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On weighted Sobolev spaces: Trudinger-Moser and isoperimetric inequalities

Leandro Gonzaga Fernandes Junior

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Leandro Gonzaga Fernandes Junior

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Advisor: Emerson Alves Mendonça de Abreu

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On weighted Sobolev spaces: Trudinger-Moser and isoperimetric inequalities

LEANDRO GONZAGA FERNANDES JUNIOR

Tese defendida e aprovada pela banca examinadora constituída pelos Senhores:

Prof. Emerson Alves Mendonça de Abreu
UFMG

Ederson Moreira dos Santos

Prof. Ederson Moreira dos Santos
USP

Prof. Ezequiel Rodrigues Barbosa
UFMG

Prof. Lucas Catão de Freitas Ferreira
UNICAMP

Marcos Montenegro

Prof. Marcos da Silva Montenegro
UFMG

Belo Horizonte, 19 de setembro de 2019.

Av. Antônio Carlos, 6627 – Campus Pampulha - Caixa Postal: 702
CEP-31270-901 - Belo Horizonte – Minas Gerais - Fone (31) 3409-5963
e-mail: pgmat@mat.ufmg.br - home page: http://www.mat.ufmg.br/pgmat

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Abstract

The main topic of the thesis is the study of Elliptic Partial Differential Equations. The thesis is divided into two Parts: (I) Trudinger-Moser Type inequality on weighted Sobolev spaces; and (II) on existence and nonexistence of isoperimetric inequalities with different monomial weights.

In part I, we establish the Trudinger-Moser inequality on weighted Sobolev spaces in the whole space, and for a class of quasilinear elliptic operators in radial form of the type $Lu := -r^{-\theta}(r^{\alpha}|u'(r)|^{\beta}u'(r))'$, where $\theta, \beta \geq 0$ and $\alpha > 0$, are constants satisfying some existence conditions. It is worth emphasizing that these operators generalize the p- Laplacian and k-Hessian operators in the radial case. Our results involve fractional dimensions, a new weighted Pólya-Szegö principle, and a boundness value for the optimal constant in a Gagliardo-Nirenberg type inequality.

In part II, we consider the monomial weight $x^A = |x_1|^{a_1} \dots |x_N|^{a_N}$, where a_i is a nonnegative real number for each $i \in \{1, \dots, N\}$, and we establish the existence and nonexistence of isoperimetric inequalities with different monomial weights. We study positive minimizers of $\int_{\partial\Omega} x^A \mathcal{H}^{N-1}(x)$ among all smooth bounded open sets Ω in $\mathbb{R}^N$ with fixed Lebesgue measure with monomial weight $\int_{\Omega} x^B dx$.

Key-words: weighted Trudinger-Moser inequality, weighted rearrangement, Schwarz symmetrization, isoperimetric inequalities, Sobolev inequalities, monomial weights.

Resumo

O objetivo geral da tese é o estudo de Equações Diferenciais Parciais Elípticas. A tese é dividida em duas Partes: (I) Desigualdade do Tipo Trudinger-Moser sobre espaços de Sobolev com pesos; e (II) A existência e não-existência de desigualdades isoperimétricas com pesos monomiais diferentes.

Na Parte I, estabelecemos uma desigualdade do tipo Trudinger-Moser sobre espaços de Sobolev com pesos sobre o intervalo $(0, +\infty)$, relacionada com a classe de operadores elípticos quasilineares cuja forma radial é dada por $Lu := -r^{-\theta}(r^{\alpha}|u'(r)|^{\beta}u'(r))'$, onde $\theta, \beta \geq 0$ e $\alpha > 0$, são constantes satisfazendo algumas condições de existência. Vale enfatizar que esses operadores generalizam o p-Laplaceano e k- Hessiana, no caso radial. Os resultados envolvem dimensão fracionária, um princípio de Pólya-Szegö com pesos e uma limitação para a constante ótima associada com a desigualdade do tipo Gagliardo-Nirenberg.

Na Parte II, consideramos pesos monomiais $x^A = |x_1|^{a_1} \dots |x_N|^{a_N}$, onde a_i é um número real não negativo para cada $i \in \{1, \dots, N\}$, e estabelecemos a existência e não-existência de desigualdades isoperimétricas com pesos monomiais diferentes. Estudamos minimizadores positivos de $\int_{\partial\Omega} x^A \mathcal{H}^{N-1}(x)$ sobre todos os conjuntos abertos, limitados e suaves cujo volume $\int_{\Omega} x^B dx$ é fixo.

Palavras-Chave: Desigualdade de Trudinger-Moser com pesos, rearranjamento com pesos, simetrização de Schwarz, desigualdades isoperimétricas, desigualdades de Sobolev, pesos monomiais.

Contents

Introduction

The thesis is divided into two Parts: (I) Trudinger-Moser type inequalities for weighted Sobolev spaces; and (II) on existence and nonexistence of isoperimetric inequalities with different monomial weights. Each Part is divided into Chapters. Each Chapter is related to a preprint, see [1, 2], as follows.

Part I:

- E. Abreu and L.G. Fernandes Jr. On a weighted trudinger-moser inequality in $\mathbb{R}^N$. arXiv:1810.12329v1, 29 October 2018.

Part II:

- E. Abreu and L.G. Fernandes Jr. On existence and nonexistence of isoperimetric inequalities with different monomial weights. arXiv:1904.01441v2, 11 April 2019.

It is worth emphasizing that each Part is completely independent.

Part I

It is well known that the classical Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for any $p \le q \le Np/(N-p)$, where $p < N$ and Ω is a domain in $\mathbb{R}^N$, see [4, 33]. If $p = N$, although the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $N \le q < \infty$, $W^{1,N}(\Omega) \not\subset L^\infty(\Omega)$. Motivated by this approach, Adams [4] proved that for every $0 < \mu \le 1$ the Sobolev space $W^{1,N}(\Omega)(\Omega\,\text{unbounded})$ is embedded in the Orlicz space $L_{\Psi_{\mu,N}}(\Omega)$, where

$$\Psi_{\mu,N}(t) = e^{\mu t^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\mu^j}{j!} t^{\frac{N}{N-1}j}.$$

Hempel, Morris and Trudinger [34] showed that the best Orlicz space $L_\Psi(\Omega)$ for the embedding of $W_0^{1,N}(\Omega)$ (where Ω is a bounded domain in $\mathbb{R}^N$) occurs when $\Psi = \phi := e^{t^{\frac{N}{N-1}}} - 1$. More precisely, the space $W_0^{1,N}(\Omega)$ may not be continuously embedded in any Orlicz space $L_\Psi(\Omega)$ whose defining function Ψ increases strictly more rapidly than the function ϕ.

The case when Ω is a bounded domain was studied by J. Moser [39], which showed the following sharp result

$$\sup_{u\in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{1}{|\Omega|}\int_\Omega e^{\mu\left(\frac{|u|}{\|\nabla u\|_{L^p}}\right)^{\frac{N}{N-1}}} dx \begin{cases} \le C(N,\mu), & \text{if } \mu \le \mu_N \\ = +\infty, & \text{if } \mu > \mu_N, \end{cases} \tag{1}$$

where $\mu_N := N\omega_{N-1}^{\frac{1}{N-1}}$, $|\Omega|$ is the Lebesgue measure of Ω, ω_{N-1} is the $(N-1)$-dimensional Hausdorff measure of the unit sphere in $\mathbb{R}^N$, and $C(N,\mu)$ is a positive constant depending on N and μ.

The case $\Omega = \mathbb{R}^N$, was studied by Ruf [42] for $N = 2$, and Li and Ruf [37] for $N \geq 3$. In all cases a sharp result was obtained. Namely, there exists $D(N,\mu)$ which depends on N and μ satisfying

$$\int_{\mathbb{R}^N} \Psi_{\mu,N}(u)dx \leq D(N,\mu) \tag{2}$$

for all $u \in W^{1,N}(\mathbb{R}^N)$ with $\|u\|_{W^{1,N}(\mathbb{R}^N)} = 1$ and $\mu \leq \mu_N$. Here, the inequality (2) is not valid if $\mu > \mu_N$.

The attainability of the best constant

$$d_{N,\mu} := \sup_{u\in W^{1,N}(\mathbb{R}^N):\|u\|_{W^{1,N}(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \Psi_{\mu,N}(u)dx, \tag{3}$$

associated with (2), was studied by Ishiwata [35] (see section 1.1, Theorem 1.5 and Theorem 1.6). A similar study was done for singular weights, see [36] .

To establish the attainability of $d_{N,\mu}$, Ishiwata proved that the compactness of a maximizing sequence to (3) happens, excluding the concentration behavior and the vanishing behavior of the maximizing sequence. He also showed that the functional $J(u) := \int_{\mathbb{R}^2} \Psi_{2,\mu}(u)dx$ does not have critical points on $M := \{u \in W^{1,2}(\mathbb{R}^2) : \|u\|_{W^{1,2}(\mathbb{R}^N)} = 1\}$ for μ sufficiently small, which implies non-existence results in this case.

Our approach for Trundiger-Moser inequality will be done for the class of quasilinear elliptic operators in radial form of the type

$$Lu := -r^{-\theta}(r^{\alpha}|u'(r)|^{\beta}u'(r))',$$

where $\theta, \beta \geq 0$ and $\alpha > 0$. For some problems involving the operator L, see [23, 27]. It is worth emphasizing that these operators generalize the p -Laplacian and k-Hessian operators in the radial case, more precisely,

(i) Laplacian	$\alpha = \theta = N-1$, $\beta = 0$
(ii) p-Laplacian ($p \geq 2$)	$\alpha = \theta = N-1$, $\beta = p-2$
(iii) k-Hessian ($1 \leq k \leq N$)	$\alpha = N-k$, $\theta = N-1$, $\beta = k-1$

where these operators act on the weighted Sobolev spaces

$$W^{1,p}_{\alpha,\theta}(0,R) := W^{1,p}((0,R), d\lambda_\alpha, d\lambda_\theta) \quad \text{for} \quad 0 < R \leq \infty$$

defined in section 2. The proposition 1.1, in section 2 (see Kufner-Opic [40]), gives us the following Sobolev-type embedding

$$W^{1,p}_{\alpha,\theta}(0,R) \hookrightarrow L^q_\theta(0,R), \quad \text{where} \quad 1 \leq q \leq q^\star, \alpha - p + 1 > 0, 0 < R < +\infty,$$

and the number $q^\star := \frac{(1+\theta)p}{\alpha-p+1}$ is the critical exponent associated with the weighted Sobolev space $W^{1,p}_{\alpha,\theta}(0,R)$. We would like to emphasize that the embedding still holds for $\alpha - p + 1 = 0$, and $1 \leq q < \infty$.

As in the classical case, a function in $W^{1,p}_{\alpha,\theta}(0,R)$ (when $\alpha-(p-1)=0$) could have a local singularity, which proves that $W^{1,p}_{p-1,\theta}(0,R)\not\subset L^\infty_\theta(0,R)$. Motivated by this approach, Oliveira and Do Ó [28] studied this embedding, and they proved some results on validity and attainability of the Trudinger-Moser inequality (for bounded domains see section 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4).

In the first part, chapters 1 and 2, our goal here is twofold: we prove a Trudinger-Moser type inequality for weighted Sobolev spaces involving fractional dimensions in the unbounded case $(0,\infty)$; and, we discuss the existence of extremal functions in such inequalities.

We attempt to replace the constant $c_{\alpha,\theta}$ (wich depends on α, θ and R) in Theorem 1.2 by a uniform constant $d(\alpha,\theta,\mu)$ (wich depends on α, θ and μ), by replacing the Dirichlet norm with weight $\|u'\|_{L^p_\alpha}$ by the Sobolev norm with weights $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}$, in the same spirit of the results stated before [37, 42]. Furthermore, we investigate the compactness on maximizing sequences for such inequalities in the same sense of the results established by Ishiwata [35].

Let

$$A_{p,\mu}(t)=e^{\mu t^{\frac{p}{p-1}}}-\sum_{j=0}^{\lfloor p\rfloor-1}\frac{\mu^j}{j!}t^{\frac{p}{p-1}j},\ \text{ with } \lfloor p\rfloor \text{ the largest integer less than p.}$$

One of our main results is:

Theorem 0.1. *Let $p\geq 2$, $\theta,\alpha\geq 0$ and $\mu>0$ be real numbers such that $\alpha-(p-1)=0$ and $\mu\leq\mu_{\alpha,\theta}:=(1+\theta)\omega_\alpha^{\frac{1}{\alpha}}$. Then there exists a constant $D(\theta,\alpha,\mu)$ which depends on θ, α and μ such that*

$$\int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x)\leq D(\theta,\alpha,\mu) \tag{4}$$

for all $u\in W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}=1$. Furthemore, the inequality (4) fails if $\mu>\mu_{\alpha,\theta}$, that is, for any $\mu>\mu_{\alpha,\theta}$ there exists a sequence $(u_j)\subset W^{1,p}_{\alpha,\theta}(0,\infty)$ such that

$$\int_0^\infty A_{p,\mu}\left(\frac{|u_j(x)|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right)d\lambda_\theta(x)\to\infty \ \text{ as } j\to\infty.$$

To state our next results, we need to define the best constant associated with the inequality (4), namely

$$d(\theta,\alpha,\mu):=\sup_{0\neq u\in W^{1,p}_{\alpha,\theta}(0,\infty)}\int_0^\infty A_{p,\mu}\left(\frac{|u(x)|}{\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right)d\lambda_\theta(x), \tag{5}$$

where $\alpha-(p-1)=0$.

Theorem 0.2. *Under the assumptions of Theorem 0.1, there exists a positive nonincreasing function u in $W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}=1$ such that*

$$d(\theta,\alpha,\mu)=\int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x),$$

in the following cases:

(i) $p > 2$ *and* $0 < \mu < \mu_{\alpha,\theta}$,

(ii) $p = 2$ *and* $\frac{2}{B(2,\theta)} < \mu < \mu_{\alpha,\theta}$,

where

$$B(2,\theta)^{-1} := \inf_{0 \neq u \in W^{1,2}_{1,\theta}(0,\infty)} \frac{\|u'\|^2_{L^2_1(0,\infty)} \cdot \|u\|^2_{L^2_\theta(0,\infty)}}{\|u\|^4_{L^4_\theta(0,\infty)}}.$$

Theorem 0.3. *Let $p = 2$, $\theta \geq 0$ and $\alpha = 1$. Then there exists μ_0 such that $d(\theta, \alpha, \mu)$ is not achieved for all $0 < \mu < \mu_0$.*

To prove the statement (1), Moser [39] used the well known Schwarz Symmetrization arguments, which provide a radially symmetric function $u^\#$ defined on the ball $B_R(0)$, where $\mathcal{L}^N(\Omega) = \mathcal{L}^N(B_R(0))$ and all the balls $\{x \in B_R(0); u^\#(x) > t\}$ have the same $\mathcal{L}^N$ measure of the sets $\{x \in \Omega; u(x) > t\}$. Furthermore, $u^\#$ satisfies the Pólya-Szegö inequality.

$$\int_{B_R(0)} |\nabla u^\#|^N dx \leq \int_\Omega |\nabla u|^N dx. \tag{6}$$

Thus, the prove of (1) was reduced to the subset of radially non-increasing symmetric functions. In our case, Pólya-Szegö inequality for $W^{1,p}_{\alpha,\theta}(0,\infty)$ was not available. That was one additional difficulty in this type of problem. See, for instance, [28].

In chapter 1, section 1.2, we present the half weighted Schwarz symmetrization with the goal of working around the problem. Thus, we will reduce again the Trudinger-Moser inequality to non-increasing functions.

The Part I is organized as follows.

Chapter 1:

In section 1.1, we define some elements and present some previous results about Trudinger-Moser inequality on $W^{1,p}_{p-1,\theta}(0,R)$, where $R < \infty$. In section 1.2, we prove a new Pólya-Szegö Principle on $W^{1,p}_{\alpha,\theta}$ using a new class of isoperimetric inequalities on $\mathbb{R}$ with respect to weights $|x|^k$. In section 1.3, we establish the Trudinger-Moser inequality on $W^{1,p}_{\alpha,\theta}(0,\infty)$, under the assumptions of Theorem 0.1.

Chapter 2:

In the section 2.1, we obtain the Theorem 0.2 studying the compactness of a maximizing sequence (u_n) for (5). In the section 2.2, we show the Theorem 0.3 proving that the functional $F(u) = \int_0^\infty A_{2,\mu}(|u(x)|) d\lambda_\theta(x)$ does not have critical points on $\{u \in W^{1,2}_{1,\theta}(0,\infty) : \|u\|_{W^{1,2}_{1,\theta}(0,\infty)} = 1\}$. Finally, in the section 2.3 we present a brief discourse about Gagliardo-Nirenberg-Sobolev type inequality and we show that $2/B(2,\theta) < 2\pi(1+\theta) = \mu_{1,\theta}$.Thus, the case (ii) of the Theorem 0.2 makes sense.

Part II

A great attention has been given recently to the isoperimetric inequalities with weights, see for instance [6], [7], [9], [11], [12], [13], [14], [15], [17], [16], [19], [24], [25], [26], [29], [32], [41] and the references therein. However, in the wide literature, most works approach volume functional and perimeter functional carrying the same weight.

It is worth emphasizing that some researchers have been studying isoperimetric inequalities when the volume and perimeter carry two different weights, see [6], [7], and [25]. In [6], motivated by some norm inequalities with weights which are well-known as Caffarelli-Kohn-Niremberg (see [18]), it was studied by Alvino et al., the following isoperimetric inequality:

$$\text{minimize} \int_{\partial\Omega} |x|^k \mathcal{H}^{N-1}(x) \text{ among all smooth sets } \Omega \subset \mathbb{R}^N \text{ satisfying } \int_\Omega |x|^l dx = 1. \tag{7}$$

The existence of an isoperimetric inequality with monomial weights was shown by Cabré, and Ros-Oton, see Theorem 1.4 in [16], namely

Theorem A (Cabré-Ros-Oton). *Let $A = (a_1, \dots, a_N)$ be a nonnegative vector in $\mathbb{R}^N$, $x^A = |x_1|^{a_1} \dots |x_N|^{a_N}$, $D = a_1 + \cdots + a_N + N$, and $\mathbb{R}^N_A = \{(x_1, \dots, x_N); x_i > 0$ whenever $a_i > 0\}$. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Denote*

$$m(\Omega) = \int_\Omega x^A dx \text{ and } P(\Omega) = \int_{\partial\Omega} x^A d\mathcal{H}^{N-1}(x).$$

Then,

$$\frac{P(\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \frac{P(B_1^A)}{m(B_1^A)^{\frac{D-1}{D}}}, \tag{8}$$

where $B_1^A := B_1(0) \cap \mathbb{R}^N_A$.

As in the classical case, the inequality (8) implies the following Sobolev Inequality with monomial weights

$$\left(\int_{\mathbb{R}^N_A} |u|^{p^\star} x^A dx \right)^{\frac{1}{p^\star}} \leq C_{p,N} \left(\int_{\mathbb{R}^N_A} |\nabla u|^p x^A dx \right)^{\frac{1}{p}}, \tag{9}$$

for every $u \in C^1_c(\Omega)$, where $p^\star = \frac{pD}{D-p}$, and $p < D$. The best constant in (9) is given by

$$C_1 = D \left(\frac{\Gamma\left(\frac{a_1+1}{2}\right) \cdots \Gamma\left(\frac{a_N+1}{2}\right)}{2^k \Gamma\left(1 + \frac{D}{2}\right)} \right)^{\frac{1}{D}} \quad \text{for } p = 1$$

and by

$$C_{p,N} = C_1 D^{\frac{1}{D}-1-\frac{1}{p}} \left(\frac{p-1}{D-p} \right)^{\frac{1}{p'}} \left(\frac{p' \Gamma(D)}{\Gamma\left(\frac{D}{p}\right) \Gamma\left(\frac{D}{p'}\right)} \right)^{\frac{1}{D}}, \quad \text{for } 1 < p < D,$$

where $p = \frac{p}{p-1}$, and k is the number of strictly positive entries of A.

Additionally, the best constant $C_{p,N}$ gives the possibility to prove a Trudinger-Moser type inequality, more especially, that there exists constants $c_1 > 0$ and $c_2 > 0$ such that

$$\int_\Omega \exp\left[\left(\frac{c_1 |u(x)|}{\|\nabla u\|_{L^D(\Omega, x^A dx)}} \right)^{\frac{D}{D-1}} \right] x^A dx \leq c_2 \int_\Omega x^A dx$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set.

Motivated by inequality (9) and the Caffarelli-Kohn-Nirenberg inequality, Castro presented in [20] the following result

Theorem B (Castro)**.** *Consider $N \geq 1$, $p \geq 1$, $F = (f_1, \ldots, f_N)$, $G = (g_1, \ldots, g_N) \in \mathbb{R}^N$. Let $f = f_1 + \cdots + f_N$ and $g = g_1 + \cdots + g_N$, for $p^* \geq 1$ defined by*

$$\frac{1}{p^*} + \frac{g+1}{N} = \frac{1}{p} + \frac{f}{N},$$

suppose

1. $\dfrac{1}{p^*} f_i + \left(1 - \dfrac{1}{p}\right) g_i > 0$ *for all* $i = 1, \ldots, N$,
2. $0 \leq f_i - g_i < 1$ *for all* $i = 1, \ldots, N$.
3. $1 - \dfrac{N}{p} < f - g \leq 1$.

Then there exists a constant $C > 0$ such that for all $u \in C_c^1\left(\mathbb{R}^N\right)$

$$\left(\int_{\mathbb{R}^N} |x^G u(x)|^{p^*} dx \right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |x^F \nabla u(x)|^p \right)^{\frac{1}{p}}.$$

For $p = 1$, we may rewrite the previous result as:

The following three conditions

i) $a_i > 0$,

ii) $0 \leq a_i - \frac{N+a-1}{N+b} b_i < 1$,

iii) $a - b \leq 1$, *where* $a := a_1 + a_2 + \cdots + a_N$ *and* $b := b_1 + b_2 + \cdots + b_N$.

are sufficient for the existence of a constant $C > 0$, that depends on a, b, and N, such that

$$\left(\int_{\mathbb{R}^N} x^B |u(x)|^{\frac{N+b}{N+a-1}} dx \right)^{\frac{N+a-1}{N+b}} \leq C \int_{\mathbb{R}^N} x^A |\nabla u(x)| dx,$$

for every $u \in C_c^1\left(\mathbb{R}^N\right)$.

Motivated by Theorem B and problem (7), we approach the existence and nonexistence of isoperimetric inequality where the volume and perimeter have different monomial weights, more specifically, we study the following isoperimetric problem:

Find the constant $C_{A,B,N} \in [0, +\infty)$, where

$$C_{A,B,N} := \inf \left\{ \frac{\displaystyle\int_{\partial\Omega} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_\Omega x^B dx\right]^{\frac{N+a-1}{N+b}}} ; \Omega \text{ is a smooth open set and } 0 < \int_\Omega x^B dx < \infty \right\}. \tag{10}$$

Even though some cases in one dimension are included, throughout the Part II we consider $N \geq 2$. For the case $N = 1$, see [6]. One of our main results is:

Theorem 0.4. *Let $N \geq 2$, and Let $A = (a_1, \ldots, a_N)$, $B = (b_1, \ldots, b_N)$ be two nonnegative vectors in $\mathbb{R}^N$. Let $a = a_1 + \cdots + a_N$, $b = b_1 + \cdots + b_N$, $\overline{a}_i = a - a_i$, and $\overline{b}_i = b - b_i$. Then, we have the following*

(I) if

$$C_{A,B,N} > 0,$$

then

$$0 \leq a_i - \frac{N+a-1}{N+b} b_i \leq \frac{N+a-1}{N+b} \tag{11}$$

or equivalently

$$0 \leq a_i - \frac{N+\overline{a}_i-1}{N+\overline{b}_i} b_i \quad and \quad \frac{a_i}{b_i+1} \leq \frac{N+\overline{a}_i-1}{N+\overline{b}_i-1}. \tag{12}$$

(II) if $a - b \leq 1$ and the condition (11) holds, then

$$C_{A,B,N} > 0.$$

For the case $a - b = 1$, on certain conditions, we present the exact value of $C_{A,B,N}$.

Theorem 0.5. *Let $N \geq 2$, and let $A = (a_1, \ldots, a_N)$, $B = (b_1, \ldots, b_N)$ be two nonnegative vectors in $\mathbb{R}^N$. Let $a = a_1 + \cdots + a_N$, $a = b_1 + \cdots + b_N$, $\overline{a}_i = a - a_i$, and $\overline{b}_i = b - b_i$. If $a_j = b_j$ for all $j \in \{1, \ldots, N\} \backslash \{i\}$, and $a_i = b_i + 1$, then*

$$C_{A,B,N} = a_i.$$

Theorem 0.6. *Let $N = 2$, $A = (0, 1)$, and $B = (0, 0)$. Then, there is no bounded, open and Lipschitz set $\Omega \subset \mathbb{R}^2$ such that*

$$C_{A,B,2} = \frac{P_A(\Omega)}{m_B(\Omega)}.$$

Our Theorem 0.4 establishes all cases of existence and nonexistence of isoperimetric inequality for two nonnegative vectors satisfying $a - b \leq 1$, which also implies the improvement and the necessity of (*ii*) in Theorem B. The condition (12), equivalent to (11), is even more general, because it shows us how to choose the entrie i of the vectors A and B, since we have already chosen the others $N - 1$ entries. For instance, if we have $N - 1$ equal entries in the vectors A and B, $a_j = b_j$ for all $j \in \{1, \ldots, N\} \backslash \{i\}$, then the condition (12) tells us that the isoperimetric inequality exists only if $a_i \leq b_i + 1$.

Theorem 0.5 is surprising, since $C_{A,B,N}$ in this case does not depend on N. It is worth emphasizing that in the proof we get a decreasing sequence $(\Omega_\varepsilon)_{\varepsilon > 0} \subset \mathbb{R}^N$, it means $\Omega_\varepsilon \subset \Omega_\delta$ whenever $\varepsilon < \delta$, such that

$$\frac{\int_{\partial\Omega_\varepsilon} x^A \mathcal{H}^{N-1}(x)}{\int_{\Omega_\varepsilon} x^B dx} \to a_i \text{ as } \varepsilon \to 0,$$

however the $\int_{\Omega_\varepsilon} x^A dx \to 0$ as $\varepsilon \to 0$.

The Theorem 0.6 shows us slightly what the Theorem 0.5 indicated. In other words, we expected that there is no minimizer set for (10) whenever A and B are nonnegative vectors in $\mathbb{R}^N$, satisfying $A = B + e_i$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$.

The Part II is organized as follows.

Chapter 3:

In section 3, we define some basic elements that we will use throughout the Part II. In section 3.2, we state some lemmata which will be used in the prove of Theorem 0.4. Finally, in section 3.3, we prove Theorem 0.5.

Chapter 4:

In section 4.1, we present Steiner symmetrization, one of the simplest and most powerful symmetrization processes ever introduced in analysis. In section 4.2, we study the case $N = 2$, $A = (0, a_2)$, and $B = (0, 0)$ based on Steiner symmetrization and elementary arguments, and establish Theorem 0.6.

Part I

Trudinger-Moser type inequality on weighted Sobolev spaces

Chapter 1

On a weighted Trudinger-Moser inequality in $\mathbb{R}^N$

We prove Theorem 0.1, which extends to the interval $(0, \infty)$ a similar inequality obtained by De Oliveira and Do Ó [28] for bounded intervals. The main ingredient in the proof is the definition of a "half"Schwarz symmetrization for functions in $W^{1,p}_{\alpha,\theta}((0,\infty))$. By proving a suitable version of the Pólya-Szegö inequality, we reduce the problem to the space of non-increasing functions. Then, for $\mu \leq \mu_{\alpha,\theta}$, we are able to obtain the finiteness of $d(\theta, \alpha, \mu)$ (5) by combining the results in [28] with sharp decay estimates.

1.1 Basic definitions and previous results

Let $0 < R \leq +\infty$, $1 \leq p < +\infty$ and $\theta \geq 0$. Let us denote by $L^p_\theta(0, R)$ the weighed Lebesgue space defined as the set of all measuable functions u on $(0, R)$ for which

$$\|u\|_{L^p_\theta(0,R)} := \left[\int_0^R |u(x)|^p d\lambda_\theta(x)\right]^{1/p} < \infty,$$

where

$$d\lambda_\theta(x) = \omega_\theta x^\theta dx, \;\; \omega_\theta = \frac{2\pi^{\frac{1+\theta}{2}}}{\Gamma\left(\frac{1+\theta}{2}\right)}, \text{ for all } \; \theta \geq 0,$$

with $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ the Gamma Function. Besides, we denote by

$$W^{1,p}_{\alpha,\theta}(0, R) := \left\{ u \in L^p_\theta(0, R); u' \in L^p_\alpha(0, R) \text{ and } \lim_{x \to R^-} u(x) = 0 \right\}$$

and

$$\|u\|_{W^{1,p}_{\alpha,\theta}(0,R)} := \left(\|u'\|^p_{L^p_\alpha(0,R)} + \|u\|^p_{L^p_\theta(0,R)} \right)^{\frac{1}{p}}.$$

In the following proposition, see [40] for more details, we collect some embedding results for the weighted spaces $W^{1,p}_{\alpha,\theta}$, which will be used in this paper.

Proposition 1.1. *Let $u : (0, R] \to \mathbb{R}$ be an absolutely continuous function. If $R < \infty$, $u(R) = 0$ and*

(1) *for $1 \le p \le q < \infty$, assume*

  (a) $\alpha > p-1$, $\theta \ge \alpha\frac{q}{p} - q\frac{p-1}{p} - 1$, *or*

  (b) $\alpha \le p-1$, $\theta > -1$.

(2) *for $1 \le q < p < \infty$, assume*

  (c) $\alpha > p-1$, $\theta > \alpha\frac{q}{p} - q\frac{p-1}{p} - 1$, *or*

  (d) $\alpha \le p-1$, $\theta > -1$

then

$$\left(\int_0^R |u|^q x^\theta dx\right)^{\frac{1}{q}} \le C\left(\int_0^R |u'|^p x^\alpha dx\right)^{\frac{1}{p}},$$

where C is a constant which does not depend on u.

Next, we present results due to Oliveira and Do Ó [28].

Theorem 1.2. *Let $\alpha, \theta \ge 0$ and $p \ge 2$ be real numbers such that $\alpha - (p-1) = 0$. Then there exists a constant $c_{\alpha,\theta}$ depending on α, θ and R such that*

$$\sup_{u \in W^{1,p}_{\alpha,\theta}(0,R)} \int_0^R e^{\mu(|u|)^{\frac{p}{p-1}}} d\lambda_\theta(r) \begin{cases} \le c_{\alpha,\theta}, & \text{if } \ \mu \le \mu_{\alpha,\theta} := (1+\theta)\omega_\alpha^{\frac{1}{\alpha}} \\ = \infty, & \text{if } \ \mu > \mu_{\alpha,\theta}, \end{cases} \tag{1.1}$$

where $\|u'\|_{L^p_\alpha} = 1$.

They also showed the existence of extremal functions for inequality (1.1), as follows

Theorem 1.3. *Under the assumptions of Theorem 1.2, there are extremal functions for $C_{\alpha,\theta,R}(\mu)$ when $\mu \le \mu_{\alpha,\theta}$; that is, there exists $u \in W^{1,p}_{\alpha,\theta}(0,R)$ such that*

$$C_{\alpha,\theta,R}(\mu) = \int_0^R e^{\mu|u|^{\frac{p}{p-1}}} d\lambda_\theta(r),$$

where

$$C_{\alpha,\theta,R}(\mu) := \sup_{u \in W^{1,p}_{\alpha,\theta}(0,\infty):\|u'\|_{L^p_\alpha}=1} \int_0^R e^{\mu(|u|)^{\frac{p}{p-1}}} d\lambda_\theta(r).$$

In the same spirit of Adachi and Tanaka (see [3]), Oliveira and Do Ó showed the following result

Theorem 1.4. *Let $\theta, \alpha \geq 0$ and $p \geq 2$ be real numbers such that $\alpha - (p-1) = 0$. Then for any $\mu \in (0, \mu_{\alpha,\theta})$ there is a constant $C_{\mu,p,\theta}$ depending on μ, p and θ such that*

$$\int_0^\infty A_{p,\mu}\left(\frac{|u(r)|}{\|u'\|_{L^p_\alpha(0,\infty)}}\right) d\lambda_\theta(r) \leq C_{\mu,p,\theta}\left(\frac{\|u\|_{L^p_\theta(0,\infty)}}{\|u'\|_{L^p_\alpha(0,\infty)}}\right)^p \tag{1.2}$$

for all $u \in W^{1,p}_{\alpha,\theta}(0,R)\backslash\{0\}$. Besides that, for any $\mu \geq \mu_{\alpha,\theta}$ there is a sequence $(u_j) \subset W^{1,p}_{\alpha,\theta}(0,\infty)$ such that $\|u_j'\|_{L^p_\alpha(0,\infty)} = 1$ and

$$\frac{1}{\|u_j'\|_{L^p_\alpha(0,\infty)}}\int_0^\infty A_{p,\mu}\left(|u_j(r)|\right) d\lambda_\theta(r) \to \infty \ \text{ as } \ j \to \infty,$$

where

$$A_{p,\mu}(t) = e^{\mu t^{\frac{p}{p-1}}} - \sum_{j=0}^{\lfloor p \rfloor - 1} \frac{\mu^j}{j!} t^{\frac{p}{p-1}j}, \ \text{ with } \lfloor p \rfloor \ \text{ is the largest integer less than } p.$$

As mentioned in the Introduction, Ishiwata [35] studied the attainability of $d_{N,\mu}$ (3) in the classical case. He emphasized the importance of evaluating vanishing behaviour on maximizing sequence in unbounded case. Next, the main results in [35] are presented.

Theorem 1.5. *Let $N \geq 2$ and*

$$B_2 := \sup_{0 \neq \psi \in W^{1,2}(\mathbb{R}^2)} \frac{\|\psi\|^4_{L^4}}{\|\nabla\psi\|^2_{L^2}\,\|\psi\|^2_{L^2}}.$$

Then $d_{N,\mu}$ is attained for $0 < \mu < \mu_N$ if $N \geq 3$ and for $2/B_2 < \mu \leq \mu_2 = 4\pi$ if $N = 2$.

Theorem 1.6. *Let $N = 2$. If $\mu \ll 1$, then $d_{2,\mu}$ is not attained.*

1.2 Pólya-Szegö Principle on $W^{1.p}_{\alpha,\theta}$

As mentioned in the introduction, we are going to define a half weighted Schwarz symmetrization to prove a Pólya-Szego Principle, see the inequality (6).

We define the measure μ_l by $d\mu_l(x) = |x|^l dx$. Besides, if $M \subset \mathbb{R}$ is a measurable set with finite μ_l-measure, then let M^* denote the interval $(0,R)$ such that

$$\mu_l((0,R)) = \mu_l(M).$$

Further, if $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that

$$\mu_l\left(\{y \in \mathbb{R}; |u(y)| > t\}\right) < \infty \ \text{ for all } \ t > 0,$$

then let u^* denote the half weighted Schawarz symmetrization of u, or in short, the half μ_l-symmetrization of u, given by

$$u^*(x) = \sup\left\{t \geq 0; \mu_l\left(\{y \in \mathbb{R}; |u(y)| > t\}\right) > \mu_l(0,x)\right\},$$

for every $x > 0$.

Remark 1.7. *The word "half" appears here because our symmetrization is slightly different in three aspects:*

(i) it is defined on $(0, \infty)$*;*

(ii) we are comparing the distribution $\rho(t) := \mu_l(\{y \in \mathbb{R}; |u(y)| > t\})$ *with the measure of* $(0, x)$*, instead of* $B_{|x|}(0)$*;*

(iii) the set M^* *is a semi ball with the same measure of* M*, instead of a ball.*

Remark 1.8. *We gather basic properties of the half* μ_l*-symmetrization of* u*:*

(i) u^* *is monotone nonincreasing;*

(ii) $\int_0^\infty |u|^p x^l dx = \int_0^\infty |u^*|^p x^l dx$*, for every* $1 \leq p < \infty$*;*

(iii) $\int_{\{|u|>t\}} x^l dx = \int_{\{u^*>t\}} x^l dx$*, for every* $0 < t < \infty$*.*

We will carry out the proof of the next result based on Isoperimetric Inequality on $\mathbb{R}$ with weight $|x|^k$ [see [6],Theorem 6.1]. Besides that, it is worth noting that Theorem 8.1 in [6] does not cover the case $k < l+1$ when $N = 1$. For negative values of k, the proof is a consequence of the well-known Hardy-Littlewood inequlaity. See also Cabré and Ros-Oton [16] for monomial weights, and Talenti [45] for some cases when $N \geq 2$.

Theorem 1.9. *Let* k, l *be real numbers such that* $0 < k \leq l+1$*, and* $1 \leq p < \infty$*. Set* $m := pk + (1-p)l$*. Then*

$$\int_0^\infty |u'|^p |x|^{pk+(1-p)l} dx \geq \int_0^\infty |(u^*)'|^p |x|^{pk+(1-p)l} dx, \tag{1.3}$$

for every $u \in W^{1,p}_{m,l}(0, \infty)$*, where* u^* *denotes the half* μ_l*-symmetrization of* u*.*

Proof. Observe that it is sufficient to consider u a non-negative function. Let

$$I := \int_0^\infty |u'|^p |x|^{pk+(1-p)l} dx \text{ and}$$
$$I^* := \int_0^\infty |(u^*)'|^p |x|^{pk+(1-p)l} dx.$$

The Coarea Formula holds

$$I := \int_0^\infty \int_{u=t} |u'|^{p-1} |x|^{pk+(1-p)l} d\mathcal{H}^0(x)dt \text{ and}$$
$$I^* := \int_0^\infty \int_{u^*=t} |(u^*)'|^{p-1} |x|^{pk+(1-p)l} d\mathcal{H}^0(x)dt.$$

If $p = 1$, we get

$$I := \int_0^\infty \int_{u=t} |x|^k d\mathcal{H}^0(x)dt \text{ and}$$
$$I^* := \int_0^\infty \int_{u^*=t} |x|^k d\mathcal{H}^0(x)dt,$$

hence, we obtain from Isoperimetric inequality on $\mathbb{R}$ with weight $|x|^k$ [see [6], Theorem 6.1] and definition of $u^\star$ that

$$\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \geq \int_{u^*=t} |x|^k \, d\mathcal{H}^0(x).$$

Therefore, $I \geq I^\star$ when $p = 1$.

Now, asssume that $1 < p < \infty$. By Holder's Inequality we have

$$\int_{u=t} |x|^k \, dH^0(x) \;\; \leq \;\; \left(\int_{u=t} |x|^{kp+(1-p)l} |u'|^{p-1} \, d\mathcal{H}^0(x) \right)^{\frac{1}{p}} \left(\int_{u=t} \frac{|x|^l}{|u'|} d\mathcal{H}^0(x) \right)^{\frac{p-1}{p}}$$

for $a.e$ $t \in [0, \infty)$, thus we get

$$I \geq \int_0^\infty \left(\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \right)^p \left(\int_{u=t} \frac{|x|^l}{|u'|} d\mathcal{H}^0(x) \right)^{1-p} dt. \tag{1.4}$$

Since that $|(u^\star)'|$ and $|x|$ are constants along with $\{u^\star = t\}$, hence, for u^* we obtain the equality, i.e,

$$I^* = \int_0^\infty \left(\int_{u^*=t} |x|^k \, d\mathcal{H}^0(x) \right)^p \left(\int_{u^*=t} \frac{|x|^l}{|(u^\star)'|} d\mathcal{H}^0(x) \right)^{1-p} dt. \tag{1.5}$$

In addition, by definition of u^*, see also Remark 1.8, we have

$$\int_{u>t} |x|^l \, dx = \int_{u^*>t} |x|^l \, dx,$$

and as a consequence of Coarea Formula we get

$$\int_{u=t} \frac{|x|^l}{|u'|} d\mathcal{H}^0(x) = \int_{u^*=t} \frac{|x|^l}{|(u^*)'|} d\mathcal{H}^0(x), \tag{1.6}$$

for $a.e$ $t \in [0, \infty)$ which is sometimes called Fleming - Rishel's Formula, see Corollary A.22.

Again, by Isoperimetric Inequality on $\mathbb{R}$ with weight $|x|^k$ [see [6],Theorem 6.1] and the definition of u^* we obtain

$$\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \geq \int_{u^*=t} |x|^k \, d\mathcal{H}^0(x). \tag{1.7}$$

Therefore, from (1.4), (1.5), (1.6), and (1.7) we have

$$I \geq I^*,$$

thus, (1.3) follows. □

1.3 Trudinger-Moser inequality on $W^{1,p}_{\alpha,\theta}(0, \infty)$

In this section, we establish a Trudinger-Moser type inequality on $W^{1,p}_{\alpha,\theta}(0, \infty)$ (Theorem 0.1) via the Pólya-Szegö Principle presented in section 1.2.

Lemma 1.10. *(i) Let u be a function in $W^{1,p}_{\alpha,\theta}(0,\infty)$. Then,*

$$|u(x)|^p \leq p\omega_\theta^{-\frac{p-1}{p}}\omega_\alpha^{-\frac{1}{p}}x^{-\frac{(p-1)\theta+\alpha}{p}}\|u\|^{p-1}_{L^p_\theta(0,\infty)}\|u'\|_{L^p_\alpha(0,\infty)} \quad \textit{for all} \ \ x>0. \tag{1.8}$$

Consequently, the embedding $W^{1,p}_{\alpha,\theta}(0,\infty)\hookrightarrow L^q_\theta(0,\infty)$ is compact for all q satisfying

$$\frac{p^2(1+\theta)}{(p-1)\theta+\alpha}\leq q<\frac{p(1+\theta)}{\alpha-(p-1)}:=p^\star,$$

where $\alpha\geq(p-1)$ and $\alpha\leq p+\theta$.

(ii) Let $u\in L^p_\theta(0,R)$ be a nonincreasing function, then

$$|u(x)|\leq\left(\frac{1+\theta}{\omega_\theta x^{1+\theta}}\right)^{1/p}\left[\int_0^R|u(s)|^p d\lambda_\theta(s)\right]^{1/p}, \ \ \textit{for all} \ \ 0<x<R. \tag{1.9}$$

Hence, if $(u_n)\subset W^{1,p}_{\alpha,\theta}(0,\infty)$ is a nonincreasing sequence converging weakly to u in $W^{1,p}_{\alpha,\theta}(0,\infty)$, then $u_n\to u$ strongly in $L^q_\theta(0,\infty)$, for each $p<q<p^\star$ $(\alpha\geq p-1)$.

Proof. It is easy to check out the inequality (1.9) for a nonincreasing function. Then, we will do only the inequality (1.8).

For every $0<x<y$ we have

$$|u(x)|^p\leq|u(y)|^p+p\int_x^y|u(t)|^{p-1}|u'(t)|\,dt.$$

By Holder Inequality and $\lim\limits_{y\to\infty}u(y)=0$, we get

$$\begin{aligned}|u(x)|^p&\leq p\int_x^\infty|u(t)|^{p-1}|u'(t)|\,dt\\&\leq p\omega_\theta^{-\frac{p-1}{p}}\omega_\alpha^{-\frac{1}{p}}x^{-\frac{(p-1)\theta+\alpha}{p}}\left(\int_0^\infty|u(t)|^p\,d\lambda_\theta(t)\right)^{\frac{p-1}{p}}\cdot\left(\int_0^\infty|u'(t)|^p\,d\lambda_\alpha(t)\right)^{\frac{1}{p}},\end{aligned}$$

which proves (1.8). □

The next remark will be used in the proof of Theorem 0.1.

Remark 1.11. *By inequality (1.8), we have $|u(x)|\leq1$, for all*

$$x\geq\left(\frac{p}{\omega_\theta^{\frac{p}{p-1}}\omega_\alpha^{\frac{1}{p}}}\right)^{\frac{p}{(p-1)(1+\theta)}}:=a_0$$

whenever $u\in W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}\leq1$ and $\alpha-(p-1)=0$. It is worth noting that a_0 depends on p, and θ.

Theorem 0.1. *Let $p \geq 2$, $\theta, \alpha \geq 0$ and $\mu > 0$ be real numbers such that $\alpha - (p-1) = 0$ and $\mu \leq \mu_{\alpha,\theta} := (1+\theta)\omega_\alpha^{\frac{1}{\alpha}}$. Then there exists a constant $D(\theta, \alpha, \mu)$ which depends on θ, α and μ such that*

$$\int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x) \leq D(\theta, \alpha, \mu) \tag{1.10}$$

for all $u \in W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$. Furthemore, the inequality (1.10) fails if $\mu > \mu_{\alpha,\theta}$, that is, for any $\mu > \mu_{\alpha,\theta}$ there exists a sequence $(u_j) \subset W^{1,p}_{\alpha,\theta}(0,\infty)$ such that

$$\int_0^\infty A_{p,\mu}\left(\frac{|u_j(x)|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right) d\lambda_\theta(x) \to \infty \ \text{ as } \ j \to \infty.$$

Proof of Theorem 0.1

We can assume by Theorem 1.9 and Remark 1.8 that u is a nonincreasing positive function on $(0,\infty)$ and we also recall that $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} \leq 1$.

Let $a \geq a_0$ (see Remark 1.11) to be chosen later. Next, we divide the integral at (1.10) into two parts, that is,

$$\int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x) = \int_0^a A_{p,\mu}(|u|)d\lambda_\theta(x) + \int_a^\infty A_{p,\mu}(|u|)d\lambda_\theta(x). \tag{1.11}$$

It follows from Lemma 1.10 and Remark 1.11 that

$$\begin{aligned}
\int_a^\infty A_{p,\mu}(|u|)d\lambda_\theta(x) \quad &= \quad \sum_{j=\lfloor p \rfloor}^\infty \frac{\mu^j}{j!} \int_a^\infty |u|^{\frac{p}{p-1}j} r^\theta \omega_\theta dr \\
&\leq \quad \omega_\theta \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor!} \int_0^\infty |u|^p r^\theta dr \\
&+ \quad \omega_\theta \sum_{j=\lfloor p \rfloor+1}^\infty \frac{\mu^j (1+\theta)^{\frac{j}{p-1}}}{j!\omega_\theta^{\frac{j}{p-1}}} \left[\omega_\theta \int_0^\infty |u|^p r^\theta dr\right]^{\frac{j}{p-1}} \\
&\cdot \quad \int_a^\infty r^{\theta - \frac{(1+\theta)j}{p-1}} dr \\
= \quad \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor!} \quad &+ \sum_{j=\lfloor p \rfloor+1}^\infty \frac{\mu^j (1+\theta)^{\frac{j}{p-1}} (p-1)\omega_\theta}{j!\omega_\theta^{j/p-1}(1+\theta)(j-(p-1))a^{\frac{(1+\theta)(j-(p+1))}{p-1}}}.
\end{aligned} \tag{1.12}$$

To estimate the other part at (1.11), let

$$v(r) = \begin{cases} u(r) - u(a), & 0 < r \leq a \\ 0, & r \geq a. \end{cases}$$

Note that if $1 < q \leq 2$ and $b \geq 0$, we have $(x+b)^q \leq |x|^q + qb^{q-1}x + b^q$ for all $x \geq -b$. Then, by

Lemma 1.10 we obtain

$$
\begin{aligned}
u(r)^{\frac{p}{p-1}} &\leq v(r)^{\frac{p}{p-1}} + \frac{p}{p-1} v(r)^{\frac{1}{p-1}} u(a) + u(a)^{\frac{p}{p-1}} \\
&\leq v(r)^{\frac{p}{p-1}} + v(r)^{\frac{p}{p-1}} u(a)^p + u(a)^{\frac{p}{p-1}} + \frac{1}{(p-1)^{1/p-1}} \\
&\leq v(r)^{\frac{p}{p-1}} \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right] + \left(\frac{1+\theta}{a^{1+\theta}\omega_\theta}\right)^{1/p-1} \\
&+ \frac{1}{(p-1)^{1/p-1}} \\
&:= v(r)^{\frac{p}{p-1}} \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right] + d(a). \qquad (1.13)
\end{aligned}
$$

Setting

$$
w(r) := v(r) \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right]^{\frac{p-1}{p}},
$$

it follows that

$$
\begin{aligned}
\omega_\alpha \int_0^a |w'|^p r^\alpha dr &= \omega_\alpha \int_0^a |u'|^p \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right]^{p-1} r^\alpha dr \\
&= \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right]^{p-1} \omega_\alpha \int_0^a |u'|^p r^\alpha dr \\
&\leq \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right]^{p-1} \left[1 - \omega_\theta \int_0^\infty |u|^p r^\theta dr\right] \\
&\leq 1, \qquad (1.14)
\end{aligned}
$$

where the last inequality comes from the non-positivity of the function $f : [0,1] \to \mathbb{R}$ defined by $f(t) = (1+\gamma t)^{p-1}(1-t) - 1$ for any γ fixed in the interval $(0, 1/(p-1))$, whence the inequality (1.14) is valid with

$$
\left(\frac{(p-1)(1+\theta)}{\omega_\theta}\right)^{1/(1+\theta)} \leq a < \infty.
$$

Next, it follows from (1.13) that

$$
u(r)^{\frac{p}{p-1}} \leq w(r)^{\frac{p}{p-1}} + d(a),
$$

and consequently, we obtain

$$
\begin{aligned}
\int_0^a A_{p,\mu}(|u(x)|) d\lambda_\theta(x) &\leq \omega_\theta \int_0^a e^{\mu |u|^{\frac{p}{p-1}}} r^\theta dr \\
&\leq \omega_\theta e^{\mu d(a)} \int_0^a e^{\mu |w|^{\frac{p}{p-1}}} r^\theta dr. \qquad (1.15)
\end{aligned}
$$

We combine (1.12), (1.14), (1.15) and Theorem 1.2 to conclude the first part of the proof of the theorem.

For the second part, we will make a change of variable as in [28]. We define $w(t) = \omega_\alpha^{\frac{1}{\alpha+1}}(1+\theta)^{\frac{\alpha}{1+\alpha}}u(Re^{-\frac{t}{1+\theta}})$ for all $u \in W^{1,p}_{\alpha,\theta}(0,R)$, where $\alpha - (p-1) = 0$. Then, we get

$$\int_0^R |u'(r)|^p d\lambda_\alpha(r) = \int_0^\infty |w'(t)|^p dt, \tag{1.16}$$

$$\int_0^R |u(r)|^p d\lambda_\theta(r) = \frac{R^{1+\theta}\omega_\theta}{(1+\theta)^p\omega_\alpha}\int_0^\infty |w(t)|^p e^{-t}dt \tag{1.17}$$

and

$$\int_0^R e^{\mu|u|^{\frac{p}{p-1}}} d\lambda_\theta(r) = \frac{\omega_\theta R^{1+\theta}}{1+\theta}\int_0^\infty e^{\frac{\mu}{\mu_{\alpha,\theta}}|w|^{\frac{p}{p-1}}-t}dt. \tag{1.18}$$

We consider Moser's functions

$$w_j(t) = \begin{cases} \frac{t}{j^{\frac{1}{p}}} & 0 \le t \le j \\ j^{\frac{p-1}{p}} & t \ge j. \end{cases}$$

Hence, we obtain from (1.16), (1.17) and (1.18) that

$$\begin{aligned} \int_0^R e^{\mu\left(\frac{|u_j|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,R)}}\right)^{\frac{p}{p-1}}} d\lambda_\theta(r) &= \frac{\omega_\theta R^{1+\theta}}{1+\theta}\int_0^\infty e^{\frac{\mu|w_j|^{\frac{p}{p-1}}}{\mu_{\alpha,\theta}\left(1+\rho(\alpha,\theta,R)a_j\right)^{\frac{1}{p-1}}}-t}dt \\ &\ge e^{\left(\frac{\mu}{\mu_{\alpha,\theta}(1+\rho(\alpha,\theta,R)a_j)^{\frac{1}{p-1}}}-1\right)j}, \end{aligned}$$

where $\rho(\alpha,\theta,R) = \frac{R^{1+\theta}\omega_\theta}{(1+\theta)^p\omega_\alpha}$, $a_j = \frac{1}{j}\displaystyle\int_0^j e^{-t}t^p dt + j^{p-1}e^{-j}$ and $w_j(t) = \omega_\alpha^{\frac{1}{1+\alpha}}(1+\theta)^{\frac{\alpha}{\alpha+1}}u_j(Re^{-\frac{t}{(1+\theta)}})$. Thus, if $\mu > \mu_{\alpha,\theta}$

$$\begin{aligned} \lim_{j\to\infty}\int_0^R e^{\mu\left(\frac{|u_j|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,R)}}\right)^{\frac{p}{p-1}}} d\lambda_\theta(r) &\ge \lim_{j\to\infty} e^{\left(\frac{\mu}{\mu_{\alpha,\theta}\left(1+\rho(\alpha,\theta,R)a_j\right)^{\frac{1}{p-1}}}-1\right)j} \\ &= +\infty, \end{aligned}$$

which concludes the theorem.

$\square$

Chapter 2

Maximizers for variational problems associated with Trudinger-Moser type inequalities

Borrowing ideas from [35], we discuss the existence of a maximizer for the maximizing problem associated with the weighted Trudinger-Moser type inequality, and we obtain both the existence and the nonexistence results. In order to show the existence of a maximizer of $d(\theta, \alpha, \mu)$, we need to avoid a lack of compactness caused by the concentration of maximizing sequences.

2.1 Concentration compactness at infinity

In this section, we are going to prove Theorem 0.2. To show the attainability, we study the behaviour of maximizing sequences to (5). Throughout this section we assume that (u_n) is a bounded sequence in $W^{1,p}_{\alpha,\theta}(0,\infty)$ satisfying

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,p}_{\alpha,\theta}(0,\infty), \quad \text{where} \quad \alpha - (p-1) = 0.$$

We begin with

Lemma 2.1. *Let $0 < \mu < \mu_{\alpha,\theta}$. Assume that (u_n) is a positive maximizing sequence to (5). Then, we have*

$$\int_0^\infty A_{p,\mu}(|u_n|) - \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor !}|u_n|^{\frac{p\lfloor p \rfloor}{p-1}}\, d\lambda_\theta - \int_0^\infty A_{p,\mu}(|u|) - \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor !}|u|^{\frac{p\lfloor p \rfloor}{p-1}}\, d\lambda_\theta \to 0$$

as $n \to \infty$. (2.1)

Proof. We can rewrite (2.1) as follows

$$\int_0^\infty B_{\lfloor p \rfloor +1,\mu}(|u_n|)\ d\lambda_\theta - \int_0^\infty B_{\lfloor p \rfloor +1,\mu}(|u|)\, d\lambda_\theta \to 0$$

as $n \to \infty$, where

$$B_{k,\mu}(t) := \sum_{j=k}^\infty \frac{\mu^j}{j!} t^{\frac{p}{p-1}j}, \ \text{ where } \ k \in \mathbb{N} \ \text{ and } \ t \in [0,\infty).$$

It follows from Mean Value Theorem and convexity of $B_{\lfloor p\rfloor+1,\mu}$ that

$$
\begin{aligned}
&\left|B_{\lfloor p\rfloor+1,\mu}(u_n(x))-B_{\lfloor p\rfloor+1,\mu}(u(x))\right|\\
&\leq \left(B_{\lfloor p\rfloor+1,\mu}\right)'(\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x))\cdot|u_n(x)-u(x)|\\
&=\mu\frac{p}{p-1}\left|\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x)\right|^{\frac{1}{p-1}}\\
&\cdot B_{\lfloor p\rfloor,\mu}(\gamma_n(x)u_n(x)+(1-\gamma_n(x))u(x))\cdot|u_n(x)-u(x)|\\
&\leq\mu\frac{p}{p-1}\left|\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x)\right|^{\frac{1}{p-1}}\\
&\cdot\left[\gamma_n(x)B_{\lfloor p\rfloor,\mu}(u_n(x))+(1-\gamma_n(x))B_{\lfloor p\rfloor,\mu}(u(x))\right]\cdot|u_n(x)-u(x)|\\
&\leq\mu\frac{p}{p-1}\left|\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x)\right|^{\frac{1}{p-1}}\cdot\left[A_{p,\mu}(u_n(x))+A_{p,\mu}(u(x))\right]\\
&\cdot|u_n(x)-u(x)| \qquad (2.2)
\end{aligned}
$$

Now, by Hölder's and Minkowski's inequalities and (2.2) we get

$$
\begin{aligned}
&\left|\int_0^\infty B_{\lfloor p\rfloor+1,\mu}\left(|u_n|\right)d\lambda_\theta-\int_0^\infty B_{\lfloor p\rfloor+1,\mu}\left(|u|\right)d\lambda_\theta\right|\\
&\leq\mu\frac{p}{p-1}\left(\int_0^\infty|\gamma_n(x)u_n(x)+(1-\gamma_n(x))u(x)|^{\frac{r}{p-1}}\,d\lambda_\theta(x)\right)^{\frac{1}{r}}\\
&\cdot\left(\int_0^\infty\left[A_{p,\mu}(u_n(x))+A_{p,\mu}(u(x))\right]^q d\lambda_\theta(x)\right)^{\frac{1}{q}}\left(\int_0^\infty|u_n(x)-u(x)|^t\,d\lambda_\theta(x)\right)^{\frac{1}{t}}\\
&\leq\mu\frac{p}{p-1}\left(\|u_n\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)}+\|u\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)}\right)\left(\int_0^\infty(A_{p,\mu}(u_n(x)))^q\,d\lambda_\theta(x)\right)^{\frac{1}{q}}\\
&\cdot\left(\int_0^\infty(A_{p,\mu}(u(x)))^q\,d\lambda_\theta(x)\right)^{\frac{1}{q}}\cdot\|u_n-u\|_{L^t_\theta(0,\infty)}\\
&\leq\mu\frac{p}{p-1}\left(\|u_n\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)}+\|u\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)}\right)\left(\int_0^\infty A_{p,q\mu}(u_n(x))d\lambda_\theta(x)\right)^{\frac{1}{q}}\\
&\cdot\left(\int_0^\infty A_{p,q\mu}(u(x))d\lambda_\theta(x)\right)^{\frac{1}{q}}\|u_n-u\|_{L^t_\theta(0,\infty)}, \qquad (2.3)
\end{aligned}
$$

where $q,r,t>1$ are real numbers satisfying $\frac{1}{r}+\frac{1}{q}+\frac{1}{t}=1$, $q\mu<\mu_{\alpha,\theta}$, $\frac{r}{p-1}\geq p$ and $t>\frac{p^2}{p-1}$. Besides, in the last inequality at 2.3) we used the following inequality

$$
\left(e^{\mu t^{\frac{p}{p-1}}}-\sum_{j=0}^{\lfloor p\rfloor-1}\frac{\mu^j}{j!}t^{\frac{p}{p-1}j}\right)^q\leq e^{q\mu t^{\frac{p}{p-1}}}-\sum_{j=0}^{\lfloor p\rfloor-1}\frac{(q\mu)^j}{j!}t^{\frac{p}{p-1}j}.
$$

Therefore, from (2.3), Lemma 1.10, and compactness embedding we conclude the proof of the Lemma. □

To study the compactness of a maximizing sequence to (5) based on the concentration-compactness type argument, we analyze the possibility of a lack of compactness which is called vanishing. For this, we will introduce some components as follows

$$\mu_0 = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_0^R |u_n(x)|^p d\lambda_\theta(x) + \int_0^R |(u_n)'(x)|^p d\lambda_\alpha(x)\right)$$

$$\mu_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_R^\infty |u_n(x)|^p d\lambda_\theta(x) + \int_R^\infty |(u_n)'(x)|^p d\lambda_\alpha(x)\right),$$

$$\nu_0 = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_0^R A_{p,\mu}(|u_n|)\, d\lambda_\theta(x)\right)$$

$$\nu_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_R^\infty A_{p,\mu}(|u_n(x)|)\, d\lambda_\theta(x)\right),$$

$$\eta_0 = \lim_{R\to\infty}\lim_{n\to\infty}\int_0^R |u_n(x)|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta(x), \text{ and}$$

$$\eta_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\int_R^\infty |u_n(x)|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta(x)$$

taking an appropriate subsequence if necessary. It is easy to see that, if $(u_n)_{n\in\mathbb{N}}$ is a maximizing sequence to (5) ($\|u_n\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$), then

$$\begin{aligned} \nu_i &\geq \frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}\eta_i,\ 1 = \mu_0 + \mu_\infty,\ d(p,\theta,\mu) = \nu_0 + \nu_\infty \text{ and} \\ 1 &\geq \eta_0 + \eta_\infty \text{ (if } p \text{ is an integer)}, \end{aligned} \tag{2.4}$$

where $i = 0$ or $i = \infty$.

Definition 2.2. (u_n) *is a normalized vanishing sequence,* (NVS) *in short, if* (u_n) *satisfies* $\|u_n\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$ *(with* $\alpha - (p-1) = 0$*),* $u = 0$ *and* $\nu_0 = 0$.

Example 2.3. *Let* ϕ *be a smooth nonincreasing function with compact support on* $[0,+\infty)$ *satisfying* $\|\phi\|_{L^p_\theta(0,\infty)} = 1$. *We set*

$$\phi_n(x) := \frac{\lambda_n \phi(\lambda_n^\gamma x)}{(1+\lambda_n^p\lambda_0)^{\frac{1}{p}}},$$

where $\gamma = \frac{p}{1+\theta}$, $\lambda_0 := \|\phi'\|^p_{L^p_\alpha(0,\infty)}$, *and* (λ_n) *is a positive sequence such that* $\lambda_n \to 0$ *as* $n \to \infty$. *Thus,* $(\phi_n)_{n\in\mathbb{N}}$ *is a normalized vanishing sequence.*

The main aim here is to show that $d(\alpha,\theta,\mu)$ is greater than the vanishing level. More precisely

$$d(\alpha,\theta,\mu) > \sup_{\left\{(u_n)\subset W^{1,p}_{\alpha,\theta}(0,\infty):(u_n) \text{ is a NVS}\right\}} \int_0^\infty A_{p,\mu}(|u_n(x)|)\, d\lambda_\theta(x).$$

Thus, we define the *normalized vanishing limit* as follows

Definition 2.4. *The number*

$$d_{nvl}(\alpha,\theta,\mu) = \sup_{\left\{(u_n)\subset W^{1,p}_{\alpha,\theta}(0,\infty):(u_n) \textit{ is a NVS}\right\}} \int_0^\infty A_{p,\mu}(|u_n(x)|)\, d\lambda_\theta(x), \tag{2.5}$$

is called a normalized vanishing limit.

The *normalized vanishing limit* will depend on α and μ.

Next, we rewrite the elements defined above. For this purpose, given a real number $R > 0$, we take a function $\phi_R \in C^\infty(\mathbb{R})$, such that

$$\begin{cases} \phi_R(x) = 1, & \text{if } 0 \le x < R, \\ 0 \le \phi_R(x) \le 1, & \text{if } R \le x \le R+1, \\ \phi_R(x) = 0, & \text{if } R+1 \le x, \\ |\phi_R'(x)| \le 2, & \text{if } x \in \mathbb{R}. \end{cases}$$

After that, we define the functions ϕ_R^0 and ϕ_R^∞ by

$$\phi_R^0(x) := \phi_R(x), \;\; \phi_R^\infty(x) := 1 - \phi_R^0(x).$$

Lemma 2.5. *Let $R > 0$, $n \in \mathbb{N}$, and $u_{n,R}^i = \phi_R^i u_n$ $(i = 0, \infty)$. We have*

$$\begin{aligned} \mu_i &= \lim_{R\to\infty} \lim_{n\to\infty} \left(\int_0^\infty |u_{n,R}^i(x)|^p d\lambda_\theta(x) + \int_0^\infty |(u_{n,R}^i(x))'|^p d\lambda_\alpha(x) \right), & (2.6) \\ \nu_i &= \lim_{R\to\infty} \lim_{n\to\infty} \int_0^\infty A_{p,\mu}\left(|u_{n,R}^i(x)|\right) d\lambda_\theta(x), \quad \textit{and} & (2.7) \\ \eta_i &= \lim_{R\to\infty} \lim_{n\to\infty} \int_0^\infty |u_{n,R}^i|^{\frac{p}{p-1}\lfloor p \rfloor} d\lambda_\theta(x). & (2.8) \end{aligned}$$

Proof. We will prove only (2.6) with $i = 0$. On the one hand,

$$\int_0^R |u_n|^p d\lambda_\theta \le \int_0^\infty |\phi_R^0 u_n|^p d\lambda_\theta \le \int_0^{R+1} |u_n|^p d\lambda_\theta. \tag{2.9}$$

On the other hand, from the Mean Value Theorem we obtain

$$\begin{aligned} \int_0^\infty |(u_{n,R}^0)'|^p d\lambda_\alpha &= \int_0^\infty |\phi_R^0 u_n' + (\phi_R^0)' u_n|^p d\lambda_\alpha \\ &= \int_0^\infty |\phi_R^0 u_n'|^p d\lambda_\alpha + \rho_{n,R}, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \rho_{n,R} = p &\int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^{p-2} \phi_R^0 u_n' (\phi_R^0)' u_n d\lambda_\alpha(x) \\ + p &\int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^{p-2} t_n(x)(\phi_R^0)' u_n \cdot (\phi_R^0)' u_n d\lambda_\alpha(x) \end{aligned}$$

and $0 \le t_n(x) \le 1$.

We get

$$\begin{aligned} |\rho_{n,R}| &\le p \left[\int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^p d\lambda_\alpha \right]^{\frac{p-1}{p}} \left[\int_0^\infty |(\phi_R^0)' u_n|^p d\lambda_\alpha \right]^{\frac{1}{p}} \\ &\le 2p[\|u_n'\|_{L_\alpha^p} + 2\|u_n\|_{L_\alpha^p(R,R+1)}]^{p-1} \|u_n\|_{L_\alpha^p(R,R+1)} \\ &\le 2p \left[1 + 2\|u_n\|_{L_\alpha^p(R,R+1)}\right]^{p-1} \|u_n\|_{L_\alpha^p(R,R+1)}. \end{aligned}$$

By compactness embedding, $W^{1,p}((R,R+1)) \hookrightarrow L^p((R,R+1))$ with $R > 0$, we have $\lim\limits_{n\to\infty} \|u_n\|_{L^p_\alpha(R,R+1)} = \|u\|_{L^p_\alpha(R,R+1)}$. Thus,

$$\lim_{R\to\infty}\lim_{n\to\infty} \rho_{n,R} = 0. \tag{2.11}$$

We conclude (2.6) (with $i = 0$) from (2.9), (2.10) and (2.11). The other cases follow from similar arguments. $\square$

Next, our goal is to determine the normalized vanishing limit defined at (2.5).

Proposition 2.6. *It holds that*

$$d_{nvl}(p,\theta,\mu) = \begin{cases} \dfrac{\mu^{p-1}}{(p-1)!} & \text{if } p \text{ is integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can suppose that u_n is non-increasing (see Theorem 1.9 and Remark 1.8), then by Lemma 1.10

$$|u_n(x)| \le \left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{1}{p}} \cdot \frac{1}{x^{\frac{1+\theta}{p}}} \left(\int_0^\infty |u_n(y)|^p d\lambda_\theta(y)\right)^{\frac{1}{p}}.$$

Assuming that $1 \le R < \infty$, then

$$\begin{aligned} \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\int_R^\infty |u_n|^{\frac{p}{p-1}j} d\lambda_\theta &\le \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{j}{p-1}} \omega_\theta \int_R^\infty x^{\theta-\frac{(1+\theta)}{p-1}j} dx \\ &\le \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{j}{p-1}} \frac{\omega_\theta(p-1)}{R^{(1+\theta)\left(\frac{j}{p-1}-1\right)}} \\ &\le \frac{\omega_\theta(p-1)}{R^{(1+\theta)\left(\frac{\lfloor p\rfloor+1}{p-1}-1\right)}} \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{j}{p-1}}. \end{aligned}$$

Thus

$$\lim_{R\to\infty}\lim_{n\to\infty} \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\int_R^\infty |u_n|^{\frac{p}{p-1}j} d\lambda_\theta = 0. \tag{2.12}$$

If p is not integer, we get

$$\int_R^\infty |u_n|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta \le \left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{\lfloor p\rfloor}{p-1}} \frac{\omega_\theta(p-1)}{\left(\lfloor p\rfloor-(p-1)\right) R^{(1+\theta)\left(\frac{\lfloor p\rfloor}{p-1}-1\right)}}. \tag{2.13}$$

Hence, using (2.12) and (2.13), we obtain $\nu_\infty = 0$, if p is not integer.

Now, if p is integer, then $\lfloor p\rfloor = p-1$ and passing to subsequence if necessary, we have

$$\begin{aligned}
\nu_\infty &= \lim_{R\to\infty}\lim_{n\to\infty}\sum_{j=\lfloor p\rfloor+1}^{\infty}\frac{\mu^j}{j!}\int_R^\infty |u_n|^{\frac{p}{p-1}j}d\lambda_\theta \\
&+ \lim_{R\to\infty}\lim_{n\to\infty}\frac{\mu^{p-1}}{(p-1)!}\left\|u_n\right\|^p_{L^p_\theta(R,\infty)} \\
&= \frac{\mu^{p-1}}{(p-1)!}\lim_{R\to\infty}\lim_{n\to\infty}\left\|u_n\right\|^p_{L^p_\theta(R,\infty)} \\
&\le \frac{\mu^{p-1}}{(p-1)!}. \qquad (2.14)
\end{aligned}$$

Taking $u_n := \phi_n$ as in the Example 2.3 we obtain (2.12) as well. Besides, we get

$$\lim_{R\to\infty}\lim_{n\to\infty}\left\|u_n\right\|^p_{L^p_\theta(R,+\infty)} = \lim_{R\to\infty}\lim_{n\to\infty}\frac{\left\|\phi\right\|^p_{L^p_\theta(\lambda_n^\sigma R,\infty)}}{(1+\lambda_n^{p\gamma}\lambda_0)} = 1. \qquad (2.15)$$

From (2.12), (2.13), (2.14) and (2.15) the proposition follows.

□

Proposition 2.7. *Let $p\ge 2$ be an integer number. Then*

$$d(p,\theta,\mu) > \begin{cases} \dfrac{\mu^{p-1}}{(p-1)!}, & \text{if } p>2 \text{ and } \mu\in(0,\mu_{\alpha,\theta}], \\[2ex] \dfrac{\mu^{p-1}}{(p-1)!}, & \text{if } p=2 \text{ and } \mu\in\left(\frac{2}{B(2,\theta)},\mu_{\alpha,\theta}\right], \end{cases}$$

where $B(2,\theta)$ is defined as in Theorem 0.2.

Proof. Let $\sigma=\frac{p}{1+\theta}$, and $v\in W^{1,p}_{\alpha,\theta}(0,\infty)$. We set

$$v_t(x) = tv(t^\sigma x), \ \text{ for all } \ t,x\in(0,\infty).$$

We get

$$\begin{aligned}
&\int_0^\infty A_{p,\mu}\left(\frac{|v_t|}{\|v_t\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right)d\lambda_\theta \\
&\ge \frac{\mu^{p-1}}{(p-1)!}\left[\frac{\|v\|^p_{L^p_\theta}}{\|v\|^p_{L^p_\theta}+t^p\|v'\|^p_{L^p_\alpha}} + \frac{\mu}{p}\frac{t^{\frac{p}{p-1}}\|v\|^{\frac{p}{p-1}p}_{L_\theta^{\frac{p}{p-1}p}}}{\left(\|v\|^p_{L^p_\theta}+t^p\|v'\|^p_{L^p_\alpha}\right)^{\frac{p}{p-1}}}\right] \\
&:= \frac{\mu^{p-1}}{(p-1)!}h_{p,\theta,\mu}(t).
\end{aligned}$$

Note that $\lim\limits_{t\to 0}h_{p,\theta,\mu}(t)=1$. Thus, it is sufficient to show that $h'_{p,\theta,\mu}(t)>0$ for $0<t\ll 1$. Through straightforward calculation we obtain

$$
\begin{aligned}
h'_{p,\theta,\mu}(t) &= \frac{pt^{\frac{p}{p-1}-1}}{\left(\|v\|^p_{L^p_\theta} + t^p \|v'\|^p_{L^p_\alpha}\right)^2} \\
&\cdot \left[\frac{\mu}{p(p-1)} \|v\|^{\frac{p}{p-1}p}_{L_\theta^{\frac{p}{p-1}p}} \left(\|v\|^p_{L^p_\theta} + t^p \|v'\|^p_{L^p_\alpha}\right)^{\frac{p-2}{p-1}}\right. \\
&- \frac{\mu}{p-1} t^p \|v\|^{\frac{p}{p-1}p}_{L_\theta^{\frac{p}{p-1}p}} \|v'\|^p_{L^p_\alpha} \left(\|v\|^p_{L^p_\theta} + t^p \|v'\|^p_{L^p_\alpha}\right)^{-\frac{1}{p-1}} \\
&- \left.\|v\|^p_{L^p_\theta} \|v'\|^p_{L^p_\alpha} t^{p-\frac{p}{p-1}}\right].
\end{aligned}
$$

Thus we get $h'_{p,\theta,\mu}(t) > 0$ for $0 < t \ll 1$ if $p > 2$. Now, for $p = 2$ it is slightly different, because

$$
h'_{2,\theta,\mu}(t) = \frac{2t}{\left(\|v\|^2_{L^2_\theta} + t^2 \|v'\|^2_{L^2_1}\right)^2} \cdot \left[\frac{\mu}{2} \|v\|^4_{L^4_\theta} - \frac{\mu t^2 \|v\|^4_{L^4_\theta} \|v'\|^2_{L^2_1}}{\left(\|v\|^2_{L^2_\theta} + t^2 \|v'\|^2_{L^2_1}\right)} - \|v\|^2_{L^2_\theta} \|v'\|^2_{L^2_1}\right].
$$

Taking $v \in W^{1,p}_{\alpha,\theta}(0,\infty)$, such that $B(2,\theta)^{-1} = B(v)^{-1}$, we obtain $h'_{2,\theta,\mu}(t) > 0$ for $0 < t \ll 1$, if $\frac{2}{B(2,\theta)} < \mu \leq 2\pi(1+\theta)$, [see Proposition 2.10]. $\square$

Lemma 2.8. *Let $\mu_i < 1$ $(i = 0, \infty)$ and let $p \geq 2$ be an integer. Then, we obtain*

$$
\begin{aligned}
& d(p,\theta,\mu) \left\|u^i_{n,R}\right\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)} \\
& \geq \int_0^\infty A_{p,\mu}\left(\left|u^i_{n,R}\right|\right) d\lambda_\theta + \left[\left(\frac{1}{\left\|u^i_{n,R}\right\|^{\frac{p}{p-1}}_{W^{1,p}_{\alpha,\theta}(0,\infty)}} - 1\right)\right. \\
& \cdot \left.\int_0^\infty A_{p,\mu}\left(\left|u^i_{n,R}\right|\right) - \frac{\mu^{p-1}}{(p-1)!} \left|u^i_{n,R}\right|^p d\lambda_\theta\right],
\end{aligned}
$$

whenever n and R are sufficiently large.

Proof. By definition, we have

$$
\begin{aligned}
d(\alpha,\theta,\mu) &\geq \sum_{j=p-1}^\infty \frac{\mu^j}{j!} \frac{\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}}{\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{W^{1,p}_{\alpha,\theta}(0,\infty)}} \\
&\geq \frac{1}{\|u^i_{n,R}\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)}} \sum_{j=p-1}^\infty \frac{\mu^j}{j!} \|u^i_{n,R}\|^{\frac{jp}{p-1}}_{L_\theta^{\frac{jp}{p-1}}} \\
&+ \frac{1}{\|u^i_{n,R}\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)}} \sum_{j=p}^\infty \left(\frac{1}{\|u^i_{n,R}\|^{\frac{p}{p-1}(j-(p-1))}_{W^{1,p}_{\alpha,\theta}(0,\infty)}} - 1\right) \frac{\mu^j}{j!} \|u^i_{n,R}\|^{\frac{jp}{p-1}}_{L_\theta^{\frac{jp}{p-1}}} \qquad (2.16)
\end{aligned}
$$

From $\mu_i < 1$ and (2.16) we obtain

$$
\begin{aligned}
& d(\alpha,\theta,\mu)\|u^i_{n,R}\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)} \\
& \geq \sum_{j=p-1}^{\infty} \frac{\mu^j}{j!}\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}} + \sum_{j=p}^{\infty}\left(\frac{1}{\|u^i_{n,R}\|^{\frac{p}{p-1}}_{W^{1,p}_{\alpha,\theta}(0,\infty)}} - 1\right)\frac{\mu^j}{j!}\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}} \\
& = \int_0^\infty A_{p,\mu}(|u^i_{n,R}|)d\lambda_\theta \\
& + \left(\frac{1}{\|u^i_{n,R}\|^{\frac{p}{p-1}}_{W^{1,p}_{\alpha,\theta}(0,\infty)}} - 1\right)\int_0^\infty A_{p,\mu}(|u^i_{n,R}|) - \frac{\mu^{p-1}}{(p-1)!}|u^i_{n,R}|^p d\lambda_\theta
\end{aligned}
$$

for large R and large n. □

Proposition 2.9. *Let $p \geq 2$ be an integer. Assume that (u_n) is a positive maximizing sequence to (5). Then,*

$$(\mu_0,\nu_0) = (1, d(p,\theta,\mu)) \ \ \textit{and} \ \ (\mu_\infty,\nu_\infty) = (0,0).$$

Proof. By contradiction, suppose that $0 < \mu_0 < 1$. Then $0 < \mu_\infty < 1$, by relation (2.4). From Lemma 2.5 and Lemma 2.8 we have

$$d(\alpha,\theta,\mu)\mu_i \geq \nu_i + \left[\frac{1}{\mu_i^{\frac{1}{p-1}}} - 1\right]\left[\nu_i - \frac{\mu^{p-1}}{(p-1)!}\eta_i\right]. \tag{2.17}$$

By relation (2.4) and together with (2.17) we get

$$d(\alpha,\theta,\mu)\mu_i \geq \nu_i, \ \text{ for } \ i = 0,\infty.$$

Thus,

$$d(\alpha,\theta,\mu) = d(\alpha,\theta,\mu)(\mu_0+\mu_\infty) \geq \nu_0+\nu_\infty = d(\alpha,\theta,\mu)$$

and consequently

$$d(\alpha,\theta,\mu)\mu_i = \nu_i.$$

From the last relation and (2.17) we obtain

$$\nu_i \leq \frac{\mu^{p-1}}{(p-1)!}\eta_i,$$

whence,

$$d(\alpha,\theta,\mu) = \nu_0+\nu_\infty \leq \frac{\mu^{p-1}}{(p-1)!}(\eta_0+\eta_\infty) \leq \frac{\mu^{p-1}}{(p-1)!},$$

which contradicts the Proposition 2.7.

Now, again, by contradiction, suppose that $\mu_0 = 0$. Thus, by Lemma 2.8

$$\begin{aligned} & d(p,\theta,\mu)\left\|u^0_{n,R}\right\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)} \\ \geq\ & \int_0^\infty A_{p,\mu}\left(\left|u^0_{n,R}\right|\right) d\lambda_\theta \\ +\ & \frac{1}{2}\int_0^\infty \left(A_{p,\mu}\left(\left|u^0_{n,R}\right|\right) - \frac{\mu^{p-1}}{(p-1)!}\left|u^0_{n,R}\right|^p\right) d\lambda_\theta. \end{aligned} \tag{2.18}$$

for large R and large n.

Taking the double limit in (2.18), $\lim_{R\to\infty}\lim_{n\to\infty}$, we obtain

$$d(\alpha,\theta,\mu)\mu_0 \geq \nu_0 + \frac{1}{2}\left(\nu_0 - \frac{\mu^{p-1}}{(p-1)!}\eta_0\right) \geq \nu_0,$$

hence, $\nu_0 = 0$ from relation (2.4), and $\mu_0 = 0$, getting a contradiction from Proposition 2.6, relation (2.4), and Proposition 2.7. Finally, using the same arguments we can get $\nu_\infty = 0$ whenever $\mu_\infty = 0$. Therefore, the proposition follows. □

Theorem 0.2. *Let $p \geq 2$, $\theta, \alpha \geq 0$ and $\mu > 0$ be real numbers such that $\alpha - (p-1) = 0$. Then there exists a positive nonincreasing function u in $W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$ such that*

$$d(\theta,\alpha,\mu) = \int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x),$$

in the following cases:

(i) $p > 2$ *and* $0 < \mu < \mu_{\alpha,\theta}$,

(ii) $p = 2$ *and* $\frac{2}{B(2,\theta)} < \mu < \mu_{\alpha,\theta}$,

where

$$B(2,\theta)^{-1} := \inf_{0\neq u\in W^{1,2}_{1,\theta}(0,\infty)} \frac{\|u'\|^2_{L^2_1(0,\infty)}\cdot\|u\|^2_{L^2_\theta(0,\infty)}}{\|u\|^4_{L^4_\theta(0,\infty)}}.$$

Proof of Theorem 0.2.

First of all, assume that (u_n) is a non-increasing positive maximizing sequence to (5). We will show that

$$\lim_{n\to\infty}\int_0^\infty |u_n|^{\frac{p}{p-1}\lfloor p\rfloor}d\lambda_\theta = \int_0^\infty |u|^{\frac{p}{p-1}\lfloor p\rfloor}d\lambda_\theta. \tag{2.19}$$

Indeed, given $R > 0$, note that

$$\begin{aligned} \left|\int_0^\infty \left(|u_n|^{\frac{p}{p-1}\lfloor p\rfloor} - |u|^{\frac{p}{p-1}\lfloor p\rfloor}\right) d\lambda_\theta\right| \ \leq\ & \left|\int_0^R \left(|u_n|^{\frac{p}{p-1}\lfloor p\rfloor} - |u|^{\frac{p}{p-1}\lfloor p\rfloor}\right) d\lambda_\theta\right| \\ +\ & \int_R^\infty |u_n|^{\frac{p}{p-1}\lfloor p\rfloor}d\lambda_\theta + \int_R^\infty |u|^{\frac{p}{p-1}\lfloor p\rfloor}dx \\ =:\ & I(n,R) + II(n,R) + III(R). \end{aligned}$$

By compact embedding we have $\lim_{R\to\infty}\lim_{n\to\infty} I(n,R)=0$. From Dominated Convergence Theorem, we obtain $\lim_{R\to\infty}\lim_{n\to\infty} III(R)=0$. If p is an integer, we get $\lim_{R\to\infty}\lim_{n\to\infty} II(n,R)=0$ from $\mu_\infty=0$ (Proposition 2.9). If $p\notin\mathbb{N}$, we obtain $\lim_{R\to\infty}\lim_{n\to\infty} II(n,R)=0$ from inequality (2.13). Hence, (2.19) follows. Now, assume that either $p>2$ and $\mu\in(0,\mu_{\alpha,\theta})$ or $p=2$ and $\mu\in(2/B(2,\theta),\mu_{1,\theta})$. Writing

$$\begin{aligned} d(p,\theta,\mu)-\int_0^\infty A_{p,\mu}(|u|)\,d\lambda_\theta &= \int_0^\infty A_{p,\mu}(|u_n|)\,d\lambda_\theta-\int_0^\infty A_{p,\mu}(|u|)\,d\lambda_\theta\\ &+ \left(d(p,\theta,\mu)-\int_0^\infty A_{p,\mu}(|u_n|)\,d\lambda_\theta\right)\\ &=: IV(n)+V(n),\end{aligned}$$

where

$$V(n):=d(p,\theta,\mu)-\int_0^\infty A_{p,\mu}(|u_n|)\,d\lambda_\theta$$

and

$$IV(n):=\int_0^\infty A_{p,\mu}(|u_n|)\,d\lambda_\theta-\int_0^\infty A_{p,\mu}(|u|)\,d\lambda_\theta.$$

We get, by definition of $d(\alpha,\theta,\mu)$, that

$$\lim_{R\to\infty}\lim_{n\to\infty} V(n)=0.$$

Since

$$\begin{aligned} IV(n)=&\int_0^\infty\left(A_{p,\mu}(|u_n|)-\frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}|u_n|^{\frac{p}{p-1}\lfloor p\rfloor}\right)d\lambda_\theta\\ &-\int_0^\infty\left(A_{p,\mu}(|u|)-\frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}|u|^{\frac{p}{p-1}\lfloor p\rfloor}\right)d\lambda_\theta\\ &+\frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}\int_0^\infty\left(|u_n|^{\frac{p}{p-1}\lfloor p\rfloor}-|u|^{\frac{p}{p-1}\lfloor p\rfloor}\right)d\lambda_\theta.\end{aligned}$$

From Lemma 2.1 and relation (2.19) we obtain

$$\lim_{n\to\infty} IV(n)=0.$$

Thus, $u\neq 0$ and

$$d(p,\theta,\mu)=\int_0^\infty A_{p,\mu}(|u|)\,d\lambda_\theta.$$

Now, we assert that $\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}=1$. Indeed, on the one hand,

$$\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}\le\liminf_{n\to\infty}\|u_n\|_{W^{1,p}_{p-1,\theta}(0,\infty)}=1.$$

On the other hand,

$$
\begin{aligned}
d(p,\theta,\mu) &\geq \int_0^\infty A_{p,\mu}\left(\left(\frac{|u|}{\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}}\right)\right) d\lambda_\theta \\
&= \sum_{j=\lfloor p\rfloor}^{\infty} \frac{\mu^j}{j!} \frac{\|u\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}}{\|u\|^{\frac{p}{p-1}j}_{W^{1,p}_{p-1,\theta}(0,\infty)}} \\
&\geq \frac{1}{\|u\|^{\frac{p}{p-1}\lfloor p\rfloor}_{W^{1,p}_{p-1,\theta}(0,\infty)}} \sum_{j=\lfloor p\rfloor}^{\infty} \frac{\mu^j}{j!} \|u\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}} \\
&\geq \frac{1}{\|u\|^{\frac{p}{p-1}\lfloor p\rfloor}_{W^{1,p}_{p-1,\theta}(0,\infty)}} \cdot d(p,\theta,\mu).
\end{aligned}
$$

Therefore, $\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1$ and

$$
d(p,\theta,\mu) = \int_0^\infty A_{p,\mu}\left(|u|\right) d\lambda_\theta.
$$

□

2.2 Lack of compactness

Throughout this section, we assume that $p = 2$ and $\mu \leq \pi(1+\theta)/3$.

By Theorem 1.4 (inequality (1.2)), we get

$$
\frac{\|u\|^{2j}_{L_\theta^{2j}}}{\|u'\|^2_{L_1^2} \cdot \|u\|^2_{L_\theta^2}} \leq C_{\gamma,2,\theta} \frac{j!}{\gamma^j} \|u'\|^{2(j-2)}_{L_1^2} \tag{2.20}
$$

for all $u \in W^{1,2}_{1,\theta}(0,\infty)$, $j \in \mathrm{N}$, and $0 < \gamma < (1+\theta)\omega_1$. We are going to use the inequality (2.20) to prove the Theorem 0.3.

Theorem 0.3. *Let $p = 2$, $\theta \geq 0$ and $\alpha = 1$. Then there exists μ_0 such that $d(\theta,\alpha,\mu)$ is not achieved for all $0 < \mu < \mu_0$.*

Proof of Theorem 0.3

Let $S := \{v \in W^{1,2}_{1,\theta}(0,\infty) : \|v\|_{W^{1,2}_{1,\theta}(0,\infty)} = 1\}$. For each $u \in S$, we define a family of functions by

$$
u_t(x) := t^{\frac{1}{2}} u(t^{\frac{1}{1+\theta}} x),
$$

where $t > 0$ is a parameter. Besides, let $v_t := u_t/\|u_t\|_{W^{1,2}_{1,\theta}(0,\infty)}$. Thus, v_t is a curve in S passing through u when $t = 1$. Then, it is sufficient to show that

$$
\frac{d}{dt} F(v_t)\,|_{t=1} < 0,
$$

where $F(w) := \displaystyle\int_0^\infty A_{2,\mu}(w(x))\, d\lambda_\theta(x)$.

Through a direct calculation we have that $\|u_t\|^{2j}_{L^{2j}_\theta} = t^{j-1}\|u\|^{2j}_{L^{2j}_\theta}$, $\|(u_t)'\|^2_{L^2_1} = t\|u'\|^2_{L^2_1}$ and

$$F(v_t) = \sum_{j=1}^\infty \frac{\mu^j}{j!} \frac{t^{j-1}\|u\|^{2j}_{L^{2j}_\theta}}{\left(\|u\|^2_{L^2_\theta} + t\|u'\|^2_{L^2_1}\right)^j}.$$

Since

$$\frac{d}{dt}F(v_t) =$$
$$\sum_{j=1}^\infty \frac{\mu^j}{j!} \frac{(j-1)t^{j-2}\|u\|^{2j}_{L^{2j}_\theta}\left(\|u\|^2_{L^2_\theta} + t\|u'\|^2_{L^2_1}\right)^j - jt^{j-1}\|u'\|^2_{L^2_1}\|u\|^{2j}_{L^{2j}_\theta}\left(\|u\|^2_{L^2_\theta} + t\|u'\|^2_{L^2_1}\right)^{j-1}}{\left(\|u\|^2_{L^2_\theta} + t\|u'\|^2_{L^2_1}\right)^{2j}}$$

we obtain

$$\begin{aligned}
\frac{d}{dt}F(v_t)\,|_{t=1} &= \sum_{j=1}^\infty \frac{\mu^j}{j!}\|u\|^{2j}_{L^{2j}_\theta}\left[(j-1)\|u\|^2_{L^2_\theta} - j\|u'\|^2_{L^2_1}\right] \\
&= -\mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1} + \sum_{j=2}^\infty \frac{\mu^j}{j!}\|u\|^{2j}_{L^{2j}_\theta}\left[(j-1)\|u\|^2_{L^2_\theta} - j\|u'\|^2_{L^2_1}\right] \\
&\le \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\left(-1 + \sum_{j=2}^\infty \frac{\mu^{j-1}}{(j-1)!}\frac{\|u\|^{2j}_{L^{2j}_\theta}}{\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}}\right). \qquad (2.21)
\end{aligned}$$

From inequality (2.20) (with $\gamma := 2\pi(1+\theta)/3$) and (2.21) we get

$$\begin{aligned}
&\frac{d}{dt}F(v_t)\,|_{t=1} \\
&\le \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\left(-1 + C_{\frac{2}{3}\pi(1+\theta),2,\theta}\sum_{j=2}^\infty \frac{\mu^{j-1}}{(j-1)!}j!\left(\frac{3}{2\pi(1+\theta)}\right)^j\right) \\
&= \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1} \\
&\cdot\left(-1 + C_{\frac{2}{3}\pi(1+\theta),2,\theta}\left(\frac{3}{2\pi(1+\theta)}\right)^2\mu\sum_{j=2}^\infty \mu^{j-2}j\left(\frac{3}{2\pi(1+\theta)}\right)^{j-2}\right) \\
&\le \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\left(-1 + C_{\frac{2}{3}\pi(1+\theta),2,\theta}\left(\frac{3}{2\pi(1+\theta)}\right)^2\mu\sum_{j=2}^\infty j\left(\frac{1}{2}\right)^{j-2}\right).
\end{aligned}$$

Thus, taking $\mu_0 := \frac{1}{a\cdot C_{\frac{2}{3}\pi(1+\theta),2,\theta}}\left(\frac{2\pi(1+\theta)}{3}\right)^2$, where $a := \displaystyle\sum_{j=2}^\infty j\left(\frac{1}{2}\right)^{j-2}$, the proof of the theorem follows.

□

2.3 Gagliardo-Nirenberg Inequalities

In this section, we discuss about the best constant of the Gagliardo-Niremberg inequality, and we will explore some ideas contained in [5, 18, 21].

It is known the interpolation inequality with weights

$$\|u\|_{L^q_\theta(0,\infty)} \leq K(p,q,\alpha,\theta)\,\|u'\|^{\gamma}_{L^p_\alpha(0,\infty)}\,\|u\|^{1-\gamma}_{L^p_\theta(0,\infty)}\,, \tag{2.22}$$

where $1 < p \leq q < p^\star = \frac{p(1+\theta)}{\alpha-(p-1)}$, $\alpha \geq p-1$, $\theta \geq 0$ and $1-\gamma = \frac{p}{q}\cdot\frac{(p^\star-q)}{(p^\star-p)}$. It is worth noting that when $\alpha = p-1$ we have $1-\gamma = \frac{p}{q}$.

Throughout this section we will assume that $\alpha \leq p+\theta$. So, we can compute the optimal $k = K(p,q,\alpha,\theta)$ in (2.22) if we determine the explicit solution of the minimization problem

$$\inf\left\{E(u) := \frac{1}{p}\int_0^\infty |u'|^p\,d\lambda_\alpha + \frac{1}{p}\int_0^\infty |u|^p\,d\lambda_\theta : \|u\|_{L^q_\theta((0,\infty))} = 1\right\}. \tag{2.23}$$

It follows from Lemma 1.10 and Theorem 1.9 (with $\alpha = m$, and $l = \theta$) the existence of a minimizer for (2.23). Now if u_∞ is a minimizer of the variational problem (2.23), then

$$E(u_\infty) \leq E(u) = \frac{1}{p}\|u'\|^p_{L^p_\alpha((0,\infty))} + \frac{1}{p}\|u\|^p_{L^p_\theta((0,\infty))}$$

for all $u \in W^{1,p}_{\alpha,\theta}((0,\infty))$ satisfying $\|u\|_{L^q_\theta((0,\infty))} = 1$. Thus,

$$E(u_\infty) \leq \frac{1}{p}\frac{\|u'\|^p_{L^p_\alpha((0,\infty))}}{\|u\|^p_{L^q_\theta((0,\infty))}} + \frac{1}{p}\frac{\|u\|^p_{L^p_\theta((0,\infty))}}{\|u\|^p_{L^q_\theta((0,\infty))}}$$

for every $0 \neq u \in W^{1,p}_{\alpha,\theta}(0,\infty)$. Scaling u as $u_t(x) = u(tx)$, we get

$$E(u_\infty) \leq t^{p-(\alpha+1)+\frac{p}{q}(1+\theta)}\frac{\|u'\|^p_{L^p_\alpha((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}} + t^{(1+\theta)\left(\frac{p}{q}-1\right)}\frac{\|u\|^p_{L^p_\theta((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}}.$$

A direct computation proves that the minimum over t is achieved at

$$t = \left[\frac{(1+\theta)(q-p)}{pq+p(1+\theta)-q(1+\alpha)}\frac{B}{A}\right]^{\frac{1}{p+\theta-\alpha}},$$

where

$$A = \frac{\|u'\|^p_{L^p_\alpha((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}} \quad\text{and}\quad B = \frac{\|u\|^p_{L^p_\alpha((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}}.$$

Therefore,

$$E(u_\infty) \leq \left[\left(\tfrac{(1+\theta)(q-p)}{qp+p(1+\theta)-q(1+\alpha)}\right)^{1-\gamma} + \left(\tfrac{(1+\theta)(q-p)}{qp+p(1+\theta)-q(1+\alpha)}\right)^{\gamma}\right]\frac{\|u'\|^{p\gamma}_{L^p_\alpha}\|u\|^{p(1-\gamma)}_{L^p_\theta}}{p\|u\|^p_{L^q_\theta}}$$

and the equality holds when $u = u_\infty$.

The next result is important in the study of attainability in the Trudinger-Moser inequality with weigth when $p = 2$.

Proposition 2.10. *If $p = 2$, $\alpha = p - 1$, and $\theta \geq 0$, then the infimum*

$$B(2,\theta)^{-1} := \inf_{0 \neq u \in W^{1,2}_{1,\theta}(0,\infty)} \frac{\|u'\|^2_{L^2_1((0,\infty))} \cdot \|u\|^2_{L^2_\theta((0,\infty))}}{\|u\|^4_{L^4_\theta((0,\infty))}},$$

is attained by a positive nonincreasing function in $W^{1,2}_{1,\theta}((0,\infty))$. Moreover,

$$B(2,\theta)^{-1} < \pi(1+\theta).$$

Proof. The first part has been discussed at the beginning of this section. Then, we focus on the second part. Set

$$B(u)^{-1} := \frac{\|u'\|^2_{L^2_1((0,\infty))} \cdot \|u\|^2_{L^2_\theta((0,\infty))}}{\|u\|^4_{L^4_\theta((0,\infty))}}.$$

Note that it is sufficient to exhibit a function $u \in W^{1,2}_{1,\theta}((0,\infty))$ such that $B(u)^{-1} = \pi(1+\theta)$ and it is not a solution of

$$-(u'x)'\omega_1 + ux^\theta\omega_\theta - \lambda u^3 x^\theta \omega_\theta = 0, \tag{2.24}$$

for all $\lambda > 0$.

On the one hand, through a direct calculation we can see that for every positive function v in $W^{1,2}_{1,\theta}((0,\infty))$ of the form

$$v(x) = a_1(1 + a_2 x^{a_3})^{a_4},$$

where a_1, a_2, a_3, a_4 are real numbers, it is not a solution for (2.24). On the other hand, choosing

$$u(x) = \frac{1}{1 + x^{1+\theta}},$$

then

$$\begin{aligned} \|u\|^4_{L^4_\theta((0,\infty))} &= \tfrac{\omega_\theta}{3(1+\theta)} \\ \|u\|^2_{L^2_\theta((0,\infty))} &= \tfrac{\omega_\theta}{(1+\theta)} \\ \|u'\|^2_{L^2_1((0,\infty))} &= \tfrac{\omega_1(1+\theta)}{6} \end{aligned}$$

Therefore, $B(u)^{-1} = \pi(1+\theta)$, and then the proposition follows. $\square$

Part II

Isoperimetric inequalities with different monomial weights

Chapter 3

Sobolev and isoperimetric inequalities with different monomial weights

We discuss the existence and nonexistence of isoperimetric inequalities where the volume and perimeter carry different monomial weights related with the Hardy-Sobolev type inequalities with monomial weights studied by Castro [20]. The Theorems 0.4 and 0.5 improve some conditions imposed by Castro [20].

3.1 Some definitions

Let us introduce some elements that we will use in the following chapters.

Given a nonnegative function $\omega : \mathbb{R}^N \to \mathbb{R}$, locally lipschitz on $\mathbb{R}^N$, we set the P_ω-Perimeter of a measurable set E by

$$P_\omega(E) := \sup \left\{ \int_E div(\omega(x)\nu(x))dx; \nu \in C^1_c(\mathbb{R}^N, \mathbb{R}^N), |\nu| \leq 1 \text{ on } \mathbb{R}^N \right\}.$$

We now consider the specific density $\omega(x) = x^A := |x_1|^{a_1} \cdot \ldots \cdot |x_N|^{a_N}$, where $a_i \geq 0$ for every $i = 1, \ldots, N$. Let $p \in [1, \infty)$, and let $\Omega \subset \mathbb{R}^N$ be an open set. We will denote by $L^p(\Omega, x^A)$ the space of all Lebesgue measurable real functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^p(\Omega,x^A)} := \left[\int_\Omega |u|^p x^A dx \right]^{\frac{1}{p}} < +\infty.$$

Then let $W^{1,p}(\Omega, x^A, x^B)$ be the weighted Sobolev space of all functions $u \in L^p(\Omega, x^A)$ possessing weak first derivatives which belong to $L^p(\Omega, x^B)$. The space will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega,x^A,x^B)} := \|\nabla u\|_{L^p(\Omega,x^A)} + \|u\|_{L^p(\Omega,x^B)}$$

Besides that, let $BV(\Omega, x^A, x^B)$ be the weighted BV-space of all functions $u \in L^1(\Omega, x^B)$ such that

$$\sup \left\{ \int_{\mathbb{R}^N} div \left(x^A \nu(x) \right) u(x)dx; \nu \in C^1_c \left(\Omega; \mathbb{R}^N \right) \text{ satisfying } \nu = 0 \text{ on } \partial\mathbb{R}^N_A, \text{and } |\nu| \leq 1 \right\} < +\infty.$$

A norm on $BV(\Omega, x^A, x^B)$ is given by

$$\|u\|_{BV(\Omega,x^A,x^B)} := \|Du\|(\Omega, x^A) + \|u\|_{L^1(\Omega,x^B)},$$

where

$$\|Du\|(\Omega, x^A) := \sup\left\{\int_{\mathbb{R}^N} div\left(x^A\nu(x)\right)u(x)dx; \nu \in C_c^1\left(\Omega;\mathbb{R}^N\right), \ \nu = 0 \,\text{on}\, \partial\mathbb{R}^N_A, \text{and } |\nu| \le 1\right\}.$$

It follows from *Riesz Representation Theorem* (see Theorem A.1 and Theorem A.27) that there exists a Radon measure $\|D^Au\|$ on Ω and a $\|D^Au\|$-measurable function $\sigma : \Omega \to \mathbb{R}^N$ such that

(*i*) $|\sigma(x)| = 1$ for $\|D^Au\|$-a.e, and

(*ii*) $\displaystyle\int_\Omega div\left(x^A\nu(x)\right)u(x)dx = -\int_\Omega \langle\nu(x), \sigma(x)\rangle d\|D^Au\|$ for all $\nu \in C_c^1\left(\Omega, \mathbb{R}^N\right)$.

Besides that, for each open set $V \subset\subset \Omega$, we have

$$\|D^Au\|(V) = \sup\left\{\int_{\mathbb{R}^N} div\left(x^A\nu(x)\right)u(x)dx; \nu \in C_c^1\left(V;\mathbb{R}^N\right), \ \nu = 0 \,\text{on}\, \partial\mathbb{R}^N_A, \text{and } |\nu| \le 1\right\}.$$

Thus, we denote $[D^Au] = \sigma \llcorner \|D^Au\|$, and the components of $[D^Au]$ by D_i^Au for each $i = 1, \ldots, N$. We say that a $\mathcal{H}^N$-measurable subset $E \subset \mathbb{R}^N_A$ has *finite perimeter with weight* x^A *and finite volume with weight* x^B, in Ω, if $\chi_E \in BV(\Omega, x^A, x^B)$. We thus denote *the weighted perimeter of* E *with weight* x^A, or *the* P_A*-perimeter of* E, or *the weighted perimeter* $P_A(E)$ by

$$P_A(E, D) = \|D^A\chi_E\|(D), \tag{3.1}$$

for every Borel set D in $\mathbb{R}^N_A$. In addition, if A is the vector 0, then the perimeter P_A coincides with the classical perimeter. When $u = \chi_E \in BV_{loc}(\Omega)$ we denote ν^E, instead of σ.

For a nonnegative measurable function $\gamma : \mathbb{R}^N \to \mathbb{R}$, we set by m_γ the Lebesgue measure with weight $\gamma(x)$, namely,

$$m_\gamma(M) = \int_M \gamma(x)dx,$$

for all $\mathcal{H}^N$-measurable set M in $\mathbb{R}^N$. If $\gamma(x) = x^B := |x|^{b_1} \cdot \ldots \cdot |x_N|^{b_N}$, we denote m_B, instead of m_{x^B}.

Next, if a measurable set M with $0 < m_\gamma(M) < \infty$, then let $\mathcal{R}_{A,B,N}(M)$ denote the isoperimetric quotient of M by

$$\mathcal{R}_{A,B,N}(M) := \frac{P_A(M)}{[m_B(M)]^{\frac{N+a-1}{N+b}}},$$

and if $u \in C_c^1\left(\mathbb{R}^N\right)\backslash\{0\}$, then let

$$\mathcal{Q}_{A,B,N}(u) := \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|x^A dx}{\left[\displaystyle\int_{\mathbb{R}^N} |u|^{\frac{N+b}{N+a-1}} x^B dx\right]^{\frac{N+a-1}{N+b}}}.$$

These two elements will be related later (see Lemma 3.3).

Let us introduce some observations.

If Ω is a smooth bounded open set, then the weighted perimeter P_ω is equivalent to the following

$$P_\omega(\Omega) = \int_{\partial\Omega} \omega(x) d\mathcal{H}^{N-1}(x),$$

and the same happens for $\omega(x) = x^A$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. We thus have for $\Omega \subset \mathbb{R}^N_A$ a smooth bounded open set that

$$\mathcal{R}_{A,B,N}(\Omega) := \frac{\displaystyle\int_{\partial\Omega} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_{\Omega} x^B dx\right]^{\frac{N+a-1}{N+b}}}. \tag{3.2}$$

It is worth emphasizing that the constant $C_{A,B,N}$ (defined in (10)) satisfies

$$C_{A,B,N} = \inf\left\{\mathcal{R}_{A,B,N}(M); M \text{ is measurable with } 0 < m_B(M) < +\infty\right\}.$$

Throughout the subsequent chapters we will use the following notation:

We say that a vector $A \in \mathbb{R}^N$ is nonnegative if all its entries are nonnegative.

For $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_N) \in \mathbb{R}^N$ a vector, and $A = (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_N) \in \mathbb{R}^N$ a nonnegative vector, let us denote by

$\overline{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$;

$\overline{A}_i := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$;

$\overline{x}_{ik} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, \dots, x_{k-1}, x_{k+1}, \dots x_N)$;

$\overline{A}_{ik} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, \dots, a_{k-1}, a_{k+1}, \dots a_N)$;

$\overline{a}_i := a - a_i = a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_N$;

$\overline{a}_{ik} := a - a_i - a_k = a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots a_{k-1} + a_{k+1} \cdots + a_N$.

Finally, when $N \in \mathbb{N}$ and $r > 0$, we denote by $B_N(r)$ the ball centered in 0 and radius r in $\mathbb{R}^N$, and $\mathbb{R}^N_A = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N; x_i > 0 \text{ whenever } a_i > 0\}$. Moreover $B^+_N(r) = B_N(r) \cap \mathbb{R}^N_+$, where $\mathbb{R}^N_+ := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N; x_i > 0 \text{ for every } i \in \{1, \dots, N\}\}$.

3.2 Existence of isoperimetric inequalities

This section contains relevant results for two theorems presented in the introduction, Part II. Here, we prove the item (I) of Theorem 0.4 based on two important lemmata, moreover we establish the sufficient condition (II) using classical arguments such as coarea formula.

Borrowing ideas from [6], we establish the following important result.

Lemma 3.1. *Let $A = (a_1, \dots, a_N)$ and $B = (b_1, \dots, a_N)$ be two nonnegative vectors in $\mathbb{R}^N$. If*

$$C_{A,B,N} > 0$$

then

$$a_i - \frac{N+a-1}{b+N} b_i \geq 0$$

or equivalently

$$a_i - \frac{N+\overline{a}_i-1}{N+\overline{b}_i} b_i \geq 0.$$

Proof. Arguing by contradiction, we assume that

$$a_i - \frac{N+a-1}{b+N} b_i < 0. \tag{3.3}$$

Consider $t > 2$ and $B(te_i, 1)$ the ball centered in te_i and radius 1.

Using the area formula, we obtain

$$\begin{aligned}
\int_{\partial B(te_i,1)} x^A d\mathcal{H}^{N-1}(x) &= \int_{x_1^2+\cdots+x_{i-1}^2+(x_i-t)^2+x_{i+1}^2+\cdots+x_N^2=1} |x_1|^{a_1} \cdot \ldots \cdot |x_N|^{a_N} d\mathcal{H}^{N-1}(x) \\
&= \int_{B_{N-1}(1)} \left| t + (1-|\overline{x}_i|^2)^{\frac{1}{2}} \right|^{a_i} \frac{\overline{x}_i^{\overline{A}_i}}{(1-|\overline{x}_i|^2)^{\frac{1}{2}}} d\overline{x}_i \\
&+ \int_{B_{N-1}(1)} \left| t - (1-|\overline{x}_i|^2)^{\frac{1}{2}} \right|^{a_i} \frac{\overline{x}_i^{\overline{A}_i}}{(1-|\overline{x}_i|^2)^{\frac{1}{2}}} d\overline{x}_i \\
&\leq (1+2^{a_i}) t^{a_i} \int_{B_{N-1}(1)} \frac{\overline{x}_i^{\overline{A}_i}}{(1-|\overline{x}_i|^2)^{\frac{1}{2}}} d\overline{x}_i
\end{aligned} \tag{3.4}$$

On the other hand, by change of variable and elementary inequalities, we get

$$\begin{aligned}
\int_{B(te_i,1)} x^B dx &= \int_{x_1^2+\cdots+x_{i-1}^2+(x_i-t)^2+x_{i+1}^2+\cdots+x_N^2<1} |x_1|^{b_1} \cdot \cdots \cdot |x_N|^{b_N} dx \\
&= \int_{t-1}^{t+1} |x_i|^{b_i} \left(\int_{B_{N-1}\left([1-(x_i-t)^2]^{\frac{1}{2}}\right)} \overline{x}_i^{\overline{B}_i} d\overline{x}_i \right) dx_i \\
&= \int_{t-1}^{t+1} |x_i|^{b_i} \left(1-(x_i-t)^2\right)^{\frac{\overline{b}_i+(N-1)}{2}} dx_i \int_{B_{N-1}(1)} \overline{x}_i^{\overline{B}_i} d\overline{x}_i \\
&= \int_{B_{N-1}(1)} \overline{x}_i^{\overline{B}_i} d\overline{x}_i \int_{-1}^{1} |y+t|^{b_i} \left(1-y^2\right)^{\frac{\overline{b}_i+(N-1)}{2}} dy \\
&\geq \int_{B_{N-1}(1)} \overline{x}_i^{\overline{B}_i} d\overline{x}_i \int_{0}^{1} |y+t|^{b_i} \left(1-y^2\right)^{\frac{\overline{b}_i+(N-1)}{2}} dy \\
&\geq t^{b_i} \int_{B_{N-1}(1)} \overline{x}_i^{\overline{B}_i} d\overline{x}_i \int_{0}^{1} \left(1-y^2\right)^{\frac{\overline{b}_i+(N-1)}{2}} dy.
\end{aligned} \tag{3.5}$$

It follows from inequalities (3.4) and (3.5) that

$$\frac{\displaystyle\int_{\partial B(te_i,1)} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_{B(te_i,1)} x^B dx\right]^{\frac{N+a-1}{N+b}}} \leq \frac{\displaystyle(1+2^{a_i})t^{a_i}\int_{B_{N-1}(1)} \frac{\overline{x}_i^{\overline{A}_i}}{(1-|\overline{x}_i|^2)^{\frac{1}{2}}} d\overline{x}_i}{\left[\displaystyle t^{b_i}\int_{B_{N-1}(1)} \overline{x}_i^{\overline{B}_i} d\overline{x}_i \int_0^1 \left(1-y^2\right)^{\frac{\overline{b}_i+(N-1)}{2}}\right]^{\frac{N+a-1}{N+b}}} \tag{3.6}$$

Thus by (3.3) and inequality (3.6), we obtain

$$\lim_{t\to\infty} \frac{\displaystyle\int_{\partial B(te_i,1)} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_{B(te_i,1)} x^A dx\right]^{\frac{N+a-1}{N+b}}} = 0.$$

Which is a contradiction with $C_{A,B,N} > 0$. $\square$

The previous lemma gives us the first behavior and huge dependence upon the vector $B = (b_1, \ldots, b_N)$ in relation to the vector $A = (a_1, \ldots, a_N)$. For instance, if $a_i = 0$, then the isoperimetric inequality exists only if $b_i = 0$.

Lemma 3.2. *Let $A = (a_1, \ldots, a_N)$ and $B = (b_1, \ldots, a_N)$ be two nonnegative vectors in $\mathbb{R}^N$. If*

$$C_{A,B,N} > 0$$

then

$$a_i - \frac{N+a-1}{N+b} b_i \leq \frac{N+a-1}{N+b}$$

or equivalently

$$\frac{a_i}{b_i+1} \leq \frac{N+\overline{a}_i-1}{N+\overline{b}_i-1}.$$

Proof. Again, by an argument of contradiction, we assume that

$$a_i - \frac{N+a-1}{N+b} b_i > \frac{N+a-1}{N+b}. \tag{3.7}$$

We define for a positive ε the set

$$\Omega_\varepsilon = \left\{x \in \mathbb{R}^N; |x| < R^N, x_j > 0 \text{ for all } j \in \{1, \ldots, N\} \text{ and } x_i < \varepsilon|\overline{x}_i|\right\}.$$

We may see that

$$
\begin{aligned}
\partial\Omega_\varepsilon = & \left\{ x\in\mathbb{R}^N; x_j>0 \text{ for all } j\in\{1,\dots,N\}, x_i=\varepsilon|\overline{x}_i|, \text{ and } |\overline{x}_i| \le \frac{R}{(1+\varepsilon^2)^{\frac12}} \right\} \\
& \bigcup \left\{ x\in\mathbb{R}^N; x_j>0 \text{ for all } j\in\{1,\dots,N\}, \frac{R}{(1+\varepsilon^2)^{\frac12}} \le |\overline{x}_i| \le R, \text{ and } x_i = \left(R^2-|\overline{x}_i|^2\right)^{\frac12} \right\} \\
& \bigcup \left\{ x\in\mathbb{R}^N; x_j>0 \text{ for all } j\in\{1,\dots,N\}\backslash\{i\}, x_i=0, |x|\le R \right\} \\
& \bigcup_{k=1,k\neq i}^{N} \left\{ x\in\mathbb{R}^N; x_j>0 \text{ for all } j\in\{1,\dots,N\}\backslash\{k\}, x_k=0, |x|\le R, \text{ and } x_i\le\varepsilon|\overline{x}_{ik}| \right\} \\
& =: A^1_\varepsilon \cup A^2_\varepsilon \cup A^3_\varepsilon \bigcup_{k=1,k\neq i}^{N} C^k_\varepsilon. \qquad (3.8)
\end{aligned}
$$

By definition of Ω_ε and change of variables, we get

$$
\begin{aligned}
\int_{\Omega_\varepsilon} x^B dx = & \int_{B^+_{N-1}\left(\frac{R}{(1+\varepsilon^2)^{\frac12}}\right)} \int_0^{\varepsilon|\overline{x}_i|} \overline{x}_i^{\overline{B}_i} x_i^{b_i} dx_i d\overline{x}_i \qquad (3.9) \\
& + \int_{B^+_{N-1}(R)\backslash B^+_{N-1}\left(\frac{R}{(1+\varepsilon^2)^{1/2}}\right)} \int_0^{\left(R^2-|\overline{x}_i|^2\right)^{1/2}} \overline{x}_i^{\overline{B}_i} x_i^{b_i} dx_i d\overline{x}_i \\
& \ge \frac{\varepsilon^{b_i+1}}{b_i+1} \int_{B^+_{N-1}\left(\frac{R}{(1+\varepsilon^2)^{\frac12}}\right)} \overline{x}_i^{\overline{B}_i} |\overline{x}_i|^{b_i+1} d\overline{x}_i \\
& = \frac{\varepsilon^{b_i+1}R^{N+b}}{(b_i+1)(1+\varepsilon^2)^{\frac{N+b}{2}}} \int_{B^+_{N-1}(1)} \overline{x}_i^{\overline{B}_i} |\overline{x}_i|^{b_i+1} d\overline{x}_i. \qquad (3.10)
\end{aligned}
$$

By (3.8), we obtain

$$
\begin{aligned}
\int_{\partial\Omega_\varepsilon} x^A d\mathcal{H}^{N-1}(x) = & \int_{A^1_\varepsilon} x^A d\mathcal{H}^{N-1}(x) + \int_{A^2_\varepsilon} x^A d\mathcal{H}^{N-1}(x) + \int_{A^3_\varepsilon} x^A d\mathcal{H}^{N-1}(x) \\
& + \sum_{k=1,k\neq i}^{N} \int_{C^k_\varepsilon} x^A d\mathcal{H}^{N-1}(x). \qquad (3.11)
\end{aligned}
$$

We now estimate the boundary area with density $x^A d\mathcal{H}^{N-1}(x)$. First, we calculate on C^k_ε's.

Let $k\neq i$. If $a_k>0$, then

$$
\int_{C^k_\varepsilon} x^A d\mathcal{H}^{N-1}(x) = 0. \qquad (3.12)
$$

Otherwise, if $a_k = 0$, then

$$
\begin{aligned}
\int_{C_\varepsilon^k} x^A d\mathcal{H}^{N-1}(x) &= \int_{B_{N-2}^+\left(\frac{R}{(1+\varepsilon^2)^{1/2}}\right)} \int_0^{\varepsilon|\overline{x}_{ik}|} \overline{x}_{ik}^{\overline{A}_{ik}} x_i^{a_i} dx_i d\overline{x}_{ik} \\
&+ \int_{B_{N-2}^+(R)\setminus B_{N-2}^+\left(\frac{R}{(1+\varepsilon^2)^{1/2}}\right)} \int_0^{\left(R^2-|\overline{x}_{ik}|^2\right)^{1/2}} \overline{x}_{ik}^{\overline{A}_{ik}} x_i^{a_i} dx_i d\overline{x}_{ik} \\
&= \frac{\varepsilon^{a_i+1}}{a_i+1} \int_{B_{N-2}^+\left(\frac{R}{(1+\varepsilon^2)^{1/2}}\right)} \overline{x}_{ik}^{\overline{A}_{ik}} |\overline{x}_{ik}|^{a_i+1} d\overline{x}_{ik} \\
&+ \frac{1}{a_i+1} \int_{B_{N-2}^+(R)\setminus B_{N-2}^+\left(\frac{R}{(1+\varepsilon^2)^{1/2}}\right)} \overline{x}_{ik}^{\overline{A}_{ik}} \left(R^2 - |\overline{x}_{ik}|^2\right)^{\frac{a_i+1}{2}} d\overline{x}_{ik} \\
&= \frac{\varepsilon^{a_i+1} R^{N+a-1}}{(a_i+1)(1+\varepsilon^2)^{\frac{N+a-1}{2}}} \int_{B_{N-2}^+(1)} \overline{x}_{ik}^{\overline{A}_{ik}} |\overline{x}_{ik}|^{a_i+1} d\overline{x}_{ik} \\
&+ \frac{R^{N+a-1}}{(a_i+1)\,(1+\varepsilon^2)^{\frac{N+\overline{a}_i-2}{2}}} \int_{B_{N-2}^+\left((1+\varepsilon^2)^{\frac{1}{2}}\right)\setminus B_{N-2}^+(1)} \left(1 - \frac{|\overline{x}_{ik}|^2}{1+\varepsilon^2}\right)^{\frac{a_i+1}{2}} \overline{x}_{ik}^{\overline{A}_{ik}} d\overline{x}_{ik} \\
&\le \frac{R^{N+a-1} O(\varepsilon^{a_i+1})}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}} + \frac{R^{N+a-1}\varepsilon^{a_i+1}}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}} \int_{B_{N-2}^+\left((1+\varepsilon^2)^{\frac{1}{2}}\right)\setminus B_{N-2}^+(1)} \overline{x}_{ik}^{\overline{A}_{ik}} d\overline{x}_{ik} \\
&\le \frac{R^{N+a-1} O(\varepsilon^{a_i+1})}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}} + \frac{R^{N+a-1}\varepsilon^{a_i+1}}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}} \left((1+\varepsilon^2)^{\frac{N+\overline{a}_i-2}{2}} - 1 \right) \int_{B_{N-2}^+(1)} \overline{x}_{ik}^{\overline{A}_{ik}} d\overline{x}_{ik} \\
&\le \frac{R^{N+a-1} O(\varepsilon^{a_i+1})}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}} + \frac{R^{N+a-1} O(\varepsilon^{a_i+3})}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}}. \qquad (3.13)
\end{aligned}
$$

We now compute the boundary area on A_ε^1. It follows from Area Formula and change of variables that

$$
\begin{aligned}
\int_{A_\varepsilon^1} x^A d\mathcal{H}^{N-1}(x) &= \int_{B_{N-1}^+\left(\frac{R}{(1+\varepsilon^2)^{\frac{1}{2}}}\right)} \overline{x}_i^{\overline{A}_i} \varepsilon^{a_i} |\overline{x}_i|^{a_i} (1+\varepsilon^2)^{\frac{1}{2}} d\overline{x}_i \\
&= \frac{\varepsilon^{a_i} R^{N+a-1}}{(1+\varepsilon^2)^{\frac{N+a-2}{2}}} \int_{B_{N-1}^+(1)} \overline{x}_i^{\overline{A}_i} |\overline{x}_i|^{a_i} d\overline{x}_i. \qquad (3.14)
\end{aligned}
$$

Finally, we estimate the last integral. By change of variable and elementary inequalities, we obtain

$$\begin{aligned}
\int_{A_\varepsilon^2} x^A d\mathcal{H}^{N-1}(x) &= \int_{B_{N-1}^+(R)\setminus B_{N-1}^+\left(\frac{R}{(1+\varepsilon^2)^{1/2}}\right)} \overline{x}_i^{\overline{A}_i}\left(R^2-|\overline{x}_i|^2\right)^{\frac{a_i}{2}} d\overline{x}_i \\
&= R^{N+a-1}\int_{B_{N-1}^+(1)\setminus B_{N-1}^+\left(\frac{1}{(1+\varepsilon^2)^{1/2}}\right)} \overline{x}_i^{\overline{A}_i}\left(1-|\overline{x}_i|^2\right)^{\frac{a_i}{2}} d\overline{x}_i \\
&= \frac{R^{N+a-1}}{(1+\varepsilon^2)^{\frac{N+\overline{a}_i-1}{2}}}\int_{B_{N-1}^+((1+\varepsilon^2)^{\frac{1}{2}}B_{N-1}^+(1)} x_i^{\overline{A}_i}\left(1-\frac{|\overline{x}_i|^2}{1+\varepsilon^2}\right)^{\frac{a_i}{2}} d\overline{x}_i \\
&\le \frac{R^{N+a-1}}{(1+\varepsilon^2)^{\frac{N+\overline{a}_i-1}{2}}}\int_{B_{N-1}^+((1+\varepsilon^2)^{\frac{1}{2}})\setminus B_{N-1}^+(1)} \overline{x}_i^{\overline{A}_i}\left(1-\frac{1}{1+\varepsilon^2}\right)^{\frac{a_i}{2}} d\overline{x}_i \\
&= \frac{R^{N+a-1}\varepsilon^{a_i}}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}}\int_{B_{N-1}^+((1+\varepsilon^2)^{\frac{1}{2}})\setminus B_{N-1}^+(1)} \overline{x}_i^{\overline{A}_i} d\overline{x}_i \\
&= \frac{R^{N+a-1}\varepsilon^{a_i}}{(1+\varepsilon^2)^{\frac{N+a-1}{2}}}\left((1+\varepsilon^2)^{\frac{N+\overline{a}_i-1}{2}}-1\right)\int_{B_{N-1}^+(1)} \overline{x}_i^{\overline{A}_i} d\overline{x}_i \\
&= R^{N+a-1}O(\varepsilon^{a_i+2}). \qquad (3.15)
\end{aligned}$$

Thus, it follows from (3.9), (3.11), (3.12) or (3.13), (3.14), and (3.15) that

$$\begin{aligned}
\frac{P_A(\Omega_\varepsilon)}{[m_B(\Omega_\varepsilon)]^{\frac{N+a-1}{N+b}}} &\le \frac{\frac{\varepsilon^{a_i}R^{N+a-1}}{(1+\varepsilon^2)^{\frac{N+a-2}{2}}}\int_{B_{N-1}^+(1)} \overline{x}_i^{\overline{A}_i}|\overline{x}_i|^{a_i} d\overline{x}_i + R^{N+a-1}\left(O(\varepsilon^{a_i+1})+O(\varepsilon^{a_i+2})+O(\varepsilon^{a_i+3})\right)}{\left[\frac{\varepsilon^{b_i+1}R^{N+b}}{(b_i+1)(1+\varepsilon^2)^{\frac{N+b}{2}}}\int_{B_{N-1}^+(1)} \overline{x}_i^{\overline{B}_i}|\overline{x}_i|^{b_i+1} d\overline{x}_i\right]^{\frac{N+a-1}{N+b}}} \\
&= \varepsilon^{a_i-\frac{N+a-1}{N+b}(b_i+1)}\frac{\frac{1}{(1+\varepsilon^2)^{\frac{N+a-2}{2}}}\int_{B_{N-1}^+(1)} \overline{x}_i^{\overline{A}_i}|\overline{x}_i|^{a_i} d\overline{x}_i}{\left[\frac{1}{(b_i+1)(1+\varepsilon^2)^{\frac{N+b}{2}}}\int_{B_{N-1}^+(1)} \overline{x}_i^{\overline{B}_i}|\overline{x}_i|^{b_i+1} d\overline{x}_i\right]^{\frac{N+a-1}{N+b}}} \\
&\quad + O\left(\varepsilon^{a_i+1-\frac{N+a-1}{N+b}(b_i+1)}\right)+O\left(\varepsilon^{a_i+2-\frac{N+a-1}{N+b}(b_i+1)}\right)+O\left(\varepsilon^{a_i+3-\frac{N+a-1}{N+b}(b_i+1)}\right) \qquad (3.16)
\end{aligned}$$

Therefore, the inequality (3.16), and (3.7) imply that

$$\lim_{\varepsilon\to 0}\frac{P_A(\Omega_\varepsilon)}{[m_B(\Omega_\varepsilon)]^{\frac{N+a-1}{N+b}}} = 0,$$

which is a contradiction with our assumption. □

The next result is expected and the proof relies on classical arguments, see for example [44]. For convenience of the reader, we sketch the proof.

Lemma 3.3. *Let Ω be a Lipschitz bounded open set. Consider ω a nonnegative locally lipschitz function and γ a nonnegative continuous function on $\mathbb{R}^N$. Then there exists a smooth and* *compactly supported sequence $(u_\varepsilon)_{\varepsilon>0}$ on $\mathbb{R}^N$ such that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|u_\varepsilon|^p\gamma(x)dx=\int_\Omega \gamma(x)dx,\ \ \textit{for each}\ p\geq 1, \tag{3.17}$$

and mainly

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|\nabla u_\varepsilon(x)|\omega(x)dx=\int_{\partial\Omega}\omega(x)dx. \tag{3.18}$$

Proof. We begin with the following assertion.

Claim 1.

$$\int_{\mathbb{R}^N}|\chi_\Omega(x+h)-\chi_\Omega(x)|\,dx\leq |h|\mathcal{H}^{N-1}(\partial\Omega),$$

where χ_Ω is the characteristic function on the set Ω, and h is any vector in $\mathbb{R}^N$.

proof of the claim 1. Let φ be a smooth and compactly supported function on $\mathbb{R}^N$. We then have

$$\int_{\mathbb{R}^N}\left[\chi_\Omega(x+h)-\chi_\Omega(x)\right]\varphi(x)dx=\int_{\mathbb{R}^N}\chi_\Omega(x)\left[\varphi(x-h)-\varphi(x)\right]dx=\int_\Omega\left[\varphi(x-h)-\varphi(x)\right]dx.$$

By fundamental theorem of calculus and divergent theorem, we get

$$\begin{aligned}\int_\Omega \varphi(x-h)-\varphi(x)dx&=-\int_\Omega\int_0^1\nabla\varphi(x-th)hdtdx\\&=-\int_\Omega\left(h\int_0^1\nabla\varphi(x-th)dt\right)dx\\&=-\int_{\partial\Omega}\left(\int_0^1\varphi(x-th)dt\right)\langle h,\eta(x)\rangle\mathcal{H}^{N-1}(x),\end{aligned}$$

where η denotes the outward unit normal vector with respect to Ω.

This gives the estimate,

$$\left|\int_{\mathbb{R}^N}\left[\chi_\Omega(x+h)-\chi_\Omega(x)\right]\varphi(x)dx\right|\leq\sup_{y\in\mathbb{R}^N}|\varphi(y)||h|\mathcal{H}^{N-1}(\partial\Omega).$$

Thus, the proof of claim 1 follows. □

Claim 2. Let a mollifier $\rho\in C_c^\infty(\mathbb{R}^N)$ supported in the unit ball $B_N(0,1)$. We define

$$u_\varepsilon(x):=\rho_\varepsilon*\chi_\Omega(x)=\int_{\mathbb{R}^N}\rho_\varepsilon(x-y)\chi_\Omega(y)dy,$$

where $\rho_\varepsilon(x)=\varepsilon^{-N}\rho\left(\frac{x}{\varepsilon}\right)$. Then

$$u_\varepsilon\to\chi_\Omega\ \text{ in }\ L^1(\Omega,dx),\text{ and }\ L^1(\Omega,\gamma(x)dx).$$

proof of the claim 2. By properties of the function ρ, we obtain

$$u_\varepsilon(x)-\chi_\Omega(x)=\int_{\mathbb{R}^N}\rho_\varepsilon(y)\left[\chi_\Omega(x-y)-\chi_\Omega(x)\right]dy.$$

By the previous inequality and claim 1, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_\varepsilon(x) - \chi_\Omega(x)|\gamma(x)dx &\le C(\gamma,\Omega)\int_{\mathbb{R}^N} |u_\varepsilon(x) - \chi_\Omega(x)|dx \\
&\le C(\gamma,\Omega)\int_{\Omega}\rho_\varepsilon(y)\int_{\mathbb{R}^N} |\chi_\Omega(x-y) - \chi_\Omega(x)|\, dxdy \\
&\le C(\gamma,\Omega)\mathcal{H}^{N-1}(\partial\Omega)\int_{\mathbb{R}^N} |y|\rho_\varepsilon(y)dy \\
&= \varepsilon C(\gamma,\Omega)\mathcal{H}^{N-1}(\partial\Omega)\int_{\mathbb{R}^N} |y|\rho(y)dy,
\end{aligned}$$

where $C(\gamma,\Omega) = \sup\{\gamma(y); y \in \mathbb{R}^N, dist(y,\Omega) < 1\}$.

Thus, the claim 2 follows, and so the equality (3.17). □

Now, we will prove equality (3.18). Taking $f \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} u_\varepsilon(x) div\,(w(x)f(x))\,dx = -\int_{\mathbb{R}^N}\langle\nabla u_\varepsilon(x), \omega(x)f(x)\rangle dx. \tag{3.19}$$

We then have

$$\left|\int_{\mathbb{R}^N} u_\varepsilon(x) div\,(\omega(x)f(x))\,dx\right| \le \sup_{y\in\mathbb{R}^N}|f(y)|\int_{\mathbb{R}^N}|\nabla u_\varepsilon(x)|\omega(x)dx.$$

Taking the supremum over all $f \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$ satistying $|f| \le 1$ on $\mathbb{R}^N$, we get

$$\int_{\partial\Omega}\omega(x)d\mathcal{H}^{N-1} \le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^N}|\nabla u_\varepsilon(x)|\omega(x)dx. \tag{3.20}$$

For the proof of the reverse inequality, we consider $\delta > 0$ arbitrary. By uniform continuity of ω on $\partial\Omega$, there exists $\theta(\delta,\partial\Omega) > 0$, that depends on δ and $\partial\Omega$, such that

$$|\omega(x+y) - \omega(x)| < \delta$$

whenever $|y| < \theta(\delta,\partial\Omega)$.

It follows from equality (3.19), divergence theorem and previous statement that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}\langle\nabla u_\varepsilon(x), f(x)\rangle\omega(x)dx\right| &= \left|\int_{\mathbb{R}^N} u_\varepsilon(x) div\,(w(x)f(x))\,dx\right| \\
&= \left|\int_{\mathbb{R}^N}\rho_\varepsilon(y)\int_{\Omega} div\,(\omega(x+y)f(x+y))\,dxdy\right| \\
&= \left|\int_{\mathbb{R}^N}\rho_\varepsilon(y)\int_{\partial\Omega}\langle f(x+y),\eta(x)\rangle\omega(x+y)d\mathcal{H}^{N-1}(x)dy\right| \\
&= \left|\int_{\mathbb{R}^N}\rho_\varepsilon(y)\int_{\partial\Omega}\langle f(x+y),\eta(x)\rangle\omega(x+y)d\mathcal{H}^{N-1}(x)dy\right| \\
&\le \sup_{y\in\mathbb{R}^N}|f(y)|\left[\int_{\mathbb{R}^N}\rho_\varepsilon(y)\int_{\partial\Omega}|\omega(x+y) - w(x)|d\mathcal{H}^{N-1}(x)dy\right. \\
&\left. + \int_{\mathbb{R}^N}\rho_\varepsilon(y)\int_{\partial\Omega}\omega(x)d\mathcal{H}^{N-1}(x)dy\right] \\
&\le \sup_{y\in\mathbb{R}^N}|f(y)|\left[\delta\mathcal{H}^{N-1}(\partial\Omega) + \int_{\partial\Omega}\omega(x)d\mathcal{H}^{N-1}(x)\right].
\end{aligned} \tag{3.21}$$

Here, η denotes the outward unit normal vector with respect to Ω, and $\varepsilon < \theta(\delta, \partial\Omega)$.

Applying the reverse Hölder inequality to the inequality (3.21), we obtain

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon(x)|\omega(x)dx \leq \delta\mathcal{H}^{N-1}(\partial\Omega) + \int_{\partial\Omega} \omega(x)\mathcal{H}^{N-1}(x), \text{ for every } \varepsilon < \theta(\delta, \partial\Omega). \tag{3.22}$$

By inequalities (3.20), and (3.22), we get the equality (3.18), and the proof of the lemma is complete. □

Remark 3.4. *Given a Lipschitz bounded open set, in order to analyze the isoperimetric quotient*

$$\frac{\displaystyle\int_{\partial\Omega} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_\Omega x^B dx\right]^{\frac{N+a-1}{N+b}}}, \tag{3.23}$$

it is sufficient to consider Ω contained in $\mathbb{R}^N_A$, if $a - b \leq 1$. The strategy below is due to Cabré and Ros-Oton, see [16].

We may assume, by symmetry, that $A = (a_1, \ldots, a_k, 0, \ldots, 0)$, where $a_i > 0$ for every $i \in \{1, \ldots, k\}$ and some $0 \leq k \leq N$. We split the domain Ω in at most 2^k disjoint subdomains Ω_j, $j \in \{1, \ldots, J\}$, where each subdomain Ω_j is contained in the cone $\{\varepsilon_i x_i > 0, i \in \{1, \ldots, k\}\}$ for different $\varepsilon_i \in \{-1, 1\}$. Thus, we have $\overline{\Omega} = \overline{\Omega}_1 \cup \ldots \cup \overline{\Omega}_J$,

$$P_A(\Omega) = \sum_{j=1}^J P_A(\Omega_j), \text{ since the weight is zero on } \{x_i = 0\}, \text{ and}$$

$$m_B(\Omega) = \sum_{j=1}^J m_B(\Omega_j).$$

Hence

$$\frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{N+a-1}{N+b}}} \geq \min\left\{\frac{P_A(\Omega_j)}{[m_B(\Omega_j)]^{\frac{N+a-1}{N+b}}}; 1 \leq j \leq J\right\} := \frac{P_A(\Omega_{j_0})}{[m_B(\Omega_{j_0})]^{\frac{N+a-1}{N+b}}}, \tag{3.24}$$

since $a - b \leq 1$, moreover, the equality in (3.24) can hold when $a - b = 1$. After reflections regarding the x_i-axis, where $i \in \{1, \ldots, k\}$, we can assume that $\Omega_{j_0} \subset \mathbb{R}^N_A$, since this movement changes neither the volume $m_B(\Omega_{j_0})$ nor the perimeter $P_A(\Omega_{j_0})$.

In addition to that, given a Lipschitz bounded open set $\Omega \subset \mathbb{R}^N_A$, the isoperimetric quotient (3.23) of Ω may be approximated on $\mathbb{R}^N_A$, namely there exists a sequence of smooth open sets $(\Omega_\delta)_{\delta>0}$ with $\overline{\Omega}_\delta \subset \Omega \subset \mathbb{R}^N_A$ satisfying

$$\frac{\displaystyle\int_{\partial\Omega_\delta} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_{\Omega_\delta} x^B dx\right]^{\frac{N+a-1}{N+b}}} \to \frac{\displaystyle\int_{\partial\Omega} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_\Omega x^B dx\right]^{\frac{N+a-1}{N+b}}} \text{ as } \delta \to 0.$$

Lemma 3.5. *Let $A = (a_1, \dots, a_N)$ and $B = (b_1, \dots, b_N)$ be two nonnegative vectors. Assume that $a - b \le 1$, then*

$$C_{A,B,N} = \inf\{\mathcal{Q}_{A,B,N}(u) : u \in C_0^1(\mathbb{R}^N)\backslash\{0\}\}$$

Proof. Consider $\varepsilon > 0$, then there exists a smooth bounded open set Ω such that $\overline{\Omega} \subset \mathbb{R}^N_A$, see Remark 3.4, satisfying

$$\mathcal{R}_{A,B,N}(\Omega) \le C_{A,B,N} + \varepsilon.$$

Applying Lemma 3.3 for the functions $\gamma(x) = x^B$, and $\omega(x) = x^A$, we then have

$$C_{A,B,N} \ge \inf\{\mathcal{Q}_{A,B,N}(u) : u \in C_0^1(\mathbb{R}^N)\backslash\{0\}\}.$$

To get the reverse inequality, without loss of generality, we may assume that u is a nonnegative function. By coarea formula, we get

$$\begin{aligned}\int_{\mathbb{R}^N} x^A|\nabla u|dx &= \int_0^\infty \int_{u=t} x^A \mathcal{H}^{N-1}(x)dt \\ &\ge C_{A,B,N} \int_0^\infty \left[\int_{u>t} x^B dx\right]^{\frac{N+a-1}{N+b}} dt. \end{aligned} \tag{3.25}$$

It follows from Minkowski's inequality for integrals and Fubini's theorem that

$$\begin{aligned}\int_{\mathbb{R}^N} x^B |u|^{\frac{N+b}{N+a-1}} dx &= \int_{\mathbb{R}^N} x^B \left[\int_0^\infty \chi_{\{z>0; u(x)>z\}}(t)dt\right]^{\frac{N+b}{N+a-1}} dx \\ &= \int_{\mathbb{R}^N} x^B \left[\int_0^\infty \chi_{\{y\in\mathbb{R}^N; u(y)>t\}}(x)dt\right]^{\frac{N+b}{N+a-1}} dx \\ &= \int_{\mathbb{R}^N} \left[\int_0^\infty \left(x^B \chi_{\{y\in\mathbb{R}^N; u(y)>t\}}(x)\right)^{\frac{N+a-1}{N+b}} dt\right]^{\frac{N+b}{N+a-1}} dx \\ &\le \left[\int_0^\infty \left(\int_{\mathbb{R}^N} x^B \chi_{\{y\in\mathbb{R}^N; u(y)>t\}}(x)dx\right)^{\frac{N+a-1}{N+b}} dt\right]^{\frac{N+b}{N+a-1}} \\ &= \left[\int_0^\infty \left(\int_{u>t} x^B dx\right)^{\frac{N+a-1}{N+b}} dt\right]^{\frac{N+b}{N+a-1}}. \end{aligned} \tag{3.26}$$

Hence, by (3.25) and (3.26), we then get

$$C_{A,B,N} \le \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u| x^A dx}{\left[\displaystyle\int_{\mathbb{R}^N} |u|^{\frac{N+b}{N+a-1}} x^B dx\right]^{\frac{N+a-1}{N+b}}}.$$

This concludes the proof of the lemma. $\square$

Theorem 0.4. *Let $N \geq 2$, and Let $A = (a_1, \ldots, a_N)$, $B = (b_1, \ldots, b_N)$ be two nonnegative vectors in $\mathbb{R}^N$. Let $a = a_1 + \cdots + a_N$, $b = b_1 + \cdots + b_N$, $\overline{a}_i = a - a_i$, and $\overline{b}_i = b - b_i$. Then, we have the following*
(I) if

$$C_{A,B,N} > 0,$$

then

$$0 \leq a_i - \frac{N+a-1}{N+b} b_i \leq \frac{N+a-1}{N+b}$$

or equivalently

$$0 \leq a_i - \frac{N+\overline{a}_i-1}{N+\overline{b}_i} b_i \quad \text{and} \quad \frac{a_i}{b_i+1} \leq \frac{N+\overline{a}_i-1}{N+\overline{b}_i-1}.$$

(II) if $a - b \leq 1$ and the condition (11) *holds, then*

$$C_{A,B,N} > 0.$$

Proof of the Theorem 0.4
The part (I) of the theorem follows from Lemmata 3.1 and 3.2.
To prove the part (II), firstly we consider that $a - b < 1$. Since the condition (11) holds, we then get

$$0 \leq a_i - \frac{N+a-1}{N+b} b_i \leq \frac{N+a-1}{N+b} < 1.$$

Thus it follows from Theorem B and Lemma 3.5 that

$$C_{A,B,N} > 0.$$

We now assume that $a - b = 1$. It follows from condition (11) that

$$0 \leq a_i - b_i \leq 1,$$

for every $i \in \{1, \ldots, N\}$.
If $a_i - b_i < 1$ for each $i \in \{1, \ldots, N\}$, then the theorem follows from Theorema A and Lemma 3.5. Otherwise, there exists $j \in \{1, \ldots, N\}$ such that $a_j - b_j = 1$ and $a_i = b_i$ for every $i \in \{1, \ldots, N\} \backslash \{j\}$, then the result relies on the proof of the Theorem 0.5 and Lemma 3.5.
$\square$

3.3 Sharp constant for $A = B + e_i$

Theorem 0.5. *Let $N \geq 2$, and let $A = (a_1, \ldots, a_N)$, $B = (b_1, \ldots, b_N)$ be two nonnegative vectors in $\mathbb{R}^N$. Let $a = a_1 + \cdots + a_N$, $a = b_1 + \cdots + b_N$, $\overline{a}_i = a - a_i$, and $\overline{b}_i = b - b_i$. If $a_j = b_j$ for all $j \in \{1, \ldots, N\} \backslash \{i\}$, and $a_i = b_i + 1$, then*

$$C_{A,B,N} = a_i.$$

Proof of the Theorem 0.5

The proof consists of showing that if $a_i = b_i + 1$, then $a_i = C_{A,B,N}$. To prove that

$$a_i \leq C_{A,B,N} \tag{3.27}$$

we will use the Lemma 3.5 and an idea contained in [20].

Given $v \in C_c^1(\mathbb{R})$, $v \geq 0$, we have, integrating by parts that

$$\begin{aligned}\int_{\mathbb{R}} |y|^{b_i} v(y)dy &= \frac{1}{b_i+1}\int_{\mathbb{R}} \left(|y|^{b_i}y\right)' v(y)dy \\ &= -\frac{1}{b_i+1}\int_{\mathbb{R}} |y|^{b_i} y v'(y)dy \\ &\leq \frac{1}{a_i}\int_{\mathbb{R}} |y|^{a_i}|v'(y)|dy. \end{aligned} \tag{3.28}$$

We now apply the inequality (3.28) to the function $v(y) = \overline{x}_i^{\overline{A}_i} u(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_N)$ with $u \geq 0$, thus we then have

$$\int_{\mathbb{R}} ||y|^{b_i}\overline{x}_i^{\overline{A}_i} u(x_1, \ldots x_{i-1}, y, x_{i+1}, \ldots, x_N)|dy \leq \frac{1}{a_i}\int_{\mathbb{R}} ||y|^{a_i}\overline{x}_i^{\overline{A}_i}\partial_y \left(u(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_N)\right)|dy.$$

Integrating with respect to the variables $x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_N$, we obtain that

$$a_i \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|x^A dx}{\displaystyle\int_{\mathbb{R}^N} |u(x)|x^B dx}. \tag{3.29}$$

Therefore, the inequality (3.27) follows from Lemma 3.5 and inequality (3.29).

To prove the reverse inequality, we will use the proof of Lemma 3.2. Indeed, by the proof of Lemma 3.2, we get

$$\begin{aligned}\frac{P_A(\Omega_\varepsilon)}{[m_B(\Omega_\varepsilon)]^{\frac{N+a-1}{N+b}}} &\leq \varepsilon^{a_i - \frac{N+a-1}{N+b}(b_i+1)} \frac{\frac{1}{(1+\varepsilon^2)^{\frac{N+a-2}{2}}}\displaystyle\int_{B^+_{N-1}(1)} \overline{x}_i^{\overline{A}_i}|\overline{x}_i|^{a_i}d\overline{x}_i}{\left[\frac{1}{(b_i+1)(1+\varepsilon^2)^{\frac{N+b}{2}}}\displaystyle\int_{B^+_{N-1}(1)} \overline{x}_i^{\overline{B}_i}|\overline{x}_i|^{b_i+1}d\overline{x}_i\right]^{\frac{N+a-1}{N+b}}} \\ &\quad + O(\varepsilon^{a_i+1-\frac{N+a-1}{N+b}(b_i+1)}) + O(\varepsilon^{a_i+2-\frac{N+a-1}{N+b}(b_i+1)}) \\ &= \frac{\frac{1}{(1+\varepsilon^2)^{\frac{N+a-2}{2}}}\displaystyle\int_{B^+_{N-1}(1)} \overline{x}_i^{\overline{A}_i}|\overline{x}_i|^{a_i}d\overline{x}_i}{\frac{1}{(b_i+1)(1+\varepsilon^2)^{\frac{N+b}{2}}}\displaystyle\int_{B^+_{N-1}(1)} \overline{x}_i^{\overline{B}_i}|\overline{x}_i|^{b_i+1}d\overline{x}_i} + O(\varepsilon) + O(\varepsilon^2) \\ &= (b_i+1)(1+\varepsilon^2)^{\frac{3}{2}} + O(\varepsilon) + O(\varepsilon^2),\end{aligned}$$

where Ω_ε is the same set as defined in Lemma 3.2. Therefore,

$$\lim_{\varepsilon \to 0} \frac{P_A(\Omega_\varepsilon)}{m_B(\Omega_\varepsilon)} = a_i,$$

which concludes the proof.

$\square$

Chapter 4

On the existence of minimizers for isoperimetric inequalities with monomial weights

The aim of this chapter is to study minimizers for the isoperimetric quotient defined at (3.2). Here each class of isoperimetric problems, we mean for each pair of vectors A and B, can involve different techniques, see for example the very elegant preprint [8]. It is worth underscoring that some ideas in this chapter can coincide with [8], because these works were done in the same direction and period.

4.1 Steiner symmetrization with weights

Here we recall the definition of the classical Steiner symmetrization. Considering A and B two nonnegative vectors in $\mathbb{R}^N$ with $a_i = b_i = 0$, we define the Steiner Symmetrization in regard to the plane $\{x_i = 0\}$ as follows. We may assume without loss of generality that $i = N$. Given a set $E \subset \mathbb{R}^N_A$ and $x \in \mathbb{R}^{N-1}$, we denote by E_x the corresponding one-dimensional section of E

$$E_x = \{y \in \mathbb{R}; (x, y) \in E\}.$$

The distribution function μ of E is defined by setting for all $x \in \mathbb{R}^{N-1}$

$$\mu(x) = \int_{E_x} d\mathcal{H}^1.$$

Moreover, denoting the essential projection of E by $\pi(E)^+ = \{x \in \mathbb{R}^{N-1}; \mu(x) > 0\}$, we set the classical Steiner symmetrization of E with respect to the hyperplane $\{x_N = 0\}$ being the set

$$E^s = \left\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}; x \in \pi(E)^+, -\frac{\mu(x)}{2} < y < \frac{\mu(x)}{2}\right\} \text{ whenever } a_N = b_N = 0.$$

Thus, by Fubini's theorem, we obtain that μ is a $\mathcal{H}^{N-1}$- measurable function in $\mathbb{R}^{N-1}$, E^s is a measurable set in $\mathbb{R}^{N-1}$ and

$$m_B(E^s) = \int_{E^s} x^{\overline{B}_N} d\mathcal{H}^N(x, y) = \int_E x^{\overline{B}_N} d\mathcal{H}^N(x, y) = m_B(E).$$

Borrowing ideas from [22], see also [10], the main aim now is to prove that

$$P_A(E^s) \le P_A(E),$$

and it requires some preliminary results that we will present here.

Given E a set of locally finite perimeter in $\mathbb{R}^N$, we briefly recall that $x \in \partial^\star E$, the reduced boundary of E, if $\|D\chi_E\|(B(x,r)) > 0$ for all $r > 0$,

$$\lim_{r\to 0} \frac{1}{\|D\chi_E\|(B(x,r))} \int_{B(x,r)} \nu^E d\|D\chi_E\| = \nu^E(x),$$

and $|\nu^E(x)| = 1$. The vector $\nu^E(x) = \left(\nu_1^E(x), \ldots, \nu_N^E(x)\right)$ is called the generalized inner normal to E at x. The reduced boundary $\partial^\star E$ of E is also the set of all points $x \in \mathbb{R}^N$ such that the vector $\nu^E(x)$ is the derivative of the measure $D\chi_E$ with respect to $\|D\chi_E\|$,

$$\nu^E(x) = \lim_{r\to 0} \frac{D\chi_E(B(x,r))}{\|D\chi_E\|(B(x,r))}$$

at every $x \in \mathbb{R}^N$ such that the indicated limit exists, and $|\nu^E(x)| = 1$.

Theorem C (De Giorgi)**.** *Assume E has locally finite perimeter in $\mathbb{R}^N$.*

(i) Then

$$\partial^\star E = \left(\bigcup_{j=1}^{\infty} K_j\right) \cup N,$$

where $\|D\chi_E\|(N) = 0$ and K_j is a compact subset of C^1-hypersurface S_j for every $j \in \mathbb{N}$.

(ii) Furthermore, $\nu^E(x)|_{S_j}$ is normal to S_j for all $j \in \mathbb{N}$, and

(iii) $\|D\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^\star E$.

The next theorem due to Vol'pert (see [46]) gives us some important information about the reduced boundary of E, the corresponding one-dimensional section of $\partial^\star E$, and the normal vector $\nu^E(x)$ at a point $x \in \partial^\star E$.

Theorem D. *Let E be a set of finite perimeter in $\mathbb{R}^N$. Then, for $\mathcal{H}^{N-1}$- a.e $x \in \mathbb{R}^{N-1}$,*

(i) E_x has finite perimeter in $\mathbb{R}$;

(ii) $\partial^\star E_x = (\partial^\star E)_x$;

(iii) $\nu_N^E(x,y) \neq 0$ for all y such that $(x,y) \in \partial^\star E$;

(iv) for $\mathcal{H}^1$-a.e. $y \in \partial^\star E_x$

$$\begin{cases} \lim\limits_{z\to y^+} \chi_E(x,z) = 1, \lim\limits_{z\to y^-} \chi_E(x,z) = 0 & \text{if } \nu_N^E(x,y) > 0, \\ \lim\limits_{z\to y^+} \chi_E(x,z) = 0, \lim\limits_{z\to y^-} \chi_E(x,z) = 1 & \text{if } \nu_N^E(x,y) < 0. \end{cases}$$

We then get a Borel set $G_E \subset \pi(E)^+$ such that the conclusions $(i) - (iv)$ of Theorem D yield for all $x \in G_E$, and $\mathcal{H}^{N-1}\left(\pi(E)^+ \backslash G_E\right) = 0$.

Lemma 4.1. *Let A and B be two nonnegative vectors in $\mathbb{R}^N$ with $a_N = b_N = 0$, let $E \subset \mathbb{R}^N_A$, and let E be a set with finite perimeter $P_A(E)$ and finite volume $m_B(E)$. Then $\mu \in BV\left(\mathbb{R}^{N-1}, x^{\overline{A}_N}, x^{\overline{B}_N}\right)$, and for any C^1 function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$, satisfying $\varphi \equiv 0$ on $\partial\mathbb{R}^{N-1}_{\overline{A}_N}$, we have*

$$\int_{\mathbb{R}^{N-1}} \varphi(x) dD_i^{\overline{A}_N}\mu = \int_{\partial^* E} x^{\overline{A}_N} \nu_i^E(x,y)\varphi(x) d\mathcal{H}^{N-1}(x,y). \tag{4.1}$$

Furthemore, for any Borel set $F \subset \mathbb{R}^{N-1}$,

$$\left\| D^{\overline{A}_N}\mu \right\| (F) \leq P_A\left(E, F \times \mathbb{R}\right). \tag{4.2}$$

Proof. Let φ be a C^1 function $\varphi : \mathbb{R}^N \to \mathbb{R}$ with compact support, satisfying $\varphi = 0$ on $\partial\mathbb{R}^{N-1}_{\overline{A}_N}$, and let $(\psi_j)_{j\in\mathbb{N}}$ be a sequence in $C_c^1(\mathbb{R})$, satisfying $0 \leq \psi_j \leq 1$ for each $j \in \mathbb{N}$, and such that

$$\lim_{j\to\infty} \psi_j(y) = 1$$

for every $y \in \mathbb{R}$.

It follows from definition of μ and the generalized Gauss-Green theorem that

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial x_i}\left(x^{\overline{A}_N}\varphi(x)\right)\mu(x)dx &= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \frac{\partial}{\partial x_i}\left(x^{\overline{A}_N}\varphi(x)\right)\chi_E(x,y)d\mathcal{H}^1(y)d\mathcal{H}^{N-1}(x) \\
&= \lim_{j\to\infty}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \frac{\partial}{\partial x_i}\left(x^{\overline{A}_N}\varphi(x)\right)\psi_j(y)\chi_E(x,y)d\mathcal{H}^1(y)d\mathcal{H}^{N-1}(x) \\
&= -\int_{\partial^* E} x^{\overline{A}_N}\nu_i^E(x,y)\varphi(x)d\mathcal{H}^{N-1}(x,y),
\end{aligned} \tag{4.3}$$

where ν^E denotes the inner normal to the boundary of E.

It follows from $m_B(\Omega) < +\infty$ and (4.3) that the distributional derivative with weight $D_i^{\overline{A}_N}\mu$, $i = 1, \ldots, N$, is a real measure with bounded variation and $\mu \in BV\left(\mathbb{R}^{N-1}, x^{\overline{A}_N}, x^{\overline{B}_N}\right)$. Besides that, the equality (4.3) proves (4.1) and, consequently, (4.2). □

Lemma 4.2. *Let A and B be two nonnegative vectors in $\mathbb{R}^N$ with $a_N = b_N = 0$, let $E \subset \mathbb{R}^N_A$, and let E be a set with finite perimeter $P_A(E)$ and finite volume $m_B(E)$. Then*

$$\frac{\partial^{\overline{A}_N}\mu}{\partial x_i}(x) = \int_{(\partial^* E)_x} \frac{x^{\overline{A}_N}\nu_i^E(x,y)}{|\nu_N^E(x,y)|} d\mathcal{H}^0(y),$$

for each $i = 1, \ldots, N-1$, and for $\mathcal{H}^{N-1}$- a.e. $x \in \pi_N(E)^+$. Here, $\frac{\partial^{\overline{A}_N}\mu}{\partial x_i}$ denotes the absolutely continuous part of $D_i^{\overline{A}_N}\mu$ with respect to $\mathcal{H}^{N-1}$.

Proof. Let G_E be the Borel set by Theorem D. Assume that g is any function in $C_c\left(\mathbb{R}^{N-1}\right)$, satisfying $g \equiv 0$ on $\partial\mathbb{R}^{N-1}_{\overline{A}_N}$, then it follows from Lemma 4.1, Theorem D, and coarea formula that

$$\begin{aligned}\int_{G_E} g(x) dD_i^{\overline{A}_N}\mu &= \int_{\mathbb{R}^{N-1}} g(x)\chi_{G_E}(x) dD_i^{\overline{A}_N}\mu \\ &= \int_{\partial^* E} x^{\overline{A}_N} g(x)\chi_{G_E}(x) d\mathcal{H}^{N-1}(x,y) \\ &= \int_{\partial^* E} g(x)\chi_{G_E}(x) x^{\overline{A}_N} \frac{\nu_i^E(x,y)}{|\nu_N^E(x,y)|}|\nu_N^E(x,y)| d\mathcal{H}^{N-1}(x) \\ &= \int_{G_E} g(x)\int_{(\partial^* E)_x} x^{\overline{A}_N}\frac{\nu_i^E(x,y)}{|\nu_N^E(x,y)|} d\mathcal{H}^0(y) d\mathcal{H}^{N-1}(x).\end{aligned}$$

Therefore, the proof of lemma follows. □

Lemma 4.3. *Let A be a nonnegative vector in $\mathbb{R}^N$ with $a_N = 0$, and let E be a bounded set of finite perimeter $P_A(E)$ in $\mathbb{R}^N_A$. Then*

$$P_A(E^s; D\times\mathbb{R}) \leq P_A(E; D\times\mathbb{R}) + \|D_N^A\chi_{E^s}\|(D\times\mathbb{R}),$$

for every Borel set $D\subset\mathbb{R}^{N-1}$.

Proof. Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a sequence of nonnegative functions from $C_c^1\left(\mathbb{R}^{N-1}\right)$ such that $\mu_j\to\mu$ $\mathcal{H}^{N-1}$ -*a.e* in $\mathbb{R}^{N-1}$ and $|D\mu_j| \rightharpoonup |D\mu|$ in the sense of measures, see Theorem A.32, and let $E_j^s := \left\{(x,y)\in\mathbb{R}^{N-1}\times\mathbb{R}; \mu_j(x)>0 \text{ and } |y|<\frac{\mu_j(x)}{2}\right\}$. Given an open set $\Omega\subset\mathbb{R}^{N-1}$, and a function $\varphi=(\varphi_1,\dots,\varphi_N)\in C_c^1(\Omega\times\mathbb{R},\mathbb{R}^N)$, satisfying $\varphi=0$ on $\partial\mathbb{R}^N_A$, and $\|\varphi\|\leq 1$. It follows from definition of E_j^s and basic properties that

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}}\chi_{E_j^s} div\left(x^{\overline{A}_N}\varphi(x,y)\right) d\mathcal{H}^N(x,y) = \int_\Omega\int_{\frac{\mu_j(x)}{2}}^{\frac{\mu_j(x)}{2}}\sum_{i=1}^{N-1}\overline{x}_i^{\overline{A}_{iN}}\frac{\partial}{\partial x_i}\left(x_i^{a_i}\varphi_i(x,y)\right) dydx \\ &+\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) d\mathcal{H}^N(x,y) \\ &= \int_\Omega\int_{-1}^1\sum_{i=1}^{N-1}\overline{x}_i^{\overline{A}_{iN}}\left[\frac{\partial}{\partial x_i}\left(x_i^{a_i}\varphi_i\left(x,\frac{\mu_j(x)}{2}y\right)\frac{\mu_j(x)}{2}\right)-\frac{1}{2}x_i^{a_i}\frac{\partial\mu_j}{\partial x_i}(x)\frac{\partial}{\partial y}\left(y\varphi_i\left(x,\frac{\mu_j(x)}{2}y\right)\right)\right] dydx \\ &+\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) dydx \\ &= -\frac{1}{2}\int_\Omega\int_{-1}^1\sum_{i=1}^{N-1}x^{\overline{A}_N}\frac{\partial\mu_j}{\partial x_i}(x)\frac{\partial}{\partial y}\left(y\varphi_i\left(x,\frac{\mu_j(x)}{2}y\right)\right) dydx + \int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) dydx \\ &\leq \int_\Omega x^{\overline{A}_N}\sqrt{\sum_{i=1}^{N-1}\frac{1}{4}\left[\varphi_i\left(x,\frac{\mu_j(x)}{2}\right)+\varphi_i\left(x,-\frac{\mu_j(x)}{2}\right)\right]^2}|\nabla\mu_j(x)| dydx \\ &+\int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) dydx \\ &\leq \int_\Omega x^{\overline{A}_N}|\nabla\mu_j(x)| dx + \int_{\Omega\times\mathbb{R}}\chi_{E_j^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) dydx. \qquad (4.4)\end{aligned}$$

The last step is a consequence of $\|\varphi\| \leq 1$.

Combining estimates (4.2) and (4.4), and taking the limit as $j \to +\infty$, yields

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \chi_{E^s} div\left(x^{\overline{A}_N}\varphi(x,y)\right) d\mathcal{H}^N(x,y) &\leq \left\|D^{\overline{A}_N}\mu\right\|(\Omega) + \int_{\Omega\times\mathbb{R}} \chi_{E^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) dydx \\ &\leq P_A(E;\Omega\times\mathbb{R}) + \int_{\Omega\times\mathbb{R}} \chi_{E^s}\frac{\partial}{\partial y}\left(x^{\overline{A}_N}\varphi_N(x,y)\right) dydx.\end{aligned}$$

Taking the supremum over all functions $\varphi \in C_c^1\left(\Omega\times\mathbb{R},\mathbb{R}^N\right)$, satisfying $\varphi \equiv 0$ on $\partial\mathbb{R}_A^N$, and $\|\varphi\| \leq 1$, we get

$$P_A(E^s;\Omega\times\mathbb{R}) \leq P_A(E;\Omega\times\mathbb{R}) + \|D_N^A\chi_{E^s}\|(\Omega\times\mathbb{R}).$$

This implies that

$$P_A(E^s;D\times\mathbb{R}) \leq P_A(E;D\times\mathbb{R}) + \|D_N^A\chi_{E^s}\|(D\times\mathbb{R}),$$

for every Borel set $D \subset \mathbb{R}^{N-1}$.

Therefore, the proof of lemma follows. □

Theorem 4.4. *Let D be a Borel set in $\mathbb{R}^{N-1}$, and let $E \subset \mathbb{R}_A^N$ be a bounded set of finite perimeter $P_A(E)$. Then*

$$P_A(E^s, D\times\mathbb{R}) \leq P_A(E, D\times\mathbb{R}).$$

Proof. By Lemma 4.2, we have

$$\frac{\partial^{\overline{A}_N}\mu}{\partial x_i}(x) = \int_{(\partial^*E)_x} \frac{x^{\overline{A}_N}\nu_i^E(x,y)}{|\nu_N^E(x,y)|}d\mathcal{H}^0(y) = 2\frac{x^{\overline{A}_N}\nu_i^E\left(x,\frac{1}{2}\mu(x)\right)}{|\nu_N^E\left(x,\frac{1}{2}\mu(x)\right)|} \text{ for } \mathcal{H}^{N-1}-a.e\ x\in\pi(E)^+. \tag{4.5}$$

We first assume that $D \subset G_E \cap G_{E^s}$. It follows from Theorem C, coarea formula, and generalized Gauss-Green theorem that

$$\begin{aligned}P_A(E^s;D\times\mathbb{R}) &= \int_{\partial^*E^s\cap(D\times\mathbb{R})} x^{\overline{A}_N}d\mathcal{H}^{N-1}(x,y) \\ &= \int_{\partial^*E^s\cap(D\times\mathbb{R})} \frac{x^{\overline{A}_N}}{|\nu_N^{E^s}(x,y)|}|\nu_N^{E^s}(x,y)|d\mathcal{H}^{N-1}(x,y) \\ &= \int_D\int_{(\partial^*E^s)_x} \frac{x^{\overline{A}_N}}{|\nu_N^{E^s}(x,y)|}d\mathcal{H}^0(y)d\mathcal{H}^{N-1}(x) \\ &= 2\int_D \frac{x^{\overline{A}_N}}{|\nu_N^{E^s}(x,\frac{1}{2}\mu(x))|}d\mathcal{H}^{N-1}(x).\end{aligned} \tag{4.6}$$

By equalities (4.5) and (4.6), we get

$$
\begin{aligned}
P_A(E^s; D\times\mathbb{R}) &= \int_D \sqrt{4x^{2\overline{A}_N} + \sum_{j=1}^{N-1}\left(\int_{(\partial E^s)_x} \frac{x^{\overline{A}_N}\nu_j^{E^s}(x,y)}{\nu_N^{E^s}(x,y)} d\mathcal{H}^0(y)\right)^2}\, d\mathcal{H}^{N-1}(x) \\
&= \int_D \sqrt{4x^{2\overline{A}_N} + \sum_{j=1}^{N-1}\left(\frac{\partial^{\overline{A}_N}\mu}{\partial x_j}(x)\right)^2}\, d\mathcal{H}^{N-1}(x) \\
&= \int_D \sqrt{4x^{2\overline{A}_N} + \sum_{j=1}^{N-1}\left(\int_{(\partial^* E)_x} x^{\overline{A}_N} \frac{\nu_j^E(x,y)}{|\nu_N^E(x,y)|} d\mathcal{H}^0(y)\right)^2}\, d\mathcal{H}^{N-1}(x) \\
&\le \int_D \sqrt{\left(\int_{(\partial^* E)_x} x^{\overline{A}_N} d\mathcal{H}^0(y)\right)^2 + \sum_{j=1}^{N-1}\left(\int_{(\partial^* E)_x} x^{\overline{A}_N} \frac{\nu_j^E(x,y)}{|\nu_N^E(x,y)|} d\mathcal{H}^0(y)\right)^2}\, d\mathcal{H}^{N-1}(x).
\end{aligned}
$$

By discrete Minkowski inequality and by inequality above, we get

$$
\begin{aligned}
P_A(E^s; D\times\mathbb{R}) &\le \int_D \int_{(\partial^\star E)_x} \sqrt{x^{2\overline{A}_N} + \sum_{j=1}^{N-1} x^{2\overline{A}_N}\left(\frac{\nu_j^E(x,y)}{|\nu_N^E(x,y)|}\right)^2}\, d\mathcal{H}^{N-1}(x) \\
&= \int_D \int_{(\partial^\star E)_x} \frac{x^{\overline{A}_N}}{|\nu_N^E(x,y)|} d\mathcal{H}^{N-1}(x,y) \\
&= \int_{\partial^\star E\cap(D\times\mathbb{R})} x^{\overline{A}_N} d\mathcal{H}^{N-1}(x,y) \\
&= P_A(E; D\times\mathbb{R}).
\end{aligned}
$$

We now consider $D \subset \mathbb{R}^N \backslash G_{E^s}\cap G_E$. Since $\mathcal{H}^{N-1}(\pi(E)^+\cap D) = 0$, we get from Theorem C, coarea formula that

$$
\begin{aligned}
|D_N^A \chi_{E^s}|(D\times\mathbb{R}) &= \int_{\partial^\star E^s\cap(D\times\mathbb{R})} x^{\overline{A}_N} |\nu_N^E(x,y)| d\mathcal{H}^{N-1}(x,y) \\
&= \int_D \int_{(\partial^\star E^s)_x} x^{\overline{A}_N} d\mathcal{H}^0(y) d\mathcal{H}^{N-1}(x,y) \\
&= 0.
\end{aligned}
$$

Hence, from the last equality and the Lemma 4.3, we get

$$
P_A(E^s, D\times\mathbb{R}) \le P_A(E, D\times\mathbb{R}).
$$

Therefore, the proof of theorem is completed. □

4.2 Case $N = 2, A = (0, a_2)$ and $B = (0, 0)$

Throughout this section, we consider $B = (0, 0)$ and $A = (0, a_2)$ with $0 < a_2 \le 1$.

The goal here is to figure out the minimizers for

$$\frac{P_A(U)}{[m_B(U)]^{\frac{a_2+1}{2}}},$$

and the best constant $C_{A,B,2}$ defined at (10), or at least to try to understand the behavior of minimizers.

Remark 4.5. *Among the possible bounded, open and Lipschitz sets $\Omega \subset \mathbb{R}^2_A$, minimizers of the isoperimetric quotient*

$$\frac{P_A(U)}{[m_B(U)]^{\frac{a_2+1}{2}}},$$

may be assumed, by translation argument, that

$$d\left(\Omega, \partial\mathbb{R}^2_A\right) = \inf\left\{\|v-w\|; v \in \Omega \ \text{ and } \ w \in \partial\mathbb{R}^2_A\right\} = 0.$$

Besides that, if there exists $(x_0, 0)$ and $(x_1, 0)$ on the boundary $\partial\Omega$ of Ω, with $x_0 < x_1$, then $\{(x,0); x_0 \le x \le x_1\} \subset \partial\Omega$. Indeed, the set Ω_1 defined as

$$\Omega_1 := \left\{(x,y) \in \mathbb{R}^2; (x,y) \in \Omega \ \text{ or } \ x \in [x_0, x_1] \ \text{ and there exists } \ z > 0, \text{ satisfying } \ (x,z) \in \Omega, \right.$$
$$\left. \text{and } \ 0 < y \le z\right\}$$

is a bounded, open and Lipschitz set with

$$\frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{a_2+1}{2}}} \ge \frac{P_A(\Omega_1)}{[m_B(\Omega_1)]^{\frac{a_2+1}{2}}}.$$

Moreover, by Theorem 4.4, we may assume that the set $\Omega \subset \mathbb{R}^2_A$ is symmetric in relation to the y-axis. However, the boundary $\partial\Omega$ of Ω may have parallel lines segments to the x-axis. Thus there exists a partition $P = \{0 < y_0 < y_1 < \cdots < y_{m-1} < y_m\}$ of the interval $[0, y_m]$ and lipschitz nonnegative functions $\varphi_j : [y_{j-1}, y_j] \to \mathbb{R}$, $j = 0, \ldots, m$, such that

$$\Omega = \bigcup_{j=0}^{m} \Omega_j,$$

where

$$\Omega_j = \left\{(x,y) \in \mathbb{R}^2_A; y_{j-1} < y < y_j \ \text{ and } \ -\varphi_j(y) < x < \varphi_j(y) \ \text{ or } \ y = y_j \ \text{ and } \ -M_j < x < M_j\right\},$$

$y_{-1} := 0$, $M_j = \min\{\varphi_j(y_j), \varphi_{j+1}(y_j)\}$, for each $j \in \{0, \ldots, m-1\}$, and

$$\Omega_m := \left\{(x,y) \in \mathbb{R}^2_A; y_{m-1} < y < y_m \ \text{ and } \ -\varphi_m(y) < x < \varphi_m(y)\right\}.$$

The next lemma gives a great information about the functions above.

Lemma 4.6. *Let $A = (0, a_2)$ be a vector in $\mathbb{R}^2$ with $a_2 > 0$, let $B = (0,0)$, let $\Omega \subset \mathbb{R}^2_A$ be a set of form*

$$\Omega = \bigcup_{j=0}^{m} \Omega_j,$$

where

$$\Omega_j = \left\{(x,y) \in \mathbb{R}^2_A; y_{j-1} < y < y_j \text{ and } -\varphi_j(y) < x < \varphi_j(y) \text{ or } y = y_j \text{ and } -M_j < x < M_j\right\},$$

$y_{-1} := 0$, $M_j = \min\{\varphi_j(y_j), \varphi_{j+1}(y_j)\}$, for each $j \in \{0, \ldots, m-1\}$, and

$$\Omega_m := \left\{(x,y) \in \mathbb{R}^2_A; y_{m-1} < y < y_m \text{ and } -\varphi_m(y) < x < \varphi_m(y)\right\}.$$

Assume that Ω minimizes the isoperimetric quotient

$$\frac{P_A(U)}{[m_B(U)]^{\frac{a_2+1}{2}}}.$$

Then, each φ_j is a nonincreasing function with $\varphi_{j+1}(y_j) \leq \varphi_j(y_j)$.

Proof. To establish the inequality $\varphi_1(y_1) \leq \varphi_0(y_1)$, one can argue as follows.

Suppose that $\overline{z}_1 := \sup\{\varphi_1(y); y \in [y_1, y_2]\} \geq \overline{z}_0 := \varphi_0(y_1)$. Setting $\overline{y}_{1,0} := \sup\{y \in [y_1, y_2]; \varphi_1(y) = \overline{z}_1\}$ and $\overline{y}_{0,0} := \inf\{y \in [y_0, y_1]; \varphi_0(y) \leq \overline{z}_1\}$. We may thus replace the partition $P := \{0 = y_0 < y_1 < y_2 < y_3 < \ldots < y_m\}$ by $P' := \{0 = y_0 < y_2 < y_3 < \ldots < y_m\}$ and the function φ_0 by $\psi_0 : [y_0, y_2] \to \mathbb{R}$ defined by

$$\psi_0(y) := \begin{cases} \varphi_0(y) \text{ if } y \in [y_0, \overline{y}_{0,0}) \\ \overline{z}_1 \text{ if } y \in [\overline{y}_{0,0}, \overline{y}_{1,0}] \\ \varphi_1(y) \text{ if } y \in [\overline{y}_{1,0}, y_2], \end{cases}$$

so that

$$\frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{a_2+1}{2}}} \geq \frac{P_A(\Omega')}{[m_B(\Omega')]^{\frac{a_2+1}{2}}},$$

where

$$\Omega' = \Omega'_0 \cup \bigcup_{j=2}^{m-1} \Omega_j,$$

$\Omega'_0 = \{(x,y) \in \mathbb{R}^2_A; y_0 < y < y_2 \text{ and } -\psi_0(y) < x < \psi_0(y) \text{ or } y = y_2 \text{ and } -\overline{m}_0 < x < \overline{m}_0\}$, and $\overline{m}_0 = \min\{\psi_0(y_2), \varphi_2(y_2)\}$.

Therefore, $\varphi_1(y_1) \leq \varphi_0(y_1)$, and the other cases follow directly from the same argument.

We now prove only that φ_0 is a non-increasing function, the others are similar. For this purpose, denote $z_0 := \sup\{\varphi_0(y); y \in [y_0, y_1]\}$, and $y_{0,0} = \sup\{z \in [y_0, y_1]; \varphi_0(z) = z_0\}$. Next, let $t_1, t_2 \in [y_{0,0}, y_1]$, $t_1 < t_2$, satisfying $\varphi_0(t) \le \varphi_0(t_1) = \varphi_0(t_2)$ for all $t \in [t_1, t_2]$. Thus, we may replace the function φ_0 by $\varphi_{0,0} : [y_0, y_1] \to \mathbb{R}$ defined by

$$\varphi_{0,0}(y) := \begin{cases} z_0 = \varphi_0(y_{0,0}) & \text{if } y \in [y_0, y_{0,0}] \\ \varphi_0(y) & \text{if } y \in [y_{0,0}, t_1] \\ \varphi_0(t_1) & \text{if } y \in [t_1, t_2] \\ \varphi_0(y) & \text{if } y \in [t_2, y_1], \end{cases}$$

so that

$$\frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{a_2+1}{2}}} \ge \frac{P_A(\Omega')}{[m_B(\Omega')]^{\frac{a_2+1}{2}}},$$

where

$$\Omega' = \Omega'_0 \cup \bigcup_{j=1}^{m-1} \Omega_j,$$

$\Omega'_0 = \{(x, y) \in \mathbb{R}^2_A; y_0 < y < y_1 \text{ and } -\varphi_{0,0}(y) < x < \varphi_{0,0}(y) \text{ or } y = y_1 \text{ and } -m_0 < x < m_0\}$, and $m_0 = \min\{\varphi_{0,0}(y_1), \varphi_1(y_1)\}$.

Since we can replace φ_0 each time there are $t_1, t_2 \in [y_0, y_1]$, $t_1 < t_2$, satisfying $\varphi_0(t) \le \varphi_0(t_1) = \varphi_0(t_2)$ for every $t \in [t_1, t_2]$ (as we showed before), then φ_0 is non-increasing.

Therefore, the proof is completed. □

Lemma 4.7. *Assume the assumptions of lemma 4.6. If Ω minimizes the isoperimetric quotient*

$$\frac{P_A(U)}{[m_B(U)]^{\frac{a_2+1}{2}}},$$

then the functions φ_i are strictly decreasing.

Proof. We consider only $i \in \{1, \ldots, m-1\}$, and the cases $i \in \{0, m\}$ can proceed similarly. Let ψ_i be a smooth nonnegative function with compact support in (y_{i-1}, y_i). It follows that the function $F : [0, +\infty) \to \mathbb{R}$ defined by

$$2^{\frac{a_2+1}{2}-1}\left(\sum_{j=0, j\neq i}^{m} \int_{y_{j-1}}^{y_j} \varphi_j(y)dy + \int_{y_{i-1}}^{y_i} (\varphi_i(y) + t\psi_i(y))\, dy\right)^{\frac{a_2+1}{2}} F(t) = \int_0^{y_1} y^{a_2}\left(1 + (\varphi_0'(y))^2\right)^{\frac{1}{2}} dy$$
$$+ y_0^{a_2}\varphi_0(y_0) + \int_{y_{m-1}}^{y_m} y^{a_2}\left(1 + (\varphi_m'(y))^2\right)^{\frac{1}{2}} dy + y_m^{a_2}\varphi_m(y_m) + \int_{y_{i-1}}^{y_i} y^{a_2}\left(1 + (\varphi_i'(y) + t\psi_i'(y))^2\right)^{\frac{1}{2}} dy$$
$$+ y_i^{a_2}(\varphi_i(y_i) - \varphi_{i+1}(y_i)) + \sum_{j=1, j\neq i}^{m-1}\left[\int_{y_{j-1}}^{y_j} y^{a_2}\left(1 + (\varphi_j'(y))^2\right)^{\frac{1}{2}} dy + y_j^{a_2}(\varphi_j(y_j) - \varphi_{j+1}(y_j))\right]$$

has a minimum point at $t = 0$. Hence,

$$2^{\frac{a_2-1}{2}} \frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{a_2+1}{2}}} \frac{a_2+1}{2} [m_B(\Omega)]^{\frac{a_2-1}{2}} \int_{y_{i-1}}^{y_i} \psi_i(y)dy = \int_{y_{i-1}}^{y_i} y^{a_2}\left(1 + (\varphi_i'(y))^2\right)^{-\frac{1}{2}} \varphi_i'(y)\psi_i'(y)dy. \tag{4.7}$$

We now claim that there is no interval $(y'_{i-1}, y'_i) \subset [y_{i-1}, y_i]$ such that $\varphi_i \mid_{(y'_{i-1}, y'_i)}$ is constant. Assume by contradiction that this assertion is not true. Then, there is an interval $(y'_{i-1}, y'_i) \subset [y_{i-1}, y_i]$ such that $\varphi'_i(y) = 0$, for every $y \in (y'_{i-1}, y'_i)$. Taking $\psi_i \in C^1_c(y'_{i-1}, y'_i)$, $\psi_i \neq 0$, a nonnegative function, we then obtain

$$\int_{y_{i-1}}^{y_i} y^{a_2} \left(1 + (\varphi'_i(y))^2\right)^{-\frac{1}{2}} \varphi'_i(y)\psi'_i(y)dy = 0,$$

and

$$\int_{y_{i-1}}^{y_i} \psi_i(y)dy > 0.$$

Which leads to a contradiction to (4.7).

Therefore, it follows from Lemma 4.6 and statement above that φ_i is strictly decreasing. $\square$

We can now apply Lemma 4.7 to ensure that

$$\Omega = \left\{(x, y) \in \mathbb{R}^2_A; x \in (-x_0, x_0) \text{ and } 0 < y < \varphi(x)\right\},$$

where $x_0 > 0$, $\varphi : [-x_0, x_0] \to \mathbb{R}$ is an even, nonnegative, and Lipschitz function with $\varphi(x_0) = 0$. Furthermore, φ is non-increasing on the interval $[0, x_0]$.

The next goal is to show that φ is a strictly decreasing function on the interval $[0, x_0]$.

Theorem 0.6. *Let $N = 2$, $A = (0, 1)$, and $B = (0, 0)$. Then, there is no bounded, open and Lipschitz set $\Omega \subset \mathbb{R}^2$ such that*

$$C_{A,B,2} = \frac{P_A(\Omega)}{m_B(\Omega)}.$$

Proof of the Theorem 0.6

Arguing by contradiction, suppose that there is a bounded, open and Lipschitz set $\Omega \subset \mathbb{R}^2$ such that

$$C_{A,B,2} = \frac{P_A(\Omega)}{m_B(\Omega)}.$$

Combining Remark 4.5, and Lemmata 4.6 and 4.7, we then have

$$\Omega = \left\{(x, y) \in \mathbb{R}^2_A; x \in (-x_0, x_0) \text{ and } 0 < y < \varphi(x)\right\},$$

where $x_0 > 0$, $\varphi : [-x_0, x_0] \to \mathbb{R}$ is an even, nonnegative, and Lipschitz function with $\varphi(x_0) = 0$.

It follows from Theorem 0.5 that

$$\frac{\displaystyle\int_{-x_0}^{x_0} \varphi(x)\sqrt{1 + (\varphi'(x))^2}dx}{\displaystyle\int_{-x_0}^{x_0} \varphi(x)dx} = 1.$$

Which leads to a contradiction.

Therefore, the proof of theorem follows. $\square$

Proposition 4.8. *Assume $0 < a_2 < 1$ and $x_0 > 0$. Let $\varphi : [-x_0, x_0] \to \mathbb{R}$ be an even, nonnegative and Lipschitz function. Further, assume that φ is nonincreasing on $[0, x_0]$ with $\varphi(x_0) = 0$. Let $\Omega \subset \mathbb{R}^2_A$ be the set*

$$\Omega = \left\{(x, y) \in \mathbb{R}^2_A; x \in (-x_0, x_0) \ \ and \ \ 0 < y < \varphi(x)\right\}.$$

Assume that Ω minimizes the isoperimetric quotient

$$\frac{P_A(U)}{[m_B(U)]^{\frac{a_2+1}{2}}}$$

then φ is strictly decreasing on the interval $[0, x_0]$.

Proof. Let us begin with a smooth nonnegative function $\psi : [-x_0, x_0] \to \mathbb{R}$ with compact support in $(-x_0, x_0)$, and $G : [0, +\infty] \to \mathbb{R}$ the function defined by

$$\left[\int_{-x_0}^{x_0} (\varphi(x) + t\psi(x))\, dx\right]^{\frac{a_2+1}{2}} G(t) = \int_{-x_0}^{x_0} (\varphi(x) + t\psi(x))^{a_2} \sqrt{1 + (\varphi'(x) + t\psi'(x))^2} dx.$$

Since $t = 0$ is a minimum point of G, it follows that

$$\begin{aligned}\frac{a_2+1}{2}\left[\int_{-x_0}^{x_0} \varphi(x)dx\right]^{\frac{a_2-1}{2}} \frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{a_2+1}{2}}} \int_{-x_0}^{x_0} \psi(x)dx &= \int_{-x_0}^{x_0} a_2 (\varphi(x))^{a_2-1} \psi(x)\sqrt{1 + \varphi'(x)^2}dx \\ &+ \int_{-x_0}^{x_0} \varphi(x)^{a_2} \left(1 + (\varphi'(x))^2\right)^{-\frac{1}{2}} \varphi'(x)\psi'(x)dx.\end{aligned} \tag{4.8}$$

Arguing by contradiction, suppose that $\varphi =$ *constant* on some interval (x_1, x_2) contained in $[0, x_0]$. Taking ψ with compact support in (x_1, x_2), $\psi \neq 0$, we obtain

$$\varphi(x_1)^{a_2-1} = \frac{a_2+1}{2a_2}\frac{P_A(\Omega)}{m_B(\Omega)}. \tag{4.9}$$

Thus, by (4.9), there are not two intervals (x_1, x_2), (x'_1, x'_2) contained in $[0, x_0]$, such that φ is constant on them and $\varphi(x_1) \neq \varphi(x'_1)$.

We have two cases:

(i) φ is constant on (x_1, x_2) with $\varphi(x_1) < \varphi(0)$,

(ii) φ is constant on (x_1, x_2) with $\varphi(x_1) = \varphi(0)$.

Firstly, suppose that the case (i) happens. It follows that, there exists a partition $P := \{0 < \varphi(x_1) < \varphi(0)\}$ of the interval $[0, \varphi(0)]$, and Lipschitz nonnegative functions $\varphi_1 : [0, \varphi(x_1)] \to \mathbb{R}$, $\varphi_2 : [\varphi(x_1), \varphi(0)] \to \mathbb{R}$ such that $\varphi_2(\varphi(x_1)) \leq \varphi_1(\varphi(x_1))$ and

$$\Omega = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 = \left\{(x, y) \in \mathbb{R}^2_A; 0 < y < \varphi(x_1) \text{ and } -\varphi_1(y) < x < \varphi_1(y) \text{ or } y = \varphi(x_1) \text{ and } -M_1 < x < M_1\right\},$$

$M_1 = \varphi_2(\varphi(x_1))$, and

$$\Omega_2 = \left\{(x,y) \in \mathbb{R}^2_A; \varphi(x_1) < y < \varphi(0) \text{ and } -\varphi_2(y) < x < \varphi_2(y)\right\}.$$

We once again define $H : [0,+\infty] \to \mathbb{R}$ by

$$2^{\frac{a_2-1}{2}}\left[\int_0^{\varphi(x_1)} (\varphi_1(y) + th(y))\,dy + \int_{\varphi(x_1)}^{\varphi(0)} \varphi_2(y)\right]^{\frac{a_2+1}{2}} H(t) = \int_0^{\varphi(x_1)} y^{a_2}\sqrt{1+(\varphi_1'(y)+th'(y))^2}dx$$
$$+\left[\varphi_1(\varphi(x_1)) + th(\varphi(x_1)) - \varphi_2(\varphi(x_1))\right]\varphi(x_1)^{a_2} + \int_0^{\varphi(x_1)} y^{a_2}\sqrt{1+(\varphi_2'(y))^2}dy,$$

where $h \in C^1[0,\varphi(x_1)]$ is a nonnegative function with $h \neq 0$.

Noting that H is a variation of the isoperimetric quotient with a minimum point at $t=0$, we then get

$$2^{\frac{a_2-1}{2}} \cdot \frac{a_2+1}{2}\left[\int_0^{\varphi(x_1)} \varphi_1(y)dy + \int_{\varphi(x_1)}^{\varphi(0)} \varphi_2(y)dy\right]^{\frac{a_2-1}{2}} \frac{P_A(\Omega)}{[m_B(\Omega)]^{\frac{a_2+1}{2}}}\int_0^{\varphi(x_1)} h(y)dy$$
$$= \int_0^{\varphi(x_1)} y^{a_2}\left(1+(\varphi_1'(y))^2\right)^{-\frac{1}{2}}\varphi_1'(y)h'(y)dx + h(\varphi(x_1))\varphi(x_1)^{a_2}. \tag{4.10}$$

Next, taking $h \equiv c_1 = \textit{constant}$ on the whole interval $[0,\varphi(x_1)]$, $c_1 > 0$, (4.10) implies that

$$\varphi(x_1)^{a_2-1} = 2^{\frac{a_2-1}{2}} \cdot \frac{a_2+1}{2}\frac{P_A(\Omega)}{m_B(\Omega)}. \tag{4.11}$$

Whence, it follows from (4.9) and (4.11) that

$$2^{\frac{a_2-1}{2}} = \frac{1}{a_2}. \tag{4.12}$$

Which leads to a contradiction, since $0 < a_2 < 1$.

Finally, if (ii) happens we can proceed similarly.

Therefore, the proof of lemma follows. □

Appendix

A.1 Basic tools

In this chapter, we gather some elementary results from geometric measure theory used in this thesis, see [30, 31, 38, 43].

Theorem A.1 (Riesz Representation Theorem). *Let $L : C_c(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(f); f \in C_c(\mathbb{R}^n;\mathbb{R}^m), \|f\| \leq 1, spt(f) \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^n$. Then there exists a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that

(i) $|\sigma(x)| = 1$ for μ-a.e x, and

(ii) $L(f) = \displaystyle\int_{\mathbf{R}^n} f \cdot \sigma d\mu$ for all $f \in C_c(\mathbb{R}^n;\mathbb{R}^m)$.

Here $spt(f)$ denotes the support of function f.

Definition A.2. *We call μ the **variation measure**, defined for each open set $V \subset \mathbb{R}^n$ by*

$$\mu(V) = \sup\{L(f); f \in C_c(\mathbb{R}^n;\mathbb{R}^m), \|f\| \leq 1, spt(f) \subset V\} < \infty$$

Theorem A.3. *Let μ, μ_k $(k = 1, 2, \dots)$ be Radon measures on $\mathbb{R}^n$. The following three statements are equivalent:*

(i) $\displaystyle\lim_{k\to\infty} \int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}^n} f d\mu$ for all $f \in C_c(\mathbb{R}^n)$.

(ii) $\displaystyle\limsup_{k\to\infty} \mu_k(K) \leq \mu(K)$ for each compact set $K \subset \mathbb{R}^n$ and $\mu(U) \leq \liminf_{k\to\infty} \mu_k(U)$ for each open set $U \subset \mathbb{R}^n$

(iii) $\displaystyle\lim_{k\to\infty} \mu_k(B) = \mu(B)$ for each bounded Borel set $B \subset\subset \mathbb{R}^n$ with $\mu(\partial B) = 0$.

Definition A.4. *If (i) through (iii) hold, we say the measures μ_k **converge weakly** to the measure μ, written*

$$\mu_k \rightharpoonup \mu.$$

Definition A.5. *Let μ and ν be Radon measures on $\mathbb{R}^N$. For each point $x \in \mathbb{R}^n$, define*

$$\overline{D}_\mu\nu(x) \equiv \begin{cases} \limsup\limits_{r\to 0} \dfrac{\nu(B(r,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0, \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0, \end{cases}$$

$$\underline{D}_\mu\nu(x) \equiv \begin{cases} \liminf\limits_{r\to 0} \dfrac{\nu(B(r,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0, \text{ for all } r > 0 \\ +\infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0. \end{cases}$$

*If $\overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x) < +\infty$, we say ν is **differentiable** with respect to μ at x and write*

$$D_\mu\nu(x) \equiv \overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x).$$

*$D_\mu\nu$ is the **derivative** of ν with respect to μ.*

Theorem A.6. *Let μ and ν be Radon measures on $\mathbb{R}^N$. Then $D_\mu\nu$ exists and is finite μ a.e. Furthermore, $D_\mu\nu$ is μ-measurable.*

Definition A.7. (i) *The measure ν is **absolutely continuous** with respect to μ, written*

$$\nu \ll \mu,$$

provided $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \subset \mathbb{R}^n$.

(ii) *The measures ν and μ are **mutually singular**, written*

$$\nu \perp \mu,$$

if there exists a Borel subset $B \subset \mathbb{R}^n$ such that

$$\mu(\mathbb{R}^n - B) = \nu(B) = 0.$$

Theorem A.8 (Lebesgue Decomposition Theorem)**.** *Let ν, μ be Radon measures on $\mathbb{R}^n$.*

(i) *Then $\nu = \nu_{ac} + \nu_s$, where ν_{ac}, ν_s are Radon measures on $\mathbb{R}^n$ with*

$$\nu_{ac} \ll \mu \ \text{ and } \ \nu_s \perp \mu.$$

(ii) *Furthermore,*

$$D_\mu\nu = D_\mu\nu_{ac} \ \text{ and } \ D_\mu\nu_s = 0 \ \ \mu \ \text{a.e.},$$

and

$$\nu(A) = \int_A D_\mu\nu d\mu + \nu_s(A)$$

for each Borel set $A \subset \mathbb{R}^n$.

Definition A.9. *We call ν_{ac} the **absolutely continuous** part, and ν_s the **singular** part, of ν with respect to μ.*

Definition A.10. *(i) Let $A \subset \mathbb{R}^n$, $0 \leq s < \infty$, $0 < \delta \leq \infty$. Define*

$$\mathcal{H}^s_\delta(A) \equiv \inf\left\{\sum_{j=1}^{\infty} \alpha(s)\left(\frac{diamC_j}{2}\right)^s ; A \subset \bigcup_{j=1}^{\infty} C_j, diamC_j \leq \delta\right\},$$

where

$$\alpha(s) \equiv \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)}.$$

Here $\Gamma(s) \equiv \displaystyle\int_0^\infty e^{-x}x^{s-1}dx$, $(0 < s < \infty)$, is the usual gamma function.

(ii) For A and s as above, define

$$\mathcal{H}^s(A) \equiv \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A).$$

*we call $\mathcal{H}^s$ s-**dimensional Hausdorff measure** on $\mathbb{R}^n$.*

Theorem A.11. *(i) $\mathcal{H}^s$ is a Borel regular measure $(0 \leq s < \infty)$.*

(ii) $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$, for each $n \in \mathbb{N}$.
Here $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Theorem A.12 (Polar Decomposition)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.*

(i) If $n \leq m$, there exists a symmetric map $S : \mathbb{R}^n \to \mathbb{R}^n$ and an orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$L = O \circ S$$

(ii) If $n \geq m$, there exists a symmetric map $S : \mathbb{R}^m \to \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$L = S \circ O^*.$$

Here O^* denotes the adjoint of O.

Definition A.13. *Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear.*

*(i) If $n \leq m$, we write $L = O \circ S$ as above, and we define the **Jacobian** of L to be*

$$[\![L]\!] = |\det S|.$$

*(ii) If $n \geq m$, we write $L = S \circ O^*$ as above, and we definte the **Jacobian** of L to be*

$$[\![L]\!] = |\det S|.$$

Remark A.14. *(i) It follows from Theorem below that the definition of $[\![L]\!]$ is independent of the particular choices of O and S.*

(ii) Clearly, $[\![L]\!] = [\![L^*]\!]$. Here L^* denotes the adjoint of L.

Theorem A.15. *(i) If $n \leq m$,*

$$[\![L]\!]^2 = \det(L^* \circ L).$$

(ii) If $n \geq m$,

$$[\![L]\!]^2 = \det(L \circ L^*).$$

Theorem A.16 (Rademacher's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$ a.e.*

Now let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. By Rademarcher's Theorem, f is differentiable $\mathcal{L}^n$ a.e., and therefore $Df(x)$ exists and can be regarded as a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ for $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$.

Notation A.17. *IF $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, we write the gradient matrix*

$$Df = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{bmatrix}_{m \times n}$$

Definition A.18. *The **Jacobian** of f is*

$$Jf(x) = [\![Df(x)]\!] \;\; (\mathcal{L}^n \, a.e. \, x).$$

Definition A.19. *A set $S \subset \mathbb{R}^n$ is said to be countably m-rectifiable if it is $\mathcal{H}^m$-measurable and*

$$S = S_0 \cup \left(\bigcup_{j \geq 1} F_j \left(\mathbb{R}^m \right) \right),$$

where

(i) $\mathcal{H}^m \left(S_0 \right) = 0$;

(ii) the functions $F_j : \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz, for all $j \geq 1$.

Proposition A.20. *Suppose that $S \subset \mathbb{R}^n$ is $\mathcal{H}^m$-measurable and countably m-rectifiable. Then*

$$S = \bigcup_{j=0}^{\infty} S_j,$$

where

(i) $\mathcal{H}^m \left(S_0 \right) = 0$;

(ii) $S_i \cap S_j = \emptyset$ if $i \neq j$;

(iii) for $j \geq 1$ there exists an m-dimensional C^1-submanifold $X_j \subset \mathbb{R}^n$ such that $S_j \subset X_j$.

Theorem A.21 (Coarea Formula)**.** *Suppose that $S \subset \mathbb{R}^n$ is $(n+k)$-rectifiable, $Z \subset \mathbb{R}^m$ is k-rectifiable and $F : S \to Z$ is a Lipschitz map. Then, for any $\mathcal{H}^{n+k}$-measurable subset $A \subset S$ we have*

$$\int_A JF(x) d\mathcal{H}^{n+k}(x) = \int_Z \mathcal{H}^n \left(F^{-1}(z) \cap A\right) d\mathcal{H}^k(z)$$

Corollary A.22. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, with*

$$ess\,inf|\nabla f| > 0.$$

Suppose also $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}^n$-summable. Then

$$\int_{\{f>t\}} g dx = \int_t^\infty \left(\int_{\{f=s\}} \frac{g}{|\nabla f|} d\mathcal{H}^{n-1}\right) ds.$$

In particular, we see

$$\frac{d}{dt}\left(\int_{\{f>t\}} g dx\right) = -\int_{\{f=t\}} \frac{g}{|\nabla f|} d\mathcal{H}^{n-1}$$

Definition A.23. *Let Ω be an open set. A function $f \in L^1(\Omega)$ has bounded variation in Ω if*

$$\sup\left\{\int_\Omega f div(\varphi); \varphi \in C_c^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1\right\} < \infty.$$

We write

$$BV(\Omega)$$

to denote the space of functions of bounded variation.

Definition A.24. *Let Ω be an open set. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has* ***finite perimeter*** *in Ω if*

$$\chi_E \in BV(\Omega).$$

It is convenient to introduce also local versions of the above concepts.

Definition A.25. *A function $f \in L^1_{loc}(\Omega)$* ***has locally bounded variation*** *in Ω if for each open set $V \subset\subset \Omega$,*

$$\sup\left\{\int_V f div(\varphi); \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \leq 1\right\} < \infty.$$

We write

$$BV_{loc}(\Omega)$$

to denote the space of such functions.

Definition A.26. *An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has locally finite perimeter in Ω if*

$$\chi_E \in BV_{loc}(\Omega).$$

The theorem below asserts that the weak first partial derivatives of a BV function are Radon measures.

Theorem A.27 (Structure Theorem for BV_{loc} function)**.** *Lef $f \in BV_{loc}(\Omega)$. Then there exists a Radon measure μ on Ω and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ *for μ-a.e., and*

(ii) $\displaystyle\int_\Omega f div(\varphi) dx = \int_\Omega \varphi \cdot \sigma d\mu$ *for all* $\varphi \in C_c^1(\Omega; \mathbb{R}^m)$.

Notation A.28. *(i) If $f \in BV_{loc}(\Omega)$, we will henceforth write*

$$\|Df\|$$

for the measure μ, and

$$[Df] \equiv \|Df\| \llcorner \sigma.$$

Hence assertion (ii) in Theorem A.27 reads

$$\int_\Omega f div(\varphi) dx = -\int_\Omega \varphi \cdot \sigma d\|Df\| = -\int_\Omega \varphi d[Df]$$

for all $\varphi \in C_c^1(\Omega; \mathbb{R}^m)$.

(ii) Similarly, if $f = \chi_E$, and E is a set of locally finite perimeter in Ω, we will hereafter write

$$\|D\chi_E\|$$

for the measure μ, and

$$\nu^E \equiv \sigma.$$

Consequently,

$$\int_E div(\varphi) dx = -\int_\Omega \varphi \cdot \nu^E d\|D\chi_E\|$$

for all $\varphi \in C_c^1(\Omega; \mathbb{R}^m)$.

Notation A.29. *If $f \in BV_{loc}(\Omega)$, we write*

$$\mu^i = \|Df\| \llcorner \sigma^i \ \ (i = 1, \ldots, n)$$

for $\sigma = (\sigma^1, \ldots, \sigma^n)$. By Lebesgue's Decomposition Theorem (Theorem A.8)), we may further set

$$\mu^i = \mu^i_{ac} + \mu^i_s,$$

where

$$\mu_{ac}^i \ll \mathcal{L}^n \text{ and } \mu_s^i \perp \mathcal{L}^n.$$

Then

$$\mu_{ac}^i = \mathcal{L}^n \llcorner f_i$$

for some function $f_i \in L^1_{loc}(\Omega)$ $(i = 1, \dots, n)$. *Write*

$$\begin{cases} \dfrac{\partial f}{\partial x_i} \equiv f_i \quad (i = 1, \dots, n) \\ Df \equiv \left(\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right), \\ [Df]_{ac} \equiv (\mu_{ac}^1, \dots, \mu_{ac}^n) = \mathcal{L}^n \llcorner Df, \\ [Df]_s \equiv (\mu_s^1, \dots, \mu_s^n). \end{cases}$$

Thus

$$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s,$$

so that $Df \in L^1_{loc}(\Omega; \mathbb{R}^n)$ *is the derivative of the absolutely continuous part of* $[Df]$.

Remark A.30. *(i)* $\|Df\|$ *is the variation measure of* f; $\|D\chi_E\|$ *is the perimeter measure of* E; $\|D\chi_E\|(\Omega)$ *is the perimeter of* E *in* Ω.

(ii) If $f \in BV_{loc}(\Omega) \cap L^1(\Omega)$, *then* $f \in BV(\Omega)$ *if and only if* $\|Df\|(\Omega) < \infty$, *in which case we define*

$$\|f\|_{BV(\Omega)} \equiv \|f\|_{L^1(\Omega)} + \|Df\|(\Omega).$$

(iii) From the proof of the Riesz Representation Theorem, we see

$$\|Df\|(V) = \sup\left\{\int_V f div(\varphi) dx; \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \leq 1\right\},$$

$$\|D\chi_E\|(V) = \sup\left\{\int_E div(\varphi) dx; \varphi \in C_c^1(V; \mathbb{R}^n), |\varphi| \leq 1\right\}$$

for each $V \subset\subset \Omega$.

Theorem A.31 (Lower semicontinuity of variation measure)**.** *Suposse* $f_k \in BV(\Omega)$ $(k = 1, \dots)$ *and* $f_k \to f$ *in* $L^1_{loc}(\Omega)$. *Then*

$$\|Df\|(\Omega) \leq \liminf_{k\to\infty} \|Df_k\|.$$

Theorem A.32 (Local approximation by smooth functions)**.** *Assume* $f \in BV(\Omega)$. *There exist functions* $\{f_k\}_{k=1}^\infty \subset BV(\Omega) \cap C^\infty(\Omega)$ *such that*

(i) $f_k \to f$ *in* $L^1(\Omega)$ *and*

(ii) $\|Df_k\|(\Omega) \to \|Df\|(\Omega)$ *as* $k \to \infty$.
Define the (vector-valued) Radon measure

$$\mu_k(B) \equiv \int_{B\cap\Omega} Df_k dx$$

for each Borel set $B \subset \mathbb{R}^n$. *Set also*

$$\mu(B) \equiv \int_{B\cap\Omega} d[Df].$$

Then

$$\mu_k \rightharpoonup \mu$$

weakly in the sense of (vector-valued) Radon measures on $\mathbb{R}^n$.

Definition A.33. *Let* $x \in \mathbb{R}^n$. *We say* $x \in \partial^\star E$, *the reduced boundary of* E, *if*
(i) $\|D\chi_E\|(B(x,r)) > 0$ *for all* $r > 0$,
(ii)

$$\lim_{r\to 0} \frac{1}{\|D\chi_E\|(B(x,r))} \int_{B(x,r)} \nu^E d\|D\chi_E\| = \nu^E(x), \quad \textit{and}$$

(iii) $\|\nu^E(x)\| = 1$.

Theorem A.34 (Structure theorem for sets of finite Perimeter)**.** *Assume* E *has locally finite perimeter in* $\mathbb{R}^n$.
(i) Then

$$\partial^\star E = \left(\bigcup_{j=1}^{\infty} K_j\right) \cup N,$$

where $\|D\chi_E\|(N) = 0$ *and* K_j *is a compact subset of* C^1*-hypersurface* S_j *for every* $j \in \mathbb{N}$.
(ii) Furthermore, $\nu^E(x)|_{S_j}$ *is normal to* S_j *for all* $j \in \mathbb{N}$, *and*
(iii) $\|D\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^\star E$.

Definition A.35. *Let* $x \in \mathbb{R}^n$. *We say* $x \in \partial_\star E$, *the* ***measure theoretic boundary*** *of* E, *if*

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} > 0$$

and

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r) - E)}{r^n} > 0.$$

Proposition A.36. *(i)* $\partial^\star E \subset \partial_\star E$.
(ii) $\mathcal{H}^{n-1}(\partial_\star E - \partial^\star E) = 0$.

Theorem A.37 (Generalized Gauss-Green Theorem)**.** *Let $E \subset \mathbb{R}^n$ have locally finite perimeter.*

(*i*) *Then $\mathcal{H}^{n-1}((\partial_\star E \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^n$.*

(*ii*) *Furthermore, for $\mathcal{H}^{n-1}$ a.e. $x \in \partial_\star E$, there is a unique measure theoretic unit inner normal $\nu^E(x)$ such that*

$$\int_E div(\varphi)dx = -\int_{\partial_\star E} \varphi \cdot \nu^E d\mathcal{H}^{n-1}$$

for each $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$.

(*iii*) $\|D\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^\star E$.

Theorem A.38. *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then, for $\mathcal{H}^{n-1}$- a.e. $x \in \mathbb{R}^{n-1}$,*

(*i*) *E_x has finite perimeter in $\mathbb{R}$;*

(*ii*) *$\partial^\star E_x = (\partial^\star E)_x$;*

(*iii*) *$\nu_n^E(x, y) \neq 0$ for all y such that $(x, y) \in \partial^\star E$;*

(*iv*) *for $\mathcal{H}^1$-a.e. $y \in \partial^\star E_x$*

$$\begin{cases} \lim_{z \to y^+} \chi_E(x, z) = 1, \lim_{z \to y^-} \chi_E(x, z) = 0 & \text{if } \nu_n^E(x, y) > 0, \\ \lim_{z \to y^+} \chi_E(x, z) = 0, \lim_{z \to y^-} \chi_E(x, z) = 1 & \text{if } \nu_n^E(x, y) < 0. \end{cases}$$

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