by *instruments*.

Definition 1.3.5 (Instrument). An instrument $\tilde{\mathcal{I}}$ with m elements is defined as

$$\tilde{\mathcal{I}} := \{\tilde{I}_i\}_{i=0}^{m-1}, \quad (1.7)$$

where $\tilde{I}_i : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ are completely positive and trace non-increasing maps such that $\sum_{i=0}^{m-1} \tilde{I}_i$ is a CPTP map.

After being subject to the action of the instrument $\tilde{\mathcal{I}}$, the quantum state ρ has a probability $p(i|\rho, \{\tilde{I}_i\}) = \text{tr}(\tilde{I}_i(\rho))$ of being transformed into $\frac{\tilde{I}_i(\rho)}{\text{tr}(\tilde{I}_i(\rho))}$.

1.3.3 The Choi-Jamiołkowski representation of Linear Maps

To begin working with linear maps, it is necessary to choose a convenient representation for them. In this work, we present the *Choi-Jamiołkowski representation*, also known as the Choi-Jamiołkowski isomorphism. The main property of such representation is that the maps are represented by operators acting on a composite system of the input and output complex spaces. For information about other representations, one can find the necessary content in Ref. [25, Section 5.2]. Let's begin by presenting the standard version of the Choi-Jamiołkowski isomorphism.

Definition 1.3.6 (Choi-Jamiołkowski isomorphism). Let $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ be a linear map and $\{|i\rangle\}_{i=0}^{d_{A_I}-1}$ be the computational basis for A_I . The Choi operator $\Lambda \in \mathcal{L}(A_I \otimes A_O)$ of the map $\tilde{\Lambda}$ is defined as

$$\Lambda := \sum_{ij} |i\rangle\langle j| \otimes \tilde{\Lambda}(|i\rangle\langle j|). \quad (1.8)$$

A common equivalent way to define the Choi operator of a map $\tilde{\Lambda}$ is given by the equation

$$\Lambda := \left[\left(\tilde{\mathbb{1}} \otimes \tilde{\Lambda} \right) (|\mathbb{1}\rangle\langle\mathbb{1}|) \right], \quad (1.9)$$

where $\tilde{\mathbb{1}} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_I)$ is an identity channel, and $|\mathbb{1}\rangle = \sum_{i=0}^{d_{A_I}-1} |ii\rangle \in A_I \otimes A_I$.

The action of the map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ on an operator $\rho \in \mathcal{L}(A_I)$ can be obtained from the Choi operator Λ by means of the relation

$$\tilde{\Lambda}(\rho) = \text{tr}_{A_I} \left[(\rho^{A_I T} \otimes \mathbb{1}^{A_O}) \cdot \Lambda \right], \quad (1.10)$$

where $\mathbb{1}^{A_O}$ is the identity operator in $\mathcal{L}(A_O)$, and $(\cdot)^T$ stands for the transposition in the computational basis.

As mentioned previously, a linear map is a quantum channel when it is completely positive and trace preserving (CPTP). In the Choi-Jamiołkowski representation, a map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ is CP iff

$$\Lambda \succeq 0, \quad (1.11a)$$

and TP iff

$$\text{tr}_{A_O}(\Lambda) = \mathbb{1}^{A_I}. \quad (1.11b)$$

The proofs of these statements can also be found in Ref. [25, Section 5.3]. For the sake of clarity, we will explicitly identify the input and output Hilbert spaces of a map with the same superscript convention mentioned previously for linear operators. For the previous situation, for example, the Choi operator is equivalently represented by Λ^{A_I/A_O} , indicating that the map $\tilde{\Lambda}$ takes an operator from $\mathcal{L}(A_I)$ to one in $\mathcal{L}(A_O)$.

In the case of an instrument $\tilde{\mathcal{I}}^{A_I/A_O}$, in the Choi-Jamiołkowski representation, the Choi operators $I_i^{A_I/A_O}$ must satisfy

$$I_i^{A_I/A_O} \succeq 0 \quad (1.12a)$$

$$\text{tr}_{A_O} \sum_i (I_i^{A_I/A_O}) = \mathbb{1}^{A_I}. \quad (1.12b)$$

1.4 Supermaps

Suppose we want to describe the experimental realization of a quantum circuit. It is possible to describe such circuit in terms of the composition of each single element of it. However, as an experimental setup has many fixed components, whereas some of them are subject to be often plugged or unplugged, this motivates one to define an object that corresponds to the composition of all fixed elements in a quantum circuit, leading to a description of the experiment as a combination of such single fixed object and the other elements that are subject to change.

The mathematical description of such object is given by the *quantum supermaps*, or simply *supermaps*, studied in Ref. [26], supermaps can be seen as a generalization of maps. By taking the idea of a linear map as mathematical objects that takes an operator that acts on an input Hilbert space A_I to another one that acts on a Hilbert space A_O , representing the mapping by $\mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$, one can extend this idea to define a supermap as the mathematical object that takes a linear mapping $\mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ to another linear mapping $\mathcal{L}(B_I) \rightarrow \mathcal{L}(B_O)$. Throughout this work, we use a double-tilde notation for representing supermaps, as

$$\tilde{\tilde{\Phi}} : \left(\mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O) \right) \rightarrow \left(\mathcal{L}(B_I) \rightarrow \mathcal{L}(B_O) \right). \quad (1.13)$$

Figure 1 illustrates this idea.

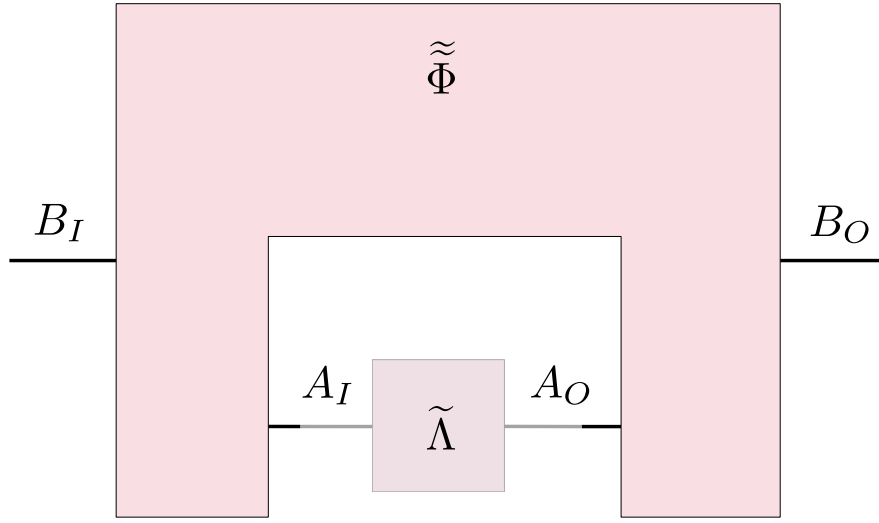


Figure 1 – Illustration of a supermap: A more general object that takes a map $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ to another mapping $\mathcal{L}(B_I) \rightarrow \mathcal{L}(B_O)$, represented by $\tilde{\Phi}$.

1.4.1 Representation of Linear Supermaps

A simple argument allows one to make use of the Choi-Jamiołkowski isomorphism to represent, not only linear maps, but also supermaps or general higher-order operations. First, consider the supermap $\tilde{\Phi} : (\mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)) \rightarrow (\mathcal{L}(B_I) \rightarrow \mathcal{L}(B_O))$. The Choi-Jamiołkowski isomorphism states that there is a one-to-one correspondence between linear maps and a linear operator that acts on the joint Hilbert spaces of input and output, *i.e.*,

$$\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O) \quad \longleftrightarrow \quad \Lambda \in \mathcal{L}(A_I \otimes A_O). \quad (1.14)$$

This argument allows us to see a supermap as a linear map of Choi operators. The Choi-Jamiołkowski isomorphism applied to such maps once again then leads us to represent the initial supermaps as Choi operators acting on the composite system of all Hilbert spaces involved, *i.e.*

$$\begin{aligned} \tilde{\Phi} : (\mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)) &\rightarrow (\mathcal{L}(B_I) \rightarrow \mathcal{L}(B_O)) \\ &\quad \updownarrow \\ \tilde{\Phi} : \mathcal{L}(A_I \otimes A_O) &\rightarrow \mathcal{L}(B_I \otimes B_O) \\ &\quad \updownarrow \\ \Phi &\in \mathcal{L}(A_I \otimes A_O \otimes B_I \otimes B_O). \end{aligned} \quad (1.15)$$

Therefore, any higher-order operation can be represented by Choi operators.

1.4.2 The link product operation

First defined in Ref. [13], the link product is a useful mathematical tool to perform compositions of elements in a quantum circuit, presented in the Choi-Jamiołkowski representation.

Definition 1.4.1 (Link product). Let $\Lambda_1 \in \mathcal{L}(A \otimes A')$ and $\Lambda_2 \in \mathcal{L}(A' \otimes A'')$ be linear operators. The link product between Λ_1 and Λ_2 is defined as

$$\Lambda_2 * \Lambda_1 := \text{tr}_{A'} \left[\left(\Lambda_1^{T_{A'}} \otimes \mathbb{1}^{A''} \right) \cdot \left(\mathbb{1}^A \otimes \Lambda_2 \right) \right], \quad (1.16)$$

where $\Lambda_1^{T_{A'}}$ is the partial transpose of Λ_1 on the space A' .

As stated previously, the link product is useful for composing linear maps and quantum objects. If $\tilde{\Lambda}_1 : \mathcal{L}(A) \rightarrow \mathcal{L}(A')$ and $\tilde{\Lambda}_2 : \mathcal{L}(A') \rightarrow \mathcal{L}(A'')$ are linear maps with Choi operators $\Lambda_1 \in \mathcal{L}(A \otimes A')$ and $\Lambda_2 \in \mathcal{L}(A' \otimes A'')$, respectively, it can be shown that the Choi operator of the composition $\tilde{\Phi} := \tilde{\Lambda}_2 \circ \tilde{\Lambda}_1 : \mathcal{L}(A) \rightarrow \mathcal{L}(A'')$ is $\Phi = \Lambda_2 * \Lambda_1$.

Considering $\rho_1 \in \mathcal{L}(A_1)$ and $\rho_2 \in \mathcal{L}(A_2)$, acting on different spaces, we have $\rho_1 * \rho_2 := \rho_1 \otimes \rho_2$. Therefore, the link product of independent systems is simply the tensor product. Additionally, when $\rho, M \in \mathcal{L}(A)$ act in the same linear space, the link product is given by $\rho * M := \text{tr}(M^T \cdot \rho)$, which is simply the trace of their product with an extra transposition. This form can also be used to write Born's rule.

In particular, when $\rho \in \mathcal{L}(A)$ is a linear operator with no components on A' , and $\Lambda \in \mathcal{L}(A \otimes A')$, we have

$$\begin{aligned} \Lambda * \rho &:= \text{tr}_A \left[\left(\rho^{T_A} \otimes \mathbb{1}^{A'} \right) \cdot (\Lambda) \right] \\ &= \tilde{\Lambda}(\rho). \end{aligned} \quad (1.17)$$

2 Optimization, semidefinite programming, and Entanglement witnesses

Many important techniques that are used in this work are presented in the context of *semidefinite programming*, which is a class of *optimization* problems. An optimization problem can be roughly described by the task of finding the optimal value of an objective function that is subject to some constraints on the values that its variables can have. This is a field with applications in many areas of knowledge, such as Economics and Finance, Engineering, Geo-localization and Machine Learning.

In the next sections, we give a brief introduction to optimization problems based on Refs. [25, 27, 28]. For a more detailed study of this topic, the reader can find the necessary content on Refs. [27, Appendix A], [28, Chapter 4] and [25, Lecture 7].

2.1 Optimization: Primal and dual problems and semidefinite programming

Consider a vector $x \in \mathbb{R}^d$ and a function $f_0 : \mathbb{R}^d \rightarrow \mathbb{R}$. An optimization problem corresponds to finding the value of x that minimizes¹ the function f_0 , subject to some conditions, which are represented by equality and inequality constraints $h_i(x) = 0$, $i = 1, \dots, p$, and $f_j(x) \leq 0$, $j = 1, \dots, q$. The notation that represent this optimization problem is

$$\begin{aligned} \min \quad & f_0(x) \\ \text{subject to} \quad & h_i(x) = 0, \quad i = 1, \dots, p, \\ & f_j(x) \leq 0, \quad j = 1, \dots, q. \end{aligned} \tag{2.1}$$

The value of x that minimizes $f_0(x)$ is called the *optimal value*. A value of x that satisfies all the constraints $h_i(x)=0$ and $f_j(x) \leq 0$, but does not necessarily minimize the objective function $f_0(x)$ is called a *feasible point*.

Every optimization problem has a primal and a dual problem. The dual problem provides a lower bound for the solution of Eq. (2.1). One can obtain the dual problem from Eq.(2.1) by following the procedure presented in Ref.[28, Chapter 5], which consists of finding its *Lagrange dual function*, then optimize it. The following steps

¹ The minimization of a function f_0 is equivalent to the maximization of $-f_0$.

summarize the procedure of obtaining the dual problem of an optimization problem from its primal.

Consider the sets $\lambda := \{\lambda_1, \dots, \lambda_p\}$ and $\nu := \{\nu_1, \dots, \nu_q\}$, λ_i and $\nu_i \in \mathbb{R} \forall i$.

- Obtain a Lagrangian functional from the relation

$$L(x, \lambda, \nu) = f_0(x) + \sum_i \nu_i h_i(x) + \sum_i \lambda_i f_i(x), \quad (2.2a)$$

with ν_i and λ_i being the Lagrange multipliers associated to the equality and inequality constraints, respectively;

- Find the Lagrange dual function from the Lagrangian using the relation

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu); \quad (2.2b)$$

- As mentioned before, the Lagrange dual function always gives a lower bound for the optimal solution of the primal problem, with ν_i having any possible value and $\lambda_i \geq 0$, *i.e.*,

$$g(\lambda, \nu) \leq \min f_0(x); \quad (2.2c)$$

- The last step is to maximize the value of the Lagrange dual function over the Lagrange multipliers, which are the new variables of the optimization problem:

$$\begin{aligned} & \max g(\lambda, \nu) \\ & \text{subject to } \lambda_i \geq 0 \forall i. \end{aligned} \quad (2.2d)$$

The Lagrange multipliers are also called the *dual variables* of the optimization problem. Whenever the equality of Eq. (2.2d) is achieved, we say that the optimization problem has *strong duality*, which will be the case for every optimization problem treated in this work.

2.2 Entanglement witness

Consider a density operator $\rho \in \mathcal{L}(A \otimes B)$ that represents a bipartite quantum state. We say that ρ is a *separable state* if it can be decomposed as the following convex combination

$$\rho = \sum_i p_i \sigma_i^A \otimes \phi_i^B, \quad (2.3)$$

with $\sigma_i^A \in \mathcal{L}(A)$, $\phi_i^B \in \mathcal{L}(B)$ being density operators, $p_i \geq 0 \forall i$ and $\sum_i p_i = 1$. We denote the set of bipartite separable states by $\mathcal{L}_{\text{sep}}(A \otimes B)$.

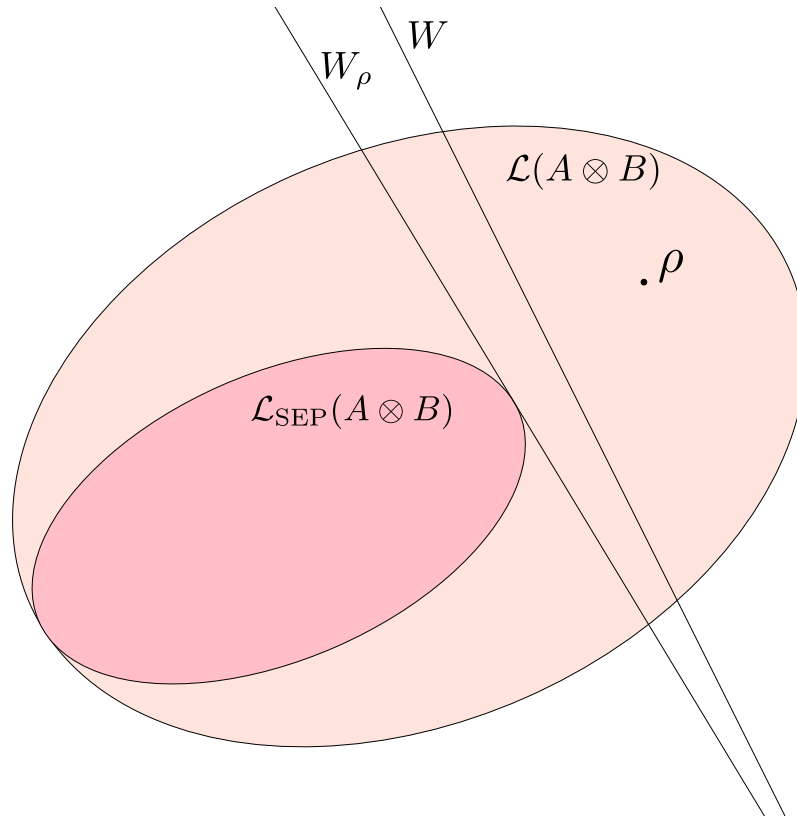


Figure 2 – Illustration of set $\mathcal{L}(A \otimes B)$ of all bipartite states and the set of separable states $\mathcal{L}_{\text{SEP}}(A \otimes B)$ as its subset. The operator $W \in \mathcal{L}(A \otimes B)$ provides a hyperplane (represented by the straight line) that splits the space in two parts, so that, for every operator on the left of the hyperplane, the relation $\text{tr}(W\rho) \geq 0$ holds, whereas $\text{tr}(W\rho) < 0$ holds for every operator on the right. The operator W_ρ is the optimal entanglement witness for ρ .

We say that ρ is *entangled* if such decomposition does not exist. By definition, the set $\mathcal{L}_{\text{SEP}}(A \otimes B)$ is a convex subset of the set of all possible states represented by density operators in $\mathcal{L}(A \otimes B)$.

A witness of bipartite entanglement is a Hermitian operator $W \in \mathcal{L}(A \otimes B)$ that satisfies $\text{tr}(W\rho_{\text{sep}}) \geq 0$ for every separable operator $\rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B)$. Non-trivial witnesses are those satisfying $\text{tr}(W\rho) < 0$ for some entangled states $\rho \in \mathcal{L}(A \otimes B)$. Since the set of separable operators is closed and convex, every entangled state can be certified by an entanglement witness W . Figure. 2 illustrates such an idea.

For a given entangled state ρ , its optimal entanglement witness W_ρ is the one satisfying

$$\text{tr}(W_\rho\rho) \leq \text{tr}(W\rho) \tag{2.4}$$

for every entanglement witness $W \in \mathcal{L}(A \otimes B)$.

2.2.1 Robustnesses of Entanglement

The task of finding the optimal entanglement witness W for a given entangled state can be represented by the following optimization problem:

$$\begin{aligned} & \text{given } \rho \\ & \min \operatorname{tr}(W\rho) \\ & \text{subject to } \operatorname{tr}(W\rho_{\text{sep}}) \geq 0 \quad \forall \rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (2.5a) \\ & \quad \mathbb{1} - W \succeq 0, \quad (2.5b) \end{aligned}$$

where ρ_{sep} is any separable state in $\mathcal{L}(A \otimes B)$. Condition (2.5b) is a choice of normalization condition taken to avoid a diverging solution for the problem of Eqs (2.5).

Eq. (2.5a) is a condition that makes the problem hard to solve, as it would be necessary, in principle, to verify if such condition is satisfied for every separable state ρ_{sep} in $\mathcal{L}_{\text{SEP}}(A \otimes B)$. We will later see that it is possible to approximate this problem by a hierarchy of SDPs that converges to the optimization problem described by Eqs. (2.5).

In appendix A, we show that the entanglement witnessing problem described in Eqs. (2.5) is a problem of *entanglement robustness*, which is associated to the minimum amount of noise one must “add” to a state to result in a separable state. This can be seen in the dual problem of Eqs. (2.5), which can be written as

$$\begin{aligned} & \text{given } \rho \\ & \min r \\ & \text{subject to } (1 - r)\rho + r\rho' = \sigma_{\text{sep}} \quad (2.6a) \\ & \quad \sigma_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (2.6b) \\ & \quad \rho', \sigma_{\text{sep}} \succeq 0, \quad (2.6c) \\ & \quad \operatorname{tr}(\rho') = \operatorname{tr}(\sigma_{\text{sep}}) = 1, \quad (2.6d) \\ & \quad r \in [0, 1]. \quad (2.6e) \end{aligned}$$

The above problem represents the *generalized robustness*, where ρ' is a general noise added to ρ in a convex combination that results in a separable state σ_{sep} .

Analogously to the generalized robustness, it is possible to evaluate the entanglement robustness as a *white noise robustness* problem, which corresponds to the minimum amount of *white noise*, represented by the maximally mixed state in $\mathcal{L}(A \otimes B)$, necessary to be added to ρ so that the result is separable. The primal problem of the white noise

robustness is given by

$$\begin{aligned} & \text{given } \rho \\ & \min \operatorname{tr}(W\rho) \\ & \text{subject to } \operatorname{tr}(W\rho_{\text{sep}}) \geq 0 \quad \forall \rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \end{aligned} \quad (2.7a)$$

$$\operatorname{tr}(W) \leq 1, \quad (2.7b)$$

and the dual problem of the white noise robustness of entanglement can be written as

$$\begin{aligned} & \text{given } \rho \\ & \min r \\ & \text{subject to } (1-r)\rho + r\frac{\mathbb{1}}{d} = \rho_{\text{sep}}, \end{aligned} \quad (2.8a)$$

$$\rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (2.8b)$$

$$\rho_{\text{sep}} \succeq 0, \operatorname{tr}(\rho_{\text{sep}}) = 1, \quad (2.8c)$$

$$0 \leq r \leq 1. \quad (2.8d)$$

In appendix A, one can find more details on how to obtain the dual problems of entanglement witnessing.

2.3 Semidefinite Programming

The main class of optimization problems considered in this work is *semidefinite programming*. Many problems in quantum mechanics can be solved in terms of finding optimal solutions for semidefinite programs. This is also convenient, as there exist efficient algorithms for solving SDPs, such as Mosek and SeDuMi [29, 30].

Consider a complex Euclidean space \mathbb{C}^d of finite dimension d , a hermiticity-preserving linear map $\tilde{\Phi} : \mathcal{L}(\mathbb{C}^d) \rightarrow \mathcal{L}(\mathbb{C}^d)$ and fixed Hermitian operators $C, D \in \mathcal{L}(\mathbb{C}^d)$. A semidefinite program is an optimization problem that can be written in the following form

$$\begin{aligned} & \text{given } \rho \\ & \min \operatorname{tr}(CX) \\ & \text{subject to } \tilde{\Phi}(X) = D, \end{aligned} \quad (2.9a)$$

$$X \succeq 0. \quad (2.9b)$$

Eq. (2.9) is the canonical form of a semidefinite program. The important elements for an optimization problem to be a SDP is that it has a positive semidefinite relation, an objective function which is linear, and linear relations between operators.

Any optimization problem that can be written in the form of Eqs. (2.9) is an SDP. However, we do not always encounter semidefinite programs in the canonical form, in

general. When considering numerical calculations for solving SDPs, it is rather useful for making use of *interpreters* such as Yalmip and CVX [31, 32], which are responsible of putting the SDPs in the canonical form, allowing the solving step.

Now, following the same steps taken in the previous section in order to obtain the dual problem of the SDP, we have the following *Lagrangian* of this problem:

$$\begin{aligned} L(X, Y, Z) &= \text{tr}(CX) + \text{tr}\left(Y\left(B - \tilde{\Phi}(X)\right)\right) - \text{tr}(ZX) \\ &= \text{tr}\left(X\left(C - \tilde{\Phi}^\dagger(Y) - Z\right)\right) + \text{tr}(YB), \end{aligned} \quad (2.10)$$

with Y and Z being Lagrange multipliers in $\mathcal{L}(\mathbb{C}^d)$ and $\tilde{\Phi}^\dagger$ is the adjoint map, which is the unique map that satisfies $\text{tr}\left(Y\tilde{\Phi}(X)\right) = \text{tr}\left(\tilde{\Phi}^\dagger(Y)X\right)$ for every Hermitian operators X and Y in $\mathcal{L}(\mathbb{C}^d)$.

By noticing that, if $Z \succeq 0$, $L(X, Y, Z) \leq \text{tr}(CX)$ for every feasible point X , *i.e.*, every value of X that satisfies $X \succeq 0$ and $\tilde{\Phi}(X) = D$. Then, if we take the infimum of L over X , we obtain the Lagrange dual function $g(Y, Z)$

$$g(Y, Z) = \inf_X L(X, Y, Z), \quad (2.11)$$

which has the non-diverging solution $g(Y, Z) = \text{tr}(YB)$ if $(C - \tilde{\Phi}^\dagger(Y) - Z) = 0$ and $Z \succeq 0$. Now, by maximizing the Lagrange dual function $g(Y, Z)$ to obtain the best possible lower bound for $\min \text{tr}(CX)$, we obtain the dual form of the SDP:

$$\max \text{tr}(YB)$$

$$\text{subject to } \tilde{\Phi}^\dagger(Y) + Z = C \quad (2.12a)$$

$$Z \succeq 0. \quad (2.12b)$$

2.3.1 Semidefinite Programming characterization of separable states

For the problem of witnessing entanglement, it is necessary to have a characterization of the set of separable states, so it becomes possible to implement the constraints related to ρ_{sep} in the optimization programs.

A simple outer approximation for the set of separable processes in $\mathcal{L}(A \otimes B)$ is the *Peres-Horodecki* criterion [33, 34]. The criterion states that if an operator $\rho \in \mathcal{L}(A \otimes B)$ is separable, then it has positive partial transposition (PPT), *i.e.*, $\rho^{T_A} \succeq 0$. This is rather easy to verify, as, by definition, a separable operator ρ_{sep} can be written as $\rho_{\text{sep}} = \sum_i p_i \rho_i^A \otimes \rho_i^B$, and the partial transposition of ρ_{sep} over A is $\rho_{\text{sep}}^{T_A} = \sum_i p_i \rho_i^{A^T} \otimes \rho_i^B \succeq 0$. For $A \otimes B$ with dimensions 2×2 or 2×3 , the Peres-Horodecki criterion is a necessary and sufficient condition for separability.

According to Refs. [35, 36], it is possible to completely characterize the set of separable states in $\mathcal{L}(A \otimes B)$ in terms of an infinite hierarchy of semidefinite programs.

Before introducing the characterization, it is necessary to define *symmetric extensions*. For $k = 1, 2, \dots$, an operator $\rho^{AB} \in \mathcal{L}(A \otimes B)$ has a symmetric k -extension (or k -symmetric extension) if there exists an operator $\rho_{\text{ext}}^{(A^{\otimes k})B}$ that satisfies

$$(P_{\text{sym}}^{A^{\otimes k}} \otimes \mathbb{1}^B) \rho_{\text{ext}}^{(A^{\otimes k})B} (P_{\text{sym}}^{A^{\otimes k}} \otimes \mathbb{1}^B) = \rho_{\text{ext}}^{(A^{\otimes k})B}, \quad (2.13a)$$

$$\text{tr}_{A^{\otimes(k-1)}} \left[\rho_{\text{ext}}^{(A^{\otimes k})B} \right] = \rho^{AB}, \quad (2.13b)$$

where $P_{\text{sym}}^{A^{\otimes k}}$ is the projector onto the symmetric subspace of $A^{\otimes k}$. The problem of finding a k -symmetric extension for an operator $\rho \in \mathcal{L}(A \otimes B)$ can be solved with an SDP.

Refs. [35, 36] state that an operator ρ^{AB} is separable if there exists, for some level k , a k -symmetric extension for it. The criterion is stronger when considering *PPT* k -symmetric extensions, *i.e.*,

$$\exists \rho_{\text{ext}}^{(A^{\otimes k})B} \text{ and } \left(\rho_{\text{ext}}^{(A^{\otimes k})B} \right)^{T_A} \succeq 0. \quad (2.14)$$

In the case where $k = 1$, the conditions in Eq. (2.14) simply reduce to the Peres-Horodecki criterion. Eq. (2.14) have necessary conditions for separability of operators in $\mathcal{L}(A \otimes B)$. They also become sufficient conditions in the limit of $k \rightarrow +\infty$, which means that the hierarchy of PPT k -symmetric extensions completely characterizes the set of separable operators in $\mathcal{L}(A \otimes B)$.

With this characterization, we can return to the problem of witnessing entanglement and re-write the conditions in terms of it. For A and B with dimensions d_A and d_B , respectively, the white noise robustness of entanglement, for example, becomes then

$$\text{given } \rho^{AB} \in \mathcal{L}(A \otimes B)$$

$$\min r$$

$$\text{subject to } (1-r)\rho^{AB} + r \frac{\mathbb{1}}{d_A d_B} = \rho_{\text{sep}}, \quad (2.15a)$$

$$0 \leq r \leq 1, \quad (2.15b)$$

$$\rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (2.15c)$$

$$\rho_{\text{sep}} \succeq 0, \text{ tr}(\rho_{\text{sep}}) = 1, \quad (2.15d)$$

$$\text{tr}_{A^{\otimes(k-1)}} \left[\rho_{\text{ext}}^{(A^{\otimes k})B} \right] = \rho_{\text{sep}}, \quad (2.15e)$$

$$(P_{\text{sym}}^{A^{\otimes k}} \otimes \mathbb{1}^B) \rho_{\text{ext}}^{(A^{\otimes k})B} (P_{\text{sym}}^{A^{\otimes k}} \otimes \mathbb{1}^B) = \rho_{\text{ext}}^{(A^{\otimes k})B}, \quad (2.15f)$$

$$\rho_{\text{ext}}^{(A^{\otimes k})B} \succeq 0, \quad (2.15g)$$

$$\left(\rho_{\text{ext}}^{(A^{\otimes k})B} \right)^{T_A} \succeq 0. \quad (2.15h)$$

The generalized robustness of entanglement can be re-written analogously.

Part II

Common-cause and Direct-cause: Framework and literature review

3 The CCDC scenario

The study of causal relations of two variables is a topic of great interest and with applications in multiple areas of knowledge. A good motivation can be brought from the situation humanity is currently facing: the coronavirus pandemic. The disease affected the world in a large scale and had consequences on each individual lifestyle and routine. The discussions about the development of vaccines attracted worldwide attention, bringing the hope that their effectiveness could bring our “normal world” back.

The study of the efficacy of a vaccine is an example of a *causal inference* problem [4]. From the conduction of an experiment with many individual separated in two groups: test group (individuals who take the actual vaccines) and control group (individuals who take placebo), one can observe a *correlation* between taking the vaccine and the development of a resistance to getting infected by the disease.

Classically, when correlations between two variables are observed, the possible origin of such correlation is either if one *causes* an effect on the other or, by the Reichenbach’s common-cause principle [3], there is a third variable that is a *common-cause* to both of them. However, such possibilities do not provide complete description of what can happen when we are dealing with quantum systems. In particular, Reichenbach’s principle fails to provide an explanation to non-local correlations present in entangled systems, *i.e.*, it is not possible to provide decomposition with a local classical random variable that recovers the joint probabilities of measurements from entangled systems in general.

Furthermore, as causal relations can be considered as analogous to communication between parties, we can also motivate this study by investigating what properties quantum theory can provide that could be somehow useful for potential applications in quantum computation. Throughout this work, one will notice that all the objects studied here have a representation as a quantum circuit, which shows that the results can be potentially useful for applications in quantum computation.

The main objects analysed in this work are the bipartite ordered processes, which may be understood as a physical dynamic which flows from Alice to Bob. For all definitions in this section, we consider a scenario where Alice is a quantum physicist in a laboratory who can perform any quantum operation that transforms states from system A_I (Alice input) to A_O (Alice output), and Bob is a quantum physicist in a laboratory who can perform any quantum measurement on states defined on system B_I (Bob’s input).

3.1 Markovian processes

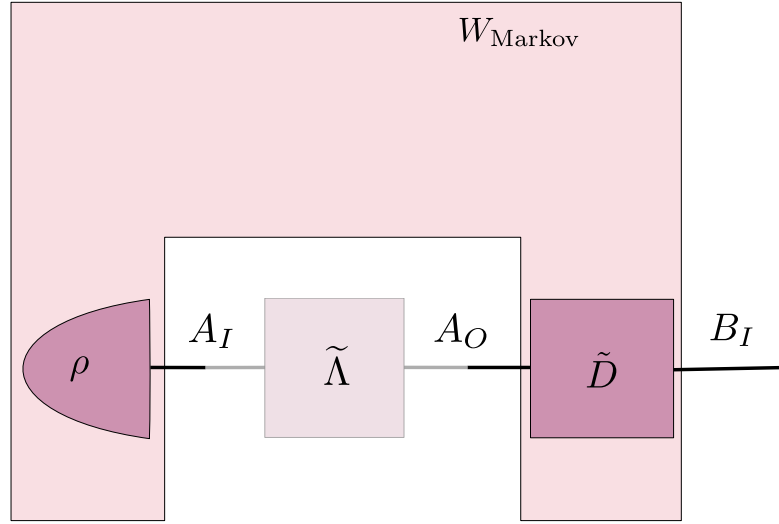


Figure 3 – Circuit representation of Markovian processes: Alice receives a fixed state $\rho \in \mathcal{L}(A_I)$ on which she can perform an arbitrary quantum operation Λ . After Alice’s operation, the system is subjected to a fixed dynamic described by a quantum channel $\tilde{D} : \mathcal{L}(A_O) \rightarrow \mathcal{L}(B_I)$, then arriving at Bob’s input space laboratory. No auxiliary system is used in this circuit, which implies that correlations between the parties are only due to the direct communication from Alice to Bob.

We start by presenting the definition of bipartite *Markovian processes*, which admits a *direct-cause* interpretation [8, 37, 38]. These processes are also found in the literature named as *quantum processes with no memory* [20], and *cause-effect causal maps* [16]. In a Markovian scenario, Alice receives a known quantum state $\rho \in \mathcal{L}(A_I)$ which acts on her input system space. Alice can then perform an arbitrary quantum operation¹ $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ to obtain a quantum state on system A_O , which will then be subjected to a known deterministic dynamics described by a channel $\tilde{D} : \mathcal{L}(A_O) \rightarrow \mathcal{L}(B_I)$. Finally, the state $\tilde{D}(\tilde{\Lambda}(\rho)) \in \mathcal{L}(B_I)$ is received by Bob, who can perform an arbitrary measurement on it. See Fig. 3 for a circuit based pictorial illustration.

A bipartite Markovian process can then be described by the known quantum state $\rho \in \mathcal{L}(A_I)$ and the known dynamic represented by the channel $\tilde{D} : \mathcal{L}(A_O) \rightarrow \mathcal{L}(B_I)$. In the Choi-Jamiołkowski representation, this can be conveniently described by $W_{\text{Markov}} := \rho^{A_I} \otimes D^{A_O/B_I}$. Under this definition, for any quantum operation $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ performed by Alice, the quantum state arriving in Bob’s input space can be obtained via the link product as

¹ Which may be a quantum channel (deterministic quantum operation) or a quantum instrument (probabilistic quantum operation).

$$\begin{aligned}
W_{\text{Markov}} * \Lambda^{A_I/A_O} &= \rho^{A_I} \otimes D^{A_O/B_I} * \Lambda^{A_I/A_O} \\
&= \rho^{A_I} * D^{A_O/B_I} * \Lambda^{A_I/A_O} \\
&= D^{A_O/B_I} * \Lambda^{A_I/A_O} * \rho^{A_I} \\
&= D^{A_O/B_I} * \tilde{\Lambda}(\rho^{A_I}) \\
&= \tilde{D}(\tilde{\Lambda}(\rho)).
\end{aligned} \tag{3.1}$$

Definition 3.1.1 (Markovian process). A linear operator $W_{\text{Markov}} \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is bipartite Markovian process if it can be written as

$$\begin{aligned}
W_{\text{Markov}} &:= \rho^{A_I} * D^{A_O/B_I}, \\
&= \rho^{A_I} \otimes D^{A_O/B_I},
\end{aligned} \tag{3.2}$$

where ρ^{A_I} is a quantum state *i.e.*, $\rho \succeq 0$, $\text{tr}(\rho) = 1$ and D^{A_O/B_I} is the Choi operator of a quantum channel from the output of Alice to Bob's input *i.e.*, $D^{A_O/B_I} \succeq 0$, $\text{tr}_{B_I}(D^{A_O/B_I}) = \mathbb{1}_{A_O}$. We denote the set of Markovian processes by $\mathcal{L}_{\text{Markov}}$.

3.2 Direct-cause processes

In a Markovian process, all correlations between Alice and Bob admit a direct-cause interpretation, since no auxiliary system or environment is required at any point. In a scenario where Alice and Bob may share prior *classical* correlations, it is natural to define *direct-cause* processes as any process which can be written as a probabilistic mixture of Markovian processes. Hence, the correlations arising from these processes can always be explained by classical mixtures of direct-cause ones.

Definition 3.2.1 (Direct-cause process). A linear operator $W_{\text{DC}} \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is direct-cause if it is a classical mixture of Markovian processes, that is, W_{DC} can be written as

$$W_{\text{DC}} := \sum_i p_i \rho_i^{A_I} * D_i^{A_O/B_I} \tag{3.3a}$$

$$= \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}, \quad p_i \in [0, 1], \tag{3.3b}$$

where $\rho_i^{A_I}$ are quantum states, *i.e.*, $\rho_i \succeq 0$ and $\text{tr}(\rho_i) = 1 \forall i$, whereas $D_i^{A_O/B_I}$ are Choi operators of quantum channels from the output of Alice to Bob's input, *i.e.*, $D_i^{A_O/B_I} \succeq 0$ and $\text{tr}_{B_I}(D_i^{A_O/B_I}) = \mathbb{1}_{A_O} \forall i$. We denote the set of direct-cause processes by \mathcal{L}_{DC} .

In the bipartite scenario, an ordered process is direct-cause if and only if it is a *process without quantum memory*, as defined and shown in Ref. [20]. See Appendix B for

details. As will be discussed later in Section 6.2, Ref. [2] discusses common-cause and direct-cause relations, but a tripartite scenario. Although Ref. [2] analyses tripartite ordered processes, when bipartite processes are considered in that context, their definition of direct-cause processes is equivalent to the one presented in Refs. [1, 16], which is also equivalent to the definition adopted in this work.

3.2.1 Quantum separability as an outer approximation for the set of DC processes

A positive semidefinite linear operator $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is separable in the bipartition $A_I|A_O B_I$ if there exist a probability distribution p_i and positive semidefinite operators $\rho_i^{A_I} \succeq 0$ and $D_i^{A_O/B_I} \succeq 0$ such that $W = \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}$, which coincides with the definition of a DC process (Def. 3.2.1) without the trace normalization condition.

It follows directly from Eq. (3.3b) that every direct-cause process is separable in the bipartition $A_I|A_O B_I$. Conversely, if a process is entangled in the bipartition $A_I|A_O B_I$, such process cannot have a DC explanation. This allows us to employ techniques from entanglement theory to certify whether a process is not direct-cause. A simple outer approximation for the set of separable states is the set of states with positive partial transpose (PPT) [33, 34]. Also, as discussed in Section 2.3.1, the set of states with a PPT k -symmetric extension forms a hierarchy of sets which converges to the set of separable states when $k \rightarrow \infty$. For every fixed k , checking if a state has a PPT symmetric k -extension can be done via SDP.

Following the conditions from Eq. (2.14), we define an outer approximation for the set of direct-cause processes as follows:

Definition 3.2.2 (Separable outer approximation for DC processes). A bipartite ordered process $W^{A_I A_O B_I}$ belongs to $\mathcal{L}_{\text{DC}}^{\text{out, PPT}^k}$ if $W^{A_I A_O B_I}$ has a PPT k -symmetric extension on the bipartition $A_I|A_O B_I$, *i.e.*, there exists a positive semidefinite operator $W_{\text{ext}}^{(A_I^{\otimes k}) A_O B_I} \in \mathcal{L}(A_I^{\otimes k} \otimes A_O \otimes B_I)$ such that

$$(P_{\text{sym}}^{A_I^{\otimes k}} \otimes \mathbb{1}^{A_O B_I}) W_{\text{ext}}^{(A_I^{\otimes k}) A_O B_I} (P_{\text{sym}}^{A_I^{\otimes k}} \otimes \mathbb{1}^{A_O B_I}) = W_{\text{ext}}^{(A_I^{\otimes k}) A_O B_I}, \quad (3.4a)$$

$$\text{tr}_{A_I^{\otimes(k-1)}} \left[W_{\text{ext}}^{(A_I^{\otimes k}) A_O B_I} \right] = W^{A_I A_O B_I}, \quad (3.4b)$$

$$\left(W_{\text{ext}}^{(A_I^{\otimes k}) A_O B_I} \right)^{T_{A_I}} \succeq 0. \quad (3.4c)$$

One could be tempted to assume that all process which are separable in the bipartition $A_I|A_O B_I$ have a direct-cause explanation. However, in Section 7.1, we construct an explicit example which proves that this is not the case. This ensures that the problem of certifying if a given process is not direct-cause is not equivalent to certifying entanglement.

3.3 Common-cause processes

Common-cause processes are those where the correlations are not due to communication from Alice to Bob, but only due to a bipartite quantum state $\rho \in \mathcal{L}(A_I \otimes B_I)$ initially shared by them. In this way, a common-cause process does not involve any particular quantum dynamic and can be fully characterized by a fixed bipartite state. In order to establish a notation which dialogues with general scenarios which may have non-trivial dynamics between Alice and Bob, we describe common-cause processes by $W_{\text{CC}} = \rho^{A_I B_I} \otimes \mathbb{1}^{A_O}$.

Definition 3.3.1 (Common-cause process). A linear operator $W_{\text{CC}} \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is common-cause process if it can be written as

$$W_{\text{CC}} = \rho^{A_I B_I} \otimes \mathbb{1}^{A_O}, \quad (3.5)$$

where $\rho^{A_I B_I}$ is a quantum state shared between Alice and Bob, *i.e.*, $\rho^{A_I B_I} \succeq 0$, $\text{tr}(\rho^{A_I B_I}) = 1$, and $\mathbb{1}^{A_O}$ is the identity operator acting on A_O . We denote the set of common-cause processes by \mathcal{L}_{CC} .

3.4 Classical CCDC processes

Classical CCDC processes are processes which can be decomposed as a convex combination of a common-cause and a direct-cause ones. Such processes can always be understood as simple classical statistical mixtures between quantum common-cause and quantum direct-cause processes [2, 16].

Definition 3.4.1 (Classical CCDC processes). A linear operator $W_{\text{CCDC}} \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is a classical CCDC process if it can be written as

$$W_{\text{CCDC}} := pW_{\text{CC}} + (1 - p)W_{\text{DC}}, \quad (3.6)$$

where $W_{\text{CC}} \in \mathcal{L}_{\text{CC}}$, $W_{\text{DC}} \in \mathcal{L}_{\text{DC}}$, and $p \in [0, 1]$. We denote the set of classical CCDC processes by $\mathcal{L}_{\text{CCDC}}$, which is the convex hull of $\mathcal{L}_{\text{CC}} \cup \mathcal{L}_{\text{DC}}$.

3.5 Bipartite ordered processes

Previously, we have only considered processes which do not require any explicit use of auxiliary systems or environments. In a more general process, the initial state ρ received by Alice can be defined in a bigger space, and Alice's lab can only operate on part of it. In order to tackle this situation, we now consider that the initial state is defined not only on Alice's input system, but also in some auxiliary system with unconstrained finite dimension, that is, $\rho \in \mathcal{L}(A_I \otimes \text{aux})$. Now, Alice can only operate on the part of ρ

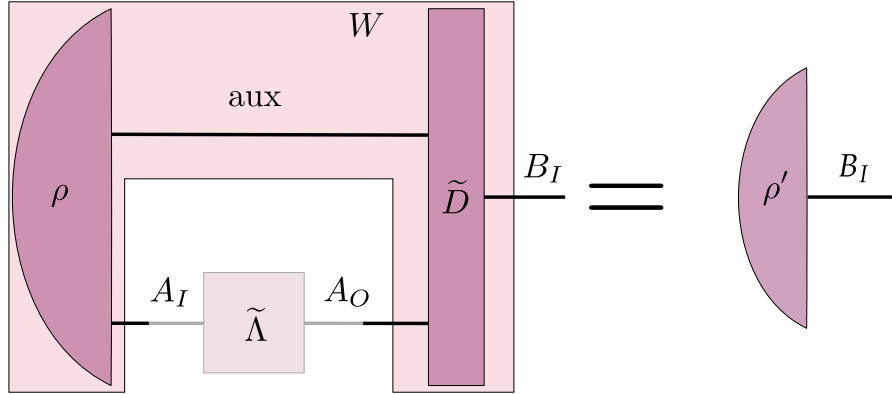


Figure 4 – A circuit illustrating a bipartite ordered process $W = \rho * D$, where $\rho \in \mathcal{L}(A_I \otimes \text{aux})$ is a quantum state and D is the Choi operator of a channel $\tilde{D} : \mathcal{L}(A_O \otimes \text{aux}) \rightarrow \mathcal{L}(B_I)$. When a quantum channel $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ with Choi operator Λ is “plugged” into the process W , the state $\rho' = W * \Lambda \in \mathcal{L}(B_I)$ is obtained.

that enters her laboratory, that is, she can perform any operation $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$, which will transform the initial state ρ into ${}^2\tilde{\Lambda}^{A_I/A_O} \otimes \tilde{\mathbb{I}}^{\text{aux}}(\rho)$, which will be subjected to a global dynamics described by a quantum channel $\tilde{D} : \mathcal{L}(A_O \otimes \text{aux}) \rightarrow \mathcal{L}(B_I)$. See Fig. 4 for a circuit based pictorial illustration.

A bipartite ordered process can then be described by the known quantum state $\rho \in \mathcal{L}(A_I \otimes \text{aux})$ and the known dynamics represented by the channel $\tilde{D} : \mathcal{L}(A_O \otimes \text{aux}) \rightarrow \mathcal{L}(B_I)$. In the Choi-Jamiołkowski representation, this can be conveniently described by $W := \rho^{A_I \text{aux}} * D^{A_O \text{aux}/B_I}$. Under this definition, for any quantum operation $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ performed by Alice, the quantum state arriving in Bob’s input space can be obtained via the link product as

$$\begin{aligned}
 W * \Lambda^{A_I/A_O} &= \rho^{A_I \text{aux}} * D^{A_O \text{aux}/B_I} * \Lambda^{A_I/A_O} \\
 &= D^{A_O/B_I} * \Lambda^{A_I/A_O} * \rho^{A_I \text{aux}} \\
 &= D^{A_O/B_I} * \tilde{\Lambda}^{A_I/A_O} \otimes \tilde{\mathbb{I}}^{\text{aux/aux}}(\rho^{A_I \text{aux}}) \\
 &= \tilde{D}\left(\tilde{\Lambda}^{A_I/A_O} \otimes \tilde{\mathbb{I}}^{\text{aux/aux}}(\rho^{A_I \text{aux}})\right).
 \end{aligned} \tag{3.7}$$

Definition 3.5.1 (Bipartite ordered process). A linear operator $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is a bipartite ordered process if there exists a quantum state $\rho^{A_I \text{aux}} \in \mathcal{L}(A_I \otimes \text{aux})$, i.e., $\rho^{A_I \text{aux}} \succeq 0$, $\text{tr}(\rho^{A_I \text{aux}}) = 1$ and a quantum channel $\tilde{D} : \mathcal{L}(A_I \otimes \text{aux}) \rightarrow \mathcal{L}(B_I)$ with Choi

² Here $\tilde{\mathbb{I}}^{\text{aux}}$ stands for the identity map on the auxiliary space, that is, for any operator $A \in \mathcal{L}(\text{aux})$ we have $\tilde{\mathbb{I}}^{\text{aux}}(A) = A$.

operator $D^{A_O \text{aux}/B_I}$, i.e., $D^{A_O \text{aux}/B_I} \succeq 0$, $\text{tr}_{B_I}(D^{A_O \text{aux}/B_I}) = \mathbb{1}^{A_O \text{aux}}$, such that

$$\begin{aligned} W &:= \rho^{A_I \text{aux}} * D^{A_O \text{aux}/B_I} \\ &= \text{tr}_{\text{aux}} \left(\left(\rho^{A_I \text{aux}} \otimes \mathbb{1}^{A_O B_I} \right)^{T_{\text{aux}}} \cdot \left(\mathbb{1}^{A_I} \otimes D^{A_O \text{aux}/B_I} \right) \right). \end{aligned} \quad (3.8)$$

We denote the set of bipartite ordered processes by $\mathcal{L}_{A \rightarrow B}$.

Bipartite ordered processes may be seen as quantum objects transforming a quantum operations into a quantum state³ and are also known in the literature as quantum non-Markovian processes [9], causal maps [16], quantum co-strategies [15], and ordered process matrices [39]. Also, ordered processes may also be seen as a particular instance of quantum channels with memory [14] and quantum combs [13].

In this general definition of bipartite ordered processes, Markovian processes form the particular case where there is no auxiliary space “aux”. Common-cause processes can also be represented as bipartite ordered processes. Indeed, if $W_{\text{CC}} = \rho^{A_I B_I} \otimes \mathbb{1}^{A_O}$, we can set the auxiliary space “aux” as isomorphic to B_I and the initial state $\rho^{A_I \text{aux}}$ as isomorphic to $\rho^{A_I B_I}$. Then, one can set the channel \tilde{D} as an operation that sends the auxiliary system to B_I and discard the system in A_O . This can be done formally via

$$\begin{aligned} W_{\text{CC}} &= \text{tr}_{\text{aux}'} \left[\rho^{A_I \text{aux}} * |\text{U}_{\text{SWAP}}\rangle\rangle\langle\langle \text{U}_{\text{SWAP}}|^{A_O \text{aux}/B_I \text{aux}'} \right] \\ &= \rho^{A_I B_I} \otimes \mathbb{1}^{A_O}, \end{aligned} \quad (3.9)$$

where $\text{U}_{\text{SWAP}} := \sum_{ij} |ij\rangle\langle ji|$ is the swap operator. A circuit illustration of common-cause process as bipartite ordered ones is presented in Fig. 5.

With all the important processes defined, Fig. 6 shows the relationship between all the sets defined in this section.

In addition to its definition, bipartite ordered processes admit a characterization in terms of linear and positive semidefinite constraints which will be useful later in this work. It follows from direct inspection that every bipartite ordered process $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ satisfies

$$W \succeq 0, \quad (3.10a)$$

$$\text{tr}_{B_I}(W) = \sigma^{A_I} \otimes \mathbb{1}^{A_O}, \quad (3.10b)$$

where σ^{A_I} is a quantum state. Interestingly, these conditions are also sufficient [13].

³ In Ref. [13, 26] the authors prove that under a set of assumptions, it can be proven that bipartite ordered processes are indeed the most general method to transform arbitrary quantum operations into quantum states.

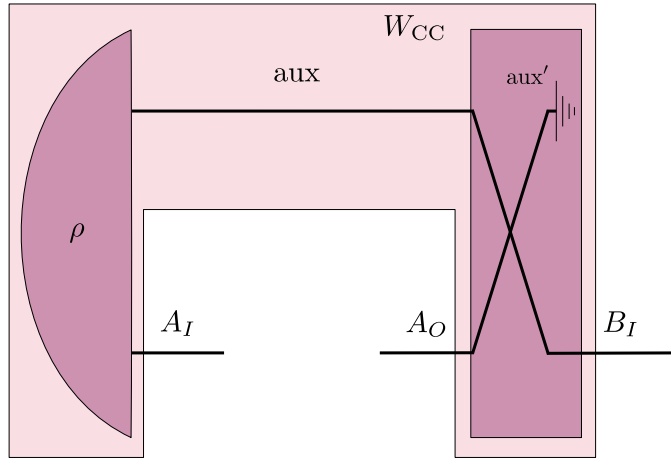


Figure 5 – A general bipartite common-cause process can be represented by a circuit of this form. The shared quantum state between Alice and Bob comes from an initial correlated state between Alice and an auxiliary system. The state of the auxiliary system is exchanged with the system coming from the output of Alice and sent to Bob, and the auxiliary system is then discarded, which makes the output state of Alice to have no impact on the input of Bob. This makes only the input states of Alice and Bob to be relevant for the process, which is why the correlations between them stem from the common-cause correlations on ρ .

For any linear operator W satisfying conditions (3.10), one can find a quantum state $\rho \in \mathcal{L}(A_I \otimes \text{aux})$ and a quantum channel $\tilde{D} : \mathcal{L}(A_O \otimes \text{aux}) \rightarrow \mathcal{L}(B_I)$ such that $\rho^{A_I \text{aux}} * D^{\text{aux} B_I / B_O} = W$. One possible construction is done by setting the auxiliary space “aux” to be isomorphic to A_I and $\rho \in \mathcal{L}(A_I \otimes \text{aux})$ to be isomorphic to a purification of σ^{A_I} , for instance,

$$\rho := \left(\mathbb{1}^{A_I} \otimes \sqrt{\sigma^{\text{aux}}^T} \right) |\mathbb{1}\rangle\langle\mathbb{1}|^{A_I \text{aux}} \left(\mathbb{1}^{A_I} \otimes \sqrt{\sigma^{\text{aux}}^T} \right). \quad (3.11)$$

Now, one can define a quantum channel via

$$D := \left(\sqrt{\sigma^{\text{aux}}^{-1}} \otimes \mathbb{1}^{A_O B_I} \right) W^{\text{aux} A_O B_I} \left(\sqrt{\sigma^{\text{aux}}^{-1}} \otimes \mathbb{1}^{A_O B_I} \right), \quad (3.12)$$

where $\sqrt{\sigma}$ is the unique positive semidefinite square root of σ , and σ^{-1} is the Moore–Penrose inverse of σ , that is, the inverse of σ on its range. In this way, direct inspection shows that D is the Choi operator of a quantum channel, and $W = \rho * D$.

Inspired by Ref. [40], we now present another characterization for bipartite ordered processes which will be useful for proving some results of this work. A linear operator $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is a bipartite ordered process if and only if it respects

$$W \succeq 0, \quad (3.13a)$$

$$W = L_{A \rightarrow B}(W), \quad (3.13b)$$

$$\text{tr}(W) = d_{A_O}, \quad (3.13c)$$

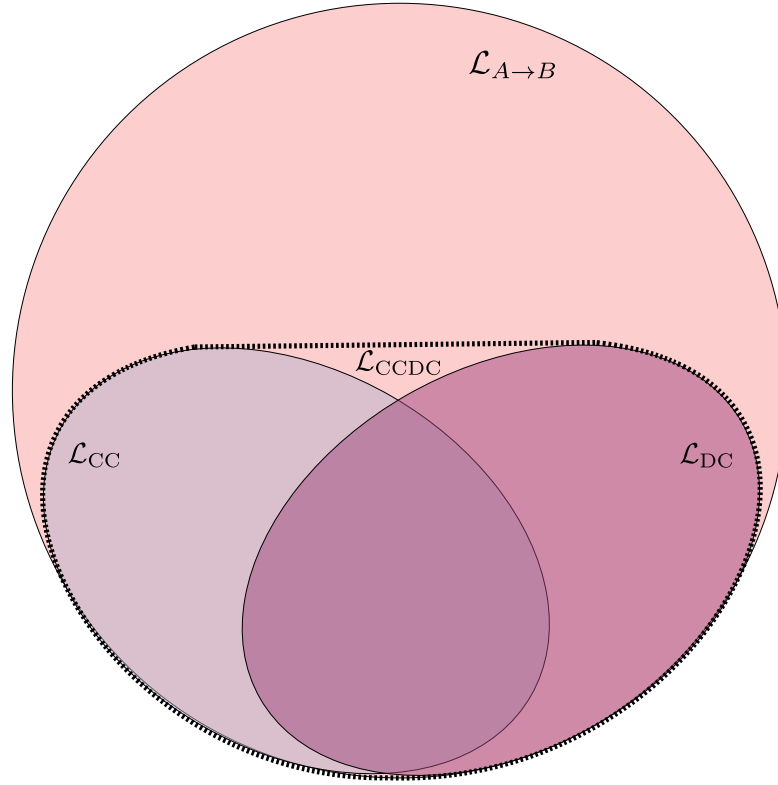


Figure 6 – Illustrative representation of the sets in the CCDC scenario. The sets of common-cause processes (\mathcal{L}_{CC}) and of direct-cause processes (\mathcal{L}_{DC}) are proper subsets of the set of bipartite ordered process $\mathcal{L}_{A \rightarrow B}$, with a non-empty intersection between \mathcal{L}_{CC} and \mathcal{L}_{DC} . Classical CCDC processes are all processes in the convex hull of $\mathcal{L}_{CC} \cup \mathcal{L}_{DC}$, represented by \mathcal{L}_{CCDC} .

where

$$L_{A \rightarrow B}(W) := W + {}_{A_O B_I} W - {}_{B_I} W, \quad (3.14)$$

with ${}_X(\cdot) = \text{tr}_X(\cdot) \otimes \frac{1_X}{d_X}$ and $L_{A \rightarrow B}(W)$ is the map projecting an operator W into the linear space spanned by bipartite ordered processes. Eqs. (3.10) and (3.13) are equivalent conditions for an operator $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ to be a valid bipartite ordered process. In particular, conditions (3.13) are rather useful for implementing numerical methods, due to the use of the projector $L_{A \rightarrow B}(W)$.

3.6 Detecting and quantifying non-classical CCDC

Following the discussion about entanglement witnessing from Section 2.2, one can quantify the violation of a classical CCDC decomposition in a process $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ in terms of its *generalized robustness of non-classical CCDC*⁴ [2] by

⁴ Note that the definition of generalized robustness R_G here has a one-to-one relation to the “non-classicality of causality” \mathcal{C} presented in Ref.[2] via $R_G(W) = \frac{\mathcal{C}(W)}{1+\mathcal{C}(W)}$.

$$R_G(W) := \min r$$

$$\text{subject to } (1-r)W + r\Omega \in \mathcal{L}_{\text{CCDC}}, \quad (3.15a)$$

$$0 \leq r \leq 1, \quad (3.15b)$$

$$\Omega \in \mathcal{L}_{A \rightarrow B}, \quad (3.15c)$$

where $\mathcal{L}_{A \rightarrow B}$ is the set of bipartite ordered processes defined in Eqs. (3.13). The generalized robustness $R_G(W)$ corresponds to how resistant is the non-classical CCDC property of the process W against its worst possible noise. Notice that this definition follows the same form of the generalized robustness of entanglement from Eqs. (2.6), which implies that $R_G(W) \in [0, 1]$ for every bipartite ordered process W .

In Appendix C, we show that the generalized robustness of all bipartite processes is upper-bounded by $R_G(W) \leq 1 - \frac{1}{d_{A_I}}$, which is attainable with the examples of non-classical CCDC processes we present in Section 5.

Once again, analogously to the robustnesses of entanglement discussed in Section 2.2, another way of quantifying non-classical CCDC is in terms of its *white-noise robustness*, given by

$$R_{\text{WN}}(W) := \min r$$

$$\text{subject to } (1-r)W + r\frac{\mathbb{1}}{d_{A_I}d_{B_I}} \in \mathcal{L}_{\text{CCDC}}, \quad (3.16a)$$

$$0 \leq r \leq 1, \quad (3.16b)$$

where $\mathbb{1}$ is the identity operator in $\mathcal{L}(A_I \otimes A_O \otimes B_I)$. The value $R_{\text{WN}}(W)$ corresponds to how resistant is the non-classical CCDC property of the process W against white noise. Figure 7 gives a representation of both types of robustness.

In Appendix D, we show that the white noise robustness of all bipartite processes is upper-bounded by $R_{\text{WN}}(W) \leq 1 - \frac{1}{d_{A_I}d_{A_O}d_{B_I}+1}$. Such bound is not tight, but is useful to demonstrate that the convex optimization problem of Eq.(3.16) respects strong duality (see Appendix E for more details).

3.6.1 Witnessing non-classical CCDC processes

A non-classical CCDC witness is a Hermitian operator $S \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ which respects $\text{tr}(SW_{\text{CCDC}}) \geq 0$ for every classical CCDC process W_{CCDC} [2]. Non-classical CCDC witnesses that are not trivial are the ones having $\text{tr}(SW) < 0$ for some non-classical CCDC process W , as the main purpose of a witness is to certify that a given process is non-classical CCDC. Since the set of classical CCDC processes is closed and convex, similarly to entanglement [41], every non-classical CCDC process may be

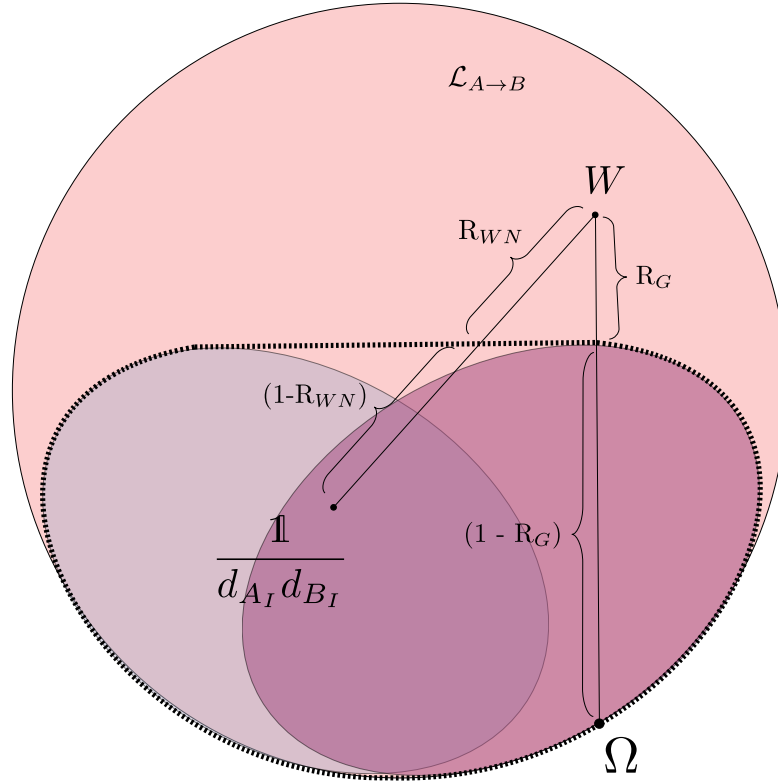


Figure 7 – Representation of the problem of determining if a given process W is non-classical CCDC. The above figure illustrates an ordered process operator $\Omega \in \mathcal{L}_{A \rightarrow B}$ such that the minimum convex combination of it with the process W results in a classical CCDC process.

certified by a non-classical CCDC witness. Also, as proven in Appendix E, the convex optimization problems used to define generalized and white noise robustness of non-classical CCDC respect a strong duality condition and the dual formulation of our outer approximation can be used to explicitly construct non-classical CCDC witness.

Similarly to entanglement witnesses, verifying if an arbitrary operator $S \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is a non-classical CCDC witness is likely to be computationally hard [42], as it would require, in principle, to verify that $\text{tr}(SW_{\text{CCDC}}) \geq 0$ for all classical CCDC processes W_{CCDC} . Despite the hardness of the general problem, we now provide simple sufficient (but not necessary) analytical conditions to ensure that S is a non-classical CCDC witness.

Any operator $S \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ respecting $\text{tr}_{A_O}(S) \succeq 0$ and $S^{T_{A_I}} \succeq 0$ is a non-classical CCDC witness. This claim follows from the fact that every CC process respects $W_{\text{CC}} = {}_{A_O}W_{\text{CC}} \succeq 0$ and every DC process respects $W_{\text{DC}}^{T_{A_I}} \succeq 0$. Hence,

$$\begin{aligned}
\text{tr}(SW_{\text{CCDC}}) &= p \text{tr}(SW_{\text{CC}}) + (1 - p)\text{tr}(SW_{\text{DC}}) \\
&= p \text{tr}(S_{A_O} W_{\text{CC}}) + (1 - p)\text{tr}(S^{T_{A_I}} W_{\text{DC}}^{T_{A_I}}) \\
&= p \text{tr}_{(A_O)}(S W_{\text{CC}}) + (1 - p)\text{tr}(S^{T_{A_I}} W_{\text{DC}}^{T_{A_I}}) \\
&\geq 0.
\end{aligned} \tag{3.17}$$

3.6.2 Entanglement approximation for non-classical CCDC certification

As mentioned in Section. 3.2.1, the separability in the bipartition $A_I|A_O B_I$ in a process $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is a necessary, but not sufficient, condition for determining whether a process is not direct-cause. This allows one to define lower bounds for the non-classical CCDC robustnesses of a process W in terms of entanglement certification of W in the bipartition $A_I|A_O B_I$. This technique was explored in other works, such as Refs. [1, 2, 20].

If we consider the hierarchy of sets $\mathcal{L}_{\text{DC}}^{\text{out,PPT}_k}$ from Def. 3.2.2 as outer approximation for the set of direct-cause processes \mathcal{L}_{DC} , we can define $\mathcal{L}_{\text{CCDC}}^{\text{out,PPT}_k} := \text{conv}(\mathcal{L}_{\text{CC}} \cup \mathcal{L}_{\text{DC}}^{\text{out,PPT}_k})$, which is an outer approximation for $\mathcal{L}_{\text{CCDC}}^{\text{out,PPT}_k}$, in order to obtain lower bounds for the generalized and white noise robustnesses of non-classical CCDC, $R_G(W)$ and $R_{\text{WN}}(W)$. Such lower bounds are called the PPT_k robustnesses and are represented by $R_G^{\text{low,PPT}_k}(W)$ and $R_{\text{WN}}^{\text{low,PPT}_k}(W)$.

Consider the problem of obtaining the PPT_k generalized robustness of a process W , *i.e.*, finding the robustness of a process against its worst noise Ω , so the resulting combination process lies in $\mathcal{L}_{\text{CCDC}}^{\text{out,PPT}_k}$. This is represented by the following optimization program:

$$R_G^{\text{low,PPT}_k}(W) := \min r \tag{3.18a}$$

$$\text{s.t. } (1 - r)W + r\Omega = qW_{\text{CC}} + (1 - q)W_{\text{DC}}^{\text{PPT}_k}, \tag{3.18b}$$

$$0 \leq r \leq 1, \tag{3.18c}$$

$$0 \leq q \leq 1, \tag{3.18d}$$

$$W_{\text{CC}} \in \mathcal{L}_{\text{CC}}, W_{\text{DC}}^{\text{PPT}_k} \in \mathcal{L}_{\text{DC}}^{\text{out,PPT}_k}, \Omega \in \mathcal{L}_{A \rightarrow B}, \tag{3.18e}$$

with the process $W_{\text{DC}}^{\text{PPT}_k}$ having a k -symmetric PPT extension $W_{\text{ext}}^{(A_I^{\otimes k})A_O B_I} \in \mathcal{L}(A_I^{\otimes k} \otimes A_O \otimes B_I)$, satisfying Eqs. (3.4). We point that, in practice, instead of Eq. (3.4a), it is more advantageous to impose the Bose k -symmetric extension condition, that is, to impose

$$W_{\text{ext}}^{(A_I^{\otimes k})A_O B_I} = (P_{\text{sym}}^{A_I^{\otimes k}} \otimes \mathbb{1}_{A_O B_I}) W_{\text{ext}}^{A_I^{\otimes k} A_O B_I}. \tag{3.19}$$

This follows from the fact that, computationally, imposing the Bose k -symmetric extension condition instead of the k -symmetric extension condition does not add any

complexity to the problem, but the Bose k -symmetric extension condition detects more entangled states than the standard k -symmetric one [43].

Following the steps from appendix A, since Eqs. (3.18) contains products of optimization variables, such as qW_{CC} , the optimization program above is not linear, hence not an SDP. This issue can be circumvented by re-writing the problem in the following equivalent form:

$$R_G^{\text{low, PPT}_k}(W) := \min \text{tr} \left(\frac{\bar{\Omega}}{d_{A_O}} \right) \quad (3.20a)$$

$$\text{s.t.} \quad \left(1 - \text{tr} \left(\frac{\bar{\Omega}}{d_{A_O}} \right) \right) W + \bar{\Omega} = \overline{W_{\text{CC}}} + \overline{W_{\text{DC}}}, \quad (3.20b)$$

$$\overline{W_{\text{CC}}} = \rho^{A_I B_I} \otimes \mathbb{1}^{A_O} \quad (3.20c)$$

$$\left(\overline{W_{\text{DC}}}^{A_I^{\otimes k} | A_O B_I} \right)^{T_{A_I}} \succeq 0, \quad (3.20d)$$

$$\overline{W_{\text{DC}}} = L_{A \rightarrow B}(\overline{W_{\text{DC}}}), \quad \bar{\Omega} = L_{A \rightarrow B}(\bar{\Omega}), \quad (3.20e)$$

$$\bar{\Omega}, \overline{W_{\text{DC}}}, \rho^{A_I B_I} \succeq 0. \quad (3.20f)$$

In the above SDP we have used the fact that valid processes should respect $\text{tr}(W) = d_{A_O}$ to embed the scalar variables into the operators. This means that the new variables relate to the variables from Eqs. (3.18) with

$$\bar{\Omega} = r\Omega, \quad \overline{W_{\text{CC}}} = qW_{\text{CC}}, \quad \overline{W_{\text{DC}}} = (1 - q)W_{\text{DC}}^{\text{PPT}_k}. \quad (3.21)$$

Analogously, the same steps are applied to the PPT white noise robustness, thus being

$$R_{\text{WN}}^{\text{low, PPT}_k}(W) := \min r \quad (3.22a)$$

$$\text{s.t.} \quad (1 - r)W + r \frac{\mathbb{1}^{A_I A_O B_I}}{d_{A_I} d_{B_I}} = \overline{W_{\text{CC}}} + \overline{W_{\text{DC}}}, \quad (3.22b)$$

$$\overline{W_{\text{CC}}} = \rho^{A_I B_I} \otimes \mathbb{1}^{A_O} \quad (3.22c)$$

$$\left(\overline{W_{\text{DC}}}^{A_I^{\otimes k} | A_O B_I} \right)^{T_{A_I}} \succeq 0, \quad (3.22d)$$

$$\overline{W_{\text{DC}}}, \rho^{A_I B_I} \succeq 0. \quad (3.22e)$$

From the Lagrangian of the SDPs, we can obtain their dual problems, which generate the optimal non-classical CCDC witnesses S for the given process W [28]. For instance, considering $k = 1$, the dual form⁵ of the PPT generalized robustness is given

⁵ Strictly speaking, the dual objective function is to “maximize $-\text{tr}(SW)$ ”, which results in $R(W) = \text{tr}(SW)$. Here, we adopted a convention of changing the sign of S and replacing this objective function by “minimize $\text{tr}(SW)$ ”. With this convention, the operator S is a non-classical CCDC witnesses as defined in Section 3.6.1, that is, it satisfies $\text{tr}(SW_{\text{CCDC}}) \geq 0$ for all classical CCDC processes W_{CCDC} .

by

$$\min \operatorname{tr}(SW) \quad (3.23a)$$

$$\text{s.t. } \mathbb{1}^{A_I A_O B_I} (1 + \operatorname{tr}(SW)) - d_{A_O} S \succeq 0 \quad (3.23b)$$

$$S - S_{\text{DC}} + S_{\text{DC}}^\perp \succeq 0 \quad (3.23c)$$

$$L_{A \rightarrow B}(S_{\text{DC}}^\perp) = 0, \quad (3.23d)$$

$$S_{\text{DC}}^{T_{A_I}} \succeq 0, \quad (3.23e)$$

$$\operatorname{tr}_{A_O}(S) \succeq 0, \quad (3.23f)$$

which is the one we use in our heuristic see-saw presented in Section F. The variable S_{DC}^\perp in Eq. (3.23c) is associated with an orthogonal projection onto $\mathcal{L}_{A \rightarrow B}$. Also, the same methods presented in Section E to prove that the robustness optimization problem satisfies strong duality can be used to show that this upper bound problem also respects strong duality. We, then, have $R_G^{\text{low, PPT}} = -\operatorname{tr}(SW)$, when S is the optimal witness for W .

We can see that Eq. (3.23c) together with Eqs. (3.23e) and (3.23f) ensure that S is a non-classical CCDC witness. Also, Eq. (3.23b) corresponds to the normalization condition, which determines that it is a witness for the generalized robustness measure.

Analogously, the dual form of the PPT white noise robustness is given by

$$\min \operatorname{tr}(SW) \quad (3.24a)$$

$$\text{s.t. } \frac{\operatorname{tr}(S)}{d_{A_I} d_{B_I}} - \operatorname{tr}(SW) \leq 1, \quad (3.24b)$$

$$S - S_{\text{DC}}^{T_{A_I}} + S_{\text{DC}}^\perp \succeq 0 \quad (3.24c)$$

$$L_{A \rightarrow B}(S_{\text{DC}}^\perp) = 0, \quad (3.24d)$$

$$S_{\text{DC}} \succeq 0, \quad (3.24e)$$

$$\operatorname{tr}_{A_O}(S) \succeq 0, \quad (3.24f)$$

where Eq. (3.24b) is the normalization condition for the witness S , which corresponds to the white noise robustness measure.

Part III

Our contribution

4 Beyond the entanglement approximation: SDP hierarchies for *tight* non-classical CCDC certification

As stated in Sec. 3.2.1, the set of separable processes in the bipartition $A_I|A_OB_I$ provides an outer approximation for the set of DC processes. As pointed in Ref. [2, 20], this provides a SDP approach that gives lower bounds for the non-DC and non-classical CCDC robustness of a bipartite ordered process. Indeed, since the relation $\mathcal{L}_{DC} \subseteq \mathcal{L}_{DC}^{\text{PPT}_k}$ holds for every natural number k , we can define the convex hull $\mathcal{L}_{\text{CCDC}}^{\text{out,PPT}_k} := \text{conv}(\mathcal{L}_{\text{CC}}, \mathcal{L}_{DC}^{\text{out,PPT}_k})$, and the quantity

$$R_G^{\text{low,PPT}_k}(W) := \min r$$

$$\text{subject to } (1-r)W + r\Omega \in \mathcal{L}_{\text{CCDC}}^{\text{out,PPT}_k} \quad (4.1a)$$

$$0 \leq r \leq 1, \quad (4.1b)$$

$$\Omega \in \mathcal{L}_{A \rightarrow B}, \quad (4.1c)$$

which gives a lower-bound for the *actual* generalized robustness (of non-classical CCDC), since the relation $R_G^{\text{low,PPT}_k}(W) \leq R_G(W)$ holds for every W . Analogously, we can also obtain a lower-bound for the white noise robustness by defining $R_{\text{WN}}^{\text{low,PPT}_k}(W)$.

As we show in Sec. 7.1, approximating the set of DC processes by the set of separable process is rather limited. In particular, there exist processes which are separable in the bipartition $A_I|A_OB_I$ but are non-classical CCDC. Hence, such processes would never be identified as non-classical CCDC by a method solely based in quantum entanglement techniques. In this section, we overcome this problem by presenting two novel SDP hierarchies of sets which converge to the set of DC processes, consequently providing a converging approximation to the set of classical CCDC processes. Also, one hierarchy is based in an inner approximation, whereas the other constitutes an outer approximation. Hence, we can obtain a sequence of converging upper and lower bounds on the non-classical CCDC robustnesses. All the sets discussed in this section are illustrated in Fig. 8.

4.1 Inner approximation for direct-cause processes

The following inner approximation leads to a hierarchy that converges to the set of direct-cause processes and is inspired by the SDP approach for entanglement detection introduced in [44].

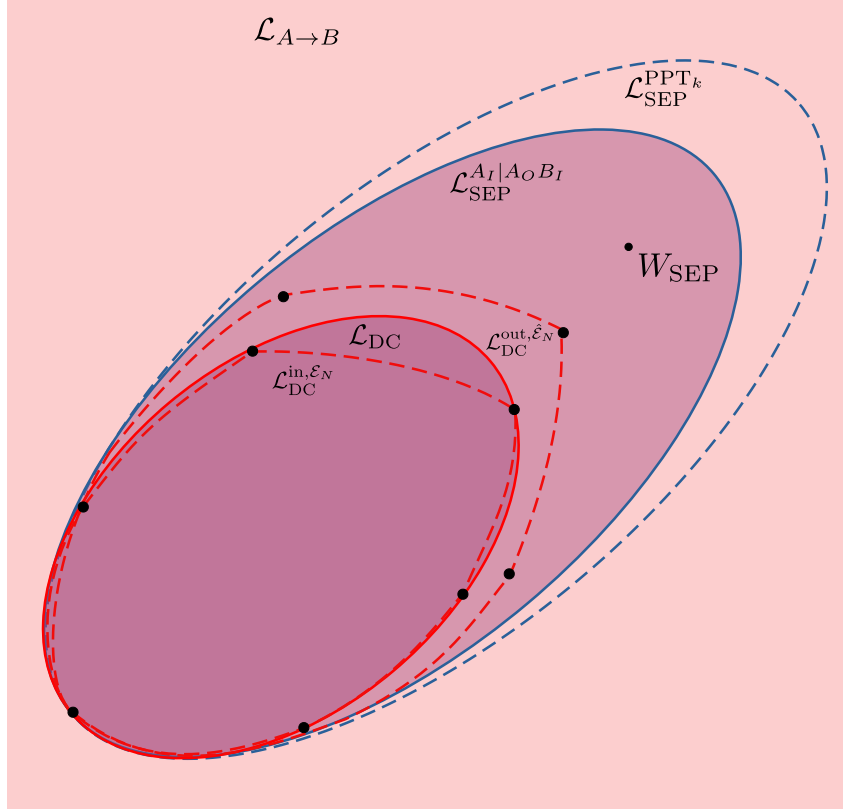


Figure 8 – Hierarchical relation between the set of processes separable in the bipartition $A_I|A_O B_I$ ($\mathcal{L}_{\text{SEP}}^{A_I|A_O B_I}$) and the set of direct-cause processes \mathcal{L}_{DC} . In the limit of large k , $\mathcal{L}_{\text{SEP}}^{\text{PPT}^k}$ converges to $\mathcal{L}_{\text{SEP}}^{A_I|A_O B_I}$. By increasing the number of points in the boundary (N), the inner approximation ($\mathcal{L}_{\text{DC}}^{\text{in}, \mathcal{E}_N}$) converges to \mathcal{L}_{DC} . Similarly, by increasing the number of points in the boundary, the outer approximation $\mathcal{L}_{\text{DC}}^{\text{out}, \mathcal{E}_N}$ also converges to \mathcal{L}_{DC} . The point W_{SEP} stands for the example of process which is separable in the bipartition $A_I|A_O B_I$, but is non-classical CCDC, defined and discussed in Sec. 7.1.

Definition 4.1.1 (Inner approximation for the direct-cause set). Let $\mathcal{E}_N := \{|\psi_i\rangle^{A_I}\}_{i=1}^N$ be a fixed set of pure quantum states $|\psi_i\rangle^{A_I}$. The set $\mathcal{L}_{\text{DC}}^{\text{in}, \mathcal{E}_N}$ corresponds to the inner approximation of the set \mathcal{L}_{DC} and is composed by every process W that can be written as

$$W = \sum_i p_i |\psi_i\rangle\langle\psi_i|^{A_I} \otimes D_i^{A_O/B_I}, \quad (4.2a)$$

where

$$\sum_i p_i = 1 \text{ and } p_i \geq 0, \forall i, \quad (4.2b)$$

and $D_i^{A_O/B_I}$ are the Choi operators of quantum channels, *i.e.*,

$$D_i^{A_O/B_I} \succeq 0 \text{ and } \text{tr}_{B_I}(D_i^{A_O/B_I}) = \mathbb{1}^{A_O}, \forall i. \quad (4.2c)$$

Note that in Def. 4.1.1, we imposed the partial trace quantum channel normalization condition $\text{tr}_{B_I}(D_i^{A_O/B_I}) = \mathbb{1}^{A_O}$, which does not appear in an entanglement-based

approach where we only impose positivity and the full trace constraint. As we will see in further sections, this corresponds to a fundamental difference, since the set of separable quantum states is not equivalent to the set of direct-cause processes.

By construction, for any set of quantum states \mathcal{E}_N , we have the relation $\mathcal{L}_{\text{DC}}^{\text{in},\mathcal{E}_N} \subseteq \mathcal{L}_{\text{DC}}$. Also, if \mathcal{E}_∞ is the set of all pure quantum states in $\mathcal{L}(A_I)$, we have $\mathcal{L}_{\text{DC}}^{\text{in},\mathcal{E}_\infty} = \mathcal{L}_{\text{DC}}$. We can now define the convex hull $\mathcal{L}_{\text{CCDC}}^{\text{in},\mathcal{E}_N} := \text{conv}(\mathcal{L}_{\text{CC}}, \mathcal{L}_{\text{DC}}^{\text{in},\mathcal{E}_N})$, and the quantity

$$\begin{aligned} R_G^{\text{up},\mathcal{E}_N}(W) &:= \min r \\ &\text{subject to } (1-r)W + r\Omega \in \mathcal{L}_{\text{CCDC}}^{\text{in},\mathcal{E}_N}, \\ &0 \leq r \leq 1, \\ &\Omega \in \mathcal{L}_{A \rightarrow B}, \end{aligned} \tag{4.3}$$

which provides an upper-bound for the generalized robustness $R_G(W)$, since the relation $R_G(W) \leq R_G^{\text{up},\mathcal{E}_N}(W)$ holds for every bipartite ordered process W and \mathcal{E}_N .

As will be discussed in details in Section 4.3, the quantity $R_G^{\text{up},\mathcal{E}_N}(W)$ can be evaluated by an SDP. Moreover, this SDP converges to the exact value of $R_G(W)$ when the set of states \mathcal{E}_N contains all pure states in $\mathcal{L}(A_I)$. Analogously, we can also obtain an upper-bound for the white noise robustness by defining $R_{\text{WN}}^{\text{up},\mathcal{E}_N}(W)$.

4.2 Outer approximation for direct-cause processes

For the outer approximation of the set of direct-cause processes, we will make use of an outer approximation for the set of quantum states in $\mathcal{L}(A_I)$. Since the set of quantum states is convex, we can always construct an outer approximation given by a polytope, similarly to the strategy used to construct polytopes providing outer approximations for the set of quantum measurements [45–49]. Let \mathcal{S}_d be the set of quantum states with dimension d , there exists a set of linear operators $\hat{\rho}_i \in \mathcal{L}(\mathbb{C}_d)$ that satisfy $\text{tr}(\hat{\rho}_i) = 1$ and such that the convex hull of $\{\hat{\rho}_i\}$ contains \mathcal{S}_d . Note that the operators $\hat{\rho}_i$ are *not* required to be positive semidefinite, hence they do not correspond to quantum states.

For the moment, let us assume that we know a set of trace-one operators $\{\hat{\rho}_i\}_{i=1}^N$ such that $\text{conv}(\{\hat{\rho}_i\}_{i=1}^N) \supseteq \mathcal{S}_d$. We can then use this finite set to construct an outer approximation for the set of DC processes, similarly to the inner approximation in Def. 4.1.1.

Definition 4.2.1 (Outer approximation for the direct-cause set). Let $\hat{\mathcal{E}}_N := \{\hat{\rho}_i^{A_I}\}_{i=1}^N$ be a fixed set of linear operators with $\text{tr}(\hat{\rho}_i) = 1 \forall i$, such that the convex hull of $\hat{\mathcal{E}}_N$ contains the set of all quantum states acting on A_I . A bipartite ordered process W is in the set

$\mathcal{L}_{\text{DC}}^{\text{out}, \hat{\mathcal{E}}_N}$ if it can be written as

$$W = \sum_i p_i \hat{\rho}_i^{A_I} \otimes D_i^{A_O/B_I}, \quad (4.4a)$$

where

$$\sum_i p_i = 1 \quad \text{and} \quad p_i \geq 0 \quad \forall i, \quad (4.4b)$$

and $D_i^{A_O/B_I}$ are the Choi operators of quantum channels *i.e.*

$$D_i^{A_O/B_I} \succeq 0 \quad \text{and} \quad \text{tr}_{B_I}(D_i^{A_O/B_I}) = \mathbb{1}^{A_O} \quad \forall i. \quad (4.4c)$$

By construction, we have that $\mathcal{L}_{\text{DC}} \subseteq \mathcal{L}_{\text{DC}}^{\text{out}, \hat{\mathcal{E}}_N}$ and the set $\mathcal{L}_{\text{CCDC}}^{\text{out}, \hat{\mathcal{E}}_N} := \text{conv}(\mathcal{L}_{\text{CC}} \cup \mathcal{L}_{\text{DC}}^{\text{out}, \hat{\mathcal{E}}_N})$ is an outer approximation for $\mathcal{L}_{\text{CCDC}}$. The generalized and white noise robustnesses are defined in a way analogous to what is done in Sec. 4.1, thus being denoted by $R_G^{\text{low}, \hat{\mathcal{E}}_N}(W)$ and $R_{\text{WN}}^{\text{low}, \hat{\mathcal{E}}_N}(W)$, and these are lower bounds for the robustness $R_G(W)$ and $R_{\text{WN}}(W)$ respectively.

We now address the problem of finding sets of trace-one operators which contain the set of quantum states. Let us first start with the simple qubit case ($d = 2$) where the set of quantum states can be faithfully represented in terms of the Bloch sphere [50]. For this case, we can construct a set of trace-one operators $\{\hat{\rho}_i\}$ such that $\text{conv}(\{\hat{\rho}_i\}) \supseteq \mathcal{S}_2$ simply by finding a polyhedron that includes a sphere of unit radius. One method to find such polyhedron goes as following:

1. Sort N normalized vectors $\{\vec{v}_i\}_{i=1}^N$ in \mathbb{R}^3 . These will be the vertices of a polyhedron inscribed by the Bloch sphere;
2. Find the radius $r_{\text{in}} < 1$ of the largest sphere inscribed by the polyhedron;
3. Construct a polyhedron with vertices given by $\frac{\vec{v}_i}{r_{\text{in}}}$. This polyhedron includes the Bloch sphere.

The last step is ensured by the symmetry of the problem. A sphere of radius r_{in} is contained in the polytope with norm-one vertices $\{\vec{v}_i\}_{i=1}^N$ if and only if a sphere of radius one contains the polytope with vertices $\frac{\vec{v}_i}{r_{\text{in}}}$. Also, the radius of largest sphere inscribed by a polytope (required in step 2) can be made by first finding the facet representation of this polytope with vertices $\{\vec{v}_i\}_{i=1}^N$, step which can be done via Fourier-Motzkin elimination and can be tackled with the aid of numerical packages such as `lrs` [51] or the Matlab code `vert2lcon` [52]. With the facet representation, the radius of the largest inscribed sphere can be found by evaluating the distance of the origin to the hyperplane represented by each facet. A concrete example for the qubit case can be found in the Appendix B: “A family of polyhedra”, of Ref. [53]. In such work, the authors provide a

family of polyhedra parametrized by $n \in \mathbb{N}$ where the radius of the larger inscribed sphere is greater than or equals to $r_n := \cos^2\left(\frac{\pi}{2n}\right)$ and if n is an odd number, the number of vertices is given by $N = 2n^2$. This family of polyhedra can be used for a systematic approach to the problem.

For the general $d > 3$, we start by pointing out that every convex set can be approximated by a polytope. Now, in order to obtain a concrete set of operators $\{\hat{\rho}_i\}$, we refer to the methods described in Appendix A: “Calculating shrinking factors”, of Ref. [46] (see also Red. [54]). In a nutshell, the method starts by sampling random pure (which are extremal) states $|\psi_i\rangle\langle\psi_i|$ in the set \mathcal{S}_d and by obtaining the facet representation for the polytope associated to this set. Then, for any fixed “shrinking factor” η , we can verify whether a noisy quantum state $\eta\rho + (1 - \eta)\mathbb{1}/d$ can be written as a convex combination of $\{|\psi_i\rangle\langle\psi_i|\}$ by means of an SDP. If all η -noisy states can be written as convex combinations of $\{|\psi_i\rangle\langle\psi_i|\}$, the convex hull of operators is given by $\hat{\rho}_i := \frac{1}{\eta}|\psi_i\rangle\langle\psi_i| + 1 - \frac{1}{\eta}\mathbb{1}/d$.

We observe that we can tighten the outer approximation presented in Def. 4.2.1 by imposing that the process W should be separable in the bipartition $A_I|A_O B_I$. Also, if we do this by means of the PPT_k -symmetric extension (as in Def. 3.2.2), the problem can still be tackled by means of an SDP.

4.3 SDPs for non-classical CCDC inner and outer approximations

The same method from Sec. 3.6.2 can be used to write our inner and outer approximations $\mathcal{L}_{\text{CCDC}}^{\text{in}, \mathcal{E}_N}$ and $\mathcal{L}_{\text{CCDC}}^{\text{out}, \hat{\mathcal{E}}_N}$ in terms of SDPs. As the only difference between them is the sets \mathcal{E}_N and $\hat{\mathcal{E}}_N$, we consider only the inner approximation for this discussion. The case for the outer approximation follows directly from the replacement of \mathcal{E}_N by $\hat{\mathcal{E}}_N$.

Consider the problem of finding the PPT_k robustness of non-classical CCDC of a process W , described by the SDP from Eqs. (3.20). It is necessary to replace the constraints over the direct-cause variable W_{DC} by condition from Eq. (4.2a). In this case, by absorbing the probabilities p_i of Eq. (3.20d) into the quantum channel D_i to obtain $\overline{D}_i := p_i D_i$, we can replace the constraint of Eq. (3.20d) by

$$W_{\text{DC}} = \sum_{i=1}^N |\psi_i\rangle\langle\psi_i|^{A_I} \otimes \overline{D}_i^{A_O/B_I}, \quad (4.5a)$$

$$\overline{D}_i^{A_O/B_I} \succeq 0, \quad (4.5b)$$

$${}_{B_I}\overline{D}_i^{A_O/B_I} = {}_{A_O B_I}\overline{D}_i^{A_O/B_I}, \quad (4.5c)$$

where $\{|\psi_i\rangle^{A_I}\}$ is a set of random states in $\mathcal{L}(A_I)$ and $\{D_i^{A_O/B_I}\}$ is a set of optimization variables in $\mathcal{L}(A_O \otimes B_I)$.

We can also obtain the dual form of the inner approximation problems of robustnesses from the Lagrangian, which results in a similar SDP to Eqs. (3.23) and Eqs. (3.24), but replacing Eqs. (3.23c) and (3.23e) by

$$\mathrm{tr}_{A_I} \left[S(|\psi_i\rangle\langle\psi_i|^{A_I} \otimes \mathbb{1}^{A_O B_I}) + {}_{B_I} S_i - {}_{A_O B_I} S_i \right] \succeq 0 \quad \forall i \in \mathbb{N}, \quad (4.6)$$

where $\{S_i\}$ is a set of variables in $\mathcal{L}(A_I \otimes A_O \otimes B_I)$.

We present a summary of all the sets defined as approximations to the set $\mathcal{L}_{\mathrm{DC}}$ in Table 1. For obtaining the corresponding inner or outer approximations for the set of classical CCDC processes $\mathcal{L}_{\mathrm{CCDC}}$, it is necessarily to take the convex hull of $\mathcal{L}_{\mathrm{CC}}$ with the chosen set for approximating $\mathcal{L}_{\mathrm{DC}}$. Before finishing this section, we state that, throughout this work, whenever we calculate the non-classical CCDC robustnesses of the inner and outer approximations for $\mathcal{L}_{\mathrm{CCDC}}$, the first lower bounds to be calculated are always provided by the outer approximation $\mathcal{L}_{\mathrm{CCDC}}^{\mathrm{out}, \mathrm{PPT}_k}$. If the lower bounds obtained with such approximation does not match the upper bounds, then we take the outer approximation provided by $\mathcal{L}_{\mathrm{CCDC}}^{\mathrm{out}, \hat{\mathcal{E}}_N}$.

Set	Robustness	Requirements for set membership
$\mathcal{L}_{\mathrm{DC}}^{\mathrm{out}, \mathrm{PPT}_k}$	$R^{\mathrm{low}, \mathrm{PPT}_k}$	W is a valid bipartite ordered process and has a PPT k -symmetric extension. (This hierarchy converges to the set of separable processes, which is strictly larger than $\mathcal{L}_{\mathrm{DC}}$)
$\mathcal{L}_{\mathrm{DC}}^{\mathrm{out}, \hat{\mathcal{E}}_N}$	$R^{\mathrm{low}, \hat{\mathcal{E}}_N}$	Given a set of operators $\hat{\mathcal{E}}_N = \{\hat{\rho}_i^{A_I}\}_{i=1}^k$ forming an outer approximation for the set of quantum states, there exist quantum channels $D_i^{A_O/B_I}$ s.t. $W = \sum_i p_i \hat{\rho}_i^{A_I} \otimes D_i^{A_O/B_I}$.
$\mathcal{L}_{\mathrm{DC}}^{\mathrm{in}, \mathcal{E}_N}$	$R^{\mathrm{up}, \mathcal{E}_N}$	Given a set of states $\mathcal{E}_N = \{\rho_i^{A_I}\}_{i=1}^k$, there exist quantum channels $D_i^{A_O/B_I}$ s.t. $W = \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}$.

Table 1 – Summary of the sets used as approximation to the set of direct-cause processes, $\mathcal{L}_{\mathrm{DC}}$, ordered from the outside to the inside of the set. The operators $D_i^{A_O/B_I}$ stands for Choi operators of quantum channels, $D_i^{A_O/B_I} \succeq 0$, $\mathrm{tr}_{B_I}(D_i^{A_O/B_I}) = \mathbb{1}$ and $\{p_i\}$ is a probability distribution. The operators $\hat{\rho}_i^{A_I}$ are not positive semidefinite, hence they are not quantum states, but the convex hull of the set $\{\hat{\rho}_i^{A_I}\}_{i=1}^k$ is a valid outer approximation for the set of quantum states. The precise definitions of these sets are given in Def. 3.2.2, Def. 4.2.1, and Def. 4.1.1. These approximations can be used for generating approximations for the set of classical CCDC processes, as $\mathcal{L}_{\mathrm{CCDC}} = \mathrm{conv}(\mathcal{L}_{\mathrm{CC}} \cup \mathcal{L}_{\mathrm{DC}})$.

5 Simplest non-classical CCDC processes

5.1 The scenario where $d_{A_I} = d_{A_O} = d$ and $d_{B_I} = d^2$

We now present a “simple” bipartite ordered process which has the highest generalized robustness in the scenario where $d_{A_I} = d_{A_O} = d$ and $d_{B_I} = d^2$. In this process, Alice shares a d -dimensional maximally entangled state $|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ with the auxiliary system, and the channel of communication to Bob (decoder) is composed of an identity channel, $D = |\mathbb{1}\rangle\rangle\langle\langle \mathbb{1}|^{\text{aux}A_O/B_I^1 B_I^2}$ (see Fig. 9). This process is mathematically represented by

$$\begin{aligned} W_{ddd^2} &:= |\phi^+\rangle\langle\phi^+|^{A_I \text{aux}} * |\mathbb{1}\rangle\rangle\langle\langle \mathbb{1}|^{\text{aux}A_O/B_I^1 B_I^2} \\ &= |\phi^+\rangle\langle\phi^+|^{A_I B_I^1} \otimes |\mathbb{1}\rangle\rangle\langle\langle \mathbb{1}|^{A_O/B_I^2}. \end{aligned} \quad (5.1)$$

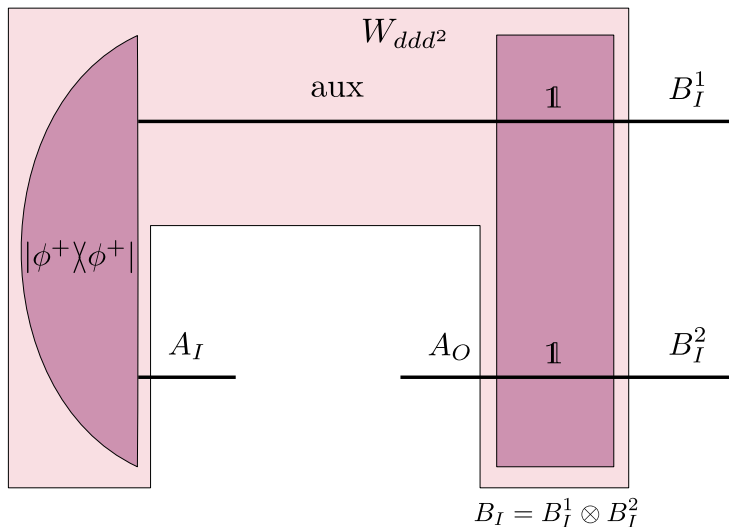


Figure 9 – Circuit representation of the process W_{ddd^2} . It consists in the preparation of a maximally entangled state between Alice and an auxiliary system and an identity channel of communication to Bob.

We argue that this process has a conceptually simple realization, since it only requires the preparation of a maximally entangled state and the identity channel. The process W_{ddd^2} has the interesting property of transforming a quantum channel $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_O)$ into a state which is proportional to its Choi operator $\Lambda = \sum_{ij} |i\rangle\langle j| \otimes$

$\tilde{\Lambda}(|i\rangle\langle j|)$, that is

$$\begin{aligned} W_{ddd^2} * \Lambda &= \left(\tilde{\mathbb{1}} \otimes \tilde{\Lambda} \right) |\phi^+\rangle\langle\phi^+|, \\ &= \frac{\Lambda}{d}. \end{aligned} \quad (5.2)$$

Despite the simplicity, we now show that W_{ddd^2} is non-classical CCDC and, moreover, it is the one with highest generalized robustness in the scenario where $d_{A_I} = d_{A_O} = d$ and $d_{B_I} = d^2$.

Theorem 5.1.1. *The bipartite ordered process $W_{ddd^2} = |\phi^+\rangle\langle\phi^+|^{A_I B_I^1} \otimes |\mathbb{1}\rangle\langle\mathbb{1}|^{A_O/B_I^2}$ attains the maximum generalized robustness of all processes with the same dimensions. That is,*

$$\begin{aligned} R_G(W_{ddd^2}) &= \max_{W \in \mathcal{L}_{A \rightarrow B}} [R_G(W)] \\ &= 1 - \frac{1}{d_{A_I}} \end{aligned} \quad (5.3)$$

Proof. We start the proof by defining the operator

$$\begin{aligned} S &:= \mathbb{1}^{A_I A_O B_I} - W_{ddd^2} \\ &= \mathbb{1}^{A_I A_O B_I} - |\phi^+\rangle\langle\phi^+|^{A_I B_I^1} \otimes |\mathbb{1}\rangle\langle\mathbb{1}|^{A_O/B_I^2} \end{aligned} \quad (5.4)$$

and showing that S is a valid non-classical CCDC witness, *i.e.*, every CCDC process W_{CCDC} satisfies $\text{tr}(S W_{\text{CCDC}}) \geq 0$. To ensure that S is a witness, we start by pointing that

$$\begin{aligned} \text{tr}_{A_O} S &= d \mathbb{1}^{A_I B_I} - |\phi^+\rangle\langle\phi^+|^{A_I B_I^1} \otimes \mathbb{1}^{B_I^2} \\ &\succeq 0, \end{aligned} \quad (5.5)$$

where the last inequality holds because the smallest eigenvalue of $|\phi^+\rangle\langle\phi^+|^{A_I B_I^1} \otimes \mathbb{1}^{B_I^2}$ is 1. Now, note that

$$\begin{aligned} S^{T_{A_I}} &= \mathbb{1} - \sum_{ij} \frac{|ij\rangle\langle ji|}{d}^{A_I B_I^1} \otimes |\mathbb{1}\rangle\langle\mathbb{1}|^{A_O/B_I^2} \\ &= \mathbb{1} - \frac{U_{\text{SWAP}}^{A_I B_I^1}}{d} \otimes |\mathbb{1}\rangle\langle\mathbb{1}|^{A_O/B_I^2} \\ &\succeq 0, \end{aligned} \quad (5.6)$$

where the last inequality holds because the eigenvalues of U_{SWAP} are $+1$ or -1 . Conditions (5.5) and (5.6) together ensure that S is a witness, as shown in Eqs. (3.17).

Direct calculation shows that $\text{tr}(S W_{ddd^2}) = d - d^2 = d(1 - d)$ and that, for any bipartite ordered process Ω , we have

$$\text{tr}(S\Omega) = \text{tr}(\Omega) - \text{tr}(\Omega W_{ddd^2}) \quad (5.7a)$$

$$\leq \text{tr}(\Omega) = d. \quad (5.7b)$$

Since S is a non-classical CCDC witness, if $(1 - r)W_{dd\bar{d}^2} + r\Omega$ is a CCDC process, it holds that $\text{tr}(S[(1 - r)W_{dd\bar{d}^2} + r\Omega]) \geq 0$. By combining $\text{tr}(SW_{dd\bar{d}^2}) = d(1 - d)$ with the inequality (5.7b), we have

$$(1 - r)d(1 - d) + rd \geq 0, \quad (5.7c)$$

thus $r \geq 1 - \frac{1}{d}$.

In appendix C, it is shown that the non-classical CCDC generalized robustness has the upper bound $R_G(W) \leq 1 - \frac{1}{d_{A_I}}$, which concludes the proof that $W_{dd\bar{d}^2}$ attains the maximum generalized robust of its scenario. \square

We now analyse the robustness of $W_{dd\bar{d}^2}$ against the white noise process. For that, we use the SDP formulation to numerically tackle the case where $d = 2$ and $d = 3$. For the inner approximation, we have used the set $\mathcal{L}_{\text{CCDC}}^{\text{in}, \mathcal{E}_N}$ with \mathcal{E}_N being a set with $N = 10^4$ uniformly random pure states for $d = 2$ and $N = 200$ uniformly random pure states for $d = 3$. For the outer approximation, it was enough to use the loose approximation $\mathcal{L}_{\text{CCDC}}^{\text{out}, \text{PPT}}$. The results obtained were

$$R_{\text{WN}}^{\text{up}, \mathcal{E}_{10^4}}(W_{222^2}) = R_{\text{WN}}^{\text{low}, \text{PPT}}(W_{222^2}) = 0.8421 \quad (5.8a)$$

$$R_{\text{WN}}^{\text{up}, \mathcal{E}_{200}}(W_{333^2}) = R_{\text{WN}}^{\text{low}, \text{PPT}}(W_{333^2}) = 0.9529, \quad (5.8b)$$

where equations hold up to numerical precision. Since the upper and lower-bound coincides, these values are the actual values of robustnesses.

For $d = 2$, we believe that W_{222^2} is the process with maximum white noise robustness on its scenario. Our conjecture is based on a heuristic see-saw technique inspired by [27, 55] and presented in details in Appendix F, which suggests that the highest value of white noise robustness in the scenario where $d = 2$ is 0.8421.

In a nutshell, our see-saw technique goes as follows: for a fixed scenario $d_{A_I} = d_{A_O} = d$ and $d_{B_I} = d^2$, we choose one of the methods described in Appendix G for sampling a random bipartite ordered process W . Then, we obtain its optimal non-classical CCDC witness S from the dual formulation of the PPT white noise robustness (see Sec. 3.6.2). We then find the bipartite ordered process which maximally violates the witness S , then finding its optimal witness after that. This process is re-iterated until it converges to a robustness value, which we expect to be considerably greater than the robustness of the initially sampled process W .

Moreover, in the scenario where $d = 3$, our see-saw method could find a bipartite ordered process W_{max} which has $R_{\text{WN}}^{\text{up}, \mathcal{E}_{200}}(W_{\text{max}}) = R_{\text{WN}}^{\text{low}, \text{PPT}}(W_{\text{max}}) = 0.9643 > R_{\text{WN}}^{\text{up}, \mathcal{E}_{200}}(W_{333^2})$. This result shows the power of the see-saw method in finding processes which are robust against white noise. It also shows that, even though $W_{dd\bar{d}^2}$ has maximum generalized robustness for every d , which we proved analytically, in the case

where $d = 3$, W_{333^2} is not the most robust process against white noise. This leads us to conjecture that W_{ddd^2} may be not the most robust process against white noise for $d > 3$.

Before finishing this section we mention that, following the same steps from the demonstration of Theorem 5.1.1, we can also obtain an analytical lower-bound for white noise robustness of W_{ddd^2} , which is

$$R_{WN}(W_{ddd^2}) \geq \frac{d^3}{(d^2 + 1)(d + 1)}. \quad (5.9)$$

Differently from the lower-bound presented for generalized robustness, the above inequality is not tight. For instance, when $d = 2$, this lower-bound provides $R_{WN}(W_{ddd^2}) \geq \frac{8}{15} \approx 0.5333$, which is considerably lower than $R_{WN}^{\text{low,PPT}}(W_{ddd^2}) = 0.8421$. However, it is interesting to point out that the above expression shows that $R_{WN}(W_{ddd^2}) \rightarrow 1$ when $d \rightarrow \infty$.

5.2 The scenario with minimum dimensions: $d_{A_I} = d_{A_O} = d_{B_I} = 2$

We now consider the scenario where $d_{A_I} = d_{A_O} = d_{B_I} = 2$, that is, the minimum non-trivial dimensions. For this scenario, we propose the process composed by a two-qubit maximally entangled state shared between Alice and the auxiliary system, and the decoder channel from Alice output to Bob's input being a control-NOT channel, where the auxiliary system is later discarded (see Fig. 10). Mathematically, such process is

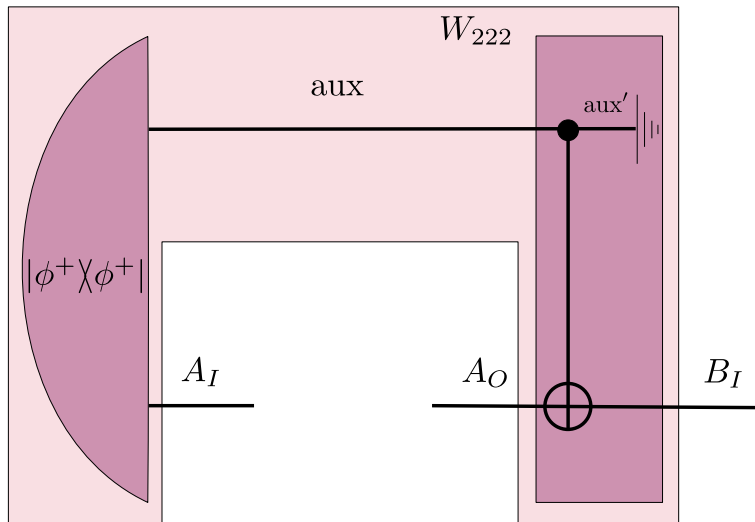


Figure 10 – Circuit representation of the process W_{222} , constructed with the smallest possible space dimensions $d_{A_I} = d_{A_O} = d_{B_I} = 2$. A maximally entangled state is initially shared between Alice and the auxiliary system. After Alice's operation, a control-NOT is applied, then aux' is discarded.

described by

$$W_{222} := \text{tr}_{\text{aux}'} \left(|\phi^+\rangle\langle\phi^+|^{A_I \text{aux}} * |U_{\text{CNOT}}\rangle\rangle\langle\langle U_{\text{CNOT}}|^{\text{aux}A_O/\text{aux}'B_I} \right) \quad (5.10)$$

where $|U_{\text{CNOT}}\rangle\rangle$ is the Choi vector of the control-NOT unitary gate U_{CNOT} , given by

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (5.11)$$

Direct calculation shows that the process W_{222} can also be written in terms of an un-normalized GHZ state $|GHZ\rangle\rangle := |000\rangle + |111\rangle \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$:

$$W_{222} = \frac{1}{2} \left(|GHZ\rangle\rangle\langle\langle GHZ| + \sigma_X^{A_O} |GHZ\rangle\rangle\langle\langle GHZ| \sigma_X^{A_O} \right), \quad (5.12)$$

where $\sigma_X^{A_O}$ is the Pauli matrix σ_X applied on subsystem A_O . The decomposition of Eq. (5.12) expresses W_{222} as a probabilistic mixture of pure ‘‘GHZ-like’’ non-normalized states¹ and it will be useful to prove that W_{222} attains the maximum generalized robustness value for its scenario.

Another decomposition of the process W_{222} is in terms of Pauli matrices:

$$W_{222} = \frac{1}{4} \left(\mathbb{1}^{A_I A_O B_I} + \sigma_Z^{A_I} \mathbb{1}^{A_O} \sigma_Z^{B_I} + \sigma_X^{A_I} \sigma_X^{A_O} \sigma_X^{B_I} - \sigma_Y^{A_I} \sigma_X^{A_O} \sigma_Y^{B_I} \right), \quad (5.13)$$

with implicit tensor product between the operators.

Theorem 5.2.1. *The bipartite ordered process W_{222} attains the maximum generalized robustness of all processes with the same dimensions. That is,*

$$R_G(W_{222}) = \max_{W \in \mathcal{L}_{A \rightarrow B}} [R_G(W)] \quad (5.14a)$$

$$= \frac{1}{2}. \quad (5.14b)$$

Proof. The proof of this theorem follows similar steps to the proof of theorem 5.1.1. We start by defining the operator

$$S := \mathbb{1} - 2W_{222} \quad (5.15a)$$

$$= \mathbb{1} - \left(|GHZ\rangle\rangle\langle\langle GHZ| + \sigma_X^{A_O} |GHZ\rangle\rangle\langle\langle GHZ| \sigma_X^{A_O} \right), \quad (5.15b)$$

¹ Interestingly, one can verify that for every $\epsilon \in (0, \frac{1}{2}]$, the operator $W = (\frac{1}{2} + \epsilon) |GHZ\rangle\rangle\langle\langle GHZ| + (\frac{1}{2} - \epsilon) \sigma_X^{A_O} |GHZ\rangle\rangle\langle\langle GHZ| \sigma_X^{A_O}$ is outside $\mathcal{L}_{A \rightarrow B}$. This ensures that, as illustrated in Fig. 11, W_{222} is on the boundary of $\mathcal{L}_{A \rightarrow B}$.

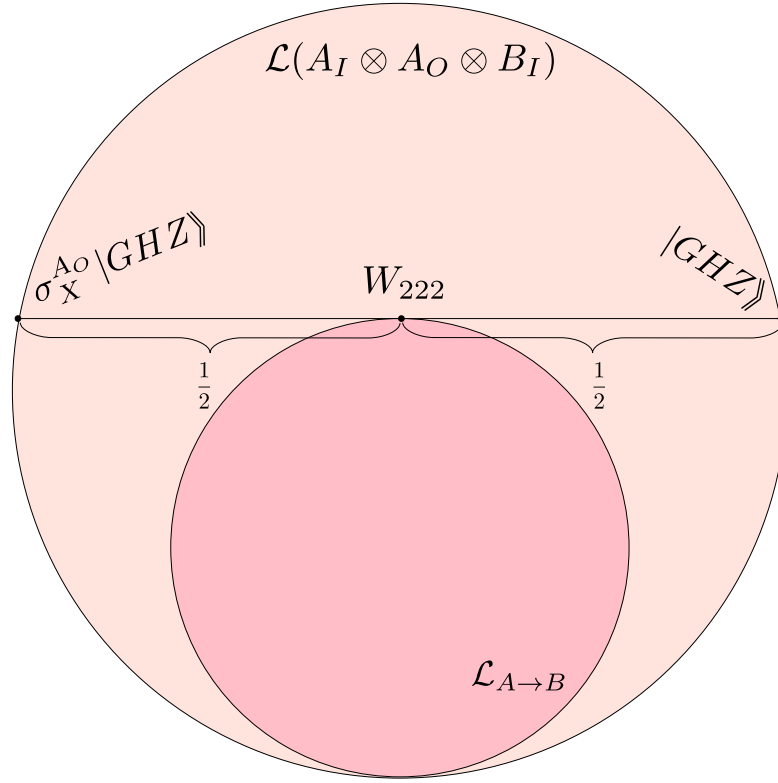


Figure 11 – The process W_{222} seen as a convex combination of two GHZ-type unnormalized states acting on A_I , A_O and B_I . Due to the ordered process causal constraints, GHZ states do not lead to valid bipartite ordered processes, but such convex combination of these two GHZ-type results in the valid bipartite ordered process W_{222} .

and showing that S is a non-classical CCDC witness. For this, we need to show that $\text{tr}_{A_O}(S) \succeq 0$ and $S^{T_{A_I}} \succeq 0$, as shown in Eqs. (3.17).

$$\begin{aligned}
 \frac{1}{2} \text{tr}_{A_O}(S) &= \mathbb{1}^{A_I B_I} - (|00\rangle\langle 00| + |11\rangle\langle 11|) \\
 &= |01\rangle\langle 01| + |10\rangle\langle 10| \\
 &\succeq 0.
 \end{aligned} \tag{5.16}$$

Now, for the second condition, we have

$$\begin{aligned}
 S^{T_{A_I}} &= \mathbb{1} - \left(|GHZ\rangle\rangle\langle\langle GHZ| + \sigma_x^{A_O} |GHZ\rangle\rangle\langle\langle GHZ| \sigma_x^{A_O} \right)^{T_{A_I}} \\
 &= \mathbb{1} - \sum_{ij} \left(|jii\rangle\langle ijj| + \sigma_x^{A_O} |jii\rangle\langle ijj| \sigma_x^{A_O} \right) \\
 &\succeq 0,
 \end{aligned} \tag{5.17}$$

where the last inequality holds true because $|GHZ\rangle\rangle\langle\langle GHZ|^{T_{A_I}}$ and $\left(\sigma_x^{A_O} |GHZ\rangle\rangle\langle\langle GHZ| \sigma_x^{A_O} \right)^{T_{A_I}}$ have orthogonal support, *i.e.*, $|GHZ\rangle\rangle\langle\langle GHZ|^{T_{A_I}} \left(\sigma_x^{A_O} |GHZ\rangle\rangle\langle\langle GHZ| \sigma_x^{A_O} \right)^{T_{A_I}} = 0$, and the eigenvalues of $\sum_{ij} |jii\rangle\langle ijj|$ and $\sum_{ij} \sigma_x^{A_O} |jii\rangle\langle ijj| \sigma_x^{A_O}$ are $+1$ and -1 .

Direct calculation shows that $\text{tr}(SW_{222}) = -2$ and, for every process Ω in this scenario, we have $\text{tr}(S\Omega) \leq 2$, by an argument analogous to Eq. (5.7b). Since S is a non-classical CCDC witness, if $(1-r)W_{222} + r\Omega$ is a CCDC process, it holds that $\text{tr}(S[(1-r)W_{222} + r\Omega]) \geq 0$. It is also true that

$$-2r + 2(1-r) \geq 0, \quad (5.18)$$

thus $r \geq \frac{1}{2}$.

In Appendix, C, it is shown that the non-classical CCDC generalized robustness has the upper bound $R_G(W) \leq \frac{1}{2}$, which concludes the proof that W_{222} attains the maximum generalized robust of its scenario. \square

We also evaluated the robustness of W_{222} against white noise, obtaining evidences that W_{222} attains the maximal white noise robustness on its scenario.

Theorem 5.2.2. *The white noise robustness of the bipartite ordered processes W_{222} is*

$$R_{WN}(W_{222}) = \frac{2}{3}. \quad (5.19)$$

Proof. We provide a lower-bound for $R_{WN}(W_{222})$ by using similar steps to the proof of theorem 5.2.1, starting with the non-classical CCDC witness

$$\begin{aligned} S &:= \mathbb{1} - 2W_{222} \\ &= \mathbb{1} - \left(|GHZ\rangle\langle GHZ| + \sigma_X^{A_0} |GHZ\rangle\langle GHZ| \sigma_X^{A_0} \right). \end{aligned} \quad (5.20)$$

We have $\text{tr}(SW_{222}) = -2$ and $\text{tr}(S) = 2^3 - 4 = 4$. As S is a non-classical CCDC witness, if $(1-r)W_{222} + r\Omega$ is a CCDC process, $\text{tr}(S[(1-r)W_{222} + r\frac{\mathbb{1}}{4}]) \geq 0$. So, it is true that

$$-2(1-r) + r \geq 0, \quad (5.21)$$

thus $r \geq \frac{2}{3}$.

We now show, using techniques which are similar to the proof of Theorem C.3, that the above lower-bound can be attained. First notice that the qubit depolarizing channel $\tilde{D}_\eta(\rho) = (1-\eta)\rho + \eta\text{tr}(\rho)\frac{\mathbb{1}}{2}$ is entanglement breaking when $\eta = \frac{2}{3}$ [56, 57]. Also, notice that, since $\text{tr}_{A_I}(W_{222}) = \frac{\mathbb{1}^{A_0 B_I}}{d_{B_I}}$, if we apply the depolarizing channel on the subspace A_I of the process W_{222} , we obtain $\tilde{D}_\eta^{A_I} \otimes \tilde{\mathbb{1}}^{A_0 B_I}(W_{222}) = (1-\eta)W_{222} + \eta\frac{\mathbb{1}}{4}$. Since \tilde{D}_η is entanglement breaking for $\eta = \frac{2}{3}$, lemma C.1 ensures that $(1-\frac{2}{3})W_{222} + \frac{2}{3}\frac{\mathbb{1}}{4}$ is CCDC, thus concluding the proof. \square

It is worth to mention that every witness introduced in the proofs of Theorems 5.1.1 and 5.2.1 are the optimal witnesses for these particular processes.

Similarly to the scenario with $d_{A_I} = d_{A_O} = d$ and $d_{B_I} = d^2$, we implemented the see-saw algorithm that indicates that, in the scenario with $d_{A_I} = d_{A_O} = d_{B_I} = 2$, the highest white noise robustness is exactly $\frac{2}{3}$. This suggests that W_{222} has also maximum white noise robustness in its scenario.

6 Relation with previous research

6.1 Comparison with the non-classical CCDC process of Ref. [1]

We now compare our proposed processes with the non-classical CCDC process presented and experimentally implemented in Ref. [1]. The process consists of the preparation of a maximally entangled state $|\phi^+\rangle$ shared between Alice and the auxiliary system, a partial swap channel from $\mathcal{L}(A_O \otimes \text{aux})$ to $\mathcal{L}(B_I \otimes \text{aux}')$, which is simply a unitary composed by a coherent mixture of an identity with a SWAP gate, and a partial trace on the output of the auxiliary system. This process can be explicitly written as

$$W_{\text{MRSR}} = \text{tr}_{\text{aux}'} \left(|\phi^+\rangle\langle\phi^+|^{A_I \text{aux}} * |\text{U}_{\text{PS}}\rangle\rangle\langle\langle\text{U}_{\text{PS}}|^{A_O \text{aux}/B_I \text{aux}'} \right), \quad (6.1)$$

where $|\text{U}_{\text{PS}}\rangle\rangle^{A_O \text{aux}/B_I \text{aux}'}$ is the Choi vector of the unitary partial SWAP¹ gate U_{PS} , given by

$$\text{U}_{\text{PS}} = \frac{1}{\sqrt{2}} (\mathbb{1} + i \text{U}_{\text{SWAP}}), \quad (6.3)$$

with U_{SWAP} being the SWAP gate for qubits, given by

$$\text{U}_{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.4)$$

The label MRSR makes reference to the names MacLean, Ried, Spekkens and Resch, authors of Ref. [1]. Fig. 12 illustrates the process W_{MRSR} .

Using our numerical methods, we can evaluate the values of robustnesses for W_{MRSR} , which are

$$R_G^{\text{low,PPT}}(W_{\text{MRSR}}) = R_G^{\text{up},\mathcal{E}_{10^4}}(W_{\text{MRSR}}) = 0.3506, \quad (6.5a)$$

$$R_{\text{WN}}^{\text{low,PPT}}(W_{\text{MRSR}}) = R_{\text{WN}}^{\text{up},\mathcal{E}_{10^4}}(W_{\text{MRSR}}) = 0.5000. \quad (6.5b)$$

When comparing the robustness values of W_{MRSR} with W_{222} (see Section 5.2 and Eqs. (6.5)), which is a process defined in the same scenario, we verify that W_{222} is strictly more robust against both generalized and white noise than W_{MRSR} .

¹ The authors from Ref. [1] use an equivalent way to represent the unitary partial SWAP gate, which is

$$\text{U}_{\text{PS}}^{A_I \text{aux}/B_I \text{aux}'} = \frac{1}{\sqrt{2}} \left(\mathbb{1}^{A_O \text{aux}/B_I \text{aux}'} + i \mathbb{1}^{A_O \text{aux}/\text{aux}' B_I} \right). \quad (6.2)$$

In the second term, the identity channel exchanges the outputs B_I and aux' , in comparison to the identity channel in the first term. This part of the channel does exactly the same as U_{SWAP} does, differing only by the fact that the output labels are not explicitly exchanged.

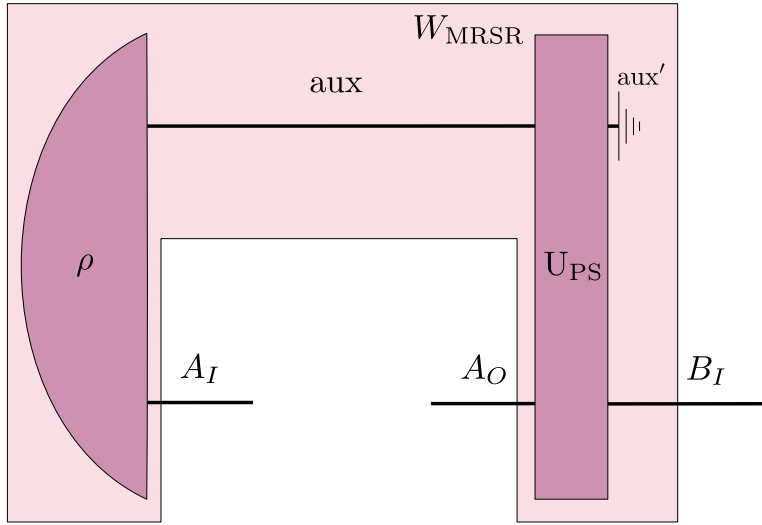


Figure 12 – Circuit representation of the process W_{MRSR} presented in Ref. [1]. A maximally entangled state is initially shared between Alice and the auxiliary system. After Alice's operation, the partial-SWAP gate U_{PS} is applied, then aux' is discarded.

For the case of the process W_{2222} , we argue that the construction of W_{2222} is simpler than W_{MRSR} . Both processes require the preparation of a maximally entangled qubit state, but W_{MRSR} requires the implementation of the control swap operation, which is a coherent superposition between the identity channel and the swap channel, while W_{2222} only requires the identity channel.

6.2 Comparison with the non-classical CCDC process of Ref. [2]

In Ref. [2], the authors proposed to study non-classical CCDC by considering quantum superpositions of both relations. Their example is the tripartite process ordered as $A \rightarrow B \rightarrow C$. A tripartite process ordered as $A \rightarrow B \rightarrow C$ is an operator $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I \otimes B_O \otimes C_I)$ which can be written as

$$W = \rho^{A_I \text{aux}} * D_A^{\text{aux} A_O / B_I \text{aux}'} * D_B^{\text{aux}' B_O / C_I}, \quad (6.6)$$

where D_A and D_B are Choi operators of quantum channels.

In a tripartite scenario, common-cause and direct-cause relations can be more complex than on the bipartite scenario. In the case of common-cause relations, more general situations are discussed in Refs. [58, 59]. However, following the definitions presented in Ref. [2], we restrict the attention to the same common-cause and direct-cause relations of the bipartite case, recovering them when taking $d_{B_O} = d_{C_I} = 1$. The following definitions are taken considering the ones presented in Ref. [2].

A tripartite ordered process W_{CC} is common-cause if

$$\text{tr}_{B_O C_I}(W_{CC}) := d_{B_O} \rho^{A_I B_I} \otimes \mathbb{1}^{A_O}, \quad (6.7)$$

where $\rho^{A_I B_I}$ is a quantum state in $\mathcal{L}(A_I \otimes B_I)$. A tripartite ordered process W_{DC} is direct-cause if

$$\text{tr}_{B_O C_I}(W_{DC}) := d_{B_O} \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}, \quad (6.8)$$

where $\rho_i^{A_I}$ are quantum states in $\mathcal{L}(A_I)$ and $D_i^{A_O/B_I}$ are quantum channels from A_O to B_I .

A tripartite ordered process W_{CCDC} is classical CCDC if it can be decomposed in a convex combination of a common-cause process and a direct-cause process. Note that when bipartite processes are considered, *i.e.*, the dimensions of B_O and C_I are equal to 1, their definition is equivalent to the ones presented in section 3.

The example of non-classical CCDC process presented by the authors is the operator² $W_{FB} = |W_{FB}\rangle\rangle\langle\langle W_{FB}|$, where

$$|W_{FB}\rangle\rangle = \frac{1}{\sqrt{2}} \left(|\phi^+\rangle^{A_I B_I} |\mathbb{1}\rangle\rangle^{A_O/C_I^1} |\mathbb{1}\rangle\rangle^{B_O/C_I^2} |0\rangle^{C_I^3} + |\phi^+\rangle^{A_I C_I^1} |\mathbb{1}\rangle\rangle^{A_O/B_I} |\mathbb{1}\rangle\rangle^{B_O/C_I^2} |1\rangle^{C_I^3} \right), \quad (6.9)$$

The label FB makes reference to the names Feix and Brukner, authors of Ref. [2].

This process can be seen as a superposition of a common-cause and a direct-cause processes, as the first term corresponds to a common-cause process, whereas the second term corresponds to a direct-cause process. We can also represent W_{FB} in terms of ordered quantum circuits³ as illustrated in Fig. 13.

In Ref. [2], the authors numerically obtained a lower-bound to the generalized robustness of W_{FB} , using the PPT outer approximation of DC processes, obtaining⁴

$$R_G^{\text{low,PPT}}(W_{FB}) = 0.1855. \quad (6.10)$$

Using our inner approximation method with $N = 200$, we could obtain, up to numerical precision, the upper-bound

$$R_G^{\text{up},\mathcal{E}_{200}}(W_{FB}) = 0.1855, \quad (6.11)$$

² The state $|\phi^+\rangle$ appearing in the expression of W_{FB} is originally written, in Ref. [2], as generic bipartite state $|\psi\rangle$. However, we verified that the numerical robustness results obtained in Ref. [2] are reproduced when $|\psi\rangle$ is a maximally entangled state, which lead us to take $|\psi\rangle = |\phi^+\rangle$, as this state is used in every other process mentioned before in this work.

³ Indeed, every ordered quantum process can be represented in terms of ordered quantum circuits by concatenating quantum states and quantum operations [13].

⁴ Strictly speaking, the authors have evaluated quantity before mentioned here, the non-classicality of causality \mathcal{C} , which has a one-to-one relation with the generalized robustness via $R_G^{\text{low,PPT}}(W) = \frac{\mathcal{C}(W)}{1+\mathcal{C}(W)}$.

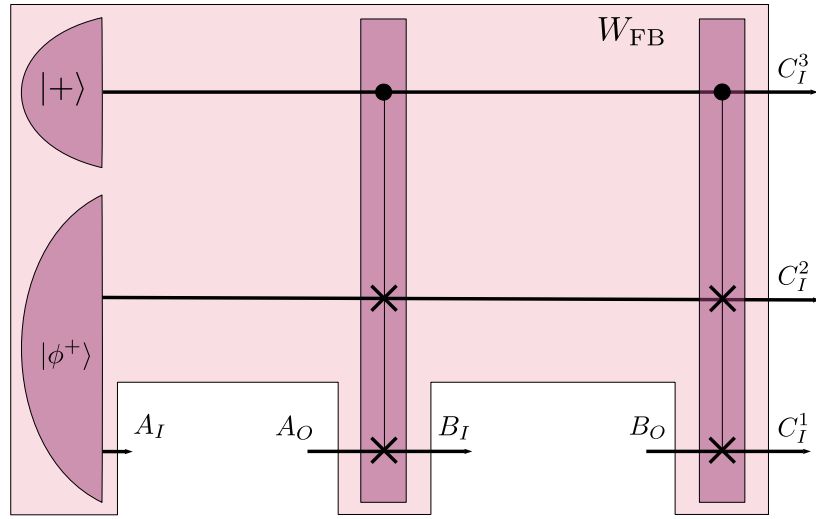


Figure 13 – Circuit representation of the process W_{FB} presented in Ref. [2]. A fixed qubit $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ in C_I^3 controls the swap operation from the auxiliary system in C_I^2 and A_O to B_I and from C_I^2 and B_O to C_I^1 . As the control qubit is fixed, the swapping operation occurs in the same way in both channels. This combination of operations generate the pure non-classical CCDC process W_{FB}

showing that $R_G(W_{\text{FB}}) = 0.1855$, up to numerical precision. For completeness, we have also computed the white noise robustness, obtaining

$$R_{\text{WN}}^{\text{low,PPT}}(W_{\text{FB}}) = R_{\text{WN}}^{\text{up},\mathcal{E}_{200}}(W_{\text{FB}}) = 0.3324. \quad (6.12)$$

6.3 The role of coherent mixture of causal relations

As discussed in this section, previous research has shown that one way to obtain processes with the non-classical CCDC property is by coherently superposing causal relations. For instance, the tripartite process presented in Ref. [2] and discussed in Section 6.2 is constructed in a way to be a coherent superposition of a purely common-cause and a purely direct-cause processes. Also, the bipartite process presented in Ref. [1] is inspired by a coherent mixture of causal relations, which is mathematically formalized by the application of the partial swap operation (see Eq. (6.2)).

Differently from previous works, we have shown in this work that the connection between non-classical CCDC and coherent superpositions of causal relations may be more subtle than it seems at first glance. In particular, although the non-classical CCDC process W_{dad^2} presented in Chapter 5 may be viewed as a superposition of a process which is initialized in $|00\rangle$ with a process which is initialized in state $|11\rangle$, it also admits a natural interpretation as a process with both common-cause and direct-cause relations simultaneously, without explicitly considering any superposition of causal relations.

We then argue that the process W_{ddd^2} does not need to be interpreted as a coherent superposition of causal relations.

7 Separability properties in non-classical CCDC processes

7.1 Separable process without a direct-cause explanation

As mentioned in previous sections, the definition of direct-cause processes (Def.3.2.1) reminds one of the definition of separable quantum states. More precisely, if we do not impose that the operators $D_i^{A_O/B_I}$ have to respect the quantum channel condition $\text{tr}_{B_I} D_i^{A_O/B_I} = \mathbb{1}^{A_O}$, equation (3.3b) is precisely the definition of a separable state on the bipartition $A_I|A_O B_I$. We now show that, despite being related, these two definitions are not equivalent.

Consider the process

$$\begin{aligned}
 W_{\text{SEP}} := & \frac{1}{2} \left(|0\rangle\langle 0|^{A_I} \otimes |0\rangle\langle 0|^{A_O} \otimes |0\rangle\langle 0|^{B_I} \right. \\
 & + |1\rangle\langle 1|^{A_I} \otimes |0\rangle\langle 0|^{A_O} \otimes |1\rangle\langle 1|^{B_I} \\
 & + |+\rangle\langle +|^{A_I} \otimes |1\rangle\langle 1|^{A_O} \otimes |+\rangle\langle +|^{B_I} \\
 & \left. + |-\rangle\langle -|^{A_I} \otimes |1\rangle\langle 1|^{A_O} \otimes |-\rangle\langle -|^{B_I} \right),
 \end{aligned} \tag{7.1}$$

which is a separable operator by construction. First, notice that

$$\text{tr}_{B_I} W_{\text{SEP}} = \frac{1}{2} \mathbb{1}^{A_I} \otimes \mathbb{1}^{A_O}, \tag{7.2}$$

which shows that W is a valid bipartite ordered process.

The process W_{SEP} can be physically realized by preparing a maximally entangled qubit state between A_I and aux, and using a “decoder” described by

$$\begin{aligned}
 D^{A_O \text{aux}/B_I} = & |0\rangle\langle 0|^{A_O} \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|)^{\text{aux}B_I} \\
 & + |1\rangle\langle 1|^{A_O} \otimes (|++\rangle\langle ++| + |--\rangle\langle --|)^{\text{aux}B_I}.
 \end{aligned} \tag{7.3}$$

In this way, we have

$$W_{\text{SEP}} = |\phi^+\rangle\langle \phi^+|^{A_I \text{aux}} * D^{A_O \text{aux}/B_I}. \tag{7.4}$$

Notice that the channel $D^{A_O \text{aux}/B_I}$ can be implemented as follows: first, perform a computational basis measurement on A_O . If the outcome is 0, perform a computational basis measurement on aux and send the output qubit to B_I . If the outcome is 1, perform a measurement on aux in the X-basis instead.

Theorem 7.1.1. *The bipartite ordered process W_{SEP} is separable in the bipartition $A_I|A_O B_I$, but is not direct-cause.*

Proof. Equation (7.1) represents W_{SEP} as a convex combination of product states, ensuring that W_{SEP} is separable in the bipartition $A_I|A_O B_I$. In order to show that W_{SEP} is not a direct-cause process, let us assume that W_{SEP} can be written as a convex combination $W_{\text{SEP}} = \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}$, where $\rho_i^{A_I}$ are normalized quantum states and every $D_i^{A_O/B_I}$ satisfy $\text{tr}_{A_I} D_i^{A_O/B_I} = \mathbb{1}^{A_O}$. Note that each $\rho_i^{A_I}$ has non-trivial overlap with, at least, 3 out of the 4 states $|0\rangle\langle 0|, |1\rangle\langle 1|, |+\rangle\langle +|, |-\rangle\langle -|$.

Indeed, let $\rho = \frac{1}{2} (\mathbb{1} + \sum_i \alpha_i \sigma_i)$ be an arbitrary state where $\{\alpha_i\}$ are real numbers that satisfy $\sum_i \alpha_i^2 \leq 1$ and σ_i are Pauli matrices. Suppose that ρ has zero overlap with some pure state $|\psi\rangle\langle\psi| = \frac{1}{2} (\mathbb{1} + \sum_i \beta_i \sigma_i)$ with $\sum_i \beta_i^2 = 1$, then we have that $(\vec{\alpha}, \vec{\beta}) = -1$, where (\cdot, \cdot) is the Euclidean inner product. By the Cauchy-Schwarz inequality, we have

$$1 = (\vec{\alpha}, \vec{\beta})^2 \leq (\vec{\alpha}, \vec{\alpha})(\vec{\beta}, \vec{\beta}) = (\vec{\alpha}, \vec{\alpha}), \quad (7.5)$$

with equality if and only if $\vec{\alpha}$ is a multiple of $\vec{\beta}$. This shows that $\vec{\alpha} = -\vec{\beta}$ and therefore ρ cannot be orthogonal to any other pure quantum state.

Let us choose some fixed index j in the sum and suppose, without lack of generality, that ρ_j has non-zero overlap with $|1\rangle, |+\rangle, |-\rangle$. Then, from the above definition of W_{SEP} , we calculate

$$\text{tr}(|1\rangle\langle 1|^{A_I} \otimes |0\rangle\langle 0|^{A_O} \otimes |0\rangle\langle 0|^{B_I} W_{\text{SEP}}) = 0. \quad (7.6)$$

From the decomposition $W_{\text{SEP}} = \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}$, we get

$$\sum_i p_i \text{tr}(|1\rangle\langle 1| \rho_i^{A_I}) \text{tr}(|0\rangle\langle 0|^{A_O} \otimes |0\rangle\langle 0|^{B_I} D_i^{A_O/B_I}) = 0. \quad (7.7)$$

By positivity of $D_i^{A_O/B_I}$ and $\rho_i^{A_I}$, each term in the sum has to be zero. Since, by assumption, $\text{tr}(|1\rangle\langle 1| \rho_j^{A_I}) \neq 0$, it must be the case that $D_j^{A_O/B_I}$ obeys

$$\text{tr}(|0\rangle\langle 0|^{A_O} \otimes |0\rangle\langle 0|^{B_I} D_j^{A_O/B_I}) = 0. \quad (7.8)$$

Similarly by calculating other projectors we get

$$\text{tr}(|1\rangle\langle 1|^{A_O} \otimes |-\rangle\langle -|^{B_I} D_j^{A_O/B_I}) = 0, \quad (7.9a)$$

$$\text{tr}(|1\rangle\langle 1|^{A_O} \otimes |+\rangle\langle +|^{B_I} D_j^{A_O/B_I}) = 0. \quad (7.9b)$$

From this we get $\text{tr}(|1\rangle\langle 1|^{A_O} \otimes \mathbb{1}^{B_I} D_j^{A_O/B_I}) = 0$, which means that $\text{tr}_{B_I} D_j^{A_O/B_I} \neq \mathbb{1}^{A_O}$. \square

By construction, W_{SEP} is separable in the bipartition $A_I|A_O B_I$, implying that W_{SEP} has a PPT k -symmetric extension for every $k \in \mathbb{N}$ and $R_G^{\text{low, PPT}_k}(W) = R_{WN}^{\text{low, PPT}_k}(W) = 0$. This means that the non-classical CCDC property of W_{SEP} cannot be certified by any purely entanglement-based criterion.

Using the inner and outer approximations presented in Secs. 4.2 and 4.2 with the family of qubits presented in the Appendix B of Ref.[53] ($n = 171$, which corresponds to $N = 2 \cdot 171^2$ states), we obtain upper and lower bounds for the generalized and white noise robustnesses of W_{SEP}

$$R_G^{\text{up}, \mathcal{E}^N}(W_{\text{SEP}}) = R_G^{\text{low}, \hat{\mathcal{E}}^N}(W_{\text{SEP}}) = 0.1465, \quad (7.10a)$$

$$R_{WN}^{\text{up}, \mathcal{E}^N}(W_{\text{SEP}}) = R_{WN}^{\text{low}, \hat{\mathcal{E}}^N}(W_{\text{SEP}}) = 0.2930, \quad (7.10b)$$

with equality holding up to numerical precision.

In Ref. [20], it was conjectured that the set of separable processes and the set of processes without quantum memory are not the same, the latter being a strict subset of the first. Since, the definitions of bipartite processes without quantum memory and bipartite direct-cause processes are equivalent (see Appendix B), we have then proven the conjecture presented in Ref. [20] by explicitly constructing the bipartite separable process W_{SEP} , which cannot be realized by processes without quantum memory.

It is interesting to point out that our numerical methods allowed us to obtain a relatively high robustness of $R_{WN}(W_{\text{SEP}}) = 0.2930$ but techniques exclusively based on entanglement would lead to the trivial lower bound $R_{WN}^{\text{sep}}(W_{\text{SEP}}) \geq 0$. Since $R_{WN}(W_{\text{SEP}}) = 0.2930$ is considerably greater than zero, we see that the approximating the set of direct-cause processes by separable processes may lead to very unsatisfactory results.

7.2 Certifying non-classical CCDC on PPT processes

In this section, we present an example of a non-classical CCDC process which has $R_G^{\text{low}, \text{PPT}}(W) = R_{WN}^{\text{low}, \text{PPT}}(W) = 0$, *i.e.*, its non-classical CCDC property cannot be certified by the PPT approximation used in Refs. [1, 2]. Such a process can be obtained by exploiting a class of entangled states with positive partial transpose, presented in Ref. [60]. The class of states of our interest is

$$\rho_a^{2 \times 4} := \frac{1}{7a+1} \begin{bmatrix} a & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+a) & 0 & 0 & \frac{1}{2}\sqrt{1-a^2} \\ a & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & 0 & \frac{1}{2}(1+a) \end{bmatrix}, \quad (7.11)$$

being entangled for $a \in (0, 1)$ and separable for $a = 0$ or $a = 1$.

We now set $a = \frac{1}{2}$ and use $\rho_{\frac{1}{2}}^{2 \times 4}$ to define:

$$W_{\text{PPT}} := d_{A_O} \cdot \rho_{\frac{1}{2}}^{2 \times 4}, \quad (7.12)$$

with $d_{A_O} = 2$. W_{PPT} is a valid bipartite ordered process, as it satisfies every condition from Eqs. (3.13) by direct inspection. Also, it has dimensions $d_{A_I} = d_{A_O} = d_{B_I} = 2$ and has bound entanglement in the bipartition $A_I | A_O B_I$.

We will now ensure that W_{PPT} is a non-classical CCDC process by using a better approximation $\mathcal{L}_{\text{DC}}^{\text{out, PPT}_k}$. In particular, we set $k = 2$, then we obtain

$$\begin{aligned} R_G^{\text{low, PPT}_2}(W_{\text{PPT}}) &= 0.0083, \\ R_{\text{WN}}^{\text{low, PPT}_2}(W_{\text{PPT}}) &= 0.0230. \end{aligned} \quad (7.13)$$

For $k = 3$, the robustnesses do not change, which indicates that using greater values of k does not improve the values of generalized and white noise robustnesses.

When using the inner and outer approximations presented in Secs. 4.2 and 4.2 with, once again, the family of qubits presented in the Appendix B of Ref.[53] ($n = 171$, corresponding to $N = 2 \cdot 171^2$ states), up to numerical precision, we obtain

$$\begin{aligned} R_G^{\text{up, } \mathcal{E}_{10^4}}(W_{\text{PPT}}) &= R_G^{\text{low, } \hat{\mathcal{E}}_{10^4}}(W_{\text{PPT}}) = 0.1085, \\ R_{\text{WN}}^{\text{up, } \mathcal{E}_{10^4}}(W_{\text{PPT}}) &= R_{\text{WN}}^{\text{low, } \hat{\mathcal{E}}_{10^4}}(W_{\text{PPT}}) = 0.2782. \end{aligned} \quad (7.14)$$

We verify that the lower bounds for the robustnesses obtained with the entanglement criterium in Eqs. (7.13) are rather loose in comparison to the actual robustnesses values from Eqs. (7.14). This example also illustrates the limitations of certifying non-classical CCDC solely based on entanglement criteria.

7.3 Summary of robustnesses to generalized and white noise

In previous sections, we presented several examples of non-classical CCDC processes with different values of generalized and white noise robustness. Table 2 summarizes the non-classical CCDC robustnesses of several process presented in this work and compare the actual value of robustness with the values obtained with methods based on entanglement criteria.

Process (Eq.)	R_G	$R_G^{\text{low, PPT}_{k=2}}$	R_{WN}	$R_{WN}^{\text{low, PPT}_{k=2}}$
W_{333^2} (5.1)	$\frac{2}{3}$	$\frac{2}{3}$	0.9529	0.9529
W_{222^2} (5.1)	$\frac{1}{2}$	$\frac{1}{2}$	0.8421	0.8421
W_{222} (5.10)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$
W_{MRSR} (6.1)	0.3506	0.3506	0.5000	0.5000
W_{FB} (6.9)	0.1855	0.1855	0.3324	0.3324
W_{SEP} (7.1)	0.1465	0	0.2930	0
W_{PPT} (7.12)	0.1085	0.0083	0.2782	0.0230

Table 2 – Table presenting generalized and white noise robustnesses for every process analysed in this work. Values represented in fractions were obtained by mathematical theorems and coincide with SDP optimization. Values with decimal digits were obtained only via SDP optimization where our upper and lower bounds are identical up to 4 decimals. The lower bounds obtained by the approximations of the CCDC set based on entanglement are the values provided in columns $R_G^{\text{low, PPT}_k}$ and $R_{WN}^{\text{low, PPT}_k}$. Lower-bounds that match the actual robustnesses are highlighted in green, while lower bounds that are rather different from the actual robustnesses are highlighted in red. We can observe that entanglement based criteria could never detect W_{SEP} as a non-classical CCDC process, while the PPT k -symmetric extension bound for W_{PPT} provides the loose lower bounds $R_G^{\text{low, PPT}_k}(W_{\text{PPT}}) \geq 0.0083$ $R_{WN}^{\text{low, PPT}_k}(W_{\text{PPT}}) \geq 0.0230$, which are values obtained both with $k = 2$ and $k = 3$.

When analysing table 2, one has the feeling that bipartite ordered processes that cannot be certified using the PPT_k robustnesses, or processes that shows differences between the PPT_k robustnesses and the other tight approximations, are “exceptions”, *i.e.*, the volume of the complement of $\mathcal{L}_{\text{CCDC}}$ inside $\mathcal{L}_{\text{CCDC}}^{\text{out, PPT}_k}$ is small, in relation to these sets. However, it is worth to mention that we evaluated the robustnesses of many processes sampled numerically, and we observed that it is rather difficult to randomly sample a process that does not show a considerable difference between the robustness obtained with the entanglement approximation and the robustnesses obtained with the tight approximations. This indicates two interesting things: the volume of processes that cannot be certified by entanglement criteria *is not* small, and the processes studied in this work, with exception to W_{SEP} and W_{PPT} , are members of some restrict class of processes that have PPT_k robustnesses coinciding with tight approximations of non-classical CCDC robustnesses. This can inspire future research about non-classical CCDC relations.

8 Conclusions

In this work, we have introduced a class of bipartite ordered processes, and a process with dimension of three qubits, which are maximally robust against general noise and very likely to be the most robust against white noise for the qubit case. This class of processes can be implemented by preparing a pair of maximally entangled states and an identity channel, admitting a natural interpretation of a process with both common-cause and direct-cause relations simultaneously. Hence, in contrast to previously known non-classical CCDC processes [2, 16], the class presented here does not require either the construction or the interpretation directly based on coherent superposition of causal relations.

Several analytical results proved in this work employed general convex analysis arguments based on witness hyper-planes, combined with entanglement theory concepts, such as entanglement breaking channels. We believe that the techniques developed here may find applications in related problems.

We have also presented a systematic semidefinite approach to characterize the set of non-classical CCDC processes. In particular, we used an entanglement-based hierarchy which, despite not converging to the set of CCDC processes, provides us several tight and non-trivial bounds. Moreover, we provided hierarchies of inner and outer approximations that converge to the set of classical CCDC processes.

In order to tackle situations where we could not prove the highest robustness of a given scenario analytically, we constructed a heuristic see-saw method, which provided numerical evidence of the highest robustnesses attainable on such scenario, also providing a valid lower-bound for the value of highest robustness.

Finally, we have shown that, although all bipartite processes that are entangled in the bipartition $A_I|A_O B_I$ do not have a direct-cause decomposition, the converse does not hold. Our proof consists in explicitly constructing a process which is separable on the bipartition $A_I|A_O B_I$, but does not have a direct-cause decomposition. Since bipartite processes without quantum memory are equivalent to bipartite direct-cause ones, our results prove a conjecture first raised in Ref. [20] and contributes towards a better understanding of the particularities of quantum memory, spacial entanglement and temporal entanglement [20–24].

All our codes are available in the public repository [61] and can be freely used under the GNU Lesser General Public License v3.0.

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Appendix

APPENDIX A – Robustnesses of entanglement

The problem of finding the optimal entanglement witness of a given state ρ can be associated with the problem of evaluating how resistant the state is to the addition of a noise, *i.e.*, to discover what is the minimum amount of noise necessary to be added to ρ so that the result is a separable state. We call this the problem of *robustness of entanglement*. Following the steps from Ref. [62] and the discussion of dualization from the previous section to obtain the dual problem of Eqs. (2.5), we are going to show that the optimal value of $\text{tr}(W\rho)$ may be viewed as an entanglement robustness.

The Lagrangian of Eqs. (2.5) can be written as

$$L(W, Z, g(\rho_{\text{sep}})) = \text{tr}(W\rho) - \text{tr}(Z(\mathbb{1} - W)) - \int_{\mathcal{L}_{\text{SEP}}} g(\rho_{\text{sep}})\text{tr}(W\rho_{\text{sep}}) d\rho_{\text{sep}}, \quad (\text{A.1})$$

where Z and $g(\rho_{\text{sep}})$ are the Lagrange multipliers associated with the constraints (2.5a) and (2.5b), respectively. To find the Lagrange dual function, we rewrite the Lagrangian in a way that is more direct to take its infimum over W .

$$\inf_W (L(W, Z, g(\rho_{\text{sep}}))) = \inf_W \left\{ \text{tr} \left[W \left(\rho + Z - \int_{\mathcal{L}_{\text{SEP}}} g(\rho_{\text{sep}})\rho_{\text{sep}} d\rho_{\text{sep}} \right) \right] + \text{tr}(Z) \right\}. \quad (\text{A.2})$$

Note that if $\left(\rho + Z - \int_{\mathcal{L}_{\text{SEP}}} g(\rho_{\text{sep}})\rho_{\text{sep}} d\rho_{\text{sep}} \right) \neq 0$, the solution diverges. The only way to have a non-diverging solution is if the term multiplying W in Eq. (A.2) is zero. Then, the Lagrange dual function is simply $\text{tr}(Z)$. Maximizing the Lagrange dual function gives the dual problem, which is

given ρ

$$\min \text{tr}(Z)$$

$$\text{subject to } \rho + Z = \int_{\mathcal{L}_{\text{SEP}}} g(\rho_{\text{sep}})\rho_{\text{sep}} d\rho_{\text{sep}}, \quad (\text{A.3a})$$

$$g(\rho_{\text{sep}}) \geq 0 \quad \forall \rho_{\text{sep}} \in \mathcal{L}_{\text{sep}}(A \otimes B), \quad (\text{A.3b})$$

$$Z \succeq 0. \quad (\text{A.3c})$$

Since $g(\rho_{\text{sep}}) \geq 0$, the integral in the above constraints correspond to a separable state. Also, any fixed separable un-normalized state $\bar{\rho}_{\text{sep}}$ is obtained with the choice of $g(\rho_{\text{sep}}) = \delta(\rho_{\text{sep}} - \bar{\rho}_{\text{sep}})$, which makes the constraint become then $\rho + Z \in \mathcal{L}_{\text{SEP}}(A \otimes B)$. This optimization problem corresponds to the *generalized robustness* of entanglement, and is interpreted as the amount of a general noise Z one must “add” to the state ρ so that the result is separable.

One can re-formulate the condition from Eq. (A.3a) as an appropriate convex combination, which leads to a re-formulation of the whole generalized robustness problem. Considering the problem of generalized robustness as the minimum value of r so that $(1 - r)\rho + r\rho' = \sigma_{\text{sep}}$, where ρ' is an arbitrary quantum state and σ_{sep} is a separable *normalized* quantum state, the optimization problem corresponding to the generalized robustness become

$$\begin{aligned} & \text{given } \rho \\ & \min r \\ & \text{subject to } (1 - r)\rho + r\rho' = \sigma_{\text{sep}} & (\text{A.4a}) \\ & \sigma_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), & (\text{A.4b}) \\ & \rho', \sigma_{\text{sep}} \succeq 0, & (\text{A.4c}) \\ & \text{tr}(\rho') = \text{tr}(\sigma_{\text{sep}}) = 1, & (\text{A.4d}) \\ & r \in [0, 1]. & (\text{A.4e}) \end{aligned}$$

As the above problem contains the product of optimization variables ($r\rho'$), it is not a linear optimization problem. An artifice to make it linear is by making the coefficient r to be absorbed by the variable ρ' and the problem becomes similar to the one from Eqs. (A.3), then being

$$\begin{aligned} & \text{given } \rho \\ & \min \text{tr}(Z) \\ & \text{subject to } (1 - \text{tr}(Z))\rho + Z = \sigma_{\text{sep}} & (\text{A.5a}) \\ & \sigma_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), & (\text{A.5b}) \\ & Z, \sigma_{\text{sep}} \succeq 0, & (\text{A.5c}) \\ & \text{tr}(\sigma_{\text{sep}}) = 1, & (\text{A.5d}) \\ & \text{tr}(Z) \leq 1. & (\text{A.5e}) \end{aligned}$$

and Eqs. (A.5) relates to Eqs. (A.4) with $r = \text{tr}(Z)$ and $\rho' = \frac{Z}{\text{tr}(Z)}$.

There are some technical differences between the generalized robustness written in the form of Eqs.(A.3) the one in the form of Eqs. (A.5). The most important point for this work is that the possible values of the objective function of Eqs. (A.3) lie between $[0, +\infty)$, while the possible values of the objective function of Eqs. (A.5) lie between $[0, 1]$. The second choice is easier to evaluate in an absolute form, as the robustnesses are higher the closer to 1 they are.

A related problem of finding an optimal entanglement witness W for a given

entangled state ρ can be casted in the form

$$\begin{aligned} & \text{given } \rho \\ & \min \operatorname{tr}(W\rho) \\ & \text{subject to } \operatorname{tr}(W\rho_{\text{sep}}) \geq 0 \quad \forall \rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (\text{A.6a}) \\ & \operatorname{tr}(W) \leq 1. \quad (\text{A.6b}) \end{aligned}$$

Following the same steps for the problem of generalized robustness introduced previously, we obtain the dual form of Eqs. (A.6)

$$\begin{aligned} & \text{given } \rho \\ & \min r \\ & \text{subject to } \rho + r\frac{\mathbb{1}}{d} = \int_{\mathcal{L}_{\text{SEP}}} g(\rho_{\text{sep}})\rho_{\text{sep}} d\rho_{\text{sep}}, \quad (\text{A.7a}) \\ & g(\rho_{\text{sep}}) \geq 0 \quad \forall \rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (\text{A.7b}) \\ & r \geq 0, \quad (\text{A.7c}) \end{aligned}$$

where $d = \dim(A \otimes B)$. The above problem corresponds to the *white noise robustness* of entanglement, and is interpreted as the minimum amount of white noise (the state $\frac{\mathbb{1}}{d}$) is needed for transforming ρ into a separable state.

Once again, it is also possible to write the problem of white noise in terms of a proper convex combination, which is

$$\begin{aligned} & \text{given } \rho \\ & \min r \\ & \text{subject to } (1-r)\rho + r\frac{\mathbb{1}}{d} = \rho_{\text{sep}}, \quad (\text{A.8a}) \\ & \rho_{\text{sep}} \in \mathcal{L}_{\text{SEP}}(A \otimes B), \quad (\text{A.8b}) \\ & \rho_{\text{sep}} \succeq 0, \operatorname{tr}(\rho_{\text{sep}}) = 1, \quad (\text{A.8c}) \\ & 0 \leq r \leq 1, \quad (\text{A.8d}) \end{aligned}$$

which also has the advantage of having the property $r \in [0, 1]$ of the generalized robustness from Eqs. (A.5).

APPENDIX B – Ordered processes without quantum memory are equivalent to direct-cause processes on the bipartite case

We now present the definition of ordered (non-Markovian) processes without quantum memory, introduced in Ref.[20].

Definition B.1 (Process without quantum memory[20]). A linear operator $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is a bipartite ordered process without quantum memory if it can be written as

$$W = \rho_{\text{SEP}}^{A_I \text{aux}} * D^{A_O \text{aux}/B_I}, \quad (\text{B.1})$$

where $\rho_{\text{SEP}}^{A_I \text{aux}}$ is a separable state and $D^{A_O \text{aux}/B_I}$ is the Choi operator of a quantum channel.

We now show that all bipartite processes without quantum memory are bipartite direct-cause, and *vice-versa*. This proof was first presented in Appendix A.3.2 of Ref. [20], but we reproduce it here for the sake of completeness.

Theorem B.1 (Appendix A.3.2 of Ref.[20]). *A bipartite ordered process W is a process without quantum memory if and only if W is direct-cause.*

Proof. We first show that every process without quantum memory is direct-cause. Since $\rho_{\text{SEP}}^{A_I \text{aux}}$ is separable, there exists $\rho_i^{A_I}$ and σ_i^{aux} and some probabilities p_i such that $\rho_{\text{SEP}}^{A_I \text{aux}} = \sum_i p_i \rho_i^{A_I} \otimes \sigma_i^{\text{aux}}$. Thus, we can write

$$\left(\sum_i p_i \rho_i^{A_I} \otimes \sigma_i^{\text{aux}} \right) * D^{\text{aux}A_O/B_I} = \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I}, \quad (\text{B.2})$$

where $D_i^{A_O/B_I} := \sigma_i^{\text{aux}} * D^{\text{aux}A_O/B_I}$ are valid quantum channels, since $D_i^{A_O/B_I} \succeq 0$ and $\text{tr}_{B_I}(D_i^{A_O/B_I}) = 1^{A_O}$.

Now, we need to show that every direct-cause process is a process without quantum memory in the bipartite case. Let us assume that W is direct-cause. Then W can be written as

$$W = \sum_i p_i \rho_i^{A_I} \otimes D_i^{A_O/B_I} \quad (\text{B.3})$$

for some states $\rho_i^{A_I}$ and channels $D_i^{A_O/B_I}$. Now, let us define the separable state $\sigma^{A_I \text{aux}} := \sum_i p_i \rho_i^{A_I} \otimes |i\rangle\langle i|^{\text{aux}}$, and the quantum channel $D^{\text{aux}A_O/B_I} := \sum_i |i\rangle\langle i|^{\text{aux}} \otimes D_i^{A_O/B_I}$. Direct

calculation shows that $\sigma^{A_I, \text{aux}} * D^{\text{aux}A_O/B_I} = W$, ensuring that W is a process without quantum memory. \square

APPENDIX C – A tight upper bound for the generalized robustness

In this section, we prove that the generalized robustness of any process is upper-bounded by $R_G(W) \leq 1 - \frac{1}{d_{A_I}}$. This bound is saturated by the processes W_{ddd^2} (Eq. (5.1)) for every dimension d and by W_{222} (Eq. (5.10)).

Lemma C.1. *Let W be a bipartite ordered process. If $\tilde{\Lambda} : \mathcal{L}(A_I) \rightarrow \mathcal{L}(A_I)$ is an entanglement breaking channel, the process $\tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W)$ is direct-cause.*

Proof. By definition, any bipartite ordered process can be written as $W = \rho^{A_I \text{aux}} * D^{\text{aux} A_O / B_I}$. Since $\tilde{\Lambda}$ is entanglement breaking, it holds that $\tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{\text{aux}}(\rho^{A_I \text{aux}})$ is a separable state. Therefore, we can use the same argument presented in the proof of theorem B.1 to ensure that $\tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W)$ is a direct-cause process. \square

Lemma C.2. *Let $\tilde{\Lambda} : \mathbb{C}_d \rightarrow \mathbb{C}_d$ be a quantum channel, $\omega = e^{\frac{2\pi\sqrt{-1}}{d}}$, and $Z := \sum_{i=0}^{d-1} \omega^i |i\rangle\langle i|$ be the d -dimensional clock operator. The channel*

$$\tilde{\Lambda}(\rho) = \frac{1}{d} \sum_{k=0}^{d-1} Z^k \rho Z^{-k} \tag{C.1}$$

is an entanglement breaking channel.

Proof. A necessary and sufficient condition [57] for $\tilde{\Lambda}$ to be entanglement-breaking is that its Choi operator Λ is separable between input and output spaces. We now show

that Λ is separable by

$$\Lambda := \sum_{ab} |a\rangle\langle b| \otimes \tilde{\Lambda}(|a\rangle\langle b|) \quad (\text{C.2a})$$

$$= \frac{1}{d} \sum_{abk} |a\rangle\langle b| \otimes Z^k (|a\rangle\langle b|) Z^{-k} \quad (\text{C.2b})$$

$$= \frac{1}{d} \sum_{abkij} |a\rangle\langle b| \otimes |i\rangle\langle i| \omega^{ik} (|a\rangle\langle b|) |j\rangle\langle j| \omega^{-jk} \quad (\text{C.2c})$$

$$= \frac{1}{d} \sum_{kij} |i\rangle\langle j| \otimes |i\rangle\langle j| \omega^{ik} \omega^{-jk} \quad (\text{C.2d})$$

$$= \frac{1}{d} \sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j| \sum_k \omega^{ik} \omega^{-jk} \quad (\text{C.2e})$$

$$= \frac{1}{d} \sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j| \sum_k \omega^{k(i-j)} \quad (\text{C.2f})$$

$$= \frac{1}{d} \sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j| d \delta_{ij} \quad (\text{C.2g})$$

$$= \sum_i |i\rangle\langle i| \otimes |i\rangle\langle i|. \quad (\text{C.2h})$$

□

Theorem C.3. *The generalized robustness of a bipartite ordered process $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is upper-bounded by $R_G(W) \leq 1 - \frac{1}{d_{A_I}}$.*

Proof. Let $\tilde{\Lambda} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ be the entanglement-breaking channel defined in Lemma C.2. Note that since $Z^0 = \mathbb{1}$, the action of $\tilde{\Lambda}$ can be written as

$$\tilde{\Lambda}(\rho) = \frac{1}{d} \sum_{k=0}^{d-1} Z^k \rho Z^{-k} \quad (\text{C.3a})$$

$$= \frac{1}{d} \rho + \frac{1}{d} \sum_{k=1}^{d-1} Z^k \rho Z^{-k} \quad (\text{C.3b})$$

$$= \frac{1}{d} \rho + \left(1 - \frac{1}{d}\right) \tilde{\Lambda}_{\setminus}(\rho), \quad (\text{C.3c})$$

where $\tilde{\Lambda}_{\setminus}(\rho) := \frac{1}{d-1} \sum_{k=1}^{d-1} Z^k \rho Z^{-k}$ is a valid quantum channel.

Lemma C.1 states that, for any bipartite ordered process W , the process $\tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W)$ is direct-cause, thus being CCDC. Hence, by making use of Eq. (C.3a), we see that the resulting process

$$\frac{1}{d_{A_I}} W + \left(1 - \frac{1}{d_{A_I}}\right) \tilde{\Lambda}_{\setminus}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W) \quad (\text{C.4})$$

is CCDC, with $\tilde{\Lambda}_{\setminus}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W)$ being a valid bipartite ordered process.

By analysing the definition of generalized robustness presented in Eq. (3.15), we see that setting $\Omega = \tilde{\Lambda}_{\setminus}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W)$ ensures that the relation

$$R_G(W) \leq 1 - \frac{1}{d_{A_I}} \quad (\text{C.5})$$

holds for any bipartite ordered process W . □

APPENDIX D – An upper bound for the white noise robustness

In this section, we present an upper bound for the white noise robustness. Differently from the generalized robustness case, this bound is not tight, but is useful for proving the strong duality relation for the problem of evaluating the white noise robustness for non-classical CCDC processes in Section E.

Theorem D.1. *The white noise robustness of a bipartite ordered process $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ is upper-bounded by $R_{\text{WN}}(W) \leq 1 - \frac{1}{d_{A_I} d_{A_O} d_{B_I} + 1}$.*

Proof. The depolarizing channel $\tilde{D}_\eta(\rho) := (1 - \eta)\rho + \eta \frac{\mathbb{1}}{d}$ is known to be entanglement breaking when $\eta \geq \frac{d}{d+1}$ [56, 57]. Hence, Lemma C.1 ensures that, for any bipartite ordered process $W \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$, the process

$$\tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W) = \frac{1}{d_{A_I} + 1} W + \frac{d_{A_I}}{d_{A_I} + 1} \left(\frac{\mathbb{1}^{A_I}}{d_{A_I}} \otimes \text{tr}_{A_I}(W) \right) \quad (\text{D.1})$$

is a direct-cause process. We now define the operator

$$W' := \frac{\mathbb{1}^{A_I}}{d_{A_I}} \otimes \frac{(d_{A_O} \mathbb{1}^{A_O B_I} - \text{tr}_{A_I}(W)^{A_O B_I})}{d_{A_O} d_{B_I} - 1}, \quad (\text{D.2})$$

which is a valid direct-cause process by direct inspection. Taking a convex combination of $\tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W)$ and W' , we obtain,

$$\begin{aligned} q \tilde{\Lambda}^{A_I} \otimes \tilde{\mathbb{I}}^{A_O B_I}(W) + (1 - q) W' &= q \left(\frac{1}{d_{A_I} + 1} W + \frac{d_{A_I}}{d_{A_I} + 1} \left(\frac{\mathbb{1}^{A_I}}{d_{A_I}} \otimes \text{tr}_{A_I}(W) \right) \right) \\ &\quad + (1 - q) \frac{\mathbb{1}^{A_I}}{d_{A_I}} \otimes \frac{(d_{A_O} \mathbb{1}^{A_O B_I} - \text{tr}_{A_I}(W)^{A_O B_I})}{d_{A_O} d_{B_I} - 1} \\ &= \frac{q}{d_{A_I} + 1} W + \left(\frac{(1 - q) d_{A_O} d_{B_I}}{d_{A_O} d_{B_I} - 1} \right) \frac{\mathbb{1}^{A_I}}{d_{A_I}} \otimes \frac{\mathbb{1}^{A_O B_I}}{d_{B_I}} \\ &\quad + \left(\frac{q d_{A_I}}{d_{A_I} + 1} - \frac{(1 - q)}{(d_{A_O} d_{B_I} - 1)} \right) \frac{\mathbb{1}^{A_I}}{d_{A_I}} \otimes \text{tr}_{A_I}(W), \end{aligned} \quad (\text{D.3})$$

which is direct-cause by construction. Now, by setting $q = \frac{d_{A_I} + 1}{d_{A_I} d_{A_O} d_{B_I} + 1}$, we obtain

$$\frac{q d_{A_I}}{d_{A_I} + 1} - \frac{(1 - q)}{(d_{A_O} d_{B_I} - 1)} = 0, \quad (\text{D.4})$$

and the process

$$\frac{1}{d_{A_I} d_{A_O} d_{B_I} + 1} W + \left(1 - \frac{1}{d_{A_I} d_{A_O} d_{B_I} + 1}\right) \frac{1}{d_{A_I} d_{B_I}} \quad (\text{D.5})$$

is guaranteed to be direct-cause. □

APPENDIX E – Strong duality for CCDC robustness problems

Theorem E.1. *The convex optimization problems for non-classical CCDC generalized robustness (Eq. (3.15)) and non-classical CCDC white noise robustness (Eq. (3.16)) satisfy strong duality.*

Proof. We recall that every convex optimization problem admitting a strictly feasible solution *i.e.*, all equality constraints are satisfied, and all inequality constraints are strictly satisfied, necessarily satisfies strong duality (Slater condition [28]).

From Theorem D.1 we see that for any value $1 - \frac{1}{d_{A_I} d_{A_O} d_{B_I} + 1} < r < 1$, for any process W , the process $\Omega := (1 - r)W + r \frac{\mathbb{1}}{d_{A_I} d_{B_I}}$ is a strictly feasible solution, ensuring that both robustness problems respect strong duality. \square

APPENDIX F – Heuristic algorithm that seeks for the maximum robustness on a given scenario

Given the dimensions $d_{A_I}, d_{A_O}, d_{B_I}$, what is the maximum robustness values $R_G(W)$ or $R_{WN}(W)$ that a process $W \in \mathcal{L}_{A \rightarrow B}$ can obtain? Inspired by the see-saw techniques of Refs. [27, 55], we now present a heuristic iterative method to seek for the highest robustness values for a given scenario. This method works either for the generalized or white noise robustness, which is why in the following we do not make explicit which robustness quantifier to work with.

Our heuristic algorithm works as follows. First, we sample a bipartite ordered process W_1 using one of the methods we describe in Appendix G. Then, we perform the dual robustness problem for W_1 and obtain its optimal non-classical CCDC witness S_1 , which gives $R(W_1) = -\text{tr}(S_1 W_1)$. Next, we find a process W_2 which maximally violates this first witness S_1 , problem that can be solved by the following SDP:

$$\min \quad \text{tr}(S_1 W_2) \quad (\text{F.1a})$$

$$\text{s.t.} \quad W_2 \succeq 0, \quad (\text{F.1b})$$

$$W_2 = L_{A \rightarrow B}(W_2), \quad (\text{F.1c})$$

$$\text{tr}(W_2) = d_{A_O}. \quad (\text{F.1d})$$

Now we repeat the previous steps, that is, we evaluate the dual robustness program for W_2 , and its optimal non-classical CCDC witness S_2 , then find the process W_3 which maximally violates S_2 . These steps are taken iteratively until some stopping criterion is satisfied. In our code, the stopping criterion used is $R(W_{i+1}) - R(W_i) \leq \epsilon = 0.0001$. In the end of this procedure we obtain a process W which attains a non-classical CCDC robustness, providing a lower bound on the maximal value for the given scenario.

In order to increase the confidence of this heuristic method, we perform this algorithm several times with multiple initial processes W_1 , randomly sampled in different ways. We have implemented this heuristic methods for three different scenarios: $d_{A_I}=d_{A_O}=d_{B_I}=2$, $d_{A_I}=d_{A_O}=2, d_{B_I}=4$, and $d_{A_I}=d_{A_O}=3, d_{B_I}=9$. For these cases, our see-saw algorithm led to the same value of robustness for several different random initial processes, suggesting that the heuristic method may have attained the global maximum.

APPENDIX G – Sampling random ordered process

In this section, we describe the methods for sampling random processes used in this work. We remark that Ref. [63] presents a method for generating uniformly distributed random process matrices which are different from the ones considered here.

Method 1

This method is similar to the technique for generating random quantum channels used in Ref. [64].

1. Sort a random density operator $\rho \in \mathcal{L}(A_I \otimes A_O \otimes B_I)$ with the Hilbert-Schmidt measure, *i.e.*, sort a random pure quantum state with the Haar measure $|\psi\rangle\langle\psi| \in \mathcal{L}(A_I \otimes A_O \otimes B_I \otimes \text{aux})$ with $d_{\text{aux}} = d_{A_I}d_{A_O}d_{B_I}$, then trace out the auxiliary space;
2. Project ρ onto the subspace of bipartite ordered process to obtain $\bar{W} = L_{A \rightarrow B}(\rho)$;
3. Evaluate the minimum eigenvalue λ_{\min} of \bar{W} and output the bipartite ordered process:

$$W = d_{A_O} \cdot \frac{\bar{W} - \lambda_{\min} \mathbb{1}^{d_{A_I}d_{A_O}d_{B_I}}}{\text{tr}(\bar{W} - \lambda_{\min} \mathbb{1}^{d_{A_I}d_{A_O}d_{B_I}})}, \quad (\text{G.1})$$

which is positive semidefinite by construction.

Method 2

This method is good for generating random processes with high values of generalized and white noise robustnesses. However, it only works when $\frac{d_{A_I}d_{A_O}}{d_{B_I}}$ is an integer;

1. Set $d_{\text{aux}} = d_{A_I}$ and sort a random pure state $|\psi\rangle\langle\psi|^{A_I \text{aux}} \in \mathcal{L}(A_I \otimes \text{aux})$ according to the Haar measure.
2. Set $d_{\text{aux}'} = \frac{d_{A_O}d_{\text{aux}}}{d_{B_I}}$ and sort a random unitary operator $U : A_O \otimes \text{aux} \rightarrow B_I \otimes \text{aux}'$ according to the Haar measure. Then, obtain its Choi operator $|\mathbb{U}\rangle\rangle\langle\langle\mathbb{U}|^{A_O \text{aux}/B_I \text{aux}'}$ and define the channel $D^{A_O \text{aux}/B_I} := \text{tr}_{\text{aux}'}(|\mathbb{U}\rangle\rangle\langle\langle\mathbb{U}|^{A_O \text{aux}/B_I \text{aux}'})$;
3. Output the operator:

$$W = \rho^{A_I \text{aux}} * D^{A_O \text{aux}/B_I}, \quad (\text{G.2})$$

which is a valid bipartite ordered process.

Method 3

1. Set $d_{\text{aux}} = d_{A_I}$ and sort a random pure state $|\psi\rangle\langle\psi|^{A_I\text{aux}}$ according to the Haar measure;
2. Sort a random density operator $\rho \in \mathcal{L}(A_O \otimes \text{aux} \otimes B_I)$ according to the Hilbert-Schmidt measure;
3. Define $D^{A_O\text{aux}/B_I} = \left(\sigma^{-\frac{1}{2}} \otimes \mathbb{1}^{B_I}\right) \rho \left(\sigma^{-\frac{1}{2}} \otimes \mathbb{1}^{B_I}\right)$, with $\sigma = \text{tr}_{B_I}(\rho)$. The resulting operator $D^{A_O\text{aux}/B_I}$ is the Choi operator of a channel $\tilde{D} : \mathcal{L}(A_I \otimes \text{aux}) \rightarrow \mathcal{L}(B_I)$, as $D^{A_O\text{aux}/B_I} \succeq 0$ and $\text{tr}_{B_I}(D^{A_O\text{aux}/B_I}) = \mathbb{1}^{A_O\text{aux}}$ by direct inspection;
4. Output the operator

$$W = |\psi\rangle\langle\psi|^{A_I\text{aux}} * D^{A_O\text{aux}/B_I}, \quad (\text{G.3})$$

which is a valid bipartite ordered process.

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