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Definable Subcategories and the Ziegler Spectrum

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Abstract

In (ZIEGLER, 1984) Ziegler associated a topological space to the category of modules over any associative ring with unit. This space, now known as the Ziegler Spectrum, has as points the isomorphism classes of pure-injective indecomposable modules. This topological space is able to give a better understanding to the category of modules.
The main objective of this text is to give some necessary definitions to understand the Ziegler spectrum and proof some important results about it. The focus of the text are definable subcategories of Mod-R, defining the Ziegler spectrum, proof some results related to it and give the example of the Ziegler Spectrum for discrete valuation rings.

Palavras-chaves: algebra, module theory, model theory, Ziegler spectrum, definable subcategories.

Resumo

Em (ZIEGLER, 1984) Ziegler associou um espaço topológico a categoria de módulos sobre qualquer anel associativo com unidade. Esse espaço, agora conhecido como Espectro de Ziegler, tem como pontos as classes de isomorfismos dos módulos puro-injetivos indecomponíveis. Este espaço topológico serve para o melhor entendimento da categoria de módulos.

O objetivo deste texto é dar algumas definições necessárias para o entendimento do Espectro de Ziegler e demonstrar resultados importantes sobre elas. Os principais focos do texto são falar sobre subcategorias definíveis de Mod-R, definir o Espectro de Ziegler, demonstrar resultados relacionados a ele e dar o exemplo do Espectro de Ziegler para anéis de valuação discreta.

Keywords: algebra , teoria de módulos, teoria de modelos, espectro de Ziegler, subcategorias definíveis.

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Introdução

In his paper (ZIEGLER, 1984) Ziegler associated a topological space with the category of modules over any ring, which is now known as the Ziegler spectrum. This space has played a central role in the model theory of modules and has also proven useful for purely algebraic reasons. The main focus of this text is to give some background so we can understand what definable subcategories are and what the Ziegler spectrum is so we can prove some important results and even give some examples.

One of the most common applications for this topological space is to prove if some theory of modules is decidable or not. In his original paper it is given a sufficient condition for a theory of modules to be decidable (ZIEGLER, 1984, Theorem 9.4). There is also a connection between the Ziegler spectrum and the Krull-Gabriel dimension of a ring which, for example, is used to prove a conjecture, in (LAKING; PREST; PUNINSKI, 2016), about the category of modules over any string algebra.

At first we want to define one of our main objects, which shall be used to define and work with the Ziegler spectrum and the definable subcategories. In Chapter 1 we will introduce positive primitive conditions (pp for short), one of our main definitions which will be used to define almost everything else. These are some special first-order sentences that talk about relations among elements in a module. Some other definitions will follow the same line of thought: pp-types will say how an n-tuple behaves with all the elements of the module and pp-pairs tell us if tuples which satisfies some pp condition must satisfy the other in a specified module. We will also show some strong relation between finitely presented modules and pp conditions.

Another central definition is the concept of purity. We will begin Chapter 2 seeing that we can see pure embeddings as a "weaker" version of split embeddings and also define some special type of exact sequence, the pure-exact sequences. With these special short exact sequences we can define pure-injective modules and pure-projective modules, in a similar way one can define injectives and projectives.

The second part of Chapter 2 will focus on pure-injective modules. These modules will be important to define our points in the Ziegler spectrum and we will also show that they can say what our definable subcategory is. Many important properties of these and some equivalent definitions are shown, so they can be used in the second part.

Chapter 4 will talk about definable subcategories, which are special subcategories of Mod-R in which all modules share some local and global properties given by a set of pp-pairs. Some equivalent conditions are given for a subcategory $\mathcal{X}$ of Mod-R to be definable and we also prove some basic results. The dual of a definable subcategory is also defined, we show it is unique and we use it to show some conditions for an inverse limit to stay in the definable subcategory. An example is also given of when that doesn't happen.

At the final chapter we define the Ziegler spectrum, show some relations between it and the definable subcategories (such as the bijection between closed sets of Zg_R and definable subcategories of Mod-R). Some important results are giving an open basis for the Ziegler spectrum and, with it, showing that this topological space is compact. After that we give an example of the Ziegler spectrum for discrete valuation rings, classifying all the points of this space and defining all the closed subsets of it.

The three appendix comes to help with some results which are not the main focus of this text.

In general we will work with right R-modules, unless it is explicitly stated. Sometimes we will write M_R to say that M is a right R-module and, in a similar way, ${}_RM$ for left modules and ${}_SM_R$ for bi-modules. Mod-R will also be used for the category of all the right R-modules and mod-R for the full subcategory of all finitely presented modules.

Part I

Main Definitions and Important Results

1 Pp Conditions

1.1 Pp conditions

In this section we will define pp conditions and pp definable subgroups for right R-modules, also giving some examples to make it easier to understand. We will also prove, in Important Properties, that pp conditions can be seen as functors from Mod-R to Ab, that $\phi(M)$ is not always an R-submodule of M and we will give a lattice structure to the set of equivalence classes of pp conditions.

1.1.1 Definition

Definition 1.1.1 (Pp condition for right modules). *Let R be an associative ring with unity and a M right R-module. Let $x_i,\ y_j$ be variables over M and $r_{ik},\ s_{jk}$ constants in R, where $1 \leq i \leq n$, $1 \leq j \leq m$ and $1 \leq k \leq c$. We define a pp condition $\phi(x_1, x_2, .., x_n)$ as:*

$$\exists y_1,\ y_2, \cdots, y_m \begin{cases} x_1 r_{11} + x_2 r_{21} + \cdots + x_n r_{n1} + y_1 s_{11} + y_2 s_{21} + \cdots + y_m s_{m1} = 0 \\ x_1 r_{12} + x_2 r_{22} + \cdots + x_n r_{n2} + y_1 s_{12} + y_2 s_{22} + \cdots + y_m s_{m2} = 0 \\ \qquad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ x_1 r_{1c} + x_2 r_{2c} + \cdots + x_n r_{nc} + y_1 s_{1c} + y_2 s_{2c} + \cdots + y_m s_{mc} = 0. \end{cases}$$

That is, pp conditions are finite homogeneous systems of R-linear equations possibly with some variables existentially bounded.

To simplify notation we will shorten our tuples, writing $\overline{x} = (x_1, x_2, \cdots, x_n)$ (in this case we say that the length of $\overline{x}$ is n), and we will use the logic symbol "and" $(\wedge)$ so we will not need to write all lines:

$$\exists \overline{y} \bigwedge_{k=1}^{c} \left(\sum_{i=1}^{n} x_i r_{ik} + \sum_{j=1}^{m} y_j s_{jk} = 0 \right)$$

We say that an element $\overline{a} = (a_1, \cdots, a_n) \in M^n$ satisfies a pp condition $\phi(\overline{x})$ of length n, where the length of ϕ is the length of $\overline{x}$, if there is $\overline{b} = (b_1, b_2, \cdots, b_m) \in M^m$ such that:

$$\bigwedge_{k=1}^{c} \left(\sum_{i=1}^{n} a_i r_{ik} + \sum_{j=1}^{m} b_j s_{jk} = 0 \right)$$

that is, $\overline{a}$ and $\overline{b}$ satisfy all linear equations from ϕ. Sometimes we use the notation $M \vDash \phi(\overline{a})$ which means "$\overline{a}$ satisfies the condition ϕ in M".

Definition 1.1.2 (Pp definable subgroup)**.** *We define the solution set for a pp condition ϕ of length n in an R-module M as:*

$$\phi(M) = \{\overline{a} \in M^n \ ; \ M \models \phi(\overline{a})\}$$

in other words, the set of all elements $\overline{a} \in M^n$ such that they satisfy all the homogeneous linear equations in the pp condition $\phi(\overline{x})$. It is easy to check that this set is an abelian group. We also refer to the solution set of a pp condition ϕ as the pp definable subgroup of ϕ in M.

To make it easier to understand the definitions here are some examples:

Example 1.1.3. *Let R be a ring, M an R-module and $r \in R$. Define $\theta(x)$ as the condition $xr = 0$. Observe that:*

$$\theta(M) = \{a \in M \ ; \ ar = 0\}$$

that is, $\theta(M) = ann_M(r)$.

Example 1.1.4. *We can see a condition $\theta(\overline{x})$, where the length of $\overline{y}$ is 0, as $\overline{x}H = 0$, where H is the matrix $(r_{ij})_{ij}$. With this we have that, for any M, $\theta(M)$ is just the kernel of the function $\overline{x} \longmapsto \overline{x}H$, which takes elements from M^n to M^m.*

Example 1.1.5. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/16\mathbb{Z}$. Let $\phi(x_1, x_2)$ be the condition $(x_1 2 = 0)$. Then $\phi(M) = \Big\{(x_1, x_2) \in M^2 \ ; \ x_1 \in \{0, \ 8\}\Big\}$. Observe that this condition is different from the one defined by $x2 = 0$ (which has solution set $\{0, \ 8\}$), because one is a subgroup of M and the other one is a subgroup of M^2.*

Using ideas as above, we call pp conditions where $s_{jk} = 0$, for all j, k, "annihilation conditions" (because we are looking at the kernel of some function).

Example 1.1.6. *Let R be a ring, M an R-module and $r \in R$. Define $\theta(x)$ as the condition $\exists y \ (x = yr)$. Observe that:*

$$\theta(M) = \{mr \ ; \ m \in M\}$$

that is, $\theta(M) = Mr$, the multiples of r in M.

Observation 1.1.7. *Any condition $\phi(\overline{x})$ may be written as $\exists \overline{y}(\overline{x}\ \overline{y})H = 0$, where $(\overline{x}\ \overline{y})$ is seen as a row vector with the entries $\overline{x}$ followed by $\overline{y}$. Then, we write H as a column matrix $H = \begin{pmatrix} A \\ -B \end{pmatrix}$ (where A is $\{r_{ij}\}_{ij}$ and $-B$ is $\{s_{jk}\}_{jk}$). This condition may be written as $\exists \overline{y}\ (\overline{x}A = \overline{y}B)$, and can be read as $B|\overline{x}A$ ("B divides $\overline{x}A$").*

Example 1.1.8. *Let $R = \Bbbk[t] = M$. Then any condition like $\exists y(x = yc)$, for some $c \in \Bbbk\backslash\{0\}$ will have $\Bbbk[t]$ as the solution set. If p is a non-constant polynomial, then $\exists y\ (x = yp)$ will define the set of multiples of p in $\Bbbk[t]$.*

We call the conditions, where the length of $\overline{y}$ is greater then zero and we have at least one $s_{jk} \neq 0$ for each k, as "divisibility conditions".

For the next example, we will use a quiver. The concept of quiver representations is not central in this text but will be used for some examples. If the reader is not familiar with this concept a good reference to understand would be (DERKSEN; WEYMAN, 2017).

Example 1.1.9. *Let $\tilde{\mathrm{A}}_1$ be the quiver shown:*

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

Let M be any representation of this quiver. Then we have that the subgroup $M\beta\alpha^{-1} = \{a \in Me_1;\ \alpha(a) \in im(\beta)\}$ is a pp-definable subgroup defined by the condition $\exists\ y\ (x\alpha = y\beta \wedge xe_2 = 0)$.

1.1.2 Important Properties

If we have a pp condition $\phi(\overline{x})$ we can rewrite it as $\exists \overline{y}\ \theta(\overline{x}\ \overline{y})$ where $\theta(\overline{x}\ \overline{y})$ has no bounded variables. Writing it like this will make it easier to work with all the variables, something that becomes explicit in the proof of the next lemma:

Lemma 1.1.10. *Let $f : M \longrightarrow N$ be an R-module homomorphism ϕ be a pp condition. Then $f(\phi(M)) \leq \phi(N)$.*

Proof. Let's rewrite $\phi(\overline{x})$ as $\exists \overline{y}\ \theta(\overline{x}\ \overline{y})$, where θ has no bounded variables. If $\overline{a} \in \phi(M)$ we know that there is an $\overline{b}$ such that $(\overline{a}\ \overline{b}) \in \theta(M)$, that is, $\sum_i a_i r_{ik} + \sum_j b_j s_{jk} = 0$ for $1 \leq k \leq c$. Observe that, by the properties of R-module homomorphism, $f(\sum_i a_i r_{ik} + \sum_j b_j s_{jk}) = \sum_i f(a_i) r_{ik} + \sum_j f(b_j) s_{jk} = 0$ for all $1 \leq k \leq c$, that is, $(f\overline{a}\ f\overline{b}) \in \theta(N)$. Then, $f\overline{a} \in \phi(N)$, where $f\overline{a}$ is defined as $(f(a_1),\ f(a_2),\ \cdots\ , f(a_n))$. □

With this result we see that each pp condition ϕ defines a functor F_ϕ from the category Mod-R, of right R-modules, to the category *Ab*, of abelian groups. The functor sends M to $\phi(M)$ and a morphism $f : M \longrightarrow N$ to the induced map from $\phi(M)$ to $\phi(N)$.

Observe that, for a module, M, its endomorphism ring, $End(M)$, sends a pp definable subgroups $\phi(M)$ of M^n to itself. By defining a left $End(M)$-module structure to M^n, given by $f\overline{a} = f(a_1, a_2, \cdots, a_n) = (f(a_1), f(a_2), \cdots, f(a_n)) \in M^n$, one can get the following corollary:

Corollary 1.1.11. *Let ϕ be a pp condition and M be any R-module. Then $\phi(M)$ is closed under the (diagonal) action of the ring* $\mathrm{End}(M)$*, that is, $\phi(M)$ is an* $\mathrm{End}(M)$*-submodule of ${}_{\mathrm{End}(M)}M^n$.*

Proof. Just make $M = N$ in the last lemma. □

Corollary 1.1.12. *If ϕ is a pp condition with one free variable then $\phi(R_R)$ is a left ideal of R.*

Proof. Let $r \in R$ be any element and define $f_r : R \longrightarrow R$ as $f_r(x) = xr$. Applying the last lemma (Lemma 1.1.10) we see that $\forall r \in R$ we have $\phi(R)r \leq \phi(R)$. □

Usually the pp-definable subgroups are not exactly the $End(M)$-submodules. For instance, a left coherent (every finitely generated left ideal is finitely presented) ring R which is not left noetherian has left ideals which are not pp-definable subgroups of the ring regarded as a right module over itself (PREST, 2009, Theorem 2.3.19).

Corollary 1.1.13. *If M is an R-module and ϕ is a pp condition then $M.\phi(R) \leq \phi(M) \leq M^n$, where n is the length of ϕ, where $M.\phi(R) = \{\sum_{i=1}^{n} m_i\overline{r_i};\ m_i \in M,\ \overline{r_i} \in \phi(R)\}$.*

Proof. By definition we already know that $\phi(M) \leq M^n$. To prove the other inequality define, for $m \in M$, $f_m : R \longrightarrow M$ as $f_m(x) = mx$ then, by the Lemma 1.1.10, if $(r_1,\ r_2,\ \cdots,\ r_n) \in \phi(R)$ we have that $(mr_1,\ mr_2,\ \cdots, mr_n) = m\overline{r} \in \phi(M)$ and, because $\phi(M)$ is an abelian group, we also have that $\forall m_i \in M$ and $\forall \overline{r_i} \in \phi(R)$ $\sum_{i=1}^{n} m_i\overline{r_i} \in \phi(M)$. □

By (PREST, 2009, Theorem 2.3.9) we have that $\phi(M) = M.\phi(R)$, for all pp conditions ϕ with one free variable if and only if M is flat, giving another way to define this class of modules.

1.1.3 Partial order and equivalence classes of pp conditions

One can see that some pp conditions are redundant, for example, if $\phi(x)$ is $(x = x)$ and $\psi(x)$ is $\exists y\ (x = y)$ both have, as solution set, the entire module (for any module in Mod-R), because every element of any module is equal to itself. To avoid this problem we will define a partial order in the set of pp conditions and, with it, define when two pp-conditions are equivalent.

We say that $\psi \leq \phi$ (or we might also write $\psi \to \phi$), where ϕ and ψ have the same length, if for every module in Mod-R we have $\psi(M) \leq \phi(M)$ and say that ψ implies or is stronger than ϕ. We will say that $\phi \equiv \psi$ if $\phi \geq \psi$ and $\phi \leq \psi$.

It is not always easy to check if $\phi \geq \psi$ with these definitions, because we need to check for a proper class of modules. The next lemma will give us a way to check these inequalities without needing to look at modules. Before proving it, here are some important observations:

Observation 1.1.14. *As we already mentioned in Example 1.1.7, we can write any pp condition as $B|\overline{x}A$, that is, $\exists \overline{y}\ (\overline{x}A = \overline{y}B)$, for some matrices A and B. It is easy to check that:*

1. *$B|\overline{x}A$ implies the pp condition $BC|\overline{x}AC$ for any matrix C;*
2. *$B|\overline{x}A$ implies the pp condition $B_0|\overline{x}A$ if $B = B_1B_0$ for some matrix B_1, B_0;*
3. *$B|\overline{x}A$ implies the pp condition $B|\overline{x}D$ if $A = A_0B + D$ for some matrix A_0, D.*

With these 3 implications between pp conditions we will be able to prove the lemma:

Lemma 1.1.15. *Let $\phi(\overline{x})$ be the pp condition $\exists \overline{y}\ (\overline{x}\ \overline{y})H_\phi = 0$ and $\psi(\overline{x})$ be the pp condition $\exists \overline{y}\ (\overline{x}\ \overline{y})H_\psi = 0$, both with the same length. We have that $\psi \leq \phi$ if and only if there are matrices $G = \begin{pmatrix} G' \\ G'' \end{pmatrix}$ and K such that $\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix} H_\phi = H_\psi K$, where I is the $n \times n$ identity matrix, where n is the length of $\overline{x}$, and 0 denotes a zero matrix with n columns.*

Proof. Just to simplify our proof, we will rewrite ψ as $B'|\overline{x}A'$ and ϕ as $B|\overline{x}A$, so $H_\psi = \begin{pmatrix} A' \\ -B' \end{pmatrix}$ and $H_\phi = \begin{pmatrix} A \\ -B \end{pmatrix}$.

($\Leftarrow$) Suppose we have the equations as in our statement, that is $\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix} \begin{pmatrix} A \\ -B \end{pmatrix} = \begin{pmatrix} A' \\ -B' \end{pmatrix} K$. So we have $A - G'B = A'K$ and $-G''B = -B'K$. With this, applying the implications from Observation 1.1.14, we get that $\begin{pmatrix} A' \\ -B' \end{pmatrix} \Rightarrow^{1.} \begin{pmatrix} A'K \\ -B'K \end{pmatrix} = \begin{pmatrix} A - G'B \\ -G''B \end{pmatrix} \Rightarrow^{2.} \begin{pmatrix} A - G'B \\ -B \end{pmatrix} \Rightarrow^{3.} \begin{pmatrix} A \\ -B \end{pmatrix}$ where we use $\Rightarrow^{i.}$ to say that one condition implies the other using rule $i.$ from the Observation 1.1.14. Then $\psi \leq \phi$.

($\Rightarrow$) Suppose that the matrices A', B' are $n \times m$ and $l \times m$ respectively. Let M be the finitely presented module generated by $x_1, \dots, x_n, y_1, \dots, y_l$ and with relations defined by $(\overline{x}\ \overline{y})\begin{pmatrix} A' \\ -B' \end{pmatrix} = 0$. That is, define $f : R^m \longrightarrow R^{n+l}$ as the function $f(\overline{z}) = \begin{pmatrix} A'\overline{z} \\ -B'\overline{z} \end{pmatrix}$ (here we write $\overline{z}$ as a collumn vector to be able to define the morphism) and M as the cokernel of this map (hence, it will be a finitely presented module with kernel generated by the rows of our matrices A' and B'). Let e_i be the i-th unit element of R^{n+l} and denote $a_i = g(e_i)$, for $i \leq n$, and $b_i = g(e_i)$, for $n+1 \leq i \leq n+l$. Because M is defined as the cokernel of f it is easy to see that $(\overline{a}\ \overline{b})\begin{pmatrix} A' \\ -B' \end{pmatrix} = 0$, so $\overline{a} \in \psi(M)$ and hence, by assumption, $\overline{a} \in \phi(M)$. So, there is some $\overline{d} \in M^l$ such that $(\overline{a}\ \overline{d})\begin{pmatrix} A \\ -B \end{pmatrix} = 0$. Since $\overline{d}$ is from M, and the projection of R^{n+l} into M is surjective, hence any element of M can be written as a finite sum $\sum_{i=1}^{n+l} \pi(e_i)r_i = \sum a_i r_i + \sum b_j r_j$, that is, $\overline{a}$ and $\overline{b}$ generates M, there are matrices G', G'' such that $\overline{d} = (\overline{a}\ \overline{b})\begin{pmatrix} G' \\ -G'' \end{pmatrix}$. Therefore we have that $(\overline{a}\ \overline{b})\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix}\begin{pmatrix} A \\ -B \end{pmatrix} = 0$ and so, because M is the cokernel of f and is generated by $\overline{a}\ \overline{b}$, we have that the matrix $\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix}\begin{pmatrix} A \\ -B \end{pmatrix}$ belongs to the submodule of R^{n+l} generated by $f(e_i')$, where e_i' is the i-th unit of R^m, that is, $\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix}\begin{pmatrix} A \\ -B \end{pmatrix} = \begin{pmatrix} A' \\ -B' \end{pmatrix} K$ as desired (because the i-th column of $\begin{pmatrix} A' \\ -B' \end{pmatrix}$ can be seen as the image of $f(e_i)$ in R^{n+l}). □

Example 1.1.16. *Suppose that $\phi(x)$ is $r|x$, that is, $\exists y(x\ y)\begin{pmatrix} 1 \\ r \end{pmatrix} = 0$, and $\psi(x)$ is $s|x$, that is, $\exists y(x\ y)\begin{pmatrix} 1 \\ s \end{pmatrix} = 0$, for some $r, s \in R$. Then $\psi \leq \phi$ if and only if there are K and G satisfying the Lemma 1.1.15, where G is 2×1 and K is 1×1. So this equation is actually $\begin{pmatrix} 1 & g' \\ 0 & g'' \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ s \end{pmatrix}(k)$. Therefore $\psi \leq \phi$ if and only if there are k, g' and $g'' \in R$ such that $k = 1 + g'r$ and $g''r = sk$. That is, $g''r = sk = s(1+g'r) \Rightarrow (g''-g')r = tr = s$, then the condition is that s is in the left ideal generated by r.*

Changing a little bit the notation from the Lemma 1.1.15 we get the following corollary:

Corollary 1.1.17. *The implication $B'|\overline{x}A' \leq B|\overline{x}A$ holds if and only if there are matrices G, H and K such that $A = A'K + GB$ and $HB = B'K$*

Definition 1.1.18 (Lattice of pp conditions)**.** *We can give a lattice structure to the partially ordered set of equivalence classes, given by $eq(\phi) = \{\psi; \psi \equiv \phi\}$, of pp condition,* *defining the conjunction and sum o pp conditions as below:*

Conjunction *Let $\phi(\overline{x})$ be $\exists \overline{y}\ \theta(\overline{x}\ \overline{y})$ and $\phi'(\overline{x})$ be $\exists \overline{y}\ \theta'(\overline{x}\ \overline{y})$, where θ and θ' have no free variables and the length of ϕ is the same as ϕ'. We define $(\phi \wedge \phi')(\overline{x})$ as $\exists \overline{y},\ \overline{y}'(\theta(\overline{x},\ \overline{y}) \wedge \theta'(\overline{x},\ \overline{y}'))$. One can see that the elements that satisfy $\phi \wedge \phi'$ are the ones that satisfy ϕ and ϕ', that is, $(\phi \wedge \phi')(M) = \phi(M) \cap \phi'(M)$;*

Sum *Let $\phi(\overline{x})$ be $\exists \overline{y}\ \theta(\overline{x}\ \overline{y})$ and $\phi'(\overline{x})$ be $\exists \overline{y}\ \theta'(\overline{x}\ \overline{y})$, where θ and θ' have no free variables and the length of ϕ is the same as ϕ'. We define $(\phi + \phi')(\overline{x})$ as $\exists \overline{z},\ \overline{z}',\ \overline{y},\ \overline{y}'(\theta(\overline{z},\ \overline{y}) \wedge \theta'(\overline{z}',\ \overline{y}') \wedge \overline{x} = \overline{z} + \overline{z}')$. One can see that the elements that satisfy $\phi + \phi'$ are the sums of elements which satisfy ϕ and with those which satisfy ϕ', that is, $(\phi + \phi')(M) = \phi(M) + \phi'(M)$.*

These functions give a structure of modular lattice to the set of equivalence classes of pp conditions. We denote this lattice of pp conditions as pp_R^n, that is, the lattice of pp conditions in n free variables over R.

With these we can also, using the Lemma 1.1.15, find a criterion for $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_t \leq \phi$:

Corollary 1.1.19. *The implication $\bigwedge_1^t (B_i | \overline{x} A_i) \leq (B | \overline{x} A)$ between pp conditions holds if and only if there are matrices G, G_i, K_i, $(i = 1, \ldots, t)$ such that $A = \sum_1^t A_i K_i + GB$ and $G_i B = B_i K_i$ $(i = 1, \ldots, t)$*

Proof. Just write this conjunction as

$$\exists \overline{y}_1, \ldots, \overline{y}_t\, (\overline{x} \quad \overline{y}_1 \ldots \overline{y_t}) \begin{pmatrix} A_1 & A_2 & \ldots & A_t \\ -B_1 & 0 & \cdots & 0 \\ 0 & -B_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -B_t \end{pmatrix} = 0$$

and apply the Lemma 1.1.15 expanding G'' and K in convenient blocks and defining G' as G. $\square$

1.2 Pp-types and free realizations

Pp-types differ from pp conditions because with pp conditions we are looking for elements in a module which satisfies a condition and with pp-types we are analyzing the opposite: given a module and a tuple from the module, the pp-type will be the set of pp-conditions which this tuple satisfies. This definition is important to show the connection between pp conditions and finitely presented modules. Another important result about pp-types, (PREST, 2009, Theorem 3.2.5), is that all the filters on the lattice of pp conditions are pp-types.

1.2.1 Pp-types

Definition 1.2.1 (Pp-type)**.** *Let M be an R-module and let $\overline{a} = (a_1, \cdots, a_n)$ be a tuple from M. We define the pp-type of $\overline{a}$ in M as:*

$$pp^M(\overline{a}) = \{\phi \ : \ \phi \text{ is a pp condition and } \overline{a} \in \phi(M)\}$$

We might also say that $pp^M(\overline{a})$ is a pp-n-type when we want to specify the length of $\overline{a}$.

By the definition of pp-types one can easily see that they are closed under conjunction of pp conditions (that is, $\phi, \ \psi \in pp^M(\overline{a})$ implies $\phi \wedge \psi \in pp^M(\overline{a})$) and under implication ($\psi \in pp^M(\overline{a})$ and $\psi \leq \phi$ implies $\phi \in pp^M(\overline{a})$). With this we see that a pp-type can be seen as a filter in the lattice pp^n_R (Definition A.1.1).

Lemma 1.2.2. *Given $n \in \mathbb{N}$ and, for each i in an index set I, an n-tuple $\overline{a_i}$ from a module M_i, define $M = \prod_i M_i$, set $\overline{a} = (\overline{a_i})_i \in \prod_i M_i^n = (\prod_i M_i)^n = M^n$. Then $pp^M(\overline{a}) = \bigcap_i pp^{M_i}(\overline{a_i})$.*

Proof. $\left(pp^M(\overline{a}) \subset \bigcap_i pp^{M_i}(\overline{a_i})\right)$ Suppose $\overline{a} \in \phi(M)$, where ϕ is $\exists \overline{y} (\overline{x} \ \overline{y}) H = 0$. Choose $\overline{b} = (\overline{b_i})_i$ from M such that $(\overline{a} \ \overline{b})H = 0$. Then $(\overline{a_i} \ \overline{b_i})H = 0$ for each $i \in I$ and so $a_i \in \phi(M_i)$.

$\left(pp^M(\overline{a}) \supset \bigcap_i pp^{M_i}(\overline{a_i})\right)$ We just need to reverse the argument above. □

Using the same idea as the one to prove the Lemma 1.2.2 we get the following result:

Lemma 1.2.3. *Given any collection of modules $(M_i)_{i \in I}$ and any pp condition ϕ one has $\phi\left(\bigoplus_i M_i\right) = \bigoplus_i \phi(M_i)$ and $\phi\left(\prod_i M_i\right) = \prod_i \phi(M_i)$.*

This result implies that the functor, F_ϕ :Mod-$R \longrightarrow Ab$, which is defined by $F_\phi(M) = \phi(M)$, commutes with direct products and direct sums.

In a similar way to pp conditions, if p a pp-n-type and $\overline{a} \in M^n$, then we write $M \models p(\overline{a})$ if $M \models \phi(\overline{a})$ for every $\phi \in p$. Here $\overline{a}$ might also satisfy pp conditions that are outside p. Then define the solution set of p in M to be:

$$p(M) = \{\overline{a} \in M^n \ : \ M \models p(\overline{a})\} = \bigcap_{\phi \in p} \phi(M).$$

1.2.2 Free realizations

Definition 1.2.4 (Finitely generated pp-type)**.** *If there is a single pp condition ϕ_0 such that $pp^M(\overline{a}) = \{\phi \in pp^n_R \ : \ \phi \geq \phi_0\}$ then we say that the pp-type of $\overline{a}$ is finitely generated by ϕ_0, and write as $pp^M(\overline{a}) = \langle \phi_0 \rangle$.*

In a similar way we define $\langle\phi_0\rangle = \{\phi\ :\phi \geq \phi_0\}$.

Observe that because the conjunction of finitely many pp conditions is also a pp condition, a pp-type being finitely generated is the same as saying that it is generated by only one element. For example, let $p = \langle\phi_1, \cdots, \phi_n\rangle$ be the closure of the set $\{\psi;\ \psi \geq \phi_i,$ for some i $1 \leq i \leq n\}$ under finite conjunctions and finite sums. Because p is closed under finite conjunctions we have that $\phi_1 \wedge \cdots \wedge \phi_n := \phi \in p$, that is, $\langle\phi\rangle \subset p$, and, for every $1 \leq i \leq n$, $\phi \leq \phi_i$, hence $\langle\phi_i\rangle \subset \langle\phi\rangle$ so $p = \langle\phi\rangle$.

Another important thing to notice is that to say that $pp^M(\overline{a}) = \langle\phi_0\rangle$ it is not enough that $\phi_0 \in pp^M(\overline{a})$ and $\phi_0(M) \leq \phi(M)$, for every $\phi \in pp^M(\overline{a})$: rather, it is necessary that for every $\phi \in pp^M(\overline{a})$ we have $\phi_0 \leq \phi$ (pp conditions that are incomparable might define the same pp-definable subgroup of a given module).

Lemma 1.2.5. *If M is a finitely presented module and $\overline{a}$ a tuple from M, then the pp-type of $\overline{a}$ in M is finitely generated.*

Proof. To prove this theorem we will first define what will be our generating pp condition ϕ and then show that it will generate the pp-type, that is, if $\psi \in pp^M(\overline{a})$ then $\phi \leq \psi$. Suppose $\overline{b}$ generates M and let $\overline{b}H = 0$, where H is a matrix with entries in the ring and such that it has as columns the generators of all relations on $\overline{b}$. Let G be such that $\overline{a} = \overline{b}G$. Define ϕ as $\exists\overline{y}\ (\overline{x}\ \overline{y})\begin{pmatrix} I & 0 \\ -G & H\end{pmatrix} = 0$. It is easy to see that $M \models \phi(\overline{a})$. Now suppose that ψ is such that $M \models \psi(\overline{a})$ and let ψ be $\exists\overline{y}\ (\overline{x}\ \overline{y})\begin{pmatrix} A \\ -B\end{pmatrix}$=0. Then,$(\overline{a}\ \overline{c})\begin{pmatrix} A \\ -B\end{pmatrix} = 0$ for some $\overline{c}$ from M. Because $\overline{b}$ generates M we have $\overline{c} = \overline{b}G'$. With this we get $(\overline{a}\ \overline{c})\begin{pmatrix} A \\ -B\end{pmatrix} = (\overline{a}\ \overline{b})\begin{pmatrix} I \\ -G'\end{pmatrix}\begin{pmatrix} A \\ -B\end{pmatrix} = 0$ and, using the same idea as the proof of Lemma 1.1.15, we get that there is a matrix K such that $\begin{pmatrix} I \\ -G'\end{pmatrix}\begin{pmatrix} A \\ -B\end{pmatrix} = \begin{pmatrix} I & 0 \\ -G & H\end{pmatrix}K$, so, again by Lemma 1.1.15, $\phi \leq \psi$. $\square$

Proposition 1.2.6. *Suppose M is a finitely presented module and that $\overline{a}$ is a tuple from M such that $pp^M(\overline{a}) = \langle\phi\rangle$. Let N be any module and let $\overline{c} \in \phi(N)$. Then there is a morphism of R-modules from M to N mapping $\overline{a}$ to $\overline{c}$.*

Proof. By the proof of the Lemma 1.2.5 we have that ϕ is equivalent to $\exists\overline{y}\ (\overline{x}\ \overline{y})\begin{pmatrix} I & 0 \\ -G & H\end{pmatrix} = 0$. Because $\overline{c} \in \phi(N)$ we have that there exists a tuple $\overline{d}$ such that $\overline{d}H = 0$ and $\overline{c} = \overline{d}G$. Since $\overline{b}$ generates M with relations defined by H one can see that the map $\overline{b} \mapsto \overline{d}$ extend to a well-defined morphism $f : M \longrightarrow N$ which maps $\overline{a} = \overline{b}G$ to $\overline{c} = \overline{d}G$, as required. $\square$

We can extend the ordering of pp conditions to pp-types: if p and q are both pp-n-types, then we say $q \leq p$ (that is, $q \to p$, "q implies p") if $p \subset q$. The ordering is that of solution sets: $q \leq p$ if and only if $q(M) \subset p(M)$ for every module M.

Lemma 1.2.7. *If $f : M \to M'$ is any morphism of R-modules and $\overline{a}$ is a tuple from M, then $pp^M(\overline{a}) \geq pp^{M'}(f\overline{a})$.*

Proof. Just observe that, by Lemma 1.1.10, we have that if $\overline{a} \in \phi(M)$ then $f\overline{a} \in \phi(M')$. □

Corollary 1.2.8. *Suppose that M is a finitely presented module, that $\overline{a}$ is a tuple from M, that N is any module and that $\overline{b}$ is a tuple from N. Then there is a morphism $f : M \longrightarrow N$ with $f\overline{a} = \overline{b}$ if, and only if, $pp^M(\overline{a}) \geq pp^N(\overline{b})$.*

Proof. Follows directly from Proposition 1.2.6 and Lemma 1.2.7. □

For the next observation, we define the p-adic integers, for p a prime number, as the abelian group $\overline{\mathbb{Z}}_{(P)}$ which has as elements $\sum_{i=0}^{\infty} a_i p^i$, where $0 \leq a_i < p$. The sum $\sum_{i=0}^{\infty} a_i p^i + \sum_{i=0}^{\infty} b_i p^i = \sum_{i=0}^{\infty} c_i p^i$ is defined as $c_i = a_i + b_i + \epsilon_i$, if $a_i + b_i + \epsilon_i < p$, and $c_i = a_i + b_i + \epsilon_i - p$, if $a_i + b_i + \epsilon_i \geq p$, where $\epsilon_0 = 0$ and, for $i \geq 1$, $\epsilon_i = 0$, if $0 \leq a_{i-1} + b_{i-1} < p$, and $\epsilon_i = 1$ otherwise. The inverse of $\sum_{i=1}^{\infty} a_i p^i$ will be an element $\sum_{i=1}^{\infty} b_i p^i$ such that, for all $i \geq 0$, $c_i = 0$. One can see that $\overline{\mathbb{Z}}_{(P)} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/\mathbb{Z}p^n$.

Another example we will use some times is the Prüfer p-group, $\mathbb{Z}_{p^\infty}$. For p a prime we define the Prüfer p-group as the abelian group generated by b_i, $i \geq 1$, and with relations $b_1 p = 0$, $b_2 p = b_1$, $\cdots$, $b_n p = b_{n-1}$, $\cdots$. Observe that the subgroup generated by b_i is isomorphic to $\mathbb{Z}/\mathbb{Z}p^i$. One can see that $\mathbb{Z}_{p^\infty} = \varinjlim_{n \in \mathbb{N}} \mathbb{Z}/\mathbb{Z}p^n$.

The criterion of the Corollary 1.2.8 fails if we replace the condition of M being a finitely presented modules with M being an arbitrary module. An example is: let $M = \overline{\mathbb{Z}}_{(P)}$, the p-adic integers, and $N = \mathbb{Z}_{(P)}$, the localization of the integers at p, embedded in M, and take $a = b$ to be any non-zero element of N. We have that $pp^M(a) = pp^N(b)$ but $Hom_{\mathbb{Z}}(\overline{\mathbb{Z}}_{(P)}, \mathbb{Z}_{(P)}) = 0$.

Definition 1.2.9 (Free realization)**.** *A free realization of a pp condition ϕ is a finitely presented module C and a tuple $\overline{c}$ from C such that $pp^C(\overline{c}) = \langle \phi \rangle$.*

Proposition 1.2.10. *Every pp condition has a free realization.*

Proof. Let ψ be a pp condition. The proof of Lemma 1.1.15($\Rightarrow$) produced a free realization, $(M, \overline{a})$ in the notation there, for ψ. In the proof we show that if $\phi \in pp^M(\overline{a})$ then we can find matrices, just like in the statement of Lemma 1.1.15, to show that $\psi \leq \phi$. □

Just as stated in the proof of Proposition 1.2.10, one can find a free realization of a pp condition using the steps from the proof ($\Rightarrow$) at Lemma 1.1.15. To create our free realization $(M,\ \overline{a})$, of ϕ, for the finitely presented module we defined a morphism $f : R^m \longrightarrow R^{n+l}$ with $f(\overline{z}) = \begin{pmatrix} A\overline{z} \\ -B\overline{z} \end{pmatrix}$, where A, an $n \times m$ matrix, and B, an $l \times m$ matrix, are matrices from our pp condition, which can be written as $\exists \overline{y}\ (\overline{x}\ \overline{y}) \begin{pmatrix} A \\ -B \end{pmatrix} = 0$. The cokernel of f, $coker(f) = R^{n+l}/Im(f)$, will be our desired module, which we shall call M. For $\overline{a} = (a_1, \cdots, a_n)$ we defined a_i as the canonical projection of $e_i \in R^{n+l}$ into M, $1 \leq i \leq n$, so $\overline{a} \in \phi(M)$ and if $\overline{a} \in \psi(M)$ we would get that $\phi \leq \psi$. Below there is an example showing this process:

Example 1.2.11. *Let $R = \mathbb{Z}$ and $\psi(x)$ be the pp condition $(x.2 = 0)$, we want to find $(M,\ a)$ a free realization of this pp condition. We can write ψ in the matrix form as $(x)(2) = 0$, where (2) is a 1×1 matrix. In this case we have $m = n = 1$, $l = 0$, $A = (2)$ and B the empty matrix, hence, $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $z \mapsto 2z$ which has as cokernel $\mathbb{Z}/\mathbb{Z}2$ is our desired M. Our a is defined as the canonical projection, $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/\mathbb{Z}2$, of $1 \in \mathbb{Z}$. Hence, the free realization of ψ shall be $(\mathbb{Z}/\mathbb{Z}2,\ 1)$.*

Corollary 1.2.12. *Let $(C_\phi, \overline{c}_\phi)$ be a free realization of ϕ and let $(C_\psi, \overline{c}_\psi)$ be a free realization of ψ. Then $\phi \geq \psi$ if, and only if, there is a morphism from C_ϕ to C_ψ taking $\overline{c}_\phi$ to $\overline{c}_\psi$.*

Proof. It follows from Corollary 1.2.8. □

An n-pointed module is a module M together with a specified n-tuple, $\overline{a} \in M^n$, of elements from M. We denote a pointed module as $(M,\ \overline{a})$. A morphism of n-pointed modules, say from $(M,\ \overline{a})$ to $(N,\ \overline{b})$, is a morphism $f : M \longrightarrow N$ of R-modules which takes $\overline{a}$ to $\overline{b}$, that is, $fa_i = b_i$ for $1 \leq i \leq n$. One can also define a forgetful functor from the category of n-pointed R-modules to the category of R-modules which takes $(M,\ \overline{a})$ and sends to M.

With Proposition 1.2.10 and Corollary 1.2.12 one can say an element $\overline{a}$ from M satisfies ϕ is there is a morphism from the pointed module $(C,\ \overline{c})$, where this pointed module is a free realization of ϕ, into the pointed module $(M,\ \overline{a})$. One can also characterize the join and meet using direct sum (PREST, 2009, Lemma 1.2.27) and pushout (PREST, 2009, Lemma 1.2.28), respectively, the solution set with the morphisms from a finitely presented modules to any module (PREST, 2009, Corollary 1.2.17).

By being able to consider pp-conditions as morphisms from the free realization to any module we can prove that pp conditions commute with direct limits, that is, the functor F_ϕ commutes with direct limits. In general it does not commute with inverse limit, as we will see in Example 3.1.13.

Proposition 1.2.13. *If ϕ is a pp condition and $M = \varinjlim_\lambda M_\lambda$, then $\phi(M) = \varinjlim_\lambda \phi(M_\lambda)$, the maps between the $\phi(M_\lambda)$ being induced by the maps between the M_λ.*

Proof. ($\phi(M) \geq \varinjlim_\lambda \phi(M_\lambda)$) By Lemma 1.1.10, $f_\lambda(\phi(M_\lambda)) \leq \phi(M)$ for all λ, then $\varinjlim_\lambda \phi(M_\lambda) \leq \phi(M)$.

($\phi(M) \leq \varinjlim_\lambda \phi(M_\lambda)$) Conversely, if $\overline{a} \in \phi(M)$, choose a morphism from $(C, \overline{c})$ to $(M, \overline{a})$ (Proposition 1.2.6), where $(C, \overline{c})$ is a free realization of ϕ. Because C is finitely presented, this morphism factors through one of the M_λ, because if ϕ is $\exists \overline{y}\ (\overline{x}\ \overline{y})H = 0$ and $\overline{b}$ is such that $(\overline{a}\ \overline{b})H = 0$ we have that, for some λ, $(\overline{a}_\lambda\ \overline{b}_\lambda)H = 0$ and $f_\lambda(\overline{a}\ \overline{b}_\lambda) = (\overline{a}\ \overline{b})$, that is, $\overline{a}_\lambda \in \phi(M_\lambda)$ and, $\forall \mu \geq \lambda$, $f_{\lambda\mu}\overline{a}_\lambda \in \phi(M_\mu)$. □

Definition 1.2.14 (Pp-pair). *We say that ϕ/ψ is a pp-pair if ϕ, ψ are pp conditions with $\phi \geq \psi$. A pp-pair ϕ/ψ is said to be open in an R-module M if $\phi(M)/\psi(M) \neq 0$, otherwise it is closed in M.*

Example 1.2.15. *Observe that if we say that a pp-pair $(x = x)/(2|x)$, for $\mathbb{Z}$-modules, is closed in a module, M, it is the same as saying for all elements $a \in M$ you can find another element b such that $a = b2$, because $M = (x = x)(M) = (2|x)M = M2$. In a similar way, if we say $(x2 = 0)/(x = 0)$ is closed is the same as saying that there is no element, different than 0, such that $x2 = 0$.*

In general when we have that a pp-pair is closed in a module we are saying that the tuples which satisfies some pp condition must satisfy the other.

Just like in pp conditions, we can define a functor of a pp-pair by defining $F_{\phi/\psi}(M) = \phi(M)/\psi(M)$. These functors will be used a lot in Chapter 2.

Corollary 1.2.16. *If $\phi > \psi$ are pp conditions, then there is a finitely presented module C such that the pair ϕ/ψ is open in C.*

Proof. Take $(C, \overline{c})$ to be a free realization of ϕ. Then $\overline{c} \in \phi(C) \setminus \psi(C)$. □

1.3 Duality

Given a pp condition for right-modules one can write down a dual condition for left modules. This duality gives some sufficient and necessary conditions for $\sum a_i \otimes b_i = 0$ in $M \otimes_R N$, which will be used to show some properties of dual modules (as defined in Subsection 1.3.3). These results will be used to show some properties of the duality of Definable Subcategories.

1.3.1 Elementary duality

Definition 1.3.1 (Dual of a pp condition). *Let ϕ be the pp condition $\exists \overline{y}\ (\overline{x}\ \overline{y}) \begin{pmatrix} H' \\ H'' \end{pmatrix} = 0$ for right R-modules. We define $D\phi$ as the pp condition $\exists \overline{z} \begin{pmatrix} I & H' \\ 0 & H'' \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = 0$ for left*

R-modules.

Dually, let ϕ be the pp condition $\exists \overline{z}\ (H'\ H'')\begin{pmatrix}\overline{x}\\ \overline{z}\end{pmatrix} = 0$ for left R-modules. We define $D\phi$ as the condition $\exists \overline{y}\ (\overline{x}\quad \overline{y})\begin{pmatrix} I & 0 \\ H' & H''\end{pmatrix}$ for right R-modules.

Example 1.3.2. *A trivial example is $x = x$ and $x = 0$ are dual of one another. To see that write $x = x$ as $(x)(0) = 0$, which has as dual the condition $\exists z\ (1\ 0)\begin{pmatrix}x\\ z\end{pmatrix} = 0$ which is $\exists z\ (x = z0) \equiv (x = 0)$. For $x = 0$ one can see this condition as $(x)(1) = 0$, which has as dual the condition $\exists z\ (1\ 1)\begin{pmatrix}x\\ z\end{pmatrix} = 0$ which is $\exists z\ (x = z) \equiv (x = x)$.*

Let $R = \mathbb{Z}$ and $\phi(x)$ be $x2 = 0$. Then, $D\phi(x)$ is $\exists z\ (1\ 2)\begin{pmatrix}x\\ z\end{pmatrix} = 0$, that is, $\exists z\ (x = -2.z)$ which is $2|x$. If we dualize this condition again, to get $D^2\phi$, we obtain $\exists y\ (x\ y)\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix} = 0$, that is, $\exists y\ (x = y \wedge y.2 = 0)$ which is equivalent to $(x.2 = 0)$.

In general, just like above, the dual of a divisibility condition is an annihilation condition and the dual of an annihilation condition is a divisibility condition. The next result shows that this duality is an anti-isomorphism of lattices:

Proposition 1.3.3. *For each $n \geq 1$ the operator D is a duality between the lattice of equivalence classes of pp conditions with n free variables for right modules and the corresponding lattice for left modules. That is, for every pp condition ϕ we have that $D^2\phi$ equivalent to ϕ and also $\psi \leq \phi$ if, and only if, $D\phi \leq D\psi$.*

Proof. (We will proof this fact using Lemma 1.1.15, it will be proven that the three basic implications are reversed in the dual).

We have $\begin{pmatrix} I & A \\ 0 & B\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & C\end{pmatrix} = \begin{pmatrix} I & AC \\ 0 & BC\end{pmatrix}$ so, because now we have left modules action, we get that $\begin{pmatrix} I & AC \\ 0 & BC\end{pmatrix} \Rightarrow \begin{pmatrix} I & A \\ 0 & B\end{pmatrix}$.

We also have $\begin{pmatrix} I & A \\ 0 & B_0B_1\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B_0\end{pmatrix}\begin{pmatrix} I & A \\ 0 & B_1\end{pmatrix}$, then, by a similar argument as above, $\begin{pmatrix} I & A \\ 0 & B_1\end{pmatrix} \Rightarrow \begin{pmatrix} I & A \\ 0 & B_0B_1\end{pmatrix}$.

We have $\begin{pmatrix} I & A_0B + D \\ 0 & B\end{pmatrix} = \begin{pmatrix} I & A_0 \\ 0 & I'\end{pmatrix}\begin{pmatrix} I & D \\ 0 & B\end{pmatrix}$ (where I' is an identity matrix with proper size), hence $\begin{pmatrix} I & D \\ 0 & B\end{pmatrix} \Rightarrow \begin{pmatrix} I & A_0B + D \\ 0 & B\end{pmatrix}$.

With this, and what we proved in Lemma 1.1.15, we get that $\phi \geq \psi$ implies

$D\psi \geq D\phi$ and $D\psi \geq D\phi$ implies $D^2\phi \geq D^2\psi$.

(Now we will prove that $D^2\phi$ is equivalent to ϕ, completing our proof)

Suppose ϕ has the form $\exists \overline{y}\ (\overline{x}\ \overline{y}) \begin{pmatrix} A \\ -B \end{pmatrix} = 0$. Then, the matrix related to $D^2\phi$ will be $\begin{pmatrix} I & 0 \\ I & A \\ 0 & -B \end{pmatrix}$. From the following equations, we can deduce (using Lemma 1.1.15) that they are indeed equivalent:

$$\begin{pmatrix} I & -I & 0 \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ I & A \\ 0 & -B \end{pmatrix} = \begin{pmatrix} 0 & -A \\ 0 & B \end{pmatrix} = \begin{pmatrix} A \\ -B \end{pmatrix} \begin{pmatrix} 0 & -I \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} A \\ -B \end{pmatrix} = \begin{pmatrix} A \\ 0 \\ B \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & A \\ 0 & -B \end{pmatrix} \begin{pmatrix} A \\ -I \end{pmatrix}.$$

□

A corollary of this result, which follows from the definitions, is that meet will be mapped to join and join will be mapped to meet via this duality.

1.3.2 Tensor product and Duality

If $\overline{a} = (a_1, a_2, \cdots, a_n)$ is an n-tuple from M_R and $\overline{b} = (b_1, b_2, \cdots, b_n)$ is an n-tuple from ${}_RN$ then by $\overline{a} \otimes \overline{b}$ we mean $\sum_{i=1}^{n} a_i \otimes b_i \in M \otimes_R N$.

Proposition 1.3.4. *Let $\overline{a}$ be a tuple from M_R and let $\overline{b}$ be a tuple from ${}_RN$. Then $\overline{a} \otimes \overline{b} = 0$ in $M \otimes_R N$ if, and only if, there are $\overline{c}$ from M_R and $\overline{d}$ from ${}_RN$ and matrices G, H such that $(\overline{a} \quad \overline{0}) = \overline{c}(G \quad H)$ and $\begin{pmatrix} G & H \end{pmatrix} \begin{pmatrix} \overline{b} \\ \overline{d} \end{pmatrix} = 0$.*

Proof. ($\Rightarrow$) If there are such matrices, then $\overline{a} = \overline{c}G$, $\overline{0} = \overline{c}H$ and $G\overline{b} + H\overline{d} = 0$. So $\overline{a} \otimes \overline{b} = \overline{c}G \otimes \overline{b} = \overline{c} \otimes G\overline{b} = \overline{c} \otimes -H\overline{d} = -\overline{c}H \otimes \overline{d} = \overline{0} \otimes \overline{d} = 0$.

($\Leftarrow$) Extend $\overline{b}$ to a (possibly infinite) generating sequence $\overline{b}\ \overline{b}'$ for N (we could, for example, make the tuple all the elements from N). So there is an exact sequence $0 \longrightarrow K \xrightarrow{j} R^{(I)} \xrightarrow{p} N \longrightarrow 0$ (where I is the set indexing $\overline{b}\ \overline{b}'$ and $p\overline{e} = \overline{b}$ and $p\overline{e}' = \overline{b}'$, where $\overline{e}\ \overline{e}'$ is the generating tuple, $(e_i)_{i \in I}$, from $R^{(I)}$).

Tensoring in M, because tensor product is right exact, gives the exact sequence $M \otimes_R K \xrightarrow{1_M \otimes j} M \otimes_R R^{(I)} \xrightarrow{1_M \otimes p} M \otimes_R N \longrightarrow 0$.

Since $\overline{a} \otimes \overline{b} = 0$ we have $\overline{a} \otimes \overline{e} \in ker(1_M \otimes p) = im(1_M \otimes j)$, say, $\overline{a} \otimes \overline{e} = \overline{c} \otimes j(\overline{k})$, where $\overline{k}$ is from K. Since $\overline{e}\ \overline{e}'$ generates $R^{(I)}$ there is a matrix $A = \begin{pmatrix} G & H \end{pmatrix}$ with only finitely many non-zero elements, such that $j(\overline{k}) = A \begin{pmatrix} \overline{e} \\ \overline{e}' \end{pmatrix}$ (here, by convention, our tuples from left modules are column vectors). So $\overline{a} \otimes \overline{e} = \overline{c} \otimes A \begin{pmatrix} \overline{e} \\ \overline{e}' \end{pmatrix} = \overline{c}A \otimes (\overline{e}\ \overline{e}') = \overline{c}A \otimes (\overline{e}\ \overline{e}')$ and so the isomorphism $M \otimes R^{(I)} \simeq M^{(I)}$ takes $\overline{a} \otimes (\overline{e})$, regarded as $(\overline{a}\ \overline{0}) \otimes (\overline{e}\ \overline{e}')$, to the same image as $\overline{c}A \otimes (\overline{e}\ \overline{e}')$, therefore $(\overline{a}\ \overline{0}) = \overline{c}A$.

The tuples $\overline{b}' = p\overline{e}'$ and $\overline{0}$ could be infinite, but A has only finitely many non-zero rows (because every element in $R^{(I)}$ may be taken as a finite sum of some scalar times an element from the basis) so, reducing A, $\overline{0}$ and $\overline{b}$ to suitable finite parts, the equations $0 = pj(\overline{k}) = A \begin{pmatrix} \overline{b} \\ p\overline{e} \end{pmatrix}$ and $(\overline{a}\ \overline{0}) = \overline{c}G$ give the desired solution. □

The next result, which will be used a lot in the next subsection, shows a good connection between pp conditions and tensor products. With this result we can characterize the solution set of $D\phi$ with the free realization of ϕ, as shown in Corollary 1.3.6.

Theorem 1.3.5 (Herzog's Criterion)**.** *Let $\overline{a}$ from M_R and $\overline{b}$ from ${}_RN$ be n-tuples. Then $\overline{a} \otimes \overline{b} = 0$ in $M \otimes_R N$ if, and only if, there is a pp condition ϕ (for right modules) such that $\overline{a} \in \phi(M)$ and $\overline{b} \in D\phi(N)$.*

Proof. ($\Rightarrow$) Suppose $\overline{a} \otimes \overline{b} = 0$ and choose G, H as in Proposition 1.3.4 and define ϕ as the condition $\exists \overline{y}\ (\overline{x} + \overline{y}G = 0\ \wedge\ \overline{y}H = 0)$. Then, by Proposition 1.3.4, $\overline{a} \in \phi(M)$. In matrix form this condition, ϕ, will be $\exists \overline{y}\ (\overline{x}\ \overline{y}) \begin{pmatrix} I & 0 \\ G & H \end{pmatrix} = 0$, which is the dual of the condition $\exists \overline{z} \begin{pmatrix} G & H \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = 0$ for left modules, that is, $D\phi$ is equivalent to this condition. By choice of G and H we get that $\overline{b} \in D\phi(N)$.

($\Leftarrow$) Suppose $\overline{a} \in \phi(M)$ and $\overline{b} \in D\phi(N)$. Suppose $D\phi$ is $\exists \overline{z} \begin{pmatrix} G & H \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = 0$, then ϕ is equivalent to $\exists \overline{y}\ (\overline{x}\ \overline{y}) \begin{pmatrix} I & 0 \\ G & H \end{pmatrix} = 0$. With these pp conditions we get that $\overline{a} \in \phi(M)$ and $\overline{b} \in D\phi(N)$ implies the existence of $\overline{c}$, $\overline{d}$, G and H satisfying the conditions of Proposition 1.3.4, which means $\overline{a} \otimes \overline{b} = 0$. □

Corollary 1.3.6. *If $(C, \overline{c})$ is a free realization of ϕ and if $\overline{l}$ is a tuple from ${}_RL$, then $\overline{c} \otimes \overline{l} = 0$ in $C \otimes_R L$ if, and only if, $\overline{l} \in D\phi(L)$.*

Proof. ($\Rightarrow$) By Theorem 1.3.5 if $\overline{c} \otimes \overline{l} = 0$ in $C \otimes_R L$ we get that there is a ψ such that $\overline{c} \in \psi(C)$ and $\overline{l} \in D\psi(L)$. Because $(C, \overline{c})$ is a free realization of ϕ we also get that $\phi \leq \psi$, which implies that $D\psi \leq D\phi$. Then we have that $\overline{l} \in D\psi(L) \leq D\phi(L)$, so $\overline{l} \in D\phi(L)$.

($\Leftarrow$) By Theorem 1.3.5 if $\overline{l} \in D\phi(L)$, because $\overline{c} \in \phi(C)$, we get that $\overline{c} \otimes \overline{l} = 0$ in $C \otimes_R L$. □

1.3.3 Character Module

Definition 1.3.7 (Dual module)**.** *Let R, S be two, not necessarily distinct, rings. Let M be a right R-module and $f : S \longrightarrow End(M)$ be any ring morphism. Regard M as a left S-module by the action $s.m = f(s)m$. Let ${}_SE$ be any injective S-module and define $M^{*^E} = Hom_S({}_SM, {}_SE)$, written just as M^* for short. The module M^* will be referred as dual of M, even though it depends on the choice of E. This has a natural left R-module structure given by $(rf).(m) = f(mr)$ for $f \in M^*$, $r \in R$, $m \in M$.*

Lemma 1.3.8. *Let $\psi(\overline{x})$ be a pp condition for right R-modules with the length of ψ being n. Then $\overline{f} \in (M^*)^n$ annihilates $\psi(M)$ if, and only if, $\overline{f} \in D\psi(M^*)$.*

Proof. ($\Leftarrow$) If $\overline{f} \in D\psi(M^*)$ and $\overline{a} \in \psi(M)$ then, by Theorem 1.3.5, $\overline{a} \otimes \overline{f} = 0$ in $M \otimes_R M^*$ so certainly the value $\overline{f}\overline{a} = \sum_i f_i a_i$, which is the image of $\overline{a} \otimes \overline{f}$ under the natural map $M \otimes_R M^* \to E$ taking $a \otimes f$ to fa, is zero. That is, $\overline{f} \in D\psi(M^*)$ annihilates $\psi(M)$.

($\Rightarrow$) Suppose ψ is the condition $\exists \overline{y}(\overline{x}A = \overline{y}B)$, where A is an $n \times m$ matrix and B is an $k \times m$ matrix. Then $D\psi$ is $\exists \overline{z}(\overline{x} = A\overline{z} \wedge B\overline{z} = 0)$. Suppose that $\overline{f}$ annihilates $\psi(M)$.

Consider the S-submodule $M^nA = \{\overline{c}A : \overline{c} \in M^n\}$ of M^m. Define $g' : M^nA \to E$ by $g'(\overline{c}A) = \overline{f}\overline{c}$. This is well defined since if $\overline{c}A = \overline{c}'A$, then, because $(\overline{c} - \overline{c}')A = 0 = \overline{0}B$, certainly $\overline{c} - \overline{c}' \in \psi(M)$, hence $\overline{f}\overline{c} = \overline{f}\overline{c}'$.

Next, consider $M^kB = \{\overline{c}B : \overline{c} \in M^k\} \leq M^m$. Define $g'' : M^kB \to E$ to be the zero map. Note that g' and g'' agree on the intersection of their domains since $M^nA \cap M^kB = \{\overline{a} \in M^m : \exists \overline{b} \in M^n\ \exists \overline{c} \in M^k\ (\overline{b}A = \overline{c}B = \overline{a})\} = \psi(M)A$, on which g' is zero. So $g' + g''$ is defined unambiguously on $M^nA + M^kB$.

By injectivity of E there is an extension of the S-linear map $g' + g''$ to a morphism $\overline{g}$, say, from M^m to E. We regard $\overline{g}$ as an m-tuple of elements of M^*.

By the definition of M^*, and it structure as an left R-module, for every $\overline{c} \in M^n$ we have $A\overline{g}.\overline{c} = \overline{g}(\overline{c}A) = g'(\overline{c}A) = \overline{f}\overline{c}$, hence $A\overline{g} = \overline{f}$. Also, for every $\overline{d} \in M^k$ we have $B\overline{g}.\overline{d} = \overline{g}(\overline{d}B) = g''(\overline{d}B) = 0$, so $B\overline{g} = 0$. Therefore $\overline{f} \in D\psi(M^*)$, as required.

□

Corollary 1.3.9. *If $\psi(M) \leq \phi(M)$ then $D\phi(M^*) \leq D\psi(M^*)$. In particular if $\phi \geq \psi$ is a pp-pair with $\phi(M) = \psi(M)$, then $D\psi(M^*) = D\phi(M^*)$.*

Proof. If $\overline{f} \in D\phi(M^*)$ then, by Theorem 1.3.5, $\overline{f}.\phi(M) = 0$ so $\overline{f}.\psi(M) = 0$ and, again by Theorem 1.3.5, $\overline{f} \in D\psi(M^*)$. □

Theorem 1.3.10. *Suppose M is a right R-module, that $S \longrightarrow End(M_R)$ is a ring morphism and that ${}_SE$ is injective. If, for every pp condition ψ, the S-module $M/\psi(M)$ embeds in a power of E (for example, if E is an injective cogenerator of S-mod, that is, an injective module such that every module embed in some power of E), then for every* *pp-pair $\phi \geq \psi$ (in n free variables, for any n) we have*

$$\phi(M) = \psi(M) \quad \textit{if, and only if,} \quad D\phi(M^*) = D\psi(M^*).$$

That is, for every n, $\phi \longleftrightarrow D\phi$ induces an anti-isomorphism between the lattice of subgroups of M^n pp-definable in M_R and the lattice of subgroups of $(M^*)^n$ pp definable in ${}_RM^*$.

Proof. Observe that, by the last corollary, we just need to prove that if $\phi > \psi$ is a pp-pair such that $\phi(M) > \psi(M)$ then $D\psi(M^*) > D\phi(M^*)$.

Suppose we have $\phi > \psi$ a pp-pair and $\phi(M) > \psi(M)$. Choose $\overline{a} \in \phi(M) \setminus \psi(M)$. By assumption, there is an S-linear map $f' : M^n/\phi(M) \to E$ with $f'(\overline{a} + \phi(M)) \neq 0$. Let $f : M^n \to E$ be $f = f' \circ \pi$, where π is the natural projection $M^n \to M^n/\phi(M)$, so $f\overline{a} \neq 0$.

Since $f.\psi(M) = 0$, Lemma 1.3.8 gives $\overline{f} \in D\psi(M^*)$ and, since $\overline{f}.\phi(M) \neq 0$, by Lemma 1.3.8, it is also the case that $\overline{f} \notin D\phi(M^*)$, so $D\psi(M) > D\phi(M)$. □

Example 1.3.11. *If $R = \mathbb{Z}$ and $E = \oplus_{p\ prime}\mathbb{Z}_{p^\infty}$ is the minimal cogenerator for $\mathbb{Z}$-modules, then the p-adic integers and the p-Prüfer group $\mathbb{Z}_{p^\infty}$ correspond to each other under this duality and the result above implies that their lattice of pp-definable subgroups are anti-isomorphic.*

2 Purity

2.1 Purity

In this chapter we will introduce the basic notions of purity and show some fundamental results. With these we will be able to define pure-injective modules, which play central role in Definable Subcategories and in the Ziegler Spectrum of a ring. The main result of this section is that a sequence is pure-exact if, and only if, it is the direct limit of split sequences.

2.1.1 Definition

Definition 2.1.1 (Pure submodule)**.** *Let M be a submodule of N. We say that M is a pure-submodule of N if for every pp condition ϕ we have that $\phi(M) = \phi(N) \cap M^n$, where n is the length of ϕ. If $f : N \to M$ is an embedding and $f(N)$ is pure in M we say that f is a pure embedding.*

The definition of purity was originally made by Prüfer (PRÜFER, 1923) in the context of abelian groups and then generalized by Cohn (COHN, 1959) for arbitrary rings. This notion might also be extended for more general categories (ADÁMEK et al., 1994).

Observation 2.1.2. *Since $\phi(M) \leq M^n \cap \phi(N)$, where n is the length of ϕ, we have the following equivalent definitions for the sentence: M is a pure submodule of N*

if, and only if, for every $\overline{a} \in M^n$ one has $pp^M(\overline{a}) = pp^N(\overline{a})$;

if, and only if, every finite system of R-linear equations with constants from M and a solution in N already has a solution in M.

(For the last one, observe that the solution of a set of R-linear equations just means that the constants, $\overline{a}$, satisfy some specific pp condition in N).

Lemma 2.1.3. *(a) If M is a direct summand of N, then the embedding of M into N is pure.*

(b) Any composition of pure embeddings is a pure embedding.

(c) Any direct limit of pure embeddings is a pure embedding.

(d) Any direct product of pure embeddings is a pure embedding.

Proof. Without loss of generality, we will work with embeddings as submodules, except in (c).

(a) Let $\overline{a} \in \phi(N) \cap M^n$ and $\pi_M : N \longrightarrow M$ be the natural projection. By Lemma 1.1.10 $\overline{a} \in \phi(M)$, which implies that $\phi(M) = \phi(N) \cap M^n$.

(b) Let $M_1 \leq M_2 \leq M_3$ be such that M_1 is pure in M_2 and M_2 is pure in M_3. If ϕ is a pp-condition we get that $\phi(M_1) = \phi(M_2) \cap M_1^n = \phi(M_3) \cap M_2^n \cap M_1^n = \phi(M_3) \cap M_1^n$, then, M_1 is pure on M_3.

(c) Take $((A_\lambda)_{\lambda\in\Lambda}, (g_{\lambda\mu} : A_\lambda \longrightarrow A_\mu)_{\lambda\leq\mu\in\Lambda})$ and $((B_\lambda)_{\lambda\in\Lambda}, (k_{\lambda\mu} : B_\lambda \longrightarrow B_\mu)_{\lambda\leq\mu\in\Lambda})$ two directed systems of modules, that is, for $\lambda \leq \mu \leq \nu$ we have $g_{\mu\nu}g_{\lambda\mu} = g_{\lambda\nu}$ and $k_{\mu\nu}k_{\lambda\mu} = k_{\lambda\nu}$. Take $\{i_\lambda : A_\lambda \to B_\lambda\}_{\lambda\in\Lambda}$ a directed system of pure embeddings with for each $\lambda \leq \mu$, $k_{\lambda\mu}i_\lambda = i_\mu g_{\lambda\mu}$, that is, the following diagram commute:

$$\begin{array}{ccc} A_\lambda & \xrightarrow{i_\lambda} & B_\lambda \\ \downarrow{\scriptstyle g_{\lambda\mu}} & & \downarrow{\scriptstyle k_{\lambda\mu}} \\ A_\mu & \xrightarrow{i_\mu} & B_\mu \end{array}$$

Let the direct limit of i_λ be $i : A \to B$. Since direct limits in Grothendieck categories are exact (Definition B.1.3) this is an embedding. We now need to prove it is a pure embedding.

Take $\overline{a}$ from A and suppose $i\overline{a} \in \phi(B)$. Say ϕ is $\exists\overline{y}\ (\overline{x}\ \overline{y})H = 0$ and suppose $\overline{b}$ from B is such that $(i\overline{a}\ \overline{b})H = 0$. Choose λ such that each element in the tuple $\overline{a}$ has preimage in A_λ and each element in $\overline{b}$ has preimage in B_λ. Each column of the matrix H gives an equation saying that a certain linear combination of the ia_j and b_k is zero: by the definition of the direct limit, that equation hold only if some preimage of this linear combination is zero. So, if λ is chosen "large enough", in a way we have our preimages satisfying these linear equations, we get $i_\lambda\overline{a}'$ and $\overline{b}'$ from B_λ in B_λ satisfying $(i_\lambda\overline{a}'\ \overline{b}')H = 0$. That is, $i_\lambda\overline{a}' \in \phi(B_\lambda)$ and hence, because i_λ is a pure embedding, $\overline{a}' \in \phi(A_\lambda)$ and, by Lemma 1.1.10, $g_{\lambda\infty}\overline{a}' = \overline{a} \in \phi(A)$, as required. (d) By Lemma 1.2.3 we get that if $\{N_i\}_{i\in I}$ and $\{M_i\}_{i\in I}$ are collections of modules such that N_i is pure in M_i then $\phi(\prod N_i) = \prod\phi(N_i) = \prod(\phi(M_i) \cap N_i^n) = \phi(\prod M_i) \cap \prod N_i^n$. □

Corollary 2.1.4. *A direct limit of split embeddings is a pure embedding.*

Proof. By (a) of Lemma 2.1.3, every split embedding is a pure embedding. The result then follows by (c). □

2.1.2 Pure-exact sequences

Definition 2.1.5 (Pure-exact sequence)**.** *Let* $0 \longrightarrow A \xrightarrow{k} B \xrightarrow{\pi} C \longrightarrow 0$ *be a short exact sequence. We say that this sequence is pure-exact if the embedding from A to B is pure.*

From this definition and Corollary 2.1.4 one can easily see that the direct limit of split exact sequences is always pure. In fact the converse is also true, as will be shown at Proposition 2.1.13, but to show this first we need to define some properties of purity.

We defined our pure-exact sequence using the embedding. We can also define it by its epimorphism, as it will be shown below.

Definition 2.1.6 (Pure-epimorphism)**.** *We say that an epimorphism $f : B \longrightarrow C$ is a pure-epimorphism if $Ker(f)$ is a pure submodule of B.*

Observe that if $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$ is an exact sequence and π is a pure-epimorphism we have that $Ker(\pi) \simeq A$ is pure in B, hence, this sequence is also pure-exact.

Proposition 2.1.7. *A morphism $\pi : B \longrightarrow C$ is a pure-epimorphism if, and only if, for every pp condition ϕ and every $\overline{c} \in \phi(C)$ there is $\overline{b} \in \phi(B)$ such that $\pi\overline{b} = \overline{c}$.*

Proof. ($\Rightarrow$) First assume that $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$ is pure-exact, where $A = ker(\pi)$. Suppose that $\overline{c} \in \phi(C)$, where ϕ is the pp condition $\exists \overline{y}\ (\overline{x}\ \overline{y})H = 0$, say $\overline{d}$ is from C with $(\overline{c},\ \overline{d})H = 0$. Choose any inverse images $\overline{e}$, $\overline{b}$ of $\overline{c}$, $\overline{d}$ respectively. Then $(\overline{e},\ \overline{b})H = \overline{a}$, for some $\overline{a}$ from $A = ker(\pi)$. Defining ψ as the pp-condition $\exists \overline{x},\ \overline{y}\ ((\overline{x},\ \overline{y})H = \overline{x}')$, it is easy to see that $\overline{a} \in \psi(B)$ which implies, by purity of A, that there are $\overline{a}$, $\overline{a}'$ from A such that $(\overline{a}'\ \overline{a}'')H = \overline{a}$. Then $(\overline{e} - \overline{a}',\ \overline{b} - \overline{a}'')H = 0$ so $\overline{e} - \overline{a}'$ is mapped to $\overline{c}$ and satisfy ϕ, as required.

($\Leftarrow$) For the converse assume that solutions lift and take ϕ a pp condition and $\overline{a}$ in $A = ker(\pi)$ such that $\phi(\overline{a}) \in \phi(B)$. Supposing that ϕ is $\exists \overline{y}\ (\overline{x}\ \overline{y})H = 0$ we have $(\overline{a}\ \overline{b})H = 0$ for some $\overline{b}$ from B. Then $\pi\overline{b}$ satisfies the condition $(\overline{0}\ \overline{x})H = 0$ in C, so, by assumption, there is $\overline{b}'$ from B with $\pi\overline{b} = \pi\overline{b}'$ and $(\overline{0}\ \overline{b}')H = 0$. Then $\overline{b} - \overline{b}'$ is a tuple from $A = ker(\pi)$ (because this sequence is exact) and $(\overline{a},\ \overline{b} - \overline{b}')H = 0$, showing that $\overline{a} \in \phi(A)$. The sequence is pure-exact, as required. $\square$

Corollary 2.1.8. *A morphism $\pi : B \to C$ is a pure epimorphism if, and only if, for every finitely presented module D, every morphism $g : D \to C$ lifts to a morphism $g' : D \to B$ with $g = \pi g'$.*

$$\begin{tikzcd} B \arrow[r, "\pi"] & C \\ D \arrow[u, dashed, "g'"] \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. ($\Leftarrow$) Let $(C_\phi, \overline{c}_\phi)$ be a free realization of a pp condition ϕ. If $\overline{c} \in \phi(C)$ then there exists a morphism $f : C_\phi \to C$ which takes $\overline{c_\phi}$ to $\overline{c}$ (Proposition 1.2.6) and, by our assumption, it factors through B. Hence $g'\overline{c_\phi} \in \phi(B)$ (Lemma 1.1.10) and $\pi g'\overline{c_\phi} = \overline{c}$ so, by Proposition 2.1.7, we have that π is a pure-epimorphism.

($\Rightarrow$) Let $g : D \to C$ be any morphism, where D is a finitely presented module. Let

$\overline{d}$ be the generating tuple of D and $\langle\phi\rangle = pp^D(\overline{d})$. With this we have that $\overline{c} = g\overline{d} \in \phi(C)$ which, by assumption, implies that there is $\overline{b} \in \phi(B)$ such that $\pi\overline{b} = \overline{c}$. Applying Proposition 1.2.6 this implies that there is an $g' : D \to B$ such that $g'\overline{d} = \overline{b}$, which is our desired morphism. □

Corollary 2.1.9. *If* $0 \longrightarrow A \xrightarrow{k} B \xrightarrow{\pi} C \longrightarrow 0$ *is pure-exact and* C *is finitely presented, then this sequence is split.*

Proof. Just apply the last corollary for $D = C$ and $g = id_C$. □

One well-known result is that the pushout of an embedding is an embedding and the pullback of an epimorphism is an epimorphism. We will see that pure-embeddings (Proposition 2.1.10) and pure-epimorphisms share this property (Proposition 2.1.11).

Proposition 2.1.10. *If* $f : M \to N$ *is a pure embedding and* $g : M \to N'$ *is any morphism, then in the pushout diagram shown* f' *is a pure embedding.*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ N' & \xrightarrow{f'} & N'' \end{array}$$

Proof. We already have that a pushout of an embedding is an embedding, so if f is a pure embedding then f' is an embedding.

Suppose ϕ is the pp condition $\exists\overline{y}\ (\overline{x}G = \overline{y}H)$, and take $\overline{a}$ from N' such that $f'\overline{a} \in \phi(N'')$. Say $\overline{b}''$ from N'' is such that $f\overline{a}'G = \overline{b}''H$. In modules, the pushout $N'' = (N' \oplus N)/\{(gc, -fc) :\ c \in M\}$ so, of $\overline{b}''$ is from N'' then it is the image of, say, $(\overline{b}, \overline{b}')$ from $N \oplus N'$. With this we get that, in $N \oplus N'$, $(\overline{a}', \overline{0})G = (\overline{b}, \overline{b}')H + (g\overline{c}, -f\overline{c})$, for some $\overline{c}$ from M. Projecting this equation on N we get that $\overline{0} = \overline{b}H - f\overline{c}$ which, by the purity of f, also gives us that there is an $\overline{c}'$ from M such that $\overline{c} = \overline{c}'H$. The projection on N' gives us that $\overline{a}' = \overline{b}'H + g\overline{c} = (\overline{b}' + g\overline{c}')H$, so $\overline{a}' \in \phi(N')$ as required. □

Proposition 2.1.11. *If* $\pi : B \to C$ *is a pure epimorphism and* $g : D \to C$ *is any morphism, then in the pullback diagram shown* π' *is a pure epimorphism.*

$$\begin{array}{ccc} P & \xrightarrow{\pi'} & D \\ \downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\pi} & C \end{array}$$

Proof. We will use the criterion from Proposition 2.1.7 to prove this fact. Suppose $\overline{d} \in \phi(D)$ and that $(C_\phi, \overline{c}_\phi)$ is a free realization of ϕ. By Proposition 1.2.6 there is $f : C_\phi \longrightarrow D$ such that $f\overline{c_\phi} = \overline{d}$. Suppose C_ϕ is generated by $\overline{a}$ with generating relations $\overline{a}H = 0$ and, say, $\overline{c_\phi} = \overline{a}G$.

Observe that $gf\overline{a}.H = 0$ in C. So, by hypothesis and Proposition 2.1.7, there is $\overline{b}$ from B such that $\overline{b}H = 0$ and $\pi\overline{b} = gf\overline{a}$. Denote h to be the morphism (Proposition 1.2.6) from C_ϕ to B which takes $\overline{a}$ to $\overline{b}$ (because $pp^{C_\phi}(\overline{a}) = \langle \overline{x}H = 0 \rangle$). Then $\pi h = gf$, so, by the pullback property, there is $k : C_\phi \longrightarrow P$ with, in particular, $\pi' k = f$, hence with $\pi'(k\overline{a}G) = \overline{d}$. Also $k\overline{a}G = k\overline{c} \in \phi(P)$, as required.

$$\begin{tikzcd}
C_\phi \arrow[drr, "f"] \arrow[dr, dotted] \arrow[ddr, "h"'] & & \\
& P \arrow[r, "\pi'"] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, "\pi"] & C
\end{tikzcd}$$

□

Lemma 2.1.12. *Every module is the direct limit of finitely presented modules.*

Proof. Let M be a module, $\overline{a}$ any finite tuple from M and H a matrix with entries in R such that $\overline{a}H = 0$ (here we are saying that our n-tuple $\overline{a}$ satisfies all the linear equations defined by the columns of the matrix H). Our directed set will be the set $\Lambda = \{(\overline{a}, H);\ \overline{a} = (a_1, a_2, \cdots, a_n) \subset M,\ n \in \mathbb{N},\ \overline{a}H = 0,\ H \in R^{n\times m}\}$ with order $(\overline{a}, H) \leq (\overline{b}, H')$ if $\overline{b} = (\overline{a}\ \overline{a}')$ and $H' = \begin{pmatrix} H & A \\ 0 & B \end{pmatrix}$. For each $\lambda = (\overline{a}, H)$ we will define M_λ as the finitely presented module generated by a tuple with the same length as $\overline{a}$ and with relations defined by H and, if $\lambda \leq \mu$, we define $f_{\lambda\mu} : M_\lambda \to M_\mu$ as the embedding which takes the generating n-tuple of M_λ to the first n-coordinates of the generating tuple of M_μ.

Let $M' = \varinjlim_\lambda M_\lambda$. By Lemma 1.1.10, we get that, for each $\lambda = (\overline{a}, H)$, there is an embedding $g_\lambda : M_\lambda \to M$ which takes the generating tuple of M_λ to $\overline{a}$ and such that if $\lambda \leq \mu$ we have $g_\lambda = g_\mu f_{\lambda\mu}$. By the direct limit property, there exists a morphism $g : M' \to M$. Because all g_λ are embeddings, g is an embedding. This morphism will be surjective because, for every $a \in M$, the generating element b of M_λ, where $\lambda = (a, 1) \in \Lambda$, satisfy $g_\lambda b = gf_\lambda b = a$. □

Proposition 2.1.13. *Every pure-exact sequence is a direct limit of split exact sequences of finitely presented modules. Hence an exact sequence is pure-exact if, and only if, it is a direct limit of split exact sequences.*

Proof. By Corollary 2.1.4 we get that the direct limit of split exact sequences is a pure-exact sequence. Now we need to prove the other direction.

Suppose that $0 \longrightarrow A \xrightarrow{k} B \xrightarrow{\pi} C \longrightarrow 0$ is pure-exact and that $((C_i)_i,\ (f_{ij} : C_i \to C_j)_{ij})$ is a directed system of finitely presented modules, with direct limit $(C,\ (f_i : C_i \to C)_i)$.

For each i set $B_i = \{(b, c) \in B \oplus C_i :\ \pi b = f_i c\}$, that is, the pullback of the morphisms $\pi : B \to C$ and $f_i : C_i \to C$. Clearly $0 \longrightarrow A_i \longrightarrow B_i \xrightarrow{\pi_i} C_i \longrightarrow 0$ is exact, where $A_i = ker(\pi_i)$. The morphisms $f_{ij} : C_i \to C_j$ induce a morphism between the corresponding pullback sequences by defining $g_{ij} : B_i \to B_j$ by $g_{ij}(b, c) = (b, f_{ij}c) \in B_j$ and taking $A_i \to A_j$ to be the restriction.

$$\begin{tikzcd}
0 \arrow[rr] & & A_i \arrow[rr] \arrow[dr] \arrow[dd] & & B_i \arrow[rr] \arrow[dr] \arrow[dd] & & C_i \arrow[rr] \arrow[dr] \arrow[dd] & & 0 & \\
& 0 \arrow[rr] & & A_j \arrow[rr] \arrow[dl] & & B_j \arrow[rr] \arrow[dl] & & C_j \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[rr] & & A \arrow[rr] & & B \arrow[rr] & & C \arrow[rr] & & 0 &
\end{tikzcd}$$

These sequences are all split because C_i is finitely presented ad, by Proposition 2.1.11, they are pure-exact and, by Corollary 2.1.9, the C_i are pure-projective. It remains to prove that the original sequence is the direct limit of these sequences.

Suppose $0 \longrightarrow A' \xrightarrow{k'} B' \xrightarrow{\pi'} C \longrightarrow 0$ is the direct limit of these exact sequences. Because we have functions from each exact sequence i to the original one, we have morphisms $A' \to A$ and $g' : B' \to B$ such that the diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, "g'"] & C \arrow[r] \arrow[d, "id_C"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Because the functions from B_i to B are embeddings, so is g'. If $b \in B$ and $\pi b = c \in C$, there is an $c_i \in C_i$ such that $f_i c_i = c = \pi b$, that is, $(b, c_i) \in B_i$ and $g' g_{i\infty} b = b$ (where $(B',\ (g_{i\infty} : B_i \to B')_i)$ is the direct limit of $((B_i)_i,\ (g_{ij})_{ij})$), which implies that g' is surjective. Then B is isomorphic to B' and, by the 5-Lemma, A is isomorphic to A'. $\square$

A consequence of Proposition 2.1.13 is that one can see purity as a "weaker" version of direct sum. We can also "weaken" the notion of projective and injective, as we will see in the next section.

We can simplify the process of checking purity by looking just at the pp conditions with one free variable, as will be seen below.

Proposition 2.1.14. *An inclusion, $M \leq N$, of modules is pure if, and only if, $\phi(M) = M \cap \phi(N)$ for every pp condition $\phi(x)$ in one free variable.*

Proof. We will prove using induction on the number, n, of free variables. Suppose that $A \leq B$ is an inclusion such that for every pp condition ψ, with $m \leq n$ free variables, we have $\psi(A) = \psi(B) \cap A^m$. Consider a pp condition $\phi(x,\ \overline{y})$ with $n + 1$ free variables and let $a,\ \overline{c}$ from A be such that $(a,\ \overline{c}) \in \phi(B)$.

Suppose that $\phi(x,\ \overline{y})$ is $\exists\overline{z}\ \theta(x,\ \overline{y},\ \overline{z})$, with θ quantifier-free, and choose $\overline{b}$ from B such that $(a,\ \overline{c},\ \overline{b}) \in \theta(B)$. Let $\phi'(\overline{y})$ be the condition $\exists x\ \phi(x,\ \overline{y})$. Then, $\overline{c} \in \phi'(B)$ and hence, by induction, $\overline{c} \in \phi'(A)$, say a', $\overline{d}$ are from A such that $(a',\ \overline{c},\ \overline{d}) \in \theta(A)$. Therefore $(a-a',\ \overline{0},\ \overline{b}-\overline{d}) \in \theta(B)$ so, if $\psi(x)$ is the condition $\exists\overline{z}\ \theta(x,\ \overline{0},\ \overline{z})$, then $a-a' \in \psi(B)\cap A = \psi(A)$, by assumption. So, there is $\overline{e}$ from A such that $(a-a',\ \overline{0},\ \overline{e}) \in \theta(A)$. Combine with $(a',\ \overline{c},\ \overline{d}) \in \theta(A)$ to get $(a,\ \overline{c},\ \overline{e}+\overline{d}) \in \theta(A)$. In particular, $(a,\ \overline{c}) \in \phi(A)$, as required. □

One can also define a dual for a left R-module, as in Subsection 1.3.3, by defining a morphism of $S \longrightarrow End({}_RL)$ (to give L a left S-module structure) and for E an injective left S-module defining the dual as $L^* = Hom_S({}_SL,\ {}_SE)$, with right R-module structure given by $fr.l = f(rl)$, where $r \in R$, $f \in L^*$ and $l \in L$. With this we can also define a bidual, M^{**}, for a module M (which will be a right R-module). The last proposition makes it easy to prove that there is a pure embedding, when choosing a suitable dualising, $M \longrightarrow M^{**}$, as seen below:

Corollary 2.1.15. *Let M be any module, S, E as in Theorem 1.3.10, $M^* = Hom_S({}_SM,{}_SE)$ and $M^{**} = Hom_S({}_SM^*,{}_SE^{**})$ (observe that we must choose S in a way we can give a left S-module structure for both M and M^*). Define $i : M \longrightarrow M^{**}$ by $ia.f = f(a)$, for $a \in M$ and $f \in M^*$. This embedding is a pure embedding.*

Proof. In the criterion of Proposition 2.1.14 we need just to check for pp conditions with one free variable ψ. Suppose $a \in \psi(M^{**})$. Then, by Lemma 1.3.8 applied to M^* and since $D^2\psi \equiv \psi$, $fa = 0$ for every $f \in D\psi(M^*)$. If $a \notin \psi(M)$ there would be, by the proof of Theorem 1.3.10, some $f \in M^*$ annihilating $\psi(M)$ and hence, by Lemma 1.3.8, with $f \in D\psi(M^*)$, but with $fa \neq 0$, a contradiction. □

Lemma 2.1.16. *For any collection $(M_i)_{i\in I}$ of modules the canonical embedding of the direct sum , $\oplus_{i\in I}M_i$, into the direct product, $\prod_{i\in I} M_i$, is pure.*

Proof. For each finite subset I' of I the embedding $\oplus_{i\in I'}M_i$ on $\prod_{i\in I} M_i$ is split. The embedding of $\oplus_{i\in I}M_i$ in $\prod_{i\in I} M_i$ is the direct limit of these embeddings which, by Lemma 2.1.3 (d), is a pure embedding. □

Example 2.1.17. *Let $R = \mathbb{Z}$, $M = \oplus_{i\in\mathbb{N}}\mathbb{Z}/\mathbb{Z}2^n$ and $N = \prod_{i\in\mathbb{N}} \mathbb{Z}/\mathbb{Z}2^n$. Suppose that this embedding is split by $\pi : M' \longrightarrow M$. Set $a = (1_2, 2_{2^2}, 2^2_{2^3}, \cdots, 2^{n-1}_{2^n}, 2^n_{2^{n+1}}, \cdots) \in M'$ and, for each n, set $a_n = (1_2, 2_{2^2}, \cdots, 2^{n-1}_{2^n}, 0, \cdots, 0, \cdots) \in M$. For each n, $2^n|a - a_n$, from which $2^n|\pi(a) - a_n$ follows. But, for some n, the n-th coordinate of $\pi(a)$ is 0 (because this has to be a finite sum in M) hence $2^n|2^{n-1}$, a contradiction.*

This is an example of an embedding which is pure but is not split.

2.2 Pure-injective modules

Pure-injective modules are central objects to define the Ziegler Spectrum and can also be used to characterize definable subcategories. We will describe some important properties of these modules that will be necessary in the next two chapters. The main focus of this section are showing some properties that pure-injective modules have which are similar to injective modules and giving some relation between pure-injective indecomposable modules and pp-types.

Definition 2.2.1 (Pure-injective modules)**.** *A module A is pure-injective if for every pure embedding $f : A \to B$ we have that $f(A)$ is a direct summand of B.*

Equivalently, we say that A is pure-injective if, and only if, every pure-exact sequence $0 \longrightarrow A \xrightarrow{k} B \xrightarrow{\pi} C \longrightarrow 0$ *is split.*

Observe that injective modules are pure-injective, by definition. One can also define pure-projectives as being the modules M such that every pure-epimorphism $\pi : N \longrightarrow M$ is split. By Corollary 2.1.9 we get that every finitely presented module is pure-projective and, by (PREST, 2009, Corollary 2.1.26), one can see that the pure-projective modules are exactly the direct summands of direct sums of finitely presented modules.

2.2.1 Pp-types with parameters

We have already defined pp conditions as finite sets of homogeneous equations. Here we will extend this definition by allowing parameters from a module, extend the notion of pp-types to pp-types with parameters and, with these new definitions, we will in the next subsection define algebraically compact modules which, by Theorem 2.2.13, are exactly the pure-injective modules.

Definition 2.2.2 (Pp condition with parameters)**.** *Let $\psi(\overline{x})$ be a condition of the form $\exists \overline{y}\ \overline{z}\ (\overline{x}\ \overline{y}\ \overline{z})G = 0$, M be any right R- module, and $\overline{a}$ from M with same length as $\overline{z}$. We define $\exists \overline{y}\ (\overline{x}\ \overline{y}\ \overline{a})G = 0$, the inhomogeneous system of linear equation, as a pp condition with parameters. We use the notation $\psi(\overline{x},\ \overline{a})$ to display the free variables and the parameters.*

We write $\psi(M,\ \overline{a}) = \{\overline{c} \in M^n;\ \exists\ \overline{b} \in M^m\ (\overline{c}\ \overline{b}\ \overline{a})G = 0\}$ as the solution set of a pp condition with parameters. If $i : M \longrightarrow N$ is any embedding, we can also define this as a pp condition with parameters in N (we usually want the case that i is a pure-embedding).

Observation 2.2.3. *(a) Suppose $\phi(\overline{x},\ \overline{a})$ is a pp condition with parameters. Then $\phi(M,\ \overline{a})$ is either empty or a coset of the pp definable group $\phi(M,\ \overline{0})$. Every coset of a pp definable subgroup is definable by a pp condition with parameters for, given a pp condition $\overline{x}$ and tuple $\overline{a}$, the coset $\overline{a} + \phi(M)$ is defined by the condition $\exists \overline{z}(\overline{x} = \overline{z} + \overline{a} \wedge \phi(\overline{z}))$ which is a pp condition with parameters;*

(b) *If $\phi(\overline{x},\ \overline{a})$ and $\psi(\overline{x},\ \overline{b})$ are pp conditions with the same number of free variables and M is any module, then $\phi(M,\ \overline{a}) \cap \psi(M,\ \overline{b})$ is either empty or a coset of $\phi(M,\ \overline{0}) \cap \psi(M,\ \overline{0})$.*

Definition 2.2.4 (Pp-type with parameters)**.** *Let M be any module. A set, p, of pp conditions with n free variables and with parameters from M is a pp-type with parameters from M, also referred as pp-type over M, if every finite subset has a solution from M. In this case we say that p is finitely satisfied (finitely solvable) in M.*

We also say pp-n-type with parameters from M when we want to specify the number of free variables of the pp conditions inside it.

Definition 2.2.5 (Solution set of a pp-type with parameters)**.** *Let p be a pp-type with parameters from M. A solution for p is a module N, such that $i : M \longrightarrow N$ is a pure-embedding, and a tuple $\overline{a}$ from N such that $\overline{a}$ satisfies all the conditions in p.*

We write $p(N) = \cap\{\phi(N,\ \overline{b});\ \phi(\overline{x},\ \overline{b}) \in p\}$ for the set of all solutions of p in N.

Observation 2.2.6. *Even though the definition of pp-types with parameters seems different from the original one, because the solution set will remain unchanged under conjunction (if $\phi_1, \phi_2, \cdots, \phi_k \in p$ then $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k \in p$) and implication ($\psi \in p$, $\psi \leq \phi$ implies $\phi \in p$), we can treat p as a filter in the partially ordered set of pp conditions with parameters from M, just like before.*

Theorem 2.2.7. *Let p be a pp-type with parameters from the module M. Then there is a pure embedding of M into a module, $M^\star$, which contains a solution for p. The module $M^\star$ may be taken such that if $\phi \geq \psi$ is a pp-pair with $\phi(M) = \psi(M)$, then $\phi(M^\star) = \psi(M^\star)$.*

Proof. Without loss of generality, as observed in Observation 2.2.6, p is closed under conjunction and implication. Let I be the set of finite subsets of p and, for each such subset, $S = \{\phi_1, \phi_2, \cdots, \phi_k\}$, choose a solution, $\overline{a}_S$ say, of $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$ in M; this solution exists because p is finitely satisfied in M. Let $\mathcal{F}$ be a filter on I (Definition A.1.1) generated by the sets $\langle\{\phi\}\rangle = \{S \in I;\ \{\phi\} \subset S\}$ as ϕ ranges over p. Let $M^\star = M^I/\mathcal{F}$ (Definition A.1.2) be the corresponding reduced product and let $\overline{a} = (\overline{a}_S)_S/\mathcal{F}$ be the tuple of $M^\star$ made from the tuples $\overline{a}_S$. If ϕ/ψ is a pp-pair such that $\phi(M) = \psi(M)$ then, by Proposition A.2.3, $\overline{a} \in \phi(M^\star)$ if, and only if, $\{S \in I;\ \overline{a}_S \in \phi(M)\} = \{S \in I;\ \overline{a}_S \in \psi(M)\} \in \mathcal{F}$ if, and only if, $\overline{a} \in \psi(M^\star)$, that is, $\phi(M^\star) = \psi(M^\star)$.

By Corollary A.2.4, the diagonal embedding of M in $M^\star$ is pure. With this embedding we are allowed to regard our pp-type, p, as a pp-type with parameters from $M^\star$. Observe that if $\phi \in p$, for each $S \in \langle\{\phi\}\rangle$ we have $\overline{a}_S \in \phi(M)$ and so, by Proposition A.2.3, $\overline{a} \in \phi(M^\star)$, that is, $\overline{a} \in p(M^\star)$. $\square$

2.2.2 Algebraically compact modules

Definition 2.2.8 (Algebraically compact modules)**.** *A module M is algebraically compact if every pp-n-type with parameters from M (that is, every pp-type which is finitely solvable in M) has a solution from M, for all $n \in \mathbb{N}$.*

By (PREST, 2009, Lemma 4.2.1) one could only check the pp-1-types. The next result shows an important property of algebraically compact modules.

One can see that applying Theorem 2.2.7 transfinitely one can get an algebraically compact module with the property of having the same closed pp-pairs as the original module.

Proposition 2.2.9. *(PREST, 2009, Proposition 4.2.2) A module M is algebraically compact if, and only if, every system of equations, with possibly infinitely many variables, with parameters from M and which is finitely solvable in M, is solvable in M, that is, if there are $(x_\lambda)_\lambda$ variables then exist $(a_\lambda)_\lambda$ from M such that, when replacing the variables by these elements, all system of equations are satisfied.*

2.2.3 Pure-injective modules

Proposition 2.2.10. *A module N is pure-injective if, and only if, given any pure-embedding $f : A \longrightarrow B$ in Mod-R, every morphism $g : A \longrightarrow N$ lifts through f, that is, exists $h : B \longrightarrow N$ such that $hf = g$.*

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "g"] & B \arrow[dl, dashed, "h"] \\ N & \end{tikzcd}$$

Proof. ($\Rightarrow$) Given $f : A \longrightarrow B$ a pure-embedding and $g : A \longrightarrow N$, form the pushout shown:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \\ N \arrow[r, "f'"] & M \end{tikzcd}$$

By Proposition 2.1.10, f' is a pure-embedding, so it is split, via $k : M \longrightarrow N$ say. Set $h = kg'$ to obtain a map with $hf = g$, as required.

($\Leftarrow$) Let $A = N$ and $g = 1_N$, as in the proposition. With this we get that this pure-embedding splits.

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "1_N"] & A \arrow[dl, dashed] \\ N & \end{tikzcd}$$

$\square$

Direct products and direct summands of injective modules are injective modules. By the next lemma we have that pure-injectives share this property.

Lemma 2.2.11. *Any direct product of pure-injective modules is pure-injective. Any direct summand of a pure-injective is a pure-injective.*

Proof. (Product is pure-injective) Let $(N_i)_i$ be a collection of pure-injective modules, let $f : A \longrightarrow B$ be a pure-embedding, $g : A \longrightarrow \prod_i N_i$ any morphism and $\pi_j : \prod_i N_i \longrightarrow N_j$ being the projections. Because each N_i is pure-injective the morphism $\pi_i \circ g$ factors through f via h_i. By the direct product property, there is some $h : B \longrightarrow \prod_i N_i$ such that $\pi_i \circ h = h_i$, for all i. Because $\pi_i \circ g = h_i \circ f = \pi_i \circ h \circ f$, and π_i is an epimorphisms, we get our desired $g = h \circ f$.

$$\begin{tikzcd}
 & B \arrow[d, dashed, "h"] \arrow[dr, "h_i"] & \\
A \arrow[ur, hook, "f"] \arrow[r, "g"] & \prod_i N_i \arrow[r, "\pi_i"] & N_i
\end{tikzcd}$$

(Direct summand is pure-injective) Let $N = N_1 \oplus N_2$ be a pure-injective module,, $f : A \longrightarrow B$ a pure embedding, $g_i : A \longrightarrow N_i$ any morphism, $i_i : N_i \longrightarrow N$ be their inclusions and $\pi_i : N \longrightarrow N_i$ their projections. Because N is pure-injective and $i_i \circ g_i$ is any morphism from A to N then there exists some $h : B \longrightarrow N$ such that $i_i \circ g = h \circ f$. With this we also get that $\pi_i \circ h \circ f = \pi_i \circ i_i \circ g = g$, making $\pi_i \circ h$ our desired morphism from B to N_i.

$$\begin{tikzcd}
 & B \arrow[drr, "h"] & & \\
A \arrow[ur, hook, "f"] \arrow[r, "g_i"] & N_i \arrow[rr, "i_i"] & & N_1 \oplus N_2 \arrow[ll, bend left, "\pi_i"]
\end{tikzcd}$$

□

Dually we also have that arbitrary direct sums and direct summands of pure-projectives are pure-projectives.

Example 2.2.12. *An infinite sum of pure-injective modules need not be pure-injective. Let $R = \mathbb{Z}$, $M = \oplus_n \mathbb{Z}/\mathbb{Z}2^n$. This module is not pure-injective because the embedding of M into $M' = \prod_n \mathbb{Z}/\mathbb{Z}2^n$ is a non-split pure-embedding (Example 2.1.17).*

Theorem 2.2.13. *A module M is pure-injective if, and only if, it is algebraically compact.*

Proof. ($\Rightarrow$) Suppose N is pure-injective and that p is a pp-type with parameters from N. By Theorem 2.2.7 there is a pure extension, $N^\star$, of N and $\overline{a}$ from $N^\star$ satisfying every condition in p. Since N is pure-injective this pure-embedding is split. Projecting $\overline{a}$ to N gives a solution to p from N so, by definition, N is algebraically compact.

($\Leftarrow$) For this proof we will use the characterization of pure-injectivity given by

Proposition 2.2.10. Suppose N is algebraically compact, let $f : A \longrightarrow B$ be a pure-embedding and let $g : A \longrightarrow N$ be any morphism. For each element $b \in B \setminus A$ introduce a variable x_b. For each relation of the form $\sum b_i r_i = a$ that holds in B, where the $b_i \in B \setminus A$, the $r_i \in R$ and the $a \in A$, form the equation (with parameters from N) $\sum_i x_{b_i} r_i = ga$. Let Θ be the set of all such equations (a set of equations with parameters from N).

A solution to Θ in N will allow us to lift the morphism g through f: if $c_b \in N$ is the value assigned by the solution to x_b, then the mapping $b(\in B \setminus A)$ to c_b will give, by construction of Θ, an R-linear map extending g.

Since N is algebraically compact there will be a solution to Θ provided that every finite subset has a solution (Proposition 2.2.9). Let $\sum_{i=1}^{n} x_{b_i} r_{ij} = ga_j$ $(j = 1, \cdots, m)$ be a finite subset of Θ (we can have a common set of variables by allowing 0 as a coefficient), so the relations $\sum b_i r_{ij} = a_j$ $(j = 1, \cdots, m)$ hold in B. That is,

$$B \models \exists y_1, \cdots, y_n \bigwedge_{j=1}^{n} \left(\sum_i y_i r_{ij} = a_j \right)$$

so, by purity of f, there are $a'_1, \cdots, a'_n$ in A with $\sum_i a'_i r_{ij} = a_j$ $(j = 1, \cdots, m)$. Then $\sum_i g a'_i r_{ij} = g a_j$ $(j = 1, \cdots, m)$ in N and Θ is indeed finite solvable in N, as required. $\square$

As a consequence of this result we get the following lemma:

Lemma 2.2.14. *If $f : R \longrightarrow S$ is any morphism of rings and the S-submodule N is pure-injective, then N, regarded as an R-module, is pure injective.*

Proof. This follows from Theorem 2.2.13 and the definition of algebraically compact modules. Because our module structure is defined as $m.r = m(f(s))$ we get that a pp-n-type which is finitely solvable as an R-module is finitely solvable as an S-module and hence, by hypothesis, has a solution as an S-module which is also a solution as an R-module, as required. $\square$

Observe that with this lemma, if we have a ring R, P a multiplicative subset of R and $P^{-1}R$ the localization of R on P. With this result we have that the pure-injective modules of $P^{-1}R$ are also pure-injective on R.

For every module there exists a morphism $i : M \longrightarrow E(M)$ which is called the injective hull of M. Two central things which we will prove in this chapter are: the existence of a pure-injective hull for every module and the uniqueness of this hull (up to isomorphism). We know that an embedding of a module into an injective module is an injective hull if, and only if, this embedding is essential (Proposition B.1.2). The next definitions and results will also show that some similar results hold for pure-injective modules and for the pure-injective hull. To show many of these results we will use the embedding of Mod-R into (R-mod, *Ab*) (where R-mod is the full subcategory of all finitely presented modules, *Ab* is the category of abelian groups and (R-mod, *Ab*) is the category of functors from R-mod to Ab) given by $M \mapsto M \otimes \_$ which, as stated in Appendix B, can be used to extract important results about the pure-injective modules in Mod-R.

Definition 2.2.15 (Pure-injective hull). *A pure-injective hull for a module M is a pure-embedding $i : M \longrightarrow N$ with N pure injective and N minimal, in the sense that there is no factorization of this map through any direct summand of N. We denote $H(M)$ for the pure-injective hull of M.*

Definition 2.2.16 (Pure-essential). *A pure-embedding $j : M \longrightarrow N$ is said to be pure-essential if, whenever $f : N \longrightarrow N'$ is a morphism such that fj is a pure embedding, then f must be a pure-embedding.*

Lemma 2.2.17. *An embedding $j : M \longrightarrow N$, of right R-modules, is pure essential if, and only if, the morphism $(j \otimes \_) : (M \otimes \_) \longrightarrow (N \otimes \_)$, of functors from the category $(mod\text{-}R, Ab)$, is an essential embedding.*

Proof. ($\Leftarrow$) Suppose that the embedding $j \otimes \_$ of functors is essential and let $f : N \longrightarrow N'$ be a morphism such that fj is a pure-embedding hence, by Theorem B.2.2, such that $(f \otimes \_)(j \otimes \_)$ is an embedding. Then, by assumption, $f \otimes \_$ is an embedding so, again by Theorem B.2.2, f is a pure embedding.

($\Rightarrow$) For the converse, suppose that j is pure essential and suppose that $\alpha : (N \otimes \_) \longrightarrow F$ is a morphism in $(R\text{-mod}, Ab)$ such that $\alpha(j \otimes \_)$ is an embedding. Let α' be the composition of α with an embedding of F into an injective hull which, by Theorem B.2.3, may be taken to have the form $N' \otimes \_$ for some $N' \in$Mod-R. Because the embedding to the functor category is full (Theorem B.2.2), $\alpha' = f \otimes \_$ for some $f : N \longrightarrow N'$. Since $(fj) \otimes \_$ is an embedding, fj is a pure embedding, so, by assumption, f is a pure embedding. Therefore $f \otimes \_$ is an embedding and so, therefore, is α, as required. □

Proposition 2.2.18. *Let $M \xrightarrow{j} N$ be a pure-embedding with N pure-injective. Then $M \longrightarrow N$ is a pure-injective hull of M if, and only if, j is pure-essential.*

Proof. ($\Rightarrow$) If j was not pure essential, then, by Lemma 2.2.17, the embedding of $M \otimes \_$ into the injective functor $N \otimes \_$ (Theorem B.2.4) would not be essential, hence would factor through a proper direct summand (Proposition B.1.2), $N' \otimes \_$ say (of this form by Theorem B.2.4), of $N \otimes \_$. Then N' would be a proper direct summand of N, which is pure-injective by Lemma 2.2.11, containing M, contradicting the definition of pure-injective hull.

($\Leftarrow$) If $f : M \longrightarrow N$ is pure-essential and N is pure-injective then (Lemma 2.2.17) $(f \otimes \_) : (M \otimes \_) \longrightarrow (N \otimes \_)$ is essential (Proposition B.1.2) $N \otimes \_$ is the injective hull of $M \otimes \_$. If N' is another pure-injective module such that there is $g : M \longrightarrow N'$ a pure-embedding, then $(g \otimes \_) : (M \otimes \_) \longrightarrow (N' \otimes \_)$ is another embedding from $(M \otimes \_)$ to an injective module (Proposition B.1.2). Because $N \otimes \_$ is the injective hull of $M \otimes \_$

there exists an embedding $(k \otimes \_) : (N \otimes \_) \longrightarrow (N' \otimes \_)$ with $(k \otimes \_)(f \otimes \_) = (g \otimes \_)$, which implies that $kf = g$ (where g is a pure-embedding by Theorem B.2.2). Because f is pure essential, then $k : N \longrightarrow N'$ is a pure embedding and, by the minimality of the pure-injective hull, we get $N = N'$ (because any other pure-injective module in which we can embed M has a pure submodule isomorphic to N'). □

Corollary 2.2.19. *A morphism $j : M \longrightarrow N$ is a pure-injective hull of M if, and only if, $(j \otimes \_) : (M \otimes \_) \longrightarrow (N \otimes \_)$ is an injective hull in the functor category $(R\text{-mod}, Ab)$.*

Proof. ($\Rightarrow$) If $j : M \longrightarrow N$ is a pure-injective hull then, by Proposition 2.2.18, it is pure essential and hence $(j \otimes \_) : (M \otimes \_) \longrightarrow (N \otimes \_)$ is essential (Lemma 2.2.17). Because $(N \otimes \_)$ is injective (Theorem B.2.3) and $(j \otimes \_)$ is essential, then $(N \otimes \_)$ is the injective hull of $(M \otimes \_)$ (Proposition B.1.2). ($\Leftarrow$) If $(j \otimes \_) : (M \otimes \_) \longrightarrow (N \otimes \_)$ is an injective hull then (Proposition B.1.2) $(j \otimes \_)$ is essential hence, by Proposition 2.2.18, $j : M \longrightarrow N$ is pure-essential and, because $(N \otimes \_)$ is injective, N is pure-injective (Theorem B.2.2) and so (Proposition 2.2.18) is the pure-injective hull of M. □

Theorem 2.2.20. *Every module M has a pure-injective hull which is unique up to isomorphism: if $j : M \longrightarrow N$ and $j' : M \longrightarrow N'$ are pure-injective hulls of M, then there is an isomorphism $f : N \longrightarrow N'$ such that $fj = j'$.*

Proof. (Existence) Let $(M \otimes \_) \longrightarrow E$ be an injective hull (existence by Theorem B.1.6). By Theorem B.2.3, $E \simeq N \otimes \_$ for some pure-injective module N an the embedding is $j \otimes \_$ for some pure-embedding $j : M \longrightarrow N$. By (Corollary 2.2.19) this is a pure-injective hull.

(Uniqueness) If $j' : M \longrightarrow N'$ is another pure-injective hull, then $j' \otimes \_ : (M \otimes \_) \longrightarrow (N' \otimes \_)$ is an injective hull, so, by uniqueness of injective hull, there is an isomorphism $f \otimes \_$ from $N \otimes \_$ to $N' \otimes \_$ such that $(f \otimes \_)(j \otimes \_) = (j' \otimes \_)$, hence f is an isomorphism (Theorem B.2.2) with $fj = j'$ as required. □

The next two results, about indecomposable pure-injective modules, will be used to talk about some properties of Definable Subcategories and the Ziegler Spectrum.

Theorem 2.2.21. *Every indecomposable pure-injective module has local endomorphism ring.*

Proof. If N is an indecomposable pure injective, then $N \otimes \_$ is an indecomposable injective in $(R\text{-mod}, Ab)$ (Theorem B.2.2), hence has local endomorphism ring (Theorem B.1.5). The fullness of the embedding of Mod-R into $(R\text{-mod}, Ab)$ (Theorem B.2.2) gives us $End(N) \simeq End(N \otimes \_)$. □

Lemma 2.2.22. *If N is an indecomposable pure-injective module and N is purely embedded in, hence, a direct summand of, $\oplus_i M_i$, where the M_i are arbitrary and this sum is finite, then N is a direct summand of M_i, for some i.*

Proof. Let $j_i : M_i \longrightarrow \oplus_j M_j = M$ and $\pi_j : M \longrightarrow M_j$ be the canonical inclusions and projections, so $\sum_i j_i \pi_i = 1_M$. Let $j : N \longrightarrow M$ be the inclusion and let $\pi : M \longrightarrow N$ be a projection, so $\pi j = 1_N$.

Then $1_N = \pi 1_M j = \sum_i (\pi j_i)(\pi_i j)$. Since $End(N)$ is local (Theorem 2.2.21) not all the $(\pi j_i)(\pi_i j)$ can lie in $JEnd(N)$ (since $End(N)$ is local we have that $JEnd(N)$ is exactly all the non-units), that is, for some i, $(\pi j_i)(\pi_i j)$ is an automorphism of N, so, for some automorphism g of N, the morphism $(g\pi j_i)(\pi_i j)$ is the identity of N. Therefore $\pi_i j : N \longrightarrow M_i$ is monic and is split, as required. □

2.2.4 Hull of pp-types

We can define a concept similar to pure-injective hulls for pp-types. This concept, by Theorem 2.2.30, is quite useful for showing that the isomorphism classes of pure-injective indecomposable modules are in bijection with indecomposable pp-types. We will also see some relations between the pure-injective hull of a module and the pure-injective hull of pp-types which can be used to show that the isomorphism classes of pure-injective indecomposable modules form a set.

If $\overline{a} = (a_1, a_2, \cdots, a_n)$ is an n-tuple from a module M we define the morphism $\overline{a} : R^n \longrightarrow M$ as $\overline{a}e_i = a_i$, where $(e_1, e_2, \cdots, e_n)$ is a generating tuple for R^n (that is, any element from R^n can be written as $\sum_i e_i r_i$, $r_i \in R$). This morphism induces a morphism between functors, $(\overline{a} \otimes \_) : (R^n \otimes \_) \longrightarrow (M \otimes \_)$, and corresponding to the embedding of $M \otimes \_$ into its injective hull $H(M) \otimes \_$, there is a morphism, induced by the embedding of $\overline{a}$ in $H(M)$, which we will denote $i\overline{a} \otimes \_$, from $R^n \otimes \_$ to $H(M) \otimes \_$. The injective hull of $im(\overline{a} \otimes \_)$ is a direct summand of $H(M) \otimes \_$ hence (Theorem B.2.3) has the form $H^M(\overline{a}) \otimes \_$, for some pure-injective module $H^M(\overline{a})$, which is a direct summand of $H(M)$.

Definition 2.2.23 (Pure-injective hull of a pp-type)**.** *Let $p = pp^M(\overline{a})$ a pp-type. We define the pure-injective hull of p as the pure-injective module $H^M(\overline{a})$. Sometimes we will write it as $H(p)$.*

Observation 2.2.24. *Observe that if $\phi > \psi$ are pp conditions and p is a pp-type with $\phi \in p$ and $\psi \notin p$, then the pp-pair ϕ/ψ is open in $H(p)$. This follows from the fact that we have a copy of $\overline{a}$ in $H(p)$ (by the construction of $H(p)$) and $H(p)$ is a pure-submodule of $H(M)$, hence $pp^{H(p)}(\overline{a}) = pp^{H(M)}(\overline{a}) = pp^M(\overline{a}) = p$. That is, $\overline{a} \in \phi(H(p)) \setminus \psi(H(p))$.*

Proposition 2.2.25. *Let M, M' be any modules and let $H(M)$, $H(M')$ be their respective pure-injective hulls. Suppose that $\overline{a}$ from M and $\overline{a}'$ from M' have the same pp-type $p = pp^M(\overline{a}) = pp^{M'}(\overline{a}')$. Then there are direct summands N of $H(M)$ and N' of $H(M')$,* *with $\overline{a}$ from N and $\overline{a}'$ from N', and an isomorphism $f : N \longrightarrow N'$ such that $f\overline{a} = \overline{a}'$. These summands may be taken to be copies of $H(p)$.*

Proof. By Proposition B.3.7, one gets that $im(\overline{a} \otimes \_) \simeq im(\overline{a} \otimes \_)$ (because they have same preimage and kernel and, by the first isomorphism theorem for abelian categories $(coim(f) \simeq im(f))$, $(R^n, \_)/F_{Dp} \simeq im(\overline{a} \otimes \_) \simeq im(\overline{a} \otimes \_))$ so, by the corresponding result for injective objects, there are direct summands $H^M(\overline{a})$ of $H(M) \otimes \_$ and $H^{M'}(\overline{a}')$ of $H(M') \otimes \_$, containing those respective images, and which are isomorphic by a morphism $\alpha : im(\overline{a} \otimes \_) \longrightarrow im(\overline{a}' \otimes \_)$ such that $\alpha(\overline{a} \otimes \_) = (\overline{a}' \otimes \_)$. Hence, $H^{M'}(\overline{a}') \simeq H^M(\overline{a})$, by the definition of the hull of a pp-type and Proposition B.1.2. By Theorem B.2.2, we get that $\alpha = f \otimes \_$ and, also, it induces an isomorphism $f : H^M(\overline{a}) \longrightarrow H^{M'}(\overline{a}')$ with $f\overline{a} = \overline{a}'$. □

Corollary 2.2.26. *If N is a pure-injective module, if $\overline{a} = (a_1, a_2, \cdots, a_n)$ is from N and $p = pp^N(\overline{a})$, then there is a copy of the hull, $H(p)$, of p which contains all the a_i and which is a direct summand of N.*

Proof. By the last proposition, $H(p)$ is a direct summand of $H(N) = N$ and $\overline{a}$ is in $H(p)$ because $im(\overline{a})$ is embedded in it. □

Corollary 2.2.27. *If N is an indecomposable pure-injective module, then there is a pp-type p such that $N \simeq H(p)$. Indeed, if $\overline{a} \neq \overline{0}$ is from N then $N = H(pp^N(\overline{a}))$*

Proof. Follows from the fact that $H(p)$ is a direct summand of N and if $\overline{a} \neq \overline{0}$ then $H^N(\overline{a}) \neq 0$. □

Corollary 2.2.28. *There is just a set of indecomposable pure-injective R-modules up to isomorphism. Indeed there are at most $2^{card(R)+\aleph_0}$.*

Proof. There is just a set of pp-pairs (without parameters) and, by Corollary 2.2.27, every indecomposable pure-injective is isomorphic to the hull of a pp-type. More precisely, there are $card(R) + \aleph_0$ pp conditions therefore no more than $2^{card(R)+\aleph_0}$ pp-types for R-modules. □

This last corollary is useful to show that $pinj_R$ (the collection of isomorphism classes of pure-injective modules) is really a set, and not a proper class. With this we will be able to define a topology on this set, which we shall call the Ziegler Spectrum.

Definition 2.2.29 (Indecomposable pp-type)**.** *A pp-type p is indecomposable if, for every $\phi \in p$ and $\psi_1,\ \psi_2 \notin p$ we have that $\phi \wedge \psi_1 + \phi \wedge \psi_2 \notin p$.*

Theorem 2.2.30 (Ziegler's Criterion)**.** *Let p be a pp-type. Then the hull of p, $H(p)$, of p is indecomposable if, and only if, p is indecomposable.*

Proof. By Lemma B.3.8 p is irreducible if, and only if $(R^n \otimes \_)/F_{Dp}$ is uniform, equivalently, if, and only if, the injective hull Proposition B.1.2, $E((R^n \otimes \_)/F_{Dp}) = H(p) \otimes \_$, is indecomposable if, and only if, $H(p)$ is indecomposable. □

Part II

Definable Subcategories and the Ziegler Spectrum

3 Definable Subcategories

3.1 Definable Subcategories

In this section we will give a definition for definable subcategories, give some equivalent conditions for a full subcategory to be definable and also show some examples. These subcategories will be important to define the closed subsets of the Ziegler Spectrum, which will be defined in the next chapter. The main focus of this chapter is giving equivalent conditions for a subcategory to be definable.

Definition 3.1.1 (Full subcategory)**.** *If $\mathcal{D}$ is a subcategory of $\mathcal{C}$ we say that it is a full subcategory if, for all objects A and B in $\mathcal{D}$, the $\mathcal{D}$-arrows between them are exactly all the $\mathcal{C}$-arrows between them.*

Definition 3.1.2 (Definable Subcategory)**.** *Let $T = \{\phi_\lambda/\psi_\lambda\}_{\lambda\in\Lambda}$ be a set of pp-pairs. Define $Mod(T)$ (here "Mod" stands for "model", not "module") to be the full subcategory of Mod-R consisting of the modules M such that $\phi_\lambda(M) = \psi_\lambda(M)$ for every $\lambda \in \Lambda$.*

Just as observed after Definition 1.2.14 saying that a pp-pair is closed in a module M is the same as saying that elements which satisfy one pp-condition must satisfy the other pp-condition. This can be used to say that every element in a module satisfies a pp condition or that only the 0 satisfies a pp condition. Below there are some examples of definable subcategories:

Example 3.1.3. *Suppose R is a domain. For each $r \in R - \{0\}$, define the pp-pair $(xr = 0)/(x = 0)$, and let $T = \{(xr = 0)/(x = 0)\}_r$. Here we are saying that if some element times r is 0 it must be 0, that is, every element is torsionfree. The definable subclass $Mod(T)$ is the class of torsionfree modules.*

Dually, if R is a domain and for each $r \in R-\{0\}$ we define the pp-pair $(x = x)/(r|x)$ and $T = \{(x = x)/(r|x)\}_r$ we will have, as the definable class $Mod(T)$, the subclass of divisible modules. In this case we are saying that for any element $a \in M$ and any scalar $r \in R$ there is another $b \in M$ such that $a = br$.

Example 3.1.4. *Let Q be a quiver, i be a vertex of Q and e_i the trivial path from i to i. Let $(x = x)/(xe_i = 0)$ be a pp-pair and $\mathcal{X}$ the definable subcategory generated by this pp-pair. This pp-pair will be closed in a representation of Q if, and only if, $M(i) = \{0\}$. Hence, the definable subcategory generated by $(x = x)/(xe_i = 0)$ has, as modules, all the representations of Q such that $M(i) = \{0\}$.*

Theorem 3.1.5. *(PREST, 2009, Theorem 4.3.21) If $\mathcal{X}$ is a definable subcategory of Mod-R and if $M \in \mathcal{X}$, then the pure-injective hull of M is in $\mathcal{X}$*

The idea used for this proof in Mike Prest book is using Theorem 2.2.7 transfinitely in a module until you get an algebraically compact module $M^\diamond$ inside $\mathcal{X}$ which, by Theorem 2.2.13, is a pure-injective module. Because the embedding from M to $M^\diamond$ is pure and $M^\diamond$ is pure-injective we have that $H(M)$ is a direct summand of $M^\diamond$ (Proposition 2.2.18) and, because definable subcategories are closed under direct summands (Theorem 3.1.6), $H(M)$ is in $\mathcal{X}$.

An alternative proof of this theorem would find an injective module as in Theorem 1.3.10 and dualise a module M twice to get M^{**}. By Corollary 2.1.15 the embedding of M into M^{**} is pure. It can also be shown that M^{**} is pure-injective (PREST, 2009, Corollary 4.3.31) so, by Theorem 2.2.20 and Proposition 2.2.10, we get that $H(M)$ is a direct summand of M^{**} and, as a consequence, is in the definable subcategory generated by M (Theorem 3.1.6).

Theorem 3.1.6. *(PREST, 2009, Theorem 3.4.7) The following conditions on a subclass $\mathcal{X}$ of Mod-R are equivalent:*

(i) $\mathcal{X}$ is definable;

(ii) $\mathcal{X}$ is closed under direct products, direct limits and pure submodules;

(iii) $\mathcal{X}$ is closed under direct products, reduced products and pure submodules;

(iv) $\mathcal{X}$ is closed under direct products, ultraproducts and pure submodules.

Proof. *(i)⇒(ii)* If a category $\mathcal{X}$ is definable, then, directly from the definitions, it is closed, under pure-submodules and, by Lemma 1.2.3, it is closed under direct products.

For closure under direct limits, let $((M_\lambda)_\lambda, (f_{\lambda\mu})_{\lambda\mu})$ be a directed system in $\mathcal{X}$, with direct limit M and canonical maps $f_{\lambda\infty} : M_\lambda \longrightarrow M$ to be the limit. Suppose ϕ/ψ is a pp-pair which is closed in every M_λ. Let $(C, \overline{c})$ be a free realization of ϕ and $\overline{a} \in \phi(M)$. Let $\overline{b}$ be a generating tuple from C with relations $\overline{b}H = 0$ and $\overline{b}G = \overline{c}$. Using this generating tuple and the fact that, if $\overline{a} \in \phi(M)$ then there is some λ such that there is $\overline{a}_\lambda$ from M_λ with $\overline{a}_\lambda \in \phi(M)$ and $f_{\lambda\infty}\overline{a}_\lambda = \overline{a}$, we can get a morphism $g' : M_\lambda \longrightarrow M$ such that $g = f_{\lambda\infty}g'$. Now $g'\overline{c} \in \phi(M_\lambda) = \psi(M_\lambda)$ by hypothesis, so, by Lemma 1.1.10, $\overline{a} \in \psi(M)$, as required.

(ii)⇒(iii) Follows directly from Lemma A.2.1.

(iii)⇒(iv) Follows from the fact that ultraproducts are a special type of reduced products.

(iv)⇒(ii) Follows directly from Theorem A.2.2.

(ii)⇒(i) (Idea of the proof) Let $\mathcal{X} \subset$Mod-R be closed under direct products, pure-submodules and direct limits. Let $T = \{\phi/\psi;\ \forall M \in \mathcal{X},\ \phi(M) = \psi(M)\}$, the set of all pp-pairs which are closed on every object of $\mathcal{X}$. Let $\mathcal{X}' =$Mod(T) be the corresponding definable subcategory which, by definition, already satisfies $\mathcal{X} \subset \mathcal{X}'$.

To prove that $\mathcal{X}' \subset \mathcal{X}$ we embed both subcategories in $(R\text{-mod}, Ab)$ with the functor which takes M and maps to $M \otimes \_$ (Theorem B.2.2). After the embedding one can prove that $\mathcal{X}'$ will be the closure of $\mathcal{X}$ under products, direct limits, injective hulls and subobjects. Because the functor $M \mapsto M \otimes \_$ commutes with products and direct limits (Theorem B.2.2) we get that $\mathcal{X}$ is already closed under products and direct limits. We also get $\mathcal{X}$ is closed under injective hulls (can be proved using Theorem 3.1.5) and is closed under embeddings (follows from the fact that f is a pure-embeddings in Mod-R if, and only if, $f \otimes \_$ is an embedding in $(R\text{-mod}, Ab)$ (Theorem B.2.2)). With this we can prove that $\mathcal{X} = \mathcal{X}'$ in $(R\text{-mod}, Ab)$ and, because this embedding is full, we get $\mathcal{X} = \mathcal{X}' \subset$Mod-$R$. □

The original proof of this theorem (ZIEGLER, 1984), which mainly uses ideas from model theory, shows the implication *(iv)⇒(i)*.

From *(ii)* of the last theorem one can see that, for R a domain, the class of torsion modules is not a definable subcategory. A clear example is that $\prod_n \mathbb{Z}/\mathbb{Z}n$ is a direct product of torsion modules but is not a torsion module.

Observation 3.1.7. *Definable subcategories are also closed under direct summands, because direct summands are pure submodules (Lemma 2.1.3), and under direct sums, because $\bigoplus_i M_i$ is pure in $\prod_i M_i$ (Lemma 2.1.16).*

Observation 3.1.8. *Note that a subcategory $\mathcal{X}$ that is not closed under pure submodules will not be definable, even if it satisfies the other conditions of Theorem 3.1.6(ii, iii and iv). A simple example is the category of $\overline{\mathbb{Z}}_{(P)}$-modules (modules over the p-adics integers) as a subcategory of abelian groups is not definable, even though it is closed under direct limits and direct products. This happens because $\mathbb{Z}_{(P)}$ is pure in $\overline{\mathbb{Z}}_{(P)}$, as an abelian group, and is not a $\overline{\mathbb{Z}}_{(P)}$-module, hence the subcategory of $\overline{\mathbb{Z}}_{(P)}$-modules in the category of abelian groups is not complete under pure subobjects.*

We will define a basis of closed sets for the Ziegler Spectrum, Zg_R with our definable subcategories and, later, show that these sets are exactly the closed subsets of Zg_R. The next result will be central to show that finite union of closed sets in Zg_R can still be defined by a definable subcategory. This will also be central to show the bijection between definable subcategories and closed subsets of Zg_R.

Definition 3.1.9 (Definable subcategory generated by a module)**.** *For any module M we can define $T_M = \{\phi/\psi \ ; \ \phi(M) = \psi(M)\}$. We write $\langle M \rangle = Mod(T_M)$ and refer to it as the definable subcategory generated by M. We can also define, in a similar way, the definable subcategory generated by a subclass of Mod-R.*

Proposition 3.1.10. *If $\mathcal{X}_1, \ \mathcal{X}_2, \cdots, \ \mathcal{X}_n$ are definable subcategories of Mod-R, then the definable subcategory generated by $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_n$ consists of all modules M which can be purely embedded into a module of the form $M_1 \oplus \cdots \oplus M_n$ with $M_i \in \mathcal{X}_i$.*

Proof. Let $\mathcal{X}$ be a class of modules as described. Certainly $\mathcal{X}_i \subset \mathcal{X}$, for all i, and any definable subcategory $\mathcal{Y}$ which contains $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_n$ must, by Theorem 3.1.6, contain $\mathcal{X}$. We check the condition *(iii)* of Theorem 3.1.6 in order to show that $\mathcal{X}$ is definable.

By definition, and because the composition of pure embeddings is a pure embedding, $\mathcal{X}$ is closed under pure submodules.

If, for $\lambda \in \Lambda$, M_λ is pure in $M_{1\lambda} \oplus \cdots \oplus M_{n\lambda}$, then $\prod_\lambda M_\lambda$ is (Lemma 1.2.3) pure in $\prod_\lambda (M_{1\lambda} \oplus \cdots \oplus M_{n\lambda}) \simeq \prod_\lambda M_{1\lambda} \oplus \cdots \oplus \prod_\lambda M_{n\lambda}$. Each $\prod_\lambda M_{i\lambda}$ is in $\mathcal{X}_i$ by Theorem 3.1.6, so $\prod_\lambda M_\lambda$ is in $\mathcal{X}$.

As for the closure under reduced products (Definition A.1.2), lets consider the same set of modules as in the proof for closure under direct products and $\mathcal{F}$ a filter on the power set $\mathcal{P}(\Lambda)$. Then there is an embedding $\prod_\lambda M_\lambda/\mathcal{F} \longrightarrow \prod_\lambda (M_{1\lambda} \oplus \cdots \oplus M_{n\lambda})/\mathcal{F}$ which, by Definition A.2.6, is pure and, since reduced product commutes with finite direct sum, the second term is isomorphic to $\prod_\lambda M_{1\lambda}/\mathcal{F} \oplus \cdots \oplus \prod_\lambda M_{n\lambda}/\mathcal{F}$. By Theorem 3.1.6 each $\prod_\lambda M_{i\lambda}/\mathcal{F}$ is in $\mathcal{X}_i$ so $\prod_\lambda M_\lambda \in \mathcal{X}$, as required. □

Observation 3.1.11. *(PREST, 2009, Proposition 3.4.10) If $\mathcal{X}$ is a definable subcategory of Mod-R, if T' is the set of pp-pairs which are closed on every module in $\mathcal{X}$ and if $T_1 = \{\phi/\psi : \ \phi/\psi \in T'$ and ϕ, ψ have just one free variable$\}$, then $Mod(T') = Mod(T_1) = \mathcal{X}$.*

Lemma 3.1.12. *If $\mathcal{X}$ is a definable subcategory of Mod-R, then there is a module $M^\diamond$ such that $\mathcal{X}$ is the definable subcategory generated by $M^\diamond$.*

Proof. Let T be the set of all pp-pairs which are closed in every module $M \in \mathcal{X}$. For each $\phi/\psi \notin T$, by Corollary 1.2.16, pick a module $M_{\phi/\psi} \in \mathcal{X}$ such that ϕ/ψ is open in $M_{\phi/\psi}$ (this module will exist because, otherwise, this pp-pair would be closed for every module in $\mathcal{X}$). Defining $M^\diamond = \oplus_{\phi/\psi \notin T} M_{\phi/\psi}$ one can see that $\langle M^\diamond \rangle = \mathcal{X}$. □

In general definable subcategories are not closed under inverse limits. Here is an example showing that:

Example 3.1.13. *Let $R = k[a,\ b;\ a^2 = b^2 = ab = 0]$ and set M_n $(n \geq 1)$ to be the k-vectorspace with basis u, v_i $(i \geq 1)$, v'_i $(1 \leq i \leq n-1)$ with $ua = 0 = ub$, $v_i a = u$ $(i \geq 1)$, $v_i b = 0$ $(i \geq n)$, $v_i b = v'_i$ $(i \leq n-1)$, $v'_i a = 0 = v'_i b$ $(1 \leq i \leq n-1)$. There are natural epimorphisms $\cdots \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1$ between these modules ($M_n \longrightarrow M_{n-1}$ is the natural map from M_n to $M_n/v'_{n-1}R$). Denote M as the inverse limit of this system and observe that M can be seen as the k-vectorspace with basis u, v_i $(i \geq 1)$, v'_i $(i \geq 1)$ and relations as in the M_n. Observe that in each M_n we have $\exists y\ (x = y.a)(M_n) = im(a) = ann(b).a = \exists y\ (yb = 0 \wedge ya = x)(M_n)$, whereas, in M, $im(a) = \exists y\ (x = ya)(M)$ is one dimensional but $\exists y\ (yb = 0 \wedge ya = x)(M) = ann(b).a \leq ann(a).a = \exists y\ (ya = 0 \wedge ya = x)(M) = 0$. That is, the pp-pair $\exists y\ (yb = 0 \wedge ya = x)/\exists y\ (x = ya)$ is closed in all M_n but is open in M. Then the definable subcategory generated by only this pp-pair has all M_n but does not contain M.*

If you have (M_λ) an iverse system of modules, all belonging to the definable subcategory $\mathcal{X}$, one condition one can work to get $M = \varprojlim_\lambda M_\lambda$ in $\mathcal{X}$ is that $\phi(\varprojlim_\lambda M_\lambda) = \varprojlim_\lambda(\phi(M_\lambda))$.

4 Ziegler Spectrum

The Ziegler spectrum is a topological space which was introduced by Ziegler in (ZIEGLER, 1984). This topological space played a big role in the model theory of modules because most broad questions about this theory can be phrased as questions about the points and topology of this space and can be tackled, and often answered, in these terms. An example is asking about the decidability of the theory of modules over a ring. The focus of this chapter is defining this topological space, giving some properties and describing this topological space for commutative discrete valuation rings, for example, showing that Zg_R is always compact and the bijection between closed subsets of Zg_R and definable subcategories of Mod-R. We will also describe the Ziegler Spectrum of a discrete valuation ring.

4.1 Ziegler Spectrum

4.1.1 Definition via definable subcategories

To simplify our notation we will denote by $pinj_R$ the set of isomorphism classes of non-zero indecomposable pure-injective R-modules. We will also abuse notation, writing $\mathcal{X} \cap pinj_R$ for the set of isomorphism classes of pure-injective modules in $\mathcal{X}$.

Definition 4.1.1 (Ziegler spectrum)**.** *Let R be an associative ring with 1. The (right) Ziegler spectrum, Zg_R, of R is a topological space with points being the elements of $pinj_R$ and the basis of closed sets being $\mathcal{X} \cap pinj_R$, for each definable subclass $\mathcal{X}$ of Mod-R.*

Usually, we will denote $\mathcal{X} \cap Zg_R$ for the closed subsets of Zg_R. There are many equivalent ways to define this topological space. In this chapter we will also define it with pp-pairs and prove some results using this equivalent definition.

For the next result we will show that every closed subset of Zg_R can be seen as a definable subcategory, that is, our basis has all the closed subsets of Zg_R.

Theorem 4.1.2. *The closed sets of the Ziegler topology are exactly those of the form $\mathcal{X} \cap Zg_R$, with $\mathcal{X} \subset$Mod-R a definable subcategory.*

Proof. To prove this fact we will show that with finite union and arbitrary intersection we get sets of the form $\mathcal{X} \cap Zg_R$. Because these sets form a basis for closed sets, we get our desired result.

(Finite union) Let $\mathcal{X}$ be the definable subcategory of Mod-R generated by the union of the definable subcategories $\mathcal{X}_1, \cdots, \mathcal{X}_n$. Clearly $X_1 \cup \cdots \cup X_n \subset X$, where $X_i = \mathcal{X}_i \cap Zg_R$

and $X = \mathcal{X} \cap Zg_R$. If $N \in X$, then, by 3.1.10, N is pure in $M_1 \oplus \cdots \oplus M_n$ for some $M_i \in \mathcal{X}_i$. But then, by 2.2.22, N is a direct summand of some of the M_i and hence is in $\mathcal{X}_i$. We conclude that $X = X_1 \cup \cdots \cup X_n$.

(Arbitrary intersection) To show closure under arbitrary intersection, take for $\lambda \in \Lambda$, $\mathcal{X}_\lambda$ to be the definable subcategory defined by a set, T_λ, of pp-pairs, that is, $\mathcal{X}_\lambda = Mod(T_\lambda)$. Let $\mathcal{Y} = Mod(\cup_\lambda T_\lambda)$ and $Y = \mathcal{Y} \cap Zg_R$. If $N \in Y$, then $N \in X_\lambda$ (since all pp-pairs in T_λ are closed on N). Conversely if $N \in \cap_\lambda X_\lambda$, then all pairs in $\cup_\lambda T_\lambda$ are closed on N, so $N \in Y$. Thus $Y = \cap_\lambda X_\lambda$, as required. □

We want to show that the closed subsets of Zg_R and the definable subcategories of Mod-R are in bijection, and that every definable subcategory can be defined just by looking at the indecomposable pure-injective modules in it. To prove these results first we will need to show that, for every pp-pair, we can find an indecomposable pure-injective module which is open in it, making it possible to define both concepts by only looking at these special modules.

Theorem 4.1.3. *Suppose that ϕ/ψ is a proper pp-pair, that is, $\phi > \psi$. Then there is an indecomposable pure-injective module N with $\phi(N) > \psi(N)$. If M is any module with $\phi(M) > \psi(M)$, then there is an indecomposable pure-injective with this property in the definable subcategory of Mod-R generated by M.*

Proof. Applying Zorn's Lemma one gets that there is a pp-type p which contains ϕ, does not contain ψ and is maximal such. First, consider the set $S = \{p;\ \phi \in p \text{ and } \psi \notin p\}$ with partial order being $p \leq q$ if $p \subset q$. This set has at least $\langle \phi \rangle$ as an element, so it is non-empty. If $\{p_i\}_i$ is a family of pp-types in S, then $p = \cup_i p_i \in S$ and is an upper bound for this family of pp-types. So with this we can get our desired pp-type p. We want now to show that p is irreducible.

To see this, let $\psi_1 \notin p$. Then the pp-type generated by p and ψ_1, that is, $\{\phi';\ \phi' \geq \phi_1 \wedge \psi_1 \text{ for some } \phi_1 \in p\}$, must, by maximality of p, contain ψ, so there is $\phi_1 \in p$ such that $\phi_1 \wedge \psi_1 \leq \psi$. In a similar way, for a $\psi_2 \notin p$ we can get that there is a $\phi_2 \in p$ such that $\psi_2 \wedge \phi_2 \leq \psi$. Hence, we get $\phi_1 \wedge \psi_1 + \phi_2 \wedge \psi_2 \leq \psi$ hence $\phi_1 \wedge \psi_1 + \phi_2 \wedge \psi_2 \notin p$. So p is indecomposable.

By 2.2.7 we can get some $M^\star$ in the definable subcategory generated by M, with $\overline{a}$ from $M^\star$ such that $pp^{M^\star}(\overline{a}) = p$. By Ziegler's Criterion 2.2.30 we get that the hull of p, $H(p)$, is indecomposable. Replacing $M^\star$ by its pure-injective hull, $H(M^\star)$ (3.1.5) we have, by 2.2.26, that $H^{H(M^\star)}(\overline{a}) = H(p)$ (the pp-type remains the same because the embedding from M to $H(M)$ is always a pure-embedding) is a direct summand of $H(M^\star)$, hence $H(p) \in \langle M \rangle$. It follows that $\phi(H(p)) > \psi(H(p))$ by 2.2.24. □

Corollary 4.1.4. *If $\mathcal{X}$ is a definable subcategory of Mod-R, then $\mathcal{X}$ is generated as such by the indecomposable pure-injectives in it.*

Proof. Let T be the set of pp-pairs closed on $\mathcal{X}$. For each $\phi/\psi \notin T$ choose, by 4.1.3, $N \in \mathcal{X} \cap Zg_R$ with $\phi(N) > \psi(N)$. Let $\mathcal{X}'$ be the definable subcategory of Mod-R generated by all these indecomposable pure-injectives. So $\mathcal{X} \subset \mathcal{X}'$. But also every pp-pair open on some member of $\mathcal{X}$ is, by construction, open on some member of $\mathcal{X}'$ so, by the definition of definable subcategory, $\mathcal{X}' \supset \mathcal{X}$, as required. □

Corollary 4.1.5. *If $\mathcal{X} \neq \emptyset$ is a definable subcategory of Mod-R, then $\mathcal{X} \cap Zg_R \neq \emptyset$. If $\mathcal{X}$, $\mathcal{X}' \subset$Mod-R are definable, then $\mathcal{X} = \mathcal{X}'$ if, and only if, $\mathcal{X} \cap Zg_R = \mathcal{X}' \cap Zg_R$.*

Proof. Follows directly from the last corollary and the Ziegler topology. □

Corollary 4.1.6. *There is a bijection between the definable subcategories of Mod-R and closed subsets of Zg_R, given by $\mathcal{X} \mapsto \mathcal{X} \cap Zg_R$ and $X \mapsto$ the definable subcategory generated by X.*

Proof. Follows directly from the last two corollaries. □

We can also give an equivalent definition for closed sets using pp-pairs directly, without talking about their definable subcategories. With this definition it will be easier to talk about the open subsets of Zg_R. This definition, using pp-pairs, will be used to talk about some properties of the topological space Zg_R.

Corollary 4.1.7. *The closed subsets of Zg_R are exactly of the form $[T] = \{N \in Zg_R\ ;\ \phi(N) = \psi(N)\ \forall \phi/\psi \in T\}$, where T is an arbitrary set of pp-pairs.*

Proof. Here instead of talking about the definable subcategory we are just defining it directly by the closed pp-pairs. □

For a pp pair ϕ/ψ, set $(\phi/\psi) = \{N \in Zg_R : \phi(N) > \psi(N)\}$ as the open set generated by this pp-pair. We can also define $(\phi/\psi) = [\phi/\psi]^c$.

Lemma 4.1.8. *A basis of open sets for the Ziegler topology consists of the (ϕ/ψ) as ϕ/ψ ranges over pp-pairs (in one free variable).*

Proof. Follows directly from definition and by the fact that all closed subsets of Zg_R can be seen as $[T]$, where T is a set of pp-pairs (then all the opens will be $[T]^c$). □

A really good property this space has is that it is compact. To prove this we will show that the open sets defined by just one pp-pair are compact and, because the whole space can be defined with one open pp-pair, we get that Zg_R is compact.

Theorem 4.1.9. *The compact sets of Zg_R are exactly the (ϕ/ψ) with ϕ/ψ a pp-pair (with an arbitrary number of free variables).*

Proof. (These sets are compact) Each such set is compact: if $(\phi/\psi) = \cup_\lambda(\phi_\lambda/\psi_\lambda)$ then, by B.4.3, $F_{\phi/\psi}$ belongs to the Serre subcategory of $(\text{mod-}R,\ Ab)^{fp}$ (of finitely presented functors) generated by the $F_{\phi_\lambda/\psi_\lambda}$, so, necessarily (B.4.2), it belongs to the Serre subcategory generated by just finitely many of them. Therefore (B.4.3) (ϕ/ψ) is the union of the corresponding finitely many open subsets.

(If a set is compact then it is (ϕ/ψ)) Since sets of this kind form a basis of the topology, an open set is compact exactly if it is a finite union of such sets. But $(\phi_1/\psi_1) \cup (\phi_2/\psi_2) \cup \cdots \cup (\phi_k/\psi_k) = (\phi/\psi)$, where ϕ is $\phi_1(\overline{x}_1) \wedge \phi_2(\overline{x}_2) \wedge \cdots \wedge \phi_k(\overline{x}_k)$ and ψ is $\psi_1(\overline{x}_1) \wedge \psi_2(\overline{x}_2) \wedge \cdots \wedge \psi_k(\overline{x}_k)$, where the sequences of free variables, $\overline{x}_i$, should be taken to be disjoint. □

Corollary 4.1.10. *For every ring R the Ziegler spectrum Zg_R is compact.*

Proof. It follows from the fact that $((x = x)/(x = 0)) = Zg_R$. □

4.2 Example

In this section we will calculate the Ziegler Spectrum of a discrete valuation ring and describe its topology.

4.2.1 Ziegler Spectrum of a Discrete Valuation ring

Definition 4.2.1 (Discrete valuation ring)**.** *A ring R is a discrete valuation ring if every ideal is a principal ideal (every ideal is generated by only one element), domain (if $r,\ s \in R$ are such that $rs = 0$, then $r = 0$ or $s = 0$) and has a unique non-zero maximal ideal.*

Theorem 4.2.2. *Let R be a commutative discrete valuation domain with maximal ideal P. The points of Zg_R are the following:*

(a) the indecomposable modules, R/P^n, of finite length, for $n \geq 1$;

(b) the completion, $\overline{R}_{(P)} = \varprojlim_n R/P^n$, of R in the P-adic topology;

(c) the Prüfer module $R_{P^\infty} = E(R/P)$;

(d) the quotient field of fractions, $Q = Q(R)$, of R.

Proof. The modules we see in (a) are indecomposable by the Structure Theorem for Finitely Generated Modules over PIDs and pure-injective by C.2.4. The module in (b) is pure-injective by C.2.9 and indecomposable by C.2.5. The last two are injective by C.1.2 and are indecomposable by C.1.3. Now it remains to prove that these are the only pure-injective indecomposable modules, up to isomorphism.

Let N be an indecomposable pure-injective module and choose any non-zero element $a \in N$. Since R is a commutative discrete valuation ring, we may get some element $p \in P$ such that $pR = \{pr;\ r \in R\} = P$. We will define the height function $h(a) = sup\{n \in \mathbb{N};\ p^n|a\}$, which will be a non-negative integer or ∞. The annihilator of a, $ann_R(a)$, is a power of P or 0. Since $ann_R(a) = P^{n+1}$ implies that $ap^n \neq 0$ and $ap^{n+1} = 0$ we can restrict to the case where $ann_R(a)$ is P or 0 by, if $ann_R(a) = P^{n+1}$, multiplying our a by p^n, which will be a non-zero element.

Discrete valuation rings are RD (PREST, 2009, Subsection 2.4.2) rings, that is, it is enough to check pp conditions of the simple form $\exists y(xr = ys)$, for $r,\ s \in R$. The idea of this proof is proving that N will be a pure-submodule of one of our listed pure-injective indecomposable modules, hence, it must be one of them. We will divide into four cases: $h(a) = n$ and $ann_R(a) = P$, which the modules R/P^{n+1} satisfies, $h(a) = n$ and $ann_R(a) = 0$, which is satisfied by $\overline{R}_{(P)}$, $h(a) = \infty$ and $ann_R(a) = P$, which is satisfied by R_{P^∞}, and $h(a) = \infty$ and $ann_R(a) = 0$, which is satisfied by $Q(R)$.

Case (i) $h(a) = n$, $ann(a) = P$. Say $a = bp^n$. Then $bR \simeq R/P^{n+1}$ is pure-injective (follows from the first isomorphism theorem and the fact that the morphism $f : R \longrightarrow bR$, defined by $fr = br$, has kernel P^n). We will show that bR is pure in N. So suppose that b satisfies $br = cs$, for some $c \in N$, where $r = p^k u$ and $s = p^l t$ with u, t units of R. Since $bp^{n+1} = 0$, we may suppose $k \leq n$. Then $a = bp^n = (bp^k u)p^{n-k}u^{-1} = (cp^l t)p^{n-k}u^{-1} = cp^{n-l+k}tu^{-1}$ so, because $h(a) = n$ we get $n - l + k \leq n$, that is, $l \leq k$. Then $b' = bp^{k-l}t^{-1}u$ is an element from bR which satisfies this condition, which implies bR pure in N. Therefore, because N and bR are pure-injective indecomposables, $N \simeq bR \simeq R/P^{n+1}$.

Case (ii) $h(a) = n$, $ann(a) = 0$. Say $a = bp^n$. Observe that $ann(b) = ann(a) = 0$, which implies $bR \simeq R$. Claim: bR is pure in N. If not, then there would exist an equality of the form $bp^k u = cp^l t$ with $l > k$, hence a non-zero torsion element $bu - cp^{l-k}t$ and, by C.0.1, we would get that N is not indecomposable (a contradiction). Now, the pure-injective hull of $bR \simeq R$ is isomorphic to $H(R) = \overline{R}_{(P)}$. Then we have a pure-embedding of bR into N and into $H(R)$, and also an arrow from $H(R)$ into N (by 2.2.10). Because the embedding of a module into its pure-injective hull is pure essential, we get that the arrow of $H(R)$ into N is a pure embedding (2.2.18), hence, $N \simeq H(R) \simeq \overline{R}_{(P)}$.

Case (iii) $h(a) = \infty$, $ann(a) = P$. For each $n \geq 1$ there is some $b_n \in N$ with $a = b_n p^n$. Therefore the set $\{x_0 = a\} \cup \{x_i p = x_{i-1};\ i \geq 1\}$ of pp conditions (with parameter a from N and infinitely many variables) is finitely satisfied in N so, by 2.2.13 and 2.2.9, has a solution in N: say there are $b_i \in N$ $(i \geq 1)$ with $b_1 p = a$, $b_2 p = b_1, \cdots$. Because $aR \simeq R/P$ and $b_i R \simeq R/P^{i+1}$ we get that these elements generate a copy of the injective module $E(R/P)$ contained in, hence a direct summand of, hence equal to, N.

Case (iv) $h(a) = \infty$, $ann(a) = 0$. By cases (i)-(iii) it may be assumed that every non-zero element of N satisfies these two conditions (because, if there exists $b \in N$ satisfying one of the conditions from case (i)-(iii), with the proofs did before, we would get a pure-injective indecomposable submodule of N different from N, a contradiction). Let $q \in Q \setminus R$ say $q = p^{-n}u$, where $n > 0$ and u is a unit of R. Because $h(a) = \infty$, there is $b \in N$ such that $a = bp^n$. Since N is torsionfree this b is unique, so we may set $aq = bu$. In this case we can define a map, $a \mapsto aq$ ($q \in Q$), which is an R-homomorphism from Q to N. Thus there is a copy of the injective module Q embedded in N so $N \simeq Q$.

□

4.2.1.1 About the topology

Now that we know about the points of our topological space, we will see which ones are isolated and which are not:

R/P^n : These points are isolated by the open set (ϕ/ψ) where ϕ is $p^{n-1}|x \wedge xp = 0$ (the elements of order p which are divisible by p^{n-1}) and ψ is $p^n|x \wedge xp = 0$ (the elements of order p which are divisible by p^{n-1}). These will be closed in the torsionfree modules $\overline{R}_{(P)}$, Q because there are no elements such that $xp = 0$ other than 0 (they are torsionfree). The pp-pairs will be closed in R_{P^∞} because all the elements inside this module can be divided by p arbitrarily (it is divisible). For the other torsion modules (R/P^m) it is either impossible to divide the elements of order p by p^{n-1} (if $m < n$) or the elements of order p are divisible by p^{n-1} and p^n (if $m > n$);

$\overline{R}_{(P)}$: The open set $((x = x)/(p|x))$ contains $\overline{R}_{(P)}$ and R/P^n for all $n \geq 1$. The sets $((p^n|x)/(p^{n+1}|x)) = \{\overline{R}_{(P)}\} \cup \{R/P^m;\ m \geq n\}$ form a neighbourhood of open sets for $\overline{R}_{(P)}$. No open neighbourhood can omit infinitely many points of the form R/P^n because then in the definable subcategory generated by the closed complement we would have that the inverse limit, $\varprojlim R/P^m$, would satisfy the property stated after 3.1.13, that is, $\overline{R}_{(P)}$ would be in the definable subcategory generated by the complement of its open set (we have the bijection of closed sets and definable subcategories given by 4.1.6) hence it is also in its complement, a contradiction;

R_{P^∞} : The open set $((xp = 0)/(x = 0))$ contains R_{P^∞} and all the R/P^n, $n \geq 1$. We can give a basis for the neighbourhood of this point with sets of the form $((xp^{n+1} = 0)/(xp^n = 0)) = \{R_{P^\infty}\} \cup \{R/P^m;\ m \geq n + 1\}$. No open neighbourhood can omit infinitely many R/P^n because, otherwise, $\varinjlim R/P^m = R_{P^\infty}$ would belong to the complement of the open set (by looking at this closed set as a definable subcategory) and we would get a contradiction;

Q : We have that Q is a direct summand of $\prod_{n\in\mathbb{N}} R_{P^\infty}$ because this is a divisible module with torsionfree elements. Also, localization in the module by the set P can be seen as a direct limit, hence we can localize $\overline{R}_{(P)}$ to get a divisible module and, because it is torsionfree, we get that Q is a direct summand of it. With this, and arguments similar to the ones before, one can see that if you have an open set with Q then it must contain R_{P^∞}, $\overline{R}_{(P)}$ and infinitely many modules of the form R/P^n.

With these observations we can finally classify all the closed sets of the module. Let X be a closed subset of Zg_R and $\mathcal{X}$ the definable subcategory generated by X, we will divide it in three cases:

(i) (There are no isolated points in X) If we have a definable subcategory which contains R_{P^∞} then it must contain Q, by what we saw before (in this case $\mathcal{X}$ would be the definable subcategory of all divisible modules). In a similar way, if you have a category which contains $\overline{R}_{(P)}$ then you must also have Q (here $\mathcal{X}$ would be the definable subcategory of torsionfree modules). So we already have the following possibilities for X: $X = \{\}$, $X = \{\overline{R}_{(P)},\ Q\}$, $X = \{R_{P^\infty},\ Q\}$, $X = \{\overline{R}_{(P)},\ R_{P^\infty},\ Q\}$ (which is the union of the last two) and $X = \{Q\}$ (which is the intersection of the ones with Q). These are the only possibilities;

(ii) (There are finitely many isolated points in X) Let X_0 be the set of isolated points and $X_1 = X \setminus X_0$. Because X_0 is closed and open we have that X_1 is closed. Hence, X_0 can be any finite subset of $\{R/P^n\}_{n\in\mathbb{N}}$ and X_1 will be one of the sets in the example above;

(iii) (There are infinitely many isolated points in X) Since the set of all isolated points is discrete, with respect to the relative topology, we don't have any restriction on the subset $X_0 \subset X$, of isolated points. As observed when we were talking about the open sets, if you have a closed set which contains infinitely many isolated points, it will also contain $\overline{R}_{(P)}$, R_{P^∞} and Q, no matter which isolated points we choose. Then, in this case, we can have any infinite subset of isolated points, $\overline{R}_{(P)}$, R_{P^∞} and Q.

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Appendix

APPENDIX A – Reduced products and Ultraproducts

A.1 Definitions

Definition A.1.1 (Filter)**.** *Let $(P, \leq)$ be a partially ordered set. A subset F of P is a filter if it satisfies the following conditions:*

(a) The subset F is nonempty;

(b) For every $x,\ y \in F$ there is some $z \in F$ such that $z \leq x$ and $z \leq y$;

(c) For every $x \in F$ if $x \leq y$ then $y \in F$.

We say that U is an ultrafilter if it is a proper subset of P and there is no filter F such that $U \subsetneq F \subsetneq P$.

Observe that, if P is a lattice, the second condition is equivalent to say that if $x,\ y \in F$ then $x \wedge y \in F$. For the next definition we say that $\mathcal{F}$ is a filter on I if $\mathcal{F}$ is a nonempty subset of the power set of I, $\mathcal{P}(I)$, closed under intersection, which will be our meet, and closed under the partial order, which will be being subset of.

Definition A.1.2 (Reduced Products and Ultraproducts)**.** *Let $(M_i)_{i\in I}$ be a collection of modules and let $\mathcal{F}$ be a filter on I. Define an equivalence relation $\sim = \sim_{\mathcal{F}}$ on the product $\prod_{i\in I} M_i$ by $(a_i)_i = \overline{a} \sim \overline{b} = (b_i)_i$ if, and only if, $\{i \in I : \ a_i = b_i\} \in \mathcal{F}$.*

Let $Z = \{\overline{a} \in \prod_i M_i : \ \overline{a} \sim 0\}$. We define our reduced product, with respect to $\mathcal{F}$, as $(\prod_i M_i)/Z$. We usually denote this reduced product as $\prod_i M_i/\mathcal{F}$. If $\mathcal{F}$ is an ultrafilter, then we say that this product is an ultraproduct.

Observation A.1.3. *One can see that, if $(M_i)_{i\in I}$ is any collection of modules, the direct sum $\oplus_i M_i$ is a reduced product with respect to the filter $\mathcal{F} = \{I' \subset I;\ |I \setminus I'| \in \mathbb{N}\}$.*

A.2 Results

Lemma A.2.1. *(PREST, 2009, Lemma 3.3.1) Let $(M_i)_{i\in I}$ be a collection of modules and let $\mathcal{F}$ be a filter on I. Then the reduced product $\prod_i M_i/\mathcal{F}$ is isomorphic to the direct limit $\varinjlim_{J\in\mathcal{F}^{op}} \prod_J M_i$. The exact sequence $0 \longrightarrow Z \longrightarrow \prod_I M_i \longrightarrow \prod_{i\in I} M_i/\mathcal{F} \longrightarrow 0$ is pure-exact.*

Theorem A.2.2. *(PREST, 2009, Theorem 3.3.2) If* $((M_i)_{i\in I},\ (g_{ij})_{ij})$ *is a directed system of modules with direct limit* $(M,\ (g_{i\infty})_i)$ *then* M *is a pure submodule of a reduced product (which may be taken to be an ultraproduct) of the* M_i*.*

Proposition A.2.3. *(PREST, 2009, Proposition 3.3.3) Let* $(M_i)_{i\in I}$ *be a collection of modules and for each* i *let* $\overline{a}_i$ *be an* n*-tuple from* M_i*. Let* $\mathcal{F}$ *be a filter on a filter on* I*. Form the reduced product* $\prod_i M_i/\mathcal{F}$ *and let* $\overline{a} = (\overline{a}_i)/\mathcal{F}$ *be the corresponding* n*-tuple from* $\prod_i M_i/\mathcal{F}$*. Let* ϕ *be a pp condition with* n *free variables.*

Then $\overline{a}/\mathcal{F} \in \phi(\prod_i M_i/\mathcal{F})$ *if, and only if,* $\{i \in I :\ \overline{a}_i \in \phi(M_i)\} \in \mathcal{F}$*.*

Corollary A.2.4. *(PREST, 2009, Corollary 3.3.4) If* M *is any module,* I *any set and* $\mathcal{F}$ *any filter on* I*, then the diagonal embedding of* M *into* $M^I/\mathcal{F}$ *is pure.*

Definition A.2.5 (Reduced products of morphisms)**.** *If for each* $i \in I$*, we have the morphism* $f_i : M_i \longrightarrow N_i$*, then we use the notation* $(f_i)_i/\mathcal{F} : \prod_i M_i/\mathcal{F} \longrightarrow \prod_i N_i/\mathcal{F}$ *for the resulting morphism, where* $\mathcal{F}$ *is a filter on* I*.*

Proposition A.2.6. *(PREST, 2009, Proposition 3.3.5) Any reduced product of pure embeddings is a pure embedding.*

APPENDIX B – Grothendieck Categories

B.1 Grothendieck and Abelian Categories

Definition B.1.1 (Essential subobjects and Uniform objects)**.** *Let $\mathcal{C}$ be a category and A be an object of this category. A subobject A' of A is said to be essential in A if, for every non-zero subobject A'' of A we have $A \cap A'' \neq 0$. We say A is uniform if every non-zero subobject of A is essential in A, that is, every two non-zero subobjects of A have non-zero intersection. We say that an embedding $j : A' \longrightarrow A$ is essential if jA' is essential in A.*

To say an embedding is essential is equivalent to say that if $j : A' \longrightarrow A$ is essential and $f : A' \longrightarrow A''$ is a morphism such that fj is an embedding then f is an embedding.

Proposition B.1.2. *(PREST, 2009, Proposition E.1.7) Let $C \xrightarrow{i} E$ be an embedding in an abelian category $\mathcal{C}$. Then the following conditions are equivalent:*

(i) $C \xrightarrow{i} E$ *is an injective hull of C;*

(ii) *E is injective and C is essential in E;*

(iii) *if $C \xrightarrow{f} E'$ is any embedding of C into an injective object E', then there is a split embedding $E \xrightarrow{k} E'$ such that $ki = j$.*

In particular, an injective is indecomposable if, and only if, it is uniform.

We say that a subclass of objects $\mathcal{G}$ of the additive category $\mathcal{C}$ generate $\mathcal{C}$ if for every non-zero morphism $f : A \longrightarrow B$ in $\mathcal{C}$ there is $G \in \mathcal{G}$ and a morphism $g : G \longrightarrow A$ such that $fg \neq 0$.

Definition B.1.3 (Grothendieck categories)**.** *An abelian category $\mathcal{C}$ is said to be a Grothendieck category if:*

(a) *It has arbitrary coproducts;*

(b) *The direct limits are exact;*

(c) *If it has a generating set of objects.*

Theorem B.1.4. *(PREST, 2009, Theorem E.1.5) The category $(\text{mod-}R, Ab)$, of additive functors from the full subcategory of finitely presented modules to the category of abelian groups, is a Grothendieck category.*

Theorem B.1.5. *(PREST, 2009, Theorem E.1.23) Suppose $\mathcal{C}$ is an abelian category and that $E \in \mathcal{C}$ is an injective object of $\mathcal{C}$. Then E is indecomposable if, and only if, $End(E)$ is a local ring.*

Theorem B.1.6. *(PREST, 2009, Theorem E.1.8) Let $\mathcal{C}$ be a Grothendieck abelian category. Then every object of $\mathcal{C}$ has an injective hull.*

Lemma B.1.7. *(PREST, 2009, Lemma 5.1.19) Suppose that $G \leq A \oplus B$ are objects of an abelian category. Then $\pi_A G/(G \cap A) \simeq \pi_B G/(G \cap B)$, where π_A, π_B denote the projections from $A \oplus B$ to A, respectively B.*

B.2 Tensor Embedding

Definition B.2.1 (Absolutely pure)**.** *An object C is said to be absolutely pure if for every embedding $i : C \longrightarrow D$ we have that iC is a pure subobject of D.*

Theorem B.2.2. *(PREST, 2009, Theorem 12.1.3) Let R be a ring. The functor ϵ :Mod-$R \longrightarrow (R\text{-mod}, Ab)$ given on objects by $M \mapsto M \otimes \_$ is a full embedding.*

An exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow N' \longrightarrow 0$ in Mod-R is pure-exact if, and only if, the image $0 \longrightarrow \epsilon M \longrightarrow \epsilon N \longrightarrow \epsilon N' \longrightarrow 0$ is exact (it is also known that it will be pure-exact). Furthermore, ϵ commutes with direct limits and direct products.

Theorem B.2.3. *(PREST, 2009, Theorem 12.1.6) If M is a right R-module, then $\epsilon M = M \otimes \_$ is an absolutely pure object of $(R\text{-mod}, Ab)$, indeed every absolutely pure functor is isomorphic to one of this form.*

Furthermore, $M \otimes \_$ is injective if, and only if, M is pure-injective.

Theorem B.2.4. *(PREST, 2009, Corollary 12.1.8) The embedding $M \longrightarrow N$ is a pure-injective hull in Mod-R if, and only if, $(M \otimes \_) \longrightarrow (N \otimes \_)$ is an injective hull in $(R\text{-mod}, Ab)$. That is, $E(M \otimes \_) \simeq (H(M) \otimes \_)$.*

Proposition B.2.5. *(PREST, 2009, Proposition 12.1.19) If F and F' are finitely presented functors and F' is a subquocient of F, then $(F') \subset (F)$ as subsets of Zg_R.*

B.3 Pp conditions as functors

For ϕ a pp condition we define the functor $F_\phi \in (\text{mod-}R, Ab)$ as $F_\phi(M) = \phi(M)$. In a similar way, if ϕ/ψ is a pp-pair we define $F_{\phi/\psi} \in (\text{mod-}R, Ab)$ as $F_{\phi/\psi}(M) = \phi(M)/\psi(M)$.

Proposition B.3.1. *(PREST, 2009, Proposition 10.1.13) In Mod-R, the representable functors generate the functor category in the sense that for every functor F :Mod$-R \longrightarrow Ab$*

there is an epimorphism $\oplus_i(A_i, \_) \longrightarrow F$ for some $A_i \in$Mod-R (here the $(A_i, \_)$ are the representable functors). A functor F is finitely generated if, and only if, this direct sum may be taken to be finite.

Lemma B.3.2. *(PREST, 2009, Corollary 10.2.3) Every finitely generated functor of a finitely presented subfunctor in (mod-R, Ab) is finitely presented.*

Lemma B.3.3. *(PREST, 2009, Corollary 10.2.7) If $F \in$ (mod-R, Ab) is finitely presented and G, $H \leq F$are finitely generated, then their intersection $G \cap H$ is finitely generated.*

Lemma B.3.4. *(PREST, 2009, Corollary 10.2.31) Every finitely presented functor in (mod-R, Ab) is isomorphic to one of the form $F_{\phi/\psi}$ for some pp-pair ϕ/ψ, and every functor of this form is finitely presented.*

Lemma B.3.5. *(PREST, 2009, Corollary 10.3.8) If $\phi \geq \psi$ is a pp-pair for right R-modules and M is a right R-module, then there is a natural isomorphism $(F_{D\psi/D\phi}, M \otimes \_) \simeq \phi(M)/\psi(M)$ as left End(M)-modules.*

Definition B.3.6 (Functor of a pp-type)**.** *Let p be a pp-type in n free variables for right R-modules. We define the functor $F_{Dp} = \sum_{\phi \in p} F_\phi$.*

Proposition B.3.7. *(PREST, 2009, Proposition 12.2.5) Let $\overline{a}$ be an n-tuple from the module M. Then the morphism $(\overline{a} \otimes \_) : (R^n \otimes \_) \longrightarrow (M \otimes \_)$ has kernel F_{Dp}, where $p = pp^M(\overline{a})$ is the pp-type of $\overline{a}$ in M.*

Lemma B.3.8. *(PREST, 2009, Corollary 12.2.3) A pp-n-type p is irreducible if, and only if, the functor $({}_RR^n, \_)/F_{Dp} = (R^n \otimes \_)/F_{Dp}$ is uniform.*

B.4 Serre subcategories

Definition B.4.1 (Serre subcategories)**.** *Let $\mathcal{C}$ be an arbitrary abelian category. We say that a subclass $\mathcal{S}$ is a Serre subcategory if, whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence in $\mathcal{C}$, then $B \in \mathcal{S}$ if, and only if, A, $C \in \mathcal{S}$.*

Observation B.4.2. *Observe that, given a subclass $\mathcal{G}$ of $\mathcal{C}$, we can get the smallest Serre subcategory which contains it by defining $\mathcal{S}_0 = \mathcal{G}$ and, for each $n \geq 1$, define $\mathcal{S}_n$ as the subclass of $\mathcal{G}$ which contains $\mathcal{S}_{n-1}$ and, if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a sequence such that A, $C \in \mathcal{S}_{n-1}$ then $B \in \mathcal{S}_n$, and if $B \in \mathcal{S}_{n-1}$ then A, $C \in \mathcal{S}_n$.*

It is easy to see that $\mathcal{S} = \bigcup \mathcal{S}_n$ is a Serre subcategory and is the smallest one which contains $\mathcal{G}$. Another important thing is that, for every $B \in \mathcal{S}$ there is some n such that $B \in \mathcal{S}_n$, that is, we can get B in a finite number of steps, that is, B can be generated by a finite subset of $\mathcal{G}$.

Theorem B.4.3. *(PREST, 2009, Lemma 12.3.19) Let $\phi_\lambda/\psi_\lambda$ $(\lambda \in \Lambda)$ and ϕ/ψ be pp-pairs and let χ the definable subcategory of Mod-R defined by $T = \{\phi_\lambda/\psi_\lambda\}_{\lambda\in\Lambda}$. Then the following are equivalent:*

(i) ϕ/ψ *is closed on* χ*;*

(ii) F_{D_ψ/D_ϕ} *belongs to the Serre subcategory of* $(R\text{-}mod, Ab)^{fp}$ *generated by the* $F_{D_{\psi_\lambda}/D_{\phi_\lambda}}$*;*

(iii) $F_{\phi/\psi}$ *belongs to the Serre subcategory of* $(R\text{-}mod, Ab)^{fp}$ *generated by the* $F_{\phi_\lambda/\psi_\lambda}$*;*

(iv) $(\phi/\psi) \subset \bigcup_{\lambda\in\Lambda}(\phi_\lambda/\psi_\lambda)$ *(inclusion of Ziegler-open sets).*

APPENDIX C – Pure-injective modules for PID

The reference for this chapter is (KAPLANSKY, 1954). Here our ring will be a PID (principal ideal domain) and we will denote it by (p) the prime ideal generated by p. The original proofs in this book are for abelian groups but, it is shown at (KAPLANSKY, 1954, Chapter 12), that all these results also work for PID.

Theorem C.0.1. *(KAPLANSKY, 1954, Theorem 10) An indecomposable module cannot be mixed; that is, it is either a torsion module or a torsion-free module.*

C.1 Injective modules

Definition C.1.1 (Divisible module)**.** *A module D is said to be a divisible module if for each $d \in D$ and each $r \in R$ there exists some $d' \in D$ such that $d'r = d$. In the language of pp conditions, a divisible module is a module such that the pp-pairs of the form $((x = x)/\exists y(x = ry))$ are closed, for each $r \in R$.*

Theorem C.1.2. *(KAPLANSKY, 1954, Theorem 2) A divisible submodule of a module is a direct summand of that module.*

That is, the divisible modules for these rings are injective modules.

Theorem C.1.3. *(KAPLANSKY, 1954, Theorem 4) A module is divisible if, and only if, it is a direct sum of modules each isomorphic to $Q(R)$, the ring of fractions of R, or to the Prüfer modules, $E(R/P) = \varinjlim_{n\in\mathbb{N}} R/P^n$.*

This also implies that $Q(R)$ and $E(R/P)$ are indecomposable modules, because if there was a submodule which was a direct summand of these it would need to be a pure submodule, hence also divisible, hence injective.

C.2 Pure-injective modules

C.2.1 Modules of bounded order

Definition C.2.1 (Module of bounded order)**.** *A module M is said to be of bounded order if there is an $r \neq 0 \in R$ such that $Mr = 0$. In the language of pp conditions this is* *equivalent to saying that there exists an $r \neq 0 \in R$ such that the pp-pair $(x = x)/(xr = 0)$ is closed in M.*

Definition C.2.2 (Cyclic module)**.** *A module M is said to be cyclic if there exists $m \in M$ such that $mR = \{mr;\ r \in R\} = M$.*

Theorem C.2.3. *(KAPLANSKY, 1954, Theorem 6) A module of bounded order is a direct sum of cyclic modules.*

Theorem C.2.4. *(KAPLANSKY, 1954, Theorem 7) Let M be a module and N a pure submodule of bounded order. Then N is a direct summand of M.*

This last result tells us that the bounded modules are pure-injective. Combining the last two we also get that the cyclic modules of the form R/P^n are pure-injective indecomposable.

C.2.2 P-adic completion

Theorem C.2.5. *(KAPLANSKY, 1954, Theorem 18) The module $\overline{R}_{(P)} = \varprojlim_n R/P^n$ is an indecomposable torsion-free module.*

Definition C.2.6 (Infinite p-height)**.** *An element m of a module M is said to have infinite p-height if, for every $n \in \mathbb{N}$ there exists some $m' \in M$ such that $m'p^n = m$.*

Definition C.2.7 (Complete discrete valuation ring)**.** *A discrete valuation ring R is said to be a complete discrete valuation ring if it is complete in the p-adic topology, which is defined by the metric $|\ .\ | : R \longrightarrow \mathbb{R}$ given by $|r| = 10^{-n}$, where $n = max\{m \in \mathbb{N};\ \exists r' \in R(r'p^m = r)\}$.*

We can extend any discrete valuation ring to a complete discrete valuation ring just by completing it under this topology.

Theorem C.2.8. *(KAPLANSKY, 1954, Theorem 23) Let R be a complete discrete valuation ring, M any R-module and S a pure submodule with no elements of infinite height which is complete in its p-adic topology. Then S is a direct summand of M.*

Thus, any module over a complete discrete valuation ring which is complete in the p-adic topology is pure injective. The next corollary follows from 2.2.14 and by the fact that there exists an embedding $i : R \longrightarrow \overline{R}_{(P)}$ of a discrete valuation ring into its completion under the p-adic topology.

Corollary C.2.9. *Let R be a discrete valuation ring and $\overline{R}_{(P)} = \varprojlim_n R/P^n$ its completion in the p-adic topology. If M is an $\overline{R}_{(P)}$-module complete in the p-adic topology, then M is pure-injective as an R-module.*

Because $\overline{R}_{(P)}$ is complete in the p-adic topology, it is pure-injective by this corollary.

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