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Tese de Doutorado

LOCAL BEHAVIOR AND EXISTENCE OF SOLUTIONS FOR PROBLEMS INVOLVING THE FRACTIONAL (p,q)-LAPLACIAN

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Tese apresentada ao corpo docente de Pós-Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais, como parte dos requisitos para a obtenção do título de Doutor em Matemática.

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"A menos que modifiquemos a nossa maneira de pensar, não seremos capazes de resolver os problemas causados pela forma como nos acostumamos a ver o mundo."

Albert Einstein

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Abstract

In the first part of this work, we study the regularity of weak solutions (in an appropriate space) of the elliptic partial differential equation

$$(-\Delta_p)^s u + (-\Delta_q)^s u = f(x)$$
 in \mathbb{R}^N ,

where 0 < s < 1 and $2 \le q \le p < N/s$, and we prove that these solutions are locally in $C^{0,\alpha}(\mathbb{R}^N)$. In the sequence, we prove the existence of solutions of the problem

$$(-\Delta_p)^s u + (-\Delta_q)^s u = |u|^{p_s^* - 2} u + \lambda g(x)|u|^{r-2} u$$
 in \mathbb{R}^N ,

where $1 < q \leq p < N/s$, λ is a parameter and g satisfies some integrability conditions. As an application of the previus result, we show that, if 0 < s < 1, $2 \leq q \leq p < N/s$ and g is bounded, then the obtained solutions are continuous and bounded.

In the final part of the work, we study the behavior as $p \to \infty$ of u_p , a positive least energy solution of the problem

$$\begin{cases} \left[(-\Delta_p)^{\alpha} + (-\Delta_{q(p)})^{\beta} \right] u = \mu_p \|u\|_{\infty}^{p-2} u(x_u) \delta_{x_u} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ |u(x_u)| = \|u\|_{\infty}, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, δ_{x_u} is the Dirac delta distribution supported at x_u ,

$$\lim_{p \to \infty} \frac{q(p)}{p} = Q \in \begin{cases} (0,1) & \text{if } 0 < \beta < \alpha < 1\\ (1,\infty) & \text{if } 0 < \alpha < \beta < 1 \end{cases}$$

and

$$\lim_{p \to \infty} \sqrt[p]{\mu_p} > R^{-\alpha},$$

with R denoting the inradius of Ω .

Introduction

In Chapter 1, we investigate the regularity of weak solutions of the (p, q)-Laplacian problem

$$(-\Delta_p)^s u + (-\Delta_q)^s u = f \quad \text{in} \quad \mathbb{R}^N$$
(1)

where 0 < s < 1, N > sp, $p_s^* = \frac{Np}{N-sp}$, $2 < q \le p < \infty$ and $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N) \cap L^{\theta}(\mathbb{R}^N)$, with $\theta > \frac{N}{sp}$. The hypothesis $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$ ensures that we can apply variational methods, and the condition $f \in L^{\theta}(\mathbb{R}^N)$ is necessary to apply the Moser's iteration technique to obtain a bound in L^{∞} -norm for a solution.

For any) $< s < 1 \le m < \infty$, the fractional *m*-Laplacian operator, under suitable smoothness condition on ϕ , can be written as

$$(-\Delta_m)^s \phi(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\phi(x) - \phi(y)|^{m-2}(\phi(x) - \phi(y))}{|x - y|^{N+sm}} \mathrm{d}y, \quad \forall x \in \mathbb{R}^N,$$
(2)

where $B_{\varepsilon}(x) := \{y \in \mathbb{R}^N; |y - x| < \varepsilon\}$, see [24, 31, 43] for more details.

There are several notions of the fractional Laplacian operator in the current literature, all of which agree when the problems are set on the whole \mathbb{R}^N . However, some of them differ in a bounded domain.

Recently, a lot of attention has been given to studying problems involving fractional operators in many different contexts, such as thin obstacle problem, finance, phase transitions, stratified materials, optimization, anomalous diffusion, semipermeable membranes, minimal surfaces. For details, see [14, 20, 24, 49].

When s = 1, (1) becomes a (p, q)-Laplacian problem of the form

$$(-\Delta_p)u + (-\Delta_q)u = f(x), \quad x \in \mathbb{R}^N,$$
(3)

which has its origin in the general reaction-diffusion problem

$$u_t = \operatorname{div}(D(u)\nabla u) + f(x, u), \quad x \in \mathbb{R}^N, \quad t > 0,$$
(4)

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. The regularity of solution of (3) has been studied by He and Li [21]. They showed that the weak solutions are locally $C^{1,\alpha}$. For a general term D(u), problem (4) has a wide range of applications in Physics and related sciences such as Biophysics, Plasma Physics, and Chemical Reaction design. In such applications, the function u describes a concentration, and the first term on the right-hand side of (4) corresponds to a diffusion process with a diffusion coefficient D(u); the term f(x, u) stands for the reaction, related to sources and energy-loss processes. Typically, in chemical and biological applications, the reaction term f(x, u) is a polynomial in u with variable coefficients (see [29, 37, 49]). Still in this case, when the solutions are local minimizers for a class of integral functionals assuming that $1 , P. Baroni, G. Mingione and M. Colombo, (see [5] and [6]), proved <math>C^{1,\alpha}$ regularity.

In the case $p, q \neq 2$, problem (1) is both non-local and non-linear. Furthermore, its leading operator $(-\Delta_p)^s$ is degenerate when p > 2. To establish optimal regularity estimates up to the boundary is not only relevant by itself, but also has useful applications to obtain multiplicity results for more general non-linear and non-local equations, such as those investigated by Ianizzotto, Liu, Perera and Squassina [35] in the framework of topological methods and Morse theory.

The first difficulty found in problem (1) is how to define a weak solution, since $W^{s,p}(\Omega)$ is not always embedded into $W^{s,q}(\Omega)$ when $p \neq q$ (see in Appendix A.2 and [41]). For this purpose, we usually consider the reflexive Banach space

$$\mathcal{W} := D^{s,p}(\mathbb{R}^N) \cap D^{s,q}(\mathbb{R}^N)$$

endowed with the norm

$$\|u\|_{\mathcal{W}} := [u]_{s,p} + [u]_{s,q}$$

where $D^{s,m}(\mathbb{R}^N) = \{ u \in L^{m_s^*}(\mathbb{R}^N); [u]_{s,m} < \infty \}$ and $[u]_{s,m}$ denotes the Gagliardo-norm

$$[u]_{s,m} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y\right)^{\frac{1}{m}}$$
(5)

for all $u \in D^{s,m}(\mathbb{R}^N)$, see [10] for details.

The non-homogeneity of the operator $(-\Delta_p)^s + (-\Delta_q)^s$ introduces technical difficulties to obtain weak solutions of problems involving this operator. The regularity of these solutions is also an issue. It is worth to mention that $[\cdot]_{s,m}$ is a norm in $D^{s,m}(\mathbb{R}^N)$, but not in $W^{s,m}(\mathbb{R}^N)$. Note that $W^{s,m}(\mathbb{R}^N) \subsetneq D^{s,m}(\mathbb{R}^N)$, so in $D^{s,m}(\mathbb{R}^N)$ we have more functions as candidates to solve problems of the type (1). Another important property that motivates us to consider the space $D^{s,m}(\mathbb{R}^N)$ is that it is the completion of $C_c^{\infty}(\mathbb{R}^N)$ with respect to the norm $[\cdot]_{s,m}$, which makes it possible to calculate some integrals, since compact support functions simplify integrations, eliminating boundary terms.

Our first and main result is concerned with local regularity of weak solutions of the problem (1), using the Moser iteration. We will show that, under integrability conditions of f, the solutions of the problem (1) are bounded in \mathbb{R}^N . Moreover, under the additional condition $f \in L^{\infty}_{loc}(\mathbb{R}^N)$ we will prove that u is locally Hölder continuous, in other words, for any compact set $\Omega \subset \mathbb{R}^N$, if $f \in L^{\infty}(\Omega)$ then $u \in C^{0,\alpha}(\overline{\Omega})$.

The continuity of the solution u is proved in Section 1.2 by adapting arguments used by Ianizzotto, Mosconi and Squassina in [34] and Serrin [45]. The main idea is to control the oscillation of the function u in any ball. In order to do that, we prove a Harnack-type inequality for weak solutions of problem (1). Viscosity solutions methods, as well as barrier arguments, are frequently used in our approach. Since this kind of argument is not valid if $1 , our proof only applies for <math>2 \le q, p < \infty$.

In Chapter 2, we will study existence and regularity of weak solutions of the following problem involving the fractional critical p_s^* -exponent

$$\begin{cases} (-\Delta_p)^s u + (-\Delta_q)^s u = |u|^{p_s^* - 2} u + \lambda g(x)|u|^{r-2} u & \text{in } \mathbb{R}^N \\ u(x) \ge 0, & x \in \mathbb{R}^N \end{cases}$$
(6)

where 0 < s < 1, N > sp, $p_s^* = \frac{Np}{N-sp}$, $1 < q \leq p < r < p_s^*$, λ is a positive parameter and g satisfies the following integrability conditions:

- (g_1) g is integrable and $g \in L^{t_s}(\mathbb{R}^N)$, with $t_s = \frac{p_s^*}{p_s^* r}$;
- (g_2) there exist an open set $\Omega_g \in \mathbb{R}^N$ and $\alpha_0 > 0$ such that $g(x) \ge \alpha_0 > 0$, for all $x \in \Omega_g$.

When s = 1, problem (6) is reduced for the (p, q)-Laplacian equation

$$(-\Delta_p)u + (-\Delta_q)u = |u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u, \quad x \in \mathbb{R}^N.$$
 (7)

The existence of a nontrivial solution of the problem (7) was studied by Chaves, Ercole and Miyagaki in [19]. They showed the existence of a nontrivial solution if λ is large enough. Using the theory of regularity developed by He and Li in [21], they showed that the weak solutions are locally $C^{1,\alpha}$, if $g \in L^{t_1}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$.

Motivated by [19], we show that there exists a nontrivial solution of the problem (6) for λ large enough and $1 < q \leq p < r < p_s^*$. Under the restrictions $1 < q < \frac{N(p-1)}{N-s} < p \leq \max\left\{p, p_s^* - \frac{q}{p-1}\right\} < r < p_s^*$, and $N > p^2s$, we show that (6) has a solution of any $\lambda > 0$, which is be done by applying a version of the Mountain Pass Theorem (see [30]) and estimates for the extremal function, (see [7, 10, 42]).

We also adapt standard arguments to prove the boundedness of Palais-Smale sequences. In order to overcome the lack of compactness of Sobolev's embedding, we prove a pointwise convergence result, which together with the Brezis-Lieb lemma yields the weak convergence. Following arguments similar to [19, 38, 51], we obtain a strict upper bound for c_{λ} , the level of the Palais-Smale sequence, which is valid for all λ large enough. Applying this fact and arguments adapted from [19, 32], we conclude that the nonnegative corresponding critical points provide nontrivial solutions of I_{λ} (the Euler Lagrange functional associated to (6)).

When the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^t(\Omega)$ for $1 \leq t < p_s^*$ is not compact, for example, when $\Omega = \mathbb{R}^N$ some concentration-compactness principle or minimization restricted methods (see [47, 50]) have been used to find weak solutions in $W^{s,p}(\mathbb{R}^N)$ of problems involving the fractional *p*-Laplacian.

In Chapter 3, we investigate the behavior of least energy solutions of a fractional (p, q(p))-Laplacian problem as p goes to infinity.

We consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$, N > 1, and the Sobolev space of fractional order $s \in (0, 1)$ and exponent m > 1,

$$W_0^{s,m}(\Omega) := \left\{ u \in L^m(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad [u]_{s,m} < \infty \right\},\$$

where $[u]_{s,m}$ is defined in (5).

As it is well known, $\left(W_0^{s,m}(\Omega), [\cdot]_{s,m}\right)$ is a uniformly convex Banach space (also characterized as the closure of $C_c^{\infty}(\Omega)$ with respect to $[\cdot]_{s,m}$), compactly embedded into $L^r(\Omega)$ whenever

$$1 \le r < m_s^* := \begin{cases} \frac{Nm}{N-sm}, & m < N/s \\ \infty, & m \ge N/s \end{cases}$$

Moreover,

$$W_0^{s,m}(\Omega) \hookrightarrow G_0(\overline{\Omega}) \quad \text{if} \quad m > N/s.$$
 (8)

(The notation $A \hookrightarrow B$ means that the continuous embedding $A \hookrightarrow B$ is compact.) It follows that the infimum

$$\lambda_{s,m} := \inf \left\{ \frac{[u]_{s,m}^m}{\|u\|_{\infty}^m} : u \in W_0^{s,m}(\Omega) \setminus \{0\} \right\}$$

is positive and, in fact, it is a minimum.

The compactness in (8) is consequence of the following Morrey's type inequality (see [24])

$$\sup_{(x,y)\neq(0,0)} \frac{|u(x) - u(y)|}{|x - y|^{s - \frac{N}{m}}} \le C[u]_{s,m}, \quad \text{for all} \quad u \in W_0^{s,m}(\Omega),$$
(9)

which holds whenever m > N/s. If m is sufficiently large, the positive constant C in (9) can be chosen uniform with respect to m (see [28, Remark 2.2]).

We consider the nonhomogeneous problem

$$\begin{cases} \left[(-\Delta_p)^{\alpha} + (-\Delta_q)^{\beta} \right] u = \mu |u(x_u)|^{p-2} u(x_u) \delta_{x_u} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ |u(x_u)| = \|u\|_{\infty} \end{cases}$$
(10)

where α, β, p, q and $\mu > 0$ satisfy suitable conditions, $x_u \in \Omega$ is a point where u attains its sup norm $(|u(x_u)| = ||u||_{\infty})$, δ_{x_u} is the Dirac delta distribution supported at x_u and Ω be a bounded, smooth domain of \mathbb{R}^N .

Proceeding as in [4] and [26], one can arrive at (10) as the limit case, as $r \to \infty$, of the problem

$$\begin{cases} \left[(-\Delta_p)^{\alpha} + (-\Delta_q)^{\beta} \right] u = \mu \|u\|_r^{p-r} |u|^{r-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\|\cdot\|_r$ denotes the standard norm in the Lebesgue space $L^r(\Omega)$.

Therefore, we define the formal energy functional associated with (10) by

$$E_{\mu}(u) := \frac{1}{p} [u]_{\alpha,p}^{p} + \frac{1}{q} [u]_{\beta,q}^{q} - \frac{\mu}{p} ||u||_{\infty}^{p}, \quad \mu > 0,$$

and formulate our hypotheses on α , β , p and q to guarantee the well-definiteness of this functional. For this, we take into account (8) and the following known facts:

- $W_0^{s,p}(\Omega) \not\hookrightarrow W_0^{s,q}(\Omega)$ for any $0 < s < 1 \le q < p \le \infty$ (see [41, Theorem 1.1]),
- $W_0^{s_2,m_2}(\Omega) \hookrightarrow W_0^{s_1,m_1}(\Omega)$, whenever $0 < s_1 < s_2 < 1 \le m_1 < m_2 < \infty$ (see [11, Lemma 2.6]).

Thus, we assume that α, β, p and q satisfy one of the following conditions:

$$0 < \alpha < \beta < 1 \quad \text{and} \quad N/\alpha < p < q \tag{11}$$

or

$$0 < \beta < \alpha < 1 \quad \text{and} \quad N/\beta < q < p.$$
(12)

The assumption (11) provides the chain of embeddings $W_0^{\beta,q}(\Omega) \hookrightarrow W_0^{\alpha,p}(\Omega) \hookrightarrow C_0(\overline{\Omega})$ whereas (12) yields $W_0^{\alpha,p}(\Omega) \hookrightarrow W_0^{\beta,q}(\Omega) \hookrightarrow C_0(\overline{\Omega})$. Therefore, the Sobolev space

$$X(\Omega) := \begin{cases} \begin{pmatrix} W_0^{\beta,q}(\Omega), [\cdot]_{\beta,q} \end{pmatrix} & \text{if } 0 < \alpha < \beta < 1 \text{ and } N/\alpha < p < q \\ \begin{pmatrix} W_0^{\alpha,p}(\Omega), [\cdot]_{\alpha,p} \end{pmatrix} & \text{if } 0 < \beta < \alpha < 1 \text{ and } N/\beta < q < p \end{cases}$$

is the natural domain for the energy functional E_{μ} . Note that

$$X(\Omega) \subset W_0^{\alpha,p}(\Omega) \cap W_0^{\beta,q}(\Omega) \text{ and } X(\Omega) \hookrightarrow C_0(\overline{\Omega}).$$

Once we have chosen $X(\Omega)$, a weak solution of (10) is defined (see Definition 3.2.2).

We conclude by observing that weak solutions of (10) are also viscosity solutions of

$$\mathcal{L}_{\alpha,p}u + \mathcal{L}_{\beta,q}u = 0 \quad \text{in } D := \Omega \setminus \{x_u\}$$

and we use this fact to argue that nonnegative least energy solutions are strictly positive in Ω .

Then we fixed the fractional orders α and β (with $\alpha \neq \beta$), allow q and μ to depend suitably on p (q = q(p) and $\mu = \mu_p$) and denote by u_p the positive least energy solution of the problem

$$\begin{cases} \left[(-\Delta_p)^{\alpha} + (-\Delta_{q(p)})^{\beta} \right] u = \mu_p |u(x_p)|^{p-2} u(x_p) \delta_{x_p} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ |u(x_p)| = \|u\|_{\infty} \,, \end{cases}$$

where $x_p = x_{u_p}$ is the point such that $|u(x_p)| = ||u||_{\infty}$.

In the sequence we determine the asymptotic behavior of the pair $(u_p, x_p) \in X(\Omega) \times \Omega$, as p goes to ∞ .

For any 0 < s < 1, we use the following notation,

$$\left(\mathcal{L}_{s}^{+}u\right)\left(x\right) := \sup_{y \in \mathbb{R}^{N} \setminus \{x\}} \frac{u(y) - u(x)}{\left|y - x\right|^{s}} \quad \text{and} \quad \left(\mathcal{L}_{s}^{-}u\right)\left(x\right) := \inf_{y \in \mathbb{R}^{N} \setminus \{x\}} \frac{u(y) - u(x)}{\left|y - x\right|^{s}}.$$
 (13)

There are a substantial amount of papers in the recent literature dealing with the asymptotic behavior of solutions as a parameter goes to infinity in problems that involve a combination of first order, local operators and nonlinearities of different homogeneity degrees (see [4], [9], [16], [17], [18], [23], [26], [40]). In [4], Alves, Ercole and Pereira determined the asymptotic behavior, as $p \to \infty$, of the following problem of order 1

$$\begin{cases} \left[-\Delta_{p}+(-\Delta_{q(p)})\right]u = \mu_{p}\left|u(x_{u})\right|^{p-2}u(x_{u})\delta_{x_{u}} & \text{in } \Omega\\ u = 0 & \text{in } \partial\Omega\\ \left|u(x_{u})\right| = \left\|u\right\|_{\infty}. \end{cases}$$
(14)

Their work motived us to formulate an adequate fractional version of (14) and study, in the present paper, the behavior of the corresponding least energy solutions as p goes to infinity.

For fractional operators, there are a few works focusing in such type of asymptotic behavior. Most of recent ones deal with the problem of determining the limit equation satisfied, in the viscosity sense, by the limit functions (as $m \to \infty$) of a family $\{u_m\}$ of minimizers. In general, such limit equation combines the operators \mathcal{L}_s^+ , \mathcal{L}_s^- and their sum

$$\mathcal{L}_s := \mathcal{L}_s^+ + \mathcal{L}_s^-.$$

We refer to this latter operator as s-Hölder infinity Laplacian, accordingly to [15], where it was introduced. In that paper, Chambolle, Lindgren and Monneau studied the problem of minimizing the functional

$$[u]_{\Omega,s,m} := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y$$

on the set

$$X_g := \left\{ u \in C(\overline{\Omega}) : u = g \quad \text{on } \partial\Omega \right\},$$

where $g \in C^{0,s}(\partial\Omega)$ is given. After showing the existence of a unique minimizer $u_m \in X_g$ for this problem (assuming m > N/s), they proved that, up to a subsequence, $u_m \to u_\infty \in C^{0,s}(\overline{\Omega})$ uniformly and that this limit function is a viscosity solution of

$$\begin{cases} \mathcal{L}_s u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial \Omega \end{cases}$$

They also showed that u_{∞} is an optimal Hölder extension of g in Ω .

In [39], Lindqvist and Lindgren characterized the asymptotic behavior (as $m \to \infty$) of the only positive, normalized first eigenfunction u_m of $(-\Delta_m)^s$ in $W_0^{s,m}(\Omega)$. Namely, $u_m > 0$ in Ω , $||u_m||_m = 1$ and $[u_m]_{s,m}^m = \Lambda_{s,m}$, where

$$\Lambda_{s,m} := \inf \left\{ [u]_{s,m}^m : u \in W_0^{s,m}(\Omega) \quad \text{and} \quad \|u\|_m = 1 \right\}$$

is the first eigenvalue of $(-\Delta_m)^s$. Among several results, they proved that

$$\lim_{m \to \infty} \sqrt[m]{\Lambda_{s,m}} = R^{-s} \le \frac{|\phi|_s}{\|\phi\|_{\infty}} \quad \forall \phi \in C_c^{\infty}(\Omega) \setminus \{0\}$$
(15)

and that any limit function u_{∞} of the family $\{u_m\}$ is a positive viscosity solution of the problem

$$\begin{cases} \max \left\{ \mathcal{L}_{\infty} u , \mathcal{L}_{\infty}^{-} u + R^{-s} u \right\} = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^{N} \setminus \Omega. \end{cases}$$

In [28], Ferreira and Pérez-Llanos studied the asymptotic behavior, as $m \to \infty$, of the solutions of the problem

$$\begin{cases} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{m-2} (u(y) - u(x))}{|x - y|^{N + sm}} \mathrm{d}y = f(x, u) & \text{in } \Omega\\ u = g & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

for the cases f = f(x) and $f = f(u) = |u|^{\theta(m)-2} u$ with $\Theta := \lim_{m \to \infty} \theta(m)/m < 1$ (the exponent of the nonlinearity goes to infinity "sublinearly"). In the first case, they obtained different limit equations involving the operators \mathcal{L}_{∞} , \mathcal{L}_{∞}^+ and \mathcal{L}_{∞}^- according to the sign of the function f(x). In the second case, they established the limit equation

$$\min\left\{-\mathcal{L}_{\infty}^{-}u-u^{\Theta},-\mathcal{L}_{\infty}u\right\}=0.$$

Such results in that paper are compatible with the ones obtained for the local operator in [8] for the first case and in [17] for the second case.

Recently, in [22], Rossi and Silva studied the problem of minimizing the Gagliardo seminorm $[\cdot]_{s,m}$ among the functions $v \in W^{s,m}(\mathbb{R}^N)$ satisfying the constraints

$$v = g \quad \text{in } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad \mathcal{L}^N \left(\{ v > 0 \} \cap \Omega \right) \le \alpha,$$
 (16)

where the function g in $\mathbb{R}^N \setminus \Omega$ and the constant $\alpha \in (0, \mathcal{L}^N(\Omega))$ are given, and $\mathcal{L}^N(D)$ denotes the *N*-dimensional Lebesgue volume of the subset $D \subset \mathbb{R}^N$. They proved that, up to subsequences, the family $\{u_m\}$ of minimizers converges uniformly to a function u_∞ , as $m \to \infty$, that solves the equation

$$\mathcal{L}_s^- u = 0 \quad \text{in } \{u > 0\} \cap \Omega$$

in the viscosity sense and also minimizes the s-Hölder seminorm $|\cdot|_s$ among the functions in $W^{s,\infty}(\mathbb{R}^N)$ satisfying (16). Further, they showed the convergence of the respective extremal values, that is: $[u_m]_{s,m} \to |u_\infty|_s$.

More recently, in [27], Ercole, Pereira and Sanchis studied the asymptotic behavior of u_m , the positive solution of the minimizing problem

$$\Lambda_m = \inf\left\{ [u]_{s,m}^m : u \in W_0^{s,m}(\Omega) \quad \text{and} \quad \int_{\Omega} (\log|u|) \omega dx = 0 \right\}$$

where $\omega \in L^1(\Omega)$ is a positive weight satisfying $\|\omega\|_1 = 1$. After showing that u_m is the positive (weak) solution of the singular problem

$$\begin{cases} -(\Delta_m)^s u = \Lambda_m \omega(x) u^{-1} & \text{in } \Omega\\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

they proved that, up to subsequence, $\{u_m\}$ converges uniformly to a function $u_{\infty} \in C_0^{0,s}(\overline{\Omega})$ and $\sqrt[m]{\Lambda_m} \to |u_{\infty}|_s$. Moreover, the limit function u_{∞} is a positive viscosity solution of

$$\left\{ \begin{array}{ll} \mathcal{L}_s^- u + \left| u \right|_s = 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega \end{array} \right.$$

satisfying

$$0 \leq \int_{\Omega} (\log |u_{\infty}|) \omega dx < \infty \quad \text{and} \quad Q_s(u_{\infty}) \leq Q_s(u) \quad \forall u \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}$$

where $Q_s(u) := |u|_s / \exp\left(\int_{\Omega} (\log |u|) \omega dx\right)$.

Chapter 1

Global Hölder regularity for the fractional (p,q)-Laplacian

In this chapter we study the regularity of weak solution of the fractional (p,q)-Laplacian problem

$$(-\Delta_p)^s u + (-\Delta_q)^s u = f \quad \text{in} \quad \mathbb{R}^N$$
(1.1)

where $s \in (0,1)$, N > sp, $2 < q \le p < \infty$, $p_s^* = \frac{Np}{N-sp}$ and $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N) \cap L^{\theta}(\mathbb{R}^N)$, with $\theta > \frac{N}{sp}$. The hypothesis $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$ guarantees that the problem is well-posed, while that the condition $f \in L^{\theta}(\mathbb{R}^N)$ is necessary for the application of Moser's iteration technique to obtain a bound in L^{∞} -norm for a solution.

We recall that, for any $1 \leq m < \infty$, the fractional *m*-Laplacian operator, under suitable smoothness condition on ϕ , can be written as

$$(-\Delta_m)^s \phi(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\phi(x) - \phi(y)|^{m-2}(\phi(x) - \phi(y))}{|x - y|^{N+sm}} \mathrm{d}y, \quad \forall x \in \mathbb{R}^N,$$
(1.2)

where $B_{\varepsilon}(x) := \{ y \in \mathbb{R}^N; |y - x| < \varepsilon \}.$

Our main result is concerned with local regularity of weak solution of the problem (1.1):

Theorem 1.0.1 Let $\theta > \frac{N}{sp}$, $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N) \cap L^{\theta}(\mathbb{R}^N)$ and $u \in D^{s,p}(\mathbb{R}^N) \cap D^{s,q}(\mathbb{R}^N)$ a solution of (1.1). Then $u \in L^{\infty}(\mathbb{R}^N)$.

Moreover, if $f \in L^{\infty}_{loc}(\mathbb{R}^N)$, then u is locally Hölder continuous with exponent α , namely, $u \in C^{\alpha}_{loc}(\mathbb{R}^N)$ with $\alpha \in \left(0, \frac{s(p-q)}{p-1}\right)$.

The additional condition $f \in L^{\infty}_{loc}(\mathbb{R}^N)$ in the above theorem is used to control the oscillations of u in a ball.

1.1 Preliminaries

1.1.1 Functions spaces

For all measurable function $u: \mathbb{R}^N \to \mathbb{R}$, let

$$[u]_{s,m,\Omega} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^m}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y\right)^{1/m}$$

be the Gagliardo semi-norm. We will consider the following spaces (see [2, 24, 12] for details):

$$W^{s,m}(\Omega) = \{ u \in L^m(\Omega) ; [u]_{s,m,\Omega} < \infty \},\$$

equipped with the norm

$$||u||_{s,m} = ||u||_{W^{s,m}(\Omega)} = ||u||_{L^m(\Omega)} + [u]_{s,m,\Omega}$$

and

$$W_0^{s,m}(\Omega) = \{ u \in W^{s,m}(\mathbb{R}^N); \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \},$$
$$W^{-s,m'}(\Omega) = (W^{s,m}(\Omega))^*, \ m' = \frac{m}{m-1} \text{ (dual space)}.$$

For any $1 < m < \frac{N}{s}$ we define the reflexive Banach space

$$D^{s,m}(\mathbb{R}^N) := \{ u \in L^{m_s^*}(\mathbb{R}^N); [u]_{s,m} < \infty \},\$$

where $m_s^* = \frac{Nm}{N-sm}$ and $[.]_{s,m} = [.]_{s,m,\mathbb{R}^N}$ is a norm in $D^{s,m}(\mathbb{R}^N)$. The so-called best Sobolev constant for the embedding $D^{s,m}(\mathbb{R}^N) \hookrightarrow L^{m_s^*}(\mathbb{R}^N)$ is given by

$$S = \inf_{u \in D^{s,m}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,m}^m}{\|u\|_{m^*_*}^m},$$
(1.3)

see [10] for details.

We will frequently make use of the following space (see [34]):

Definition 1.1.1 Let $\Omega \subset \mathbb{R}^N$ be bounded. We set ¹

$$\widetilde{W}^{s,m}(\Omega) := \left\{ u \in L^m_{loc}(\mathbb{R}^N) : \exists \ U \supseteq \Omega, \ \|u\|_{W^{s,m}(U)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{m-1}}{(1+|x|)^{N+sm}} \mathrm{d}x < \infty \right\}$$

If Ω is unbounded, we set

$$\widetilde{W}^{s,m}_{loc}(\Omega) := \{ u \in L^m_{loc}(\mathbb{R}^N) : u \in \widetilde{W}^{s,m}(\Omega') \text{ for any bounded } \Omega' \subseteq \Omega \}.$$

 $^{{}^1\}Omega \Subset U$ means that Ω is a compact subset of U.

For all $\alpha \in (0, 1]$ and all measurable $u : \overline{\Omega} \to \mathbb{R}$ we set

$$|u|_{C^{\alpha}(\overline{\Omega})} = \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}$$
$$C^{0,\alpha}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : |u|_{C^{\alpha}(\overline{\Omega})} < \infty\}.$$

Throughout the chapter we assume that $0 < \alpha < 1$ and

$$C^{\alpha}(\overline{\Omega}) = C^{0,\alpha}(\overline{\Omega}),$$

which is a Banach space under the norm

$$||u||_{C^{\alpha}(\overline{\Omega})} = ||u||_{L^{\infty}\Omega} + |u|_{C^{\alpha}(\overline{\Omega})}.$$

We recall, (see [34]), that the nonlocal tail centered at $x \in \mathbb{R}^N$ with radius R > 0, is defined as

$$Tail_m(u;x;R) = \left(R^{sm} \int_{B_R^c(x)} \frac{|u(y)|^{m-1}}{|x-y|^{N+sm}} \mathrm{d}y \right)^{1/(m-1)}.$$
 (1.4)

We will also set $Tail_m(u; 0; R) = Tail_m(u; R)$.

Remark 1.1.2 Note that, if $u \in L_{\infty}(\mathbb{R}^N)$ and $m \ge 1$ then

$$[Tail_{m}(u;R)]^{m-1} = R^{sm} \int_{B_{R}^{c}(0)} \frac{|u(y)|^{m-1}}{|y|^{N+sm}} dy$$

$$\leq R^{sm} ||u||_{\infty}^{m-1} \int_{B_{R}^{c}(0)} |y|^{-N-sm} dy$$

$$= R^{sm} N\omega_{N} ||u||_{\infty}^{m-1} \int_{R}^{\infty} \rho^{-1-sm} d\rho$$

$$= \frac{N\omega_{N} ||u||_{\infty}^{m-1}}{sm}.$$

Thus, if $u \in L_{\infty}(\mathbb{R}^N)$ we have $Tail_m(u; R)^{m-1} \leq C$, where C = C(u, m, N, s) is independent of R.

1.1.2 Some elementary inequalities

For all $m \geq 1$ and $t \in \mathbb{R}$, we set

$$J_m(t) = |t|^{m-2}t.$$

We recall a few well-known inequalities

$$(a+b)^m \le 2^{m-1}(a^m+b^m), \ a,b \ge 0, \ m \ge 1;$$
 (1.5)

$$(a+b)^m \le a^m + b^m \ a, b \ge 0, \ m \in (0,1];$$
(1.6)

$$\left|J_{m+1}(a) - J_{m+1}(b)\right| \le m \left(J_m(a) + J_m(b)\right) |a - b|, \ a, b \in \mathbb{R}, \ q \ge 1.$$
(1.7)

Using the Taylor's formula and Young's inequality, we can prove that, for all $\theta > 0$ exists $C_{\theta} > 0$ such that

$$(a+b)^q - a^q \le \theta a^q + C_\theta b^q, \ a,b \ge 0, \ q > 0, \text{ and } C_\theta \to \infty \text{ as } \theta \to 0^+.$$
 (1.8)

For any b > 0, consider the function $f(t) = J_m(t) - J_m(t-b)$. Its global minimum is $f(b/2) = 2^{2-m}b^{m-1}$, and hence we obtain the inequality

$$J_m(a) - J_m(a-b) \ge 2^{2-m} b^{m-1}, \ \forall a \in \mathbb{R}, \ b \ge 0, \ \text{and} \ q \ge 1.$$
 (1.9)

Finally, in order to apply Moser iteration process, we will use the following lemma:

Lemma 1.1.3 Let $1 < m < \infty$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function. Defining

$$G(t) = \int_0^t (g'(\tau))^{\frac{1}{m}} \mathrm{d}\tau, \ t \in \mathbb{R},$$

we have that

$$J_m(a-b)(g(a)-g(b)) \ge |G(a)-G(b)|^m, \quad \forall a, b \in \mathbb{R}.$$

Proof. We will present the proof basead in an idea that can be found at [12, Lemma A.2]. Observe that we can suppose a > b without loss of generality. Then, the fundamental theorem of calculus yields

$$J_{m}(a-b)(g(a) - g(b)) = (a-b)^{m-2}(a-b)(g(a) - g(b))$$

= $(a-b)^{m-1} \int_{b}^{a} g'(\tau) d\tau$
= $(a-b)^{m-1} \int_{b}^{a} (G'(\tau))^{m} d\tau$
 $\geq \left(\int_{b}^{a} G'(\tau) d\tau\right)^{m}$

thanks to Jensen's inequality. \blacksquare

1.1.3 Some basic properties of $(-\Delta_p)^s + (-\Delta_q)^s$

The following result describes a fundamental non-local feature of the fractional (p, q)-Laplacian operator $(-\Delta_p)^s + (-\Delta_q)^s$.

Given $1 \leq q \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ we denote by

$$\mathcal{W}(\Omega) = \widetilde{W}^{s,p}(\Omega) \cap \widetilde{W}^{s,q}(\Omega).$$

Definition 1.1.4 Let $\Omega \subset \mathbb{R}^N$ be a domain bounded. We say that $u \in \mathcal{W}(\Omega)$ is a weak solution of $(-\Delta_p)^s u + (-\Delta_q)^s u = f$ in Ω if, for all $\varphi \in C_0^{\infty}(\Omega)$,

$$\sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y = \int_{\Omega} f\varphi \mathrm{d}x.$$

The weak inequality $(-\Delta_p)^s u + (-\Delta_q)^s u \leq f$ in Ω will mean that

$$\sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \le \int_{\Omega} f\varphi \mathrm{d}x,$$

for all $\varphi \in C_0^{\infty}(\Omega)$, $\varphi \ge 0$. Similarly for $(-\Delta_p)^s u + (-\Delta_q)^s u \ge f$.

Remark 1.1.5 By Lemma 2.3 in [34] the functional

$$W_0^{s,m}(\Omega) \ni \varphi \mapsto (u,\varphi) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y$$

is finite and belongs to $W^{-s.m'}(\Omega)$, which implies that the Definition 1.1.4 makes sense.

Lemma 1.1.6 Suppose that $u \in \mathcal{W}(\Omega)$ satisfies $(-\Delta_p)^s u + (-\Delta_q)^s u = f$ weakly in Ω for some $f \in L^1_{loc}(\Omega)$. Let $v \in L^1_{loc}(\mathbb{R}^N)$ be such that

dist(supp
$$(v), \Omega) > 0$$
, $\int_{\Omega^c} \frac{|v(x)|^{m-1}}{(1+|x|)^{N+sm}} \mathrm{d}x < \infty$, for $m \in \{p, q\}$

Then, $u + v \in \mathcal{W}(\Omega)$ and satisfies $(-\Delta_p)^s(u + v) + (-\Delta_q)^s(u + v) = f + h$ weakly in Ω , where

$$h(x) = 2\sum_{m=p,q} \int_{\text{supp}(v)} \frac{J_m(u(x) - u(y) - v(y)) - J_m(u(x) - u(y))}{|x - y|^{N+sm}} dx$$

Proof. It suffices to consider the case when Ω is bounded. Define K = supp(v) and consider $U \subset \mathbb{R}^N$ such that,

$$\Omega \Subset U \quad \text{and} \quad ||u||_{W^{s,m}(U)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{m-1}}{(1+|x|)^{N+sm}} \mathrm{d}x < \infty$$

for $m \in \{p, q\}$.

Without loss of generality we can assume that $\Omega \in U \in K^c$, since dist $(\Omega, K) = d > 0$. Clearly u + v = u in U, and thus $u + v \in W^{s,m}(U)$ for $m \in \{p,q\}$. Moreover, for $m \in \{p,q\}$ we have

$$\int_{\mathbb{R}^N} \frac{|u(x) + v(x)|^{m-1}}{(1+|x|)^{N+sm}} \mathrm{d}x \le C \int_{\mathbb{R}^N} \frac{|u(x)|^{m-1}}{(1+|x|)^{N+sm}} \mathrm{d}x + C \int_{\mathbb{R}^N} \frac{|v(x)|^{m-1}}{(1+|x|)^{N+sm}} \mathrm{d}x < \infty.$$

Therefore, $u + v \in \mathcal{W}(\Omega)$.

Now assume that $(-\Delta_p)^s u + (-\Delta_q)^s u = f$ weakly in Ω . Choose $\varphi \in C_0^{\infty}(\Omega)$ and compute

$$\begin{split} \sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) + v(x) - u(y) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &= \sum_{m=p,q} \int_{\Omega} \int_{\Omega} \frac{J_m(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &+ \sum_{m=p,q} \int_{\Omega^c} \int_{\Omega} \frac{J_m(u(x) - u(y) - v(y))\varphi(x)}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &- \sum_{m=p,q} \int_{\Omega} \int_{\Omega^c} \frac{J_m(u(x) - u(y) + v(y))\varphi(y)}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &= \sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &+ \sum_{m=p,q} \int_{\Omega} \int_{\Omega^c} \frac{J_m(u(x) - u(y))\varphi(y)}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &+ \sum_{m=p,q} \int_{\Omega} \int_{\Omega^c} \int_{\Omega} \frac{J_m(u(x) - u(y) - v(y))\varphi(x)}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &+ 2\sum_{m=p,q} \int_{\Omega} \int_{\Omega} \int_{\Omega^c} \frac{J_m(u(x) - u(y) - v(y))\varphi(x)}{|x - y|^{N + sm}} \mathrm{d}y \mathrm{d}x \mathrm{d}y \end{split}$$

thus, we obtain

$$\begin{split} &\sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) + v(x) - u(y) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y \\ &= \int_{\Omega} \left(f(x) + 2\sum_{m=p,q} \int_{\Omega^c} \frac{J_m(u(x) - u(y) - v(y)) - J_m(u(x) - u(y))}{|x - y|^{N+sm}} \mathrm{d}y \right) \varphi(x) \mathrm{d}x \\ &= \int_{\Omega} (f(x) + h(x))\varphi(x) \mathrm{d}x, \end{split}$$

where in the end we have used Fubini's theorem. The density of $C^{\infty}(\Omega)$ in $W_0^{s,m}(\Omega)$ allows to conclude.

The arguments used to show the next lemma are in [34, Proposition 2.10], we will make an adaptation.

Lemma 1.1.7 Let Ω be bounded, and let $u, v \in W(\Omega)$ satisfy $u \leq v$ in Ω^c . Then $(u - v)_+ \in W(\Omega)$.

Proof. It is enough to prove that, if $u, v \in W_0^{s,m}(\Omega)$ satisfy $u \leq v$ in Ω^c , then $(u-v)^+ \in W_0^{s,m}(\Omega)$. Denote $w = (u-v)^+$ and let $U \supseteq \Omega$ be as in Definition 1.1.1 for both u and v. We split the Gagliardo norm in \mathbb{R}^N as

$$\begin{split} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \\ &= \int_U \int_U \frac{|w(x) - w(y)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y + 2 \int_\Omega \int_{U^c} \frac{|w(x)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \end{split}$$

where we used that w = 0 in Ω^c , since $u \leq v$ in Ω^c . The first term is bounded, since $u, v \in W^{s,p}(U)$. For the second term we have

$$|x-y| \ge C_{\Omega,U}(1+|y|), \text{ for all } x \in \Omega \text{ and } y \in U^c$$

since $dist(\Omega, U^c) > 0$. Thus we obtain,

$$\int_{\Omega} \int_{U^c} \frac{|w(x)|^m}{|x-y|^{N+sm}} \mathrm{d}x \mathrm{d}y$$

$$\leq C_{\Omega,U} \left(\int_{\Omega} (|u(x)|^m + |v(x)|^m) \mathrm{d}x \right) \left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|)^{N+sm}} \mathrm{d}y \right)$$

$$\leq C_{\Omega,U} \int_{\Omega} (|u(x)|^m + |v(x)|^m) \mathrm{d}x.$$

Taking, $m = \{p, q\}$, we conclude the Lemma.

Proposition 1.1.8 (Comparison Principle) Let Ω be bounded, and let $u, v \in \mathcal{W}(\Omega)$ satisfy $u \leq v$ in Ω^c . Suppose that, for all $\varphi \in W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega), \varphi \geq 0$ in Ω , it is valid

$$\sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y$$
$$\leq \sum_{m=p,q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_m(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y$$

Then $u \leq v$ in Ω .

Proof. The proof is a straightforward calculus, but for convenience of the reader we sketch the details. Subtracting the above equations and adjusting the terms, we obtain

$$0 \ge \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{J_p(u(x) - u(y)) - J_p(v(x) - v(y))}{|x - y|^{N + sp}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{J_q(u(x) - u(y)) - J_q(v(x) - v(y))}{|x - y|^{N + sq}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y,$$
(1.10)

since $\varphi \geq 0$.

We show that the integrand is non-negative for $\varphi = (u - v)^+$ which belongs to $\mathcal{W}(\Omega)$ thanks to Lemma 1.1.7. Taking a = v(x) - v(y) and b = u(x) - u(y), the identity

$$J_m(b) - J_m(a) = (m-1)(b-a) \int_0^1 |a+t(b-a)|^{m-2} \mathrm{d}t$$

yields

$$J_m(u(x) - u(y)) - J_m(v(x) - v(y)) = (m-1) \left[(u-v)(x) - (u-v)(y) \right] Q_m(x,y),$$

where $Q_m(x,y) = \int_0^1 |(v(x) - v(y)) + t[(u-v)(x) - (u-v)(y)]|^{m-2} dt.$

We have $Q_m(x,y) \ge 0$ and $Q_m(x,y) = 0$ only if v(x) = v(y) and u(x) = u(y).

Rewriting the integrands in (1.10) we obtain

$$\int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \left(\frac{(p-1) \left[(u-v)(x) - (u-v)(y) \right] Q_{p}(x,y)}{|x-y|^{N+sp}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y \\
+ \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \left(\frac{(q-1) \left[(u-v)(x) - (u-v)(y) \right] Q_{q}(x,y)}{|x-y|^{N+sq}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y \le 0. \quad (1.11)$$

We now choose the test function $\varphi = (u - v)^+$ and define

$$\psi = u - v = (u - v)^{+} - (u - v)^{-}, \quad \varphi = (u - v)^{+} = \psi^{+}$$

From (1.11) results that

$$\begin{split} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \left(\frac{(p-1)(\psi(x) - \psi(y))(\psi^{+}(x) - \psi^{+}(y))Q_{p}(x,y)}{|x-y|^{N+sp}} \right) \mathrm{d}x \mathrm{d}y \\ &+ \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \left(\frac{(q-1)(\psi(x) - \psi(y))(\psi^{+}(x) - \psi^{+}(y))Q_{q}(x,y)}{|x-y|^{N+sq}} \right) \mathrm{d}x \mathrm{d}y \leq 0 \end{split}$$

Using the inequality

$$(\xi - \eta)(\xi^+ - \eta^+) \ge |\xi^+ - \eta^+|^2, \quad \forall \xi, \eta \in \mathbb{R},$$

we can see that

$$\begin{split} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(p-1)|\psi^+(x) - \psi^+(y)|^2 Q_p(x,y)}{|x-y|^{N+sp}} \mathrm{d}x \mathrm{d}y \\ &+ \frac{(q-1)|\psi^+(x) - \psi^+(y)|^2 Q_q(x,y)}{|x-y|^{N+sq}} \mathrm{d}x \mathrm{d}y \leq 0. \end{split}$$

Thus

$$\psi^+(x) = \psi^+(y)$$
 or $Q_m(x,y) = 0$,

at a. e. point (x, y). Also the latter alternative implies that $\psi^+(x) = \psi^+(y)$, and so

$$(u-v)^+(x) = C \ge 0, \quad \forall x \in \mathbb{R}^N.$$

The boundary condition implies that C = 0 and consequently $v \ge u$ in \mathbb{R}^N .

Proposition 1.1.9 Suppose Ω is bounded and $u \in \widetilde{W}^{s,m}(\Omega) \cap C^{1,\gamma}_{loc}(\Omega)$, with $\gamma \in [0,1]$ such that

$$\gamma > \begin{cases} 1 - m(1 - s), & \text{if } m \ge 2, \\ \frac{1 - m(1 - s)}{m - 1}, & \text{if } m < 2. \end{cases}$$

Then $(-\Delta_m)^s u = f$ strongly in Ω for some $f \in L^{\infty}_{loc}(\Omega)$.

Proof. See Proposition 2.12, [34]. \blacksquare

1.2 Interior Hölder regularity

Now we assume that $2 \leq q \leq p < \infty$ and we will prove a weak Harnack type inequality for non-negative supersolutions and then we will obtain an estimate of the oscillation of a bounded weak solution in a ball. In the sequence $B_R = B(0; R)$ and $\mathcal{W}(\Omega) = \widetilde{W}^{s,p}(\Omega) \cap \widetilde{W}^{s,q}(\Omega)$.

Theorem 1.2.1 Let $2 \le q \le p < \infty$ and $u \in \mathcal{W}(B_{R/3})$ satisfying weakly

$$\begin{cases} (-\Delta_p)^s u + (-\Delta_q)^s u \ge -K & em \quad B_{R/3} \\ u(x) \ge 0, \quad x \in \mathbb{R}^N \end{cases}$$
(1.12)

for some $K \geq 0$. Then there $\sigma \in (0,1)$ and $\overline{C} > 0$ such that

$$\inf_{B_{R/4}} u \ge \sigma \left(\oint_{B_R \setminus B_{R/2}} u^{q-1} \mathrm{d}x \right)^{1/(q-1)} - \overline{C} \left(KR^{sp} \right)^{1/(p-1)}$$

Proof. Choose a function $\varphi \in C^{\infty}(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$ in \mathbb{R}^N , $\varphi \equiv 1$ in $B_{3/4}$ and $\varphi \equiv 0$ in B_1^c . By Proposition 1.1.9, $|(-\Delta_m)^s \varphi| \leq C_1$ weakly in B_1 , for $m \geq 2$. Set $\varphi_R(x) = \varphi(3x/R)$, so that $\varphi_R \in C^{\infty}(\mathbb{R}^N)$, $0 \leq \varphi_R \leq 1$ in \mathbb{R}^N , $\varphi_R \equiv 1$ in $B_{R/4}$, $\varphi_R \equiv 0$ in $B_{R/3}^c$ and $|(-\Delta_m)^s \varphi| \leq C_1 R^{-sm}$ weakly in $B_{R/3}$. Given $\sigma \in (0, 1)$, consider

$$L(m) = \left(\oint_{B_R \setminus B_{R/2}} u^{m-1} \mathrm{d}x \right)^{1/(m-1)} \text{ and } w = \sigma L(q)\varphi_R + \chi_{B_R \setminus B_{R/2}} u$$

Thus $w \in \mathcal{W}(B_{R/3})$, and by Lemma 1.1.6 we have weakly in $B_{R/3}$,

$$\begin{split} (-\Delta_p)^s w(x) + (-\Delta_q)^s w(x) &= (-\Delta_p)^s (\sigma L(q)\varphi_R(x)) + (-\Delta_q)^s (\sigma L(q)\varphi_R(x)) \\ &+ 2\sum_{m=p,q} \int_{B_R \setminus B_{R/2}} \frac{J_m \big(\sigma L(q)\varphi_R(x) - u(y) \big) - J_m \big(\sigma L(q)\varphi_R(x) \big)}{|x - y|^{N + sm}} \mathrm{d}y \\ &\leq (\sigma L(q))^{p-1} (-\Delta_p)^s \varphi_R(x) + (\sigma L(q))^{q-1} (-\Delta_q)^s \varphi_R(x) \\ &+ 2\sum_{m=p,q} \int_{B_R \setminus B_{R/2}} \frac{J_m \big(\sigma L(q)\varphi_R(x) - u(y) \big) - J_m \big(\sigma L(q)\varphi_R(x) \big)}{|x - y|^{N + sm}} \mathrm{d}y. \end{split}$$

Thus using the inequality (1.9) set

$$\begin{aligned} (-\Delta_p)^s w(x) + (-\Delta_q)^s w(x) &\leq \frac{C_1(\sigma L(q))^{p-1}}{R^{sp}} + \frac{C_1(\sigma L(q))^{q-1}}{R^{sq}} \\ &- 2^{3-p} \int_{B_R \setminus B_{R/2}} \frac{(u(y))^{p-1}}{|x-y|^{N+sp}} \mathrm{d}y - 2^{3-q} \int_{B_R \setminus B_{R/2}} \frac{(u(y))^{q-1}}{|x-y|^{N+sq}} \mathrm{d}y \\ &\leq \frac{C_1(\sigma L(q))^{p-1}}{R^{sp}} + \frac{C_1(\sigma L(q))^{q-1}}{R^{sq}} - \frac{C_2(L(p))^{p-1}}{R^{sp}} - \frac{C_1(L(q))^{q-1}}{R^{sq}} \end{aligned}$$

(1.13)

Applying the Hölder's inequality we have $L(q) \leq L(p)$, for $p \geq q$ and thus, since $\sigma \in (0, 1)$ the above inequality yields

$$(-\Delta_p)^s w(x) + (-\Delta_q)^s w(x) \le \left(C_1 \sigma^{q-1} - C_2\right) \left(\frac{(L(q))^{p-1}}{R^{sp}} + \frac{(L(q))^{q-1}}{R^{sq}}\right)$$

Choosing $0 < \sigma < \min\left\{1, \left(\frac{C_2}{2C_1}\right)^{1/(q-1)}\right\}$ we get the upper estimate $(-\Delta_p)^s w(x) + (-\Delta_q)^s w(x) \le -\frac{C_2}{2} \frac{(L(q))^{p-1}}{R^{sp}}.$

We set $\overline{C} = \left(\frac{2}{C_2}\right)^{1/(p-1)}$ and distinguish two cases:

• If $L(q) \leq \overline{C}(KR^{sp})^{1/(p-1)}$, then

$$\inf_{B_{R/4}} u \ge 0 \ge \sigma L(q) - \overline{C} (KR^{sp})^{1/(p-1)};$$

• If $L(q) > \overline{C}(KR^{sp})^{1/(p-1)}$ then using (1.13) we obtain weakly in $B_{R/3}$

$$\begin{cases} (-\Delta_p)^s w(x) + (-\Delta_q)^s w(x) \le -K \le (-\Delta_p)^s u(x) + (-\Delta_q)^s u(x) \\ w = \chi_{B_R \setminus B_{R/2}} u \ge u, \ x \in B_{R/3}^c. \end{cases}$$
(1.14)
(1.15)

Using the Proposition 1.1.8, we obtain that $w \leq u$ in \mathbb{R}^N , in particular

$$\inf_{B_{R/4}} u \ge \inf_{B_{R/4}} w \ge \sigma L(q) \inf_{B_{R/4}} \varphi_R = \sigma L(q) \ge \sigma L(q) - \overline{C} (KR^{sp})^{1/(p-1)}$$

and so the we concludes proof. \blacksquare

Lemma 1.2.2 Let R < 1, $2 \le q \le p < \infty$ and $u \in \mathcal{W}(B_{R/3})$ such that

$$\begin{cases} (-\Delta_p)^s u + (-\Delta_q)^s u \ge -K & \text{in } B_{R/3} \\ u \ge 0, & \text{in } B_R, \end{cases}$$
(1.16)

for some $K \ge 0$. If $u \in L^{\infty}(\mathbb{R}^N)$ then there exist $\sigma \in (0,1)$, $K_0 > 0$, $\overline{C} > 0$ and for all $\varepsilon > 0$ a constant $C_{\varepsilon} > 0$ such that

$$\inf_{B_R} u \ge \sigma \left(\int_{B_R \setminus B_{R/2}} u^{q-1} \mathrm{d}x \right)^{\frac{1}{q-1}} - \overline{C} (K_0 R^{s(p-q)})^{\frac{1}{p-1}} - \varepsilon \sup_{B_R} u - C_{\varepsilon} Tail_p(u_-; R).$$

Proof. Let us apply Lemma 1.1.6 for the functions u and $v = u_{-}$, so that $u_{+} = u + v$ and $\Omega = B_{R/3}$. Then we have weakly in $B_{R/3}$,

$$\begin{split} (-\Delta_p)^s u_+(x) + (-\Delta_q)^s u_+(x) &= (-\Delta_p)^s u(x) + (-\Delta_q)^s u(x) \\ &+ 2 \sum_{m=p,q} \int_{B_{R/3}^c} \frac{J_m(u(x) - u(y) - u_-(y)) - J_m(u(x) - u(y))}{|x - y|^{N+sm}} \mathrm{d}y \\ &\geq -K + C \sum_{m=p,q} \int_{\{u < 0\}} \frac{(u(x))^{m-1} - (u(x) - u(y))^{m-1}}{|x - y|^{N+sm}} \mathrm{d}y \\ &\geq -K - C \sum_{m=p,q} \int_{\{u < 0\}} \frac{(u(x) - u(y))^{m-1} - (u(x))^{m-1}}{|y|^{N+sm}} \mathrm{d}y \end{split}$$

where in the end we have used that $|x - y| \ge \frac{2}{3}|y|$, for all $y \in \{u < 0\} \subset B_R^c$ and $x \in B_{R/3}$. By inequality (1.8), for any $\theta > 0$ there exists $C_{\theta} > 0$ such that weakly in $B_{R/3}$,

$$\begin{split} \big((-\Delta_{p})^{s} + (-\Delta_{q})^{s}\big)u_{+}(x) &\geq -K - C\sum_{m=p,q} \int_{\{u<0\}} \frac{\theta(u(x))^{m-1} - C_{\theta}(u(y))^{m-1}}{|y|^{N+sm}} \mathrm{d}y \\ &\geq -K - C\theta \sum_{m=p,q} (\sup_{B_{R}})^{m-1} \int_{B_{R}^{c}} \frac{1}{|y|^{N+sm}} \mathrm{d}y - C_{\theta} \sum_{m=p,q} \int_{B_{R}^{c}} \frac{(u(y))^{m-1}}{|y|^{N+sm}} \mathrm{d}y \\ &\geq -K - \frac{C\theta}{R^{sp}} \left(\sup_{B_{R}} u\right)^{p-1} - \frac{C_{\theta}}{R^{sp}} \left(Tail_{p}(u_{-};R)\right)^{p-1} \\ &- \frac{C\theta}{R^{sq}} \left(\sup_{B_{R}} u\right)^{q-1} - \frac{C_{\theta}}{R^{sq}} \left(Tail_{q}(u_{-};R)\right)^{q-1} \\ &=: -\widetilde{K}. \end{split}$$

Using the Remark 1.1.2 we can see that $Tail_q(u_-; R) \leq C_0$, where C_0 is independent of R > 0. We also have $R^{sp} \leq R^{s(p-q)}$ for $R \in (0, 1]$, since $q \leq p$. Thus,

$$\widetilde{K}R^{sp} \leq KR^{sp} + C\theta \left(\sup_{B_R} u\right)^{p-1} + C_{\theta} \left(Tail_p(u_-; R)\right)^{p-1} + CR^{s(p-q)}\theta \left(\sup_{B_R} u\right)^{q-1} + C_{\theta}R^{s(p-q)} \left(Tail_q(u_-; R)\right)^{\frac{q-1}{p-1}}\right)^{q-1} \leq KR^{s(p-q)} + C\theta \left(\sup_{B_R} u\right)^{p-1} + C_{\theta} \left(Tail_p(u_-; R)\right)^{p-1} + (C\theta + C_{\theta}) M_0 R^{s(p-q)}.$$

where $M_0 > 0$ is a constant independent of R > 0, that depend on $||u||_{L^{\infty}(\mathbb{R}^N)}$.

Consequently, given $\varepsilon > 0$ we can take $\theta < \min\left\{1, \frac{\varepsilon}{C^{p-1}}\right\}$ to obtain

$$\left(\widetilde{K}R^{sp}\right)^{\frac{1}{p-1}} \le (K_0 R^{s(p-q)})^{\frac{1}{p-q}} + \varepsilon \sup_{B_R} u + C_{\varepsilon} Tail_p(u_-; R)$$

where $K_0 = K_0(K, ||u||_{L^{\infty}(\mathbb{R}^N)}) > 0$ is independent of R > 0.

Therefore, applying the Lemma 1.2.1 for u_+ results

$$\inf_{B_{R/4}} = \inf_{B_{R/4}} u_{+} \ge \sigma \left(\oint_{B_{R} \setminus B_{R/2}} u^{q-1} dx \right)^{1/(q-1)} - (\widetilde{K}R^{sp})^{\frac{1}{p-1}} \\
\ge \sigma \left(\oint_{B_{R} \setminus B_{R/2}} u^{q-1} dx \right)^{\frac{1}{q-1}} - (K_{0}R^{s(p-q)})^{\frac{1}{p-1}} - \varepsilon \sup_{B_{R}} u - C_{\varepsilon}Tail_{p}(u_{-};R)$$

which concludes the the proof \blacksquare

Now we use the above results to produce an estimate of the oscillation of a bounded function u such that $(-\Delta_p)^s u + (-\Delta_q)^s u$ is locally bounded. We set for all $R > 0, x_0 \in \mathbb{R}^N$

$$Q(u; x_0; R) = ||u||_{L^{\infty}(B_R(x_0))} + Tail_p(u; x_0; R), \quad Q(u; R) = Q(u; 0; R).$$

Theorem 1.2.3 Let $2 < q \leq p < \infty$, $R_0 \in (0,1]$ and $u \in \mathcal{W}(B_{R_0}) \cap L^{\infty}(\mathbb{R}^N)$ be a function such that

$$|(-\Delta_p)^s u + (-\Delta_q)^s u| \le K \text{ weakly in } B_{R_0},$$

for some $K \ge 0$. Then there exist $\alpha \in (0,1)$ and C > 0 such that, for all $r \in (0,R_0)$ we have

$$\underset{B_r}{\operatorname{osc}} u \leq C \left[\left(K_0 R_0^{s(p-q)} \right)^{\frac{1}{p-1}} + Q(u; R_0) \right] \left(\frac{r}{R_0} \right)^{\alpha},$$

where $K_0 = K_0(K, ||u||_{L^{\infty}(\mathbb{R}^N)}) > 0$ is independent of R > 0.

Proof. For all integer $j \ge 0$ we set $R_j = \frac{R_0}{4^j}$, $B_j = B_{r_j}$ and $\frac{1}{2}B_j = B_{R_j/2}$. We claim that there are $\alpha \in (0, 1)$ and $\lambda > 0$, a non-decreasing sequence (m_j) and a non-increasing sequence (M_j) , such that

$$m_j \leq \inf_{B_j} u \leq \sup_{B_j} \leq M_j, \ M_j - m_j = \lambda R_j^{\alpha}, \text{ for any } j \geq 0.$$

We argue by induction on j.

Step zero: We set $M_0 = \sup_{B_{R_0}} u$ and $m_0 = M_0 - \lambda R_j^{\alpha}$, where $0 < \lambda < \frac{2||u||_{L^{\infty}(B_{R_0})}}{R_0^{\alpha}}$.

Inductive step: Assume that the sequences (m_j) and (M_j) are constructed up to the index j. Then

$$M_{j} - m_{j} = \int_{B_{R} \setminus B_{R/2}} (M_{j} - u) dx + \int_{B_{R} \setminus B_{R/2}} (u - m_{j}) dx$$
$$\leq \left(\int_{B_{R} \setminus B_{R/2}} (M_{j} - u)^{q-1} dx \right)^{\frac{1}{q-1}} + \left(\int_{B_{R} \setminus B_{R/2}} (u - m_{j})^{q-1} dx \right)^{\frac{1}{q-1}}.$$

Since (M_j) is non-increasing and (m_j) is non-decreasing, $M_j - u$ and $u - m_j$ are bounded in \mathbb{R}^N . Moreover, for all $j \ge 0$ we have

 $M_j - u \le M_0 - u$ and $u - m_j \le u - m_0$.

Let $\sigma \in (0, 1)$, $\tilde{C} > 0$ be as in Lemma 1.2.2. Multiply the previous inequality by σ to obtain, via Lemma 1.2.2,

$$\sigma(M_{j} - m_{j}) \leq \inf_{B_{j+1}} (M_{j} - u) + \inf_{B_{j+1}} (u - m_{j}) + 2\overline{C} (K_{0} R_{j}^{s(p-q)})^{\frac{1}{p-1}} + \varepsilon \left[\sup_{B_{j}} (M_{j} - u) + \sup_{B_{j}} (u - m_{j}) \right] + C_{\varepsilon} Tail_{p} ((M_{j} - u)_{-}; R_{j}) + C_{\varepsilon} Tail_{p} ((u - m_{j})_{-}; R_{j}).$$

Setting universally $\varepsilon = \frac{\sigma}{4}$, $C = \max\{2\tilde{C}, C_{\varepsilon}\}$ and rearranging, we have

$$\sup_{B_{j+1}} u \leq (1 - \frac{b}{2})(M_j - m_j)$$

+ $C\left[\left(K_0 R_0^{s(p-q)}\right)^{\frac{1}{p-1}} + Tail_p((M_j - u)_-; R_j) + Tail_p((u - m_j)_-; R_j)\right]$

In Appendix A.2, using the arguments of [34], we estimate both non-local tails,

$$Tail_p((M_j - u)_-; R_j) \le C \left[\lambda S(\alpha)^{\frac{1}{p-1}} + \frac{Q(u; R_0)}{R_0^{\alpha}} \right] R_j^{\alpha}, \ S(\alpha) \to 0 \text{ as } \alpha \to 0,$$

the same being valid for $Tail_p((u-m_j)_-)$. Therefore

$$\underset{B_{j+1}}{\operatorname{osc}} u \le \left(1 - \frac{\sigma}{2}\right) (M_j - m_j) + C \left(K_0 R_0^{s(p-q)}\right)^{\frac{1}{p-1}} + C \left[\lambda S(\alpha)^{\frac{1}{p-1}} + \frac{Q(u; R_0)}{R_0^{\alpha}}\right] R_j^{\alpha}.$$

Recalling that $M_j - m_j = \lambda R_j^{\alpha}$ and $R_j = \frac{R_0}{4^j}$, it follows that

$$\sup_{B_{j+1}} u \leq 4^{\alpha} \left[\left(1 - \frac{\sigma}{2} \right) + CS(\alpha)^{\frac{1}{p-1}} \right] \lambda R_{j+1}^{\alpha}$$

$$+ 4^{\alpha} C \left[K_0^{\frac{1}{p-1}} R_j^{\frac{s(p-q)}{p-1} - \alpha} + \frac{Q(u; R_0)}{R_0^{\alpha}} \right] R_{j+1}^{\alpha}$$

Now we choose $\alpha \in \left(0, \frac{s(p-q)}{p-1}\right)$ universally such that

$$4^{\alpha} \left[\left(1 - \frac{\sigma}{2} \right) + CS(\alpha)^{\frac{1}{p-1}} \right] < \left(1 - \frac{\sigma}{4} \right)$$

which is possible because $S(\alpha) \to 0$ as $\alpha \to 0$. Now, setting

$$\lambda = \frac{4^{\alpha+1}}{\sigma} C \left[K_0^{\frac{1}{p-1}} R_0^{\frac{s(p-q)}{p-1} - \alpha} + \frac{Q(u; R_0)}{R_0^{\alpha}} \right]$$
(1.17)

we have $\lambda \geq \frac{2||u||_{L^{\infty}(B_{R_0}(x_0))}}{R_0^{\alpha}}$, since $4^{\alpha+1}C/\sigma > 2$ and

$$\underset{B_{j+1}}{\operatorname{osc}} u \leq (1 - \frac{\sigma}{4})\lambda R_{j+1}^{\alpha} + \frac{\sigma}{4}\lambda R_{j+1}^{\alpha}$$
$$= \lambda R_{j+1}^{\alpha}.$$

We may pick m_{j+1} , M_{j+1} such that

$$m_j \le m_{j+1} \le \inf_{B_{j+1}} u \le \sup_{B_{j+1}} u \le M_{j+1} \le M_j, \quad M_{j+1} - m_{j+1} = \lambda R_{j+1}^{\alpha},$$

which completes the induction and proves the claim.

Now fix $r \in (0, R_0)$ and find an integer $j \ge 0$ such that $R_{j+1} \le r \le R_j$. Thus $R_j \le 4r$. Hence, by the claim and (1.17), we have

$$\underset{B_r}{\operatorname{osc}} \leq \underset{B_j}{\operatorname{osc}} \leq \lambda R_j^{\alpha} \leq C \left[\left(K_0 R_0^{s(p-q)} \right)^{\frac{1}{p-1}} + Q(u; R_0) \right] \left(\frac{r}{R_0} \right)^{\alpha},$$

which concludes the argument.

Corollary 1.2.4 Let $u \in \mathcal{W}(B_{2R_0}(x_0)) \cap L^{\infty}(\mathbb{R}^N)$ such that

 $|(-\Delta_p)^s u + (-\Delta_q)^s u| \le K \text{ weakly in } B_{2R_0}(x_0),$

for some $K \ge 0$ and $R_0 \in (0,1]$. Then there exist C > 0 and $\alpha \in (0,1)$ such that

$$|u|_{C^{0,\alpha}(B_{R_0}(x_0))} \le C \left[(K_0 R_0^{s(p-q)})^{\frac{1}{p-1}} + Q(u; x_0; 2R_0) \right] R_0^{-\alpha}.$$

Proof. Given $x, y \in B_{R_0}(x_0)$. Let $r = |x - y| \le R_0$. Let us apply the Theorem 1.2.3 to the ball $B_{R_0}(x) \subset B_{2R_0}(x_0)$. Clearly $||u||_{L^{\infty}(B_{R_0}(x))} \le ||u||_{L^{\infty}(B_{2R_0}(x_0))}$ and

$$Tail_{p}(u; x; R_{0})^{p-1} = R_{0}^{sp} \int_{B_{R_{0}}^{c}(x)} \frac{|u(y)|^{p-1}}{|x-y|^{N+sp}} dy$$
$$\leq C \|u\|_{L^{\infty}(B_{2R_{0}}(x_{0}))}^{p-1} + CR_{0}^{sp} \int_{B_{2R_{0}}^{c}(x_{0})} \frac{|u(y)|^{p-1}}{|x_{0}-y|^{N+sp}} dy$$

for a universal C, where as usual we used $|x - y| \ge |x_0 - y|/2$ for $y \in B_{2R_0}^c(x_0)$ and $x \in B_{R_0}(x)$. This implies that,

$$Q(u; x; R_0) \le CQ(u; x_0; 2R_0)$$

and thus we obtain the desired estimate on the Hölder seminorm. \blacksquare

1.3 Proof of Theorem 1.0.1

In this section, we present the proof of the Theorem 1.0.1. Replacing u by |u|, we can assume that $u \ge 0$.

Given a $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$, consider the problem

$$\begin{cases} (-\Delta_p)^s u + (-\Delta_q)^s u = f \text{ in } \mathbb{R}^N \\ u(x) \ge 0, \quad x \in \mathbb{R}^N, \end{cases}$$
(1.18)

where $s \in (0, 1)$, N > sp, $1 < q \le p < \infty$ and $p_s^* = \frac{Np}{N-sp}$. We will denote

$$\mathcal{W} := D^{s,p}(\mathbb{R}^N) \cap D^{s,q}(\mathbb{R}^N)$$

which is a Banach space with the induced norm

$$||u||_{\mathcal{W}} = [u]_{s,p} + [u]_{s,q}.$$

Definition 1.3.1 We say that $u \in W$ is a weak solution of (1.18) if

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{J_p(u(x) - u(y))}{|x - y|^{N + sp}} + \frac{J_q(u(x) - u(y))}{|x - y|^{N + sq}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}^N} f\varphi \mathrm{d}x,$$

for all $\varphi \in \mathcal{W}$.

The following remark is a direct consequence of the spaces involved and the key to concluding the continuity of the solutions of (1.18).

Remark 1.3.2 1) Note that, $\mathcal{W} \subseteq \mathcal{W}(\Omega)$ for any $\Omega \subset \mathbb{R}^N$ a bounded domain. 2) The condition $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$ ensures for the functional $\varphi \mapsto \int_{\mathbb{R}^N} f\varphi dx$ to be well defined for any $\varphi \in \mathcal{W}$. Proof of Theorem 1.0.1. By Remark 1.3.2, if $u \in \mathcal{W}$ satisfies (1.18) with $f \in L^{\infty}_{loc}(\mathbb{R}^N)$, then given $x_0 \in \mathbb{R}^N$ and $0 < R_0 \leq 1$ we have $u \in \mathcal{W}(B_{2R_0}(x_0))$ and $|(-\Delta_p)^s u + (-\Delta_q)^s u| \leq K =$ $||f||_{L^{\infty}(B_{2R_0}(x_0))}$. Now, by applying Corollary 1.2.4 we have

$$|u|_{C^{0,\alpha}(B_{R_0}(x_0))} \le C \left[(K_0 R_0^{s(p-q)})^{\frac{1}{p-1}} + Q(u; x_0; 2R_0) \right] R_0^{-\alpha}.$$

Given $\Omega \subset \mathbb{R}^N$ compact, we consider a covering $\Omega \subset \bigcup_i B_{R_i}(x)$ with $x \in \Omega$ and $0 \leq R_i < 1$. We use the same arguments of the proof of Theorem 1.1 in [34], to conclude that $u \in C^{\alpha}(\Omega)$.

To show that $u \in L^{\infty}(\mathbb{R}^N)$, we assume that $f \in L^{\frac{P_s^*}{p_s^*-1}}(\mathbb{R}^N) \cap L^{\theta}(\mathbb{R}^N)$ and use the Moser iteration process.

Let M > 0 and $\beta > 1$, we set for simplicity $u_M = \min\{u, M\}$ and

$$g_{\beta,M}(t) = (\min\{t, M\})^{\beta} = \begin{cases} t^{\beta}, & \text{se } t \le M, \\ M^{\beta}, & \text{se } t > M. \end{cases}$$

We can see that $g_{\beta,M}$ is continuous and has bounded derivative. Hence,

$$u_M = g_{\beta,M}(u) \in \mathcal{W} \cap L^{\infty}(\mathbb{R}^N).$$

Then we consider the test function $\varphi = g_{\beta,M}(u)$ in the Definition (1.3.1) and use Hölder's inequality, to set

$$\int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \left(\frac{J_{p}(u(x) - u(y))}{|x - y|^{N + sp}} + \frac{J_{q}(u(x) - u(y))}{|x - y|^{N + sq}} \right) (g_{\beta,M}(u(x)) - g_{\beta,M}(u(y))) dxdy
= \int_{\mathbb{R}^{N}} f(x) g_{\beta,M}(u(x)) dx = \int_{\mathbb{R}^{N}} f(x) u_{M}^{\beta}(x) dx \leq \|f\|_{\theta} \|u_{M}^{\beta}\|_{\theta'}.$$
(1.19)

Setting

$$G_{\beta,M}(t) = \int_0^t (g'_{\beta,M}(\tau))^{\frac{1}{m}} \mathrm{d}\tau = \frac{\beta^{\frac{1}{m}}m}{\beta+m-1} (\min\{t,M\})^{\frac{\beta+m-1}{m}}$$
(1.20)

and using Lemma 1.1.3 for $m \in \{p, q\}$, a = u(x), and b = u(y), results of (1.19) that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|G_{\beta,M}(u(x)) - G_{\beta,M}(u(y))|^p}{|x - y|^{N + sp}} \mathrm{d}x \mathrm{d}y \le \|f\|_{\theta} \|u_M^{\beta}\|_{\theta'}$$

By Sobolev inequality (1.3) we get

$$S\left(\int_{\mathbb{R}^N} |G_{\beta,M}(u(x))|^{p^*_s} \mathrm{d}x\right)^{\frac{p}{p^*_s}} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|G_{\beta,M}(u(x)) - G_{\beta,M}(u(y))|^p}{|x - y|^{N + sp}} \mathrm{d}x \mathrm{d}y$$
$$\leq ||f||_{\theta} ||u^{\beta}_M||_{\theta'}.$$

From (1.20)

$$S\left(\frac{\beta^{\frac{1}{p}}p}{\beta+p-1}\right)^{p}\left(\int_{\mathbb{R}^{N}}u_{M}^{\frac{(\beta+p-1)p_{s}^{*}}{p}}\mathrm{d}x\right)^{\frac{p}{p_{s}^{*}}}\leq||f||_{\theta}\|u_{M}^{\beta}\|_{\theta'}.$$

Equivalently using $\beta > 1$

$$\left(\int_{\mathbb{R}^N} u_M^{\frac{(\beta+p-1)p_s^*}{p}} \mathrm{d}x\right)^{\frac{p}{p_s^*}} \le C_1 \left(p_s^* \frac{\beta+p-1}{p}\right)^p \|u_M^\beta\|_{\theta'},\tag{1.21}$$

where $C_1 = C_1(s, p, N, ||f||_{\theta}) > 0$. By setting

$$\beta_{n+1} = p_s^* \frac{\beta_n + p - 1}{p\theta'}, \ \beta_0 = \frac{p_s^*}{\theta'} > 1 \text{ and } \sigma_n = \frac{\beta_n}{\beta_n + p - 1} < 1,$$

we can see that (β_n) is increasing, and we obtain of (1.21) for $\beta = \beta_n > 1$

$$\|u_M\|_{L^{\theta'\beta_{n+1}}(\mathbb{R}^N)} \le C^{\frac{1}{\beta_{n+1}}} \beta_{n+1}^{\frac{\beta_0}{\beta_{n+1}}} \|u_M\|_{L^{\theta'\beta_n}(\mathbb{R}^N)}^{\sigma_n}$$

Iterating this inequality and using that $\sigma_n < 1$, we get for any $n \ge 1$

$$\|u_M\|_{L^{\theta'\beta_{n+1}}(\mathbb{R}^N)} \le C^{j=1} \frac{1}{\beta_j} \left(\prod_{j=1}^{n+1} \beta_j^{\frac{1}{\beta_j}}\right)^{\beta_0} \|u_M\|_{L^{p_s^*}(\mathbb{R}^N)}^{n}.$$
 (1.22)

Setting $\gamma = \frac{p_s^*}{\theta' p} = \frac{N(\theta-1)}{(N-sp)\theta}$, we have $\gamma > 1$ since that $\theta > \frac{N}{sp}$, $\beta_n = \gamma^n \beta_0 + (p-1) \frac{\gamma^{n+1} - \gamma}{\gamma - 1}.$

Therefore,

$$\lim_{n \to \infty} \frac{\beta_n}{\gamma^n} = \beta_0 + (p-1) \lim_{n \to \infty} \frac{\gamma^{n-1} - \gamma}{\gamma^n (\gamma - 1)} = \beta_0 + (p-1) \frac{\gamma}{\gamma - 1} = \frac{p_s^* (p_s^* - \theta')}{\theta' (p_s^* - \theta' p)}.$$

Thus, using the limit comparison test, we conclude that

$$\sum_{j=1}^{\infty} \frac{1}{\beta_j} < \infty.$$

On the other hand, for any $n \in \mathbb{N}$ consider $a_n = \prod_{j=1}^n \beta_j^{\frac{1}{\beta_j}}$. Thus,

$$\ln a_n = \ln \left(\prod_{j=1}^n \beta_j^{\frac{1}{\beta_j}} \right) = \sum_{i=1}^n \frac{1}{\beta_i} \ln \beta_i,$$

that is

$$a_n = e^{\left(\sum_{i=1}^n \frac{1}{\beta_i} \ln \beta_i\right)}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\lim_{t\to\infty} \frac{\ln t}{\sqrt{t}} = 0$, there exists a constant K > 0 such that

$$\ln t \le K\sqrt{t}, \quad \forall t > 0. \tag{1.23}$$

Using that $\beta_i \geq \beta_0 \gamma^i$ with $\gamma > 1$ and the inequality (1.23), we have

$$\sum_{i=1}^{\infty} \frac{1}{\beta_i} \ln \beta_i \le K \sum_{i=0}^{\infty} \frac{1}{\beta_i} \sqrt{\beta_i} = K \sum_{i=0}^{\infty} \frac{1}{\sqrt{\beta_i}} \le \frac{K}{\sqrt{\beta_0}} \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{\gamma}}\right)^i < \infty.$$

Therefore a_n is convergent, namely,

$$\prod_{j=1}^{+\infty} \beta_j^{\frac{1}{\beta_j}} < \infty.$$

Moreover,

$$\lim_{n \to \infty} \prod_{j=0}^n \sigma_j = \lim_{n \to \infty} \prod_{j=0}^n \gamma \frac{\beta_j}{\beta_{j+1}} = \lim_{n \to \infty} \gamma^{n+1} \frac{\beta_0}{\beta_{n+1}} = \lim_{n \to \infty} \frac{\gamma^{n+1}}{\beta_{n+1}} \frac{p_s^*}{\theta'} = \frac{p_s^* - \theta' p}{p_s^* - \theta'}.$$

Using these estimates and taking $n \to +\infty$ in (1.22) we obtain

$$\|u_M\|_{L^{\infty}(\mathbb{R}^N)} \le C \|u_M\|_{L^{p_s^*-\theta'}(\mathbb{R}^N)}^{\frac{p_s^*-\theta'p}{p_s^*-\theta'}} \le C \|u\|_{L^{p_s^*}(\mathbb{R}^N)}^{\frac{p_s^*-\theta'p}{p_s^*-\theta'}}$$
(1.24)

for some $C = C(s, p, N, ||f||_{\theta}) > 0$. Letting $M \to +\infty$, we conclude that $u \in L^{\infty}(\mathbb{R}^N)$, and

$$||u||_{L^{\infty}(\mathbb{R}^N)} \leq C||u||_{L^{p_s^*-\theta'}}^{\frac{p_s^*-\theta'}{p_s^*-\theta'}}.$$

Remark 1.3.3 The condition $2 \le q \le p$ is necessary only to prove the continuity of u. To prove the boundedness u we can assume that $1 \le q \le p$.

Chapter 2

Existence and regularity of solution to an equation involving the fractional (p,q) -Laplacian in \mathbb{R}^N

In this chapter, we first study the existence and regularity of nontrivial weak solutions for the following nonlinear elliptic problem of fractional (p, q)-Laplacian type involving the critical Sobolev exponent

$$\begin{cases} (-\Delta_p)^s u + (-\Delta_q)^s u = |u|^{p_s^* - 2} u + \lambda g(x)|u|^{r-2} u & \text{in } \mathbb{R}^N \\ u(x) \ge 0 & x \in \mathbb{R}^N. \end{cases}$$
(2.1)

where $s \in (0,1)$, $1 < q \leq p < r < p_s^*$, N > sp, $p_s^* = \frac{Np}{N-sp}$, $\lambda > 0$ is a parameter and $(-\Delta)_p^s + (-\Delta)_q^s$ is the fractional (p,q)-Laplacian operator.

The function g will satisfy some hypothesis amoong the following

- (g_1) g is integrable and $g \in L^t(\mathbb{R}^N)$, with $t = \frac{p_s^*}{p_s^* r}$;
- (g_2) there is an open set $\Omega_g \subset \mathbb{R}^N$ and $\alpha_0 > 0$ such that $g(x) \ge \alpha_0 > 0, \forall x \in \Omega_g;$
- $(g_3) g \in L^{\infty}(\mathbb{R}^N).$

About existence our main results are as follows.

Theorem 2.0.1 Assume that $g : \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions (g_1) and (g_2) .

(i) If $1 < q \leq p < r < p_s^*$, then there exists $\lambda^* > 0$ such that, for any $\lambda > \lambda^*$, problem (2.1) has at least one nontrivial and nonnegative weak solution in \mathcal{W} .

(ii) If
$$1 < q < \frac{N(p-1)}{N-s} < p \le \max\left\{p, p_s^* - \frac{q}{p-1}\right\} < r < p_s^*$$
, $N > p^2s$, then (2.1) has a non-trivial weak solution in \mathcal{W} for any $\lambda > 0$.

In a second moment, we will use the regularity theory of Chapter 1 to show that the solutions of (2.1) are of class $C_{loc}^{1,\alpha}$. More precisely, we have

Theorem 2.0.2 Let $2 < q \leq p < r < p_s^*$, $\lambda > 0$ and 0 < s < 1, be such that N > sp. Assume that (g_1) and (g_3) is holds. If $u \in D^{s,p}(\mathbb{R}^N) \cap D^{s,p}(\mathbb{R}^N)$ is a solution of (2.1), then $u \in L^{\infty}(\mathbb{R}^N) \cap C^{\alpha}_{loc}(\mathbb{R}^N)$.

2.1 Preliminaries

Let $1 < m < \frac{N}{s}$ and $u : \mathbb{R}^N \to \mathbb{R}$ measurable function. The quantity

$$[u]_{s,m} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^m}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y\right)^{\frac{1}{m}}$$

defines a uniformly convex norm on the reflexive Banach space

$$D^{s,m}(\mathbb{R}^N) = \{ u \in L^{m^*_s}(\mathbb{R}^N); [u]_{s,m} < \infty \} \text{ with } m^*_s = \frac{Nm}{N-sm}$$

In our context, we denote by $||.||_t$ the norm of $L^t(\mathbb{R}^N)$ for any $t \in (1, \infty)$ and the Sobolev constant given by

$$S = \inf_{u \in D^{s,m}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,m}^m}{||u||_{m^*}^m}$$
(2.2)

is the associated Rayleigh quotient.

The constant S is well defined and is positive by the fractional Sobolev inequality. Very recently, Brasco, Mosconi and Squassina obtained in [10] that there exists a radially symmetric nonnegative decreasing minimizer U = U(r) for S. The authors also showed that U satisfies

$$[U]_{s,m}^{m} = ||U||_{m_{s}^{*}}^{m_{s}^{*}} = S^{\frac{N}{sp}}.$$
(2.3)

Moreover, for any $\varepsilon > 0$ the function

$$U_{\varepsilon}(x) = \frac{1}{\varepsilon^{(N-sm)/m}} U(|x|/\varepsilon)$$

is also a minimizer for S satisfying (2.3).

Let $\mathcal{W} := D^{s,p}(\mathbb{R}^N) \cap D^{s,q}(\mathbb{R}^N)$ endowed with the norm

$$||u||_{\mathcal{W}} := [u]_{s,p} + [u]_{s,q}.$$

The following lemma can be found in [36, lemma 4.8].

Lemma 2.1.1 Let $\Omega \subset \mathbb{R}^N$, $1 and <math>\{u_n\} \subset L^p(\Omega)$ be a bounded sequence converging to u almost everywhere in Ω . Then $u_n \rightharpoonup u$ in $L^p(\Omega)$.

In the sequel will prove a result related to the compactness.

Lemma 2.1.2 Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in \mathcal{W} . Then there exists $u \in \mathcal{W}$ such that up to a subsequence, $u_n(x) \to u(x)$ a.e. in \mathbb{R}^N . Moreover, for $m \in \{p,q\}$ we have

$$\lim_{n \to \infty} \left[u_n - u \right]_{s,m}^m = \lim_{n \to \infty} \left(\left[u_n \right]_{s,m}^m - \left[u \right]_{s,m}^m \right).$$

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{W} such that,

$$[u_n]_{\mathcal{W}} = [u_n]_{s,p} + [u_n]_{s,q} \le C, \quad \forall n \in \mathbb{N}.$$
(2.4)

It is standart to show that \mathcal{W} is an uniformly convex Banach space (\mathcal{W} is reflexive Banach space). Thus there exists $u \in \mathcal{W}$ such that $u_n \rightharpoonup u$ in \mathcal{W} .

On the other hand, given $\Omega_0 \subset \mathbb{R}^N$ compact, using Hölder's inequality we have

$$\begin{split} \int_{\Omega_0} |u_n|^p \mathrm{d}x + \int_{\Omega_0} \int_{\Omega_0} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} \mathrm{d}x \mathrm{d}y &\leq |\Omega_0|^{\frac{N}{sp}} \left(\int_{\mathbb{R}^N} |u_n|^{p_s^*} \mathrm{d}x \right)^{\frac{r}{p_s^*}} + [u_n]_{s,p}^p \\ &\leq \left(\frac{|\Omega_0|^{\frac{N}{sp}}}{S} + 1 \right) [u_n]_{s,p}^p \leq C. \end{split}$$

Therefore $u_n \in W^{s,p}(\Omega_0)$ for each $n \in \mathbb{N}$ and all Ω_0 compact. Since the embedding $W^{s,p}(\Omega_0) \hookrightarrow L^p(\Omega_0)$ is compact, it follows that the embedding $\mathcal{W} \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ is compact. Hence up to a subsequence, $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ and consequently $u_n(x) \to u(x)$ a.e. in \mathbb{R}^N .

In the second part of the lemma, let $m \in \{p, q\}$ and define

$$\mathcal{U}_n(x,y) = \frac{u_n(x) - u_n(y)}{|x - y|^{\frac{N}{m} + s}} \in L^m(\mathbb{R}^N \times \mathbb{R}^N)$$

By the first part of the lemma

$$\mathcal{U}_n(x,y) \to \mathcal{U}(x,y) = \frac{u(x) - u(y)}{|x - y|^{\frac{N}{m} + s}}, \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

Since (u_n) is bounded in \mathcal{W} it follows that $(\mathcal{U}_n)_{n \in \mathbb{N}}$ is bounded in $L^m(\mathbb{R}^N \times \mathbb{R}^N)$, Lemma 2.1.1 then implies that

$$\mathcal{U}_n \rightharpoonup \mathcal{U} \text{ in } L^m(\mathbb{R}^N \times \mathbb{R}^N).$$

By applying the Brezis Lieb Lemma we complete the proof. \blacksquare

Throughout the text, for any $1 \le m < \infty$, we will constantly use the notation,

$$J_m(t) = |t|^{m-2}t$$
, for all $t \in \mathbb{R}$.

2.2 Mountain Pass Geometry

In this section we will use the mountain pass theorem to show the existence of a solution to (2.1). To commodity of the reader, let us recall the mountain pass theorem.

Theorem 2.2.1 Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$. Suppose that $\Phi(0) = 0$ and that there exist $\beta, \rho > 0$ and $x_1 \in X \setminus \overline{B}_{\rho}(0)$ such that

- (i) $\Phi(u) \ge \beta$ for all $u \in X$ with $||u||_X = \rho$;
- (*ii*) $\Phi(x_1) < \beta$.

There exists a sequence $(u_n) \subset X$ satisfying

$$\Phi(u_n) \to c \text{ and } \Phi'(u_n) \to 0$$

where c is the minimax level, defined by

$$c := \inf \left\{ \max_{t \ge 0} \Phi(\gamma(t)) : \gamma \in C([0,1],\mathbb{R}), \gamma(0) = 0 \quad and \quad \gamma(1) = x_1 \right\}.$$

For a proof and applications of this theorem, see [30, 3, 46, 48].

Definition 2.2.2 We say that $u \in W$ is a weak solution to the problem (2.1) if

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{J_p(u(x) - u(y))}{|x - y|^{N + sp}} + \frac{J_q(u(x) - u(y))}{|x - y|^{N + sq}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y$$
$$= \int_{\mathbb{R}^N} (u^+)^{p_s^* - 2} u^+ \varphi \mathrm{d}x + \lambda \int_{\mathbb{R}^N} g(u^+)^{r - 2} u^+ \varphi \mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{W}.$$

In this way, the Euller-Lagrange functional for (2.1) is given by,

$$I_{\lambda}(u) = \frac{1}{p} \left[u \right]_{s,p}^{p} + \frac{1}{q} \left[u \right]_{s,q}^{q} - \frac{1}{p_{s}^{*}} \int_{\mathbb{R}^{N}} (u^{+})^{p_{s}^{*}} \mathrm{d}x - \frac{\lambda}{r} \int_{\mathbb{R}^{N}} g(u^{+})^{r} \mathrm{d}x,$$
(2.5)

where $u^{\pm} = \max\{\pm u, 0\}$ and $u \in \mathcal{W}$.

Lemma 2.2.3 Let (g_1) hold. Then I_{λ} is well defined, for all $\lambda > 0$, $I_{\lambda} \in C^1(\mathcal{W}, \mathbb{R})$ and for all $u, \varphi \in \mathcal{W}$ we have

$$I_{\lambda}(u)\varphi = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{J_p(u(x) - u(y))}{|x - y|^{N + sp}} + \frac{J_q(u(x) - u(y))}{|x - y|^{N + sq}} \right) (\varphi(x) - \varphi(y)) \mathrm{d}x \mathrm{d}y - \int_{\mathbb{R}^N} (u^+)^{p_s^* - 2} u^+ \varphi \mathrm{d}x - \lambda \int_{\mathbb{R}^N} g(u^+)^{r - 2} u^+ \varphi \mathrm{d}x.$$
(2.6)

Proof. The proof of this fact can be found in [43], but we give an idea of it. Thanks to the (g_1) and the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$, we have I_{λ} is well defined, Gâteaux-differentiable in \mathcal{W} and its Gâteaux-derivate is given by (2.6).

Now, let $u_n \to u \in \mathcal{W}$ as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ a.e. in \mathbb{R}^N . Then for $m \in \{p, q\}$ the sequence

$$\left\{\frac{|u_n(x) - u_n(y)|^{m-2}(u_n(x) - u_n(y))}{|x - y|^{(N+sm)/m'}}\right\}_{n \in \mathbb{N}} \text{ is bounded in } L^{m'}(\mathbb{R}^{2N})$$
and

$$\mathcal{U}_n(x,y) := \frac{|u_n(x) - u_n(y)|^{m-2}(u_n(x) - u_n(y))}{|x - y|^{(N+sm)/m'}} \longrightarrow \mathcal{U}(x,y) := \frac{|u(x) - u(y)|^{m-2}(u(x) - u(y))}{|x - y|^{(N+sm)/m'}}.$$

a.e. in \mathbb{R}^{2N}

Thus, the Brezis-Lieb Lemma yields

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{U}_n(x,y) - \mathcal{U}(x,y)|^{m'} dx dy$$
$$= \lim_{n \to \infty} \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{U}_n(x,y)|^{m'} dx dy - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{U}(x,y)|^{m'} dx dy \right]$$
$$= \lim_{n \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \frac{|u_n(x) - u_n(y)|^m}{|x - y|^{N+sm}} - \frac{|u(x) - u(y)|^m}{|x - y|^{N+sm}} \right\} dx dy.$$
(2.7)

The fact that $u_n \to u$ strongly in \mathcal{W} implies that

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \frac{|u_n(x) - u_n(y)|^m}{|x - y|^{N + sm}} - \frac{|u(x) - u(y)|^m}{|x - y|^{N + sm}} \right\} \mathrm{d}x \mathrm{d}y = 0.$$
(2.8)

From (2.7) it follows that $\mathcal{U}_n \to \mathcal{U}$ in $L^{m'}(\mathbb{R}^{2N})$.

Given $\varphi \in \mathcal{W}$ using (1.3), (g_1) and the dominate convergence theorem we can see that

$$\int_{\mathbb{R}^N} (u_n^+)^{p_s^* - 1} \varphi \mathrm{d}x \to \int_{\mathbb{R}^N} (u^+)^{p_s^* - 1} \varphi \mathrm{d}x \tag{2.9}$$

and

$$\int_{\mathbb{R}^N} g(x)(u_n^+)^{r-1} \varphi \mathrm{d}x \to \int_{\mathbb{R}^N} g(x)(u^+)^{r-1} \varphi \mathrm{d}x.$$
(2.10)

Note that $\frac{\varphi(x) - \varphi(y)}{|x - y|^{(N+sm)/m}} \in L^m(\mathbb{R}^{2N})$, for $m \in \{p, q\}$. From (2.8), (2.9) and (2.10) we have

$$(I'_{\lambda}(u_n) - I'_{\lambda}(u))\varphi \to 0 \text{ as } n \to \infty.$$

Therefore,

$$||I'_{\lambda}(u_n) - I'_{\lambda}(u)||_{\mathcal{W}^*} = \sup_{\varphi \in \mathcal{W}, \varphi \neq 0} \frac{|(I'_{\lambda}(u_n) - I'_{\lambda}(u))\varphi|}{\|\varphi\|_{\mathcal{W}}}$$
$$= \sup_{\varphi \in \mathcal{W}, \|\varphi\|_{\mathcal{W}} \le 1} |(I'_{\lambda}(u_n) - I'_{\lambda}(u))\varphi| \to 0$$

Thus, $I'_{\lambda}(u_n) \to I'_{\lambda}(u)$, as $n \to \infty$, that is, $I'_{\lambda} \in C^1(\mathcal{W}, \mathbb{R})$.

In the sequel, we show that, if $1 < q \leq p < r < p_s^*$ and g satisfy the conditions (g_1) , (g_2) then I_{λ} satisfies the Mountain Pass geometry.

Lemma 2.2.4 There exist $\beta, \rho > 0$ and $u_0 \in W$ satisfying:

- (i) $||u_0||_{\mathcal{W}} > \rho$ and $I_{\lambda}(u_0) < 0$,
- (ii) $I_{\lambda}(u) \geq \beta$ for any $u \in \mathcal{W}$ with $||u||_{\mathcal{W}} = \rho$.

Proof. First note that by Hölder's inequality, with $t = \frac{p_s^*}{p_s^* - r}$

$$\frac{\lambda}{r} \int_{\mathbb{R}^N} g(u^+)^r \mathrm{d}x \le \frac{\lambda}{r} ||g||_{L^t(\mathbb{R}^N)} [u]_{s,p}^r \le C_1 ||u||_{\mathcal{W}}^r$$
(2.11)

and (1.3) yields,

$$\frac{1}{p_s^*} \int_{\mathbb{R}^N} (u^+)^{p_s^*} \mathrm{d}x \le \frac{1}{p_s^* S^{\frac{p_s^*}{p}}} [u]_{s,p}^{p_s^*} \le C_2 ||u||_{\mathcal{W}}^{p_s^*}.$$
(2.12)

Let us suppose $[u]_{s,q} \leq ||u||_{\mathcal{W}} \leq 1$. Since $q \leq p < r < p_s^*$ it follows that

$$I_{\lambda}(u) \geq \frac{1}{p} \left([u]_{s,p}^{p} + [u]_{s,q}^{q} \right) - C_{2} ||u||_{\mathcal{W}}^{p_{s}^{*}} - C_{1} ||u||_{\mathcal{W}}^{r}$$

$$\geq \frac{1}{p} \left([u]_{s,p}^{p} + [u]_{s,q}^{p} \right) - C_{2} ||u||_{\mathcal{W}}^{p_{s}^{*}} - C_{1} ||u||_{\mathcal{W}}^{r}$$

$$\geq \frac{1}{2^{p-1}p} ||u||_{\mathcal{W}}^{p} - C_{2} ||u||_{\mathcal{W}}^{p_{s}^{*}} - C_{1} ||u||_{\mathcal{W}}^{r}.$$

Thus, there are $\rho, \beta > 0$ such that $I_{\lambda}(u) \ge \beta$, for all $u \in \mathcal{W}$, with $||u||_{\mathcal{W}} = \rho$.

Now, let $v_0 \in \mathcal{W} \setminus \{0\}$ be such that $v_0 \ge 0$. Then, for any t > 0, one has

$$I_{\lambda}(tv_0) = \frac{t^p}{p} \left[v_0 \right]_{s,p}^p + \frac{t^q}{q} \left[v_0 \right]_{s,q}^q - \frac{t^r}{r} \int_{\mathbb{R}^N} g v_0^r \mathrm{d}x - \frac{t^{p_s^*}}{p_s^*} \int_{\mathbb{R}^N} v_0^{p_s^*} \mathrm{d}x$$

Since $1 < q \le p < r < p_s^*$, it follows that $I_{\lambda}(tu) \longrightarrow -\infty$ as $t \longrightarrow \infty$.

Consequently, there exists $u_0 = t_0 v_0 \in \mathcal{W}$ such that $||u_0||_{\mathcal{W}} > \rho$ and $I_{\lambda} < 0$.

Lemma 2.2.5 Let $(u_n) \subset W$ be a Palais-Smale sequence. Then $(u_n)_{n \in \mathbb{N}}$ is bounded in W.

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be such that

$$I_{\lambda}(u_n) \leq d_0 \text{ and } I'_{\lambda}(u_n) \to 0 \text{ in } \mathcal{W}^*.$$

Thus, for all n large

$$d_{0} + d_{1}||u_{n}||_{\mathcal{W}} \geq I_{\lambda}(u_{n}) - \frac{1}{r}I_{\lambda}'(u_{n})$$

= $\left(\frac{1}{p} - \frac{1}{r}\right)[u_{n}]_{s,p}^{p} + \left(\frac{1}{q} - \frac{1}{r}\right)[u_{n}]_{s,q}^{q} + \left(\frac{1}{r} - \frac{1}{p_{s}^{*}}\right)\int_{\mathbb{R}^{N}}|u_{n}|^{p_{s}^{*}}dx$
$$\geq \left(\frac{1}{p} - \frac{1}{r}\right)\left([u_{n}]_{s,p}^{p} + [u_{n}]_{s,q}^{q}\right).$$

That is, for all n large, we have

$$k_0(1 + [u_n]_{\mathcal{W}}) \ge k_1 [u_n]_{s,p}^p + k_2 [u_n]_{s,q}^q,$$

where k_0, k_1 and k_2 are positive constants that do not depend on n. Suppose $||u_n||_{\mathcal{W}} \to \infty$. Then we have three cases to consider:

- 1. $[u_n]_{s,p}^p \to \infty$ and $[u_n]_{s,q}^q \to \infty$.
- 2. $[u_n]_{s,p}^p \to \infty$ and $[u_n]_{s,q}^q$ is bounded.
- 3. $[u_n]_{s,p}^p$ is bounded and $[u_n]_{s,q}^q \to \infty$.

The first case cannot occur. Indeed, it implies that $[u_n]_{s,p}^p > [u_n]_{s,p}^q$ for all n large, and thus

$$k_0(1 + ||u_n||_{\mathcal{W}}) \ge k_1 [u_n]_{s,p}^p + k_2 [u_n]_{s,q}^q \ge k_1 [u_n]_{s,p}^q + k_2 [u_n]_{s,q}^q \ge \frac{k_3}{2^{q-1}} ||u_n||_{\mathcal{W}}^q$$

which contradicts the fact that $||u_n||_{\mathcal{W}} \to \infty$.

If the second case occurs, we have, for all n large,

$$k_0 \left(1 + [u_n]_{s,p} + [u_n]_{s,q} \right) \ge k_1 \left[u_n \right]_{s,p}^p + k_2 \left[u_n \right]_{s,q}^q \ge k_1 \left[u_n \right]_{s,p}^p$$

and hence we arrive at the absurd

$$0 < \frac{k_1}{k_0} \le \lim_{n \to \infty} \left(\frac{1}{[u_n]_{s,p}^p} + \frac{1}{[u_n]_{s,p}^{p-1}} + \frac{[u_n]_{s,q}}{[u_n]_{s,p}^p} \right) = 0.$$

Proceeding as in the second case, one can check that the third case cannot happen as well.

2.3 Estimate for the energy level

For each $\lambda > 0$ we denote by,

$$\bar{c}_{\lambda} := \inf_{u \in \mathcal{W} \setminus \{0\}} \max_{t \ge 0} I_{\lambda}(tu).$$
(2.13)

Lemma 2.3.1 Let (g_1) and (g_2) hold. If $1 < q \le p < r < p_s^*$, then there exists $\lambda^* > 0$ such that $0 < \overline{c}_{\lambda} < \frac{s}{N} S^{\frac{n}{sp}}$ for all $\lambda > \lambda^*$.

Proof. It follows from Lemma 2.2.4 that $I_{\lambda}(u) \geq \beta > 0$ whenever $||u||_{\mathcal{W}} = \rho$. This fact implies that $\overline{c}_{\lambda} \geq \beta > 0$.

We recall that $\Omega_g = \{x \in \mathbb{R}^N; g(x) > 0\}$. Let $u_0 \in \mathcal{W} \setminus \{0\}$ with support in Ω_g such that $u_0 \ge 0$ and $||u_0||_{p_s^*} = 1$. Since

$$I_{\lambda}(tu_0) = \frac{t^p}{p} \left[u_0 \right]_{s,p}^p + \frac{t^q}{q} \left[u_0 \right] s, q^q - \frac{\lambda t^r}{r} \int_{\mathbb{R}^N} g u_0^r \mathrm{d}x - \frac{t^{p_s^*}}{p_s^*}, \ t > 0$$

we can see that $I_{\lambda}(tu_0) \to -\infty$ as $t \to \infty$ and that $I_{\lambda}(tu_0) \to 0$ as $t \to 0^+$. These facts imply that there exists $t_{\lambda} > 0$ such that

$$\max_{t \ge 0} I_{\lambda}(tu_0) = I_{\lambda}(t_{\lambda}u_0).$$

Hence

$$0 = \frac{d}{dt} [I_{\lambda}(tu_{0})]_{t=t_{\lambda}}$$

= $t_{\lambda}^{p-1} [u_{0}]_{s,p}^{p} + t_{\lambda}^{q-1} [u_{0}]_{s,q}^{q} - \lambda t_{\lambda}^{r-1} \int_{\mathbb{R}^{N}} gu_{0}^{r} \mathrm{d}x - t_{\lambda}^{p_{s}^{*}-1}$

so that

$$0 < \lambda \int_{\mathbb{R}^N} g u_0^r \mathrm{d}x = \frac{\left[u_0\right]_{s,p}^p}{t_\lambda^{r-p}} + \frac{\left[u_0\right]_{s,q}^q}{t_\lambda^{r-q}} - t_\lambda^{p_s^*-r}, \text{ for all } \lambda > 0.$$

It follows that $t_{\lambda} \to 0$ as $\lambda \to \infty$. Since $I_{\lambda}(t_{\lambda}u_0) \to 0$ as $t_{\lambda} \to 0^+$, there exists $\lambda^* > 0$ such that

$$\max_{t \ge 0} I_{\lambda}(tu_0) = I_{\lambda}(t_{\lambda}u_0) < \frac{s}{N} S^{\frac{N}{sp}}, \text{ for all } \lambda > \lambda^*.$$

Since $\overline{c}_{\lambda} \leq \max_{t \geq 0} I_{\lambda}(tu_0)$, we conclude that

$$\overline{c}_{\lambda} < \frac{s}{N} S^{\frac{N}{sp}}$$
 for all $\lambda > \lambda^*$

Now, we use the Appendix A.1 and the condition

$$N > p^2 s \text{ and } 1 < q < \frac{N(p-1)}{N-s} < p \le \max\left\{p, p_s^* - \frac{q}{p-1}\right\} < r < p_s^*.$$
(2.14)

to show that $\overline{c}_{\lambda} < \frac{s}{N}S^{\frac{N}{sp}}$, for all $\lambda > 0$.

Lemma 2.3.2 Assume that (2.14), (g₁) and (g₂) hold. Then, $\overline{c}_{\lambda} \in \left(0, \frac{s}{N}S^{\frac{n}{sp}}\right)$, for any $\lambda > 0$, where \overline{c}_{λ} is defined in (2.13).

Proof. We can assume without loss of generality that $0 \in \Omega_g$. Fix R > 0 such that $B_{\theta R}(0) \subset \Omega_g$ and denote $u_{\varepsilon} = u_{\varepsilon,R}$ as in Appendix A.1. Define $v_{\varepsilon} = \frac{u_{\varepsilon}}{||u_{\varepsilon}||_{p_s^*}}$, so that $||v_{\varepsilon}||_{p_s^*} = 1$ and $v_{\varepsilon}(x) = 0$ for all $x \in \mathbb{R}^N \setminus \Omega_q$.

Consider the function $\psi : [0, \infty) \to \mathbb{R}$ given by

$$\psi(t) := I_{\lambda}(tv_{\varepsilon})$$

= $\frac{t^p}{p} [v_{\varepsilon}]_{s,p}^p + \frac{t^q}{q} [v_{\varepsilon}]_{s,q}^q - \frac{t^{p_s^*}}{p_s^*} - \frac{\lambda t^r}{r} \int_{\mathbb{R}^N} gv_{\varepsilon}^r \mathrm{d}x.$

In this way, ψ it is continuous, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = -\infty$. Then there exists $t_{\varepsilon} > 0$ such that,

$$\psi(t_{\varepsilon}) = \sup_{t \ge 0} \psi(t) = \sup_{t \ge 0} I_{\lambda}(tv_{\varepsilon}).$$

Thus, $\psi'(t_{\varepsilon}) = 0$, that is

$$t_{\varepsilon}^{p-1} \left[v_{\varepsilon} \right]_{s,p}^{p} + t_{\varepsilon}^{q-1} \left[v_{\varepsilon} \right]_{s,q}^{q} - t_{\varepsilon}^{p_{s}^{*}-1} - \lambda t_{\varepsilon}^{r-1} \int_{\mathbb{R}^{N}} g v_{\varepsilon}^{r} \mathrm{d}x = 0.$$

$$(2.15)$$

Using the condition (g_2) we obtain

$$t_{\varepsilon}^{p_{s}^{*}-1} < t_{\varepsilon}^{p-1} \left[v_{\varepsilon} \right]_{s,p}^{p} + t_{\varepsilon}^{q-1} \left[v_{\varepsilon} \right]_{s,q}^{q}$$

As $q < \frac{N(p-1)}{N-s}$, combining (A.6), Lemma A.1.2 and (A.9) we get

$$[v_{\varepsilon}]_{s,p}^{p} \leq S + O(\varepsilon^{\frac{N-sp}{p}}) \quad \text{and} \quad [v_{\varepsilon}]_{s,q}^{q} \leq O(\varepsilon^{\frac{q(N-sp)}{p^{2}}}).$$
 (2.16)

Using (2.16) we have

$$t_{\varepsilon}^{p_{s}^{*}-p} < [v_{\varepsilon}]_{s,p}^{p} + t_{\varepsilon}^{q-p} [v_{\varepsilon}]_{s,q}^{q} \\ \leq S + O(\varepsilon^{\frac{N-sp}{p}}) + t_{\varepsilon}^{q-p} O(\varepsilon^{\frac{q(N-sp)}{p^{2}}})$$

Therefore for any $\overline{\varepsilon} > 0$ small enough, there exists $t^0_{\overline{\varepsilon}} > 0$ such that $t_{\varepsilon} \leq t^0_{\overline{\varepsilon}}, \forall \varepsilon \leq \overline{\varepsilon}.$

From (2.15), (g_1) and Hölder inequality, we have

$$t_{\varepsilon}^{p-1} [v_{\varepsilon}]_{s,p}^{p} < t_{\varepsilon}^{p_{s}^{*}-1} + \lambda t_{\varepsilon}^{r-1} \int_{\mathbb{R}^{N}} g v_{\varepsilon}^{r} \mathrm{d}x$$
$$\leq t_{\varepsilon}^{p_{s}^{*}-1} + \lambda ||g||_{\gamma} t_{\varepsilon}^{r-1} ||v_{\varepsilon}||_{r}.$$
(2.17)

Moreover, as $||v_{\varepsilon}||_{p_s^*} = 1$, by Sobolev inequality

$$S = S||v_{\varepsilon}||_{p_s^*}^p \le [v_{\varepsilon}]_{s,p}^p.$$

Hence, from (2.17)

$$S \le \left[v_{\varepsilon}\right]_{s,p}^{p} < t_{\varepsilon}^{p_{s}^{*}-p} + \lambda ||g||_{\gamma} t_{\varepsilon}^{r-p} ||v_{\varepsilon}||_{r}.$$

$$(2.18)$$

Using (2.16)-(2.18) there exist T > 0 such that for any $\varepsilon > 0$, $t_{\varepsilon} \ge T$.

Let $f(t) = \frac{t^p}{p} [v_{\varepsilon}]_{s,p}^p - \frac{t^{p_s^*}}{p_s^*}$, then f attains its maximum at $t_0 = [v_{\varepsilon}]_{s,p}^{\frac{p}{p_s^* - p}}$. Note also that $N > p^2 s > sp$ implies N(p-1) < p(N-sp) and thus $\frac{N(p-1)}{N-sp} .$

Hence, for $\varepsilon \leq \overline{\varepsilon}$, applying Lemma A.1.2 for $q < \frac{N(p-1)}{N-sp} < r$ and using (g_2) we obtain

$$\begin{split} \psi(t_{\varepsilon}) &= f(t_{\varepsilon}) + \frac{t_{\varepsilon}^{q}}{q} \left[v_{\varepsilon} \right]_{s,q}^{q} - \lambda \frac{t_{\varepsilon}^{r}}{r} \int_{\mathbb{R}^{N}} g v_{\varepsilon}^{r} \mathrm{d}x \\ &= f(t_{\varepsilon}) + \frac{t_{\varepsilon}^{q}}{q} \left[v_{\varepsilon} \right]_{s,q}^{q} - \lambda \frac{t_{\varepsilon}^{r}}{r} \int_{\mathbb{R}^{N}} g v_{\varepsilon}^{r} \mathrm{d}x \\ &\leq f(t_{0}) + \frac{(t_{\varepsilon}^{0})^{q}}{q} \left[v_{\varepsilon} \right]_{s,q}^{q} - \lambda \frac{T^{r}}{r} \alpha_{0} \| v_{\varepsilon} \|_{r}^{r} \\ &\leq \frac{s}{N} S^{\frac{N}{sp}} + c_{1} \varepsilon^{\frac{N-sp}{p}} + c_{2} \varepsilon^{\frac{q(N-sp)}{p^{2}}} - c_{3} \varepsilon^{\frac{(p-1)}{p}(N - \frac{r(N-sp)}{p})}, \end{split}$$

with $c_1, c_2, c_3 > 0$ (independent of ε). Since, q < p and $\max\left\{p, p_s^* - \frac{q}{p-1}\right\} < r$ we have

$$\frac{N - sp}{p} > \frac{q(N - sp)}{p^2} > \frac{(p - 1)}{p} (N - \frac{r(N - sp)}{p})$$

from which we can choose $\varepsilon > 0$ small so that $\psi(t_{\varepsilon}) < \frac{s}{N}S^{\frac{N}{s_p}}$.

Hence,
$$\bar{c}_{\lambda} = \inf_{u \in \mathcal{W}/\{0\}} \sup_{t \ge 0} I_{\lambda}(tu) \le \sup_{t \ge 0} I_{\lambda}(tv_{\varepsilon}) = \psi(t_{\varepsilon}) < \frac{s}{N} S^{\frac{N}{sp}}.$$

2.4 Existence of Solution

Proof of Theorem 2.0.1. We know that the functional I_{λ} has the structure of the mountain pass theorem, and from Lemma 2.2.5 its (PS) sequence is bounded. Let $(u_n) \subset \mathcal{W}$ be a (PS) sequence satisfying

$$I_{\lambda}(u_n) \to c_{\lambda} \text{ and } I'_{\lambda}(u_n) \to 0,$$

where c_{λ} is the minimax level of the mountain pass theorem associated with I_{λ} . Using the same arguments of [48, Theorem 4.2] (see also [44]) we conclude that $c_{\lambda} \leq \overline{c}_{\lambda}$.

Since that (u_n) is bounded in \mathcal{W} , up to a subsequence one has $u_n \rightharpoonup u$ in \mathcal{W} and applying Lemma 2.1.2 we have $u_n \rightarrow u$ a.e. in \mathbb{R}^N .

To prove case (i) in Theorem 2.0.1, we will use Lemma 2.3.1 to get $\lambda^* > 0$ such that $0 < c_{\lambda} \leq \overline{c}_{\lambda} < \frac{s}{N}S^{\frac{N}{sp}}$ for all $\lambda > \lambda^*$, while for case (ii) we use Lemma 2.3.2 to get $0 < c_{\lambda} \leq \overline{c}_{\lambda} < \frac{s}{N}S^{\frac{N}{sp}}$ for all $\lambda > 0$.

Claim: Let $u_n^- = \max\{-u_n, 0\}$. Then $u_n^- \to 0$ in \mathcal{W} . In particular, $u_n^+ \to u$ a.e. in \mathbb{R}^N . Indeed, since $I'_{\lambda}(u_n)u_n^- \to 0$,

$$\begin{split} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{J_p(u_n(x) - u_n(y))}{|x - y|^{N + sp}} + \frac{J_q(u_n(x) - u_n(y))}{|x - y|^{N + sq}} \right) (u_n^-(x) - u_n^-(y)) \mathrm{d}x \mathrm{d}y \\ &= \int_{\mathbb{R}^N} (u_n^+)^{p_s^* - 2} u_n^+ u_n^- \mathrm{d}x + \lambda \int_{\mathbb{R}^N} g(u_n^+)^{r - 2} u_n^+ u_n^- \mathrm{d}x + o(1) \end{split}$$

Using the elementary inequality, for $m = \{p, q\}$,

$$|v^{-}(x) - v^{-}(y)|^{m} \le J_{m}(v(x) - v(y))(v^{-}(x) - v^{-}(y)), \text{ for all } x, y \in \mathbb{R}^{N},$$

it follows that $u_n^- \to 0$ in \mathcal{W} . Hence $u_n^- \to 0$ a.e. in \mathbb{R}^N , in particular $u_n^+ \to u$ a.e. in \mathbb{R}^N .

Applying the Lemma 2.1.1 for (u_n^+) which is bounded in $L^{p_s^*}(\mathbb{R}^N)$ results

$$(u_n^+)^{p_s^*-1} \rightharpoonup u^{p_s^*-1} \text{ in } L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$$
 (2.19)

and

$$(u_n^+)^{r-1} \rightharpoonup u^{r-1} \text{ in } L^{\frac{p_s}{r-1}}(\mathbb{R}^N).$$
 (2.20)

Let $m \in \{p, q\}$ and denote

$$\mathcal{U}_n(x,y) = \frac{|u_n(x) - u_n(y)|^{m-2}(u_n(x) - u_n(y))}{|x - y|^{(N+sm)/m'}}$$

Since $u_n \to u$ a.e. in \mathbb{R}^N we have

$$\mathcal{U}_n(x,y) \longrightarrow \mathcal{U}(x,y) := \frac{|u(x) - u(y)|^{m-2}(u(x) - u(y))}{|x - y|^{(N+sm)/m'}}$$
 a.e. in \mathbb{R}^N .

Moreover,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\mathcal{U}_n(x,y)|^{m'} \mathrm{d}x \mathrm{d}y \le [u_n]_{s,m}^m$$

Thus (\mathcal{U}_n) is bounded in $L^{m'}(\mathbb{R}^{2N})$ for $m \in \{p,q\}$. By Lemma 2.1.1 yields

$$\mathcal{U}_n \rightharpoonup \mathcal{U} \text{ in } L^{m'}(\mathbb{R}^{2N}).$$
 (2.21)

Now note that given $\varphi \in \mathcal{W}$ we have $\frac{\varphi(x) - \varphi(y)}{|x - y|^{(N+sm)/m}} \in L^m(\mathbb{R}^{2N})$. Hence, it follows from (2.21) that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{U}_n(x,y) \frac{\varphi(x) - \varphi(y)}{|x - y|^{(N+sm)/m}} \mathrm{d}x \mathrm{d}y \longrightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{U}(x,y) \frac{\varphi(x) - \varphi(y)}{|x - y|^{(N+sm)/m}} \mathrm{d}x \mathrm{d}y.$$
(2.22)

It follows from (2.19), (2.20) and (2.22) that $I'_{\lambda}(u_n)\varphi \to I'_{\lambda}(u)\varphi$ and so u is a solution (weak) of (2.1).

We know that $u \ge 0$. It remains to verify that $u \ne 0$. Let

$$\lim_{n \to \infty} \left[u_n \right]_{s,p}^p =: a \ge 0 \text{ and } \lim_{n \to \infty} \left[u_n \right]_{s,q}^q =: b \ge 0.$$

and suppose that $u \equiv 0$.

Since $I'_{\lambda}(u_n)u_n \to 0$, we also have

$$[u_n]_{s,p}^p + [u_n]_{s,q}^q = \lambda \int_{\mathbb{R}^N} g(u_n^+)^r \mathrm{d}x + \int_{\mathbb{R}^N} (u_n^+)^{p_s^*} \mathrm{d}x + o(1)$$

Using the condition (g_1) and the weak convergence $(u_n^+)^r \rightharpoonup u^r$ in $L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)$ we get

$$\lambda \int_{\mathbb{R}^N} g(u_n^+)^r \mathrm{d}x \to 0.$$

Thus,

$$[u_n]_{s,p}^p = a + o(1), \quad [u_n]_{s,q}^q = b + o(1), \text{ and } ||u_n||_{p_s^*}^{p_s^*} = a + b + o(1)$$

By taking into account that $I_{\lambda}(u_n) \to c_{\lambda}$, we have

$$\frac{a}{p} + \frac{b}{q} - \frac{a+b}{p_s^*} = c_\lambda > 0.$$
 (2.23)

Hence

$$c_{\lambda} = a \left(\frac{1}{p} - \frac{N - sp}{Np}\right) + b \left(\frac{1}{q} - \frac{1}{p_s^*}\right)$$
(2.24)

$$\geq a \frac{s}{N}.\tag{2.25}$$

The equality (2.23) shows that $a + b \neq 0$. The definition of S show that

$$S(a+b)^{\frac{p}{p_s}} \le a \Rightarrow a > 0.$$

Thus

$$Sa^{\frac{p}{p_s^*}} \le S(a+b)^{\frac{p}{p_s^*}} \le a \Rightarrow a \ge S^{\frac{N}{sp}}.$$

Then by (2.25) we have

$$c_{\lambda}\frac{N}{s} \ge a \ge S^{\frac{N}{sp}}$$

which is a contradiction, because $c_{\lambda} < \frac{s}{N}S^{\frac{N}{sp}}$. This concludes the proof.

2.5 Regularity of solutions

In this section we will apply the regularity results proved in Section 1.2 to show that if $u \in \mathcal{W}$ satisfies (2.1), then $u \in L^{\infty}(\mathbb{R}^N) \cap C^{\alpha}_{loc}(\mathbb{R}^N)$.

Proof of Theorem 2.0.2. Due to Theorem 1.0.1, it is enough to show that $u \in L^{\theta(p_s^*-1)}(\mathbb{R}^N)$ for some $\theta > \frac{N}{sp}$. In fact, if this is true, then since $u \in L^{p_s^*}(\mathbb{R}^N)$ and $g \in L^t(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$, where t > 0 is give in (g_1) , we have by the Hölder's inequality for $\gamma = \frac{p_s^*-1}{r-1} > 1$ and $\gamma' = \frac{p_s^*-1}{p_s^*-r}$, that

$$\int_{\mathbb{R}^N} |g|^{\theta} u^{\theta(r-1)} \mathrm{d}x \le \|g\|_{L^{\infty}(\mathbb{R}^N)}^{\frac{\theta(p_s^*-1)-p_s^*}{p_s^*-1}} \|g\|_{L^t(\mathbb{R}^N)}^{\frac{p_s^*}{p_s^*-r}} \left(\int_{\mathbb{R}^N} u^{\theta(p_s^*-1)} \mathrm{d}x\right)^{\frac{r-1}{p_s^*-1}} < \infty,$$

for $\theta > \frac{N}{sp} > \frac{Np}{Np-N+sp}$ (which implies that $\frac{\theta(p_s^*-1)-p_s^*}{p_s^*-1} > 0$). Therefore $f = \lambda g u^{r-1} + u^{p_s^*-1} \in L^{\theta}(\mathbb{R}^N)$, with $\theta > \frac{N}{sp}$, which jointly with Theorem 1.0.1 give us $u \in L^{\infty}(\mathbb{R}^N)$.

Let us show that $u \in L^{\theta(p_s^*-1)}(\mathbb{R}^N)$ for some $\theta > \frac{N}{sp}$. Let M > 0 and $\beta > 1$, and denote as before $u_M = \min\{u, M\}$.

Define $h_{\beta,M}(t) = t(\min\{t, M\})^{\beta-1}$, thus,

$$h_{\beta,M}(t) = \begin{cases} t^{\beta}, & \text{se } t \leq M, \\ tM^{\beta-1}, & \text{se } t \geq M. \end{cases}$$

We have that $h_{\beta,M}$ is increasing, continuous and has bounded derivative. Hence, if $u \in \mathcal{W}$, then $h_{\beta,M}(u) \in \mathcal{W}$. Using the test function $\varphi = h_{\beta,M}(u)$ in (2.6) we get

$$\int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \left(\frac{J_{p}(u(x) - u(y))}{|x - y|^{N + sp}} + \frac{J_{q}(u(x) - u(y))}{|x - y|^{N + sq}} \right) (h_{\beta,M}(u(x)) - h_{\beta,M}(u(y))) \, \mathrm{d}x \mathrm{d}y \\
= \lambda \int_{\mathbb{R}^{N}} g u^{r-1} h_{\beta,M}(u) \, \mathrm{d}x + \int_{\mathbb{R}^{N}} u^{p_{s}^{*}-1} h_{\beta,M}(u) \, \mathrm{d}x \\
= \lambda \int_{\mathbb{R}^{N}} g u^{r} u_{M}^{\beta-1} \, \mathrm{d}x + \int_{\mathbb{R}^{N}} u^{p_{s}^{*}} u_{M}^{\beta-1} \, \mathrm{d}x =: J_{1} + J_{2},$$
(2.26)

where

$$J_1 := \lambda \int_{\mathbb{R}^N} g u^r u_M^{\beta - 1} \mathrm{d}x$$

and

$$J_2 := \int_{\mathbb{R}^N} u^{p_s^*} u_M^{\beta - 1} \mathrm{d}x.$$

To estimate the term J_2 we proceed as in [12]. Using Hölder's inequality and the fact that $u_M \leq u$, we obtain

$$J_{2} = \int_{\mathbb{R}^{N}} u^{p_{s}^{*}-p} u_{M}^{\beta-1} u^{p} dx$$

$$= \int_{\{u \leq K_{0}\}} u^{p_{s}^{*}} u_{M}^{\beta-1} dx + \int_{\{u \geq K_{0}\}} u^{p_{s}^{*}} u_{M}^{\beta-1} dx$$

$$\leq K_{0}^{\beta-1} \int_{\mathbb{R}^{N}} u^{p_{s}^{*}} dx + \left(\int_{\{u \geq K_{0}\}} u^{p_{s}^{*}} dx\right)^{\frac{p_{s}^{*}-p}{p_{s}^{*}}} \left(\int_{\mathbb{R}^{N}} u^{p_{s}^{*}} u_{M}^{(\beta-1)\frac{p_{s}^{*}}{p}} dx\right)^{\frac{p}{p_{s}^{*}}},$$
(2.27)

where $K_0 > 1$ is a constant that will be choosen later. To estimate J_1 , note that, since $u_M \leq u$ we get

$$\lambda \int_{\{u < K_0\}} g u^r u_M^{\beta - 1} \mathrm{d}x \le \lambda K_0^{\beta - 1} \|g\|_{L^t(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} u^{p_s^*} \mathrm{d}x \right)^{\frac{r}{p_s^*}}$$

and on the other hand, since $K_0 > 1$ and $r < p_s^*$, using that $g \in L^{\infty}(\mathbb{R}^N)$ and applying Hölder's inequality we obtain

$$\begin{split} \lambda \int_{\{u \ge K_0\}} g u^r u_M^{\beta-1} \mathrm{d}x &\leq \lambda \int_{\{u \ge K_0\}} g u^{p_s^*} u_M^{\beta-1} \mathrm{d}x \leq \lambda \|g\|_{L^{\infty}(\mathbb{R}^N)} \int_{\{u \ge K_0\}} u^{p_s^*} u_M^{\beta-1} \mathrm{d}x \\ &\leq C \left(\int_{\{u \ge K_0\}} u^{p_s^*} \mathrm{d}x \right)^{\frac{p_s^*-p}{p_s^*}} \left(\int_{\mathbb{R}^N} u^{p_s^*} u_M^{(\beta-1)\frac{p_s^*}{p}} \mathrm{d}x \right)^{\frac{p}{p_s^*}}. \end{split}$$

Then

$$J_1 \le CK_0^{\beta-1} ||u||_{L^{p_s^*}(\mathbb{R}^N)}^r + C\left(\int_{\{u \ge K_0\}} u^{p_s^*} \mathrm{d}x\right)^{\frac{p_s^* - p}{p_s^*}} \left(\int_{\mathbb{R}^N} u^{p_s^*} u_M^{(\beta-1)\frac{p_s^*}{p}} \mathrm{d}x\right)^{\frac{p}{p_s^*}}.$$

Let

$$G_{\beta,M}(t) = \int_0^t (h'_{\beta,M}(\tau))^{\frac{1}{p}} \mathrm{d}\tau \ge \frac{p}{\beta + p - 1} t(\min\{t, M\})^{\frac{\beta - 1}{p}}.$$
 (2.28)

By Sobolev inequality (2.2), and Lemma 1.1.3 we can see that

$$S\left(\int_{\mathbb{R}^N} |G_{s,M}(u(x))|^{p_s^*} \mathrm{d}x\right)^{\frac{p}{p_s^*}} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|G_{\beta,M}(u(x)) - G_{\beta,M}(u(y))|^p}{|x - y|^{N + sp}} \mathrm{d}x \mathrm{d}y$$
$$\leq J_1 + J_2.$$

Consequently,

$$S\left(\int_{\mathbb{R}^{N}} |G_{s,M}(u(x))|^{p_{s}^{*}} \mathrm{d}x\right)^{\frac{p}{p_{s}^{*}}} \leq C_{1}K_{0}^{\beta-1}\left(||u||_{L^{p_{s}^{*}}(\mathbb{R}^{N})}^{r}+||u||_{L^{p_{s}^{*}}(\mathbb{R}^{N})}^{p_{s}^{*}}\right)$$
$$+C_{2}\left(\int_{\{u\geq K_{0}\}} u^{p_{s}^{*}} \mathrm{d}x\right)^{\frac{p_{s}^{*}-p}{p_{s}^{*}}} \left(\int_{\mathbb{R}^{N}} u^{p_{s}^{*}} u_{M}^{(\beta-1)\frac{p_{s}^{*}}{p}} \mathrm{d}x\right)^{\frac{p}{p_{s}^{*}}}.$$

From (2.28), and the above inequality, we get

$$S\left(\frac{p}{p+\beta-1}\right)^{p} \left(\int_{\mathbb{R}^{N}} u^{p_{s}^{*}} u_{M}^{(\beta-1)\frac{p_{s}^{*}}{p}} \mathrm{d}x\right)^{\frac{p}{p_{s}^{*}}} \leq C_{1} K_{0}^{\beta-1} \left(||u||_{L^{p_{s}^{*}}(\mathbb{R}^{N})}^{r}+||u||_{L^{p_{s}^{*}}(\mathbb{R}^{N})}^{p_{s}^{*}}\right) + C_{2} \left(\int_{\{u \geq K_{0}\}} u^{p_{s}^{*}} \mathrm{d}x\right)^{\frac{p_{s}^{*}-p}{p_{s}^{*}}} \left(\int_{\mathbb{R}^{N}} u^{p_{s}^{*}} u_{M}^{(\beta-1)\frac{p_{s}^{*}}{p}} \mathrm{d}x\right)^{\frac{p}{p_{s}^{*}}}.$$

$$(2.29)$$

Fixing $\theta > \frac{N}{sp}$, we take $\beta > 1$ such that

$$(\beta - 1)\frac{p_s^*}{p} + p_s^* = \theta(p_s^* - 1)$$
 i.e. $\beta = p\theta \frac{(p_s^* - 1)}{p_s^*} - (p - 1).$

and as $u \in L^{p_s^*}(\mathbb{R}^N)$ we can choose $K_0 = K_0(\beta, u) > 0$ such that

$$\left(\int_{\{u\geq K_0\}} u^{p_s^*} \mathrm{d}x\right)^{\frac{p_s^*-p}{p_s^*}} \leq \frac{S}{2} \left(\frac{p}{\beta+p-1}\right)^p$$

Hence from (2.29) we get

$$\left(\int_{\mathbb{R}^N} u_M^{\theta(p_s^*-1)} \mathrm{d}x\right)^{\frac{p}{p_s^*}} \le C\left(\frac{p+\beta-1}{p}\right)^p K_0^{\beta-1}\left(||u||_{L^{p_s^*}(\mathbb{R}^N)}^r + ||u||_{L^{p_s^*}(\mathbb{R}^N)}^{p_s^*}\right).$$

If we now take the limit as M goes to ∞ , we finally get that $u \in L^{\theta(p_s^*-1)}(\mathbb{R}^N)$.

Consequently, $u \in L^{\infty}(\mathbb{R}^N)$. The hypothesis $g \in L^{\infty}(\mathbb{R}^N)$, implies that $f = |u|^{p_s^*-2}u + \lambda g|u|^{r-2}u \in L^{\infty}(\mathbb{R}^N)$. Therefore, by Theorem 1.0.1 results $u \in C^{\alpha}_{loc}(\mathbb{R}^N)$.

Chapter 3

On the behavior of least energy solutions of a fractional (p, q(p))problem as p goes to infinity

In this chapter, first, we consider the nonhomogeneous problem

$$\begin{cases} \left[\left(-\Delta_p \right)^{\alpha} + \left(-\Delta_q \right)^{\beta} \right] u = \mu \left| u(x_u) \right|^{p-2} u(x_u) \delta_{x_u} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ \left| u(x_u) \right| = \left\| u \right\|_{\infty} \end{cases}$$
(3.1)

where α, β, p, q and $\mu > 0$ satisfy suitable conditions, $x_u \in \Omega$ is a point where u attains its sup norm $(|u(x_u)| = ||u||_{\infty}), \delta_{x_u}$ is the Dirac delta distribution supported at x_u and Ω be a bounded, smooth domain of \mathbb{R}^N .

Proceeding as in [4] and [26], one can arrive at (3.1) as the limit case, as $r \to \infty$, of the problem

$$\begin{cases} \left[\left(-\Delta_p\right)^{\alpha} + \left(-\Delta_q\right)^{\beta} \right] u = \mu \left\| u \right\|_r^{p-r} \left| u \right|^{r-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\|\cdot\|_r$ denotes the standard norm in the Lebesgue space $L^r(\Omega)$.

In Section 3.3, we fix the fractional orders α and β (with $\alpha \neq \beta$), allow q and μ to depend suitably on p (q = q(p) and $\mu = \mu_p$) and denote by u_p the positive least energy solution of the problem

$$\begin{cases} \left[\left(-\Delta_p \right)^{\alpha} + \left(-\Delta_{q(p)} \right)^{\beta} \right] u = \mu_p \left| u(x_p) \right|^{p-2} u(x_p) \delta_{x_p} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \left| u(x_p) \right| = \| u \|_{\infty} \,. \end{cases}$$

In the sequence we determine the asymptotic behavior of the pair $(u_p, x_p) \in X(\Omega) \times \Omega$, as p goes to ∞ .

Our main results are stated in Theorem 3.0.1 below, where, for each $s \in (0, 1]$,

$$C_0^{0,s}(\overline{\Omega}) := \left\{ u \in C_0(\overline{\Omega}) : \left| u \right|_s < \infty \right\},\,$$

with $|\cdot|_s$ denoting the s-Hölder seminorm, defined by

$$|u|_{s} := \sup\left\{\frac{|u(x) - u(y)|}{|x - y|^{s}} : x, y \in \overline{\Omega} \quad \text{and} \quad x \neq y\right\}.$$
(3.2)

Theorem 3.0.1 Assume that

$$\lim_{p \to \infty} \frac{q(p)}{p} =: Q \in \left\{ \begin{array}{ll} (0,1) & \text{if } 0 < \beta < \alpha < 1\\ (1,\infty) & \text{if } 0 < \alpha < \beta < 1 \end{array} \right.$$

and

$$\Lambda := \lim_{p \to \infty} \sqrt[p]{\mu_p} > R^{-\alpha}$$

where R is the inradius of Ω (i.e. the radius of the largest ball inscribed in Ω).

Let $p_n \to \infty$. There exist $x_{\infty} \in \Omega$ and $u_{\infty} \in C_0^{0,\beta}(\overline{\Omega})$ such that, up to a subsequence, $x_{u_{p_n}} \to x_{\infty}$ and $u_{p_n} \to u_{\infty}$ uniformly in $\overline{\Omega}$. Moreover:

- (i) $0 < u_{\infty}(x) \leq (\Lambda R^{\beta})^{\frac{1}{Q-1}} (\operatorname{dist}(x, \partial \Omega))^{\beta} \quad \forall x \in \Omega,$
- (ii) $\operatorname{dist}(x_{\infty}, \partial \Omega) = R$,
- (iii) $u_{\infty}(x_{\infty}) = ||u_{\infty}||_{\infty} = R^{\beta} (\Lambda R^{\beta})^{\frac{1}{Q-1}},$

(iv)
$$|u_{\infty}|_{\beta} = (\Lambda R^{\beta})^{\frac{1}{Q-1}}$$
,

(v)
$$\frac{|u_{\infty}|_{\beta}}{\|u_{\infty}\|_{\infty}} = R^{-\beta} = \min\left\{\frac{|v|_{\beta}}{\|v\|_{\infty}} : v \in C_0^{0,\beta}(\overline{\Omega}) \setminus \{0\}\right\},$$

(vi) u_{∞} is a viscosity solution of

$$\max\left\{\mathcal{L}_{\alpha}^{+}u,\left(\mathcal{L}_{\beta}^{+}u\right)^{Q}\right\}=\max\left\{-\mathcal{L}_{\alpha}^{-}u,\left(-\mathcal{L}_{\beta}^{-}u\right)^{Q}\right\}\quad\text{in }\Omega\setminus\left\{x_{\infty}\right\},$$

where in the above equation the operators are defined according to the following notation, where 0 < s < 1:

$$\left(\mathcal{L}_{s}^{+}u\right)(x) := \sup_{y \in \mathbb{R}^{N} \setminus \{x\}} \frac{u(y) - u(x)}{|y - x|^{s}} \quad \text{and} \quad \left(\mathcal{L}_{s}^{-}u\right)(x) := \inf_{y \in \mathbb{R}^{N} \setminus \{x\}} \frac{u(y) - u(x)}{|y - x|^{s}}.$$
 (3.3)

Our approach in this Chapter is inspired by the arguments and techniques developed in some of the works above mentioned and can be applied to the fractional version of [26] and also for studying a fractional version for the system considered in [40].

3.1 Preliminaries

Let Ω be a bounded, smooth domain of \mathbb{R}^N , N > 1, and consider the Sobolev space of fractional order $s \in (0, 1)$ and exponent m > 1,

$$W_0^{s,m}(\Omega) := \left\{ u \in L^m(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad [u]_{s,m} < \infty \right\},$$

where

$$[u]_{s,m} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{m}}$$

is the Gagliardo seminorm.

As it is well known, $\left(W_0^{s,m}(\Omega), [\cdot]_{s,m}\right)$ is a uniformly convex Banach space (also characterized as the closure of $C_c^{\infty}(\Omega)$ with respect to $[\cdot]_{s,m}$), compactly embedded into $L^r(\Omega)$ whenever

$$1 \le r < m_s^* := \begin{cases} \frac{Nm}{N-sm}, & m < N/s, \\ \infty, & m \ge N/s. \end{cases}$$

Moreover,

$$W_0^{s,m}(\Omega) \hookrightarrow C_0(\overline{\Omega}) \quad \text{if} \quad m > N/s.$$
 (3.4)

(The notation $A \hookrightarrow B$ means that the continuous embedding $A \hookrightarrow B$ is compact.) It follows that the infimum

$$\lambda_{s,m} := \inf \left\{ \frac{[u]_{s,m}^m}{\|u\|_{\infty}^m} : u \in W_0^{s,m}(\Omega) \setminus \{0\} \right\}$$

is positive and, in fact, a minimum.

The compactness in (3.4) is consequence of the following Morrey's type inequality (see [24])

$$\sup_{(x,y)\neq(0,0)} \frac{|u(x) - u(y)|}{|x - y|^{s - \frac{N}{m}}} \le C[u]_{s,m}, \quad \forall u \in W_0^{s,m}(\Omega)$$
(3.5)

which holds whenever m > N/s. If m is sufficiently large, the positive constant C in (3.5) can be chosen uniform with respect to m (see [28, Remark 2.2]).

In [39], Lindqvist and Lindgren characterized the asymptotic behavior (as $m \to \infty$) of the only positive, normalized first eigenfunction u_m of $(-\Delta_m)^s$ in $W_0^{s,m}(\Omega)$. Namely, $u_m > 0$ in Ω , $||u_m||_m = 1$ and $[u_m]_{s,m}^m = \Lambda_{s,m}$, where

$$\Lambda_{s,m} := \inf \left\{ \begin{bmatrix} u \end{bmatrix}_{s,m}^m : u \in W_0^{s,m}(\Omega) \quad \text{and} \quad \|u\|_m = 1 \right\}$$

is the first eigenvalue of $(-\Delta_m)^s$. Among several results, they proved that

$$\lim_{m \to \infty} \sqrt[m]{\Lambda_{s,m}} = R^{-s} \le \frac{|\phi|_s}{\|\phi\|_{\infty}} \quad \forall \phi \in C_c^{\infty}(\Omega) \setminus \{0\}$$
(3.6)

Let $(-\Delta_m)^s$ be the s-fractional m-Laplacian, the operator acting from $W_0^{s,m}(\Omega)$ into its topological dual, defined for all $u, \varphi \in W_0^{s,m}(\Omega)$ by

$$\left\langle \left(-\Delta_{m}\right)^{s} u, \varphi\right\rangle_{s,m} := \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{\left|u(x) - u(y)\right|^{m-2} \left(u(x) - u(y)\right) (\varphi(x) - \varphi(y))}{\left|x - y\right|^{N+sm}} \mathrm{d}x \mathrm{d}y$$

We recall that $(-\Delta_m)^s u$ is the Gâteaux derivative at a function $u \in W_0^{s,m}(\Omega)$ of the Fréchet differentiable functional $v \mapsto m^{-1} [v]_{s,m}^m$.

An alternative pointwise expression for $(-\Delta_m)^s u$ is

$$(\mathcal{L}_{s,m}u)(x) := 2 \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{m-2} (u(y) - u(x))}{|x - y|^{N+sm}} \mathrm{d}y.$$
(3.7)

As argued in [39], this expression appears formally as follows

$$\begin{split} \left\langle \left(-\Delta_{m}\right)^{s} u, \varphi \right\rangle_{s,m} &= \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{|u(x) - u(y)|^{m-2} \left(u(x) - u(y)\right)(\varphi(x) - \varphi(y))}{|x - y|^{N+sm}} \mathrm{d}x \mathrm{d}y \\ &= \int_{\mathbb{R}^{N}} \varphi(x) \left(\int_{\mathbb{R}^{N}} \frac{|u(x) - u(y)|^{m-2} \left(u(x) - u(y)\right)}{|x - y|^{N+sm}} \mathrm{d}y \right) \mathrm{d}x \\ &- \int_{\mathbb{R}^{N}} \varphi(y) \left(\int_{\mathbb{R}^{N}} \frac{|u(x) - u(y)|^{m-2} \left(u(x) - u(y)\right)}{|x - y|^{N+sm}} \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{\mathbb{R}^{N}} \varphi(x) (\mathcal{L}_{s,m} u)(x) \mathrm{d}x. \end{split}$$

As usual, we interpret (3.1) as an identity between functionals applied to the (weak) solution u. Thus,

$$\left\langle \left(-\Delta_{p}\right)^{\alpha} u, \varphi\right\rangle_{\alpha, p} + \left\langle \left(-\Delta_{q}\right)^{\beta} u, \varphi\right\rangle_{\beta, q} = \mu \left|u(x_{u})\right|^{p-2} u(x_{u})\varphi(x_{u}) \quad \forall \varphi \in X(\Omega),$$
(3.8)

where $X(\Omega)$ is an appropriate Sobolev space (that will be derived in the sequence). The functional at the left-hand side of (3.8) is the Gâteaux derivative of the Fréchet differentiable functional $v \mapsto p^{-1} [v]_{\alpha,p}^p + q^{-1} [v]_{\beta,q}^q$ at u. However, the functional at the right-hand side is merely related to the right-sided Gâteaux derivative of the functional $\varphi \mapsto p^{-1} \|\varphi\|_{\infty}^p$ whenever u assumes its sup norm at a unique point x_u . This has to do with the following fact (see Lemma 3.2.5 and Remark 3.2.6): if $u \in C(\overline{\Omega})$ assumes its sup norm only at $x_u \in \Omega$, then

$$\lim_{\epsilon \to 0^+} \frac{\|u + \epsilon \varphi\|_{\infty}^p - \|u\|_{\infty}^p}{p\epsilon} = |u(x_u)|^{p-2} u(x_u)\varphi(x_u), \quad \forall \varphi \in C(\overline{\Omega}).$$

Therefore, we define the formal energy functional associated with (3.1) by

$$E_{\mu}(u) := \frac{1}{p} \left[u \right]_{\alpha,p}^{p} + \frac{1}{q} \left[u \right]_{\beta,q}^{q} - \frac{\mu}{p} \left\| u \right\|_{\infty}^{p}, \quad \mu > 0,$$

and formulate our hypotheses on α , β , p and q to guarantee the well-definiteness of this functional. For this, we take into account (3.4) and the following known facts:

- $W_0^{s,p}(\Omega) \not\hookrightarrow W_0^{s,q}(\Omega)$ for any $0 < s < 1 \le q < p \le \infty$ (see Appendix A.2 and [41, Theorem 1.1]),
- $W_0^{s_2,m_2}(\Omega) \hookrightarrow W_0^{s_1,m_1}(\Omega)$, whenever $0 < s_1 < s_2 < 1 \le m_1 < m_2 < \infty$ (see [11, Lemma 2.6]).

Thus, we assume that α, β, p and q satisfy one of the following conditions:

$$0 < \alpha < \beta < 1 \quad \text{and} \quad N/\alpha < p < q \tag{3.9}$$

or

$$0 < \beta < \alpha < 1 \quad \text{and} \quad N/\beta < q < p. \tag{3.10}$$

The assumption (3.9) provides the chain of embeddings $W_0^{\beta,q}(\Omega) \hookrightarrow W_0^{\alpha,p}(\Omega) \hookrightarrow C_0(\overline{\Omega})$ whereas (3.10) yields $W_0^{\alpha,p}(\Omega) \hookrightarrow W_0^{\beta,q}(\Omega) \hookrightarrow C_0(\overline{\Omega})$. Therefore, the Sobolev space

$$X(\Omega) := \begin{cases} \begin{pmatrix} W_0^{\beta,q}(\Omega), [\cdot]_{\beta,q} \end{pmatrix} & \text{if } 0 < \alpha < \beta < 1 \text{ and } N/\alpha < p < q \\ \begin{pmatrix} W_0^{\alpha,p}(\Omega), [\cdot]_{\alpha,p} \end{pmatrix} & \text{if } 0 < \beta < \alpha < 1 \text{ and } N/\beta < q < p \end{cases}$$

is the natural domain for the energy functional E_{μ} . Note that

$$X(\Omega) \subset W_0^{\alpha,p}(\Omega) \cap W_0^{\beta,q}(\Omega) \text{ and } X(\Omega) \hookrightarrow C_0(\overline{\Omega}).$$

Once we have chosen $X(\Omega)$, a weak solution of (3.1) is defined (see Definition 3.2.2) by means of (3.8).

As for the parameter μ , we assume that

$$\mu > \lambda_{\alpha, p},\tag{3.11}$$

where

$$\lambda_{\alpha,p} := \inf\left\{\frac{[u]_{\alpha,p}^{p}}{\|u\|_{\infty}^{p}} : u \in W_{0}^{\alpha,p}(\Omega) \setminus \{0\}\right\} = \frac{[e]_{\alpha,p}^{p}}{\|e\|_{\infty}^{p}} > 0$$
(3.12)

for some function $e \in W_0^{\alpha,p}(\Omega) \setminus \{0\}$. The existence of e is a consequence of the compact embedding of $W_0^{\alpha,p}(\Omega)$ into $C_0(\overline{\Omega})$ that holds in both cases (3.9) and (3.10).

It turns out that (3.11) is also a necessary condition for the existence of weak solutions (see Remark 3.2.3).

Assuming the above conditions on α, β, p, q and μ we show the existence of at least one positive weak solution that minimizes the energy functional either on $W_0^{\beta,q}(\Omega) \setminus \{0\}$, when (3.9) holds, or on the following Nehari-type set

$$\mathcal{N}_{\mu} := \left\{ u \in W_0^{\alpha, p}(\Omega) \setminus \{0\} : [u]_{\alpha, p}^p + [u]_{\beta, q}^q = \mu \, \|u\|_{\infty}^p \right\},\tag{3.13}$$

when (3.10) holds. Both type of minimizers are referred in this work as *least energy solutions* of (3.1). The reason behind the appearance of the Dirac delta is that the set where a minimizer of E_{μ} attains its sup norm is a singleton (as we will show).

3.2 Existence of a positive least energy solution

In this section, we assume that μ satisfy (3.11) and that α, β, p and q are related by one of the conditions (3.9) or (3.10). Our goal is to prove the existence of at least one positive least energy solution $u_{\mu} \in X(\Omega) \setminus \{0\}$ for the problem (3.1).

Remark 3.2.1 We recall that $[|u|]_{s,p} \leq [u]_{s,p}$ for all $u \in W_0^{s,m}(\Omega)$ since

$$||u(x)| - |u(y)|| < |u(x) - u(y)|$$
 if $u(x)u(y) < 0$.

Definition 3.2.2 We say that a function $u \in X(\Omega)$ is a weak solution of (3.1) if $||u||_{\infty} = |u(x_u)|$ and

$$\left\langle \left(-\Delta_{p}\right)^{\alpha}u,\varphi\right\rangle_{\alpha,p}+\left\langle \left(-\Delta_{q}\right)^{\beta}u,\varphi\right\rangle_{\beta,q}=\mu\left|u(x_{u})\right|^{p-2}u(x_{u})\varphi(x_{u}),\quad\forall\varphi\in X(\Omega)$$

Remark 3.2.3 If $u \in X(\Omega)$ is a weak solution of (3.1), then (by taking $\varphi = u$)

$$[u]_{\alpha,p}^{p} + [u]_{\beta,q}^{q} = \mu \, \|u\|_{\infty}^{p}.$$

If, in addition, $u \neq 0$ the definition of $\lambda_{\alpha,p}$ yields

$$\mu \|u\|_{\infty}^{p} = [u]_{\alpha,p}^{p} + [u]_{\beta,q}^{q} > [u]_{\alpha,p}^{p} \ge \lambda_{\alpha,p} \|u\|_{\infty}^{p}.$$

This shows that (3.11) is a necessary condition for the existence of a nontrivial weak solution.

Proposition 3.2.4 Suppose that α, β, p and q satisfy (3.9). There exists at least one nonnegative function $u_{\mu} \in X(\Omega) \setminus \{0\}$ such that

$$E_{\mu}(u_{\mu}) \leq E_{\mu}(u) \quad \forall u \in X(\Omega).$$

Proof. Let

$$\lambda_{\beta,q} := \inf\left\{\frac{[u]_{\beta,q}^q}{\|u\|_{\infty}^q} : u \in W_0^{\beta,q}(\Omega) \setminus \{0\}\right\} > 0.$$

$$(3.14)$$

Since $X(\Omega) = W_0^{\beta,q}(\Omega)$ we have

$$E_{\mu}(u) \ge \frac{1}{q} [u]_{\beta,q}^{q} - \frac{\mu}{p} \|u\|_{\infty}^{p} \ge \frac{1}{q} [u]_{\beta,q}^{q} - \frac{\mu}{p} [u]_{\beta,q}^{p} \left(\sqrt[q]{\lambda_{\beta,q}}\right)^{-p} = h([u]_{\beta,q}) \quad \forall u \in X(\Omega),$$

where $h: [0, \infty) \mapsto \mathbb{R}$ is given by

$$h(t) := \frac{1}{q} t^q - \frac{\mu}{p} \left(\sqrt[q]{\lambda_{\beta,q}} \right)^{-p} t^p.$$

Noting that $\lim_{t\to\infty} h(t) = \infty$ and

$$h(t) \ge h\left(\left[\mu\left(\sqrt[q]{\lambda_{\beta,q}}\right)^{-p}\right]^{\frac{1}{q-p}}\right) = -\left(\frac{1}{p} - \frac{1}{q}\right)\left[\mu\left(\sqrt[q]{\lambda_{\beta,q}}\right)^{-p}\right]^{\frac{q}{q-p}}$$

we conclude that E_{μ} is coercive and bounded from below. Hence, by standards arguments of the Calculus of Variations (recall that $X(\Omega) = W_0^{\beta,q}(\Omega) \hookrightarrow W_0^{\alpha,p}(\Omega) \hookrightarrow C_0(\overline{\Omega})$) we can show that the functional E_{μ} assumes the global minimum value at a function $u_{\mu} \in X(\Omega)$.

Now, in order to verify that $u_{\mu} \not\equiv 0$ we show that $E_{\mu}(v) < 0 = E_{\mu}(0)$ for some $v \in W_0^{\beta,q}(\Omega)$. Let $u \in W_0^{\alpha,p}(\Omega) \setminus \{0\}$ be such that

$$\lambda_{\alpha,p} \le \frac{\left[u\right]_{\alpha,p}^p}{\left\|u\right\|_{\infty}^p} < \frac{\lambda_{\alpha,p} + \mu}{2}$$

By density and compactness, there exists a sequence $\{\varphi_n\} \subset C_c^{\infty}(\Omega)$ such that $[\varphi_n]_{\alpha,p} \to [u]_{\alpha,p}$ and $\|\varphi_n\|_{\infty} \to \|u\|_{\infty}$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\left[\varphi_{n_0}\right]_{\alpha,p}^p}{\left\|\varphi_{n_0}\right\|_{\infty}^p} \le \frac{\lambda_{\alpha,p} + \mu}{2} < \mu.$$

Since $\varphi_{n_0} \in X(\Omega)$ we have

$$E_{\mu}(t\varphi_{n_0}) \leq \frac{t^q}{q} \left[\varphi_{n_0}\right]_{\beta,q}^q - \frac{t^p}{p} \frac{\|\varphi_{n_0}\|_{\infty}^p}{2} \left(\mu - \lambda_{\alpha,p}\right) < 0$$

for some t > 0 sufficiently small. Thus, $v := t\varphi_{n_0}$ is such that $E_{\mu}(v) < 0$.

According to Remark 3.2.1, $E_{\mu}(|u_{\mu}|) \leq E_{\mu}(u_{\mu})$. Therefore, we can assume $u_{\mu} \geq 0$ in Ω .

In the sequence we show that under (3.9) any minimizer of the energy functional E_{μ} is a weak solution of (3.1). For this we need the following result proved in [33] and we will reproduce here.

Lemma 3.2.5 Let $u \in C(\overline{\Omega})$ and $\Gamma_u := \{x \in \Omega : |u(x)| = ||u||_{\infty}\}$. Then,

$$\lim_{\epsilon \to 0^+} \frac{\|u + \epsilon \varphi\|_{\infty}^p - \|u\|_{\infty}^p}{p\epsilon} = \max\left\{ |u(x)|^{p-2} u(x)\varphi(x) : x \in \Gamma_u \right\}, \quad \forall \varphi \in C(\overline{\Omega})$$

Remark 3.2.6 According to the notation of the Lemma 3.2.5, if for some $L \in \mathbb{R}$,

$$\max\left\{\left|u(x)\right|^{p-2}u(x)\varphi(x):x\in\Gamma_{u}\right\}\leq L\leq\min\left\{\left|u(x)\right|^{p-2}u(x)\varphi(x):x\in\Gamma_{u}\right\},$$

for all $\varphi \in C(\overline{\Omega})$, then

$$|u(x)|^{p-2}u(x)\varphi(x) = L = |u(y)|^{p-2}u(y)\varphi(y), \quad \forall \varphi \in C(\overline{\Omega}) \quad \text{and} \quad x, y \in \Gamma_u.$$

Of course this implies that Γ_u is a singleton, say $\Gamma_u = \{x_u\}$, and therefore the Lemma 3.2.5 yields

$$\lim_{\epsilon \to 0^+} \frac{\|u + \epsilon \varphi\|_{\infty}^p - \|u\|_{\infty}^p}{p\epsilon} = L = |u(x_u)|^{p-2} u(x_u)\varphi(x_u), \quad \forall \varphi \in C(\overline{\Omega})$$

Proposition 3.2.7 Suppose that α, β, p and q satisfy (3.9). If $u \in X(\Omega)$ satisfies

$$E_{\mu}(u) \le E_{\mu}(v) \quad \forall v \in X(\Omega),$$

then u is a weak solution of (3.1).

Proof. Let $\varphi \in X(\Omega)$ and $\epsilon > 0$. By hypothesis,

$$0 \le \frac{E_{\mu}(u + \epsilon \varphi) - E_{\mu}(\varphi)}{\epsilon} = A(\epsilon) - \mu B(\epsilon),$$

where

$$A(\epsilon) := \frac{[u + \epsilon \varphi]_{\alpha, p}^{p} - [u]_{\alpha, p}^{p}}{\epsilon p} + \frac{[u + \epsilon \varphi]_{\beta, q}^{q} - [u]_{\beta, q}^{q}}{\epsilon q}$$

and

$$B(\epsilon) = \frac{\|u + \epsilon\varphi\|_{\infty}^{p} - \|u\|_{\infty}^{p}}{\epsilon p}$$

As we already know (from the Introduction)

$$L := \lim_{\epsilon \to 0^+} A(\epsilon) = \left\langle \left(-\Delta_p \right)^{\alpha} u, \varphi \right\rangle_{\alpha, p} + \left\langle \left(-\Delta_q \right)^{\beta} u, \varphi \right\rangle_{\beta, q}.$$

According to Lemma 3.2.5

$$\lim_{\epsilon \to 0^+} B(\epsilon) = \max\left\{ \left| u(x) \right|^{p-2} u(x)\varphi(x) : x \in \Gamma_u \right\}.$$

Consequently,

$$\mu \cdot \max\left\{\left|u(x)\right|^{p-2} u(x)\varphi(x) : x \in \Gamma_u\right\} \le L.$$

Now, repeating the above arguments with φ replaced with $-\varphi$ we also conclude that

$$L \le \mu \min\left\{ |u(x)|^{p-2} u(x)\varphi(x) : x \in \Gamma_u \right\}.$$

It follows that (see Remark 3.2.6) $\Gamma_u = \{x_u\}$ and

$$L = \mu \left| u(x_u) \right|^{p-2} u(x_u) \varphi(x_u).$$

Now, let us analyze E_{μ} under the hypothesis (3.10). First we observe that E_{μ} is unbounded from below in $X(\Omega)$. In fact, this follows from the identity (where $e \in W_0^{\alpha,p}(\Omega)$ is given in (3.12))

$$E_{\mu}(te) = \frac{t^{q}}{q} \left[e \right]_{\beta,q}^{q} - \frac{t^{p}}{p} \left(\mu - \lambda_{\alpha,p} \right) \| e \|_{\infty}^{p}, \quad \forall t > 0.$$
(3.15)

Thus, as usual, we look for a minimizer of E_{μ} restricted to Nehari-type set \mathcal{N}_{μ} given by (3.13).

Taking (3.10) into account, the following properties for a function $u \in X(\Omega) \setminus \{0\}$ can be easily verified

$$u \in \mathcal{N}_{\mu} \iff E_{\mu}(u) = \left(\frac{1}{q} - \frac{1}{p}\right) [u]_{\beta,q}^{q}$$
 (3.16)

and

$$tu \in \mathcal{N}_{\mu} \iff [u]_{\alpha,p}^{p} < \mu \, \|u\|_{\infty}^{p} \quad \text{and} \quad t = \left(\frac{[u]_{\beta,q}^{q}}{\mu \, \|u\|_{\infty}^{p} - [u]_{\alpha,p}^{p}}\right)^{\frac{1}{p-q}}.$$
(3.17)

The latter property shows that $\mathcal{N}_{\mu} \neq \emptyset$, since

$$\left[e\right]_{\alpha,p}^{p} = \lambda_{\alpha,p} \left\|e\right\|_{\infty}^{p} < \mu \left\|e\right\|_{\infty}^{p}.$$

Moreover, combining (3.14) and (3.16) we obtain,

$$\mu \|u\|_{\infty}^{p} = [u]_{\alpha,p}^{p} + [u]_{\beta,q}^{q} > [u]_{\beta,q}^{q} \ge \lambda_{\beta,q} \|u\|_{\infty}^{q},$$

for an arbitrary $u \in \mathcal{N}_{\mu}$. Consequently,

$$\|u\|_{\infty} > \left(\frac{\lambda_{\beta,q}}{\mu}\right)^{\frac{1}{p-q}} > 0, \quad \forall u \in \mathcal{N}_{\mu}$$

and

$$E_{\mu}(u) \ge \left(\frac{1}{q} - \frac{1}{p}\right) \lambda_{\beta,q} \|u\|_{\infty}^{q} > \left(\frac{1}{q} - \frac{1}{p}\right) \lambda_{\beta,q} \left(\frac{\lambda_{\beta,q}}{\mu}\right)^{\frac{q}{p-q}} > 0, \quad \forall u \in \mathcal{N}_{\mu}.$$

Another property is that

$$[u]_{\alpha,p} < \frac{\sqrt[p]{\mu}}{\sqrt[q]{\lambda_{\beta,q}}} [u]_{\beta,q}, \quad \forall u \in \mathcal{N}_{\mu},$$
(3.18)

which also follows from (3.14), since

$$[u]_{\alpha,p}^{p} < [u]_{\alpha,p}^{p} + [u]_{\beta,q}^{q} = \mu \, \|u\|_{\infty}^{p} \le \mu \left(\frac{[u]_{\beta,q}^{q}}{\lambda_{\beta,q}}\right)^{\frac{p}{q}} = \mu \left(\lambda_{\beta,q}\right)^{-\frac{p}{q}} [u]_{\beta,q}^{p}.$$

Proposition 3.2.8 Suppose that α, β, p and q satisfy (3.10). There exists at least one nonnegative function $u_{\mu} \in W_0^{\alpha,p}(\Omega) \setminus \{0\}$ such that

$$E_{\mu}(u_{\mu}) \leq E_{\mu}(u) \quad \forall u \in \mathcal{N}_{\mu}.$$

Proof. Let $\{u_n\} \in \mathcal{N}_{\mu}$ be a minimizing sequence:

$$E_{\mu}(u_n) = \left(\frac{1}{q} - \frac{1}{p}\right) [u_n]_{\beta,q}^q \to m_{\mu} := \inf \left\{ E_{\mu}(u) : u \in \mathcal{N}_{\mu} \right\}.$$

Taking (3.18) into account and using compactness arguments, we can assume that u_n converges to a function $u_{\mu} \in W_0^{\alpha,p}(\Omega)$ uniformly in $C(\overline{\Omega})$ and weakly in both Sobolev spaces $W_0^{\alpha,p}(\Omega)$ and $W_0^{\beta,q}(\Omega)$. Of course, $u_{\mu} \neq 0$ since

$$\left\|u_{\mu}\right\|_{\infty} > \left(\frac{\lambda_{\beta,q}}{\mu}\right)^{\frac{1}{p-q}} > 0.$$

Hence,

$$[u_{\mu}]_{\alpha,p}^{p} < [u_{\mu}]_{\alpha,p}^{p} + [u_{\mu}]_{\beta,q}^{q} \le \liminf_{n} \left([u_{n}]_{\alpha,p}^{p} + [u_{n}]_{\beta,q}^{q} \right) = \mu \liminf_{n} \|u_{n}\|_{\infty}^{p} = \mu \|u_{\mu}\|_{\infty}^{p},$$

thus implying that $\theta u_{\mu} \in \mathcal{N}_{\mu}$, where

$$\theta := \left(\frac{[u_{\mu}]_{\beta,q}^{q}}{\mu \|u_{\mu}\|_{\infty}^{p} - [u_{\mu}]_{\alpha,p}^{p}}\right)^{\frac{1}{p-q}} \le 1.$$

Consequently,

$$m_{\mu} \leq E_{\mu}(\theta u_{\mu})$$

= $\theta^{q} \left(\frac{1}{q} - \frac{1}{p}\right) \left[u_{\mu}\right]_{\beta,q}^{q} \leq \left(\frac{1}{q} - \frac{1}{p}\right) \liminf_{n} \left[u_{n}\right]_{\beta,q}^{q} = \lim_{n} E_{\mu}(u_{n}) = m_{\mu},$

that is, $\theta = 1$, $u_{\mu} \in \mathcal{N}_{\mu}$ and $m_{\mu} = E_{\mu}(u_{\mu})$.

Remark 3.2.1 and (3.17) show that $|u_{\mu}| \in \mathcal{N}_{\mu}$ and $E_{\mu}(|u_{\mu}|) \leq E_{\mu}(u_{\mu})$. Thus, we can assume that $u_{\mu} \geq 0$ in Ω .

Proposition 3.2.9 Suppose that α, β, p and q satisfy (3.10). If $u \in \mathcal{N}_{\mu}$ is such that

$$E_{\mu}(u) \leq E_{\mu}(v), \quad \forall v \in \mathcal{N}_{\mu},$$

then u is a weak solution of (3.1).

Proof. Let $\varphi \in X(\Omega)$ be fixed. Since $u \in \mathcal{N}_{\mu}$ we have $\mu \|u\|_{\infty}^p - [u]_{\alpha,p}^p = [u]_{\beta,q}^q > 0$. Thus, by continuity there exists $\epsilon > 0$ such that

$$\mu \left\| u + s\varphi \right\|_{\infty}^{p} - \left[u + s\varphi \right]_{\alpha, p}^{p} > 0, \quad \forall \, s \in (-\epsilon, \epsilon).$$

It follows that

$$\tau(s)(u+s\varphi) \in N_{\mu}, \quad \forall s \in (-\epsilon,\epsilon),$$

where

$$\tau(s) := \left(\frac{[u+s\varphi]^q_{\beta,q}}{\mu \, \|u+s\varphi\|^p_{\infty} - [u+s\varphi]^p_{\alpha,p}}\right)^{\frac{1}{p-q}}, \quad s \in (-\epsilon,\epsilon).$$

Therefore, the function

$$\gamma(s) := E_{\mu}\left(\tau(s)u + s\varphi\right)$$
$$= \frac{\tau(s)^{p}}{p}\left[u + s\varphi\right]_{\alpha,p}^{p} + \frac{\tau(s)^{q}}{q}\left[u + s\varphi\right]_{\beta,q}^{q} - \mu\frac{\tau(s)^{p}}{p}\left\|u + s\varphi\right\|_{\infty}^{p}, \quad s \in (-\epsilon, \epsilon)$$

assumes a minimum value at s = 0. This implies that

$$\gamma'(0^+) := \lim_{s \to 0^+} \frac{\gamma(s) - \gamma(0)}{s} \ge 0.$$
(3.19)

Using Lemma 3.2.5 and observing that $\tau(0^+) = 1$ and $u \in \mathcal{N}_{\mu}$ we compute

$$\gamma'(0^+) = \left\langle \left(-\Delta_p\right)^{\alpha} u, \varphi \right\rangle_{\alpha, p} + \left\langle \left(-\Delta_q\right)^{\beta} u, \varphi \right\rangle_{\beta, q} - \mu \max\left\{ \left|u(x)\right|^{p-2} u(x)\varphi(x) : x \in \Gamma_u \right\}.$$

Hence, (3.19) yields,

$$\mu \max\left\{\left|u(x)\right|^{p-2} u(x)\varphi(x) : x \in \Gamma_u\right\} \le \left\langle\left(-\Delta_p\right)^{\alpha} u, \varphi\right\rangle_{\alpha, p} + \left\langle\left(-\Delta_q\right)^{\beta} u, \varphi\right\rangle_{\beta, q}.$$

Replacing φ by $-\varphi$ we obtain

$$\left\langle \left(-\Delta_{p}\right)^{\alpha}u,\varphi\right\rangle_{\alpha,p}+\left\langle \left(-\Delta_{q}\right)^{\beta}u,\varphi\right\rangle_{\beta,q}\leq\mu\min\left\{\left|u(x)\right|^{p-2}u(x)\varphi(x):x\in\Gamma_{u}\right\}.$$

Hence, according to Remark 3.2.6, $\Gamma_u = \{x_u\}$ and

$$\left\langle \left(-\Delta_{p}\right)^{\alpha}u,\varphi\right\rangle_{\alpha,p}+\left\langle \left(-\Delta_{q}\right)^{\beta}u,\varphi\right\rangle_{\beta,q}=\mu\left|u(x_{u})\right|^{p-2}u(x_{u})\varphi(x_{u}).$$

We gather the results above in the following theorem.

Theorem 3.2.10 Suppose that α, β, p and q satisfy either (3.9) or (3.10), and that μ satisfies (3.11). Then (3.1) has at least one nonnegative least energy solution $u_{\mu} \in X(\Omega) \setminus \{0\}$.

We remark that $u_{\mu} \in X(\Omega) \setminus \{0\}$ given by Theorem 3.2.10 is a nonnegative weak solution of the fractional harmonic-type equation

$$\left[\left(-\Delta_p\right)^{\alpha} + \left(-\Delta_q\right)^{\beta}\right] u = 0 \tag{3.20}$$

in the punctured domain $\Omega \setminus \{x_u\}$, since

$$\left\langle \left(-\Delta_{p}\right)^{\alpha}u_{\mu},\varphi\right\rangle_{\alpha,p}+\left\langle \left(-\Delta_{q}\right)^{\beta}u_{\mu},\varphi\right\rangle_{\beta,q}=0\quad\forall\varphi\in C_{c}^{\infty}(\Omega\setminus\left\{x_{u_{\mu}}\right\}).$$
(3.21)

Consequently, if $p > \frac{1}{1-\alpha}$ and $q > \frac{1}{1-\beta}$ (see Remark 3.2.11) one can adapt the arguments developed in [28, Lemma 3.9] and [39, Proposition 11] to verify that u_{μ} is also a viscosity solution of

$$\mathcal{L}_{\alpha,p}u + \mathcal{L}_{\beta,q}u = 0 \quad \text{in } \Omega \setminus \left\{ x_{u_{\mu}} \right\}, \tag{3.22}$$

(recall the definition of $\mathcal{L}_{s,m}$ in (3.7)). This means that u_{μ} is both a supersolution and a subsolution of (3.22), that is, u_{μ} meets the (respective) requirements:

- $(\mathcal{L}_{\alpha,p}\varphi)(x_0) + (\mathcal{L}_{\beta,q}\varphi)(x_0) \leq 0$ for every pair $(x_0,\varphi) \in (\Omega \setminus \{x_{u_\mu}\}) \times C_c^1(\mathbb{R}^N)$ satisfying $\varphi(x_0) = u_\mu(x_0)$ and $\varphi(x) \leq u_\mu(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_{u_\mu}, x_0\},$
- $(\mathcal{L}_{\alpha,p}\varphi)(x_0) + (\mathcal{L}_{\beta,q}\varphi)(x_0) \ge 0$ for every pair $(x_0,\varphi) \in D \times C_c^1(\mathbb{R}^N)$ satisfying $\varphi(x_0) = u_\mu(x_0)$ and $\varphi(x) \ge u_\mu(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_{u_\mu}, x_0\}.$

Remark 3.2.11 As observed in [39], if D is a bounded domain of \mathbb{R}^N , $m > \frac{1}{s-1}$ and $\varphi \in C_c^1(\mathbb{R}^N)$, then the function $\mathcal{L}_{s,m}\varphi$ given by (3.7) is well defined and continuous at each point $x \in D$. Obviously, the same holds for $\psi = \varphi + k$, where k is an arbitrary constant and $\varphi \in C_c^1(\mathbb{R}^N)$, since

$$\left(\mathcal{L}_{s,m}\psi\right)(x) = \left(\mathcal{L}_{s,m}\varphi\right)(x).$$

Moreover, it is simple to check that u_{μ} fulfills both requirements above even for test functions of the form $\psi = \varphi + k$.

It is interesting to notice that $u_{\mu} > 0$ in $\Omega \setminus \{x_{u_{\mu}}\}$ as consequence of u_{μ} being a supersolution of (3.22). The argument comes from [39, Lemma 12]: by supposing that $u_{\mu}(x_0) = 0$ for some $x_0 \in \Omega \setminus \{x_{u_{\mu}}\}$ and noting that $0 \not\equiv u \geq 0$, we can find a nonnegative and nontrivial test function $\varphi \in C_c^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = 0 \le \varphi(x) \le u_\mu(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_u, x_0\}.$$

Hence,

$$0 \leq \int_{\mathbb{R}^N} \frac{2 |\varphi(y)|^{p-1}}{|x_0 - y|^{N+\alpha p}} \mathrm{d}y = \int_{\mathbb{R}^N} \frac{2 |\varphi(y)|^{p-2} \varphi(y)}{|x_0 - y|^{N+\alpha p}} \mathrm{d}y \leq (\mathcal{L}_{\alpha, p}\varphi)(x_0) + (\mathcal{L}_{\beta, q}\varphi)(x_0) \leq 0,$$

which leads to the contradiction $\varphi \equiv 0$.

3.3 Asymptotic behavior as p goes to infinity

Let D be a bounded smooth (at least Lipschitz) domain of \mathbb{R}^N . We recall that $(C_0^{0,s}(\overline{D}), |\cdot|_s)$ is a Banach space, but

$$C_0^{0,s}(\overline{D}) \neq \overline{C_c^{\infty}(D)}^{|\cdot|_s}$$

That is, $C_c^{\infty}(D)$ is not $|\cdot|_s$ -dense in $C_0^{0,s}(\overline{D})$.

However, we have the following lemma that follows from [27, Lemma 9].

Lemma 3.3.1 Let $v \in C_0^{0,s}(\overline{D})$. There exists a sequence $\{v_k\} \subset C_c^{\infty}(D)$ such that

$$\lim_{k \to \infty} \|v_k\|_{\infty} = \|v\|_{\infty} \quad \text{and} \quad \limsup_{k \to \infty} |v_k|_s \le |v|_s.$$

Now, returning to our bounded domain Ω , let

$$R := \max_{x \in \overline{\Omega}} \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega).$$

It is the inradius of Ω : the radius of the largest ball inscribed in Ω .

Let $B_R(x_0)$ be a ball centered at $x_0 \in \Omega$ with radius R and let $\phi_R : \overline{B_R(x_0)} \to [0, R]$ be the distance function to the boundary $\partial B_R(x_0)$, that is,

$$\phi_R(x) := R - |x - x_0|$$

It is simple to verify that $\phi_R \in C_0^{0,s}(\overline{B_R(x_0)})$, for every $s \in (0, 1]$, with

$$\|\phi_R\|_{\infty} = R \text{ and } |\phi_R|_s = R^{1-s}.$$
 (3.23)

Moreover, it is clear that ϕ_R extended by zero outside $B_R(x_0)$ belongs to $C_0^{0,s}(\overline{\Omega})$ and its *s*-Hölder seminorm is preserved. In particular, such an extension is a Lipschitz function vanishing outside Ω . Hence,

 $\phi_R \in W_0^{1,m}(\Omega) \hookrightarrow W_0^{s,m}(\Omega) \quad \forall s \in (0,1) \text{ and } m \ge 1.$

(Note that we are considering Ω at least a Lipschitz domain.) Consequently, we can apply [27, Lemma 7] to conclude that

$$\lim_{m \to \infty} [\phi_R]_{s,m} = |\phi_R|_s = R^{1-s}, \quad \text{for each } s \in (0,1).$$
(3.24)

The proof of the following proposition is adapted from [39] where (3.6) is proved.

Proposition 3.3.2 For each $s \in (0, 1]$ one has

$$\lim_{m \to \infty} \sqrt[m]{\lambda_{s,m}} = R^{-s} = \frac{|\phi_R|_s}{\|\phi_R\|_\infty} = \min_{v \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}} \frac{|v|_s}{\|v\|_\infty}.$$
(3.25)

Proof. The second equality in (3.25) follows from (3.23). Since $\phi_R \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}$ to prove the third equality in (3.25) it suffices to verify that

$$R^{-s} \le \frac{|v|_s}{\|v\|_{\infty}} \quad \forall v \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}.$$
(3.26)

Let $v \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}$. According to Lemma 3.3.1, there exists a sequence $\{v_k\} \subset C_c^{\infty}(\Omega)$ such that

$$\lim_{k \to \infty} \|v_k\|_{\infty} = \|v\|_{\infty} \quad \text{and} \quad \limsup_{k \to \infty} |v_k|_s \le |v|_s.$$

Hence, (3.6) yields

$$R^{-s} \le \limsup_{k \to \infty} \frac{|v_k|_s}{\|v_k\|_\infty} \le \frac{|v|_s}{\|v\|_\infty},$$

concluding the proof of the third equality in (3.25)

Now, let us prove that

$$\lim_{m \to \infty} \sqrt[m]{\lambda_{s,m}} = R^{-s}.$$

First, observing that

$$\sqrt[m]{\lambda_{s,m}} \le \frac{[\phi_R]_{s,m}}{\|\phi_R\|_{\infty}}$$

we obtain from (3.23) and (3.24) that

$$\limsup_{m \to \infty} \sqrt[m]{\lambda_{s,m}} \le \lim_{m \to \infty} \frac{[\phi_R]_{s,m}}{\|\phi_R\|_{\infty}} = \frac{|\phi_R|_s}{\|\phi_R\|_{\infty}} = R^{-s}.$$
(3.27)

To prove that

$$R^{-s} \le \liminf_{m \to \infty} \left(\sqrt[m]{\lambda_{s,m}}\right) \tag{3.28}$$

we fix $m_0 > \frac{N}{s}$ and take, for each *m* sufficiently large, $u_m \in W_0^{s,m}(\Omega)$ such that $||u_m||_{\infty} = 1$ and

$$\lambda_{s,m} = \left[u_m\right]_{s,m}^m$$

According to (3.5), we have

$$\begin{aligned} |u_m|_{s-\frac{N}{m_0}} &= \sup_{(x,y)\neq(0,0)} \frac{|u_m(x) - u_m(y)|}{|x - y|^{s-\frac{N}{m_0}}} \\ &= \sup_{(x,y)\neq(0,0)} \frac{|u_m(x) - u_m(y)|}{|x - y|^{s-\frac{N}{m}}} |x - y|^{(\frac{N}{m_0} - \frac{N}{m})} \\ &\leq (\operatorname{diam}(\Omega))^{(\frac{N}{m_0} - \frac{N}{m})} \sup_{(x,y)\neq(0,0)} \frac{|u_m(x) - u_m(y)|}{|x - y|^{s-\frac{N}{m}}} \\ &\leq (\operatorname{diam}(\Omega))^{(\frac{N}{m_0} - \frac{N}{m})} C [u_m]_{s,m} = C(\operatorname{diam}(\Omega))^{(\frac{N}{m_0} - \frac{N}{m})} \sqrt[m]{\lambda_{s,m}}. \end{aligned}$$

The estimate (3.27) implies that $\{u_m\}$ is uniformly bounded in the Hölder space $C_0^{0,\beta-\frac{N}{m_0}}(\overline{\Omega})$, which is compactly embedded in $C_0(\overline{\Omega})$. It follows that, up to a subsequence, $\{u_m\}$ converges uniformly in $\overline{\Omega}$ to a function $u \in C_0(\overline{\Omega})$ such that $||u||_{\infty} = 1$.

For each 1 < k < m, we have, by Hölder's inequality,

$$\int_{\Omega} \int_{\Omega} \frac{|u_m(x) - u_m(y)|^k}{|x - y|^{(\frac{N}{m} + s)k}} \mathrm{d}x \mathrm{d}y \le |\Omega|^{2(1 - \frac{k}{m})} \left(\int_{\Omega} \int_{\Omega} \frac{|u_m(x) - u_m(y)|^m}{|x - y|^{N + sm}} \mathrm{d}x \mathrm{d}y \right)^{\frac{k}{m}} \le |\Omega|^{2(1 - \frac{k}{m})} \left([u_m]_{s,m} \right)^k = |\Omega|^{2(1 - \frac{k}{m})} \left(\sqrt[m]{\lambda_{s,m}} \right)^k$$

Making $m \to \infty$, using the uniform convergence, Fatou's Lemma and the above estimate we obtain

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{k}}{|x - y|^{sk}} \mathrm{d}x \mathrm{d}y \leq \liminf_{m \to \infty} \int_{\Omega} \int_{\Omega} \frac{|u_{m}(x) - u_{m}(y)|^{k}}{|x - y|^{\frac{Nk}{m} + sk}} \mathrm{d}x \mathrm{d}y$$
$$\leq |\Omega|^{2} \liminf_{m \to \infty} \left(\sqrt[m]{\lambda_{s,m}}\right)^{k}.$$

Therefore,

$$|u|_{s} = \lim_{k \to \infty} \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{k}}{|x - y|^{sk}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{k}} \le \liminf_{m \to \infty} \left(\sqrt[m]{\lambda_{s,m}} \right)$$

Since $R^{-s} \leq |u|_s$ (according to (3.26)) we obtain (3.28).

In the remaining of this section we fix $\alpha, \beta \in (0, 1)$, with $\alpha \neq \beta$, and consider q a continuous function of p satisfying

$$\lim_{p \to \infty} \frac{q}{p} =: Q \in \begin{cases} (0,1) & \text{if } 0 < \beta < \alpha < 1\\ (1,\infty) & \text{if } 0 < \alpha < \beta < 1. \end{cases}$$
(3.29)

We maintain the notation q instead of q(p) to simplify the presentation. Note that (3.29) implies that

$$\lim_{p \to \infty} q = \infty$$

Moreover, q < p if $Q \in (0, 1)$ and p < q if $Q \in (1, \infty)$.

Our goal is to study the asymptotic behavior, as $p \to \infty$, of the least energy solution u_p of the problem

$$\|u\|_{\infty} = u(x_p) \quad \text{and} \quad \begin{cases} \left[(-\Delta_p)^{\alpha} + (-\Delta_q)^{\beta} \right] u = \mu_p \|u\|_{\infty}^{p-1} \delta_{x_p} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$
(3.30)

where μ_p satisfies

$$\Lambda := \lim_{p \to \infty} \sqrt[p]{\mu_p} > R^{-\alpha}, \tag{3.31}$$

with R denoting the inradius of Ω .

This condition guarantees that

$$\mu_p > \lambda_{\alpha, p} \tag{3.32}$$

for all p sufficiently large, say $p > p_0$. Moreover, by taking a larger p_0 one of the conditions (3.9) or (3.10) is fulfilled. So, according to Theorem 3.2.10, for each $p > p_0$ the problem (3.30) has at least one positive least energy solution

$$u_p \in X_p(\Omega) := \begin{cases} W_0^{\alpha, p}(\Omega) & \text{if } 0 < \beta < \alpha < 1\\ W_0^{\beta, q}(\Omega) & \text{if } 0 < \alpha < \beta < 1 \end{cases}$$

Remark 3.3.3 Combining (3.24) and (3.31) we have

$$\lim_{p \to \infty} \frac{[\phi_R]_{\alpha,p}}{\|\phi_R\|_{\infty}} = R^{-\alpha} < \Lambda := \lim_{p \to \infty} \sqrt[p]{\mu_p}.$$

Consequently, $\mu_p \|\phi_R\|_{\infty}^p > [\phi_R]_{\alpha,p}^p$ for all p large enough.

Proposition 3.3.4 Suppose (3.29) and (3.31) hold. Then,

$$\lim_{p \to \infty} \left[u_p \right]_{\beta,q} = \left(\Lambda R^{\beta} \right)^{\frac{1}{Q-1}} \quad \text{and} \quad \lim_{p \to \infty} \left\| u_p \right\|_{\infty} = R^{\beta} (\Lambda R^{\beta})^{\frac{1}{Q-1}}, \tag{3.33}$$

Proof. We assume that p is large enough so that u_p exists according to Theorem 3.2.10.

Since u_p is a weak solution of (3.30) and $W_0^{\beta,q}(\Omega)$ is continuously embedded into $C(\overline{\Omega})$ we have

$$[u_p]^q_{\beta,q} \le [u_p]^p_{\alpha,p} + [u_p]^q_{\beta,q} = \mu_p \, \|u_p\|^p_{\infty} \le \mu_p \frac{[u_p]^r_{\beta,q}}{(\sqrt[q]{\lambda_{\beta,q}})^p},\tag{3.34}$$

so that

$$\frac{\left(\left[u_p\right]_{\beta,q}\right)^{\frac{q}{p}}}{\sqrt[p]{\mu_p}} \le \left\|u_p\right\|_{\infty} \le \frac{\left[u_p\right]_{\beta,q}}{\sqrt[q]{\lambda_{\beta,q}}}$$

Hence, taking into account the first equality in (3.25) and (3.31) we easily check that the second limit in (3.33) is a consequence of the first one.

Let us then prove the first limit (3.33).

We start with the case $Q \in (1, \infty)$, where necessarily p < q (and $0 < \alpha < \beta$). After isolating $[u_p]_{\beta,q}$ in (3.34) we obtain

$$\limsup_{p \to \infty} \left[u_p \right]_{\beta,q} \le \lim_{p \to \infty} \left(\frac{\sqrt[p]{\mu_p}}{\sqrt[q]{\lambda_{\beta,q}}} \right)^{\frac{p}{q-p}} = \left(\Lambda R^{\beta} \right)^{\frac{1}{Q-1}}.$$
(3.35)

Let

$$t = \left(\frac{\mu_p \|\phi_R\|_{\infty}^p - [\phi_R]_{\alpha,p}^p}{[\phi_R]_{\beta,q}^q}\right)^{\frac{1}{q-p}} = \left(\frac{\mu_p R^p - [\phi_R]_{\alpha,p}^p}{[\phi_R]_{\beta,q}^q}\right)^{\frac{1}{q-p}}$$

(Note from Remark 3.3.3 that t is well-defined). It is simple to verify that

$$E_{\mu_p}(t\phi_R) = \left(\frac{1}{q} - \frac{1}{p}\right) t^q \left[\phi_R\right]_{\beta,q}^q.$$

Noticing that

$$\left(\frac{1}{q} - \frac{1}{p}\right) [u_p]_{\beta,q}^q = E_{\mu_p}(u_p) \le E_{\mu_p}(t\phi_R) = \left(\frac{1}{q} - \frac{1}{p}\right) t^q [\phi_R]_{\beta,q}^q < 0$$

we obtain

$$[u_p]_{\beta,q} \ge t \, [\phi_R]_{\beta,q} = \left(\frac{\mu_p R^p - [\phi_R]_{\alpha,p}^p}{[\phi_R]_{\beta,q}^q}\right)^{\frac{1}{q-p}} [\phi_R]_{\beta,q} = \left(\frac{\sqrt[p]{\mu_p} R}{[\phi_R]_{\beta,q}} \sqrt[p]{1 - (a_p)^p}\right)^{\frac{p}{q-p}},$$

where

$$a_p := \frac{[\phi_R]_{\alpha,p}}{\sqrt[p]{\mu_p}R}.$$
(3.36)

.

Since

$$\lim_{p \to \infty} a_p = \frac{R^{1-\alpha}}{\Lambda R} = \frac{R^{-\alpha}}{\Lambda} < 1,$$

we can verify that

$$\lim_{p \to \infty} \sqrt[p]{1 - (a_p)^p} = 1.$$

Hence,

$$\liminf_{p \to \infty} [u_p]_{\beta,q} \ge \lim_{p \to \infty} \left(\frac{\sqrt[p]{\mu_p}R}{[\phi_R]_{\beta,q}} \right)^{\frac{p}{q-p}} \lim_{p \to \infty} \left(\sqrt[p]{1 - (a_p)^p} \right)^{\frac{p}{q-p}} = \left(\frac{\Lambda R}{R^{1-\beta}} \right)^{\frac{1}{Q-1}} = (\Lambda R^\beta)^{\frac{1}{Q-1}}.$$

Combining this with (3.35) we obtain the first limit in (3.33).

Now, let us analyze the case $Q \in (0,1)$, where necessarily q < p (and $0 < \beta < \alpha$). In this case,

$$0 < \left(\frac{1}{q} - \frac{1}{p}\right) [u_p]_{\beta,q}^q = E_{\mu_p}(u_p) \le E_{\mu_p}(t\phi_R) = \left(\frac{1}{q} - \frac{1}{p}\right) t^q [\phi_R]_{\beta,q}^q,$$

where

$$t = \left(\frac{[\phi_R]_{\beta,q}^q}{\mu_p \|\phi_R\|_{\infty}^p - [\phi_R]_{\alpha,p}^p}\right)^{\frac{1}{p-q}} = \left(\frac{[\phi_R]_{\beta,q}^q}{\mu_p R^p - [\phi_R]_{\alpha,p}^q}\right)^{\frac{1}{p-q}}$$

(which is also well-defined according to Remark 3.3.3). It follows that

$$[u_p]_{\beta,q} \le t \, [\phi_R]_{\beta,q} = \left(\frac{[\phi_R]_{\beta,q}}{\sqrt[p]{\mu_p}R \sqrt[p]{1-(a_p)^p}}\right)^{\frac{p}{p-q}}$$

where a_p is also given by (3.36). Consequently,

$$\limsup_{p \to \infty} [u_p]_{\beta,q} \le \lim_{p \to \infty} \left(\frac{[\phi_R]_{\beta,q}}{\sqrt[p]{\mu_p} R \sqrt[p]{1 - (a_p)^p}} \right)^{\frac{p}{p-q}} = \left(\frac{R^{1-\beta}}{\Lambda R} \right)^{\frac{1}{1-Q}} = \left(\Lambda R^{\beta} \right)^{\frac{1}{Q-1}}.$$
 (3.37)

After isolating $[u_p]_{\beta,q}$ in (3.34) we obtain

$$\liminf_{p \to \infty} \left[u_p \right]_{\beta,q} \ge \lim_{p \to \infty} \left(\frac{\sqrt[q]{\lambda_{\beta,q}}}{\sqrt[p]{\mu_p}} \right)^{\frac{p}{p-q}} = \left(\frac{R^{-\beta}}{\Lambda} \right)^{\frac{1}{1-Q}} = \left(\Lambda R^{\beta} \right)^{\frac{1}{Q-1}},$$

which combined with (3.37) provides the first limit in (3.33).

Corollary 3.3.5 Suppose (3.29) and (3.31) hold. Then,

$$\frac{1}{\Lambda R^{\alpha}} (\Lambda R^{\beta})^{\frac{Q}{Q-1}} \leq \liminf_{p \to \infty} \left[u_p \right]_{\alpha, p} \leq \limsup_{p \to \infty} \left[u_p \right]_{\alpha, p} \leq (\Lambda R^{\beta})^{\frac{Q}{Q-1}}$$

Proof. It follows from the second limit in (3.33) combined with the estimates

$$\lambda_{\alpha,p} \|u_p\|_{\infty}^p \le [u_p]_{\alpha,p}^p \le [u_p]_{\alpha,p}^p + [u_p]_{\beta,q}^q = \mu_p \|u_p\|_{\infty}^p.$$

In the next proposition we prove that the limit functions of the family $\{u_p\}_{p>p_0}$, as $p \to \infty$, belongs to $C_0^{0,\beta}(\overline{\Omega})$ and minimize the quotient $|u|_{\beta} / ||u||_{\infty}$ in $C_0^{0,\beta}(\overline{\Omega}) \setminus \{0\}$.

Proposition 3.3.6 Let $\{p_n\}$ and $\{q_n\}$ satisfying (3.29), with $p_n \to \infty$, and let $\mu_n := \mu_{p_n}$ satisfying (3.31). Then, there exist $u_{\infty} \in C_0^{0,\beta}(\overline{\Omega})$ and $x_{\infty} \in \Omega$ such that, up to subsequences, $u_{p_n} \to u_{\infty}$ uniformly in $\overline{\Omega}$ and $x_{p_n} \to x_{\infty} \in \Omega$, with

$$u_{\infty}(x_{\infty}) = \left\| u_{\infty} \right\|_{\infty} = R^{\beta} (\Lambda R^{\beta})^{\frac{1}{Q-1}}.$$

Moreover,

$$u_{\infty}|_{\beta} = (\Lambda R^{\beta})^{\frac{1}{Q-1}} = \lim \left[u_{p_n} \right]_{\beta,q_n}$$

and

$$\frac{|u_{\infty}|_{\beta}}{\|u_{\infty}\|_{\infty}} = \frac{1}{R^{\beta}} = \min_{u \in C_0^{0,\beta}(\overline{\Omega}) \setminus \{0\}} \frac{|u|_{\beta}}{\|u\|_{\infty}}.$$
(3.38)

Proof. Since Ω is bounded, we can assume that (passing to a subsequence) x_{p_n} converges to a point $x_{\infty} \in \overline{\Omega}$. Fix $m_0 > N/\beta$ and assume that n is large enough so that $m_0 < \{p_n, q_n\}$.

Taking into account the inequality (3.5), we have (as in Proposition 3.3.2)

$$\begin{aligned} |u_{p_n}|_{\beta-\frac{N}{m_0}} &= \sup_{(x,y)\neq(0,0)} \frac{|u_{p_n}(x) - u_{p_n}(y)|}{|x - y|^{\beta-\frac{N}{m_0}}} \\ &\leq (\operatorname{diam}(\Omega))^{(\frac{N}{m_0} - \frac{N}{q_n})} \sup_{(x,y)\neq(0,0)} \frac{|u_{p_n}(x) - u_{p_n}(y)|}{|x - y|^{\beta-\frac{N}{q_n}}} \\ &\leq (\operatorname{diam}(\Omega))^{(\frac{N}{m_0} - \frac{N}{q_n})} C [u_{p_n}]_{\beta,q_n} \,. \end{aligned}$$

The first limit in (3.33) implies that $\{u_{p_n}\}$ is uniformly bounded in the Hölder space $C_0^{0,\beta-\frac{N}{m_0}}(\overline{\Omega})$, which is compactly embedded in $C_0(\overline{\Omega})$. It follows that, up to a subsequence, $\{u_{p_n}\}$ converges uniformly in $\overline{\Omega}$ to a function $u_{\infty} \in C_0(\overline{\Omega})$. Of course, $||u_{\infty}|| = u_{\infty}(x_{\infty})$ and, by virtue of the second limit in (3.33),

$$u_{\infty}(x_{\infty}) = R^{\beta} (\Lambda R^{\beta})^{\frac{1}{Q-1}} > 0,$$

so that $x_{\infty} \notin \partial \Omega$.

Now, if $m > m_0$ and n is sufficiently large such that $q_n > m$, Hölder's inequality yields

$$\int_{\Omega} \int_{\Omega} \frac{|u_{p_n}(x) - u_{p_n}(y)|^m}{|x - y|^{\frac{Nm}{q_n} + \beta m}} \mathrm{d}x \mathrm{d}y \le |\Omega|^{2(1 - \frac{m}{q_n})} \left(\int_{\Omega} \int_{\Omega} \frac{|u_{p_n}(x) - u_{p_n}(y)|^{q_n}}{|x - y|^{N + \beta q_n}} \mathrm{d}x \mathrm{d}y \right)^{\frac{m}{q_n}} \le |\Omega|^{2(1 - \frac{m}{q_n})} \left([u_{p_n}]_{\beta, q_n} \right)^m.$$

Hence, combining the first limit in (3.33) and Fatou's Lemma,

$$\int_{\Omega} \int_{\Omega} \frac{|u_{\infty}(x) - u_{\infty}(y)|^m}{|x - y|^{\beta m}} \mathrm{d}x \mathrm{d}y \leq \liminf_{n \to \infty} \int_{\Omega} \int_{\Omega} \frac{|u_{p_n}(x) - u_{p_n}(y)|^m}{|x - y|^{\frac{Nm}{q_n} + \beta m}} \mathrm{d}x \mathrm{d}y$$
$$\leq |\Omega|^2 \liminf_{n \to \infty} \left([u_{p_n}]_{\beta, q_n} \right)^m = |\Omega|^2 \left(\Lambda R^{\beta} \right)^{\frac{m}{Q-1}}.$$

Therefore,

$$|u_{\infty}|_{\beta} = \lim_{m \to \infty} \left(\int_{\Omega} \int_{\Omega} \frac{|u_{\infty}(x) - u_{\infty}(y)|^m}{|x - y|^{\beta m}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{m}} \le (\Lambda R^{\beta})^{\frac{1}{Q-1}}.$$

It follows that $u_{\infty} \in C_0^{0,\beta}(\overline{\Omega})$. Hence, observing that

$$\frac{1}{R^{\beta}} = \min_{v \in C_0^{0,\beta}(\overline{\Omega}) \setminus \{0\}} \frac{|v|_{\beta}}{\|v\|_{\infty}} \le \frac{|u_{\infty}|_{\beta}}{\|u_{\infty}\|_{\infty}} = \frac{|u_{\infty}|_{\beta}}{R^{\beta} (\Lambda R^{\beta})^{\frac{1}{Q-1}}}$$

we obtain

$$|u_{\infty}|_{\beta} \ge (\Lambda R^{\beta})^{\frac{1}{Q-1}}$$

Therefore,

$$|u_{\infty}|_{\beta} = (\Lambda R^{\beta})^{\frac{1}{Q-1}} = \lim \left[u_{p_n} \right]_{\beta,q_n}$$

and

$$\frac{|u_{\infty}|_{\beta}}{\|u_{\infty}\|_{\infty}} = \frac{\left(\Lambda R^{\beta}\right)^{\frac{1}{Q-1}}}{R^{\beta}(\Lambda R^{\beta})^{\frac{1}{Q-1}}} = \frac{1}{R^{\beta}} = \min_{v \in C_{0}^{0,\beta}(\overline{\Omega}) \setminus \{0\}} \frac{|v|_{\beta}}{\|v\|_{\infty}}.$$

Remark 3.3.7 Considering Corollary 3.3.5 we can reproduce the proof of Proposition 3.3.6 to conclude that, in the case $Q \in (0,1)$, the limit function is more regular: $u_{\infty} \in C_0^{0,\alpha}(\overline{\Omega})$ and, moreover,

$$R^{-\alpha} \le \frac{|u_{\infty}|_{\alpha}}{\|u_{\infty}\|_{\infty}} \le \Lambda.$$

These estimates are also valid in the complementary case $Q \in (1, \infty)$, where obviously the β -regularity is better that α -regularity since $0 < \alpha < \beta$.

Corollary 3.3.8 One has

$$u_{\infty}(x) \leq \left(\Lambda R^{\beta}\right)^{\frac{1}{Q-1}} \left(\operatorname{dist}(x,\partial\Omega)\right)^{\beta} \quad \forall x \in \Omega$$

and, therefore, the maximum point x_{∞} of u_{∞} is also a maximum point of the distance function to the boundary $\partial \Omega$.

Proof. For each $x \in \Omega$ let $y_x \in \partial \Omega$ be such

$$\operatorname{dist}(x,\partial\Omega) = |x - y_x|.$$

Then, since $u_{\infty}(y_x) = 0$ and $|u_{\infty}|_{\beta} = (\Lambda R^{\beta})^{\frac{1}{Q-1}}$, we get

$$u_{\infty}(x) = |u_{\infty}(x) - u_{\infty}(y_x)| \le |u_{\infty}|_{\beta} |x - y_x|^{\beta} = (\Lambda R^{\beta})^{\frac{1}{Q-1}} (\operatorname{dist}(x, \partial \Omega))^{\beta}.$$

Hence, observing that $\operatorname{dist}(x,\partial\Omega) = |x - y_x| \leq R$ and $u_{\infty}(x_{\infty}) = R^{\beta}(\Lambda R^{\beta})^{\frac{1}{Q-1}}$, we obtain

$$R^{\beta}(\Lambda R^{\beta})^{\frac{1}{Q-1}} \leq \left(\Lambda R^{\beta}\right)^{\frac{1}{Q-1}} \left(\operatorname{dist}(x_{\infty},\partial\Omega)\right)^{\beta} \leq \left(\Lambda R^{\beta}\right)^{\frac{1}{Q-1}} R^{\beta},$$

so that

$$\operatorname{dist}(x_{\infty}, \partial \Omega) = R = \left\| \operatorname{dist}(\cdot, \partial \Omega) \right\|_{\infty}.$$

In the sequel, we argue that the function u_{∞} is a viscosity solution of the equation

$$\max\left\{\mathcal{L}_{\alpha}^{+}u,\left(\mathcal{L}_{\beta}^{+}u\right)^{Q}\right\} = \max\left\{-\mathcal{L}_{\alpha}^{-}u,\left(-\mathcal{L}_{\beta}^{-}u\right)^{Q}\right\}$$
(3.39)

in $\Omega \setminus \{x_{\infty}\}$ (the operators \mathcal{L}_{α}^+ and \mathcal{L}_{α}^- are defined in (3.3)). This means that u_{∞} is both a supersolution and a subsolution of (3.39) or, equivalently, u_{∞} meets the (respective) requirements:

• $\max \left\{ (\mathcal{L}^{+}_{\alpha}\varphi)(x_{0}), \left((\mathcal{L}^{+}_{\beta}\varphi)(x_{0}) \right)^{Q} \right\} \leq \max \left\{ (-\mathcal{L}^{-}_{\alpha}\varphi)(x_{0}), \left((-\mathcal{L}^{-}_{\beta}\varphi)(x_{0}) \right)^{Q} \right\}$ for every the pair $(x_{0}, \varphi) \in (\Omega \setminus \{x_{\infty}\}) \times C^{1}_{c}(\mathbb{R}^{N})$ satisfying

 $\varphi(x_0) = u(x_0)$ and $\varphi(x) \le u(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_0, x_\infty\},\$

• $\max \left\{ (\mathcal{L}^{+}_{\alpha}\varphi)(x_{0}), \left((\mathcal{L}^{+}_{\beta}\varphi)(x_{0}) \right)^{Q} \right\} \geq \max \left\{ (-\mathcal{L}^{-}_{\alpha}\varphi)(x_{0}), \left((-\mathcal{L}^{-}_{\beta}\varphi)(x_{0}) \right)^{Q} \right\}$ for every the pair $(x_{0}, \varphi) \in (\Omega \setminus \{x_{\infty}\}) \times C^{1}_{c}(\mathbb{R}^{N})$ satisfying

$$\varphi(x_0) = u(x_0) \text{ and } \varphi(x) \ge u(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_0, x_\infty\}.$$

A proof of the following result (where $t^{\pm} = \max{\{\pm t, 0\}}$), adapted from [15, Lemma 6.5], can be found in [28, Lemma 6.1].

Lemma 3.3.9 If $s \in (0,1)$, $\varphi \in C_c^1(\mathbb{R}^N)$, $\lim_{m \to \infty} z_m \to x$, then,

$$\lim_{m \to \infty} A_m(\varphi(z_m)) = (\mathcal{L}_{s,\infty}^+ \varphi)(x_0) \quad \text{and} \quad \lim_{m \to \infty} B_m(\varphi(z_m)) = (-\mathcal{L}_{s,\infty}^- \varphi)(x_0),$$

where

$$(A_m(\varphi(z_m)))^{m-1} := 2 \int_{\mathbb{R}^N} \frac{|\varphi(y) - \varphi(z_m)|^{m-2} (\varphi(y) - \varphi(z_m))^+}{|y - z_m|^{N+sm}} \mathrm{d}y.$$

and

$$(B_m(\varphi(z_m)))^{m-1} := 2 \int_{\mathbb{R}^N} \frac{|\varphi(y) - \varphi(z_m)|^{m-2} (\varphi(y) - \varphi(z_m))^-}{|y - z_m|^{N+sm}} \mathrm{d}y.$$

The proof of the next result is based on [28] and [39].

Proposition 3.3.10 The function u_{∞} is a viscosity solution of (3.39) in the punctured domain $\Omega \setminus \{x_{\infty}\}$. Moreover, $u_{\infty} > 0$ in Ω .

Proof. In order to verify that u_{∞} is a supersolution of (3.39) in $\Omega \setminus \{x_{\infty}\}$ we fix a pair $(x_0, \varphi) \in (\Omega \setminus \{x_{\infty}\}) \times C_c^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = u_{\infty}(x_0) \text{ and } \varphi(x) \le u_{\infty}(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_0, x_{\infty}\}$$

Since $x_0 \neq x_{\infty} = \lim x_n$, we can assume that there exist $n_0 \in \mathbb{N}$ and a ball $B_{\rho}(x_0)$, centered at x_0 and with radius ρ , such that

$$x_n \notin B_{\rho}(x_0) \subset (\Omega \setminus \{x_\infty\}) \quad \forall n \ge n_0.$$

Hence,

$$\mathcal{L}_{\alpha,p_n}u_n + \mathcal{L}_{\beta,q_n}u_n = 0 \quad \text{in } B_{\rho}(x_0), \quad \forall n \ge n_0,$$
(3.40)

in the viscosity sense.

By standard arguments, we can construct a sequence $\{z_n\} \subset B_\rho(x_0)$ such that $z_n \to x_0$ and

$$k_n := \min_{B_{\rho}(x_0)} (u_n(x) - \varphi(x)) = u_n(z_n) - \varphi(z_n) < u_n(x) - \varphi(x) \quad \forall x \neq x_n.$$

It follows that the function $\psi_n := \varphi + k_n$ satisfies

 $\psi(z_n) = u_n(z_n)$ and $\psi(x) < u_n(x) \quad \forall x \in B_\rho(x_0).$

Consequently, (see Remark 3.2.11)

$$(\mathcal{L}_{\alpha,p_n}\psi_n)(z_n) + (\mathcal{L}_{\beta,q_n}\psi_n)(z_n) = (\mathcal{L}_{\alpha,p_n}\varphi)(z_n) + (\mathcal{L}_{\beta,q_n}\varphi)(z_n) \le 0, \quad \forall n \ge n_0.$$

The inequality can be write as

$$A_{p_n}^{p_n-1} + A_{q_n}^{q_n-1} \le B_{p_n}^{p_n-1} + B_{q_n}^{q_n-1},$$
(3.41)

where $A_{p_n} := A_{p_n}(\varphi(z_n)), A_{q_n} := A_{q_n}(\varphi(z_n)), B_{p_n} := B_{p_n}(\varphi(z_n))$ and $B_{q_n} := B_{q_n}(\varphi(z_n)).$

We have

$$\lim_{p_{n} \to 1} \sqrt{A_{p_{n}}^{p_{n}-1} + A_{q_{n}}^{q_{n}-1}} = \lim_{p_{n} \to 1} A_{p_{n}} \left(\sqrt[p_{n}-1]{1 + \left((A_{q_{n}})^{\frac{q_{n}-1}{p_{n}-1}} / A_{p_{n}} \right)^{p_{n}-1}} \right)$$
$$= \max_{p_{n} \to 1} \left\{ \lim_{p_{n} \to 1} A_{p_{n}}, (\lim_{p_{n} \to 1} A_{q_{n}})^{\lim_{p_{n} \to 1} \frac{q_{n}-1}{p_{n}-1}} \right\}$$
$$= \max_{p_{n} \to 1} \left\{ (\mathcal{L}_{\alpha}^{+}\varphi)(x_{0}), \left[(\mathcal{L}_{\beta}^{+}\varphi)(x_{0}) \right]^{Q} \right\},$$

where the latter equality follows from Lemma 3.3.9. Analogously, we compute

$$\lim \sqrt[p_{p_n}]{}^{p_n-1}\sqrt{B_{p_n}^{p_n-1}+B_{q_n}^{q_n-1}} = \max\left\{\left(-\mathcal{L}_{\alpha}^{-}\varphi\right)(x_0), \left[\left(-\mathcal{L}_{\beta}^{-}\varphi\right)(x_0)\right]^Q\right\}$$

Therefore, (3.41) yields

$$\max\left\{ (\mathcal{L}_{\alpha}^{+}\varphi)(x_{0}), \left[\left(\mathcal{L}_{\beta}^{+}u\right)(x_{0}) \right]^{Q} \right\} \leq \max\left\{ (-\mathcal{L}_{\alpha}^{-}\varphi)(x_{0}), \left[\left(-\mathcal{L}_{\beta}^{-}u\right)(x_{0}) \right]^{Q} \right\},$$

which shows that u_{∞} is a viscosity supersolution of (3.39) in $\Omega \setminus \{x_{\infty}\}$.

Similarly, by symmetric arguments, we can prove that u_{∞} is a viscosity subsolution of (3.39) in $\Omega \setminus \{x_{\infty}\}$.

The positivity of u_{∞} in Ω comes from the fact that u_{∞} is a supersolution of (3.39). Indeed, adapting the argument of [39, Lemma 22], if $u_{\infty}(x_0) = 0$, then either

$$(\mathcal{L}^+_{\alpha}\varphi)(x_0) \le (-\mathcal{L}^-_{\alpha}\varphi)(x_0) \quad \text{or} \quad (\mathcal{L}^+_{\beta}\varphi)(x_0) \le (-\mathcal{L}^-_{\beta}\varphi)(x_0)$$

for a nonnegative, nontrivial $\varphi \in C_c^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = u_{\infty}(x_0) = 0 \le \varphi(x) \le u_{\infty}(x) \quad \forall x \in \mathbb{R}^N \setminus \{x_0, x_{\infty}\}.$$

In the first case, this yields

$$\frac{\varphi(z)}{|x_0 - y|^{\alpha}} \le \sup_{y \in \mathbb{R}^N \setminus \{x_0\}} \frac{\varphi(y)}{|x_0 - y|^{\alpha}} + \inf_{y \in \mathbb{R}^N \setminus \{x_0\}} \frac{\varphi(y)}{|x_0 - y|^{\alpha}} \le 0 \quad \forall z \in \mathbb{R}^N \setminus \{x_0\}$$

and leads to the contradiction $\varphi \equiv 0$. Obviously, in the second case we arrive at the same contradiction.

Proof of Theorem 3.0.1. It follows by gathering Proposition 3.3.6, Corollary 3.3.8 and Proposition 3.3.10. ■

Appendix

A.1 Estimates for $u_{\delta,R}$

Let U be a radially symmetric and decreasing minimizer for the Sobolev constant defined in (1.3) for m = p and it is know from [12] that there exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that

$$\frac{c_1}{|x|^{\frac{N-sp}{p-1}}} \le U(|x|) \le \frac{c_1}{|x|^{\frac{N-sp}{p-1}}}, \quad \forall |x| \ge 1,$$
(A.1)

$$\frac{U(\theta r)}{U(r)} \le \frac{1}{2}, \quad \forall r \ge 1.$$
(A.2)

Multiplying U by a positive constant if necessary, we may assume that U satisfies the following:

(i)
$$(-\Delta_p)^s U = U^{p_s^* - 1}$$
 in \mathbb{R}^N
(ii) $[U]_{s,p}^p = ||U||_{p_s^*}^{p_s^*} = S^{\frac{N}{sp}}.$

For any $\delta > 0$, the function

$$U_{\delta}(x) = \delta^{-\frac{N-sp}{p}} U(|x|/\delta)$$

is also a minimizer for S, satisfying (i) and (ii).

We may assume that $0 \in \Omega_g$. For any $\delta > 0$ and R > 0 consider the radially symmetric non-increasing function $\overline{u}_{\delta,R} : [0, \infty) \to \mathbb{R}$ by

$$\overline{u}_{\delta,R} = \begin{cases} U_{\delta}(r), & \text{se } r \leq R, \\ 0, & \text{se } r \geq \theta R. \end{cases}$$

Therefore, we have the following estimates from [42].

Lemma A.1.1 For any R > 0, exist C = C(N, p, s) > 0 such that for any $\delta \leq \frac{R}{2}$

$$\left[\overline{u}_{\delta,R}\right]_{s,p}^{p} \le S^{\frac{N}{sp}} + C\left(\frac{\delta}{R}\right)^{\frac{n_{sp}}{p-1}},\tag{A.3}$$

$$\|\overline{u}_{\delta,R}\|_p^p \ge \begin{cases} \frac{1}{C} \delta^{sp} log(R/\delta), & se \ N = sp^2, \\ \frac{1}{C} \delta^{sp}, & se \ N > sp^2. \end{cases}$$
(A.4)

$$\|\overline{u}_{\delta,R}\|_{p_{s}^{*}}^{p_{s}^{*}} \ge S^{\frac{N}{sp}} - C(\frac{\delta}{R})^{N/(p-1)}.$$
(A.5)

Let $\varepsilon > 0$. Since Ω_g is open, we can taking R > 0 fixed such that $B_{\theta R}(0) \subset \Omega_g$ and let us define the function $u_{\varepsilon,R} : [0,\infty) \to \mathbb{R}$ by

$$u_{\varepsilon,R}(r) = \varepsilon^{-\frac{N-sp}{p^2}} \overline{u}_{\delta,R}(r), \text{ with } \delta = \varepsilon^{\frac{p-1}{p}}.$$

Therefore applying (A.3)-(A.5) yield

$$\left[u_{\varepsilon,R}\right]_{s,p}^{p} \le S^{\frac{N}{sp}} \varepsilon^{-\frac{N-sp}{p}} + O(1).$$
(A.6)

The demonstrations of the following lemma can be found in [7].

Lemma A.1.2 Let $u_{\varepsilon,R}$ be defined as above. Then the following estimates hold for $t \ge 1$,

$$\begin{aligned} \|u_{\varepsilon,R}\|_{p_{s}^{*}}^{p} &= S^{\frac{N-sp}{sp}} \varepsilon^{-\frac{N-sp}{p}} + O(1). \end{aligned}$$
(A.7)
$$\|u_{\varepsilon,R}\|_{t}^{t} \geq \begin{cases} k\varepsilon^{\frac{N(p-1)-t(N-sp)}{p}} + O(1), & se \ t > \frac{N(p-1)}{N-sp}, \\ k|ln\varepsilon| + O(1), & se \ t = \frac{N(p-1)}{N-sp}. \\ O(1), & se \ t < \frac{N(p-1)}{N-sp} \end{cases}$$
(A.8)

The next result is a very important because it compares the different Gagliardo seminorms of the function $u_{\varepsilon,R}$.

Lemma A.1.3 The following estimates hold for $1 \le t \le \frac{N(p-1)}{N-s}$,

$$[u_{\varepsilon,R}]_{s,t}^t \le O(1), \text{ for } 1 \le t < \frac{N(p-1)}{N-s},$$
(A.9)

where k is a positive constant independent of ε .

Proof. Using the notation of [7], we have $\bar{u}_{\delta,R} = G_{\delta,R}(U_{\delta}(r))$ and thus

$$\begin{aligned} \left[u_{\varepsilon,R}\right]_{s,t}^{t} &= \varepsilon^{-\frac{(N-sp)t}{p^{2}}} \left[\bar{u}_{\delta,R}\right]_{s,t}^{t} \\ &= \varepsilon^{-\frac{(N-sp)t}{p^{2}}} \left[G_{\delta,R}(U_{\delta})\right]_{s,t}^{t} \\ &= \varepsilon^{-\frac{(N-sp)t}{p^{2}}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{\left|G_{\delta,R}(U_{\delta}(x)) - G_{\delta,R}(U_{\delta}(y))\right|^{t}}{|x-y|^{N+st}} \mathrm{d}x\mathrm{d}y \end{aligned}$$
(A.10)

Using the mean value theorem there exists $\tau \in (0, 1)$ such that

$$|G_{\delta,R}(U_{\delta}(x)) - G_{\delta,R}(U_{\delta}(y))| \le |G'_{\delta,R}(U_{\delta}(x) + \tau(U_{\delta}(y) - U_{\delta}(x))||U_{\delta}(y) - U_{\delta}(x)|$$
(A.11)

and from [7, Section 5] we have

$$G'_{\delta,R}\left(U_{\delta}(x) + \tau(U_{\delta}(y) - U_{\delta}(x)) \le 1 + \frac{c_2}{c_1}\theta^{\frac{N+sp}{p-1}} = c_3.$$
(A.12)

Substituting (A.11) and (A.12) into (A.10) we have

$$\begin{split} \left[u_{\varepsilon,R}\right]_{s,t}^{t} &\leq \varepsilon^{-\frac{(N-sp)t}{p^{2}}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{|G_{\delta,R}'(U_{\delta}(x) + \tau(U_{\delta}(y) - U_{\delta}(x))|^{t}|U_{\delta}(y) - U_{\delta}(x)|^{t}}{|x - y|^{N+st}} \mathrm{d}x\mathrm{d}y\\ &\leq \varepsilon^{-\frac{(N-sp)t}{p^{2}}} c_{3}^{t} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{|U_{\delta}(y) - U_{\delta}(x)|^{t}}{|x - y|^{N+st}} \mathrm{d}x\mathrm{d}y\\ &\leq C\varepsilon^{-\frac{(N-sp)t}{p^{2}}} \frac{\delta^{N-st}}{\delta^{\frac{(N-sp)t}{p}}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{|U(x) - U(y)|^{t}}{|x - y|^{N+st}} \mathrm{d}x\mathrm{d}y\\ &= C\varepsilon^{-\frac{(N(p-1)(p-t) - (N-sp)t}{p^{2}}} \left[U\right]_{s,t}^{t} \end{split}$$

where we have used that $\delta = \varepsilon^{\frac{p-1}{p}}$. Note that if $t < \frac{N(p-1)}{N-s}$ then

$$(N(p-1)(p-t) - (N-sp)t > 0$$

and hence

$$[u_{\varepsilon,R}]_{s,t}^t \le O(1), \text{ for all } t < \frac{N(p-1)}{N-s}$$

A.2 A non embedding

The example presented in this appendix are in [41]. We will only do one review.

If Ω is a bounded domain in \mathbb{R}^N , $1 \leq q and <math>s = 0, 1, 2, ...,$ then $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$. Let us consider the case s = 1. Using the Hölder inequality we have

$$\|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q} - \frac{1}{p}} \|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega).$$

In the same way we obtain

$$\|\nabla u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q} - \frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

Thus, $W^{1,p}(\Omega) \subset W^{1,q}$ for any $1 \leq q .$

However this property does not hold when s in not an integer. It is against our intuition at first, since if Ω is bounded set then $L^p(\Omega) \subset L^q(\Omega)$ and $W^{1,p}(\Omega) \subset W^{1,q}(\Omega)$. Thus, using a "s-interpolation" it was to be expected that $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$. But this is not true.

It should be noted , for $\Omega \subset \mathbb{R}^N$ we have

$$||f||_{W^{s,r}(\Omega)} := ||f||_{L^{r}(\Omega)} + \left(\int_{\Omega} \int_{\Omega} \frac{|f(x+h) - f(x)|^{r}}{|h|^{N+sr}} \mathrm{d}h \mathrm{d}x\right)^{\frac{1}{r}}, \quad \text{for all} \quad f \in W^{s,r}(\Omega).$$

We consider the case $s \in (0,1)$. Consider $\Omega = (0,2\pi)$ and for any $n \in \mathbb{N}$ the function $\varphi_n : \Omega \to \mathbb{C}$ give for

$$\varphi_n(x) = e^{inx}$$

Let us use the sequence φ_n to construct a function $g \in W^{s,p} \setminus W^{s,q}$, for $s \in (0,1)$ and $1 \leq q .$

Lemma A.2.1 For any $n \in \mathbb{N}$ we have

$$\|\varphi_n\|_{W^{s,r}(\Omega)} \sim n^s, \quad as \quad n \to \infty.$$

Proof. Let $1 \leq r < \infty$ and $s \in (0, 1)$. We have,

$$\int_{\Omega} |\varphi_n|^r \mathrm{d}x = \int_0^{2\pi} |e^{inx}|^r \mathrm{d}x = 2\pi.$$

Moreover,

$$I_n = \int_{\Omega} \int_{\Omega} \frac{|\varphi_n(x+h) - \varphi_n(x)|^r}{|h|^{1+sr}} dh dx = \int_0^{2\pi} \int_0^{2\pi} \frac{|e^{irnx}||e^{inh} - 1|^r}{|h|^{1+sr}} dh dx$$
$$= 2\pi n^{sr} \int_0^{2\pi n} \frac{|e^{i\xi} - 1|^r}{\xi^{sr+1}} d\xi.$$

Note that, since $1 - \cos t \le t^2$ for all $t \ge 0$ we have

$$\begin{split} \int_{0}^{2\pi n} \frac{|e^{i\xi} - 1|^{r}}{\xi^{sr+1}} \mathrm{d}\xi &= 2^{\frac{r}{2}} \int_{0}^{2\pi n} \frac{(1 - \cos\xi)^{\frac{r}{2}}}{\xi^{sr+1}} \mathrm{d}\xi \\ &\leq 2^{\frac{r}{2}} \int_{0}^{2\pi} \frac{\xi^{r}}{\xi^{sr+1}} \mathrm{d}\xi + 2^{\frac{r}{2}} \int_{2\pi}^{2\pi n} \frac{1 - \cos\xi}{\xi^{sr+1}} \mathrm{d}\xi \\ &= 2^{\frac{r}{2}} \int_{0}^{2\pi} \xi^{r(1-s)-1} \mathrm{d}\xi + 2^{\frac{r}{2}} \int_{2\pi}^{2\pi n} \xi^{-sr-1} \mathrm{d} \\ &= \frac{2^{\frac{r}{2}} (2\pi)^{r(1-s)}}{r(1-s)} + \frac{2^{\frac{r}{2}}}{rs} \left[\frac{1}{(2\pi)^{rs}} - \frac{1}{(2\pi n)^{rs}} \right] \end{split}$$

Therefore,

$$I_n \sim n^{sr}$$
, as $n \to \infty$.

Thus, $\|\varphi_n\|_{W^{s,r}(\Omega)} \sim n^s$, as $n \to \infty$.

We assume that $p < \infty$ and we will construct by induction on j sequences λ_j and n_j such that $g(x) = \sum \lambda_j \varphi_{n_j}(x)$, belongs to $W^{s,p}(\Omega)$ but not to $W^{s,q}(\Omega)$.

$$g(x) = \sum_{j \ge 1} \lambda_j \varphi_{n_j}(x)$$
, belongs to $W^{s,p}(\Omega)$ but not to $W^{s,q}(\Omega)$

We choose $\lambda_1 = 1$ and $n_1 = 1$. Assume that $\lambda_1, ..., \lambda_j, n_1, ..., n_j$ already constructed, let

$$f_n(x) := \frac{1}{n^s j^{1/q}} e^{inx}$$
 and $g_j(x) = \sum_{k=1}^j \lambda_k \varphi_{n_k}(x).$

By Lemma A.2.1 we have

$$||f_n||_{W^{s,r}(\Omega)}^r \sim \frac{1}{j^{r/q}}, \quad \text{for all} \quad 1 \le r < \infty.$$

On the other hand, $f_n \to 0$ pointwise as $n \to \infty$. By Brezis-Lieb lemma, for $1 \le r < \infty$ we have as $n \to \infty$

$$||f_n||_{W^{s,r}(\Omega)}^r = ||f_n + g_j - g_j||_{W^{s,r}(\Omega)}^r = ||f_n + g_j||_{W^{s,r}(\Omega)}^r - ||g_j||_{W^{s,r}(\Omega)}^r + o(1)$$

that is

$$|f_n + g_j|_{W^{s,r}(\Omega)}^r = ||g_j||_{W^{s,r}(\Omega)}^r + ||f_n||_{W^{s,r}(\Omega)}^r + o(1), \text{ as } n \to \infty.$$

Thus, for n sufficiently large, we have

$$\|f_n + g_j\|_{W^{s,p}(\Omega)}^p \le \|g_j\|_{W^{s,p}(\Omega)}^p + \frac{C_1}{j^{p/q}}$$
(A.13)

and

$$\|f_n + g_j\|_{W^{s,r}(\Omega)}^q \ge \|g_j\|_{W^{s,r}(\Omega)}^q + \frac{C_2}{j} \ge \frac{C_2}{j}$$
(A.14)

Therefore, using the hypothesis , we have

$$g_j \in W^{s,p}(\Omega) \Rightarrow ||g_j||_{W^{s,p}(\Omega)}^p = C(p).$$

Thus, we choose $\lambda_j = \frac{1}{j^{1/q}n^s}$ and $n_{j+1} = n$. So

$$f_n + g_j = g_{j+1} = \sum_{k=1}^{j+1} \lambda_k \varphi_{n_k}(x).$$

Adding up in $j \ge 1$ we have

$$||g||_{W^{s,p}(\Omega)} \le C(p) + \sum_{j\ge 1} \frac{C_1}{j^{p/q}} < \infty$$

and

$$||g||_{W^{s,q}(\Omega)} \ge \sum_{j\ge 1} \frac{C_1}{j}.$$

Therefore, $g \in W^{s,p}(\Omega) \setminus W^{s,q}(\Omega)$.

A.3 Estimates for non-local tails

We present here the estimates of both non-local tails proved by Iannizzoto, Mosconi and Squassina in [34].

Lemma A.3.1 Consider $R_0 > 0$ and $R_j = \frac{R_0}{4^j}$, $B_j = B_{R_j}$. Assume that, there exists $\alpha \in (0, 1)$, $\lambda > 0$, numbers m_j and M_j such that

$$m_j \leq \inf_{B_j} u \leq \sup_{B_j} u \leq M_j, \quad M_j - m_j = \lambda R_j^{\alpha}.$$

Then,

$$Tail_p((u-m_j)_-; R_j) \le C \left[\lambda S(\alpha)^{1/(p-1)} + \frac{Q(u; R_0)}{R_0^{\alpha}}\right] R_j^{\alpha}.$$
Proof. We have,

$$Tail_{p}((u - m_{j}); R_{j})^{p-1} = R_{j}^{sp} \sum_{k=0}^{j-1} \int_{B_{k} \setminus B_{k+1}} \frac{(u(y) - m_{j})_{-}^{p-1}}{|y|^{N+sp}} dy$$
$$+ R^{sp} \int_{B_{0}^{c}} \frac{(u(y) - m_{j})_{-}^{p-1}}{|y|^{N+sp}} dy$$

By hypothesis, for all $0 \le k \le j - 1$ we have, in $B_k \setminus B_{k+1}$,

$$(u - m_j)_{-} \le m_j - m_k \le (m_j - M_j) + (M_k - m_k) = \lambda (R_k^{\alpha} - R_j^{\alpha}),$$

hence

$$\begin{split} \sum_{k=0}^{j-1} \int_{B_k \setminus B_{k+1}} \frac{(u(y) - m_j)_{-}^{p-1}}{|y|^{N+sp}} \mathrm{d}y &\leq \lambda^{p-1} \sum_{k=0}^{j-1} \int_{B_k \setminus B_{k+1}} \frac{(R_k^{\alpha} - R_j^{\alpha})^{p-1}}{|y|^{N+sp}} \mathrm{d}y \\ &= \lambda^{p-1} R_j^{\alpha(p-1)} \sum_{k=0}^{j-1} \int_{B_k \setminus B_{k+1}} \frac{(4^{\alpha(j-k)} - 1)^{p-1}}{|y|^{N+sp}} \mathrm{d}y \\ &\leq C \lambda^{p-1} S(\alpha) R_j^{\alpha(p-1)-sp}, \end{split}$$

where we have set, for all $\alpha \in (0, 1)$,

$$S(\alpha) = \sum_{n=1}^{\infty} \frac{(4^{\alpha n} - 1)^{p-1}}{4^{spn}}$$

noting $S(\alpha) \to 0$ as $\alpha \to 0^+$.

On the other hand, we have

$$m_j \leq \inf_{B_j} u \leq \sup_{B_j} u \leq ||u||_{L^{\infty}(B_0)},$$

hence

$$\int_{B_0^c} \frac{(u(y) - m_j)_{-}^{p-1}}{|y|^{N+sp}} \mathrm{d}y \le \int_{B_0^c} \frac{(||u||_{L^{\infty}(B_0)} + |u(y)|)^{p-1}}{|y|^{N+sp}} \mathrm{d}y \le C \frac{Q(u, R_0)^{p-1}}{R_0^{sp}}.$$

Therefore, choosing $\alpha < sp/(p-1)$ and using the inequalities above we have

$$Tail_p((u-m_j)_{-};R_j)^{p-1} \le C\lambda^{p-1}S(\alpha)R_j^{\alpha(p-1)} + C\frac{Q(u,R_0)^{p-1}R_j^{sp}}{R_0^{sp}}$$

namely,

$$Tail_{p}((u-m_{j})_{-};R_{j}) \leq C \left[\lambda^{p-1}S(\alpha)R_{j}^{\alpha(p-1)} + \frac{Q(u,R_{0})^{p-1}R_{j}^{sp}}{R_{0}^{sp}}\right]^{1/(p-1)}$$
$$\leq C \left[\lambda S(\alpha)^{1/(p-1)} + \frac{Q(u;R_{0})}{R_{0}^{\alpha}}\right]R_{j}^{\alpha}.$$

In an analogous way we show that

$$Tail_p((M_j - u)_-; R_j) \le C \left[\lambda S(\alpha)^{1/(p-1)} + \frac{Q(u; R_0)}{R_0^{\alpha}}\right] R_j^{\alpha}.$$

with $S(\alpha) \to 0$ as $\alpha \to 0^+$.

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