UNIVERSIDADE FEDERAL DE MINAS GERAIS

PhD Thesis

Mathematical Theory of Incompressible Flows: Local Existence, Uniqueness, Blow-up and Asymptotic Behavior of Solutions in Sobolev-Gevrey and Lei-Lin Spaces

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A thesis submitted to the Department of Mathematics - UFMG, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the subject of Mathematics.

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Abstract

This research project has as main objective to generalize and improve recently developed methods to establish existence, uniqueness and blow-up criteria of local solutions in time for the Navier-Stokes equations involving Sobolev-Gevrey and Lei-Lin spaces; as well as assuming the existence of a global solution in time for this same system, present decay rates of these solutions in these spaces.

Keywords: Navier-Stokes equations; local existence; blow-up criteria; decay rates; Sobolev-Gevrey spaces; Lei-Lin spaces.

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Contents

Introduction

Let us present a study on the mathematical theory of incompressible flows. More specifically, we will address the existence, uniqueness and blow-up criteria for local (in time) solutions of equations described by these flows; as well as the decay of global solutions in time for these same systems, considering the Sobolev-Gevrey and Lei-Lin spaces. Below, we detail the procedures adopted in each chapter.

Initially, using as main inspiration J. Benameur and L. Jlali [4, 7], this thesis presents results related to local existence, uniqueness and blow-up criteria for solutions of the classical Navier-Stokes equation:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \mu \Delta u, & x \in \mathbb{R}^3, t \in [0, T^*), \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3, t \in [0, T^*), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $T^* > 0$ denotes the solution existence time, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, and $p(x,t) \in \mathbb{R}$ the hydrostatic pressure. The positive constant μ is the kinematic viscosity and the initial data for the velocity field, given by u_0 in (1), is assumed to be divergence free, i.e., $\operatorname{div} u_0 = 0$.

The existence of solutions for this system has been intensively studied in the literature see, for example, [2, 3, 4, 7, 10, 13, 14, 27, 28, 29, 32]. It is important to add that finding smooth global solutions for the Navier-Stokes equations (1) is still an open problem. On the other hand, it is well known that there exists a maximal time $T > 0$ for which the system (1) has a classical solution $u(x,t)$, defined for all $(x,t) \in [0,T) \times \mathbb{R}^3$.

J. Benameur and L. Jlali [7] guarantee the existence of a unique $u \in C([0,T^*), H^1_{a,\sigma}(\mathbb{R}^3))$ solution of (1), provided that the initial data u_0 is properly chosen in the appropriate Sobolev-Gevrey space, specifically:

Theorem 0.0.1 (see [7]). *Let $a > 0$ and $\sigma > 1$. Let $u_0 \in (H^1_{a,\sigma}(\mathbb{R}^3))^3$ be such that $\operatorname{div} u_0 = 0$, then there is a unique $T^* \in (0,\infty]$ and a unique $u \in C([0,T^*), H^1_{a,\sigma}(\mathbb{R}^3))$ solution to system* (1) *such that $u \notin C([0,T^*], H^1_{a,\sigma}(\mathbb{R}^3))$. If $T^* < \infty$, then*

$$\frac{c_1}{(T^*-t)^{\frac{2\sigma_0+1}{3\sigma}+\frac{1}{3}}} exp\left[\frac{ac_2}{(T^*-t)^{\frac{1}{3\sigma}}}\right] \leq \|u(t)\|_{H^1_{a,\sigma}(\mathbb{R}^3)}, \tag{2}$$

where $c_1 = c_1(u_0, a, \sigma) > 0$, $c_2 = c_2(u_0, \sigma) > 0$ *and* $2\sigma_0$ *is the integer part of* 2σ.

In Chapter 2, some extensions and improvements, for Theorem 0.0.1, have been obtained. Briefly, we prove that given $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$, with $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$, we obtain a unique local solution $u \in C([0, T^*), H^s_{a,\sigma}(\mathbb{R}^3))$ for the system (1), for all $s \leq s_0$, defined in some maximal interval $[0, T^*)$. Besides, the Theorem 0.0.1 presents blow-up criterion (2) for the Sobolev-Gevrey norm $\|\cdot\|_{H^1_{a,\sigma}(\mathbb{R}^3)}$, which is also valid for the norm $\|\cdot\|_{H^s_{a,\sigma}(\mathbb{R}^3)}$, if $s \leq s_0$.

In Chapter 3, results of existence, uniqueness and blow-up for local solutions of the Navier-Stokes equations, analogous to that above, were also obtained for the homogeneous Sobolev-Gevrey spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$, for $s \in (\frac{1}{2}, \frac{3}{2})$. As one of the main reasons for attempting to achieve this goal, it highlights the inclusion $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$.

It is important to note that, considering the critical cases $s = \frac{1}{2}$ e $s = \frac{3}{2}$, the local existence, uniqueness and blow-up of the solution (1) are not discussed here and are of still open problems in the mathematical theory of incompressible flows. Complementing this theory, J. Benameur [4] showed a similar result to the Theorem 0.0.1 in $H^s_{a,\sigma}(\mathbb{R}^3)$, with $s > \frac{3}{2}$.

Theorem 0.0.2 (see [4]). *Let* a, s, $\sigma \in \mathbb{R}$ *such that* $a > 0$, $s > \frac{3}{2}$ *e* $\sigma > 1$. *Let* $u_0 \in (H^s_{a,\sigma}(\mathbb{R}^3))^3$ *such that* $\operatorname{div} u_0 = 0$. *Then, there is a unique time* $T^* \in (0, \infty]$ *and a unique solution* $u \in C([0, T^*), H^s_{a,\sigma}(\mathbb{R}^3))$ *of Navier-Stokes equations (1) such that* $u \notin C([0, T^*], H^s_{a,\sigma}(\mathbb{R}^3))$. *Moreover, if* $T^* < \infty$, *then*

$$C_1(T^* - t)^{-\frac{s}{3}} \exp(aC_2(T^* - t)^{-\frac{1}{3\sigma}}) \leq \|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}, \ \forall t \in [0, T^*), \tag{3}$$

where $C_1 = C_1(u_0, s, \sigma) > 0$ *and* $C_2 = C_2(u_0, s, \sigma) > 0$.

In addition to the Navier-Stokes equations, the Magneto-Hydrodynamics equations (MHD) will also be the source of research in this thesis:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla(p + \frac{1}{2}|\, b\,|^2) = \mu \Delta u + b \cdot \nabla b, & x \in \mathbb{R}^3, t \in [0, T^*), \\ b_t + u \cdot \nabla b = \nu\, \Delta b + b \cdot \nabla u, & x \in \mathbb{R}^3, t \in [0, T^*), \\ \operatorname{div} u = \operatorname{div} b = 0, & x \in \mathbb{R}^3, t \in (0, T^*), \\ u(\cdot, 0) = u_0(\cdot), \ b(\cdot, 0) = b_0(\cdot), & x \in \mathbb{R}^3. \end{cases} \tag{4}$$

Here $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, $b(x, t) = (b_1(x, t), b_2(x, t), b_3(x, t)) \in \mathbb{R}^3$ the magnetic field and $p(x, t) \in \mathbb{R}$ the hydrostatic pressure. The positive constants μ and ν are associated with specific properties of the fluid. The initial data for the velocity and magnetic fields are assumed to be divergence free. Actually, the MHD equations (4) reduce to the classical Navier-Stokes equations, with velocity field $u(x, t)$, pressure $p(x, t)$, and viscosity μ, provided that $b = 0$ (the existence of solutions for this system has been intensively studied in the literature – see e.g. [4, 7, 13, 14, 27, 28, 29, 32] and references therein).

In Chapter 4, extensions of the Theorem 0.0.1 were obtained for the more general case of the system (4) in homogeneous Sobolev-Gevrey spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$, for $s \in (\frac{1}{2}, \frac{3}{2})$. Moreover in Chapter 5, in addition to the Theorem 0.0.1, the Theorem 0.0.2 was also extended to the MHD equations (4) in Sobolev-Gevrey spaces. It is important to note that the main results obtained by J. Benameur and L. Jlali [4, 7] becomes particular cases of the Theorems presented here (see Theorems 5.1.1, 5.2.9, 5.2.10, 5.2.11 and 5.2.12), since we have extended all the results stated in [4, 7] from the classical Navier-Stokes equations to the MHD system (4).

The research developed in this thesis also seeks results of existence, uniqueness and decay rates of global solutions in time for the 2D Micropolar system:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = (\mu+\chi)\Delta u + \chi\nabla\times w, \quad x\in\mathbb{R}^2, t\geq 0,\\ w_t + u\cdot\nabla w = \gamma\,\Delta w + \chi\nabla\times u - 2\chi w, \quad x\in\mathbb{R}^2, t\geq 0,\\ \operatorname{div} u = 0, \quad x\in\mathbb{R}^2, t>0,\\ u(\cdot,0) = u_0(\cdot),\ w(\cdot,0) = w_0(\cdot), \quad x\in\mathbb{R}^2, \end{cases} \tag{5}$$

where $u(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ denotes the incompressible velocity field, $w(x,t) \in \mathbb{R}$ the microrotational velocity field and $p(x,t) \in \mathbb{R}$ the hydrostatic pressure. The positive constants μ, χ, γ and ν are associated with specific properties of the fluid. The initial data for the velocity field is assumed to be divergence-free.

In the literature, results involving blow-up criteria for local solution at the time of systems (1) and (4) have been developed in numerous papers of great relevance. In order to make the theory as complete as possible and by using as our main reference J. Benameur and L. Jlali [6], Chapter 6 will consider the global existence in time of solutions obtained in Sobolev-Gevrey spaces for the system (5). Our goal is to analyze these decay rate of the solutions. To cite some references, we give the examples [6, 15, 16, 23, 35, 37]. The decay rate analysis for the equations (1) and (4) was made by R. H. Guterres, W. G. Melo, J. R. Nunes e C. F. Perusato [35], in the following result:

Teorema 1 (see [35])**.** *Assume that $a > 0$, $\sigma > 1$, and $s > 1/2$ with $s \neq 3/2$. Consider that $(u, b) \in C([0,\infty); H^s_{a,\sigma}(\mathbb{R}^3))$ is a global solution for the MHD equations* (4)*. Then,*

i) $\lim_{t\to\infty} \|(u,b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0;$

ii) $\lim_{t\to\infty} t^{\frac{s}{2}} \|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0.$

Finally, in Chapter 7, we present a study related to the local existence, uniqueness and properties at potential blow-up times for solutions of the following generalized Magnetohydrodynamics (GMHD) equations:

$$\begin{cases} u_t + (-\Delta)^\alpha u + u\cdot\nabla u + \nabla(p + \frac{1}{2}|\,b\,|^2) = b\cdot\nabla b, \quad x\in\mathbb{R}^3, t\in[0,T^*),\\ b_t + (-\Delta)^\beta b + u\cdot\nabla b = b\cdot\nabla u, \quad x\in\mathbb{R}^3, t\in[0,T^*),\\ \operatorname{div} u = \operatorname{div} b = 0, \quad x\in\mathbb{R}^3, t\in(0,T^*),\\ u(\cdot,0) = u_0(\cdot),\ b(\cdot,0) = b_0(\cdot), \quad x\in\mathbb{R}^3. \end{cases} \tag{6}$$

Here, it is assumed that $\alpha, \beta \in (\frac{1}{2}, 1]$ and the initial data for the velocity and magnetic fields are assumed to be divergence free. The existence of global solutions in time for the GMHD equations (6) is still an open problem; thus, this issue has become a fruitful field in the study of the incompressible fluids (see e.g. [44, 45] and references therein).

In [5], J. Benameur and M. Benhamed have studied local existence, uniqueness and blow-up times for solutions to the quasi-geostrophic equations in Lei-Lin spaces $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. Applying the techniques contained in [5], we guarantee local existence and uniqueness for GMHD equations (6), assuming $(u_0, b_0) \in \mathcal{X}^s(\mathbb{R}^3)$, with $\max\left\{1-2\alpha, 1-2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\right\} \leq s < 0$ and $\alpha, \beta \in (\frac{1}{2}, 1]$. Moreover, if we assume that the maximal time of existence $T^* > 0$ is finite, we conclude

$$\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} = \infty.$$

In Chapter 1, we establish some notations and definitions that will be used throughout the text and we also present some fundamental lemmas used in the proof of results presented later.

Chapter 1

Preliminary

This chapter presents notations, definitions as well as lemmas that will be needed for the proofs of the main theorems.

1.1 Notations and Definitions

The main notations and definitions of this PhD thesis are listed below:

- We denote the standard inner product in $\mathbb{C}^n$ by

$$x \cdot y := x_1\overline{y}_1 + x_2\overline{y}_2 + ... + x_n\overline{y}_n$$

  and let the norm induced by this product be

$$|x| := \sqrt{|x_1|^2 + |x_2|^2 + ... + |x_n|^2},$$

  with $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{C}^n$ $(n \in \mathbb{N})$.

- The vector fields are denoted by

$$f = f(t) = f(x,t) = (f_1(x,t), f_2(x,t), ..., f_n(x,t)),$$

  where $x \in \mathbb{R}^i$ $(i = 1, 2, 3)$, $t \in [0, T^*)$ and $n \in \mathbb{N}$.

- The i-th spatial derivative is denoted by $D_i = \partial/\partial x_i$ $(i = 1, 2, 3)$.

- The gradient field is defined by $\nabla f = (\nabla f_1, \nabla f_2, ..., \nabla f_n)$, where $\nabla f_j = (D_1 f_j, ..., D_i f_j)$ $(j = 1, 2, ..., n$ and $i = 1, 2, 3)$.

- The usual Laplacian $f = (f_1, f_2, .., f_n)$ is given by $\Delta f = (\Delta f_1, \Delta f_2, ..., \Delta f_n)$, where $\Delta f_j = \sum_{k=1}^{i} D_k^2 f_j$ $(i = 1, 2, 3)$.

- The standard divergent is given by $\operatorname{div} f = \sum_{k=1}^{i} D_k f_k$, provided that $f = (f_1, ..., f_i)$ $(i = 1, 2, 3)$.

- The notation $f \cdot \nabla g$ means $\sum_{i=1}^{3} f_i D_i g$, where $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$. However, in the particular case of the 2D Micropolar equations (6.1), $f \cdot \nabla g = \sum_{i=1}^{2} f_i D_i g$, where $f = (f_1, f_2)$.

- Define Fourier transform of f by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x)\, dx, \quad \forall\, \xi \in \mathbb{R}^n,$$

  and its inverse by

$$\mathcal{F}^{-1}(f)(\xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(x)\, dx, \quad \forall\, \xi \in \mathbb{R}^n.$$

- The fractional Laplacian $(-\Delta)^\gamma$, $\gamma > 0$, is defined by

$$\mathcal{F}[(-\Delta)^\gamma f](\xi) = |\xi|^{2\gamma} \hat{f}(\xi), \quad \forall f \in S'(\mathbb{R}^n),$$

  where $S'(\mathbb{R}^n)$ is the set of tempered distributions.

- Here $L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space, where

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad \forall p \in [1, \infty),$$

  and $\|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname{esssup}_{x \in \mathbb{R}^n} \{|f(x)|\}$.

- Assuming that $(X, \|\cdot\|)$ is a normed vector space and $T > 0$, the space $L^p([0,T]; X)$ (or simply $L^p_T(X)$), $1 \leq p \leq \infty$, contains all measurable functions $f : [0,T] \to X$ for which the following norms are finite:

$$\|f\|_{L^\infty([0,T];X)} = \|f\|_{L^\infty_T(X)} := \operatorname{esssup}_{t \in [0,T]} \{\|f(t)\|\}$$

  and

$$\|f\|_{L^p([0,T];X)} = \|f\|_{L^p_T(X)} := \Big[\int_0^T \|f(t)\|^p \, dt \Big]^{\frac{1}{p}}, \quad \forall\, 1 \leq p < \infty.$$

  Analogously, $C([0,T]; X) = C_T(X) = \{f : [0,T] \to X \text{ continuous}\}$ is endowed with the norm $\|\cdot\|_{L^\infty_T(X)}$.

- Let $s \in \mathbb{R}$. $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space

$$\dot{H}^s(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi < \infty \Big\}.$$

It is assumed that the $\dot{H}^s(\mathbb{R}^n)$-norm is given by

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi,$$

Furthermore, the $\dot{H}^s(\mathbb{R}^n)$-inner product is given by

$$\langle f, g \rangle_{\dot{H}^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} \widehat{f}(\xi) \cdot \widehat{g}(\xi) \, d\xi.$$

- Assume $s \in \mathbb{R}$. The nonhomogeneous Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi < \infty \Big\}.$$

This space is assumed to be endowed with the $H^s(\mathbb{R}^n)$-norm

$$\|f\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi.$$

Moreover, the $H^s(\mathbb{R}^n)$-inner product is given by

$$\langle f, g \rangle_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \widehat{f}(\xi) \cdot \widehat{g}(\xi) \, d\xi.$$

- Let $a > 0, \sigma \geq 1$ and $s \in \mathbb{R}$. The Sobolev-Gevrey space

$$\dot{H}^s_{a,\sigma}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2 \, d\xi < \infty \Big\}$$

is endowed with the $\dot{H}^s_{a,\sigma}(\mathbb{R}^n)$-norm

$$\|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2 \, d\xi.$$

Moreover, the $\dot{H}^s_{a,\sigma}(\mathbb{R}^n)$-inner product is given by

$$\langle f, g \rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} \widehat{f}(\xi) \cdot \widehat{g}(\xi) \, d\xi.$$

- Assume $a > 0$, $\sigma \geq 1$ and $s \in \mathbb{R}$. The nonhomogeneous Sobolev-Gevrey space is given by
$$H^s_{a,\sigma}(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2 \, d\xi < \infty\}.$$
It is assumed that the $H^s_{a,\sigma}(\mathbb{R}^n)$-norm is given by
$$\|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi)|^2 \, d\xi$$
and $H^s_{a,\sigma}(\mathbb{R}^n)$-inner product by
$$\langle f, g \rangle_{H^s_{a,\sigma}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} \widehat{f}(\xi) \cdot \widehat{g}(\xi) \, d\xi.$$

- For $s \in \mathbb{R}$, the Lei-Lin space is given by
$$\mathcal{X}^s(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\xi|^s |\widehat{f}(\xi)| \, d\xi < \infty\},$$
which is equipped with the $\mathcal{X}^s(\mathbb{R}^n)$-norm
$$\|f\|_{\mathcal{X}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^s |\widehat{f}(\xi)| \, d\xi.$$

- The tensor product is given by $f \otimes g := (g_1 f, ..., g_n f)$, where $f : \mathbb{R}^k \to \mathbb{R}^m$ and $g : \mathbb{R}^k \to \mathbb{R}^n$ $(k, m, n \in \mathbb{N})$.

- The convolution is defined by $\varphi * \psi(x) = \displaystyle\int_{\mathbb{R}^n} \varphi(x - y)\psi(y) \, dy$, where $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$.

- Let given $v : \mathbb{R}^3 \to \mathbb{R}^3$; then, there exist w and $\nabla\phi$ such that
$$v = w - \nabla\phi, \quad \text{div } w = 0.$$
In this case, $w = P_H(v)$ is called Helmontz's projector (see e.g. Section 7.2 in [32] and references therein).

- The gamma function is defined by $\Gamma(z) = \displaystyle\int_0^\infty x^{z-1} e^{-x} \, dx$, for all $z = x + iy \in \mathbb{C}$, with $x > 0$.

- Let $A \subseteq Y$. The indicator function $\chi_A : Y \to \mathbb{R}$ is defined by $\chi_A(x) = 1$, if $x \in A$, and $\chi_A(x) = 0$, if $x \notin A$.

- As usual, constants that appear in this thesis may change in value from line to line without change of notation. Here $C_{q,r,s}$ denotes a constant that depends on q, r and s, for example.

1.2 Auxiliary Results

1.2.1 Auxiliary Results for Chapters 2 to 6

In this section, we presenting some auxiliary results that will be useful in the demonstration of the statement in Chapters 2 to 6.

The first two Lemmas listed below are the results that will guarantee the existence of a fixed point for the equations presented in the chapters mentioned above. The Lemma 1.2.1 guarantees the existence and uniqueness of solution for the Navier-Stokes (2.1) and MHD equations (4.1).

Lemma 1.2.1 (see [13])**.** *Let $(X, \|\cdot\|)$ be a Banach space and $B : X \times X \to X$ a continuous bilinear operator, i.e., there exists a positive constant C such that*

$$\|B(x, y)\| \leq C\|x\|\|y\|, \quad \forall\, x, y \in X.$$

Then, for each $x_0 \in X$ that satisfies $4C\|x_0\| < 1$, one has that the equation

$$a = x_0 + B(a, a) \tag{1.1}$$

admits a solution $x = a \in X$. Moreover, x obeys the inequality $\|x\| \leq 2\|x_0\|$ and it is the only one such that $\|x\| \leq \frac{1}{2C}$.

Also the next Lemma is the main ingredient to prove the existence and uniqueness of solution for the Micropolar equations (6.1).

Lemma 1.2.2 (See [13])**.** *Let $(X, \|\cdot\|)$ be a Banach space, $L : X \to X$ continuous linear operator and $B : X \times X \to X$ continuous bilinear operator, i.e., there exist positive constants C_1 and C_2 such that*

$$\|L(x)\| \leq C_1\|x\|,\ \|B(x, y)\| \leq C_2\|x\|\|y\|, \quad \forall\, x, y \in X.$$

Then, for each $C_1 \in (0, 1)$ and $x_0 \in X$ that satisfy $4C_2\|x_0\| < (1 - C_1)^2$, one has that the equation

$$a = x_0 + B(a, a) + L(a), \quad a \in X,$$

admits a solution $x \in X$. Moreover, x obeys the inequality $\|x\| \leq \frac{2\|x_0\|}{1-C_1}$ and it is the only one such that $\|x\| \leq \frac{1-C_1}{2C_2}$.

The following result has been proved by [4] and it is useful in order to obtain some important inequalities related to the elementary exponential function.

Lemma 1.2.3. *The following inequality holds:*

$$(a + b)^r \leq ra^r + b^r, \quad \forall\, 0 \leq a \leq b, r \in (0, 1]. \tag{1.2}$$

Proof. First of all, notice that if $b = 0$ then $a = 0$ and, consequently, (1.2) holds. Thus, assume that $b > 0$ and let $c = a/b \in [0, 1]$. Now, apply Taylor's Theorem to the function $t \mapsto (1 + t)^r$, with $t \in [0, c]$, in order to obtain $\gamma \in [0, c]$ such that

$$(1 + c)^r = 1 + rc + \frac{r(r-1)(1+\gamma)^{r-2}}{2} c^2.$$

By using the fact that $r, \gamma \in [0, 1]$, one has $(1 + c)^r \leq 1 + rc$. Moreover, $c, r \in [0, 1]$ implies that $c \leq c^r$. As a result, $(1 + c)^r \leq 1 + rc^r$. Replace $c = a/b$ in this last inequality to prove (1.2).

□

Now, let us introduce two consequences of Lemma 1.2.3.

Lemma 1.2.4. *The inequality below is valid for all $n \in \mathbb{N}$:*

$$e^{a|\xi|^{\frac{1}{\sigma}}} \leq e^{a \max\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}} e^{\frac{a}{\sigma} \min\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}}, \quad \forall\, \xi, \eta \in \mathbb{R}^n, a > 0, \sigma \geq 1.$$

Proof. Lemma 1.2.3 assures that

$$\begin{aligned} a|\xi|^{\frac{1}{\sigma}} &= a|\xi - \eta + \eta|^{\frac{1}{\sigma}} \leq a(|\xi - \eta| + |\eta|)^{\frac{1}{\sigma}} \leq a(\max\{|\xi - \eta|, |\eta|\} + \min\{|\xi - \eta|, |\eta|\})^{\frac{1}{\sigma}} \\ &\leq a \max\{|\xi - \eta|, |\eta|\}^{\frac{1}{\sigma}} + \frac{a}{\sigma} \min\{|\xi - \eta|, |\eta|\}^{\frac{1}{\sigma}}. \end{aligned}$$

Hence, one has

$$e^{a|\xi|^{\frac{1}{\sigma}}} \leq e^{a \max\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}} + \frac{a}{\sigma} \min\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}} = e^{a \max\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}} e^{\frac{a}{\sigma} \min\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}}.$$

The proof of Lemma 1.2.4 is complete.

□

Lemma 1.2.5. *Let $a > 0$, $\sigma \geq 1$ and $\xi, \eta \in \mathbb{R}^n$ with $n \in \mathbb{N}$. Then,*

$$e^{a|\xi|^{\frac{1}{\sigma}}} \leq e^{a|\xi-\eta|^{\frac{1}{\sigma}}} e^{a|\eta|^{\frac{1}{\sigma}}}. \tag{1.3}$$

Proof. It is a direct implication of Lemma 1.2.4 and the fact that $\sigma \geq 1$.

□

The next lemma presents an interpolation property involving the space $\dot{H}^s(\mathbb{R}^n)$ with $n = 1, 2, 3$, and it has been proved by J.-Y. Chemin.

Lemma 1.2.6 (see [14]). *Let $(s_1, s_2) \in \mathbb{R}^2$, such that $s_1 < \frac{n}{2}$ and $s_1 + s_2 > 0$. Then, there exists a positive constant C_{s_1,s_2} such that, for all $f, g \in \dot{H}^{s_1}(\mathbb{R}^n) \cap \dot{H}^{s_2}(\mathbb{R}^n)$, we have*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{n}{2}}(\mathbb{R}^n)} \leq C_{s_1,s_2}[\|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}\|g\|_{\dot{H}^{s_2}(\mathbb{R}^n)} + \|f\|_{\dot{H}^{s_2}(\mathbb{R}^n)}\|g\|_{\dot{H}^{s_1}(\mathbb{R}^n)}].$$

If $s_1 < \frac{n}{2}$, $s_2 < \frac{n}{2}$ and $s_1 + s_2 > 0$, then there is a positive constant C_{s_1,s_2} such that

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{n}{2}}(\mathbb{R}^3)} \leq C_{s_1,s_2}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}\|g\|_{\dot{H}^{s_2}(\mathbb{R}^n)}.$$

J. Benameur and L. Jlali [7] have proved a version of Chemin's Lemma (see [14]) by considering Sobolev-Gevrey space $\dot{H}^s_{a,\sigma}(\mathbb{R}^n)$ with $n = 1, 2, 3$. Let us introduce this result exactly as it has been stated and proved in [7].

Lemma 1.2.7 (see [7]). *Let $a > 0$, $\sigma \geq 1$ and $(s_1, s_2) \in \mathbb{R}^2$, such that $s_1 < \frac{n}{2}$ and $s_1 + s_2 > 0$. Then, there exists a positive constant C_{s_1,s_2} such that, for all $f, g \in \dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^n) \cap \dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^n)$, we have*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{n}{2}}_{a,\sigma}(\mathbb{R}^3)} \leq C_{s_1,s_2}[\|f\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^n)} + \|f\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^n)}]. \tag{1.4}$$

If $s_1 < \frac{n}{2}$, $s_2 < \frac{n}{2}$ and $s_1 + s_2 > 0$, then there is a positive constant C_{s_1,s_2} such that

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{n}{2}}_{a,\sigma}(\mathbb{R}^n)} \leq C_{s_1,s_2}\|f\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^n)}. \tag{1.5}$$

Proof. We aim to apply Lemma 1.2.6. Thereby, to accomplish this goal, it is necessary to use the elementary equality

$$\widehat{fg}(\xi) = (2\pi)^{-n}(\widehat{f} * \widehat{g})(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Thus, we estimate the expression on the left hand side of the inequalities (1.4) and (1.5) as follows:

$$\begin{aligned}
\|fg\|^2_{\dot{H}^{s_1+s_2-\frac{n}{2}}_{a,\sigma}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\xi|^{2s_1+2s_2-n} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{fg}(\xi)|^2 \, d\xi \\
&= (2\pi)^{-2n} \int_{\mathbb{R}^n} |\xi|^{2s_1+2s_2-n} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{f} * \widehat{g}(\xi)|^2 \, d\xi \\
&\leq (2\pi)^{-2n} \int_{\mathbb{R}^n} |\xi|^{2s_1+2s_2-n} e^{2a|\xi|^{\frac{1}{\sigma}}} \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi - \eta)||\widehat{g}(\eta)| \, d\eta \right)^2 d\xi \\
&= (2\pi)^{-2n} \int_{\mathbb{R}^n} |\xi|^{2s_1+2s_2-n} \left(\int_{\mathbb{R}^n} e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{f}(\xi - \eta)||\widehat{g}(\eta)| \, d\eta \right)^2 d\xi.
\end{aligned}$$

Moreover, the inequality (1.3) implies the following results:

$$\begin{aligned}\|fg\|^2_{\dot H^{s_1+s_2-\frac{n}{2}}_{a,\sigma}(\mathbb{R}^n)} &\le (2\pi)^{-2n}\int_{\mathbb{R}^n}|\xi|^{2s_1+2s_2-n}\left(\int_{\mathbb{R}^n} e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\right)^2 d\xi\\ &= (2\pi)^{-2n}\int_{\mathbb{R}^n}|\xi|^{2s_1+2s_2-n}\{[(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)*(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)](\xi)\}^2\,d\xi\\ &= \int_{\mathbb{R}^n}|\xi|^{2s_1+2s_2-n}\{\mathcal{F}[\mathcal{F}^{-1}(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi)|)\mathcal{F}^{-1}(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|)]\}^2\,d\xi\\ &= \|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)\|^2_{\dot H^{s_1+s_2-\frac{n}{2}}(\mathbb{R}^n)}.\end{aligned}$$

Now, we are ready to apply Lemma 1.2.6 and, consequently, deduce (1.4). In fact, one has

$$\begin{aligned}\|fg\|_{\dot H^{s_1+s_2-\frac{n}{2}}_{a,\sigma}(\mathbb{R}^n)} &\le \|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)\|_{\dot H^{s_1+s_2-\frac{n}{2}}(\mathbb{R}^n)}\\ &\le C_{s_1,s_2}[\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)\|_{\dot H^{s_1}(\mathbb{R}^n)}\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)\|_{\dot H^{s_2}(\mathbb{R}^n)}\\ &\quad+\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)\|_{\dot H^{s_2}(\mathbb{R}^n)}\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)\|_{\dot H^{s_1}(\mathbb{R}^n)}]\\ &= C_{s_1,s_2}[\|f\|_{\dot H^{s_1}_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot H^{s_2}_{a,\sigma}(\mathbb{R}^n)}+\|f\|_{\dot H^{s_2}_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot H^{s_1}_{a,\sigma}(\mathbb{R}^n)}].\end{aligned}$$

On the other hand, if $s_1, s_2 < \frac{n}{2}$ and $s_1+s_2>0$, use Lemma 1.2.6 again in order to obtain

$$\begin{aligned}\|fg\|_{\dot H^{s_1+s_2-\frac{n}{2}}_{a,\sigma}(\mathbb{R}^n)} &\le \|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)\|^2_{\dot H^{s_1+s_2-\frac{n}{2}}(\mathbb{R}^n)}\\ &\le C_{s_1,s_2}\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|)\|_{\dot H^{s_1}(\mathbb{R}^n)}\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|)\|_{\dot H^{s_2}(\mathbb{R}^n)}\\ &= C_{s_1,s_2}\|f\|_{\dot H^{s_1}_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot H^{s_2}_{a,\sigma}(\mathbb{R}^n)},\end{aligned}$$

which proves inequality.

□

One of the application of Lemma 1.2.4 is to obtain interpolation inequalities related to the space $H^s_{a,\sigma}(\mathbb{R}^n)$, with $n\in\mathbb{N}$. More precisely, the lemma below is an improvement of a similar result from [4] (see Proposition 4.1 in [4]).

Lemma 1.2.8. *Let $a\ge 0$, $\sigma\ge 1$ and $n\in\mathbb{N}$. For every $f,g\in H^s_{a,\sigma}(\mathbb{R}^n)$ with $s\ge 0$, we have $fg\in H^s_{a,\sigma}(\mathbb{R}^n)$. More precisely, one obtains*

i) $\|fg\|_{H^s_{a,\sigma}(\mathbb{R}^n)} \le 2^{\frac{2s-2n+1}{2}}\pi^{-n}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|_{L^1(\mathbb{R}^n)}\|g\|_{H^s_{a,\sigma}(\mathbb{R}^n)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^n)}\|f\|_{H^s_{a,\sigma}(\mathbb{R}^n)}]$.

Moreover, for $s>n/2$, one obtains

ii) $\|fg\|_{H^s_{a,\sigma}(\mathbb{R}^n)} \le 2^{\frac{2s-2n+1}{2}}\pi^{-n}C_{s,n}(\|f\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^n)}\|g\|_{H^s_{a,\sigma}(\mathbb{R}^n)}+\|f\|_{H^s_{a,\sigma}(\mathbb{R}^n)}\|g\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^n)})$, *where* $C_{s,n} = \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}\,d\xi\right)^{\frac{1}{2}} = (\pi^{n/2}\Gamma(s-n/2)/\Gamma(s))^{\frac{1}{2}}$. *Here* $\Gamma(\cdot)$ *is the standard gamma function.*

Proof. This result is a consequence of Lemma 1.2.4. First of all, let us estimate the left hand side of the inequalities given in **i)** and **ii)**. Thus,

$$
\begin{aligned}
\|fg\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{fg}(\xi)|^2\,d\xi \\
&= (2\pi)^{-2n}\int_{\mathbb{R}^n}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}*\widehat{g}(\xi)|^2\,d\xi \\
&\le (2\pi)^{-2n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{\frac{s}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta\right)^2 d\xi \\
&\le (2\pi)^{-2n}\int_{\mathbb{R}^n}\Big(\int_{|\eta|\le|\xi-\eta|}(1+|\xi|^2)^{\frac{s}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta \\
&\quad + \int_{|\eta|>|\xi-\eta|}(1+|\xi|^2)^{\frac{s}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta\Big)^2 d\xi.
\end{aligned}
$$

By using basic arguments, it is easy to check that

$$
\begin{aligned}
(1+|\xi|^2)^{\frac{s}{2}} &\le [1+(|\xi-\eta|+|\eta|)^2]^{\frac{s}{2}} \le [1+(2\max\{|\xi-\eta|,|\eta|\})^2]^{\frac{s}{2}} \\
&\le 2^s[1+\max\{|\xi-\eta|,|\eta|\}^2]^{\frac{s}{2}}.
\end{aligned}
\tag{1.6}
$$

Now, we apply Lemma 1.2.4 to obtain

$$
\begin{aligned}
\|fg\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} &\le (2\pi)^{-2n}2^{2s}\int_{\mathbb{R}^n}\Big(\int_{|\eta|\le|\xi-\eta|}(1+|\xi-\eta|^2)^{\frac{s}{2}}e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta \\
&\quad + \int_{|\eta|>|\xi-\eta|}e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|(1+|\eta|^2)^{\frac{s}{2}}e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\Big)^2 d\xi \\
&\le (2\pi)^{-2n}2^{2s+1}\Big[\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{\frac{s}{2}}e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\right)^2 d\xi \\
&\quad + \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|(1+|\eta|^2)^{\frac{s}{2}}e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\right)^2 d\xi\Big].
\end{aligned}
$$

Rewriting the last inequality above, we deduce

$$
\begin{aligned}
\|fg\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} &\le (2\pi)^{-2n}2^{2s+1}\|[(1+|\cdot|^2)^{\frac{s}{2}}e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|]*[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|]\|^2_{L^2(\mathbb{R}^n)} \\
&\quad + (2\pi)^{-2n}2^{2s+1}\|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|]*[(1+|\cdot|^2)^{\frac{s}{2}}e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|]\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Consequently, it follows from Young's inequality for convolutions[1] that

$$
\begin{aligned}
\|fg\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} &\le 2^{2s-2n+1}\pi^{-2n}[\|(1+|\cdot|^2)^{\frac{s}{2}}e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^2(\mathbb{R}^n)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^1(\mathbb{R}^n)} \\
&\quad + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^1(\mathbb{R}^n)}\|(1+|\cdot|^2)^{\frac{s}{2}}e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^2(\mathbb{R}^n)}].
\end{aligned}
\tag{1.7}
$$

[1]Let $1\le p,q,r\le\infty$ such that $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Assume that $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$; then, $\|f*g\|_{L^r(\mathbb{R}^n)}\le\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$.

Notice that the $L^2(\mathbb{R}^n)$-norm of $(1+|\xi|^2)^{\frac{s}{2}} e^{a|\xi|^{\frac{1}{\sigma}}} \widehat{f}(\xi)$ presented above can be replaced with the $H^s_{a,\sigma}(\mathbb{R}^n)$-norm of f. More precisely, we have

$$\|(1+|\cdot|^2)^{\frac{s}{2}} e^{a|\cdot|^{\frac{1}{\sigma}}} \widehat{f}\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 \, d\xi = \|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)}. \tag{1.8}$$

This same process can be applied to the equivalent term related to g in (1.7). Thus, it is true that

$$\|(1+|\cdot|^2)^{\frac{s}{2}} e^{a|\cdot|^{\frac{1}{\sigma}}} \widehat{g}\|^2_{L^2(\mathbb{R}^n)} = \|g\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)}. \tag{1.9}$$

As a consequence, replace (1.8) and (1.9) in (1.7) in order to get

$$\|fg\|_{H^s_{a,\sigma}(\mathbb{R}^n)} \leq 2^{\frac{2s-2n+1}{2}} \pi^{-n} [\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{f}\|_{L^1(\mathbb{R}^n)} \|g\|_{H^s_{a,\sigma}(\mathbb{R}^n)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{g}\|_{L^1(\mathbb{R}^n)} \|f\|_{H^s_{a,\sigma}(\mathbb{R}^n)}].$$

This concludes the proof of **i)**.

It is important to point out that **ii)** follows directly from results established above and Cauchy-Schwarz's inequality. In fact,

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{g}\|^2_{L^1(\mathbb{R}^n)} \leq \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} \, d\xi\right) \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s e^{\frac{2a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{g}(\xi)|^2 \, d\xi\right) =: C_{s,n} \|g\|^2_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^n)}, \tag{1.10}$$

and, consequently,

$$\|fg\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} \leq (2\pi)^{-2n} 2^{2s+1} C_{s,n} [\|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} \|g\|^2_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^n)} + \|f\|^2_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^n)} \|g\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)}],$$

which proves **ii)**.

□

Let us observe that Lemma 1.2.8 **ii)** also imply

$$\|fg\|_{H^s_{a,\sigma}(\mathbb{R}^n)} \leq (2\pi)^{-n} 2^{s+1} C_{s,n} \|f\|_{H^s_{a,\sigma}(\mathbb{R}^n)} \|g\|_{H^s_{a,\sigma}(\mathbb{R}^n)}, \tag{1.11}$$

since $H^s_{a,\sigma}(\mathbb{R}^n) \hookrightarrow H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^n)$ $(a/\sigma \leq a)$.

The next result gives our extension of Lemma 2.5 given in [7].

Lemma 1.2.9. *Let $a > 0$, $\sigma > 1$, and $s \in [0, \frac{3}{2})$. For every $f, g \in H^s_{a,\sigma}(\mathbb{R}^3)$, we have $fg \in H^s_{a,\sigma}(\mathbb{R}^3)$. More precisely, one obtains*

$$\|fg\|_{H^s_{a,\sigma}(\mathbb{R}^3))} \leq 2^{s-2} \pi^{-3} C_{a,\sigma,s} \|f\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \|g\|_{H^s_{a,\sigma}(\mathbb{R}^3)},$$

where $C_{a,\sigma,s} := \sqrt{\frac{4\pi\sigma\Gamma(\sigma(3-2s))}{[2(a-\frac{a}{\sigma})]^{\sigma(3-2s)}}}$. As before, $\Gamma(\cdot)$ is the standard gamma function.

Proof. By applying Cauchy-Schwarz's inequality, one infers

$$\begin{aligned}
\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|\,d\xi \\
&\leq \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}e^{2(\frac{a}{\sigma}-a)|\xi|^{\frac{1}{\sigma}}}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^3}|\xi|^{-2s}e^{2(\frac{a}{\sigma}-a)|\xi|^{\frac{1}{\sigma}}}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&=: C_{a,\sigma,s}\|g\|_{H^s_{a,\sigma}(\mathbb{R}^3)}, \qquad (1.12)
\end{aligned}$$

where

$$C^2_{a,\sigma,s} = \frac{4\pi\sigma\Gamma(\sigma(3-2s))}{[2(a-\frac{a}{\sigma})]^{\sigma(3-2s)}},$$

since $\sigma > 1$ and $0 \leq s < 3/2$. Similarly, we obtain

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|_{L^1(\mathbb{R}^3)} \leq C_{a,\sigma,s}\|f\|_{H^s_{a,\sigma}(\mathbb{R}^3)}. \qquad (1.13)$$

Hence, by combining (1.7), (1.8), (1.9), (1.12) and (1.13), we have

$$\|fg\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq 2^{2s-4}\pi^{-6}C^2_{a,\sigma,s}\|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\|g\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

□

The next result is our version of Lemma 2.8 in [7], once this last lemma is the same as Lemma 1.2.10 below, whether it is considered $s = 1$.

Lemma 1.2.10. *Let $s \geq 0, a > 0, \sigma \geq 1$ and $f \in H^s_{a,\sigma}(\mathbb{R}^n)$ with $n \in \mathbb{N}$. Then, the following inequalities hold:*

$$\|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} \leq 2^s[e^{2a}(2\pi)^n\|f\|^2_{L^2(\mathbb{R}^n)} + \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}] \leq 2^s[e^{2a}+1]\|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)}. \qquad (1.14)$$

Proof. This result follows directly from the definition of the spaces $H^s_{a,\sigma}(\mathbb{R}^n)$, $\dot{H}^s_{a,\sigma}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. In fact, note that, by using Parseval's identity, i.e.,

$$\|f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n}\|\hat{f}\|^2_{L^2(\mathbb{R}^n)},$$

one obtains

$$\begin{aligned}
\|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi \\
&= \int_{|\xi|\leq 1}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi + \int_{|\xi|>1}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi \\
&\leq 2^s e^{2a}\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\,d\xi + 2^s\int_{\mathbb{R}^n}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi \\
&= 2^s e^{2a}(2\pi)^n\|f\|^2_{L^2(\mathbb{R}^n)} + 2^s\|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)},
\end{aligned}$$

which implies the first inequality in (1.14).

By applying the last equality above, one infers

$$\begin{aligned} 2^s e^{2a}(2\pi)^n \|f\|^2_{L^2(\mathbb{R}^n)} + 2^s \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} &= 2^s e^{2a} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi + 2^s \int_{\mathbb{R}^n} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 \, d\xi \\ &\leq 2^s [e^{2a}+1] \int_{\mathbb{R}^n} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 \, d\xi \\ &= 2^s [e^{2a}+1] \|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^n)}. \end{aligned}$$

Therefore, the proof of the second inequality in (1.14) is complete.

□

Remark 1.2.11. It is worth to observe that the proof of the lemma above establishes, for instance, the standard embeddings $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ and $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ $(s \geq 0)$. In fact, note that in the proof of Lemma 1.2.10, we have proved

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^3)} &= (2\pi)^{-3} \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \, d\xi \leq (2\pi)^{-3} \int_{\mathbb{R}^3} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 \, d\xi \\ &= (2\pi)^{-3} \|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}. \end{aligned}$$

Consequently, the continuous embedding $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ $(s \geq 0)$ is given by the inequality

$$\|f\|_{L^2(\mathbb{R}^3)} \leq (2\pi)^{-\frac{3}{2}} \|f\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

The other embedding follows directly from the following results:

$$\begin{aligned} \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^3} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(\xi)|^2 \, d\xi \\ &= \|f\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}. \end{aligned}$$

To guarantee the veracity of the blow–up criteria, it will be necessary to present two basic tools. The first one was obtained by J. Benameur [4] and we shall prove it for convenience of the reader.

Lemma 1.2.12. *Let $\delta > 3/2$, and $f \in \dot{H}^\delta(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then, the following inequality is valid:*

$$\|\widehat{f}\|_{L^1(\mathbb{R}^3)} \leq C_\delta \|f\|^{1-\frac{3}{2\delta}}_{L^2(\mathbb{R}^3)} \|f\|^{\frac{3}{2\delta}}_{\dot{H}^\delta(\mathbb{R}^3)},$$

where

$$C_\delta = 2(2\pi)^{\frac{3}{2}(1-\frac{3}{2\delta})} \sqrt{\frac{\pi}{3}} \Big[\Big(\frac{2\delta}{3}-1\Big)^{\frac{3}{4\delta}} + \Big(\frac{2\delta}{3}-1\Big)^{-1+\frac{3}{4\delta}} \Big].$$

Moreover, for each $\delta_0 > 3/2$ there exists a positive constant C_{δ_0} such that $C_\delta \leq C_{\delta_0}$, for all $\delta \geq \delta_0$.

Proof. Consider $\epsilon > 0$ arbitrary. Thereby, by using Cauchy-Schwarz's inequality, we deduce

$$\begin{aligned}\|\widehat{f}\|_{L^1(\mathbb{R}^3)} &= \int_{|\xi|\leq\epsilon} |\widehat{f}(\xi)|\, d\xi + \int_{|\xi|>\epsilon} |\widehat{f}(\xi)|\, d\xi \\ &\leq \left(\int_{|\xi|\leq\epsilon} d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|\leq\epsilon} |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \left(\int_{|\xi|>\epsilon} \frac{1}{|\xi|^{2\delta}} d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|>\epsilon} |\xi|^{2\delta}|\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

Now, apply Parseval's identity and the fact that $\delta > 3/2$ to reach

$$\begin{aligned}\|\widehat{f}\|_{L^1(\mathbb{R}^3)} &\leq 2\sqrt{\frac{\pi}{3}}\epsilon^{\frac{3}{2}}(2\pi)^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R}^3)} + 2\sqrt{\frac{\pi}{2\delta-3}}\epsilon^{\frac{3}{2}-\delta}\|f\|_{\dot{H}^\delta(\mathbb{R}^3)} \\ &= 2\sqrt{\frac{\pi}{3}}\left[\epsilon^{\frac{3}{2}}(2\pi)^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R}^3)} + \frac{\epsilon^{\frac{3}{2}-\delta}}{\sqrt{\frac{2\delta}{3}-1}}\|f\|_{\dot{H}^\delta(\mathbb{R}^3)}\right].\end{aligned}$$

Thus, we can guarantee that the function given by

$$\epsilon \mapsto \epsilon^{\frac{3}{2}}(2\pi)^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R}^3)} + \frac{\epsilon^{\frac{3}{2}-\delta}}{\sqrt{\frac{2\delta}{3}-1}}\|f\|_{\dot{H}^\delta(\mathbb{R}^3)}$$

attains its minimum at

$$\left[\frac{\sqrt{\frac{2\delta}{3}-1}\|f\|_{\dot{H}^\delta(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}\|f\|_{L^2(\mathbb{R}^3)}}\right]^{\frac{1}{\delta}}.$$

Consequently, we have

$$\|\widehat{f}\|_{L^1(\mathbb{R}^3)} \leq 2(2\pi)^{\frac{3}{2}(1-\frac{3}{2\delta})}\sqrt{\frac{\pi}{3}}\left[\left(\frac{2\delta}{3}-1\right)^{\frac{3}{4\delta}} + \left(\frac{2\delta}{3}-1\right)^{-1+\frac{3}{4\delta}}\right]\|f\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2\delta}}\|f\|_{\dot{H}^\delta(\mathbb{R}^3)}^{\frac{3}{2\delta}}.$$

It is easy to check that

$$\lim_{\delta\to\infty} 2(2\pi)^{\frac{3}{2}(1-\frac{3}{2\delta})}\sqrt{\frac{\pi}{3}}\left[\left(\frac{2\delta}{3}-1\right)^{\frac{3}{4\delta}} + \left(\frac{2\delta}{3}-1\right)^{-1+\frac{3}{4\delta}}\right] = 2(2\pi)^{\frac{3}{2}}\sqrt{\frac{\pi}{3}}.$$

As a consequence, for each $\delta_0 > 3/2$, one deduces that

$$2(2\pi)^{\frac{3}{2}(1-\frac{3}{2\delta})}\sqrt{\frac{\pi}{3}}\left[\left(\frac{2\delta}{3}-1\right)^{\frac{3}{4\delta}} + \left(\frac{2\delta}{3}-1\right)^{-1+\frac{3}{4\delta}}\right]$$

is bounded in the interval $[\delta_0, \infty)$.

This concludes the proof of Lemma 1.2.12.

□

It is important to point out that the next lemma can also be used in order to assure that inequality (2.6) is not trivial.

Lemma 1.2.13. *Let $a > 0$, $\sigma \geq 1$, $s \in [0, \frac{3}{2})$ and $\delta \geq \frac{3}{2}$. For every $f \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$, we have that $f \in \dot{H}^\delta(\mathbb{R}^3)$. More precisely, one concludes that there is a positive constant $C_{a,s,\delta,\sigma}$ such that*

$$\|f\|_{\dot{H}^\delta(\mathbb{R}^3)} \leq C_{a,s,\delta,\sigma}\|f\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

Proof. Notice that $\mathbb{R}_+ \subseteq \cup_{n\in\mathbb{N}\cup\{0\}}[n, n+1)$ and since $2\sigma(\delta - s) \in \mathbb{R}_+$, there exists a $n_0 \in \mathbb{N}\cup\{0\}$ that depends on σ, δ and s such that $\frac{n_0}{\sigma} \leq 2\delta - 2s < \frac{n_0+1}{\sigma}$. Consequently, one obtains $t \in [0, 1]$ such that, by Young's inequality[2], we infer

$$\begin{aligned}|\xi|^{2\delta-2s} &= |\xi|^{t\cdot\frac{n_0}{\sigma}+(1-t)\cdot\frac{n_0+1}{\sigma}} = |\xi|^{t\cdot\frac{n_0}{\sigma}}|\xi|^{(1-t)\cdot\frac{n_0+1}{\sigma}}\\ &\leq t|\xi|^{\frac{n_0}{\sigma}} + (1-t)|\xi|^{\frac{n_0+1}{\sigma}} \leq |\xi|^{\frac{n_0}{\sigma}} + |\xi|^{\frac{n_0+1}{\sigma}}.\end{aligned}$$

Therefore, one has

$$\begin{aligned}\|f\|^2_{\dot{H}^\delta(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2\delta}|\hat{f}(\xi)|^2\,d\xi \leq \int_{\mathbb{R}^3}[|\xi|^{\frac{n_0}{\sigma}} + |\xi|^{\frac{n_0+1}{\sigma}}]|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi\\ &\leq \int_{\mathbb{R}^3}\Big[\frac{(2a+1)(2a)^{n_0}(n_0+1)!}{(2a)^{n_0+1}n_0!}|\xi|^{\frac{n_0}{\sigma}} + \frac{(2a+1)(2a)^{n_0+1}(n_0+1)!}{(2a)^{n_0+1}(n_0+1)!}|\xi|^{\frac{n_0+1}{\sigma}}\Big]\\ &\quad\times |\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi.\end{aligned}$$

As a result, we get

$$\|f\|^2_{\dot{H}^\delta(\mathbb{R}^3)} \leq \frac{(n_0+1)!(2a+1)}{(2a)^{n_0+1}}\int_{\mathbb{R}^3}\Big[\frac{(2a|\xi|^{\frac{1}{\sigma}})^{n_0}}{n_0!} + \frac{(2a|\xi|^{\frac{1}{\sigma}})^{n_0+1}}{(n_0+1)!}\Big]|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi.$$

Hence, we deduce

$$\|f\|^2_{\dot{H}^\delta(\mathbb{R}^3)} \leq \frac{(n_0+1)!(2a+1)}{(2a)^{n_0+1}}\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi = \frac{(n_0+1)!(2a+1)}{(2a)^{n_0+1}}\|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)},$$

which completes the proof of Lemma 1.2.13.

□

Lemma 1.2.14. *Let $a > 0$, $\sigma \geq 1$, $s \in [0, \frac{3}{2})$ and $\delta \geq \frac{3}{2}$. For every $f \in H^s_{a,\sigma}(\mathbb{R}^3)$, we have that $f \in \dot{H}^\delta(\mathbb{R}^3)$. More precisely, there exists a positive constant $C_{a,s,\delta,\sigma}$ such that*

$$\|f\|_{\dot{H}^\delta(\mathbb{R}^3)} \leq C_{a,s,\delta,\sigma}\|f\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

[2]Let p and q be positive real numbers such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for all $a, b > 0$.

Proof. It is sufficient to apply Lemma 1.2.13 and the standard continuous embedding $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$.

$\square$

Remark 1.2.15. Observe that the proof above assures that Lemma 1.2.14 is still valid if it is assumed that $\delta \geq s > 3/2$.

The next result is another version of the Lemmas 1.2.8 **i)** and 1.2.9 in the spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$.

Lemma 1.2.16. *Let $a > 0$, $\sigma > 1$, and $s \in [0, \frac{n}{2})$, for each $n \in \mathbb{N}$. For every $f, g \in \dot{H}^s_{a,\sigma}(\mathbb{R}^n)$, we obtain $fg \in \dot{H}^s_{a,\sigma}(\mathbb{R}^n)$. More specifically, one has*

i) $\|fg\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} \leq 2^{\frac{2s+1-2n}{2}}\pi^{-n}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|_{L^1(\mathbb{R}^n)}\|g\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^n)}\|f\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}]$;

ii) $\|fg\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n))} \leq 2^{s+1-n}\pi^{-n}C_{a,\sigma,s}\|f\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}\|g\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}$,

where $C_{a,\sigma,s,n} := \sqrt{\frac{2\pi^{\frac{n}{2}}\sigma\Gamma(\sigma(n-2s))}{\Gamma(\frac{n}{2})[2(a-\frac{a}{\sigma})]^{\sigma(n-2s)}}} < \infty$. *Here $\Gamma(\cdot)$ is the gamma function.*

Proof. It is easy to check that

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{fg}(\xi)|^2\, d\xi = (2\pi)^{-2n}\int_{\mathbb{R}^n} |\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{f} * \widehat{g}(\xi)|^2\, d\xi$$
$$\leq (2\pi)^{-2n}\int_{\mathbb{R}^n}\Big(\int_{|\eta|\leq|\xi-\eta|}|\xi|^s e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\, d\eta + \int_{|\eta|>|\xi-\eta|}|\xi|^s e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\, d\eta\Big)^2 d\xi.$$

By using the inequality $|\xi|^s \leq 2^s[\max\{|\xi-\eta|, |\eta|\}]^s$ and Lemma 1.2.4, one deduces

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} \leq (2\pi)^{-2n}2^{2s+1}\Big[\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\xi-\eta|^s e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\, d\eta\Big)^2 d\xi$$
$$+ \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\eta|^s e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\, d\eta\Big)^2 d\xi\Big],$$

or equivalently,

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} \leq (2\pi)^{-2n}2^{2s+1}\{\|[|\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|] * [e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|]\|^2_{L^2(\mathbb{R}^n)} + \|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|] * [|\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|]\|^2_{L^2(\mathbb{R}^n)}\}.$$

Therefore, Young's inequality for convolutions implies

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)} \leq 2^{2s+1-2n}\pi^{-2n}[\||\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^2(\mathbb{R}^n)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^1(\mathbb{R}^n)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^1(\mathbb{R}^n)}\||\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^2(\mathbb{R}^n)}]$$
$$\leq 2^{2s+1-2n}\pi^{-2n}[\|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^1(\mathbb{R}^n)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^1(\mathbb{R}^n)}\|g\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}]. \quad (1.15)$$

The proof of **i)** is completed. Now, let us give the proof of **ii)**. Applying Cauchy-Schwarz's inequality, it results

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|\,d\xi \le \left(\int_{\mathbb{R}^n} |\xi|^{-2s}e^{2(\frac{a}{\sigma}-a)|\xi|^{\frac{1}{\sigma}}}\,d\xi\right)^{\frac{1}{2}}\|g\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^n)}. \tag{1.16}$$

Hence, by replacing (1.16) in (1.15), **ii)** follows.

□

As a consequence of Lemma 1.2.16 **i)**, we have the following result.

Lemma 1.2.17. *Let $a \ge 0$, $\sigma \ge 1$, and $s > 1$. Then, the following inequality holds:*

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} \le \frac{2^{3s}\pi^{-1}e^{2a}}{s-1}(\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} + \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)} + \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}\|g\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}). \tag{1.17}$$

Proof. By applying the Cauchy-Schwarz's inequality, one infers

$$\begin{aligned}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|\,d\xi \le \left(\int_{\mathbb{R}^2}(1+|\xi|^2)^{-s}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\ &= \sqrt{\frac{\pi}{s-1}}\|g\|_{H^s_{a,\sigma}(\mathbb{R}^2)}.\end{aligned} \tag{1.18}$$

It is important to point out that (1.17) follows directly from Lemma 1.2.16 **i)**, (1.18) and Lemma 1.2.10. □

The next lemma is based on the paper [6].

Lemma 1.2.18. *Let $(s_1, s_2) \in \mathbb{R}^2$ such that $s_1 < 1 < s_2$. Then, there is a positive constant C_{s_1,s_2} such that*

$$\|\widehat{f}\|_{L^1(\mathbb{R}^2)} \le C_{s_1,s_2}\|f\|^{\frac{s_2-1}{s_2-s_1}}_{\dot{H}^{s_1}(\mathbb{R}^2)}\|f\|^{\frac{1-s_1}{s_2-s_1}}_{\dot{H}^{s_2}(\mathbb{R}^2)}.$$

Proof. Let c be an arbitrary positive constant. Note that

$$\|\widehat{f}\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2}|\widehat{f}(\xi)|\,d\xi = \int_{|\xi|>c}|\widehat{f}(\xi)|\,d\xi + \int_{|\xi|\le c}|\widehat{f}(\xi)|\,d\xi.$$

By applying Cauchy-Schwarz's inequality, we have

$$\int_{|\xi|\le c}|\widehat{f}(\xi)|\,d\xi \le \left(\int_{|\xi|\le c}|\xi|^{-2s_1}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|\xi|^{2s_1}|\widehat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} = \sqrt{\frac{2\pi}{2-2s_1}}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^2)}c^{1-s_1}.$$

Similarly, one obtains

$$\int_{|\xi|>c} |\widehat{f}(\xi)|\, d\xi \le \left(\int_{|\xi|>c} |\xi|^{-2s_2}\, d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\xi|^{2s_2}|\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} = \sqrt{\frac{2\pi}{2s_2-2}}\|f\|_{\dot{H}^{s_2}(\mathbb{R}^2)}c^{1-s_2}.$$

Consequently, we can write

$$\|\widehat{f}\|_{L^1(\mathbb{R}^2)} \le \sqrt{\frac{2\pi}{2-2s_1}}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^2)}c^{1-s_1} + \sqrt{\frac{2\pi}{2s_2-2}}\|f\|_{\dot{H}^{s_2}(\mathbb{R}^2)}c^{1-s_2} =: g(c).$$

It is easy to see that g attains its minimum at $c_0 = \left[\frac{(s_2-1)\sqrt{\frac{2\pi}{2s_2-2}}\|f\|_{\dot{H}^{s_2}(\mathbb{R}^2)}}{(1-s_1)\sqrt{\frac{2\pi}{2-2s_1}}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^2)}}\right]^{\frac{1}{s_2-s_1}}$. Thus,

$$\|\widehat{f}\|_{L^1(\mathbb{R}^2)} \le g(c_0) \le C_{s_1,s_2}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^2)}^{\frac{s_2-1}{s_2-s_1}}\|f\|_{\dot{H}^{s_2}(\mathbb{R}^2)}^{\frac{1-s_1}{s_2-s_1}}.$$

□

It is important to point out that if $s \in (0,1)$; then, one can assume $s_1 = s$ and $s_2 = s+1$ in Lemma 1.2.18 in order to obtain the following interpolation inequality

$$\|\widehat{f}\|_{L^1(\mathbb{R}^2)} \le C_s\|f\|_{\dot{H}^s(\mathbb{R}^2)}^{s}\|f\|_{\dot{H}^{s+1}(\mathbb{R}^2)}^{1-s}. \tag{1.19}$$

Lastly, we present an elementary result, which follows from basic Calculus tools.

Lemma 1.2.19 (see [8])**.** *Let $a, b > 0$. Then, $\lambda^a e^{-b\lambda} \le a^a(eb)^{-a}$ for all $\lambda > 0$.*

Proof. Consider the real function f defined by $f(\lambda) = \lambda^a e^{-b\lambda}$, for all $\lambda > 0$. It is easy to verify that f attains its maximum at a/b since

$$f'(\lambda) = \lambda^a e^{-b\lambda}\left[\frac{a}{\lambda} - b\right] \text{ and } f''(\lambda) = \lambda^a e^{-b\lambda}\left[\left(\frac{a}{\lambda} - b\right)^2 - \frac{a}{\lambda^2}\right], \quad \forall\, \lambda > 0.$$

Therefore, the proof of the lemma is complete.

□

1.2.2 Auxiliary Results for the Chapter 7

In this section, we present some results that will play an important role in Chapter 7. The first one is a result well-known as Banach Fixed Point Theorem.

Lemma 1.2.20 (See [21]). *Let Y be a nonempty complete metric space and let $S : Y \to Y$ be a strict contraction, i.e.,*

$$d(Sx, Sy) \le K d(x, y), \quad \forall\, x, y \in Y,$$

where $0 < K < 1$. Then, S has a unique fixed point.

Now, let us cite an improvement of Lemma 3 in [5] (it is enough to assume $s = 1 - 2\gamma$).

Lemma 1.2.21. *The following inequalities are valid:*

i) $\|f\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} \le \|f\|_{\mathcal{X}^{s}(\mathbb{R}^3)}^{1-\frac{1}{2\gamma}} \|f\|_{\mathcal{X}^{s+2\gamma}(\mathbb{R}^3)}^{\frac{1}{2\gamma}}$, *provided that $\gamma \ge \frac{1}{2}$ and $s \in \mathbb{R}$;*

ii) $\|f\|_{\mathcal{X}^{0}(\mathbb{R}^3)} \le \|f\|_{\mathcal{X}^{s}(\mathbb{R}^3)}^{1+\frac{s}{2\gamma}} \|f\|_{\mathcal{X}^{s+2\gamma}(\mathbb{R}^3)}^{-\frac{s}{2\gamma}}$, *if $\gamma > \frac{1}{2}$ and $-2\gamma \le s \le 0$.*

Proof. By applying Hölder's inequality[3], it is true that

$$\begin{aligned}
\|f\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{s+1} |\hat{f}(\xi)|\, d\xi \\
&\le \left(\int_{\mathbb{R}^3} |\xi|^{s} |\hat{f}(\xi)|\, d\xi \right)^{1-\frac{1}{2\gamma}} \left(\int_{\mathbb{R}^3} |\xi|^{s+2\gamma} |\hat{f}(\xi)|\, d\xi \right)^{\frac{1}{2\gamma}} \\
&= \|f\|_{\mathcal{X}^{s}(\mathbb{R}^3)}^{1-\frac{1}{2\gamma}} \|f\|_{\mathcal{X}^{s+2\gamma}(\mathbb{R}^3)}^{\frac{1}{2\gamma}}.
\end{aligned}$$

Therefore, item **i)** is established. The proof of **ii)** is also a consequence of Hölder's inequality. In fact,

$$\begin{aligned}
\|f\|_{\mathcal{X}^{0}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\hat{f}(\xi)|\, d\xi \\
&\le \left(\int_{\mathbb{R}^3} |\xi|^{s} |\hat{f}(\xi)|\, d\xi \right)^{1+\frac{s}{2\gamma}} \left(\int_{\mathbb{R}^3} |\xi|^{s+2\gamma} |\hat{f}(\xi)|\, d\xi \right)^{-\frac{s}{2\gamma}} \\
&= \|f\|_{\mathcal{X}^{s}(\mathbb{R}^3)}^{1+\frac{s}{2\gamma}} \|f\|_{\mathcal{X}^{s+2\gamma}(\mathbb{R}^3)}^{-\frac{s}{2\gamma}}.
\end{aligned}$$

□

The next result is our version of Lemma 4 obtained by J. Benameur and M. Benhamed in [5].

[3] Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ Consider that $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then, $\|fg\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$.

Lemma 1.2.22. *Assume that f, $g \in \mathcal{X}^{s+1}(\mathbb{R}^3) \cap \mathcal{X}^0(\mathbb{R}^3)$, with $s \geq -1$. Then, the inequality below is valid:*

$$\|fg\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} \leq 2^{s-2}\pi^{-3}(\|f\|_{\mathcal{X}^0(\mathbb{R}^3)}\|g\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} + \|f\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\|g\|_{\mathcal{X}^0(\mathbb{R}^3)}).$$

Proof. Notice that

$$\begin{aligned}
\|fg\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{s+1}|\widehat{fg}(\xi)|\, d\xi \\
&= (2\pi)^{-3}\int_{\mathbb{R}^3} |\xi|^{s+1}|\hat{f} * \hat{g}(\xi)|\, d\xi \\
&\leq (2\pi)^{-3}\int_{\mathbb{R}^3} |\xi|^{s+1}\left(\int_{\mathbb{R}^3} |\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|\, d\eta\right)\, d\xi \\
&= (2\pi)^{-3}\int_{\mathbb{R}^3} \Big(\int_{|\eta|\leq|\xi-\eta|} |\xi|^{s+1}|\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|\, d\eta \\
&\quad + \int_{|\eta|>|\xi-\eta|} |\xi|^{s+1}|\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|\, d\eta\Big) d\xi.
\end{aligned}$$

By using basic arguments, it is easy to check that

$$|\xi|^{s+1} \leq (|\xi-\eta|+|\eta|)^{s+1} \leq (2\max\{|\xi-\eta|,|\eta|\})^{s+1} = 2^{s+1}\max\{|\xi-\eta|,|\eta|\}^{s+1}, \quad \forall\, s \geq -1.$$

Hence, we deduce

$$\begin{aligned}
\|fg\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} &\leq 2^{s+1}(2\pi)^{-3}\int_{\mathbb{R}^3} [|\hat{f}(\xi)| * (|\xi|^{s+1}|\hat{g}(\xi)|) + (|\xi|^{s+1}|\hat{f}(\xi)|) * |\hat{g}(\xi)|]\, d\xi \\
&= 2^{s+1}(2\pi)^{-3}[\||\hat{f}| * (|\cdot|^{s+1}|\hat{g}|)\|_{L^1(\mathbb{R}^3)} + \|(|\cdot|^{s+1}|\hat{f}|) * |\hat{g}|\|_{L^1(\mathbb{R}^3)}].
\end{aligned}$$

Consequently, it follows from Young's inequality for convolutions that

$$\begin{aligned}
\|fg\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} &\leq 2^{s+1}(2\pi)^{-3}[\|\hat{f}\|_{L^1(\mathbb{R}^3)}\||\cdot|^{s+1}|\hat{g}|\|_{L^1(\mathbb{R}^3)} + \||\cdot|^{s+1}|\hat{f}|\|_{L^1(\mathbb{R}^3)}\|\hat{g}\|_{L^1(\mathbb{R}^3)}] \\
&= 2^{s+1}(2\pi)^{-3}[\|f\|_{\mathcal{X}^0(\mathbb{R}^3)}\|g\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} + \|f\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\|g\|_{\mathcal{X}^0(\mathbb{R}^3)}].
\end{aligned}$$

□

The next two lemmas will be applied in the proof of Theorems 7.1.1, 7.2.1 and 7.3.1. Moreover, these results were inspired by Lemmas 5 and 6 in [5].

Lemma 1.2.23. *Let $\gamma > 0$, $s \in \mathbb{R}$, and $\theta, \lambda \in S'(\mathbb{R}^3)$ such that $\operatorname{div}\lambda = 0$. Then,*

i) $\int_0^t \|e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda\cdot\nabla\theta)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|\theta\otimes\lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau$;

ii) $\displaystyle\int_0^t \|e^{-(t-\tau)(-\Delta)^\gamma} \lambda \cdot \nabla\theta\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|\theta \otimes \lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau;$

iii) $\displaystyle\left\|\int_0^t e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda \cdot \nabla\theta)\, d\tau\right\|_{L^1_T(\mathcal{X}^{s+2\gamma}(\mathbb{R}^3))} \leq \int_0^t \|\theta \otimes \lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau;$

iv) $\displaystyle\left\|\int_0^t e^{-(t-\tau)(-\Delta)^\gamma} \lambda \cdot \nabla\theta\, d\tau\right\|_{L^1_T(\mathcal{X}^{s+2\gamma}(\mathbb{R}^3))} \leq \int_0^t \|\theta \otimes \lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau,$

for all $T > 0$ and $t \in [0, T]$, where $P_H(\cdot)$ is the Helmontz projector.

Proof. Let us begin with the proof of item **i)**. Notice that

$$\begin{aligned}
\int_0^t \|e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda \cdot \nabla\theta)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau &= \int_0^t \int_{\mathbb{R}^3} |\xi|^s |\mathcal{F}\{e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda \cdot \nabla\theta)\}|\, d\xi\, d\tau \\
&= \int_0^t \int_{\mathbb{R}^3} e^{-(t-\tau)|\xi|^{2\gamma}} |\xi|^s |\mathcal{F}\{P_H(\lambda \cdot \nabla\theta)\}|\, d\xi\, d\tau \\
&\leq \int_0^t \int_{\mathbb{R}^3} |\xi|^s |\mathcal{F}\{\lambda \cdot \nabla\theta\}|\, d\xi\, d\tau.
\end{aligned}$$

The inequality above follow from (2.11). On the other hand, since div $\lambda = 0$, one can write

$$\begin{aligned}
\int_{\mathbb{R}^3} |\xi|^s |\mathcal{F}\{\lambda \cdot \nabla\theta\}|\, d\xi &= \int_{\mathbb{R}^3} |\xi|^s \Big|\sum_{j=1}^3 \mathcal{F}\{D_j(\lambda_j\theta)\}\Big|\, d\xi = \int_{\mathbb{R}^3} |\xi|^s |\xi \cdot \mathcal{F}\{\theta \otimes \lambda\}|\, d\xi \\
&\leq \int_{\mathbb{R}^3} |\xi|^{s+1} |\mathcal{F}\{\theta \otimes \lambda\}|\, d\xi = \|\theta \otimes \lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}, \qquad (1.20)
\end{aligned}$$

Therefore, this completes the proof of **i)**.

Let us mention that the proof of **ii)** is similar to the one described above, without using (2.11). Now, we are going to prove **iii)**. Observe that

$$\begin{aligned}
&\left\|\int_0^t e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda \cdot \nabla\theta)\, d\tau\right\|_{L^1_T(\mathcal{X}^{s+2\gamma}(\mathbb{R}^3))} \\
&\leq \int_0^T \int_0^t \int_{\mathbb{R}^3} |\xi|^{s+2\gamma} |\mathcal{F}\{e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda \cdot \nabla\theta)\}|\, d\xi\, d\tau\, dt \\
&= \int_0^T \int_0^t \int_{\mathbb{R}^3} |\xi|^{s+2\gamma} e^{-(t-\tau)|\xi|^{2\gamma}} |\mathcal{F}\{P_H(\lambda \cdot \nabla\theta)\}|\, d\xi\, d\tau\, dt.
\end{aligned}$$

By using (2.11) and following a similar process to the one presented in (1.20), we get

$$\begin{aligned}
&\left\|\int_0^t e^{-(t-\tau)(-\Delta)^\gamma} P_H(\lambda\cdot\nabla\theta)\,d\tau\right\|_{L^1_T(\mathcal{X}^{s+2\gamma}(\mathbb{R}^3))}\\
&\leq \int_0^T\int_0^t\int_{\mathbb{R}^3}|\xi|^{s+2\gamma+1}e^{-(t-\tau)|\xi|^{2\gamma}}|\widehat{\theta\otimes\lambda}(\xi)|\,d\xi\,d\tau\,dt\\
&=\int_{\mathbb{R}^3}|\xi|^{s+2\gamma+1}\left(\int_0^T\int_0^t e^{-(t-\tau)|\xi|^{2\gamma}}|\widehat{\theta\otimes\lambda}(\xi)|d\tau\,dt\right)d\xi. \qquad (1.21)
\end{aligned}$$

On the other hand, it is easy to see

$$\begin{aligned}
\int_0^T\int_0^t e^{-(t-\tau)|\xi|^{2\gamma}}|\widehat{\theta\otimes\lambda}(\xi)|d\tau\,dt &= \int_0^T\int_\tau^T e^{-(t-\tau)|\xi|^{2\gamma}}|\widehat{\theta\otimes\lambda}(\xi)|\,dt\,d\tau\\
&=\int_0^T|\widehat{\theta\otimes\lambda}(\xi)|\left(\int_\tau^T e^{-(t-\tau)|\xi|^{2\gamma}}\,dt\right)d\tau\\
&=\int_0^T\left(\frac{1-e^{-(T-\tau)|\xi|^{2\gamma}}}{|\xi|^{2\gamma}}\right)|\widehat{\theta\otimes\lambda}(\xi)|\,d\tau.
\end{aligned}$$

By replacing this last result in (1.21), we infer

$$\begin{aligned}
\left\|\int_0^t e^{-(t-\tau)(-\Delta)^\gamma}P_H(\lambda\cdot\nabla\theta)\,d\tau\right\|_{L^1_T(\mathcal{X}^{s+2\gamma}(\mathbb{R}^3))} &\leq \int_{\mathbb{R}^3}|\xi|^{s+1}\left(\int_0^T|\widehat{\theta\otimes\lambda}(\xi)|\,d\tau\right)d\xi\\
&=\int_0^T\int_{\mathbb{R}^3}|\xi|^{s+1}|\widehat{\theta\otimes\lambda}(\xi)|\,d\xi\,d\tau\\
&=\int_0^T\|\theta\otimes\lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau.
\end{aligned}$$

The proof of **iii)** is complete. Moreover, the proof of **iv)** is analogous to the one obtained in **iii)**, without using (2.11). □

The next result presents sufficient conditions to prove that $\theta\otimes\lambda\in L^1_T(\mathcal{X}^{s+1}(\mathbb{R}^3))$.

Lemma 1.2.24. *Assume that* $T>0$ *and* $\max\left\{\frac{\alpha(1-2\beta)}{\beta},\frac{\beta(1-2\alpha)}{\alpha}\right\}\leq s<0$*, where* $\alpha,\,\beta\in(\frac{1}{2},1]$*. If* $\lambda\in L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))\cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))$ *and* $\theta\in L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))\cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))$*, then*

$$\begin{aligned}
\int_0^T\|\theta\otimes\lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau \leq C_s[&T^{1+\frac{s}{2\alpha}-\frac{1}{2\beta}}\|\lambda\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|\theta\|^{1-\frac{1}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|\lambda\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|\theta\|^{\frac{1}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}\\
&+T^{1+\frac{s}{2\beta}-\frac{1}{2\alpha}}\|\lambda\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|\theta\|^{1+\frac{s}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|\lambda\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|\theta\|^{-\frac{s}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}],
\end{aligned}$$

where $C_s=9(2^{s-2}\pi^{-3})$.

Proof. Notice that

$$\begin{aligned}\int_0^T \|\theta\otimes\lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau &= \int_0^T\int_{\mathbb{R}^3}|\xi|^{s+1}\left[\sum_{j,k=1}^3|\widehat{\lambda_j\theta_k}(\xi)|^2\right]^{\frac{1}{2}}\,d\xi\,d\tau\\ &\leq \sum_{j,k=1}^3\int_0^T\int_{\mathbb{R}^3}|\xi|^{s+1}|\widehat{\lambda_j\theta_k}(\xi)|\,d\xi\,d\tau\\ &= \sum_{j,k=1}^3\int_0^T\|\lambda_j\theta_k\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau.\end{aligned}$$

Hence, by applying Lemma 1.2.22, one obtains

$$\int_0^T \|\theta\otimes\lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau \leq 2^{s-2}\pi^{-3}\sum_{j,k=1}^3\int_0^T(\|\lambda_j\|_{\mathcal{X}^0(\mathbb{R}^3)}\|\theta_k\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}+\|\lambda_j\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\|\theta_k\|_{\mathcal{X}^0(\mathbb{R}^3)})\,d\tau, \tag{1.22}$$

since $s\geq -1$. Now, by using Lemma 1.2.21, we deduce

- $\|\lambda_j\|_{\mathcal{X}^0(\mathbb{R}^3)}\leq \|\lambda_j\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\lambda_j\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}$;
- $\|\theta_k\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\leq \|\theta_k\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\theta_k\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}$;
- $\|\lambda_j\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\leq \|\lambda_j\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\lambda_j\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}$;
- $\|\theta_k\|_{\mathcal{X}^0(\mathbb{R}^3)}\leq \|\theta_k\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\theta_k\|^{-\frac{s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}$,

provided that $\max\{-2\alpha,-2\beta\}\leq s<0$ and α, $\beta\in(\frac{1}{2},1]$. Therefore, by replacing the inequalities above in (1.22), one concludes

$$\begin{aligned}\int_0^T \|\theta\otimes\lambda\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau \leq C_s\int_0^T(&\|\lambda\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\lambda\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\|\theta\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\theta\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\\ &+\|\lambda\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\lambda\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\|\theta\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\theta\|^{-\frac{s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)})\,d\tau,\end{aligned} \tag{1.23}$$

where $C_s=9(2^{s-2}\pi^{-3})$. Moreover, since $\lambda\in L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))\cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))$ and $\theta\in L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))\cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))$, the last two terms in the right hand side of (1.23) can be estimated as follows.

$$\begin{aligned}&\int_0^T\|\lambda\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\lambda\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\|\theta\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\theta\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\,d\tau\\ &\leq \|\lambda\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|\theta\|^{1-\frac{1}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\int_0^T\|\lambda\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\|\theta\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\,d\tau.\end{aligned}$$

By using Hölder's inequality twice, it follows

$$\begin{aligned}
&\int_0^T \|\lambda\|_{\mathcal{X}^s(\mathbb{R}^3)}^{1+\frac{s}{2\alpha}} \|\lambda\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{s}{2\alpha}} \|\theta\|_{\mathcal{X}^s(\mathbb{R}^3)}^{1-\frac{1}{2\beta}} \|\theta\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{1}{2\beta}} \, d\tau \\
&\leq \|\lambda\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{1+\frac{s}{2\alpha}} \|\theta\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{1-\frac{1}{2\beta}} \|\lambda\|_{L_T^1(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}^{-\frac{s}{2\alpha}} \left(\int_0^T \|\theta\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{\alpha}{\beta(s+2\alpha)}} \, d\tau \right)^{\frac{s+2\alpha}{2\alpha}} \\
&\leq T^{1+\frac{s}{2\alpha}-\frac{1}{2\beta}} \|\lambda\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{1+\frac{s}{2\alpha}} \|\theta\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{1-\frac{1}{2\beta}} \|\lambda\|_{L_T^1(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}^{-\frac{s}{2\alpha}} \|\theta\|_{L_T^1(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}^{\frac{1}{2\beta}}, \qquad (1.24)
\end{aligned}$$

provided that $\max\left\{\frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\right\} \leq s < 0$.

A similar process to the one presented above, yields

$$\begin{aligned}
&\int_0^T \|\lambda\|_{\mathcal{X}^s(\mathbb{R}^3)}^{1-\frac{1}{2\alpha}} \|\lambda\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{1}{2\alpha}} \|\theta\|_{\mathcal{X}^s(\mathbb{R}^3)}^{1+\frac{s}{2\beta}} \|\theta\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{s}{2\beta}} \, d\tau \\
&\leq T^{1+\frac{s}{2\beta}-\frac{1}{2\alpha}} \|\lambda\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{1-\frac{1}{2\alpha}} \|\theta\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{1+\frac{s}{2\beta}} \|\lambda\|_{L_T^1(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}^{\frac{1}{2\alpha}} \|\theta\|_{L_T^1(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}^{-\frac{s}{2\beta}}. \qquad (1.25)
\end{aligned}$$

Finally replacing (1.24) and (1.25) in (1.23), the proof is complete.

□

Chapter 2

Navier-Stokes equations: local existence, uniqueness and blow-up of solutions in $H^s_{a,\sigma}(\mathbb{R}^3)$

This Chapter presents a study that determines the local existence, uniqueness and blow–up criteria of solutions for the following Navier–Stokes equations:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \mu\Delta u, \quad x\in\mathbb{R}^3, t\in[0,T^*), \\ \operatorname{div} u = 0, \quad x\in\mathbb{R}^3, t\in[0,T^*), \\ u(x,0) = u_0(x), \quad x\in\mathbb{R}^3, \end{cases} \tag{2.1}$$

where $T^*>0$ denotes the solution existence time, $u(x,t)=(u_1(x,t),u_2(x,t),\, u_3(x,t))\in\mathbb{R}^3$ denotes the incompressible velocity field, and $p(x,t)\in\mathbb{R}$ the hydrostatic pressure. The positive constant μ is the kinematic viscosity and the initial data for the velocity field, given by u_0 in (2.1), is assumed to be divergence free, i.e., $\operatorname{div} u_0=0$.

We shall study the above system with initial data in the Sobolev–Gevrey spaces $H^s_{a,\sigma}(\mathbb{R}^3)$, with $a>0, \sigma\geq 1$ and $s\in\mathbb{R}$.

It is important to emphasize that there are two main goals to be accomplished in this chapter: prove the local existence and uniqueness of a solution $u(x,t)$ for the Navier-Stokes equations (2.1) and establish a blow–up criteria for $u(x,t)$. It is important to point out that the results were mainly inspired by J. Benameur and L. Jlali [7].

Assuming that the initial data u_0 belongs to $H^{s_0}_{a,\sigma}(\mathbb{R}^3)$, with $s_0\in(\frac{1}{2},\frac{3}{2})$, $a>0$ and $\sigma\geq 1$, we prove that there exist a positive time T and a unique solution $u\in C([0,T];H^s_{a,\sigma}(\mathbb{R}^3))$ of the Navier-Stokes equations (2.1) for all $s\leq s_0$ (let us recall that it is not known if $T=\infty$ always holds for these famous equations). Besides, the local existence and uniqueness result obtained in [7] is a particular case of ours; in fact, it is enough to take $s=s_0=1$.

Under the same assumptions adopted above for s_0 and a, and moreover if $\sigma > 1$, $s \in (\frac{1}{2}, s_0]$ and the maximal time interval of existence, $0 \leq t < T^*$, is finite; then, the blow–up inequality

$$\|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \geq \frac{C_2 \exp\{C_3(T^* - t)^{-\frac{1}{3\sigma}}\}}{(T^* - t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}}, \quad \forall\, t \in [0, T^*), \tag{2.2}$$

holds, where C_2 and C_3 are positive constants that depend only on a, μ, s, σ and u_0, and $2\sigma_0$ is the integer part of 2σ. As a consequence, it is easy to check that (2.2) implies

$$\|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \geq \frac{C_2}{(T^* - t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}}, \quad \forall\, t \in [0, T^*).$$

In order to give more details on what it is going to be done in this chapter, we shall also prove the following blow–up criteria related to the space $L^1(\mathbb{R}^3)$

$$\int_t^{T^*} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty, \tag{2.3}$$

and

$$\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{\sqrt{T^* - t}}, \tag{2.4}$$

for all $t \in [0, T^*), n \in \mathbb{N} \cup \{0\}$. Note that the criteria (2.3) follows from the limit superior

$$\limsup_{t \nearrow T^*} \|u(t)\|_{H^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)} = \infty, \quad \forall\, n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

Notice that (2.4) is not trivial; provided that, $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1(\mathbb{R}^3)}$ is finite for all $t \in [0, T^*), n \in \mathbb{N} \cup \{0\}$. It can be concluded due to the estimate (1.12) and the standard continuous embedding $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)$. Furthermore, by applying the Dominated Convergence Theorem in (2.4), one obtains

$$\|\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} = \lim_{n\to\infty} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{\sqrt{T^* - t}}, \quad \forall\, t \in [0, T^*). \tag{2.6}$$

Besides, the inequality (2.6) is not trivial as well. In fact, it follows from Lemmas 1.2.12 and 1.2.14, and (2.52) below.

It is also important to clarify that the lower bound given in (2.2) is not the only one that is obtained assuming the $H^s_{a,\sigma}(\mathbb{R}^3)$–norm. More specifically, we shall assure that

$$\|u(t)\|_{H^s_{\frac{a}{(\sqrt{\sigma})^n},\sigma}(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{C_1\sqrt{T^* - t}}, \quad \forall\, t \in [0, T^*), n \in \mathbb{N} \cup \{0\}, \tag{2.7}$$

where C_1 depends only on a, σ, s and n.

Notice that all the blow–up criteria obtained in [7] are particular cases of ours, it suffices to assume $s = s_0 = 1$.

2.1 Local Existence and Uniqueness of Solutions

In this section, we shall assume that the initial data u_0 belongs to $H^{s_0}_{a,\sigma}(\mathbb{R}^3)$, with $s_0 \in (\frac{1}{2}, \frac{3}{2})$, to show the existence of an instant $T > 0$ and a unique solution $u \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ for the Navier–Stokes system (2.1) provided that $s \leq s_0$, $a > 0$ and $\sigma \geq 1$.

Let us to establish our first result that presents the local existence and uniqueness of solutions for the Navier-Stokes equations 2.1.

Theorem 2.1.1. *Assume that $a > 0$, $\sigma \geq 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Then, there exist an instant $T = T_{s,a,\mu,u_0} > 0$ and a unique solution $u \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ for the Navier–Stokes equations* (2.1).

Proof. Our aim in this proof is to assure that all the assumptions presented in Lemma 1.2.1 are satisfied if (2.13) and (2.14) below hold; thus, first of all, let us rewrite the Navier-Stokes equations (2.1) as in (1.1).

Use the heat semigroup $e^{\mu\Delta(t-\tau)}$, with $\tau \in [0,t]$, in the first equation given in (2.1), and; then, integrate the obtained result over the interval $[0,t]$ to reach

$$\int_0^t e^{\mu\Delta(t-\tau)} u_\tau \, d\tau + \int_0^t e^{\mu\Delta(t-\tau)} [u \cdot \nabla u + \nabla p] \, d\tau = \mu \int_0^t e^{\mu\Delta(t-\tau)} \Delta u \, d\tau.$$

By applying integration by parts to the first integral above and using the properties of the heat semigroup, one deduces

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} [u \cdot \nabla u + \nabla p] \, d\tau. \tag{2.8}$$

Let us recall that Helmontz's projector P_H is well defined and it is a linear operator such that

$$P_H(u \cdot \nabla u) = u \cdot \nabla u + \nabla p, \tag{2.9}$$

and also

$$\mathcal{F}[P_H(f)](\xi) = \widehat{f}(\xi) - \frac{\widehat{f}(\xi) \cdot \xi}{|\xi|^2} \xi. \tag{2.10}$$

Notice that the equality (2.10) implies that

$$|\mathcal{F}[P_H(f)](\xi)|^2 = \left| \widehat{f}(\xi) - \frac{\widehat{f}(\xi) \cdot \xi}{|\xi|^2} \xi \right|^2 = |\widehat{f}(\xi)|^2 - \frac{|\widehat{f}(\xi) \cdot \xi|^2}{|\xi|^2} \leq |\widehat{f}(\xi)|^2. \tag{2.11}$$

On the other hand, by replacing (2.9) in (2.8), it follows that

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H[u \cdot \nabla u] \, d\tau.$$

Since $u \cdot \nabla u = \sum_{j=1}^{3} u_j D_j u$, one has

$$\begin{aligned} u(t) &= e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H(u \cdot \nabla u)\, d\tau \\ &= e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H\left[\sum_{j=1}^{3}(u_j D_j u)\right] d\tau \\ &= e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H\left[\sum_{j=1}^{3} D_j(u_j u)\right] d\tau, \end{aligned}$$

provided that $\operatorname{div} u = 0$. Rewriting this last equality above, we get

$$u(t) = e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H\left[\sum_{j=1}^{3} D_j(u_j u)\right] d\tau, \tag{2.12}$$

or equivalently,

$$u(t) = e^{\mu\Delta t}u_0 + B(u,u)(t), \tag{2.13}$$

where

$$B(w,v)(t) = -\int_0^t e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(v_j w)]\, d\tau. \tag{2.14}$$

In order to apply Lemma 1.2.1, let X be the Banach space $C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ ($T > 0$ will be chosen later). It is important to notice that (2.13) is the same equation as (1.1) if it is considered that $a = u$ and $x_0 = e^{\mu\Delta t}u_0$. Moreover, it is easy to check that B is a bilinear operator. Therefore, we shall prove that B is continuous by choosing T small enough.

At first, let us estimate $B(w,v)(t)$ in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$. It follows from the definition of the space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ that

$$\begin{aligned} &\|e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(v_j w)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \\ &= \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(v_j w)]\}(\xi)|^2\, d\xi. \end{aligned}$$

It is also well known that

$$\mathcal{F}\{e^{\Delta t} f\}(\xi) = e^{-t|\xi|^2} \hat{f}(\xi), \quad \forall\, \xi \in \mathbb{R}^3, t \geq 0.$$

As a consequence, we have

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\
&= \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}\{P_H[\sum_{j=1}^{3}D_j(v_jw)]\}(\xi)|^2\,d\xi.
\end{aligned}$$

By applying (2.11), one can write

$$\begin{aligned}
\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\sum_{j=1}^{3}\mathcal{F}[D_j(v_jw)](\xi)|^2\,d\xi\\
&\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes v)(\xi)\cdot\xi|^2\,d\xi\\
&\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s+2}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes v)(\xi)|^2\,d\xi.
\end{aligned}$$

Rewriting the last integral above with the goal of applying Lemma 1.2.19, we have

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq \int_{\mathbb{R}^3} |\xi|^{5-2s}e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes v)(\xi)|^2\,d\xi.
\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows

$$\begin{aligned}
\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\leq \frac{\left(\frac{5-2s}{4e\mu}\right)^{\frac{5-2s}{2}}}{(t-\tau)^{\frac{5-2s}{2}}}\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes v)(\xi)|^2\,d\xi\\
&=: \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{2}}}\|w\otimes v\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)}, \qquad (2.15)
\end{aligned}$$

where $C_{s,\mu}=[(5-2s)/4e\mu]^{\frac{5-2s}{2}}$ $(s<3/2)$. On the other hand, let us estimate the term $\|w\otimes v\|_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)}$ presented in the last equality above. Lemma 1.2.7 is the tool that provides a suitable result related to our goal in this proof. Thus, by using this lemma, one infers

$$\begin{aligned}
\|w\otimes v\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{w\otimes v}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{v_jw_k}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\|v_jw_k\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)}\\
&\leq C_s\|w\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|v\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}, \qquad (2.16)
\end{aligned}$$

provided that $0 < s < 3/2$. Therefore, by replacing (2.16) in (2.15), one deduces

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{4}}}\|w\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|v\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

By integrating over $[0,t]$, the above estimate, we conclude

$$\int_0^t \|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau \leq C_{s,\mu}\int_0^t \frac{\|w\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|v\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}}{(t-\tau)^{\frac{5-2s}{4}}}\,d\tau$$
$$\leq C_{s,\mu}T^{\frac{2s-1}{4}}\|w\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|v\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}, \tag{2.17}$$

for all $t \in [0,T]$ (recall that $s > 1/2$).

By (2.14), we can assure that (2.17) implies

$$\|B(w,v)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,\mu}T^{\frac{2s-1}{4}}\|w\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|v\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}, \tag{2.18}$$

for all $t \in [0,T]$. It is important to observe that (2.18) presents our estimate to the operator B related to the space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$.

Now, let us estimate $B(w,v)(t)$ in $H^s_{a,\sigma}(\mathbb{R}^3)$. By Lemma 1.2.10 and (2.18), it is enough to get a upper bound to $B(w,v)(t)$ in $L^2(\mathbb{R}^3)$. Following a similar process to the one presented above, we have

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)](\xi)|^2\,d\xi.$$

Parseval's identity implies the following equality:

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^{-3}\int_{\mathbb{R}^3} |\mathcal{F}\{e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\}(\xi)|^2\,d\xi.$$

As a result, we obtain

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\|^2_{L^2(\mathbb{R}^3)}$$
$$= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\mathcal{F}\{P_H[\sum_{j=1}^{3} D_j(v_j w)]\}(\xi)|^2\,d\xi.$$

By using (2.11), it is true that

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(v_j w)]\|^2_{L^2(\mathbb{R}^3)} \leq (2\pi)^{-3}\int_{\mathbb{R}^3} |\xi|^2 e^{-2\mu(t-\tau)|\xi|^2}|\mathcal{F}(w\otimes v)(\xi)|^2\,d\xi.$$

Rewriting the last integral in order to apply Lemma 1.2.19, one has

$$\begin{aligned}&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|^2_{L^2(\mathbb{R}^3)}\\&\leq (2\pi)^{-3}\int_{\mathbb{R}^3}|\xi|^{5-2s}e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s-3}|\mathcal{F}(w\otimes v)(\xi)|^2\,d\xi.\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|^2_{L^2(\mathbb{R}^3)}\leq \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{2}}}\|w\otimes v\|^2_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)},$$

since $s<3/2$. Now we are interested in estimating the term $\|w\otimes v\|_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)}$ above. Lemma 1.2.6 is the tool that lets us obtain this specific bound. More precisely, by using this lemma, one has

$$\begin{aligned}\|w\otimes v\|^2_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)}&=\int_{\mathbb{R}^3}|\xi|^{2s-3}|\widehat{w\otimes v}(\xi)|^2\,d\xi=\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\xi|^{2s-3}|\widehat{v_jw_k}(\xi)|^2\,d\xi\\&=\sum_{j,k=1}^{3}\|v_jw_k\|^2_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)}\leq C_s\|w\|^2_{\dot{H}^s(\mathbb{R}^3)}\|v\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

By the continuous embedding $H^s_{a,\sigma}(\mathbb{R}^3)\hookrightarrow \dot{H}^s(\mathbb{R}^3)$ $(s\geq 0)$ holds and by applying Lemma 1.2.10, we deduce

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|_{L^2(\mathbb{R}^3)}\leq \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{4}}}\|w\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|v\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

By integrating over $[0,t]$, the above estimate, we conclude

$$\begin{aligned}&\int_0^t\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(v_jw)]\|_{L^2(\mathbb{R}^3)}\,d\tau\\&\leq C_{s,\mu}T^{\frac{2s-1}{4}}\|w\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|v\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))},\end{aligned}\tag{2.19}$$

for all $t\in[0,T]$ (since that $s>1/2$).

By using the definition (2.14) and applying (2.19), one concludes

$$\|B(w,v)(t)\|_{L^2(\mathbb{R}^3)}\leq C_{s,\mu}T^{\frac{2s-1}{4}}\|w\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|v\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))},\tag{2.20}$$

for all $t\in[0,T]$.

Finally, by using Lemma 1.2.10, (2.18), (2.20) and the fact that $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ $(s \geq 0)$, it follows that

$$\|B(w,v)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,a,\mu} T^{\frac{2s-1}{4}} \|w\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))} \|v\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}, \tag{2.21}$$

for all $t \in [0,T]$.

To use Lemma 1.2.1, it is enough to guarantee that

$$4C_{s,a,\mu} T^{\frac{2s-1}{4}} \|e^{\mu\Delta t} u_0\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))} < 1.$$

Thus, first of all, as we did before, one concludes

$$\begin{aligned}\|e^{\mu\Delta t} u_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\mu\Delta t} u_0\}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^3} e^{-2\mu t|\xi|^2} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}_0(\xi)|^2 \, d\xi \\ &\leq \int_{\mathbb{R}^3} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}_0(\xi)|^2 \, d\xi \\ &= \|u_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}$$

As a result, we write

$$\|e^{\mu\Delta t} u_0\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))} \leq \|u_0\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

Now, choosing

$$T < \frac{1}{[4C_{s,a,\mu} \|u_0\|_{H^s_{a,\sigma}(\mathbb{R}^3)}]^{\frac{4}{2s-1}}},$$

where $C_{s,a,\mu}$ is given in (2.21), and apply Lemma 1.2.1 in order to obtain a unique solution $u \in C([0,T];\, H^s_{a,\sigma}(\mathbb{R}^3))$ for the equation (2.13).

The arguments given above also establish the local existence of a unique solution for the Navier-Stokes equations (2.1). □

Now, let us enunciate precisely our main result related to the local existence and uniqueness of solutions for the Navier-Stokes equations (2.1).

Theorem 2.1.2. *Assume that $a > 0$, $\sigma \geq 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Then, there exist an instant $T = T_{s_0,a,\mu,u_0} > 0$ and a unique solution $u \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \leq s_0$, for the Navier–Stokes equations* (2.1).

Proof. By applying Theorem 2.1.1, one has $T = T_{s_0,a,\mu,u_0} > 0$ and a unique solution $u \in C([0,T]; H^{s_0}_{a,\sigma}(\mathbb{R}^3))$ of the Navier-Stokes system (2.1). On the other hand, we also have that $s \leq s_0$. As a result, one obtains the standard embedding $H^{s_0}_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^s_{a,\sigma}(\mathbb{R}^3)$ and, consequently, $u \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$.

□

2.2 Blow–up Criteria for the Solution

In this section, we establish some blow–up criteria for the solution of the Navier-Stokes equations (2.1). We will argue similarly as in references [2, 3, 4, 7, 10, 11, 34].

2.2.1 Limit Superior Related to $H^s_{a,\sigma}(\mathbb{R}^3)$

The first blow–up criterion is related to the limit superior given in (2.5) (case $n = 1$).

Theorem 2.2.1. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that* $\operatorname{div} u_0 = 0$. *Consider that $u \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 2.1.2. *If $T^* < \infty$, then*

$$\limsup_{t \nearrow T^*} \|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = \infty. \tag{2.22}$$

Proof. Suppose by contradiction that (2.22) is not valid, i.e., assume that

$$\limsup_{t \nearrow T^*} \|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} < \infty. \tag{2.23}$$

As a result, we shall prove that the solution $u(\cdot, t)$ can be extended beyond $t = T^*$ (it is the absurd that we shall obtain). Let us prove this statement.

Assuming (2.23) holds, and using Theorem 2.1.2, there is a non negative constant C such that

$$\|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C, \quad \forall\, t \in [0, T^*). \tag{2.24}$$

Integrating over $[0, t]$ the inequality (2.40) below, and applying (2.24) and (1.12), one concludes

$$\|u(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \mu \int_0^t \|\nabla u(\tau)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq \|u_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\mu} C^4 T^*,$$

for all $t \in [0, T^*)$. As we are interested in using the fact that the integral above is bounded, we can write

$$\int_0^t \|\nabla u(\tau)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{\mu} \|u_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\mu} C^4 T^* =: C_{s,a,\sigma,\mu,u_0,T^*}, \tag{2.25}$$

for all $t \in [0, T^*)$.

Now, let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence such that $\kappa_n \nearrow T^*$, where $\kappa_n \in (0, T^*)$, for all $n \in \mathbb{N}$ (choose $\kappa_n = T^* - 1/n$, for n large enough, for instance). We will show that $(u(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $H^s_{a,\sigma}(\mathbb{R}^3)$, that is,

$$\lim_{n,m\to\infty} \|u(\kappa_n) - u(\kappa_m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0. \tag{2.26}$$

Let us mention that the limit (2.26) does not depend on the sequence $(\kappa_n)_{n\in\mathbb{N}}$. This fact will be shown later. First of all, we begin with the proof of (2.26). Thereby, one can apply (2.13) and (2.14) in order to obtain

$$u(\kappa_n) - u(\kappa_m) = I_1(n,m) + I_2(n,m) + I_3(n,m), \tag{2.27}$$

where

$$I_1(n,m) = [e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}]u_0, \tag{2.28}$$

$$I_2(n,m) = \int_0^{\kappa_m} [e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H[u\cdot\nabla u]\,d\tau, \tag{2.29}$$

and also

$$I_3(n,m) = -\int_{\kappa_m}^{\kappa_n} e^{\mu\Delta(\kappa_n-\tau)}P_H[u\cdot\nabla u]\,d\tau. \tag{2.30}$$

Let us prove that $I_j(n,m)\to 0$ in $H^s_{a,\sigma}(\mathbb{R}^3)$, as $n,m\to\infty$, for $j=1,2,3$.

In order to prove the validity of the limit related to $I_1(n,m)$, notice that

$$\begin{aligned}
\|I_1(n,m)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \|[e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}]u_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&= \int_{\mathbb{R}^3}[e^{-\mu\kappa_n|\xi|^2} - e^{-\mu\kappa_m|\xi|^2}]^2(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\,d\xi\\
&\leq \int_{\mathbb{R}^3}[e^{-\mu\kappa_n|\xi|^2} - e^{-\mu T^*|\xi|^2}]^2(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\,d\xi.
\end{aligned}$$

By using the fact that $u_0\in H^s_{a,\sigma}(\mathbb{R}^3)$ and that $e^{-\mu\kappa_n|\xi|^2} - e^{-\mu T^*|\xi|^2}\leq 1$, for all $n\in\mathbb{N}$, it results from Dominated Convergence Theorem that

$$\lim_{n,m\to\infty}\|I_1(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

Now, our next goal is to establish the limit $\lim_{n,m\to\infty}\|I_2(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0$. Thus, we have

$$\|I_2(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq \int_0^{\kappa_m}\|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau =$$
$$\int_0^{\kappa_m}\Big(\int_{\mathbb{R}^3}[e^{-\mu(\kappa_m-\tau)|\xi|^2} - e^{-\mu(\kappa_n-\tau)|\xi|^2}]^2(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[P_H(u\cdot\nabla u)](\xi)|^2 d\xi\Big)^{\frac{1}{2}}\,d\tau.$$

By (2.11), we can write $|\mathcal{F}[P_H(f)](\xi)|\leq|\hat{f}(\xi)|$ and, consequently,

$$\begin{aligned}
&\|I_2(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq \int_0^{T^*}\Big(\int_{\mathbb{R}^3}[1-e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla u](\xi)|^2 d\xi\Big)^{\frac{1}{2}}\,d\tau.
\end{aligned}$$

Use Cauchy-Schwarz's inequality in order to obtain

$$\|I_2(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}$$
$$\leq \sqrt{T^*}\Big(\int_0^{T^*}\int_{\mathbb{R}^3}[1-e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla u](\xi)|^2 d\xi d\tau\Big)^{\frac{1}{2}}.$$

On the other hand, observe that by Lemma 1.2.9, (2.24) and (2.25), one infers

$$\begin{aligned}\int_0^{T^*}\|u\cdot\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau &\leq C^2_{s,a,\sigma}\int_0^{T^*}\|u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\ &\leq C^2_{s,a,\sigma}C^2\int_0^{T^*}\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau<\infty. \qquad (2.31)\end{aligned}$$

As $1-e^{-\mu(T^*-\kappa_m)|\xi|^2}\leq 1$, for all $m\in\mathbb{N}$; then, by Dominated Convergence Theorem, we deduce

$$\lim_{n,m\to\infty}\|I_2(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}=0.$$

Lastly, we show that $\lim_{n,m\to\infty}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}=0$. Indeed

$$\begin{aligned}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} &\leq \int_{\kappa_m}^{\kappa_n}\|e^{\mu\Delta(\kappa_n-\tau)}P_H(u\cdot\nabla u)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\ &= \int_{\kappa_m}^{\kappa_n}\Big(\int_{\mathbb{R}^3}e^{-2\mu(\kappa_n-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[P_H(u\cdot\nabla u)](\xi)|^2 d\xi\Big)^{\frac{1}{2}}\,d\tau.\end{aligned}$$

By (2.11), we can write $|\mathcal{F}[P_H(f)](\xi)|\leq|\hat{f}(\xi)|$ and, consequently,

$$\begin{aligned}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} &\leq \int_{\kappa_m}^{\kappa_n}\Big(\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla u](\xi)|^2 d\xi\Big)^{\frac{1}{2}}\,d\tau\\ &\leq \int_{\kappa_m}^{T^*}\|u\cdot\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau.\end{aligned}$$

By Cauchy-Schwarz's inequality, (2.31) and (2.25), one infers

$$\begin{aligned}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} &\leq \sqrt{T^*-\kappa_m}\left(\int_{\kappa_m}^{T^*}\|u\cdot\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\right)^{\frac{1}{2}}\\ &\leq CC_{s,a,\sigma}\sqrt{T^*-\kappa_m}\left(\int_{\kappa_m}^{T^*}\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\right)^{\frac{1}{2}}\\ &\leq C_{s,a,\sigma,\mu,u_0,T^*}\sqrt{T^*-\kappa_m}.\end{aligned}$$

As a result, we infer that $\lim_{n,m\to\infty}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}=0$. Thus, (2.27) implies (2.26). In addition, (2.26) means that $(u(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $H^s_{a,\sigma}(\mathbb{R}^3)$. Hence, there exists $u_1\in H^s_{a,\sigma}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty}\|u(\kappa_n)-u_1\|_{H^s_{a,\sigma}(\mathbb{R}^3)}=0.$$

Now, we shall prove that the above limit does not depend on the sequence $(\kappa_n)_{n\in\mathbb{N}}$. Thus, choose an arbitrary sequence $(\rho_n)_{n\in\mathbb{N}} \subseteq (0, T^*)$ such that $\rho_n \nearrow T^*$ and

$$\lim_{n\to\infty} \|u(\rho_n) - u_2\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0,$$

for some $u_2 \in H^s_{a,\sigma}(\mathbb{R}^3)$. Let us show that $u_2 = u_1$. In fact, define $(\varsigma_n)_{n\in\mathbb{N}} \subseteq (0, T^*)$ by $\varsigma_{2n} = \kappa_n$ and $\varsigma_{2n-1} = \rho_n$, for all $n \in \mathbb{N}$. It is easy to check that $\varsigma_n \nearrow T^*$. By rewriting the process above, we guarantee that there is $u_3 \in H^s_{a,\sigma}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|u(\varsigma_n) - u_3\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

As a consequence, one has

$$\lim_{n\to\infty} \|u(\kappa_n) - u_3\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = \lim_{n\to\infty} \|u(\varsigma_{2n}) - u_3\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0$$

and also

$$\lim_{n\to\infty} \|u(\rho_n) - u_3\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = \lim_{n\to\infty} \|u(\varsigma_{2n-1}) - u_3\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

By uniqueness of the limit, one infers $u_1 = u_3 = u_2$. Therefore, the limit (2.26) does not depend on the sequence $(\kappa_n)_{n\in\mathbb{N}}$.

It means that $\lim_{t\nearrow T^*} \|u(t) - u_1\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0$. Thereby, assuming (2.1) with the initial data u_1, instead of u_0, we assure, by Theorem 2.1.2, the local existence and uniqueness of $\bar{u} \in C([0, \bar{T}]; H^s_{a,\sigma}(\mathbb{R}^3))$ $(\bar{T} > 0)$ for the system (2.1). Hence, $\tilde{u} \in C([0, \bar{T} + T^*]; H^s_{a,\sigma}(\mathbb{R}^3))$ defined by

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [0, T^*); \\ \bar{u}(t - T^*), & t \in [T^*, \bar{T} + T^*], \end{cases}$$

solves (2.1) in $[0, \bar{T} + T^*]$. Thus, the solution of (2.1) can be extended beyond $t = T^*$. It is a contradiction. Consequently, one must have

$$\limsup_{t\nearrow T^*} \|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = \infty.$$

□

2.2.2 Blow–up of the Integral Related to $L^1(\mathbb{R}^3)$

Now, we present the proof of the inequality (2.3) in the case $n = 1$. It is important to let the reader know that the next theorem might be written as a corollary of Theorem 2.2.1 since the first one follows from this last result.

Theorem 2.2.2. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the Navier-Stokes equations (2.1) obtained in Theorem 2.1.2. If $T^* < \infty$, then*

$$\int_t^{T^*} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty.$$

Proof. This result follows from the limit superior presented in Theorem 2.2.1. Thus, let us start taking the $H^s_{a,\sigma}(\mathbb{R}^3)$-inner product, with $u(t)$, in the first equation of (2.1) to get

$$\langle u, u_t\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)} = \langle u, -u\cdot\nabla u - \nabla p + \mu\Delta u\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)}. \tag{2.32}$$

In order to study some terms on the right hand side of the equality above, use the fact that

$$\mathcal{F}(D_j f)(\xi) = i\xi_j \hat{f}(\xi), \quad \forall\, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,$$

to get

$$\begin{aligned}\mathcal{F}(u)\cdot\mathcal{F}[\nabla p](\xi) &= -i\sum_{j=1}^{3}\mathcal{F}(u_j)(\xi)\xi_j\overline{\hat{p}(\xi)} = -\sum_{j=1}^{3}\mathcal{F}(D_j u_j)(\xi)\overline{\hat{p}(\xi)}\\ &= -\mathcal{F}(\operatorname{div} u)(\xi)\overline{\hat{p}(\xi)} = 0,\end{aligned} \tag{2.33}$$

because u is divergence free (see (2.1)). Thereby, the term related to the pressure in (2.32) is null, namely

$$\langle u, \nabla p\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)} = \int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}\,\mathcal{F}(u)\cdot\mathcal{F}[\nabla p](\xi)\,d\xi = 0. \tag{2.34}$$

On the other hand, following a similar argument, one infers

$$\begin{aligned}\widehat{u}\cdot\widehat{\Delta u}(\xi) &= \sum_{j=1}^{3}\widehat{u}\cdot\widehat{D_j^2 u}(\xi) = -i\sum_{j=1}^{3}\widehat{u}\cdot[\xi_j\widehat{D_j u}(\xi)]\\ &= -\sum_{j=1}^{3}\widehat{D_j u}\cdot\widehat{D_j u}(\xi) = -|\widehat{\nabla u}(\xi)|^2.\end{aligned} \tag{2.35}$$

Therefore, the term related to Δu in (2.32) satisfies

$$\begin{aligned}\langle u, \Delta u\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}\,\widehat{u}\cdot\widehat{\Delta u}(\xi)\,d\xi\\ &= -\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}\,|\widehat{\nabla u}(\xi)|^2\,d\xi\\ &= -\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned} \tag{2.36}$$

By replacing (2.34) and (2.36) in (2.32), we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \mu\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq |\langle u, u\cdot\nabla u\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)}|. \tag{2.37}$$

Now, let us study the inner product above. Since $\operatorname{div} u = 0$, one obtains

$$\begin{aligned}
\mathcal{F}(\nabla u)\cdot\mathcal{F}(u\otimes u)(\xi) &= \sum_{j=1}^{3}\mathcal{F}(\nabla u_j)\cdot\mathcal{F}(u_j u)(\xi) = \sum_{j,k=1}^{3}\mathcal{F}(D_k u_j)(\xi)\overline{\mathcal{F}(u_j u_k)(\xi)}\\
&= i\sum_{j,k=1}^{3}\xi_k\mathcal{F}(u_j)(\xi)\overline{\mathcal{F}(u_j u_k)(\xi)} = -\sum_{j,k=1}^{3}\mathcal{F}(u_j)(\xi)\overline{\mathcal{F}(D_k(u_j u_k))(\xi)}\\
&= -\sum_{j,k=1}^{3}\mathcal{F}(u_j)(\xi)\overline{\mathcal{F}(u_k D_k u_j)(\xi)} = -\sum_{j=1}^{3}\mathcal{F}(u_j)(\xi)\overline{\mathcal{F}(u\cdot\nabla u_j)(\xi)}\\
&= -\mathcal{F}(u)\cdot\mathcal{F}(u\cdot\nabla u)(\xi).
\end{aligned}$$

As a result, by using the tensor product, it follows that

$$\begin{aligned}
\langle u, u\cdot\nabla u\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(u)\cdot\mathcal{F}(u\cdot\nabla u)(\xi)\,d\xi\\
&= -\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(\nabla u)\cdot\mathcal{F}(u\otimes u)(\xi)\,d\xi\\
&= -\langle\nabla u, u\otimes u\rangle_{H^s_{a,\sigma}(\mathbb{R}^3)}. \qquad (2.38)
\end{aligned}$$

Hence, using Cauchy-Schwarz's inequality, (2.37) and (2.38) imply

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \mu\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \le \|\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}. \qquad (2.39)$$

Now, our interest is to find an estimate for the term $\|u\otimes u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}$ obtained above. Thus, by applying Lemma 1.2.8 **i)**, one has

$$\begin{aligned}
\|u\otimes u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(u\otimes u)(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(u_j u_k)(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\|u_j u_k\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\le C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_j}\|_{L^1(\mathbb{R}^3)}\|u_k\|_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|_{L^1(\mathbb{R}^3)}\|u_j\|_{H^s_{a,\sigma}(\mathbb{R}^3)}]^2\\
&\le C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_j}\|^2_{L^1(\mathbb{R}^3)}\|u_k\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|^2_{L^1(\mathbb{R}^3)}\|u_j\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}]\\
&\le C_s\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)},
\end{aligned}$$

or equivalently,

$$\|u\otimes u\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_s \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}\|u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

By replacing this inequality in (2.39), we deduce

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \mu\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_s \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}\|u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

By Young's inequality, it results that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \frac{\mu}{2}\|\nabla u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,\mu} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}. \tag{2.40}$$

Consider $0 \leq t \leq T < T^*$ in order to obtain, by Gronwall's inequality (differential form)[1], the following estimate:

$$\|u(T)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq \|u(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \exp\{C_{s,\mu}\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau\}.$$

Passing to the limit superior, as $T \nearrow T^*$, Theorem 2.2.1 implies

$$\int_t^{T^*} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty, \quad \forall\, t \in [0, T^*).$$

The proof of Theorem 2.2.2 is completed.

□

2.2.3 Blow–up Inequality Involving $L^1(\mathbb{R}^3)$

Below, we present the proof of blow–up inequality (2.4) in the case $n = 1$.

Theorem 2.2.3. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that* $\operatorname{div} u_0 = 0$. *Consider that $u \in C([0,T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 2.1.2. *If $T^* < \infty$, then*

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{\sqrt{T^*-t}}, \quad \forall\, t \in [0, T^*).$$

Proof. Let us mention that this result is a consequence of Theorem 2.2.2. Indeed, apply the Fourier Transform and take the scalar product in $\mathbb{C}^3$ of the first equation of (2.1), with $\widehat{u}(t)$, in order to obtain

$$\widehat{u}\cdot\widehat{u}_t = -\mu|\widehat{\nabla u}|^2 - \widehat{u}\cdot\widehat{u\cdot\nabla u},$$

[1]Let $f, g : [t,T] \to \mathbb{R}$ be differential functions in (t,T) such that $f'(s) \leq g(s)f(s)$, for all $s \in [t,T]$. Then, $f(s) \leq f(t)\exp\left\{\int_t^s g(\tau)\,d\tau\right\}$, for all $s \in [t,T]$.

see (2.33) and (2.35). Consequently, one infers

$$\frac{1}{2}\partial_t|\widehat{u}(t)|^2+\mu|\widehat{\nabla u}|^2\le|\widehat{u}\cdot\widehat{u\cdot\nabla u}|. \tag{2.41}$$

For $\delta>0$ arbitrary, by applying Cauchy-Schwarz's inequality, it is easy to check that

$$\partial_t\sqrt{|\widehat{u}(t)|^2+\delta}+\mu\frac{|\widehat{\nabla u}|^2}{\sqrt{|\widehat{u}|^2+\delta}}\le\frac{|\hat{u}|}{\sqrt{|\widehat{u}|^2+\delta}}|\widehat{u\cdot\nabla u}|\le|\widehat{u\cdot\nabla u}|.$$

By integrating from t to T, with $0\le t\le T<T^*$, one has

$$\sqrt{|\widehat{u}(T)|^2+\delta}+\mu|\xi|^2\int_t^T\frac{|\widehat{u}(\tau)|^2}{\sqrt{|\widehat{u}(\tau)|^2+\delta}}\,d\tau\le\sqrt{|\widehat{u}(t)|^2+\delta}+\int_t^T|(\widehat{u\cdot\nabla u})(\tau)|\,d\tau,$$

since $|\widehat{\nabla u}|=|\xi||\widehat{u}|$. Passing to the limit, as $\delta\to 0$, multiplying by $e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}$ and integrating over $\xi\in\mathbb{R}^3$, we obtain

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(T)\|_{L^1(\mathbb{R}^3)}+\mu\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{\Delta u}(\tau)\|_{L^1(\mathbb{R}^3)}\,d\tau\le\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)}\\&+\int_t^T\int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u\cdot\nabla u})(\tau)|\,d\xi d\tau,\end{aligned} \tag{2.42}$$

since $|\widehat{\Delta u}|=|\xi|^2|\widehat{u}|$. Studying the last term above, we can assure

$$\begin{aligned}|(\widehat{u\cdot\nabla u})(\xi)|&=\Big|\sum_{j=1}^3\widehat{u_jD_ju}(\xi)\Big|=(2\pi)^{-3}\Big|\sum_{j=1}^3\widehat{u_j}*\widehat{D_ju}(\xi)\Big|\\&=(2\pi)^{-3}\Big|\sum_{j=1}^3\int_{\mathbb{R}^3}\widehat{u_j}(\eta)\widehat{D_ju}(\xi-\eta)\,d\eta\Big|\\&\le(2\pi)^{-3}\Big|\int_{\mathbb{R}^3}\widehat{u}(\eta)\cdot\widehat{\nabla u}(\xi-\eta)\,d\eta\Big|\\&\le(2\pi)^{-3}\int_{\mathbb{R}^3}|\widehat{u}(\eta)||\widehat{\nabla u}(\xi-\eta)|\,d\eta.\end{aligned}$$

Therefore, by (1.3), the last integral in (2.42) can be estimated as follows:

$$\begin{aligned}\int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u\cdot\nabla u})(\xi)|\,d\xi&\le(2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\eta)||\widehat{\nabla u}(\xi-\eta)|\,d\eta d\xi\\&\le(2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{u}(\eta)|e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{\nabla u}(\xi-\eta)|\,d\eta d\xi\\&=(2\pi)^{-3}\int_{\mathbb{R}^3}[e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|]*[e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{\nabla u}(\xi)|]\,d\xi\\&=(2\pi)^{-3}\|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{u}|]*[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{\nabla u}|]\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Applying Young's inequality for convolutions we obtain the following inequality.

$$\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{(u\cdot\nabla u)}(\xi)|\, d\xi \leq (2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|_{L^1(\mathbb{R}^3)} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\nabla u}\|_{L^1(\mathbb{R}^3)}. \tag{2.43}$$

Let us obtain an estimate for the term $\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\nabla u}\|_{L^1(\mathbb{R}^3)}$ above. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\nabla u}\|_{L^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{\nabla u}(\xi)|\, d\xi = \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\xi||\widehat{u}(\xi)|\, d\xi \\ &\leq \left(\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\xi|^2|\widehat{u}(\xi)|\, d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)|\, d\xi\right)^{\frac{1}{2}} \\ &= \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta u}\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)}, \end{aligned} \tag{2.44}$$

since $|\xi|^2|\widehat{u}| = |\widehat{\Delta u}|$ and $|\widehat{\nabla u}| = |\xi||\widehat{u}|$. Then, by replacing (2.44) in (2.43), one deduces

$$\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{(u\cdot\nabla u)}(\xi)|\, d\xi \leq (2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|^{\frac{3}{2}}_{L^1(\mathbb{R}^3)} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta u}\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)}.$$

By using Cauchy-Schwarz's inequality once again, we conclude

$$\begin{aligned} &(2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|^{\frac{3}{2}}_{L^1(\mathbb{R}^3)} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta u}\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)} \leq \frac{1}{128\pi^6\mu} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|^3_{L^1(\mathbb{R}^3)} \\ &+ \frac{\mu}{2} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta u}\|_{L^1(\mathbb{R}^3)}. \end{aligned}$$

Consequently, (2.42) can be rewritten as follows:

$$\begin{aligned} &\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(T)\|_{L^1(\mathbb{R}^3)} + \frac{\mu}{2} \int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta u}(\tau)\|_{L^1(\mathbb{R}^3)}\, d\tau \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \\ &+ \frac{1}{128\pi^6\mu} \int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|^3_{L^1(\mathbb{R}^3)} d\tau. \end{aligned}$$

By Gronwall's inequality (integral form)[2], one gets

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(T)\|^2_{L^1(\mathbb{R}^3)} \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|^2_{L^1(\mathbb{R}^3)} \exp\left\{\frac{1}{64\pi^6\mu} \int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)} d\tau\right\},$$

for all $0 \leq t \leq T < T^*$, or equivalently,

$$\left(-64\pi^6\mu\right) \frac{d}{dT}\left[\exp\left\{-\frac{1}{64\pi^6\mu} \int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)} d\tau\right\}\right] \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}(t)\|^2_{L^1(\mathbb{R}^3)}.$$

[2]Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions in $[a, b]$ such that $f(s) \leq f(a) + \int_a^s g(\tau)f(\tau)\, d\tau$, for all $s \in [a, b]$. Then, $f(s) \leq f(a) \exp\left\{\int_a^s g(\tau)\, d\tau\right\}$, for all $s \in [a, b]$.

Integrate from t to t_0, with $0 \leq t \leq t_0 < T^*$, in order to get

$$\begin{aligned}&\big(-64\pi^6\mu\big)\exp\left\{-\frac{1}{64\pi^6\mu}\int_t^{t_0}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\}+64\pi^6\mu\\&\leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|^2_{L^1(\mathbb{R}^3)}(t_0-t).\end{aligned}$$

By passing to the limit, as $t_0 \nearrow T^*$, and using Theorem 2.2.2, we have

$$64\pi^6\mu \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|^2_{L^1(\mathbb{R}^3)}(T^*-t), \quad \forall\, t\in[0,T^*),$$

which proves Theorem 2.2.3.

□

2.2.4 Blow–up Inequality involving $H^s_{a,\sigma}(\mathbb{R}^3)$

The lower bound (2.7), in the case $n = 1$, can be rewritten as below. From now on $T^*_\omega < \infty$ denotes the first blow-up time for the solution $u \in C([0,T^*_\omega); H^s_{\omega,\sigma}(\mathbb{R}^3))$, where $\omega > 0$.

Theorem 2.2.4. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2},\frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0,T^*_a); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 2.1.2*. If $T^*_a < \infty$, then*

$$\|u(t)\|_{H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{C_1\sqrt{T^*_a-t}}, \quad \forall\, t\in[0,T^*_a),$$

where $C_1 := \left\{4\pi\sigma\Big[2a\Big(\dfrac{1}{\sqrt{\sigma}}-\dfrac{1}{\sigma}\Big)\Big]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))\right\}^{\frac{1}{2}}.$

Proof. This theorem is a direct implication of Theorem 2.2.3. First of all, notice that $\frac{a}{\sqrt{\sigma}} \in (0,a)$. As a result, it holds the following continuous embedding $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)$ that comes from the inequality

$$\|u\|_{H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} \leq \|u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

Then, we can guarantee, by Theorem 2.1.2 and inequality above, that $u \in C([0,T^*_a), H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3))$ (since $u \in C([0,T^*_a), H^s_{a,\sigma}(\mathbb{R}^3))$) and also that

$$T^*_{\frac{a}{\sqrt{\sigma}}} \geq T^*_a. \tag{2.45}$$

Moreover, by applying Theorem 2.2.3 and Cauchy-Schwarz's inequality, it follows that

$$
\begin{aligned}
\frac{8\pi^3\sqrt{\mu}}{\sqrt{T_a^*-t}} &\leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|\,d\xi \\
&\leq \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}e^{2(\frac{a}{\sigma}-\frac{a}{\sqrt{\sigma}})|\xi|^{\frac{1}{\sigma}}}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{s}e^{2\frac{a}{\sqrt{\sigma}}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^3}|\xi|^{-2s}e^{2(\frac{a}{\sigma}-\frac{a}{\sqrt{\sigma}})|\xi|^{\frac{1}{\sigma}}}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{s}e^{2\frac{a}{\sqrt{\sigma}}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&\leq C_{a,\sigma,s}\|u(t)\|_{H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)}, \qquad (2.46)
\end{aligned}
$$

for all $t \in [0, T_a^*)$, where

$$
C^2_{a,\sigma,s} := \int_{\mathbb{R}^3}\frac{1}{|\xi|^{2s}}e^{-2a(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma})|\xi|^{\frac{1}{\sigma}}}\,d\xi = 4\pi\sigma\left[2a\left(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s)).
$$

(Recall that $s < 3/2$ and $\sigma > 1$). This concludes the proof of Theorem 2.2.4.

□

2.2.5 Generalization of the Blow–up Criteria

We are ready to prove the blow–up criteria given in (2.3), (2.4), (2.5) and (2.7) with $n > 1$. Actually, it is enough to show the case $n = 2$; since, the proof of the general case follows by applying a simple argument of induction.

Theorem 2.2.5. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0,T_a^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the Navier-Stokes equations (2.1) obtained in Theorem 2.1.2. If $T_a^* < \infty$, then*

i) $\limsup_{t\nearrow T_a^*}\|u(t)\|_{H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} = \infty$;

ii) $\displaystyle\int_t^{T_a^*}\|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty$;

iii) $\|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \geq \dfrac{8\pi^3\sqrt{\mu}}{\sqrt{T_a^*-t}}$;

iv) $\|u(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} \geq \dfrac{8\pi^3\sqrt{\mu}}{C_1\sqrt{T_a^*-t}}$, *for all* $t \in [0, T_a^*)$, *where*

$$C_1 = C_{a,\sigma,s} := \Big\{4\pi\sigma\Big[2\frac{a}{\sqrt{\sigma}}\Big(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\Big)\Big]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))\Big\}^{\frac{1}{2}}.$$

Proof. First of all, let us mention that this result is, in its most part, an adaptation of the proofs of the theorems established before. Understood this, notice that (2.46) implies

$$\limsup_{t\nearrow T_a^*} \|u(t)\|_{H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} = \infty. \tag{2.47}$$

This demonstrates **i)**.

By applying **i)**, as in the proof of in Theorem 2.2.2, one can infer that

$$\int_t^{T_a^*} \|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty, \quad \forall\, t \in [0, T_a^*).$$

It proves **ii)**.

Consequently, **iii)** follows from **ii)** and the proof of Theorem 2.2.3.

Moreover, as an immediate consequence of (2.47), one obtains

$$T_a^* \geq T^*_{\frac{a}{\sqrt{\sigma}}}. \tag{2.48}$$

Thus, using the inequalities (2.45) and (2.48), we reach

$$T_a^* = T^*_{\frac{a}{\sqrt{\sigma}}}. \tag{2.49}$$

Then, as in (2.46), by Cauchy-Schwuarz's inequality, we obtain

$$\begin{aligned}\frac{8\pi^3\sqrt{\mu}}{\sqrt{T^*_{\frac{a}{\sqrt{\sigma}}} - t}} &\leq \|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} e^{\frac{a}{\sigma\sqrt{\sigma}}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|\,d\xi \\ &\leq \Big(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}e^{-2(\frac{a}{\sigma}-\frac{a}{\sigma\sqrt{\sigma}})|\xi|^{\frac{1}{\sigma}}}\,d\xi\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{\frac{2a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \\ &\leq C_{a,\sigma,s}\|u(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)},\end{aligned} \tag{2.50}$$

for all $t \in [0, T^*_{\frac{a}{\sqrt{\sigma}}})$, where

$$C^2_{a,\sigma,s} = \int_{\mathbb{R}^3}\frac{1}{|\xi|^{2s}}e^{-2a(\frac{1}{\sigma}-\frac{1}{\sigma\sqrt{\sigma}})|\xi|^{\frac{1}{\sigma}}}\,d\xi = 4\pi\sigma\left[2\frac{a}{\sqrt{\sigma}}\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s)).$$

By (2.49) and (2.50), one has

$$\|u(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{C_{a,\sigma,s}\sqrt{T_a^* - t}}, \quad \forall\, t \in [0, T_a^*). \tag{2.51}$$

This completes the proof of **iv)**.

□

Remark 2.2.6. Passing to the limit superior, as $t \nearrow T_a^*$, in (2.51), we deduce

$$\limsup_{t \nearrow T_a^*} \|u(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} = \infty.$$

Consequently, the inequality (2.5), with $n = 3$, holds and the process above established can be rewritten in order to guarantee the veracity of (2.3), (2.4), (2.5) and (2.7) with $n = 3$. Therefore, inductively, one concludes that our blow–up criteria are valid for all $n > 1$.

2.2.6 Main Blow-up criterion Involving $H^s_{a,\sigma}(\mathbb{R}^3)$

To end this chapter, let us prove the lower bound given in (2.2). This inequality is our main blow–up criterion of the solution obtained in Theorem 2.1.2.

Theorem 2.2.7. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 2.1.2*. If $T^* < \infty$, then*

$$\|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \geq \frac{a^{\sigma_0+\frac{1}{2}} C_2 \exp\{aC_3(T^*-t)^{-\frac{1}{3\sigma}}\}}{(T^*-t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}}, \quad \forall\, t \in [0, T^*),$$

where $C_2 = C_{\mu,s,\sigma,u_0}, C_3 = C_{\mu,s,\sigma,u_0}$ and $2\sigma_0$ is the integer part of 2σ.

Proof. This result follows from Lemma 1.2.12. In fact, choose $\delta = s + \frac{k}{2\sigma}$, with $k \in \mathbb{N} \cup \{0\}$ and $k \geq 2\sigma$, and $\delta_0 = s + 1$. By using Lemmas 1.2.12 and 1.2.14, and (2.6), we obtain

$$\frac{8\pi^3\sqrt{\mu}}{\sqrt{T^*-t}} \leq \|\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \leq C_s \|u(t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2(s+\frac{k}{2\sigma})}} \|u(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}^{\frac{3}{2(s+\frac{k}{2\sigma})}}.$$

By using the energy estimate

$$\|u(t)\|_{L^2(\mathbb{R}^3)} \leq \|u(t_0)\|_{L^2(\mathbb{R}^3)}, \quad \forall\, 0 \leq t_0 \leq t < T^*, \tag{2.52}$$

see (2) in [11], one has

$$\frac{C_{\mu,s,u_0}}{(T^*-t)^{\frac{2s}{3}}} \left(\frac{D_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}} \right)^k \leq \|u(t)\|^2_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}, \tag{2.53}$$

where $C_{\mu,s,u_0} = (C_s^{-1} 8\pi^3 \sqrt{\mu})^{\frac{4s}{3}} \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{6-4s}{3}}$ and $D_{\sigma,s,\mu,u_0} = (C_s^{-1} 8\pi^3 \sqrt{\mu} \|u_0\|_{L^2(\mathbb{R}^3)}^{-1})^{\frac{2}{3\sigma}}$. Multiplying (2.53) by $\frac{(2a)^k}{k!}$, one obtains

$$\frac{C_{\mu,s,u_0}}{(T^*-t)^{\frac{2s}{3}}} \frac{\left(\frac{2aD_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k}{k!} \leq \int_{\mathbb{R}^3} \frac{(2a)^k}{k!} |\xi|^{2(s+\frac{k}{2\sigma})} |\widehat{u}(t)|^2 \, d\xi$$
$$= \int_{\mathbb{R}^3} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!} |\xi|^{2s} |\widehat{u}(t)|^2 \, d\xi.$$

By summing over the set $\{k \in \mathbb{N}; k \geq 2\sigma\}$ and applying Monotone Convergence Theorem, it results

$$\frac{C_{\mu,s,u_0}}{(T^*-t)^{\frac{2s}{3}}} \Big[\exp\Big\{ \frac{2aD_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}} \Big\} - \sum_{0 \leq k < 2\sigma} \frac{\left(\frac{2aD_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k}{k!} \Big]$$
$$\leq \int_{\mathbb{R}^3} \Big[e^{2a|\xi|^{\frac{1}{\sigma}}} - \sum_{0 \leq k < 2\sigma} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!} \Big] |\xi|^{2s} |\widehat{u}(t)|^2 \, d\xi$$
$$\leq \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(t)|^2 \, d\xi$$
$$\leq \|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}^2,$$

for all $t \in [0, T^*)$. Finally, if we define

$$f(x) = \Big[e^x - \sum_{k=0}^{2\sigma_0} \frac{x^k}{k!} \Big] [x^{-(2\sigma_0+1)} e^{-\frac{x}{2}}], \quad \forall\, x \in (0, \infty),$$

where $2\sigma_0$ is the integer part of 2σ; then, f is continuous on $(0,\infty)$, $f > 0$, $\lim_{x\to\infty} f(x) = \infty$ (it means that f is bounded below as $x \to \infty$) and $\lim_{x \nearrow 0} f(x) = \dfrac{1}{(2\sigma_0+1)!}$ (it implies that f is bounded below as $x \nearrow 0$). Hence, there is a positive constant C_{σ_0} such that $f(x) \geq C_{\sigma_0}$, for all $x > 0$. Therefore, we can write

$$\|u(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}^2 \geq \frac{C_{\mu,s,\sigma_0,u_0}}{(T^*-t)^{\frac{2s}{3}}} \left(\frac{2aD_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}} \right)^{2\sigma_0+1} \exp\left\{ \frac{aD_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}} \right\}$$
$$= \frac{a^{2\sigma_0+1} C_{\mu,s,\sigma,\sigma_0,u_0}}{(T^*-t)^{\frac{2(s\sigma+\sigma_0)+1}{3\sigma}}} \exp\left\{ \frac{aD_{\sigma,s,\mu,u_0}}{(T^*-t)^{\frac{1}{3\sigma}}} \right\},$$

for all $t \in [0, T^*)$. Therefore, the proof of Theorem 2.2.7 is completed. □

Chapter 3

Navier-Stokes equations: local existence, uniqueness and blow-up of solutions in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

Our goal is to improve the results of existence, uniqueness and some blow-up criteria obtained by J. Benameur and L. Jlali [7] for the Navier-Stokes equations (2.1) in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ with $s \in (\frac{1}{2}, \frac{3}{2})$. By this we mean that if we take $s = 1$ in Theorem 3.1.1, the results presented by J. Benameur and L. Jlali [7] for the space $\dot{H}^1_{a,\sigma}(\mathbb{R}^3)$ are immediately obtained.

3.1 Local Existence and Uniqueness of Solutions

This section deals with existence and uniqueness solution of the Navier-Stokes equations (2.1) in Sobolev-Gevrey spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ with $s \in (\frac{1}{2}, \frac{3}{2})$. More specifically, we prove the following result.

Theorem 3.1.1. *Assume that* $a > 0$ *and* $s \in (\frac{1}{2}, \frac{3}{2})$. *Let* $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = 0$. *If* $\sigma \geq 1$*; then, there exist an instant* $T = T_{s,\mu,u_0} > 0$ *and a unique solution* $u \in C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ *for the Navier-Stokes equations* (2.1).

Proof. Our goal is to apply the Lemma 1.2.1 in Navier-Stokes equations (2.1), for this, we use (2.13) and (2.14) to rewrite (2.1), namely,

$$u(t) = e^{\mu\Delta t}u_0 + B(u,u)(t),$$

where

$$B(w,v)(t) = -\int_0^t e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(v_j w)]\, d\tau.$$

Moreover, we use (2.17) to guarantee that B, defined in $C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2},\frac{3}{2})$, is continuous, more specifically

$$\|B(w,v)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,\mu} T^{\frac{2s-1}{4}} \|w\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))} \|v\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}, \tag{3.1}$$

for all $t \in [0,T]$. Therefore, we have proved that $B : C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3)) \times C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3)) \to C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$, where $s \in (\frac{1}{2},\frac{3}{2})$, is a continuous bilinear operator. Thus, consider

$$T < [4C_{s,\mu}\|u_0\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}]^{-\frac{4}{2s-1}},$$

where $C_{s,\mu}$ is given in (3.1) (use the estimative $\|e^{\mu\Delta t}u_0\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \|u_0\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$, which comes from arguments previously established), and apply Lemma 1.2.1 to obtain a unique solution $u \in C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ for the Navier-Stokes equations (2.1).

□

3.2 Blow–up Criteria for the Solution

In this section, we establish the blow–up criteria for the solution of the Navier- Stokes equations (2.1) presented in Theorem 3.1.1, by proving appropriate theorems. It is worth pointing out the difference between the theorems presented this section and in section 2.2, in those presented in previous chapter we have a solution the Navier-Stokes equations (2.1) in nonhomogeneous Sobolev-Gevrey space $H^s_{a,\sigma}(\mathbb{R}^3)$ and the one shown below in Sobolev-Gevrey space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$.

3.2.1 Limit Superior Related to $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

Here, we generalize the arguments presented in the Appendix of [3], where it is considered the space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ ($s \in (1/2, 3/2)$).

Theorem 3.2.1. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2},\frac{3}{2})$. Let $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that* $\operatorname{div} u_0 = 0$. *Consider that $u \in C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem 3.1.1. If $T^* < \infty$, than*

$$\limsup_{t \nearrow T^*} \|u(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \infty. \tag{3.2}$$

Proof. By contradiction, consider that this result is not valid. As a result, from the existence of solution for (2.1) proved above, there exists an absolute positive constant C such that

$$\|u(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C, \quad \forall\, t \in [0, T^*). \tag{3.3}$$

By integrating over the interval $[0,t]$ the estimate (3.9) below, and using (3.3) and (1.16), we have

$$\int_0^t \|\nabla u(\tau)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{\mu}\|u_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\mu}C^4 T^* =: C_{s,a,\sigma,\mu,u_0,T^*}, \quad \forall\, t \in [0,T^*). \tag{3.4}$$

On the other hand, let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence such that $\kappa_n \nearrow T^*$, where $\kappa_n \in (0,T^*)$, for all $n \in \mathbb{N}$. As follows, it will be proved that

$$\lim_{n,m\to\infty} \|u(\kappa_n) - u(\kappa_m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0. \tag{3.5}$$

In fact, (2.12) implies

$$\begin{aligned} u(\kappa_n) - u(\kappa_m) = [e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}]u_0 + \int_0^{\kappa_m} [e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H\,[u\cdot\nabla u]\; d\tau \\ - \int_{\kappa_m}^{\kappa_n} e^{\mu\Delta(\kappa_n-\tau)}P_H\,[u\cdot\nabla u]\; d\tau =: I_1(n,m) + I_2(n,m) + I_3(n,m). \end{aligned} \tag{3.6}$$

Let us estimate each integral $I_j(n,m)$, where $j = 1,2,3$. Starting with $I_1(n,m)$, one deduces

$$\begin{aligned} \|I_1(n,m)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \|[e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}]u_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \\ &\leq \int_{\mathbb{R}^3} [e^{-\mu\kappa_n|\xi|^2} - e^{-\mu T^*|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u}_0(\xi)|^2\, d\xi. \end{aligned}$$

Dominated Convergence Theorem guarantees that $\lim\limits_{n,m\to\infty} \|I_1(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (recall that $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$). Now, let us estimate $I_2(n,m)$ by applying the inequality in (2.11) in order to obtain

$$\begin{aligned} &\|I_2(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \\ &\quad\leq \int_0^{\kappa_m} \|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \\ &\quad= \int_0^{\kappa_m} \Big(\int_{\mathbb{R}^3} [e^{-\mu(\kappa_m-\tau)|\xi|^2} - e^{-\mu(\kappa_n-\tau)|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[P_H(u\cdot\nabla u)](\xi)|^2 d\xi\Big)^{\frac{1}{2}} d\tau \\ &\quad\leq \sqrt{T^*}\Big(\int_0^{T^*}\int_{\mathbb{R}^3} [1 - e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla u](\xi)|^2 d\xi d\tau\Big)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, by Lemma 1.2.16 **ii)** and (3.3), one concludes

$$\|u\cdot\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{a,\sigma,s}\sum_{j=1}^3 \|u_j\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|D_j u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{a,\sigma,s}C\|\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \tag{3.7}$$

As a result, $\int_0^{T^*} \|u\cdot\nabla u\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} d\tau < \infty$. Therefore, Dominated Convergence Theorem implies $I_2(n,m) \to 0$ in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ (see (3.4)).

Now, following an analogous argument to the one presented above to obtain Fourier transform of the heat semigroup and, furthermore, by using (2.11), (3.7), Cauchy-Schwarz's inequality and (3.4), we have

$$\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \int_{\kappa_m}^{\kappa_n} \|u\cdot\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq CC_{s,a,\sigma,\mu,u_0,T^*}\sqrt{T^*-\kappa_m}.$$

As a consequence, $\lim_{n,m\to\infty} \|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$. At last, (3.5) holds and, by applying this limit, there is $u_1 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\lim_{n\to\infty} \|u(\kappa_n) - u_1\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (recall that $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ is a Hilbert space if $s < 3/2$). Notice that the independence of $(\kappa_n)_{n\in\mathbb{N}}$ follows the same process presented in proof of Theorem 2.2.1. Besides, a similar proof shows us how to extend our solution beyond $t = T^*$. It is a contradiction. Hence, we must have $\limsup_{t\nearrow T^*} \|u(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \infty$. In addition, this limit superior also proves that $u \notin C([0,T^*];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2},\frac{3}{2})$.

□

3.2.2 Blow–up of the Integral Related to $L^1(\mathbb{R}^3)$

The next theorem might be written as a corollary of Theorem 3.2.1 since the first one follows from this last result.

Theorem 3.2.2. *Assume that* $a > 0$, $\sigma > 1$ *and* $s \in (\frac{1}{2},\frac{3}{2})$. *Let* $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = 0$. *Consider that* $u \in C([0,T^*);\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ *is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 3.1.1. *If* $T^* < \infty$, *then*

$$\int_t^{T^*} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty.$$

Proof. Arguing as in proof of Theorem 2.2.2, we can write

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + \mu\|\nabla u(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \|\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \tag{3.8}$$

Now, our goal is to find an estimate for the term $\|u\otimes u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$ obtained above. Thus, by applying Lemma 1.2.16 **i)** ($s \in (1/2,3/2)$), one has

$$\|u\otimes u\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \sum_{j,k=1}^{3} \|u_j u_k\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C_s\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

Replacing this inequality in (3.8) and using Young's inequality, it results that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \frac{\mu}{2}\|\nabla u(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le C_{s,\mu}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}. \tag{3.9}$$

Let $0 \le t \le T < T^*$ in order to get, by Gronwall's inequality (differential form), the inequality

$$\|u(T)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le \|u(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \exp\{C_{s,\mu}\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau\}.$$

Passing to the limit superior, as $T \nearrow T^*$, Theorem 3.2.1 implies

$$\int_t^{T^*} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty, \quad \forall\, t \in [0,T^*).$$

□

3.2.3 Blow–up Inequality Involving $L^1(\mathbb{R}^3)$

The theorem below could be stated as a corollary of Theorem 3.2.2.

Theorem 3.2.3. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in \dot H^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0,T^*); \dot H^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 3.1.1*. If $T^* < \infty$, then*

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \ge \frac{8\pi^3\sqrt{\mu}}{\sqrt{T^*-t}}, \quad \forall\, t \in [0,T^*).$$

Proof. Arguing as in proof of Theorem 2.2.3 and using Theorem 3.2.2, we have

$$64\pi^6\mu \le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|^2_{L^1(\mathbb{R}^3)}(T^*-t), \quad \forall\, t \in [0,T^*). \tag{3.10}$$

□

3.2.4 Blow–up Inequality involving $\dot H^s_{a,\sigma}(\mathbb{R}^3)$

Let us recall the following notation: $T^*_\omega < \infty$ denotes the first blow-up time for the solution $u \in C([0,T^*_\omega); \dot H^s_{\omega,\sigma}(\mathbb{R}^3))$, where $\omega > 0$.

Theorem 3.2.4. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0, T^*_a); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 3.1.1. *If $T^*_a < \infty$, then*

$$\|u(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{C_1\sqrt{T^*_a - t}}, \quad \forall\, t \in [0, T^*_a),$$

where $C_1 := \Big\{4\pi\sigma\Big[2a\Big(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\Big)\Big]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))\Big\}^{\frac{1}{2}}$.

Proof. The following embedding $\dot{H}^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)$ holds; then, we can guarantee, by the existence of solution for (2.1), that $u \in C([0, T^*_a), \dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3))$; since, $u \in C([0, T^*_a), \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$. On the other hand, the inequality $\|u(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} \leq \|u(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$, implies that $T^*_{\frac{a}{\sqrt{\sigma}}} \geq T^*_a$. Moreover, by applying (3.10) and Cauchy-Schwarz's inequality (similarly to (1.16)), it follows that

$$\frac{8\pi^3\sqrt{\mu}}{\sqrt{T^*_a - t}} \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \leq C_{a,\sigma,s}\|u(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)}, \quad \forall\, t \in [0, T^*_a), \tag{3.11}$$

where $C^2_{a,\sigma,s} = 4\pi\sigma\left[2a\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s)) < \infty$.

□

3.2.5 Generalization of the Blow–up Criteria

Now, let us apply a simple argument of induction to prove the blow–up criteria given in Theorems 3.2.1, 3.2.2, 3.2.3 and 3.2.4, for $n > 1$.

Theorem 3.2.5. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = 0$. Consider that $u \in C([0, T^*_a); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 3.1.1. *If $T^*_a < \infty$, then*

i) $\limsup_{t \nearrow T^*_a} \|u(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)} = \infty$;

ii) $\displaystyle\int_t^{T^*_a} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty$;

iii) $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \geq \dfrac{8\pi^3\sqrt{\mu}}{\sqrt{T^*_a - t}}$;

iv) $\|u(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^n},\sigma}(\mathbb{R}^3)} \geq \dfrac{8\pi^3\sqrt{\mu}}{C_1\sqrt{T_a^*-t}}$,

for all $t \in [0, T_a^*)$, $n \in \mathbb{N}$; *where*

$$C_1 = C_{a,\sigma,s} := \Big\{4\pi\sigma\Big[2\frac{a}{(\sqrt{\sigma})^{(n-1)}}\Big(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\Big)\Big]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))\Big\}^{\frac{1}{2}}.$$

Proof. Notice that (3.11) implies $\limsup_{t\nearrow T_a^*} \|u(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} = \infty$. This limit superior is **i)** with $n = 2$. As it was discussed before, we can infer that

$$\int_t^{T^*} \|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty, \quad \forall\, t \in [0, T^*).$$

(It proves **ii)** with $n = 2$). It concludes **iii)** with $n = 2$. Moreover, as an immediate consequence of the limit superior above, one obtains $T_a^* \geq T^*_{\frac{a}{\sqrt{\sigma}}}$. Hence, we deduce $T_a^* = T^*_{\frac{a}{\sqrt{\sigma}}}$. Now, reexamining the above steps with $\frac{a}{\sqrt{\sigma}}$ instead of a, as in (3.11), one has

$$\frac{8\pi^3\sqrt{\mu}}{\sqrt{T^*_{\frac{a}{\sqrt{\sigma}}} - t}} \leq \|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \leq C_{\frac{a}{\sqrt{\sigma}},\sigma,s}\|u(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)}, \quad \forall\, t \in [0, T^*_{\frac{a}{\sqrt{\sigma}}}). \tag{3.12}$$

The equality $T_a^* = T^*_{\frac{a}{\sqrt{\sigma}}}$ and (3.12) imply

$$\|u(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} \geq \frac{8\pi^3\sqrt{\mu}}{C_1\sqrt{T^*-t}},$$

for all $t \in [0, T_a^*)$. It proves **iv)** with $n = 2$. Passing to the limit superior, as $t \nearrow T_a^*$, one can get $\limsup_{t\nearrow T_a^*} \|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} = \infty$. Thereby, **i)** with $n = 3$ is established. It is easy to observe that the rest of the proof follows by induction.

□

3.2.6 Main Blow-up criterion Involving $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

At last, let us prove the inequality that is our main blow–up criterion of the solution obtained in Theorem 3.1.1.

Theorem 3.2.6. *Assume that* $a > 0$, $\sigma > 1$ *and* $s \in (\frac{1}{2}, \frac{3}{2})$. *Let* $u_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = 0$. *Consider that* $u \in C([0,T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ *is the maximal solution for the Navier-Stokes equations* (2.1) *obtained in Theorem* 3.1.1. *If* $T^* < \infty$, *then*

$$\frac{a^{\sigma_0+\frac{1}{2}}C_2\exp\{aC_3(T^*-t)^{-\frac{1}{3\sigma}}\}}{(T^*-t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}} \leq \|u(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}, \text{ provided that } u_0 \in L^2(\mathbb{R}^3),$$

for all $t \in [0, T^*)$, *where* $C_2 = C_{\mu,s,\sigma,u_0}$, $C_3 = C_{\mu,s,\sigma,u_0}$ *and* $2\sigma_0$ *is the integer part of* 2σ.

Proof. Take $\delta = s + \frac{k}{2\sigma}$, with $k \in \mathbb{N} \cup \{0\}$ and $k \geq 2\sigma$, and $\delta_0 = s + 1$. Now, using Lemmas 1.2.12 and 1.2.13, and Dominated Convergence Theorem in Theorem 3.2.5 **iii)**, we deduce

$$\frac{8\pi^3\sqrt{\mu}}{\sqrt{T^* - t}} \leq \|\widehat{u}(t)\|_{L^1(\mathbb{R}^3)} \leq C_s \|u(t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2(s+\frac{k}{2\sigma})}} \|u(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}^{\frac{3}{2(s+\frac{k}{2\sigma})}}.$$

Consequently, by using the inequality $\|u(t)\|_{L^2(\mathbb{R}^3)} \leq \|u_0\|_{L^2(\mathbb{R}^3)}$, for all $0 \leq t < T^*$ (see (4) in [4]), one infers

$$\frac{C_{\mu,s,u_0,b_0}}{(T^* - t)^{\frac{2s}{3}}} \left(\frac{D_{\sigma,s,\mu,u_0}}{(T^* - t)^{\frac{1}{3\sigma}}} \right)^k \leq \|u(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}^2, \tag{3.13}$$

where $C_{\mu,s,u_0} = (C_s^{-1} 8\pi^3 \sqrt{\mu})^{\frac{4s}{3}} \|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{6-4s}{3}}$ and $D_{\sigma,s,\mu,u_0} = (C_s^{-1} 8\pi^3 \sqrt{\mu} \|u_0\|_{L^2(\mathbb{R}^3)}^{-1})^{\frac{2}{3\sigma}}$. From this point, just follow the same steps as in proof of Theorem 2.2.7.

□

Chapter 4

The Magneto–Hydrodynamic equations: local existence, uniqueness and blow-up of solutions in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

Consider the unforced Magneto–Hydrodynamic (MHD) equations for incompressible flows on all space $\mathbb{R}^3$:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla(p + \frac{1}{2}|b|^2) = \mu\Delta u + b \cdot \nabla b, \quad x \in \mathbb{R}^3, \quad t \geq 0, \\ b_t + u \cdot \nabla b = \nu\,\Delta b + b \cdot \nabla u, \quad x \in \mathbb{R}^3, \quad t \geq 0, \\ \operatorname{div} u = \operatorname{div} b = 0, \quad x \in \mathbb{R}^3, \quad t \geq 0, \\ u(x,0) = u_0(x), \; b(x,0) = b_0(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

Here $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, $b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t)) \in \mathbb{R}^3$ the magnetic field and $p(x,t) \in \mathbb{R}$ the hydrostatic pressure. The positive constants μ and ν are associated with specific properties of the fluid: The constant μ is the kinematic viscosity and ν^{-1} is the magnetic Reynolds number. The initial data for the velocity and magnetic fields, given by u_0 and b_0 in (4.1), are assumed to be divergence free, i.e., $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Note that the MHD system reduces to the classical incompressible Navier–Stokes system if $b = 0$.

We shall study the above system using the Sobolev–Gevrey spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$. More precisely, we shall obtain solutions with $(u,b) \in C([0,T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ where $\frac{1}{2} < s < \frac{3}{2}$, $a > 0$ and $\sigma \geq 1$. Even in the Navier–Stokes case it is not known if $T^* = \infty$ always holds. In this paper we shall derive blow–up rates for the solution if T^* is finite.

In a recent paper, J. Benameur and L. Jlali [7] proved blow–up criteria for the Navier–Stokes equations in Sobolev–Gevrey spaces. This chapter extends the results of [7] from the Navier–Stokes to the MHD system. Also, we prove the blow–up inequality for $\frac{1}{2} < s < \frac{3}{2}$

whereas only the value $s = 1$ was considerded in [7]. For further blow–up results for the Navier–Stokes and MHD systems we refer to [2, 3, 7, 10, 11, 12, 28, 29, 32, 34, 42] and references therein.

4.1 Local Existence and Uniqueness of Solutions

The following Theorem one guarantees the existence of a finite time $T > 0$ and a unique solution $(u, b) \in C([0, T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2}, \frac{3}{2})$, $a > 0$ and $\sigma \geq 1$, for the MHD equations (4.1).

Theorem 4.1.1. *Assume that $a > 0$, $\sigma \geq 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Then, there exist an instant $T = T_{s,\mu,\nu,u_0,b_0} > 0$ and a unique solution $(u, b) \in C([0, T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ for the MHD equations* (4.1).

Proof. We first proceed formally and apply the heat semigroup $e^{\mu\Delta(t-\tau)}$, with $\tau \in [0, t]$, to the velocity equation in (4.1). Integration in time yields

$$\int_0^t e^{\mu\Delta(t-\tau)} u_\tau \, d\tau + \int_0^t e^{\mu\Delta(t-\tau)} \Big(u \cdot \nabla u - b \cdot \nabla b + \nabla(p + \frac{1}{2}|b|^2) \Big) \, d\tau = \mu \int_0^t e^{\mu\Delta(t-\tau)} \Delta u \, d\tau.$$

Using integration by parts one deduces

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} \Big(u \cdot \nabla u - b \cdot \nabla b + \nabla(p + \frac{1}{2}|b|^2) \Big) \, d\tau.$$

Let us recall that the Helmholtz's projector P_H (see Section 7.2 in [32] and references therein) is well defined, yielding

$$P_H(u \cdot \nabla u - b \cdot \nabla b) = u \cdot \nabla u - b \cdot \nabla b + \nabla(p + \frac{1}{2}|b|^2).$$

As a result, it follows that

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H(u \cdot \nabla u - b \cdot \nabla b) \, d\tau.$$

Therefore,

$$\begin{aligned} u(t) &= e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)}P_H(u\cdot\nabla u - b\cdot\nabla b)\,d\tau \\ &= e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3(u_jD_ju - b_jD_jb)]\,d\tau \\ &= e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(u_ju - b_jb)]\,d\tau, \end{aligned}$$

provided that $\operatorname{div} u = \operatorname{div} b = 0$. Hence,

$$u(t) = e^{\mu\Delta t}u_0 - \int_0^t e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(u_ju - b_jb)]\,d\tau. \tag{4.2}$$

Next, our goal is to present an equality for the field b analogous to (4.2). By applying the heat semigroup $e^{\nu\Delta(t-\tau)}$, with $\tau\in[0,t]$, to the second equation in (4.1) and integrating in time, we obtain

$$\int_0^t e^{\nu\Delta(t-\tau)}b_\tau\,d\tau + \int_0^t e^{\nu\Delta(t-\tau)}[u\cdot\nabla b - b\cdot\nabla u]\,d\tau = \nu\int_0^t e^{\nu\Delta(t-\tau)}\Delta b\,d\tau.$$

Using integrating by parts again, we have

$$b(t) = e^{\nu\Delta t}b_0 - \int_0^t e^{\nu\Delta(t-\tau)}[u\cdot\nabla b - b\cdot\nabla u]\,d\tau.$$

As u and b are divergence free (see (4.1)), it follows that

$$\begin{aligned} b(t) &= e^{\nu\Delta t}b_0 - \int_0^t e^{\nu\Delta(t-\tau)}[\sum_{j=1}^3(u_jD_jb - b_jD_ju)]\,d\tau \\ &= e^{\nu\Delta t}b_0 - \int_0^t e^{\nu\Delta(t-\tau)}[\sum_{j=1}^3 D_j(u_jb - b_ju)]\,d\tau, \end{aligned}$$

that is

$$b(t) = e^{\nu\Delta t}b_0 - \int_0^t e^{\nu\Delta(t-\tau)}[\sum_{j=1}^3 D_j(u_jb - b_ju)]\,d\tau. \tag{4.3}$$

By (4.2) and (4.3), one obtains

$$(u,b)(t) = (e^{\mu\Delta t}u_0, e^{\nu\Delta t}b_0) + B((u,b),(u,b))(t), \tag{4.4}$$

where

$$B((w,v),(\gamma,\phi))(t) = \int_0^t (-e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)], -e^{\nu\Delta(t-\tau)}[\sum_{j=1}^3 D_j(w_j\phi - v_j\gamma)])\, d\tau. \tag{4.5}$$

Here w, v, γ, and ϕ belong to a suitable function space that we now discuss.

Let us estimate $B((w,v),(\gamma,\phi))(t)$ in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ with $1/2 < s < 3/2$, $a > 0$ and $\sigma \geq 1$. It follows from the definition of the space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ that

$$\begin{aligned}&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\quad= \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2\, d\xi.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} =\\ &\quad\int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2\, d\xi.\end{aligned}$$

By applying (2.11), we can write

$$\begin{aligned}&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\quad\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\sum_{j=1}^3 \mathcal{F}[D_j(\gamma_j w - v_j\phi)](\xi)|^2\, d\xi\\ &\quad\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)\cdot\xi|^2\, d\xi\\ &\quad\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} |\xi|^{2s+2} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)|^2\, d\xi.\end{aligned}$$

Rewriting the last integral, we have

$$\begin{aligned}&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\quad\leq \int_{\mathbb{R}^3} |\xi|^{5-2s} e^{-2\mu(t-\tau)|\xi|^2} |\xi|^{4s-3} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)|^2\, d\xi.\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows that

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq \frac{(\frac{5-2s}{4e\mu})^{\frac{5-2s}{2}}}{(t-\tau)^{\frac{5-2s}{2}}}\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi\\
&=: \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{2}}}\|w\otimes\gamma-\phi\otimes v\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)},
\end{aligned}$$

since $s < 3/2$.

On the other hand, by using Lemma 1.2.7, one infers

$$\begin{aligned}
\|w\otimes\gamma\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{w\otimes\gamma}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{\gamma_j w_k}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\|\gamma_j w_k\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)}\\
&\leq C_s\|w\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|\gamma\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)},
\end{aligned} \tag{4.6}$$

provided that $0 < s < 3/2$. Therefore, one deduces

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{4}}}\|(w,v)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|(\gamma,\phi)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

By integrating the above estimate over time from 0 to t, we conclude

$$\begin{aligned}
&\int_0^t \|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\
&\leq C_{s,\mu}\int_0^t \frac{\|(w,v)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|(\gamma,\phi)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}}{(t-\tau)^{\frac{5-2s}{4}}}\,d\tau\\
&\leq C_{s,\mu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned} \tag{4.7}$$

for all $t \in [0,T]$ (recall that $s > 1/2$).

Analogously, we can write

$$\begin{aligned}
&\int_0^t \|e^{\nu\Delta(t-\tau)}[\sum_{j=1}^{3} D_j(w_j\phi - v_j\gamma)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\
&\qquad\leq C_{s,\nu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned} \tag{4.8}$$

for all $t \in [0, T]$.

By (4.5), we can assure that (4.7) and (4.8) imply the bound

$$\|B((w,v),(\gamma,\phi))(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,\mu,\nu} T^{\frac{2s-1}{4}} \|(w,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))} \|(\gamma,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}, \tag{4.9}$$

for all $t \in [0, T]$.

To summarize, it has been shown that

$$\begin{aligned} \|e^{\nu\Delta t} b_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\nu\Delta t} b_0\}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^3} e^{-2\nu t|\xi|^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2 \, d\xi \\ &\leq \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2 \, d\xi = \|b_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \end{aligned} \tag{4.10}$$

Therefore, we have established the following estimate:

$$\|(e^{\mu\Delta t} u_0, e^{\nu\Delta t} b_0)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \|(u_0, b_0)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

Notice that $B : C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3)) \times C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3)) \to C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ (with $s \in (\frac{1}{2}, \frac{3}{2})$, $a > 0$ and $\sigma \geq 1$) is a bilinear operator, which is continuous (see (4.5) and (4.9)). Choosing a time $T > 0$ with

$$T < \frac{1}{[4C_{s,\mu,\nu}\|(u_0, b_0)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}]^{\frac{4}{2s-1}}},$$

where $C_{s,\mu,\nu}$ is given in (4.9), we can apply Lemma 1.2.1 to obtain a unique solution $(u, b) \in C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ for the equation (4.4).

This completes the proof of Theorem 4.1.1.

□

4.2 Blow–up Criteria for the Solution

4.2.1 Limit Superior Related to $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

By assuming that $[0, T^*)$ is the maximal interval of existence for the solution $(u, b)(x, t)$ obtained in Theorem 4.1.1 with T^* finite, let us present our blow-up criteria for the solution $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2}, \frac{3}{2})$ of the MHD equations (4.1).

Theorem 4.2.1. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $div\, u_0 = div\, b_0 = 0$. Assume that $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the solution for the MHD equations* (4.1) *in the maximal time interval $0 \le t < T^*$. If $T^* < \infty$, then*

$$\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \infty.$$

Proof. We first generalize the arguments given in the Appendix of [7]. We prove this theorem by contradiction. Suppose the solution $(u, b)(t)$ exists only in the finite time interval $0 \le t < T^*$ and

$$\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} < \infty. \tag{4.11}$$

We shall prove that the solution can be extended beyond $t = T^*$.

By (4.11) and Theorem 4.1.1 (since $s \in (\frac{1}{2}, \frac{3}{2})$), there exists an absolute constant C with

$$\|(u, b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \le C, \quad \forall\, t \in [0, T^*). \tag{4.12}$$

Integrating the inequality (4.28) below in time and applying (4.12) and (1.16), one concludes

$$\|(u, b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + \theta \int_0^t \|\nabla(u, b)(\tau)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \le \|(u_0, b_0)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\theta} C^4 T^*,$$

for all $t \in [0, T^*)$, where $\theta = \min\{\mu, \nu\}$. Consequently,

$$\begin{aligned} \int_0^t \|\nabla(u, b)(\tau)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau &\le \frac{1}{\theta}\|(u_0, b_0)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\theta} C^4 T^* \\ &=: C_{s,a,\sigma,\theta,u_0,b_0,T^*}, \end{aligned} \tag{4.13}$$

for all $t \in [0, T^*)$.

Let $(\kappa_n)_{n\in\mathbb{N}}$ denote a sequence if times with $0 < \kappa_n < T^*$ and $\kappa_n \nearrow T^*$. We shall prove that

$$\lim_{n,m\to\infty} \|(u, b)(\kappa_n) - (u, b)(\kappa_m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0. \tag{4.14}$$

The following equality holds:

$$(u, b)(\kappa_n) - (u, b)(\kappa_m) = I_1(n, m) + I_2(n, m) + I_3(n, m), \tag{4.15}$$

where

$$I_1(n, m) = ([e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}]u_0, [e^{\nu\Delta\kappa_n} - e^{\nu\Delta\kappa_m}]b_0), \tag{4.16}$$

$$
\begin{aligned}
I_2(n,m) = \Big(\int_0^{\kappa_m} & [e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}] P_H \left[u\cdot\nabla u - b\cdot\nabla b\right] \, d\tau, \\
& \int_0^{\kappa_m} [e^{\nu\Delta(\kappa_m-\tau)} - e^{\nu\Delta(\kappa_n-\tau)}] (u\cdot\nabla b - b\cdot\nabla u) \, d\tau \Big), \qquad (4.17)
\end{aligned}
$$

and also

$$
I_3(n,m) = -\Big(\int_{\kappa_m}^{\kappa_n} e^{\mu\Delta(\kappa_n-\tau)} P_H \left[u\cdot\nabla u - b\cdot\nabla b\right] \, d\tau, \int_{\kappa_m}^{\kappa_n} e^{\nu\Delta(\kappa_n-\tau)} (u\cdot\nabla b - b\cdot\nabla u) \, d\tau \Big). \tag{4.18}
$$

(See (4.4) and (4.5)). On the other hand, it is easy to check that

$$
\begin{aligned}
\| [e^{\nu\Delta\kappa_n} - e^{\nu\Delta\kappa_m}] b_0 \|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} [e^{-\nu\kappa_n|\xi|^2} - e^{-\nu\kappa_m|\xi|^2}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^3} [e^{-\nu\kappa_n|\xi|^2} - e^{-\nu T^*|\xi|^2}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2 \, d\xi.
\end{aligned}
$$

Since $b_0 \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ and $e^{-\nu\kappa_n|\xi|^2} - e^{-\nu T^*|\xi|^2} \leq 1$ for all $n \in \mathbb{N}$ the Dominated Convergence Theorem yields that

$$
\lim_{n,m\to\infty} \| [e^{\nu\Delta\kappa_n} - e^{\nu\Delta\kappa_m}] b_0 \|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0.
$$

Similarly,

$$
\lim_{n,m\to\infty} \| [e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}] u_0 \|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0.
$$

Consequently, $\lim_{n,m\to\infty} \|I_1(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (see (4.16)).

We also have:

$$
\begin{aligned}
&\int_0^{\kappa_m} \| [e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}] P_H(u\cdot\nabla u - b\cdot\nabla b) \|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau \\
&= \int_0^{\kappa_m} \Big(\int_{\mathbb{R}^3} [e^{-\mu(\kappa_m-\tau)|\xi|^2} - e^{-\mu(\kappa_n-\tau)|\xi|^2}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[P_H(u\cdot\nabla u - b\cdot\nabla b)](\xi)|^2 d\xi \Big)^{\frac{1}{2}} \, d\tau.
\end{aligned}
$$

By applying (2.11), we obtain that

$$
\begin{aligned}
&\int_0^{\kappa_m} \| [e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}] P_H(u\cdot\nabla u - b\cdot\nabla b) \|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau \\
&\leq \int_0^{T^*} \Big(\int_{\mathbb{R}^3} [1 - e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[u\cdot\nabla u - b\cdot\nabla b](\xi)|^2 d\xi \Big)^{\frac{1}{2}} \, d\tau.
\end{aligned}
$$

The Cauchy-Schwarz's inequality yields that

$$
\begin{aligned}
&\int_0^{\kappa_m} \| [e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}] P_H(u\cdot\nabla u - b\cdot\nabla b) \|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau \\
&\leq \sqrt{T^*} \Big(\int_0^{T^*} \int_{\mathbb{R}^3} [1 - e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2 |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[u\cdot\nabla u - b\cdot\nabla b](\xi)|^2 d\xi d\tau \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Observe that $1 - e^{-\mu(T^*-\kappa_m)|\xi|^2} \leq 1$ for all $m \in \mathbb{N}$ and $\int_0^{T^*} \|u\cdot\nabla u - b\cdot\nabla b\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} d\tau < \infty$ since that

$$\begin{aligned}
\|u\cdot\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\leq C_{a,\sigma,s}\sum_{j=1}^{3}\|u_j\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|D_j u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq C_{a,\sigma,s}C\|\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.
\end{aligned} \tag{4.19}$$

(See Lemma 1.2.16 **ii)** ($0 \leq s < 3/2$ and $\sigma > 1$), (4.12) and (4.13)). Application of the Dominated Convergence Theorem yields that

$$\lim_{n,m\to\infty}\int_0^{\kappa_m}\|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau = 0.$$

Analogously, we obtain

$$\lim_{n,m\to\infty}\int_0^{\kappa_m}\|[e^{\nu\Delta(\kappa_m-\tau)} - e^{\nu\Delta(\kappa_n-\tau)}](u\cdot\nabla b - b\cdot\nabla u)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau = 0.$$

Therefore, $\lim_{n,m\to\infty}\|I_2(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (see (4.17)).

Finally, note that

$$\begin{aligned}
\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq &\int_{\kappa_m}^{\kappa_n}\|e^{\mu\Delta(\kappa_n-\tau)}P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\
&+\int_{\kappa_m}^{\kappa_n}\|e^{\mu\Delta(\kappa_n-\tau)}(u\cdot\nabla b - b\cdot\nabla u)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.
\end{aligned}$$

Following a similar process to the one proved in (4.10) and applying (2.11), one gets

$$\begin{aligned}
\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq &\int_{\kappa_m}^{\kappa_n}\|u\cdot\nabla u - b\cdot\nabla b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\
&+\int_{\kappa_m}^{\kappa_n}\|u\cdot\nabla b - b\cdot\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.
\end{aligned}$$

Use (4.19) to obtain

$$\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq CC_{a,\sigma,s}\int_{\kappa_m}^{T^*}\|\nabla(u,b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.$$

Therefore, by the Cauchy-Schwarz's inequality and (4.13), one has

$$\begin{aligned}
\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\leq C_{a,\sigma,s}\sqrt{T^*-\kappa_m}\left(\int_{\kappa_m}^{T^*}\|\nabla(u,b)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\right)^{\frac{1}{2}}\\
&\leq C_{s,a,\sigma,\theta,u_0,b_0,T^*}\sqrt{T^*-\kappa_m}.
\end{aligned}$$

This implies that $\lim_{n,m\to\infty}\|I_3(n,m)\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} = 0$. To summarize, we have derived the limit statement of (4.14) from equality (4.15). In other words, we have proved that $((u,b)(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\dot H^s_{a,\sigma}(\mathbb{R}^3)$ (recall that $s < 3/2$). Therefore, there is $(u_1, b_1) \in \dot H^s_{a,\sigma}(\mathbb{R}^3)$ with

$$\lim_{n\to\infty}\|(u,b)(\kappa_n) - (u_1,b_1)\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

Notice that the independence of $(\kappa_n)_{n\in\mathbb{N}}$ follows the same process presented in proof of Theorem 2.2.1.

Finally, consider the MHD equations (4.1) with the initial data (u_1, b_1) in instead of (u_0, b_0) and apply Theorem 4.1.1. As usual, we can piece the two solutions together to obtain a solution in an extended time interval, $0 \le t \le T^* + T$ with $T > 0$. This contradiction proves that

$$\limsup_{t\nearrow T^*}\|(u,b)(t)\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} = \infty.$$

□

4.2.2 Blow–up of the Integral Related to $L^1(\mathbb{R}^3)$

The next result generalizes (4.1) of [7]. In fact, taking $s = 1$ in Theorem 4.2.2 yields (4.1) in [7].

Theorem 4.2.2. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot H^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u,b) \in C([0,T^*); \dot H^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the MHD equations (4.1) obtained in Theorem 4.1.1. If $T^* < \infty$, then*

$$\int_t^{T^*}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty.$$

Proof. Taking the $\dot H^s_{a,\sigma}(\mathbb{R}^3)$-inner product of the velocity equation of (4.1) with $u(t)$ yields

$$\langle u, u_t\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} = \langle u, -u\cdot\nabla u + b\cdot\nabla b - \nabla(p + \frac{1}{2}|b|^2) + \mu\Delta u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}. \tag{4.20}$$

On the Fourier side, the second term on the right hand side of the above equation is

$$\begin{aligned}\mathcal{F}(u)\cdot\mathcal{F}[\nabla(p+\frac{1}{2}|b|^2)](\xi) &= -i\sum_{j=1}^{3}\mathcal{F}(u_j)(\xi)\xi_j\overline{\mathcal{F}[(p+\frac{1}{2}|b|^2)](\xi)}\\ &= -\sum_{j=1}^{3}\mathcal{F}(D_ju_j)(\xi)\overline{\mathcal{F}[(p+\frac{1}{2}|b|^2)](\xi)}\\ &= -\mathcal{F}(\operatorname{div} u)(\xi)\overline{\mathcal{F}[(p+\frac{1}{2}|b|^2)](\xi)} = 0,\end{aligned} \tag{4.21}$$

because u is divergence free. As a consequence, we have

$$\langle u, \nabla(p + \frac{1}{2}|b|^2)\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} \mathcal{F}(u) \cdot \mathcal{F}[\nabla(p + \frac{1}{2}|b|^2)](\xi)\, d\xi = 0. \tag{4.22}$$

Furthermore, it happens that

$$\widehat{u} \cdot \widehat{\Delta u}(\xi) = \sum_{j=1}^{3} \widehat{u} \cdot \widehat{D_j^2 u}(\xi) = -i \sum_{j=1}^{3} \widehat{u} \cdot [\xi_j \widehat{D_j u}(\xi)] = -\sum_{j=1}^{3} \widehat{D_j u} \cdot \widehat{D_j u}(\xi) = -|\widehat{\nabla u}(\xi)|^2. \tag{4.23}$$

Therefore,

$$\begin{aligned} \langle u, \Delta u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} \widehat{u} \cdot \widehat{\Delta u}(\xi)\, d\xi = -\int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{\nabla u}(\xi)|^2\, d\xi \\ &= -\|\nabla u\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \end{aligned} \tag{4.24}$$

Using (4.22) and (4.24) in (4.20), we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + \mu\|\nabla u(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq |\langle u, u \cdot \nabla u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle u, b \cdot \nabla b\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}|. \tag{4.25}$$

Next we consider the magnetic field equation of (4.1) and derive an estimate for $b(t)$ similar to the velocity estimate (4.25). Taking the $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$-inner product of the magnetic field equation with $b(t)$ yields that

$$\langle b, b_t\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \langle u, -u \cdot \nabla b + b \cdot \nabla u + \nu\Delta b\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

By applying (4.24), with b instead of u, it follows that

$$\frac{1}{2}\frac{d}{dt}\|b(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + \nu\|\nabla b(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq |\langle b, u \cdot \nabla b\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle b, b \cdot \nabla u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}|. \tag{4.26}$$

Combining (4.25) and (4.26), we conclude that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + \theta\|\nabla(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \\ &\quad \leq |\langle u, u \cdot \nabla u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle u, b \cdot \nabla b\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle b, u \cdot \nabla b\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle b, b \cdot \nabla u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}|, \end{aligned}$$

where $\theta = \min\{\mu, \nu\}$. Furthermore, since $\operatorname{div} b = 0$, we have

$$\begin{aligned} \mathcal{F}(\nabla b) \cdot \mathcal{F}(b \otimes u)(\xi) &= \sum_{j=1}^{3} \mathcal{F}(\nabla b_j) \cdot \mathcal{F}(u_j b)(\xi) = \sum_{j,k=1}^{3} \mathcal{F}(D_k b_j)(\xi)\overline{\mathcal{F}(u_j b_k)(\xi)} \\ &= i \sum_{j,k=1}^{3} \xi_k \mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(u_j b_k)(\xi)} \\ &= -\sum_{j,k=1}^{3} \mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(D_k(u_j b_k))(\xi)}, \end{aligned}$$

that is

$$\begin{aligned}\mathcal{F}(\nabla b)\cdot\mathcal{F}(b\otimes u)(\xi) &= -\sum_{j,k=1}^{3}\mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(b_k D_k u_j)(\xi)}\\ &= -\sum_{j=1}^{3}\mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(b\cdot\nabla u_j)(\xi)}\\ &= -\mathcal{F}(b)\cdot\mathcal{F}(b\cdot\nabla u)(\xi).\end{aligned}$$

It follows that

$$\begin{aligned}\langle b, b\cdot\nabla u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(b)\cdot\mathcal{F}(b\cdot\nabla u)(\xi)\,d\xi\\ &= -\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(\nabla b)\cdot\mathcal{F}(b\otimes u)(\xi)\,d\xi\\ &= -\langle\nabla b, b\otimes u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}$$

Using that u is divergence free and applying the Cauchy-Schwarz's inequality yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\theta\|\nabla(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\quad\le\|\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|b\otimes b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\quad+\|\nabla b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|\nabla b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|b\otimes u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}\tag{4.27}$$

We have to estimate the term $\|u\otimes b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$ appearing above. Applying Lemma 1.2.16 **i)** $(0\le s<3/2)$ yields that

$$\begin{aligned}\|u\otimes b\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(u\otimes b)(\xi)|^2\,d\xi\\ &= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(b_j u_k)(\xi)|^2\,d\xi\\ &= \sum_{j,k=1}^{3}\|b_j u_k\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\le C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b_j}\|_{L^1(\mathbb{R}^3)}\|u_k\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|_{L^1(\mathbb{R}^3)}\|b_j\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}]^2\\ &\le C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b_j}\|^2_{L^1(\mathbb{R}^3)}\|u_k\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|^2_{L^1(\mathbb{R}^3)}\|b_j\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}]\\ &\le C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|b\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}],\end{aligned}$$

or, equivalently,

$$\|u\otimes b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\le C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}\|u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}\|b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}].$$

Usinging this inequality in (4.27), we infer that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}+\theta\|\nabla(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\\&\qquad\le C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}][\|u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}+\|b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}]\|\nabla(u,b)(t)\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}$$

By Young's inequality:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}+\frac{\theta}{2}\|\nabla(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\\&\quad\le C_{s,\mu,\nu}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}]^2[\|u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}+\|b\|_{\dot H^1_{a,\sigma}(\mathbb{R}^3)}]^2\\&\quad\le C_{s,\mu,\nu}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^2_{L^1(\mathbb{R}^3)}\|(u,b)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}\tag{4.28}$$

Consider $0\le t\le T<T^*$ and apply the Gronwall's inequality (differential form) to obtain:

$$\|(u,b)(T)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\le\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\exp\{C_{s,\mu,\nu}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau\}.$$

Passing to the limit superior, as $T\nearrow T^*$, Theorem 4.2.1 yields that

$$\int_t^{T^*}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau=\infty,\quad\forall\,t\in[0,T^*).$$

□

4.2.3 Blow–up Inequality Involving $L^1(\mathbb{R}^3)$

In this section, we point out that (4.2) in [7] is a particular case of Theorem 4.2.3 obtained for $s=1$ and $b=0$ in (4.1).

Theorem 4.2.3. *Assume that $a>0$, $\sigma>1$ and $s\in(\frac{1}{2},\frac{3}{2})$. Let $(u_0,b_0)\in\dot H^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div}u_0=\operatorname{div}b_0=0$. Consider that $(u,b)\in C([0,T^*);\dot H^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the MHD equations (4.1) obtained in Theorem 4.1.1. If $T^*<\infty$, then*

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}\ge\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}},$$

for all $t\in[0,T^*)$, where $\theta=\min\{\mu,\nu\}$.

Proof. Using Fourier transformation and taking the scalar product in $\mathbb{C}^3$ with $\widehat{u}(t)$, we obtain for the velocity equation of the MHD system:

$$\widehat{u}\cdot\widehat{u}_t=-\mu|\widehat{\nabla u}|^2-\widehat{u}\cdot\widehat{u\cdot\nabla u}+\widehat{u}\cdot\widehat{b\cdot\nabla b}.$$

We have used (4.21) and (4.23). Consequently,

$$\frac{1}{2}\partial_t|\widehat{u}(t)|^2+\mu|\widehat{\nabla u}|^2\leq|\widehat{u}\cdot\widehat{u\cdot\nabla u}|+|\widehat{u}\cdot\widehat{b\cdot\nabla b}|. \tag{4.29}$$

Similarly, by applying Fourier transformation and taking the scalar product in $\mathbb{C}^3$ with $\widehat{b}(t)$, we obtain from the magnetic field equation of the MHD system:

$$\widehat{b}\cdot\widehat{b}_t=-\nu|\widehat{\nabla b}|^2-\widehat{b}\cdot\widehat{u\cdot\nabla b}+\widehat{b}\cdot\widehat{b\cdot\nabla u}.$$

Therefore,

$$\frac{1}{2}\partial_t|\widehat{b}(t)|^2+\nu|\widehat{\nabla b}|^2\leq|\widehat{b}\cdot\widehat{u\cdot\nabla b}|+|\widehat{b}\cdot\widehat{b\cdot\nabla u}|. \tag{4.30}$$

Combining (4.29) and (4.30), it follows that

$$\frac{1}{2}\partial_t|(\widehat{u},\widehat{b})(t)|^2+\theta|(\widehat{\nabla u},\widehat{\nabla b})|^2\leq|\widehat{u}||\widehat{u\cdot\nabla u}|+|\widehat{u}||\widehat{b\cdot\nabla b}|+|\widehat{b}||\widehat{u\cdot\nabla b}|+|\widehat{b}||\widehat{b\cdot\nabla u}|,$$

where $\theta=\min\{\mu,\nu\}$. For $\delta>0$ arbitrary, it is easy to check that

$$\partial_t\sqrt{|(\widehat{u},\widehat{b})(t)|^2+\delta}+\theta\frac{|(\widehat{\nabla u},\widehat{\nabla b})|^2}{\sqrt{|(\widehat{u},\widehat{b})|^2+\delta}}\leq|\widehat{u\cdot\nabla u}|+|\widehat{b\cdot\nabla b}|+|\widehat{u\cdot\nabla b}|+|\widehat{b\cdot\nabla u}|.$$

Integrating from t to T (where $0\leq t\leq T<T^*<\infty$), one obtains that

$$\begin{aligned}&\sqrt{|(\widehat{u},\widehat{b})(T)|^2+\delta}+\theta|\xi|^2\int_t^T\frac{|(\widehat{u},\widehat{b})(\tau)|^2}{\sqrt{|(\widehat{u},\widehat{b})(\tau)|^2+\delta}}\,d\tau\\&\leq\sqrt{|(\widehat{u},\widehat{b})(t)|^2+\delta}+\int_t^T[|(\widehat{u\cdot\nabla u})(\tau)|+|(\widehat{b\cdot\nabla b})(\tau)|+|(\widehat{u\cdot\nabla b})(\tau)|+|(\widehat{b\cdot\nabla u})(\tau)|]\,d\tau,\end{aligned}$$

since $|(\widehat{\nabla u},\widehat{\nabla b})|=|\xi||(\widehat{u},\widehat{b})|$. Passing to the limit, as $\delta\to 0$, multiplying by $e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}$ and integrating over $\xi\in\mathbb{R}^3$, we obtain

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|_{L^1(\mathbb{R}^3)}+\theta\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}\,d\tau\\&\quad\leq\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}\\&\qquad+\int_t^T\int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}[|(\widehat{u\cdot\nabla u})(\tau)|+|(\widehat{b\cdot\nabla b})(\tau)|+|(\widehat{u\cdot\nabla b})(\tau)|+|(\widehat{b\cdot\nabla u})(\tau)|]\,d\xi d\tau,\end{aligned}$$

because $|(\widehat{\Delta u},\widehat{\Delta b})| = |\xi|^2|(\widehat{u},\widehat{b})|$. Moreover, we have

$$
\begin{aligned}
|(\widehat{u\cdot\nabla b})(\xi)| &= \Big|\sum_{j=1}^{3}\widehat{u_j D_j b}(\xi)\Big| = (2\pi)^{-3}\Big|\sum_{j=1}^{3}\widehat{u_j} * \widehat{D_j b}(\xi)\Big| \\
&= (2\pi)^{-3}\Big|\sum_{j=1}^{3}\int_{\mathbb{R}^3}\widehat{u_j}(\eta)\widehat{D_j b}(\xi-\eta)\,d\eta\Big| \\
&\le (2\pi)^{-3}\Big|\int_{\mathbb{R}^3}\widehat{u}(\eta)\cdot\widehat{\nabla b}(\xi-\eta)\,d\eta\Big| \le (2\pi)^{-3}\int_{\mathbb{R}^3}|\widehat{u}(\eta)||\widehat{\nabla b}(\xi-\eta)|\,d\eta.
\end{aligned}
$$

Using the estimate (1.3), we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u\cdot\nabla b})(\xi)|\,d\xi &\le (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\eta)||\widehat{\nabla b}(\xi-\eta)|\,d\eta d\xi \\
&\le (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{u}(\eta)|e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{\nabla b}(\xi-\eta)|\,d\eta d\xi \\
&= (2\pi)^{-3}\int_{\mathbb{R}^3}[e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{u}(\xi)|] * [e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{\nabla b}(\xi)|]\,d\xi \\
&= (2\pi)^{-3}\|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{u}|] * [e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{\nabla b}|]\|_{L^1(\mathbb{R}^3)}.
\end{aligned}
$$

Applying Young's inequality for convolution it follows that

$$
\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u\cdot\nabla b})(\xi)|\,d\xi \le (2\pi)^{-3}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{\nabla b}\|_{L^1(\mathbb{R}^3)}. \tag{4.31}
$$

Furthermore, the Cauchy-Schwarz's inequality implies that

$$
\begin{aligned}
\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{\nabla b}\|_{L^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{\nabla b}(\xi)|\,d\xi = \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\xi||\widehat{b}(\xi)|\,d\xi \\
&\le \left(\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\xi|^2|\widehat{b}(\xi)|\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{b}(\xi)|\,d\xi\right)^{\frac{1}{2}} \\
&= \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{\Delta b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}},
\end{aligned} \tag{4.32}
$$

since $|\xi|^2|\widehat{b}| = |\widehat{\Delta b}|$ and $|\widehat{\nabla b}| = |\xi||\widehat{b}|$. Using the estimate (4.32) in (4.31) yields that

$$
\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u\cdot\nabla b})(\xi)|\,d\xi \le (2\pi)^{-3}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{\Delta b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}}.
$$

Consequently,

$$
\begin{aligned}
&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|_{L^1(\mathbb{R}^3)} + \theta\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}\,d\tau \\
&\quad \le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} + 4(2\pi)^{-3}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|_{L^1(\mathbb{R}^3)}^{\frac{3}{2}}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}}\,d\tau.
\end{aligned}
$$

By using the Cauchy-Schwarz's inequality again, we conclude that

$$\begin{aligned}4(2\pi)^{-3}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^{\frac{3}{2}}_{L^1(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)}\\ \leq \frac{1}{8\pi^6\theta}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^3_{L^1(\mathbb{R}^3)}+\frac{\theta}{2}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Hence,

$$\begin{aligned}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|_{L^1(\mathbb{R}^3)}+\frac{\theta}{2}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}\,d\tau\\ \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}+\frac{1}{8\pi^6\theta}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^3_{L^1(\mathbb{R}^3)}d\tau.\end{aligned}$$

By the Gronwall's inequality (integral form), it follows that

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|^2_{L^1(\mathbb{R}^3)}\leq\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}\exp\left\{\frac{1}{4\pi^6\theta}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\},$$

for all $0\leq t\leq T<T^*$, or equivalently,

$$\left(-4\pi^6\theta\right)\frac{d}{dT}\left[\exp\left\{-\frac{1}{4\pi^6\theta}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\}\right]\leq\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}.$$

Integrate from t to t_0, with $0\leq t\leq t_0<T^*$, to obtain that

$$\left(-4\pi^6\theta\right)\exp\left\{-\frac{1}{4\pi^6\theta}\int_t^{t_0}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\}+4\pi^6\theta\leq\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}(t_0-t).$$

By passing to the limit, as $t_0\nearrow T^*$, and using Theorem 4.2.2, we have

$$4\pi^6\theta\leq\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}(T^*-t),\quad\forall\,t\in[0,T^*).$$

□

4.2.4 Blow–up Inequality involving $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

Here, $T^*_\omega<\infty$ denotes the first blow-up time for the solution $(u,b)\in C([0,T^*_\omega);\dot{H}^s_{\omega,\sigma}(\mathbb{R}^3))$ of the MHD system, where $\omega>0$.

Theorem 4.2.4. *Assume that $a>0$, $\sigma>1$ and $s\in(\frac{1}{2},\frac{3}{2})$. Let $(u_0,b_0)\in\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div}u_0=\operatorname{div}b_0=0$. Consider that $(u,b)\in C([0,T^*_a);\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the MHD equations (4.1) obtained in Theorem 4.1.1. If $T^*_a<\infty$, then*

$$\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)}\geq\frac{2\pi^3\sqrt{\theta}}{C_1\sqrt{T^*_a-t}},\quad\forall\,t\in[0,T^*_a),$$

where $\theta=\min\{\mu,\nu\}$ and $C_1:=\left\{4\pi\sigma\left[2a\left(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))\right\}^{\frac{1}{2}}.$

Proof. To demonstrate this result it is sufficient to follow analogous steps those presented in proof of Theorem 3.2.4.

□

4.2.5 Generalization of the Blow–up Criteria

Notice that the Theorems 4.2.1, 4.2.2, 4.2.3 and 4.2.4 prove the Theorem 4.2.5 in case $n = 1$. From this, it is sufficient to use the induction process presented in proof of Theorem 3.2.5 to guarantee the veracity of theorem below.

Theorem 4.2.5. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*_a); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the MHD equations (4.1) obtained in Theorem 4.1.1. If $T^*_a < \infty$, then*

i) $\limsup_{t \nearrow T^*_a} \|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)} = \infty$;

ii) $\displaystyle\int_t^{T^*_a} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} (\widehat{u}, \widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty$;

iii) $\displaystyle\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}} (\widehat{u}, \widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \geq \frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*_a - t}}$;

iv) $\displaystyle\|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^n},\sigma}(\mathbb{R}^3)} \geq \frac{2\pi^3\sqrt{\theta}}{C_1\sqrt{T^*_a - t}}$,

for all $t \in [0, T^*_a)$, $n \in \mathbb{N}$; where $\theta = \min\{\mu, \nu\}$ and

$$C_1 = C_{a,\sigma,s} := \Big\{4\pi\sigma\Big[2\frac{a}{(\sqrt{\sigma})^{(n-1)}}\Big(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\Big)\Big]^{-\sigma(3-2s)}\Gamma(\sigma(3 - 2s))\Big\}^{\frac{1}{2}}.$$

4.2.6 Main Blow-up criterion Involving $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$

Lastly, observe also that Theorem 4.2.6, by assuming $s = 1$ and $b = 0$, gives the same lower bound as the one determined in [7].

Theorem 4.2.6. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution for the MHD equations (4.1) obtained in Theorem 4.1.1. If $T^* < \infty$, then*

$$\frac{a^{\sigma_0+\frac{1}{2}}C_2 \exp\{aC_3(T^* - t)^{-\frac{1}{3\sigma}}\}}{(T^* - t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}} \leq \|(u, b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}, \text{ provided that } u_0 \in L^2(\mathbb{R}^3),$$

for all $t \in [0,T^*)$, where $C_2 = C_{\mu,\nu,s,\sigma,u_0,b_0}, C_3 = C_{\mu,\nu,s,\sigma,u_0,b_0}$ and $2\sigma_0$ is the integer part of 2σ.

Proof. Choose $\delta = s+\frac{k}{2\sigma}$ with $k \in \mathbb{N}\cup\{0\}$ and $k \geq 2\sigma$ and set $\delta_0 = s+1$. By using Lemmas 1.2.12 and 1.2.13, and Dominated Convergence Theorem in Theorem 4.2.5 **iii)**, we obtain

$$\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}} \leq \|(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \leq C_s\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2(s+\frac{k}{2\sigma})}}\|(u,b)(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}^{\frac{3}{2(s+\frac{k}{2\sigma})}},$$

$\theta = \min\{\mu,\nu\}$. Hence, using the inequality

$$\|(u,b)(t)\|_{L^2(\mathbb{R}^3)} \leq \|(u,b)(t_0)\|_{L^2(\mathbb{R}^3)}, \quad \forall\, 0 \leq t_0 \leq t < T^*, \tag{4.33}$$

(see (2) in [11]) we obtain that

$$\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}}\left(\frac{D_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k \leq \|(u,b)(t)\|^2_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)},$$

where

$$D_{\sigma,s,\theta,u_0,b_0} = (C_s^{-1}2\pi^3\sqrt{\theta}\|(u_0,b_0)\|^{-1}_{L^2(\mathbb{R}^3)})^{\frac{2}{3\sigma}}$$

and

$$C_{\theta,s,u_0,b_0} = (C_s^{-1}2\pi^3\sqrt{\theta})^{\frac{4s}{3}}\|(u_0,b_0)\|_{L^2(\mathbb{R}^3)}^{\frac{6-4s}{3}}.$$

Now, just follow the same steps as in proof of Theorem 2.2.7.

□

Chapter 5

The Magneto–Hydrodynamic equations: local existence, uniqueness and blow-up of solutions in $H^s_{a,\sigma}(\mathbb{R}^3)$

This chapter has two main goals: the first one is to generalize all the improvements obtained in Chapter 2 to the MHD equations; the second one is to extend all the results established by J. Benameur [4] from the Navier-Stokes equations to the MHD system (4.1).

5.1 Local Existence and Uniqueness of Solutions

Now, let us list our main results related to the space $H^s_{a,\sigma}(\mathbb{R}^3)$. The first one regards to the existence of an instant $t = T > 0$ and a unique solution $(u, b) \in C([0, T]; H^s_{a,\sigma}(\mathbb{R}^3))$ for the MHD equations (4.1). More precisely, we state the following theorem.

Theorem 5.1.1. *Let $a > 0$, $\sigma \geq 1$ and $s > \frac{1}{2}$ with $s \neq \frac{3}{2}$. Let $(u_0, b_0) \in H^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. If $s > \frac{3}{2}$ (respectively $s \in (\frac{1}{2}, \frac{3}{2})$), then there exist an time $T = T_{s,\mu,\nu,u_0,b_0}$ (respectively $T = T_{s,a,\mu,\nu,u_0,b_0}$) and a unique solution $(u, b) \in C([0, T]; H^{\varpi}_{a,\sigma}(\mathbb{R}^3))$, for all $\varpi \leq s$, of the MHD equations given in (4.1).*

Proof. We know that the Magneto-Hydrodynamic equations can be rewritten as follows (see (4.4) and (4.5)):

$$(u, b)(t) = (e^{\mu\Delta t}u_0, e^{\nu\Delta t}b_0) + B((u, b), (u, b))(t),$$

where

$$B((w, v), (\gamma, \phi))(t) = \int_0^t (-e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)], -e^{\nu\Delta(t-\tau)}[\sum_{j=1}^{3} D_j(w_j\phi - v_j\gamma)])\, d\tau.$$

From this point, we analyze the cases $s \in (\frac{1}{2}, \frac{3}{2})$ and $s > \frac{3}{2}$ separately.

1º Case: Assume that $s > 3/2$.

Here w, v, γ, and ϕ belong to an appropriate space that will be revealed next. In order to examine $(u, b)(t)$ in $H^s_{a,\sigma}(\mathbb{R}^3)$, let us estimate $B((w, v), (\gamma, \phi))(t)$ in this same space. Thus, we deduce

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2 \, d\xi.
\end{aligned}$$

By applying (2.11), one can write

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \\
&\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\sum_{j=1}^{3} \mathcal{F}[D_j(\gamma_j w - v_j\phi)](\xi)|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w \otimes \gamma - \phi \otimes v)(\xi) \cdot \xi|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^3} |\xi|^2 e^{-2\mu(t-\tau)|\xi|^2} (1 + |\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w \otimes \gamma - \phi \otimes v)(\xi)|^2 \, d\xi.
\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows

$$\|e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_\mu (t-\tau)^{-\frac{1}{2}} \|w \otimes \gamma - \phi \otimes v\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

Similarly, we can write

$$\|\sum_{j=1}^{3} D_j e^{\nu\Delta(t-\tau)} (w_j\phi - v_j\gamma)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_\nu (t-\tau)^{-\frac{1}{2}} [\|\phi \otimes w\|_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|\gamma \otimes v\|_{H^s_{a,\sigma}(\mathbb{R}^3)}].$$

Consequently, one gets

$$\begin{aligned}
&\|B((w, v), (\gamma, \phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \\
&\leq C_{\mu,\nu} \Big[\int_0^t (t-\tau)^{-\frac{1}{2}} \|w \otimes \gamma\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau + \int_0^t (t-\tau)^{-\frac{1}{2}} \|\phi \otimes v\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau \\
&\quad + \int_0^t (t-\tau)^{-\frac{1}{2}} \|\phi \otimes w\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau + \int_0^t (t-\tau)^{-\frac{1}{2}} \|\gamma \otimes v\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \, d\tau \Big]. \qquad (5.1)
\end{aligned}$$

On the other hand, it follows

$$\begin{aligned}
\|w\otimes\gamma\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{w\otimes\gamma}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{\gamma_j w_k}(\xi)|^2\,d\xi\\
&= (2\pi)^{-6}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{\gamma_j}*\widehat{w_k}(\xi)|^2\,d\xi\\
&\leq C\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}[(|\widehat{\gamma}|*|\widehat{w}|)(\xi)]^2\,d\xi.
\end{aligned}$$

Therefore, one deduces

$$\|w\otimes\gamma\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C\int_{\mathbb{R}^3}(1+|\xi|^2)^s\left[\int_{\mathbb{R}^3}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{w}(\eta)||\widehat{\gamma}(\xi-\eta)|\,d\eta\right]^2 d\xi.$$

By using (1.3), we conclude

$$\begin{aligned}
\|w\otimes\gamma\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &\leq C\int_{\mathbb{R}^3}(1+|\xi|^2)^s\left[\int_{\mathbb{R}^3}e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{w}(\eta)|e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{\gamma}(\xi-\eta)|\,d\eta\right]^2 d\xi\\
&= C\int_{\mathbb{R}^3}(1+|\xi|^2)^s[e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{w}(\xi)|*e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{\gamma}(\xi)|]^2\,d\xi\\
&= C\int_{\mathbb{R}^3}(1+|\xi|^2)^s\{\mathcal{F}[\mathcal{F}^{-1}(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{w}(\xi)|)\mathcal{F}^{-1}(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{\gamma}(\xi)|)]\}^2\,d\xi\\
&= C\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{w}|)\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{\gamma}|)\|^2_{H^s(\mathbb{R}^3)}.
\end{aligned}$$

Hence, by following a similar process to that used in the proof of Lemma 1.2.8, one has

$$\|w\otimes\gamma\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_s\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{w}|)\|_{H^s(\mathbb{R}^3)}\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{\gamma}|)\|_{H^s(\mathbb{R}^3)} = C_s\|w\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|\gamma\|_{H^s_{a,\sigma}(\mathbb{R}^3)},$$

since $s>3/2$. Replacing this result in (5.1), one obtains

$$\begin{aligned}
&\|B((w,v),(\gamma,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq C_{s,\mu,\nu}\Big[\int_0^t(t-\tau)^{-\frac{1}{2}}\|w\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|\gamma\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau+\int_0^t(t-\tau)^{-\frac{1}{2}}\|\phi\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|v\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\
&\quad+\int_0^t(t-\tau)^{-\frac{1}{2}}\|\phi\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|w\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau+\int_0^t(t-\tau)^{-\frac{1}{2}}\|\gamma\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|v\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\Big].
\end{aligned}$$

As a consequence, if we consider $T>0$, we get

$$\begin{aligned}
&\|B((w,v),(\gamma,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq C_{s,\mu,\nu}T^{\frac{1}{2}}[\|w\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3)}\|\gamma\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}+\|\phi\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|v\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\\
&\quad+\|\phi\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|w\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}+\|\gamma\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|v\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}],
\end{aligned}$$

for all $t \in [0,T]$. Therefore, we deduce

$$\|B((w,v),(\gamma,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,\mu,\nu}T^{\frac{1}{2}}\|(w,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}, \tag{5.2}$$

for all $t \in [0,T]$. By noticing that $B : C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))^2 \to C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ is a bilinear operator and continuous (see (4.5) and (5.2)), it is enough to apply Lemma 1.2.1 and consider T small enough in order to obtain a unique solution $(u,b) \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ for the equation (4.4). More specifically, choose

$$T < \frac{1}{[4C_{s,\mu,\nu}\|(u_0,b_0)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}]^2},$$

where $C_{s,\mu,\nu}$ is given in (5.2) and

$$\|(e^{\mu\Delta t}u_0, e^{\nu\Delta t}b_0)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq \|(u_0,b_0)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

(This estimate comes from a similar process to the one described above).

2º Case: Consider that $s \in (1/2, 3/2)$.

Let us estimate $B((w,v),(\gamma,\phi))(t)$ in $H^s_{a,\sigma}(\mathbb{R}^3)$. It is enough to get a lower bound to $B((w,v),(\gamma,\phi))(t)$ in $L^2(\mathbb{R}^3)$, because (4.9) ensures a lower bound in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ (see Lemma 1.2.10). Following a similar process to the one presented above, we have

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)](\xi)|^2\, d\xi.$$

By using Parseval's identity, one gets

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{L^2(\mathbb{R}^3)}\\
&\quad = (2\pi)^{-3}\int_{\mathbb{R}^3} |\mathcal{F}\{e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2\, d\xi\\
&\quad = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\mathcal{F}\{P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2\, d\xi.
\end{aligned}$$

By using (2.11), we get

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^3 D_j(\gamma_j w - v_j\phi)]\|^2_{L^2(\mathbb{R}^3)} \leq (2\pi)^{-3}\int_{\mathbb{R}^3} |\xi|^2 e^{-2\mu(t-\tau)|\xi|^2}|\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)|^2\, d\xi.$$

Rewriting the last integral, one has

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(\gamma_j w - v_j\phi)]\|^2_{L^2(\mathbb{R}^3)} \\
&\leq (2\pi)^{-3}\int_{\mathbb{R}^3}|\xi|^{5-2s}e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s-3}|\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi.
\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(\gamma_j w - v_j\phi)]\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{2}}}\|w\otimes\gamma-\phi\otimes v\|^2_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)},$$

since $1/2 < s < 3/2$. On the other hand, by utilizing Lemma 1.2.6, one has

$$\|w\otimes\gamma\|^2_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)} = \sum_{j,k=1}^{3}\|\gamma_j w_k\|^2_{\dot{H}^{s-\frac{3}{2}}(\mathbb{R}^3)} \leq C_s\|w\|^2_{\dot{H}^s(\mathbb{R}^3)}\|\gamma\|^2_{L^2(\mathbb{R}^3)}.$$

Thereby, as $H^s_{a,\sigma}(\mathbb{R}^3)\hookrightarrow \dot{H}^s(\mathbb{R}^3)$ $(s\geq 0)$ and Lemma 1.2.10, we deduce

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(\gamma_j w - v_j\phi)]\|_{L^2(\mathbb{R}^3)} \leq \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{4}}}\|(w,v)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|(\gamma,\phi)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

By integrating the above estimate over $[0,t]$, we conclude

$$\begin{aligned}
&\int_0^t \|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3}D_j(\gamma_j w - v_j\phi)]\|_{L^2(\mathbb{R}^3)}\,d\tau \\
&\leq C_{s,\mu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned} \tag{5.3}$$

for all $t\in[0,T]$ (since that $1/2 < s < 3/2$). Similarly, we can obtain

$$\begin{aligned}
&\int_0^t \|e^{\nu\Delta(t-\tau)}[\sum_{j=1}^{3}D_j(w_j\phi - v_j\gamma)]\|_{L^2(\mathbb{R}^3)}\,d\tau \\
&\leq C_{s,\nu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned} \tag{5.4}$$

for all $t\in[0,T]$. By using the definition (4.5) and applying (5.3) and (5.4), one concludes

$$\|B((w,v),(\gamma,\phi))(t)\|_{L^2(\mathbb{R}^3)} \leq C_{s,\mu,\nu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}, \tag{5.5}$$

for all $t\in[0,T]$. Finally, by using Lemma 1.2.10, (4.9), (5.5) and the fact that $H^s_{a,\sigma}(\mathbb{R}^3)\hookrightarrow \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ $(s\geq 0)$, it results

$$\|B((w,v),(\gamma,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,a,\mu,\nu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^3))}, \tag{5.6}$$

for all $t \in [0,T]$. By noticing that $B : C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))^2 \to C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ is a bilinear operator and continuous (see (4.5) and (5.6)), it is enough to apply Lemma 1.2.1 and consider T small enough in order to obtain a unique solution $(u,b) \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^3))$ for the equation (4.4). More specifically, choose

$$T < \frac{1}{[4C_{s,a,\mu,\nu}\|(u_0,b_0)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}]^{\frac{4}{2s-1}}},$$

where $C_{s,a,\mu,\nu}$ is given in (5.6); since,

$$\|(e^{\mu\Delta t}u_0, e^{\nu\Delta t}b_0)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq \|(u_0,b_0)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}.$$

(This estimate follows the steps present above).

Lastly, by assuming that $\varpi \leq s$, it follows that $u \in C([0,T]; H^{\varpi}_{a,\sigma}(\mathbb{R}^3))$ since $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^{\varpi}_{a,\sigma}(\mathbb{R}^3)$.

□

5.2 Blow–up Criteria for the Solution

Assuming that the maximal time of existence of the solution for the MHD equation (4.1), obtained in Theorem 5.1.1, is finite, it is possible to establish some blow-up criteria for this same solution.

5.2.1 Limit Superior Related to $H^s_{a,\sigma}(\mathbb{R}^3)$

Here, we generalize the arguments presented in subsection 2.2.1 . Moreover, it is important to point out that Theorem 5.2.2 is a generalization of the limit superior obtained in Theorem 2.2.1 and Theorem 5.2.1 is an extension for this same limit determined in [4].

Theorem 5.2.1. *Assume that* $s_0 > 3/2$, $a > 0$, *and* $\sigma \geq 1$. *Let* $(u_0,b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Consider that* $(u,b) \in C([0,T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, *for all* $s \in (\frac{3}{2}, s_0]$, *is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If* $T^* < \infty$, *then*

$$\limsup_{t \nearrow T^*} \|(u,b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = \infty.$$

Proof. Suppose by contradiction that Theorem 5.2.1 is not valid, i.e., assume that $T^* < \infty$ is the maximal time of existence of the solution $(u,b)(x,t)$ and consider that

$$\limsup_{t \nearrow T^*} \|(u,b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} < \infty. \tag{5.7}$$

As a result, we can extend the solution obtained above beyond $t = T^*$. It is an absurd. Let us prove these assertions as follows.

Assuming (5.7), and using Theorem 5.1.1, there is $C \geq 0$ such that

$$\|(u,b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C, \quad \forall\, t \in [0, T^*). \tag{5.8}$$

As a consequence, integrating over $[0,t]$ the inequality (5.14) below, and applying (5.8), (1.10) and the fact that $H^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)$, one concludes

$$\|(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \theta \int_0^t \|\nabla(u,b)(\tau)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq \|(u_0,b_0)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,\mu,\nu} C^4 T^*,$$

for all $t \in [0,T^*)$, where $s > 3/2$, $\sigma \geq 1$ and $\theta = \min\{\mu,\nu\}$. As a result, we infer

$$\int_0^t \|\nabla(u,b)(\tau)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{\theta}\|(u_0,b_0)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,\mu,\nu} C^4 T^* =: C_{s,\mu,\nu,u_0,b_0,T^*}, \quad \forall\, t \in [0,T^*), \tag{5.9}$$

where $s > 3/2$ and $\sigma \geq 1$. Now, let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence such that $\kappa_n \nearrow T^*$, where $\kappa_n \in (0,T^*)$, for all $n \in \mathbb{N}$. We claim that

$$\lim_{n,m\to\infty} \|(u,b)(\kappa_n) - (u,b)(\kappa_m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0. \tag{5.10}$$

In fact, let us begin with the following equality:

$$(u,b)(\kappa_n) - (u,b)(\kappa_m) = I_1(n,m) + I_2(n,m) + I_3(n,m), \tag{5.11}$$

where I_1, I_2, I_3 were defined in (4.16), (4.17) and (4.18), respectively. On the other hand, notice that

$$\begin{aligned}\|[e^{\nu\Delta\kappa_n} - e^{\nu\Delta\kappa_m}]b_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} [e^{-\nu\kappa_n|\xi|^2} - e^{-\nu\kappa_m|\xi|^2}]^2 (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2\, d\xi \\ &\leq \int_{\mathbb{R}^3} [e^{-\nu\kappa_n|\xi|^2} - e^{-\nu T^*|\xi|^2}]^2 (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2\, d\xi.\end{aligned}$$

By utilizing the fact that $b_0 \in H^s_{a,\sigma}(\mathbb{R}^3)$ (since $H^{s_0}_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^s_{a,\sigma}(\mathbb{R}^3)$ and $b_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$) and that $e^{-\nu\kappa_n|\xi|^2} - e^{-\nu T^*|\xi|^2} \leq 1$, for all $n \in \mathbb{N}$, it results from the Dominated Convergence Theorem that

$$\lim_{n,m\to\infty} \|[e^{\nu\Delta\kappa_n} - e^{\nu\Delta\kappa_m}]b_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

Analogously, one has

$$\lim_{n,m\to\infty} \|[e^{\mu\Delta\kappa_n} - e^{\mu\Delta\kappa_m}]u_0\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

Thus, we have proved that $\lim_{n,m\to\infty} \|I_1(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (see (4.16)). It is also true that

$$\int_0^{\kappa_m} \|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau =$$
$$\int_0^{\kappa_m} \Big(\int_{\mathbb{R}^3} [e^{-\mu(\kappa_m-\tau)|\xi|^2} - e^{-\mu(\kappa_n-\tau)|\xi|^2}]^2 (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[P_H(u\cdot\nabla u - b\cdot\nabla b)](\xi)|^2 d\xi\Big)^{\frac{1}{2}}\, d\tau.$$

By (2.11), we can write

$$\begin{aligned}&\int_0^{\kappa_m} \|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\ &\leq \int_0^{T^*} \Big(\int_{\mathbb{R}^3} [1 - e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2 (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[u\cdot\nabla u - b\cdot\nabla b](\xi)|^2 d\xi\Big)^{\frac{1}{2}}\, d\tau.\end{aligned}$$

Use Cauchy-Schwarz's inequality in order to obtain

$$\begin{aligned}&\int_0^{\kappa_m} \|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\ &\leq \sqrt{T^*}\Big(\int_0^{T^*}\int_{\mathbb{R}^3} [1 - e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2 (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}[u\cdot\nabla u - b\cdot\nabla b](\xi)|^2 d\xi d\tau\Big)^{\frac{1}{2}}.\end{aligned}$$

Observe that $1 - e^{-\mu(T^*-\kappa_m)|\xi|^2} \leq 1$, for all $m \in \mathbb{N}$, and $\int_0^{T^*} \|u\cdot\nabla u - b\cdot\nabla b\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} d\tau < \infty$; provided that,

$$\|u\cdot\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_s \sum_{j=1}^{3} \|u_j\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \|D_j u\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_s C \|\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}, \tag{5.12}$$

Lemma 1.2.8 **ii)**, (5.8) and (5.9) hold. Then, by Dominated Convergence Theorem, we deduce

$$\lim_{n,m\to\infty} \int_0^{\kappa_m} \|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau = 0.$$

Following a similar argument, one reaches

$$\lim_{n,m\to\infty} \int_0^{\kappa_m} \|[e^{\nu\Delta(\kappa_m-\tau)} - e^{\nu\Delta(\kappa_n-\tau)}](u\cdot\nabla b - b\cdot\nabla u)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau = 0.$$

We have proved that $\lim_{n,m\to\infty} \|I_2(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (see (4.17)). At last, notice that

$$\begin{aligned}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq &\int_{\kappa_m}^{\kappa_n} \|e^{\mu\Delta(\kappa_n-\tau)}P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\ &+ \int_{\kappa_m}^{\kappa_n} \|e^{\mu\Delta(\kappa_n-\tau)}(u\cdot\nabla b - b\cdot\nabla u)\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.\end{aligned}$$

By using an analogous process to the described above; moreover, by applying (5.12), Cauchy-Schwarz's inequality and (5.9), one obtains

$$\begin{aligned}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} &\leq \int_{\kappa_m}^{\kappa_n} \|u\cdot\nabla u - b\cdot\nabla b\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau + \int_{\kappa_m}^{\kappa_n} \|u\cdot\nabla b - b\cdot\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\ &\leq CC_s\sqrt{T^*-\kappa_m}\left(\int_{\kappa_m}^{T^*}\|\nabla(u,b)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\right)^{\frac{1}{2}}\\ &\leq C_{s,\mu,\nu,u_0,b_0,T^*}\sqrt{T^*-\kappa_m}.\end{aligned}$$

where $s > 3/2$ and $\sigma \geq 1$. As a result, we infer that $\lim_{n,m\to\infty}\|I_3(n,m)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (see (4.18)). Thus, (5.11) implies (5.10). In addition, (5.10) means that $((u,b)(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $H^s_{a,\sigma}(\mathbb{R}^3)$. Hence, there exists $(u_1,b_1)\in H^s_{a,\sigma}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty}\|(u,b)(\kappa_n)-(u_1,b_1)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = 0.$$

From this point, just follow the same steps as in proof of Theorem 2.5

□

Theorem 5.2.2. *Assume that* $a > 0$, $\sigma > 1$ *and* $s_0 \in (\frac{1}{2},\frac{3}{2})$. *Let* $(u_0,b_0)\in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Consider that* $(u,b)\in C([0,T^*);H^s_{a,\sigma}(\mathbb{R}^3))$, *for all* $s\in(\frac{1}{2},s_0]$, *is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If* $T^* < \infty$, *then*

$$\limsup_{t\nearrow T^*}\|(u,b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} = \infty.$$

Proof. The proof is analogous to that of the Theorem 5.2.1, except for the use of Lemma 1.2.9 instead of Lemma 1.2.8 **ii)** and the fact that the constants $C_{s,\mu,\nu,u_0,b_0,T^*}$ and C_s given in (5.9) and (5.12), respectively, depends also on a and σ.

□

5.2.2 Blow–up of the Integral Related to $L^1(\mathbb{R}^3)$

It is important to emphasize that Theorem 5.2.4 is a generalization of Theorem 2.2.2 (since $(u_0,b_0)\in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $1/2 < s_0 < 3/2$) and Theorem 5.2.3 is an extension by considering [4] (provided that $(u_0,b_0)\in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $s_0 > 3/2$).

Theorem 5.2.3. *Assume that* $s_0 > 3/2$, $a > 0$, *and* $\sigma \geq 1$. *Let* $(u_0,b_0)\in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ *be such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Consider that* $(u,b)\in C([0,T^*);H^s_{a,\sigma}(\mathbb{R}^3))$, *for all* $s\in(\frac{3}{2},s_0]$, *is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If* $T^* < \infty$, *then*

$$\int_t^{T^*}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty.$$

Proof. Arguing as in proof of Theorem 4.2.2, we can write

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \theta\|\nabla(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\qquad \leq \|\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes u\|_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|\nabla u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|b\otimes b\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\qquad\quad + \|\nabla b\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes b\|_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|\nabla b\|_{H^s_{a,\sigma}(\mathbb{R}^3)}\|b\otimes u\|_{H^s_{a,\sigma}(\mathbb{R}^3)}. \qquad (5.13)
\end{aligned}$$

Now, our goal is to find an estimate for the term $\|u\otimes b\|_{H^s_{a,\sigma}(\mathbb{R}^3)}$ obtained above. Thus, by applying Lemma 1.2.8 **i)** ($s\geq 0$), one has

$$\begin{aligned}
\|u\otimes b\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} &= \sum_{j,k=1}^{3}\|b_j u_k\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\\
&\leq C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\sigma}}\widehat{b_j}\|_{L^1(\mathbb{R}^3)}\|u_k\|_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|_{L^1(\mathbb{R}^3)}\|b_j\|_{H^s_{a,\sigma}(\mathbb{R}^3)}]^2\\
&\leq C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|b\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}].
\end{aligned}$$

Replacing this last result in (5.13) and using Young's inequality, one gets

$$\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} + \frac{\theta}{2}\|\nabla(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq C_{s,\mu,\nu}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^2_{L^1(\mathbb{R}^3)}\|(u,b)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}. \qquad (5.14)$$

Assume $0\leq t\leq T<T^*$ in order to obtain, by Gronwall's inequality (differential form), the following inequality:

$$\|(u,b)(T)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)} \leq \|(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)}\exp\{C_{s,\mu,\nu}\int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau\}.$$

By applying Theorem 5.2.1, we infer

$$\int_t^{T^*}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty, \quad \forall\, t\in[0,T^*).$$

□

Theorem 5.2.4. *Assume that $a>0$, $\sigma>1$ and $s_0\in(\frac{1}{2},\frac{3}{2})$. Let $(u_0,b_0)\in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u,b)\in C([0,T^*);H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s\in(\frac{1}{2},s_0]$, is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If $T^*<\infty$, then*

$$\int_t^{T^*}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty.$$

Proof. It is enough to remake the proof of Theorem 5.2.3 replacing the use to Theorem 5.2.1 by Theorem 5.2.2.

□

5.2.3 Blow–up Inequality Involving $L^1(\mathbb{R}^3)$

As written before, it is important to make sure that Theorem 5.2.6 is a generalization of Theorem 2.2.3 (since $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $1/2 < s_0 < 3/2$) and Theorem 5.2.5 is an extension by considering [4] (provided that $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $s_0 > 3/2$). Moreover, the proofs of theorems below use the same arguments of the proof of Theorem 4.2.3.

Theorem 5.2.5. *Assume that $s_0 > 3/2$, $a > 0$, and $\sigma \geq 1$. Let $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{3}{2}, s_0]$, is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If $T^* < \infty$, then*

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \geq \frac{2\pi^3\sqrt{\theta}}{\sqrt{T^* - t}},$$

for all $t \in [0, T^*)$, where $\theta = \min\{\mu, \nu\}$.

Theorem 5.2.6. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If $T^* < \infty$, then*

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \geq \frac{2\pi^3\sqrt{\theta}}{\sqrt{T^* - t}},$$

for all $t \in [0, T^*)$, where $\theta = \min\{\mu, \nu\}$.

5.2.4 Blow–up Inequality involving $H^s_{a,\sigma}(\mathbb{R}^3)$

As mentioned before, it is important to inform that Theorem 5.2.8 is a generalization of this same blow-up criterion obtained in Theorem 2.2.4 (since $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $1/2 < s_0 < 3/2$ and the blow–up inequality is valid for $1/2 < s \leq s_0$) and Theorem 5.2.7 is an extension by considering [4] (provided that $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $s_0 > 3/2$ and the blow–up inequality is valid for $3/2 < s \leq s_0$).

Theorem 5.2.7. *Assume that $s_0 > 3/2$, $a > 0$, and $\sigma \geq 1$. Let $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{3}{2}, s_0]$, is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If $T^* < \infty$, then*

$$\|(u, b)(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} \geq \frac{2\pi^3 C_s\sqrt{\theta}}{\sqrt{T^* - t}},$$

for all $t \in [0, T^*)$; where $C_s = (\int_{\mathbb{R}^3}(1 + |\xi|^2)^{-s}\, d\xi)^{-\frac{1}{2}}$ and $\theta = \min\{\mu, \nu\}$.

Proof. This result follows directly from Theorem 5.2.5 and Cauchy-Schwarz's inequality. In fact,

$$\begin{aligned}
\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}} &\leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \\
&= \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u},\widehat{b})(t)|\, d\xi \\
&\leq \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}\, d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{s}e^{\frac{2a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u},\widehat{b})(t)|^2\, d\xi\right)^{\frac{1}{2}} \\
&=: C_s\|(u,b)(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)}, \quad \forall\, t\in[0,T^*),
\end{aligned} \tag{5.15}$$

recall that $s > 3/2$. □

From now on, $T^*_\omega < \infty$ denotes the first blow-up time for the solution $(u,b) \in C([0,T^*_\omega); H^s_{\omega,\sigma}(\mathbb{R}^3))$, where $\omega > 0$.

Theorem 5.2.8. *Assume that* $a > 0$, $\sigma > 1$ *and* $s_0 \in (\frac{1}{2}, \frac{3}{2})$. *Let* $(u_0,b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Consider that* $(u,b) \in C([0,T^*_a); H^s_{a,\sigma}(\mathbb{R}^3))$, *for all* $s \in (\frac{1}{2}, s_0]$, *is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If* $T^* < \infty$, *then*

$$\|(u,b)(t)\|_{H^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} \geq \frac{2\pi^3\sqrt{\theta}}{C_1\sqrt{T^*_a-t}}, \quad \forall\, t \in [0,T^*_a),$$

where $\theta = \min\{\mu,\nu\}$ *and* $C_1^2 = C^2_{a,\sigma,s,n} := 4\pi\sigma\left[\frac{2a}{(\sqrt{\sigma})^{(n-1)}}\left(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))$.

Proof. It is sufficient to argue as in proof of Theorem 3.2.4.

□

5.2.5 Generalization of the Blow–up Criteria

As informed previously, it is important to let the reader know that Theorem 5.2.10 is a generalization of this same blow-up criterion obtained in Theorem 2.2.5 (since $(u_0,b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $1/2 < s_0 < 3/2$ and the blow–up inequality is valid for $1/2 < s \leq s_0$) and Theorem 5.2.9 is an extension by considering [4] (provided that $(u_0,b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ with $s_0 > 3/2$ and the blow–up inequality is valid for $3/2 < s \leq s_0$).

Theorem 5.2.9. *Assume that* $s_0 > 3/2$, $a > 0$, *and* $\sigma \geq 1$. *Let* $(u_0,b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ *such that* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Consider that* $(u,b) \in C([0,T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, *for all* $s \in (\frac{3}{2}, s_0]$, *is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If* $T^* < \infty$, *then*

i) $\limsup_{t\nearrow T^*} \|(u,b)(t)\|_{H^s_{\frac{a}{\sigma^{n-1}},\sigma}(\mathbb{R}^3)} = \infty;$

ii) $\displaystyle\int_t^{T^*} \|e^{\frac{a}{\sigma^n}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty;$

iii) $\|e^{\frac{a}{\sigma^n}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \geq \dfrac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}};$

iv) $\|(u,b)(t)\|_{H^s_{\frac{a}{\sigma^n},\sigma}(\mathbb{R}^3)} \geq \dfrac{2\pi^3 C_s\sqrt{\theta}}{\sqrt{T^*-t}},$

for all $t\in[0,T^*), n\in\mathbb{N}$*; where* $C_s = (\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}\,d\xi)^{-\frac{1}{2}}$ *and* $\theta=\min\{\mu,\nu\}$*.*

Proof. Notice that the Theorems 5.2.1, 5.2.3, 5.2.5 and 5.2.7 guarantee the veracity of Theorem 5.2.9 in the case $n=1$. Moreover, (5.15) assures that $(u,b)\in C([0,T^*),H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3))$ (since $H^s_{a,\sigma}(\mathbb{R}^3)\hookrightarrow H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)$) and

$$\limsup_{t\nearrow T^*} \|(u,b)(t)\|_{H^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} = \infty. \tag{5.16}$$

The limit above guarantees the veracity of Theorem 5.2.9 **i)** in the case $n=2$. By following a similar process to the one described above and applying (5.16), instead of Theorem 5.2.1, one infers

$$\int_t^{T^*} \|e^{\frac{a}{\sigma^2}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty, \quad \forall\, t\in[0,T^*),$$

(see Theorem 5.2.9 **ii)** with $n=2$) and, consequently

$$4\pi^6\theta \leq \|e^{\frac{a}{\sigma^2}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}(T^*-t), \quad \forall\, t\in[0,T^*),$$

which is Theorem 5.2.9 **iii)** with $n=2$. This implies that

$$\limsup_{t\nearrow T^*} \|(u,b)(t)\|_{H^s_{\frac{a}{\sigma^2},\sigma}(\mathbb{R}^3)} = \infty, \tag{5.17}$$

since

$$\begin{aligned}\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}} &\leq \|e^{\frac{a}{\sigma^2}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}\\ &\leq \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{\frac{2a}{\sigma^2}|\xi|^{\frac{1}{\sigma}}}|(\widehat{u},\widehat{b})(t)|^2\,d\xi\right)^{\frac{1}{2}}\\ &=: C_s\|(u,b)(t)\|_{H^s_{\frac{a}{\sigma^2},\sigma}(\mathbb{R}^3)},\end{aligned} \tag{5.18}$$

for all $t\in[0,T^*)$ (recall that $s>3/2$). Notice that (5.17) is Theorem 5.2.9 **i)**, in the case $n=3$, and (5.18) assures the veracity of Theorem 5.2.9 **iv)**, with $n=2$. Thus, inductively, one proves Theorem 5.2.9 with $n>1$.

□

Theorem 5.2.10. *Assume that $a > 0$, $\sigma > 1$ and $s_0 \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*_a); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{1}{2}, s_0]$, is the maximal solution for the MHD equations (4.1) obtained in Theorem 5.1.1. If $T^*_a < \infty$, then*

i) $\limsup_{t \nearrow T^*_a} \|(u, b)(t)\|_{H^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)} = \infty$;

ii) $\int_t^{T^*_a} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty$;

iii) $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \geq \dfrac{2\pi^3\sqrt{\theta}}{\sqrt{T^*_a - t}}$;

iv) $\|(u, b)(t)\|_{H^s_{\frac{a}{(\sqrt{\sigma})^{n}},\sigma}(\mathbb{R}^3)} \geq \dfrac{2\pi^3\sqrt{\theta}}{C_1\sqrt{T^*_a - t}}$,

for all $t \in [0, T^*_a)$, $n \in \mathbb{N}$, where $C_1^2 = C^2_{a,\sigma,s,n} := 4\pi\sigma \left[\frac{2a}{(\sqrt{\sigma})^{(n-1)}}\left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)} \Gamma(\sigma(3 - 2s))$ and $\theta = \min\{\mu, \nu\}$.

Proof. First of all, notice that the Theorems 5.2.2, 5.2.4, 5.2.6 and 5.2.8 prove the Theorem 5.2.10 in case $n = 1$. The remainder of prove follows a induction process similar to the one in proof of Theorem 3.2.5.

□

5.2.6 Main Blow-up criterion Involving $H^s_{a,\sigma}(\mathbb{R}^3)$

As it was written previously, it is important to inform that Theorems 5.2.11 and 5.2.12 are generalizations of the same blow-up criteria obtained in [4] and Theorem 2.2.7, respectively.

Theorem 5.2.11. *Assume that $s_0 > 3/2$, $a > 0$, and $\sigma \geq 1$. Let $(u_0, b_0) \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u, b) \in C([0, T^*); H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s \in (\frac{3}{2}, s_0]$, is the maximal solution for the MHD equations (4.1) obtained in Theorem 5.1.1. If $T^* < \infty$, then*

$$\|(u, b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)} \geq C_1 \|(u, b)(t)\|^{1-\frac{2s}{3}}_{L^2(\mathbb{R}^3)} \exp\{aC_2 \|(u, b)(t)\|^{-\frac{2}{3\sigma}}_{L^2(\mathbb{R})} (T^* - t)^{-\frac{1}{3\sigma}}\} (T^* - t)^{-\frac{s}{3}},$$

for all $t \in [0, T^*)$; where C_1 and C_2 depend only on μ, ν, s and μ, ν, s, σ, respectively.

Proof. In fact, by choosing $\delta_0 = s$ $(> 3/2)$ and $\delta = s + \frac{m}{2\sigma}$ $(\geq \delta_0)$ in Lemma 1.2.12, Remark 1.2.15 and Dominated Convergence Theorem in Theorem 5.2.9 **iii)**, where $m \in \mathbb{N} \cup \{0\}$, we obtain

$$\frac{4\pi^6\theta}{T^* - t} \leq \|(\widehat{u}, \widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)} \leq C_s \|(u, b)(t)\|^{2-\frac{3}{s+\frac{m}{2\sigma}}}_{L^2(\mathbb{R}^3)} \|(u, b)(t)\|^{\frac{3}{s+\frac{m}{2\sigma}}}_{\dot{H}^{s+\frac{m}{2\sigma}}(\mathbb{R}^3)}.$$

Hence, using (4.33) below, one has

$$\frac{C_{s,\theta}\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{2-\frac{4s}{3}}}{(T^*-t)^{\frac{2s}{3}}}\left(\frac{D_{s,\sigma,\theta}\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{-\frac{2}{3\sigma}}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^m \leq \|(u,b)(t)\|^2_{\dot{H}^{s+\frac{m}{2\sigma}}(\mathbb{R}^3)}, \tag{5.19}$$

where $C_{s,\theta} = (C_s 4\pi^6\theta)^{\frac{2s}{3}}$ and $D_{s,\sigma,\theta} = (C_s 4\pi^6\theta)^{\frac{1}{3\sigma}}$. Multiplying (5.19) by $\frac{(2a)^m}{m!}$, one gets

$$\frac{C_{s,\theta}\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{2-\frac{4s}{3}}}{(T^*-t)^{\frac{2s}{3}}}\frac{\left(\frac{2aD_{s,\sigma,\theta}\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{-\frac{2}{3\sigma}}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^m}{m!} \leq \int_{\mathbb{R}^3}\frac{(2a|\xi|^{\frac{1}{\sigma}})^m}{m!}|\xi|^{2s}|(\widehat{u},\widehat{b})(t)|^2\,d\xi.$$

By summing over $m\in\mathbb{N}$ and applying the Monotone Convergence Theorem, it results

$$\begin{aligned}\frac{C_{s,\theta}\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{2-\frac{4s}{3}}}{(T^*-t)^{\frac{2s}{3}}}\exp\left\{\frac{2aD_{s,\sigma,\theta}\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{-\frac{2}{3\sigma}}}{(T^*-t)^{\frac{1}{3\sigma}}}\right\} &\leq \int_{\mathbb{R}^3} e^{2a|\xi|^{\frac{1}{\sigma}}}|\xi|^{2s}|(\widehat{u},\widehat{b})(t)|^2\,d\xi \\ &\leq \|(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)},\end{aligned}$$

for all $t\in[0,T^*)$.

$\square$

Theorem 5.2.12. *Assume that $a>0$, $\sigma>1$ and $s_0\in(\frac{1}{2},\frac{3}{2})$. Let $(u_0,b_0)\in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Consider that $(u,b)\in C([0,T^*);H^s_{a,\sigma}(\mathbb{R}^3))$, for all $s\in(\frac{1}{2},s_0]$, is the maximal solution for the MHD equations* (4.1) *obtained in Theorem* 5.1.1. *If $T^*<\infty$, then*

$$\frac{a^{\sigma_0+\frac{1}{2}}C_2\exp\{aC_3(T^*-t)^{-\frac{1}{3\sigma}}\}}{(T^*-t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}} \leq \|(u,b)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^3)},$$

for all $t\in[0,T^*)$, where $C_2 = C_{\mu,\nu,s,\sigma,u_0,b_0}$, $C_3 = C_{\mu,\nu,\sigma,s,u_0,b_0}$ and $2\sigma_0$ is the integer part of 2σ.

Proof. Choose $\delta = s+\frac{k}{2\sigma}$, with $k\in\mathbb{N}\cup\{0\}$ and $k\geq 2\sigma$, and $\delta_0 = s+1$. By using Lemmas 1.2.12 and 1.2.14, and Dominated Convergence Theorem in Theorem 5.2.10 **iii)**, we obtain

$$\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}} \leq \|(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \leq C_s\|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2(s+\frac{k}{2\sigma})}}\|(u,b)(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}^{\frac{3}{2(s+\frac{k}{2\sigma})}}.$$

Hence, using (4.33), one has

$$\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}}\left(\frac{D_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k \leq \|(u,b)(t)\|^2_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}, \tag{5.20}$$

where $C_{\theta,s,u_0,b_0} = (C_s^{-1}2\pi^3\sqrt{\theta})^{\frac{4s}{3}}\|(u_0,b_0)\|_{L^2(\mathbb{R}^3)}^{\frac{6-4s}{3}}$ and $D_{\sigma,s,\theta,u_0,b_0} = (C_s^{-1}2\pi^3\sqrt{\theta}\|(u_0,b_0)\|_{L^2(\mathbb{R}^3)}^{-1})^{\frac{2}{3\sigma}}$. Multiplying (5.20) by $\frac{(2a)^k}{k!}$, one gets

$$\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}}\frac{\left(\frac{2aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k}{k!} \leq \int_{\mathbb{R}^3}\frac{(2a)^k}{k!}|\xi|^{2(s+\frac{k}{2\sigma})}|(\widehat{u},\widehat{b})(t)|^2\,d\xi = \int_{\mathbb{R}^3}\frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!}|\xi|^{2s}|(\widehat{u},\widehat{b})(t)|^2\,d\xi.$$

By summing over the set $\{k\in\mathbb{N}; k\geq 2\sigma\}$ and applying the Monotone Convergence Theorem, it results

$$\begin{aligned}
&\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}}\left[\exp\left\{\frac{2aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right\} - \sum_{0\leq k<2\sigma}\frac{\left(\frac{2aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k}{k!}\right]\\
&\leq \int_{\mathbb{R}^3}\left[e^{2a|\xi|^{\frac{1}{\sigma}}} - \sum_{0\leq k<2\sigma}\frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!}\right]|\xi|^{2s}|(\widehat{u},\widehat{b})(t)|^2\,d\xi\\
&\leq \int_{\mathbb{R}^3}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|(\widehat{u},\widehat{b})(t)|^2\,d\xi = \|(u,b)(t)\|^2_{H^s_{a,\sigma}(\mathbb{R}^3)},
\end{aligned}$$

for all $t\in[0,T^*)$. Finally, this proof follows the same steps as in proof of Theorem 2.2.7.

□

Chapter 6

2D Micropolar equations: local existence, uniqueness and asymptotic behavior of solutions in Sobolev-Gevrey spaces

This chapter studies local existence of a unique solution, as well decay rates (if it is assumed that this solution is global in time), for the following 2D Micropolar equations in Sobolev-Gevrey spaces:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p \;=\; (\mu+\chi)\Delta u + \chi\nabla\times w, \quad x\in\mathbb{R}^2, t\geq 0,\\ w_t + u\cdot\nabla w \;=\; \gamma\,\Delta w + \chi\nabla\times u - 2\chi w, \quad x\in\mathbb{R}^2, t\geq 0,\\ \operatorname{div} u \;=\; 0, \quad x\in\mathbb{R}^2, t>0,\\ u(\cdot,0) \;=\; u_0(\cdot),\;\; w(\cdot,0) \;=\; w_0(\cdot), \quad x\in\mathbb{R}^2, \end{cases} \tag{6.1}$$

where $u(x,t)=(u_1(x,t),u_2(x,t))\in\mathbb{R}^2$ denotes the incompressible velocity field, $w(x,t)\in\mathbb{R}$ the microrotational velocity field and $p(x,t)\in\mathbb{R}$ the hydrostatic pressure. The positive constants μ,χ,γ and ν are associated with specific properties of the fluid; more precisely, $\mu>0,\gamma>0,\chi\geq 0$ are the kinematic, spin and vortex viscosities, respectively. The initial data for the velocity field, given by u_0 in (6.1), is assumed to be divergence free, i.e., $\operatorname{div} u_0=0$. Here $\nabla\times u=D_1u_2-D_2u_1$ and $\nabla\times w=(D_2w,-D_1w)$.

The local existence, uniqueness and blow-up of solutions for the micropolar system (6.1) and for its periodic version have been extensively studied in the literature, see for instance [9, 11, 15, 16, 19, 23, 30, 33, 34, 37, 38, 40] and references therein.

6.1 Local Existence and Uniqueness of Solutions

By considering $w = \chi = 0$ and $\mathbb{R}^3$ instead of $\mathbb{R}^2$ in (6.1), in the Chapter 2 it was proved that there are a positive instant $t = T$ and a unique solution $u \in C([0,T], H^s_{a,\sigma}(\mathbb{R}^3))$ (with $u_0 \in H^{s_0}_{a,\sigma}(\mathbb{R}^3)$, $s_0 > 1/2$, $s_0 \neq 3/2$, $a > 0$, $\sigma \geq 1$ and $s \leq s_0$) for the Navier-Stokes system (we also cite [4, 6, 7, 10, 28, 29, 32] and references therein). Motivated by the previous chapters, we are interested in showing which are the assumptions that are necessary in order to guarantee the local existence and uniqueness of solutions for the equations given in (6.1) in nonhomogenous Sobolev-Gevrey spaces. More precisely, we present our first result.

Theorem 6.1.1. *Let $a > 0$, $\sigma \geq 1$ and $s > 0$ with $s \neq 1$. Let $(u_0, w_0) \in H^s_{a,\sigma}(\mathbb{R}^2)$ such that $\operatorname{div} u_0 = 0$. If $s > 1$ (respectively $s \in (0,1)$), then there exist a time $T = T_{s,\mu,\chi,\gamma,u_0,w_0}$ (respectively $T = T_{s,a,\mu,\chi,\gamma,u_0,w_0}$) and a unique solution $(u,w) \in C([0,T]; H^{\varpi}_{a,\sigma}(\mathbb{R}^2))$, for all $\varpi \leq s$, of the micropolar equations given in (6.1).*

Proof. By applying the heat semigroup $e^{(\mu+\chi)\Delta(t-\tau)}$, with $\tau \in [0,t]$, in the first equation of (6.1), and, subsequently, integrating the result over the interval $[0,t]$, we obtain

$$\int_0^t e^{(\mu+\chi)\Delta(t-\tau)} u_\tau \, d\tau + \int_0^t e^{(\mu+\chi)\Delta(t-\tau)} [u \cdot \nabla u + \nabla p - \chi(\nabla \times w)] \, d\tau = (\mu+\chi) \int_0^t e^{(\mu+\chi)\Delta(t-\tau)} \Delta u \, d\tau.$$

Now, use integration by parts in order to deduce

$$u(t) = e^{(\mu+\chi)\Delta t} u_0 - \int_0^t e^{(\mu+\chi)\Delta(t-\tau)} [u \cdot \nabla u + \nabla p] \, d\tau + \chi \int_0^t e^{(\mu+\chi)\Delta(t-\tau)} (\nabla \times w) \, d\tau.$$

Consequently, by (2.9), one can write

$$u(t) = e^{(\mu+\chi)\Delta t} u_0 - \int_0^t e^{(\mu+\chi)\Delta(t-\tau)} P_H[\sum_{j=1}^2 D_j(u_j u)] \, d\tau + \chi \int_0^t e^{(\mu+\chi)\Delta(t-\tau)} (\nabla \times w) \, d\tau, \quad (6.2)$$

since u is divergence free.

Analogously, by considering the field w, we deduce the equality below.

$$w(t) = e^{\gamma\Delta t} w_0 - \int_0^t e^{\gamma\Delta(t-\tau)} \sum_{j=1}^2 D_j(u_j w) \, d\tau + \chi \int_0^t e^{\gamma\Delta(t-\tau)} (\nabla \times u) \, d\tau - 2\chi \int_0^t e^{\gamma\Delta(t-\tau)} w \, d\tau. \quad (6.3)$$

By (6.2) and (6.3), one obtains

$$(u,w)(t) = (e^{(\mu+\chi)\Delta t} u_0, e^{\gamma\Delta t} w_0) + B((u,w),(u,w))(t) + L(u,w)(t), \quad (6.4)$$

where

$$B((z,v),(\varphi,\phi))(t) = \int_0^t (-e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)], -e^{\gamma\Delta(t-\tau)}[\sum_{j=1}^{2} D_j(z_j\phi)])\, d\tau, \quad (6.5)$$

and

$$L(z,v)(t) = \int_0^t (\chi e^{(\mu+\chi)\Delta(t-\tau)}(\nabla\times v), \chi e^{\gamma\Delta(t-\tau)}(\nabla\times z) - 2\chi e^{\gamma\Delta(t-\tau)}v)\, d\tau. \quad (6.6)$$

Notice that L is a linear operator and B is a bilinear operator. Furthermore, φ, $z:\mathbb{R}^2\to\mathbb{R}^2$ and ϕ, $v:\mathbb{R}^2\to\mathbb{R}$ belong to appropriate spaces that will be give next. In order to examine $(u,w)(t)$ in $H^s_{a,\sigma}(\mathbb{R}^2)$, let us estimate $B((w,v),(\gamma,\phi))(t)$ and $L(z,v)(t)$ in this same space.

At first, let us estimate $L(z,v)(t)$ in $H^s_{a,\sigma}(\mathbb{R}^2)$. Thus, we deduce

$$\begin{aligned}
\|e^{(\mu+\chi)\Delta(t-\tau)}(\nabla\times v)\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)} &= \int_{\mathbb{R}^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}\{e^{(\mu+\chi)\Delta(t-\tau)}(\nabla\times v)\}(\xi)|^2\, d\xi \\
&= \int_{\mathbb{R}^2} e^{-2(\mu+\chi)(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{\nabla\times v}(\xi)|^2\, d\xi \\
&\leq \int_{\mathbb{R}^2}|\xi|^2 e^{-2(\mu+\chi)(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{v}(\xi)|^2\, d\xi,
\end{aligned}$$

since $|\widehat{\nabla\times v}| = |\xi||\widehat{v}|$. As a result, by using Lemma 1.2.19, it follows that

$$\|e^{(\mu+\chi)\Delta(t-\tau)}(\nabla\times v)\|_{H^s_{a,\sigma}(\mathbb{R}^2)} \leq C_{\mu,\chi}(t-\tau)^{-\frac{1}{2}}\|(z,v)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}.$$

By integrating over $[0,t]$ ($t\in[0,T]$) the above estimate, we conclude

$$\int_0^t \|\chi e^{(\mu+\chi)\Delta(t-\tau)}(\nabla\times v)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\, d\tau \leq C_{\mu,\chi}T^{\frac{1}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}. \quad (6.7)$$

Similarly, one gets

$$\int_0^t \|\chi e^{\gamma\Delta(t-\tau)}(\nabla\times z)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\, d\tau \leq C_{\chi,\gamma}T^{\frac{1}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}. \quad (6.8)$$

since $|\widehat{\nabla\times z}| \leq |\xi||\widehat{z}|$. On the other hand, it is valid that

$$\|e^{\gamma\Delta(t-\tau)}v\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)} \leq \int_{\mathbb{R}^2} e^{-2\gamma(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{v}(\xi)|^2\, d\xi \leq \|(z,v)\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)}.$$

By integrating over $[0,t]$, the above estimative, we conclude

$$\int_0^t \|2\chi e^{\gamma\Delta(t-\tau)}v\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\, d\tau \leq C_\chi T^{\frac{1}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}, \quad \forall\, t\in[0,T], \quad (6.9)$$

if it is assumed that $T<1$. By (6.6), we can assure that (6.7), (6.8) and (6.9) imply

$$\|L(z,v)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)} \leq C_{\mu,\chi,\gamma} T^{\frac{1}{2}} \|(z,v)\|_{L^\infty([0,T],H^s_{a,\sigma}(\mathbb{R}^2))}, \quad \forall\, t\in[0,T]. \tag{6.10}$$

Now, let us estimate $B((z,v),(\varphi,\phi))(t)$ in $H^s_{a,\sigma}(\mathbb{R}^2)$. To this end, we shall divide the proof into two cases:

1º Case: Assume that $s>1$.

Notice that

$$\begin{aligned}
&\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^2 D_j(\varphi_j z)]\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)}\\
&= \int_{\mathbb{R}^2} (1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^2 D_j(\varphi_j z)]\}(\xi)|^2\, d\xi\\
&= \int_{\mathbb{R}^2} e^{-2(\mu+\chi)(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{P_H[\sum_{j=1}^2 D_j(\varphi_j z)]\}(\xi)|^2\, d\xi.
\end{aligned}$$

By applying (2.11), one can write

$$\begin{aligned}
&\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^2 D_j(\varphi_j z)]\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)}\\
&\leq \int_{\mathbb{R}^2} e^{-2(\mu+\chi)(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\sum_{j=1}^2 \mathcal{F}[D_j(\varphi_j z)](\xi)|^2\, d\xi\\
&= \int_{\mathbb{R}^2} e^{-2(\mu+\chi)(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(z\otimes\varphi)(\xi)\cdot\xi|^2\, d\xi\\
&\leq \int_{\mathbb{R}^2} |\xi|^2 e^{-2(\mu+\chi)(t-\tau)|\xi|^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(z\otimes\varphi)(\xi)|^2\, d\xi.
\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^2 D_j(\varphi_j z)]\|_{H^s_{a,\sigma}(\mathbb{R}^2)} \leq C_{\mu,\chi}(t-\tau)^{-\frac{1}{2}}\|z\otimes\varphi\|_{H^s_{a,\sigma}(\mathbb{R}^2)}.$$

Similarly, we can write

$$\|e^{\gamma\Delta(t-\tau)}[\sum_{j=1}^2 D_j(z_j\phi)]\|_{H^s_{a,\sigma}(\mathbb{R}^2)} \leq C_{\gamma}(t-\tau)^{-\frac{1}{2}}\|\phi\otimes z\|_{H^s_{a,\sigma}(\mathbb{R}^2)}.$$

Consequently, one gets

$$\begin{aligned}&\|B((z,v),(\varphi,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\\&\leq C_{\mu,\chi,\gamma}\Big[\int_0^t (t-\tau)^{-\frac{1}{2}}\|z\otimes\varphi\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\,d\tau+\int_0^t (t-\tau)^{-\frac{1}{2}}\|\phi\otimes z\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\,d\tau\Big].\end{aligned}\tag{6.11}$$

On the other hand, it results

$$\begin{aligned}\|z\otimes\varphi\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)}&=\int_{\mathbb{R}^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{z\otimes\varphi}(\xi)|^2\,d\xi\\&=(2\pi)^{-4}\sum_{j,k=1}^{2}\int_{\mathbb{R}^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{\varphi_j}*\widehat{z_k}(\xi)|^2\,d\xi\\&\leq C\int_{\mathbb{R}^2}(1+|\xi|^2)^s e^{2a|\xi|^{\frac{1}{\sigma}}}[(|\widehat{\varphi}|*|\widehat{z}|)(\xi)]^2\,d\xi.\end{aligned}$$

Therefore, one deduces

$$\|z\otimes\varphi\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)}\leq C\int_{\mathbb{R}^2}(1+|\xi|^2)^s\left[\int_{\mathbb{R}^2}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{z}(\eta)||\widehat{\varphi}(\xi-\eta)|\,d\eta\right]^2 d\xi.$$

By using (1.3), we conclude

$$\begin{aligned}\|z\otimes\varphi\|^2_{H^s_{a,\sigma}(\mathbb{R}^2)}&\leq C\int_{\mathbb{R}^2}(1+|\xi|^2)^s\left[\int_{\mathbb{R}^2}e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{z}(\eta)|e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{\varphi}(\xi-\eta)|\,d\eta\right]^2 d\xi\\&=C\int_{\mathbb{R}^2}(1+|\xi|^2)^s[(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{z}(\xi)|)*(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{\varphi}(\xi)|)]^2\,d\xi\\&=C\int_{\mathbb{R}^2}(1+|\xi|^2)^s\{\mathcal{F}[\mathcal{F}^{-1}(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{z}(\xi)|)\mathcal{F}^{-1}(e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{\varphi}(\xi)|)]\}^2\,d\xi\\&=C\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{z}|)\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{\varphi}|)\|^2_{H^s(\mathbb{R}^2)}.\end{aligned}$$

Hence, by the inequality (1.11) with $a=0$ and $n=2$, one has

$$\|z\otimes\varphi\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\leq C_s\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{z}|)\|_{H^s(\mathbb{R}^2)}\|\mathcal{F}^{-1}(e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{\varphi}|)\|_{H^s(\mathbb{R}^2)}=C_s\|z\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\|\varphi\|_{H^s_{a,\sigma}(\mathbb{R}^2)},$$

since $s>1$. Replacing this result in (6.11), one obtains

$$\begin{aligned}\|B((z,v),(\varphi,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}&\leq C_{s,\mu,\chi,\gamma}\Big[\int_0^t (t-\tau)^{-\frac{1}{2}}\|z\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\|\varphi\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\,d\tau\\&+\int_0^t (t-\tau)^{-\frac{1}{2}}\|\phi\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\|z\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\,d\tau\Big].\end{aligned}$$

Therefore, we deduce

$$\|B((z,v),(\varphi,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\leq C_{s,\mu,\chi,\gamma}T^{\frac{1}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))},\tag{6.12}$$

for all $t \in [0,T]$. By noticing that $L : X \times Y \to X \times Y$ is a continuous linear operator (see (6.6) and (6.10)) and $B : (X \times Y)^2 \to X \times Y$ is a continuous bilinear operator (see (6.5) and (6.12)), where $Y = C([0,T]; H^s_{a,\sigma}(\mathbb{R}^2))$ and $X = Y^2$ (with $s > 1$), it is enough to apply Lemma 1.2.2 and consider T small enough in order to obtain a unique solution $(u,w) \in C([0,T]; H^s_{a,\sigma}(\mathbb{R}^2))$ for the equation (6.4). More specifically, choose

$$T < \min\left\{[(4C_{s,\mu,\chi,\gamma}\|(u_0,w_0)\|_{H^s_{a,\sigma}(\mathbb{R}^2)})^{\frac{1}{2}} + C_{\mu,\chi,\gamma}]^{-4}, C^{-2}_{\mu,\chi,\gamma}, 1\right\},$$

where $C_{\mu,\chi,\gamma}$ and $C_{s,\mu,\chi,\gamma}$ are given in (6.10) and (6.12), respectively; and

$$\|(e^{(\mu+\chi)\Delta t}u_0, e^{\gamma\Delta t}w_0)\|_{H^s_{a,\sigma}(\mathbb{R}^2)} \leq \|(u_0,w_0)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}.$$

2º Case: Consider that $s \in (0,1)$.

At first, let us estimate $B((z,v),(\varphi,\phi))(t)$ in $\dot{H}^s_{a,\sigma}(\mathbb{R}^2)$. By applying (2.11) and the Cauchy-Schwarz's inequality, one can write

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} \leq \int_{\mathbb{R}^2} |\xi|^{4-2s}e^{-2(\mu+\chi)(t-\tau)|\xi|^2}|\xi|^{4s-2}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(z\otimes\varphi)(\xi)|^2\,d\xi.$$

As a result, by using Lemma 1.2.19, it follows

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} \leq \frac{C_{s,\mu,\chi}}{(t-\tau)^{2-s}}\|z\otimes\varphi\|^2_{\dot{H}^{2s-1}_{a,\sigma}(\mathbb{R}^2)},$$

since $0 < s < 1$. On the other hand, by using Lemma 1.2.7, one infers

$$\|z\otimes\varphi\|^2_{\dot{H}^{2s-1}_{a,\sigma}(\mathbb{R}^2)} = \sum_{j,k=1}^{2}\|\varphi_j z_k\|^2_{\dot{H}^{2s-1}_{a,\sigma}(\mathbb{R}^2)} \leq C_s\|z\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}\|\varphi\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)},$$

provided that $0 < s < 1$. Therefore, one deduces

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} \leq \frac{C_{s,\mu,\chi}}{(t-\tau)^{\frac{2-s}{2}}}\|(z,v)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}\|(\varphi,\phi)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}.$$

By integrating over $[0,t]$ ($t \in [0,T]$) the above estimate, we conclude

$$\begin{aligned}&\int_0^t \|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}\,d\tau \\ &\leq C_{s,\mu,\chi}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^2))}.\end{aligned} \tag{6.13}$$

Similarly, we can write

$$\int_0^t \|e^{\gamma\Delta(t-\tau)}[\sum_{j=1}^{2} D_j(z_j\phi)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}\, d\tau \le C_{s,\gamma}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^2))}. \tag{6.14}$$

By (6.5), we can assure that (6.13) and (6.14) imply

$$\|B((z,v),(\varphi,\phi))(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} \le C_{s,\mu,\chi,\gamma}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^2))}. \tag{6.15}$$

Let us estimate $B((z,v),(\varphi,\phi))(t)$ in $L^2(\mathbb{R}^2)$. Following a similar process to the one presented above, we have, by using the Parseval's identity, that

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|^2_{L^2(\mathbb{R}^2)} = (2\pi)^{-2}\int_{\mathbb{R}^2} |\mathcal{F}\{e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\}(\xi)|^2\, d\xi.$$

As a result, by using (2.11), we get

$$\begin{aligned}\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|^2_{L^2(\mathbb{R}^2)} &\le (2\pi)^{-2}\int_{\mathbb{R}^2} |\xi|^2 e^{-2(\mu+\chi)(t-\tau)|\xi|^2}|\mathcal{F}(z\otimes\varphi)(\xi)|^2\, d\xi\\ &\le (2\pi)^{-2}\int_{\mathbb{R}^2} |\xi|^{4-2s} e^{-2(\mu+\chi)(t-\tau)|\xi|^2}|\xi|^{2s-2}|\mathcal{F}(z\otimes\varphi)(\xi)|^2\, d\xi.\end{aligned}$$

As a result, by using Lemma 1.2.19, it follows

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|_{L^2(\mathbb{R}^2)} \le \frac{C_{s,\mu,\chi}}{(t-\tau)^{\frac{2-s}{2}}}\|z\otimes\varphi\|_{\dot{H}^{s-1}(\mathbb{R}^2)},$$

On the other hand, by utilizing Lemma 1.2.6 (provided that $0 < s < 1$), one has

$$\|z\otimes\varphi\|^2_{\dot{H}^{s-1}(\mathbb{R}^2)} = \sum_{j,k=1}^{2}\|\varphi_j z_k\|^2_{\dot{H}^{s-1}(\mathbb{R}^2)} \le C_s\|z\|^2_{\dot{H}^s(\mathbb{R}^2)}\|\varphi\|^2_{L^2(\mathbb{R}^2)}.$$

Thereby, as $H^s_{a,\sigma}(\mathbb{R}^2) \hookrightarrow \dot{H}^s(\mathbb{R}^2)$ $(s \ge 0)$ and by applying Lemma 1.2.10, we deduce

$$\|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|_{L^2(\mathbb{R}^2)} \le \frac{C_{s,\mu,\chi}}{(t-\tau)^{\frac{2-s}{2}}}\|(z,v)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}\|(\varphi,\phi)\|_{H^s_{a,\sigma}(\mathbb{R}^2)}.$$

By integrating the above estimate over $[0,t]$ $(t \in [0,T])$, we conclude

$$\begin{aligned}&\int_0^t \|e^{(\mu+\chi)\Delta(t-\tau)}P_H[\sum_{j=1}^{2} D_j(\varphi_j z)]\|_{L^2(\mathbb{R}^2)}\, d\tau\\ &\le C_{s,\mu,\chi}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}.\end{aligned} \tag{6.16}$$

Similarly, we can obtain

$$\int_0^t \|e^{\gamma\Delta(t-\tau)}[\sum_{j=1}^{2} D_j(z_j\phi)]\|_{L^2(\mathbb{R}^2)}\,d\tau \le C_{s,\gamma}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}. \tag{6.17}$$

By using the definition (6.5) and applying (6.16) and (6.17), one concludes

$$\|B((z,v),(\varphi,\phi))(t)\|_{L^2(\mathbb{R}^2)} \le C_{s,\mu,\chi,\gamma}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}, \tag{6.18}$$

for all $t \in [0,T]$. Finally, by using Lemma 1.2.10, (6.15), (6.18) and the fact that $H^s_{a,\sigma}(\mathbb{R}^2) \hookrightarrow \dot{H}^s_{a,\sigma}(\mathbb{R}^2)$ $(s \ge 0)$, it results

$$\|B((z,v),(\varphi,\phi))(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)} \le C_{s,a,\mu,\chi,\gamma}T^{\frac{s}{2}}\|(z,v)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}\|(\varphi,\phi)\|_{L^\infty([0,T];H^s_{a,\sigma}(\mathbb{R}^2))}, \tag{6.19}$$

for all $t \in [0,T]$. Choose

$$T < \min\left\{[(4C_{s,a,\mu,\chi,\gamma}\|(u_0,w_0)\|_{H^s_{a,\sigma}(\mathbb{R}^2)})^{\frac{1}{2}} + C_{\mu,\chi,\gamma}]^{-\frac{4}{5}}, C^{-2}_{\mu,\chi,\gamma}, 1\right\},$$

where $C_{\mu,\chi,\gamma}$ and $C_{s,a,\mu,\chi,\gamma}$ is given in (6.10) and (6.19), respectively, and apply Lemma 1.2.2 to obtain the desired result.

Lastly, by assuming that $\varpi \le s$, it follows that $(u,w) \in [C([0,T];H^{\varpi}_{a,\sigma}(\mathbb{R}^2))]^3$ since $H^s_{a,\sigma}(\mathbb{R}^2) \hookrightarrow H^{\varpi}_{a,\sigma}(\mathbb{R}^2)$.

□

6.2 Asymptotic Behavior for the Solution

In this section, we establish the asymptotic behavior of the solution (by assuming its global existence in time) obtained in Theorem 6.1.1 by extending and improving the steps presented by J. Benameur and L. Jlali [6]. More specifically, we suppose that the solution (u,w) obtained above is global in order to present decay rates related to the spaces $H^s_{a,\sigma}(\mathbb{R}^2)$ and $\dot{H}^s_{a,\sigma}(\mathbb{R}^2)$ (where $\sigma > 1$, $a > 0$ and $s > 0$ with $s \neq 1$).

Let us inform that theses rates will be accomplished by applying the following result established by R. H. Guterres, W. G. Melo, J. R. Nunes and C. F. Perusato [23].

Theorem 6.2.1 (See [23]). *Let* $(u_0,w_0) \in L^2(\mathbb{R}^2)$ *such that* $\operatorname{div} u_0 = 0$. *For a Leray global solution* (u,w) *of the Micropolar equations* (6.1), *one has*

i) $\lim_{t\to\infty} \|(u,w)(t)\|_{L^2(\mathbb{R}^2)} = 0$;

ii) $\lim_{t\to\infty} t^{\frac{s}{2}} \|(u,w)(t)\|_{\dot{H}^s(\mathbb{R}^2)} = 0$, *for all* $s \geq 0$.

Moreover, if $\chi > 0$, *one obtains*

iii) $\lim_{t\to\infty} t^{\frac{s+1}{2}} \|w(t)\|_{\dot{H}^s(\mathbb{R}^2)} = 0$ *for all* $s \geq 0$.

Remark 6.2.2. Under the same assumptions in Theorems 6.1.1 and 6.2.1, it is important to point out the following observations:

1. It is easy to check that Theorem 6.2.1 **ii)** implies the following limit:

$$\lim_{t\to\infty} \|(u,w)(t)\|_{\dot{H}^s(\mathbb{R}^2)} = 0, \tag{6.20}$$

   since

$$\lim_{t\to\infty} \|(u,w)(t)\|_{\dot{H}^s(\mathbb{R}^2)} = \lim_{t\to\infty} t^{-\frac{s}{2}}[t^{\frac{s}{2}}\|(u,w)(t)\|_{\dot{H}^s(\mathbb{R}^2)}] = 0, \quad \forall s \geq 0; \tag{6.21}$$

2. Notice also that the limit

$$\lim_{t\to\infty} \|(u,w)(t)\|_{H^s(\mathbb{R}^2)} = 0, \quad \forall s \geq 0, \tag{6.22}$$

   is a direct consequence of Theorem 6.2.1 **i)**, (6.20), and the elementary inequality

$$\|f\|^2_{H^s(\mathbb{R}^2)} \leq 2^s[(2\pi)^2\|f\|^2_{L^2(\mathbb{R}^2)} + \|f\|^2_{\dot{H}^s(\mathbb{R}^2)}], \quad \forall s \geq 0.$$

6.2.1 Estimates Involving $\dot{H}^s(\mathbb{R}^2)$

In this section, we give some lemmas that will play a key role in the proof of the decay rates given in Theorems 6.2.6, 6.2.7 and 6.2.8.

Lemma 6.2.3. *Consider that* $(u,w) \in C([0,\infty); H^s_{a,\sigma}(\mathbb{R}^2))$ *is a global solution for the Micropolar equations* (6.1). *Then, there is an instant* $t = T$ *that depends only on* s, μ, γ *and* $\|(u_0,w_0)\|_{H^s(\mathbb{R}^2)}$ *such that*

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)} \leq [1 + 2\|(u_0,w_0)\|^2_{\dot{H}^s(\mathbb{R}^2)}]^{\frac{1}{2}}, \quad \forall\, t \in [0,T].$$

Proof. By applying the Fourier Transform and taking the scalar product in $\mathbb{C}^2$ of the first equation of (6.1) with $\widehat{u}(t)$, one has

$$\frac{1}{2}\partial_t|\widehat{u}(t)|^2 + (\mu+\chi)|\widehat{\nabla u}|^2 = -\mathrm{Re}\,[\widehat{u}\cdot\widehat{u\cdot\nabla u}] + \chi\mathrm{Re}\,[\widehat{u}\cdot\widehat{\nabla\times w}]. \tag{6.23}$$

Similarly, considering the second equation of (6.1), one obtains

$$\frac{1}{2}\partial_t|\widehat{w}(t)|^2+\gamma|\widehat{\nabla w}|^2+2\chi|\widehat{w}|^2=-\mathrm{Re}\,[\widehat{w}\cdot\widehat{u\cdot\nabla w}]+\chi\mathrm{Re}\,[\widehat{w}\cdot\widehat{\nabla\times u}]. \tag{6.24}$$

By using (6.23) and (6.24) and the fact that $\widehat{u}\cdot\widehat{\nabla\times w}=\widehat{\nabla\times u}\cdot\widehat{w}$, it follows that

$$\begin{aligned}&\frac{1}{2}\partial_t|(\widehat{u},\widehat{w})(t)|^2+(\mu+\chi)|\widehat{\nabla u}|^2+\gamma|\widehat{\nabla w}|^2+2\chi|\widehat{w}|^2\\&=-\mathrm{Re}\,[\widehat{u}\cdot\widehat{u\cdot\nabla u}+\widehat{w}\cdot\widehat{u\cdot\nabla w}-2\chi\widehat{\nabla\times u}\cdot\widehat{w}].\end{aligned}$$

By applying Cauchy-Schwarz's inequality, we obtain

$$2\mathrm{Re}\,[\widehat{\nabla\times u}\cdot\widehat{w}]\leq 2|\widehat{\nabla u}||\widehat{w}|\leq|\widehat{\nabla u}|^2+|\widehat{w}|^2.$$

Therefore,

$$\frac{1}{2}\partial_t|(\widehat{u},\widehat{w})(t)|^2+\mu|\widehat{\nabla u}|^2+\gamma|\widehat{\nabla w}|^2+\chi|\widehat{w}|^2\leq-\mathrm{Re}\,[\widehat{u}\cdot\widehat{u\cdot\nabla u}+\widehat{w}\cdot\widehat{u\cdot\nabla w}].$$

As a consequence, one has

$$\partial_t|(\widehat{u},\widehat{w})(t)|^2+2\theta|(\widehat{\nabla u},\widehat{\nabla w})|^2\leq-2\mathrm{Re}\,[\widehat{u}\cdot\widehat{u\cdot\nabla u}+\widehat{w}\cdot\widehat{u\cdot\nabla w}],$$

where $\theta=\min\{\mu,\gamma\}$. Multiplying the inequality above by $|\xi|^{2s}e^{2t|\xi|}$, where $t\geq 0$, and integrating over $\xi\in\mathbb{R}^2$, we have

$$\begin{aligned}&2\mathrm{Re}\,\langle\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t)),\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u}_t,\widehat{w}_t)(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}+2\theta\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq\\&-2\mathrm{Re}\,[\langle\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u}(t)),\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\cdot\nabla u}(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}+\langle\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w}(t)),\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\cdot\nabla w}(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}].\end{aligned} \tag{6.25}$$

On the other hand, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}&=\mathrm{Re}\,\langle\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t)),\mathcal{F}^{-1}(|\cdot|e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}\\&+\mathrm{Re}\,\langle\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t)),\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u}_t,\widehat{w}_t)(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}.\end{aligned}$$

Therefore, by applying Cauchy-Schwarz's inequality, it results that

$$\begin{aligned}&2\mathrm{Re}\,\langle\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t)),\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u}_t,\widehat{w}_t)(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}\geq\\&\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}-2\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|_{\dot{H}^s(\mathbb{R}^2)},\end{aligned}$$

since $|(\widehat{\nabla u},\widehat{\nabla w})|=|\xi||(\widehat{u},\widehat{w})|$. Once again, by using Cauchy-Schwarz's inequality, one has

$$\begin{aligned}&2\mathrm{Re}\,\langle\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t)),\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u}_t,\widehat{w}_t)(t))\rangle_{\dot{H}^s(\mathbb{R}^2)}\geq\\&\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}-\frac{1}{\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}-\theta\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}.\end{aligned} \tag{6.26}$$

Thus, replace the inequality (6.26) in (6.25) in order to get

$$\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot H^s(\mathbb{R}^2)} + \theta\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|^2_{\dot H^s(\mathbb{R}^2)} \leq \frac{1}{\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot H^s(\mathbb{R}^2)}$$
$$- 2\mathrm{Re}\,[\langle \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u}(t)), \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\cdot\nabla u}(t))\rangle_{\dot H^s(\mathbb{R}^2)} + \langle \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w}(t)), \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\cdot\nabla w}(t))\rangle_{\dot H^s(\mathbb{R}^2)}]. \tag{6.27}$$

On the other hand, notice that

$$\widehat{\nabla w}\cdot\widehat{u\otimes w}(\xi) = -\widehat{w}\cdot\widehat{u\cdot\nabla w}(\xi),$$

provided that $\mathrm{div}\, u = 0$. As a result, we get

$$\langle \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w}(t)), \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\cdot\nabla w}(t))\rangle_{\dot H^s(\mathbb{R}^2)} = -\langle \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{\nabla w}(t)), \mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\rangle_{\dot H^s(\mathbb{R}^2)}.$$

Hence, (6.27) can be rewritten, by using Cauchy-Schwarz's inequality, as follows:

$$\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot H^s(\mathbb{R}^2)} + \theta\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|^2_{\dot H^s(\mathbb{R}^2)} \leq \frac{1}{\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot H^s(\mathbb{R}^2)}$$
$$+ 2\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|_{\dot H^s(\mathbb{R}^2)}[\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes u}(t))\|_{\dot H^s(\mathbb{R}^2)} + \|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\|_{\dot H^s(\mathbb{R}^2)}]. \tag{6.28}$$

From now on, we shall continue this demonstration by studying two cases.

1° Case: Assume that $s > 1$:

By using Lemma 1.2.17, one has

$$\begin{aligned}
\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\|^2_{\dot H^s(\mathbb{R}^2)} &= \sum_{j,k=1}^{2}\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w_j u_k}(t))\|^2_{\dot H^s(\mathbb{R}^2)}\\
&\leq C_s e^{2t}[\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w}(t))\|^2_{\dot H^s(\mathbb{R}^2)}\|u(t)\|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u})\|^2_{\dot H^s(\mathbb{R}^2)}\|w(t)\|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w}(t))\|^2_{\dot H^s(\mathbb{R}^2)}\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u}(t))\|^2_{\dot H^s(\mathbb{R}^2)}].
\end{aligned}$$

where C_s is a positive constant. It is well known that the following inequality holds for the micropolar equations (6.1) (see [15]):

$$\|(u,w)(t)\|_{L^2(\mathbb{R}^2)} \leq \|(u_0,w_0)\|_{L^2(\mathbb{R}^2)}, \quad \forall\, t \geq 0. \tag{6.29}$$

From (6.29), it results

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\|_{\dot H^s(\mathbb{R}^2)} \leq C_{s,u_0,w_0}e^t[\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot H^s(\mathbb{R}^2)} + \|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot H^s(\mathbb{R}^2)}].$$

where C_{s,u_0,w_0} is a positive constant that depends only on s and $\|(u_0,w_0)\|_{L^2(\mathbb{R}^2)}$. Consider that $T^*>0$ is fixed and apply Young's inequality in order to get

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\|_{\dot{H}^s(\mathbb{R}^2)} \leq C_{s,u_0,w_0,T^*}[1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}], \quad \forall\, t\in[0,T^*].$$

Consequently, (6.28) becomes

$$\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}+\theta\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)} \leq \frac{1}{\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}$$
$$+\,C_{s,u_0,w_0,T^*}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}[1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}],$$

for all $t\in[0,T^*]$. By using Cauchy-Schwarz's inequality, one infers

$$\frac{d}{dt}[1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}] \leq C_{s,\theta,u_0,w_0,T^*}[1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}]^2,$$

for all $t\in[0,T^*]$. Thus, by integrating the inequality above over $[0,t]$ ($t\in[0,T^*]$), we reach

$$\varphi(t)\leq\varphi(0)+C_{s,\theta,u_0,w_0,T^*}\int_0^t\varphi(\tau)^2\,d\tau, \quad \forall\, t\in[0,T^*], \tag{6.30}$$

where

$$\varphi(t)=1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}, \quad \forall\, t\in[0,T^*].$$

Let us denote $T'=[8C_{s,\theta,u_0,w_0,T^*}\varphi(0)]^{-1}$ (where C_{s,θ,u_0,w_0,T^*} is given in (6.30)) and $T''=\sup\{t\in[0,T^*):\varphi(\tau)\leq 2\varphi(0),\forall\,\tau\in[0,t]\}$. As a consequence, we assure that

$$\begin{aligned}\varphi(t)&\leq\varphi(0)+C_{s,\theta,u_0,w_0,T^*}\int_0^t\varphi(\tau)^2\,d\tau\leq\varphi(0)+4C_{s,\theta,u_0,w_0,T^*}\varphi(0)^2t\\&\leq\varphi(0)[1+4C_{s,\theta,u_0,w_0,T^*}\varphi(0)T']\leq 2\varphi(0),\end{aligned}$$

for all $t\in[0,T]$, where $T=\frac{1}{2}\min\{T'',T'\}$. Rewriting the result above obtained, we have

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq 1+2\|(u_0,w_0)\|^2_{\dot{H}^s(\mathbb{R}^2)}, \quad \forall\, t\in[0,T].$$

(Notice that T depends only on $s,\theta,\|(u_0,w_0)\|_{H^s(\mathbb{R}^2)}$).

2º Case: Assume that $s\in(0,1)$:

Note that, by utilizing Lemma 1.2.16 **i)**, we get

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}=\sum_{j,k=1}^{2}\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w_ju_k}(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}$$

$$\leq C_s\sum_{j,k=1}^{2}[\|e^{t|\cdot|}\widehat{w}_j(t)\|_{L^1(\mathbb{R}^2)}\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u}_k(t))\|_{\dot{H}^s(\mathbb{R}^2)}+\|e^{t|\cdot|}\widehat{u}_k(t)\|_{L^1(\mathbb{R}^2)}\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{w}_j(t))\|_{\dot{H}^s(\mathbb{R}^2)}]^2.$$

where C_s is a positive constant. Now, apply the inequality (1.19) in order to obtain

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}\widehat{u\otimes w}(t))\|_{\dot{H}^s(\mathbb{R}^2)} \leq C_s\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^{s+1}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^{s+1}(\mathbb{R}^2)}^{1-s}. \tag{6.31}$$

On the other hand, it is also true that

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{\nabla u},\widehat{\nabla w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2 = \|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^{s+1}(\mathbb{R}^2)}^2, \tag{6.32}$$

since $|(\widehat{\nabla u},\widehat{\nabla w})| = |\xi||(\widehat{u},\widehat{w})|$. By replacing (6.31) and (6.32) in (6.28), we obtain

$$\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2 + \theta\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^{s+1}(\mathbb{R}^2)}^2 \leq \frac{1}{\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2$$
$$+ C_s\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^{s+1}(\mathbb{R}^2)}^{2-s}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^{s+1}.$$

By using Young's inequality, one infers

$$\frac{d}{dt}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2 + \frac{\theta}{2}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^{s+1}(\mathbb{R}^2)}^2 \leq \frac{1}{\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2$$
$$+ C_{s,\theta}\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^{\frac{2s+2}{s}}.$$

Once again, by applying Young's inequality, we have

$$\frac{d}{dt}[1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2] \leq C_{s,\theta}[1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2]^{\frac{s+1}{s}}.$$

Thus, by integrating the inequality above over $[0,t]$ $(t\geq 0)$, we reach

$$1+\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2$$
$$\leq [1+\|(u_0,w_0)\|_{\dot{H}^s(\mathbb{R}^2)}^2] + C_{s,\theta}\int_0^t [1+\|\mathcal{F}^{-1}(e^{\tau|\cdot|}(\widehat{u},\widehat{w})(\tau))\|_{\dot{H}^s(\mathbb{R}^2)}^2]^{\frac{s+1}{s}}\,d\tau.$$

As it was done in the first case, we obtain

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}^2 \leq 1+2\|(u_0,w_0)\|_{\dot{H}^s(\mathbb{R}^2)}^2, \quad \forall\, t\in[0,T].$$

(Notice that T depends only on $s,\theta,\|(u_0,w_0)\|_{\dot{H}^s(\mathbb{R}^2)}$).

□

Remark 6.2.4. Now, recall that the limit (6.22) (since $(u,w)\in C([0,\infty);H^s(\mathbb{R}^2))$) assures that there is a positive constant M such that

$$\|(u,w)(t)\|_{H^s(\mathbb{R}^2)} \leq M, \quad \forall\, t\geq 0. \tag{6.33}$$

Lemma 6.2.5. *Consider that* $(u,w)\in C([0,\infty);H^s_{a,\sigma}(\mathbb{R}^2))$ *is a global solution for the Micropolar equations* (6.1). *Then, there is an instant* $t=T$ *that depends only on* s,μ,γ *and* M *such that*

$$\|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)}\leq[1+2M^2]^{\frac{1}{2}},\quad \forall\, t\geq T,$$

where M *is given in* (6.33).

Proof. By considering the system

$$\begin{cases} v_t\,+\,v\cdot\nabla v\,+\,\nabla p\;=\;(\mu+\chi)\Delta v\,+\,\chi\nabla\times b, & x\in\mathbb{R}^2, t\geq 0,\\ b_t\,+\,v\cdot\nabla b\;=\;\gamma\,\Delta b\,+\,\chi\nabla\times v-2\chi b, & x\in\mathbb{R}^2, t\geq 0,\\ \operatorname{div} v\;=\;0, & x\in\mathbb{R}^2, t>0,\\ v(\cdot,0)\;=\;u(\cdot,T_1),\;\; b(\cdot,0)\;=\;w(\cdot,T_1), & x\in\mathbb{R}^2,\end{cases}\tag{6.34}$$

where $T_1\geq 0$ is arbitrary, we obtain, by following the proof of Lemma 6.2.3, a constant T (which depends only on s,θ,M) such that

$$\|\mathcal{F}^{-1}(e^{t|\cdot|}(\widehat{v},\widehat{b})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq 1+2\|(v,b)(0)\|^2_{\dot{H}^s(\mathbb{R}^2)}=1+2\|(u,w)(T_1)\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq 1+2M^2,$$

for all $t\in[0,T]$. In particular, we infer

$$\|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{v},\widehat{b})(T))\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq 1+2M^2,$$

that is,

$$\|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{u},\widehat{w})(T+T_1))\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq 1+2M^2.$$

Now, suppose that $t\geq T$ in order to obtain (for $T_1=t-T\geq 0$)

$$\|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)}\leq 1+2M^2,\quad \forall\, t\geq T.$$

□

6.2.2 Decay Rate Related to $\dot{H}^s_{a,\sigma}(\mathbb{R}^2)$

Now, let us establish the asymptotic behavior in $\dot{H}^s_{a,\sigma}(\mathbb{R}^2)$ of the solution (by assuming its global existence in time) obtained in Theorem 6.1.1 by extending and improving the steps presented by J. Benameur and L. Jlali [6].

Theorem 6.2.6. *Let* $a>0$, $\sigma>1$, *and* $s>0$ *with* $s\neq 1$. *Consider that* $(u,w)\in C([0,\infty);H^s_{a,\sigma}(\mathbb{R}^2))$ *is a global solution for the Micropolar equations* (6.1). *Then,*

$$\lim_{t\to\infty} t^{\frac{s}{2}}\|(u,w)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)}=0.$$

Proof. By Lemma 6.2.5, it follows that

$$\|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{u},\widehat{w})(t))\|^2_{\dot{H}^s(\mathbb{R}^2)} \leq 1 + 2M^2 =: M_1^2, \quad \forall\, t \geq T. \tag{6.35}$$

By applying Young's inequality, we have

$$\|(u,w)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |(\widehat{u},\widehat{w})(t)|^2\, d\xi \leq C_{T,\sigma,a} \int_{\mathbb{R}^2} |\xi|^{2s} e^{T|\xi|} |(\widehat{u},\widehat{w})(t)|^2\, d\xi.$$

Now, use Cauchy-Schwarz's inequality and (6.35) in order to conclude that

$$\begin{aligned} t^{\frac{s}{2}}\|(u,w)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} &\leq C_{T,\sigma,a} t^{\frac{s}{2}} \left(\int_{\mathbb{R}^2} |\xi|^{2s} |(\widehat{u},\widehat{w})(t)|^2\, d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\xi|^{2s} e^{2T|\xi|} |(\widehat{u},\widehat{w})(t)|^2\, d\xi\right)^{\frac{1}{2}} \\ &= C_{T,\sigma,a} \|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)} [t^{\frac{s}{2}} \|(u,w)(t)\|_{\dot{H}^s(\mathbb{R}^2)}] \\ &\leq C_{T,\sigma,a} M_1 [t^{\frac{s}{2}} \|(u,w)(t)\|_{\dot{H}^s(\mathbb{R}^2)}], \end{aligned}$$

for all $t \geq T$. Lastly, by applying Theorem 6.2.1 **ii)**, it results that

$$\lim_{t\to\infty} t^{\frac{s}{2}} \|(u,w)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} = 0.$$

□

6.2.3 Decay Rate of the Microrotational Velocity in $\dot{H}^s_{a,\sigma}(\mathbb{R}^2)$

The next theorem assures that the micro-rotational velocity field $w(t)$ decays faster than the velocity field $u(t)$ (see Theorem 6.2.1), provided that $\chi > 0$.

Theorem 6.2.7. *Let $a > 0$, $\sigma > 1$, and $s > 0$ with $s \neq 1$. Consider that $(u,w) \in C([0,\infty); H^s_{a,\sigma}(\mathbb{R}^2))$ is a global solution for the Micropolar equations* (6.1)*. If $\chi > 0$, one has*

$$\lim_{t\to\infty} t^{\frac{s+1}{2}} \|w(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} = 0.$$

Proof. By applying Young's inequality, we have

$$\|(u,w)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |(\widehat{u},\widehat{w})(t)|^2\, d\xi \leq C_{T,\sigma,a} \int_{\mathbb{R}^2} |\xi|^{2s} e^{T|\xi|} |(\widehat{u},\widehat{w})(t)|^2\, d\xi.$$

Now, use Cauchy-Schwarz's inequality and (6.35) in order to conclude that

$$\begin{aligned} t^{\frac{s+1}{2}}\|w(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} &\leq C_{T,\sigma,a} t^{\frac{s+1}{2}} \left(\int_{\mathbb{R}^2} |\xi|^{2s} |\widehat{w}(t)|^2\, d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\xi|^{2s} e^{2T|\xi|} |(\widehat{u},\widehat{w})(t)|^2\, d\xi\right)^{\frac{1}{2}} \\ &= C_{T,\sigma,a} \|\mathcal{F}^{-1}(e^{T|\cdot|}(\widehat{u},\widehat{w})(t))\|_{\dot{H}^s(\mathbb{R}^2)} [t^{\frac{s+1}{2}} \|w(t)\|_{\dot{H}^s(\mathbb{R}^2)}] \\ &\leq C_{T,\sigma,a} M_1 [t^{\frac{s+1}{2}} \|w(t)\|_{\dot{H}^s(\mathbb{R}^2)}], \end{aligned}$$

for all $t \geq T$. Lastly, by applying Theorem 6.2.1 **iii)**, one has

$$\lim_{t\to\infty} t^{\frac{s+1}{2}} \|w(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} = 0,$$

since $\chi > 0$.

□

6.2.4 Decay Rate Related to $H^s_{a,\sigma}(\mathbb{R}^2)$

Finally, let us guarantee the asymptotic behavior in $H^s_{a,\sigma}(\mathbb{R}^2)$ of the solution (by assuming its global existence in time) obtained in Theorem 6.1.1 by extending and improving the steps presented by J. Benameur and L. Jlali [6].

Theorem 6.2.8. *Let $a > 0$, $\sigma > 1$, and $s > 0$ with $s \neq 1$. Consider that $(u, w) \in C([0,\infty); H^s_{a,\sigma}(\mathbb{R}^2))$ is a global solution for the Micropolar equations* (6.1)*. Then,*

$$\lim_{t\to\infty} \|(u, w)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)} = 0.$$

Proof. By applying Theorem 6.2.6, the same way as in (6.21), we obtain

$$\lim_{t\to\infty} \|(u, w)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^2)} = 0. \tag{6.36}$$

As a result, by using Lemma 1.2.10, Theorem 6.2.1 **i)** and the limit (6.36), one deduces

$$\lim_{t\to\infty} \|(u, w)(t)\|_{H^s_{a,\sigma}(\mathbb{R}^2)} = 0.$$

□

Chapter 7

Generalized MHD equations: local existence, uniqueness and blow-up of solutions in Lei-Lin spaces

This chapter presents a study related to the local existence, uniqueness and properties at potential blow-up times for solutions of the following generalized Magnetohydrodynamics (GMHD) equations:

$$\begin{cases} u_t \,+\, (-\Delta)^{\alpha}\, u \,+\, u\cdot\nabla u \,+\, \nabla(p+\frac{1}{2}|\, b\,|^2) \;=\; b\cdot\nabla b, \quad x\in\mathbb{R}^3, t\in[0,T^*), \\ b_t \,+\, (-\Delta)^{\beta}\, b \,+\, u\cdot\nabla b \;=\; b\cdot\nabla u, \quad x\in\mathbb{R}^3, t\in[0,T^*), \\ \mathrm{div}\, u \;=\; \mathrm{div}\, b \;=\; 0, \quad x\in\mathbb{R}^3, t\in(0,T^*), \\ u(\cdot,0) \;=\; u_0(\cdot),\;\; b(\cdot,0) \;=\; b_0(\cdot), \quad x\in\mathbb{R}^3, \end{cases} \tag{7.1}$$

where $T^* > 0$ gives the solution's existence time, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, $b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t)) \in \mathbb{R}^3$ the magnetic field and $p(x,t) \in \mathbb{R}$ the hydrostatic pressure. Furthermore, we consider that $\alpha, \beta \in (\frac{1}{2}, 1]$. Lastly, the initial data for the velocity and magnetic fields, given by u_0 and b_0 in (7.1), are assumed to be divergence free, i.e., $\mathrm{div}\, u_0 = \mathrm{div}\, b_0 = 0$.

Notice that the GMHD equations (7.1) are an extension of the MHD equations; in fact, it is enough to consider $\alpha = \beta = 1$ in (7.1). Let us mention that some papers in the literature have presented a study related to the local existence, uniqueness and blow-up criteria for solutions of the MHD equations in Sobolev-Gevrey spaces (these ones are defined by a slight variation of the usual Sobolev spaces as well as Lei-Lin spaces). Here we refer to [24, 25, 26, 31] (and references therein). Although our interest is only connected with the mathematical theory of incompressible fluids, it is important to point out that "Magnetohydrodynamics is a branch of Physics devoted to the study of the dynamics of electrically conducting fluids in the presence of magnetic fields. In addition, MHD applies to most astrophysical plasmas, some laboratory plasmas, and liquid metals (e.g. mercury, sodium, gallium). More specifically, some applications of the MHD are the following: the machinery of the sun, stars, stellar winds, black holes with the formation of extragalactic jets, interstellar clouds, and planetary magnetospheres" (for more details see [22, 39] and references therein).

If we assume $\alpha = 1$ and $b = 0$ in (7.1), it is also easy to note that the famous Navier-Stokes equations are a particular case of the GMHD equations (7.1). More specifically, the Navier-Stokes system has been vastly discussed by various authors in order to establish new blow-up criteria for local solutions in Lei-Lin, Sobolev-Gevrey and the usual Sobolev spaces (see, for instance, [5, 10, 29] and included references).

The existence of global solutions in time for the GMHD equations (7.1) is still an open problem; thus, this issue has become a fruitful field in the study of the incompressible fluids (see e.g. [45] and references therein). More precisely, this chapter investigates the local existence and uniqueness of a classical solution $(u, b)(x, t)$ for the GMHD equations (7.1) in Lei-Lin spaces $\mathcal{X}^s(\mathbb{R}^3)$, provided that $\max\left\{\frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\right\} \leq s < 0$. Lastly, by assuming that the maximal time $T^* > 0$ of existence for the solution $(u, b)(x, t)$ is finite, we guarantee that the limit superior, as t tends to T^*, of the norm $\|(u, b)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)}$ blows up (whether $\max\left\{1 - 2\alpha, 1 - 2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\right\} < s < 0$).

7.1 Existence of Local Solutions

Below, we shall present one of our main results that establishes the existence of a time $T > 0$ and a solution $(u, b) \in [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))] \times [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))]$ for the GMHD equations (7.1), provided that the initial data is in the appropriate Lei-Lin space.

Theorem 7.1.1. *Assume that* $\max\left\{1 - 2\alpha, 1 - 2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\right\} \leq s < 0$, *with* $\alpha, \beta \in (\frac{1}{2}, 1]$. *If* $(u_0, b_0) \in \mathcal{X}^s(\mathbb{R}^3)$ *then, there exist a time* $T > 0$ *and a solution* $(u, b) \in [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))] \times [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))]$ *of the GMHD equations* (7.1).

Proof. The proof of local existence for solutions of the GMHD equations (7.1) is based on Lemma 1.2.20. Initially, we must assume that r is a positive constant such that

$$0 < r < \frac{1}{66C_s}, \tag{7.2}$$

where C_s is given by Lemma 1.2.24. Also, let us choose $N \in \mathbb{N}$ that satisfies

$$\int_{|\xi|>N} |\xi|^s |\hat{u}_0(\xi)| \, d\xi < \frac{r}{33}, \quad \int_{|\xi|>N} |\xi|^s |\hat{b}_0(\xi)| \, d\xi < \frac{r}{33}. \tag{7.3}$$

It comes from the fact that $(u_0, b_0) \in \mathcal{X}^s(\mathbb{R}^3)$.

Now, define $V_0 = \mathcal{F}^{-1}(\chi_{\{|\xi|>N\}}\hat{u}_0)$ and $W_0 = \mathcal{F}^{-1}(\chi_{\{|\xi|>N\}}\hat{b}_0)$. By using (7.3), one obtains

$$\begin{aligned}
\|V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^s |\widehat{V}_0(\xi)|\, d\xi \\
&= \int_{\mathbb{R}^3} |\xi|^s \chi_{\{|\xi|>N\}}(\xi) |\hat{u}_0(\xi)|\, d\xi \\
&= \int_{|\xi|>N} |\xi|^s |\hat{u}_0(\xi)|\, d\xi \\
&< \frac{r}{33}
\end{aligned} \tag{7.4}$$

and, analogously, we have

$$\|W_0\|_{\mathcal{X}^s(\mathbb{R}^3)} < \frac{r}{33}.$$

Let us also define $U(t) = e^{-t(-\Delta)^\alpha} U_0$ and $B(t) = e^{-t(-\Delta)^\beta} B_0$, where $U_0 = \mathcal{F}^{-1}(\chi_{\{|\xi|\leq N\}}\hat{u}_0)$ and $B_0 = \mathcal{F}^{-1}(\chi_{\{|\xi|\leq N\}}\hat{b}_0)$. By definition, $U(t)$ and $B(t)$ are the unique solutions of the systems

$$\left\{ \begin{aligned} U_t + (-\Delta)^\alpha U &= 0; \\ U(0) &= U_0, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} B_t + (-\Delta)^\beta B &= 0; \\ B(0) &= B_0, \end{aligned} \right. \tag{7.5}$$

respectively, for all $t > 0$. Moreover, one concludes

$$\begin{aligned}
\|U(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^s e^{-t|\xi|^{2\alpha}} |\widehat{U}_0(\xi)|\, d\xi \leq \int_{\mathbb{R}^3} |\xi|^s \chi_{\{|\xi|\leq N\}} |\hat{u}_0(\xi)|\, d\xi \\
&= \int_{|\xi|\leq N} |\xi|^s |\hat{u}_0(\xi)|\, d\xi \leq \|u_0\|_{\mathcal{X}^s(\mathbb{R}^3)},
\end{aligned} \tag{7.6}$$

for all $t \geq 0$. By following a similar process, we get

$$\|B(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|b_0\|_{\mathcal{X}^s(\mathbb{R}^3)}, \quad \forall t \geq 0. \tag{7.7}$$

On the other hand, one can write

$$\begin{aligned}
\|U\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} &= \int_0^T \int_{\mathbb{R}^3} |\xi|^{s+2\alpha} |\widehat{U}(\xi)|\, d\xi\, dt \\
&= \int_0^T \int_{\mathbb{R}^3} |\xi|^{s+2\alpha} e^{-t|\xi|^{2\alpha}} |\widehat{U}_0(\xi)|\, d\xi\, dt \\
&\leq \int_0^T \int_{\mathbb{R}^3} |\xi|^{s+2\alpha} e^{-t|\xi|^{2\alpha}} |\hat{u}_0(\xi)|\, d\xi\, dt \\
&= \int_{\mathbb{R}^3} |\xi|^{s+2\alpha} |\hat{u}_0(\xi)| \left(\int_0^T e^{-t|\xi|^{2\alpha}}\, dt \right) d\xi \\
&\leq \int_{\mathbb{R}^3} (1 - e^{-T|\xi|^{2\alpha}}) |\xi|^s |\hat{u}_0(\xi)|\, d\xi.
\end{aligned} \tag{7.8}$$

Similarly, it is possible to obtain

$$\|B\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \leq \int_{\mathbb{R}^3} (1 - e^{-T|\xi|^{2\beta}})|\xi|^s|\hat{u}_0(\xi)|\, d\xi. \tag{7.9}$$

As $(u_0, b_0) \in \mathcal{X}^s(\mathbb{R}^3)$, by applying the Dominated Convergence Theorem, one reaches

$$\lim_{T\nearrow 0} \|U\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} = 0, \quad \lim_{T\nearrow 0} \|B\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} = 0. \tag{7.10}$$

By using (7.8) and (7.9), it is also true that

$$\|U\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \leq \|u_0\|_{\mathcal{X}^s(\mathbb{R}^3)}, \quad \|B\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} \leq \|b_0\|_{\mathcal{X}^s(\mathbb{R}^3)}. \tag{7.11}$$

Let us choose $\varepsilon > 0$ small enough satisfying the following conditions:

- $C_s(\|u_0\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\alpha}} + \|u_0\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{-\frac{s}{2\alpha}}) < \dfrac{1}{33};$ (7.12)
- $2C_s\|u_0\|^{2+\frac{s-1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1-s}{2\alpha}} < \dfrac{r}{33};$ (7.13)
- $C_s(\|b_0\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\beta}} + \|b_0\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{-\frac{s}{2\beta}}) < \dfrac{1}{33};$ (7.14)
- $2C_s\|b_0\|^{2+\frac{s-1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1-s}{2\beta}} < \dfrac{r}{33};$ (7.15)
- $C_s(\|u_0\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|b_0\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\beta}-\frac{s}{2\alpha}} + \|u_0\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|b_0\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\alpha}-\frac{s}{2\beta}}) < \dfrac{r}{33}.$ (7.16)

On the other hand, as a consequence of (7.10), there exists a time $T = T(\varepsilon) \in (0, 1)$ such that

$$\|U\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} < \varepsilon, \quad \|B\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} < \varepsilon. \tag{7.17}$$

Now, define $V = u - U$ e $W = b - B$. Notice that, if (u, b) is a solution of (7.1), then (V, W) is a solution of the following system:

$$\begin{cases} V_t \ + (-\Delta)^\alpha V \ + (V+U)\cdot\nabla(V+U) + \nabla(p + \frac{1}{2}|W+B|^2) = (W+B)\cdot\nabla(W+B), \\ W_t \ + (-\Delta)^\beta W \ + (V+U)\cdot\nabla(W+B) \ = \ (W+B)\cdot\nabla(V+U), \\ \operatorname{div} V \ = \ \operatorname{div} W \ = \ 0, \\ V(\cdot,0) \ = \ V_0(\cdot), \ W(\cdot,0) \ = \ W_0(\cdot). \end{cases} \tag{7.18}$$

Our aim is to assure the existence and uniqueness of local solutions for the equations (7.18). To do this, we will use Lemma 1.2.20.

First of all, use the heat semigroup $e^{-(t-\tau)(-\Delta)^\alpha}$, with $\tau \in [0, t]$ $(t \in [0, T])$, in the first equation given of the system (7.18), and, after that, integrate the obtained result over the interval $[0, t]$ to reach

$$\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} V_t\, d\tau + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha}(-\Delta)^\alpha V\, d\tau$$
$$= -\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}[(V+U)\cdot\nabla(V+U) + \nabla(p + \frac{1}{2}|W+B|^2) - (W+B)\cdot\nabla(W+B)]\, d\tau.$$

By applying integration by parts to the first integral above and using the proprieties of the heat semigroup, one deduces

$$\begin{aligned}V(t) &= e^{-t(-\Delta)^{\alpha}}V_0\\ &-\int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}}[(V+U)\cdot\nabla(V+U)+\nabla(p+\frac{1}{2}|W+B|^2)-(W+B)\cdot\nabla(W+B)]\,d\tau.\end{aligned}$$

Let us recall that Helmontz's projector P_H is a linear operator such that

$$\begin{aligned}&P_H[(V+U)\cdot\nabla(V+U)-(W+B)\cdot\nabla(W+B)]\\ &=(V+U)\cdot\nabla(V+U)+\nabla(p+\frac{1}{2}|W+B|^2)-(W+B)\cdot\nabla(W+B).\end{aligned}$$

Consequently, one can write

$$V(t)=e^{-t(-\Delta)^{\alpha}}V_0-\int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}}P_H[(V+U)\cdot\nabla(V+U)-(W+B)\cdot\nabla(W+B)]\,d\tau. \tag{7.19}$$

Now, we are interested in obtaining an equality analogous to (7.19) related to the field W. Thus, by using the heat semigroup $e^{-(t-\tau)(-\Delta)^{\beta}}$, with $\tau\in[0,t]$, integrating over $[0,t]$, and integrating by parts, one gets

$$W(t)=e^{-t(-\Delta)^{\beta}}W_0-\int_0^t e^{-(t-\tau)(-\Delta)^{\beta}}[(V+U)\cdot\nabla(W+B)-(W+B)\cdot\nabla(V+U)]\,d\tau, \tag{7.20}$$

see the second equation of the system (7.18).

On the other hand, let us define the operator

$$\Psi(V,W)(t)=(\Psi_1(V,W)(t),\Psi_2(V,W)(t)),\quad \forall t\in[0,T],$$

where

$$\begin{aligned}\Psi_1(V,W)(t)&=e^{-t(-\Delta)^{\alpha}}V_0\\ &\quad-\int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}}P_H[(V+U)\cdot\nabla(V+U)-(W+B)\cdot\nabla(W+B)]\,d\tau\end{aligned}$$

and

$$\begin{aligned}\Psi_2(V,W)(t)&=e^{-t(-\Delta)^{\beta}}W_0\\ &\quad-\int_0^t e^{-(t-\tau)(-\Delta)^{\beta}}[(V+U)\cdot\nabla(W+B)-(W+B)\cdot\nabla(V+U)]\,d\tau,\end{aligned}$$

for all $t\in[0,T]$.

Moreover, let us consider the space

$$\mathcal{X}_T = \mathcal{X}_{T,\alpha,\beta,s}(\mathbb{R}^3) = [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))] \times [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))]$$

endowed with the norm

$$\|(f,g)\|_{\mathcal{X}_T} = \|f\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} + \|f\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} + \|g\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} + \|g\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))},$$

for all $(f,g) \in \mathcal{X}_T$. Our goal here is to prove that $\Psi : \mathcal{X}_T \to \mathcal{X}_T$ admits a fixed point, for a suitable $T > 0$.

Initially, we will prove that $\Psi(\mathcal{X}_T) \subseteq \mathcal{X}_T$. In fact, consider that $(V,W) \in \mathcal{X}_T$. Note that

$$\begin{aligned}\|\Psi_1(V,W)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|e^{-t(-\Delta)^\alpha}V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} &+ \|\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} P_H[(V+U)\cdot\nabla(V+U)]\, d\tau\|_{\mathcal{X}^s(\mathbb{R}^3)} \\ &+ \|\int_0^t e^{-(t-\tau)(-\Delta)^\alpha} P_H[(W+B)\cdot\nabla(W+B)]\, d\tau\|_{\mathcal{X}^s(\mathbb{R}^3)}.\end{aligned}$$

By applying Lemma 1.2.23, it results

$$\begin{aligned}\|\Psi_1(V,W)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} &+ \int_0^t \|(V+U)\otimes(V+U)\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau \\ &+ \int_0^t \|(W+B)\otimes(W+B)\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau.\end{aligned}$$

Thus, use Lemma 1.2.24 to obtain

$$\begin{aligned}\|\Psi_1(V,W)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} &+ 2C_s[\|V+U\|^{2+\frac{s-1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V+U\|^{\frac{1-s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \\ &+ \|W+B\|^{2+\frac{s-1}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|W+B\|^{\frac{1-s}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}].\end{aligned}$$

Consequently, by using (7.6), (7.7) and (7.11), we deduce

$$\|\Psi_1(V,W)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} + 4C_s[\|(V,W)\|_{\mathcal{X}_T} + \|(u_0,b_0)\|_{\mathcal{X}^s(\mathbb{R}^3)}]^2, \quad \forall t \in [0,T].$$

Lastly, (7.4) lets us conclude that

$$\|\Psi_1(V,W)\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} < \frac{r}{33} + 4C_s[\|(V,W)\|_{\mathcal{X}_T} + \|(u_0,b_0)\|_{\mathcal{X}^s(\mathbb{R}^3)}]^2 < \infty,$$

provided that $(V,W) \in \mathcal{X}_T$ and $(u_0,b_0) \in \mathcal{X}^s(\mathbb{R}^3)$. It is important to point out here that, by following a similar process, we have

- $\|\Psi_1(V,W)\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} < \frac{r}{33} + 4C_s[\|(V,W)\|_{\mathcal{X}_T} + \|(u_0,b_0)\|_{\mathcal{X}^s(\mathbb{R}^3)}]^2 < \infty$;
- $\|\Psi_2(V,W)\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} < \frac{r}{33} + 4C_s[\|(V,W)\|_{\mathcal{X}_T} + \|(u_0,b_0)\|_{\mathcal{X}^s(\mathbb{R}^3)}]^2 < \infty$;
- $\|\Psi_2(V,W)\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} < \frac{r}{33} + 4C_s[\|(V,W)\|_{\mathcal{X}_T} + \|(u_0,b_0)\|_{\mathcal{X}^s(\mathbb{R}^3)}]^2 < \infty$.

Thereby, we have $\Psi(\mathcal{X}_T) \subseteq \mathcal{X}_T$.

Now, denote

$$
\begin{aligned}
B_r &= B_{r,\alpha,\beta,s} \\
&= \{(V,W) \in \mathcal{X}_T :\ \|V\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}, \|W\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}, \|V\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}, \|W\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} \leq r\}.
\end{aligned}
$$

Thus, we shall show that $\Psi(B_r) \subseteq B_r$. In fact, let us consider $(V,W) \in B_r$ to infer

$$
\begin{aligned}
\|\Psi_1(V,W)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} &\leq \|e^{-t(-\Delta)^\alpha} V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(V\cdot\nabla V)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \\
&+ \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(V\cdot\nabla U)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau + \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(U\cdot\nabla V)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \\
&+ \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(U\cdot\nabla U)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau + \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(W\cdot\nabla W)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \\
&+ \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(W\cdot\nabla B)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau + \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(B\cdot\nabla W)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \\
&+ \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(B\cdot\nabla B)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau =: \sum_{j=0}^{8} I_j(t). \qquad (7.21)
\end{aligned}
$$

By (7.4), we have

$$
I_0(t) = \|e^{-t(-\Delta)^\alpha} V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|V_0\|_{\mathcal{X}^s(\mathbb{R}^3)} < \frac{r}{33}, \quad \forall t \in [0,T]. \qquad (7.22)
$$

On the other hand, by applying Lemmas 1.2.23 and 1.2.24, it follows that

$$
\begin{aligned}
I_1(t) &= \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(V\cdot\nabla V)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|V\otimes V\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau \\
&\leq 2C_s \|V\|^{2+\frac{s-1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} \|V\|^{\frac{1-s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \leq 2C_s r^2,
\end{aligned}
$$

since $(V,W) \in B_r$. Then, use (7.2) to conclude that

$$
I_1(t) < \frac{r}{33}, \quad \forall t \in [0,T]. \qquad (7.23)
$$

Again, by applying Lemmas 1.2.23 and 1.2.24, one checks that

$$
\begin{aligned}
I_2(t) &= \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(V\cdot\nabla U)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|U\otimes V\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau \\
&\leq C_s[\|V\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} \|U\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} \|V\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \|U\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \\
&\quad + \|V\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} \|U\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} \|V\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} \|U\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}].
\end{aligned}
$$

By using (7.6), (7.12) and (7.17), we infer

$$I_2(t) \le C_s r[\|u_0\|_{\mathcal{X}^s(\mathbb{R}^3)}^{1-\frac{1}{2\alpha}} \varepsilon^{\frac{1}{2\alpha}} + \|u_0\|_{\mathcal{X}^s(\mathbb{R}^3)}^{1+\frac{s}{2\alpha}} \varepsilon^{-\frac{s}{2\alpha}}] < \frac{r}{33}, \quad \forall t \in [0,T], \tag{7.24}$$

provided that $(V,W) \in B_r$. Analogously, we get the following estimate

$$I_3(t) < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.25}$$

Moreover, let us observe that Lemmas 1.2.23 and 1.2.24 also imply that

$$\begin{aligned} I_4(t) &= \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H(U \cdot \nabla U)\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau \le \int_0^t \|U \otimes U\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)} \, d\tau \\ &\le 2C_s \|U\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))}^{2+\frac{s-1}{2\alpha}} \|U\|_{L_T^1(\mathcal{X}^s(\mathbb{R}^3))}^{\frac{1-s}{2\alpha}}. \end{aligned}$$

As a result, by (7.6), (7.13) and (7.17), it follows

$$I_4(t) < 2C_s \|u_0\|_{\mathcal{X}^s(\mathbb{R}^3)}^{2+\frac{s-1}{2\alpha}} \varepsilon^{\frac{1-s}{2\alpha}} < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.26}$$

By following a similar process to the one presented above and applying (7.14) and (7.15), we have

$$I_5(t), I_6(t), I_7(t), I_8(t) < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.27}$$

Therefore, by replacing (7.22)–(7.27) in (7.21), we can guarantee that

$$\|\Psi_1(V,W)\|_{L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))} < r. \tag{7.28}$$

Now, let us estimate $\Psi_2(V,W)$ in $L_T^\infty(\mathcal{X}^s(\mathbb{R}^3))$-norm. First of all, notice that

$$\|\Psi_2(V,W)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \le \sum_{i=0}^{8} J_i(t), \quad \forall t \in [0,T], \tag{7.29}$$

where

- $J_0(t) = \|e^{-t(-\Delta)^\beta} W_0\|_{\mathcal{X}^s(\mathbb{R}^3)}, \quad J_1(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} V \cdot \nabla W\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau;$
- $J_2(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} V \cdot \nabla B\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau, \quad J_3(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} U \cdot \nabla W\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau;$
- $J_4(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} U \cdot \nabla B\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau, \quad J_5(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} W \cdot \nabla V\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau;$
- $J_6(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} W \cdot \nabla U\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau, \quad J_7(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} B \cdot \nabla V\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau;$
- $J_8(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} B \cdot \nabla U\|_{\mathcal{X}^s(\mathbb{R}^3)} \, d\tau,$

for all $t \in [0,T]$. Initially, by applying (7.2), one obtains

$$J_0(t) = \|e^{-t(-\Delta)^\beta} W_0\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \|W_0\|_{\mathcal{X}^s(\mathbb{R}^3)} < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.30}$$

Moreover, use Lemmas 1.2.23, 1.2.24 and (7.2) in order to deduce

$$\begin{aligned} J_1(t) &= \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} V\cdot\nabla W\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|W\otimes V\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau \\ &\leq C_s[\|V\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|W\|^{1-\frac{1}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|W\|^{\frac{1}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} \\ &\quad + \|V\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|W\|^{1+\frac{s}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|W\|^{-\frac{s}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}] \\ &\leq 2C_s r^2 < \frac{r}{33}, \end{aligned} \tag{7.31}$$

for all $t \in [0,T]$; since, $(V,W) \in B_r$. Now, let us present an estimate related to $J_2(t)$. Thus, one has

$$\begin{aligned} J_2(t) &= \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} V\cdot\nabla B\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|B\otimes V\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau \\ &\leq C_s[\|V\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|B\|^{1-\frac{1}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|B\|^{\frac{1}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} \\ &\quad + \|V\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|B\|^{1+\frac{s}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|B\|^{-\frac{s}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}]. \end{aligned}$$

Hence, by applying (7.7), (7.14) and (7.17), one deduces

$$J_2(t) \leq C_s r[\|b_0\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\beta}} + \|b_0\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{-\frac{s}{2\beta}}] < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.32}$$

Similarly, we get

$$J_3(t) < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.33}$$

At last, Lemmas 1.2.23 and 1.2.24 imply

$$\begin{aligned} J_4(t) &= \int_0^t \|e^{-(t-\tau)(-\Delta)^\beta} U\cdot\nabla B\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \leq \int_0^t \|B\otimes U\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\, d\tau \\ &\leq C_s[\|U\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|B\|^{1-\frac{1}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|U\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|B\|^{\frac{1}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} \\ &\quad + \|U\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|B\|^{1+\frac{s}{2\beta}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|U\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|B\|^{-\frac{s}{2\beta}}_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}], \end{aligned}$$

for all $t \in [0,T]$. Thus, use the estimates obtained in (7.6), (7.7) and (7.17) in order to conclude that

$$J_4(t) \leq C_s[\|u_0\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|b_0\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\beta}-\frac{s}{2\alpha}} + \|u_0\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|b_0\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\alpha}-\frac{s}{2\beta}}].$$

Consequently, (7.16) assures that

$$J_4(t) < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.34}$$

In an analogous way, we are able to prove that

$$J_5(t), J_6(t), J_7(t), J_8(t) < \frac{r}{33}, \quad \forall t \in [0,T]. \tag{7.35}$$

Thereby, by replacing (7.30)–(7.35) in (7.29), we conclude

$$\|\Psi_2(V,W)\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} < r. \tag{7.36}$$

It is easy to check, by applying Lemma 1.2.23 and the process established above, that

$$\|\Psi_1(V,W)\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} < r, \quad \|\Psi_2(V,W)\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} < r. \tag{7.37}$$

The estimates (7.28), (7.36) and (7.37) complete the proof of the fact that $\Psi(B_r) \subseteq B_r$.

Finally, let us prove that the operator Ψ satisfies the following inequality:

$$\|\Psi(V_1,W_1) - \Psi(V_2,W_2)\|_{\mathcal{X}_T} \leq K\|(V_1,W_1)-(V_2,W_2)\|_{\mathcal{X}_T}, \quad \forall (V_1,W_1),(V_2,W_2) \in B_r,$$

provided that $0 < K < 1$. First of all let us write

$$\|\Psi_1(V_1,W_1)(t) - \Psi_1(V_2,W_2)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq \sum_{j=1}^{8} L_j(t), \quad \forall t \in [0,T], \tag{7.38}$$

where

- $L_1(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[(V_1 - V_2)\cdot \nabla U]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau;$
- $L_2(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[U \cdot \nabla (V_1 - V_2)]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau;$
- $L_3(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[(V_1 - V_2)\cdot \nabla V_1]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau;$
- $L_4(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[V_2 \cdot \nabla (V_1 - V_2)]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau,$

and also

- $L_5(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[(W_1 - W_2)\cdot \nabla B]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau;$
- $L_6(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[B \cdot \nabla (W_1 - W_2)]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau;$
- $L_7(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[(W_1 - W_2)\cdot \nabla W_1]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau;$
- $L_8(t) = \int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha} P_H[W_2 \cdot \nabla (W_1 - W_2)]\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau,$

for all $t \in [0,T]$. Thus, by applying Lemmas 1.2.23 and 1.2.24, one has

$$\begin{aligned}L_1(t), L_2(t) \leq C_s[&\|V_1 - V_2\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|U\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V_1 - V_2\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|U\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\\ &+ \|V_1 - V_2\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|U\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V_1 - V_2\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|U\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}],\end{aligned}$$

for all $t \in [0,T]$. Thus, by applying Lemmas 1.2.23 and 1.2.24, one has

$$\begin{aligned}L_1(t), L_2(t) \leq C_s[&\|V_1 - V_2\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|U\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V_1 - V_2\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|U\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\\ &+ \|V_1 - V_2\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|U\|^{1+\frac{s}{2\alpha}}_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))}\|V_1 - V_2\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}\|U\|^{-\frac{s}{2\alpha}}_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}].\end{aligned}$$

Use the estimates obtained in (7.6) and (7.17) in order to conclude that

$$\begin{aligned}L_1(t), L_2(t) &< C_s\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T}[\|u_0\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{\frac{1}{2\alpha}} + \|u_0\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\varepsilon^{-\frac{s}{2\alpha}}]\\ &< \frac{1}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T}, \quad \forall t \in [0,T],\end{aligned} \tag{7.39}$$

see (7.12). It is also true that

$$L_3(t), L_4(t) \leq 2C_s r\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T} < \frac{1}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T}, \quad \forall t \in [0,T]. \tag{7.40}$$

It is enough to apply Lemma 1.2.23, Lemma 1.2.24 and (7.2), since that $(V_1, W_1), (V_2, W_2) \in B_r$. Analogously, we can estimate $L_5(t)$, $L_6(t)$, $L_7(t)$ and $L_8(t)$ in order to infer

$$L_5(t), L_6(t), L_7(t), L_8(t) < \frac{1}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T}, \quad \forall t \in [0,T]. \tag{7.41}$$

Therefore, by replacing (7.39)–(7.41) in (7.38), we conclude

$$\|\Psi_1(V_1, W_1) - \Psi_1(V_2, W_2)\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} < \frac{8}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T}, \tag{7.42}$$

for all $(V_1, W_1), (V_2, W_2) \in B_r$. By following a similar process, we guarantee that

- $\|\Psi_2(V_1, W_1) - \Psi_2(V_2, W_2)\|_{L^\infty_T(\mathcal{X}^s(\mathbb{R}^3))} < \frac{8}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T};$ (7.43)
- $\|\Psi_1(V_1, W_1) - \Psi_1(V_2, W_2)\|_{L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))} < \frac{8}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T};$ (7.44)
- $\|\Psi_2(V_1, W_1) - \Psi_2(V_2, W_2)\|_{L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))} < \frac{8}{33}\|(V_1 - V_2, W_1 - W_2)\|_{\mathcal{X}_T},$ (7.45)

for all $(V_1, W_1), (V_2, W_2) \in B_r$. Thereby, by (7.42)–(7.45), one concludes

$$\|\Psi(V_1, W_1) - \Psi(V_2, W_2)\|_{\mathcal{X}_T} < \frac{32}{33}\|(V_1, W_1) - (V_2, W_2)\|_{\mathcal{X}_T}, \quad \forall (V_1, W_1), (V_2, W_2) \in B_r. \tag{7.46}$$

Lastly, by noticing that $\Psi : B_r \to B_r$ is a contraction mapping (see (7.46)), it is enough to apply Lemma 1.2.20 in order to obtain a unique solution $(V, W) \in B_r$ for the equations (7.18). Thus, $(u, b) = (V, W) + (U, B) \in \mathcal{X}_T$ is a local solution of the GMHD system (7.1), where (V, W) is a solution of (7.18) and (U, B) of the heat equations presented in (7.5).

□

7.2 Uniqueness of Local Solutions

The Theorem below guarantee uniqueness for solutions of the GMHD equations, obtained in Theorem 7.1.1.

Theorem 7.2.1. *Assume that* $\max\left\{1-2\alpha, 1-2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\right\} \leq s < 0$, *with* $\alpha, \beta \in (\frac{1}{2}, 1]$. *If* $(u_0, b_0) \in \mathcal{X}^s(\mathbb{R}^3)$ *then, the solution* $(u, b) \in [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))] \times [C_T(\mathcal{X}^s(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))]$ *for the GMHD equations* (7.1) *obtained in Theorem* 7.1.1 *is unique.*

Proof. Suppose that (u_1, b_1), $(u_2, b_2) \in \mathcal{X}_T$ are local solutions of the GMHD equations (7.1), related to the pressures p_1 and p_2 respectively. It is important to emphasize that we are interested in proving that $(u_1, b_1)(t) = (u_2, b_2)(t)$ for all $t \in [0, T]$ (here T is given in Theorem 7.1.1) Thus, it is true that

$$\begin{cases} \delta_t + (-\Delta)^\alpha \delta + \delta \cdot \nabla u_1 + u_2 \cdot \nabla \delta + \nabla(p_1 - p_2 + \frac{1}{2}|\, b_1\,|^2 - \frac{1}{2}|\, b_2\,|^2) = \rho \cdot \nabla b_1 + b_2 \cdot \nabla \rho, \\ \rho_t + (-\Delta)^\beta \rho + \delta \cdot \nabla b_1 + u_2 \cdot \nabla \rho = \rho \cdot \nabla u_1 + b_2 \cdot \nabla \delta, \\ \operatorname{div} \delta = \operatorname{div} \rho = 0, \\ \delta(\cdot, 0) = \rho(\cdot, 0) = 0, \end{cases} \tag{7.47}$$

where $\delta = u_1 - u_2$, and $\rho = b_1 - b_2$. By applying Fourier Transform and taking the scalar product in $\mathbb{C}^3$ of the first equation of (7.47) with $\widehat{\delta}(t)$, one has

$$\widehat{\delta} \cdot \widehat{\delta}_t + \widehat{\delta} \cdot (|\xi|^{2\alpha} \widehat{\delta}) + \widehat{\delta} \cdot \widehat{\delta \cdot \nabla u_1} + \widehat{\delta} \cdot \widehat{u_2 \cdot \nabla \delta} = \widehat{\delta} \cdot \widehat{\rho \cdot \nabla b_1} + \widehat{\delta} \cdot \widehat{b_2 \cdot \nabla \rho}.$$

Thereby, it follows that

$$\frac{1}{2}\partial_t |\widehat{\delta}(t)|^2 + |\xi|^{2\alpha} |\widehat{\delta}(t)|^2 \leq |\widehat{\delta}(t)| [|\widehat{\delta \cdot \nabla u_1}| + |\widehat{u_2 \cdot \nabla \delta}| + |\widehat{\rho \cdot \nabla b_1}| + |\widehat{b_2 \cdot \nabla \rho}|]. \tag{7.48}$$

Considering $\epsilon > 0$ arbitrary, we can write

$$\frac{1}{2}\partial_t |\widehat{\delta}(t)|^2 = \frac{1}{2}\partial_t (|\widehat{\delta}(t)|^2 + \epsilon) = \sqrt{|\widehat{\delta}(t)|^2 + \epsilon}\ \partial_t \sqrt{|\widehat{\delta}(t)|^2 + \epsilon}.$$

By replacing this last equality in (7.48), one gets

$$\partial_t \sqrt{|\widehat{\delta}(t)|^2 + \epsilon} + \frac{|\xi|^{2\alpha} |\widehat{\delta}(t)|^2}{\sqrt{|\widehat{\delta}(t)|^2 + \epsilon}} \leq \frac{|\widehat{\delta}(t)|}{\sqrt{|\widehat{\delta}(t)|^2 + \epsilon}} [|\widehat{\delta \cdot \nabla u_1}| + |\widehat{u_2 \cdot \nabla \delta}| + |\widehat{\rho \cdot \nabla b_1}| + |\widehat{b_2 \cdot \nabla \rho}|].$$

Moreover, by integrating from 0 to t, we obtain

$$\sqrt{|\widehat{\delta}(t)|^2+\epsilon}+\int_0^t \frac{|\xi|^{2\alpha}|\widehat{\delta}(\tau)|^2}{\sqrt{|\widehat{\delta}(\tau)|^2+\epsilon}}\,d\tau \leq \sqrt{\epsilon}+\int_0^t (|\widehat{\delta\cdot\nabla u_1}|+|\widehat{u_2\cdot\nabla\delta}|+|\widehat{\rho\cdot\nabla b_1}|+|\widehat{b_2\cdot\nabla\rho}|)\,d\tau.$$

Taking the limit as $\epsilon \to 0$, multiplying by $|\xi|^s$ and integrating over $\xi \in \mathbb{R}^3$, we get the following estimate:

$$\begin{aligned}&\|\delta(t)\|_{\mathcal{X}^s(\mathbb{R}^3)}+\int_0^t \|\delta(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\,d\tau\\ &\leq \int_0^t (\|\delta\cdot\nabla u_1\|_{\mathcal{X}^s(\mathbb{R}^3)}+\|u_2\cdot\nabla\delta\|_{\mathcal{X}^s(\mathbb{R}^3)}+\|\rho\cdot\nabla b_1\|_{\mathcal{X}^s(\mathbb{R}^3)}+\|b_2\cdot\nabla\rho\|_{\mathcal{X}^s(\mathbb{R}^3)})\,d\tau. \end{aligned}\tag{7.49}$$

Analogously, we guarantee a similar estimative to (7.49) by considering the second equation of (7.47). More precisely, we infer

$$\begin{aligned}&\|\rho(t)\|_{\mathcal{X}^s(\mathbb{R}^3)}+\int_0^t \|\rho(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\,d\tau\\ &\leq \int_0^t (\|\delta\cdot\nabla b_1\|_{\mathcal{X}^s(\mathbb{R}^3)}+\|u_2\cdot\nabla\rho\|_{\mathcal{X}^s(\mathbb{R}^3)}+\|\rho\cdot\nabla u_1\|_{\mathcal{X}^s(\mathbb{R}^3)}+\|b_2\cdot\nabla\delta\|_{\mathcal{X}^s(\mathbb{R}^3)})\,d\tau. \end{aligned}\tag{7.50}$$

Hence, by following a similar process as in the proof of Lemma 1.2.23, we can rewrite (7.49) as follows:

$$\begin{aligned}&\|\delta(t)\|_{\mathcal{X}^s(\mathbb{R}^3)}+\int_0^t \|\delta(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\,d\tau\\ &\leq \int_0^t (\|u_1\otimes\delta\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}+\|\delta\otimes u_2\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}+\|b_1\otimes\rho\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}+\|\rho\otimes b_2\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)})\,d\tau.\end{aligned}$$

Now, by using the proof of Lemma 1.2.24, it follows that

$$\begin{aligned}&\|\delta(t)\|_{\mathcal{X}^s(\mathbb{R}^3)}+\int_0^t \|\delta(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\,d\tau\\ &\leq C_s\sum_{i=1}^{2}\int_0^t (\|\delta\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|u_i\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\delta\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\|u_i\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\\ &+\|\delta\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|u_i\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\delta\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\|u_i\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\\ &+\|\rho\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|b_i\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\rho\|^{-\frac{s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\|b_i\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\\ &+\|\rho\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|b_i\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)}\|\rho\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\|b_i\|^{-\frac{s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)})\,d\tau.\end{aligned}\tag{7.51}$$

By using Young's inequality, we infer

$$
\begin{aligned}
&\|\delta(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|\delta(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{4}\int_0^t \|\delta\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau + \frac{1}{4}\int_0^t \|\rho\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \\
&+ C_{s,\alpha,\beta}\sum_{i=1}^{2}\int_0^t [\|\delta(\tau)\|_{\mathcal{X}^s(\mathbb{R}^3)}(\|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{2\alpha-1}{s+2\alpha}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{1}{s+2\alpha}} + \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{s+2\alpha}{2\alpha-1}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{s}{2\alpha-1}}) \\
&+ \|\rho(\tau)\|_{\mathcal{X}^s(\mathbb{R}^3)}(\|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{2\beta-1}{s+2\beta}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{1}{s+2\beta}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{s+2\beta}{2\beta-1}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{s}{2\beta-1}})]\, d\tau. \qquad (7.52)
\end{aligned}
$$

Moreover, by using (7.50) and an analogous argument as in (7.52), one concludes

$$
\begin{aligned}
&\|\rho(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|\rho(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{4}\int_0^t \|\delta\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau + \frac{1}{4}\int_0^t \|\rho\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \\
&+ C_{s,\alpha,\beta}\sum_{i=1}^{2}\int_0^t [\|\delta(\tau)\|_{\mathcal{X}^s(\mathbb{R}^3)}(\|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\alpha(2\beta-1)}{\beta(s+2\alpha)}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{\alpha}{\beta(s+2\alpha)}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\alpha(s+2\beta)}{\beta(2\alpha-1)}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{\alpha s}{\beta(2\alpha-1)}}) \\
&+ \|\rho(\tau)\|_{\mathcal{X}^s(\mathbb{R}^3)}(\|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\beta(2\alpha-1)}{\alpha(s+2\beta)}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{\beta}{\alpha(s+2\beta)}} + \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\beta(s+2\alpha)}{\alpha(2\beta-1)}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{\beta s}{\alpha(2\beta-1)}})]\, d\tau. \qquad (7.53)
\end{aligned}
$$

Therefore, by combining the inequalities (7.52) and (7.53), we deduce

$$
\begin{aligned}
&\|(\delta,\rho)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \frac{1}{2}\int_0^t \|\delta\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau + \frac{1}{2}\int_0^t \|\rho\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \leq C_{s,\alpha,\beta}\int_0^t \|(\delta,\rho)(\tau)\|_{\mathcal{X}^s(\mathbb{R}^3)} \\
&\times \sum_{i=1}^{2}(\|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{2\alpha-1}{s+2\alpha}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{1}{s+2\alpha}} + \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{s+2\alpha}{2\alpha-1}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{s}{2\alpha-1}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{2\beta-1}{s+2\beta}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{1}{s+2\beta}} \\
&+ \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{s+2\beta}{2\beta-1}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{s}{2\beta-1}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\alpha(2\beta-1)}{\beta(s+2\alpha)}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{\alpha}{\beta(s+2\alpha)}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\alpha(s+2\beta)}{\beta(2\alpha-1)}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{\alpha s}{\beta(2\alpha-1)}} \\
&+ \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\beta(2\alpha-1)}{\alpha(s+2\beta)}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{\beta}{\alpha(s+2\beta)}} + \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\beta(s+2\alpha)}{\alpha(2\beta-1)}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{\beta s}{\alpha(2\beta-1)}})\, d\tau.
\end{aligned}
$$

Thereby, apply Gronwall's inequality (integral form) in order to obtain $(\delta,\rho)(t) = 0$, for all $0 \leq t \leq T$, provided that

$$
\begin{aligned}
&\int_0^t \sum_{i=1}^{2}(\|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{2\alpha-1}{s+2\alpha}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{1}{s+2\alpha}} + \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{s+2\alpha}{2\alpha-1}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{s}{2\alpha-1}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{2\beta-1}{s+2\beta}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{1}{s+2\beta}} \\
&+ \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{s+2\beta}{2\beta-1}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{s}{2\beta-1}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\alpha(2\beta-1)}{\beta(s+2\alpha)}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{\alpha}{\beta(s+2\alpha)}} + \|b_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\alpha(s+2\beta)}{\beta(2\alpha-1)}}\|b_i\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{-\frac{\alpha s}{\beta(2\alpha-1)}} \\
&+ \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\beta(2\alpha-1)}{\alpha(s+2\beta)}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{\beta}{\alpha(s+2\beta)}} + \|u_i\|_{\mathcal{X}^s(\mathbb{R}^3)}^{\frac{\beta(s+2\alpha)}{\alpha(2\beta-1)}}\|u_i\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{-\frac{\beta s}{\alpha(2\beta-1)}})\, d\tau
\end{aligned}
$$

is finite, for all $0 \leq t \leq T$; provided that, $(u_i, b_i) \in \mathcal{X}_T$ (with $i = 1, 2$). In fact, it is sufficient to apply Hölder's inequality and a similar process as in (1.24); since that, $\max\big\{1 - 2\alpha, 1 - 2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\big\} \leq s < 0$ (with $\alpha, \beta \in (\frac{1}{2}, 1]$).

This completes the proof of Theorem 7.2.1.

□

7.3 Blow-up Criterion for the Solution

The next theorem assures a blow-up criterion for solutions of the GMHD equations (7.1) if we assume that the maximal time of existence is finite.

Theorem 7.3.1. *Assume that $\alpha, \beta \in (\frac{1}{2}, 1]$, $\max\big\{1 - 2\alpha, 1 - 2\beta, \frac{\alpha(1-2\beta)}{\beta}, \frac{\beta(1-2\alpha)}{\alpha}\big\} < s < 0$, and $(u_0, b_0) \in \mathcal{X}^s(\mathbb{R}^3)$. Consider that $(u, b) \in C([0, T^*); \mathcal{X}^s(\mathbb{R}^3))$ is the maximal solution for the GMHD equations* (7.1) *obtained in Theorem* 7.1.1. *If $T^* < \infty$, then*

$$\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} = \infty. \tag{7.54}$$

Proof. Consider that $(u, b) \in C([0, T^*); \mathcal{X}^s(\mathbb{R}^3))$ is the maximal solution for the GMHD equations (7.1) obtained in Theorem 7.1.1, with $T^* < \infty$. Thus, let us prove that the blow-up criterion (7.54) is valid. It is important to point out here that we have used the techniques presented in [5].

Suppose by contradiction that Theorem 7.3.1 does not hold, i.e., consider that

$$\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} < \infty. \tag{7.55}$$

Thus, by (7.55) and Theorem 7.1.1, there exists an absolute constant $C > 0$ such that

$$\|(u, b)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} \leq C, \quad \forall\, t \in [0, T^*). \tag{7.56}$$

On the other hand, we can show analogously to (7.51) and by assuming the GMHD system (7.1), that

$$\begin{aligned}
&|u(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|u(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau \\
&\leq C_s \int_0^t (\|u\|^{2+\frac{s-1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)} \|u\|^{\frac{1-s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)} + \|b\|^{2+\frac{s-1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)} \|b\|^{\frac{1-s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)})\, d\tau
\end{aligned} \tag{7.57}$$

and also

$$
\begin{aligned}
&\|b(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|b(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \\
&\leq C_s \int_0^t (\|u\|^{1+\frac{s}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)} \|b\|^{1-\frac{1}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)} \|u\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)} \|b\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)} \\
&+ \|u\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^s(\mathbb{R}^3)} \|b\|^{1+\frac{s}{2\beta}}_{\mathcal{X}^s(\mathbb{R}^3)} \|u\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)} \|b\|^{-\frac{s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)})\, d\tau. \qquad (7.58)
\end{aligned}
$$

As a consequence, by replacing (7.56) in (7.57), one has

$$
\begin{aligned}
&\|u(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|u(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau \\
&\leq C_s C^{2+\frac{s-1}{2\alpha}} \int_0^t \|u(\tau)\|^{\frac{1-s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau + C_s C^{2+\frac{s-1}{2\beta}} \int_0^t \|b(\tau)\|^{\frac{1-s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau.
\end{aligned}
$$

Similarly, by using (7.56) in (7.58), we obtain

$$
\begin{aligned}
&\|b(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|b(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \leq C_s C^{2+\frac{s}{2\alpha}-\frac{1}{2\beta}} \int_0^t \|u\|^{-\frac{s}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)} \|b\|^{\frac{1}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)} \\
&+ C_s C^{2+\frac{s}{2\beta}-\frac{1}{2\alpha}} \int_0^t \|u\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)} \|b\|^{-\frac{s}{2\beta}}_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau.
\end{aligned}
$$

Apply Young's inequality in order to obtain

$$
\begin{aligned}
&\|u(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|u(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{4} \int_0^t \|u(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau + C_{s,\alpha}(T^*)^{1-\frac{1-s}{2\alpha}} \\
&+ \frac{1}{4} \int_0^t \|b(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau + C_{s,\beta}(T^*)^{1-\frac{1-s}{2\beta}} \qquad (7.59)
\end{aligned}
$$

and also

$$
\begin{aligned}
&\|b(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} + \int_0^t \|b(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \leq \frac{1}{4} \int_0^t \|u(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau + C_{s,\alpha,\beta}(T^*)^{1-\frac{\alpha}{\beta(s+2\alpha)}} \\
&+ \frac{1}{4} \int_0^t \|b(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau + C_{s,\alpha,\beta}(T^*)^{1-\frac{\beta}{\alpha(s+2\beta)}}. \qquad (7.60)
\end{aligned}
$$

Thereby, by combining (7.59) and (7.60), we get

$$
\int_0^t \|u(\tau)\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}\, d\tau \leq C_{s,\alpha,\beta,T^*}, \quad \int_0^t \|b(\tau)\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}\, d\tau \leq C_{s,\alpha,\beta,T^*}, \quad \forall t \in [0,T^*). \qquad (7.61)
$$

Now, consider that $(\kappa_n)_{n\in\mathbb{N}}$ is a sequence such that $\kappa_n \nearrow T^*$, where $\kappa_n \in (0,T^*)$, for all $n \in \mathbb{N}$. Let us show that

$$
\lim_{n,m\to\infty} \|(u,b)(\kappa_m) - (u,b)(\kappa_n)\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0. \qquad (7.62)
$$

First of all, analogously to (7.19) and (7.20) and by using (7.1), we obtain

$$u(t) = e^{-t(-\Delta)^\alpha}u_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} P_H(u \cdot \nabla u - b \cdot \nabla b)\, d\tau$$

and

$$b(t) = e^{-t(-\Delta)^\beta}b_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\beta} (u \cdot \nabla b - b \cdot \nabla u)\, d\tau.$$

Therefore, we can write

$$(u,b)(\kappa_m) - (u,b)(\kappa_n) = Q_1(m,n) + Q_2(m,n) + Q_3(m,n),$$

where

$$Q_1(m,n) = ([e^{-\kappa_m(-\Delta)^\alpha} - e^{-\kappa_n(-\Delta)^\alpha}]u_0, [e^{-\kappa_m(-\Delta)^\beta} - e^{-\kappa_n(-\Delta)^\beta}]b_0),$$

$$\begin{aligned} Q_2(m,n) = -(&\int_0^{\kappa_m} [e^{-(\kappa_m-\tau)(-\Delta)^\alpha} - e^{-(\kappa_n-\tau)(-\Delta)^\alpha}]P_H(u \cdot \nabla u - b \cdot \nabla b)\, d\tau, \\ &\int_0^{\kappa_m} [e^{-(\kappa_m-\tau)(-\Delta)^\beta} - e^{-(\kappa_n-\tau)(-\Delta)^\beta}](u \cdot \nabla b - b \cdot \nabla u)\, d\tau) \end{aligned}$$

and

$$Q_3(m,n) = (\int_{\kappa_m}^{\kappa_n} [e^{-(\kappa_n-\tau)(-\Delta)^\alpha} P_H(u \cdot \nabla u - b \cdot \nabla b)\, d\tau, \int_{\kappa_m}^{\kappa_n} e^{-(\kappa_n-\tau)(-\Delta)^\beta}(u \cdot \nabla b - b \cdot \nabla u)\, d\tau).$$

Notice that

$$\begin{aligned} \|[e^{-\kappa_m(-\Delta)^\alpha} - e^{-\kappa_n(-\Delta)^\alpha}]u_0\|_{\mathcal{X}^s(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^s (e^{-\kappa_m|\xi|^{2\alpha}} - e^{-\kappa_n|\xi|^{2\alpha}})|\widehat{u}_0(\xi)|\, d\xi \\ &\leq \int_{\mathbb{R}^3} |\xi|^s (e^{-\kappa_m|\xi|^{2\alpha}} - e^{-T^*|\xi|^{2\alpha}})|\widehat{u}_0(\xi)|\, d\xi, \end{aligned}$$

provided that $\kappa_n < T^*$, for all $n \in \mathbb{N}$. Thus, by using the fact that $u_0 \in \mathcal{X}^s(\mathbb{R}^3)$, it results from Dominated Convergence Theorem that

$$\lim_{n,m\to\infty} \|[e^{-\kappa_m(-\Delta)^\alpha} - e^{-\kappa_n(-\Delta)^\alpha}]u_0\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0.$$

By following a similar argument, one reaches

$$\lim_{n,m\to\infty} \|[e^{-\kappa_m(-\Delta)^\beta} - e^{-\kappa_n(-\Delta)^\beta}]b_0\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0.$$

Therefore, we have $\lim\limits_{n,m\to\infty}\|Q_1(m,n)\|_{\mathcal{X}^s(\mathbb{R}^3)}=0$. Moreover, by using (2.11), one deduces

$$\begin{aligned}
&\int_0^{\kappa_m}\|[e^{-(\kappa_m-\tau)(-\Delta)^\alpha}-e^{-(\kappa_n-\tau)(-\Delta)^\alpha}]P_H(u\cdot\nabla u-b\cdot\nabla b)\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau\\
&\le\int_0^{\kappa_m}\int_{\mathbb{R}^3}|\xi|^s[e^{-(\kappa_m-\tau)|\xi|^{2\alpha}}-e^{-(\kappa_n-\tau)|\xi|^{2\alpha}}]|\mathcal{F}(u\cdot\nabla u-b\cdot\nabla b)(\xi)|\,d\xi\,d\tau\\
&\le\int_0^{T^*}\int_{\mathbb{R}^3}|\xi|^s[1-e^{-(T^*-\kappa_m)|\xi|^{2\alpha}}]|\mathcal{F}(u\cdot\nabla u-b\cdot\nabla b)(\xi)|\,d\xi\,d\tau,
\end{aligned}$$

since $\kappa_n<T^*$, for all $n\in\mathbb{N}$. On the other hand, by following a similar process to the one applied in the proof of Lemma 1.2.23, one infers

$$\int_0^{T^*}\|u\cdot\nabla u-b\cdot\nabla b\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau\le\int_0^{T^*}\|u\otimes u\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau+\int_0^{T^*}\|b\otimes b\|_{\mathcal{X}^{s+1}(\mathbb{R}^3)}\,d\tau.$$

Consequently, by applying Lemma 1.2.24, it follows

$$\begin{aligned}
&\int_0^{T^*}\|u\cdot\nabla u-b\cdot\nabla b\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau\\
&\le 2C_s[(T^*)^{1+\frac{s}{2\alpha}-\frac{1}{2\beta}}\|u\|_{L^\infty_{T^*}(\mathcal{X}^s(\mathbb{R}^3))}^{2+\frac{s-1}{2\alpha}}\|u\|_{L^1_{T^*}(\mathcal{X}^{s+2\alpha}(\mathbb{R}^3))}^{\frac{1-s}{2\alpha}}+(T^*)^{1+\frac{s}{2\beta}-\frac{1}{2\alpha}}\|b\|_{L^\infty_{T^*}(\mathcal{X}^s(\mathbb{R}^3))}^{2+\frac{s-1}{2\beta}}\|b\|_{L^1_{T^*}(\mathcal{X}^{s+2\beta}(\mathbb{R}^3))}^{\frac{1-s}{2\beta}}]\\
&\le 2C_s[(T^*)^{1+\frac{s}{2\alpha}-\frac{1}{2\beta}}C^{2+\frac{s-1}{2\alpha}}C_{s,\alpha,\beta,T^*}^{\frac{1-s}{2\alpha}}+(T^*)^{1+\frac{s}{2\beta}-\frac{1}{2\alpha}}C^{2+\frac{s-1}{2\beta}}C_{s,\alpha,\beta,T^*}^{\frac{1-s}{2\beta}}]<\infty,
\end{aligned}$$

provided that the estimates (7.56) and (7.61) are valid. As $\int_0^{T^*}\|u\cdot\nabla u-b\cdot\nabla b\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau<\infty$; then, by Dominated Convergence Theorem, we deduce

$$\lim_{n,m\to\infty}\int_0^{\kappa_m}\|[e^{-(\kappa_m-\tau)(-\Delta)^\alpha}-e^{-(\kappa_n-\tau)(-\Delta)^\alpha}]P_H(u\cdot\nabla u-b\cdot\nabla b)\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau=0.$$

Analogously, one obtains

$$\lim_{n,m\to\infty}\int_0^{\kappa_m}\|[e^{-(\kappa_m-\tau)(-\Delta)^\beta}-e^{-(\kappa_n-\tau)(-\Delta)^\beta}](u\cdot\nabla b-b\cdot\nabla u)\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau=0.$$

Hence, $\lim\limits_{n,m\to\infty}\|Q_2(m,n)\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau=0$. Lastly, by applying Lemma 1.2.23 and the proof of Lemma 1.2.24, we infer

$$\begin{aligned}
&\int_{\kappa_m}^{\kappa_n}\|e^{-(\kappa_n-\tau)(-\Delta)^\alpha}P_H(u\cdot\nabla u-b\cdot\nabla b)\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau\le\\
&2C_s\int_{\kappa_m}^{T^*}[\|u\|_{\mathcal{X}^s(\mathbb{R}^3)}^{2+\frac{s-1}{2\alpha}}\|u\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{1-s}{2\alpha}}+\|b\|_{\mathcal{X}^s(\mathbb{R}^3)}^{2+\frac{s-1}{2\beta}}\|b\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{1-s}{2\beta}}]\,d\tau.
\end{aligned}$$

Apply (7.56) in order to obtain

$$\int_{\kappa_m}^{\kappa_n}\|e^{-(\kappa_n-\tau)(-\Delta)^\alpha}P_H(u\cdot\nabla u-b\cdot\nabla b)\|_{\mathcal{X}^s(\mathbb{R}^3)}\,d\tau\le C_{s,\alpha,\beta}\int_{\kappa_m}^{T^*}(\|u\|_{\mathcal{X}^{s+2\alpha}(\mathbb{R}^3)}^{\frac{1-s}{2\alpha}}+\|b\|_{\mathcal{X}^{s+2\beta}(\mathbb{R}^3)}^{\frac{1-s}{2\beta}})\,d\tau.$$

By Hölder's inequality and using (7.61), one checks that

$$\int_{\kappa_m}^{\kappa_n} \|e^{-(\kappa_n-\tau)(-\Delta)^\alpha} P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau \le C_{s,\alpha,\beta,T^*}[(T^*-\kappa_m)^{1-\frac{1-s}{2\alpha}} + (T^*-\kappa_m)^{1-\frac{1-s}{2\beta}}].$$

Consequently, taking $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \int_{\kappa_m}^{\kappa_n} \|e^{-(\kappa_n-\tau)(-\Delta)^\alpha} P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau = 0,$$

where $\max\{1-2\beta, 1-2\alpha\} < s < 0$ and $\alpha, \beta \in (\frac{1}{2}, 1]$. Moreover, by applying an analogous process, we conclude that

$$\lim_{n,m\to\infty} \int_{\kappa_m}^{\kappa_n} \|e^{-(\kappa_n-\tau)(-\Delta)^\beta} (u\cdot\nabla b - b\cdot\nabla u)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau = 0.$$

Consequently, $\lim\limits_{n,m\to\infty} \|Q_3(m,n)\|_{\mathcal{X}^s(\mathbb{R}^3)}\, d\tau = 0$. Therefore, (7.62) is proved.

In addition, (7.62) means that $((u,b)(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{X}^s(\mathbb{R}^3)$. Thus, there exists $(u_1, b_1) \in \mathcal{X}^s(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|(u,b)(\kappa_n) - (u_1,b_1)\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0.$$

Now, we are going to prove that the limit above does not depend on $(\kappa_n)_{n\in\mathbb{N}}$. Thus, choose $(\rho_n)_{n\in\mathbb{N}} \subseteq (0, T^*)$ such that $\rho_n \nearrow T^*$ and

$$\lim_{n\to\infty} \|(u,b)(\kappa_n) - (u_2,b_2)\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0,$$

for some $(u_2, b_2) \in \mathcal{X}^s(\mathbb{R}^3)$ (repeat the same process). Let us verify that $(u_2, b_2) = (u_1, b_1)$. In fact, define $(\varsigma_n)_{n\in\mathbb{N}} \subseteq (0, T^*)$ by $\varsigma_{2n} = \kappa_n$ and $\varsigma_{2n-1} = \rho_n$, for all $n \in \mathbb{N}$. It is easy to check that $\varsigma_n \nearrow T^*$. By rewriting the process above, we guarantee that there is $(u_3, b_3) \in X^s(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|(u,b)(\varsigma_n) - (u_3,b_3)\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0.$$

By uniqueness of limit, one infers $(u_1, b_1) = (u_3, b_3) = (u_2, b_2)$. This means that

$$\lim_{t\nearrow T^*} \|(u,b)(t) - (u_1,b_1)\|_{\mathcal{X}^s(\mathbb{R}^3)} = 0.$$

Thereby, by assuming (7.1) with the initial data (u_1, b_1), instead of (u_0, b_0), we assure, by Theorems 7.1.1 and 7.2.1, the existence and uniqueness of $(\bar{u}, \bar{b}) \in C_{\bar{T}}(\mathcal{X}^s(\mathbb{R}^3))$ $(\bar{T} > 0)$ for the GMHD system (7.1). Therefore, $(\tilde{u}, \tilde{b}) \in C_{\bar{T}+T^*}(\mathcal{X}^s(\mathbb{R}^3))$ given by

$$(\tilde{u}, \tilde{b})(t) = \begin{cases} (u,b)(t), & t \in [0, T^*); \\ (\bar{u}, \bar{b})(t - T^*), & t \in [T^*, \bar{T} + T^*], \end{cases}$$

solves (7.1) in $[0, \bar{T} + T^*]$. This is a contradiction. Consequently, one must have

$$\limsup_{t\nearrow T^*} \|(u,b)(t)\|_{\mathcal{X}^s(\mathbb{R}^3)} = \infty.$$

□

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