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Generic dynamics for evolution groups and C_0-semigroups on Hilbert spaces

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À memória de meu avô

Incipit vita nova

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Papers

[**A**] M. Aloisio, S. L. Carvalho, and C. R. de Oliveira, Quantum quasiballistic dynamics and thick point spectrum. To appear in Lett. Math. Phys, (2019).

[**B**] M. Aloisio, A note on spectrum and quantum dynamics. Submitted for publication, (2018).

[**C**] M. Aloisio, S. L. Carvalho, and C. R. de Oliveira, Refined scales of decaying rates of operator semigroups on Hilbert spaces: typical behaviour. Submitted for publication, (2018).

[**D**] M. Aloisio, S. L. Carvalho, and C. R. de Oliveira, Category theorems for Schrödinger semigroups. Submitted for publication, (2019).

Abstract

In this thesis, we study the typical behaviour (from the topological viewpoint) of the decaying rates of the orbits of unitary evolution groups and C_0-semigroups on Hilbert spaces. We have found that: (1) some dynamical quantities related to evolution groups have an oscillating behaviour between a polynomially rapid decay and an arbitrarily slow decay; (2) the decaying rates of each typical orbit, in Baire's sense, of C_0-semigroups which are stable but not exponentially stable depend on sequences of time going to infinity. The proofs are based on the relations between such decaying rates and some spectral properties of their respective generators.

Resumo

Nesta tese, estudamos o comportamento típico (do ponto de vista topológico) das taxas de decaimento das órbitas de grupos unitários de evolução e C_0-semigrupos em espaços de Hilbert. Encontramos que: (1) algumas quantidades dinâmicas relacionadas aos grupos de evolução tem um comportamento oscilando entre um decaimento polinomialmente rápido e um decaimento arbitrariamente lento; (2) as taxas de decaimento de cada órbita típica, no sentido de Baire, de C_0-semigrupos que são estáveis mas não exponencialmente estáveis dependem de sequências do tempo que vão para infinito. As provas são baseadas nas relações existentes entre essas taxas de decaimento e propriedades espectrais dos respectivos geradores.

Contents

Introduction

This thesis is divided the following parts:

Part I

In Part I, based on works by Simon [56] and by Carvalho and de Oliveira [15, 16], we study the behaviour of the decaying rates (from the topological viewpoint) of dynamical quantities related to the wave packet solutions of the Schrödinger equation ; in particular, we show that, in the same vein of Simon's Wonderland Theorem [56], Baire generically, the rates for which the solutions of the Schrödinger equation escape, in time average, from each finite-dimensional subspace depend on sequences of time going to infinity. In this part of the thesis, we also discuss a result about dynamical lower bounds and dense point spectrum, of independent interest.

Part II

In Part II, we propose a new (and original) approach to the problem of obtaining lower bounds for the decaying rates of C_0-semigroups on Hilbert spaces, and then show that the decaying rates of the orbits of C_0-semigroups which are stable but not exponentially stable, typically in Baire's sense, depend on sequences of time going to infinity. Namely, in order to obtain lower bounds for the decaying rates of C_0-semigroups, many authors (see [8, 9, 11, 50] and references therein) usually have related estimates on the norm of the resolvent of the generator to quantitative decaying rates of the form

$$\|T(t)A^{-1}\|_{\mathcal{B}(\mathcal{H})} = O(r(t)), \quad t \to \infty,$$

with $\lim_{t\to\infty} r(t) = 0$, which implies that all classical solutions of the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t), \ t \geq 0, \\ x(0) = x, \ x \in \mathcal{H}, \end{cases} \tag{ACP}$$

converge uniformly (on the unit ball of $\mathcal{D}(A)$ endowed with the graph norm) to zero at infinity with rate r. Since $\mathcal{D}(A) \subset \mathcal{H}$ is dense, one could argue that such solutions display typical behaviour. In the present thesis, we consider a different notion of typical behaviour, in terms of dense G_δ subsets of initial values $x \in \mathcal{H}$. In this setting, we show that there exist dense G_δ sets of initial values $x \in \mathcal{H}$ such that the orbit $(T(t)x)_{t\geq 0}$ contains a sequence that decays to zero no faster than a fixed but arbitrarily slow rate, and a sequence that decays to zero at a fixed rate arbitrarily close to r. In this sense, we show that typical orbits display unexpected and erratic behaviour.

Appendices

In the appendices we recall some important concepts and results on spectral theory, theory of unitary evolution groups and C_0-semigroups used in this work.

Part I Some results on quantum dynamics

Selected Notation

$\mathcal{H}$	Separable complex Hilbert space
T	Self-adjoint operator in $\mathcal{H}$
$\varrho(T)$	Resolvent set of T
$R(\lambda, T)$	Resolvent operator of T at $\lambda \in \varrho(T) \subset \mathbb{C}$
$\sigma(T)$	Spectrum of T
E^T	Resolution of the identity of T
μ	Finite positive Borel measure on $\mathbb{R}$
μ_ξ^T	Spectral measure of T with respect to $\xi \in \mathcal{H}$
$B(x, \epsilon)$	Open interval $(x - \epsilon, x + \epsilon)$ centered at $x \in \mathbb{R}$

Contextualization and main results

Contextualization

There is a vast literature concerning the large time asymptotic behaviour of the solutions to the Schrödinger equation

$$\begin{cases} \partial_t \xi = -iT\xi, \ t \in \mathbb{R}, \\ \xi(0) = \xi, \ \xi \in \mathcal{H}, \end{cases} \tag{SE}$$

where T is a self-adjoint operator in a separable complex Hilbert space $\mathcal{H}$. Namely, the relations between the quantum dynamics of solutions of (SE) and the spectral properties of T are a classical subject of the mathematics and physics literatures. In this context, we refer to [5, 12, 14, 15, 16, 17, 23, 20, 29, 31, 32, 55, 56, 57, 59], among others.

We recall that, for each $\xi \in \mathcal{H}$, the unitary evolution group $\mathbb{R} \ni t \mapsto e^{-itT}$ is so that the curve $e^{-itT}\xi$, in some sense (see Remark B.1), solves (SE). The state ξ, in the context of quantum mechanics, is called wave packet and describes the "non-relativistic quantum state" of a one-particle system. Next, we list some quantities usually considered to probe the large time behaviour of the dynamics $e^{-itT}\xi$.

1. The (time-average) quantum return probability, which gives the (time-average) probability of finding the particle at time $t > 0$ in its initial state ξ, is defined as

$$\langle p_\xi \rangle(t) := \frac{1}{t} \int_0^t |\langle \xi, e^{-isT}\xi \rangle|^2 \, \mathrm{d}s. \tag{a}$$

2. Let A be a positive operator such that, for each $t \in \mathbb{R}$, $e^{-itT}\mathcal{D}(A) \subset \mathcal{D}(A)$. For each $\xi \in \mathcal{D}(A)$, the (time-average) expectation value of A in the state ξ at time $t > 0$ is defined as

$$\langle A_\xi^T \rangle_t := \frac{1}{t} \int_0^t \langle e^{-isT}\xi, Ae^{-isT}\xi \rangle \, \mathrm{d}s. \tag{b}$$

3. Let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$. The (time-average) q-moment, $q > 0$, of the position operator at time $t > 0$, with initial condition ξ, is defined as

$$\langle\langle |X|^q \rangle\rangle_{t,\xi} := \frac{1}{t} \int_0^t \sum_n |n|^q |\langle e^{-isT}\xi, e_n\rangle|^2 \, \mathrm{d}s. \tag{c}$$

Each one of the quantities defined in (c) is a special case of (b), where for each $q > 0$, A represents the q-moment of the position operator:

$$|X|^q \equiv \sum_n |n|^q \langle e_n, \cdot\rangle e_n.$$

These quantities describe the (time-average) behaviour of the "basis position" of the wave packet $e^{-itT}\xi$, as t goes to infinity (see [5, 20, 29, 31, 32, 38] and references therein). Actually, in the specific case (which is more relevant from a physical point of view) where $\mathcal{H} = \ell^2(\mathbb{Z}^\nu)$, $\nu \in \mathbb{N}$, and $\{e_n\}$ is the canonical basis $\{\delta_n\}$, such quantities characterize the spreading of the wave packet $e^{-isT}\xi$.

Next, we present well known results that relate such quantities to some properties of the spectral measure μ_ξ^T of T associated with ξ (see Definition A.3). Firstly, we refer to Wiener's Lemma [22].

Theorem (Wiener's Lemma)**.** *Let $\xi \in \mathcal{H}$. Then,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\langle \xi, e^{-isT}\xi\rangle|^2 \, \mathrm{d}s = \sum_{\lambda\in\mathbb{R}} |\mu_\xi^T(\{\lambda\})|^2.$$

Now we refer to the notorious RAGE's Theorem, named after Ruelle, Amrein, Georgescu, and Enss [22].

Theorem (RAGE's Theorem)**.** *Let A be a compact operator on $\mathcal{H}$. Then, for every $\xi \in \mathcal{H}$,*

$$\lim_{t\to\infty} \langle | A_\xi^T | \rangle_t = 0$$

if and only if μ_ξ^T is purely continuous.

Taking into account RAGE's Theorem, special cases of interest are projectors onto finite-dimensional subspaces of $\mathcal{H}$. Namely, let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$ and let P_N be the (compact) projection on a sphere of radius $N \in \mathbb{N}$, that is,

$$P_N \equiv \sum_{|n|\le N} \langle e_n, \cdot\rangle e_n.$$

We note that

$$\begin{aligned}\langle e^{-isT}\xi, P_N e^{-isT}\xi\rangle &= \langle e^{-isT}\xi, \sum_{|n|\leq N}\langle e_n, e^{-isT}\xi\rangle e_n\rangle \\ &= \sum_{|n|\leq N}|\langle e^{-isT}\xi, e_n\rangle|^2.\end{aligned}$$

Thus, if μ_ξ^T is purely continuous, that is, if μ_ξ^T has no atoms (for each $\lambda\in\mathbb{R}$, $\mu_\xi^T(\{\lambda\})=0$), then, by RAGE's Theorem,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sum_{|n|\leq N}|\langle e^{-isT}\xi, e_n\rangle|^2\,\mathrm{d}s = 0.$$

Since N is arbitrary and the dynamics $e^{-isT}\xi$ is unitary, one has the following result [38].

Corollary (RAGE's Corollary)**.** *If μ_ξ^T is purely continuous, then, for every $q>0$,*

$$\lim_{t\to\infty}\langle\langle|X|^q\rangle\rangle_{t,\xi} = \infty.$$

Remark.

1. We note that, by Wiener's Lemma, the (average) probability of finding the particle at time $t>0$ in its initial state ξ is asymptotically null if and only if μ_ξ^T is purely continuous.

2. Since any projector onto a finite-dimensional subspace of $\mathcal{H}$ satisfies the hypotheses of RAGE's Theorem, initial states whose spectral measures are purely continuous can be interpreted as those whose trajectories that escape, in time average, from every finite-dimensional subspace. Actually, by RAGE's Corollary, in this case, it can be said that there is a spreading of the wave packet.

Now we recall some basic definitions.

Definition. A sequence of bounded linear operators (T_n) strongly converges to T in $\mathcal{H}$ if, for every $\xi\in\mathcal{H}$, $T_n\xi\longrightarrow T\xi$ in $\mathcal{H}$.

We recall that the resolvent set of T, $\varrho(T)$, is the set of all $\lambda\in\mathbb{C}$ for which the resolvent operator of T at λ,

$$R(\lambda,T):\mathcal{H}\longrightarrow\mathcal{D}(T),\ \ R(\lambda,T):=(\lambda I-T)^{-1},$$

exists and is bounded. The spectrum of T is the set $\sigma(T)=\mathbb{C}\backslash\rho(T)$.

Definition. Let T be a self-adjoint operator and let (T_n) be a sequence of self-adjoint operators. One says that T_n converges to T in the strong resolvent sense if $R(i, T_n)$ strongly converges to $R(i, T)$.

We also recall that T is said to have purely continuous spectrum if, for every $\xi \in \mathcal{H}$, μ_ξ^T is purely continuous.

A complete metric space (X, d) of self-adjoint operators, acting in $\mathcal{H}$, is said to be regular if convergence with respect to d implies strong resolvent convergence of operators. One of the results stated in [56], the so-called Wonderland Theorem, says that, for some regular spaces X, $\{T \in X \mid T$ has purely continuous spectrum$\}$ is a dense G_δ set in X. Hence, for these spaces, by Wiener's Lemma and RAGE's Theorem, for each compact operator A and each $0 \neq \xi \in \mathcal{H}$,

$$\{T \in X \mid \lim_{t\to\infty} \langle p_\xi \rangle(t) = \lim_{t\to\infty} \langle |A_\xi^T| \rangle_t = 0 \text{ and } \lim_{t\to\infty} \langle \langle |X|^q \rangle \rangle_{t,\xi} = \infty \text{ for each } q > 0\}$$

contains a dense G_δ set in X. In this context, it is quite natural to study the behaviour of the decaying rates (from the topological viewpoint) of these dynamical quantities. Consider the following classes of self-adjoint operators.

Jacobi matrices. For every fixed $a > 0$, consider the family of Jacobi matrices, M, given on $\ell^2(\mathbb{Z})$ by the action

$$(Mu)_j := u_{j-1} + u_{j+1} + v_j u_j,$$

where (v_j) is a sequence in $\ell^\infty(\mathbb{Z})$, such that, for each $j \in \mathbb{Z}$, $|v_j| \leq a$. Denote by X_a the set of these matrices endowed with the topology of pointwise convergence on (v_j). Then, X_a is (by Tychonoff's Theorem) a compact metric space such that convergence in metric implies strong resolvent convergence. Actually, $M_k \longrightarrow M$ in X_a if and only if, for each $j \in \mathbb{Z}$, $\lim_{k\to\infty} v_j^k = v_j$ and, therefore, if and only if M_k converges strongly to M.

Schrödinger operators. Fix $C > 0$ and consider the family of Schrödinger operators, H_V, defined in the Sobolev space $\mathcal{H}^2(\mathbb{R})$ by the action

$$(H_V u)(x) := -\triangle u(x) + V(x)u(x),$$

with $V \in \mathcal{B}^\infty(\mathbb{R})$ (the space of bounded Borel functions) so that, for every $x \in \mathbb{R}$, $|V(x)| \leq C$. Denote by X_C the set of these operators endowed with the topology of pointwise convergence on $(V(x))$. Then, X_C is (again by Tychonoff's Theorem) a compact metric space so that convergence in metric implies strong resolvent convergence. Namely, if $H_{V_k} \to H_V$ in X_C, then, for each $x \in \mathbb{R}$, one has that $\lim_{k\to\infty} V_k(x) = V(x)$. Thus, for each $u \in \mathrm{L}^2(\mathbb{R})$, by the second resolvent identity and dominated convergence,

$$\begin{aligned} \|(R_i(H_{V_k}) - R_i(H_V))u\|_{\mathrm{L}^2(\mathbb{R})} &= \|R_i(H_{V_k})(V_k - V)R_i(H_V)u\|_{\mathrm{L}^2(\mathbb{R})} \\ &\leq \|(V_k - V)R_i(H_V)u\|_{\mathrm{L}^2(\mathbb{R})} \longrightarrow 0 \end{aligned}$$

as $k \to \infty$.

In this part of the work, we study the decaying rates of $\langle |A_\xi^T|\rangle_t$ for these two different classes of self-adjoint operators (see [15, 16, 56] for another classes). Namely, we use RAGE's Theorem and an argument involving separability to obtain some results about the typical behaviour of such decaying rates (Theorems II and III). In this part, we also prove a result about dynamical lower bounds and dense point spectrum (Theorem I), of independent interest.

Brief discussion of our main results

Carvalho and de Oliveira showed in [15] that for several classes of discrete Schrödinger operators (from the topological viewpoint), $\langle p_\xi\rangle(t)$ has an oscillating behaviour between a (maximum) polynomial rapid decay and a (minimum) polynomial slow decay. In [16], they have confirmed this polynomial dual behaviour for $\langle\langle |X|^q\rangle\rangle_{t,\xi}$. In this setting, here we discuss some results about the decaying rates of $\langle |A_\xi^T|\rangle_t$ for X_a and X_C (Theorems II and III). For X_a, we also say something about $\langle\langle |X|^q\rangle\rangle_{t,\xi}$ (Theorem I).

Let T be a self-adjoint operator in $\mathcal{H}$. We recall that T has dense point spectrum if the set of eigenvalues of T is dense in $\sigma(T)$.

Taking into account some ideas of Simon [56] (which were also explored by Carvalho and de Oliveira in [15, 16]), it is natural, in X_a, to consider the density of the set of Jacobi matrices with dense point spectrum. It is well known that this dense subset can be obtained by using Anderson's localization. Namely, for every fixed $a > 0$, let $\Omega = [-a, a]^{\mathbb{Z}}$ be endowed with the product topology and with the respective Borel σ-algebra. Assume that $(\omega_j)_{j\in\mathbb{Z}} = \omega \in \Omega$ is a set of independent, identically distributed real-valued random variables with a common probability measure ρ not concentrated on a single point and such that $\int |\omega_j|^\theta \mathrm{d}\rho(w_j) < \infty$, for some $\theta > 0$. Denote by $\nu := \rho^{\mathbb{Z}}$ the probability measure on Ω. The Anderson model is a random Hamiltonian on $\ell^2(\mathbb{Z})$, defined for each $\omega \in \Omega$ by

$$(h_\omega u)_j := u_{j-1} + u_{j+1} + \omega_j u_j.$$

It turns out that [20, 59]

$$\sigma(h_\omega) = [-2, 2] + \mathrm{supp}(\rho),$$

and ν-a.s. ω, h_ω has pure point spectrum [14, 61]. Thus, if μ denotes the product of infinite copies of the normalized Lebesgue measure on $[-a, a]$, that is, $(2a)^{-1}\ell$, then

$$D = \{M \in X_a \mid \sigma(M) = [-a-2, a+2],\ \sigma(M) \text{ is pure point}\}$$

is so that $\mu(X_a \backslash D) = 0$ and therefore, D is a dense subset of X_a.

Now we recall that to describe the algebraic growth $\langle\langle |X|^q \rangle\rangle_{t,\xi} \sim t^{\alpha(q)}$ for large t, one usually considers the lower and upper transport exponents, respectively, given by

$$\alpha^-(\xi, q) := \liminf_{t\to\infty} \frac{\ln\langle\langle |X|^q \rangle\rangle_{t,\xi}}{\ln t},$$

$$\alpha^+(\xi, q) := \limsup_{t\to\infty} \frac{\ln\langle\langle |X|^q \rangle\rangle_{t,\xi}}{\ln t}.$$

There are some relations between such exponents and the behaviour (as function of the time) of the spreading of the wave packet. Namely, if there exists a $q > 0$ such that $\alpha^+(\xi, q) > 0$, then there is no dynamical localization (the system (T, ξ) is said to be dynamical localized if, for every $q > 0$ and every $\gamma > 0$, $\lim_{t\to\infty} t^{-\gamma}\langle\langle |X|^q \rangle\rangle_{t,\xi} = 0$). In this case, one says that there is transport, since for at least one temporal sequence, part of the wave packet is not contained in a bounded region of the space. For more details about the wave packet spreading phenomenon, see [5]; for a discussion about various types of localization, see [23, 30].

We note that if $\alpha^-(\xi, q) = q$ for $q > 0$, then one says that the transport is ballistic, since the"time law" that describes the behavior of the wave packet refers to the uniform rectilinear movement. If $\alpha^+(\xi, q) = q$ for $q > 0$, then one says that the transport is quasiballistic. In this context, our first result says that for every $T \in D \subset X_a$, the dynamics of every initial condition in a robust set have quasiballistic behaviour.

Theorem I. *Let $-\infty < a < b < \infty$, and let T be a self-adjoint operator with purely dense point spectrum equal to $[a, b]$. So, there exists a dense G_δ set G^T in $\mathcal{H}$ such that, for each $\xi \in G^T$, if $\langle\langle |X|^q \rangle\rangle_{t,\xi}$ is well defined (finite) for each $t, q > 0$, then*

$$\alpha^+(\xi, q) \geq q \ \textit{ for each } q > 0.$$

Remark.

1. There are rather general sufficient conditions for which $\langle\langle |X|^q \rangle\rangle_{t,\xi}$ is well defined [29]. Namely, let T be a bounded self-adjoint operator on $\mathcal{H}$ and let $B = \{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$. Then,

   1.1. $\langle\langle |X|^q \rangle\rangle_{t,e_0}$ is well defined (finite) for all $t, q > 0$;

1.2. $\alpha^+(e_0, q)$ are increasing functions of q;

1.3. $\alpha^+(e_0, q) \in [0, q]$, for all $q > 0$.

2. Theorem I gives a rather general sufficient condition for an operator with pure point spectrum to present non-trivial dynamical lower bounds. The main ingredient in the proof of this result involves a fine analysis of the generalized fractal dimensions (see Definition 1.2) of spectral measures of operators with pure point spectrum. Namely, in order to prove Theorem I, we explore relations between such dimensions and the spacing properties of its eigenvalues (see Theorem 1.1), and then apply a result due to Barbaroux et. al. [5] (see Theorem 1.2).

3. There are in the literature numerous other examples of operators satisfying the hypotheses of Theorem I (see [14, 20, 23, 25, 55, 57, 61]), showing that this is a result of independent interest.

Our next results are about the behaviour of the decaying rates of $\langle |A_\xi^M| \rangle_t$.

Theorem II. *Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be such that*

$$\limsup_{t\to\infty} \alpha(t) = \infty.$$

Then, for each compact operator A and each $0 \neq \xi \in \mathcal{H}$,

$$\{M \in X_a \mid M \text{ has purely continuous spectrum}, \limsup_{t\to\infty} \alpha(t)\langle |A_\xi^M| \rangle_t = \infty \text{ and } \liminf_{t\to\infty} t\langle |A_\xi^M| \rangle_t = 0\}$$

is a dense G_δ set in X_a.

Theorem III. *Let $C > 0$ and let α be as in the statement of Theorem II. Then, for every compact operator A, there exists a dense G_δ set $G_\alpha(A)$ in $\mathrm{L}^2(\mathbb{R})$ such that, for every $\xi \in G_\alpha(A)$,*

$$\{H \in X_C \mid H \text{ has purely continuous spectrum on } (0,\infty), \limsup_{t\to\infty} \alpha(t)\langle |A_\xi^H| \rangle_t = \infty\}$$

is a dense G_δ set in X_C.

Remark.

1. It is possible to check that the behaviour of lim inf in the statement of Theorem II follows from Theorem 1.2 in [15]. Thus, our main contribution here refers to the behaviour of lim sup. In order to prove the behaviour of lim sup, we use the density of the set of Jacobi matrices in X_a with dense point spectrum, combined with RAGE's Theorem.

2. Theorem III is a partial version of Theorem II to the class of (unbounded) Schrödinger operators X_C. In order to prove Theorem III, we use RAGE's Theorem, a theory of existence of negative eigenvalues for Schrödinger operators [22, 53], and an argument involving separability.

Organization of the text

In Chapter 1, we discuss in details the proof of a result, of independent interest, about dynamical lower bounds and dense point spectrum (Theorem I).

In Chapter 2, we prove Theorems II and III.

Chapter 1

Dynamics and dense point spectrum

Our main goal in this chapter is to present a proof of Theorem I.

1.1 Weakly-spaced sequences

In order to properly present our proof of Theorem I, we need the following notion.

Definition 1.1. Let $(a_j) \subset \mathbb{R}$. One says that (a_j) is weakly-spaced if, for each $\alpha > 0$, there exists a subsequence (a_{j_l}) of (a_j) such that

1. $c_l := a_{j_l} - a_{j_{l+1}} > 0$ is monotone and $\lim_{l\to\infty}(a_{j_l} - a_{j_{l+1}}) = 0$.
2. There exists $C_\alpha > 0$ so that, for every $l \geq 1$, $a_{j_l} - a_{j_{l+1}} \geq C_\alpha / l^{1+\alpha}$.

Proposition 1.1. *Let $-\infty < a < b < \infty$. If $\cup_j\{a_j\}$ is a dense subset of $[a, b]$, then (a_j) is weakly-spaced.*

Proof. Let $\alpha > 0$. Firstly, we note that, for each $x > 1$,

$$\left(\frac{x}{x-1}\right)^\alpha + \left(\frac{x}{x+1}\right)^\alpha > 2. \tag{1.1}$$

Namely, set

$$f(\alpha) := \left(\frac{x}{x-1}\right)^\alpha + \left(\frac{x}{x+1}\right)^\alpha.$$

So,

$$\begin{aligned}\left(\frac{x-1}{x}\right)^\alpha f'(\alpha) &= \ln\left(\frac{x}{x-1}\right) - \left(\frac{x-1}{x+1}\right)^\alpha \ln\left(\frac{x+1}{x}\right) \\ &> \ln\left(\frac{x}{x-1}\right)\left(1 - \left(\frac{x-1}{x+1}\right)^\alpha\right) > 0.\end{aligned}$$

Since $f(0) = 2$, the inequality in (1.1) follows.

For each $l \geq 1$, set

$$b_l := a + \frac{1}{l^\alpha};$$

by (1.1), for $l \geq 2$ one has $K_l := b_{l-1} - 2b_l + b_{l+1} > 0$. Note that

$$\lim_{l\to\infty} l^{1+\alpha}(b_l - b_{l+1}) = \alpha. \tag{1.2}$$

Now, for l sufficiently large such that $b_l \in [a, b)$, pick a_{j_l} satisfying

$$0 \leq a_{j_l} - b_l \leq \min\left\{\frac{K_l}{2}, \frac{\alpha}{4l^{1+\alpha}}\right\}. \tag{1.3}$$

Then, by (1.2) and (1.3), for l sufficiently large, one has

$$\begin{aligned} a_{j_l} - a_{j_{l+1}} &= (a_{j_l} - b_l) - (a_{j_{l+1}} - b_{l+1}) + (b_l - b_{l+1}) \\ &\geq -\frac{\alpha}{4(l+1)^{1+\alpha}} + \frac{3\alpha}{4l^{1+\alpha}} \geq \frac{\alpha}{2l^{1+\alpha}}, \end{aligned}$$

$$\begin{aligned} a_{j_l} - a_{j_{l+1}} &= (a_{j_l} - b_l) - (a_{j_{l+1}} - b_{l+1}) + (b_l - b_{l+1}) \\ &\leq \frac{\alpha}{4l^{1+\alpha}} + \frac{7\alpha}{4l^{1+\alpha}} = \frac{2\alpha}{l^{1+\alpha}}. \end{aligned}$$

Hence,

$$\frac{\alpha}{2l^{1+\alpha}} \leq a_{j_l} - a_{j_{l+1}} \leq \frac{2\alpha}{l^{1+\alpha}}.$$

Moreover,

$$\begin{aligned} (a_{j_l} - a_{j_{l+1}}) - (a_{j_{l+1}} - a_{j_{l+2}}) &= (a_{j_l} - 2a_{j_{l+1}} + a_{j_{l+2}}) \\ &= a_{j_l} - b_l - 2(a_{j_{l+1}} - b_{l+1}) + a_{j_{l+2}} - b_{l+2} + (b_l - 2b_{l+1} + b_{l+2}) \\ &\geq -2(a_{j_{l+1}} - b_{l+1}) + K_{l+1} \geq 0, \end{aligned}$$

which implies that $a_{j_l} - a_{j_{l+1}}$ goes to zero monotonically. Therefore, (a_j) is weakly-spaced. $\square$

1.2 Fractal dimensions and Proof of Theorem I

The study of fractal dimensions of spectral measures in the context of quantum mechanics appeared as an attempt to answer the following question: "What determines the spreading of a wave packet?" In this context, we highlight the works [4, 5, 31, 32]. Here, we use a notorious result due to Barbaroux et. al. [5] (Theorem 1.2) in order to prove Theorem I.

Definition 1.2. Let μ be a finite positive Borel measure on $\mathbb{R}$ and let $q \in \mathbb{R} \setminus \{1\}$. The lower and upper q-generalized fractal dimensions of μ are defined, respectively, as

$$D_\mu^-(q) := \liminf_{\epsilon\downarrow 0} \frac{\ln[\int \mu(B(x,\epsilon))^{q-1} \mathrm{d}\mu(x)]}{(q-1)\ln\epsilon} \quad \text{and} \quad D_\mu^+(q) := \limsup_{\epsilon\downarrow 0} \frac{\ln[\int \mu(B(x,\epsilon))^{q-1} \mathrm{d}\mu(x)]}{(q-1)\ln\epsilon},$$

where the integration is performed over $\mathrm{supp}(\mu)$.

These q-generalized fractal dimensions give the average lower and upper polynomial behaviour (in the measure itself) of the measure of balls (weighted by the exponent $(q-1)$) as their radii go to zero. Now we consider the mean-q dimension, which, for every $q > 0$, $q \neq 1$, coincide with q-generalized fractal dimensions.

Definition 1.3. Let μ be a finite positive Borel measure on $\mathbb{R}$ and let $q \in \mathbb{R} \setminus \{1\}$. The lower and upper mean-q dimensions of μ are defined, respectively, as

$$m_\mu^-(q) := \liminf_{\epsilon \downarrow 0} \frac{\ln[\epsilon^{-1} \int_{\mathbb{R}} \mu(B(x,\epsilon))^q \, \mathrm{d}x]}{(q-1)\ln \epsilon} \quad \text{and} \quad m_\mu^+(q) := \limsup_{\epsilon \downarrow 0} \frac{\ln[\epsilon^{-1} \int_{\mathbb{R}} \mu(B(x,\epsilon))^q \, \mathrm{d}x]}{(q-1)\ln \epsilon}.$$

The next result lists some properties of the above dimensions.

Proposition 1.2 (Theorem 2.1. and Propositions 3.1 and 3.3 in [6])**.** *Let μ be a finite positive Borel measure on $\mathbb{R}$. Then,*

1. *For every $q > 0$, $q \neq 1$, $D_\mu^\mp(q) = m_\mu^\mp(q)$.*
2. *$D_\mu^-(q)$ and $D_\mu^+(q)$ are nonincreasing functions of $q \in \mathbb{R} \setminus \{1\}$.*
3. *If μ has bounded support, then for all $q \in (0,1)$, $0 \leq D_\mu^-(q) \leq D_\mu^+(q) \leq 1$.*

For a more detailed discussion on such dimensions, see [6].

Our next result relates such spacing properties (Definition 1.1) of the eigenvalues of self-adjoint operators with purely point spectrum, to the generalized fractal dimensions of their spectral measures.

Theorem 1.1. *Let T be a self-adjoint operator with purely point spectrum. Suppose that the sequence of eigenvalues of T is weakly-spaced. Then,*

$$\{\xi \in \mathcal{H} \mid D^-_{\mu_\xi^T}(q) = 0 \ \textit{and} \ D^+_{\mu_\xi^T}(q) = 1 \, \textit{for each} \, 0 < q < 1\}$$

is a dense G_δ set in $\mathcal{H}$.

Consider the following result due to Barbaroux et. al. [5].

Theorem 1.2 (Theorem 2.1 in [5])**.** *Let T be a self-adjoint operator in $\mathcal{H}$. Then, for each $\xi \in \mathcal{H}$ and each $q > 0$,*

$$\alpha^+(\xi, q) \geq D^+_{\mu_\xi^T}\left(\frac{1}{1+q}\right) q.$$

Remark 1.1. We note that Theorem I is a consequence of Proposition 1.1, Theorems 1.1 and 1.2. Namely, since, in this case, T has purely dense point spectrum equal to $[a,b]$, if follows from Proposition 1.1 that the sequence of eigenvalues of T is weakly-spaced. Thus, Theorem I is a direct consequence of Theorems 1.1 and 1.2. Therefore, it remains to prove only Theorem 1.1.

In order to prove Theorem 1.1, we need of some preparation. Let $r > 0$ and let μ be a finite positive Borel measure on $\mathbb{R}$ so that $\mathrm{supp}(\mu) \subset [-r, r]$. Consider, for every $t > 0$ and every $q \in \mathbb{R}$,

$$C_\mu(q,t) := t \int_{-r-1}^{r+1} \left(\int_{\mathbb{R}} e^{-t|x-y|} \mathrm{d}\mu(y) \right)^q \mathrm{d}x.$$

Lemma 1.1. *Let μ be as before and $q > 0$, $q \neq 1$. Then,*

$$\liminf_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} = -D_\mu^+(q),$$

$$\limsup_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} = -D_\mu^-(q).$$

Proof. We show that

$$\liminf_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} = -m_\mu^+(q), \tag{1.4}$$

$$\limsup_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} = -m_\mu^-(q). \tag{1.5}$$

Since $\mathrm{supp}(\mu) \subset [-r, r]$, one has, for each $t > 1$ and each $x \in [-r-1, r+1]^c$, $\mu(B(x, \frac{1}{t})) = 0$. Hence, it follows that, for $t > 1$,

$$\begin{aligned} C_\mu(q,t) &= t \int_{-r-1}^{r+1} \left(\int_{\mathbb{R}} e^{-t|x-y|} \mathrm{d}\mu(y) \right)^q \mathrm{d}x \geq t \int_{-r-1}^{r+1} \left(\int_{|x-y|<\frac{1}{t}} e^{-t|x-y|} \mathrm{d}\mu(y) \right)^q \mathrm{d}x \\ &\geq \frac{t}{e^q} \int_{-r-1}^{r+1} \mu(B(x, \frac{1}{t}))^q \mathrm{d}x = \frac{t}{e^q} \int_{\mathbb{R}} \mu(B(x, \frac{1}{t}))^q \mathrm{d}x \end{aligned}$$

and, therefore,

$$\liminf_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} \leq -m_\mu^+(q), \qquad \limsup_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} \leq -m_\mu^-(q).$$

Let $0 < \delta < 1$. Then, for each $x \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}} e^{-t|x-y|} \mathrm{d}\mu(y) &= \int_{|x-y|<\frac{1}{t^{1-\delta}}} e^{-t|x-y|} \mathrm{d}\mu(y) + \int_{|x-y|\geq\frac{1}{t^{1-\delta}}} e^{-t|x-y|} \mathrm{d}\mu(y) \\ &\leq \mu\big(B\big(x, \frac{1}{t^{1-\delta}}\big)\big) + e^{-t^\delta} \mu(\mathbb{R}). \end{aligned}$$

Thus,

$$\begin{aligned} \left(\int_{\mathbb{R}} e^{-t|x-y|} \mathrm{d}\mu(y) \right)^q &\leq 2^q \max\left\{ \mu\big(B\big(x, \frac{1}{t^{1-\delta}}\big)\big), \mu(\mathbb{R}) e^{-t^\delta} \right\}^q \\ &\leq 2^q \mu\big(B\big(x, \frac{1}{t^{1-\delta}}\big)\big)^q + 2^q \mu(\mathbb{R})^q e^{-qt^\delta}. \end{aligned} \tag{1.6}$$

Since, by Proposition 1.2, $m_\mu^-(q) \geq 0$, it follows from (1.6) that, for sufficiently large t,

$$\begin{aligned} C_\mu(q,t) &\leq 2^q t \int_{\mathbb{R}} \mu\big(B\big(x, \frac{1}{t^{1-\delta}}\big)\big)^q \mathrm{d}x + (2r+2)2^q \mu(\mathbb{R})^q t e^{-qt^\delta} \\ &\leq 2^{q+1} t \int_{\mathbb{R}} \mu\big(B\big(x, \frac{1}{t^{1-\delta}}\big)\big)^q \, \mathrm{d}x, \end{aligned}$$

and then,

$$\frac{1}{(1-\delta)} \liminf_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} \geq -m_\mu^+(q),$$

$$\frac{1}{(1-\delta)} \limsup_{t\to\infty} \frac{\ln C_\mu(q,t)}{(q-1)\ln t} \geq -m_\mu^-(q).$$

Since $0 < \delta < 1$ is arbitrary, the complementary inequalities in (1.4) and (1.5) follow. The results are now a consequence of Proposition 1.2. □

Lemma 1.2. *Let T be a bounded self-adjoint operator on $\mathcal{H}$ and $q \in (0,1)$. Then, for each $\gamma \geq 0$,*

1. $G^T_{\gamma^-} := \{\xi \in \mathcal{H} \mid D^-_{\mu^T_\xi}(q) \leq \gamma\}$ *is a G_δ set in $\mathcal{H}$,*
2. $G^T_{\gamma^+} := \{\xi \in \mathcal{H} \mid D^+_{\mu^T_\xi}(q) \geq \gamma\}$ *is a G_δ set in $\mathcal{H}$.*

Proof. We just present the proof of item *1*. For each $j \geq 1$, let $g_j : (0,\infty) \to (0,\infty)$, $g_j(t) := t^{\frac{1}{j}+\gamma}$. Since, for each $j \geq 1$ and each $t > 0$, the mapping

$$\mathcal{H} \ni \xi \mapsto g_j(t) C_{\mu^T_\xi}(q,t)^{1/(q-1)}$$

is continuous (by dominated convergence), it follows that, for each $j,k,n \in \mathbb{N}$, the set

$$\bigcup_{t\geq k} \{\xi \in \mathcal{H} \mid g_j(t) C_{\mu^T_\xi}(q,t)^{1/(q-1)} > n\}$$

is open; thus, by Lemma 1.1,

$$\begin{aligned} G^T_{\gamma^-} &= \bigcap_{j\geq 1} \{\xi \in \mathcal{H} \mid \limsup_{t\to\infty} g_j(t) C_{\mu^T_\xi}(q,t)^{1/(q-1)} = \infty\} \\ &= \bigcap_{j\geq 1} \bigcap_{n\geq 1} \bigcap_{k\geq 1} \bigcup_{t\geq k} \{\xi \in \mathcal{H} \mid g_j(t) C_{\mu^T_\xi}(q,t)^{1/(q-1)} > n\} \end{aligned}$$

is a G_δ set in $\mathcal{H}$. □

Proof (Theorem 1.1). Fix $0 < q < 1$ and let (e_j) be an orthonormal family of eigenvectors of T, that is, $Te_j = \lambda_j e_j$ for every $j \geq 1$. Let $(b_j) \subset \mathbb{C}$ be a sequence such that $|b_j| > 0$, for all $j \geq 1$, and $\sum_{j=1}^{\infty} |b_j|^{2q} < \infty$. Given $\xi \in \mathcal{H}$, write $\xi = \sum_{j=1}^{\infty} a_j e_j$, and then consider, for each $k \geq 1$,

$$\xi_k := \sum_{j=1}^{k} a_j e_j + \sum_{j=k+1}^{\infty} b_j e_j.$$

It is clear that $\xi_k \to \xi$. Moreover, for $k \geq 1$ and each $\epsilon > 0$,

$$\begin{aligned} \int_{\mathrm{supp}(\mu^T_{\xi_k})} \mu^T_{\xi_k}(B(x,\epsilon))^{q-1} \mathrm{d}\mu^T_{\xi_k}(x) &= \sum_{j=1}^{\infty} \mu^T_{\xi_k}(B(\lambda_j,\epsilon))^{q-1} \mu^T_{\xi_k}(\{\lambda_j\}) \\ &\leq \sum_{j=1}^{\infty} \mu^T_{\xi_k}(\{\lambda_j\})^q = \sum_{j=1}^{k} |a_j|^{2q} + \sum_{j=k+1}^{\infty} |b_j|^{2q}, \quad (1.7) \end{aligned}$$

from which follows that $D^{\mp}_{\mu^T_{\xi_k}}(q) = 0$. Hence, $G^T_{0^-} = \{\xi \in \mathcal{H} \mid D^-_{\mu^T_\xi}(q) = 0\}$ is a dense set and, therefore, by Lemma 1.2, a dense G_δ set in $\mathcal{H}$.

Now we discuss the upper dimensions. Fix an $n \in \mathbb{N}$ with $n > \frac{q}{1-q}$ and let (λ_{j_l}) be a subsequence of (λ_j) so that: 1. $\lim_{l\to\infty}(\lambda_{j_l} - \lambda_{j_{l+1}}) = 0$ monotonically; 2. there exists a $C_n > 0$ such that, for every $l \geq 1$, $\lambda_{j_l} - \lambda_{j_{l+1}} \geq C_n / l^{1+\frac{1}{n}}$. Consider, for each $k \geq 1$,

$$\xi_k := \sum_{l=1}^{k} a_l e_l + \sum_{l=r(k)}^{\infty} \frac{1}{\sqrt{l^{1+\frac{1}{n}}}} e_{j_l},$$

where we set $r(k)$ large enough so that $\{e_1, ..., e_k, e_{j_{r(k)}}, e_{j_{r(k)+1}}, ...\}$ is an orthonormal set. Again, $\xi_k \to \xi$ in $\mathcal{H}$.

For each $m \geq 1$, put $\epsilon_m := |\lambda_{j_m} - \lambda_{j_{m+1}}|/2$. Then, for each $m > M(k)$ and each $1 \leq l \leq m$,

$$\mu^T_{\xi_k}(B(\lambda_{j_l}, \epsilon_m)) = \mu^T_{\xi_k}(\{\lambda_{j_l}\}),$$

where $M(k)$ is large enough so that for each $m > M(k)$, each $l \geq 1$ and each $1 \leq i \leq k$, $\lambda_i \notin B(\lambda_{j_l}, \epsilon_m)$. Hence, for $m > \max\{M(k), r(k)\} =: s(k)$,

$$\begin{aligned} \int_{\mathrm{supp}(\mu^T_{\xi_k})} \mu^T_{\xi_k}(B(x,\epsilon_m))^{q-1} \mathrm{d}\mu^T_{\xi_k}(x) &= \sum_{l=1}^{\infty} \mu^T_{\xi_k}(B(\lambda_l,\epsilon_m))^{q-1} \mu^T_{\xi_k}(\{\lambda_l\}) \\ &\geq \sum_{l=s(k)}^{m} \mu^T_{\xi_k}(B(\lambda_{j_l},\epsilon_m))^{q-1} \mu^T_{\xi_k}(\{\lambda_{j_l}\}) \\ &= \sum_{l=s(k)}^{m} \mu^T_{\xi_k}(\{\lambda_{j_l}\})^q = \sum_{l=s(k)}^{m} \frac{1}{l^{(1+\frac{1}{n})q}} \\ &\geq E_k\, m^{1-(1+\frac{1}{n})q} \geq E_k \left(\frac{C_n}{2\epsilon_m}\right)^{(1-(1+\frac{1}{n})q)/(1+\frac{1}{n})}, \end{aligned}$$

where E_k is a constant depending only of k, which results in

$$D^+_{\mu^T_{\xi_k}}(q) \geq \frac{1-(1+\frac{1}{n})q}{(1-q)(1+\frac{1}{n})} =: t_{n,q}.$$

Thus, $G^T_{(t_{n,q})^+}$ is a dense set and, therefore, by Lemma 1.2, a dense G_δ set in $\mathcal{H}$. Since

$$G^T_{1^+} = \bigcap_{n>\frac{q}{1-q}} G^T_{(t_{n,q})^+}$$

and, by Proposition 1.2, $G^T_{1^+} = \{\xi \in \mathcal{H} \mid D^+_{\mu^T_\xi}(q) = 1\}$, follows from Baire's Theorem that

$$\{\xi \in \mathcal{H} \mid D^-_{\mu^T_\xi}(q) = 0 \;\; and \;\; D^+_{\mu^T_\xi}(q) = 1\}$$

is a dense G_δ set in $\mathcal{H}$. Finally, let $\mathbb{Q}_+ := \{x \in \mathbb{Q} \mid x > 0\}$. Since, by Proposition 1.2,

$$\begin{aligned} &\{\xi \in \mathcal{H} \mid D^-_{\mu^T_\xi}(q) = 0 \;\; and \;\; D^+_{\mu^T_\xi}(q) = 1 \, for\, each \, 0 < q < 1\} \\ &= \bigcap_{q \in \mathbb{Q}_+ \cap (0,1)} \{\xi \in \mathcal{H} \mid D^-_{\mu^T_\xi}(q) = 0 \;\; and \;\; D^+_{\mu^T_\xi}(q) = 1\}, \end{aligned}$$

the result is proven. □

Chapter 2

Dynamics for compact operators

In this chapter, we prove Theorems II and III.

2.1 Proof of Theorem II

In order to proof Theorem II, some preparation is required.

Let T be a self-adjoint operator in $\mathcal{H}$. Now, for every measurable $f : \mathbb{R} \longrightarrow \mathbb{C}$, we denote $E^T(f)$ simply by $f(T)$, where E^T represents the resolution of the identity of T.

Proposition 2.1 (Proposition 10.1.9 in [22])**.** *A sequence of self-adjoint operators (T_n) converges to a self-adjoint operator T in the strong resolvent sense if and only if $f(T_n)$ strongly converges to $f(T)$ in $\mathcal{H}$ for every bounded and continuous $f : \mathbb{R} \longrightarrow \mathbb{C}$.*

Definition 2.1. Let μ be a σ-finite positive Borel measure on $\mathbb{R}$. One says that μ is (uniformly) Lipschitz continuous if there exists a constant $C > 0$ such that, for each interval I with $\ell(I) < 1$, $\mu(I) < C\,\ell(I)$, where $\ell(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$.

Theorem 2.1 (Theorem 3.2 in [38])**.** *If μ_ξ^T is Lipschitz continuous, then there exists a constant C_ξ such that for any compact operator A and any $t > 0$,*

$$\langle |A_\xi^T| \rangle_t \, < \, C_\xi \|A\|_1 t^{-1},$$

where $\|A\|_1$ denotes the trace norm of A.

Theorem 2.2. *Let $\xi \in \mathcal{H}$. Then, the set $L := \{M \in X_a \mid \mu_\xi^M \text{ is Lipschitz continuous}\}$ is a dense set in X_a.*

Proof. See the proof of Theorem 1.2 in [15]. □

Proof (Theorem II). The proof that

$$\{M \in X_a \mid \liminf_{t\to\infty} t\langle |A_\xi^M|\rangle_t = 0\}$$

is a dense G_δ set in X_a is a direct consequence of Theorem 2.2 in [4], Lemma 3.2 and Theorem 3.2 in [38], and Theorem 1.2 in [15]. For the convenience of the reader, we present a simple proof of this fact in details.

Since, by Proposition 2.1 and dominated convergence, for each $t \in \mathbb{R}$ the mapping

$$X_a \ni M \mapsto \alpha(t)\langle |A_\xi^M|\rangle_t$$

is continuous, it follows that, for each $k \geq 1$ and each $n \geq 1$, the set

$$\bigcup_{t\geq k}\{M \in X_a \mid \alpha(t)\langle |A_\xi^M|\rangle_t > n\}$$

is open, so

$$\{M \in X_a \mid \limsup_{t\to\infty} \alpha(t)\langle |A_\xi^M|\rangle_t = \infty\} = \bigcap_{n\geq 1}\bigcap_{k\geq 1}\bigcup_{t\geq k}\{M \in X_a \mid \alpha(t)\langle |A_\xi^M|\rangle_t > n\}$$

is a G_δ set in X_a.

Now, as previously discussed, it is well known that

$$D = \{M \in X_a \mid \sigma(M) = [-a-2, a+2],\ \sigma(M) \text{ is pure point}\}$$

is a dense subset of X_a. Thus, by RAGE's Theorem,

$$D \subset \{M \in X_a \mid \limsup_{t\to\infty} \alpha(t)\langle |A_\xi^M|\rangle_t = \infty\}$$

is a dense G_δ set in X_a.

We note that, for each $j \geq 1$,

$$L_j := \{M \in X_a \mid \liminf_{t\to\infty} t^{1-\frac{1}{j}}\langle |A_\xi^M|\rangle_t = 0\}$$

is also a G_δ set in X_a. Since, by Theorem 2.1, for each $j \geq 1$, $L \subset L_j$, it follows from Baire's Theorem that

$$\{M \in X_a \mid \liminf_{t\to\infty} t\langle |A_\xi^M|\rangle_t = 0\} = \bigcap_{j\geq 1} L_j$$

is a dense G_δ set in X_a, concluding the proof of the theorem. □

2.2 Proof of Theorem III

In order to prove Theorem III, we need the following results.

Theorem 2.3 (Theorem 4.5 in [56]). *The set*

$$\{H_V \in X_C \mid H_V \text{ has purely continuous spectrum on } (0,\infty)\}$$

is a dense G_δ set in X_C.

Theorem 2.4 (Corollary 4.6.1 in [53]). *Let $V : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded Borel function. If there exists $c \geq 0$ such that, for every $x \geq c$, $V(x) \leq 0$ and*

$$\int_c^\infty V(x)dx = -\infty,$$

then $H_V = \Delta + V$ has at least one negative eigenvalue.

If T is a self-adjoint operator, denote the set of its eigenvalues by $\Sigma(T)$.

Lemma 2.1. *Let T be a self-adjoint operator such that $\Sigma(T) \neq \emptyset$, and let α be as in the statement of* Theorem II *. Then, for any compact operator A,*

$$G_\alpha(A,T) := \{\xi \in \mathcal{H} \mid \limsup_{t\to\infty} \alpha(t)\langle |A_\xi^T|\rangle_t = \infty\}$$

is a dense G_δ set in $\mathcal{H}$.

Proof. Since, for each $t \in \mathbb{R}$, the mapping

$$\mathcal{H} \ni \xi \mapsto \alpha(t)\langle |A_\xi^T|\rangle_t$$

is continuous (by dominated convergence), it follows that

$$G_\alpha(T,A) = \bigcap_{n\geq 1}\bigcap_{k\geq 1}\bigcup_{t\geq k}\{\xi \in \mathcal{H} \mid \alpha(t)\langle |A_\xi^T|\rangle_t > n\}$$

is a G_δ set in $\mathcal{H}$.

Given $\xi \in \mathcal{H}$, write $\xi = \xi_1 + \xi_2$, with $\xi_1 \in Span\{\xi_0\}^\perp$ and $\xi_2 \in Span\{\xi_0\}$, where ξ_0, with $\|\xi_0\|_{\mathcal{H}} = 1$, is an eigenvector of T associated with an eigenvalue λ. If $\xi_2 \neq 0$, then

$$\begin{aligned}\mu_\xi^T(\{\lambda\}) &= \|E^T(\{\lambda\})\xi\|_{\mathcal{H}}^2\\ &\geq 2Re\langle E^T(\{\lambda\})\xi_1, E^T(\{\lambda\})\xi_2\rangle + \|E^T(\{\lambda\})\xi_2\|_{\mathcal{H}}^2\\ &= \|\xi_2\|_{\mathcal{H}}^2 > 0,\end{aligned}$$

where $E^T(\{\lambda\})$ represents the resolution of the identity of T over the set $\{\lambda\}$. Now, if $\xi_2 = 0$, define, for each $k \geq 1$,

$$\xi_k := \xi + \frac{\xi_0}{k}.$$

It is clear that $\xi_k \to \xi$. Moreover, by the previous arguments, for each $k \geq 1$, one has

$$\mu^T_{\xi_k}(\{\lambda\}) > 0.$$

Thus, $G := \{\xi \in \mathcal{H} \mid \mu^T_\xi$ has an atom$\}$ is a dense set in $\mathcal{H}$. Nevertheless, by RAGE's Theorem, $G \subset G_\alpha(T, A)$, proving that $G_\alpha(T, A)$ is a dense G_δ set in $\mathcal{H}$. □

Proof (Theorem III). By the arguments presented in the proof of Theorem II, for every $\xi \in \mathrm{L}^2(\mathbb{R})$,

$$\{H_V \in X_C \mid \limsup_{t\to\infty} \alpha(t)\langle |A^{H_V}_\xi|\rangle_t = \infty\}$$

is a G_δ set in X_C.

Now, given $H_V \in X_C$, we define for every $k \geq 1$,

$$V_k(x) := \frac{k}{k+1}\chi_{B(0,k)}V(x) - \frac{C}{(k+1)(|x|+1)}.$$

We note that, for each $k \geq 1$ and each $x \geq k$, $V_k(x) \leq 0$. Moreover, for each $k \geq 1$,

$$\int_k^\infty V_k(x)dx = -\infty.$$

Therefore, by Theorem 2.4, for every $k \geq 1$, H_{V_k} has at least one negative eigenvalue; in particular, $\Sigma(H_{V_k}) \neq \emptyset$. Since $H_{V_k} \to H_V$ in X_C, it follows that

$$Y := \{H_V \in X_C \mid \Sigma(H_V) \neq \emptyset\}$$

is a dense set in X_C.

Now, let (H_{V_k}) be a countable dense subset in Y (which is separable, since X_C is separable); then, by Lemma 2.1 and Baire's Theorem, $\bigcap_{k\geq1} G_\alpha(H_{V_k}, A)$ is a dense G_δ set in $\mathrm{L}^2(\mathbb{R})$. Moreover, for every $\xi \in \bigcap_{k\geq1} G_\alpha(H_{V_k}, A)$,

$$\{H_V \in X_C \mid \limsup_{t\to\infty} \alpha(t)\langle |A^{H_V}_\xi|\rangle_t = \infty\} \supset \bigcup_{k\geq1}\{H_{V_k}\}$$

is a dense G_δ set in X_C. The theorem is now a consequence of Theorem 2.3 and Baire's Theorem. □

Remark 2.1. Note that this separability argument used in the proof of Theorem I has allowed, in some sense, the use of the typical behaviour in $\mathrm{L}^2(\mathbb{R})$ (Lemma 2.1) in the determination of the typical behaviour in X_C.

Part II Some results on asymptotic of C_0-semigroups

Selected Notation

$\mathcal{H}$	Complex Hilbert space
$\mathcal{B}(\mathcal{H})$	Space of all bounded linear operators on $\mathcal{H}$
I	Identity operator on $\mathcal{H}$
$\mathcal{D}(A)$	Domain of the linear operator A in $\mathcal{H}$
$\mathrm{rng}(A)$	Range of A
$\mathrm{N}(A)$	Kernel of A
$\varrho(A)$	Resolvent set of A
$R(\lambda, A)$	Resolvent operator of A at $\lambda \in \varrho(A) \subset \mathbb{C}$
$\sigma(A)$	Spectrum of A
N	Normal operator in $\mathcal{H}$
E^N	Resolution of the identity of N
μ_x^N	Spectral measure of N with respect to $x \in \mathcal{H}$
$(T(t))_{t\geq 0}$	C_0-semigroup
$\omega_0(T)$	Exponential growth bound of $(T(t))_{t\geq 0}$
$\mathbb{C}_+$	The set $\{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$

Contextualization and main results

Contextualization

A central question in the theory of differential equations refers to the asymptotic behaviour of their solutions; for instance, whether they reach an equilibrium and, if so, with which speed. This kind of question is addressed by the asymptotic theory of C_0-semigroups. More specifically, here we consider the theory of stability for solutions of the abstract Cauchy problem on a Hilbert space $\mathcal{H}$, that is,

$$\begin{cases} \dot{x}(t) = Ax(t), \ t \geq 0, \\ x(0) = x, \ x \in \mathcal{H}, \end{cases} \tag{ACP}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$.

Definition. Let $A : \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator. The resolvent set of A, denoted by $\varrho(A)$, is the set of all $\lambda \in \mathbb{C}$ for which the resolvent operator of A at λ,

$$R(\lambda, A) : \mathcal{H} \longrightarrow \mathcal{D}(A), \ \ R(\lambda, A) := (\lambda I - A)^{-1},$$

exists and is bounded.

Definition. The spectrum of A is the set $\sigma(A) = \mathbb{C} \backslash \rho(A)$.

We recall that a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$ is said to be bounded if there exists a constant $C > 0$ so that, for each $t \geq 0$, $\|T(t)\|_{\mathcal{B}(\mathcal{H})} \leq C$; if $C = 1$, then it is called a C_0-semigroup of contractions.

We also recall that $(T(t))_{t\geq 0}$ is (strongly) stable if, for every $x \in \mathcal{H}$,

$$\lim_{t\to\infty} \|T(t)x\|_{\mathcal{H}} = 0;$$

$(T(t))_{t\geq 0}$ is exponentially stable if there exist constants $C > 0$ and $a > 0$ such that, for every $t \geq 0$,

$$\|T(t)\|_{\mathcal{B}(\mathcal{H})} \leq C\, e^{-ta}.$$

Over the three last decades, the asymptotic theory of C_0-semigroups on Hilbert spaces had a fast development, with a large number of long-standing open problems being solved. Among such problems, one can highlight the characterization of exponential stability for C_0-semigroups of contractions on Hilbert spaces (Gearhart's Theorem), due to Herbst, Howland and Prüss [33, 36, 48].

Theorem (Gearhart's Theorem)**.** *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$, with generator A. Then, $(T(t))_{t\geq 0}$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(A) \quad \textit{and} \quad \limsup_{|\lambda|\to\infty} \|R(i\lambda, A)\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

The stability theorem, by Arendt, Batty, Lyubich and Vũ [3, 43], states that a bounded C_0-semigroup on a reflexive Banach space is (strongly) stable if the spectrum of its generator is countable and contains no residual spectrum.

Theorem (Arendt-Batty-Lyubich-Vũ's Theorem)**.** *Let X be a reflexive Banach space and let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on X with generator A. Assume that the eigenvalues of A do not intercept the imaginary axis. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is stable.*

We also highlight the recent results obtained by Borichev and Tomilov [11], by Batty, Chill and Tomilov [8], and very recently by Rozendaal, Seifert, and Stahn [50], which relate estimates on the norm of the resolvent of the generator to quantitative decaying rates of the form

$$\|T(t)A^{-1}\|_{\mathcal{B}(\mathcal{H})} = O(r(t)), \quad t \to \infty,$$

with $\lim_{t\to\infty} r(t) = 0$, developed in order to explore polynomial and logarithmic scales, among others, of decaying rates of C_0-semigroups (see Batty-Chill-Tomilov's Theorem ahead). As it is known, this strategy has allowed numerous applications of the theory to PDEs; namely, estimates on the norm of the resolvent of the generator are often easier to compute than the estimates on the norm of the semigroup itself. In this context, we refer to [1, 2, 8, 13, 18, 19, 21, 24, 27, 28, 39, 41, 42, 45, 50], among others.

An important intermediate step between Gearhart's Theorem [33, 36, 48] and the results by Rozendaal et al. [50] is Batty-Duyckaerts's Theorem [9], which relates the decaying rates of $\|T(t)A^{-1}\|_{\mathcal{B}(X)}$, $i\mathbb{R} \subset \varrho(A)$, with the arbitrary growth of the norm of the resolvent of the generator. In order to properly recall such result, some preparation is required.

For every A, the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, with $i\mathbb{R} \subset \varrho(A)$, we define a continuous non-decreasing function

$$M(y) := \max_{\lambda\in[-y,y]} \|R(i\lambda, A)\|_{\mathcal{B}(X)}, \quad y \geq 0,$$

and the associated function

$$M_{\log}(y) := M(y)(\log(1 + M(y)) + \log(1 + y)), \quad y \geq 0.$$

We denote by $M_{\log}^{-1} : [M_{\log}(0), \infty) \to \mathbb{R}$ the inverse of $M_{\log}$.

Theorem (Batty-Duyckaerts's Theorem)**.** *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X, with generator A such that $i\mathbb{R} \subset \varrho(A)$. Then, there exists $C > 0$ such that*

$$\|T(t)A^{-1}\|_{\mathcal{B}(X)} = O(C(M_{\log}^{-1}(t/C))^{-1}), \quad t \to \infty. \tag{d}$$

For a refinement of (d) on Hilbert spaces, see [8, 11] (see also Batty-Chill-Tomilov's Theorem ahead).

The proof of the theorem presented above by Batty and Duyckaerts's [9], which uses a technique developed by Korevaar [37], makes use of Cauchy's Theorem and Neumann series expansions. Usually, the problem of obtaining lower bounds for the decaying rates of stable bounded C_0-semigroups passes through the understanding of some theory of integral representation (like, for instance, Cauchy's theory and the functional calculus of sectorial operators [8, 9]). In this part of the thesis, we use the joint resolution of the identity for normal operators [10, 52] to find the typical asymptotic behaviour, in Baire's sense, of the orbits of normal C_0-semigroups of contractions. We also use recent results of the asymptotic theory of C_0-semigroups [8, 44] to say something about non-normal semigroups, and then discuss applications to some evolution equations. To the best of our knowledge, none of this has been detailed in the literature yet.

Brief discussion of our main results

Exact asymptotic behaviour of normal semigroups

Let, for every $\lambda \in \mathbb{C}$ and every $t \geq 0$, $g_t(\lambda) = e^{t\lambda}$, and let N be a normal operator in $\mathcal{H}$; denote by $Re(\lambda)$ the real part of λ. If $\mathbb{C}_+ \subset \varrho(N)$, then, by the (Spectral) Functional Calculus, $(e^{tN})_{t\geq 0} := (g_t(N))_{t\geq 0}$ is a normal C_0-semigroup of contractions generated by N. It is well known that every normal C_0-semigroup of contractions is of this form [51]. Namely, if $(T(t))_{t\geq 0}$ is a normal C_0-semigroup of contractions on $\mathcal{H}$ and A is its generator, then A is normal and $\mathbb{C}_+ \subset \varrho(A)$; in this case, $(T(t))_{t\geq 0}$ can be rewritten as $(e^{tA})_{t\geq 0}$.

As discussed previously, after Batty and Duyckaerts [9] have related the decay of stable bounded C_0-semigroups to the arbitrary growth of the norms of the respective resolvents, the study of polynomial and logarithmic scales of such rates has been the subject of many recent papers (for instance, [8, 11, 50]). In contrast with this setting, our next result says that the decaying rates of the orbits of normal C_0-semigroups of contractions, typically in Baire's sense, may depend on sequences of time going to infinity.

Theorem IV. *Let N be a normal operator in $\mathcal{H}$ such that $\sup\{Re(\lambda) : \lambda \in \sigma(N)\} = 0$ and let $\alpha, \beta : \mathbb{R}_+ \longrightarrow (0, \infty)$ be real functions so that*

$$\lim_{t\to\infty} \alpha(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} \beta(t) e^{-t\epsilon} = 0, \ \forall \epsilon > 0.$$

Suppose that $(\mathrm{e}^{tN})_{t\geq 0}$ is stable. Then,

$$\mathcal{G}_N(\alpha, \beta) := \{x \mid \limsup_{t\to\infty} \alpha(t) \|\mathrm{e}^{tN} x\|_{\mathcal{H}} = \infty \quad \textit{and} \quad \liminf_{t\to\infty} \beta(t) \|\mathrm{e}^{tN} x\|_{\mathcal{H}} = 0\}$$

is a dense G_δ set in $\mathcal{H}$. Moreover, the assumption on β is optimal, that is, β can not be chosen to grow faster than sub-exponentially.

Remark. We note that for every N satisfying the hypotheses of Theorem IV, the set $\mathcal{G}_N(\alpha, \beta)$ has empty interior (see Remark 4.1 ahead); so, it is always a proper subset of $\mathcal{H}$.

Example. Let $\varphi : [2, \infty) \longrightarrow \mathbb{C}$ be given by the action $\varphi(y) := \frac{1}{\ln y} + iy$, and then define $M_\varphi : \mathcal{D}(M_\varphi) \subset \mathrm{L}^2([2, \infty)) \longrightarrow \mathrm{L}^2([2, \infty))$,

$$(M_\varphi f)(y) = -\varphi(y) f(y),$$

where $f \in \mathcal{D}(M_\varphi) := \{u \in \mathrm{L}^2([2, \infty)) \mid \varphi u \in \mathrm{L}^2([2, \infty))\}$.

We note that M_φ is a normal operator and $\sigma(M_\varphi) = \{-\frac{1}{\ln y} - iy, \ y \geq 2\}$, which implies that $\sup\{Re(\lambda) : \lambda \in \sigma(M_\varphi)\} = 0$. Moreover, it is possible show that

$$\|e^{tM_\varphi} M_\varphi^{-1}\|_{\mathcal{B}(\mathrm{L}^2([2,\infty)))} = O(e^{-2\sqrt{t}}) \tag{e}$$

(Example 5.2 in [8]). We also note that M_φ satisfies the hypotheses of Theorem IV. Therefore, although in this case, by (e), all classical solutions of (ACP) do go to zero with sub-exponential rate, typically in Baire's sense, the orbits of this semigroup do not have this asymptotic behaviour. Namely, by Theorem IV, each typical orbit goes arbitrarily slow to zero for a sequence of time going to infinity and sub-exponentially fast for another one. We note that, in this case, $\mathcal{D}(M_\varphi) \cap \mathcal{G}_N(\alpha, \beta) = \emptyset$.

Example. Let $A : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be the linear operator given by

$$(Au)_n = u_{n-1} + u_{n+1}, \;\; n \in \mathbb{Z}.$$

It is known that A is unitarily equivalent to the multiplication operator $\mathcal{M}_\phi$ on $\mathrm{L}^2[0, 2\pi)$, with $\phi(x) = 2\cos(x)$, from which follows that A is a bounded self-adjoint operator with continuous spectrum $\sigma(A) = \sigma(\mathcal{M}_\phi) = \sigma_c(\mathcal{M}_\phi) = [-2, 2]$ (see [22] for details).

Now we consider the discrete Laplacian, $\triangle$, given on $\ell^2(\mathbb{Z})$ by the action

$$(\triangle u)_n = (Au)_n - 2u_n.$$

So, $\triangle$ is a bounded self-adjoint operator with continuous spectrum $\sigma(\triangle) = [-4, 0]$, from which follows that $(e^{t\triangle})_{t\geq 0}$ is stable but not exponentially stable; therefore, $\triangle$ satisfies the hypotheses of Theorem IV.

It is clear that all orbits of the semigroup $(e^{t\triangle})_{t\geq 0}$ are infinitely differentiable, since the discrete Laplacian is a bounded linear operator. This illustrates, in Theorem IV, that in some cases very regular initial data may belong to the typical set $\mathcal{G}_N(\alpha, \beta)$.

Now we recall that every normal operator N can be written as $N = N_R + iN_I$, where N_R and N_I are self-adjoint operators such that $N_R N_I = N_I N_R$. The next theorem, a new spectral classification of (strong) stability for normal C_0-semigroups of contractions, is a direct application of Gearhart's Theorem and Theorem IV.

Theorem V. *Let N be a normal operator in $\mathcal{H}$ so that $\mathbb{C}_+ \subset \varrho(N)$. Then:*

1. *All orbits of $(e^{tN})_{t\geq 0}$ converge to zero with exponential rate if and only if $0 \notin \sigma(N_R)$.*

2. *There is a dense G_δ set $\mathcal{G}_N \subset \mathcal{H}$ so that for each $x \in \mathcal{G}_N$, $(e^{tN}x)_{t\geq 0}$ goes arbitrarily slow to zero for some sequence of time going to infinity, and sub-exponentially fast for another sequence if and only if $0 \in \sigma(N_R)$ but 0 is not an eigenvalue of N_R.*

3. *There is a dense G_δ set $\mathcal{F}_N \subset \mathcal{H}$ such that for each $x \in \mathcal{F}_N$, $(e^{tN}x)_{t\geq 0}$ does not converge to zero if and only if 0 is an eigenvalue of N_R.*

Remark. Theorem V is optimal in the sense that, if $N_R \neq 0$, then the dense G_δ sets given by cases *2.* and *3.* are necessarily proper. For case *2.*, this follows from Remark 4.1 stated ahead. For case *3.*, this follows from the fact that, for each $x \in \mathrm{N}(N_R)^\perp$,

$$\lim_{t\to\infty} \|e^{tN}x\|_{\mathcal{H}} = 0.$$

Non-normal semigroups

Our next result is a partial extension of Theorem IV to non-normal semigroups. We recall that the exponential growth bound of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$ is defined as [60]

$$\omega_0(T) := \lim_{t\to\infty} \frac{\ln \|T(t)\|_{\mathcal{B}(\mathcal{H})}}{t}.$$

Theorem VI. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on $\mathcal{H}$ such that $\omega_0(T) = 0$, and let A be its generator. Suppose that, for some $k \geq 1$,*

$$\|T(t)A^{-k}\|_{\mathcal{B}(\mathcal{H})} = O(r(t)), \quad t \to \infty, \tag{H}$$

with $\lim_{t\to\infty} r(t) = 0$. Let $\alpha, \beta : \mathbb{R}_+ \longrightarrow (0, \infty)$ be real functions so that

$$\lim_{t\to\infty} \alpha(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} \beta(t)r(t) = 0.$$

Then,

$$\mathcal{G}_A(\alpha, \beta) = \{x \mid \limsup_{t\to\infty} \alpha(t)\|T(t)x\|_{\mathcal{H}} = \infty \text{ and } \liminf_{t\to\infty} \beta(t)\|T(t)x\|_{\mathcal{H}} = 0\}$$

is a dense G_δ set in $\mathcal{H}$.

Remark.

1. The difference between Theorems IV and VI is that the former, under the perspective of this work, describes the exact asymptotic behaviour of normal C_0-semigroups of contractions. Moreover, thanks to the Spectral Theorem, we do not need to use hypothesis (H) in order to prove Theorem IV.

2. We note that, by Batty-Duyckaerts's Theorem, if $i\mathbb{R} \subset \varrho(A)$, then hypothesis (H) is satisfied.

3. Suppose that there exists an $a > 0$ such that $\|T(t)A^{-1}\|_{\mathcal{B}(\mathcal{H})} = O(t^{-a})$. Since, for every $k \geq 1$,
$$\|T(t)A^{-k}\|_{\mathcal{B}(\mathcal{H})} = \|[T(t/k)A^{-1}]^k\|_{\mathcal{B}(\mathcal{H})},$$
one has
$$\|T(t)A^{-k}\|_{\mathcal{B}(\mathcal{H})} = O(t^{-ka}).$$
Thus, it follows from Theorem VI that
$$\bigcap_{k\geq 1} \{x \mid \limsup_{t\to\infty} \alpha(t)\|T(t)x\|_{\mathcal{H}} = \infty \text{ and } \liminf_{t\to\infty} t^{ka/2}\|T(t)x\|_{\mathcal{H}} = 0\}$$
is a dense G_δ set in $\mathcal{H}$. Therefore, in this case, Baire generically in $\mathcal{H}$, the orbits of the semigroup have an arbitrarily slow decaying rate for some sequence of time going to infinity and a super-polynomially fast decaying rate for another one.

Consider the following result [8].

Theorem (Batty-Chill-Tomilov Theorem's)**.** *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$, with generator A, so that $i\mathbb{R} \subset \varrho(A)$. Then, given $a > 0$ and $b \geq 0$, the following assertions are equivalent:*

1. $\|(is\mathbf{1} - A)^{-1}\|_{\mathcal{B}(\mathcal{H})} = O(|s|^a(\ln|s|)^{-b}), \quad |s| \to \infty,$
2. $\|T(t)A^{-1}\|_{\mathcal{B}(\mathcal{H})} = O(t^{-\frac{1}{a}}(\ln t)^{-b/a}), \quad t \to \infty.$

Theorem VI can be naturally combined with Batty-Chill-Tomilov Theorem's in order to produce refined scales of decay of C_0-semigroups. Namely, if we replace condition (H) in Theorem VI by the condition depicted in item *2.* of Batty-Chill-Tomilov Theorem's, then, typically, every typical orbit of the semigroup have an arbitrarily slow decaying rate for some sequence of time going to infinity, and a polynomially fast decaying rate for another one. There are in the literature numerous examples of bounded C_0-semigroups satisfying these assumptions (see Chapter 3, [1, 2, 8, 11, 27, 50] and references therein).

Schrödinger semigroups

Stimulated by the category theorems of Eisner and Serény in [26], which show that the set of all weakly stable unitary groups (isometric semigroups) is of first category, while the set of all almost weakly stable unitary groups is residual for an appropriate topology, we also prove some category theorems for Schrödinger semigroups. Specifically, we show that, for a given class of Schrödinger semigroups, they are, Baire generically, stable but not exponentially stable.

Fix $l > 0$ and let the family of (negative continuous) Schrödinger operators, H_V, defined in $\mathcal{H}^2(\mathbb{R}^\nu)$, $\nu \in \mathbb{N}$, by the action

$$(H_V u)(x) := \triangle u(x) + V(x)u(x),$$

with $V \in \mathcal{B}^\infty(\mathbb{R}^\nu)$ (the space of bounded Borel functions) such that, for each $x \in \mathbb{R}^\nu$, $-l \leq V(x) \leq 0$. Denote by X_l^ν the set of these operators endowed with the topology of pointwise convergence on $(V(x))$. Then, X_l^ν is (by Tychonoff's Theorem) a compact metric space, so that convergence in metric implies strong resolvent convergence.

Theorem VII. *For every $l > 0$ and every $\nu \in \mathbb{N}$,*

$$\{H \in X_l^\nu \mid (e^{tH})_{t\geq 0} \ \ \textit{is stable but not exponentially stable}\}$$

is a dense G_δ set in X_l^ν.

Remark. It follows from the Theorems IV and VII that, for every $l > 0$ and every $\nu \in \mathbb{N}$, typically in X_l^ν, the orbits of each Schrödinger semigroup $(e^{tH})_{t\geq 0}$, typically in $\mathrm{L}^2(\mathbb{R}^\nu)$, have decaying rates depending on sequences of time going to infinity. Hence, for every X_l^ν, the dynamics is typically (from the topological viewpoint) nontrivial.

Definition. Let μ be a finite (positive) Borel measure on $\mathbb{R}$. The pointwise lower and upper scaling exponents of μ at $w \in \mathbb{R}$ are defined, respectively, by

$$d_\mu^-(w) := \liminf_{\epsilon\downarrow 0} \frac{\ln \mu(B(w,\epsilon))}{\ln \epsilon} \quad \text{and} \quad d_\mu^+(w) := \limsup_{\epsilon\downarrow 0} \frac{\ln \mu(B(w,\epsilon))}{\ln \epsilon}\,,$$

if, for all small enough $\epsilon > 0$, $\mu(B(w,\epsilon)) > 0$; $d_\mu^\mp(w) := \infty$, otherwise.

Our next result says something about the local scale spectral properties of this class of Schrödinger semigroups. It indicates the subtlety of the relation between the dynamics of a Schrödinger semigroup and the local scale spectral properties of its generator.

Theorem VIII. *For each $l > 0$ and each $\nu \in \mathbb{N}$, there exists a dense G_δ set G_l^ν in $\mathrm{L}^2(\mathbb{R}^\nu)$, such that, for every $f \in G_l^\nu$,*

$$J_l^\nu(f) := \{H \in X_l^\nu \mid d^-_{\mu_f^H}(0) = 0 \text{ and } d^+_{\mu_f^H}(0) = \infty\}$$

is a dense G_δ set in X_l^ν.

Organization of the text

In Chapter 3, we discuss explicit applications of Theorem VI to some specific evolution equations.

Chapter 4 contains a detailed study of the relation between the decaying rates of a normal semigroup and the local scale spectral properties of its generator; in particular, in this chapter we prove Theorems IV, V and VI.

In Chapter 5, we prove Theorems VII and VIII.

Chapter 3

Applications

Next, we illustrate Theorem VI by presenting applications to specific evolution equations. We gather below equations whose associated semigroups are polynomially stable, that is, all classical solutions of (ACP) converge uniformly (on the unit ball of $\mathcal{D}(A)$ endowed with the graph norm) to zero at infinity with polynomial rate, but not exponentially stable [1, 2, 27, 58]; therefore, they are examples of semigroups for which the hypotheses of Theorem VI are satisfied. Hence, the (mild) solutions of these equations, typically in Baire's sense, depend on sequences of time going to infinity.

3.1 Damped wave equation on the torus

Let M be a smooth compact connected Riemannian manifold with boundary ∂M. The respective damped wave equation (one of the basic models in control theory) is given by

$$\begin{aligned} u_{tt} - \triangle u + a(x)u_t &= 0 \text{ in } \mathbb{R}_+ \times M, \\ u &= 0 \text{ in } \mathbb{R}_+ \times \partial M, \\ u(0,\cdot) &= u_0 \text{ in } M, \\ u_t(0,\cdot) &= u_1 \text{ in } M. \end{aligned} \tag{3.1}$$

The study of the asymptotic behaviour as $t \to \infty$ of the solutions of such equation has attracted significant interest. An approach that has been successfully used to address this problem is an involved asymptotic theory of C_0-semigroups. In this context, we refer to [2, 13, 39, 58], among others.

We note that if one multiplies (3.1) by u_t and integrates on M, one gets the following dissipation identity

$$\frac{1}{2}\frac{d}{dt}E(u,t) = -\int_M a|u_t(t)|^2 dx,$$

where the energy of a solution is defined by

$$E(u,t) := \|\nabla u(t)\|^2_{\mathrm{L}^2(M)} + \|u_t(t)\|^2_{\mathrm{L}^2(M)}.$$

We also note that if one sets

$$U = \begin{bmatrix} u \\ u_t \end{bmatrix},$$

then

$$\frac{dU}{dt} = \begin{bmatrix} u_t \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_t \\ \triangle u - a u_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ \triangle & -a \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} = AU.$$

Thus, such equation can be rewritten as an abstract Cauchy problem (ACP) in the Hilbert space $\mathcal{H} := H^1_0(M) \times \mathrm{L}^2(M)$, with the wave operator A defined by

$$\mathcal{D}(A) := (H^2(M) \cap H^1_0(M)) \times H^1_0(M),$$

$$A := \begin{bmatrix} 0 & I \\ \triangle & -a \end{bmatrix}.$$

It is not hard to show, through Lumer-Philips's Theorem, that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions. We note that any estimate on the decaying rates of the norm of the semigroup is an estimate on the decaying rates of the energy of the system (since the natural norm on $\mathcal{H}$ corresponds to such energy).

For the case $M = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, the 2-dimensional torus with the standard flat metric, the damped wave equation reduces to

$$\begin{aligned} u_{tt} - \triangle u + a(x)u_t &= 0 \text{ in } \mathbb{R}_+ \times \mathbb{T}^2, \\ u(0,\cdot) &= u_0 \text{ in } \mathbb{T}^2, \\ u_t(0,\cdot) &= u_1 \text{ in } \mathbb{T}^2, \end{aligned} \tag{3.2}$$

where $a \in \mathrm{L}^\infty(\mathbb{T}^2)$, $a \geq 0$. It was shown in [2] that, under some conditions on a, $(T(t))_{t\geq 0}$ is polynomially stable with decay between $1/t^{1/2}$ and $1/t^{2/3}$; recently, it was proven [58] that, also under conditions on a, this decay is exactly $t^{-4/3}$.

3.2 Wave equation with localized viscoelasticity

The system below corresponds to the wave equation with localized viscoelasticity of Kelvin-Voigt type [1],

$$\rho_1 u_{tt} - k_1 u_{xx} - k_2 u_{xxt} = 0 \text{ in }]-L,0[\times]0,\infty[, \tag{3.3}$$

$$\rho_2 v_{tt} - k_3 v_{xx} = 0 \text{ in }]0,L[\times]0,\infty[, \tag{3.4}$$

with k_1, k_2 and k_3 denoting positive elastic constants; ρ_1, ρ_2 stand for the mass and densities. Here, we consider Dirichlet boundary conditions, which can be written as

$$u(-L,t)=0, \quad v(L,t)=0, \quad t\geq 0. \tag{3.5}$$

The transmission conditions are given by

$$u(0,t)=v(0,t), \quad k_1u_x(0,t)+k_2u_{xt}(0,t)=k_3v_x(0,t), \quad t\geq 0. \tag{3.6}$$

Finally, the initial data are given by

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0)=u_1(x) \ \text{ in } \ ]-L,0[\,, \\ v(x,0) &= v_0(x), \quad v_t(x,0)=v_1(x) \ \text{ in } \ ]0,L[\,. \end{aligned} \tag{3.7}$$

This equation is related to a transmission problem with localized Kelvin-Voigt viscoelastic damping; see [1] for details. We note that this equation was studied by several authors (see [18, 41, 45] and references therein).

It is easy to see that (3.3)-(3.7) can be rewritten as an abstract Cauchy problem (ACP) in the Hilbert space

$$\mathcal{H}=\mathbb{H}^1_L\times\mathbb{L}^2,$$

where

$$\mathbb{H}^m=H^m(-L,0)\times H^m(0,L), \quad m=1,2, \quad \mathbb{L}^2=\mathrm{L}^2(-L,0)\times\mathrm{L}^2(0,L),$$

$$\mathbb{H}^1_L=\{(u,v)\in\mathbb{H}^1 : u(-L)=v(L)=0, u(0)=v(0)\},$$

equipped with the inner product

$$\begin{aligned} \langle(u_1,v_1,\eta_1,\mu_1),(u_2,v_2,\eta_2,\mu_2)\rangle_{\mathcal{H}} = \ & k_1\int_{-L}^{0} u_{1x}\overline{u_{2x}}\,dx + k_3\int_0^L v_{1x}\overline{v_{2x}}\,dx \\ + \ & \rho_1\int_{-L}^{0}\eta_1\overline{\eta_2}\,dx+\rho_2\int_0^L\mu_1\overline{\mu_2}\,dx. \end{aligned}$$

In this case, the linear operator A is given by

$$\mathcal{D}(A)=\{U\in\mathcal{H} : (\eta,\mu)\in\mathbb{H}^1_L,\ (k_1u+k_2\eta,v)\in\mathbb{H}^2,\ k_1u_x(0)+k_2\eta_x(0)=k_3v_x(0)\},$$

where $U=(u,v,\eta,\mu)$ and

$$A=\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \frac{k_1}{\rho_1}\partial_{xx}(\cdot) & 0 & \frac{k_2}{\rho_1}\partial_{xx}(\cdot) & 0 \\ 0 & \frac{k_3}{\rho_2}\partial_{xx}(\cdot) & 0 & 0 \end{bmatrix}.$$

It was shown [1], using Hille-Yosida's Theorem, that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions on $\mathcal{H}$. It was also shown that $(T(t))_{t\geq 0}$ is polynomially stable with decay $1/t^2$ and that this rate is polynomially optimal.

3.3 Thermoelastic systems of Bresse type

Now we consider a systems of Bresse type with frictional damping effective in one of its equations [27],

$$\begin{aligned} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) &= 0 \text{ in }]0,L[\times]0,\infty[, && (3.8)\\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma\psi_t &= 0 \text{ in }]0,L[\times]0,\infty[, && (3.9)\\ \rho_1\omega_{tt} + k_0 l(\omega_x - l\varphi)_x - kl(\varphi_x + \psi + l\omega) &= 0 \text{ in }]0,L[\times]0,\infty[, && (3.10)\end{aligned}$$

with positive constants $\rho_1, \rho_2, k, k_0, b, l$ and γ. We also consider the Dirichlet-Neumann-Neumann boundary conditions

$$\begin{aligned} \varphi(t,0) = \varphi(t,L) &= \psi_x(t,0) \\ &= \psi_x(t,L) = \omega_x(t,0) = \omega_x(t,L) = 0 \text{ in }]0,\infty[, \qquad (3.11)\end{aligned}$$

with the following initial conditions:

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x) \text{ in }]0,L[,\\ &\psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x) \text{ in }]0,L[,\\ &\omega(0,x) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x) \text{ in }]0,L[. \qquad (3.12)\end{aligned}$$

This system, also known as circular arc problem, have been the subject of studies by many authors (see [27, 28, 42] and references therein).

Once more, (3.8)-(3.12) can be rewritten as an abstract Cauchy problem (ACP) in the Hilbert space

$$\mathcal{H} = H_0^1(0,L) \times \mathrm{L}^2(0,L) \times H_*^1(0,L) \times \mathrm{L}_*^2(0,L) \times H_*^1(0,L) \times \mathrm{L}_*^2(0,L),$$

with norm given by

$$\|U\|_{\mathcal{H}}^2 = \rho_1\|\varphi\|_{L^2}^2 + \rho_2\|\psi\|_{L^2}^2 + \rho_1\|\omega\|_{L^2}^2 + b\|\psi_x\|_{L^2}^2 + k\|\varphi_x + \psi + l\omega\|_{L^2}^2 + k_0\|\omega_x - l\varphi\|_{L^2}^2,$$

where

$$\mathrm{L}_*^2(0,1) = \{u \in \mathrm{L}^2(0,L) : \int_0^L u(x)dx = 0\} \text{ and } H_*^1(0,L) = \mathrm{L}_*^2(0,L) \cap H_0^1(0,L).$$

The corresponding linear operator A is given by

$$\mathcal{D}(A) = \{U \in \mathcal{H} : \varphi \in H^2(0,L) \cap H_0^1(0,L),\ \psi, \omega \in H^2(0,L),$$
$$\psi_x, \omega_x \in H_0^1(0,L),\ \tilde{\varphi} \in H_0^1(0,L),\ \tilde{\psi}_x, \tilde{\omega}_x \in H_*^1(0,L)\},$$

where $U = (\varphi, \psi, \omega, \tilde{\varphi}, \tilde{\psi}, \tilde{\omega})$ and

$$A = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_{xx}(\cdot) - \frac{k_0 l^2}{\rho_1} I & 0 & \frac{k}{\rho_1}\partial_x(\cdot) & 0 & \frac{(k+k_0)l}{\rho_1}\partial_x(\cdot) & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{k}{\rho_2}\partial_x(\cdot) & 0 & \frac{b}{\rho_2}\partial_{xx}(\cdot) - \frac{k}{\rho_2} I & -\frac{\gamma}{\rho_2} I & -\frac{kl}{\rho_2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ -\frac{(k+k_0)l}{\rho_1}\partial_x(\cdot) & 0 & -\frac{kl}{\rho_1} I & 0 & \frac{k_0}{\rho_1}\partial_{xx}(\cdot) - \frac{kl^2}{\rho_1} I & 0 \end{bmatrix}.$$

It is well known that, by Hille-Yosida's Theorem, A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions on $\mathcal{H}$ [27]. It was shown in [27] that if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad K \neq K_0,$$

then $(T(t))_{t\geq 0}$ is polynomially stable with decay $1/t^{1/2}$ and such decay is polynomially optimal.

Remark 3.1. There are in the literature numerous other examples of evolution equations whose associated semigroups satisfy the assumptions in the statement of Theorem VI (see [8, 19, 50] and references therein); particularly, such result also applies to some thermoelastic systems of Timoshenko type (see also [21, 24] and references therein).

Chapter 4

Fine scales of decaying rates

4.1 Normal semigroups: polynomial decaying rates × spectral properties

It follows from the Spectral Theorem that every normal operator N on a Hilbert space $\mathcal{H}$, with $\mathbb{C}_+ \subset \varrho(N)$, generates a normal C_0-semigroup of contractions; namely,

$$e^{tN} = \int_{\sigma(N)} e^{t\lambda} \quad dE^N(\lambda),$$

where E^N is the resolution of the identity of N. It is well known that every normal C_0-semigroup of contractions is of this form [51].

We recall that every normal operator N can be written as $N = N_R + iN_I$, where

$$N_R = \frac{N + N^*}{2} \quad \text{and} \quad N_I = -i\frac{N - N^*}{2}$$

are self-adjoint operators and $N_R N_I = N_I N_R$. In this case, E^N corresponds to a joint resolution of the identity associated with the operator pair $\{N_R, N_I\}$ [10, 52]. Thus, for every $x \in \mathcal{H}$, $\|x\|_{\mathcal{H}} = 1$,

$$\begin{aligned}
\|e^{tN}x\|^2_{\mathcal{H}} &= \|e^{t(N_R+iN_I)}x\|^2_{\mathcal{H}} \\
&= \int_{\sigma(N_R)\times\sigma(N_I)} |e^{t(y+iv)}|^2 \quad d\mu_x^{N_R}(y)\, d\mu_x^{N_I}(v) \\
&= \int_{\sigma(N_I)} 1\, d\mu_x^{N_I}(v) \int_{\sigma(N_R)} e^{2ty} \quad d\mu_x^{N_R}(y) \\
&= \int_{-\infty}^{0} e^{2ty} d\mu_x^{N_R}(y)\,, \qquad (4.1)
\end{aligned}$$

where $\mu_x^{N_R}$ denotes the spectral measure of N_R associated with x; the last equality in (4.1) is a consequence from fact that $(e^{tN})_{t\geq 0}$ is a semigroup of contractions. Thus, at least when $\mu_x^{N_R}$ has a certain local regularity (with respect to the Lebesgue measure), we expect that

$$\|e^{tN}x\|_{\mathcal{H}}^2 = \int_{-\infty}^{0} e^{2ty} d\mu_x^{N_R}(y) \ \sim \ \mu_x^{N_R}(B(0, \frac{1}{t}))\,.$$

If f and g are two real-value functions, $f \sim g$ means that f and g are asymptotically equivalent, that is,

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1\,.$$

We recall that if μ be a finite (positive) Borel measure on $\mathbb{R}$, then the pointwise lower and upper scaling exponents of μ at $w \in \mathbb{R}$ are defined, respectively, as

$$d_\mu^-(w) := \liminf_{\epsilon\downarrow 0} \frac{\ln \mu(B(w,\epsilon))}{\ln \epsilon} \quad \text{and} \quad d_\mu^+(w) := \limsup_{\epsilon\downarrow 0} \frac{\ln \mu(B(w,\epsilon))}{\ln \epsilon},$$

if, for all small enough $\epsilon > 0$, $\mu(B(w,\epsilon)) > 0$; $d_\mu^{\mp}(w) := \infty$, otherwise.

Taking into account (4.1), the following result is expected.

Proposition 4.1. *Let N be a normal operator so that $\mathbb{C}_+ \subset \varrho(N)$, and let $x \in \mathcal{H}$, with $x \neq 0$. Then,*

$$\liminf_{t\to\infty} \frac{\ln \|e^{tN}x\|_{\mathcal{H}}^2}{\ln t} = -d^+_{\mu_x^{N_R}}(0) \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\ln \|e^{tN}x\|_{\mathcal{H}}^2}{\ln t} = -d^-_{\mu_x^{N_R}}(0).$$

We note that Proposition 4.1 relates, for every $x \in \mathcal{H}$, the polynomial decaying rates of $\|e^{tN}x\|_{\mathcal{H}}$ to dimensional properties of the spectral measure $\mu_x^{N_R}$. Namely, this result establishes an explicit relation between the dynamics of the semigroup and the local scale spectral properties of its generator.

We also note that Proposition 4.1 indicates that the polynomial decaying rates of an orbit $(e^{tN}x)_{t\geq 0}$ may depend on sequences of time going to infinity; by Proposition 4.1, this will occur if $d^-_{\mu_x^{N_R}}(0) < d^+_{\mu_x^{N_R}}(0)$. We will show (Corollary 4.1) that if $(e^{tN})_{t\geq 0}$ is stable but not exponentially stable, then, Baire generically in $\mathcal{H}$,

$$d^-_{\mu_x^{N_R}}(0) = 0 \quad \text{and} \quad d^+_{\mu_x^{N_R}}(0) = \infty.$$

Proof (Proposition 4.1). Let μ be a finite (positive) Borel measure on $\mathbb{R}$. We show that, for each $w \in \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} = -d_\mu^+(w), \tag{4.2}$$

$$\limsup_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} = -d_\mu^-(w); \tag{4.3}$$

so, Proposition 4.1 becomes a direct consequence of (4.1).

Fix $w \in \mathbb{R}$. If there exists $\epsilon > 0$ such that $\mu(B(w,\epsilon)) = 0$, then, for each $t \geq 0$,

$$\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y) = \int_{B(w,\epsilon)^c} e^{-2t|w-y|} d\mu(y) \leq \mu(\mathbb{R}) e^{-2t\epsilon};$$

thus,

$$\liminf_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} = \limsup_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} = -\infty$$

and the result follows. Suppose that, for each $\epsilon > 0$, $\mu(B(w,\epsilon)) > 0$. Since

$$\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y) \geq \int_{B(w,\frac{1}{t})} e^{-2t|w-y|} d\mu(y) \geq e^{-2} \mu(B(w,\frac{1}{t})),$$

it follows that

$$\liminf_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} \geq -\limsup_{t\to\infty} \frac{\ln \mu(B(w,\frac{1}{t}))}{\ln t^{-1}} = -d_\mu^+(w)$$

and that

$$\limsup_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} \geq -\liminf_{t\to\infty} \frac{\ln \mu(B(w,\frac{1}{t}))}{\ln t^{-1}} = -d_\mu^-(w).$$

Now, let $0 < \delta < 1$. Then, for each $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y) &= \int_{B(w,\frac{1}{t^{1-\delta}})} e^{-2t|w-y|} d\mu(y) + \int_{B(w,\frac{1}{t^{1-\delta}})^c} e^{-2t|w-y|} d\mu(y) \\ &\leq \mu\big(B\big(w,\frac{1}{t^{1-\delta}}\big)\big) + e^{-t^\delta} \mu(\mathbb{R}). \end{aligned} \tag{4.4}$$

Given that it is not possible to compare directly the two terms on the right-hand side of (4.4), one needs to analyse two distinct cases.

Case $d_\mu^-(w) < \infty$: One has, from the definition of $d_\mu^-(w)$, that

$$\liminf_{t\to\infty} \frac{\ln \mu(B(w,\frac{1}{t^{1-\delta}}))}{\ln t^{-(1-\delta)}} = d_\mu^-(w) < \max\{2d_\mu^-(w), 1\} =: \gamma$$

(γ can be defined as any positive number greater than $d_\mu^-(w)$), so

$$\limsup_{t\to\infty} \frac{\ln \mu(B(w,\frac{1}{t^{1-\delta}}))}{\ln t^{(1-\delta)}} > -\gamma.$$

Hence, there exists a sequence (t_k), with $\lim_{k\to\infty} t_k = \infty$, such that, for sufficiently large k,

$$\mu(B(w,\frac{1}{t_k^{1-\delta}})) \geq t_k^{-\gamma(1-\delta)} \geq e^{-t_k^\delta} \mu(\mathbb{R}). \tag{4.5}$$

Now, combining (4.4) and (4.5) one has, for sufficiently large k,

$$\int_{\mathbb{R}} e^{-2t_k|w-y|} d\mu(y) \leq 2\,\mu\big(B\big(w, \frac{1}{t_k^{1-\delta}}\big)\big),$$

which results in

$$\frac{1}{(1-\delta)} \liminf_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} \leq -\limsup_{t\to\infty} \frac{\ln \mu(B(w, \frac{1}{t^{1-\delta}}))}{\ln t^{-(1-\delta)}} = -d_\mu^+(w).$$

Since $0 < \delta < 1$ is arbitrary, the complementary inequality in (4.2) follows.

It remains to prove the complementary inequality in (4.3). This is trivial if $d_\mu^-(w) = 0$, since $\limsup_{t\to\infty} \ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]/\ln t \leq 0$. So, let $d_\mu^-(w) > 0$; it follows from the definition of $d_\mu^-(w)$ that, for each $0 < \epsilon < d_\mu^-(w)$, there exists $t_{\delta,\epsilon} > 0$ such that, for $t > t_{\delta,\epsilon}$,

$$\mu\big(B(w, \frac{1}{t^{1-\delta}})\big) \leq t^{-(1-\delta)(d_\mu^-(w)-\epsilon)}. \tag{4.6}$$

Combining (4.4) with (4.6), one gets, for sufficiently large t,

$$\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y) \leq 2t^{-(1-\delta)(d_\mu^-(w)-\epsilon)}.$$

Thus,

$$\limsup_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} \leq -(1-\delta)(d_\mu^-(w)-\epsilon),$$

and since $0 < \delta < 1$ and $0 < \epsilon < d_\mu^-(w)$ are arbitrary, the result follows.

Case $d_\mu^-(w) = \infty$: One has

$$\lim_{t\to\infty} \frac{\ln \mu(B(w, \frac{1}{t}))}{\ln t} = -\infty\,;$$

given an arbitrary $\alpha > 0$, there is $t_\alpha > 0$ so that, for each $t > t_\alpha$, $\mu(B(w, \frac{1}{t})) \leq t^{-2\alpha}$. Combining this inequality with (4.4) (taking $\delta = \frac{1}{2}$), one obtains, for sufficiently large t,

$$\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y) \leq \mu\big(B\big(w, \frac{1}{t^{1/2}}\big)\big) + e^{-t^{1/2}}\mu(\mathbb{R}) \leq 2t^{-\alpha},$$

from which follows that

$$\liminf_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]}{\ln t} \leq -\alpha$$

and that

$$\limsup_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-y|} d\mu(y)]]}{\ln t} \leq -\alpha;$$

since $\alpha > 0$ is arbitrary, the result follows. □

4.2 Proof of Theorem IV

Next, we present a proof of Theorem IV. The main ingredient of this proof is the relation, given by the Spectral Theorem, between the decaying rates of the semigroup $(e^{tN})_{t\geq 0}$ and the local scale properties of the corresponding spectral measures of N_R. However, some preparation is required.

We recall that a C_0-semigroup $(T(t))_{t\geq 0}$ is weakly stable if it converges to zero as $t \to \infty$ in the weak operator topology.

Theorem 4.1 (Theorem 5.3 in [44]). *Let $(T(t))_{t\geq 0}$ be a weakly stable C_0-semigroup on a Hilbert space $\mathcal{H}$, with generator A, such that $\omega_0(T) = 0$. Let $g : \mathbb{R}_+ \longrightarrow (0, \infty)$ be a bounded function such that $\lim_{t\to\infty} g(t) = 0$ and let $\epsilon > 0$. Then, there exists $x_0 \in \mathcal{H}$ so that $\|x_0\|_{\mathcal{H}} < \sup_{t\geq 0}\{g(t)\} + \epsilon$ and*

$$|\langle T(t)x_0, x_0\rangle| > g(t), \quad \forall t \geq 0.$$

The next result is a particular case of Gearhart's Theorem [33, 36, 48].

Proposition 4.2. *Let N be a normal operator so that $\mathbb{C}_+ \subset \varrho(N)$. Then, $(e^{tN})_{t\geq 0}$ is exponentially stable if and only if $0 \notin \sigma(N_R)$.*

Lemma 4.1. *Let A be a negative self-adjoint operator such that $0 \in \sigma(A)$ and let also $\alpha : \mathbb{R}_+ \longrightarrow (0, \infty)$ such that*

$$\lim_{t\to\infty} \alpha(t) = \infty.$$

Then, there exist $x \in \mathcal{H}$ and a sequence $t_j \to \infty$ such that, for sufficiently large j,

$$\mu_x^A\big(B(0, \frac{1}{t_j})\big) \geq \frac{1}{\alpha(t_j)}.$$

Proof. We note that, by Proposition 4.2, $(e^{tA})_{t\geq 0}$ is not exponentially stable. We also note that it is sufficient to prove the case in which there exists a sequence $s_j \to \infty$ such that $\alpha(s_j) \leq e^{s_j}$, for sufficiently large j. Set $g : \mathbb{R}_+ \longrightarrow (0, \infty)$ such that it satisfies the hypotheses of Theorem 4.1, $\sup_{t\geq 0}\{g(t)\} < 1$ and

$$\lim_{t\to\infty} \frac{1}{g(t)\alpha(\sqrt{t})} = 0. \tag{4.7}$$

Since $\sqrt{g}$ also satisfies the hypotheses of Theorem 4.1, there exists $x \in \mathcal{H}$, $\|x\|_{\mathcal{H}} \leq 1$, such that for every $t > 0$,

$$\begin{aligned} g(t) \leq \|e^{tA}x\|_{\mathcal{H}}^2 &= \int\limits_{\mathbb{R}} e^{2ty} d\mu_x^A(y) \\ &= \int\limits_{B(0,\frac{1}{\sqrt{t}})} e^{2ty} d\mu_x^A(y) + \int\limits_{B(0,\frac{1}{\sqrt{t}})^c} e^{2ty} d\mu_x^A(y) \\ &\leq \mu_x^A\big(B\big(0, \frac{1}{\sqrt{t}}\big)\big) + e^{-\sqrt{t}}. \end{aligned} \tag{4.8}$$

Now, if there does not exist a sequence $t_j \to \infty$ so that, for large enough j,

$$\mu_x^A(B(0, 1/t_j)) \geq \frac{1}{\alpha(t_j)},$$

then, by (4.8), for large enough t,

$$g(t) \leq \frac{1}{\alpha(\sqrt{t})} + e^{-\sqrt{t}},$$

which implies, for large enough j,

$$\frac{2}{g(s_j)\alpha(\sqrt{s_j})} \geq 1;$$

since this contradicts (4.7), the result is proven. □

Proof (Theorem IV). Since $\sup\{Re(\lambda) : \lambda \in \sigma(N)\} = 0$ and $\sigma(N_R) \subset \mathbb{R}_-$ is closed, one has $0 \in \sigma(N_R)$. Therefore, by (4.1), one can assume without loss of generality that N is a self-adjoint operator such that $0 \in \sigma(N) \subset \mathbb{R}_-$; thus, it follows from Proposition 4.2 that $(e^{tN})_{t\geq 0}$ is not exponentially stable.

Since, for each $t \geq 0$, the mapping

$$\mathcal{H} \ni x \longmapsto \alpha(t)\, \|e^{tN}x\|_{\mathcal{H}}$$

is continuous, one has that

$$\begin{aligned} \mathcal{G}_N(\alpha) &:= \{x \mid \limsup_{t\to\infty} \alpha(t)\|e^{tN}x\|_{\mathcal{H}} = \infty\} \\ &= \bigcap_{n\geq 1} \bigcap_{k\geq 1} \bigcup_{t\geq k} \{x \mid \alpha(t)\|e^{tN}x\|_{\mathcal{H}} > n\} \end{aligned}$$

is a G_δ set in $\mathcal{H}$. The proof that

$$\mathcal{G}_N(\beta) := \{x \mid \liminf_{t\to\infty} \beta(t)\|e^{tN}x\|_{\mathcal{H}} = 0\}$$

is a G_δ set in $\mathcal{H}$ is completely analogous, being, therefore, omitted.

Let $\overline{x} \in \mathcal{H}$, let (t_j) be the sequence given by Lemma 4.1 and set, for every $x \in \mathcal{H}$ and every $k \geq 1$,

$$x_k := E^N(D_k)x + \frac{1}{k}\overline{x},$$

where $D_k := (-\infty, -1/k) \cup \{0\} \cup (1/k, \infty)$. It is clear that $x_k \to x$ in $\mathcal{H}$.

Now, since $(e^{tN})_{t\geq 0}$ is stable, $E^N(\{0\}) = 0$. Namely, it follows from the Spectral Theorem and dominated convergence that, for each $x \in \mathcal{H}$,

$$\lim_{t\to\infty} \|e^{tN}x\|^2 = \mu_x^N(\{0\}) + \lim_{t\to\infty} \int_{\mathbb{R}_-\setminus\{0\}} e^{2ty} d\mu_x^N(y) = \mu_x^N(\{0\});$$

thus, $(e^{tN})_{t\geq 0}$ is stable if and only if 0 is not an eigenvalue of N, that is, if and only if $E^N(\{0\}) = 0$. Hence, for each $k \geq 1$ and each j such that $\frac{1}{t_j} < \frac{1}{k}$, one has

$$\begin{aligned}
\mu_{x_k}^N(B(0, \frac{1}{t_j})) &= \langle E^N(B(0, \frac{1}{t_j}))E^N(D_k)x, E^N(D_k)x\rangle + \frac{1}{k^2}\langle E^N(B(0, \frac{1}{t_j}))\overline{x}, \overline{x}\rangle \\
&= \langle E^N(\{0\})x, E^N(D_k)x\rangle + \frac{1}{k^2}\langle E^N(B(0, \frac{1}{t_j}))\overline{x}, \overline{x}\rangle \\
&= \frac{1}{k^2}\mu_{\overline{x}}^N(B(0, \frac{1}{t_j})),
\end{aligned}$$

from which follows that, for sufficiently large j,

$$\begin{aligned}
\alpha(t_j)\|e^{t_jN}x_k\|_{\mathcal{H}} &\geq \alpha(t_j)\Big(\int_{B(0,\frac{1}{t_j})} e^{2t_jy} d\mu_{x_k}^N(y)\Big)^{1/2} \\
&\geq \frac{\alpha(t_j)}{e}\Big(\mu_{x_k}^N(B(0, \frac{1}{t_j}))\Big)^{1/2} \\
&= \frac{\alpha(t_j)}{ke}\Big(\mu_{\overline{x}}^N(B(0, \frac{1}{t_j}))\Big)^{1/2} \\
&\geq \frac{\sqrt{\alpha(t_j)}}{ke}.
\end{aligned}$$

Consequently, for every $k \geq 1$,

$$\limsup_{t\to\infty} \alpha(t)\|e^{tN}x_k\|_{\mathcal{H}} = \infty;$$

this proves that $\mathcal{G}_N(\alpha)$ is a dense set in $\mathcal{H}$.

Now we prove that

$$\mathcal{G}_N(\beta) := \{x \in \mathcal{H} \mid \liminf_{t\to\infty} \beta(t)\|e^{tN}x\|_{\mathcal{H}} = 0\}$$

is a dense set in $\mathcal{H}$. Given $x \in \mathcal{H}$, define, for each $k \geq 1$,

$$x_k := E^N(D_k)x.$$

Then, $x_k \to x$ in $\mathcal{H}$. Moreover, since $E^N(\{0\}) = 0$, it follows that for each $\epsilon < \frac{1}{k}$, $\mu^N_{x_k}(B(0,\epsilon)) = 0$. Thus,

$$\lim_{t\to\infty} \beta(t)\|e^{tN}x_k\|_{\mathcal{H}} \le \lim_{t\to\infty} \beta(t)e^{-t\epsilon}\|x_k\|_{\mathcal{H}} = 0. \tag{4.9}$$

It remains to prove that the assumption on β is optimal. Let $x \in \mathcal{H}$ be such that there exits $\epsilon > 0$ with $\mu^N_x([-\epsilon, 0]) = 0$; then, $\|e^{tN}x\|_{\mathcal{H}} = O(e^{-t\epsilon})$. Set $\alpha(t) = t$ and consider $\mathcal{G}_N(\alpha)$, which is a dense G_δ in $\mathcal{H}$. Then, for each $x \in \mathcal{G}_N(\alpha)$ and each $\epsilon > 0$, $\mu^N_x([-\epsilon, 0]) > 0$. It follows from the Spectral Theorem and Jensen's inequality that

$$\begin{aligned}\frac{1}{\|e^{tN}x\|^2_{\mathcal{H}}} &= \left(\int_{-\infty}^{0} e^{2ty}d\mu^N_x(y)\right)^{-1} \le \left(\int_{-\epsilon}^{0} e^{2ty}d\mu^N_x(y)\right)^{-1} \\ &\le \frac{1}{(\mu^N_x([-\epsilon,0]))^2}\int_{-\epsilon}^{0} e^{-2ty}d\mu^N_x(y) \le \frac{e^{2t\epsilon}}{\mu^N_x([-\epsilon,0])}.\end{aligned}$$

Thus, for each $x \in \mathcal{G}_N(\alpha)$, one has that $\|e^{tN}x\|_{\mathcal{H}}$ vanishes slower than exponential as $t \to \infty$. □

The next result, which is a direct consequence of Proposition 4.1 and Theorem IV, indicates how delicate is the relation between the dynamics of the semigroup and the spectral properties of its generator.

Corollary 4.1. *Let N be as in the statement of Theorem IV. Suppose that $(e^{tN})_{t\ge0}$ is stable. Then,*

$$G_N := \{x \mid d^-_{\mu^{N_R}_x}(0) = 0 \quad \textit{and} \quad d^+_{\mu^{N_R}_x}(0) = \infty\}$$

is a dense G_δ set in $\mathcal{H}$.

Proof. For every $n \ge 1$ and $t > 0$, consider $\alpha_n(t) := t^{1/n}$ and $\beta_n(t) := t^n$. Then, by Proposition 4.1,

$$G_N = \bigcap_{n\ge1} \mathcal{G}_N(\alpha_n, \beta_n).$$

It follows from Theorem IV that G_N is a dense G_δ set in $\mathcal{H}$. □

Remark 4.1.

i) We note that for each N, α and β satisfying the hypotheses of Theorem IV, the set $\mathcal{G}_N(\alpha,\beta)$ has empty interior. Namely, by (4.9)

$$\{x \mid \lim_{t\to\infty} \beta(t)\|e^{tN}x\|_{\mathcal{H}} = 0\}$$

is a dense set in $\mathcal{H}$.

ii) In proof of Theorem VI we exhibit another proof that $\mathcal{G}_N(\alpha)$ is a dense set in $\mathcal{H}$. The above proof explicitly indicates the relation between the decaying rates of the semigroup $(e^{tN})_{t\geq 0}$ and the local scale properties of the corresponding spectral measures of N_R (Lemma 4.1).

4.3 Proof of Theorem V

Proof (Theorem V). Again, by (4.1), we assume without loss of generality that N is a self-adjoint operator such that $N \leq 0$.

1.. This is a direct consequence of Proposition 4.2.

2.. This is a consequence of Theorem IV and the fact that $(e^{tN})_{t\geq 0}$ is stable if and only if 0 is not an eigenvalue of N.

3.. If there exists an $x \in \mathcal{H}$ such that

$$\lim_{t\to\infty} \|e^{tN}x\|^2 = \mu_x^N(\{0\}) > 0,$$

then it follows that 0 is an eigenvalue of N. Therefore, it remains to prove that, generically in $\mathcal{H}$, each orbit of $(e^{tN})_{t\geq 0}$ does not converge to zero if 0 is an eigenvalue of N. Since, for each $t \geq 0$, the mapping

$$\mathcal{H} \ni x \longmapsto \|e^{tN}x\|_{\mathcal{H}}$$

is continuous, it follows that

$$\begin{aligned} \mathcal{F}_N := \{x \mid \lim_{t\to\infty} \|e^{tN}x\|_{\mathcal{H}} > 0\} &= \{x \mid \limsup_{t\to\infty} \|e^{tN}x\|_{\mathcal{H}} > 0\} \\ &= \bigcap_{k\geq 1} \bigcup_{t\geq k} \{x \mid \|e^{tN}x\|_{\mathcal{H}} > 0\} \end{aligned}$$

is a G_δ set in $\mathcal{H}$.

Now, given $x \in \mathcal{H}$, write $x = x_1 + x_2$, with $x_1 \in \operatorname{Span}\{x_0\}^\perp$ and $x_2 \in \operatorname{Span}\{x_0\}$, where x_0, with $\|x_0\|_{\mathcal{H}} = 1$, is an eigenvector of N associated with the eigenvalue 0. If $x_2 \neq 0$, then

$$\begin{aligned} \|e^{tN}x\|_{\mathcal{H}}^2 &= \mu_x^N(\{0\}) + \int_{\mathbb{R}_-\setminus\{0\}} e^{2ty} d\mu_x^N(y) \geq \mu_x^N(\{0\}) = \|E^N(\{0\})x\|_{\mathcal{H}}^2 \\ &\geq 2\, R\langle E^N(\{0\})x_1, E^N(\{0\})x_2\rangle + \|E^N(\{0\})x_2\|_{\mathcal{H}}^2 \\ &= \|x_2\|_{\mathcal{H}}^2, \end{aligned}$$

from which follows that

$$\lim_{t\to\infty} \|e^{tN}x\|_{\mathcal{H}} > 0.$$

Now, if $x_2 = 0$, define, for each $k \geq 1$,

$$x_k := x + \frac{x_0}{k}.$$

It is clear that $x_k \to x$. Moreover, by the previous arguments, one has, for each $k \geq 1$,

$$\lim_{t\to\infty} \|e^{tN}x_k\|_{\mathcal{H}} > 0.$$

This proves that $\mathcal{F}_N$ is a dense set in $\mathcal{H}$. □

4.4 Proof of Theorem VI

The proof of Theorem VI relies, once more, on Theorem 4.1.

Proof (Theorem V). The proof that each one of the sets

$$\mathcal{G}_A(\alpha) := \{x \mid \limsup_{t\to\infty} \alpha(t)\|T(t)x\|_{\mathcal{H}} = \infty\}$$

and

$$\mathcal{G}_A(\beta) := \{x \mid \liminf_{t\to\infty} \beta(t)\|T(t)x\|_{\mathcal{H}} = 0\}$$

is a G_δ set in $\mathcal{H}$ follows the same reasoning presented in the proof of Theorem IV.

Since, by hypothesis (H), there exists $C > 0$ such that, for every $x \in \mathcal{D}(A^k)$ and every sufficiently large t,

$$\|T(t)x\|_{\mathcal{H}} \leq Cr(t)\|A^k x\|_{\mathcal{H}},$$

it follows that, for every $x \in \mathcal{D}(A^k)$,

$$\liminf_{t\to\infty} \beta(t)\|T(t)x\|_{\mathcal{H}} = 0.$$

Thus, $\mathcal{G}_A(\beta) \supset \mathcal{D}(A^k)$ is a dense set in $\mathcal{H}$.

It remains to prove that $\mathcal{G}_A(\alpha)$ is dense in $\mathcal{H}$. We note that, by Theorem 4.1, there exists $x_0 \in \mathcal{H}$ such that $\|x_0\|_{\mathcal{H}} \leq 1$ and, for each $t \geq 0$,

$$\|T(t)x_0\|_{\mathcal{H}} > \frac{1}{\sqrt{\alpha(t)+2}}. \tag{4.10}$$

So, given $x \in \mathcal{H}$, suppose now that, for each $k \geq 1$, there exists a sequence $t_j \to \infty$ so that, for each j,

$$\|T(t_j)x\|_{\mathcal{H}} < \frac{\|T(t_j)x_0\|_{\mathcal{H}}}{4k}; \tag{4.11}$$

otherwise, it follows from (4.10) that $x \in \mathcal{G}_A(\alpha)$. Set, for each $k \geq 1$,

$$x_k := x + \frac{x_0}{k}.$$

It is clear that $x_k \to x$ in $\mathcal{H}$. Moreover, by (4.11), for each $k \geq 1$ and each j,

$$\begin{aligned} \|T(t_j)x_k\|_{\mathcal{H}}^2 &\geq -\frac{2\|T(t_j)x\|_{\mathcal{H}}\|T(t_j)x_0\|_{\mathcal{H}}}{k} + \frac{\|T(t_j)x_0\|_{\mathcal{H}}^2}{k^2} \\ &\geq \frac{\|T(t_j)x_0\|_{\mathcal{H}}^2}{2k^2}. \end{aligned} \tag{4.12}$$

Thus, combining (4.10) and (4.12) it follows that, for each $k \geq 1$,

$$\limsup_{t\to\infty} \alpha(t)\|T(t)x_k\|_{\mathcal{H}} = \infty.$$

□

Chapter 5

Category theorems for Schrödinger semigroups

5.1 Proof of Theorem VII

We recall that a metric space (X, d) of negative self-adjoint operators, acting in $\mathcal{H}$, is called regular if it is complete and convergence with respect to d implies strong resolvent convergence of operators.

Proposition 5.1. *Let (X, d) be a regular space of negative self-adjoint operators. Suppose that*

1. *$\{A \in X \mid 0 \in \sigma(A)\}$ is dense in X,*
2. *$\{A \in X \mid 0 \notin \sigma(A)\}$ is dense in X.*

Then,

$$\{A \in X \mid (e^{tA})_{t\geq 0} \text{ is stable but not exponentially stable}\}$$

is a dense G_δ set in X.

Proof. To prove Proposition 5.1, it is enough to show that

1. $E := \{A \in X \mid (e^{tA})_{t\geq 0} \text{ is exponentially stable}\}$ is meager in X,
2. $Y := \{A \in X \mid (e^{tA})_{t\geq 0} \text{ is stable}\}$ is a dense G_δ set in X.

Firstly, let us show that E is an F_σ set in X. It follows from Proposition 2.1 that each section of the mapping

$$\mathbb{R}_+ \times \mathcal{H} \times X \ni (t, x, A) \mapsto \|e^{tA}x\|_{\mathcal{H}}$$

is continuous. Thus, for each $n \geq 1$, the mapping

$$X \ni A \mapsto \sup_{t\geq 0} \sup_{\|x\|_{\mathcal{H}}=1} e^{\frac{t}{n}} \|e^{tA}x\|_{\mathcal{H}} = \sup_{t\geq 0} e^{\frac{t}{n}} \|e^{tA}\|_{\mathcal{B}(\mathcal{H})}$$

is lower semicontinuous, from which it follows that, for each $n \geq 1$, the set

$$F_n = \{A \in X \mid \sup_{t\geq 0} e^{\frac{t}{n}} \|e^{tA}\|_{\mathcal{B}(\mathcal{H})} \leq 1\}$$

is closed.

Since the inclusion $\cup_{n\geq 1} F_n \subset E$ is immediate, we just need to prove that $E \subset \cup_{n\geq 1} F_n$. Let $A \in E$; then, by Proposition 4.2, one has that $a := -\sup \sigma(A) > 0$. Nevertheless, $\sup_{t\geq 0} e^{ta} \|e^{tA}\|_{\mathcal{B}(\mathcal{H})} \leq 1$, from which it follows that $A \in \cup_{n\geq 1} F_n$. Thus, E is an F_σ in X.

Now, since $\{A \in X \mid 0 \in \sigma(A)\}$ is dense in X, it follows from Proposition 4.2 that E^c is dense in X; therefore, E is meager in X and 1. is proven.

It remains to prove 2. By the previous arguments, given $x \in \mathcal{H}$, for each $k \geq 1$ and each $n \geq 1$,

$$\bigcup_{t\geq k} \{A \in X \mid \|e^{tA}x\|_{\mathcal{H}} < \frac{1}{n}\}$$

is open, hence

$$\begin{aligned} Y_x := \{A \in X \mid \lim_{t\to\infty} \|e^{tA}x\|_{\mathcal{H}} = 0\} &= \{A \in X \mid \liminf_{t\to\infty} \|e^{tA}x\|_{\mathcal{H}} = 0\} \\ &= \bigcap_{n\geq 1} \bigcap_{k\geq 1} \bigcup_{t\geq k} \{A \in X \mid \|e^{tA}x\|_{\mathcal{H}} < \frac{1}{n}\} \end{aligned}$$

is a G_δ set in X. Since, by Proposition 4.2, $\{A \in X \mid 0 \notin \sigma(A)\} \subset Y_x$ is dense in X, it follows that Y_x is a dense G_δ set in X.

Finally, let $\cup_{k\geq 1}\{x_k\}$ be a dense subset in $\mathcal{H}$ (which is separable). Then,

$$Y = \bigcap_{k\geq 1} Y_{x_k}.$$

Namely, the inclusion $Y \subset \bigcap_{k\geq 1} Y_{x_k}$ is obvious, and the reciprocal one follows from the fact that, for each $A \in \bigcap_{k\geq 1} Y_{x_k}$ and each $x \in \mathcal{H}$, by Moore-Osgood Theorem,

$$\lim_{t\to\infty} \|e^{tA}x\|_{\mathcal{H}} = \lim_{k\to\infty} \lim_{t\to\infty} \|e^{tA}x_k\|_{\mathcal{H}} = 0.$$

Thus, by Baire's Theorem, Y is a dense G_δ set in X and 2. is proven. □

Definition 5.1. Let T be a self-adjoint operator in $\mathcal{H}$. The essential spectrum of T is the set $\sigma_{\text{ess}}(T)$ of the accumulation points of $\sigma(T)$ together with the eigenvalues of T of infinite multiplicity.

Proposition 5.2 (Weyl's criterion)**.** *Let $V : \mathbb{R}^\nu \longrightarrow \mathbb{C}$ be a mensurable Borel function such that*

$$\lim_{|x|\to\infty} |V(x)| = 0.$$

Then, the essential spectrum of $H_V = \Delta + V$ is $\sigma_{\rm ess}(H_V) = (-\infty, 0]$.

Theorem VII is a consequence of Proposition 5.1 and Proposition 5.2.

Proof (Theorem VII). By Proposition 5.1, we just need to show that, for every $l > 0$ and every $\nu \in \mathbb{N}$,

1. $\{H_V \in X_l^\nu \mid 0 \in \sigma(H_V)\}$ is dense in X_l^ν,
2. $\{H_V \in X_l^\nu \mid 0 \notin \sigma(H_V)\}$ is dense in X_l^ν.

Let $H_V \in X_l^\nu$ and define, for each $k \geq 1$, $V_k := \chi_{B(0,k)}V$. Then, by Weyl's criterion, the essential spectrum of H_{V_k} is given by $\sigma_{\rm ess}(H_{V_k}) = (-\infty, 0]$; moreover, $H_{V_k} \to H_V$ in X_l^ν. Thus, $\{H_V \in X_l^\nu \mid 0 \in \sigma(H_V)\}$ is dense in X_l^ν.

Now, let (H_{V_j}) be a sequence in X_l^ν such that, for each $j \geq 1$,

$$V_j := \frac{j}{j+1}V - \frac{l}{j+1}.$$

It is clear that, for each $j \geq 1$, $0 \notin \sigma(H_{V_j})$. Moreover, $H_{V_j} \to H_V$ in X_l^ν. Therefore, $\{H_V \in X_l^\nu \mid 0 \notin \sigma(H_V)\}$ is dense in X_l^ν. □

5.2 Proof of Theorem VIII

Proof (Theorem VIII). Since, by Proposition 4.1,

$$\begin{aligned} J_l^\nu(f) &= \{H \mid d^-_{\mu_f^H}(0) = 0 \text{ and } d^+_{\mu_f^H}(0) = \infty\} \\ &= \bigcap_{n\geq 1} \big\{H \mid \limsup_{t\to\infty} t^{1/n} \|e^{tH}f\|_{\mathrm{L}^2(\mathbb{R}^\nu)} = \infty \text{ and } \liminf_{t\to\infty} t^n \|e^{tH}f\|_{\mathrm{L}^2(\mathbb{R}^\nu)} = 0\big\}, \end{aligned}$$

it follows from the arguments presented in the proof of Proposition 5.1 that, for every $f \in \mathrm{L}^2(\mathbb{R}^\nu)$, $J_l^\nu(f)$ is a G_δ set in X_l^ν.

Now, let

$$C_l^\nu = \{H \in X_l^\nu \mid (e^{tH})_{t\geq 0} \text{ is stable but not exponentially stable}\}.$$

It follows from Theorem VII that C_l^ν is a dense G_δ set in X_l^ν. Let $\cup_{k\geq 1}(H_k)$ be an enumerable dense subset of C_S^ν (which is separable, since X_l^ν is separable). Then, by Corollary 4.1,

$$G_l^\nu := \bigcap_{k\geq 1} \{f \mid d^-_{\mu_f^{H_k}}(0) = 0 \text{ and } d^+_{\mu_f^{H_K}}(0) = \infty\}$$

is a dense G_δ set in $\mathrm{L}^2(\mathbb{R}^\nu)$. Nevertheless, for every $f \in G_l^\nu$, $J_l^\nu(f) \supset \cup_{k\geq 1} H_k$ is a dense G_δ set in X_l^ν. □

The next example, together with Theorem VIII, says that for each $f \in C_l^\nu$ and each $H \in J_l^\nu(f)$, $f \notin \operatorname{rng} H$, or equivalent, that the partial differential equation

$$Hu = f$$

does not have a solution in $\mathcal{D}(H)$.

Example 5.1. Let A be a negative self-adjoint operator. Then, for $u \in \operatorname{rng} A$, $d^{\mp}_{\mu_u^A}(0) \geq 2$.

Example 5.1 can be seen as a statement from the fact that every spectral measure associated with every vector of the range of a negative self-adjoint operator has a certain local regularity with respect to the Lebesgue measure. We note that this is a direct consequence of Propositions 4.1 and 5.3.

Proposition 5.3. *Let A be a negative self-adjoint operator. Then, for $u \in \operatorname{rng} A$ and every $x \in A^{-1}\{u\}$, one has, for each $t > 0$,*

$$\|e^{tA}u\|_{\mathcal{H}} \leq \frac{\|x\|_{\mathcal{H}}}{e\, t}.$$

Proof. Let $x \in A^{-1}\{u\}$. Then, by the Spectral Theorem, for each $t > 0$,

$$\begin{aligned} t^2\|e^{tA}u\|^2_{\mathcal{H}} &= t^2\|Ae^{tA}x\|^2_{\mathcal{H}} = \int_{-\infty}^{0} (ty)^2\ e^{2ty}\, \mathrm{d}\mu_x^A(y) \\ &\leq \frac{1}{e^2}\int_{-\infty}^{0} 1\, \mathrm{d}\mu_x^A(y) = \frac{\|x\|^2_{\mathcal{H}}}{e^2}. \end{aligned}$$

□

Remark 5.1.

i) We note that the polynomial decaying rate obtained in Proposition 5.3 is optimal. Namely, define $M : \mathcal{D}(M) \subset \mathrm{L}^2[0,\infty) \longrightarrow \mathrm{L}^2[0,\infty)$ by the action

$$(Mu)(y) = -yu(y),$$

where $u \in \mathcal{D}(M) := \{u \in \mathrm{L}^2[0,\infty) \mid yu \in \mathrm{L}^2[0,\infty)\}$. Consider $\frac{1}{2} < \delta < 1$, and then define $f_\delta : [0,\infty) \to \mathbb{R}$ by the action $f_\delta(y) = \chi_{[0,1]}y^\delta$; f_δ clearly belongs to $\operatorname{rng} M$. Moreover, for every $0 < \epsilon \leq 1$,

$$\mu^M_{f_\delta}(B(0,\epsilon)) = \int_0^\epsilon |f_\delta|^2 dy = \int_0^\epsilon y^{2\delta} dy = \frac{\epsilon^{2\delta+1}}{2\delta+1}.$$

Thus, by Proposition 4.1, $\|e^{tM} f_\delta\|_{\mathrm{L}^2[0,\infty)} \geq C_{f_\delta} t^{-1/2-\delta}$, for all $t \geq 1$, where C_{f_δ} is a constant depending only on f_δ.

ii) Let A be as in the statement of Proposition 5.3. If $a = -\sup \sigma(A) > 0$, then $(e^{tA})_{t\geq 0}$ is exponentially stable. Actually, $\|e^{tA}\|_{\mathcal{B}(\mathcal{H})} = O(e^{-ta})$, which implies that, for each $x \in \mathcal{H}$, $\|e^{tA}x\|_{\mathcal{H}} = O(e^{-ta})$. Proposition 5.3 presents more information about the decay of $\|e^{tA}u\|_{\mathcal{H}}$ in case $u \in \mathrm{rng}(A + a\mathbf{1})$, since, in this case, it shows that there exists $C_u > 0$, depending only on u, such that, for every $t > 0$,

$$\|e^{tA}u\|_{\mathcal{H}} \leq C_u \, \frac{e^{-ta}}{t}.$$

Namely, let $t \in \mathbb{R}$ and $v \in \mathcal{H}$; then,

$$\|v\|_{\mathcal{H}} = \|e^{-ta\mathbf{1}} e^{ta\mathbf{1}} v\|_{\mathcal{H}} \leq \|e^{-ta\mathbf{1}}\|_{\mathcal{B}(\mathcal{H})} \|e^{ta\mathbf{1}} v\|_{\mathcal{H}} \leq e^{-ta} \|e^{ta\mathbf{1}} v\|_{\mathcal{H}}.$$

If $u \in \mathrm{rng}(A + a\mathbf{1})$ and $x \in \mathcal{D}(A)$ with $(A + a\mathbf{1})x = u$, then, by Proposition 5.3,

$$\|x\|_{\mathcal{H}} \, \frac{1}{e\,t} \geq \|e^{t(A+a\mathbf{1})} u\|_{\mathcal{H}} = \|e^{ta\mathbf{1}} e^{tA} u\|_{\mathcal{H}} \geq e^{ta} \|e^{tA} u\|_{\mathcal{H}}.$$

Appendix A

Spectral theorem for normal operators

In this Appendix, we present some results in the spectral theory of normal operators. The material presented here is based on [22, 51].

We recall that, for each densely defined linear operator A in $\mathcal{H}$, corresponds a unique linear operator A^* in $\mathcal{H}$, the so-called Hilbert adjoint of A, whose domain $\mathcal{D}(A^*)$ consist of all $\eta \in \mathcal{H}$ for which the linear functional

$$\mathcal{D}(A) \ni \xi \mapsto \langle A\xi, \eta\rangle$$

is continuous; by the Hahn-Banach theorem, such functional can be continuously extended to $\mathcal{H}$ and, therefore, there exists an element unique $A^*\eta \in \mathcal{H}$ so that

$$\langle A\xi, \eta\rangle = \langle \xi, A^*\eta\rangle, \;\; \xi \in \mathcal{D}(A).$$

Definition A.1. Let $A : \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator. One says that A is a symmetric operator if

$$\langle A\xi, \eta\rangle = \langle \xi, A\eta\rangle, \;\; \xi, \eta \in \mathcal{D}(A).$$

If A is densely defined and $A = A^*$, then one says that A is a self-adjoint operator.

Definition A.2. A densely defined linear operator $A : \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is said to be normal if is closed and if

$$A^*A = AA^*.$$

Now we present some examples of normal and self-adjoint operators [22].

Example A.1 (Multiplication operator)**.** Let μ be a σ-finite positive Borel measure over a metric space X. Let $\varphi : F \subset X \longrightarrow \mathbb{C}$ be a measurable function. The respective multiplication operator by φ, M_φ, is the linear operator given by

$$\mathcal{D}(M_\varphi) := \{\psi \in \mathrm{L}^2_\mu(F) : \varphi\psi \in \mathrm{L}^2_\mu(F)\},$$

$$(M_\varphi\psi)(x) := \varphi(x)\psi(x),\ \ \psi \in \mathcal{D}(M_\varphi).$$

It is not hard to show that $\mathcal{D}(M_\varphi)$ is dense in $\mathrm{L}^2_\mu(F)$ and that $M^*_\varphi = M_{\overline{\varphi}}$, from which follows that M_φ is a normal operator. If φ is a real function, then M_φ is a self-adjoint operator.

Example A.2 (Schrödinger operator)**.** A Schrödinger operator defined in $\mathcal{H}^2(\mathbb{R}^\nu)$, $\nu \in \mathbb{N}$, is a linear operator of the form

$$H = -\Delta + V,$$

where Δ is the (self-adjoint) Laplacian and V a real-valued multiplication operator (the so-called potential), so that $-\Delta + V$ is a self-adjoint operator in $\mathcal{H}^2(\mathbb{R}^\nu)$ (for instance, by the Kato-Rellich theorem, V can be any bounded Borel function [22]).

We note that for every measurable Borel $V : \mathbb{R}^\nu \longrightarrow \mathbb{C}$, $V \in \mathrm{L}^2_{Loc}(\mathbb{R}^\nu)$

$$-\Delta + V : C_0^\infty(\mathbb{R}^\nu) \subset \mathrm{L}^2(\mathbb{R}^\nu) \longrightarrow \mathrm{L}^2(\mathbb{R}^\nu)$$

is always a symmetric operator, but not necessarily self-adjoint.

A.1 Resolution of the identity

Every normal operator corresponds to a unique resolution of the identity (and vice versa), and this is what allows us to present a complete description of each normal operator.

Definition A.3. Denote by $\mathcal{A}$ the Borel σ-algebra in $\Omega \subset \mathbb{C}$. A resolution of the identity is a mapping

$$\mathcal{A} \ni \Lambda \mapsto E(\Lambda) \in \mathcal{B}(\mathcal{H})$$

with the following properties:

1. $E(\emptyset) = 0$ and $E(\Omega) = I$.

2. Every $E(\Lambda)$ is an orthogonal projection.

3. If $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $E(\Lambda_1 \cup \Lambda_2) = E(\Lambda_1) + E(\Lambda_2)$.

4. $E(\Lambda_1 \cap \Lambda_2) = E(\Lambda_1)E(\Lambda_2)$.

5. For every $\xi \in \mathcal{H}$ and every $\eta \in \mathcal{H}$,

$$\mathcal{A} \ni \Lambda \mapsto \langle E(\Lambda)\xi, \eta \rangle$$

is a regular Borel complex measure, the so-called spectral measure of E with respect to ξ, η, denoted by $\mu^E_{\xi,\eta}$.

We note that for $\xi = \eta$, the spectral measure of E with respect to ξ is always a real-valued measure, since $E(\Lambda) \geq 0$; in this case, we denote it by μ^E_ξ.

Example A.3. Consider the normal operator M_φ, for $\varphi : F \subset X \longrightarrow \mathbb{C}$, acting in $\mathrm{L}^2_\mu(F)$, defined in Example A.1. The mapping [22]

$$\mathcal{A} \ni \Lambda \mapsto E^{M_\varphi}(\Lambda) := \chi_{\varphi^{-1}(\Lambda)}$$

is the resolution of the identity of M_φ.

Example A.4. Let N be a normal operator on a finite-dimensional complex Hilbert space $\mathcal{H}$. It is well know that there exists an orthonormal basis of $\mathcal{H}$ whose elements are the eigenvectors of N, corresponding to the eigenvalues λ_j. The mapping [22]

$$\mathcal{A} \ni \Lambda \mapsto E^N(\Lambda) := \sum_{j:\lambda_j \in \Lambda} E_j,$$

where each E_j represents the orthogonal projection onto the eigenspace corresponding to the eigenvalue λ_j, defines the resolution of the identity of N. We note that

$$N = \sum_{j:\lambda_j \in \mathbb{C}} \lambda_j E_j. \tag{A.1}$$

The next result lists some properties of a resolution of the identity [51].

Theorem A.1 (Functional Calculus)**.** *Let E be a resolution of the identity. Then:*

1. *For each measurable $f : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ corresponds a closed densely defined linear operator*

$$E(f) = \int_\Omega f(\lambda)\ dE(\lambda)$$

in $\mathcal{H}$, such that, for every $\xi \in \mathcal{D}(E(f))$,

$$\|E(f)\xi\|^2 = \int_\Omega |f(\lambda)|^2\ d\mu^E_\xi(\lambda).$$

2. If $f, g : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ are measurable, then

$$E(f)E(g) \subset E(fg) \text{ and } \mathcal{D}(E(f)E(g)) = \mathcal{D}(E(g)) \cap \mathcal{D}(E(fg)).$$

Hence, $E(f)E(g) = E(fg)$ if and only if $\mathcal{D}(E(fg)) \subset \mathcal{D}(E(g))$.

3. For every measurable $f : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$,

$$E(f)^* = E(\overline{f}) \text{ and } E(f)E(f)^* = E(|f|^2) = E(f)^*E(f).$$

A.2 Spectral theorem

We present below a version of the Spectral Theorem, which says that any normal operator, in some sense, can be written as (A.1) [51].

Theorem A.2 (Spectral Theorem)**.** *Every normal operator N corresponds to a unique resolution E^N of the identity such that*

$$N = \int_{\sigma(N)} \lambda \; dE^N(\lambda).$$

Moreover, E^N is supported on $\sigma(N) \subset \mathbb{C}$, in the sense that $E^N(\sigma(N)) = I$.

Appendix B

Unitary evolution groups

In this Appendix, we introduce some concepts concerning unitary evolution groups [22]. A major interest here is the in solutions of the Schrödinger equation

$$\begin{cases} \partial_t \xi = -iT\xi, \ t \in \mathbb{R}, \\ \xi(0) = \xi, \ \xi \in \mathcal{H}, \end{cases} \tag{SE}$$

where T is a self-adjoint operator in a separable complex Hilbert space $\mathcal{H}$.

B.1 Definitions and examples

Definition B.1. A transformation $G : \mathbb{R} \longrightarrow \mathcal{B}(\mathcal{H})$ is a one-parameter unitary evolution group, or simply a unitary evolution group, on $\mathcal{H}$ if $G(t)$ is a unitary operator onto $\mathcal{H}$ and $G(t+s) = G(t)G(s)$, $\forall\, t, s \in \mathbb{R}$.

Definition B.2. The generator of a unitary evolution group $G(t)$ is the linear operator T defined by

$$\mathcal{D}(T) := \{\xi \in \mathcal{H} \mid \lim_{h\to 0} \frac{G(h)\xi - \xi}{h} \text{ exists}\},$$

$$T\xi := i \lim_{h\to 0} \frac{G(h)\xi - \xi}{h}, \ \ \xi \in \mathcal{D}(T).$$

Example B.1 (Multiplication group)**.** Consider M_φ, acting in $\mathrm{L}^2_\mu(F)$, the self-adjoint operator defined in Example A.1 for $\varphi : F \subset X \longrightarrow \mathbb{C}$ real. It is easy to check that $e^{-iM_\varphi t} : \mathrm{L}^2_\mu(F) \longrightarrow \mathrm{L}^2_\mu(F)$ defined, for each $t \in \mathbb{R}$ and each $f \in \mathrm{L}^2_\mu(F)$, by

$$(e^{-iM_\varphi t} f)(x) = e^{-i\varphi(x)t} f(x), \ \ x \in F,$$

is a unitary evolution group whose generator is M_φ.

Example B.2 (Schrödinger group)**.** Consider a Schrödinger operator $H = -\Delta + V$, acting in $\mathcal{H}^2(\mathbb{R}^\nu)$, defined in Example A.2. By the Functional Calculus, e^{-iHt} is a unitary evolution group whose generator is H.

B.2 Stone theorem

The next results show that there exists a one-to-one relation between self-adjoint operators and unitary evolution groups.

Theorem B.1. *If T is self-adjoint, there exists a unitary evolution group $G(t)$ for which T is its generator. In this case, one writes $G(t) = e^{-iTt}$, $t \in \mathbb{R}$.*

Theorem B.2 (Stone)**.** *If $G(t)$ is a unitary evolution group on $\mathcal{H}$, then its generator T is self-adjoint, that is, $G(t) = e^{-iTt}$, $t \in \mathbb{R}$.*

Corollary B.1. *Let $G : \mathbb{R} \longrightarrow \mathcal{B}(\mathcal{H})$ be a unitary evolution group. Then, its generator T is self-adjoint and, therefore, $G(t) = e^{-iTt}$, $t \in \mathbb{R}$. Moreover, for every $\xi \in \mathcal{D}(T)$, the curve $\xi(t) := e^{-iTt}\xi$ in $\mathcal{H}$ is the unique solution of* (SE).

Remark B.1. We note that since $\mathcal{D}(T)$ is dense in $\mathcal{H}$, for every $\xi \in \mathcal{H}$, there exists a sequence $(\xi_n) \subset \mathcal{D}(T)$ such that for every n, (SE_{ξ_n}) has a unique solution $\xi(t, \xi_n)$ with $\lim_{n\to\infty} \xi(t, \xi_n) = \xi(t)$ uniformly for $t \geq 0$. Thus, if $\xi \notin \mathcal{D}(T)$, one says that $\xi(t)$, although not differentiable, is a (weak) solution of (SE).

Example B.3. Consider a Schrödinger operator $H = -\Delta + V$, acting in $\mathcal{H}^2(\mathbb{R}^\nu)$, and the respective Schrödinger group e^{-iHt} that is directly connected to the Schrödinger equation. Namely, if $f \in \mathrm{L}^2(\mathbb{R}^\nu)$, then

$$u(x,t) = (e^{-iHt} f)(x), \quad t \in \mathbb{R},$$

is the solution to the Schrödinger equation

$$\begin{cases} i\partial_t u = -\Delta u + V u, \; t \in \mathbb{R}, \\ u_0(x) = f(x) \quad \text{for all } x \in \mathbb{R}^\nu. \end{cases}$$

Appendix C

C_0-semigroups and their generators

Finally, we recall the main concepts of the theory of C_0-semigroups [34, 46, 60]. The most basic results of the theory are presented. Throughout this section, X represents a Banach space.

C.1 Definitions and examples

Definition C.1. A family $(T(t))_{t\geq 0}$ of bounded linear operators acting on X is called a C_0-semigroup if the following properties are satisfied:

1. $T(0) = I$ and $T(t+s) = T(t)T(s), \quad t, s \geq 0,$
2. $\lim_{t\downarrow 0} \|T(t)x - x\|_X = 0, \ \forall\, x \in X.$

Remark C.1. It is easy to see that there are constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{B}(X)} \leq Me^{\omega t}, \ \ t \geq 0,$$

and that, for each $x \in X$, the mapping $[0,\infty) \ni t \longmapsto T(t)x$ is continuous.

Definition C.2. The generator of a C_0-semigroup $(T(t))_{t\geq 0}$ is the linear operator A defined by

$$\mathcal{D}(A) := \{x \in X \mid \lim_{h\downarrow 0} \frac{T(h)x - x}{h} \text{ exists}\},$$

$$Ax := \lim_{h\downarrow 0} \frac{T(h)x - x}{h}, \ \ x \in \mathcal{D}(A).$$

We recall that a C_0-semigroup $(T(t))_{t\geq 0}$ on X is said to be bounded if there exists a constant $C > 0$ so that, for each $t \geq 0$, $\|T(t)\|_{\mathcal{B}(\mathcal{H})} \leq C$; if $C = 1$, then it is called a C_0-semigroup of contractions.

Example C.1. Let $a \in \mathbb{R}$. It follows from the Spectral Theorem that every normal operator N in a Hilbert space $\mathcal{H}$, with $\{\lambda \in \mathbb{C} \mid Re(\lambda) > a\} \subset \varrho(N)$, generates the normal C_0-semigroup [46]

$$e^{tN} = \int_{\sigma(N)} e^{t\lambda} \; dE^N(\lambda).$$

We note that $(e^{tN})_{t\geq 0}$ is of contractions if and only if $a \leq 0$.

Example C.2. Consider $X = \mathrm{L}^p[0,1]$ with the norm $\|u\|_X = \|e^{-y}u(y)\|_{\mathrm{L}^p(0,1)}$, $1 \leq p < \infty$. Define the interacted fractional integral of order t of $u \in X$ by [34]

$$(I^t u)(y) := \frac{1}{\Gamma(t)} \int_0^y (y-w)^{t-1} u(w)\, dw, \;\; y \in [0,1], \;\; t > 0,$$

where $\Gamma(\cdot)$ denotes the Gamma function; $I^0 := I$. $(I^t)_{t\geq 0}$ is an unbounded C_0-semigroup whose generator has empty spectrum.

Now we recall that a classical solution of the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t),\, t \geq 0, \\ x(0) = x,\, x \in X, \end{cases} \tag{ACP}$$

is a continuously differentiable function $u : [0,\infty) \longrightarrow X$, taking its values in $\mathcal{D}(A)$, which satisfies (ACP). A continuous function $u : [0,\infty) \longrightarrow X$ is a mild solution of (ACP) if there exists a sequence $(x_n) \subset \mathcal{D}(A)$ such that for each n, the problem (ACP$_{x_n}$) has a classical solution $u(t,x_n)$ with $\lim_{n\to\infty} u(t,x_n) = u(t)$ locally uniformly for $t \geq 0$.

The next theorem says that (ACP) has a solution for every $x \in X$, and this solution is $T(t)x$ [46].

Theorem C.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup and let A be its generator. Then,*

i) A is a closed densely defined linear operator.

ii) For each $x \in \mathcal{D}(A)$, $t \longmapsto T(t)x$ is continuously differentiable for $t \geq 0$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \;\; t \geq 0.$$

C.2 Hille-Yosida and Lumer-Phillips theorems

Hille-Yosida's and Lumer-Phillips's Theorems [46] give conditions on the behaviour of the resolvent of a linear operator A, which are necessary and sufficient for A to be the generator of a C_0-semigroup.

Theorem C.2 (Hille-Yosida's Theorem)**.** *A linear operator* $A : \mathcal{D}(A) \subset X \longrightarrow X$ *is the generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *such that, for constants* $\omega \geq 0$ *and* $M \geq 1$*,*

$$\|T(t)\|_{\mathcal{B}(X)} \leq Me^{\omega t}, \ \ t \geq 0,$$

if and only if

1. *A is closed and densely defined;*

2. *The resolvent set* $\varrho(A)$ *of* A *contains* (ω, ∞)*, and for each* $\lambda > \omega$*,*

$$\|R(\lambda, A)\|_{\mathcal{B}(X)} \leq \frac{1}{\lambda - \omega}.$$

Let X^* be the dual to X. We denote the value of $x^* \in X$ at $x \in X$ by $\langle x, x^* \rangle$. For every $x \in X$, we define the duality set $J(x) \subset X^*$ by

$$J(x) = \{x^* : \langle x, x^* \rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2\}.$$

It follows from the Hahn-Banach theorem that, for every $x \in X$, $J(x) \neq \emptyset$.

Definition C.3. A linear operator A is dissipative if, for every $x \in \mathcal{D}(A)$, there exists a $x^* \in J(x)$ such that $Re\langle Ax, x^* \rangle \leq 0$.

Theorem C.3 (Lumer-Phillips's Theorem)**.** *Let* A *be a linear operator with dense domain* $\mathcal{D}(A)$ *in* X*.*

1. *If* A *is dissipative and there exists a* $\lambda_0 > 0$ *such that* $\mathrm{rng}(\lambda_0 I - A) = X$*, then* A *is the generator of a* C_0*-semigroup of contractions.*

2. *If* A *is the generator of a* C_0*-semigroup of contractions, then* $\mathrm{rng}(\lambda I - A) = X$ *for every* $\lambda > 0$ *and* A *is dissipative. Moreover, for every* $x \in \mathcal{D}(A)$ *and every* $x^* \in J(x)$*,* $Re\langle Ax, x^* \rangle \leq 0$*.*

Hille-Yosida's and Lumer-Phillips's Theorems are employed to show the existence of solutions for evolution equations with energy dissipation, for instance, wave equations, thermoelastic systems of Bresse type and Timoshenko type. We note that in the literature, there are other results, similar to such theorems, which are often applied to systems without energy dissipation (see [54] and references therein).

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