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To my mother and my family

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ABSTRACT

This Thesis is devoted to the study of the dimensions of the strata of the moduli of smooth pointed curves by fixing Weierstrass semigroups at the marked point. We provide a new lower bound for the dimension of each stratum by diminishing Deligne–Pinkham's bound by the dimension of the positive graded part of the first module of the cotangent complex associated to a suitable monomial curve. This new lower bound is better than a lower bound given recently by N. Pflueger. We also prove a conditional result which provides the exact dimension when the Weierstrass semigroup is symmetric.

Keywords: Weierstrass Points, Deformation Theory, Gorenstein Curves, Complex Cotangent, Deligne–Pinkham's upper bound.

CONTENTS

0 INTRODUCTION

Once introduced the term *moduli* by B. Riemann in 1857 to designate the number of parameters on which the complex structure of a Riemann surface of genus $g > 1$ depends (he proved that is $3g - 3$), *moduli spaces* started to play a central role in Algebraic and Complex Geometries. The naive idea is that a moduli space parametrize classes of algebraic and/or geometric objects. To describe the structure of this set of classes is, in general, a rather non trivial problem. In regard to moduli space of compact Riemann surfaces of genus g, or equivalently, moduli of smooth algebraic curves of genus g, its structure was understood in a celebrated work by P. Deligne and D. Mumford in the late 60's, where they realized a compactification of the moduli spaces of genus-g curves as a *moduli stack* by allowing *stable curves* at the boundary.

A current open question in Algebraic Geometry is to describe the Chow ring of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of the moduli space of smooth curves with fixed genus $g > 1$. Excellent papers were published trying to answer this great question. Starting with the remarkable works due to C. Faber [Fa1, Fa2] and later by many mathematicians, trying among other things to answer some conjectures proposed by Faber, as we can see in the works of L. Gatto [Ga1, Ga2], Gatto-Ponza [GaPz], Penev-Vakil [PeVa], Pandharipande-Pixton [PanPix], A. Pixton [Pix], and many others. However, a definitive answer seems to be far from being attained. More precisely, we know how to describe the Chow ring of $\overline{\mathcal{M}}_g$ for $g \leq 6$.

As we can see in the above cited papers, some of the cycles of $\overline{\mathcal{M}}_g$ can be described in terms of Weierstrass points. In this way the study of the loci of curves with prescribed Weierstrass gaps becomes very interesting. This leads us to introduce and study the spaces:

$$\mathcal{M}^{\mathcal{S}}_{g,1} := \{[\mathcal{C}, p] \in \mathcal{M}_{g,1} | \mathcal{S}_p = \mathcal{S}\}$$

where $\mathcal{S}$ denotes a fixed numeric semigroup of genus g, while $\mathcal{S}_p$ denotes the Weierstrass semigroup of the curve $\mathcal{C}$ at p. Here $\mathcal{M}_{g,1}$ stands for the moduli space of smooth pointed curves of genus g. So, we should investigate the dimension and global structure of $\mathcal{M}^{\mathcal{S}}_{g,1}$.

About the dimension of $\mathcal{M}^{\mathcal{S}}_{g,1}$ we must highlight two classical and two others recent results. The first one comes from two classical works, by P. Deligne [D] and by H. Pinkham [Pi], to get the following upper bound

$$\dim \mathcal{M}^{\mathcal{S}}_{g,1} \leq 2g - 2 + [\mathrm{End}(\mathcal{S}) : \mathcal{S}]. \tag{1}$$

The above upper bound is attained and Rim-Vittuli [RV] studied classes of semigroups that attain this bound, such classes contain the negatively graded semigroups. Later on, Eisenbud-Harris [EH1, EH2], using Limit Linear Series on stable curves, obtained a lower bound, namely:

$$3g-2-\mathrm{wt}(\mathcal{S})\leq \dim \mathcal{M}_{g,1}^{\mathcal{S}} \tag{2}$$

where $\mathrm{wt}(\mathcal{S})$ denotes the weight of the semigroup $\mathcal{S}$. Although the dimension of Eisenbud-Harris is attained, for example when $\mathrm{wt}(\mathcal{S})\leq \frac{1}{2}g$, it provides negative values for certain classes of semigroups, even when the dimension of the moduli space $\mathcal{M}_{g,1}^{\mathcal{S}}$ is positive.

Through deformations of monomial curves and their syzygies, Contiero and Stöhr [CS] introduced a method to obtain good upper bounds for $\dim \mathcal{M}_{g,1}^{\mathcal{S}}$ when $\mathcal{S}$ is a nonhyperelliptic and nontrigonal symmetric semigroup. The bounds obtained are in examples and in families of semigroups, better than those given by Deligne–Pinkham. The symmetric semigroups are examples of semigroups where the upper and lower bounds given by Deligne–Pinkham and Eisenbud–Harris seem to be far from the exact dimension of $\mathcal{M}_{g,1}^{\mathcal{S}}$, see Chapter 3 of this thesis.

Recently, N. Pflueger [Pf1] improved Eisenbud–Harris lower bound, he also used limit linear series, now with the advances on the theory given by B. Osserman [Os1, Os2, Os3, Os4]. The lower bound given by N. Pflueger is:

$$3g-2-\mathrm{ewt}(\mathcal{S})\leq \dim \mathcal{M}_{g,1}^{\mathcal{S}} \tag{3}$$

where $\mathrm{ewt}(\mathcal{S})$ denotes the effective weight of $\mathcal{S}$. Even obtaining classes of semigroups where their bound is attained, Pflueger constructed classes of semigroups where his bound does not provide the exact dimension of $\mathcal{M}_{g,1}^{\mathcal{S}}$, see [Pf1, 2.6] and [Pf2].

There are few works in the literature regarding to the structure of $\mathcal{M}_{g,1}^{\mathcal{S}}$. We must emphasize here the work of H. Pinkham [Pi] on equivariant deformation theory that provides a tool to study these spaces.

Kontsevich-Zorich [KZ] constructed the space of moduli of pointed curves not fixing the semigroup but only its greater gap, namely $l_g=2g-1$, i.e. they constructed the space $\cup\mathcal{M}_{g,1}^{\mathcal{S}}$ where $\mathcal{S}$ runs over the symmetric semigroups. They also showed that this space has 3 connected components. It is worth emphasizing the importance of this space in the theory of dynamic systems, cf. [FoMa] and [Z]. Indeed billards in convex polygons correspond to billards in suitable compact Riemann surfaces of genus g with a holomorphic differential having zero of order $2g-2$ at a point. A simple application of the Riemann-Roch Theorem shows that the Weierstrass semigroup at this point in the Riemann surface is symmetric. E. Bullock [Bu1] classified the generic curve of each connect component of the Kontsevich-Zorich space in terms of Weierstrass semigroups. Furthermore, he showed in [Bu2] that $\mathcal{M}_{g,1}^{\mathcal{S}}$ is stably rational for $g\leq 6$, except for a few cases. His study was done case by case.

A compactification of $\mathcal{M}_{g,1}^{\mathcal{S}}$ was obtained by Stöhr [S] and then by Contiero-Stöhr [CS], in the symmetric non-trigonal case, by considering Gorenstein curves at its boundary. The compactification is obtained through an explicit construction and realizes it as a closed subset in a suitable projective space. Contiero and his former student A. Fontes [CF], extended the construction also to trigonal numerical semigroups.

Recently N. Pflueger described some particular cases of numerical semigroups, called Castelnuovo semigroups, where $\mathcal{M}_{g,1}^{\mathcal{S}}$ is not irreducible and is not of pure dimension, these are the first examples in the literature.

Using some classical works,we can show that if $\mathcal{S}$ is symmetric and generated by at most 4 elements, then a compactification of $\mathcal{M}_{g,1}^{\mathcal{S}}$ is a projective space whose dimension is well known, see Equation (3.1).

The present thesis is organized in three main chapters as follows: in Chapter 1 we provide the necessary tools to the development the others chapters, such as algebraic and formal deformations of schemes. We also summarize the construction of the $\overline{\mathcal{M}_{g,1}^{\mathcal{S}}}$ made by Contiero, Fontes and Stöhr [CS, CF, S]. We conclude this chapter with a clearer proof of Deligne's formula and point out how to obtain the upper bound for $\dim \mathcal{M}_{g,1}^{\mathcal{S}}$ given by Deligne–Pinkham, which is the main upper bound until now and the main subject of study here.

In the chapter 2 we apply the techniques developed by [CS, CF, S] to construct the moduli $\mathcal{M}_{g,1}^{\mathcal{S}}$ when $\mathcal{S}$ runs over the family $\mathcal{S} =< 6, 7+6\tau, 8+6\tau, 9+6\tau, 10+6\tau >$, cf. Theorem 13. Additionally, we also compute the dimension of $\mathcal{M}_{g,1}^{\mathcal{S}}$ for all semigroups in this family, see Corollary 3. In this same chapter we also collect the known dimensions of $\mathcal{M}_{g,1}^{\mathcal{S}}$ for semigroups and families of semigroups, see Tables 1, 2 and Corollary 4. With this collecting data in hands, a completely natural question emerges:

What is the role that $T^{1,+}(\mathbf{k}[\mathcal{S}])$ plays on the dimension of $\mathcal{M}_{g,1}^{\mathcal{S}}$?

Here $T^{1,+}(\mathbf{k}[\mathcal{S}])$ stand for the positive graded part of the first cotangent complex associated to the $\mathbf{k}$-algebra $\mathbf{k}[\mathcal{S}]$.

A partial answer for the above question is the main result of Chapter 4. We show that $2g - 2 - [\mathrm{End}(\mathcal{S}) : \mathcal{S}] - \dim \mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}])$ is a lower bound for the dimension of $\mathcal{M}_{g,1}^{\mathcal{S}}$, when $\mathcal{M}_{g,1}^{\mathcal{S}}$ is non empty. The proof uses deeply the theory of equivariant deformation of Pinkham and a result in Ph.D. Thesis of Schlessinger, cf. Theorem 16. This new lower bound is never smaller than Pflueger's bound in [Pf1], so we get an improvement of it, see Proposition 4. As a final result we investigate the case when the monomial curve associated do $\mathcal{S}$ is Gorenstein, i.e. $\mathcal{S}$ is symmetric, see Section 3.1. In this case we provide a conditional result that our new lower bound is also an upper bound. The condition depends on finding a tight upper bound for the degree of the normal sheaf of a canonical Gorenstein (monomial) curve. It is known that this tight upper bound is attained when the curve is locally complete intersection. We strongly believe that this tight bound is an achievable result, we do not know any counterexample and we keep working on it.

1 PRELIMINARIES

1.1 A GLIMPSE ON DEFORMATION

We review some of the relevant elements of deformation theory, details can be found in [Es], [Js] and/or [Rh]. We fixed once and for all an algebraically closed field **k**.

Deformation theory is closely related to the problem of classification of algebro-geometric objects. For example, if we consider a class $\mathcal{M}$ of algebro-geometric objects:

$$\mathcal{M} = \{\text{projective nonsingular curves of genus } g\}/(\text{isomorphism}),$$

$$\mathcal{M} = \{\text{closed subschemes of } \mathbb{P}^r \text{with given Hilbert polynomial}\}.$$

The basic problem, but far from having an easy answer, is the following: What is the structure of $\mathcal{M}$? Would be wonderful if the following could be true

Postulate. *Classes of algebro-geometric objects are parametrized by also algebro-geometric objects.*

The interest and the difficulty of this problem come from the existence of families. The existence of *families of objects* in $\mathcal{M}$ implies that $\mathcal{M}$ is not just a set but has some kind of "structure", hopefully will be a scheme, which will be the *moduli space* of the classification problem. In most cases $\mathcal{M}$ is not a scheme but has a weaker structure. So the above postulate is not true, but can be true if we consider another classes of algebro-geometric spaces, e.g. *algebraic stacks.*

The basic notion of *family* is related to the natural fact that all objects of algebraic geometry can be "*deformed*" by *varying the coefficients of their defining equations.*

If for example we want to consider a class $\mathcal{M}$ of algebraic varieties (e.g., smooth and/or projective curves), then a family will be a *flat* morphism

$$\begin{array}{c} \mathcal{X} \\ \downarrow{\scriptstyle \pi} \\ S \end{array}$$

whose fibres $\mathcal{X}(s) = \pi^{-1}(s), s \in S$, are elements of $\mathcal{M}$. If the class $\mathcal{M}$ consists of complete and/or nonsingular varieties, then π will be also required to be proper and/or smooth. Here $\mathcal{X}$ and S are called, respectively, the *total space* and the *parameter space* of the family. If S is connected then π is called a *family of deformations* of $\mathcal{X}(s_0)$ for any $s_0 \in S$.

Example 1. Let $\mathcal{M}$ be a class of closed subschemes of $\mathbb{P}^r$. A family will be a diagram:

$$\begin{array}{c} \mathcal{X} \subset S \times \mathbb{P}^r \\ \big\downarrow \pi \\ S \end{array}$$

where π is the restriction of the first projection, the inclusion is closed, and all fibres of π are in $\mathcal{M}$. Typically, a family of hypersurfaces of degree d in $\mathbb{P}^r$ parametrized by an affine space $\mathbb{A}^n = \mathrm{Spec}(\mathbf{k}[t_1, ..., t_n])$, $\mathbf{k}$ being a field, will be a hypersurface $H \subset \mathbb{A}^n \times \mathbb{P}^r$ defined by a homogeneous polynomial $P(t, X) \in \mathbf{k}[t_1, ..., t_n, X_0, ..., X_r]$ of degree d in $X_0, ..., X_r$.

In order to provide a more formal definition of what deformation is, we will consider the following categories of $\mathbf{k}$-algebras:

$\mathcal{A}$:= the category of local artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$;
$\hat{\mathcal{A}}$:= the category of complete local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$;
$\mathcal{A}^*$:= the category of local noetherian k-algebras with residue field $\mathbf{k}$,
where morphisms are unitary $\mathbf{k}$-homomorphisms, which are local in $\mathcal{A}, \hat{\mathcal{A}}$ and $\mathcal{A}^*$. Note $\mathcal{A}^* \subset \hat{\mathcal{A}} \subset \mathcal{A}$. An algebraic scheme means a scheme over $\mathbf{k}$ of finite type.

Let X be an algebraic scheme. A *deformation* of X parametrized by S (or over S) is a cartesian diagram of morphisms of schemes

$$\eta : \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array} \tag{1.1}$$

where π is flat and surjective, and S is connected. We call S and $\mathcal{X}$, respectively, the *parameter scheme* and the *total scheme* of the deformation.

If S is algebraic, for each $\mathbf{k}$-rational point $t \in S$ the scheme-theoretic fibre $\mathcal{X}(t)$ is also called a *deformation* of X. When $S = \mathrm{Spec}(A)$ with A in $\mathrm{ob}(\mathcal{A}^*)$ and $s \in S$ is the closed point we have a *local family of deformations* (shortly, a *local deformation*) of X over A. The deformation η will be also denoted (S, η) or (A, η) when $S = \mathrm{Spec}(A)$. The local deformation (A, η) is *infinitesimal* (resp., *first order*) if $A \in \mathrm{ob}(\mathcal{A})$ (resp., $A = \mathbf{k}[\epsilon]$, with $\epsilon^2 = 0$).

Remark 1. If X is nonsingular and/or projective we will require π to be smooth and/or projective.

A morphism between two deformations (S, η) and (S, β) of X is a morphism $\Phi : \mathcal{X} \to \mathcal{Y}$ such that the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ \big\downarrow \pi & & \big\downarrow \pi' \\ S & \xrightarrow{id} & S \end{array}$$

is commutative.

A deformation of X over S is called *trivial* if the cartesian diagram is of the form:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \cong X \times_{\mathrm{Spec}(\mathbf{k})} S \\ \downarrow & & \downarrow \pi' \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & S \end{array}.$$

All fibres over **k**-rational points of a trivial deformation of X parametrized by an algebraic scheme are isomorphic to X. The converse is not true: there are deformations which are not trivial but have isomorphic fibres over all the **k**-rational points (see Example (2) below). The scheme X is called *rigid* if every infinitesimal deformation of X over A is trivial for every A in $\mathrm{ob}(\mathcal{A})$.

An infinitesimal deformation η of X is called *locally trivial* if every point $x \in X$ has an open neighbourhood $U_x \subset X$ such that

$$\eta|_{U_x}: \quad \begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}|_{U_x} \\ \downarrow & & \downarrow \pi' \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & S \end{array}$$

is a trivial deformation of U_x.

Example 2. Let $\mathrm{Spec}(\mathbf{k}[x,y]/(xy))$ be the local equation of a simple node in the plane $\mathbb{A}^2 = \mathrm{Spec}(\mathbf{k}[x,y])$. We consider the family $\mathcal{X}$ given by $\mathrm{Spec}(\mathbf{k}[x,y,s]/(xy-s))$ in $\mathbb{A}^3 = \mathrm{Spec}(\mathbf{k}[x,y,s])$, together with its map to the parameter space $S = \mathbb{A}^1 = \mathrm{Spec}(\mathbf{k}[s])$. Thus, via the projection

$$\begin{array}{ccc} \mathbb{A}^3 & \to & \mathbb{A}^1 \\ (x,y,s) & \mapsto & s \end{array}$$

we obtain a diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathbf{k}[x,y]/(xy)) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & S = \mathbb{A}^1 \end{array}$$

i.e. we have a flat family $\mathcal{X} \to \mathbb{A}^1$ whose fibres are affine conics. For $s \neq 0$ the fiber is a nonsingular hyperbola. For $s = 0$ we recover the original nodal singularity. Note that this family is not trivial since the fibre $\mathcal{X}(0)$ is singular, hence not isomorphic to the fibres $\mathcal{X}(s)$, $s \neq 0$, which are nonsingular.

Example 3. Let $f : X \to Y$ be a surjective morphism of algebraic schemes, with X integral and Y an irreducible and nonsingular curve. Then f is flat. This is a special case of Prop. III.9.7 of [Rh1]. Therefore f defines a family of deformations of any of its closed fibres.

Due to basis extension properties of flat morphism, we are able to extend the parameter space of the deformation. Let us consider a deformation of X as (1.1) and a morphism $\psi : S' \to S$. The *induced deformation* by ψ is the cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \times_S S' \\ \downarrow & & \downarrow{\scriptstyle \psi^*(\pi)} \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & S' \end{array} .$$

Definition 1. *A deformation (S, η) of X is called* miniversal *or* semi-universal *if every deformation (S', η') of X is isomorphic to a deformation $\psi^*(\pi)$, for some map $\psi : S' \to S$.*

One of the main classical results on first order deformation, that is useful for dealing with rigid/non-rigidness is the well known:

Kodaira–Spencer Correspondence ([Es]). *Let X be an algebraic variety and $T_X \cong Hom(\Omega^1_X, \mathcal{O}_X) \cong \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ its tangent sheaf. There is a 1-1 correspondence*

$$\alpha : \left\{ \begin{array}{c} \text{isomorphism classes of first order} \\ \text{locally trivial deformations of X} \end{array} \right\} \longleftrightarrow H^1(X, T_X), \tag{1.2}$$

such that $\alpha(\eta) = 0$ if and only if η is the trivial deformation class. Since every first order deformation of a smooth algebraic variety is locally trivial, we get in particular that if X is nonsingular, then α is a 1-1 correspondence

$$\alpha : \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{first order deformations of X} \end{array} \right\} \longleftrightarrow H^1(X, T_X).$$

Proposition 1 ([Js], Prop. page 23)**.** *The $\mathcal{O}_X$- module of first-order deformations is isomorphic to the normal bundle $N_X = Hom_X(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$.*

1.2 FORMAL DEFORMATION

Since first order deformations of smooth varieties are locally trivial, the most interesting study of (local) deformations lies on deformation of singularities. The naive idea is that we may think of the localized deformation as a deformation by restricting the singular variety and the base space to **arbitrary small neighbourhoods** of the singularity and of the closed point of the base space.

Let us start with a typical model of this thesis, that is the case of a projective curve $\mathcal{C}$ with a unique (isolated) singularity. Take a (global) deformation of $\mathcal{C}$, that is a cartesian diagram

$$\eta : \begin{array}{ccc} \mathcal{C} \cong \mathcal{X} \times_S \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{s_0\} := \mathrm{Spec}(\mathbf{k}) & \longrightarrow & S \end{array} ,$$

where π is a proper flat morphism and s_0 is a closed point of S.
We first localize the base space of above global deformation of $\mathcal{C}$, getting the local deformation

$$\begin{array}{ccccc} \mathcal{X}\times_S \operatorname{Spec}(\mathbf{k})\cong\mathcal{C} & \longrightarrow & \mathcal{X}\times_S \operatorname{Spec}(\mathcal{O}_{S,s_0}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \{s_0\}=\operatorname{Spec}(\mathbf{k}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{S,s_0}) & \longrightarrow & S \end{array}.$$

The above localized deformation $\operatorname{Spec}(\mathcal{O}_{S,s_0})\times_S\mathcal{X}\to\operatorname{Spec}(\mathcal{O}_{S,s_0})$ is trivial if and only if the global deformation $\mathcal{X}\to S$ of $\mathcal{C}$ becomes trivial if we restrict to a sufficiently small neighbourhood of s_0. More generally, a second global deformation

$$\begin{array}{ccc} \mathcal{X}'\times_{S'} \operatorname{Spec}(\mathbf{k})\cong\mathcal{C} & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \{s_0'\}:=\operatorname{Spec}(\mathbf{k}) & \longrightarrow & S' \end{array}$$

induces the same local deformation if and only if they are isomorphic after eventually restricting the bases S and S' to some open neighbourhood of s_0 and s_0', respectively.

In order to study localizations of the global deformations, using a powerful microscope, we have to restrict the base space to an **arbitrary small neighbourhood** of its closed point s_0. Hence, we may even consider for each $n\geq 1$ the infinitesimal deformations:

$$\begin{array}{ccccccc} \mathcal{X}\times_S \operatorname{Spec}(\mathbf{k})\cong\mathcal{C} & \longrightarrow & \mathcal{X}\times_S \operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n+1}) & \longrightarrow & \mathcal{X}\times_S \operatorname{Spec}(\mathcal{O}_{S,s_0}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{s_0\}=\operatorname{Spec}(\mathbf{k}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n+1}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{S,s_0}) & \longrightarrow & S \end{array}$$

where by properties of base extensions the special fibre of the infinitesimal deformation $\mathcal{X}\times_S\operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n+1})\to\operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n+1})$ is isomorphic to $\mathcal{C}$.

We may even obtain a **formal deformation** defined as the inverse system of the above infinitesimal deformations, as can be read of from the following diagram.

$$\begin{array}{ccccccccc} \mathcal{C} & \longrightarrow & \cdots & \longrightarrow & \mathcal{X}\times_S \operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n}) & \longrightarrow & \mathcal{X}\times_S \operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n+1}) & \longrightarrow & \cdots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ \{s_0\} & \longrightarrow & \cdots & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{S,s_0}/\mathfrak{m}_{s_0}^{n+1}) & \longrightarrow & \cdots \end{array}$$

Remark 2. *There is a technical difficulty just on the first line of the above diagram, because the tensor product need not be compatible with the projective limit. So it is required a more suggestive language to deal with it.*

For a moment let us ignore this technical difficult and let us restrict the curve $\mathcal{C}$ to small open neighbourhoods of its singularity p. The composed morphism $\mathcal{C}\to\mathcal{X}\to S$ induces local morphisms $\operatorname{Spec}(\mathcal{O}_{\mathcal{C},p})\to\operatorname{Spec}(\mathcal{O}_{\mathcal{X},x_0})\to\operatorname{Spec}(\mathcal{O}_{S,s_0})$ of local schemes and so we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec}(\mathcal{O}_{\mathcal{C},p}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{\mathcal{X},x_0}) & \searrow & \\
\downarrow & & \downarrow & & \\
\mathcal{C} \cong \mathrm{Spec}(\mathbf{k}) \times_S \mathcal{X} & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s_0}) \times_S \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
\{t_0\} = \mathrm{Spec}\,\mathbf{k} & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s_0}) & \longrightarrow & S
\end{array} .$$

It follows by base extensions properties that we get a cartesian diagram

$$\begin{array}{ccc}
\mathrm{Spec}(\mathcal{O}_{\mathcal{C},p}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{\mathcal{X},x_0}) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s_0})
\end{array} .$$

Thus we have that the above deformation of the local scheme $\mathrm{Spec}(\mathcal{O}_{C,p})$ and the deformation of $\mathcal{C}$ have the same base space.

$$\begin{array}{ccc}
\mathrm{Spec}(\mathcal{O}_{\mathcal{C},p}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{\mathcal{X},x_0}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s_0})
\end{array} .$$

Now we discuss when a series of infinitesimal deformation comes from a deformation, that is, a deformation over some $A \in \mathrm{ob}(\hat{\mathcal{A}})$, by considering the general setting of the above typical case including the suggestive language to avoid (or ignore) the technical difficult.

Let's consider X an algebraic scheme and $A \in \mathrm{ob}(\hat{\mathcal{A}})$. A formal deformation of X over A is $\hat{\eta} \in \mathrm{Def}_X(A)$,(see example (6) page 20) i.e.

$$\hat{\eta} = \left\{ \eta_n : \begin{array}{ccc}
X & \xrightarrow{f_n} & \mathcal{X}_n \\
\downarrow & & \downarrow{\scriptstyle \pi_n} \\
\mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(A_n)
\end{array} \right\}_{n \geq 0} ,$$

where $A_n = A/m_A^{n+1}$, each η_n is a fiber product, and the pullback $\mathcal{X}_n \otimes_{\mathrm{Spec}(A_n)} \mathrm{Spec}(A_{n-1})$ is isomorphic to $\mathcal{X}_{n-1}$.

A natural question is: is it true that all these fiber products are pullbacks of some deformation over A? That is, we ask for the existence of a deformation $\pi : \mathcal{X} \to \mathrm{Spec}(A)$, making the fiber product diagram:

$$\begin{array}{ccccccccc}
X & \xrightarrow{f_0} & \cdots & \longrightarrow & \mathcal{X}_{n-1} & \longrightarrow & \mathcal{X}_n & \longrightarrow \cdots \longrightarrow & \mathcal{X} \\
\downarrow & & & & \downarrow{\scriptstyle \pi_{n-1}} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi} \\
\mathrm{Spec}(\mathbf{k}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Spec}(A_{n-1}) & \longrightarrow & \mathrm{Spec}(A_n) & \longrightarrow \cdots \longrightarrow & \mathrm{Spec}(A)
\end{array} .$$

We say that (A, η_n) is *effective* if such a deformation

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

exists.

Remark 3. A formal deformation is the same as giving a morphism of formal schemes

$$\bar{\pi} : \bar{\mathcal{X}} \to \mathrm{Specf}(A)$$

where $\bar{\mathcal{X}} = (X; \varprojlim \mathcal{O}_{\mathcal{X}_n})$, $\mathrm{Specf}(A)$ is a formal spectrum; and $\bar{\pi} = \varprojlim \pi_n$

Remark 4. A formal deformation is effective if and only if there exists a deformation

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

such that $\bar{\mathcal{X}}$ is the completation of $\mathcal{X}$ along X, i.e.. $\bar{\mathcal{X}} = \hat{\mathcal{X}} = (X; \varprojlim \mathcal{O}_{\mathcal{X}/\mathcal{I}^n_{\mathcal{X}}})$

Example 4. For $X = \mathbb{P}^r$ the formal deformation

$$\left\{ \eta_n : \begin{array}{ccc} \mathbb{P}^r & \longrightarrow & \mathbb{P}^r_{A_n} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(A_n) \end{array} \right\}_{n \geq 0},$$

is effective; the associated formal scheme[1] is the completation of $\mathbb{P}^r_A$ along $\mathbb{P}^r$, denoted by $\mathcal{P}^r_A = (\mathbb{P}^r; \varprojlim \mathcal{O}_{\mathbb{P}^r_{A_n}})$.

Theorem 1 (Grothendieck, [Es], Thm 2.5.13)**.** *Let X be a projective scheme.*

1. *Let $A \in \mathrm{ob}\hat{A}$ and $\bar{\pi} : \bar{\mathcal{X}} \to \mathrm{Specf}(A)$ be a formal deformation of X over A. Assume that there exists j such that the diagram*

$$\begin{array}{ccc} \bar{\mathcal{X}} & \xrightarrow{\ j\ } & \mathcal{P}^r_A \\ & {\scriptstyle \bar{\pi}} \searrow & \downarrow p \\ & & \mathrm{Specf}(A) \end{array}$$

   is commutative, where p is the projection. Then $\bar{\pi}$ is effective.

2. *Assume that $H^2(X, \mathcal{O}_X) = 0$. Then every formal deformation of X is effective.*

1 [Rh1] pag 190

Now we recall the concept and basic properties of functors of Artin rings that was developed by M. Schlessinger in his Ph.D. thesis.

For a given $\Lambda \in \mathrm{ob}(\mathcal{A}^*)$ we will consider the following:

$\mathcal{A}_\Lambda$ = the category of local artinian Λ-algebras with residue field $\mathbf{k}$

$\mathcal{A}^*_\Lambda$ = the category of local noetherian Λ-algebras with residue field $\mathbf{k}$

They are subcategories of $\mathcal{A}$ and $\mathcal{A}^*$ respectively. If Λ is in $\mathrm{ob}(\hat{\mathcal{A}})$ then we will let

$\hat{\mathcal{A}}_\Lambda$ = the category of complete local noetherian Λ-algebras with residue field $\mathbf{k}$. This is a subcategory of $\hat{\mathcal{A}}$=category of complete local artinian rings with residual field $\mathbf{k}$.

A *functor of Artin rings* is a covariant functor

$$F : \mathcal{A}_\Lambda \to (\text{sets})$$

where $\Lambda \in \mathrm{ob}(\hat{\mathcal{A}})$. Let $A \in \mathrm{ob}(\mathcal{A}_\Lambda)$. An element $\xi \in F(A)$ will be called an *infinitesimal deformation* of $\xi_0 \in F(\mathbf{k})$ if $\xi \to \xi_0$ under the map $F(A) \to F(\mathbf{k})$; if $A = \mathbf{k}[\epsilon]$ then ξ is called a *first order deformation* of ξ_0.

Example 5. Functors of Artin rings are obtained by fixing an R in $\mathrm{ob}(\hat{\mathcal{A}}_\Lambda)$ and letting:

$$\begin{array}{rcl} h_{R/\Lambda} : \mathcal{A}_\Lambda & \to & (\text{sets}) \\ A & \mapsto & h_{R/\Lambda}(A) = \mathrm{Hom}_{\hat{\mathcal{A}}_\Lambda}(R, A). \end{array}$$

Such a functor is clearly nothing but the restriction to $\mathcal{A}_\Lambda$ of a representable functor on $\hat{\mathcal{A}}_\Lambda$.

Definition 2. *A functor of Artin rings F is called* prorepresentable *if is isomorphic to $h_{R/\Lambda}$ for some $R \in \mathrm{ob}(\hat{\mathcal{A}}_\Lambda)$. In case $\Lambda = \mathbf{k}$ we write h_R instead of $h_{R/\mathbf{k}}$.*

Every *representable* functor $h_{R/\Lambda}$, $R \in \mathrm{ob}(\mathcal{A}_\Lambda)$, is a trivial example of prorepresentable functor.

Typically a prorepresentable functor of Artin rings arises as follows. We consider a scheme M and the restriction

$$\begin{array}{rcl} \Psi : \mathcal{A} & \to & (\text{sets}) \\ A & \mapsto & \Psi(A) = \mathrm{Hom}(\mathrm{Spec}(A), M) \end{array}$$

of the functor of points

$$\mathrm{Hom}(-, M) : (\text{schemes})^\circ \to (\text{sets})$$

Ψ is a functor of Artin rings; if $\phi : \mathrm{Spec}(A) \to M$ is an element of $\Psi(A)$ then ϕ is an infinitesimal deformation of the composition

$$\mathrm{Spec}(\mathbf{k}) \to \mathrm{Spec}(A) \xrightarrow{\phi} M$$

where the first morphism corresponds to $A \to A/m_A = \mathbf{k}$. For a fixed $\mathbf{k}$-rational point $m \in M$, we may consider the subfunctor

$$F : \mathcal{A} \to (\text{sets})$$

of Ψ defined as follows:

$$F(A) = \text{Hom}(\text{Spec}(A), M)_m = \left\{ \begin{array}{c} \text{morphism Spec}(A) \to M \\ \text{whose image is } \{m\} \end{array} \right\}.$$

Note that, an element of $F(A)$, i.e., a morphism

$$\begin{array}{rrcl} \phi : & \text{Spec}(A) & \to & M \\ & \text{closed point} & \mapsto & \{m\} \end{array}$$

corresponds to a homomorphism of local $\mathbf{k}$-algebras

$$\tilde{\phi} : \mathcal{O} = \mathcal{O}_{M,m} \to A.$$

Since A is artinian, $\tilde{\phi}$ factors through the completion $\mathcal{O} \to \hat{\mathcal{O}}$ with respect to the maximal ideal and therefore the properties of $\mathcal{O}$ detected by the study of infinitesimal deformations will be preserved under completion. We have $F = h_R$, where $R = \hat{\mathcal{O}}_{M,m}$, so F is prorepresentable.

A prorepresentable functor $F = h_{R/\Lambda}$ satisfies the following conditions:

H_0) $F(\mathbf{k})$ consists of one element (the canonical quotient $R \to R/m_R = \mathbf{k}$).

Let

$$\begin{array}{ccc} & & A'' \\ & & \downarrow \\ A' & \longrightarrow & A. \end{array} \tag{1.3}$$

be a diagram in $\mathcal{A}_\Lambda$ and consider the natural map

$$\alpha : \ F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \tag{1.4}$$

induced by the commutative diagram:

$$\begin{array}{ccc} F(A' \times_A A'') & \longrightarrow & F(A'') \\ \downarrow & & \downarrow \pi \\ F(A') & \longrightarrow & F(A), \end{array}$$

then

H_l) (*left exactness*) For every diagram (1.3) α is bijective.

H_f) $F(\mathbf{k}[\epsilon])$ has a structure (see [Es] page 52) of finite dimensional $\mathbf{k}$-vector space.

A property weaker than H_l satisfied by a prorepresentable functor F is the following:

H_ϵ) α is bijective if $A = \mathbf{k}$ and $A'' = k[\epsilon]$.

Lemma 1 ([Es], Lemma 2.2.1). *If F is a functor of Artin rings having properties H_0 and H_ϵ then the set $F(\mathbf{k}[\epsilon])$ has a structure of $\mathbf{k}$-vector space in a functorial way. This vector space is called the* tangent space of the functor F , *and denoted t_F. If $F = h_{R/\Lambda}$ then $t_F = t_{R/\Lambda}$. For a natural transformation $f : F \to G$ between such functors, $df : t_F \to t_G$ is called the* differential of f.

Every functor of Artin rings F can be extended to a functor

$$\hat{F} : \hat{\mathcal{A}}_\Lambda \to (\text{sets})$$

defined by

$$\hat{F}(R) = \varprojlim F(R/m_R^{n+1})$$

for every $R \in \text{ ob}(\hat{A}_\Lambda)$, and for every $\varphi : R \to S$:

$$\hat{F}(\varphi) : \hat{F}(R) \to \hat{F}(S)$$

to be the map induced by the maps $F(R/m_R^n) \to F(S/m_S^n), n \geq 1$.

Definition 3. *An element $\hat{u} \in \hat{F}(R)$ is called a* formal element *of F.*

By definition $\hat{u}$ can be represented as a system of elements $\{u_n \in F(R/m_R^{n+1})\}_{n\geq 0}$ such that for every $n \geq 1$ the map

$$F(R/m_R^{n+1}) \to F(R/m_R^n)$$

induced by the projection

$$R/m_R^{n+1} \to R/m_R^n \quad (1.5)$$
$$u_n \mapsto u_{n-1}, \quad (1.6)$$

$\hat{u}$ is also called a *formal deformation* of u_0.

If for example F is the functor of infinitesimal deformations of a nonsingular variety X, each u_n is an infinitesimal deformation of X parametrized by Spec(R/m^{n+1}). The compatibility condition (1.6) is that u_n pulls back to u_{n-1} under the closed embedding

$$\text{Spec}(R/m^n) \subset \text{Spec}(R/m^{n+1}).$$

In this case the formal element $\hat{u}$ is also called a *formal family of deformations* of X.

Lemma 2 ([Es], Lemma 2.2.2)**.** *Let $R \in \mathrm{ob}(\mathcal{A}_\Lambda)$. Then there exist a bijection*

$$\left\{\hat{u} \in \hat{F}(R)\right\} \longleftrightarrow \{\text{natural transformations } h_{R/\Lambda} \to F\}.$$

Definition 4. *1. $f : F \to G$ is smooth if for every surjection $B \to A, F(B) \to F(A) \times_{G(A)} G(B)$ is also surjective.*

2. F is smooth if for every surjection $B \to A, F(B) \to F(A)$ is surjective (i.e., the natural transformation from F to the trivial functor is smooth).

Definition 5. *Fix $\hat{u} \in \hat{F}(R)$, where $R \in \mathrm{ob}(\hat{\mathcal{A}}_\Lambda)$. Recall that $\hat{u}$ induces $\hat{u} : h_{R/\Lambda} \to F$.*

1. We call $\hat{u}$ semi-universal if $\hat{u} : h_{R/\Lambda} \to F$ is smooth, and $t_{R/\Lambda} \to t_F$ is bijective (where $t_{R/\Lambda}$ is the tangent space for h_R/Λ).

2. We call $\hat{u}$ universal if $\hat{u} : h_{R/\Lambda} \to F$ is an isomorphism.

Definition 6. *Let R be a local $\mathbf{k}$-algebra with residue field $\mathbf{k}$. A small extension of R is a $\mathbf{k}$-extension of R by $\mathbf{k}$:*

$$0 \to \mathbf{k} \to R' \to R \to 0$$

such that $\mathbf{k}^2 = 0$ in R'.

Theorem 2 (Schlessinger, [Ms1])**.** *Let $F : \mathcal{A}_\Lambda \to$ (sets) be a functor of Artin rings satisfying condition H_0. Let $A' \to A$ and $A'' \to$ be homomorphism in $\mathcal{A}_\Lambda$ and let*

$$\alpha : \; F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \tag{1.7}$$

be the natural map. Then:

i) F has a semi-universal formal element if and only if satisfies the following conditions:

$\bar{H}$) if $A'' \to A$ is a small extension, then the map (1.7) is surjective.

H_ϵ) if $A = \mathbf{k}$ and $A'' = \mathbf{k}[\epsilon]$, then the map (1.7) is bijective.

H_f) $\dim_{\mathbf{k}}(t_F) < \infty$.

ii) F has a universal element if and only if also satisfies the following condition:

H) the natural map

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

is bijective for every small extension $A' \to A$ in $\mathcal{A}_\Lambda$.

Example 6. For an algebraic scheme X, define

$$\begin{aligned} \mathrm{Def}_X : \mathcal{A} &\to \text{(sets)} \\ A &\mapsto \mathrm{Def}_X(A) = \{\text{deformation of } X \text{ over } A\}/\text{isomorphism}. \end{aligned}$$

Then $\mathrm{Def}_X(\mathbf{k}) = \{\text{point}\}$, and Def_X satisfies $\bar{H}$ and H_ϵ.

1.3 THE COTANGENT SHEAF T^1_X

Let $X \to S$ be a morphism of schemes. An *extension* of X/S is a closed immersion $X \subset X'$, where X' is an S-scheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ such that $\mathcal{I}^2 = 0$. To give an extension $X \subset X'$ of X/S is equivalent to giving an exact sequence on X:

$$\xi : \quad 0 \to \mathcal{I} \to \mathcal{O}_{X'} \xrightarrow{\phi} \mathcal{O}_X \to 0$$

where $\mathcal{I}$ is an $\mathcal{O}_X$-module, ϕ is a homomorphism of $\mathcal{O}_S$-algebras and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$. In this way, ξ is called an *extension of* X/S *by* $\mathcal{I}$ or *with kernel* $\mathcal{I}$.

Given two extensions $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$, they are called *isomorphic* if there is an $\mathcal{O}_S$-homomorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X''}$ inducing the identity on both $\mathcal{I}$ and $\mathcal{O}_X$. It follows that α must necessarily be an S-isomorphism.

Notation 1.

1. We denote by $\mathrm{Ex}(X/S, \mathcal{I})$ the set of isomorphism classes of extensions of X/S with kernel $\mathcal{I}$. In case $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes and $\mathcal{I} = \tilde{N}$ we have the following identification:

$$\mathrm{Ex}_A(B, N) = \mathrm{Ex}(X/S, \mathcal{I}).$$

2. If $S = \mathrm{Spec}(A)$ is affine we will sometimes write $\mathrm{Ex}_A(X, \mathcal{I})$ instead of $\mathrm{Ex}(X/\mathrm{Spec}(A), \mathcal{I})$.

Exactly as in the affine case (see Appendix A) we can prove that $\mathrm{Ex}(X/S, \mathcal{I})$ is a $\Gamma(X, \mathcal{O}_X)$-module with identity element the class of the trivial extension:

$$0 \to \mathcal{I} \to \mathcal{O}_X \tilde{\oplus} \mathcal{I} \to \mathcal{O}_X \to 0$$

where $\mathcal{O}_X \tilde{\oplus} \mathcal{I}$ is defined similarly as in the module case (see Appendix (A)). The correspondence

$$\mathcal{I} \to \mathrm{Ex}(X/S, \mathcal{I})$$

defines a covariant functor from $\mathcal{O}_X$-modules to $\Gamma(X, \mathcal{O}_X)$-modules.

The most important case in deformation theory is when $\mathcal{I} = \mathcal{O}_X$, being related to first order deformations.

Note that given a morphism of finite type of schemes $f : X \to S$ we can define a quasi-coherent sheaf $T^1_{X/S}$ on X with the following properties. If $U = \mathrm{Spec}(A)$ is an affine open subset of S and $V = \mathrm{Spec}(B)$ is an affine open subset of $f^{-1}(U)$, then

$$\Gamma(V, T^1_{X/S}) = T^1_{B/A}.$$

It follows from the properties of the first cotangent modules that $T^1_{X/S}$ is coherent. $T^1_{X/S}$ is called the *first cotangent sheaf* of X/S.

Proposition 2 ([Es], Prop. 1.1.9). *1. If X is an algebraic scheme, then T^1_X is supported on the singular locus of X. More generally if $X \to S$ is a morphism of finite type of algebraic schemes, then $T^1_{X/S}$ is supported on the locus where X is not smooth over S.*

2. If we have a closed embedding $X \subset Y$ with Y nonsingular, then we have an exact sequence of coherent sheaves on X

$$0 \to T_X \to T_{Y|X} \to N_{X/Y} \to T^1_X \to 0 \tag{1.8}$$

so that, letting $N'_{X/Y} = \ker[N_{X/Y} \to T^1_X]$, we have the short exact sequence

$$0 \to T_X \to T_{Y|X} \to N'_{X/Y} \to 0 \tag{1.9}$$

$N'_{X/Y}$ is called the equisingular normal sheaf of X in Y.

Theorem 3 ([Es], Thm. 1.1.10). *Let $X \to S$ be a morphism of finite type of algebraic schemes and $\mathcal{I}$ be a coherent locally free sheaf on X. Assume that X is reduced and S-smooth on a dense open subset. Then*

$$\mathrm{Ex}(X/S, \mathcal{I}) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}).$$

An immediate consequence of this theorem is:

Corollary 1 ([Es], Cor. 1.1.11). *Let $X \to S$ be a morphism of finite type of algebraic schemes, smooth on a dense open subset of X. Assume X to be reduced. Then there is a canonical isomorphism of coherent sheaves on X:*

$$T^1_{X/S} \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X).$$

In particular, if $S = \mathrm{Spec}(\mathbf{k})$, then

$$T^1_X \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

and if moreover $X = \mathrm{Spec}(B_0)$, then

$$T^1_{B_0} \cong \mathrm{Ext}^1_{\mathbf{k}}(\Omega^1_{B_0/\mathbf{k}}, B_0).$$

Remark 5. A analysis of the proof of Theorem (3) shows that without assuming X reduced we only have an inclusion

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \subset T^1_X.$$

Theorem 4 ([Ma], Thm. 6.2). *The first order deformation of X are in one to one correspondence with T^1_X.*

Proposition 3 ([Es], Thm. 2.4.1 (iii)). *For a* $\mathbf{k}$*-algebra* B_0, $\mathrm{Def}_{B_0}(\mathbf{k}[\epsilon]) \cong T^1_{B_0}$.

Remark 6. Suppose $B_0 = \mathbf{k}[x_1, ..., x_d]/J$, with J prime. Then there is an exact sequence:

$$0 \to \mathrm{Hom}(\Omega_{B_0/k}, B_0) \to \mathrm{Hom}(\Omega_{\mathbf{k}[x_1,...,x_d]/\mathbf{k}} \otimes B_0, B_0) \to \mathrm{Hom}(J/J^2, B_0) \to T^1_{B_0} \to 0,$$

and thus $T^1_{B_0}$ can be computed. The result is: If J is generated by a regular sequence $(f_1, ..., f_n)$, then

$$T^1_{B_0} \cong \frac{\mathbf{k}[x_1, ..., x_d]^n}{\left(\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \end{pmatrix}, \dots, \begin{pmatrix} \frac{\partial f_1}{\partial x_d} \\ \vdots \\ \frac{\partial f_n}{\partial x_d} \end{pmatrix} \right)} \otimes_{\mathbf{k}[x_1,...,x_d]} B_0$$

Example 7. *1.* For a hypersurface $B_0 = V(f)$, $T^1_{B_0} \cong \frac{\mathbf{k}[x_1,...,x_d]}{\left(f, \frac{\partial f}{\partial x_d}, ..., \frac{\partial f}{\partial x_d}\right)}$.

2. For $f = x^2 - y^2$ and $g = x^3 - y^2$, we have $T^1_{B_0} \cong \mathbf{k}$ and $T^1_{B_0'} \cong \mathbf{k}^2$.

Remark 7. Note that if X is smooth, then $T^1_X = 0$. This implies that any two classes of first order deformation of X are isomorphic over $\mathbf{k}[\epsilon]$ (in fact obtainable by a change of coordinates in $\mathbb{A}^n_{\mathbf{k}[\epsilon]}$).

Remark 8. If X has isolated singularities, T^1_X as vector space has finite dimension.

Remark 9. Suppose the singularities of X admit a good $\mathbf{k}^*$-action, so $\mathcal{O}_X$ is a (positive) graded module. Then all modules considered above inherit a grading.

Theorem 5 (Pinkham 1974 [Pi])**.** *A singularity X with a good $\mathbf{k}^*$-action has a $\mathbf{k}^*$-equivariant versal deformation $\pi : X \to S$. The restriction $\pi_- : X_- \to S_-$ to the subspace of negative weight is versal for deformations of X with negative weight.*

1.4 BACKGROUND ON $\mathcal{M}^{\mathcal{S}}_{g,1}$

Let $\mathcal{C}$ be a complete projective curve of genus $g > 1$ and P a smooth point of $\mathcal{C}$. We may consider an ascending chain of $\mathbf{k}$-vector spaces

$$\mathbf{k} = \mathrm{H}^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(0 \cdot P)) \subseteq \mathrm{H}^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(1 \cdot P)) \subseteq \mathrm{H}^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(2 \cdot P)) \subseteq \dots$$

where

$$\mathrm{H}^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(n \cdot P)) = \{f \in \mathbf{k}(\mathcal{C}) \mid f \text{ has a pole of order at most } n \text{ at } P\}.$$

A positive integer n is called a *gap* associated to the pair $(\mathcal{C}, P)$ if

$$\mathrm{H}^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(n \cdot P)) = \mathrm{H}^0(\mathcal{C}, \mathcal{O}_\mathcal{C}((n+1) \cdot P)),$$

otherwise we say that n is a *nongap*.

By virtue of Riemann-Roch Theorem for singular curves, $\dim H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(d \cdot P)) = d+1-g$ for each $d \geq 2g-1$, i.e. there are precisely g integers $l_1 < \cdots < l_g$ between 0 and $2g-1$ for which does not exists a rational function on $\mathbf{k}(\mathcal{C})$ with pole divisor $l_i\, P$. Therefore, we associate to the point P the set $\mathcal{S}_P := \mathbb{N} - \{l_1, \cdots, l_g\}$, that is, by properties of valuation at P, $\mathcal{S}_P$ is a sub-semigroup of the positive integers, i.e., contains the zero number and it is closed under addition.

A point $P \in \mathcal{C}$ is *ordinary* if $\mathcal{S}_P = \{0, g{+}1, g{+}2, ...\}$, otherwise P is called a *Weierstrass* point. For any Weierstrass point $P \in \mathcal{C}$, let $0 = n_0 < n_1 < \cdots$ be the nongap sequence of $\mathcal{C}$ at P. So for each n_i we can take a function $x_{n_i} \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n_i{\cdot}P)) \backslash H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}((n_i{-}1){\cdot}P))$ for which the pole order at P is n_i, hence

$$H^0(C, \mathcal{O}_{\mathcal{C}}(n_i \cdot P)) = \mathbf{k}x_{n_0} \oplus \mathbf{k}x_{n_1} \oplus \cdots \oplus \mathbf{k}x_{n_i}.$$

In particular, $\dim H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n_i \cdot P)) = n_i + 1$.

Let $(\mathcal{L}, V)$ be a linear system of degree d and dimension r on a smooth curve $\mathcal{C}$, thus $\mathcal{L}$ is a line bundle of degree d on $\mathcal{C}$ and V is a sub-vector space of $\mathrm{H}^0(\mathcal{C}, \mathcal{L})$ of dimension $r{+}1$. We recall that a point $P \in \mathcal{C}$ is a ramification point of $(\mathcal{L}, V)$ if there exists a section $z \in V$ such that $\mathrm{ord}_P(z) \geq r+1$. In this case, an equivalent way to define Weierstrass points is that $P \in \mathcal{C}$ is a ramification point of the canonical linear system $(K_{\mathcal{C}}, \mathrm{H}^0(\mathcal{C}, K_{\mathcal{C}}))$ where $K_{\mathcal{C}}$ is the canonical line bundle of $\mathcal{C}$. As we see above, this equivalent definition fits nicely when $\mathcal{C}$ is singular, we just interchange the canonical system by the the dualizing system.

Given a numerical semigroup $\mathcal{S} \subset \mathbb{N}$ of genus $g := \#(\mathbb{N} \setminus \mathcal{S}) > 1$, we may ask for a complete projective curve that realizes this semigroup as a Weierstrass semigroup. If we restrict to only is smooth curve, there are numerical semigroups which are not Weierstrass semigroups, see for example [Ft]. But if we assume that $\mathcal{C}$ may be singular, then every numerical semigroup is the Weierstrass semigroup of a suitable monomial curve, namely, we take a projective closure of the affine monomial curve $\mathcal{C}_{\mathcal{S}} := \{(t^{m_1}, \ldots, t^{m_r})\}$ where $\mathcal{S}$ is generated by $m_1, \ldots, m_r$, thus its unique point at infinite is smooth and realizes $\mathcal{S}$.

Let us now consider the set

$$\mathcal{M}_{g,1}^{\mathcal{S}} := \{(C, P) |\ \mathcal{S}_P = \mathcal{S}\}/_{\cong}$$

where $\cong$ stands for pointed isomorphism of curve, i.e., isomorphisms of curves that send marked points to marked points.

There are two powerful tools for dealing with $\mathcal{M}_{g,1}^{\mathcal{S}}$, both based on deformations of suitable singular curves. On the one hand, since the ℓ-th gap defines an upper semi-continuous function, the set $\mathcal{M}_{g,1}^{\mathcal{S}}$ is locally closed in the moduli space $\mathcal{M}_{g,1}$ of smooth pointed curves of genus $g = g(\mathcal{S})$. Hence we get an appropriate ambient to embed (locally closed) $\mathcal{M}_{g,1}^{\mathcal{S}}$. Thus we can use the theory of Limit Linear Series of curves of compact type to study $\mathcal{M}_{g,1}^{\mathcal{S}}$. In this approach the definition of Weierstrass points as ramification points of the canonical sheaf is more useful. Here we must cite the works of Eisenbud–Harris [EH1, EH2] and Esteves–Medeiros [EM] on limit linear series with applications to Weierstrass points, and more recently the works of Bullock [Bu1, Bu2] and Pflueger [Pf1, Pf2]. We see in this approach a fundamental problem, we have to find a suitable curve whose "Weierstrass semigroup"is the fixed one $\mathcal{S}$, but we do not know what is a Weierstrass point on a nodal curve with more than one component. Of course that we always consider the case where $\mathcal{M}_{g,1}^{\mathcal{S}}$ is non-empty and then study the limit of Weierstrass semigroups.

On the other hand, we have the theory of (formal versal) deformations of singularities. In particular, deformations of affine monomial curves which are curves far from being stable, because their singularities are unibranched. There are fundamental works using this approach to study $\mathcal{M}_{g,1}^{\mathcal{S}}$, for example the Ph.D. thesis of Pinkham [Pi], the work of Stoehr [S] and the Contiero–Stöhr [CS]. The advantage with these works is that we already have a suitable curve with a required Weierstrass point to deform. However, there are some disadvantages, the first one is that we do not have a natural ambient space to include $\mathcal{M}_{g,1}^{\mathcal{S}}$, a second is that we have to deal with rather abstract objects and rather less intuitive techniques than Limit Linear Series. But the cited works in this paragraph point out to a computational approach to study the spaces $\mathcal{M}_{g,1}^{\mathcal{S}}$, and this is our approach that we will try to convince the readers that it is until now the most appropriated.

Using the approach of Limit Linear Series, N. Pflueger made a substantial improvement of a lower bound for $\mathcal{M}_{g,1}^{\mathcal{S}}$ given by Eisenbud–Harris. He showed the following:

Theorem 6 (Pflueger's lower bound, [Pf1])**.** *Let $\mathcal{S}$ be a numerical semigroup of genus $g > 1$. Set $\operatorname{ewt}(\mathcal{S}) := \sum_{\ell \text{ gaps}}(\#\{a_i \mid a_i \leq \ell\})$ the effective weight of $\mathcal{S}$ where $\{a_1, \dots, a_r\}$ is a system of generators of $\mathcal{S}$. If $\mathcal{M}_{g,1}^{\mathcal{S}}$ is nonempty then*

$$3g - 2 - \operatorname{ewt}(\mathcal{S}) \leq \dim \mathcal{M}_{g,1}^{\mathcal{S}}.$$

Remark 10. Pflueger's lower bound is attained, for example if $\operatorname{ewt}(\mathcal{S}) < \frac{1}{2}g$, then $3g - 2 - \operatorname{ewt}(\mathcal{S}) = \dim \mathcal{M}_{g,1}^{\mathcal{S}}$. However, as we notice in the introduction, there are numerical semigroups where Pflueger's lower bound is not attained, for example if $\mathcal{S}$ is a Castelnuovo semigroup introduced by Pflueger in [Pf2].

1.4.1 Weierstrass points on canonical curves

We recall that a point $P \in \mathcal{C}$ is said to be a *Gorenstein point* (see [S1]) if the stalk of the dualizing sheaf $\omega_{\mathcal{C},P}$ is a free $\mathcal{O}$-module. The curve $\mathcal{C}$ is Gorenstein if all of its points are Gorenstein, or equivalently, if ω is an invertible sheaf.

Now, let $x_0, ..., x_n$ be $\mathbf{k}$-linear independent elements of $\mathbf{k}(\mathcal{C})$, so that for $n \geq 1$ we have the morphism

$$(x_0, ..., x_n) : \mathcal{C} \to \mathbb{P}^n,$$

whose image by the extension theorem of valuation theory is a projective algebraic curve (see [S2]). Thus we obtain a morphism $\mathcal{C} \to \mathbb{P}^n$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \mathbb{P}^n \\ & \searrow & \uparrow \\ & & \mathcal{C} \end{array}$$

(here $\widetilde{\mathcal{C}} \to \mathcal{C}$ is the normalization of $\mathcal{C}$) commutes if and only if the O_P-ideal $\sum_{i=0}^{n} O_P x_i$ is principal. Let β be a non-zero differential one form such that $\omega_{\mathcal{C}} = \omega_\beta \cdot \beta$. By choosing a basis $\beta_0, ..., \beta_{g-1}$ for the space of the regular differentials on $\mathcal{C}$, we can write $\beta_i = x_i\beta(i = 0, ..., g-1)$, where $x_0, ..., x_{g-1}$ is a basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}})$. In this way, we have $(\beta_0, ..., \beta_{g-1}) = (x_0, ..., x_{g-1})$.

The following two theorems are well known in the literature.

Theorem 7 ([S1]). *Let $\mathcal{C}$ be a curve of genus $g \geq 1$. For each $P \in \mathcal{C}$, we have $\omega_{\mathcal{C},P} = \mathcal{O}x_0 + \cdots + \mathcal{O}_P x_{g-1}$. The morphism $(\beta_0, \cdots, \beta_{g-1}) : \widetilde{\mathcal{C}} \to \mathbb{P}^{g-1}$ induces a morphism $\mathcal{C} \to \mathbb{P}^{g-1}$ if and only if the curve $\mathcal{C}$ is Gorenstein.*

Theorem 8 ([R]). *Let $\mathcal{C}$ be a Gorenstein curve. The morphism $\mathcal{C} \to \mathbb{P}^{g-1}$ is an isomorphism onto the image curve if only if $\mathcal{C}$ is non-hyperelliptic.*

Let us now recall the compactification given by Contiero–Stoehr in [CS] of $\mathcal{M}_{g,1}^{\mathcal{S}}$ by assuming that $\mathcal{S}$ is a symmetric semigroup.

Let $\mathcal{C}$ be a complete integral Gorenstein curve of arithmetic genus $g > 1$ defined over $\mathbf{k}$. For each smooth point P of $\mathcal{C}$, let $\mathcal{S}$ be the Weierstrass semigroup of $\mathcal{C}$ at P. By the very definition, for each $n \in \mathcal{S}$ there is a rational function x_n on $\mathcal{C}$ with pole divisor nP. Let us assume that the semigroup $\mathcal{S}$ is symmetric, i.e. the last gap ℓ_g is the biggest possible, $l_g = 2g-1$. Equivalently, $n \in \mathcal{S}$ if, and only if, $\ell_g - n \notin \mathcal{S}$. Let ω be the dualizing sheaf of $\mathcal{C}$. A basis for the vector space $H^0(\mathcal{C}, \omega)$ is $\{x_{n_0}, x_{n_1}, \ldots, x_{n_{g-1}}\}$, and thus $\omega \cong \mathcal{O}_{\mathcal{C}}((2g-2)P)$. By assuming that $\mathcal{C}$ is nonhyperelliptic, the canonical morphism

$$(x_{n_0} : x_{n_1} : \ldots : x_{n_{g-1}}) : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}$$

is an embedding. Thus $\mathcal{C}$ becomes a curve of degree $2g-2$ in $\mathbb{P}^{g-1}$ and the integers $l_i - 1$ are the contact orders of the curve with the hyperplanes at $P = (0 : \ldots : 0 : 1)$. Conversely, any nonhyperelliptic symmetric semigroup $\mathcal{S}$ can be realized as the Weierstrass semigroup of the Gorenstein canonical *monomial curve*

$$\mathcal{C}_{\mathcal{S}} := \{(s^{n_0}t^{\ell_g - 1} : s^{n_1}t^{\ell_{g-1}-1} : \ldots : s^{n_{g-2}}t^{\ell_2 - 1} : s^{n_{g-1}}t^{\ell_1 - 1}) \,|\, (s:t) \in \mathbb{P}^1\} \subset \mathbb{P}^{g-1}$$

at its unique point P at the infinity.

Since $\mathcal{S}$ is symmetric, each nongap $s \in \mathcal{S}, s \leq 4g-4$ can be written as a sum of two other nongaps (see [Ol, theorem 1.3]),

$$s = a_s + b_s, \; a_s \leq b_s \leq 2g-2.$$

By taking a_s as the smallest possible, the $3g-3$ rational functions $x_{a_s}x_{b_s}$ form a P-hermitian basis of the space of global sections $H^0(\mathcal{C},\omega^2)$ of the bicanonical divisor. If $r \geq 3$, then a P-hermitian basis of the vector space $H^0(\mathcal{C},\omega^r)$ (cf. [CS, Lemma 2.1]) is

$$\begin{array}{lr} x_{n_0}^{r-1}x_{n_i} & (i=0,\ldots,g-1), \\ x_{n_0}^{r-2-i}x_{a_s}x_{b_s}x_{n_{g-1}}^i & (i=0,\ldots,r-2,\ s=2g,\ldots,4g-4), \\ x_{n_0}^{r-3-i}x_{n_1}x_{2g-n_1}x_{n_{g-2}}x_{n_{g-1}}^i & (i=0,\ldots,r-3). \end{array}$$

A consequence of the existence of a P-hermitian basis of $H^0(\mathcal{C},\omega^r)$ for any $r \geq 1$ is a Max-Noehter's theorem, namely the following homomorphism

$$\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r \longrightarrow H^0(\mathcal{C},\omega^r)$$

induced by the substitutions $X_{n_i} \longmapsto x_{n_i}$ is surjective for each $r \geq 1$, where $\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r$ is the vector space of r-forms.

Let $I(\mathcal{C}) = \bigoplus_{r=2}^{\infty} I_r(\mathcal{C}) \subset \mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]$ be the ideal of $\mathcal{C} \subset \mathbb{P}^{g-1}$. The codimension of $I_r(\mathcal{C})$ in $\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r$ is equal to $(2r-1)(g-1)$, in particular,

$$\dim I_2(\mathcal{C}) = (g-2)(g-3)/2.$$

For $r \geq 2$, let Λ_r be the vector space in $\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r$ spanned by the lifting of the P-hermitian basis of $H^0(\mathcal{C},\omega^r)$. Since $\Lambda_r \cap I_r(\mathcal{C}) = 0$ and

$$\dim \Lambda_r = \dim H^0(\mathcal{C},\omega^r) = \operatorname{codim} I_r(\mathcal{C}),$$

it follows that

$$\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r = \Lambda_r \oplus I_r(\mathcal{C}),\ \ r \geq 2.$$

For each nongap $s \leq 4g-4$, let us consider all the partitions of s as sum of two nongaps not greater than $2g-2$,

$$s = a_{si} + b_{si},\ \text{ with } a_{si} \leq b_{si},\ (i=1,\ldots,\nu_s),\ \text{ where } a_{s0} := a_s.$$

Hence, given a nongap $s \leq 4g-4$ and $i=1,\ldots,\nu_s$ we can write

$$x_{a_{si}}x_{b_{si}} = \sum_{n=0}^{s} c_{sin}x_{a_n}x_{b_n},$$

where a_n and b_n are nongaps of $\mathcal{S}$ whose sum is equal to s, and c_{sin} are suitable constants in $\mathbf{k}$. By normalizing the coefficients $c_{sis}=1$, it follows that the $\binom{g+1}{2} - (3g-3) = \frac{(g-2)(g-3)}{2}$ quadratic forms

$$F_{si} = X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s} - \sum_{n=0}^{s-1} c_{sin}X_{a_n}X_{b_n}$$

vanish identically on the canonical curve $\mathcal{C}$, where the coefficients c_{sin} are uniquely determined constants. They are linearly independent, hence they form a basis for the space of quadratic relations $I_2(\mathcal{C})$.

It is necessary to make some assumptions on the symmetric semigroup $\mathcal{S}$ to assure that the ideal $I(\mathcal{C})$ is generated by quadratic relations. More precisely we suppose that $\mathcal{S}$ satisfies $3 < n_1 < g$ and $\mathcal{S} \neq \langle 4,5 \rangle$. According to [CF, Lemma 3.1], both the conditions $n_1 \neq 3$ and $n_1 \neq g$ on $\mathcal{S}$ are to avoid possible trigonal Gorenstein curves whose Weierstrass semigroup at P equal to $\mathcal{S} = \langle 3, g+1 \rangle$ and $\mathcal{S} = \langle g, g+1, \ldots, 2g-2 \rangle$, respectively. This two avoided cases are treated by Contiero and Fontes in [CF]. By the assumptions on the semigroup $\mathcal{S}$ it follows by the Enriques–Babbage theorem that $\mathcal{C}$ is nontrigonal and it is not isomorphic to a plane quintic.

If $\mathcal{C}$ is smooth, then Petri's theorem [ACGH] assure that the ideal of $\mathcal{C}$ is generated by the quadratic relations. Given a canonical curve $\mathcal{C}$, not necessarily smooth, an algorithmic proof that the ideal of $\mathcal{C}$ is generated by the quadratic forms F_{si} was done by Contiero and Stöhr in [CS, Theorem 2.5].

On the other hand, for each symmetric semigroup $\mathcal{S}$ with $3 < n_1 < g$ and $\mathcal{S} \neq \langle 4,5 \rangle$, we can take the following $(g-2)(g-3)/2$ quadractic forms

$$F_{si} = X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} - \sum_{n=0}^{s-1} c_{sin} X_{a_n} X_{b_n}, \tag{1.10}$$

where c_{sin} are constants to be determined in order that the intersection $\cap V(F_{si}) \subset \mathbb{P}^{g-1}$ is a canonical Gorenstein curve of genus g whose Weierstrass semigroup at P is $\mathcal{S}$. Analogously, let

$$F_{si}^{(0)} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} \tag{1.11}$$

be the quadratic forms that generate the ideal of the canonical monomial curve $\mathcal{C}_{\mathcal{S}}$, cf. [CS, Lemma 2.2]. One of the keys to construct a compactification of $\mathcal{M}_{g,1}^{\mathcal{S}}$ is the following lemma.

Syzygy Lemma (cf. [CS], Lemma 2.3)**.** *For each of the* $\frac{1}{2}(g-2)(g-5)$ *quadratic binomials* $F_{s'i'}^{(0)}$ *different from* $F_{n_i+2g-2,1}^{(0)}$ $(i = 0, \ldots, g-3)$ *there is a syzygy of the form*

$$X_{2g-2} F_{s'i'}^{(0)} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si}^{(0)} = 0$$

where the coefficients $\varepsilon_{nsi}^{(s'i')}$ *are integers equal to* 1, -1 *or* 0, *and where the sum is taken over the nongaps* $n < 2g-2$ *and the double indices* si *with* $n + s = 2g - 2 + s'$.

Let us described briefly the algorithmic construction of a compactification of $\mathcal{M}_{g,1}^{\mathcal{S}}$ which was done by Stöhr [S] and Contiero–Stöhr [CS]. First, we replace the binomials $F_{s'i'}^{(0)}$ and $F_{si}^{(0)}$ on the left hand side of the Syzygy Lemma by the corresponding quadratic forms $F_{s'i'}$ and F_{si}. Hence we obtain a linear combination of cubic monomials of weight less than $s'+2g-2$. By virtue of [CS, Lemma 2.4] this linear combination of cubic monomials admits the following decomposition.

$$X_{2g-2}F_{s'i'} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si} = \sum_{nsi} \eta_{nsi}^{(s'i')} X_n F_{si} + R_{s'i'}$$

where the sum on the right hand side is taken over the nongaps $n \le 2g-2$ and the double indices si with $n+s < s'+2g-2$, where the coefficients $\eta_{nsi}^{(s'i')}$ are constants, and where $R_{s'i'}$ is a linear combination of cubic monomials of pairwise different weights $< s'+2g-2$.

For each nongap $m < s'+2g-2$ we denote by $\varrho_{s'i'm}$ the unique coefficient of $R_{s'i'}$ of weight m. Finally, let us consider the following quasi-homogeneous polynomial in the constants c_{sin},

$$R_{s'i'}(t^{n_0}, t^{n_1}, \ldots, t^{n_{g-1}}) = \sum_{m=0}^{s'+2g-3}{}' \varrho_{s'i'm} t^m \ .$$

Since the coordinates functions x_n, $n \in \mathcal{S}$ and $n \le 2g-2$, are not uniquely determined by their pole divisor nP by assuming the characteristic of the field $\mathbf{k}$ to be zero (or a prime not dividing any of the differences $m-n$ where n, m are nongaps of $\mathcal{S}$ such that $n, m \le 2g-2$), we transform

$$X_{n_i} \longmapsto X_{n_i} + \sum_{j=0}^{i-1} c_{n_i n_{i-j}} X_{n_{i-j}},$$

for each $i = 1, \ldots, g-1$, and so we can normalize $\frac{1}{2}g(g-1)$ of the coefficients c_{sin} to be zero, see [S, Proposition 3.1]. Due to these normalizations and the normalizations of the coefficients $c_{sin} = 1$ with $n = s$, the left to us is to transform $x_{n_i} \mapsto c^{n_i} x_{n_i}$ for $i = 1, \ldots, g-1$. Summarizing, we get

Theorem 9. *[CS, Theorem 2.6] Let $\mathcal{S}$ be a symmetric semigroup of genus g satisfying $3 < n_1 < g$ and $\mathcal{S} \neq \langle 4, 5\rangle$. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{S}$ correspond bijectively to the orbits of the $\mathbb{G}_m(k)$-action*

$$(c, \ldots, c_{sin}, \ldots) \longmapsto (\ldots, c^{s-n} c_{sin}, \ldots)$$

on the affine quasi-cone of the vectors whose coordinates are the coefficients c_{sin} of the normalized quadratic F_{si} satisfying the quasi-homogeneous equations $\varrho_{s'i'm} = 0$.

Remark 11. *Roughly speaking, the compactification of Deligne–Mumford of the moduli space of genus g smooth curves says that a general model for curves are the stable ones, which leads us to think the same about the study of curves and Weierstrass points. However, the compactification of Contiero–Stoehr of $\mathcal{M}_{g,1}^{\mathcal{S}}$ says that the natural model for curves with symmetric Weierstrass points are the Gorenstein ones, which are not stable.*

1.5 DELIGNE–PINKHAM'S BOUND

The purpose of this section is to give a clearer proof of the important Deligne's formula for the smoothing component of the formal versal deformation space of a singularity, and so derive the main upper bound for $\mathcal{M}^{\mathcal{S}}_{g,1}$ using a result due to Pinkham.

Let $\mathcal{C}$ be a reduced projective algebraic curve defined over $\mathbf{k}$ and $q \in \mathcal{C}$ a closed point. We fix the following notations:

1) $\mathcal{O}$ denotes the local ring of $\mathcal{C}$ at q, and $\widetilde{\mathcal{O}}$ its normalization;

2) $\delta := \dim_{\mathbf{k}} \widetilde{\mathcal{O}}/\mathcal{O}$ is the singularity degree of $\mathcal{C}$ at q;

3) $\mathrm{Der}(\widetilde{\mathcal{O}}) := \mathrm{Der}(\widetilde{\mathcal{O}}, \widetilde{\mathcal{O}})$ is the module of $\mathbf{k}$ derivations of $\widetilde{\mathcal{O}}$, and similarly $\mathrm{Der}(\mathcal{O}) := \mathrm{Der}(\mathcal{O}, \mathcal{O})$;

4) $\mu := \dim_{\mathbf{k}} \frac{\mathrm{Der}(\widetilde{\mathcal{O}})}{\mathrm{Der}(\widetilde{\mathcal{O}}) \cap \mathrm{Der}(\mathcal{O})} - \dim_{\mathbf{k}} \frac{\mathrm{Der}(\mathcal{O})}{\mathrm{Der}(\widetilde{\mathcal{O}}) \cap \mathrm{Der}(\mathcal{O})}$

Deligne's Formula. *Let E be an irreducible component of the formal versal deformation of* $\mathrm{Spec}(\mathcal{O})$*. If the fiber above the generic point of E is smooth , then*

$$\dim E = 3\delta - \mu. \tag{1.12}$$

Proof. Since the statement of the theorem just depends on the completion of $\mathcal{O}$, we use the same symbol for the local ring $\mathcal{O}$ and its completion $\widehat{\mathcal{O}}$. We also assume that q is the unique singular point and so the universal deformation of $\mathcal{C}$ does exist.

Let us fix the formal (semiuni)versal deformation of $\mathrm{Spec}(\mathcal{O})$:

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & T \end{array} \tag{1.13}$$

and denote by t the only closed point of T. By the (semiuni)versal property, the universal deformation of $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ t_0 = \mathrm{Spec}(\mathbf{k}) & \longrightarrow & T_0 \end{array} \tag{1.14}$$

is given by a morphism $\alpha : T_0 \to T$ and an isomorphism $\mathcal{X}_0 \cong \mathcal{X} \times_T T_0$.

From a theorem of [Ri, Cor 2.10 Exposè IV] the morphism α is smooth, i.e. flat with smooth fibers. By the universal property, the first order deformation of $\mathcal{C}$, i.e. the deformation of $\mathcal{C}$ over $\mathrm{Spec}(\mathbf{k}[\epsilon])$, say:

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{Y}_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

corresponds bijectively to the morphims $\mathrm{Spec}(\mathbf{k}[\epsilon]) \longrightarrow T_0$ and hence to the elements of the tangent space Θ_{T_0,t_0}. Since q is the only singular point of $\mathcal{C}$, the first order deformation of $\mathcal{C}$ is locally trivial if and only if the induced first order deformation of Spec($\mathcal{O}$) is trivial, i.e. the induced deformation

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}_0,y_0}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

is trivial. Equivalently, the composite morphism $\mathrm{Spec}(\mathbf{k}[\epsilon]) \longrightarrow T_0 \longrightarrow T$ defines the zero-vector in the tangent space $\Theta_{T,t}$ of T at t. Thus we have a bijection

$$\left\{ \begin{array}{c} \text{trivial first order} \\ \text{deformations of } \mathcal{C} \end{array} \right\} \longleftrightarrow \mathrm{Kernel}\left(\Theta_{T_0,t_0} \to \Theta_{T,t}\right).$$

By the Kodaira-Spencer correspondence (1.2) we get a bijection

$$H^1(\mathcal{C}, \mathcal{T}_{\mathcal{C}}) \longleftrightarrow \mathrm{Kernel}\left(\Theta_{T_0,t_0} \to \Theta_{T,t}\right).$$

where $\mathcal{T}_{\mathcal{C}} := \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{C}}}(\Omega_{\mathcal{C}|\mathbf{k}}, \mathcal{O}_{\mathcal{C}}) \cong \mathrm{Der}_{\mathbf{k}}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}})$ is the tangent sheaf of $\mathcal{C}$.

Since $\alpha : T_0 \longrightarrow T$ is smooth, the following sequence is exact

$$0 \longrightarrow \Theta_{\alpha^{-1}(t),t_0} \longrightarrow \Theta_{T_0,t_0} \longrightarrow \Theta_{T,t} \longrightarrow 0$$

and hence

$$\dim H^1(\mathcal{C},\, \mathcal{T}_{\mathcal{C}}) = \dim \Theta_{\alpha^{-1}(t),t_0} = \dim \Theta_{T_0,t_0} - \dim \Theta_{T,t}. \tag{1.15}$$

By the smooth property of α we also obtain that $\dim \Theta_{\alpha^{-1}(t),t_0} = \dim \alpha^{-1}(t)$.

Let E be an irreducible component of the base space T of the formal versal deformation of Spec($\mathcal{O}$) (1.13). Let $E_0 := \alpha^{-1}(E) \cong E \times_T T_0$. Then, from above equation (1.15) we obtain

$$\dim H^1(\mathcal{C},\, \mathcal{T}_{\mathcal{C}}) = \dim E_0 - \dim E$$

Since the morphism $X_0 \to T_0$ is flat, all its fibers have the same arithmetic genus g. By assumption, the fiber of the morphism $X \to T$ over a generic point e of E is smooth and so the fiber of the morphism $X_0 \to T_0$ over the generic point of E_0 is also smooth. The formal moduli scheme of smooth projective curves of genus g is smooth of dimension $3g - 3$, that means

$$\dim E_0 = 3g - 3$$

The differential sheaf $\Omega_{\mathcal{C}|\mathbf{k}}$ of $\mathcal{C}$ may not be torsion-free, but the tangent sheaf $\mathcal{T}_{\mathcal{C}}$ is a coherent fractional ideal sheaf (and therefore torsion-free of rank 1) and by the Riemann-Roch Theorem we have

$$\chi(\mathcal{T}_{\mathcal{C}}) = \deg(\mathcal{T}_{\mathcal{C}}) + 1 - g.$$

Let $\widetilde{\mathcal{C}}$ be the nonsingular model of $\mathcal{C}$ and $\widetilde{g}$ its genus. Since q is the only singular point of $\mathcal{C}$ we get $\widetilde{g} = g - \delta$. Since $\Omega_{\widetilde{\mathcal{C}}|\mathbf{k}}$ is locally free of rank 1 and degree $2\widetilde{g} - 2$, we conclude that $\mathcal{T}_{\widetilde{\mathcal{C}}}$ is locally free of rank 1 and has degree $2 - 2\widetilde{g}$. Thus

$$\begin{aligned} \deg(\mathcal{T}_{\mathcal{C}}) = \deg(\mathcal{T}_{\widetilde{\mathcal{C}}}) - m_1 + \delta &= 2 - 2\widetilde{g} - \mu + \delta \\ &= 2 - 2g - \mu + 3\delta \end{aligned}$$

and hence

$$\chi(\mathcal{T}_{\mathcal{C}}) = 3 - 3g - \mu + 3\delta.$$

Since $\dim E = \dim E_0 - \dim H^0(\mathcal{C}, \mathcal{T}_{\mathcal{C}})$, we obtain

$$\dim E = 3\delta - \mu - \dim H^0(\mathcal{C}, \mathcal{T}_{\mathcal{C}}).$$

If we assume that $g \gg 0$, then $\deg(\mathcal{T}_{\mathcal{C}}) < 0$ and the results follows. □

Now we point out how to link the formal (semiuni)versal deformation of Deligne's Formula and the moduli space $\mathcal{M}^{\mathcal{S}}_{g,1}$. Everything here is in much more detailed in [RV, §6, pg. 474–476]. Denote by (S, R) the formal (semiuni)versal deformation of $\mathbf{k}[\mathcal{S}]$. Pinkham[2] proved that there is an ideal N of R such that (S', R') with $R' = R/NR$ and $S' = S/NS$ is an infinitesimal deformation of $\mathbf{k}[\mathcal{S}]$. Let us set

$$U = \{x \in \mathrm{Spec}(R') | \text{the fiber above } x \text{ in } S' \text{ is smooth}\}$$

Pinkham also has shown that U is invariant under the $G_m(\mathbf{k})$ action.

Theorem 10 (c.f. [Pi] Theorem 13.9 page 103.)**.** *There exists a morphism $U \longrightarrow \mathcal{M}_{g,1}$ that factors through the quotient $\overline{U}$ of U by the action of $\mathbb{G}_m(\mathbf{k})$. Additionally, this morphism induces a bijection between $\overline{U}$ and $\mathcal{M}^{\mathcal{S}}_{g,1}$.*

By virtue of the above Pinkham's theorem, Deligne's formula says that $\dim U \leq 3\delta - \mu$. Since $\dim \overline{U} = \dim U - 1$, we get $\dim \mathcal{M}^{\mathcal{S}}_{g,1} \leq 3\delta - \mu - 1$. Now, it can be verified that

$$3\delta - \mu - 1 = 2g - 2 + \lambda(\mathcal{S}),$$

where $\lambda(\mathcal{S})$ is the number of gaps l such that $l + n \in S$ whenever n is a nongap. Hence we get the main general upper bound for the dimension of $\mathcal{M}^{\mathcal{S}}_{g,1}$, namely:

Theorem 11 (Deligne–Pinkham's upper bound)**.** *For any numerical semigroup $\mathcal{S}$,*

$$\dim \mathcal{M}^{\mathcal{S}}_{g,1} \leq 2g - 2 + \lambda(\mathcal{S}).$$

[2] See [Pi] chapter I section 2 for the general case and chapter IV section 13 for monomial curves.

Deligne–Pinkham's bound is attained. Rim and Vitulli showed, [RV], that if the semigroup $\mathcal{S}$ is negatively graded, i.e. if the first module $T^1(\mathbf{k}[\mathcal{S}])$ of the cotangent complex associated to $\mathbf{k}[\mathcal{S}]$ does not have elements of positive degree, then $\mathcal{S}$ can be negatively smoothable and so $\dim \mathcal{M}_{g,1}^{\mathcal{S}} = 2g - 2 + \lambda(\mathcal{S})$. Additionally, Rim and Vitulli classified all numerical semigroups that are negatively graded, (see [RV, Thm. 4.7]), namely:

Theorem 12 (Negatively graded semigroups)**.** *A numerical semigroup $\mathcal{S}$ of genus g is negatively graded if and only if:*

1. $\mathcal{S} = \mathcal{S}_g := \{0, g+1, g+2, \dots\}$ *the ordinary one, or*
2. $\mathcal{S}$ *is hyper-ordinary, i.e.* $m\mathbb{N} + \mathcal{S}_g$*, or*
3. *if* $\lambda := \lambda(\mathcal{S}) > 1$*, then*

$$\mathcal{S} = \{0, g, g+1, \dots, 2g-\lambda-1, \widehat{2g-\lambda}, 2g-\lambda+1, 2g-\lambda+2, \dots\},$$

if $\lambda = 1$*, then*

$$\mathcal{S} = \{0, g, g+1, \dots, 2g-2, \widehat{2g-1}, 2g, 2g+1, \dots\}$$

or

$$\mathcal{S} = \{0, g-1, \widehat{g}, g+1, g+2, \dots\}.$$

Pflueger noticed that if $\mathcal{S}$ is a negatively graded semigroup, then his lower bound in Theorem 6 is equal to the Deligne–Pinkham's upper bound, see [Pf1, Prop. 2.11].

2 COMPUTATIONS

2.1 FAMILIES OF SYMMETRIC SEMIGROUPS

In this section we apply the techniques developed by in [CS] and [S] (briefly described in the preliminaries) to deal with families of symmetric semigroups. We note that if the symmetric semigroup is generated by less than five elements, the dimension of the moduli variety $\mathcal{M}_{g,1}^{\mathcal{S}}$ is well known, as we noted in Introduction of this thesis or in equation (3.1) in the beginning of Chapter 3. So, we must consider symmetric semigroups of multiplicity bigger than 5, just because a symmetric semigroup of multiplicity m can be generated by $m-1$ elements. The main idea is to adapt the techniques developed in [CS] and [S] to deal with a projection of the (affine) canonical monomial curve over an (affine) ambient space whose dimension does not depend on the genus g, *but only on the multiplicity of the semigroup*. Thus, we are able to perform with a family of symmetric semigroups with a given multiplicity. It is clear this approach is very related to the equivariant deformation developed by Pinkham [Pi].

For each $\tau \geq 1$, let

$$\begin{aligned}\mathcal{S}: &= <6, 7+6\tau, 8+6\tau, 9+6\tau, 10+6\tau> \\ &= 6\mathbb{N} \sqcup \bigsqcup_{i=7}^{9}(i+6\tau+6\mathbb{N}) \sqcup (17+12\tau+6\mathbb{N}).\end{aligned}$$

The nongaps of $\mathcal{S}$ are:

$$\begin{aligned}&i+6j,\ \ j=0,...,\tau,\ \ i=1,2,3,4\\ &6j+5,\ \ j=0,...,2\tau+1.\end{aligned}$$

By counting the nongaps of $\mathcal{S}$ we obtain the genus of $\mathcal{S}$ $g=6+6\tau$ and the largest nongap is $l_g = 12\tau+11 = 2g-1$, and so $\mathcal{S}$ is a symmetric semigroup.

Suppose that $\mathcal{C}$ is a complete integral projective Gorenstein curve of arithmetic genus g and let P be a nonsingular point of $\mathcal{C}$ such that the Weierstrass semigroup of $(\mathcal{C}, P)$ is $\mathcal{S}$.

Since $\mathcal{S}$ is Weierstrass semigroup, we have that for each $n \in \mathcal{S}$ there is a meromorphic function $x_n \in H^0(\mathcal{C}, (2g-2)P) = H^0(\mathcal{C}, (10+12\tau)P)$, whose pole order at P is exactly n. We introduce the following notation:

$$x := x_6,\ \ y_i := x_{i+6\tau}\ \ (i=7,8,9,10) \tag{2.1}$$

with normalizations

$$x_{6i} = x^i,\ \ x_{j+6\tau+6i} = x^i y_j.$$

Provided that $l_2 = 2$, $\mathcal{S}$ is nonhyperelliptic, it follows that we can identify $\mathcal{C}$ with its image under the canonical embedding

$$(x_0 : ... : x_{n_{g-1}}) : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}.$$

Thus, we can assume that $\mathcal{C}$ is a canonical curve in $\mathbb{P}^{g-1}$ and $P = (0 : ... : 0 : 1)$ the Weierstrass point of $\mathcal{C}$.

A P-Hermitian basis for the vector space $H^0(\mathcal{C}, (2g-2)P)$ consists of the following functions

$$x^i \ (i = 0, ..., 2\tau + 1)$$
$$x^i y_j \ (i = 0, ..., \tau, \ j = 7, 8, 9, 10).$$

Now we will study the quadratic relations of the canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$. For this, we consider a P-Hermitian basis of the vector space $H^0(\mathcal{C}, 2(2g-2)P)$, which consist of the $3g-3$ functions:

$$x^i \ (i = 0, ..., 4\tau + 3)$$
$$x^i y_j \ (i = 0, ..., 3\tau + 2, \ j = 7, 8)$$
$$x^i y_j \ (i = 0, ..., 3\tau + 1, \ j = 9, 10)$$
$$x^i y_7 y_{10} \ (i = 0, ..., 2\tau).$$

Let X, Y_7, Y_8, Y_9, Y_{10} be indeterminates with respective weights $6, 7+6\tau, 8+6\tau, 9+6\tau$ and $10+6\tau$. For each $n \in \mathcal{S}$, we define a monomial Z_n of weight n as follows

$$Z_{6i} = X^i, \ Z_{j+6\tau+6i} = Y_j X^i \text{ and } Z_{17+12\tau+6i} = Y_7 Y_{10} X^i.$$

By considering the $(g-2)(g-3)/2$ quadratic forms as in (1.11), and making implosion[1], we obtain nine quadratic forms

$$\begin{array}{lll} F_{14}^{(0)} = Y_7^2 - X^{\tau+1} Y_8 & F_{15}^{(0)} = Y_7 Y_8 - X^{\tau+1} Y_9 & F_{16}^{(0)} = Y_7 Y_9 - X^{\tau+1} Y_{10}, \\ G_{16}^{(0)} = Y_8^2 - X^{\tau+1} Y_{10} & F_{17}^{(0)} = Y_8 Y_9 - Y_7 Y_{10} & F_{18}^{(0)} = Y_8 Y_{10} - X^{2\tau+3}, \\ G_{18}^{(0)} = Y_9^2 - X^{2\tau+3} & G_{19}^{(0)} = Y_9 Y_{10} - X^{\tau+2} Y_7 & G_{20}^{(0)} = Y_{10}^2 - X^{\tau+2} Y_8. \end{array}$$

Writing the nine products $F_i^{(0)}, G_j^{(0)}$ as linear combination of the basis elements of the k-vector space $H^0(\mathcal{C}, 2(2g-2)P)$. we obtain, in the indeterminates X, Y_7, Y_8, Y_9, Y_{10}, the polynomials

$$\begin{aligned} F_i &= F_i^{(0)} + \sum_{j=0}^{12\tau+i} f_{ij} Z_{12\tau+i-j} \ (i = 14, 15, 16, 17, 18) \\ G_i &= G_i^{(0)} + \sum_{j=0}^{12\tau+i} g_{ij} Z_{12\tau+i-j} \ (i = 16, 18, 19, 20), \end{aligned}$$

[1] by considering redundancies, for example $X_6^2 = X_{12}$, then the quadratic form $F_{12}^{(0)} = X_{12} - X_6^2 = 0$.

that vanish identically on the affine curve $\mathcal{D} \cap \mathbb{A}^5$, where $\mathcal{D}$ is the image of $\mathcal{C}$ on the map

$$(1 : x : y_7 : y_8 : y_9 : y_{10}) : \mathcal{C} \hookrightarrow \mathbb{P}^5$$

which defines an isomorphism of the canonical curve $\mathcal{C}$ onto a curve $\mathcal{D} \subset \mathbb{P}^5$ of degree $10 + 6\tau$.

Lemma 3. *The ideal of the affine curve $\mathcal{D} \cap \mathbb{A}^5$ is equal to the ideal $\mathcal{I}$ generated by the above forms F_i and G_i. In particular, if $\mathcal{C}$ is the canonical monomial curve $\mathcal{C}_{\mathcal{S}}$, then the ideal of the affine monomial curve*

$$\mathcal{D}_S \cap \mathbb{A}^5 = \{(t^6, t^{7+6\tau}, t^{8+6\tau}, t^{9+6\tau}, t^{10+6\tau}) : t \in \mathbf{k}\}$$

is generated by the initial forms $F_i^{(0)}$ and $G_i^{(0)}$.

Proof. Suppose that f is a polynomial in variables X, Y_7, Y_8, Y_9, Y_{10}. Notice that, this polynomial f modulo the ideal $\mathcal{I}$ (generated by the nine quadratic forms F_i, G_i) has monomials not divisible by the nine products $Y_iY_j, (i, j) \neq (7, 10)$, i.e., $f\text{mod}(\mathcal{I}) = \sum c_n Z_n$, where $c_n \in \mathbf{k}$ and the monomials Z_n have pairwise different weight with $n \in \mathcal{S}$. Thus, such a linear combination $\sum c_n Z_n$ vanishes identically on the curve $\mathcal{D} \cap \mathbb{A}^5$ if and only if the corresponding linear combination $\sum c_n x_n$ of rational functions $x_n \in \mathbf{k}(\mathcal{C})$ is equal to zero, that is, $c_n = 0$ for each $n \in \mathcal{S}$. $\square$

We can write more appropriately the functions F_i, G_i in such a way that the constants f_{ij}, g_{ij} are more easy to normalize:

$$F_{14} = Y_7^2 - X^{\tau+1}Y_8 - \sum_{i=0}^{\tau+1} f_{14,1+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} f_{14,2+6i}X^{2\tau+2-i} - \sum_{i=0}^{\tau} f_{14,4+6i}X^{\tau-i}Y_{10} - \sum_{i=0}^{\tau} f_{14,5+6i}X^{\tau-i}Y_9 - \sum_{i=0}^{\tau} f_{14,6+6i}X^{\tau-i}Y_8$$

$$F_{15} = Y_7Y_8 - X^{\tau+1}Y_9 - \sum_{i=0}^{\tau+1} f_{15,1+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} f_{15,2+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} f_{15,3+6i}X^{2\tau+2-i} - \sum_{i=0}^{\tau} f_{15,5+6i}X^{\tau-i}Y_{10} - \sum_{i=0}^{\tau} f_{15,6+6i}X^{\tau-i}Y_9$$

$$F_{16} = Y_7Y_9 - X^{\tau+1}Y_{10} - \sum_{i=0}^{\tau+1} f_{16,1+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} f_{16,2+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} f_{16,3+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} f_{16,4+6i}X^{2\tau+2-i} - \sum_{i=0}^{\tau} f_{16,6+6i}X^{\tau-i}Y_{10}$$

$$G_{16} = Y_8^2 - X^{\tau+1}Y_{10} - \sum_{i=0}^{\tau+1} g_{16,1+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} g_{16,2+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} g_{16,3+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} g_{16,4+6i}X^{2\tau+2-i} - \sum_{i=0}^{\tau} g_{16,6+6i}X^{\tau-i}Y_{10}$$

$$F_{17} = Y_8Y_9 - Y_7Y_{10} - \sum_{i=0}^{\tau+1} f_{17,1+6i}X^{\tau+1-i}Y_{10} - \sum_{i=0}^{\tau+1} f_{17,2+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} f_{17,3+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} f_{17,4+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} f_{17,5+6i}X^{2\tau+2-i}$$

$$F_{18} = Y_8Y_{10} - X^{2\tau+3} - f_{18,1}Y_7Y_{10} - \sum_{i=0}^{\tau+1} f_{18,2+6i}X^{\tau+1-i}Y_{10} - \sum_{i=0}^{\tau+1} f_{18,3+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} f_{18,4+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} f_{18,5+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} f_{18,6+6i}X^{2\tau+2-i}$$

$$G_{18} = Y_9^2 - X^{2\tau+3} - g_{18,1}Y_7Y_{10} - \sum_{i=0}^{\tau+1} g_{18,2+6i}X^{\tau+1-i}Y_{10} - \sum_{i=0}^{\tau+1} g_{18,3+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} g_{18,4+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} g_{18,5+6i}X^{\tau+1-i}Y_7 - \sum_{i=0}^{2\tau+2} g_{18,6+6i}X^{2\tau+2-i}$$

$$G_{19} = Y_9Y_{10} - X^{\tau+2}Y_7 - \sum_{i=0}^{2\tau+3} g_{19,1+6i}X^{2\tau+3-i} - g_{19,2}Y_7Y_{10} - \sum_{i=0}^{\tau+1} g_{19,3+6i}X^{\tau+1-i}Y_{10} - \sum_{i=0}^{\tau+1} g_{19,4+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} g_{19,5+6i}X^{\tau+1-i}Y_8 - \sum_{i=0}^{\tau+1} g_{19,6+6i}X^{\tau+1-i}Y_7$$

$$G_{20} = Y_{10}^2 - X^{\tau+2}Y_8 - \sum_{i=0}^{\tau+2} g_{20,1+6i}X^{\tau+2-i}Y_7 - \sum_{i=0}^{2\tau+3} g_{20,2+6i}X^{2\tau+3-i} - g_{20,3}Y_7Y_{10} - \sum_{i=0}^{\tau+1} g_{20,4+6i}X^{\tau+1-i}Y_{10} - \sum_{i=0}^{\tau+1} g_{20,5+6i}X^{\tau+1-i}Y_9 - \sum_{i=0}^{\tau+1} g_{20,6+6i}X^{\tau+1-i}Y_8.$$

In order to normalize some of the coefficients f_{ij} and g_{ij}, we note that we have just the freedom to transform:

$$\begin{aligned}
x &\mapsto x+\alpha_6,\\
y_7 &\mapsto y_7+\sum_{i=0}^{\tau+1}\beta_{1+6i}x^{\tau+1-i},\\
y_8 &\mapsto y_8+\gamma_1 y_7+\sum_{i=0}^{\tau+1}\gamma_{2+6i}x^{\tau+1-i},\\
y_9 &\mapsto y_9+\nu_1 y_8+\nu_2 y_7+\sum_{i=0}^{\tau+1}\nu_{3+6i}x^{\tau+1-i},\\
y_{10} &\mapsto y_{10}+\mu_1 y_9+\mu_2 y_8+\mu_3 y_7+\sum_{i=0}^{\tau+1}\mu_{4+6i}x^{\tau+1-i}
\end{aligned}$$

where $\gamma_1, \nu_1, \mu_1, \nu_2, \mu_2, \mu_3, \alpha_6 \in \mathbf{k}$ with weight $1, 2, 3$ and 6 respectively. Thus, by linear changes of variables we may normalize

$$f_{18,1}=g_{18,1}=g_{19,2}=g_{20,3}=0,\quad f_{15,6}=f_{16,2}=g_{16,1}=0$$

and

$$f_{16,1+6i}=f_{17,4+6i}=f_{18,2+6i}=g_{19,3+6i}=0,\qquad (i=0,...,\tau+1).$$

Stoehr construction ensures that the isomorphism class of the pointed Gorenstein curve $(\mathcal{C}, P)$ determines uniquely the coefficients up to $\mathbb{G}_m(\mathbf{k})-$action

$$f_{ij}\mapsto \eta^j f_{ij} \text{ and } g_{ij}\mapsto \eta^j g_{ij}\quad \text{where}\quad \eta\in\mathbb{G}_m=\mathbf{k}^\star$$

and we attached to the coefficients of f_{ij} and g_{ij} the weight j.

Now applying the syzygy lemma (1.4.1) to the ideal of $\mathcal{D}_\mathcal{S}\cap\mathbb{A}^5$ we get seven quasi-homogeneous binomials

$$\begin{aligned}
&Y_{10}F_{14}^{(0)}-Y_8F_{16}^{(0)}+Y_7F_{17}^{(0)}=0,\\
&Y_{10}F_{15}^{(0)}-Y_9G_{16}^{(0)}+Y_8F_{17}^{(0)}=0,\\
&Y_{10}G_{16}^{(0)}-Y_8F_{18}^{(0)}-X^{\tau+1}G_{20}=0,\\
&Y_{10}F_{17}^{(0)}-Y_8G_{19}^{(0)}+Y_7G_{20}^{(0)}=0,\\
&Y_{10}F_{18}^{(0)}-X^{\tau+2}G_{16}^{(0)}-Y_8G_{20}^{(0)}=0,\\
&Y_{10}G_{18}^{(0)}-X^{\tau+2}F_{16}^{(0)}-Y_9F_{19}^{(0)}=0,\\
&Y_{10}G_{19}^{(0)}-X^{\tau+2}F_{17}^{(0)}-Y_9G_{20}^{(0)}=0.
\end{aligned}$$

The seven syzygy of the monomial curve $\mathcal{D}_\mathcal{S}\cap\mathbb{A}^5$ give rise to seven syzygy of the curve $\mathcal{D}\cap\mathbb{A}^5$. And thus we obtain the seven polynomial equations module Λ_3

$$\begin{aligned}
&Y_{10}F_{14}-Y_8F_{16}+Y_7F_{17}\equiv\\
&\quad-\sum_{i=0}^{\tau+1}X^{\tau+1-i}[(f_{17,3+6i}-f_{16,3+6i})F_{15}+f_{17,2+6i}F_{16}-f_{16,2+6i}G_{16}]\\
&\quad-\sum_{i=0}^{\tau}X^{\tau-i}[(f_{14,6+6i}-f_{16,6+6i})F_{18}+f_{14,5+6i}G_{19}+f_{14,4+6i}G_{20}],
\end{aligned}$$

$$Y_{10}F_{15} - Y_9G_{16} + Y_8F_{17} \equiv$$
$$\sum_{i=0}^{\tau} X^{\tau-i}[(g_{16,6+6i} - f_{15,6+6i})G_{19} - f_{15,5+6i}G_{20}]$$
$$+\sum_{i=0}^{\tau+1} X^{\tau+1-i}[g_{16,3+6i}F_{16} - f_{17,3+6i}G_{16} - (f_{17,2+6i} - g_{16,2+6i})F_{17}$$
$$-(f_{15,1+6i} + f_{17,1+6i})F_{18} + g_{16,1+6i}G_{18}],$$
$$Y_{10}G_{16} - Y_8F_{18} + X^{\tau+1}G_{20} \equiv -\sum_{i=0}^{\tau} X^{\tau-i}g_{16,6+6i}G_{20}+$$
$$\sum_{i=0}^{\tau+1} X^{\tau+1-i}[f_{18,5+6i}F_{15} + f_{18,4+6i}G_{16} + f_{18,3+6i}F_{17} - g_{16,2+6i}F_{18} - g_{16,1+6i}G_{19}],$$
$$Y_{10}F_{17} - Y_8G_{19} + Y_7G_{20} \equiv$$
$$+\sum_{i=0}^{\tau+1} X^{\tau+1-i}[(g_{19,6+6i} - g_{20,6+6i})F_{15} - g_{20,5+6i}F_{16} + g_{19,5+6i}G_{16} + g_{19,4+6i}F_{17}$$
$$-f_{17,3+6i}F_{18} - f_{17,2+6i}G_{19} - f_{17,1+6i}G_{20}] - \sum_{i=0}^{\tau+2} X^{\tau+2-i}g_{20,1+6i}F_{14},$$
$$Y_{10}F_{18} - X^{\tau+2}G_{16} - Y_8G_{20} \equiv \sum_{i=0}^{\tau+2} X^{\tau+2-i}g_{20,1+6i}F_{15}$$
$$+\sum_{i=0}^{\tau+1} X^{\tau+1-i}[g_{20,6+6i}G_{16}+g_{20,5+6i}F_{17}-(f_{18,4+6i}-g_{20,4+6i})F_{18}-f_{18,3+6i}G_{19}],$$
$$Y_{10}G_{18} - X^{\tau+2}F_{16} - Y_9G_{19} \equiv \sum_{i=0}^{\tau+1} X^{\tau+1-i}[g_{19,6+6i}F_{16} + g_{19,5+6i}F_{17}$$
$$+g_{19,4+6i}G_{18} - g_{18,4+6i}F_{18} - g_{18,3+6i}G_{19} - g_{18,2+6i}G_{20}],$$
$$Y_{10}G_{19} - X^{\tau+2}F_{17} - Y_9G_{20} \equiv \sum_{i=0}^{\tau+2} X^{\tau+2-i}g_{20,1+6i}F_{16}$$
$$+\sum_{i=0}^{\tau+1} X^{\tau+1-i}[g_{20,6+6i}F_{17}+g_{20,5+6i}G_{18}-g_{19,5+6i}F_{18}-(g_{19,4+6i}-g_{20,4+6i})G_{19}].$$

In order to simplify these equations, we introduce the following polynomials in $\mathbf{k}[t]$:

$$g_i := \sum_{r=1}^{12\tau+i} g_{ir}t^r = G_i(t^{-6}, t^{-7-6\tau}, t^{-8-6\tau}, t^{-9-6\tau}, t^{-10-6\tau}) \quad (i = 16, 18, 19, 20),$$

and write each one as sum of its partial polynomials

$$g_i^{(j)} := \sum_{r\equiv j \bmod 6} g_{ir}t^r \quad (j = 1, ..., 6).$$

similarly for each f_i. By using the normalizations of f_{ij}, g_{ij}, we may express each f_i, g_i in terms of 41 partial polynomials. More precisely, we can write

$$
\begin{aligned}
f_{14} &= f_{14}^{(1)} + f_{14}^{(2)} + f_{14}^{(4)} + f_{14}^{(5)} + f_{14}^{(6)}\\
f_{15} &= f_{15}^{(1)} + f_{15}^{(2)} + f_{15}^{(3)} + f_{15}^{5()} + f_{15}^{6()}\\
f_{16} &= f_{16}^{(2)} + f_{16}^{(3)} + f_{16}^{(4)} + f_{14}^{(6)}\\
f_{17} &= f_{17}^{(1)} + f_{17}^{(2)} + f_{17}^{(3)} + f_{14}^{(5)}\\
f_{18} &= f_{18}^{(3)} + f_{18}^{(4)} + f_{18}^{(5)} + f_{18}^{(6)}\\
g_{16} &= g_{16}^{(1)} + g_{16}^{(2)} + g_{16}^{(3)} + g_{16}^{(4)} + g_{16}^{(6)}\\
g_{18} &= g_{18}^{(2)} + g_{18}^{(3)} + g_{18}^{(4)} + g_{18}^{(5)} + g_{18}^{(6)}\\
g_{19} &= g_{16}^{(1)} + g_{19}^{(4)} + g_{19}^{(5)} + g_{19}^{(6)}\\
g_{20} &= g_{20}^{(1)} + g_{20}^{(2)} + g_{20}^{(4)} + g_{20}^{(5)} + g_{20}^{(6)}
\end{aligned}
$$

By computing the degrees of the partial polynomials we can counting the number of coefficients that are still involved, thus we obtain $50\tau + 84$ coefficients. With everything done so far we get an explicit description of the compactified moduli space $\overline{\mathcal{M}_{g,1}^{\mathcal{S}}}$.

Theorem 13. *Let $\mathcal{S}$ be the semigroup generated by $6, 7+6\tau, 8+6\tau, 9+6\tau$ and $10+6\tau$ where τ is a positive integer. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{S}$ correspond bijectively to the orbits of the $\mathbb{G}_m$-action on the quasi-cone of the vectors of length $50\tau + 84$ whose coordinates are the coefficients f_{ij}, g_{ij} of the 41 partial polynomials that satisfy the seven equations:*

$$
\begin{aligned}
f_{18} - g_{16} - g_{20} &= g_{20}^{(6)} g_{16} + g_{20}^{(5)} f_{17} - (f_{18}^{(4)} - g_{20}^{(4)}) f_{18} - f_{18}^{(3)} g_{19} - f_{18}^{(2)} g_{20} + g_{20}^{(1)} f_{15},\\
g_{18} - f_{16} - g_{19} &= g_{19}^{(6)} f_{16} + g_{19}^{(5)} f_{17} + g_{19}^{(4)} g_{18} - g_{18}^{(4)} f_{18} - g_{18}^{(3)} g_{19} - g_{18}^{(2)} g_{20},\\
g_{19} - f_{17} - g_{20} &= g_{20}^{(6)} f_{17} + g_{20}^{(5)} g_{18} - g_{19}^{(5)} f_{18} - (g_{19}^{(4)} - g_{20}^{(4)}) g_{19} + g_{20}^{(1)} f_{16},\\
g_{16} - f_{18} + g_{20} &= \ f_{18}^{(5)} f_{15} + f_{18}^{(4)} g_{16} + f_{18}^{(3)} f_{17} - g_{16}^{(2)} f_{18} - g_{16}^{(1)} g_{19} - g_{16}^{(6)} g_{20},\\
f_{14} - f_{16} + f_{17} &= \ f_{16}^{(2)} g_{16} - (f_{17}^{(3)} - f_{16}^{(3)}) f_{15} - f_{17}^{(2)} f_{16} - (f_{14}^{(6)} - f_{16}^{(6)}) f_{18}\\
&\quad - f_{14}^{(5)} g_{19} - f_{14}^{(4)} g_{20},\\
f_{15} - g_{16} + f_{17} &= \ g_{16}^{(1)} g_{18} - (f_{15}^{(1)} + f_{17}^{(1)}) f_{18} + g_{16}^{(3)} f_{16} - f_{17}^{(3)} g_{16} - f_{15}^{(5)} g_{20}\\
&\quad - (f_{17}^{(2)} - g_{16}^{(2)}) f_{17} - (f_{15}^{(6)} - g_{16}^{(6)}) g_{19},\\
f_{17} - g_{19} + g_{20} &= \ g_{19}^{(4)} f_{17} + g_{19}^{(5)} g_{16} - g_{20}^{(5)} f_{16} - (g_{20}^{(6)} - g_{19}^{(6)}) f_{15}\\
&\quad - f_{17}^{(3)} f_{18} - f_{17}^{(2)} g_{19} - f_{17}^{(1)} g_{20} - g_{20}^{(1)} f_{14}.
\end{aligned}
$$

As consequence of the above theorem, we have that the moduli space $\overline{\mathcal{M}_{g,1}^{\mathcal{S}}}$ admits an embedding into a $50\tau + 84$-dimensional weighted projective space.

Since the vector space $T^{1,-}_{\mathbf{k}[\mathcal{S}],\mathbf{k}}$ corresponds bijectively to the locus of the linearization of the 41 equations of the linear system obtained by replacing the quadratic terms on the right sides of the equations of the theorem (13) by zeros, when we solving this system follows that the vector space $T^{1,-}_{\mathbf{k}[\mathcal{S}],\mathbf{k}}$ can be identified with the space whose entries are the coefficients of the remaining partial polynomials. Thus, after some computations, we see that the linearizations depend only on the 11 following partial polynomials

$$f_{14}^{(1)}, f_{17}^{(2)}, f_{16}^{(2)}, g_{16}^{(3)}, f_{14}^{(4)}, g_{20}^{(4)}, f_{14}^{(5)}, f_{18}^{(5)}, f_{14}^{(6)}, f_{15}^{(6)} \text{ and } g_{20}^{(6)}.$$

Counting its coefficients and discounting the three normalizations $f_{15,6} = f_{16,2} = g_{16,1} = 0$, we obtain $11\tau + 15$ coefficients. Thus

$$\dim T^{1,-}_{\mathbf{k}[\mathcal{S}],\mathbf{k}} = 11\tau + 15.$$

More precisely, we can obtain the dimension of the graded component of $T^{1,-}_{\mathbf{k}[\mathcal{S}]}$ of negative weight $-j$ by counting the coefficients of a given weight j

$$j = -1-6i \quad dimT^1_j = \begin{cases} 0, & \text{if } i \geq \tau+2, \\ 1, & \text{if } 1 \leq i \leq \tau+1, \\ 0, & \text{if } i = 0 \end{cases}$$

$$j = -2-6i \quad dimT^1_j = \begin{cases} 0, & \text{if } i \geq \tau+2, \\ 2, & \text{if } 1 \leq i \leq \tau+1, \\ 1, & \text{if } i = 0 \end{cases}$$

$$j = -3-6i \quad dimT^1_j = \begin{cases} 0, & \text{if } i \geq \tau+2, \\ 1, & \text{if } 1 \leq i \leq \tau+1, \\ 1, & \text{if } i = 0 \end{cases}$$

$$j = -4-6i \quad dimT^1_j = \begin{cases} 0, & \text{if } i \geq \tau+2, \\ 1, & \text{if } i = \tau+1, \\ 2, & \text{if } 1 \leq i \leq \tau+1 \end{cases}$$

$$j = -5-6i \quad dimT^1_j = \begin{cases} 0, & \text{if } i \geq \tau+2, \\ 1, & \text{if } i = \tau+1, \\ 2, & \text{if } 1 \leq i \leq \tau+1 \end{cases}$$

$$j = -6-6i \quad dimT^1_j = \begin{cases} 0, & \text{if } i \geq \tau+2, \\ 1, & \text{if } i = \tau+1, \\ 3, & \text{if } 1 \leq i \leq \tau \\ 2, & \text{if } i = 0. \end{cases}$$

Thus, the compactified moduli space $\overline{\mathcal{M}^{\mathcal{S}}_{g,1}}$ has been realized as a closed subspace of the $(11\tau+14)$-dimensional weighted projective space $\mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]})$.

Now we compute the quadratic quasi-cone and its dimension. Entering with our solution of the system of 41 linear equations in the quadratic terms of degree at most two(quadratic approximation) and eliminate the same partial polynomials that the linear case, the quadratic quasi-cone $\mathfrak{Q}_{\mathcal{S}}$ is a subvariety of $T^{1,-}_{\mathbf{k}[\mathcal{S}],\mathbf{k}}$ whose equations are

$$\begin{aligned} &\pi_{13+6\tau}(f_{14}^{(1)}\tilde{g}_{20}^{(6)} + f_{14}^{(4)}g_{16}^{(3)} + f_{14}^{(5)}f_{17}^{(2)} - f_{17}^{(2)}f_{18}^{(5)} + g_{16}^{(3)}g_{20}^{(4)}) = 0 \\ &\pi_{7+6\tau}(-f_{14}^{(1)}\tilde{f}_{15}^{(6)} - f_{14}^{(4)}g_{16}^{(3)} - f_{14}^{(5)}f_{17}^{(2)}) = 0 \\ &\pi_{9+6\tau}(-f_{14}^{(1)}f_{16}^{(2)} - f_{14}^{(4)}f_{18}^{(5)} - f_{14}^{(5)}g_{20}^{(4)}) = 0 \\ &\pi_{10+6\tau}(f_{16}^{(2)}f_{17}^{(2)} + f_{14}^{(4)}(\tilde{g}_{20}^{(6)} - \tilde{f}_{15}^{(6)}) - g_{20}^{(4)}\tilde{f}_{15}^{(6)}) = 0 \\ &\pi_{11+6\tau}(f_{14}^{(5)}\tilde{f}_{15}^{(6)} - f_{14}^{(5)}\tilde{g}_{20}^{(6)} - \tilde{f}_{15}^{(6)}f_{18}^{(5)} + f_{16}^{(2)}g_{16}^{(3)}) = 0, \end{aligned} \tag{2.2}$$

where $\tilde{g}_{20}^{(6)} = g_{20}^{(6)} - f_{14}^{(6)}$, $\tilde{f}_{15}^{(6)} = f_{15}^{(6)} - f_{14}^{(6)}$ and π_i denotes the projection operator in t that annihilates the terms of degree not larger than i. We can observe that these equations does not depend of the coefficients $f_{14,1}, f_{17,2}, g_{16,3}, g_{20,4}, f_{18,5}$ and $f_{14,6i}, i = 2, \dots, \tau+1$. By considering the $(\tau+1)$-dimensional artinian algebra

$$A := k[\epsilon] = \bigoplus_{j=0}^{\tau} k\epsilon^j, \text{ where } \epsilon^{\tau+1} = 0,$$

we can write the equations in (2.2) in terms of five polynomial equations between $\tau+1$ elements of the A.

Theorem 14. *The quadratic quasi-cone $\mathcal{Q}$ is isomorphic to the direct product*

$$\mathcal{Q} = M \times N,$$

where M is the $(\tau+5)$-dimensional weighted space of weights $1,2,3,4,5$ and $6i, i = 2, \dots, \tau+1$, and N is the quadratic quasi-cone consisting of vectors

$$(\omega_1, \dots, \omega_{10}) = \left(\sum_{j=0}^{\tau} \omega_{1j}\epsilon^j, \dots, \sum_{j=0}^{\tau} \omega_{10,j}\epsilon^j \right),$$

satisfying the five equations

$$\begin{aligned}
&\omega_1\omega_{10} + \omega_4\omega_5 + \omega_3\omega_7 - \omega_3\omega_8 + \omega_4\omega_6 = 0,\\
&-\omega_1\omega_9 - \omega_4\omega_5 - \omega_3\omega_7 = 0,\\
&-\omega_1\omega_2 - \omega_5\omega_8 - \omega_6\omega_7 = 0,\\
&\omega_2\omega_3 + \omega_5(\omega_{10} - \omega_9) - \omega_6\omega_9 = 0,\\
&\omega_7\omega_9 - \omega_7\omega_{10} - \omega_8\omega_9 + \omega_2\omega_4 = 0,
\end{aligned}$$

in the artinian algebra A.

Proof. The five conditions (2.2) are equivalent to the five quadratic equations in the Artinian algebra A when we define ω_{ij} of the following form

$$\omega_{1j} := f_{14,1+6\tau-6j}, \omega_{2j} := f_{16,8+6\tau-6j}, \omega_{3j} := f_{17,2+6\tau-6j}, \omega_{4j} := g_{16,3+6\tau-6j},$$

$$\omega_{5j} := f_{14,4+6\tau-6j}, \omega_{6j} := g_{20,4+6\tau-6j}, \omega_{7j} := f_{14,5+6\tau-6j}, \omega_{8J} := f_{18,5+6\tau-6j},$$

$$\omega_{9j} := \tilde{f}_{15,12+6\tau-6j}, \omega_{10,j} := \tilde{g}_{20,6+6\tau-6j}.$$

□

Corollary 2. *We have* $\dim \mathfrak{Q}_{\mathcal{S}} = 8\tau + 12$. *Thus*

$$\dim \mathcal{M}_{g,1}^{\mathcal{S}} \leq 8\tau + 11.$$

Proof. Since the dimension of M is $\tau + 5$, we only have to show that

$$\dim N = 7\tau + 7$$

Let $U_i (i = 1, ..., 10)$ be the open subset of N defined by the equation $\omega_{i0} \neq 0$, and since A is Artinian algebra we have that ω_i is a unit. Now suppose that the vector $(\omega_1, ..., \omega_{10}) \in U_1$, then from (14) we may eliminate ω_2, ω_9, and ω_{10} from the first three quadratic equations and the remaining two equations become trivial. Therefore, U_1 has dimension $7(\tau + 1)$ in $A^{10(\tau+1)}$. Similarly we see that

$$\dim(U_i) = 7(\tau + 1) \ (i = 1, ..., 10).$$

Thus, if $\tau = 1$, then $N = U_1 \cup \cdots \cup U_{10}$ and $\dim N = 7(\tau + 1)$. Now we assume that $\tau > 1$ and $(\omega_1, ..., \omega_{10}) \in N$ but does not belong to the union $U_1 \cup \cdots \cup U_{10}$, meaning that $\omega_{ij} = 0$ whenever $j = 0$, and then the ten coefficients ω_{ij} with $j = \tau$ do not enter into five (14), and by induction we have

$$\dim(N \backslash (U_1 \cup \cdots \cup U_{10})) = 7(\tau - 1) + 10 = 7\tau + 3 < 7(\tau + 1),$$

and therefore $\dim N = 7(\tau + 1)$. Thus, we conclude that

$$\mathcal{M}^{\mathcal{S}}_{g,1} \leq 8\tau + 11$$

□

On the other hand, computing the effective weight of $\mathcal{S}$ we have $\mathrm{ewt}(\mathcal{S}) = 10\tau + 5$, and so by Theorem 6 we get

$$\dim \mathcal{M}^{\mathcal{S}}_{g,1} \geq 3(6 + 6\tau) - 2 - (5 + 10\tau) = 8\tau + 11.$$

Corollary 3. *Let $\mathcal{S}$ be a semigroup generated by $6, 7 + 6\tau, 8 + 6\tau, 9 + 6\tau$ and $10 + 6\tau$ with $\tau \geq 1$, then*

$$\dim \mathcal{M}^{\mathcal{S}}_{g,1} = 8\tau + 11$$

2.2 COLLECTING KNOWN DIMENSIONS

As cited in Section 1.5 of this thesis, if a numerical semigroup $\mathcal{S}$ is negatively graded then the dimension of $\mathcal{M}^{\mathcal{S}}_{g,1}$ is equal to Deligne–Pinkham's upper bound $2g - 2 + \lambda(\mathcal{S})$ which is also equal to Pflueger's bound $3g - 2 - \mathrm{ewt}(\mathcal{S})$ in this case. Additionally, we know the dimension of $\mathcal{M}^{\mathcal{S}}_{g,1}$ for all numerical semigroups whose genus is not bigger than 6, it is equal to Pflueger's bound.

In the below table 1 we collect the bounds due to Deligne–Pinkham and Pflueger for all numerical semigroups of genus $g \leq 6$. Clearly, we just consider the non-negatively graded numerical semigroups. In table 1 D–P stands for the Deligne–Pinkham's bound,

Table 1 – negatively graded semigroups of genus ≤ 6

gaps	NP	$\dim \mathcal{M}^{\mathcal{S}}_{g,1}$	D–P	$\dim T^{1,+}$
1, 2, 4, 5, 8	9	9	10	1
1, 2, 3, 5, 7	10	10	11	1
1, 2, 3, 6, 7	9	9	10	1
1, 2, 4, 5, 7, 10	11	11	12	1
1, 2, 4, 5, 8, 11	10	10	11	1
1, 2, 3, 5, 6, 9	12	12	13	1
1, 2, 3, 5, 6, 10	11	11	12	1
1, 2, 3, 5, 7, 9	11	11	13	2
1, 2, 3, 5, 7, 11	10	10	11	1
1, 2, 3, 6, 7, 11	10	10	11	1
1, 2, 3, 4, 6, 8	13	13	14	1
1, 2, 3, 4, 6, 9	12	12	13	1
1, 2, 3, 4, 7, 8	12	12	13	1
1, 2, 3, 4, 7, 9	11	11	12	1
1, 2, 3, 4, 8, 9	10	10	12	2

NP for Pflueger's bound, and finally $\dim T^{1,+}$ for the dimension of the positive graded part of the first cohomology module of the cotangent complex associated to $\mathbf{k}[\mathcal{S}]$, namely $\dim T^{1,+} := \sum_{s=1}^{\infty} \dim T^1(\mathbf{k}[\mathcal{S}])_s$.
We also collect in table 2 the bounds for the dimensions of $\mathcal{M}^{\mathcal{S}}_{g,1}$ for families of symmetric semigroups, including $\langle 6, 7+6\tau, 8+6\tau, 9+6\tau, 10+6\tau\rangle$ with $\tau \geq 1$, the upper bound obtained by Contiero and Stoehr in [CS, Cor. 4.5] and A. Fontes [CF] for the symmetric semigroups $\langle 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau\rangle$ and $\langle 6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau\rangle$ with $\tau \geq 1$, respectively.

Table 2 – $\dim \mathcal{M}^{\mathcal{S}}_{g,1}$ for three families of semigroups

semigroup	NP	CFV-CS	D–P	$\dim T^{1,+}$
$\langle 6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau\rangle$	$8\tau+7$	$8\tau+7$	$12\tau+5$	$4\tau-2$
$\langle 6, 7+6\tau, 8+6\tau, 9+6\tau, 10+6\tau\rangle$	$8\tau+11$	$8\tau+11$	$12\tau+11$	4τ
$\langle 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau\rangle$	$8\tau+5$	$8\tau+5$	$12\tau+1$	$4\tau-4$

By comparing the bounds which appears in table 1 and table 2, and using the theorem due Rim and Vitulli on negatively graded semigroups, we can conclude the following.

Corollary 4. *For $\mathcal{S}$ each numerical semigroup of genus $g \leq 6$, or any negatively graded numerical semigroup $\mathcal{S}$, or one of the following symmetric semigroups $\langle 6, 7+6\tau, 8+6\tau, 9+6\tau, 10+6\tau\rangle$, $\langle 6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau\rangle$ or $\langle 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau\rangle$, we get*

$$3g-2-\mathrm{ewt}(\mathcal{S}) = \dim \mathcal{M}^{\mathcal{S}}_{g,1} = 2g-2+\lambda(\mathcal{S}) - \dim T^{1,+}(\mathbf{k}[\mathcal{S}]).$$

Due to above computations and results it is just natural to ask:

What is the role that $T^{1,+}(\mathbf{k}[\mathcal{S}])$ plays on the dimension of $\mathcal{M}_{g,1}^{\mathcal{S}}$?

The following question is a conjecture proposed by A. Contiero.

Question 2.2.1. *For which numerical semigroups $\mathcal{S}$ it is true that*

$$\dim \mathcal{M}_{g,1}^{\mathcal{S}} \leq 2g - 2 + \lambda(\mathcal{S}) - \dim T^{1,+}(\mathbf{k}[\mathcal{S}])\,?$$

In the next chapter we provide some contributions involving the above Question.

3 A NEW LOWER BOUND

To address Question 2.2.1, we start this chapter recalling how to compute the dimension of the homogeneous part of degree ℓ of the cotangent complex $\mathrm{T}^1(\mathbf{k}[\mathcal{S}])$ using the description of the cotangent complex given by Buchweitz in [B].

Let $\mathcal{S} := \langle a_1, \ldots, a_r \rangle$ be a numerical semigroup of genus $g > 1$. By a theorem due to Herzog the ideal of the affine monomial curve associated do $\mathcal{S}$

$$C_{\mathcal{S}} := \{(t^{a_1}, \ldots, t^{a_r})\,;\, t \in \mathbf{k}\} \subset \mathbb{A}^r$$

can be generated by isobaric polynomials F which are differences of two monomials

$$F_i := X_1^{\alpha_{i1}} \ldots X_r^{\alpha_{ir}} - X_1^{\beta_{i1}} \ldots X_r^{\beta_{ir}}$$

with $\alpha_i \cdot \beta_i = 0$. As usual, the weight of F_i is $d_i := \sum_j a_j \alpha_{ij} = \sum_j a_j \beta_{ij}$. For each i let $v_i := (\alpha_{i1} - \beta_{i1}, \ldots, \alpha_{ir} - \beta_{ir})$ be a vector in $\mathbf{k}^r$.

Theorem 15 (cf. Thm. 2.2.1 of [B])**.** *For each $\ell \notin \mathrm{End}(\mathcal{S})$,*

$$\dim T^1(\mathbf{k}[\mathcal{S}])_\ell = \#\{i \in \{1, \ldots, r\}\,;\, a_i + \ell \notin \mathcal{S}\} - \dim V_\ell - 1$$

where V_ℓ is the subvector space of $\mathbf{k}^r$ generated by the vectors v_i such that $d_i + \ell \notin \mathcal{S}$. It also true that

$$\dim T^1(\mathbf{k}[\mathcal{S}])_s = 0, \ \forall\, s \in \mathrm{End}(\mathcal{S}).$$

In a recent preprint [CFQ, section 5] by A. Contiero, A.Fontes, J. Stevens and myself, we proved the following result concerning to the Question 2.2.1.

Theorem 16. *If $\mathcal{M}_{g,1}^{\mathcal{S}}$ is nonempty, then for any irreducible component* X *of $\mathcal{M}_{g,1}^{\mathcal{S}}$ we have*

$$2g - 2 + \lambda(\mathcal{S}) - \dim \mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}]) \leq \dim \mathrm{X}$$

Proof. Let $\mathcal{Y}$ be the formal versal deformation space of the local ring at the singular point of the monomial curve $\mathcal{C}_{\mathcal{S}}$. Adding $\dim \mathrm{T}^{1,+}$ linear equations, in order that each component E intersect the subspace of $\mathcal{Y}$ of negative weight, the dimension of the intersection is at least $\dim E - \dim T^{1,+}(\mathbf{k}[\mathcal{S}])$. So $2g - 2 + \lambda(S) - \dim \mathrm{T}^{1,+}$ is a lower bound for $\dim \mathrm{X}$, if nonempty. □

The next result shows that the above lower bound is not bigger than of Pflueger's bound.

Proposition 4. *For any numerical semigroup $\mathcal{S}$ of genus $g \geq 1$,*

$$3g - 2 - \mathrm{ewt}(\mathcal{S}) \leq 2g - 2 + \lambda(\mathcal{S}) - \dim T^{1,+}(\mathcal{S}).$$

Proof. For each $\ell \in \mathbb{Z}$, set $A_\ell := \{i \in \{1, \ldots, r\}\,;\, i + \ell \notin \mathcal{S}\}$. Using Theorem 15, we obtain

$$\dim T^{1,+}(\mathbf{k}[\mathcal{S}]) = \sum_{\ell \notin \mathrm{End}(\mathcal{S})} (\#A_\ell - \dim_{\mathbf{k}} V_\ell) - g + \lambda(\mathcal{S}).$$

Hence, we just have to prove that $\mathrm{ewt}(\mathcal{S}) - \sum_{\ell \notin \mathrm{End}(\mathcal{S})} \sharp A_\ell \geq 0$. We proceed by induction on the genus g of $\mathcal{S}$. The statement is trivial for $g = 1$. If $\mathcal{S}$ is a numerical semigroup of genus $g > 1$, whose biggest gap is ℓ_g, then consider the numerical semigroup $\mathcal{S}' := \mathcal{S} \cup \{\ell_g\}$, whose genus is $g - 1 \geq 1$. It is clear that $\{\ell \notin \mathrm{End}(\mathcal{S})\} = \{\ell \notin \mathrm{End}(\mathcal{S}')\} \coprod \{\ell \,|\, \ell + a_i = \ell_g \text{ and } \ell + a_j \in \mathcal{S}, \forall\, j \neq i\}$. Now the result follows easily. □

Now, by virtue [CFQ, Lemma 5.4] we get an improvement of a lower bound given by Pflueger in [Pf1] as follows

Corollary 5. *If $\mathcal{M}^{\mathcal{S}}_{g,1}$ is nonempty, then*

$$3g - 2 - \mathrm{ewt}(\mathcal{S}) \leq 2g - 2 + \lambda(\mathcal{S}) - \mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}]) \leq \dim \mathrm{X}.$$

where X *is any one of its irreducible component.*

The above bound given in Theorem 16 is an effective improvement of Pflueger's bound, for the example for symmetric semigroup $\mathcal{S} :=< 6, 7, 8 >$ Pflueger's bound gives 14 while lower bound in Theorem 16 gives 15. This example can be considered in a more general case via classical works as follows.

Recall that the unbranched monomial curve $\mathrm{Spec}\,\mathbf{k}[\mathcal{S}]$ is Gorenstein if and only if the semigroup $\mathcal{S}$ is symmetric. In this case, a compactification of $\mathcal{M}^{\mathcal{S}}_{g,1}$ when $\mathcal{S}$ is nontrigonal was done by Stoehr in [S] and also by Contiero and Stoehr in [CS], and generalized by Contiero and Fontes [CF] for all symmetric semigroups including the trigonal one. Hence, the moduli space $\mathcal{M}^{\mathcal{S}}_{g,1}$ is an open subspace of $\overline{\mathcal{M}^{\mathcal{S}}_{g,1}}$. If the symmetric semigroup $\mathcal{S}$ is generated by 4 elements, say $\mathcal{S}$, then by using Pinkham's equivariant deformation theory [Pi], complete intersection theory and a quasi-homogeneous version of Buchsbaum-Eisenbud's structure theorem for Gorenstein ideals of codimension 3 (see [BE, p. 466]), one can deduce that the affine monomial curve $\mathrm{Spec}\,\mathbf{k}[\mathcal{S}]$ can be negatively smoothed without any obstructions (see [B], [W1] [W2, Satz 7.1]), hence $\dim \mathcal{M}^{\mathcal{S}}_{g,1} = \dim \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}})$, and therefore

$$\overline{\mathcal{M}^{\mathcal{S}}_{g,1}} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}) \tag{3.1}$$

and so $\mathcal{M}^{\mathcal{S}}_{g,1}$ is a dense open subvariety of $\overline{\mathcal{M}^{\mathcal{S}}_{g,1}}$.

3.1 ON THE GORENSTEIN CASE

In [S] K-O Stoehr showed that if the multiplicity m of the numerical symmetric semigroup $\mathcal{S}$ satisfies $3 < m < g$, then the moduli space $\mathcal{M}^{\mathcal{S}}_{g,1}$ admits a compactification, by allowing

Gorenstein singular curves at its bordering, which can be realized as a closed subset of $\mathbb{P}(T^{1,-}(\mathbf{k}[\mathcal{S}]))$, see [S, Thm. ? and Appendix]. Later on, Contiero–Stoehr [CS] and Contiero–Fontes [CF] extended the techniques in [S] in order that the construction of a compactification of $\mathcal{M}^{\mathcal{S}}_{g,1}$ is completely implementable and include all nonhyperelliptic symmetric semigroups, i.e.. $2 \notin \mathcal{S}$. We already notice that $\mathcal{S}$ is symmetric if and only if the affine monomial curve associated to $\mathcal{C}_{\mathcal{S}}$ is Gorenstein. The conclusion is that:

Theorem 17 (Contiero-Fontes–Stoehr)**.** *Let $\mathcal{S}$ be a symmetric nonhyperelliptic numerical semigroup. There is a locally closed embedding $\mathcal{M}^{\mathcal{S}}_{g,1} \hookrightarrow \mathbb{P}(\mathrm{T}^{1,-}(\mathbf{k}[\mathcal{S}]))$.*

Hence, in order to get an upper bound for $\mathcal{M}^{\mathcal{S}}_{g,1}$ we may try to compute a tight upper bound for the dimension of the negatively graded part $\mathrm{T}^{1,-}(\mathbf{k}[\mathcal{S}])$ of the cotangent complex associated to $\mathcal{S}$.

In this last section we will give a conditional proof that the dimension of $\mathcal{M}^{\mathcal{S}}_{g,1}$ is exactly the lower bound in Theorem 16 in the Gorenstein nonhyperelliptic case. We strongly believe that our approach can be successful. Additionally, we do not know any counter-example.

Let $\mathcal{C}$ be a complete reduced Gorenstein curve defined over $\mathbf{k}$ and $q \in \mathcal{C}$ its only singular point. Let $\omega : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}$ be the closed immersion induced by the dualizing sheaf. By the fundamental exact sequence for the T^1s, see [LS], we have the following exact sequence:

$$0 \to \mathcal{T}_{\mathcal{C}} \to \mathcal{T}_{\mathbb{P}|\mathcal{C}} \to \mathcal{N} \to \mathrm{T}^1 \to 0, \tag{3.2}$$

where $\mathcal{T}_{\mathcal{C}} = \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\Omega_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}})$ is the tangent sheaf of $\mathcal{C}$, $\mathcal{T}_{\mathbb{P}^{g-1}|\mathcal{C}} = \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\omega^*\Omega_{\mathbb{P}^{g-1}/\mathbf{k}}, \mathcal{O}_{\mathcal{C}})$ the restriction to $\mathcal{C}$ of the tangent sheaf of $\mathbb{P}^{g-1}$ induced by $\mathcal{C} \hookrightarrow \mathbb{P}^{g-1}$, $\mathcal{N} = \underline{Hom}_{\mathcal{O}_{\mathcal{C}}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{C}})$ the normal sheaf and $\mathrm{T}^1 = \underline{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\Omega_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) = coker(\mathcal{T}_{\mathbb{P}^{g-1}|\mathcal{C}} \to \mathcal{N})$ the cotangent complex sheaf associated to $\mathcal{C}$.

Taking Euler characteristic, we get

$$\chi(\mathrm{T}^1) = \chi(\mathcal{T}_{\mathcal{C}}) + \chi(\mathcal{N}) - \chi(\mathcal{T}_{\mathbb{P}^{g-1}|\mathcal{C}}) \tag{3.3}$$

The differential sheaf $\Omega_{C|\mathbf{k}}$ of C may not be torsion-free, but the tangent sheaf $\mathcal{T}_C$ is a coherent fractional ideal sheaf, hence the Riemann–Roch theorem for singular curves assure, that

$$\chi(\mathcal{T}_C) = \deg(\mathcal{T}_C) + 1 - g.$$

where g is the arithmetical genus of $\mathcal{C}$. Let $\widetilde{C}$ be the nonsingular model of C and $\widetilde{g}$ its geometrical genus. Since P is the only singular point of C we get $\widetilde{g} = g - \delta$. The differential sheaf $\Omega_{\widetilde{C}|\mathbf{k}}$ of the nonsingular model is locally free of rank 1 and has degree $2\widetilde{g} - 2$. Thus $\mathcal{T}_{\widetilde{C}}$ is locally free of rank 1 of degree $2 - 2\widetilde{g}$ and $\deg(\mathcal{T}_C) = \deg(\mathcal{T}_{\widetilde{C}}) - \mu + \delta = 2 - 2g - \mu + 3\delta$. Hence

$$\chi(\mathcal{T}_{\mathcal{C}}) = 3 - 3g - \mu + 3\delta \tag{3.4}$$

where δ and μ are as in Deligne's Formula.

We recall that for any torsion free sheaf $\mathcal{F}$ of rank r over a reduced curve, we have $\chi(F) = \mathrm{rank}(\mathcal{F})(1-g) + \deg(\mathcal{F})$. By definition, $\mathcal{N} = \underline{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_\mathcal{C})$ is torsion free of rank $g-2$ and $\mathcal{T}_{\mathbb{P}^{g-1}|_\mathcal{C}} = \underline{Hom}_{\mathcal{O}_\mathcal{C}}(\Omega_{\mathbb{P}^{g-1}|_\mathcal{C}}, \mathcal{O}_\mathcal{C}))$ is locally free rank $g-1$, hence

$$\chi(\mathcal{N}) - \chi(\mathcal{T}_{\mathbb{P}^{g-1}|\mathcal{C}}) = g - 1 + \deg(\mathcal{N}) - \deg(\mathcal{T}_{\mathbb{P}^{g-1}|\mathcal{C}}) \tag{3.5}$$

We note that if a curve $\mathcal{C}$ is locally complete intersection in $\mathbb{P}^n$, then $\deg(\mathcal{N}) - \deg(\mathcal{T}_{\mathbb{P}^n|_\mathcal{C}}) = 2g-2$, just because $\wedge^{n-1}\mathcal{N} \otimes \wedge^n \Omega_{\mathbb{P}^n|_\mathcal{C}}$ is isomorphic to the dualizing sheaf $\omega_\mathcal{C}$, cf. [Rh1, Thm. 7.11]. Hence $\chi(\mathcal{T}^1) = 3\delta - \mu$. Recall that every locally complete intersection curve is Gorenstein. This fact and the philosophy on Remark 11 lead us to formulate the following question.

Question 3.1.1. *Is it true that if $\mathcal{C}$ is a (monomial) canonical Gorenstein curve, then*

$$\deg(\mathcal{N}) - \deg(\mathcal{T}_{\mathbb{P}^n|_\mathcal{C}}) \leq 2g - 2\,?$$

Conditional Result. *Let $\mathcal{S}$ be a symmetric nonhyperelliptic numerical semigroup of genus $g > 1$. Assume that the answer to the above question is YES for the canonical monomial curve $\mathcal{C} := \mathcal{C}_\mathcal{S} \subseteq \mathbb{P}^{g-1}$, then*

$$\dim \mathcal{M}^{\mathcal{S}}_{g,1} = 2g - 2 - \lambda(\mathcal{S}) - \dim \mathrm{T}^{1,+}(\mathbf{k}[N])$$

if $\mathcal{M}^{\mathcal{S}}_{g,1}$ is nonempty.

Proof. By hypothesis $\deg(\mathcal{N}) - \deg(\mathcal{T}_{\mathbb{P}^n|_\mathcal{C}}) \leq 2g-2$, hence from equations (3.4) and (3.5) we conclude that

$$\dim \mathrm{H}^0(\mathcal{C}, \mathcal{T}^1) \leq 3\delta - \mu + \dim \mathrm{H}^1(\mathcal{C}, \mathcal{T}^1).$$

In his Ph.D. thesis, Schlessinger [Ms1, pg. 66], proved that $\dim \mathrm{H}^1(\mathcal{C}, \mathcal{T}^1) = 0$. Using the facts that the cotangent complex is supported in the unique singular point of $\mathcal{C}$ and that the global sections of the cotangent complex is a graded module, so we can split it into the positively and negatively graded parts, we conclude that

$$\dim \mathrm{T}^{1,-}(\mathbf{k}[\mathcal{S}])) \leq 3\delta - \mu - \dim \mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}]).$$

Now the results follows by noting that $\dim \mathbb{P}(\mathrm{T}^{1,-}(\mathbf{k}[\mathcal{S}]))) \leq 3\delta - \mu - 1 - \dim \mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}]) = 2g - 2 - \lambda(\mathbf{k}[\mathcal{S}]) - \dim \mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}])$ and using the lower bound in Theorem 16. □

As a final observation, we note that is easy to compute $\deg(\mathcal{T}_{\mathcal{P}^{g-1}|\mathcal{C}})$ as follows. Since $\mathcal{C}$ is a canonical Gorenstein curve, it is a projective curve of genus g and has degree $2g-2$. The cotangent sheaf of $\mathbb{P}^{g-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{g-1}}((-g)H)$ where H is a hyperplane section. So we get

$$\deg \Omega_{\mathbb{P}^{g-1}|_\mathcal{C}} = \deg(\mathcal{O}_{\mathbb{P}^{g-1}}(-g\,H) \otimes \mathcal{O}_\mathcal{C}) = -g \deg \mathcal{C}.$$

So we conclude that $\deg \mathrm{T}_{\mathbb{P}^{g-1}|_{\mathcal{C}}} = g(2g-2)$.

Hence, it only remains to compute de degree of the normal sheaf $\mathcal{N}$. In general this is a not a simple question. There are a few works in the literature devoted to this subject on non-locally complete intersection cases. Even in smooth case (that is l.c.i.) there are deep conjectures involving the normal bundle, see for example a conjecture due to Aprodu, Farkas and Ortega in [AFO] and some results in [Bruns].

A THE FIRST COTANGENT COMPLEX MODULE

A.1 EXTENSIONS

Let $A \to R$ be a ring homomorphism. An A-*extension* of R (or of R by I) is an exact sequence:

$$(R', \phi) : 0 \to I \to R' \xrightarrow{\phi} R \to 0$$

where R' is an A-algebra and ϕ is a homomorphism of A-algebras whose kernel I is an ideal of R' satisfying $I^2 = (0)$. This condition implies that I has a structure of R-module. (R', ϕ) is also called *an extension of A-algebras.*

If (R', ϕ) and (R'', ψ) are A-extensions of R by I, an A-homomorphism $\xi : R' \to R''$ is called an *isomorphism of extensions* if the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0\,. \\ & & \| & & \downarrow{\scriptstyle \xi} & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & R'' & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

Such a ξ is necessarily an isomorphism of A-algebras. More generally, given A-extensions (R', ϕ) and (R'', ψ) of R, not necessarily having the same kernel, a homomorphism of A-algebras $r : R' \to R''$ such that $\psi r = \phi$ is called a *homomorphism of extensions.*

Lemma 4. *Let (R', ϕ) be an extension as above. Given an A-algebra B and two A-homomorphisms $f_1, f_2 : B \to R'$ such that $\phi f_1 = \phi f_2$ the induced map $f_2 - f_1 : B \to I$ is an A-derivation. In particular, given two homomorphisms of extensions $r_1, r_2 : (R', \phi) \to (R'', \psi)$ the induced map $r_2 - r_1 : R' \to ker(\psi)$ is an A-derivation.*

The A-extension (R', ϕ) is called *trivial* if it has a *section*, that is, if there exists a homomorphism of A-algebras $\sigma : R \to R'$ such that $\phi\sigma = 1_R$. We also say that (R', ϕ) splits, and we call σ a *splitting.* Given an R-module I, a trivial A-extension of R by I can be constructed whose underlying A-module is $R\tilde{\oplus}I$ and by considering the A-algebra $R \oplus I$ with multiplication defined by:

$$(r, i)(s, j) = (rs, rj + si).$$

The first projection

$$p : \; R\tilde{\oplus}I \to R$$

defines an A-extension of R by I which is trivial: a section q is given by $q(r) = (r, 0)$.

We can identified the section of p with the A-derivations $d : R \to I$. Indeed, if we have a section $\sigma : R \to R\tilde{\oplus}I$ with $\sigma(r) = (r, d(r))$ then for all $r, r' \in R$:

$$\sigma(rr') = (rr', d(rr')) = \sigma(r)\sigma(r') = (r, d(r))(r', d(r')) = (rr', rd(r') + r'd(r))$$

and if $a \in A$ then:

$$\sigma(ar) = (ar, d(ar)) = a\sigma(r) = a(r, d(r)) = (ar, ad(r))$$

hence $d : R \to I$ is an A-derivation. Conversely every A-derivation $d : R \to I$ defines a section $\sigma_d : R \to R\tilde{\oplus}I$ by $\sigma_d(r) = (r, d(r))$.

Every trivial A-extension (R', ϕ) of R by I is isomorphic to $(R\tilde{\oplus}I, p)$.

Example 8.

1. Every A-extension of A is trivial because by definition it has a section. Therefore it is of the form $A\tilde{\oplus}V$ for a A-module V. In particular, if t is an indeterminate the A-extension $A[t]/(t^2)$ of A is trivial, and is denoted $A[\epsilon]$ (where $\epsilon = t \bmod(t^2)$ satisfies $\epsilon^2 = 0$). The corresponding exact sequence is:

$$0 \to (\epsilon) \to A[\epsilon] \to A \to 0$$

$A[\epsilon]$ is called the *algebra of dual numbers* over A.

2. Assume that K is a field. If R is a local K-algebra with residue field K a K-extension of R by K is called a *small extension* of R. Let

$$(R', f) :\ 0 \to (t) \to R' \xrightarrow{f} R \to 0$$

be a small K-extension; in other words $t \in m_{R'}$ is annihilated by $m_{R'}$ so that (t) is a K-vector space of dimension one. (R', f) is trivial if and only if the surjective linear map induced by f:

$$f_1 :\ \frac{m_{R'}}{m_{R'}^2} \to \frac{m_R}{m_R^2}$$

is not bijective. Indeed for the trivial K-extension

$$0 \to (t) \to R\tilde{\oplus}(t) \to R \to 0$$

we have $t \in m_{R\tilde{\oplus}(t)}/m^2_{R\tilde{\oplus}(t)}$, hence the map f_1 is not injective because $f_1(\bar{t}) = 0$. Conversely, if f_1 is not injective then $f_1(\bar{t}) = 0$; choose a vector subspace $U \subset m_{R'}/m^2_{R'}$ such that $m_{R'}/m^2_{R'} = U \oplus (\bar{t})$ and let $V \subset R'$ be the subring generated by U. Then V is a subring mapped isomorphically onto R by f. The inverse of $f|V$ is a section of f, therefore (R', f) is trivial.

For example, it follows from this criterion that the extension of K-algebras

$$0 \to \frac{(t^n)}{(t^{n+1})} \to \frac{K[t]}{(t^{n+1})} \to \frac{K[t]}{(t^n)} \to 0.$$

Notation 2. Given an A-algebra R and an A-module I, we denote by $\mathrm{Ex}_A(R; I)$ the space of isomorphism classes of A-extensions of R by I, and by $[R', \phi]$ the class of (R', ϕ).

A.2 MODULE STRUCTURE ON $\text{Ex}_A(R, I)$

Let $A \to R$ be a ring homomorphism. In this subsection we will see how to give an R-module structure to $\text{Ex}_A(R, I)$

Let (R', ϕ) be an A-extension of R by I and $f : S \to R$ a homomorphism of A-algebras. The module structure on $\text{Ex}_A(R, I)$ is based on two operations:

(*pullback*) Given

$$0 \to I \xrightarrow{\alpha} R' \xrightarrow{\phi} R \to 0$$

and $f : S \to R$ an A-algebra homomorphism, the *pullback* of (R', ϕ) by f is the A-extension $f^*(R', \phi)$:

$$f^*(R', \phi) : \ 0 \to I \to R' \times_R S \to S \to 0 \ \in \text{Ex}_A(S, I)$$

where $R' \times_R S$ denotes the fibered product defined in the usual way.

(*pushout*) Given (R', ϕ) and $\lambda : I \to J$ an R-module homomorphism, the pushout of (R', ϕ) by λ is the A-extension $\lambda_*(R', \phi)$:

$$\lambda_*(R', \phi) : \ 0 \to I \to R' \coprod_I J \to R \to 0 \ \in \text{Ex}(R, J)$$

where

$$R' \coprod_I J := \frac{R' \tilde{\oplus} J}{(-\alpha(i) : \lambda(i))|i \in I}$$

Definition 7. *Given* $[R', \phi]$ *and* $[R'', \psi] \in \text{Ex}_A(R; I)$*, we have the following diagram: which defines an* A*-extension*

$$(R' \times_R R'', \xi) : 0 \to I \oplus I \to R' \times_R R'' \xrightarrow{\xi} R \to 0.$$

Let $\delta : I \oplus I \to I$ *be defined by* $(i, j) \mapsto i + j$*. Then the addition is*

$$[R'\phi] + [R'', \psi] := [\delta_*(R' \times_R R'', \xi)].$$

On the other hand, for $[R', \phi] \in \text{Ex}_A(R, I), r \in R$*, let* $r : I \to I$ *be the multiplication by* r*. Define* $r \cdot [R', \phi] := [r_*(R', \phi)]$*. The identity element in* $\text{Ex}_A(R, I)$ *is the trivial extension* $[R\tilde{\oplus}I, p]$.

Definition 8. $\text{Ex}_A(R, I)$ *is an* R*-module with the pullback and pushout operations defined above.*

Remark 12. If $f : R \to S$ is a homomorphism of A-algebras and I is an S-module, then by the operation of pullback we get a homomorphism of S-modules

$$f* : \text{Ex}_A(S, I) \to \text{Ex}_A(R, I).$$

We have the following useful result:

Proposition 5. *Let A be a ring, $f : S \to R$ a homomorphism of A-algebras and let I be an R-module. Then there is an exact sequence of R-modules:*

$$\begin{aligned} 0 \quad \to \quad & \mathrm{Der}_S(R, I) \to \mathrm{Der}_A(R, I) \to \mathrm{Der}_A(S, I) \otimes_S R \xrightarrow{\alpha} \\ \to \quad & \mathrm{Ex}_S(R, I) \xrightarrow{\nu} \mathrm{Ex}_A(R, I) \xrightarrow{f_*} \mathrm{Ex}_A(S, I) \otimes_S R \end{aligned}$$

Proof. See [Es] page 13. □

Definition 9. *The R-module $\mathrm{Ex}_A(R, R)$ is called the* first cotangent module *of R over A and it is denoted by $T^1_{R/A}$. If $A = \mathbf{k}$ we will write T^1_R instead of $T^1_{R/\mathbf{k}}$.*

Proposition 6. *Let $A \to B$ be an essentially of finite type ring homomorphism and let $B = P/J$ where P is a smooth A-algebra. Then for every B-module N we have an exact sequence:*

$$\mathrm{Der}_A(P, N) \to \mathrm{Hom}_B(J/J^2, N) \to \mathrm{Ex}_A(B, N) \to 0. \tag{A.1}$$

If $A \to B$ is a smooth homomorphism then $\mathrm{Ex}_A(B, N) = 0$ for every B-module N.

Proof. See [Es] page 14. □

The following result is a direct consequence of the exact sequence (A.1).

Corollary 6. *Suppose $A \to B$ is an essentially finite type ring homomorphism and N is a finitely generated B-module then $\mathrm{Ex}_A(B, N)$ is a finitely generated B-module. In particular, if $B = P/J$ for a smooth A-algebra P and an ideal $J \subset P$, $T^1_{B/A}$ is a finitely generated B-module and we have an exact sequence:*

$$0 \to \mathrm{Hom}_B(\Omega_{B/A}, N) \to \mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, N) \to \mathrm{Hom}_B(I/I^2, N) \to T^1(B/A, N) \to 0.$$

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