UNIVERSIDADE FEDERAL DE MINAS GERAIS

TESE DE DOUTORADO

Long-time behavior of solutions to nonlinear Schrödinger-type equations

Luccas Cassimiro Campos

Belo Horizonte - MG

2019

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Long-time behavior of solutions to nonlinear Schrödinger-type equations

Tese apresentada ao Programa de Pós-Graduação em Matemática da UFMG, como requisito parcial para a obtenção do título de doutor em Matemática.

Orientador: Luiz Gustavo Farah Dias
Coorientadora: Svetlana Roudenko

Universidade Federal de Minas Gerais

Dezembro de 2019

Ficha catalográfica elaborada pela bibliotecária Belkiz Inez Rezende Costa CRB 6ª Região nº 1510

Campos, Luccas Cassimiro.

C198l Long-time behavior of solutions to nonlinear Schrödinger-type equations / Luccas Cassimiro Campos.— Belo Horizonte, 2019.
190 f. il.; 29 cm.

Tese(doutorado) - Universidade Federal de Minas Gerais – Departamento de Matemática.
Orientador: Luiz Gustavo Farah Dias.
Coorientador: Svetlana Roudenko

1. Matemática - Teses. 2. Schrodinger, Equação de. .3. Espalhamento. 4.Explosão (blow-up).I. Orientador. II. Coorientador. III.Título.

CDU 51 (043)

ATA DA CENTÉSIMA TRIGÉSIMA QUINTA DEFESA DE TESE DO ALUNO LUCCAS CASSIMIRO CAMPOS, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 12 DE DEZEMBRO DE 2019.

Aos doze dias do mês de dezembro de 2019, às 16h15, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Luccas Cassimiro Campos**, intitulada: *"Long-time behavior of solutions to nonlinear Schrödinger-type equations"*, requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Luiz Gustavo Farah Dias, após dar conhecimento aos presentes o teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 12 de dezembro de 2019.

Luiz Gustavo Farah Dias

PROF. DR. LUIZ GUSTAVO FARAH DIAS
Orientador (UFMG)

Svetlana Roudenko

PROFA. DRA. SVETLANA ROUDENKO
Coorientadora (FIU-EUA)

Ademir Pastor

PROF. DR. ADEMIR PASTOR FERREIRA
Examinador (UNICAMP)

Alex Hernandez Ardila

PROF. DR. ALEX JAVIER HERNANDEZ ARDILA
Examinador (UFMG)

José Felipe Linares R

PROF. DR. JOSÉ FELIPE LINARES RAMIREZ
Examinador (IMPA)

Paulo Cupertino Lima

PROF. DR. PAULO CUPERTINO DE LIMA
Examinador (UFMG)

Às pessoas que amo

Acknowledgments

Agradeço e dedico esta tese aos meus pais, Telma e Waldir, aos meus irmãos, Laísa, Leonardo e Lígia, bem como à minha família, por 28 anos de apoio incondicional e em todos os momentos.

I am grateful to my advisors, Luiz Gustavo Farah and Svetlana Roudenko, for their guidance, their support and for all the confidence they had in me.

I thank the members of this thesis defense committee, for generously offering their time, support and good will upon reviewing this document.

Agradeço aos amigos da UFMG, em especial a Mykael e Moacir, pelas companhias no giz e na cerveja. Um abraço especial também para Lázaro, Alan, Isabella, Leandro, Aldo e Natã.

Special thanks to Tomas Duyckaerts and Nicola Visciglia, for valuable discussions about the threshold problem.

I thank Florida International University for having received me as a visitor for one year, from where many projects have started. I also thank my friends and collaborators Andy, Kai, Oussama and Debdeep.

Sou grato à OBMEP, ao PIC-Jr e ao PICME, bem como a todas as pessoas envolvidas nesses projetos.

Agradeço à CAPES pelo apoio financeiro concedido, tanto no Brasil, por meio da bolsa de doutorado, como no exterior, pelo programa PDSE. Agradeço também à UFMG, por ter sido minha segunda casa desde o início da graduação.

"Ah, not in knowledge is happiness, but in the acquisition of knowledge"

(Edgar Allan Poe)

Resumo

Neste trabalho, apresentamos diversos resultados relacionados ao comportamento assintótico de soluções de equações do tipo *Schrödinger*.

Para o caso clássico (e do tipo *focusing*) da equação de *Schrödinger* não-linear (NLS), descrevemos as soluções no limiar massa-energia, tanto no caso intercrítico quanto no caso H^1-crítico. O comportamento dessas soluções é completamente classificado, mostrando que há uma certa rigidez quanto aos tipos de solução possíveis nesse regime. No contexto H^1-crítico, estendemos o trabalho de Duyckaerts e Merle [24] para dimensões $N \geq 6$ (c.f. Li e Zhang [63] para uma abordagem diferente), e no caso intercrítico, o trabalho de Duyckaerts e Roudenko [25].

Para a equação de *Schrödinger* não-linear e não-homogênea (INLS), apresentamos uma prova do *scattering* (espalhamento) abaixo do *ground state* (estado estacionário), adaptando a abordagem de Dodson e Murphy [22] para a INLS, bem como estendendo resultados anteriores de Farah e Guzmán [31, 30].

Discutimos também o comportamento de soluções da INLS que estão *acima* do limiar massa-energia. Exibimos um cenário em que há uma dicotomia entre *scattering* e *blow-up* (explosão), além de provar diferentes critérios de *blow-up*. Estendemos, assim, o trabalho de Duyckaerts e Roudenko [26] para a INLS.

Palavras-chave: Equações não-lineares do tipo *Schrödinger*; Comportamento global; Espalhamento (*scattering*); Explosão (*blow-up*).

Abstract

We show several results regarding long-time behavior of solutions to Schrödinger-type equations.

For the focusing (classical) nonlinear Schrödinger (NLS) equation, we study solutions at the mass-energy threshold in the intercritical and energy-critical setting. We completely identify and classify the behavior of such solutions, showing that there is some rigidity in this regime. In the energy-critical setting, we extend the works of Duyckaerts and Merle [24] to dimensions $N \geq 6$ (see also Li and Zhang [63] for a different approach), and in the intercritical range, we extend the work of Duyckaerts and Roudenko [25].

For the focusing inhomogeneous nonlinear Schrödinger (INLS) equation, we present a proof of scattering below the ground state, adapting the approach of Dodson and Murphy [22] to the INLS, and extending the previous results of Farah and Guzmán [31, 30].

We also discuss the behavior of solutions to the INLS that are *above* the mass-energy threshold. We give a dichotomy between scattering and blow-up in this scenario, and also some blow-up criteria. This chapter extends the works of Duyckaerts and Roudenko [26] to the INLS equation.

Keywords: Nonlinear Schrödinger-type equations; Global behavior; Scattering; Blow-up.

Contents

1 Introduction

In this work, we consider the Cauchy problem for nonlinear Schrödinger-type equations

$$\begin{cases} iu_t + \Delta u + F(x,u) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.0.1}$$

where $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $N \geq 1$.

If F has the form $F(x,u) = \mu|u|^{p-1}u$, with $\mu \in \{+1,-1\}$, the initial-value problem

$$\begin{cases} iu_t + \Delta u + \mu|u|^{p-1}u = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.0.2}$$

is called the (classic) nonlinear Schrödinger equation (NLS). If $\mu = +1$, the problem is called *focusing*, and if $\mu = -1$, it is called *defocusing*. The global behavior of *focusing* and *defocusing* equations can be very different, and we are mainly interested here in the *focusing* case. Unless specified otherwise, we assume $\mu = +1$ in equation (1.0.2) throughout all the text.

Different versions of (1.0.1) are obtained as models in physics, as in the Hartree-type equation:

$$iu_t + \Delta u + (|x|^{-(N-\gamma)} * |u|^p)|u|^{p-2}u = 0,$$

which models boson systems interacting via a non-local potential of convolution type, see Ginibre and Velo [43] and Hepp [49]. One also has examples in optics, in the form

$$iu_t + \Delta u + V(x)|u|^{p-1}u = 0.$$

In this case, the potential $V(x)$ is proportional to the electronic density of the medium. We refer to the works of Gill [41] and Liu and Tripathi [68] for a physical point of view. The particular case $V(x) = |x|^{-b}$ appears naturally as a limit case of potentials $V(x)$ that decay at infinity as $|x|^{-b}$ (Genoud and Stuart [39]), and will be called inhomogeneous non-linear Schrödinger equation (INLS):

$$\begin{cases} iu_t + \Delta u + |x|^{-b}|u|^{p-1}u = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.0.3}$$

We are interested in L^2-supercritical and H^1-subcritical (or simply intercritical) case

$$1 + \frac{4-2b}{N} < p < \begin{cases} 1 + \frac{4-2b}{N-2}, & N \geq 3 \\ +\infty, & N \leq 2, \end{cases}$$

as well as in the energy-critical case

$$p = 1 + \frac{4-2b}{N-2}, \quad N \geq 3,$$

for $0 \leq b < \min\{N, 2\}$.

The classic case (NLS) has been extensively studied in the past decades. For a textbook treatment, we refer the reader to the works of Sulem-Sulem [80], Bourgain [10], Cazenave [13], Linares-Ponce [65], Fibich [32], Tao [83] and the references therein.

The INLS model has received more attention recently (see for instance, [19], [38], [29], [18], [48], [31], [30]).

Structure of this work

In the next chapter, we review the common background necessary to this work.

In Chapter 3, we present a proof of scattering below the ground state, adapting the approach of Dodson and Murphy [22] to the INLS (1.0.3), and extending the previous results of Farah and Guzmán [31, 30].

Chapter 4 is devoted to describing solutions at the mass-energy threshold for the NLS (1.0.2). We completely classify the behavior of solutions in the intercritical and energy-critical setting, extending the works of Duyckaerts and Merle [24] to dimensions $N \geq 6$

(see also Li and Zhang [63]), and of Duyckaerts and Roudenko [25] to all the intercritical range.

In Chapter 5, we discuss the behavior of solutions to the INLS that are *above* the mass-energy threshold. We give a dichotomy between scattering and blow-up in this scenario, and also some blow-up criteria. This chapter extends the works of Duyckaerts and Roudenko [26] to the INLS equation.

Finally, in Chapter 6, we list possible future research directions related to this work.

2 Preliminaries

In this chapter, we give some definitions and basic estimates, that are used in the subsequent chapters, and review the state-of-the-art.

2.1 Notation

We denote by p' the Hölder conjugate of $p \geq 1$. We use $X \lesssim Y$ to denote $X \leq CY$, where the constant C only depends on the parameters (such as N, p and b) and exponents, but never on the solution u or on t. The notations a^+ and a^- denote, respectively, $a + \eta$ and $a - \eta$, for a fixed $0 < \eta \ll 1$. We use p^* to denote the critical exponent of the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, that is, $p^* = 2N/(N-2)$ if $N > 2$, and $p^* = +\infty$ if $N \leq 2$.

Schwartz functions and tempered distributions

We define the Schwartz space of rapidly decaying functions as

$$\mathcal{S}(\mathbb{R}^N) := \left\{ f : \mathbb{R}^N \to \mathbb{C} : \sup_x |x^\alpha \partial^\beta f(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{Z}^N_{\geq 0} \right\},$$

and the space of tempered distributions as the linear functionals that satisfy

$$S'(\mathbb{R}^N) := \left\{ T \in L(\mathcal{S}(\mathbb{R}^N), \mathbb{C}) : T(f_n) \to 0 \text{ if } \sup_x |x^\alpha \partial^\beta f_n(x)| \to 0 \text{ for all } \alpha, \beta \in \mathbb{Z}^N_{\geq 0} \right\}.$$

The Fourier transform

We denote the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^N)$ as

$$\hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\, dx.$$

If $1 \le p \le 2$, the Fourier transform $\wedge : L^p \to L^{p'}$ is defined, by density, as the natural extension of the previous definition.

Sobolev Spaces

For $s \ge 0$ and $p \ge 1$, we define the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^N)$ as the completion of $\mathcal{S}(\mathbb{R}^N)$ with the norm

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^N)} := \|D^s f\|_{L^p(\mathbb{R}^N)},$$

where $\widehat{D^s f}(\xi) := |\xi|^s \hat{f}(\xi)$ is the inverse Riesz potential of order s. The inhomogeneous space $W^{s,p}(\mathbb{R}^N)$ is defined as the completion of $\mathcal{S}(\mathbb{R}^N)$ with the norm

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} := \|(1-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^N)},$$

where $[(1-\Delta)^{s/2} f]^\wedge(\xi) := (1+|\xi|^2)^{s/2}\hat{f}(\xi)$ is the Bessel potential of order s. If $p = 2$, we denote $\dot{W}^{s,2}(\mathbb{R}^N) = \dot{H}^s(\mathbb{R}^N)$ and $W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$.

Mixed Lebesgue spaces

Let $1 \le q, r \le +\infty$ and $I \subset \mathbb{R}$. We define the $L^q_I L^r_x$ spaces as

$$L^q_I L^r_x = \left\{ f : I \times \mathbb{R}^N \to \mathbb{C} : \|f\|_{L^q_I L^r_x} := \left(\int_I \|f(t,\cdot)\|^q_{L^r_x}\, dt \right)^{\frac{1}{q}} \right\}.$$

If $I = \mathbb{R}$, we will denote $L^q_I L^r_x$ as $L^q_t L^r_x$. The mixed Lebesgue/Sobolev spaces $L^q_I W^{k,r}_x$ are defined analogously.

The free Schrödinger operator

For $t \in \mathbb{R}$, we define the (free) Schrödinger operator $e^{it\Delta}$ as

$$[e^{it\Delta} f]^\wedge(\xi) = e^{-4\pi^2 it|\xi|^2} \hat{f}(\xi).$$

If $f \in H^2(\mathbb{R}^N)$, then $u(x,t) = e^{it\Delta} f(x)$ satisfies the (linear) Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u(x,0) = f(x) \end{cases}$$

2.2 Useful inequalities

We now present some lemmas that will be heavily used in the next chapters.

2.2.1 Smoothness and integrability

The next lemmas show that smoothness can be exchanged into integrability on higher L^p norms. Let us start with a definition.

Definition 2.2.1. For $0 < \alpha < N$, the Riesz potential of order α is defined as

$$I_\alpha f(x) = \int \frac{f(y)}{|x-y|^{N-\alpha}}\, dy.$$

The next lemma shows that I_α is bounded on some L^p spaces.

Lemma 2.2.2 (Hardy-Littlewood-Sobolev, see Stein [77, p. 119, Theorem 1])**.** *Let* $0 < \alpha < N$ *and* $1 < p, q < \infty$. *If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N},$$

then

$$\|I_\alpha f\|_{L^q(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}.$$

The Hardy-Littlewood-Sobolev lemma can be used, among other applications, to prove the Sobolev inequality:

Lemma 2.2.3 (Sobolev inequality, see Stein [77, p. 124, Theorem 2])**.** *If* $0 < \rho - \sigma < N$, $1 < q < p < \infty$, *and*

$$\frac{1}{p} = \frac{1}{q} - \frac{\rho - \sigma}{N},$$

then the following estimate holds

$$\|D^\sigma u\|_{L^p(\mathbb{R}^N)} \lesssim \|D^\rho u\|_{L^q(\mathbb{R}^N)}.$$

If one works with radial H^1 functions, then it is possible to show some localization.

Lemma 2.2.4 (Strauss [78])**.** *If* $f \in H^1_{rad}(\mathbb{R}^N)$, $N \geq 2$, *then, for any* $R > 0$,

$$\|f\|_{L^\infty_{\{|x|\geq R\}}} \lesssim R^{-\frac{N-1}{2}} \|f\|_{H^1(\mathbb{R}^N)}. \tag{2.2.1}$$

As an immediate corollary, we get

Corollary 2.2.5. *If* $f \in H^1_{rad}(\mathbb{R}^N)$, $N \geq 2$ *and* $p \geq 1$, *then, for any* $R > 0$,

$$\|f\|^{p+1}_{L^{p+1}_{\{|x|\geq R\}}} \lesssim R^{-\frac{(N-1)(p-1)}{2}} \|f\|^{p+1}_{H^1(\mathbb{R}^N)}.$$

One can also use complex interpolation to allow embedding into inhomogeneous spaces. This gives rise to the Gagliardo-Nirenberg-Sobolev inequalities, whose proof was given independently by Gagliardo [34] and Nirenberg [73].

Lemma 2.2.6 (Gagliardo-Nirenberg-Sobolev)**.** *Let* $1 \leq p, q, r \leq \infty$, $j, m \in \mathbb{Z}_{\geq 0}$, $j < m$, $j/m \leq \theta \leq 1$ *and suppose that the following cases do not hold:*

$$\begin{cases} j = 0, \\ rm < N, \\ q = \infty \end{cases} \quad \text{and} \quad \begin{cases} \theta = 1, \\ 1 < r < \infty, \\ m - j - \frac{N}{r} \text{ is a non-negative integer,} \end{cases}$$

then

$$\|D^j u\|_{L^p} \lesssim \|D^m u\|^{\theta}_{L^r} \|u\|^{1-\theta}_{L^q},$$

where

$$\frac{1}{p} - \frac{j}{N} = \theta\left(\frac{1}{r} - \frac{m}{N}\right) + (1-\theta)\frac{1}{q}.$$

In particular, we often make use of the following case, valid for $1 \leq p \leq 1 + 4/(N-2)$ if $N \geq 3$, for $1 \leq p < \infty$ if $N = 2$, and for $1 \leq p \leq \infty$ if $N = 1$:

$$\|f\|_{L^{p+1}}^{p+1} \leq C_{N,p} \|\nabla f\|_{L^2}^{\frac{N(p-1)}{2}} \|f\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}}, \tag{2.2.2}$$

for which, for instance, the best constant $C_{N,p}$ is known and plays a crucial role on classifying the long-time behavior for the NLS. We postpone this discussion to the subsequent sections.

Tailored for the INLS (1.0.3), there is also a weighted version of (2.2.2), proved by Genoud [38] for $p = 1 + (4-2b)/N$ and posteriorly generalized by Farah [29]:

$$\int |x|^{-b}|f|^{p+1} \leq C_{N,p} \|\nabla f\|_{L^2}^{\frac{N(p-1)+2b}{2}} \|f\|_{L^2}^{2-\frac{(N-2)(p-1)+2b}{2}}. \tag{2.2.3}$$

The inequality holds for $1 + (4-2b)/N \leq p < 1 + (4-2b)/(N-2)$ if $N \geq 3$, and for $1 + (4-2b)/N \leq p < +\infty$ if $N \leq 2$. In any case, b is allowed to belong to the interval $(0, \min\{2, N\})$.

There is also another way to spend smoothness: to get rid of a singularity. This is the essence of the so-called Hardy (or Hardy-Sobolev) inequalities, as in the following lemma.

Lemma 2.2.7 (Hardy inequality, see Kufner and Opic [59])**.** *Let* $1 < r < N$*. If* $f \in W^{1,r}(\mathbb{R}^N)$*, then*

$$\int |\nabla f|^r \geq \left(\frac{N-r}{r}\right)^r \int \frac{|f|^r}{|x|^r}.$$

2.2.2 Acquiring smoothness

The uncertainty principle allows us to spend derivatives to get higher integrability, but forbids the converse. However, the dispersive nature of the NLS-type equations causes some regularity gain, due to the linear operator $e^{it\Delta}$. This smoothing effect is exploited in the so-called *Strichartz estimates*. They were first proved in the NLS context by Strichartz [79], later extended to the (possible) endpoints by Keel and Tao [54], and generalized by Kato [53] and Foschi [33] in the inhomogeneous case. To state the estimates, we first define *admissibility* and *acceptability*.

Definition 2.2.8. If $N \geq 1$ and $s \in (-1,1)$, the pair (q,r) is called $\dot{H}^s$*-admissible* if it satisfies the condition

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s, \tag{2.2.4}$$

where

$$2 \leq q, r \leq \infty, \text{ and } (q,r,N) \neq (2,\infty,2).$$

In particular, if $s = 0$, we say that the pair is L^2-admissible.

Definition 2.2.9. If $N \geq 1$, the pair (q,r) is called acceptable if

$$\frac{2}{q} < N\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{or } (q,r) = (\infty, 2).$$

Remark 2.2.10. For $s \in (-1,1)$, every $\dot{H}^s$-admissible pair is acceptable.

In this work, we use the following versions of the Strichartz estimates:

Lemma 2.2.11 (Homogeneous Strichartz estimates, see Cazenave [13], Strichartz [79], Keel and Tao [54])**.** *Let $s \in [0,1)$. If (q,r) is an $\dot{H}^s$-admissible pair, then*

$$\|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s}. \tag{2.2.5}$$

Lemma 2.2.12 (Kato-Strichartz inequalities, see Cazenave [13], Kato [53], Foschi [33], Keel and Tao [54])**.** *Let $N \geq 1$ and $1 \leq q_i, r_i \leq \infty$, $i = 1,2$. If the pairs (q_1,r_1) and (q_2,r_2) are acceptable, satisfy*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{N}{2}\left(1 - \frac{1}{r_1} - \frac{1}{r_2}\right)$$

and:

- *If $N = 2$, we require that $r_1, r_2 < +\infty$,*
- *If $N > 2$, we consider two subcases*
  - *non sharp case:*

$$\frac{1}{q_1} + \frac{1}{q_2} < 1,$$

$$\frac{N-2}{r_1} \leq \frac{N}{r_2}, \quad \frac{N-2}{r_1} \leq \frac{N}{r_2},$$

– *sharp case:*

$$\frac{1}{q_1} + \frac{1}{q_2} = 1,$$
$$\frac{N-2}{r_1} < \frac{N}{r_2}, \quad \frac{N-2}{r_1} < \frac{N}{r_2},$$
$$\frac{1}{r_1} \leq \frac{1}{q_1}, \quad \frac{1}{r_2} \leq \frac{1}{q_2}.$$

Then the following estimate holds

$$\left\| \int_{s>t} e^{i(t-s)\Delta} F(s)\, ds \right\|_{L_t^{q_1} L_x^{r_1}} + \left\| \int_{s<t} e^{i(t-s)\Delta} F(s)\, ds \right\|_{L_t^{q_1} L_x^{r_1}} \lesssim \|F\|_{L_t^{q_2'} L_x^{r_2'}}. \tag{2.2.6}$$

Remark 2.2.13. In particular, for $s \in (0,1)$, if (q_1, r_1) is $\dot{H}^s$-admissible, (q_2, r_2) is $\dot{H}^{-s}$-admissible, and we have, for $i = 1, 2$,

$$\frac{2N}{N-2s} < r_i < \begin{cases} \frac{2N}{N-2}, & \text{if } N > 2, \\ \infty, & \text{if } N \leq 2, \end{cases}$$

then estimate (2.2.6) holds. Note that we do not have all endpoints available and, in practical applications, we have to restrict ourselves to a closed subset of acceptable pairs. That usually does not pose a problem, since any reasonable argument usually requires only a finite number of pairs.

2.2.3 Fractional calculus

We often deal with fractional derivatives of products and compositions of functions. Remarkably, the fractional calculus enjoys rules akin to the product and chain rules for smooth functions, as we state in the next lemmas.

Lemma 2.2.14 (Leibniz rule, see Grafakos and Seungli [45] and Kenig, Ponce and Vega

[56]). *Let* $s \in (0,1)$, $p_j, q_j \in (1,\infty)$, *with* $\frac{1}{p} = \frac{1}{p_j} + \frac{1}{q_j}$, $j = 1,2$. *Then*

$$\|D^s(fg)\|_{L^p(\mathbb{R}^N)} \lesssim \left(\|D^s f\|_{L^{p_1}(\mathbb{R}^N)}\|g\|_{L^{q_1}(\mathbb{R}^N)} + \|f\|_{L^{p_2}(\mathbb{R}^N)}\|D^s g\|_{L^{q_2}(\mathbb{R}^N)}\right). \tag{2.2.7}$$

Lemma 2.2.15 (Fractional chain rule for C^1 functions, see Christ and Weinstein [16]). *Suppose* $F \in C^1$, $s \in (0,1]$, *and* $1 < p, p_1, p_2 < \infty$ *are such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then*

$$\|D^s F(u)\|_{L^p} \lesssim \|F'(u)\|_{L^{p_1}(\mathbb{R}^N)}\|u\|_{L^{p_2}(\mathbb{R}^N)}.$$

If one does not have $F \in C^1$, but only Hölder continuous (such as $F(u) = |u|^{p-1}u$, for $1 < p < 2$), we still have a chain rule, given that the order of the derivative is lower than the Hölder order:

Lemma 2.2.16 (Fractional chain rule for Hölder continuous functions, see Visan [85]). *Let* F *be a Hölder continuous function of order* $0 < \alpha < 1$. *Then, for every* $0 < s < \alpha$, $1 < p < \infty$, *and* $\frac{s}{\alpha} < \nu < 1$ *we have*

$$\|D^s F(u)\|_{L^p(\mathbb{R}^N)} \lesssim \||u|^{\alpha - \frac{s}{\nu}}\|_{L^{p_1}(\mathbb{R}^N)}\|D^\nu u\|^{\frac{s}{\nu}}_{L^{\frac{s}{\nu}q_1}(\mathbb{R}^N)}, \tag{2.2.8}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1}$ *and* $\left(1 - \frac{s}{\nu\alpha}\right)p_1 > 1$.

2.3 Symmetries, conserved quantities and a monotonicity formula

Equations of Schrödinger type can be seen as infinite-dimensional Hamiltonian systems. As such, invariances of the Hamiltonian give rise to conserved quantities. At the H^1 level, the following conserved quantities for the NLS (1.0.2) and INLS (1.0.2) are of interest to us.

The *mass*:

$$M(u(t)) := \int |u(t)|^2 = M(u_0),$$

the *energy*:

$$E(u(t)) := \frac{1}{2}\int |\nabla u(t)|^2 - \frac{1}{p+1}\int |x|^{-b}|u(t)|^{p+1} = E(u_0),$$

and, only in the NLS (1.0.2) case, the *linear momentum*:

$$P(u(t)) := \text{Im}\int \bar{u}(t)\nabla u(t) = P(u_0).$$

.

The NLS and the INLS equations also present several symmetries. If u is a solution to the NLS (1.0.2), then also are:

(a) $v_1(x,t) = \lambda^{\frac{2}{p-1}}u(\lambda x, \lambda^2 t)$, $\lambda > 0$ (scaling),

(b) $v_2(x,t) = u(x, t+t_0)$, $t_0 \in \mathbb{R}$ (time translation),

(c) $v_3(x,t) = e^{i\theta_0}u(x,t)$, $\theta_0 \in [0, 2\pi)$ (phase),

(d) $v_4(x,t) = \bar{u}(x,-t)$ (time-reversal),

(e) $v_5(x,t) = u(x+x_0, t)$, $x_0 \in \mathbb{R}^N$ (spatial translation),

(f) $v_6(x,t) = e^{ix\cdot v - i|v|^2 t}u(x+vt, t)$, $v \in \mathbb{R}^N$ (Galilean invariance).

The INLS equation (1.0.3), in turn, presents the following symmetries:

(a) $v_1(x,t) = \lambda^{\frac{2-b}{p-1}}u(\lambda x, \lambda^2 t)$, $\lambda > 0$ (scaling),

(b) $v_2(x,t) = u(x, t+t_0)$, $t_0 \in \mathbb{R}$ (time translation),

(c) $v_3(x,t) = e^{i\theta_0}u(x,t)$, $\theta_0 \in [0, 2\pi)$ (phase),

(d) $v_4(x,t) = \bar{u}(x,-t)$ (time reversal).

Note that the space inhomogeneity causes the symmetries (e) and (f) to be broken in the INLS (1.0.3).

Besides the conserved quantities and symmetries, there is also an important tool on studying the long-time behavior of solutions, called *the Virial identity*. It is used in the works of Vlasov-Petrischev-Talanov [86], Zakharov[90] and Glassey [44] to show the *blow-up* phenomenon to the NLS in the focusing case. A more general version of the Virial identity was adapted by Morawetz [72], in the context of wave equation, to show *scattering* (see

Section 2.5). The argument of Morawetz was adapted to the NLS by Lin and Strauss [64]. Both arguments are a particular case of a more general principle, which can be stated using the following definitions and identities.

Definition 2.3.1. If u is a solution to (1.0.2) or (1.0.3), we define the *Virial quantity* associated to the weight a as

$$V_a(t) = \int a(x)|u(x,t)|^2\, dx,$$

and the *Morawetz action* as

$$Z_a(t) = 2\,\mathrm{Im} \int \nabla a(x) \cdot \nabla u(x,t)\bar{u}(x,t) dx.$$

Using the associated PDE, we have the following identities (see, for example, Glassey [44] and Farah [29]).

$$V_a'(t) = Z_a(t), \tag{2.3.1}$$

and

$$\begin{aligned} Z_a'(t) = &\left(2 - \frac{4}{p+1}\right)\mu \int |x|^{-b}|u(x,t)|^{p+1}\Delta a\, dx \\ &- \frac{4\mu b}{p+1}\int |x|^{-b-2}|u(x,t)|^{p+1} x\cdot \nabla a\, dx \\ &- \int |u(x,t)|^2 \Delta^2 a + 4\,\mathrm{Re}\sum_{i,j}\int \partial_{ij}^2 a_{ij}\, \partial_i \bar{u}\, \partial_j u. \end{aligned} \tag{2.3.2}$$

In particular, for the NLS, if we take

- $a(x) = |x|^2$, we have $\nabla a = 2x$, $\Delta a = 2N$, $\Delta^2 a = 0$ and $\partial_{ij}^2 a = 2\delta_{ij}$. Therefore,

$$\begin{aligned} Z_a'(t) &= 8\left[\int |\nabla u|^2 - \left(\frac{N}{2} - \frac{N}{p+1}\right)\mu \int |u|^{p+1}\right] \\ &= 4E[u_0] - \frac{2\mu}{p+1}[N(p-1)-4]\int |u|^{p+1}. \end{aligned}$$

In the focusing case, as $\mu = +1$, we see that, if $p > 1 + 4/N$, an initial condition with negative energy and that decays rapidly enough in space cannot exist for all positive times. Indeed, the quantity $V_a(t)$ is non-negative, and must lie below the graph of an inverted parabola. This is known as the convexity argument (see, for example, [44]) to prove *blow-up*.

- $a(x) = |x|$, we have

$$Z_a'(t) = \int \frac{|\nabla\!\!\!/ u|^2}{|x|} - \frac{2\mu(N-1)(p-1)}{p+1} \int \frac{|u|^{p+1}}{|x|} - \frac{1}{4} \int |u|^2 \Delta^2 |x|,$$

where $\nabla\!\!\!/ := \nabla - \frac{x}{|x|}\left(\frac{x}{|x|} \cdot \nabla\right)$ denotes the angular gradient of u. Using the fact that the distribution $\Delta^2|x| \leq 0$ if $N \geq 3$, we have, in the case $\mu = -1$ (*defocusing*):

$$\int \frac{|\nabla\!\!\!/ u(x,t)|^2}{|x|} \, dx + \int \frac{|u(x,t)|^{p+1}}{|x|} \, dx \lesssim \sup_t \|u(t)\|_{H^1}.$$

This information about u around the origin is used to prove *scattering* in the *defocusing* case (see Lin and Strauss [64]).

In particular, both methods above rely on some kind of *monotonicity* to prove the desired results. Monotonicity-based proofs are a fundamental tool in the context of dispersive equations, and have been widely used in the last years.

2.4 Local theory

We now review the local well-posedness theory for the NLS (1.0.2) and INLS (1.0.3).

Definition 2.4.1. By local well-posedness in a Sobolev space S, we mean that for every $u_0 \in S$, there exists $T > 0$, a subspace X of $C([-T;T];S)$ and a unique function u such that

1. u is a solution,
2. $u \in X$,
3. the solution depends continuousy on the initial data.

The precise definition of solution and the spaces S and X will be given in the next subsection for each equation.

The Sobolev index in which the $\dot{H}^s$ norm is invariant is given by scaling. Indeed, if $u(x,t)$ is a solution to NLS or INLS with initial data ϕ, then $u_\lambda(x,t) = \lambda^{\frac{2-b}{p-1}} u(\lambda x, \lambda^2 t)$ is also a solution, with initial data $\phi_\lambda(x) = \lambda^{\frac{2-b}{p-1}} \phi(\lambda x)$. Computing the homogeneous $\dot{H}^s$ norm:

$$\|\phi_\lambda\|_{\dot{H}^s} = \lambda^{s-\left(\frac{N}{2}-\frac{2-b}{p-1}\right)}\|\phi\|_{\dot{H}^s}.$$

The Sobolev index that lets the $\dot{H}^s$ norm invariant is called the *critical index*, and it is given by

$$s_c := \frac{N}{2} - \frac{2-b}{p-1}.$$

Note that $s_c = 0 \iff p = 1 + \frac{4-2b}{N}$ and $s_c = 1 \iff p = 1 + \frac{4-2b}{N-2}$, $N \geq 3$. One can then classify the NLS and INLS problems. If

- $s_c < 0$, the problems are called *mass-subcritical* or L^2*-subcritical*,
- $s_c = 0$, the problems are called *mass-critical* or L^2*-critical*,
- $0 < s_c < 1$, the problems are called *intercritical*,
- $s_c = 1$, the problems are called *energy-critical* or H^1*-critical*,
- $s_c > 1$, the problems are called *energy-supercritical* or H^1*-supercritical*.

In this work, we are interested in the intercritical case for the NLS and INLS, and in the energy-critical case for the NLS. As for the energy-critical INLS, the local well-posedness in H^1 is still an open problem, although Lee and Seo [62] claim to have solved it on weighted spaces.

2.4.1 Intercritical NLS and INLS

Let us start with the NLS. The Cauchy problem for the intercritical NLS was studied by Ginibre and Velo [42]. They showed that, for initial data $u_0 \in H^1(\mathbb{R}^N)$, there exists a non-empty interval I, whose size depends on the norm $\|u_0\|_{H^1}$, and a unique local-in-time solution $u : \mathbb{R}^N \times I \to \mathbb{C}$.

By *solution* here, we mean that u belongs to $C_t^0 H_x^1(\mathbb{R}^N \times J) \cap L_t^q W_x^{1,r}(\mathbb{R}^N \times J)$ for any L^2-admissible pair (q, r) (c.f. Section 2.2.2) and for every compact $J \subset I$, and satisfies the Duhamel formula

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}|u|^{p-1}u(s)\,ds$$

for all $t \in I$. Moreover, it is known that the map $u_0 \mapsto u$ is uniformly continuous in both

$C_t^0 H_x^1$ and $L_t^q W_x^{1,r}$ norms.

For the case $b > 0$, Genoud and Stuart [39] proved that the intercritical INLS is locally well-posed in $H^1(\mathbb{R}^N)$, $N \geq 1$ for $0 < b < \min\{2, N\}$, in the sense of distributions. That is, they showed that the solution belongs to $C_t^0 H_x^1(\mathbb{R}^N \times I) \cap C_t^1 H_x^{-1}(\mathbb{R}^N \times I)$. More recently, Guzmán [48] established the local well-posedness of (1.0.3) based on Strichartz estimates. In particular, defining

$$b^* = \begin{cases} \frac{N}{3}, & N \leq 3 \\ 2, & N \geq 4, \end{cases}$$

he proved that, for $N \geq 2$ and $0 < b < b^*$, the initial value problem (1.0.3) is locally well-posed in $H^1(\mathbb{R}^N)$. Dinh [20] extended Guzmán's results in dimension $N = 3$ for $0 < b < \frac{3}{2}$ and $1 + \frac{4-2b}{N} < p < \frac{5-2b}{2b-1}$. Note that, in the results of Guzmán [48] and Dinh [20], the ranges of b are more restricted than those in the results of Genoud and Stuart [39] (mainly due to the natural restrictions on Sobolev embeddings). However, Guzmán and Dinh give more detailed information on the solutions, showing that there exists a nonempty interval I, whose size depends on $\|u_0\|_{H^1}$, such that $u \in L_t^q W_x^{1,r}(\mathbb{R}^N \times J)$, for any L^2-admissible pair (q, r) and every compact $J \subset I$.

2.4.2 Energy-critical NLS

In the energy-critical case, the Cauchy problem for (1.0.2) was studied by Cazenave and Weissler [14] and Tao and Visan [84]. For initial data $u_0 \in \dot{H}^1$, there exists a non-empty interval I and a unique local-in-time solution $u : \mathbb{R}^N \times I \to \mathbb{C}$ that belongs to $C_t^0 \dot{H}_x^1(\mathbb{R}^N \times J) \cap L_{t,x}^{2(N+2)/(N-2)}(\mathbb{R}^N \times J)$ for every compact $J \subset I$. The map from the initial data to the solution is also uniformly continuous and the solution satisfies the corresponding Duhamel formula.

In this case, the size of the interval I does not depend only on the norm $\|u_0\|_{\dot{H}^1}$, but on u_0 itself. That is, different initial data with the same norm may have a very different maximal interval of existence. Indeed, for any $\lambda > 0$, since the $\dot{H}^1$ norm is invariant by scaling, if the maximal time of existence of the initial data u_0 is the interval $(-T, T)$, the maximal interval of existence of $u_{0,\lambda}(x) = \lambda^{\frac{N-2}{2}} u_0(\lambda x)$, is $(-T/\lambda, T/\lambda)$, although all have the same norm.

Note that, since u_0 belongs to the *homogeneous* Sobolev space $\dot{H}^1$, it does not necessarily has a finite mass. Also, its linear momentum may not be defined. Nevertheless, if these quantities are defined for the initial data, then they are defined on the interval of existence of the solution, and are conserved by the NLS flow.

2.5 Finite-time blowup and scattering

We say that a solution to NLS (1.0.2) or INLS (1.0.3) blows up in finite positive time $T > 0$ if it is defined on $[0, T)$ and

$$\|u\|_{S(\dot{H}^{s_c},[0,T))} = +\infty.$$

Blowup in finite negative time is defined analogously.

Solutions defined in, at least, one half-line can also exhibit a *scattering* behavior. We say that a solution to NLS or INLS scatters forward in time in $H^1(\mathbb{R}^N)$ if there exists $u_+ \in H^1(\mathbb{R}^N)$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{H^1} = 0. \tag{2.5.1}$$

In the energy-critical case for the NLS, the definition is the same, except that the $\dot{H}^1(\mathbb{R}^N)$ norm is used instead. Scattering backward in time is defined analogously.

The $S(\dot{H}^{s_c})$ norm also plays a role in the scattering theory (see Kenig and Merle [55], Holmer and Roudenko [51], Cazenave [12], Farah and Guzmán [31]): one can prove that spacetime bounds are a sufficient condition for scattering. Namely, solutions to either NLS or INLS on $[0, +\infty)$ that are uniformly bounded in H^1 scatter forward in time if

$$\|u\|_{S(\dot{H}^{s_c},[0,+\infty))} < +\infty.$$

In this case, it is a standard procedure to show that, if $F(u)$ is the corresponding nonlinearity, then

$$u_+ := u_0 + i\int_0^\infty e^{-is\Delta}F(u(s))\,ds$$

is well-defined and (2.5.1) holds.

2.6 Ground states, standing waves and the mass-energy threshold

Besides between finite-time blowup and scattering, there is the concept of *standing waves*. Consider the elliptic equation

$$\Delta\psi - (1 - s_c)\psi + |\psi|^{p-1}\psi = 0. \tag{2.6.1}$$

It is known that, for $0 < s_c < 1$, this equation admits a unique radial, positive solution in $H^1(\mathbb{R}^N)$, which we call *ground state* and denote by $Q = Q_{p,N}$ (see Strauss [78], Berestycki and Lions [7] and Kwong [60]). If Q solves (2.6.1), then the *standing wave*

$$u(x,t) = e^{it}Q(x)$$

is a solution to NLS (1.0.2) that neither blows up in finite time, nor scatters, in any time direction. If $s_c = 1$, since the equation (2.6.1) is invariant by scaling, the radial, positive solution to (2.6.1) is not unique. An explicit solution is given by

$$Q_{\frac{2N}{N-2},N}(x) := \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}}.$$

This solution is commonly denoted by W, and we shall often do so.

A simple calculation shows that $W \in \dot{H}^1(\mathbb{R}^N)$ for any $N \geq 3$, and that $W \in L^2(\mathbb{R}^N)$ if, and only if, $N \geq 5$. As its subcritical counterpart, the static solution $u(x,t) = W(x)$ to NLS neither blows up in finite time, nor scatters, in any time direction.

Also, the following *Pohozaev* identities follow from (2.6.1):

$$\begin{gathered}\int |Q|^{p+1} = \frac{2(p+1)}{N(p-1)} \int |\nabla Q|^2, \\ \int |Q|^2 = \frac{4-(N-2)(p-1)}{(1-s_c)N(p-1)} \int |\nabla Q|^2, \text{ if } 0 < s_c < 1.\end{gathered} \tag{2.6.2}$$

Remark 2.6.1. The choice of the constant $(1 - s_c)$ in (2.6.1) is only for convenience. If $0 < s_c < 1$, we can modify Q and replace $(1 - s_c)$ by any positive constant by scaling.

Similarly, if $s_c = 1$, the choice of $Q_{\frac{2N}{N-2},N} = W$ is arbitrary, and we could have used any rescaled version of W. Since we always state our results up to scaling (among other symmetries) or using scale-invariant quantities, there is no loss on generality.

The works of Weinstein [88], in the case $0 < s_c < 1$, and of Aubin [4] and Talenti [81], for $s_c = 1$, give the characterization of the ground state as the minimizer of

$$\|f\|_{L^{p+1}}^{p+1} \leq C_{N,p} \|\nabla f\|_{L^2}^{\frac{N(p-1)}{2}} \|f\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}}, \tag{2.6.3}$$

with equality if, and only if, $f(x) = e^{i\theta_0} Q(x + x_0)$, if $0 < s_c < 1$ or $f(x) = e^{i\theta_0} \lambda_0^{\frac{N-2}{2}} W(\lambda_0 x + x_0)$, if $s_c = 1$, for some $\theta_0 \in [0, 2\pi)$, $x_0 \in \mathbb{R}^N$ and $\lambda_0 > 0$. Here, $C_{N,p}$ is the sharp constant of inequality (2.6.3), also seen in Section 2.2.1.

In the INLS case, consider the elliptic problem

$$\Delta\phi - \phi + |x|^{-b} |\phi|^{p-1} \phi = 0.$$

The existence of a unique radial, positive solution $Q = Q_{p,N,b}$ in $H^1(\mathbb{R}^N)$ was proved in Genoud [35, 36], and Genoud and Stuart [39], while uniqueness was handled in Yanagida [89] and Genoud [37]. Existence and uniqueness hold for $N \geq 1$, $s_c < 1$ and $0 \leq b < \min\{2, N\}$.

Moreover, the ground state for the INLS satisfies the following corresponding Pohozaev's identities (see relations (1.9)-(1.10) in Farah [29])

$$\int |x|^{-b} |Q|^{p+1} \, dx = \frac{2(p+1)}{N(p-1) + 2b} \|\nabla Q\|_{L^2}^2$$

and

$$\int |Q|^2 = \frac{4 - (N-2)(p+1) - 2b}{N(p-1) + 2b} \int |\nabla Q|^2. \tag{2.6.4}$$

In [29], Farah proved the sharp Gagliardo-Nirenberg inequality for the INLS (also discussed in Section 2.2.1), valid for $0 \leq s_c < 1$ and $0 < b < \min\{2, N\}$

$$\int_{\mathbb{R}^N} |x|^{-b} |f(x)|^{p+1} \, dx \leq C_{p,N} \|\nabla f\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)+2b}{2}} \|f\|_{L^2(\mathbb{R}^N)}^{2-\frac{(N-2)(p-1)+2b}{2}}, \tag{2.6.5}$$

with equality if, and only if, $f(x) = e^{i\theta_0} Q(x)$, for some $\theta_0 \in [0, 2\pi)$. Note the absence of the translation parameter here, due to the fact that the corresponding symmetry is broken. This inequality can be seen as an extension to the case $b > 0$ of the classical Gagliardo-Nirenberg inequality. It is also an extension of the inequality obtained by Genoud [38], who showed its validity for $p = 1 + \frac{4-2b}{N}$.

The ground state is also associated with the *threshold* for a dichotomy between finite-time blow-up and scattering. The behavior of solutions with $E(u_0) < E(W)$ was studied first by Kenig and Merle [55] for radial solutions in the energy-critical setting, and $N = 3$, 4 and 5. Later, Killip and Visan [57] extended the result for $N \geq 5$, without assuming radiality. We summarize their results in the following theorem.

Theorem 2.6.2. *Let u be a solution to (1.0.2) such that $E(u_0) < E(W)$. If $N = 3$ or 4, assume also that u is radial. Then, exactly one of the following alternatives hold.*

- *If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then u is defined for all positive and negative times. Moreover, u scatters in both time directions.*
- *If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and either u is radial and has finite mass or $|x|u_0 \in L^2(\mathbb{R}^N)$, then u blows up in finite positive and negative times.*

Note that $\|\nabla u_0\|_{L^2}$ and $E(u_0)$ are scale-invariant quantities if $s_c = 1$. In the intercritical setting, consider the following scale-invariant conserved quantity, introduced by Holmer and Roudenko [50],

$$\mathcal{ME}(u(t)) := M(u(t))^{1-s_c} E(u(t))^{s_c} = \mathcal{ME}(u_0), \tag{2.6.6}$$

and the scaling invariant, but not necessarily conserved quantity

$$\mathcal{MK}(u(t)) := \|u_0\|_{L^2}^{1-s_c} \|\nabla u(t)\|_{L^2}^{s_c}.$$

Holmer and Roudenko [51] in the 3d cubic radial case, later being joined by Duyckaerts [27] and removing the radial assumption, adapted the Kenig-Merle concentration-compactness approach [55] to prove scattering. Fang, Xie and Cazenave [28] and Guevara [47] (see also Guevara and Carreon [46]) extended the result to all intercritical ranges and dimensions. The blow-up *versus* global existence part of Theorem 2.6.3 below was proved by Holmer and Roudenko in [50] Their result is summarized as follows.

Theorem 2.6.3. *For $0 < s_c < 1$, let u be a solution to (1.0.2) such that $\mathcal{ME}(u_0) < \mathcal{ME}(Q)$. Then we have the following alternatives.*

- *If $\mathcal{MK}(u_0) < \mathcal{MK}(Q)$, then u is defined for all positive and negative times. Moreover, u scatters in both time directions.*
- *If $\mathcal{MK}(u_0) > \mathcal{MK}(Q)$ and either u is radial or $|x|u_0 \in L^2(\mathbb{R}^N)$, then u blows up in finite positive and negative times.*

This dichotomy does not hold above the ground state mass-energy threshold. In [52], blow-up criteria that included solutions above the mass-energy threshold In [26], Duyckaerts and Roudenko showed, for $0 < s_c \leq 1$, the existence of solutions to NLS that are above the threshold and that scatter in one time direction and blow up in finite time in the other time direction. In fact, they showed that it suffices to multiply the ground state by a quadratic phase to produce such result. In the same paper, they proved a dichotomy-type result if one has a restriction on the mass-energy, provided certain conditions on the variance of the initial data are satisfied.

Recently, another method to prove scattering in the intercritical case for $N > 2$ was developed by Dodson and Murphy [22, 23], based on Morawetz estimates instead of concentration-compactness. Their method requires much less machinery than using profile decomposition, but it gives slightly weaker space-time bounds (see [23]). This approach has proven to be versatile, as it extends to other equations, such as the INLS, which we discuss in Chapter 3 (see the work of the author [11]) and the Hartree-type equation in Arora [2]. Due to the slow decay in time of solutions to the linear problem, extending this result to the lower-dimensional case is harder, although a very recent result by Arora, Dodson and Murphy [3] proves it in the radial case for $N = 2$.

3 A new proof of scattering for the INLS below the threshold

3.1 Introduction

The argument of Farah and Guzmán to prove scattering for the radial INLS (1.0.3) is based on the concentration-compactness-rigidity method introduced by Kenig and Merle [55] for the energy-critical case, which was adapted to an intercritical case (3d cubic) of the NLS by Holmer and Roudenko [51]. Recently, Dodson and Murphy [3] revisited this result, but using a different approach, with a scattering criterion proved by Tao [82] and a Virial/Morawetz estimate.

Tao's scattering criterion and Dodson-Murphy's approach for the radial NLS

The method used by Dodson and Murphy to prove scattering to the radial NLS is based on Ogawa and Tsutsumi's [74] argument, by combining the weights $a(x) = |x|^2$ and $a(x) = |x|$ in order to establish control of the solution on large balls around the origin. Dodson and Murphy used the following scattering criterion.

Theorem 3.1.1 (Scattering criterion)**.** *Let $N = p = 3$. Let u be a radial solution in $H^1(\mathbb{R}^N)$ to (1.0.2) defined on $[0, +\infty)$ and assume the a priori bound*

$$\sup_{t\in[0,+\infty)} \|u(t)\|_{H^1_x} = E < +\infty.$$

There exist constants $R > 0$ and $\epsilon > 0$ depending only on E (but never on u or t) such that, if

$$\liminf_{t\to+\infty} \int_{B(0,R)} |u(x,t)|^2 \, dx \leq \epsilon^2, \tag{3.1.1}$$

then there exists a function $u_+ \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t\to+\infty} \left\| u(t) - e^{it\Delta}u_+ \right\|_{H^1(\mathbb{R}^3)} = 0,$$

that is, u scatters forward in time in $H^1(\mathbb{R}^3)$.

According to this criterion, it is enough to ensure that the mass escapes from a large ball around the origin to guarantee scattering. Relying on Virial/Morawetz-type inequalities, and on the compact embedding $H^1_{rad}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$, for $2 \leq p \leq 6$, Dodson and Murphy proved that, if $\mathcal{ME}[u_0] < 1$ and $\mathcal{MK}[u_0] < 1$, then, for R large enough,

$$\frac{1}{T}\int_0^T \int_{|x|\leq R} |u(x,t)|^4 \, dx \, dt \lesssim \frac{R}{T} + \frac{1}{R^2}.$$

This inequality is enough to establish the bound (3.1.1), therefore, proving scattering. Note that this result was already known, but the previous proofs are based on the Kenig-Merle's concentration-compactness-rigidity method which, albeit powerful, requires a lot of machinery to be applied.

We show here that the method used by Dodson and Murphy can be adapted to the INLS. Naturally, Tao's scattering criterion, which is a crucial part of the argument, has to be proved for this different nonlinearity. Due to the nature of the potential $|x|^{-b}$, the proof of the criterion has to be carefully adapted, avoiding the use of Sobolev embeddings (which may worsen the singularity at the origin). To this end, we prove a different Strichartz-type estimate, which we call *local-in-time*. All the proofs here extend immediately to the case $b = 0$, and therefore, we include it in the statement of the theorems.

Theorem 3.1.2 (Scattering criterion)**.** *Let $N > 2$, $1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2}$ and $0 \leq b < 2$. Consider a spherically symmetric $H^1(\mathbb{R}^N)$ solution u to (1.0.3) defined on $[0, +\infty)$ and assume the a priori bound*

$$\sup_{t\in[0,+\infty)} \|u(t)\|_{H^1_x} = E < +\infty. \tag{3.1.2}$$

There exist constants $R > 0$ and $\epsilon > 0$ depending only on E, N, p and b (but never on u or t) such that if

$$\liminf_{t\to+\infty} \int_{B(0,R)} |u(x,t)|^2 \, dx \leq \epsilon^2, \tag{3.1.3}$$

then there exists a function $u_+ \in H^1(\mathbb{R}^N)$ such that

$$\lim_{t\to+\infty} \left\| u(t) - e^{it\Delta}u_+ \right\|_{H^1(\mathbb{R}^n)} = 0,$$

i.e., u scatters forward in time in $H^1(\mathbb{R}^N)$.

Remark 3.1.3. The notation $N > 2$ instead of $N \geq 3$ is intentional, since we allow N to be arbitrarily close to 2. At least in the radial case, it is possible to define Sobolev spaces with non-integer N, as in this case the dimension becomes just a parameter. It is also mathematically convenient, as this flexibility is useful in some harder proofs. We mention here the work of Landman, Papanicolau, Sulem and Sulem [61] (see also Sulem and Sulem [80]), in which self-similar solutions were computed numerically for dimension approaching 2 as the relative rate of the scaling parameter approaches zero. Later on, Kopell and Landman [58] constructed a *blow-up profile* for equation (1.0.2) in the cubic case when the dimension N is exponentially asymptotically close to 2. In [71], Merle, Raphael and Szeftel constructed stable blow-up solutions in the cubic case when $d \gtrapprox 2$. Later, Rottshafer and Kaper [76] improved the construction in [58] to allow the dimension to be polynomially close to 2.

The criterion above is used to prove scattering in H^1 below the mass-energy threshold, as in the following theorem. We emphasize that the main aim of this chapter is to show that a different approach, based on Dodson-Murphy's method, instead of the classic Kenig-Merle's concentration-compactness-rigidity technique, can be applied to the INLS equation. Moreover, our method extends the range of parameters in which scattering can be proved (see Remark 3.1.5).

Theorem 3.1.4. *Let $N > 2$, $1+\frac{4-2b}{N} < p < 1+\frac{4-2b}{N-2}$, $0 \leq b < \min\{N/2, 2\}$, and $u_0 \in H^1_{rad}$ be such that*

$$\mathcal{ME}[u_0] < \mathcal{ME}[Q]$$

and

$$\mathcal{MK}[u_0] < \mathcal{MK}[Q].$$

Then the solution $u(t)$ to (1.0.3) is defined on $\mathbb{R}$ and scatters in H^1 in both time directions.

Remark 3.1.5. The above result is known for $b = 0$ and proved in Holmer and Roudenko [51] Duyckaerts et al. [27], Fang et al. [28], Guevara [47]. The case $b > 0$ is considered by Farah and Guzmán [31] with the assumption $0 < b < \min\{N/3, 1\}$, for $N \geq 2$. In the theorem above, not only we employ a new method to prove scattering, but we actually extend the range of b in dimensions $N > 2$, allowing $0 < b < \min\{N/2, 2\}$ in this case. Moreover, we extend the range of p in the case $N = 3$. Indeed, the result proved in Farah and Guzmán [31] considered $p < 4 - 2b$, while here we allow p to be in the entire intercritical range for the 3d case. The proofs in [51, 27, 28, 31, 47] use the so-called concentration-compactness-rigidity approach, pionereed by Kenig and Merle [55] in the context of the energy-critical ($s_c = 1$) NLS equation.

This chapter is organized as follows: in the next section, we introduce some notation and basic estimates. In Section 3, we prove the scattering criterion (Theorem 3.1.2). In Section 4, we apply this criterion, together with Morawetz/Virial estimates to prove Theorem 3.1.4.

3.2 Preliminaries

We start defining the Strichartz norms, which are used together with the Strichartz estimates.

Definition 3.2.1. Given $N > 2$, consider the set

$$\mathcal{A}_0 = \left\{(q,r) \text{ is } L^2\text{-admissible} \,\middle|\, 2 \leq r \leq \frac{2N}{N-2}\right\}.$$

For $N > 2$ and $s \in (0,1)$, consider also

$$\mathcal{A}_s = \left\{(q,r) \text{ is } \dot{H}^s\text{-admissible} \,\middle|\, \left(\frac{2N}{N-2s}\right)^+ \leq r \leq \left(\frac{2N}{N-2}\right)^-\right\}$$

and

$$\mathcal{A}_{-s} = \left\{(q,r) \text{ is } \dot{H}^{-s}\text{-admissible} \,\middle|\, \left(\frac{2N}{N-2s}\right)^+ \leq r \leq \left(\frac{2N}{N-2}\right)^-\right\}.$$

We define the following Strichartz norm

$$\|u\|_{S(\dot{H}^s,I)} = \sup_{(q,r)\in\mathcal{A}_s} \|u\|_{L^q_I L^r_x},$$

and the dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-s},I)} = \inf_{(q,r)\in\mathcal{A}_{-s}} \|u\|_{L_I^{q'}L_x^{r'}}.$$

If $s=0$, we shall write $S(\dot{H}^0,I)=S(L^2,I)$ and $S'(\dot{H}^0,I)=S'(L^2,I)$. If $I=\mathbb{R}$, we will often omit I.

We now turn to a *local-in-time* Strichartz estimate, which is the key point to prove Theorem 3.1.2.

Lemma 3.2.2 (Local-in-time Strichartz estimate, see the work of this author [11])**.**

$$\left\|\int_a^b e^{i(t-\tau)\Delta}g(\cdot,\tau)\,d\tau\right\|_{S(\dot{H}^s,\mathbb{R})} \lesssim \|g\|_{S(\dot{H}^{-s},[a,b])}.$$

Proof of Lemma 3.2.2. Recall the decay of the linear operator (see, for instance, Linares and Ponce [65, Lemma 4.1])

$$\|e^{it\Delta}f\|_{L_x^p} \lesssim \frac{1}{|t|^{\frac{N}{2}\left(\frac{1}{p'}-\frac{1}{p}\right)}}\|f\|_{L_x^{p'}}, \quad p\geq 2. \tag{3.2.1}$$

For $s\in[0,1)$, let q, $\tilde{q}$ and r be such that (q,r) is an $\dot{H}^s$-admissible pair, and $(\tilde{q},r)$ is an $\dot{H}^{-s}$-admissible pair. If $s=0$, assume additionally that $2<q<\infty$. Consider $\alpha:=(N/2)(1/r'-1/r)=2/\tilde{q}+s=2/q-s$ and note that $0<\alpha<1$ and $\frac{1}{q}+\frac{1}{\tilde{q}}=\alpha$. From Minkowski's inequality, and the decay of the linear Schrödinger operator (3.2.1):

$$\begin{aligned}
\left\|\int_a^b e^{i(t-\tau)\Delta}g(\cdot,\tau)\,d\tau\right\|_{L_x^r} &\leq \int_a^b \left\|e^{i(t-\tau)\Delta}g(\cdot,\tau)\right\|_{L_x^r}\,d\tau\\
&\lesssim \int_a^b \frac{1}{|t-\tau|^\alpha}\|g(\tau)\|_{L_x^{r'}}\,d\tau\\
&= \int_{-\infty}^{+\infty} \frac{1}{|t-\tau|^\alpha}\chi_{[a,b]}(\tau)\,\|g(\tau)\|_{L_x^{r'}}\,d\tau\\
&= I_{1-\alpha}\left(\chi_{[a,b]}\,\|g\|_{L_x^{r'}}\right)(t),
\end{aligned}$$

where I_α is the Riesz potential of order α (see Definition 2.2.1). From Lemma 2.2.2 (Hardy-Littlewood-Sobolev), we get

$$\left\|\int_a^b e^{i(t-\tau)\Delta}g(\cdot,\tau)\,d\tau\right\|_{L_t^qL_x^r} \lesssim \left\|\chi_{[a,b]}\,\|g\|_{L_x^{r'}}\right\|_{L_t^{\tilde{q}'}} = \|g\|_{L_{[a,b]}^{\tilde{q}'}L_x^{r'}}. \tag{3.2.2}$$

In particular, if $s = 0$, then $q = \tilde{q}$ and

$$\left\| \int_a^b e^{i(t-\tau)\Delta} g(\cdot,\tau)\, d\tau \right\|_{L_t^q L_x^r} \lesssim \|g\|_{L^{q'}_{[a,b]} L_x^{r'}}. \tag{3.2.3}$$

Note that (3.2.3) also immediately holds in the case $(s, q, r) = (0, \infty, 2)$. Now observe that, if $s = 0$ and $g \in C_0^\infty(\mathbb{R}^{N+1})$,

$$\begin{aligned}
\left\| \int_a^b e^{i(t-\tau)\Delta} g(\cdot,\tau)\, d\tau \right\|^2_{L_x^2} &= \int \int_a^b e^{i(t-\tau)\Delta} g(\cdot,\tau)\, d\tau \overline{\int_a^b e^{i(t-\tau')\Delta} g(\cdot,\tau')\, d\tau'}\, dx \\
&= \int \int_a^b g(\cdot,\tau) \overline{\int_a^b e^{i(\tau-\tau')\Delta} g(\cdot,\tau')\, d\tau'}\, d\tau\, dx \\
&\leq \int_a^b \|g(\tau)\|_{L_x^{r'}} \left\| \int_a^b e^{i(\tau-\tau')\Delta} g(\cdot,\tau')\, d\tau' \right\|_{L_x^r} d\tau \\
&\leq \|g\|_{L^{q'}_{[a,b]} L_x^{r'}} \left\| \int_a^b e^{i(\tau-\tau')\Delta} g(\cdot,\tau')\, d\tau' \right\|_{L_\tau^q L_x^r} \\
&\lesssim \|g\|^2_{L^{q'}_{[a,b]} L_x^{r'}}.
\end{aligned} \tag{3.2.4}$$

Therefore, as in Kato [53, Theorem 2.1], we can interpolate (3.2.2) and (3.2.4) and use a density argument to obtain (3.2.2). □

In what follows we also use the following standard estimates.

Lemma 3.2.3 (See Guzmán [48, Section 4], Farah and Guzmán [31] and the work of this author [11]). *Let $N > 2$, $u, v \in C_0^\infty(\mathbb{R}^{N+1})$, $1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2}$ and $0 \leq b < \min\{2, N/2\}$. Then there exists $0 \leq \theta = \theta(N,p,b) \ll p-1$ such that the following inequalities hold*

$$\||x|^{-b}|u|^{p-1}u\|_{L_I^\infty L_x^r} \lesssim \|u\|^p_{L_I^\infty H_x^1}, \quad 1 \leq r < \frac{2N}{N+2}, \tag{3.2.5}$$

$$\left\||x|^{-b}|u|^{p-1}u\right\|_{S'(\dot{H}^{-s_c},I)} \lesssim \|u\|^\theta_{L_t^\infty H_x^1} \|u\|^{p-\theta}_{S(\dot{H}^{s_c},I)}, \tag{3.2.6}$$

$$\left\||x|^{-b}|u|^{p-1}u\right\|_{S'(L^2,I)} \lesssim \|u\|^\theta_{L_t^\infty H_x^1} \|u\|^{p-1-\theta}_{S\left(\dot{H}^{s_c},I\right)} \|u\|_{S(L^2,I)}, \tag{3.2.7}$$

$$\left\|\nabla\left(|x|^{-b}|u|^{p-1}u\right)\right\|_{S'(L^2,I)} \lesssim \|u\|^\theta_{L_t^\infty H_x^1} \|u\|^{p-1-\theta}_{S\left(\dot{H}^{s_c},I\right)} \|\nabla u\|_{S(L^2,I)}. \tag{3.2.8}$$

Proof. Inequality (3.2.5) follows immediately from the Gagliardo-Nirenberg-type inequality (2.2.3). To prove the remaining inequalities, consider the exponents

$$\hat{q} = \frac{4(p-1)(p+1)}{(p-1)[N(p-1)+2b] - \theta[N(p-1)-4+2b]}, \quad \hat{r} = \frac{N(p-1)(p+1)}{(p-1)(N-b) - \theta(2-b)},$$
$$\tilde{a} = \frac{2(p-1)(p+1-\theta)}{(p-1)[N(p-\theta)-2+2b)] - (4-2b)(1-\theta)}, \quad \hat{a} = \frac{2(p-1)(p+1-\theta)}{4-2b-(N-2)(p-1)}.$$

Choosing $\theta = 0$ if $b = 0$, and $0 < \theta \ll 1$ if $b > 0$, we have that $(\hat{q}, \hat{r}) \in \mathcal{A}_0$, $(\hat{a}, \hat{r}) \in \mathcal{A}_{s_c}$ and $(\tilde{a}, \hat{r}) \in \mathcal{A}_{-s_c}$. By Hölder and Sobolev inequalities (see [48, Lemmas 4.1 and 4.2] for details), we have

$$\||x|^{-b}|u|^{p-1}v\|_{L^{\hat{r}'}} \lesssim \|u\|_{H^1_x}^{\theta} \|u\|_{L^{\hat{r}}x}^{p-1-\theta} \|v\|_{L^{\hat{r}}_x}, \tag{3.2.9}$$

so that (3.2.6) and (3.2.7) follow.

Consider now (3.2.8). If $b = 0$, then it follows directly from (3.2.9). For $b > 0$, define the pairs

$$\bar{q} = \frac{4(p-1)(p-\theta)}{(p-1)[N(p-1)+2b-2] - \theta[N(p-1)-4+2b]},$$
$$\bar{r} = \frac{2N(p-1)(p-\theta)}{(p-1)(N+2-2b) - \theta(4-2b)},$$
$$\bar{a} = \frac{4(p-1)(p-\theta)}{4-2b-(N-2)(p-1)}.$$

It is immediate to check that $(2, 2N/(N-2), (\bar{q}, \bar{r}) \in \mathcal{A}_0$, and that $(\bar{a}, \bar{r}) \in \mathcal{A}_{s_c}$. Let B be the unit ball centered at the origin, $B^c = \mathbb{R}^N \backslash B$ and let A denote B or B^c. Since

$$|\nabla(|x|^{-b}|u|^{p-1}u)| \lesssim |x|^{-b}|u|^{p-1}|\nabla u| + |x|^{-b}|x|^{-1}(|u|^{p-1}|u|),$$

we estimate, by Hölder inequality

$$\|\nabla(|x|^{-b}|u|^{p-1}u)\|_{L_A^{\frac{2N}{N+2}}} \lesssim \||x|^{-b}\|_{L_A^{r_1}} \left(\||u|^{p-1}\nabla u\|_{L^{r_2}} + \||x|^{-1}|u|^{p-1}u\|_{L^{r_2}}\right), \tag{3.2.10}$$

where we choose

$$\frac{1}{r_1} = \frac{b}{N} + l, \text{ with } l := \begin{cases} \frac{\theta(1-s_c)}{N}, & \text{if } A = B, \\ -\frac{\theta s_c}{N}, & \text{if } A = B^c, \end{cases}$$

and

$$\frac{1}{r_2} = \frac{N+2}{2N} - \frac{1}{r_1}.$$

Since $1 < \frac{2N}{N+2-2b} < N$ for $N > 2$ and $0 < b < N/2$, if we choose θ (and thus l) small enough, we conclude that $\||x|^{-b}\|_{L_A^{r_1}} < +\infty$, and that $1 < r_2 < N$. In view of Hardy's inequality (see [59]),

$$\int |\nabla f|^r \geq \left(\frac{N-r}{r}\right)^r \int \frac{|f|^r}{|x|^r}, \quad f \in W^{1,r}(\mathbb{R}^N),\ 1 < r < N,$$

we have

$$\||x|^{-1}|u|^{p-1}u\|_{L^{r_2}} \lesssim \|\nabla(|u|^{p-1}u)\|_{L^{r_2}} \lesssim \||u|^{p-1}\nabla u\|_{L^{r_2}}.$$

Therefore, (3.2.10) becomes

$$\|\nabla(|x|^{-b}|u|^{p-1}u)\|_{L^{\frac{2N}{N+2}}} \lesssim \||u|^{p-1}\nabla u\|_{L^{r_2}}.$$

Now, by splitting

$$\frac{1}{r_2} = \underbrace{\theta\left(\frac{1}{2} - \frac{s_c}{N}\right) - l}_{\frac{1}{r_3}} + \underbrace{\frac{p-1-\theta}{\bar{r}}}_{\frac{1}{r_4}} + \underbrace{\frac{1}{\bar{r}}}_{\frac{1}{r_5}},$$

it is easy to see that $2 \leq \theta r_3 \leq 2N/(N-2)$. By Hölder and Sobolev inequalities

$$\||u|^{p-1}\nabla u\|_{L^{r_2}} \lesssim \|u\|^{\theta}_{L^{\theta r_2}} \|u\|^{p-\theta}_{L^{\bar{r}}} \|\nabla u\|_{L^{\bar{r}}} \lesssim \|u\|^{\theta}_{H^1} \|u\|^{p-\theta}_{L^{\bar{r}}} \|\nabla u\|_{L^{\bar{r}}}.$$

Therefore, by Hölder inequality on the time variable:

$$\|\nabla(|x|^{-b}|u|^{p-1}u)\|_{L_t^2 L_x^{\frac{2N}{N+2}}} \lesssim \|u\|^{\theta}_{L_t^\infty H_x^1} \|u\|^{p-1-\theta}_{L_t^{\bar{a}} L_x^{\bar{r}}} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}},$$

which finishes the proof of the lemma. □

Remark 3.2.4. Inequalities (3.2.6)-(3.2.8) were proved in [48] for $0 < b < b^*$ (see Definition 2.4.1) and with the additional restriction $p < 4 - 2b$ instead of $p < 5 - 2b$ in the 3d case. The proof we give here extends the range of b to $\min\{N/2, 2\}$ and of p to the whole range where local well-posedness is proved. We expect that Lemma 3.2.3 can be used to extend the results in [48] via the concentration-compactness-rigidity tecnique.

The next lemma was proved in [48] with the same restrictions mentioned in Remark 3.2.4. In view of Lemma 3.2.3, the proof in [48] immediately extends to the new range of p and b.

Lemma 3.2.5 (Small data theory, see Guzmán [48, Theorem 1.8], and the work of this author [11]). *Let $N \geq 1$, $1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2}$ and $0 \leq b < \min N/2, 2$. Suppose $\|u_0\|_{H^1} \leq E$. Then there exists $\delta_{sd} = \delta_{sd}(E) > 0$ such that if*

$$\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c},[0,+\infty))} \leq \delta_{sd},$$

then the solution u to (1.0.3) with initial condition $u_0 \in H^1(\mathbb{R}^N)$ is globally defined on $[0, +\infty)$. Moreover,

$$\|u\|_{S(\dot{H}^{s_c},[0,+\infty))} \leq 2\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c},[0,+\infty))},$$

and

$$\|u\|_{S(L^2,[0,+\infty))} + \|\nabla u\|_{S(L^2,[0,+\infty))} \lesssim \|u_0\|_{H^1}.$$

3.3 Proof of the scattering criterion

We start this section with a remark.

Remark 3.3.1. Under Definition 3.2.1, there exists a small $\delta > 0$ (possibly depending on N, p, s and b) such that, for a fixed $0 < s < 1$

$$2 + \delta \leq r \leq p^* - \delta, \text{ and}$$

$$2 + \delta \leq \frac{2}{1-s} < q \leq \frac{1}{\delta},$$

for any pair $(q, r) \in \mathcal{A}_s$.

For $N > 2$, fix the parameters

$$\alpha = \frac{\delta(2+\delta)}{(p^* - \delta)(p^* - 2)} > 0$$

and

$$\gamma = \min\left\{\frac{\delta(p-\theta)}{(p^*-\delta)(p^*-2)}, \alpha\delta s_c\right\} > 0,$$

Where $0 \leq \theta \ll p-1$ is given in Lemma 3.2.3. The following result is the key to prove Theorem 3.1.2.

Lemma 3.3.2. *Let $N > 2$, $1+\frac{4-2b}{N} < p < 1+\frac{4-2b}{N-2}$, $0 \leq b < 2$ and u be a radial $H^1(\mathbb{R}^N)$-solution to (1.0.3) satisfying (3.1.2). If u satisfies (3.1.3) for some $0<\epsilon<1$, then there exists $T>0$ such that the following estimate is valid*

$$\left\| e^{i(\cdot - T)\Delta} u(T) \right\|_{S\left(\dot{H}^{s_c},[T,+\infty)\right)} \lesssim \epsilon^{\gamma}.$$

Proof. From (2.2.5), there exists $T_0 > \epsilon^{-\alpha}$ such that

$$\left\| e^{it\Delta} u_0 \right\|_{S\left(\dot{H}^{s_c},[T_0,+\infty)\right)} \leq \epsilon^{\gamma}. \tag{3.3.1}$$

For $T \geq T_0$ to be chosen later, define $I_1 := [T-\epsilon^{-\alpha}, T]$, $I_2 := [0, T-\epsilon^{-\alpha}]$ and let η denote a smooth, spherically symmetric function which equals 1 on $B(0,1/2)$ and 0 outside $B(0,1)$. For any $R>0$ use η_R to denote the rescaling $\eta_R(x) := \eta(x/R)$.

From Duhamel's formula

$$u(T) = e^{iT\Delta}u_0 + \int_0^T e^{i(T-s)\Delta}|x|^{-b}|u|^{p-1}u(s)\,ds,$$

we obtain

$$e^{i(t-T)\Delta}u(T) = e^{it\Delta}u_0 + F_1 + F_2,$$

where, for $i=1,2$,

$$F_i = \int_{I_i} e^{i(t-s)\Delta}|x|^{-b}|u|^{p-1}u(s)\,ds.$$

We refer to F_1 as the "recent past", and to F_2 as the "distant past". By (3.3.1), it remains to estimate F_1 and F_2.

Step 1. Estimate on recent past.

By (3.1.3), we can fix $T \geq T_0$ such that

$$\int \eta_R(x)\,|u(T,x)|^2\,dx \lesssim \epsilon^2. \tag{3.3.2}$$

Given the relation (obtained by multiplying (1.0.3) by $\eta_R\bar{u}$, taking the imaginary part and integrating by parts, see Tao [82, Section 4] for details)

$$\partial_t \int \eta_R |u|^2 \, dx = 2 \operatorname{Im} \int \nabla \eta_R \cdot \nabla u \bar{u},$$

we have, from (3.1.2), for all times,

$$\left| \partial_t \int \eta_R(x) |u(t,x)|^2 dx \right| \lesssim \frac{1}{R},$$

so that, by (3.3.2), for $t \in I_1$,

$$\int \eta_R(x) \, |u(t,x)|^2 \, dx \lesssim \epsilon^2 + \frac{\epsilon^{-\alpha}}{R}.$$

If $R > \epsilon^{-(\alpha+2)}$, then we have $\|\eta_R u\|_{L^\infty_{I_1} L^2_x} \lesssim \epsilon$.

Let $(q,r) \in \mathcal{A}_{s_c}$. Recalling that $2+\delta \leq r \leq p^* - \delta$ (see Remark 3.3.1), using interpolation and Sobolev inequalities and the decay of the L^∞ norm of radial functions outside the ball (2.2.1), we get

$$\begin{aligned} \|u\|_{L^\infty_{I_1} L^r_x} &\lesssim \|\eta_R u\|^{\frac{2(p^*-r)}{r(p^*-2)}}_{L^\infty_{I_1} L^2_x} \|\eta_R u\|^{1-\frac{2(p^*-r)}{r(p^*-2)}}_{L^\infty_{I_1} L^{p^*}_x} + \|(1-\eta_R)u\|^{\frac{r-2}{r}}_{L^\infty_{I_1} L^\infty_x} \|(1-\eta_R)u\|^{\frac{2}{r}}_{L^\infty_{I_1} L^2_x} \\ &\lesssim \epsilon^{\frac{2(p^*-r)}{r(p^*-2)}} \|u\|^{1-\frac{2(p^*-r)}{r(p^*-2)}}_{L^\infty_{I_1} L^{p^*}_x} + R^{-\frac{N-1}{2}\left(\frac{r-2}{r}\right)} \|u\|^{\left(\frac{r-2}{r}\right)}_{L^\infty_t H^1_x} \|u_0\|^{\frac{2}{r}}_{L^2_x} \\ &\lesssim \epsilon^{\frac{2\delta}{(p^*-\delta)(p^*-2)}} + R^{-\frac{N-1}{2}\frac{\delta}{p^*-\delta}} \lesssim \epsilon^{\frac{2\delta}{(p^*-\delta)(p^*-2)}}, \end{aligned} \tag{3.3.3}$$

if R is large enough. Note that, in the penultimate step, we used the $H^1 \hookrightarrow L^{p^*}$ embedding. Using the local-in-time Strichartz estimate (3.2.2), together with estimates (3.2.6) and (3.3.3), we bound

$$\begin{aligned} \left\| \int_{I_1} e^{i(t-s)\Delta} |x|^{-b} |u|^{p-1} u(s) \, ds \right\|_{S(\dot{H}^{s_c},[T,+\infty))} &\leq \| \, |x|^{-b} |u|^{p-1} u \|_{S'(\dot{H}^{-s_c}, I_1)} \\ &\leq \|u\|^\theta_{L^\infty_t H^1_x} \|u\|^{p-\theta}_{S(\dot{H}^{s_c}, I_1)} = \|u\|^\theta_{L^\infty_t H^1_x} \sup_{(q,r)\in\mathcal{A}_{s_c}} \|u\|^{p-\theta}_{L^q_{I_1} L^r_x} \\ &\leq \|u\|^\theta_{L^\infty_t H^1_x} \sup_{2+\delta\leq r\leq p^*-\delta} \|u\|^{p-\theta}_{L^\infty_{I_1} L^r_x} \epsilon^{-\alpha\left(\frac{p-\theta}{q}\right)} \\ &\leq \|u\|^\theta_{L^\infty_t H^1_x} \epsilon^{\frac{2\delta}{(p^*-\delta)(p^*-2)}(p-\theta)} \epsilon^{-\alpha\left(\frac{p-\theta}{2+\delta}\right)} \lesssim \epsilon^{\frac{\delta(p-\theta)}{(p^*-\delta)(p^*-2)}}, \end{aligned}$$

where we used the definition of $\alpha > 0$ and the fact that $q \geq 2 + \delta$.

Step 2. Estimate on distant past.

Let $(q, r) \in \mathcal{A}_{s_c}$. Define

$$\frac{1}{c} = \left(\frac{1}{1 - s_c}\right)\left[\frac{1}{q} - \delta s_c\right]$$

and

$$\frac{1}{d} = \left(\frac{1}{1 - s_c}\right)\left[\frac{1}{r} - s_c\left(\frac{N - 2 - 4\delta}{2N}\right)\right]$$

We claim that $(c, d) \in \mathcal{A}_0$. Indeed, it is immediate to check that (c, d) satisfies (2.2.4) with $s = 0$. Moreover, since

$$q > \frac{2}{1 - s_c},$$

we see, since $\delta > 0$ is small, that $2 < c < +\infty$, so that the pair (c, d) is L^2-admissible. We have

$$\|F_2\|_{L^q_{[T,+\infty)} L^r_x} \leq \|F_2\|^{1-s_c}_{L^c_{[T,+\infty)} L^d_x} \|F_2\|^{s_c}_{L^{\frac{1}{\delta}}_{[T,+\infty)} L_x^{\frac{2N}{N-2-4\delta}}}.$$

Using Duhamel's principle, write

$$F_2 = e^{it\Delta}\left[e^{i(-T+\epsilon^{-\alpha})\Delta} u(T - \epsilon^{-\alpha}) - u(0)\right].$$

Thus, by the Strichartz estimate (2.2.5),

$$\begin{aligned}
\|F_2\|_{L^q_{[T,+\infty)} L^r_x} &\leq \left\| e^{it\Delta}\left[e^{i(-T+\epsilon^{-\alpha})\Delta} u(T - \epsilon^{-\alpha}) - u(0)\right]\right\|^{1-s_c}_{L^c_{[T,+\infty)} L^d_x} \|F_2\|^{s_c}_{L^{\frac{1}{\delta}}_{[T,+\infty)} L_x^{\frac{2N}{N-2-4\delta}}} \\
&\leq \left(\|u\|_{L^\infty_t L^2_x}\right)^{1-s_c} \|F_2\|^{s_c}_{L^{\frac{1}{\delta}}_{[T,+\infty)} L_x^{\frac{2N}{N-2-4\delta}}} \lesssim \epsilon^{\alpha\delta s_c},
\end{aligned}$$

since, by (3.2.1) and (3.2.5),

$$\begin{aligned}
\|F_2\|_{L^{\frac{1}{\delta}}_{[T,+\infty)} L_x^{\frac{2N}{N-2-4\delta}}} &\lesssim \left\| \int_{I_2} |\cdot - s|^{-(1+2\delta)} \left\| |x|^{-b} |u|^{p-1} u(s) \right\|_{L_x^{\frac{2N}{N+2+4\delta}}} ds \right\|_{L^{\frac{1}{\delta}}_{[T,+\infty)}} \\
&\lesssim \|u\|^p_{L^\infty_{[T,+\infty)} H^1_x} \left\| \left(\cdot - T + \epsilon^{-\alpha}\right)^{-2\delta} \right\|_{L^{\frac{1}{\delta}}_{[T,+\infty)}} \\
&\lesssim \epsilon^{\alpha\delta}.
\end{aligned}$$

Therefore, recalling that

$$e^{i(t-T)\Delta}u(T)=e^{it\Delta}u_0+F_1+F_2,$$

we have

$$\left\|e^{i(\cdot-T)\Delta}u(T)\right\|_{S\left(\dot H^{s_c},[T,+\infty)\right)}\lesssim\epsilon^{\gamma}.$$

Hence, Lemma 3.3.2 is proved. □

Proof of Theorem 3.1.2. Choose ϵ is small enough so that, by Lemma 3.3.2,

$$\left\|e^{i(\cdot)\Delta}u(T)\right\|_{S\left(\dot H^{s_c},[0,+\infty)\right)}=\left\|e^{i(\cdot-T)\Delta}u(T)\right\|_{S\left(\dot H^{s_c},[T,+\infty)\right)}\leq c\epsilon^{\gamma}\leq\delta_{sd},$$

where δ_{sd} is given in Lemma 3.2.5. Thus, by small data theory, we have

$$\|u\|_{S\left(\dot H^{s_c},[T,+\infty)\right)}\lesssim\epsilon^{\gamma},\text{ and }\ \|(1+|\nabla|)u\|_{S(L^2,[T,+\infty))}\lesssim 1.$$

Define $u_+=e^{-iT\Delta}u(T)+i\displaystyle\int_T^{+\infty}e^{-is\Delta}|x|^{-b}|u|^{p-1}u(s)\,ds$. Using (3.2.7) and (3.2.8), we estimate

$$\begin{aligned}
\|u(t)-e^{it\Delta}u_+\|_{H^1_x}&=\left\|\int_t^{+\infty}e^{i(t-s)\Delta}|x|^{-b}|u|^{p-1}u(s)\,ds\right\|_{H^1_x}\\
&\lesssim\left\|(1+|\nabla|)\int_t^{+\infty}e^{i(t-s)\Delta}|x|^{-b}|u|^{p-1}u(s)\,ds\right\|_{L^2_x}\\
&\lesssim\sup_{\tau\in[t,+\infty)}\left\|(1+|\nabla|)\int_\tau^{+\infty}e^{i(\tau-s)\Delta}|x|^{-b}|u|^{p-1}u(s)\,ds\right\|_{L^2_x}\\
&\lesssim\left\|\int_\tau^{+\infty}e^{i(\tau-s)\Delta}(1+|\nabla|)\left(|x|^{-b}|u|^{p-1}u(s)\right)\,ds\right\|_{S(L^2,[t,+\infty))}\\
&\lesssim\left\|(1+|\nabla|)\left(|x|^{-b}|u|^{p-1}u(s)\right)\right\|_{S'(L^2,[t,+\infty))}\\
&\lesssim\|u\|^{p-1-\theta}_{S\left(\dot H^{s_c},[t,+\infty)\right)}.
\end{aligned}$$

(Note that the same estimate ensures that $u_+\in H^1$). Hence, we conclude that

$$\lim_{t\to+\infty}\|u(t)-e^{it\Delta}u_+\|_{H^1_x}=0$$

as desired. □

3.4 Proof of the main result

We now turn to Theorem 3.1.4. The main idea behind the proof is to combine radial decay with a truncated Virial identity. By choosing the right weight, and using bounds given by coercivity in large balls around the origin, one can control a time-averaged L^p norm on these balls. Averaging is necessary due to the lack uniform estimates in time, since we are not employing concentration-compactness as in Holmer-Roudenko [51, 27].

We start with the following "trapping" lemmas, which follow immediately from the sharp Gagliardo-Nirenberg inequality, and can be found in Farah and Guzmán [31, Lemma 4.2].

Lemma 3.4.1 (Energy trapping). *Let $N \geq 1$ and $0 < s_c < 1$. If*

$$M[u_0]^{\frac{1-s_c}{s_c}} E[u_0] < (1-\delta) M[u_0]^{\frac{1-s_c}{s_c}} E[u_0]$$

for some $\delta > 0$ and

$$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L^2} \leq \|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2},$$

then there exists $\delta' = \delta'(\delta) > 0$ such that

$$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L^2} < (1-\delta') \|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2}.$$

for all $t \in I$, where $I \subset \mathbb{R}$ is the maximal interval of existence of the solution $u(t)$ to (1.0.3)*. Moreover, $I = \mathbb{R}$ and u is uniformly bounded in H^1.*

Lemma 3.4.2. *Suppose, for $f \in H^1(\mathbb{R}^N)$, $N \geq 1$, that*

$$\|f\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla f\|_{L^2} < (1-\delta) \|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2}.$$

Then there exists $\delta' = \delta'(\delta) > 0$ so that

$$\int |\nabla f|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int |x|^{-b} |f|^{p+1} \geq \delta' \int |x|^{-b} |f|^{p+1}.$$

From now on, we consider u to be a solution to (1.0.3) satisfying the conditions

$$M[u_0]^{\frac{1-s_c}{s_c}} E[u_0] < M[Q]^{\frac{1-s_c}{s_c}} E[Q]$$

and

$$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L^2} \leq \|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2}.$$

In particular, by Lemma 3.4.1, u is global and uniformly bounded in H^1. Moreover, there exists $\delta > 0$ such that

$$\sup_{t\in\mathbb{R}} \|u_0\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_{L^2} < (1-2\delta)\|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2} \tag{3.4.1}$$

In the spirit of Dodson and Murphy [22], we prove a local coercivity estimate. We start with a preliminary result.

Lemma 3.4.3. *For $N \geq 1$, let ϕ be a smooth cutoff to the set $\{|x| \leq \frac{1}{2}\}$ and define $\phi_R(x) = \phi\left(\frac{x}{R}\right)$. If $f \in H^1(\mathbb{R}^N)$, then*

$$\int |\nabla(\phi_R f)|^2 = \int \phi_R^2 |\nabla f|^2 - \int \phi_R \Delta(\phi_R)|f|^2. \tag{3.4.2}$$

In particular,

$$\left| \int |\nabla(\phi_R f)|^2 - \int \phi_R^2 |\nabla f|^2 \right| \leq \frac{c}{R^2}\|f\|_{L^2}^2. \tag{3.4.3}$$

Proof. We first calculate directly

$$|\nabla(\phi_R f)|^2 = \ |\nabla \phi_R f + \phi_R \nabla f|^2 = |\nabla\phi_R|^2|f|^2 + 2\,\mathrm{Re}(\nabla\phi_R \cdot \nabla f\, \phi_R\, \bar{f}) + \phi_R^2|\nabla f|^2.$$

Now, integrating by parts, we have

$$2\,\mathrm{Re}\int(\nabla\phi_R \cdot \nabla f\, \phi_R\, \bar{f}) = -\int \phi_R\Delta(\phi_R)|f|^2 - \int |\nabla\phi_R|^2|f|^2.$$

Using the last two identities, we conclude (3.4.2). To obtain (3.4.3), we note that

$$\|\phi_R\Delta(\phi_R)\|_{L^\infty} \leq \frac{c}{R^2}.$$

□

Lemma 3.4.4 (Local coercivity)**.** *For $N \geq 1$, let u be a globally defined $H^1(\mathbb{R}^N)$ solution to (1.0.3) satisfying (3.4.1). There exists $\bar{R} = \bar{R}(\delta, M[u_0], Q, s_c) > 0$ such that, for any $R \geq \bar{R}$,*

$$\sup_{t\in\mathbb{R}} \|\phi_R u(t)\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla(\phi_R u(t))\|_{L^2} \leq (1-\delta)\|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2}.$$

In particular, by Lemma 3.4.2, there exists $\delta' = \delta'(\delta) > 0$ such that

$$\int |\nabla(\phi_R u(t))|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int |x|^{-b}|\phi_R u(t)|^{p+1} \geq \delta' \int |x|^{-b}|\phi_R u(t)|^{p+1}.$$

Proof. First note that

$$\|\phi_R u(t)\|_{L^2}^2 \leq \|u(t)\|_{L^2}^2 = M[u_0],$$

for all $t \in \mathbb{R}$. Thus, we only need to control the $\dot{H}^1$ term. Using Lemma 3.4.3 and (3.4.1), we conclude

$$\begin{aligned}\|\phi_R u(t)\|_{L^2}^{\frac{2(1-s_c)}{s_c}} \|\phi_R u(t)\|_{\dot{H}^1}^2 &\leq M[u_0]^{\frac{1-s_c}{s_c}} \left(\|\nabla u(t)\|_{L^2}^2 + \frac{c}{R^2} M[u_0]\right) \\ &< (1-2\delta)^2 \|Q\|_{L^2}^{\frac{2(1-s_c)}{s_c}} \|\nabla Q\|_{L^2}^2 + \frac{c}{R^2} M[u_0]_{L^2}^{\frac{1}{s_c}}.\end{aligned}$$

Thus, by choosing R large enough, depending on δ, $M[u_0]$, Q and s_c, we bound the last expression by $\left[(1-\delta)\|Q\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q\|_{L^2}\right]^2$, which finishes the proof.

□

We exploit the coercivity given by the previous lemma by making use of the Virial identity. Recalling (2.3.2), if $a : \mathbb{R}^N \to \mathbb{R}$ is a smooth weight and $|\nabla a| \in L^\infty$, define

$$Z(t) = 2\,\mathrm{Im} \int \bar{u} \nabla u \cdot \nabla a \, dx.$$

Then, if u is a solution to INLS (1.0.3), we have the following identity

$$\begin{aligned}\frac{d}{dt} Z(t) = \left(\frac{4}{p+1} - 2\right) \int |x|^{-b}|u|^{p+1} \Delta a - \frac{4b}{p+1} \int |x|^{-b-2}|u|^{p+1} x \cdot \nabla a \\ - \int |u|^2 \Delta\Delta a + 4\,\mathrm{Re} \sum_{i,j} \int a_{ij} \bar{u}_i u_j.\end{aligned}$$

We now have all the basic tools needed to prove scattering. Let $R \gg 1$ to be determined below. We take a to be a radial function satisfying

$$a(x) = \begin{cases} |x|^2 & |x| \leq \frac{R}{2}, \\ 2R|x| - R^2 & |x| > R. \end{cases}$$

In the intermediate region $\frac{R}{2} < |x| \leq R$, we impose that

$$\partial_r a \geq 0, \;\; \partial_r^2 a \geq 0, \;\; |\partial^\alpha a(x)| \lesssim_\alpha R|x|^{-|\alpha|+1} \;\text{ for }\; |\alpha| \geq 1.$$

Here, ∂_r denotes the radial derivative, i.e., $\partial_r a = \nabla a \cdot \frac{x}{|x|}$. Note that for $|x| \leq \frac{R}{2}$, we have

$$a_{ij} = 2\delta_{ij}, \;\; \Delta a = 2N, \;\; \Delta\Delta a = 0,$$

while, for $|x| > R$, we have

$$a_{ij} = \frac{2R}{|x|}\left[\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right], \;\; \Delta a = \frac{2(N-1)R}{|x|}, \;\; |\Delta\Delta a(x)| \lesssim \frac{R}{|x|^3}.$$

Proposition 3.4.5 (Virial/Morawetz estimate)**.** *For $N > 2$, let u be a radial H^1-solution to (1.0.3) satisfying (3.4.1). Then, for $R = R(\delta, M[u_0], Q)$ sufficiently large, and $T > 0$,*

$$\frac{1}{T}\int_0^T \int_{|x|\leq R} |u(x,t)|^{p+1}\, dx\, dt \lesssim_{u,\delta} \frac{R^{b+1}}{T} + \frac{1}{R^{(2-b)\left(\frac{N-1}{N}\right)}}.$$

Proof. Choose $R \geq \bar{R}(\delta, M[u_0], Q, s_c)$ as in Lemma 3.4.4. We define the weight a as above and define $Z(t)$ as in Lemma 3.4. Using Cauchy-Schwarz inequality, and the definition of $Z(t)$, we have

$$\sup_{t\in\mathbb{R}} |Z(t)| \lesssim R. \tag{3.4.4}$$

As in Dodson and Murphy [22, Proposition 3.4], we compute

$$\begin{aligned}\frac{d}{dt}Z(t) = 8&\left[\int_{|x|\leq\frac{R}{2}} |\nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right)\int_{|x|\leq\frac{R}{2}} |x|^{-b}|u|^{p+1}\right]\\ &+ \int_{|x|>\frac{R}{2}} \left[\left(\frac{4}{p+1} - 2\right)(N-1)\Delta a - \frac{4b}{p+1}\frac{x\cdot\nabla a}{|x|^2}\right]|x|^{-b}|u|^{p+1}\\ &+ \int_{|x|>\frac{R}{2}} 4\partial_r^2 a|\partial_r u|^2 - \int_{|x|>\frac{R}{2}} |u|^2\Delta\Delta a,\end{aligned}$$

where we used the radiality of u and a. By the definition of a, and the fact that $\partial_r^2 a \geq 0$,

$$\frac{d}{dt}Z(t) \geq 8\left[\int_{|x|\leq\frac{R}{2}} |\nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right)\int_{|x|\leq\frac{R}{2}} |x|^{-b}|u|^{p+1}\right] \tag{3.4.5}$$

$$- \frac{c}{R^b} \int_{|x|>\frac{R}{2}} |u|^{p+1} - \frac{c}{R^2} M[u_0].$$

Define ϕ^A , $A > 0$, as a smooth cutoff to the set $\{|x| \le \frac{1}{2}\}$ that vanishes outside the set $\{|x| \le \frac{1}{2} + \frac{1}{A}\}$, and define $\phi_R^A(x) = \phi^A\left(\frac{x}{R}\right)$. We will now estimate the first term in the last inequality.

$$\begin{aligned}
&\int_{|x|\le\frac{R}{2}} |\nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int_{|x|\le\frac{R}{2}} |x|^{-b}|u|^{p+1} = \\
&= \left[\int (\phi_R^A)^2 |\nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int (\phi_R^A)^2 |x|^{-b}|u|^{p+1}\right] \\
&- \underbrace{\left[\int_{\frac{R}{2}<|x|\le\frac{R}{2}+\frac{R}{A}} (\phi_R^A)^2 |\nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int_{\frac{R}{2}<|x|\le\frac{R}{2}+\frac{R}{A}} (\phi_R^A)^2 |x|^{-b}|u|^{p+1}\right]}_{I_A} \\
&= \left[\int |\phi_R^A \nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int |x|^{-b}|\phi_R^A u|^{p+1}\right] \\
&-I_A - \underbrace{\left(\frac{N}{2} - \frac{N-b}{p+1}\right) \int \left((\phi_R^A)^{p+1} - (\phi_R^A)^2\right) |x|^{-b}|u|^{p+1}}_{II_A}.
\end{aligned} \tag{3.4.6}$$

Using Lemma 3.4.3, we can write

$$\begin{aligned}
&\int |\phi_R^A \nabla u|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int |x|^{-b}|\phi_R^A u|^{p+1} \ge \\
&\int |\nabla(\phi_R^A u)|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int |x|^{-b}|\phi_R^A u|^{p+1} - \frac{c}{R^2} M[u_0].
\end{aligned} \tag{3.4.7}$$

The inequalities (3.4.5), (3.4.6) and (3.4.7) can be rewritten as

$$\begin{aligned}
\frac{d}{dt} Z(t) \ge 8 &\left[\int |\nabla(\phi_R^A u)|^2 + \left(\frac{N-b}{p+1} - \frac{N}{2}\right) \int |x|^{-b}|\phi_R^A u|^{p+1}\right] \\
&- \frac{c}{R^b} \int_{|x|>\frac{R}{2}} |u|^{p+1} - \frac{c}{R^2} M[u_0] - 8I_A - 8II_A.
\end{aligned} \tag{3.4.8}$$

By Corollary 2.2.5 and by Lemma 3.4.4, we can write (3.4.8) as

$$\int |x|^{-b}|\phi_R^A u(t)|^{p+1} \lesssim \frac{d}{dt} Z(t) + \frac{1}{R^{\frac{(N-1)(p-1)}{2}+b}} + \frac{1}{R^2} + 8I_A + 8II_A.$$

We can now make $A \to +\infty$ to obtain $I_A + II_A \to 0$ by dominated convergence. Hence,

$$R^{-b}\int_{|x|\leq\frac{R}{2}}|u(t)|^{p+1}\lesssim\int_{|x|\leq\frac{R}{2}}|x|^{-b}|u(t)|^{p+1}\lesssim\frac{d}{dt}Z(t)+\frac{1}{R^{\frac{(N-1)(p-1)}{2}+b}}+\frac{1}{R^2}.$$

We finish the proof integrating over time, and using (3.4.4). We have

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{|x|\leq\frac{R}{2}}|u(t)|^{p+1}&\lesssim\frac{R^b}{T}\sup_{t\in[0,T]}|Z(t)|+\frac{1}{R^{\frac{(N-1)(p-1)}{2}}}+\frac{1}{R^{2-b}}\\&\lesssim\frac{R^{b+1}}{T}+\frac{1}{R^{(2-b)\frac{(N-1)}{N}}},\end{aligned}$$

since $p>1+\frac{4-2b}{N}$. □

We are now able to prove the *energy evacuation*.

Proposition 3.4.6 (Energy evacuation). *Under the hypotheses of Proposition 3.4.5, there exist a sequence of times $t_n\to+\infty$ and a sequence of radii $R_n\to+\infty$ such that*

$$\lim_{n\to+\infty}\int_{|x|\leq R_n}|u(t_n)|^{p+1}=0 \tag{3.4.9}$$

Proof. Using Proposition 3.4.5, choose $T_n\to+\infty$ and $R_n=T_n^{\frac{N}{3N-2+b}}$, so that

$$\frac{1}{T_n}\int_0^{T_n}\int_{|x|\leq R_n}|u(t)|^{p+1}\lesssim\frac{1}{T_n^{\frac{(2-b)(N-1)}{3N-2+b}}}\to 0\text{ as }n\to+\infty.$$

Therefore, by the Mean Value Theorem, there is a sequence $t_n\to+\infty$ such that (3.4.9) holds. The proof is complete. □

Using Proposition 3.4.6, we can prove Theorem 3.1.4. We will prove only the case $t\to+\infty$, as the case $t\to-\infty$ is entirely analogous.

Proof of Theorem 3.1.4. Take $t_n\to+\infty$ and $R_n\to+\infty$ as in Proposition 3.4.6. Fix $\epsilon>0$ and $R>0$ as in Theorem 3.1.2. Choosing n large enough, such that $R_n\geq R$, Hölder's inequality and (3.4.9) yield

$$\int_{|x|\leq R}|u(x,t_n)|^2\lesssim R^{\frac{N(p-1)}{p+1}}\left(\int_{|x|\leq R_n}|u(x,t_n)|^{p+1}\right)^{\frac{2}{p+1}}\to 0\text{ as }n\to+\infty.$$

Therefore, by Theorem 3.1.2, u scatters forward in time. □

4 Classifications of solutions to the NLS at the threshold

4.1 Introduction

In this chapter, we consider solutions to NLS (1.0.2), in the intercritical and energy-critical case, for initial data at the mass-energy threshold, that is,

$$\mathcal{ME}[u_0] = \mathcal{ME}[Q]$$

(see (2.6.6)).

Solutions to NLS at the threshold level were first studied by Duyckaerts and Merle in [24], in the radial case, for $N = 3$, 4 and 5. Later, the result was proved for $N \geq 6$ by Li and Zhang [63]. Their results can be summarized as following.

Theorem 4.1.1. *Let $N \geq 3$. There exist two radial solutions W^+ and W^- to* (1.0.2) *in $\dot{H}^1(\mathbb{R}^N)$ such that*

- *$E[W^\pm] = E[W]$, $W^\pm$ is defined at least in $[0, +\infty)$ and there exist C, $e_0 > 0$ such that*
$$\|W^\pm(t) - W\|_{H^1} \leq Ce^{-e_0 t}, \text{ for all } t \geq 0,$$
- *$\|\nabla W_0^+\|_2 > \|\nabla W\|_2$ and, if $N \geq 5$, W^+ blows-up in finite negative time,*
- *$\|\nabla W_0^-\|_2 < \|\nabla W\|_2$ and W^- is globally defined and scatters backward in time.*

In the next theorems, by *u equals v up to the symmetries of the equation*, we mean that there exist $t_0 \in \mathbb{R}$, $\lambda_0 > 0$ and $\theta_0 \in \mathbb{R}$ such that

$$u(x,t) = \frac{e^{i\theta_0}}{\lambda_0^{\frac{N-2}{2}}} v\left(\frac{x}{\lambda_0}, \frac{t+t_0}{\lambda_0^2}\right) \text{ or } u(x,t) = \frac{e^{i\theta_0}}{\lambda_0^{\frac{N-2}{2}}} \bar{v}\left(\frac{x}{\lambda_0}, \frac{-t+t_0}{\lambda_0^2}\right).$$

Theorem 4.1.2. *For $N \geq 3$, let u be a radial solution to* (1.0.2) *such that $E(u_0) = E(W)$.*

Then, the following holds.

- *If* $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, *then* u *is defined for all times. Moreover, either* u *scatters in both time directions, or* $u = W^-$ *up to the symmetries of the equation.*
- *If* $\|u_0\|_{\dot{H}^1} = \|W\|_{\dot{H}^1}$, *then* $u = W$ *up to the symmetries of the equation.*
- *If* $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, *and* $u_0 \in L^2$, *then either* u *blows-up in finite positive and negative time, or* $u = W^+$ *up to the symmetries of the equation.*

In the intercritical case, the following similar result in the 3d cubic equation was proved by Duyckaerts and Roudenko [25]:

Theorem 4.1.3. *Let* $N = p = 3$. *There exist two radial solutions* Q^+ *and* Q^- *to* (1.0.2) *in* $H^1(\mathbb{R}^N)$ *such that*

- $M[Q^\pm] = M[Q]$, $E[Q^\pm] = E[Q]$, $Q^\pm$ *is defined at least in* $[0, +\infty)$ *and there exist* C, $e_0 > 0$ *such that*

$$\|Q^\pm(t) - Q\|_{H^1} \leq Ce^{-e_0 t}, \text{ for all } t \geq 0,$$

- $\|\nabla Q_0^+\|_2 > \|\nabla Q\|_2$ *and* Q^+ *blows-up in finite negative time,*
- $\|\nabla Q_0^-\|_2 < \|\nabla Q\|_2$ *and* Q^- *is globally defined and scatters backward in time.*

Theorem 4.1.4. *Let* $N = p = 3$, *and* u *be a solution to* (1.0.2) *such that* $M(u_0)E(u_0) = M(Q)E(Q)$. *Then, the following holds.*

- *If* $\|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2}$, *then* u *is defined for all times. Moreover, either* u *scatters in both time directions, or* $u = Q^-$ *up to the symmetries of the equation.*
- *If* $\|\nabla u_0\|_{L^2} = \|\nabla Q\|_{L^2}$, *then* $u = Q$ *up to the symmetries of the equation.*
- *If* $\|\nabla u_0\|_{L^2} > \|\nabla Q\|_{L^2}$, *and* u_0 *is radial or* $|x|u_0 \in L^2$, *then either* u *blows-up in finite positive and negative time, or* $u = Q^+$ *up to the symmetries of the equation.*

The aim of this chapter is to generalize the results in [25] to all possible values of N and p in the intercritical range. Since the proof can be readily applied to energy-critical case, and is considerably different from the proof given in [63] for higher dimensions, we also state and prove our results when $s_c = 1$, for $N \geq 6$. Our main results are the following.

Theorem 4.1.5 (Critical case)**.** *For* $N \geq 6$, *there exist two radial solutions* W^+ *and* W^-

to (1.0.2) *in* $\dot{H}^1(\mathbb{R}^N)$ *such that*

- $E[W^\pm] = E[W]$, $W^\pm$ *is defined at least in* $[0, +\infty)$ *and there exist* C, $e_0 > 0$ *such that*
$$\|W^\pm(t) - W\|_{H^1} \leq Ce^{-e_0 t}, \text{ for all } t \geq 0,$$
- $\|\nabla W_0^+\|_2 > \|\nabla W\|_2$ *and, if* $N \geq 5$, W^+ *blows-up in finite negative time,*
- $\|\nabla W_0^-\|_2 < \|\nabla W\|_2$ *and* W^- *is globally defined and scatters backward in time.*

Theorem 4.1.6 (Critical case)**.** *For* $N \geq 6$, *let* u *be a radial solution to* (1.0.2) *such that* $E(u_0) = E(W)$. *Then, the following holds.*

- *If* $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, *then* u *is defined for all times. Moreover, either* u *scatters in both time directions, or* $u = W^-$ *up to the symmetries of the equation.*
- *If* $\|u_0\|_{\dot{H}^1} = \|W\|_{\dot{H}^1}$, *then* $u = W$ *up to the symmetries of the equation.*
- *If* $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, *and* $u_0 \in L^2$, *then either* u *blows-up in finite positive and negative time, or* $u = W^+$ *up to the symmetries of the equation.*

Theorem 4.1.7 (Intercritical case)**.** *For* $N \geq 1$ *and* $0 < s_c < 1$, *there exist two radial solutions* Q^+ *and* Q^- *to* (1.0.2) *in* $H^1(\mathbb{R}^N)$ *such that*

- $M[Q^\pm] = M[Q]$, $E[Q^\pm] = E[Q]$, $Q^\pm$ *is defined at least in* $[0, +\infty)$ *and there exist* C, $e_0 > 0$ *such that*
$$\|Q^\pm(t) - e^{it}Q\|_{H^1} \leq Ce^{-e_0 t}, \text{ for all } t \geq 0,$$
- $\|\nabla Q_0^+\|_2 > \|\nabla Q\|_2$ *and* Q^+ *blows-up in finite negative time,*
- $\|\nabla Q_0^-\|_2 < \|\nabla Q\|_2$ *and* Q^- *is globally defined and scatters backward in time.*

Theorem 4.1.8 (Intercritical case)**.** *For* $N \geq 1$ *and* $0 < s_c < 1$, *let* u *be a solution to* (1.0.2), *such that* $\mathcal{ME}(u_0) = \mathcal{ME}(Q)$. *Then, the following holds.*

- *If* $\mathcal{MK}(u_0) < \mathcal{MK}(Q)$, *then* u *is defined for all times. Moreover, either* u *scatters in both time directions, or* $u = Q^-$ *up to the symmetries of the equation.*
- *If* $\mathcal{MK}(u_0) = \mathcal{MK}(Q)$, *then* $u = Q$ *up to the symmetries of the equation.*
- *If* $\mathcal{MK}(u_0) > \mathcal{MK}(Q)$ *and* u_0 *is either radial or* $|x|u_0 \in L^2(\mathbb{R}^N)$, *then either* u

blows-up in finite positive and negative time, or $u = Q^+$ up to the symmetries of the equation.

There are two major difficulties on extending the previous results. The first one is to deal with low powers of the parameter p. If $p < 3$, then the nonlinearity $|u|^{p-1}u$ is not a smooth function of $(u, \bar{u})$. Moreover, as the power of the nonlinearity is not an odd integer, the difference $|u|^{p-1}u - |v|^{p-1}v$ cannot be written as a polynomial. Therefore we cannot use the same estimates as in [25], as they rely heavily on $H^s(\mathbb{R}^N)$ estimates, for large values of s. Moreover, if $p \leq 2$, then the nonlinearity is not twice real-differentiable. In the energy-critical case, $p_c \leq 2$ happens exactly when $N \geq 6$. In order to perform the necessary estimates, we employ the fractional calculus tools introduced by Christ and Weinstein [16] and Visan [85]. This approach is different from Li and Zhang [63], which used weighted Sobolev estimates to prove Theorems 4.1.5 and 4.1.6 in the energy-critical setting.

Another problem arises from the fast decay of the ground state Q, for $0 < s_c < 1$. When constructing the solutions $Q^\pm$, we must deal with some estimates that involve terms on the form $\|Q^{-1}f\|_{L^\infty}$. Even though $(Q^{-1}f)(x)$ is pontwise defined for any function f, the exponential decay of Q makes it harder to obtain good estimates. Therefore, we have to carefully study the desired functions f to ensure that they have the necessary decay. We establish the decay via several bootstrap arguments, and by making use of resolvent convolution kernels associated to the corresponding elliptic equations.

It is worth mentioning that, in order to prove the classification results in the intercritical case for all dimensions, one has to change the orthogonality conditions that were used by Duyckaerts and Roudenko [25], as in lower dimensions they would not necessarily ensure coercivity. See Remark 4.3.4 and the proof of Lemma 4.3.5 for details.

Remark 4.1.9. By scaling, the condition $\mathcal{ME}(u_0) = \mathcal{ME}(Q)$ can be read, without loss of generality, as

$$\begin{cases} M(u_0) = M(Q) \\ E(u_0) = E(Q). \end{cases}$$

Indeed, considering $u_{0,\delta}(x) = \delta^{\frac{2}{p-1}} u_0(\delta x)$, with $\delta = (M(u_0)/M(Q))^{\frac{1}{2s_c}}$, gives the above

[0]To be precise, at least $s > N/2$, to make use of the fact that $H^s(\mathbb{R}^N)$ is an algebra.

condition for $u_{0,\delta}$. Similarly, the condition

$$\mathcal{MK}(u_0) < \mathcal{MK}(Q)$$

(resp. "=", ">") can be read as

$$\|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2}$$

(resp. "=", ">"). Unless stated otherwise, we shall adopt this simplification throughout the whole chapter.

4.2 Notation

We make use of the following Strichartz norms, defined separately for the energy-critical and intercritical cases.

Definition 4.2.1 (Critical case). Let I be a (possibly unbounded) time interval. Given $0 < \varepsilon \ll \frac{4}{N-2}$, $N \geq 6$, define the spaces

$$\begin{aligned}
S(\dot{H}^1, I) &= L_I^\infty L_x^{\frac{2N}{N-2}} \\
S(\dot{H}^{1-\varepsilon}, I) &= L_I^\infty L_x^{\frac{2N}{N-2+2\varepsilon}} \cap L_I^{\frac{4}{\varepsilon}} L_x^{\frac{2N}{N-2+\varepsilon}} \cap L_I^{\frac{2(N-2)}{\varepsilon(N-4)}} L_x^{\frac{2N(N-2)}{(N-2)^2+4\varepsilon}}, \\
S'(\dot{H}^{-(1-\varepsilon)}, I) &= L_I^{\frac{2}{\varepsilon}} L_x^{\frac{2N}{N+2}}, \\
S(L^2, I) &= \left\{ L_I^q L_x^r \,\middle|\, (q, r) \text{ is } L^2-\text{admissible} \right\}, \\
S'(L^2, I) &= L_I^2 L_x^{\frac{2N}{N+2}}.
\end{aligned}$$

Remark 4.2.2. In particular, we make use of the following spaces in $S(L^2)$: $L_I^\infty L_x^2$, $L_I^{\frac{4}{\varepsilon}} L_x^{\frac{2N}{N-\varepsilon}}$, $L_I^{\frac{2(N-2)}{\varepsilon(N-4)}} L_x^{\frac{2N(N-2)}{N(N-2)-2\varepsilon(N-4)}}$, $L_I^2 L_x^{\frac{2N}{N-2}}$, $L_I^{\frac{2(N+2)}{N-2}} L_x^{\frac{2(N+2)}{N^2+4}}$, and $L_I^{\frac{16}{\varepsilon(N-2)}} L_x^{\frac{8N}{4N-\varepsilon(N-2)}}$.

Remark 4.2.3. By Sobolev embedding, if $f \in S(\dot{H}^1, I) \cap \nabla^{-1} S(L^2, I)$,

$$\|f\|_{S(\dot{H}^1,I)} + \|D^\varepsilon f\|_{S(\dot{H}^{1-\varepsilon},I)} \lesssim \|\nabla f\|_{S(L^2,I)}.$$

And by Kato-Strichartz estimates,

$$\left\| \int_{s>t} e^{i(t-s)\Delta} F(s)\, ds \right\|_{S(L^2)} \lesssim \|F\|_{S'(L^2)},$$
$$\left\| \int_{s>t} e^{i(t-s)\Delta} F(s)\, ds \right\|_{S(\dot{H}^{1-\varepsilon})} \lesssim \|F\|_{S'(\dot{H}^{-(1-\varepsilon)})}.$$

Remark 4.2.4. Note that the pairs in $S(\dot{H}^1, I)$ are $\dot{H}^1$-admissible, the pairs in $S(\dot{H}^{1-\varepsilon}, I)$ are $\dot{H}^{1-\varepsilon}$-admissible, the pairs in $S(L^2, I)$ and in the dual space of $S'(L^2, I)$ are L^2-admissible, and the pair corresponding to the dual space of $S'(\dot{H}^{-(1-\varepsilon)}, I)$ is $\dot{H}^{-(1-\varepsilon)}$-admissible.

Definition 4.2.5 (Intercritical case)**.** Define the set

$$\mathcal{A}_0 = \Big\{(q,r) | (q,r) \text{ is } L^2\text{-admissible}\Big\}.$$

For $s \in (0,1)$, define $\mathcal{A}_s$ as the $\dot{H}^s$-admissible pairs that satisfy

$$\begin{cases} \frac{2N}{N-2s} \le r \le \left(\frac{2N}{N-2}\right)^- , & N \ge 3, \\ \frac{2}{1-s} \le r \le \left(\left(\frac{2}{1-s}\right)^+\right)' , & N = 2, \\ \frac{2}{1-2s} \le r \le \infty , & N = 1, \end{cases}$$

and $\mathcal{A}_{-s}$ as the $\dot{H}^{-s}$-admissible pairs that satisfy

$$\begin{cases} \left(\frac{2N}{N-2s}\right)^+ \le r \le \left(\frac{2N}{N-2}\right)^- , & N \ge 3, \\ \left(\frac{2}{1-s}\right)^+ \le r \le \left(\left(\frac{2}{1-s}\right)^+\right)' , & N = 2, \\ \left(\frac{2}{1-2s}\right)^+ \le r \le \infty , & N = 1. \end{cases}$$

Let I be a (possibly unbounded) time interval. For $s \in [0,1)$, we define the following Strichartz norms

$$\|u\|_{S(L^2,I)} = \sup_{(q,r)\in\mathcal{A}_0} \|u\|_{L_I^q L_x^r},$$
$$\|u\|_{S(\dot{H}^{s_c},I)} = \sup_{(q,r)\in\mathcal{A}_{s_c}} \|u\|_{L_I^q L_x^r},$$

and the dual Strichartz norms

$$\|u\|_{S'(L^2,I)} = \inf_{(q,r)\in\mathcal{A}_0} \|u\|_{L_I^{q'} L_x^{r'}},$$

$$\|u\|_{S'(\dot{H}^{-s_c},I)} = \inf_{(q,r)\in\mathcal{A}_{-s_c}} \|u\|_{L_I^{q'}L_x^{r'}}.$$

Remark 4.2.6. By Sobolev embedding, if $f \in S(\dot{H}^{s_c},\, I) \cap \langle\nabla\rangle^{-1}S(L^2,\, I)$,

$$\|f\|_{S(\dot{H}^{s_c},\,I)} \lesssim \||\nabla|^{s_c} f\|_{S(L^2,\,I)} \lesssim \|\langle\nabla\rangle f\|_{S(L^2,\,I)}.$$

And by Kato-Strichartz estimates,

$$\left\|\int_{s>t} e^{i(t-s)\Delta}F(s)\,ds\right\|_{S(L^2)} \lesssim \|F\|_{S'(L^2)}\,,$$
$$\left\|\int_{s>t} e^{i(t-s)\Delta}F(s)\,ds\right\|_{S(\dot{H}^{s_c})} \lesssim \|F\|_{S'(\dot{H}^{-s_c})}\,.$$

4.3 The linearized equation

In order to prove the main theorems of this chapter, we need to carefully study NLS (1.0.2) around the ground state. We will often identify the complex number $a + bi$ with the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. Also, if f is a complex-valued function, we will write its real part as f_1, and its imaginary part as f_2, i.e., $f = f_1 + if_2$. We now introduce some definitions.

Definition 4.3.1. For $0 < s_c \leq 1$, we define

$$\begin{aligned} L_+ &:= (1-s_c) - \Delta - pQ^{p-1}, \\ L_- &:= (1-s_c) - \Delta - Q^{p-1}, \\ \mathcal{L} &:= \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \\ R(f) &:= |Q+f|^{p-1}(Q+f) - Q^p - pQ^{p-1}f_1 - iQ^{p-1}f_2, \\ K(f) &:= pQ^{p-1}f_1 + iQ^{p-1}f_2. \end{aligned}$$

If u is a solution to NLS (1.0.2), write $u = e^{i(1-s_c)t}(Q+v)$. Then v must satisfy

$$\partial_t v + \mathcal{L}v = iR(v), \tag{4.3.1}$$

or, writing it as a Schrödinger equation,

$$i\partial_t v + \Delta v - (1 - s_c)v + K(v) = -R(v). \tag{4.3.2}$$

In the next two sections we recall some properties of the operator $\mathcal{L}$.

4.3.1 The linearized operator

We will need the spectral theory for $\mathcal{L}$, as well as estimates on solutions to the linearized equation. For $0 < s_c \leq 1$, we have, by direct calculation,

$$\mathcal{L}(\partial_k Q) = \mathcal{L}(iQ) = 0, \quad 1 \leq k \leq N.$$

Also, defining Q_1 as $\frac{2}{p-1}Q + x \cdot \nabla Q$, we have

$$\mathcal{L}(Q_1) = -2(1 - s_c)Q.$$

Note that, in the energy-critical case, $\mathcal{L}(W_1) = 0$. These directions are obtained from Q by the symmetries of the NLS equation. Indeed, defining

$$f_{[x_0,\lambda_0,\theta_0]}(x) = e^{i\theta_0}\frac{1}{\lambda_0^{\frac{2}{p-1}}}f\left(\frac{x}{\lambda_0} + x_0\right),$$

we have

$$(\nabla Q, Q_1, iQ) = \left.\frac{\partial Q_{[x_0,\lambda_0,\theta_0]}}{\partial(x_0, \lambda_0, \theta_0)}\right|_{(x_0,\lambda_0,\theta_0)=(0,1,0)}.$$

The following result is well-known and will be proved in Section 4.10.

Lemma 4.3.2 (see [24],[15])**.** *Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$, defined in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ and let $\sigma_{ess}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{ess}(\mathcal{L}) = \{iy; y \in \mathbb{R}, |y| \geq 1 - s_c\}, \quad \sigma \cap \mathbb{R} = \{-e_0, 0, e_0\} \quad \textit{with } e_0 > 0.$$

Moreover, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions $\mathcal{Y}_+$ and $\mathcal{Y}_- = \overline{\mathcal{Y}}_+ \in \mathcal{S}$, respectively. The null space of $\mathcal{L}$ is spanned by iQ and $\partial_k Q$, $1 \leq k \leq N$ (and, in the energy-critical case, also by W_1).

Remark 4.3.3. By Lemma 4.3.2, if $\mathcal{Y}_1 = \mathrm{Re}(\mathcal{Y}_+)$ and $\mathcal{Y}_2 = \mathrm{Im}(\mathcal{Y}_+)$, then

$$L_+\mathcal{Y}_1 = e_0\mathcal{Y}_2 \quad \text{and } L_-\mathcal{Y}_2 = -e_0\mathcal{Y}_1.$$

Furthermore, the null space of L_+ is spanned by the vectors $\partial_k Q$, $k \leq N$ (and by W_1, if $s_c = 1$) and the null space of L_- is spanned by Q.

Consider the bilinear form

$$\begin{aligned} B(f,g) &:= \frac{1}{2}(L_+f_1, g_1) + \frac{1}{2}(L_-f_2, g_2) \\ &= \frac{1-s_c}{2}\int f_1 \cdot g_1 + \frac{1}{2}\int \nabla f_1 \cdot \nabla g_1 - \frac{p}{2}\int Q^{p-1} f_1 g_1 + \\ &\quad + \frac{1-s_c}{2}\int f_2 \cdot g_2 + \frac{1}{2}\int \nabla f_2 \cdot \nabla g_2 - \frac{1}{2}\int Q^{p-1} f_2 g_2, \end{aligned}$$

and define the *linearized energy*

$$\begin{aligned} \Phi(f) := B(f,f) &= \frac{1}{2}(L_+f_1, f_1) + \frac{1}{2}(L_-f_2, f_2) \\ &= \frac{1-s_c}{2}\int |\nabla f|^2 + \frac{1}{2}\int |\nabla f|^2 - \frac{1}{2}\int Q^{p-1}(p|f_1|^2 + |f_2|^2). \end{aligned}$$

If $0 < s_c \leq 1$, one can check directly that, for any f, $g \in S(\mathbb{R}^N)$,

$$\begin{gathered} B(f,g) = B(g,f), \\ B(\mathcal{L}f, g) = -B(f, \mathcal{L}g), \\ B(iQ, f) = 0, \\ B(\partial_k Q, f) = 0,\ 1 \leq k \leq N, \\ B(Q_1, f) = -\frac{(1-s_c)(p-1)}{2}\int Q^p f_1, \\ \Phi(\mathcal{Y}_+) = \Phi(\mathcal{Y}_-) = 0. \end{gathered} \tag{4.3.3}$$

In the energy-critical case, note that we have

$$B(W_1, f) = 0.$$

If $0 < s_c < 1$, consider the following orthogonality relations in the real Hilbert space $L^2(\mathbb{R}^N, \mathbb{C})$

$$\int Qv_2 = \int \partial_k Q v_1 = 0, \quad 1 \leq k \leq N, \tag{4.3.4}$$

$$\int Q^p v_1 = 0, \tag{4.3.5}$$

$$\int \mathcal{Y}_1 v_2 = \int \mathcal{Y}_2 v_1 = 0. \tag{4.3.6}$$

Denote by $G^\perp$ the set of $v \in H^1$ satisfying (4.3.4) and (4.3.5), and $\tilde{G}^\perp$ the set of $v \in H^1$ satisfying (4.3.4) and (4.3.6).

Remark 4.3.4. Differently than Duyckaerts and Roudenko [25], we use the orthogonality condition (4.3.5) instead of $\int \Delta Q v_1 = 0$. We make this choice in order to be able to prove coercivity in all dimensions, specially in dimension $N = 1$.

By direct calculations, one sees that

$$\Phi_{|\text{span}\{\nabla Q, iQ\}} = 0$$

and

$$\Phi(Q) = -\frac{p+1}{2} \int Q^{p+1} < 0. \tag{4.3.7}$$

If $s_c = 1$, consider the mutually orthogonal directions W, iW, $W_1 = \frac{N-2}{2} W + x \cdot W$ and $1 \leq \partial_k W$, $1 \leq k \leq N$ in the real Hilbert space $\dot{H}^1 = \dot{H}^1(\mathbb{R}^N, \mathbb{C})$. Denote by $G := \text{span}\{W, \nabla W, iW, W_1\}$ and by $G^\perp$ its orthogonal complement in $\dot{H}^1$ for the canonical scalar product, which is given by

$$(f, g)_{\dot{H}^1} = \int \nabla f_1 \cdot \nabla g_1 + \int \nabla f_2 \cdot \nabla g_2 = \text{Re} \int \nabla f \cdot \nabla \overline{g}.$$

Let $\tilde{G}^\perp$ be the set $\left\{ v \in \dot{H}^1;\, v \perp \text{span}\{\nabla W, iW, W_1\},\, B(\mathcal{Y}_+, v) = B(\mathcal{Y}_-, v) = 0 \right\}$.

By direct calculations, one sees that

$$\Phi_{|\text{span}\{\nabla W, iW, W_1\}} = 0$$

and

$$\Phi(W) = -\frac{2}{(N-2)C_N^N} < 0, \tag{4.3.8}$$

where C_N is the sharp constant for Sobolev inequality for the embedding $\dot{H}^1 \hookrightarrow L^{\frac{2N}{N-2}}$. The following lemma shows that Φ is coercive in $G^\perp \cup \tilde{G}^\perp$.

Lemma 4.3.5. *For $0 < s_c \leq 1$, there is a constant $\tilde{c} > 0$ such that, for any $f \in G^\perp \cup \tilde{G}^\perp$*

$$\Phi(f) \geq \tilde{c}\|f\|^2_{\dot{H}^1}.$$

This result is well-known, and its proof will be given in Section 4.10.

We now prove results for the ground state in the case $0 < s_c < 1$. For convenience, from now on we rescale Q in the intercritical case as to solve

$$\Delta Q - Q + Q^p = 0.$$

This is in order to simplify the exposition, avoiding unnecessary parameters in the calculations. The term $(1 - s_c)$ must be replaced by 1 in the definition of $\mathcal{L}$ and in the standing wave solution $e^{i(1-s_c)t}Q$ as well.

Unlike the energy-critical case, the ground state decays exponentially if $0 < s_c < 1$. In the next sections, we need sharp bounds on the decay of Q and its derivatives. We start recalling the following result, proved by Gidas, Ni and Nirenberg.

Lemma 4.3.6 (See Gidas et al. [40, Theorem 2, p. 370])**.** *For $N \geq 1$, $1 + \frac{4}{N} \leq p < 2^* - 1$, let $Q \in S(\mathbb{R}^N)$ be the radial, positive solution of the equation*

$$\Delta Q - Q + Q^p = 0.$$

Then there exists $C > 0$ such that

$$\lim_{|x|\to+\infty} |x|^{\frac{N-1}{2}} e^{|x|} Q(x) = C.$$

We next study the decay of solutions

Lemma 4.3.7. *Let $f \in S(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$. If f solves*

$$(1 - \Delta + \lambda i) f = G,$$

with

$$|G(x)| \lesssim \frac{e^{-a|x|}}{\left(1 + |x|^{\frac{N-1}{2}}\right)^b},$$

for $0 < a \neq \operatorname{Re}\sqrt{1+\lambda i}$, $0 < b \neq 1$, then

$$|f(x)| \lesssim \frac{1}{\left(1 + |x|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}} \left(e^{-|x|}\right)^{\min\{a, \operatorname{Re}\sqrt{1+\lambda i}\}}.$$

Proof. Let $c = \operatorname{Re}\sqrt{1+\lambda i} \geq 1$. We recall the integral form of the resolvent (see [1])

$$(1 - \Delta + \lambda i)^{-1} G = K * G,$$

where $K \in L^1(\mathbb{R}^N)$ is such that, for $|x| \gg 1$,

$$K(x) \lesssim \frac{e^{-c|x|}}{1 + |x|^{\frac{N-1}{2}}}, \tag{4.3.9}$$

and, for $|x| \ll 1$,

$$K(x) \lesssim \begin{cases} \frac{1}{|x|^{\frac{N-1}{2}}} & \text{for } N > 2, \\ \ln \frac{1}{|x|} & \text{for } N = 2, \\ 1 & \text{for } N < 2. \end{cases} \tag{4.3.10}$$

Consider first the case $0 < a < c$. We estimate

$$|K * G(x)| \lesssim \int K(y) \frac{e^{-a|x-y|}}{\left(1 + |x-y|^{\frac{N-1}{2}}\right)^b}$$

$$\lesssim \frac{e^{-a|x|}}{\left(1+|x|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}} \int K(y)e^{a|y|} \frac{\left(1+|x-y|^{\frac{N-1}{2}}+|y|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}}{\left(1+|x-y|^{\frac{N-1}{2}}\right)^{b}}$$
$$\lesssim \frac{e^{-a|x|}}{\left(1+|x|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}} \int K(y)e^{a|y|} \left(1+|y|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}.$$

By (4.3.9) and (4.3.10), the integral in the last inequality is $O(1)$. For $a > c$, the estimate is

$$|K * G(x)| \lesssim \frac{e^{-c|x|}}{\left(1+|x|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}} \int K(y)e^{c|y|} \left(1+|y|^{\frac{N-1}{2}}\right)^{\min\{b,1\}} e^{-(a-c)|x-y|}$$
$$\lesssim \frac{e^{-c|x|}}{\left(1+|x|^{\frac{N-1}{2}}\right)^{\min\{b,1\}}} \left[\int e^{-(a-c)|x-y|} + \int_{|y|\leq 1} K(y)e^{c|y|} \left(1+|y|^{\frac{N-1}{2}}\right)\right].$$

Since the first integral in the last inequality is bounded uniformly in x, the lemma is proved. □

Corollary 4.3.8. *The following estimates hold, for any multi-index* $\alpha \in \mathbb{Z}_+^N$.

(i) $\|Q^{-1}\partial^\alpha Q\|_{L^\infty} < +\infty$,

(ii) $\|Q^{-1}e^{\eta|x|}\partial^\alpha \mathcal{Y}_\pm\|_{L^\infty} < +\infty$, *for some* $0 < \eta \ll 1$,

(iii) $\|Q^{-1}e^{\eta|x|}\partial^\alpha[(\mathcal{L}-\lambda)^{-1}f]\|_{L^\infty} < +\infty$, *for every* $\lambda \in \mathbb{R}\backslash\sigma(\mathcal{L})$ *and every* $f \in S(\mathbb{R}^N)$ *such that* $\|Q^{-1}e^{\eta|x|}\partial^\beta f\|_{L^\infty} < +\infty$ *for some* $0 < \eta < \mathrm{Re}(\sqrt{1+\lambda i})$ *and any* $\beta \in \mathbb{Z}_+^N$.

Proof. We first remark that Q is strictly positive, and thus Q^{-1} is well-defined. Recalling Lemma 4.3.6, we have, for all x,

$$Q(x) \approx \frac{e^{-|x|}}{1+|x|^{\frac{N-1}{2}}}.$$

We differentiate (2.6.1) to obtain

$$(1-\Delta)\nabla Q = pQ^{p-1}\nabla Q.$$

Since $Q \in S$, by Lemma 4.3.7 and a bootstrap argument, we conclude (i) for $|\alpha| = 1$. By repeatedly differentiating 2.6.1 and repeating the argument, we conclude (i) for any α.

To prove (ii), recall the differential equation for $\mathcal{Y}_1 = \mathrm{Re}(\mathcal{Y}_+)$

$$(1-\Delta-pQ^{p-1})(1-\Delta-Q^{p-1})\mathcal{Y}_1 = -e_0^2\mathcal{Y}_1.$$

By factoring $[(1-\Delta)^2+e_0^2] = (1-\Delta+ie_0)(1-\Delta-ie_0)$ and using item (i), this equation can be rewritten as

$$(1-\Delta+ie_0)(1-\Delta-ie_0)\mathcal{Y}_1 = G_2(\mathcal{Y}_1),$$

where we define $G_k(f)$ as a linear function on f and its derivatives up to order k that satisfies, for any $k \geq 1$,

$$|G_k(f)| \lesssim Q^{p-1}\sum_{|\alpha|\leq k}|\partial^\alpha f|.$$

Writing $g = (1-\Delta-ie_0)\mathcal{Y}_1$, we have, for any multi-indices α, β,

$$\begin{cases}(1-\Delta+ie_0)\partial^\alpha g = G_{|\alpha|+2}(\mathcal{Y}_1)\\ (1-\Delta-ie_0)\partial^\beta\mathcal{Y}_1 = \partial^\beta g.\end{cases}$$

Therefore, using Lemma 4.3.7 and bootstrapping, we prove that $Q^{-1}e^{\eta|x|}\partial^\alpha\mathcal{Y}_1 \in L^\infty$ for any α, where $0<\eta\ll \mathrm{Re}(\sqrt{1+ie_0})-1$. The estimate on $\mathcal{Y}_2$ is entirely analogous, and hence (ii) holds. We now turn to estimate (iii). If $g = (\mathcal{L}-\lambda)^{-1}f$, then, for any α,

$$\begin{cases}-\partial^\alpha(1-\Delta-Q^{p-1})g_2 - \lambda\partial^\alpha g_1 = \partial^\alpha f_1\\ \partial^\alpha(1-\Delta-pQ^{p-1})g_1 - \lambda\partial^\alpha g_2 = \partial^\alpha f_2.\end{cases}$$

We can rewrite this system as

$$[(1-\Delta)^2+\lambda^2]\partial^\alpha g_1 = G_{|\alpha|+2}(g_1) + H_{|\alpha|+2}(f),$$

where we define $H_k(f)$ as a linear function on f and its derivatives up to order k that satisfies, for any $k\geq 1$,

$$|H_k(f)| \lesssim \sum_{|\alpha|\leq k}|\partial^\alpha f|.$$

By bootstrapping similarly to the previous items, and noting that the argument to g_2 is analogous, we finish the proof of Lemma 4.3.7. $\square$

4.3.2 Estimates on the linearized equation

We now prove some estimates that will be used in the next sections. We start with estimates for the energy-critical case.

Lemma 4.3.9 (Preliminar estimates). *Let $s_c = 1$, $N \geq 6$ (and then $p_c - 1 \leq 1$), $0 < \epsilon \ll \frac{4}{N-2}$ and I be a bounded time interval with $|I| \leq 1$, and consider $f, g \in S(\dot{H}^1, I)$ such that $\nabla f, \nabla g \in S(L^2, I)$. The following estimates hold.*

(i) $\|\nabla K(f)\|_{S'(L^2,I)} \lesssim |I|^{\frac{1}{2}} \|\nabla f\|_{S(L^2,I)}$,

(ii) $$\|\nabla(R(f) - R(g))\|_{S'(L^2,I)} \lesssim \|\nabla(f-g)\|_{S(L^2,I)} \left(\|\nabla f\|^{p_c-1}_{S(L^2,I)} + \|\nabla g\|^{p_c-1}_{S(L^2,I)}\right) + \|D^\varepsilon(f-g)\|^{p_c-1}_{S(\dot{H}^{1-\varepsilon},I)} \left(\|\nabla f\|_{S(L^2,I)} + \|\nabla g\|_{S(L^2,I)}\right).$$

If $N > 6$, then also

(iii) $\|D^\varepsilon K(f)\|_{S'(\dot{H}^{-(1-\varepsilon)},I)} \lesssim |I|^{\frac{\varepsilon}{N-2}} \|D^\varepsilon f\|_{S(\dot{H}^{1-\varepsilon},I)}$,

(iv) $\|D^\varepsilon(R(f) - R(g))\|_{S'(\dot{H}^{-(1-\varepsilon)},I)} \lesssim \|D^\varepsilon(f-g)\|_{S(\dot{H}^{1-\varepsilon},I)} \left(\|\nabla f\|^{p_c-1}_{S(L^2,I)} + \|\nabla g\|^{p_c-1}_{S(L^2,I)}\right)$.

Remark 4.3.10. It is necessary to treat the case $N > 6$ differently due to the low power of the nonlinearity. If $N \leq 6$, then it is possible to estimate $\|\nabla(R(f) - R(g))\|_{S'(L^2,I)}$ at least linearly in terms of $\|\nabla(f-g)\|_{S(L^2,I)}$. In higher dimensions, one of the terms must be in the form $\|\nabla(f-g)\|^{p_c-1}_{S(L^2,I)}$, which is not good enough for the fixed-point argument carried on in the next section. The use of less that one derivative enables us to keep the desired linearity.

Proof of Lemma 4.3.9. We start by proving the following claim:

Claim 4.3.11. *Let H be a map such that $H(0) = 0$ and $|H(f) - H(g)| \leq C|f-g|^{\frac{4}{N-2}}$ for all functions $f, g : \mathbb{R}^N \to \mathbb{C}$, $N > 6$. Then, for all $f, g \in S(\dot{H}^1, I) \cap \nabla^{-1} S(L^2, I)$,*

$$\|D^\varepsilon(H(f)g)\|_{L_I^{\frac{2}{\varepsilon}} L_x^{\frac{2N}{N+2}}} \lesssim \|\nabla f\|^{\frac{4}{N-2}}_{\frac{16}{\varepsilon(N-2)}, \frac{8N}{4N-\varepsilon(N-2)}} \|D^\varepsilon g\|_{\frac{4}{\varepsilon}, \frac{2N}{N-2+\varepsilon}} + \|\nabla f\|^{\frac{4}{N-2}}_{\frac{4}{\varepsilon}, \frac{2N}{N-\varepsilon}} \|D^\varepsilon g\|_{\frac{2(N-2)}{\varepsilon(N-4)}, \frac{2N(N-2)}{(N-2)^2+4\varepsilon}}.$$

In other words,

$$\|D^\varepsilon(H(f)g)\|_{S'(\dot{H}^{-(1-\varepsilon)},I)} \lesssim \|\nabla f\|^{\frac{4}{N-2}}_{S(L^2,I)} \|D^\varepsilon g\|_{S(\dot{H}^{1-\varepsilon},I)}.$$

Proof of Claim 4.3.11. By Leibniz Rule (2.2.7) and Holder's inequality, we can write

$$\begin{aligned}\|D^\varepsilon(H(f)g)\|_{L_I^{\frac{2}{\varepsilon}}L_x^{\frac{2N}{N+2}}} &\lesssim \left\| \|D^\varepsilon H(f)\|_{L_x^{\frac{2N}{4+\varepsilon}}} \|g\|_{L_x^{\frac{2N}{N-2-\varepsilon}}} \right\|_{L_I^{\frac{2}{\varepsilon}}} \\ &\quad + \left\| \|H(f)\|_{L_x^{\frac{N(N-2)}{2(N-2-\varepsilon)}}} \|D^\varepsilon g\|_{L_x^{\frac{2N(N-2)}{(N-2)^2+4\varepsilon}}} \right\|_{L_I^{\frac{2}{\varepsilon}}} \\ &\lesssim \|D^\varepsilon H(f)\|_{L_I^{\frac{4}{\varepsilon}}L_x^{\frac{2N}{4+\varepsilon}}} \|g\|_{L_I^{\frac{4}{\varepsilon}}L_x^{\frac{2N}{N-2-\varepsilon}}} \\ &\quad + \|H(f)\|_{L_I^{\frac{N-2}{\varepsilon}}L_x^{\frac{N(N-2)}{2(N-2-\varepsilon)}}} \|D^\varepsilon g\|_{L_I^{\frac{2(N-2)}{\varepsilon(N-4)}}L_x^{\frac{2N(N-2)}{(N-2)^2+4\varepsilon}}}.\end{aligned}$$

By Sobolev inequality,

$$\|g\|_{L_I^{\frac{4}{\varepsilon}}L_x^{\frac{2N}{N-2-\varepsilon}}} \lesssim \|D^\varepsilon g\|_{L_I^{\frac{4}{\varepsilon}}L_x^{\frac{2N}{N-2+\varepsilon}}},$$

and since, by assumption on H, $|H(f)| \lesssim |f|^{\frac{4}{N-2}}$, we have, by Sobolev,

$$\|H(f)\|_{L_I^{\frac{N-2}{\varepsilon}}L_x^{\frac{N(N-2)}{2(N-2-\varepsilon)}}} \lesssim \|f\|^{\frac{4}{N-2}}_{\frac{4}{\varepsilon},\frac{2N}{N-2-\varepsilon}} \lesssim \|\nabla f\|^{\frac{4}{N-2}}_{\frac{4}{\varepsilon},\frac{2N}{N-\varepsilon}}.$$

It remains to estimate $\|D^\varepsilon H(f)\|_{L_x^{\frac{2N}{4+\varepsilon}}}$. Choosing ν such that $\frac{(N-2)\varepsilon}{4} < \nu < 1$, by fractional chain rule[1] (2.2.8) and Sobolev embeddings, we have

$$\begin{aligned}\|D^\varepsilon H(f)\|_{L_x^{\frac{2N}{4+\varepsilon}}} &\lesssim \|f\|^{\frac{4}{N-2}-\frac{\varepsilon}{\nu}}_{L_x^{\left(\frac{4}{N-2}-\frac{\varepsilon}{\nu}\right)p_1}} \|D^\nu f\|^{\frac{\varepsilon}{\nu}}_{L_x^{\frac{\varepsilon}{\nu}q_1}} \\ &\lesssim \|\nabla f\|^{\frac{4}{N-2}-\frac{\varepsilon}{\nu}}_{L_x^{p_2}} \|\nabla f\|^{\frac{\varepsilon}{\nu}}_{L_x^{q_2}},\end{aligned}$$

where we choose $p_2 = q_2 = \frac{8N}{4N-\varepsilon(N-2)} \in (1,+\infty)$, and p_1 and q_1 must satisfy

$$1 < p_1, q_1 < \infty,$$

$$\frac{1}{p_2} = \frac{1}{\left(\frac{4}{N-2}-\frac{\varepsilon}{\nu}\right)p_1} + \frac{1}{N}, \quad \frac{1}{q_2} = \frac{1}{\frac{\varepsilon}{\nu}q_1} + \frac{1-\nu}{N},$$

and

$$\left(1 - \frac{\varepsilon(N-2)}{4\nu}\right)p_1 > 1.$$

[1]That is where the hypothesis $N > 6$ is used, as the fractional chain rule requires $0 < 4/(N-2) < 1$.

These conditions can be easily satisfied if ε is small enough (depending only on the dimension). The claim is now proved. $\square$

Estimate (iii) of Lemma 4.3.9 follows directly from Sobolev inequality and Claim 4.3.11, by taking $H(W) = |W|^{\frac{4}{N-2}}$, and from the fact that $|\nabla W| \in L^2_x \cap L^\infty_x$, if $N > 6$.

To prove (iv), note that $R(f) = W^{p_c} J(W^{-1}f)$, where $J(z) = |1+z|^{p_c-1}(1+z) - 1 - \frac{p_c+1}{2} z - \frac{p_c-1}{2}\bar{z}$ is $C^1(\mathbb{C})$. Its derivatives J_z and $J_{\bar{z}}$ satisfy $J_z(0) = J_{\bar{z}}(0) = 0$ and, if $N > 6$, are Hölder continuous of order $p_c - 1 < 1$. Therefore, writing

$$R(f)-R(g) = W^{p_c-1}\int_0^1 J_z(W^{-1}(sf+(1-s)g))(f-g)+J_{\bar{z}}(W^{-1}(sf+(1-s)g))(\overline{f-g})ds, \tag{4.3.11}$$

we can apply Claim 4.3.11 to estimate each term in (4.3.11), taking $H(f) = W^{p_c-1}J_z(W^{-1}f)$ or $H(f) = W^{p_c-1}J_{\bar{z}}(W^{-1}f)$. Estimate (iv) then follows directly.

To prove (i), we write

$$|\nabla K(f)| \lesssim |W|^{p_c-2}|\nabla W||f| + |W|^{p_c-1}|\nabla f|.$$

Using the fact that $|\partial^\alpha W(x)| \le C_\alpha |W(x)|$ for every multi-index $\alpha \in \mathbb{Z}^N_+$ and all x, we have, by Hölder inequality

$$\begin{aligned}\|\nabla K(f)\|_{L^2_I L_x^{\frac{2N}{N+2}}} &\lesssim \left\| \|W\|^{\frac{4}{N-2}}_{L_x^{\frac{2N}{N-2}}} \|f\|_{L_x^{\frac{2N}{N-2}}} \right\|_{L^2_t} + \left\| \|W\|^{\frac{4}{N-2}}_{L_x^{\frac{4N}{N-2}}} \|\nabla f\|_{L^2_x} \right\|_{L^2_t} \\ &\lesssim |I|^{\frac{1}{2}} \left(\|f\|_{L^\infty_t L_x^{\frac{2N}{N-2}}} + \|\nabla f\|_{L^\infty_t L^2_x} \right).\end{aligned}$$

Note that we used that $W \in L_x^{\frac{2N}{N-2}} \cap L_x^{\frac{4N}{N-2}}$, which follows from the fact that $W \in L^2_x \cap L^\infty_x$, if $N > 6$. The inequality follows from Sobolev embedding.

We finally turn to estimate (ii). Write

$$\begin{aligned}\nabla(R(f)-R(g)) = &\underbrace{p_c W^{p_c-1}\nabla W(J(W^{-1}f) - J(W^{-1}g))}_{(a)} \\ &+ \underbrace{W^{p_c-1}J_z(W^{-1}f)\nabla f - W^{p_c-1}J_z(W^{-1}g)\nabla g}_{(b)}\end{aligned}$$

$$+\underbrace{W^{p_c-1}J_{\bar{z}}(W^{-1}f)\nabla\bar{f}-W^{p_c-1}J_{\bar{z}}(W^{-1}g)\nabla\bar{g}}_{(c)}$$
$$+\underbrace{W^{p_c-2}\nabla W J_z(W^{-1}f)f-W^{p_c-2}\nabla W J_z(W^{-1}g)g}_{(d)}$$
$$+\underbrace{W^{p_c-2}\nabla W J_{\bar{z}}(W^{-1}f)\bar{f}-W^{p_c-2}\nabla W J_{\bar{z}}(W^{-1}g)\bar{g}}_{(e)}$$

To estimate (a), note that

$$\begin{aligned}|(a)| &\lesssim W^{p_c-1}\int_0^1 |J_z(W^{-1}(sf+(1-s)g))(f-g)+J_{\bar{z}}(W^{-1}(sf+(1-s)g))(\overline{f-g})|ds\\ &\lesssim \left(|f|^{p_c-1}+|g|^{p_c-1}\right)|f-g|.\end{aligned}$$

Thus, by the Hölder and Sobolev inequalities,

$$\begin{aligned}\|(a)\|_{L_I^2L_x^{\frac{2N}{N+2}}} &\lesssim |I|^{\frac{1}{2}}\left(\|f\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2}}}+\|g\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2}}}\right)\|f-g\|_{L_I^\infty L_x^{\frac{2N}{N-2}}}\\ &\lesssim |I|^{\frac{1}{2}}\left(\|\nabla f\|^{p_c-1}_{L_I^\infty L_x^2}+\|\nabla g\|^{p_c-1}_{L_I^\infty L_x^2}\right)\|\nabla(f-g)\|_{L_I^\infty L_x^2}.\end{aligned}$$

We now estimate (b). By triangle inequality,

$$\begin{aligned}|(b)| &\le W^{p_c-1}|J_z(W^{-1}f)||\nabla f-\nabla g|+W^{p_c-1}|J_z(W^{-1}f)-J_z(W^{-1}g)||\nabla g|\\ &\le |f|^{p_c-1}|\nabla(f-g)|+|f-g|^{p_c-1}|\nabla g|.\end{aligned}$$

So that, by the Hölder and Sobolev inequalities,

$$\begin{aligned}\|(b)\|_{L_I^2L_x^{\frac{2N}{N+2}}} &\lesssim \|f\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2}}}\|\nabla(f-g)\|_{L_I^2L_x^{\frac{2N}{N-2}}}+\|f-g\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2}}}\|\nabla g\|_{L_I^2L_x^{\frac{2N}{N-2}}}\\ &\lesssim \|\nabla f\|^{p_c-1}_{L_I^\infty L_x^2}\|\nabla(f-g)\|_{L_I^2L_x^{\frac{2N}{N-2}}}+\|D^\varepsilon(f-g)\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2+2\varepsilon}}}\|\nabla g\|_{L_I^2L_x^{\frac{2N}{N-2}}}.\end{aligned}$$

The estimate for (c) is analogous. To estimate (d), we write

$$\begin{aligned}|(d)| &\leq W^{p_c-1}|J_z(W^{-1}f)||f-g| + W^{p_c-1}|J_z(W^{-1}f) - J_z(W^{-1}g)||g| \\ &\leq |f|^{p_c-1}|f-g| + |f-g|^{p_c-1}|g|.\end{aligned}$$

Therefore, by Hölder and Sobolev,

$$\begin{aligned}\|(d)\|_{L_I^2 L_x^{\frac{2N}{N+2}}} &\lesssim |I|^{\frac{1}{2}}\|f\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2}}}\|f-g\|_{L_I^\infty L_x^{\frac{2N}{N-2}}} + |I|^{\frac{1}{2}}\|f-g\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2}}}\|g\|_{L_I^\infty L_x^{\frac{2N}{N-2}}} \\ &\lesssim |I|^{\frac{1}{2}}\|\nabla f\|^{p_c-1}_{L_I^\infty L_x^2}\|\nabla(f-g)\|_{L_I^\infty L_x^2} + |I|^{\frac{1}{2}}\|D^\varepsilon(f-g)\|^{p_c-1}_{L_I^\infty L_x^{\frac{2N}{N-2+2\varepsilon}}}\|\nabla g\|_{L_I^\infty L_x^2}.\end{aligned}$$

Since the estimate for (e) is analogous, the proof of Lemma 4.3.9 is complete. □

The following Strichartz-type continuity argument follows from Lemma 4.3.9 and will be useful on proving the main results of this chapter.

Lemma 4.3.12. *Let h be a solution to (4.3.2). If, for some $c>0$, and all $t>0$,*

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-ct}, \tag{4.3.12}$$

then, for all $t>0$,

$$\|\nabla h\|_{S(L^2,\,[t,+\infty))} \lesssim e^{-ct}. \tag{4.3.13}$$

Proof. Differentiating (4.3.2), we get

$$i\partial_t(\nabla h) + \Delta(\nabla h) + \nabla(K(h) + R(h)) = 0.$$

By Duhamel formula, Strichartz estimates and items (i) and (ii) of Lemma 4.3.9, if $0<\tau<1$,

$$\|\nabla h\|_{S(L^2,\,[t,t+\tau])} \lesssim \|h(t)\|_{\dot{H}^1} + \tau^{\frac{1}{2}}\|\nabla h\|_{S(L^2,\,[t,t+\tau])} + \|\nabla h\|^{p_c}_{S(L^2,\,[t,t+\tau])}.$$

By (4.3.12), we get, for some $K > 0$,

$$\|\nabla h\|_{S(L^2,\,[t,t+\tau])} \leq K(e^{-ct} + \tau^{\frac{1}{2}}\|\nabla h\|_{S(L^2,\,[t,t+\tau])} + \|\nabla h\|^{p_c}_{S(L^2,\,[t,t+\tau])}). \tag{4.3.14}$$

This implies, for large t,

$$\|\nabla h\|_{S(L^2,\,[t,t+\tau_0])} < 2Ke^{-ct}, \quad \tau_0 = \frac{1}{9K^2}.$$

Indeed, assume by contradiction that there exists $\tau \in (0, \tau_0]$ such that $\|h\|_{S(L^2,\,[t,t+\tau])} = 2Ke^{-ct}$, for fixed $t > 0$. Then, by (4.3.14),

$$2Ke^{-ct} \leq Ke^{-ct} + 2K^2\tau^{\frac{1}{2}}e^{-ct} + (2K)^{p_c}Ke^{-cp_ct} \leq \frac{5}{3}Ke^{-ct} + (2K)^{p_c}Ke^{-cp_ct},$$

which is a contradiction if t is large. Therefore, by decomposing $[t, +\infty) = \bigcup_{j=0}^{\infty}[t + j\tau_0, t + (j+1)\tau_0]$ and using the triangle inequality, we see that (4.3.13) holds. □

The following lemma is the intercritical version of Lemma 4.3.9, and its proof is analogous.

Lemma 4.3.13 (Preliminar estimates, subcritical case)**.** *Let $0 < s_c < 1$ and I be a bounded time interval such that $|I| \leq 1$, and consider $f, g \in S(L^2,\, I)$ such that $\nabla f, \nabla g \in S(L^2,\, I)$. There exists $\alpha > 0$ such that the following estimates hold.*

For $p > 1$:

(i) $\|\langle\nabla\rangle K(f)\|_{S'(L^2,I)} \lesssim |I|^\alpha \|\langle\nabla\rangle f\|_{S(L^2,I)}$,

(ii) $\|K(f)\|_{S'(\dot{H}^{-s_c},I)} \lesssim |I|^\alpha \|f\|_{S(\dot{H}^{s_c},I)}$.

For $p > 2$:

(iii) $$\|\langle\nabla\rangle(R(f) - R(g))\|_{S'(L^2,I)} \lesssim \|\langle\nabla\rangle(f-g)\|_{S(L^2,I)}\left[\|\langle\nabla\rangle f\|_{S(L^2,I)} + \|\langle\nabla\rangle g\|_{S(L^2,I)} + \|\langle\nabla\rangle f\|^{p-1}_{S(L^2,I)} + \|\langle\nabla\rangle g\|^{p-1}_{S(L^2,I)}\right],$$

(iv) $$\|R(f) - R(g)\|_{S'(\dot{H}^{-s_c},I)} \lesssim \|f-g\|_{S(\dot{H}^{s_c},I)}\left[\|f\|_{S(\dot{H}^{s_c},I)} + \|g\|_{S(\dot{H}^{s_c},I)} + \|f\|^{p-1}_{S(\dot{H}^{s_c},I)} + \|g\|^{p-1}_{S(\dot{H}^{s_c},I)}\right],$$

For $1 < p \leq 2$:

(v) $$\|\langle\nabla\rangle(R(f) - R(g))\|_{S'(L^2,I)} \lesssim \|\langle\nabla\rangle(f-g)\|_{S(L^2,I)}\left(\|f\|^{p-1}_{S(\dot{H}^{s_c},I)} + \|g\|^{p-1}_{S(\dot{H}^{s_c},I)}\right) + \|f-g\|^{p-1}_{S(\dot{H}^{s_c},I)}\left(\|\langle\nabla\rangle f\|_{S(L^2,I)} + \|\langle\nabla\rangle g\|_{S(L^2,I)}\right),$$

(vi) $\|R(f) - R(g)\|_{S'(\dot{H}^{-s_c}, I)} \lesssim \|f - g\|_{S(\dot{H}^{s_c}, I)} \left(\|f\|^{p-1}_{S(\dot{H}^{s_c}, I)} + \|g\|^{p-1}_{S(\dot{H}^{s_c}, I)}\right)$.

Proof. The estimates are very similar as the ones in the proof of the energy-critical case. We use the following classical inequalities

$$\||a|^{p-1}b\|_{S'(L^2)} \leq \|a\|^{p-1}_{S(\dot{H}^{s_c})}\|b\|_{S(L^2)} \lesssim \|\langle\nabla\rangle a\|^{p-1}_{S(L^2)}\|b\|_{S(L^2)},$$

and

$$\||a|^{p-1}b\|_{S'(\dot{H}^{-s_c})} \leq \|a\|^{p-1}_{S(\dot{H}^{s_c})}\|b\|_{S(\dot{H}^{s_c})} \lesssim \|\langle\nabla\rangle a\|^{p-1}_{S(L^2)}\|\langle\nabla\rangle b\|_{S(L^2)},$$

which can be verified using the pairs $\left(\frac{4(p+1)}{N(p-1)}, p+1\right) \in \mathcal{A}_0$, $\left(\frac{2(p-1)(p+1)}{4-(N-2)(p-1)}, p+1\right) \in \mathcal{A}_{s_c}$, and $\left(\frac{2(p-1)(p+1)}{(p-1)(Np-2)-4}, p+1\right) \in \mathcal{A}_{-s_c}$, together with Sobolev inequality. Let us estimate, for example, $\|\nabla(R(f) - R(g))\|_{S'(L^2,I)}$. Write

$$\begin{aligned}
\nabla(R(f) - R(g)) = &\underbrace{pQ^{p-1}\nabla Q(J(Q^{-1}f) - J(Q^{-1}g))}_{(a)} \\
&+ \underbrace{Q^{p-1}J_z(Q^{-1}f)\nabla f - Q^{p-1}J_z(Q^{-1}g)\nabla g}_{(b)} \\
&+ \underbrace{Q^{p-1}J_{\bar{z}}(Q^{-1}f)\nabla\bar{f} - Q^{p-1}J_{\bar{z}}(Q^{-1}g)\nabla\bar{g}}_{(c)} \\
&+ \underbrace{Q^{p-2}\nabla QJ_z(Q^{-1}f)f - Q^{p_c-2}\nabla QJ_z(Q^{-1}g)g}_{(d)} \\
&+ \underbrace{Q^{p-2}\nabla QJ_{\bar{z}}(Q^{-1}f)\bar{f} - Q^{p-2}\nabla QJ_{\bar{z}}(Q^{-1}g)\bar{g}}_{(e)}
\end{aligned}$$

Making use of $|\nabla Q| \lesssim Q$ (which follows from Corollary 4.3.8), we write (a) as

$$|(a)| \lesssim Q^{p-1}\int_0^1 |J_z(Q^{-1}(sf + (1-s)g))(f-g) + J_{\bar{z}}(Q^{-1}(sf + (1-s)g))(\overline{f-g})|ds.$$

Now, since

$$|J_z(z_1) - J_z(z_2)| + |J_{\bar{z}}(z_1) - J_{\bar{z}}(z_2)| \lesssim \begin{cases} |z_1 - z_2|(1 + |z_1|^{p-2} + |z_2|^{p-2}), & p \geq 2, \\ |z_1 - z_2|^{p-1}, & 1 < p < 2, \end{cases}$$

we have

$$|(a)| \lesssim \begin{cases} (Q^{p-2}|f| + Q^{p-2}|g| + |f|^{p-1} + |g|^{p-1})|f-g|, & p \geq 2, \\ (|f|^{p-1} + |g|^{p-1})|f-g|, & 1 < p < 2. \end{cases}$$

Thus, since $Q \in \mathcal{S}(\mathbb{R}^N)$ and $|I| \leq 1$,

$$\|(a)\|_{S'(L^2,I)} \lesssim \begin{cases} (\|f\|_{S(\dot{H}^{s_c},I)} + \|g\|_{S(\dot{H}^{s_c},I)} \\ \qquad + \|f\|^{p-1}_{S(\dot{H}^{s_c},I)} + \|g\|^{p-1}_{S(\dot{H}^{s_c},I)})\|f-g\|_{S'(L^2,I)}, & p \geq 2, \\ (\|f\|^{p-1}_{S(\dot{H}^{s_c},I)} + \|g\|^{p-1}_{S(\dot{H}^{s_c},I)})\|f-g\|_{S'(L^2,I)}, & 1 < p < 2. \end{cases}$$

We also have

$$\|(b)\|_{S'(L^2,I)} + \|(d)\|_{S'(L^2,I)} \lesssim \begin{cases} \|\nabla(f-g)\|_{S(L^2,I)}(\|\langle\nabla\rangle f\|_{S(L^2,I)} + \|\langle\nabla\rangle f\|^{p-1}_{S(L^2,I)} \\ \qquad + \|\langle\nabla\rangle g\|_{S(L^2,I)} + \|\langle\nabla\rangle g\|^{p-1}_{S(L^2,I)}), \\ \qquad\qquad p \geq 2, \\ \|\langle\nabla\rangle(f-g)\|_{S(L^2,\,I)} \left(\|f\|^{p-1}_{S(\dot{H}^{s_c},\,I)} + \|g\|^{p-1}_{S(\dot{H}^{s_c},\,I)}\right) \\ \qquad + \|f-g\|^{p-1}_{S(\dot{H}^{s_c},\,I)} \left(\|\langle\nabla\rangle f\|_{S(L^2,\,I)} + \|\langle\nabla\rangle g\|_{S(L^2,\,I)}\right), \\ \qquad\qquad 1 < p < 2, \end{cases}$$

with the same bounds for (d) and (e). □

Remark 4.3.14. We do not employ the same estimates as Duyckaerts and Roudenko [25], since the nonlinearity $|u|^{p-1}u$ is not a polynomial in $(u, \bar{u})$ if p is not an odd integer. Therefore, instead of using H^s estimates, we rely on $S(L^2)$ and $S(\dot{H}^{s_c})$ estimates, that are more suitable to generalizing the result to all possible dimensions and powers of the nonlinearity.

Remark 4.3.15. We also employ a different approach than Li and Zhang [63], that divide all estimates in regions where $|f| > W$ or $|f| \leq W$. Instead, we use fractional derivatives to avoid some sublinear estimates, resulting in a simpler proof.

4.4 Construction of special solutions

In this section, we construct *special* solutions to NLS (1.0.2), in the sense that they are on the same energy level of the ground state, converge to the standing wave in $\dot{H}^1$ as $t \to +\infty$, but have kinetic energy different from $\|\nabla Q\|_{L^2}$.

4.4.0.1 Construction of a family of approximate solutions

We start with a proposition that was first proved by Duyckaerts and Merle in [24], for $s_c = 1$. We extend here their proof to the intercritical case. The main difference from the energy-critical case is that Q decays exponentially if $0 < s_c < 1$, so we need to be careful with its spatial decay, as we make use of estimates of the type $\|Q^{-1}f\|_{L^\infty}$. To this end, we make use of the sharp decay estimate for Q given by 4.3.6 and of the control on the spatial decay given by Corollary 4.3.8.

Proposition 4.4.1. *Let $0 < s_c \leq 1$ and $A \in \mathbb{R}$. There exists a sequence $(Z_k^A)_{k\geq 1}$ of functions in $\mathcal{S}(\mathbb{R}^N)$ such that $Z_1^A = A\mathcal{Y}_+$ and, if $k \geq 1$ and $\mathcal{V}_k^A = \sum_{j=1}^k e^{-je_0 t} Z_j^A$, then as $t \to +\infty$ we have*

$$\partial_t \mathcal{V}_k^A + \mathcal{L}\mathcal{V}_k^A = iR(\mathcal{V}_k^A) + O\left(e^{-(k+1)e_0 t}\right) \ \text{in } \mathcal{S}(\mathbb{R}^N), \tag{4.4.1}$$

where $\mathcal{L}$ and R are given in Definition 4.3.1.

Proof. We prove this proposition by induction. For simplicity, we often omit the superscript A.

Define $Z_1 = A\mathcal{Y}_+$ and $\mathcal{V}_1 = e^{-e_0 t} Z_1$. Thus

$$\partial_t \mathcal{V}_1 + \mathcal{L}\mathcal{V}_1 - iR(\mathcal{V}_1) = -iR(\mathcal{V}_1).$$

Note now that $R(f) = Q^p J(Q^{-1}f)$, where $J(z) = |1+z|^{p-1}(1+z) - 1 - \frac{p+1}{2}z - \frac{p-1}{2}\bar{z}$ is real-analytic in the disc $\{z; |z| < 1\}$, and satisfies $J(0) = \partial_z J(0) = \partial_{\bar{z}} J(0) = 0$. Write its Taylor expansion as

$$J(z) = \sum_{i+j\geq 2} a_{ij} z^i \bar{z}^j \tag{4.4.2}$$

with normal convergence of the series and all of its derivatives in the compact disc $\{z; |z| \leq \frac{1}{2}\}$.

Now, if $s_c = 1$, since $Z_1 \in \mathcal{S}(\mathbb{R}^N)$ and W decays polynomially, we have that $\|W^{-1}Z_1\|_{L^\infty} < +\infty$. For $0 < s_c < 1$, we make use of the Corollary 4.3.8.(ii), to conclude that $\|Q^{-1}Z_1\|_{L^\infty} < +\infty$. In any case, we can choose t_0 such that $|\mathcal{V}_1(t)| \leq \frac{1}{2}Q$, for any $t \geq t_0$. Therefore, for large t, we have

$$|R(\mathcal{V}_1)| \leq \|Q\|^p_{L^\infty} \left(\sum_{i+j\geq 0} |a_{ij}| \frac{1}{2^{i+j}} \right) |Q^{-1}\mathcal{V}_1|^2 = C|Q^{-1}\mathcal{V}_1|^2.$$

In the same fashion, we can use Leibiniz rule, equation (4.4.2) and items (i) and (ii) of Corollary 4.3.8 to bound all the derivatives of $R(\mathcal{V}_1)$. Using that $\mathcal{V}_1 = e^{-e_0 t}Z_1$, we conclude that $R(\mathcal{V}_1) = O(e^{-2e_0 t})$ in $\mathcal{S}(\mathbb{R}^N)$. Moreover, by Corollary 4.3.8.(ii), we have $\|Q^{-1}e^{\eta|x|}\partial^\alpha Z_1\|_{L^\infty} < +\infty$.

Now let $k \geq 1$ and assume that $\mathcal{V}_i$ is defined and satisfy (4.4.1) for all $i \leq k$. For $0 < s_c < 1$, assume furthermore that, for all $i \leq k$, and all α,

$$\|Q^{-1}e^{\eta|x|}\partial^\alpha Z_i\|_{L^\infty} < +\infty. \tag{4.4.3}$$

Defining

$$\epsilon_k = \partial_t \mathcal{V}_k + \mathcal{L}\mathcal{V}_k - R(\mathcal{V}_k), \tag{4.4.4}$$

note that

$$\partial_t \mathcal{V}_k = \sum_{j=1}^{k} (-je_0)e^{-je_0 t} Z_k,$$

so that (4.4.4) can be written as

$$\epsilon_k(x,t) = \sum_{j=1}^{k} e^{-je_0 t} \left(-je_0 Z_k(x) + \mathcal{L}Z_k(x)\right) - R(\mathcal{V}_k(x,t)). \tag{4.4.5}$$

Recall that, for all k, $Z_k \in \mathcal{S}(\mathbb{R}^N)$. If $0 < s_c < 1$, we also have (4.4.3). Therefore, for large t, and all x, $|\mathcal{V}_k(x,t)| \leq \frac{1}{2}Q(x)$. Writing $R(\mathcal{V}_k) = Q^p J(Q^{-1}\mathcal{V}_k)$ and using again the expansion (4.4.2), we get by (4.4.5) that there exist functions $F_j \in \mathcal{S}(\mathbb{R}^N)$ such that for large t

$$\epsilon_k(x,t) = \sum_{j=1}^{k+1} e^{-je_0 t} F_j(x) + O(e^{-e_0(k+2)t}) \text{ in } \mathcal{S}(\mathbb{R}^N).$$

By (4.4.1), we conclude that $F_j = 0$ for $j \leq k$, which shows

$$\epsilon_k(x,t) = e^{-(k+1)e_0 t} F_{k+1} + O(e^{-(k+2)e_0 t}). \tag{4.4.6}$$

Noting that $(k+1)e_0$ is not in the spectrum of $\mathcal{L}$, define $Z_{k+1} = -(\mathcal{L} + (k+1)e_0)^{-1} F_{k+1}$, which belongs to $\mathcal{S}$ (see Section 4.10). Moreover, if $0 < s_c < 1$, Z_{k+1} satisfies (4.4.3) with k replaced by $k+1$. By definition, we have $\mathcal{V}_{k+1} = \mathcal{V}_k + e^{-(k+1)e_0 t} Z_{k+1}$. Furthermore,

$$\epsilon_{k+1} = \epsilon_k - e^{-(k+1)e_0 t} F_{k+1} - i(R(\mathcal{V}_{k+1}) - R(\mathcal{V}_k)).$$

By (4.4.6), $\epsilon_k - e^{-(k+1)e_0 t} F_{k+1} = O(e^{-(k+2)e_0 t})$. Writing again $R(f) = Q^p J(Q^{-1} f)$, and using the expansion (4.4.2), we conclude that $R(\mathcal{V}_{k+1}) - R(\mathcal{V}_k) = O(e^{-(k+2)e_0 t})$. The proof is complete. □

4.4.1 Contraction argument near an approximate solution

We now prove the key result of this subsection. The propositions are stated for the energy-critical and for the intercritical cases separately.

4.4.1.1 Energy-critical case

We only treat here the case $N \geq 6$, as in the lower-dimensional cases this result is proved in [24]. The main difference here from [24] is that $0 < p_c - 1 < 1$ if $N > 6$, so that the nonlinearity is no longer C^2, and its derivative is only Hölder-continuous of order $p_c - 1$. This introduces difficulties, as the control of the convergence of ∇U^A to ∇W is not enough to close the contraction argument, and we need to ensure that the higher order terms $D^\varepsilon (U^A - W - \mathcal{V}_k)$ converges faster to 0, for a small $\varepsilon > 0$. The fractional derivative D^ε is needed here to avoid certain end-point Strichartz estimates, which are not proved for any combination of $\dot{H}^1$-admissible and $\dot{H}^{-1}$-admissible pairs.

Proposition 4.4.2. *Let $N \geq 6$. There exists $k_0 > 0$ such that for any $k \geq k_0$, there exists*

$t_k \geq 0$ and a solution U^A to (1.0.2) such that for $t \geq t_k$ and $l(k) = \left\lceil \frac{N-2}{4}k + \frac{N-6}{4} \right\rceil$,

$$\begin{aligned} \|D^\varepsilon(U^A - W - \mathcal{V}^A_{l(k)})\|_{S(\dot{H}^{1-\varepsilon},\,[t,+\infty))} &\leq e^{-(k+\frac{1}{2})\frac{N-2}{4}e_0 t}, \text{ and} \\ \|\nabla(U^A - W - \mathcal{V}^A_{l(k)})\|_{S(L^2,\,[t,+\infty))} &\leq e^{-(k+\frac{1}{2})e_0 t}. \end{aligned} \tag{4.4.7}$$

Furthermore, U^A is the unique solution to (1.0.2) satisfying (4.4.7) for large t. Finally, U^A is independent of k and satisfies for large t,

$$\|U^A(t) - W - Ae^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} \leq e^{-2e_0 t}. \tag{4.4.8}$$

Proof. Since $A \in \mathbb{R}$ will be fixed in the proof, we will omit the superscripts A. Define

$$h = U^A - W - \mathcal{V}^A_{l(k)},$$

so that U^A is a solution to (1.0.2) if, and only if, h satisfies

$$i\partial_t h + \Delta h = -K(h) - (R(\mathcal{V}_{l(k)} + h) - R(\mathcal{V}_{l(k)})) + i\epsilon_{l(k)},$$

where $\epsilon_{l(k)} = O\left(e^{-(l(k)+1)e_0 t}\right)$ in $\mathcal{S}(\mathbb{R}^N)$ for all $k \geq 0$. Therefore, the existence of U^A can be written as the fixed-point problem

$$h(t) = \mathcal{M}(h)(t),$$

where

$$\mathcal{M}(h)(t) = -i\int_t^{+\infty} e^{i(t-s)\Delta}\left[-K(h) - (R(\mathcal{V}_{l(k)} + h) - R(\mathcal{V}_{l(k)})) + i\epsilon_{l(k)}\right].$$

Let first $N > 6$. We will show that $\mathcal{M}$ is a contraction on B defined by

$$\begin{aligned} \|h\|_E &:= \sup_{t\geq t_k} e^{\left(k+\frac{1}{2}\right)\frac{N-2}{4}e_0 t}\|D^\varepsilon h\|_{S(\dot{H}^{1-\varepsilon},\,[t,+\infty))} + \sup_{t\geq t_k} e^{\left(k+\frac{1}{2}\right)e_0 t}\|\nabla h\|_{S(L^2,\,[t,+\infty))}, \\ E = E(k,t_k) &:= \Big\{h \in S(\dot{H}^1,\,[t_k,+\infty)), D^\varepsilon h \in S(\dot{H}^{1-\varepsilon}\,[t_k,+\infty)), \\ &\qquad \nabla h \in S(L^2,\,[t_k,+\infty)); \|h\|_E < +\infty\Big\}, \\ B = B(k,t_k) &:= \{h \in E; \|h\|_E \leq 1\}, \end{aligned}$$

equipped with the metric

$$\rho(u,v) = \sup_{t\geq t_k} e^{\left(k+\frac{1}{2}\right)\frac{N-2}{4}e_0 t}\|D^{\varepsilon}(u-v)\|_{S(\dot{H}^{1-\varepsilon},\,[t,+\infty))}.$$

Let $\{h_n\} \subset B$ and $h \in S(\dot{H}^{1-\varepsilon}, I)$, with $\rho(h_n, h) \to 0$. By reflexiveness and uniqueness between weak and strong limits, $h \in B$. This shows that (B, d) is a complete metric space. We will show that $\mathcal{M}(B) \subset B$ and that $\mathcal{M}$ is a contraction.

By Strichartz estimates, there exists $C^* > 0$ such that

$$\begin{aligned}\|\nabla \mathcal{M}(h)\|_{S(L^2,\,[t,+\infty))} \leq C^* \Big[&\|\nabla K(h)\|_{S'(L^2,\,[t,+\infty))} \\ &+\|\nabla(R(\mathcal{V}_{l(k)}+h) - R(\mathcal{V}_{l(k)}))\|_{S'(L^2,\,[t,+\infty))} + \|\nabla \epsilon_{l(k)}\|_{S'(L^2,\,[t,+\infty))}\Big],\end{aligned} \tag{4.4.9}$$

$$\begin{aligned}\|D^{\varepsilon}\mathcal{M}(h)\|_{S(\dot{H}^{1-\varepsilon},\,[t,+\infty))} \leq C^* \Big[&\|D^{\varepsilon}K(h)\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))} \\ &+ \|D^{\varepsilon}(R(\mathcal{V}_{l(k)}+h) - R(\mathcal{V}_{l(k)}))\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))} \\ &+ \|D^{\varepsilon}\epsilon_{l(k)}\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))}\Big],\end{aligned} \tag{4.4.10}$$

and

$$\begin{aligned}\|D^{\varepsilon}(\mathcal{M}(g) - \mathcal{M}(h))\|_{S(\dot{H}^{1-\varepsilon},\,[t,+\infty))} \leq C^* \Big[&\|D^{\varepsilon}K(g-h)\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))} \\ &+ \|D^{\varepsilon}(R(\mathcal{V}_{l(k)}+g) - R(\mathcal{V}_{l(k)}+h))\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))}\Big].\end{aligned} \tag{4.4.11}$$

To finish the argument, we just need the following estimates.

Lemma 4.4.3. *For every $\eta > 0$, there exists $\tilde{k}(\eta) > 0$ such that, if $k \geq \tilde{k}(\eta)$, then for any $g,\ h \in B$ the following inequalities hold for all $t \geq t_k$.*

(i) $\|\nabla K(h)\|_{S'(L^2,\,[t,+\infty))} \leq \eta e^{-\left(k+\frac{1}{2}\right)e_0 t}\|h\|_E$,

(ii) $\|\nabla(R(\mathcal{V}_{l(k)}+h) - R(\mathcal{V}_{l(k)}))\|_{S'(L^2,\,[t,+\infty))} \leq C_k e^{-\left(k+\frac{1}{2}+\frac{4}{N-2}\right)e_0 t}$,

(iii) $\|D^{\varepsilon}K(h)\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))} \le \eta e^{-\left(k+\frac{1}{2}\right)\frac{N-2}{4}e_0 t}\rho(h,0)$,

(iv) $\|D^{\varepsilon}(R(\mathcal{V}_{l(k)}+g)-R(\mathcal{V}_{l(k)}+h))\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))} \le C_k e^{-\left(k+\frac{1}{2}+\frac{16}{(N-2)^2}\right)\frac{N-2}{4}e_0 t}\rho(g,h)$,

(v) $\|\nabla\epsilon_{l(k)}\|_{S'(L^2,\,[t,+\infty))} + \|D^{\varepsilon}\epsilon_{l(k)}\|_{S'(\dot{H}^{-(1-\varepsilon)},\,[t,+\infty))} \le C_k e^{-(k+1)\frac{N-2}{4}e_0 t}$.

Indeed, assuming this lemma, choosing first $\eta > 0$ small enough, and then a large enough t_k, we see by (4.4.9) and (4.4.10) that $\mathcal{M}(B) \subset B$. Moreover, by (4.4.11), $\mathcal{M}$ is a contraction on B. Thus, for every $k \ge k_0 = \tilde{k}(\eta)$, there is a unique solution U^A to (1.0.2) satisfying (4.4.7) for $t \ge t_k$. Note that the uniqueness still holds in the class of solutions to (1.0.2) satisfying (4.4.7) for $t \ge t'_k$, where $t'_k \ge t_k$. Thus, uniqueness of solutions to (1.0.2) shows that U^A does not depend on k.

It remains to show Lemma 4.4.3. By Lemma 4.3.9.(i), if $\tau_0 > 0$ and $t \ge t_k$, then

$$\|\nabla K(h)\|_{S'(L^2,\,[t,t+\tau_0])} \le C\tau_0^{\frac{1}{2}} e^{-\left(k+\frac{1}{2}\right)e_0 t}\|h\|_E.$$

Summing up this equation at times $t_j = t + j\tau_0$, and using triangle inequality, we get a geometric series, whose sum is

$$\|\nabla K(h)\|_{S'(L^2,\,[t,+\infty))} \le C\frac{\tau_0^{\frac{1}{2}}}{1-e^{-\left(k+\frac{1}{2}\right)\frac{4}{N-2}e_0\tau_0}} e^{-\left(k+\frac{1}{2}\right)e_0 t}\|h\|_E.$$

Choosing τ_0 and k_0 such that $\tau_0^{\frac{1}{2}} = \frac{\eta}{2C}$ and $e^{-\left(k+\frac{1}{2}\right)e_0\tau_0} \le \frac{1}{2}$, we get estimate (i) of Lemma 4.4.3. Estimate (iii) follows analogously from Lemma 4.3.9.(iii).

We now turn to item (ii). By Lemma 4.3.9, item (ii), we have

$$\|\nabla(R(\mathcal{V}_{l(k)}+h)-R(\mathcal{V}_{l(k)}))\|_{S'(L^2,\,[t,t+1])} \le (A)\|\nabla h\|_{S(L^2,\,[t,t+1])} + (B)\|D^{\varepsilon}h\|^{\frac{4}{N-2}}_{S(\dot{H}^{1-\varepsilon},\,[t,t+1])},$$

where $(A) \lesssim \|\nabla\mathcal{V}_{l(k)}\|^{\frac{4}{N-2}}_{S(L^2,\,[t,t+1])} + \|\nabla h\|^{\frac{4}{N-2}}_{S(L^2,\,[t,t+1])}$ and $(B) \lesssim \|\nabla\mathcal{V}_{l(k)}\|_{S(L^2,\,[t,t+1])} + \|\nabla h\|_{S(L^2,\,[t,t+1])}$. By the explicit form of $\mathcal{V}_k$ and the fact that $h \in B$, we get

$$(A)+(B) \le C_k e^{-e_0\frac{4}{N-2}t}.$$

Therefore,

$$\|(R(\mathcal{V}_{l(k)}+h)-R(\mathcal{V}_{l(k)}))\|_{S'(L^2,\,[t,t+1])} \leq C_k e^{-e_0\frac{4}{N-2}t}(\|\nabla h\|_{S(L^2,\,[t,t+1])}+\|D^\varepsilon h\|_{S(\dot{H}^{1-\varepsilon},\,[t,t+1])}^{\frac{4}{N-2}})$$

$$\leq C_k e^{-\left(k+\frac{1}{2}+\frac{4}{N-2}\right)e_0 t}.$$

Since $h\in B$, triangle inequality and the sum of the resulting geometric series gives us (ii). As for item (iv), it follows analogously from Lemma 4.3.9.(iv). Estimate (v) of Lemma 4.4.3 follows immediately from (4.4.6) and from the bound

$$l(k)+1\geq (k+1)\,\frac{N-2}{4}.$$

Finally, given that $U^A=W+\mathcal{V}_k+h$, with $h\in B$, we see that, for large k,

$$\|\nabla h(t)\|_{L^2}\leq e^{-\frac{5}{2}e_0 t}\|h\|_E\leq e^{-\frac{5}{2}e_0 t},$$

and recalling the definition of $\mathcal{V}_k$ given in Proposition 4.4.1, we have, for all k,

$$\mathcal{V}_{l(k)}=Ae^{-e_0 t}\mathcal{Y}_+ + O(e^{-2e_0 t}) \text{ in } \mathcal{S}(\mathbb{R}^N),$$

which proves (4.4.8), and finishes the case $N>6$.

For the case $N=6$, note that $\frac{N-2}{4}=1$, so that, by Sobolev embedding, only the space $S(L^2,I)$ is enough for the contraction argument. Therefore, defining the space B as

$$\|h\|_E := \sup_{t\geq t_k} e^{\left(k+\frac{1}{2}\right)e_0 t}\|\nabla h\|_{S(L^2,\,[t,+\infty))},$$

$$E=E(k,t_k):=\Big\{h\in C_t\dot{H}^1([t_k,+\infty))\cap S(\dot{H}^1,\,[t_k,+\infty)),$$

$$\nabla h\in S(L^2,\,[t_k,+\infty));\,\|h\|_E<+\infty\Big\},$$

$$B=B(k,t_k):=\{h\in E;\,\|h\|_E\leq 1\},$$

equipped with the metric

$$\rho(u,v) = \sup_{t \geq t_k} e^{\left(k+\frac{1}{2}\right)e_0 t} \|\nabla(u-v)\|_{S(L^2,\,[t,+\infty))},$$

we see, by Lemma 4.3.9, that the analogous estimates of Lemma 4.4.3 hold. Hence, the conclusion of Proposition 4.4.2 also holds for $N = 6$, finishing its proof.

□

4.4.1.2 Intercritical case

Proposition 4.4.4. *There exists $k_0 > 0$ such that for any $k \geq k_0$, there exists $t_k \geq 0$ and a solution U^A to NLS (1.0.2) such that for $t \geq t_k$ and $l(k) = \max\{\left\lceil \frac{k+1}{p-1} - 1 \right\rceil, k\}$,*

$$\|U^A - e^{it}Q - e^{it}\mathcal{V}^A_{l(k)}\|_{S(\dot{H}^{s_c},[t,+\infty))} \leq e^{-(k+\frac{1}{2})\max\{\frac{1}{p-1},1\}e_0 t}, \text{ and} \tag{4.4.12}$$

$$\|\langle\nabla\rangle(U^A - e^{it}Q - e^{it}\mathcal{V}^A_{l(k)})\|_{Z(t,+\infty)} \leq e^{-(k+\frac{1}{2})e_0 t}. \tag{4.4.13}$$

Furthermore, U^A is the unique solution to NLS satisfying (4.4.13) for large t. Finally, U^A is independent of k and satisfies for large t,

$$\|U^A(t) - e^{it}Q - Ae^{-e_0 t + it}\mathcal{Y}_+\|_{H^1} \leq e^{-2e_0 t}$$

In view of Lemma 4.3.13, the proof of Proposition 4.4.4 is essentially the same as in the energy-critical case, and we state Lemma 4.4.5 below for completeness. Note that (4.4.12) is a consequence of (4.4.13) in the case $p \geq 2$, due to the Sobolev inequalities.

Lemma 4.4.5. *For every $\eta > 0$, there exists $\tilde{k}(\eta) > 0$ such that, if $k \geq \tilde{k}(\eta)$, then for any g, $h \in B$ the following inequalities hold for all $t \geq t_k$.*

(i) $\|\nabla K(h)\|_{S'(L^2,\,[t,+\infty))} \leq \eta e^{-\left(k+\frac{1}{2}\right)e_0 t}\|h\|_E$,

(ii) $\|\nabla(R(\mathcal{V}_{l(k)} + h) - R(\mathcal{V}_{l(k)}))\|_{S'(L^2,\,[t,+\infty))} \leq C_k e^{-\min\{p-1,1\}e_0 t} e^{-\left(k+\frac{1}{2}\right)e_0 t}$,

(iii) $\|K(h)\|_{S'(\dot{H}^{-s_c},\,[t,+\infty))} \leq \eta e^{-\left(k+\frac{1}{2}\right)\max\{\frac{1}{p-1},1\}e_0 t}\rho(h,0)$,

(iv) $\|(R(\mathcal{V}_{l(k)}+g) - R(\mathcal{V}_{l(k)}+h))\|_{S'(\dot{H}^{-s_c},\,[t,+\infty))} \leq C_k e^{-\min\{p-1,1\}e_0 t} e^{-\left(k+\frac{1}{2}\right)\max\{\frac{1}{p-1},1\}e_0 t}\rho(g,h)$,

(v) $\|\nabla \epsilon_{l(k)}\|_{S'(L^2,[t,+\infty))} + \|\epsilon_{l(k)}\|_{S'(\dot{H}^{-s_c},[t,+\infty))} \leq C_k e^{-(k+1)\max\{\frac{1}{p-1},1\}e_0 t}$.

4.5 Modulation

Throughout the rest of the chapter, we will write, for $0 < s_c \leq 1$,

$$d(f) = \left| \|\nabla f\|_{L^2} - \|\nabla Q\|_{L^2} \right|.$$

If u is a solution to (1.0.2) and if there is no risk of ambiguity, we will also often write

$$d(t) = d(u(t)).$$

4.5.1 Energy-critical case

The variational characterization of W [4, 81, 67] shows that, if $E(f) = E(W)$, then

$$\inf_{\substack{x\in\mathbb{R}^N\\ \lambda>0\\ \theta\in\mathbb{R}}} \|f_{[x,\lambda,\theta]} - W\|_{\dot{H}^1} \leq \epsilon(d(f)),$$

with $\epsilon = \epsilon(\delta)$ such that

$$\lim_{\delta\to 0^+} \epsilon(\delta) = 0.$$

The goal of this section is to construct modulation parameters x_0, λ_0 and θ_0 such that the quantity $d(f)$ controls linearly $\|f_{[x_0,\lambda_0,\theta_0]} - W\|_{\dot{H}^1}$ as well as the parameters and its derivatives. By making use of the Implicit Function Theorem, we can construct appropriate modulation parameters. The proof of the next two lemmas is very similar to the ones in [24], with the introduction of a translation parameter, and will be given in Section 4.10.

Lemma 4.5.1. *There exist $\delta_0 > 0$ and a positive function $\epsilon(d)$ defined for $0 < d < \delta_0$, which tends to 0 as d tends to 0, such that, for all $f \in \dot{H}^1$ satisfying $E(f) = E(W)$ and $d(f) < \delta_0$, there existss (x, λ, θ) such that*

$$\|f_{[x,\lambda,\theta]} - W\|_{\dot{H}^1} \leq \epsilon(d(u)),$$

$$f_{[x,\lambda,\theta]} \perp span\{\nabla W, iW, W_1\}.$$

The parameters (x,λ,θ) are unique in $\mathbb{R}^N \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}_+$ and the mapping $u \mapsto (x,\lambda,\theta)$ is C^1.

Let u be a solution to (1.0.2) and I be a time interval such that

$$d(t) < \delta_0 \text{ for all } t \in I.$$

For each $t \in I$, choose the parameters $(x(t),\lambda(t),\theta(t))$ according to Lemma 4.5.1. Write

$$u_{[x(t),\lambda(t),\theta(t)]}(t) = (1+\alpha(t))W + h(t), \tag{4.5.1}$$

where

$$\alpha(t) = \frac{1}{\|W\|_{\dot{H}^1}}(u_{[x(t),\lambda(t),\theta(t)]}, W)_{\dot{H}^1} - 1.$$

Note that $\alpha(t)$ is chosen so that $h(t) \in G^{\perp}$. By Lemma 4.5.1 and a standard regularization argument (see, for instance [70, Lemma 4] for details in a similar context), the map $t \mapsto (x(t),\lambda(t),\theta(t))$ is C^1. We are now able to prove estimates on the modulation.

Lemma 4.5.2. *Let u be a solution to (1.0.2) satisfying $E(u_0) = E(W)$. Taking a smaller δ_0, if necessary, the following estimates hold on I:*

$$|\alpha(t)| \approx \|h(t)\|_{\dot{H}^1} \approx d(t) < \delta_0 \tag{4.5.2}$$

$$|\alpha'(t)| + |x'(t)| + |\theta'(t)| + \left|\frac{\lambda'(t)}{\lambda(t)}\right| \lesssim \lambda^2(t)d(t). \tag{4.5.3}$$

In the next two sections, we mainly consider *radial* solutions to (1.0.2). Since the translation parameter is not needed, we write

$$f_{[\lambda_0,\theta_0]}(x) = e^{i\theta_0}\frac{1}{\lambda_0^{\frac{N-2}{2}}} f\left(\frac{x}{\lambda_0}\right).$$

In some results regarding compactness, the parameter θ_0 can also be omitted. In this case, we write

$$f_{[\lambda_0]}(x) = \frac{1}{\lambda_0^{\frac{N-2}{2}}} f\left(\frac{x}{\lambda_0}\right).$$

For solutions to the intercritical NLS, since the scaling parameter is fixed (see Remark 4.1.9), we use the notation

$$f_{[x_0,\theta_0]}(x) = e^{i\theta_0} f(x + x_0).$$

When θ_0 can be omitted, we write

$$f_{[x_0]}(x) = f(x + x_0).$$

4.5.2 Intercritical case

For $0 < s_c < 1$, the variational characterization of Q [66] shows that, if $M(f) = M(Q)$ and $E(f) = E(Q)$, then

$$\inf_{\substack{x\in\mathbb{R}^N \\ \theta\in\mathbb{R}}} \|f_{[x,\theta]} - Q\|_{H^1} \leq \epsilon(d(f)), \tag{4.5.4}$$

with

$$\lim_{\delta\to 0^+} \epsilon(\delta) = 0.$$

As in the previous subsection, the goal of here is to construct modulation parameters x_0 and θ_0 such that the quantity $d(f)$ controls linearly $\|f_{[x_0,\theta_0]} - Q\|_{H^1}$, as well as the parameters and its derivatives. We follow mainly [25] here.

Lemma 4.5.3. *There exist $\delta_0 > 0$ and a positive function $\epsilon(d)$ defined for $0 < d < \delta_0$, which tends to 0 as d tends to 0, such that, for all $f \in H^1$ satisfying $M(f) = M(Q)$, $E(f) = E(Q)$ and $d(f) < \delta_0$, there exist (x, θ) such that*

$$\|f_{[x,\theta]} - Q\|_{H^1} \leq \epsilon(d(f)),$$
$$f_{[x,\theta]} \perp \textit{span}\{\nabla Q, iQ\}.$$

The parameters (x, θ) are unique in $\mathbb{R}^N \times \mathbb{R}/2\pi\mathbb{Z}$ and the mapping $u \mapsto (x, \theta)$ is C^1.

Let u be a solution to NLS (1.0.2) and I be a time interval such that $d(t) := d(u(t)) < \delta_0$ for all $t \in I$. For each $t \in I$, choose the parameters $(x(t), \theta(t))$ according to Lemma 4.5.3. Write

$$u_{[x(t),\theta(t)]}(t) = (1 + \alpha(t))Q + h(t), \tag{4.5.5}$$

where

$$\alpha(t) = \frac{\mathrm{Re}(e^{-it} \int \nabla u_{[x(t),\theta(t)]} \cdot \nabla Q)}{\|\nabla Q\|_{L^2}^2} - 1.$$

Note that $\alpha(t)$ is chosen so that $h(t) \in G^{\perp}$. Recall that the parameters (x, θ) are C^1. We are now able to prove estimates on the modulation.

Lemma 4.5.4. *Let u be a solution to (1.0.2) satifying $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$. Taking a smaller δ_0, if necessary, the following estimates hold on I:*

$$|\alpha(t)| \approx \|h(t)\|_{H^1} \approx \left| \int Q h_1 \right| \approx d(t), \tag{4.5.6}$$
$$|\alpha'(t)| \approx |x'(t)| \approx |\theta'(t)| \lesssim d(t).$$

We finish this section with a lemma that will be useful in the next sections, in the intercritical case.

Lemma 4.5.5. *Let u be a solution to (1.0.2) such that $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$. Assume that u is defined on $[0, +\infty)$ and that there exists $c > 0$ such that*

$$\int_t^{+\infty} d(s) ds \lesssim e^{-ct}. \tag{4.5.7}$$

Then there exists (x_0, θ_0) such that

$$\|u_{[x_0,\theta_0]} - e^{it} Q\|_{H^1} \lesssim e^{-ct}.$$

Proof. Step 1. Convergence of $\delta(t)$. We claim that

$$\lim_{t \to +\infty} d(t) = 0. \tag{4.5.8}$$

To prove this, first note that (4.5.7) implies that there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ such that

$$\lim_{n \to +\infty} d(t_n) = 0.$$

Suppose now, by contradiction, that (4.5.8) does not hold. In this case, we can find another sequence $\{t_n'\}$ and $0 < \epsilon_1 < \delta_0$ such that

$$\begin{aligned} t_n &< t_n' \quad \forall n \\ d(t_n') &= \epsilon_1, \end{aligned} \tag{4.5.9}$$

and

$$d(t) < \epsilon_1 \quad \forall t \in [t_n, t_n').$$

Since $\epsilon_1 < \delta_0$, the parameter $\alpha(t)$ is well-defined on $[t_n, t_n')$. By Lemma 4.5.4, $|\alpha'(t)| \lesssim d(t)$, so that $\int_{t_n}^{t_n'} |\alpha'(t)|dt \lesssim e^{-ct_n}$, by (4.5.7). Therefore,

$$\lim_{n\to\infty} |\alpha(t_n) - \alpha(t_n')| = 0.$$

Since, by Lemma (4.5.4), $|\alpha(t)| \approx d(t)$, we get that $\alpha(t_n)$ tends to 0, which contradicts (4.5.9) and proves (4.5.8). Recalling the decomposition (4.5.5), to conclude the proof of Lemma 4.5.5, it is sufficient to prove that there exists (x_0, θ_0) such that

$$d(t) + |\alpha(t)| + \|h(t)\|_{\dot{H}^1} + |x(t) - x_0| + |\theta(t) - \theta_0| \lesssim e^{-ct}.$$

By Lemma 4.5.4, $|\alpha(t)| \approx d(t) \to 0$, as $t \to +\infty$. Therefore,

$$|\alpha(t)| \leq \int_t^{+\infty} |\alpha'(s)|ds \lesssim \int_t^{+\infty} d(s)ds \lesssim e^{-ct},$$

since $|\alpha'(t)| \approx d(t)$. Again by Lemma 4.5.4, $d(t) \approx \|h(t)\|_{\dot{H}^1} \approx |\alpha(t)|$, we get the bounds on $d(t)$ and $\|h(t)\|_{\dot{H}^1}$. To obtain the bounds on $x(t)$ and $\theta(t)$, it is sufficient to recall that Lemma 4.5.4 says $|x'(t)| + |\theta'(t)| \lesssim d(t) \lesssim e^{-ct}$. □

4.6 Solutions with high kinetic energy

4.6.1 Energy-critical case

In this and in the next section, we prove that radial solutions to (1.0.2) on the same energy level as W that do not blow-up in finite positive time (and have finite mass), and that do not scatter forward in time must converge exponentially to W, as $t \to +\infty$. We follow closely [24].

Proposition 4.6.1. *Let u be a solution to (1.0.2) defined on $[0, +\infty)$ satisfying*

$$E(u_0) = E(W) \quad \textit{and} \quad \|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}. \tag{4.6.1}$$

Assume furthermore that u_0 is radial and belongs to L^2. Then there exist (λ_0, θ_0) and $c > 0$ such that

$$\|u - W_{[\lambda_0, \theta_0]}\|_{\dot{H}^1} \lesssim e^{-ct}. \tag{4.6.2}$$

Moreover, u blows up in finite negative time.

We will work with a truncated variance. Consider a radial function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\phi(r) \geq 0 \quad \forall r \geq 0,$$

$$\phi(r) = \begin{cases} r^2, & r \leq 1, \\ 0, & r \geq 3, \end{cases}$$

and

$$\frac{d^2\phi}{dr^2}(r) \leq 2, r \geq 0.$$

Define $\phi_R(x) = R^2 \phi(\frac{x}{R})$ and

$$F_R(t) = \int \phi_R |u(t)|^2.$$

By virial identities, if $E(u_0) = E(W)$, we have

$$\begin{aligned} F_R'(t) &= 2 \operatorname{Im} \int \nabla\phi \cdot \nabla u \, \bar{u} \\ F_R''(t) &= -\frac{16}{N-2} d(t) + A_R(t), \end{aligned} \tag{4.6.3}$$

where

$$A_R(u(t)) = \int_{|x| \geq R} |\nabla u(t)|^2 \left(4\partial_r^2 \phi_R - 8\right) + \int_{|x| \geq R} |u|^{2^*} \left(8 - \frac{4}{N} \Delta\phi_R\right) - \int_{|x| \geq R} |u|^2 \Delta^2 \phi_R. \tag{4.6.4}$$

Recall that, if $\|\nabla u_0\|_{\dot{H}^1} > \|\nabla W\|_{\dot{H}^1}$, then, for all t in the interval of definition of u,

$$d(t) = |\|\nabla u(t)\|_{\dot{H}^1} - \|\nabla W\|_{\dot{H}^1}| = \|\nabla u(t)\|_{\dot{H}^1} - \|\nabla W\|_{\dot{H}^1}.$$

In order to prove Proposition 4.6.1, we start with the following lemma.

Lemma 4.6.2. *Let u be a radial solution to (1.0.2) defined on $[0, +\infty)$ and satisfying (4.6.1). Assume furthermore that the mass $M(u_0)$ of u_0 is finite. Then, there exists a constant $R_0 > 0$ such that, for all t in the interval of existence of u and all $R \geq R_0$,*

$$F_R'(t) > 0, \tag{4.6.5}$$

and there exists $c > 0$ such that

$$\int_t^{+\infty} d(s)ds \lesssim e^{-ct}, \quad \forall t \geq 0. \tag{4.6.6}$$

Proof of Lemma 4.6.2.

Step 1. A general bound on A_R

By the definition of ϕ_R, we have the bounds $4\partial_r^2\phi_R \leq 8$, $|\Delta^2\phi_R| \lesssim 1$ and $|\Delta^2\phi_R(r)| \lesssim \frac{1}{r^2}$. Therefore,

$$A_R(u(t)) \lesssim \int_{|x|\geq R} |u(x,t)|^{2^*} + \frac{1}{R^2}|u(x,t)|^2\, dx.$$

Now, making use of the decay given by radiality in H^1 from Lemma 2.2.4, we can bound

$$\int_{|x|\geq R} |u(x,t)|^{2^*} dx \leq \|u(t)\|_{L^\infty_{\{|x|\geq R\}}}^{\frac{4}{N-2}} \|u_0\|_{L^2}^2 \leq \frac{C}{R^{\frac{2N-2}{N-2}}} \|\nabla u(t)\|_{L^2}^{\frac{2}{N-2}} \|u_0\|_{L^2}^{\frac{2N-2}{N-2}},$$

to obtain

$$A_R(u(t)) \leq C_0 \left[\frac{1}{R^2} + \frac{1}{R^{\frac{2N-2}{N-2}}}(d(t) + \|W\|_{\dot{H}^1})^{\frac{1}{N-2}}\right],$$

where C_0 depends only on $M(u_0)$.

Step 2. A bound on A_R when $d(t)$ is small.

Taking a small δ_1, write the decomposition (4.5.1) as $u_{[x(t),\lambda(t),\theta(t)]} = W + v$, with $\|v\|_{\dot{H}1} \lesssim$

$d(t)$, by Lemma 4.5.2. We will start by proving

$$\lambda_- := \inf\{\lambda(t),\, t \geq 0,\, d(t) \leq \delta_1\} > 0. \tag{4.6.7}$$

Indeed, by mass conservation,

$$\|u_0\|_{L^2} \geq \int_{|x|\leq\lambda(t)} |u(t)|^2 = \frac{1}{\lambda^2(t)} \left(\int_{|x|\leq 1} W^2(x)dx - Cd^2(t) \right).$$

If $d(t) \leq \delta_1$ and δ_1 is small enough, then (4.6.7) holds. We now give an estimate on A_R when $d(t)$ is small. Since W is a static solution to (1.0.2), $\frac{d}{dt}\int \phi_R|W|^2 = 0$, so that $A_R(W) = 0$, for all $R > 0$, by (4.6.3). If we assume $R \geq 1$, by a change of variables, Hölder, Hardy and Sobolev inequalities, we can write (4.6.4) as

$$\begin{aligned}
|A_R(u(t))| &= |A_{R\lambda(t)}(W+v)| = |A_{R\lambda(t)}(W+v) - A_{R\lambda(t)}(W)| \qquad (4.6.8)\\
&\leq C\left[\int_{|x|\geq R\lambda(t)} |\nabla v|^2 + |\nabla W\cdot\nabla v| + W^{2^*-1}|v| + |v|^{2^*} + \frac{1}{(R\lambda(t))^2}(W|v| + |v|^2)\right]\\
&\leq C\left[\|v\|^2_{\dot H^1} + \frac{1}{(R\lambda(t))^{\frac{N-2}{2}}}\|v\|_{\dot H^1} + \|v\|^{2^*}_{\dot H^1} + \frac{1}{(R\lambda(t))^{\frac{N+2}{2}}}\|v\|_{\dot H^1} + \frac{1}{\lambda^2(t)}\|v\|^2_{\dot H^1}\right]\\
&\leq C_1\left[d^2(t) + \frac{1}{R^{\frac{N-2}{2}}}d(t)\right], \qquad (4.6.9)
\end{aligned}$$

where we used the fact that $\|\nabla W\|_{L^2_{\{|x|\geq r\}}} \approx \|W\|_{L^{2^*}_{\{|x|\geq r\}}} \approx \frac{1}{r^{\frac{N-2}{2}}}$, which can be verified by explicit computation. Note that the constant C_1 depends only on λ_-, which in turn depends only on $M(u_0)$.

Step 3. Bounds on A_R prove bounds on $d(t)$.

We now claim the bound

$$A_R(t) \leq \frac{8}{N-2}d(t). \tag{4.6.10}$$

This is clear from the bound (4.6.9), if $d(t) \leq \delta_1$ and $R \geq R_1$, where R_1 is a large constant depending only on $M(u_0)$. Now, if $d(t) > \delta_1$, consider the function

$$\varphi_R(\delta) = \frac{C_0}{R^2} + \frac{C_0}{R^{\frac{2N-2}{N-2}}}(\delta + \|W\|_{\dot H^1})^{\frac{1}{N-2}} - \frac{8}{N-2}\delta.$$

By direct computation, we see that $\varphi_R''(\delta) < 0$ for any $\delta > 0$. We can choose a large $R_2 \geq R_1$ (depending again only on $M(u_0)$) such that $\varphi_{R_2}(\delta_1) \leq 0$ and $\varphi_{R_2}'(\delta_1) \leq 0$, so that $\varphi_{R_2}(\delta) \leq 0$ for all $\delta \geq \delta_1$. The bound (4.6.10) is now proved.

Bound (4.6.10), together with (4.6.3), gives, for $R \geq R_2$ and any $t \geq 0$,

$$F_R''(t) \leq -\frac{8}{N-2} d(t) < 0. \tag{4.6.11}$$

Note that we must have $F_R'(t) > 0$, for all $t \geq 0$, as (4.6.11) would otherwise contradiction the positivity of F_R. Therefore, (4.6.5) is proved.

We will now make use of the following lemma, in the spirit of Banica [5, Lemma 2.1] and Duyckaerts-Roudenko [25, Claim 5.4], whose proof is given in Section 4.10.

Lemma 4.6.3. *Let $\phi \in C^1(\mathbb{R}^N)$ and $f \in H^1(\mathbb{R}^N)$. Assume that $\int |\nabla\phi|^2|f|^2 < +\infty$, and $E(f) = E(W)$. Then*

$$\left(\operatorname{Im}\int \nabla\phi \cdot \nabla f\, \bar{f}\right)^2 \lesssim d^2(f) \int |\nabla\phi|^2|f|^2.$$

By Lemma 4.6.3 and the fact that $F_R'(t) > 0$ and $F''(t) < 0$, we can write

$$\frac{F_R'(t)}{\sqrt{F_R(t)}} \lesssim -F_R''(t),$$

so that

$$\int_t^{+\infty} d(s)\, ds \lesssim e^{-ct},$$

which proves (4.6.6) and finishes the proof of Lemma 4.6.2. □

Proof of Proposition 4.6.1. We first prove that

$$\lim_{t\to+\infty} d(t) = 0. \tag{4.6.12}$$

Indeed, by Lemma 4.6.2, there exists $t_n \to +\infty$ such that $d(t_n) \to 0$. Assume, by contradiction, that there exists a sequence $t_n' > t_n$ such that

$$d(t_n') = \delta_0, \text{ and } 0 < d(t) < \delta_0 \quad \forall t \in (t_n, t_n'), \tag{4.6.13}$$

where δ_0 is given by Lemmas 4.5.1 and 4.5.2. Recall the decomposition (4.5.1):

$$u_{[\lambda(t),\theta(t)]}(t) = (1+\alpha(t))W + h(t), \text{ with } h \in G^{\perp}.$$

By taking subsequences, if necessary, we can assume

$$\lim \lambda(t_n) = \lambda_\infty \in (0, +\infty].$$

We now prove that $\lambda_\infty < +\infty$.

If $\lambda_\infty = +\infty$, as $u_{[\lambda(t_n),\theta(t_n)]}$ converges to W in $\dot{H}^1$, we have, for any $C > 0$,

$$\int_{|x|\geq C} |u(t_n)|^{2^*} \to 0.$$

For any $\epsilon > 0$ we have, by Hölder inequality,

$$|F_R(t_n)| \lesssim \epsilon \|u(t_n)\|_{\dot{H}^1} + \int_{|x|\geq C_\epsilon} |u(t_n)|^{2^*},$$

so that

$$\lim F_R(t_n) = 0.$$

However, by (4.6.5), $F_R'(t) > 0$. This implies $F_R(t) < 0$ for all $t \geq 0$, contradicting the fact that F_R is positive. Therefore, $\lambda(t_n)$ must be bounded.

Now, by Lemma 4.5.2, we have $\left|\frac{\lambda'(t)}{\lambda^3(t)}\right| \lesssim d(t)$. This implies, if $t \in (t_n, t_n')$,

$$\left|\frac{1}{\lambda^2(t)} - \frac{1}{\lambda^2(t_n)}\right| \lesssim e^{-ct_n}.$$

Therefore, $\lambda(t) \leq 2\lambda_\infty$ on $\cup_n (t_n, t_n')$, for large t. Turning to the bound on α' in Lemma 4.5.2,

$$|\alpha'(t)| \lesssim \lambda^2(t) d(t) \lesssim d(t).$$

This implies $|\alpha(t_n) - \alpha(t_n')| \to 0$. Moreover, again by Lemma 4.5.2, $|\alpha(t)| \approx d(t)$, which contradicts (4.6.13) and proves (4.6.12).

To finish the proof of Proposition 4.6.1, we must refine the estimate on $d(t)$. Since $d(t) \to 0$ as $t \to +\infty$, the decomposition (4.5.1) is well-defined for all large times. Therefore, by

(4.6.12) and (4.6.7), we have

$$\lim_{t\to+\infty} \lambda(t) = \lambda_\infty \in (0,+\infty), \quad \lim_{t\to+\infty} \alpha(t) = \lim_{t\to+\infty} d(t) = 0,$$

and

$$\|h(t)\|_{\dot{H}^1} \approx |\alpha(t)| = \int_t^{+\infty} |\alpha'(t)|ds \lesssim \int_t^{+\infty} \lambda^2(s)d(s)ds \lesssim e^{-ct}.$$

Furthermore, the bound $|\theta'(t)| \lesssim \lambda^2(t)d(t)$ implies that there exists θ_∞ such that

$$\lim_{t\to+\infty} |\theta(t) - \theta_\infty| = 0.$$

Therefore, (4.6.2) is proved.

It remains to prove the finite-time blow-up. This is a corollary of Lemma 4.6.2, applied to the time-reversed solution $v(t) := \bar{u}(-t)$. If u is defined on $\mathbb{R}$, by (4.6.5), we have

$$\operatorname{Im} \int \nabla\phi \cdot \nabla u_0\, \bar{u}_0 > 0$$

and

$$\operatorname{Im} \int \nabla\phi \cdot \nabla v_0\, \bar{v}_0 > 0,$$

which clearly contradicts the fact that

$$\operatorname{Im} \int \nabla\phi \cdot \nabla u_0\, \bar{u}_0 = -\operatorname{Im} \int \nabla\phi \cdot \nabla v_0\, \bar{v}_0.$$

This finishes the proof of Proposition 4.6.1. □

4.6.2 Intercritical case

We state here the corresponding results for the intercritical case. Since the proofs are very similar to the energy-critical case, we mainly sketch them.

Proposition 4.6.4. *Let u be a solution to (1.0.2) defined on $[0,+\infty)$ satisfying*

$$M(u_0) = M(Q), \quad E(u_0) = E(Q) \quad \text{and} \quad \|\nabla u_0\|_{L^2} > \|\nabla Q\|_{L^2}. \tag{4.6.14}$$

Assume furthermore that either u_0 is radial or $|x|u_0 \in L^2(\mathbb{R}^N)$. Then there exists (x_0, θ_0) and $c > 0$ such that

$$\|u - e^{it}Q_{[x_0,\theta_0]}\|_{H^1} \lesssim e^{-ct}.$$

Moreover, u blows up in finite negative time.

Proof. We divide the argument in two cases: the finite-variance case, and the radial case. Using the same proof of the finite-time blow-up as in the energy-critical case, and in view of Lemmas 4.5.5, 4.6.5 and 4.6.7 in the next subsections, Proposition 4.6.4 follows. □

4.6.2.1 Finite variance solutions

Lemma 4.6.5. *Let u be a solution to (1.0.2) defined on $[0, +\infty)$ and satisfying (4.6.14) and $\||x|u_0\|_{L^2} < +\infty$. Then, for all t in the interval of existence of u,*

$$\operatorname{Im} \int x \cdot \nabla u(t)\overline{u(t)} > 0, \tag{4.6.15}$$

and there exists $c > 0$ such that

$$\int_t^{+\infty} d(s)ds \lesssim e^{-ct}, \quad \forall t \geq 0. \tag{4.6.16}$$

Proof. Let $F(t) = \int |x|^2|u(x,t)|^2\, dx$. Then, by the virial identities, we have, for all $t \geq 0$,

$$F'(t) = 4 \operatorname{Im} \int x \cdot \nabla u(t)\overline{u}(t).$$

Note that, by Cauchy-Schwarz, $F'(t)$ is well-defined. Furthermore, since $E(u) = E(Q)$,

$$F''(t) = -[2N(p-1) - 8] \left(\int |\nabla u|^2 - \int |u|^{p+1} \right) = -[2N(p-1) - 8]d(u(t)).$$

Now, if (4.6.15) does not hold, there exists t_1 such that $F'(t_1) \leq 0$. Since $F'' \leq 0$, for any $t_0 > t_1$,

$$F'(t) \leq F'(t_0) < 0, \quad \forall t \geq t_0.$$

This implies that $F(t) < 0$ for large t, contradicting the definition of F.

We now claim that

$$[F'(t)]^2 \lesssim F(t)[F''(t)]^2, \tag{4.6.17}$$

which is a consequence of the following lemma, whose proof we postpone to Section 4.10.

Lemma 4.6.6. *Let* $\phi \in C^1(\mathbb{R}^N)$ *and* $f \in H^1(\mathbb{R}^N)$*. Assume that* $\int |\nabla\phi|^2|f|^2 < +\infty$*,* $M(f) = M(Q)$ *and* $E(f) = E(Q)$*. Then*

$$\left(\operatorname{Im}\int \nabla\phi\cdot\nabla f\,\bar{f}\right)^2 \lesssim d^2(f)\int |\nabla\phi|^2|f|^2.$$

Taking $\phi(x) = |x|^2$, (4.6.17) is proved. Since $F'(t) > 0$ and $F''(t) < 0$ for all $t \geq 0$, equation (4.6.17) can be rewritten as

$$\frac{F'(t)}{\sqrt{F(t)}} \lesssim -F''(t).$$

Integrating from 0 to $t \geq 0$,

$$\sqrt{F(t)} - \sqrt{F(0)} \lesssim -(F'(t) - F'(0)) \leq F'(0).$$

From (4.6.17), we deduce

$$F'(t) \lesssim -\left(\sqrt{F(0)} + F'(0)\right)F''(t) \lesssim -F''(t),$$

which shows

$$F'(t) \lesssim e^{-ct}.$$

Finally,

$$F'(t) = -\int_t^{+\infty} F''(s)ds = 4\int_t^{+\infty} d(s)ds,$$

and we obtain (4.6.16). Lemma 4.6.5 is proved in the finite-variance case. □

4.6.2.2 Radial solutions

We will work with a truncated variance. Consider a radial function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\phi(r) \geq 0 \quad \forall r \geq 0,$$

$$\phi(r) = \begin{cases} r^2, & r \le 1, \\ 0, & r \ge 3, \end{cases}$$

and

$$\frac{d^2\phi}{dr^2}(r) \le 2, r \ge 0.$$

Define $\phi_R(x) = R^2\phi(\frac{x}{R})$ and

$$F_R(t) = \int \phi_R |u(t)|^2.$$

By virial identities, if $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$, we have

$$F_R'(t) = 2\,\mathrm{Im} \int \nabla\phi_R \cdot \nabla u\, \bar{u}$$
$$F_R''(t) = -[2N(p-1) - 8]d(t) + A_R(t),$$

where

$$A_R(u(t)) = \int_{|x|\ge R} |\nabla u(t)|^2 \left(4\partial_r^2\phi_R - 8\right) + \frac{2(p-1)}{(p+1)} \int_{|x|\ge R} |u|^{p+1} (2N - \Delta\phi_R) - \int_{|x|\ge R} |u|^2 \Delta^2\phi_R. \tag{4.6.18}$$

The following lemma holds.

Lemma 4.6.7. *Let u be radial a solution to (1.0.2) defined on $[0, +\infty)$ and satisfying (4.6.14). Then, there exists a constant $R_0 > 0$ such that, for all t in the interval of existence of u and all $R \ge R_0$,*

$$F_R'(t) > 0, \tag{4.6.19}$$

and there exists $c > 0$ such that

$$\int_t^{+\infty} d(s)ds \lesssim e^{-ct}, \quad \forall t \ge 0.$$

Moreover, u_0 has finite variance.

Proof. The proof of Lemma 4.6.7 is essentially the same as in the energy-critical case, and will be omitted, except for the finite-variance part.

By hypothesis, there is a sequence $t_n \to +\infty$ such that $d(t_n) \to 0$. By (4.5.4), extracting a subsequence, if necessary, we have $u_n \to e^{i\theta_0}Q$ in H^1 for some $\theta_0 \in \mathbb{R}$. Since F_R is increasing by (4.6.19), we have

$$\int \phi_R |u_0|^2 = F_R(0) \leq \int \phi_R Q^2.$$

Thus, we can make $R \to +\infty$, which proves the finite variance of u_0. □

4.7 Solutions with low kinetic energy

4.7.1 Energy-critical case

In this section, we will consider solutions such that

$$E(u_0) = E(W), \text{ and } \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}. \tag{4.7.1}$$

Definition 4.7.1. A solution u to (1.0.2) with lifespan I is said to be *almost periodic modulo symmetries* on $J \subset I$ if there exist functions $x : J \to \mathbb{R}^N$, $\lambda : J \to \mathbb{R}_+^*$ and $C : \mathbb{R}_*^+ \to \mathbb{R}_*^+$ such that for all $t \in I$ and all $\eta > 0$

$$\int_{|x-x(t)| \geq C(\eta)/\lambda(t)} |\nabla u(x,t)|^2 \, dx \leq \eta$$

and

$$\int_{|\xi| \geq C(\eta)\lambda(t)} |\xi|^2 |\hat{u}(\xi,t)|^2 \, d\xi \leq \eta.$$

Remark 4.7.2. By Arzelà-Ascoli's theorem, almost periodicity modulo symmetries is equivalent to the set

$$\left\{ u_{[x(t),\lambda(t),0]};\, t \in J \right\}$$

being precompact in $\dot{H}^1$.

Remark 4.7.3. If the solution is radial, $x(t)$ can be chosen as zero.

Proposition 4.7.4. *Let u be a solution to* (1.0.2) *and $I = (T^-, T^+)$ be its maximal* *interval of existence. If u satisfies (4.7.1) then*

$$I = \mathbb{R}.$$

Furthermore, if

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u(x,t)|^{\frac{2(N+2)}{N-2}} dx\, dt = +\infty, \tag{4.7.2}$$

then u is almost periodic modulo symmetries on $[0, +\infty)$.

The proof of Proposition 4.7.4 is essentially contained in the proof in [57, Proposition 3.1], which extended the work in [55] to dimensions $N \geq 6$.

Remark 4.7.5. By time-reversal symmetry, the analogous version of (4.7.2) for the interval $(-\infty, 0]$ holds.

The next theorem is the main result proved in [57, Theorem 1.7].

Theorem 4.7.6. *For $N \geq 5$, let $u : I \times \mathbb{R}^N \to \mathbb{C}$ be a solution to (1.0.2) satisfying*

$$E_* := \sup_{t \in I} \|u(t)\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}.$$

Then,

$$\int_I \int_{\mathbb{R}^N} |u(x,t)|^{\frac{2(N+2)}{N-2}} dx\, dt = C(E_*) < +\infty.$$

In particular, by uniqueness of solutions and continuity of the flow of (1.0.2), we have the following corollary.

Corollary 4.7.7. *For $N \geq 5$, let u be a solution to (1.0.2) satisfying (4.7.1) and (4.7.2). Then there exists a sequence $t_n \to +\infty$ such that*

$$\lim_{n \to +\infty} d(u(t_n)) = 0.$$

The main aim of this section is to prove the following proposition.

Proposition 4.7.8. *Let u be a radial solution to 1.0.2 satisfying (4.7.1) and (4.7.2). Then there exist (λ_0, θ_0) and $c > 0$ such that, for all $t \geq 0$,*

$$\|u - W_{[\lambda_0, \theta_0]}\|_{\dot{H}^1} \lesssim e^{-ct}. \tag{4.7.3}$$

Moreover, u scatters backward in time.

As in the proof of Proposition 4.6.1, we need to show that

$$d(t) = \|W\|_{\dot{H}^1} - \|u(t)\|_{\dot{H}^1} \to 0, \text{ as } t \to +\infty.$$

We start by stating the following monotonicity results.

Lemma 4.7.9. *Consider* $\{t_n\}_n$ *and* $\{t'_n\}_n$, $t_n < t'_n$ *two real sequences, and* $\{u_n\}_n$ *a sequence of radial solutions to* (1.0.2) *on* $[t_n, t'_n]$ *satisfying* (4.7.1). *Assume that there exist* $\{\lambda_n(t)\}_n \subset R^*_+$ *such that the set*

$$K = \left\{(u_n(t))_{[\lambda_n(t)]};\, n \in \mathbb{N},\, t \in [t_n, t'_n]\right\}$$

is relatively compact in $\dot{H}^1$. *If*

$$\lim_n d(u_n(t_n)) + d(u_n(t'_n)) = 0,$$

then

$$\lim_n \left\{ \sup_{t\in[t_n,t'_n]} d(u_n(t)) \right\} = 0. \tag{4.7.4}$$

Lemma 4.7.10. *Under the assumptions of Lemma 4.7.9, if n is large enough so that* $d(u_n(t)) \le \delta_0$ *on* $[t_n, t'_n]$ *and if* θ_n, μ_n *and* α_n *are the parameters of the decomposition* (4.5.1), *then*

$$\lim_n \frac{\sup_{t\in[t_n,t'_n]} \mu_n(t)}{\inf_{t\in[t_n,t'_n]} \mu_n(t)} = 1 \tag{4.7.5}$$

Remark 4.7.11. In Lemmas 4.7.9 and 4.7.10, it is sufficient to assume

$$\inf_{t\in[t_n,t'_n]} \lambda(t) = 1 \quad \forall n \in \mathbb{N}.$$

In fact, if $\tilde{\lambda}_n := \inf_{t\in[t_n,t'_n]} \lambda(t)$, then

$$u^*_n(x,t) := \frac{1}{\tilde{\lambda}_n^{\frac{N-2}{2}}} u_n\left(\frac{x}{\tilde{\lambda}_n}, \frac{t}{\tilde{\lambda}^2_n}\right), \quad \lambda^*_n(t) := \frac{\lambda_n(t)}{\tilde{\lambda}_n}, \quad t^*_n := \frac{t_n}{\tilde{\lambda}^2_n}, \quad t'^*_n := \frac{t'_n}{\tilde{\lambda}^2_n},$$

$$K^* := \left\{(u^*_n(t))_{[\lambda^*_n(t)]};\, n \in \mathbb{N},\, t \in [t^*_n, t'^*_n]\right\}$$

satisfy the assumptions of Lemma 4.7.9. Moreover, the conclusions of the Lemmas are unchanged under these transformations.

Before proving Lemmas 4.7.9 and 4.7.10, we prove two auxiliary lemmas.

Lemma 4.7.12. *Consider* $\{t_n\}_n$ *and* $\{t'_n\}_n$, $t_n < t'_n$ *two real sequences, and* $\{u_n\}_n$ *a sequence of radial solutions to* (1.0.2) *on* $[t_n, t'_n]$ *satisfying* (4.7.1)*. Assume that there exist* $\{\lambda_n(t)\}_n \subset R^*_+$ *such that the set*

$$K = \left\{(u_n(t))_{[\lambda_n(t)]};\, n \in \mathbb{N},\, t \in [t_n, t'_n]\right\}$$

is relatively compact in $\dot{H}^1$*. Assume furthermore that*

$$\inf_{t\in[t_n,t'_n]} \lambda(t) = 1 \quad \forall n \in \mathbb{N}. \tag{4.7.6}$$

Then, for all $n \in \mathbb{N}$,

$$\int_{t_n}^{t'_n} d(u(t)) \lesssim d(u(t_n)) + d(u(t'_n)). \tag{4.7.7}$$

Proof of Lemma 4.7.12. For $R > 0$, consider the function

$$F_{R,n}(t) = \int \phi_R |u_n(t)|^2.$$

By Hölder and Sobolev inequalities, and recalling that $\|u(t)\|_{\dot{H}^1} \leq \|W\|_{\dot{H}^1}$, we have

$$F_{R,n}(t) \lesssim R^4.$$

By Lemma 4.6.3,

$$|F'_{R,n}(t)| \lesssim d(u_n(t))\sqrt{F_{R,n}(t)} \lesssim R^2 d(u_n(t)). \tag{4.7.8}$$

By (4.7.6), $\lambda(t) \geq 1$ on $[t_n, t'_n]$. We claim that, whenever defined, μ_n is bounded away from zero. In fact, by the precompactness of K and decomposition (4.5.1), we have

$$(u_n(t))_{[\lambda_n(t)]} = (1 + \alpha_n(t))W_{[\lambda_n(t)/\mu_n(t)]} + (h_n(t))_{[\lambda_n(t)/\mu_n(t)]},$$

with $(h_n(t))_{[\lambda_n(t)/\mu_n(t)]} \perp W_{[\lambda_n(t)/\mu_n(t)]}$ and $\alpha_n(t) \leq \|u_{[\lambda_n(t)]}\|_{\dot{H}^1} + 1$. Therefore, the set

$$\bigcup_n \left\{ W_{[\lambda_n(t)/\mu_n(t)]};\, t \in [t_n, t'_n], d(u_n(t)) \leq \delta_0 \right\} \tag{4.7.9}$$

must be precompact. Since W does not depend on time, we get

$$\lambda_n(t) \approx \mu_n(t) \text{ on } \{t \in [t_n, t'_n], d(u_n(t)) \leq \delta_0\}.$$

(Note that the constant does not depend on n). Thus, $\mu_- := \inf_{\substack{t\in[t_n,t'_n],\\ d(u_n(t))\leq\delta_0}} \mu_n(t) \gtrsim 1$.

We will now give a lower bound to $F''_{R,n}(t)$. Recalling (4.6.8), if $d(u_n(t)) \leq \delta_0$ and $R \geq \frac{1}{\mu_-}$, we have

$$|A_R(u_n(t))| \lesssim \left[d^2(u_n(t)) + \frac{1}{(R\mu_-)^{\frac{N-2}{2}}} d(u_n(t)) \right].$$

Therefore, there exist $\delta_1 > 0$ and $R_1 > 0$ such that, if $d(u_n(t)) \leq \delta_1$, then

$$|A_R(u_n(t))| \leq \frac{8}{N-2} d(u_n(t)).$$

Now, by almost periodicity modulo symmetries and (4.7.6), if $\eta > 0$ and $R \geq C(\eta)$, then

$$|A_R(u_n(t))| \lesssim \eta.$$

Thus, we can choose $\eta_1 = \eta_1(\delta_1)$ such that, if $d(u_n(t)) \geq \delta_1$ and $R \geq C(\eta_1)$, then

$$|A_R(u_n(t))| \leq \frac{8}{N-2}\delta_1 \leq \frac{8}{N-2} d(u_n(t)).$$

Finally, since

$$F''_{R,n}(t) = \frac{16}{N-2} d(u_n(t)) + A_R(u_n(t)),$$

we get, if $R \geq \max\{R_1, C(\eta_1)\}$,

$$F''_{R,n}(t) \geq \frac{8}{N-2} d(u_n(t)). \tag{4.7.10}$$

Integrating (4.7.10) and using (4.7.8), we obtain (4.7.7). □

Lemma 4.7.13. *Under the assumptions of Lemma 4.7.9 and Remark 4.7.11, if* $s_n \in [t_n, t'_n]$

and the sequence $\lambda_n(s_n)$ is bounded, then

$$\lim_n d(u_n(s_n)) = 0. \tag{4.7.11}$$

Proof of Lemma 4.7.13. By Remark 4.7.11, we have $\lambda_n(s_n) \approx 1$, so we can assume that the sequence $\{u_n(s_n)\}_n$ converges to some $v_0 \in \dot{H}^1$. If (4.7.11) does not hold, then

$$d(v_0) > 0, \text{ and } \|v_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}. \tag{4.7.12}$$

By strong convergence, we have $E(v_0) = E(W)$. Let v be the solution to (1.0.2) with initial condition v_0. By Proposition 4.7.4, v is defined on $\mathbb{R}$.

We claim that, for large n, $s_n + 1 \leq t'_n$. Indeed, if $t'_n \in (s_n, s_n + 1)$ for an infinite number of n, after extracting a subsequence, we have that $t'_n - s_n$ converges to some $\tau \in [0, 1]$. By continuity of the flow, $u_n(t'_n) \to v(\tau)$. But since $d(u_n(t'_n)) \to 0$, $d(v(\tau)) = 0$, which implies that $v = W_{[\lambda_0,\theta_0]}$, for some fixed λ_0, θ_0. Uniqueness of solutions to (1.0.2) then contradicts (4.7.12). Therefore, for large n, $t_n \leq s_n \leq s_n + 1 \leq t'_n$. Again by continuity of the flow,

$$\lim_n \int_{s_n}^{s_n+1} d(u_n(t))\, dt = \int_0^1 d(v(t))\, dt > 0.$$

But Lemma 4.7.12 gives

$$\lim_n \int_{s_n}^{s_n+1} d(u_n(t))\, dt \leq \lim_n \int_{t_n}^{t'_n} d(u_n(t))\, dt \lesssim \lim_n d(u_n(t_n)) + d(u_n(t'_n)) = 0.$$

Lemma 4.7.13 is now proven. □

We now prove Lemmas 4.7.9 and 4.7.10.

Proof of Lemma 4.7.9. By Remark 4.7.11, we can choose, for every n, $b_n \in [t_n, t'_n]$ such that

$$\lim_n \lambda_n(b_n) = 1.$$

This implies, by Lemma 4.7.13, that

$$\lim_n d(u_n(b_n)) = 0.$$

Assume, by contradiction, that (4.7.4) does not hold. Without loss of generality, there exists $\delta_1 > 0$ such that

$$\sup_{t\in[t_n,b_n]} d(u_n(t)) \geq \delta_1, \quad \forall n \in \mathbb{N}. \tag{4.7.13}$$

Choosing $\delta_2 < \min\{\delta_0, \delta_1\}$, by continuity there exists $a_n \in [t_n, b_n)$ such that

$$d(u_n(t)) < \delta_2 \text{ on } (a_n, b_n) \text{ and } d(u_n(a_n)) = \delta_2.$$

Since $\delta_2 < \delta_0$, the modulation parameter μ_n is well-defined. Recalling that the set defined by (4.7.9) is precompact, we must have $\lambda_n \approx \mu_n$, where the constants do not depend on n. Thus, up to a subsequence, we can assume

$$\mu_n(b_n) \to \mu_0 \in (0, +\infty).$$

We will now show that the μ_n are uniformly bounded on $\cup_n[a_n, b_n]$. Suppose, by contradiction, that there exists $c_n \in [a_n, b_n)$ such that, for large n,

$$\mu_n(t) < 2\mu_0 \text{ on } (c_n, b_n) \text{ and } \mu_n(c_n) = 2\mu_0. \tag{4.7.14}$$

Since $\mu_n(c_n)$ is bounded, so is $\lambda_n(c_n)$. Therefore, by Lemma 4.7.13, $\lim d(u_n(c_n)) = 0$. Recalling Lemma 4.5.2, we have

$$\left|\frac{1}{\mu_n^2(c_n)} - \frac{1}{\mu_n^2(b_n)}\right| \leq \int_{c_n}^{b_n} \left|\frac{\mu_n'(t)}{\mu_n^3(t)}\right| \lesssim \int_{c_n}^{b_n} d(u_n(t))\,dt.$$

By Lemma 4.7.12, the last integral converges to 0, contradicting (4.7.14). Therefore,

$$\sup_{\substack{t\in[a_n,b_n]\\ n\in\mathbb{N}}} \mu_n(t) < +\infty.$$

We conclude that $\mu_n(a_n)$ must be bounded, and so must be $\lambda_n(a_n)$. Invoking again Lemma 4.7.13, we have $\lim d(u_n(a_n)) = 0$, contradicting (4.7.13). Lemma 4.7.9 is proven. □

Proof of Lemma 4.7.10. As in the proof of the previous Lemma, by Remark 4.7.11 and Lemmas 4.7.12 and 4.7.13, we have that $\mu_n \approx 1$, where the constant does not depend on

n. Let a_n and b_n be such that

$$\mu_n(a_n) = \inf_{t\in[t_n,t'_n]} \mu_n(t), \text{ and } \mu_n(b_n) = \sup_{t\in[t_n,t'_n]} \mu_n(t).$$

Let $\bar{a}_n = \min\{a_n, b_n\}$ and $\bar{b}_n = \max\{a_n, b_n\}$. Then,

$$\left|\frac{1}{\mu_n^2(a_n)} - \frac{1}{\mu_n^2(b_n)}\right| \leq \int_{\bar{a}_n}^{\bar{b}_n} \left|\frac{\mu_n'(t)}{\mu_n^3(t)}\right| \lesssim \int_{\bar{a}_n}^{\bar{b}_n} d(u_n(t))\,dt \to 0, \text{ as } n \to +\infty.$$

Since $\mu_n(b_n)$ is bounded, we get (4.7.5), and Lemma 4.7.10 is proven. □

We now have all the tools to prove Proposition 4.7.8.

Proof of Proposition 4.7.8. By Corollary 4.7.7, there exists a sequence $t_n \to +\infty$ such that

$$\lim_n d(u(t_n)) = 0.$$

By Lemma (4.7.9), with $u_n = u$, $\lambda_n = \lambda$ (where λ is the frequency scale obtained from Proposition 4.7.4) and $t'_n = t_{n+1}$, this implies

$$\lim_{t\to+\infty} d(t) = 0. \tag{4.7.15}$$

Therefore, the modulation parameters $\alpha(t)$, $\mu(t)$, $\theta(t)$ are defined for large t. We now prove that

$$\lim_{t\to+\infty} \mu(t) = \mu_\infty \in (0, +\infty). \tag{4.7.16}$$

Indeed, if not, then as $t \to +\infty$, $\log(\mu(t))$ does not satisfy the Cauchy criterion. Therefore, there must exist sequences $T_n < T'_n$ such that

$$\lim_n \frac{\mu(T'_n)}{\mu(T_n)} \neq 1. \tag{4.7.17}$$

But $d(T_n) + d(T'_n) \to 0$, by (4.7.15). By Lemma (4.7.10), with $u_n = u$, $\lambda_n = \lambda$, $t_n = T_n$ and $t'_n = T'_n$, we have

$$\lim_n \frac{\sup_{t\in[T_n,T'_n]} \mu(t)}{\inf_{t\in[T_n,T'_n]} \mu(t)} = 1,$$

contradicting (4.7.17). Turning to the proof of (4.7.3), we claim the following inequality:

$$\int_t^{+\infty} d(u(s))\,ds \lesssim d(u(t)). \tag{4.7.18}$$

Suppose by contradiction that (4.7.18) does not hold. Then there exists a sequence $T_n \to +\infty$ such that

$$\int_{T_n}^{+\infty} d(u(s))\,ds \geq 2n\,d(u(T_n)).$$

Moreover, there exists a sequence $\{S_n\}_n$ such that $S_n > T_n$ for all n, and

$$\int_{T_n}^{S_n} d(u(s))\,ds \geq n\,d(u(T_n)).$$

By (4.7.16), for any sequence $\{T_n'\}_n$ such that $T_n' \geq S_n$ for all n, we are under the assumptions of Lemma 4.7.12, with $u_n = u$, $\lambda_n = \lambda$, $T_n = t_n$ and $t_n' = T_n'$, Hence,

$$n\,d(u(T_n)) \leq \int_{T_n}^{T_n'} d(u(s))\,ds \lesssim d(u(T_n)) + d(u(T_n')).$$

Since T_n' can be taken arbitrarily large, and the implicit constant is independent of the choice of a particular $\{T_n'\}_n$ (given the function u itself does not change), we have a contradiction.

Note that (4.7.18) is equivalent to the existence of $c > 0$ such that

$$\int_t^{+\infty} d(u(s))\,ds \lesssim e^{-ct}.$$

By Lema 4.5.2, since $|\alpha(t)| \approx d(u(t))$ and μ is bounded, there exist θ_∞ such that

$$|\alpha(t)| + |\theta(t) - \theta_\infty| + \|h(t)\|_{\dot{H}^1} \lesssim e^{-ct}.$$

Therefore, the bound (4.7.3) is proven. The assertion about scattering for negative times is a corollary of Lemma 4.7.12. Indeed, if

$$\|u\|_{S(0,+\infty)} = \|u\|_{S(-\infty,0)} = +\infty,$$

by time-reversal and (4.7.3), we see that the set

$$\{u(t);\, t \in \mathbb{R}\}$$

is relatively compact and that

$$\lim_{t\to\pm\infty} d(t) = 0.$$

Therefore, by Lemma 4.7.12, with $u_n = u$, $\lambda_n = 1$, $t_n = -n$ and $t'_n = n$, we have

$$\int_{-\infty}^{+\infty} d(t)\,dt = \lim_{n\to+\infty} \int_{-n}^{n} d(t)\,dt \lesssim d(-n) + d(n) = 0.$$

Therefore, $d(u_0) = 0$, contradicting (4.7.1). Proposition 4.7.8 is proven. □

4.7.2 Intercritical case

In this section, we will consider solutions such that

$$M(u_0) = M(Q),\ E(u_0) = E(Q),\ \text{and } \|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2}. \tag{4.7.19}$$

Since the scaling parameter is fixed *a priori* in the intercritical regime, controlling scaling is no longer an issue. We can then use the fact that the solution has finite mass, together with information given by virial-type and compactness arguments, to control the translation parameter, allowing us to prove results in the non-radial setting. We start with a definition.

Definition 4.7.14. A solution u to (1.0.2) with lifespan I is said to be *almost periodic modulo symmetries* on $J \subset I$ if there exist functions $x : J \to \mathbb{R}^N$ and $C : \mathbb{R}^+_* \to \mathbb{R}^+_*$ such that for all $t \in J$ and all $\eta > 0$

$$\int_{|x-x(t)|\geq C(\eta)} |\nabla u(x,t)|^2 + |u(x,t)|^2\,dx \leq \eta.$$

Remark 4.7.15. By Arzelà-Ascoli's theorem, almost periodicity modulo symmetries is equivalent to the set

$$\left\{u_{[x(t)]};\, t \in J\right\}$$

being precompact in H^1.

Proposition 4.7.16. *Let u be a solution to (1.0.2) and $I = (T^-, T^+)$ be its maximal interval of existence. If u satisfies (4.7.19) then*

$$I = \mathbb{R}.$$

Furthermore, if

$$\|u\|_{S(0,+\infty)} = +\infty, \tag{4.7.20}$$

then u is almost periodic modulo symmetries on $[0, +\infty)$, and we have

$$P(u) = \operatorname{Im} \int \bar{u} \nabla u = 0, \text{ and}$$

$$\lim_{t \to \infty} \frac{x(t)}{t} = 0.$$

The proof of Proposition 4.7.16 is now classical, and it is essentially the same as in Duyckaerts and Roudenko [25, Lemma 6.2, Corollary 6.3 and Lemma 6.4].

Remark 4.7.17. By time-reversal symmetry, the analogous version of (4.7.20) for the interval $(-\infty, 0]$ holds.

Remark 4.7.18. As in Duyckaerts and Roudenko [25, Lemma 6.2], the function $x(t)$ can be chosen as to be continuous on $\mathbb{R}$ and the same as the one given in Lemmas 4.5.3 and 4.5.4, if $d(t) < \delta_0$.

Proposition 4.7.19. *Let u be a solution to (1.0.2) satisfying (4.7.19) and (4.7.20). Then there exist (x_0, θ_0) and $c > 0$ such that, for all $t \geq 0$,*

$$\|u - e^{it} Q_{[x_0, \theta_0]}\|_{H^1} \lesssim e^{-ct}.$$

Moreover, u scatters backward in time.

As in the proof of Proposition 4.6.4, we need to show that

$$\int_t^{+\infty} d(s)\, ds \lesssim e^{-ct}, \; \forall t \geq 0. \tag{4.7.21}$$

We start with the following lemmas.

Lemma 4.7.20. *Let u be a solution to (1.0.2) satisfying (4.7.19) and (4.7.20). Then*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T d(t)\,dt = 0.$$

Proof. Let $R>0$ to be chosen later and let ϕ_R and F_R be as in the previous section. Then, by Hölder and inequatity,

$$|F_R'(t)| \lesssim R. \tag{4.7.22}$$

Moreover, we have

$$F_R''(t) = [2N(p-1)-8]d(t) + A_R(u(t)), \tag{4.7.23}$$

where A_R is given by (4.6.18).

Fix $\eta>0$. By definition of ϕ_R and almost periodicity modulo symmetries, if $R\geq C(\eta)$, we have

$$|A_R(u(t))| \lesssim \int_{|x|\geq R} |\nabla u(x,t)|^2 + |u(x,t)|^{p+1} + \frac{1}{|x|^2}|u(x,t)|^2\,dx. \tag{4.7.24}$$

Choose $T_0(\eta)\geq 0$ such that, for any $t\geq T_0$,

$$|x(t)| \leq \eta t.$$

For $T\geq T_0$, choose $R := \eta T + C(\eta) + 1$. With this choice of R, we have

$$\begin{aligned}
|A_R(u(t))| &\lesssim \int_{|x-x(t)|+|x(t)|\geq R} |\nabla u(x,t)|^2 + |u(x,t)|^{p+1} + |u(x,t)|^2\,dx\\
&\lesssim \int_{|x-x(t)|\geq C(\eta)} |\nabla u(x,t)|^2 + |u(x,t)|^{p+1} + |u(x,t)|^2\,dx\\
&\lesssim \eta.
\end{aligned}$$

By (4.7.22), (4.7.23), and (4.7.24),

$$\begin{aligned}
[2N(p-1)-8]\int_{T_0}^T d(t)\,dt &\lesssim |F_R'(T)| + |F_R'(T_0)| + \eta(T-T_0)\\
&\lesssim R + \eta(T-T_0)\\
&= \eta T + \eta(T-T_0).
\end{aligned}$$

Letting $T \to +\infty$,
$$\limsup_{T\to+\infty} \frac{1}{T}\int_0^T d(t)\,dt \lesssim \eta.$$
Since η is arbitrary, we conclude the proof of Lemma 4.7.20. □

We next state a key result to prove Proposition 4.7.21.

Lemma 4.7.21. *Let u be a solution to (1.0.2) satisfying (4.7.19) and (4.7.20), and $x(t)$ as in Proposition 4.7.16 and Remark 4.7.18. Then, for any $0 \le \sigma \le \tau$,*
$$\int_\sigma^\tau d(u(t)) \lesssim \left[1 + \sup_{\sigma\le t\le\tau} |x(t)|\right](d(u(\sigma)) + d(u(\tau))), \tag{4.7.25}$$
and, if $\tau \ge \sigma + 1$,
$$|x(\tau) - x(\sigma)| \lesssim \int_\sigma^\tau d(u(t)). \tag{4.7.26}$$

The proof of (4.7.25) is similar to the energy-critical setting (it is in fact easier, since there is no scaling involved). We refer to [25, Lemma 6.7] for the argument in the 3d cubic case. The proof of (4.7.26) follows verbatim from the proof in [25, Lemma 6.8].

We are now able to prove Proposition 4.7.19. We follow closely the proof in [25].

Proof of Proposition 4.7.19. We first show that $x(t)$ is bounded. By Lemma 4.7.20, there exists a sequence $\{t_n\}_n$ such that $t_{n+1} \ge t_n + 1$ for all n, and $d(u(t_n)) \to 0$. By Lemma 4.7.21, there exists $C_0 > 0$ such that, if $n > n_0$ and $1 + t_{n_0} \le t \le t_n$, then
$$|x(t) - x(t_{n_0})| \le C_0\left[1 + \sup_{t_{n_0}\le s\le t_n} |x(s)|\right](d(u(t_n)) + d(u(t_{n_0}))).$$

If n_0 is large enough so that $d(u(t_n)) + d(u(t_{n_0})) \le 1/(2C_0)$, and t is chosen in $[t_{n_0} + 1, t_n]$ so that $\sup_{t_{n_0}+1\le s\le t_n} |x(s)| = |x(t)|$, then
$$\sup_{t_{n_0}+1\le s\le t_n} |x(s)| \le C(n_0) + \frac{1}{2}\sup_{t_{n_0}+1\le s\le t_n} |x(s)|,$$
where $C(n_0) = |x(t_{n_0})| + \frac{1}{2}\sup_{t_{n_0}\le s\le t_{n_0}+1} |x(t)| + \frac{1}{2}$. Therefore, $x(t)$ is bounded on $[t_{n_0} + 1, +\infty)$, and hence, by continuity, on $[0, +\infty)$.

By the boundedness of $x(t)$ and (4.7.25), we have

$$\int_{\sigma}^{\tau} d(u(t)) \lesssim d(u(\sigma)) + d(u(\tau)).$$

For a fixed $\sigma \geq 0$ and choosing $\tau = t_n$, we let $n \to +\infty$ to obtain

$$\int_{\sigma}^{\infty} d(u(t)) \lesssim d(u(\sigma)).$$

By Gronwall's Lemma, we have (4.7.21) and, by Lemma 4.5.5, we finish the proof of Proposition 4.7.19. □

4.8 Estimates on exponentially decaying solutions

According to the previous sections, we must study the behavior of solutions approaching $e^{it}Q$ exponentially fast in time. We start with the energy-critical setting.

4.8.1 Energy critical case

In contrast to the previous two sections, the radiality assumption is not needed to prove the results in this subsection. We consider the linearized approximate equation

$$\partial_t h + \mathcal{L} h = \epsilon \tag{4.8.1}$$

with h and ϵ such that, for $t \geq 0$,

$$\begin{aligned} \|h(t)\|_{\dot{H}^1} &\lesssim e^{-c_0 t}, \\ \|\epsilon(t)\|_{\frac{2N}{N+2}} + \|\nabla \epsilon\|_{S'(L^2,[t,+\infty))} &\lesssim e^{-c_1 t}, \end{aligned} \tag{4.8.2}$$

where $c_1 > c_0 > 0$. The following self-improving estimate was proved for radial data in [24]. We give the proof without the radial assumption in Section 4.10.

Lemma 4.8.1. *Under the assumptions* (4.8.2), *(i) if* $e_0 \notin [c_0, c_1)$, *then*

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-c_1^- t}, \tag{4.8.3}$$

(ii) if $e_0 \in [c_0, c_1)$, *then there exists* $A \in \mathbb{R}$ *such that*

$$\|h(t) - Ae^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} \lesssim e^{-c_1^- t}. \tag{4.8.4}$$

To further improve the convergence in the case $N \geq 6$, we study the linearized equation around $W + \mathcal{V}_k^A$, for $A \neq 0$, which was defined in (4.4.1). For simplicity, we omit the superscripts A. Defining, for every k,

$$\tilde{\mathcal{L}}_k := \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} + \frac{(p_c+1)}{2}|W + \mathcal{V}_k|^{p_c-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$+ \frac{(p_c-1)}{2}|W + \mathcal{V}_k|^{p_c-3}\begin{pmatrix} \operatorname{Im}(W+\mathcal{V}_k)^2 & -\operatorname{Re}(W+\mathcal{V}_k)^2 \\ -\operatorname{Re}(W+\mathcal{V}_k)^2 & -\operatorname{Im}(W+\mathcal{V}_k)^2 \end{pmatrix},$$

$$\tilde{K}_k(h) = \frac{(p_c+1)}{2}|W + \mathcal{V}_k|^{p_c-1}h + \frac{(p_c-1)}{2}|W + \mathcal{V}_k|^{p-3}(W + \mathcal{V}_k)^2\bar{h},$$

and

$$\tilde{R}_k(h) := |W + \mathcal{V}_k|^{p_c-1}(W + \mathcal{V}_k)J((W + \mathcal{V}_k)^{-1}h),$$

where

$$J(z) = |1+z|^{p_c-1}(1+z) - 1 - \frac{(p_c+1)}{2}z - \frac{(p_c-1)}{2}\bar{z},$$

we have that if, $u = W + \mathcal{V}_k + h$ satisfies (1.0.2), then h satisfies

$$\partial_t h + \tilde{\mathcal{L}}_k h = i\tilde{R}_k(h) + \epsilon_k, \tag{4.8.5}$$

or in the form of a Schrödinger equation,

$$i\partial_t h + \Delta h + \tilde{K}_k h = -\tilde{R}_k(h) + i\epsilon_k,$$

where ϵ_k are $O(e^{-(k+1)e_0 t})$ in $\mathcal{S}(\mathbb{R}^N)$. Note that the operator $\tilde{\mathcal{L}}_k$ is time-dependent and that, by the construction of $\mathcal{V}_k$, we have, for all $t \geq 0$,

$$|\mathcal{V}_k(t)| \lesssim e^{-e_0 t}|W|,$$

and

$$|\nabla \mathcal{V}_k(t)| \lesssim e^{-e_0 t}|\nabla W| \lesssim e^{-e_0 t}|W|.$$

This implies that the estimates in Lemmas 4.3.9 and 4.3.12 hold with the same proof if we replace K by $\tilde{K}_k$ and R by $\tilde{R}_k$. Therefore, we have the following results.

Lemma 4.8.2. *Let $N \geq 6$, $k \geq 1$ and I be a bounded time interval, and consider $f \in S(\dot{H}^1, I)$ such that $\nabla f \in S(L^2, I)$. The following estimates hold*

(i) $\|\nabla \tilde{K}_k(f)\|_{S'(L^2,I)} \lesssim |I|^{\frac{1}{2}} \|\nabla f\|_{S(L^2,I)}$,

(ii) $\|\nabla \tilde{R}_k(f)\|_{S'(L^2,I)} + \|\tilde{R}_k(f)\|_{\frac{2N}{N+2}} \lesssim \left(1 + |I|^{\frac{1}{2}}\right) \|\nabla f\|^{p_c}_{S(L^2,I)}$.

Lemma 4.8.3. *Let h be a solution to (4.8.5). If, for some $c > 0$ and for any $t \geq 0$,*

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-ct},$$

then

$$\|\nabla h\|_{S(L^2,[t,+\infty))} \lesssim e^{-\min\{c,(k+1^-)e_0\}t}.$$

In the spirit of Lemma 4.8.1, we prove the following estimate.

Lemma 4.8.4. *For $N \geq 6$, let h be a solution to*

$$\partial_t h + \tilde{\mathcal{L}}_k h = \epsilon, \tag{4.8.6}$$

with h and ϵ such that, for $t \geq 0$,

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-c_0 t},$$

$$\|\epsilon(t)\|_{\frac{2N}{N+2}} + \|\nabla \epsilon\|_{S'(L^2,[t,+\infty))} \lesssim e^{-c_1 t},$$

where $(k+1)e_0 > c_1 > c_0 > e_0$. Then,

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-c_1^- t}. \tag{4.8.7}$$

Proof of Lemma 4.8.4. Since the subscript k will be fixed in this proof, it will be omitted. By Lemma 4.8.3, we have

$$\|\nabla h(t)\|_{S(L^2,[t,+\infty))} \lesssim e^{-c_0 t}.$$

We first note that (4.8.6) can be written as

$$\partial_t h + \mathcal{L}h = \epsilon + (\mathcal{L} - \tilde{\mathcal{L}}_k)h.$$

Now, if $N > 6$ and $h \in \dot{H}^1$,

$$|(\mathcal{L} - \tilde{\mathcal{L}})h| \lesssim |\mathcal{V}(t)|^{p_c-1}|h| \lesssim e^{-(p_c-1)e_0 t}|W|^{p_c-1}|h|,$$

and

$$\begin{aligned}|\nabla[(\mathcal{L} - \tilde{\mathcal{L}})h]| &\lesssim |W|^{p_c-2}\left[e^{-e_0 t}|\nabla W| + |\nabla \mathcal{V}(t)|\right]|h| + |\mathcal{V}(t)|^{p_c-1}|\nabla h| \\ &\lesssim e^{-(p_c-1)e_0 t}\left[|W|^{p_c-1}|h| + |\nabla h|\right],\end{aligned}$$

where we used the fact that $\mathcal{V}(t) \in \mathcal{S}(\mathbb{R}^N)$, $\|\mathcal{V}(t)\|_{L^\infty} \lesssim e^{-e_0 t}$ and $|\nabla W| \lesssim |W|$.

Thus,

$$\|(\mathcal{L} - \tilde{\mathcal{L}})h\|_{\frac{2N}{N+2}} + \|\nabla[(\mathcal{L} - \tilde{\mathcal{L}})h]\|_{S'(L^2,[t,+\infty))} \lesssim e^{-[\min\{c_0,(k+1^-)e_0\}+(p_c-1)e_0]t}.$$

Therefore, by Lemma 4.8.1, since $c_0 > e_0$ by hypothesis,

$$\|h\|_{\dot{H}^1} \lesssim e^{-\min\{[c_0+(p_c-1)e_0],\,c_1\}^- t}$$

By iterating this argument, we get (4.8.7).

□

We now improve the convergence given by Propositions 4.6.1 and 4.7.8.

Lemma 4.8.5. *For $N \geq 6$, if u is a solution to (1.0.2) satisfying, for all $t \geq 0$,*

$$\|u(t) - W\|_{\dot{H}^1} \lesssim e^{-ct}, \quad E(u_0) = E(W), \tag{4.8.8}$$

then there exists a unique $A \in \mathbb{R}$ such that $u = U^A$.

Proof. Step 1. Linearize around W to improve the decay on time.

If u is a solution to (1.0.2), write $u = h + W$. Recall that h is a solution to (4.3.1). We will first show the bound

$$\|\nabla(R(h))\|_{S'(L^2,\,[t,+\infty))} + \|R(h(t))\|_{\frac{2N}{N+2}} \lesssim e^{-cp_c t} \text{ for all } t \geq 0. \tag{4.8.9}$$

Indeed, by Lemmas 4.3.12 and 4.3.9.((ii)),

$$\|\nabla(R(h))\|_{S'(L^2,\,[t,t+1])} \lesssim e^{-cp_c t}.$$

Therefore, triangle inequality gives

$$\|\nabla(R(h))\|_{S'(L^2,\,[t,+\infty))} \lesssim e^{-cp_c t}.$$

Now, by (4.3.11), we have

$$|R(h(t))| \leq |h(t)|^{p_c},$$

so that, by Sobolev inequality,

$$\|R(h(t))\|_{\frac{2N}{N+2}} \lesssim \|h(t)\|_{2^*}^{p_c} \lesssim e^{-cp_c t}.$$

Therefore, the bound (4.8.9) is proved.

We are now under the hypotheses of Lemma 4.8.1, with $c_0 = c$ and $c_1 = cp_c > c$. The conclusion of this Lemma gives

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-e_0 t} + e^{-cp_c^- t}.$$

If $c > e_0/p_c$, we get

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-e_0 t},$$

and, by the same argument used to prove (4.8.9),

$$\|\nabla(R(h))\|_{S'(L^2,\,[t,+\infty))} + \|R(h(t))\|_{\frac{2N}{N+2}} \lesssim e^{-e_0 p_c t}.$$

Thus, (4.8.4) gives

$$\|h(t) - Ae^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} \lesssim e^{-p_c^- e_0 t}. \tag{4.8.10}$$

If, however, $c \le e_0/p_c$, then assumption (4.8.8) holds with $\frac{1+p_c}{2}c > c$ instead of c. By iteration, we get (4.8.10).

Step 2. Linearize around $W + \mathcal{V}_k$ to improve higher order convergence to U^A.

For $k \ge 2$ to be chosen later, write $\tilde{h} = h - \mathcal{V}_k$, so that $\tilde{h}$ is a solution to (4.8.5). Since k is fixed throughout the proof, it will often be omitted. By Lemma 4.8.2, we have

$$\|\nabla(\tilde{R}(\tilde{h}))\|_{S'(L^2,[t,+\infty))} + \|\tilde{R}(\tilde{h}(t))\|_{\frac{2N}{N+2}} \lesssim_k \|\nabla\tilde{h}\|^{p_c}_{S(L^2,[t,+\infty))}.$$

Therefore, by (4.8.5),

$$\partial_t h + \tilde{\mathcal{L}}_k h = \eta,$$

with

$$\|\nabla\eta\|_{S'(L^2,[t,+\infty))} + \|\eta\|_{\frac{2N}{N+2}} \lesssim_k \|\nabla\tilde{h}\|^{p_c}_{S(L^2,[t,+\infty))} + e^{-(k+1)e_0 t}. \tag{4.8.11}$$

By (4.8.10) and the definition of $\mathcal{V}_k$, we have

$$\|\tilde{h}\|_{\dot{H}^1} \lesssim \|h - Ae^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} + O(e^{-2e_0 t}) \lesssim_k e^{-p_c^- e_0 t}. \tag{4.8.12}$$

By iteration, starting with (4.8.12), and repeatedly applying Lemmas 4.8.4 and 4.8.3, as well as estimate (4.8.11), we have, for any $k \ge 2$,

$$\|\tilde{h}\|_{\dot{H}^1} \lesssim_k e^{-(k+1^-)e_0 t}.$$

Therefore, choosing $k = l(k_0)$ where k_0 and l are defined in Proposition 4.4.2, we have, for $N \ge 6$ and $t \ge 0$,

$$\|D^\varepsilon(u - W - \mathcal{V}_{l(k_0)})\|_{S(\dot{H}^{1-\varepsilon},[t,+\infty))} \lesssim \|\nabla(u - W - \mathcal{V}_{l(k_0)})\|_{S(L^2,[t,+\infty))} \lesssim e^{-(k_0+\frac{3}{4})\frac{N-2}{4}e_0 t}.$$

Hence, by uniqueness in Proposition 4.4.2, we get that $u = U^A$.

□

Corollary 4.8.6. *Let $N \geq 6$. For any $A \neq 0$, there exists $T_A \in \mathbb{R}$ such that either*

$$U^A(t) = W^+(t + T_A), \text{ if } A > 0,$$

or

$$U^A(t) = W^-(t + T_A), \text{ if } A < 0.$$

Proof. Choose T_A such that $|A|e^{-e_0 T_A} = 1$. We have, by (4.4.8),

$$\|U^A(t + T_A) - W \mp e^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} \lesssim e^{-2e_0 t}. \tag{4.8.13}$$

Note that $U^A(t + T_A)$ satisfies the hypotheses of Lemma 4.8.5. Thus, there exists $\bar{A} \in \mathbb{R}$ such that $U^A(t + T_A) = U^{\bar{A}}$. But (4.8.13) implies that $\bar{A} = 1$, if $A > 0$, and $\bar{A} = -1$, if $A < 0$, finishing the proof of the corollary. □

4.8.2 Intercritical case

For $0 < s_c < 1$, we study the linearized approximate equation

$$\partial_t h + \mathcal{L}h = \epsilon \tag{4.8.14}$$

with h and ϵ such that, for $t \geq 0$,

$$\begin{aligned} \|h(t)\|_{H^1} &\lesssim e^{-c_0 t}, \\ \|\langle\nabla\rangle\epsilon(t)\|_{N(t,+\infty)} &\lesssim e^{-c_1 t}, \end{aligned} \tag{4.8.15}$$

where $c_1 > c_0 > 0$. We merely state the results in this case, as their proof is very close to the energy-critical case (in fact, some proofs are easier, since the L^2 norm of the solution is finite).

Lemma 4.8.7. *Under the assumptions* (4.8.15),

(i) if $e_0 \notin [c_0, c_1)$, then

$$\|h(t)\|_{H^1} \lesssim e^{-c_1^- t}, \tag{4.8.16}$$

(ii) if $e_0 \in [c_0, c_1)$, then there exists $A \in \mathbb{R}$ such that

$$\|h(t) - Ae^{-e_0 t}\mathcal{Y}_+\|_{H^1} \lesssim e^{-c_1^- t}. \tag{4.8.17}$$

Omitting A for simplicity and defining, for every k,

$$\begin{aligned}\tilde{\mathcal{L}}_k := &\begin{pmatrix} 0 & 1-\Delta \\ -(1-\Delta) & 0 \end{pmatrix} + \frac{(p+1)}{2}|Q+\mathcal{V}_k|^{p-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &+ \frac{(p-1)}{2}|Q+\mathcal{V}_k|^{p-3}\begin{pmatrix} \operatorname{Im}(Q+\mathcal{V}_k)^2 & -\operatorname{Re}(Q+\mathcal{V}_k)^2 \\ -\operatorname{Re}(Q+\mathcal{V}_k)^2 & -\operatorname{Im}(Q+\mathcal{V}_k)^2 \end{pmatrix},\end{aligned}$$

$$\tilde{K}_k(h) = \frac{(p+1)}{2}|Q+\mathcal{V}_k|^{p-1}h + \frac{(p-1)}{2}|Q+\mathcal{V}_k|^{p-3}(Q+\mathcal{V}_k)^2\bar{h},$$

and

$$\tilde{R}_k(h) := |Q+\mathcal{V}_k|^{p-1}(Q+\mathcal{V}_k)J((Q+\mathcal{V}_k)^{-1}h),$$

where

$$J(z) = |1+z|^{p-1}(1+z) - 1 - \frac{(p+1)}{2}z - \frac{(p-1)}{2}\bar{z},$$

we have that if, $u = e^{it}(Q+\mathcal{V}_k+h)$ satisfies (1.0.2), then h satisfies

$$\partial_t h + \tilde{\mathcal{L}}_k h = i\tilde{R}_k(h) + \epsilon_k, \tag{4.8.18}$$

or in the form of a Schrödinger equation,

$$i\partial_t h + \Delta h - h + \tilde{K}_k h = -\tilde{R}_k(h) + i\epsilon_k,$$

where ϵ_k are $O(e^{-(k+1)e_0 t})$ in $\mathcal{S}(\mathbb{R}^N)$. By the construction of $\mathcal{V}_k$, we have, for all $t \geq 0$,

$$|\mathcal{V}_k(t)| \lesssim e^{-e_0 t}|Q|,$$

and

$$|\nabla\mathcal{V}_k(t)| \lesssim e^{-e_0 t}|\nabla Q| \lesssim e^{-e_0 t}|Q|.$$

Therefore, as in the energy-critical case, we have the following results.

Lemma 4.8.8. *Let $p > 1$, $k \geq 1$ and I be a bounded time interval, and consider $f \in S(L^2, I)$ such that $\nabla f \in S(L^2, I)$. There exists $\alpha > 0$ such that the following estimates hold.*

For all $p > 1$:

(i) $\|\langle\nabla\rangle \tilde{K}_k(f)\|_{S'(L^2,\,I)} \lesssim |I|^\alpha \|\langle\nabla\rangle f\|_{S(L^2,\,I)}$.

For $p > 2$:

(ii) $\|\langle\nabla\rangle \tilde{R}_k(f))\|_{S'(L^2,\,I)} \lesssim \|\langle\nabla\rangle f)\|_{S(L^2,\,I)} \left(|I|^\alpha \|\langle\nabla\rangle f\|_{S(L^2,\,I)} + \|\langle\nabla\rangle f\|^{p-1}_{S(L^2,\,I)}\right)$.

For $1 < p \leq 2$:

(iii) $\|\langle\nabla\rangle \tilde{R}_k(f)\|_{S'(L^2,\,I)} \lesssim (1 + |I|^\alpha)\, \|\langle\nabla\rangle f\|^p_{S(L^2,\,I)}$.

Lemma 4.8.9. *Let h be a solution to (4.8.18). If, for some $c > 0$ and for any $t \geq 0$,*

$$\|h(t)\|_{H^1} \lesssim e^{-ct},$$

then

$$\|\langle\nabla\rangle h\|_{S(L^2,\,[t,+\infty))} \lesssim e^{-\min\{c\,,(k+1^-)e_0\}t}.$$

Lemma 4.8.10. *Let h be a solution to*

$$\partial_t h + \tilde{\mathcal{L}}_k h = \epsilon,$$

with h and ϵ such that, for $t \geq 0$,

$$\|h(t)\|_{H^1} \lesssim e^{-c_0 t},$$
$$\|\langle\nabla\rangle \epsilon\|_{S'(L^2,\,[t,+\infty))} \lesssim e^{-c_1 t},$$

where $(k+1)e_0 > c_1 > c_0 > e_0$. Then,

$$\|h(t)\|_{H^1} \lesssim e^{-c_1^- t}.$$

Lemma 4.8.11. *If u is a solution to (1.0.2) satisfying*

$$\|u(t) - e^{it}Q\|_{H^1} \lesssim e^{-ct}, \quad M(u_0) = M(Q), \quad E(u_0) = E(Q),$$

then there exists a unique $A \in \mathbb{R}$ such that $u = U^A$.

Corollary 4.8.12. *Let $1 + 4/N < p < 2^* - 1$. For any $A \neq 0$, there exists $T_A \in \mathbb{R}$ such that either*

$$U^A(t) = Q^+(t + T_A), \text{ if } A > 0,$$

or

$$U^A(t) = Q^-(t + T_A), \text{ if } A < 0.$$

4.9 Closure of the main theorems

Having proved Propositions 4.6.1 and 4.7.8, and Lemma 4.8.5, we can proceed as in [24].

Proof of Theorem 4.1.5. Recall the notation $\mathcal{Y}_1 = \operatorname{Re}\mathcal{Y}_+ = \operatorname{Re}\mathcal{Y}_-$. We claim that $(W, \mathcal{Y}_1)_{\dot{H}^1} \neq 0$. If not, since W solves the equation $\Delta W = -W^{p_c}$, we would have

$$B(W, \mathcal{Y}_\pm) = \frac{1}{2}\int \nabla W \cdot \nabla \mathcal{Y}_1 - \frac{p_c}{2}\int W^{p_c}\mathcal{Y}_1 = \frac{p_c}{2}\int \Delta W \mathcal{Y}_1 = 0,$$

so that $W \in G^\perp$. But, by Lemma 4.3.5, Φ is nonnegative (in fact, it is coercive) on $G^\perp$, which contradicts (4.3.8).

Replacing $\mathcal{Y}_\pm$, if necessary, we may assume

$$(W, \mathcal{Y}_1)_{\dot{H}^1} > 0.$$

Defining

$$W^\pm := U^{\pm 1},$$

we claim that the conclusions of Theorem 4.1.5 hold. By the strong convergence $U^A(t) \to W$ in $\dot{H}^1$ and energy conservation, we conclude $E(W^\pm) = E(W)$. Moreover, by (4.4.7),

$$\|U^A(t)\|^2_{\dot{H}^1} = \|W\|^2_{\dot{H}^1} + 2Ae^{-2e_0t}(W, \mathcal{Y}_1)_{\dot{H}^1} + O(e^{-3e_0t}),$$

which shows that $\|U^A(t)\|_{\dot H^1} - \|W\|_{\dot H^1}$ has the same sign as A, for large t. By uniqueness and continuity of the flow, this sign must remain the same for every t in the intervals of existence of $W^\pm$. By Proposition 4.7.4, W^- is defined on $\mathbb{R}$, and by Proposition 4.7.8, W^- scatters backward in time.

We now show that U^A has finite mass, if $N \geq 5$. Let, as in the proof of Proposition 4.7.4, ϕ be a smooth, positive, radial cutoff to the set $\{|x| \leq 1\}$. Define, for $R > 0$ and large t,

$$F_R(t) = \int |U^A(x,t)|^2 \phi(\frac{x}{R})\, dx.$$

Since U^A is a solution to (1.0.2), by Lemma 4.6.3 and Hardy's inequality, we have

$$|F_R'(t)| \lesssim \|U^A(t) - W\|_{\dot H^1} \left(\int \frac{1}{|x|^2} |U^A(t)|^2 \right)^{\frac{1}{2}} \lesssim \|U^A(t) - W\|_{\dot H^1} \|U^A(t)\|_{\dot H^1} \lesssim e^{-e_0 t}.$$

Hence, integrating from a large t to $+\infty$,

$$\left| F_R(t) - \int |W|^2 \phi(\frac{x}{R})\, dx \right| \lesssim e^{-e_0 t}.$$

Recalling that $W \in L^2(\mathbb{R}^N)$ if $N \geq 5$, we can make $R \to +\infty$ to obtain $M(U^A) = M(W) < +\infty$. In particular, $W^\pm \in L^2(\mathbb{R}^N)$ and, by Proposition 4.6.1, W^+ blows up in finite negative time. This finishes the proof of Theorem 4.1.5. □

Proof of Theorem 4.1.6. The case $\|u_0\|_{\dot H^1} < \|W\|_{\dot H^1}$ follows immediately from Proposition 4.7.8, Lemma 4.8.5 and Corollary 4.8.6. Case $\|u_0\|_{\dot H^1} = \|W\|_{\dot H^1}$ is a consequence of the variational characterization of W. Finally, $\|u_0\|_{\dot H^1} > \|W\|_{\dot H^1}$ follows from Proposition 4.6.1, Lemma 4.8.5 and Corollary 4.8.6. □

Proof of Theorem 4.1.7. Recall the notation $\mathcal{Y}_1 = \operatorname{Re} \mathcal{Y}_+ = \operatorname{Re} \mathcal{Y}_-$. We claim that $(Q, \mathcal{Y}_1)_{H^1} \neq 0$. If not, since Q solves the equation $Q - \Delta Q = -Q^p$, we would have

$$B(Q, \mathcal{Y}_\pm) = \frac{1}{2} \int Q \mathcal{Y}_1 + \frac{1}{2} \int \nabla Q \cdot \nabla \mathcal{Y}_1 - \frac{p}{2} \int Q^{p_c} \mathcal{Y}_1 = \frac{p+1}{2} (Q, \mathcal{Y}_1)_{H^1} = 0,$$

so that $Q \in \tilde G^\perp$. But, by Lemma 4.3.5, Φ is nonnegative (in fact, it is coercive) on $\tilde G^\perp$, which contradicts (4.3.7).

Replacing $\mathcal{Y}_\pm$, if necessary, we may assume

$$(Q, \mathcal{Y}_1)_{H^1} > 0.$$

Defining

$$Q^\pm := U^{\pm 1},$$

we claim that the conclusions of Theorem 4.1.7 hold. By the strong convergence $e^{-it}U^A(t) \to Q$ in H^1 and energy conservation, we conclude $M(Q^\pm) = M(Q)$ and $E(Q^\pm) = E(Q)$. Moreover, by (4.4.13),

$$\|U^A(t)\|_{H^1}^2 = \|Q\|_{H^1}^2 + 2Ae^{-2e_0 t}(Q, \mathcal{Y}_1)_{H^1} + O(e^{-2p_c^- e_0 t}),$$

which shows that $\|U^A(t)\|_{H^1} - \|Q\|_{H^1}$ has the same sign as A, for large t. By uniqueness and continuity of the flow, this sign must remain the same for every t in the intervals of existence of $Q^\pm$. By Proposition 4.7.16, Q^- is defined on $\mathbb{R}$, and by Proposition 4.7.19, Q^- scatters backward in time. Finally, by Proposition 4.6.4, Q^+ blows up in finite negative time. This finishes the proof of Theorem 4.1.7. □

Proof of Theorem 4.1.8. The case $\|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2}$ follows immediately from Proposition 4.7.19, Lemma 4.8.11 and Corollary 4.8.12. Case $\|\nabla u_0\|_{L^2} = \|\nabla Q\|_{L^2}$ is a consequence of the variational characterization of of Q. Finally, $\|\nabla u_0\|_{L^2} > \|\nabla Q\|_{L^2}$ follows from Proposition 4.6.4, Lemma 4.8.11 and Corollary 4.8.12. □

4.10 Auxiliary results

4.10.1 Spectral properties of the linearized operator

We prove here some results about the operator $\mathcal{L}$, following closely [25].

Proof of Lemma 4.3.2. In this proof, we will write $V = Q^{p-1}$ for $0 < s_c \leq 1$. Note that V defines a compact operator from H^1 to L^2.

Intercritical case. The operator $\mathcal{L}$ is a relatively compact perturbation of $i(1 - \Delta)$, and therefore has the same essential spectrum. We now prove the existence of exactly one negative eigenvalue to $\mathcal{L}$. From the proof of Lemma 4.3.5, we see that L_- on L^2 with domain H^2 is nonnegative. Since it is also self-adjoint, it has a unique square root $L_-^{\frac{1}{2}}$ with domain H^1. It is equivalent to show that the the self-adjoint operator $P := L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ on L^2 with domain H^4 has a unique negative eigenvalue. Indeed, consider the function

$$Z = Q_1 - \frac{(Q_1, Q)_{L^2}}{(Q, Q)_{L^2}} Q.$$

One can check that $Z \in H^2$, $Z \in \{Q\}^\perp$ and, for $0 < s_c \leq 1$,

$$(L_+ Z, Z)_{L^2} = -\frac{N^2(p-1)}{4(p+1)} \left[p - \left(1 + \frac{4}{N} \right) \right] \int Q^{p-1} < 0.$$

Defining $h := L_-^{-\frac{1}{2}} Z \in Q^\perp$, one also has

$$h = (L_-^{\frac{1}{2}} L_-^{-1})(L_-^{-1} L_-) Z = L_-^{-1} L_-^{-\frac{1}{2}} L_- Z \in H^3.$$

For $\varepsilon > 0$, choose $\tilde{h}_\varepsilon \in H^4$ such that $\tilde{h}_\varepsilon \perp Q$ and $\|h - \tilde{h}_\varepsilon\|_{H^3} < \varepsilon$. We have

$$\inf_{f \in H^4} \frac{(Pf, f)_{L^2}}{\|f\|_{L^2}^2} \leq \frac{(L_+ L_-^{\frac{1}{2}} \tilde{h}_\varepsilon, L_-^{\frac{1}{2}} \tilde{h}_\varepsilon)_{L^2}}{\|\tilde{h}_\varepsilon\|_{L^2}^2} < 0,$$

if ε is small enough.

Hence, by the mini-max principle, P has a negative eigenvalue $-e_0^2$ and an associated eigenfunction g. Defining $\mathcal{Y}_1 := L_-^{\frac{1}{2}} g$, $\mathcal{Y}_2 := \frac{1}{e_0} L_+ \mathcal{Y}_1$, and $\mathcal{Y}_\pm := \mathcal{Y}_1 \pm i \mathcal{Y}_2$ we have $\mathcal{L} \mathcal{Y}_\pm = \pm e_0$. Uniqueness of the negative eigenfunction of P follows from the non-negativity of L_+ on $\{Q^p\}^\perp$. The assertions about the kernel of $\mathcal{L}$ follow from the coercivity given by Lemma 4.3.5.

It remains to prove that $\mathcal{Y}_\pm \in S(\mathbb{R}^N)$. It suffices to prove this assertion for $\mathcal{Y}_1 = \operatorname{Re} \mathcal{Y}_+$. The differential equation for $\mathcal{Y}_1$ is

$$[(1-\Delta)^2 + e_0^2] \mathcal{Y}_1 = [pV^2 + V(1-\Delta)] \mathcal{Y}_1 - p(1-\Delta)[V \mathcal{Y}_1]. \tag{4.10.1}$$

Since the Fourier symbol of $(1-\Delta)^2 + e_0^2$ is $(1+|\xi|^2)^2 + e_0^2 \approx (1+|\xi|^2)^2$, and $V, \mathcal{Y}_1 \in H^2(\mathbb{R}^N)$, we have that $\mathcal{Y}_1 \in H^s$ for all $s \geq 0$. As in [24], we show that for all non-negative integers k, s and all $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\|\varphi(x/R)\mathcal{Y}_1\|_{H^s} \leq \frac{C(\varphi, s, k)}{R^k}, \text{ for all } R \geq 1. \tag{4.10.2}$$

The inequality (4.10.2) holds if $k = 0$, for any $s \geq 0$. By induction, we show that if it holds for (k, s), it also holds for $(k+1, s+1)$. Given φ, consider $\tilde{\varphi} \in C_0^\infty \mathbb{R}^N$ such that $\tilde{\varphi}$ is 1 on the support of φ, so that we have $\tilde{\varphi}\partial^\alpha\varphi = \partial^\alpha\varphi$ for any α. Since Q and its derivatives decay (more than) polynomially, (4.10.1) gives, for $s \geq 3$,

$$\|\varphi(x/R)[(1-\Delta)^2 + e_0^2]\mathcal{Y}_1\|_{H^{s-3}} \leq \frac{C}{R}\|\tilde{\varphi}(x/R)\mathcal{Y}_1\|_{H^{s-1}} \leq \frac{C}{R}\|\tilde{\varphi}(x/R)\mathcal{Y}_1\|_{H^s}.$$

Using the straightforward commutator estimate $\|[(1-\Delta)^2 + e_0^2; \phi(x/R)]\|_{H^{s-3} \to H^{s-3}} \leq C(N)/R$, we get

$$\|\varphi(x/R)\mathcal{Y}_1\|_{H^{s+1}} \approx \|[(1-\Delta)^2 + e_0^2](\varphi(x/R)\mathcal{Y}_1)\|_{H^{s-3}} \leq \frac{C}{R}\|\tilde{\varphi}(x/R)\mathcal{Y}_1\|_{H^s}.$$

By the induction hypothesis, we get $\|\varphi(x/R)\mathcal{Y}_1\|_{H^{s+1}} \leq C/R^{k+1}$, as desired. The same argument shows that, if $\lambda \in \mathbb{R}\backslash\sigma(\mathcal{L})$, then $(\lambda - \mathcal{L})^{-1}S(\mathbb{R}^N) \subset S(\mathbb{R}^N)$.

Critical case.

The range of the operator L_- is no longer closed, but the operator $1 + L_-$ is invertible on $\{Q\}^\perp$. Therefore, for any $\varepsilon > 0$, one can take $G_\varepsilon \in H^2$ such that

$$\|L_-G_\varepsilon - (1 + L_-)Z\|_{L^2} < \varepsilon.$$

Letting $h_\varepsilon := (1+L_-)^{-1}L_-^{\frac{1}{2}}G_\varepsilon = L_-^{\frac{1}{2}}(1+L_-)^{-1}G_\varepsilon = (1+L_-)^{-1}(1+L_-)^{-\frac{1}{2}}L_-^{\frac{1}{2}}(1+L_-)^{-\frac{1}{2}}G_\varepsilon \in H^3$, we have

$$\begin{aligned}\|L_-^{\frac{1}{2}}h_\varepsilon - Z\|_{H^2} &= \|(1-\Delta)(1+L_-)^{-1}[L_-G_\varepsilon - (1+L_-)Z]\|_{L^2} \\ &\leq \varepsilon\|[1 - V(1-\Delta)^{-1}]^{-1}\|_{L^2 \to L^2}.\end{aligned}$$

Choosing $\tilde{h}_\varepsilon \in H^4$ such that $\tilde{h}_\varepsilon \perp Q$ and $\|h_\varepsilon - \tilde{h}_\varepsilon\|_{H^3} < \varepsilon$, we get

$$(P\tilde{h}_\varepsilon, \tilde{h}_\varepsilon)_{L^2} = (L_+ Z, Z)_{L^2} + O(\varepsilon).$$

Thus, if ε is small enough, the conclusion follows. The regularity and the decay of $\mathcal{Y}_\pm$ follow analogously from the argument for the intercritical case. □

Proof of Lemma 4.3.5, energy-critical case.

Step 1. Coercivity in $G^\perp$. We adapt here the proof in [75] to our context.

Let $\Pi : S^N \to \mathbb{R}^N$ be the "stretched" stereographic projection of the sphere S^N onto $\mathbb{R}^N$, with respect to the North pole, defined by

$$y_i = \frac{1}{N(N-2)} \frac{x_i}{1 - x_{N+1}}, \quad 1 \leq i \leq N.$$

If $y = \Pi x$ and v is a real function is defined on $\mathbb{R}^N$, we define a function u on S^N by

$$u(x) = W^{-1}(y) v(y).$$

By integration by parts, one can check that

$$\int_{S^N} |\nabla_{S_N} u|^2 d\sigma = 2^{N-2} \int_{\mathbb{R}^N} |\nabla v|^2 - W^{p_c} v^2 dy,$$

and

$$\int_{S^N} u^2 d\sigma = \frac{2^N}{N(N-2)} \int_{\mathbb{R}^N} W^{p_c} v^2 dy.$$

The spectrum of Δ_{S^N} is well-known [8]. Namely, for the first eigenvalues λ_α, with multiplicity n_α and associated eigenfunctions u_α, we have

$$\begin{aligned}
&\lambda_0 = 0, && n_0 = 1, && u_0 = 1, \\
&\lambda_1 = N, && n_1 = N+1, && u_{1,j} = x_i, \quad 1 \leq i \leq N+1, \\
&\lambda_2 = 2(N+1).
\end{aligned}$$

Therefore, if $v \perp W$ in $\dot{H}^1$, then u is orthogonal to u_0, and we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 - W^{p_c} v^2 dy \geq \frac{4\lambda_1}{N(N-2)} \int_{\mathbb{R}^N} W^{p_c} v^2 dy,$$

which is equivalent to

$$\int_{\mathbb{R}^N} |\nabla v|^2 - W^{p_c} v^2 dy \geq \frac{4}{N+2} \int_{\mathbb{R}^N} |\nabla v|^2 dy.$$

Similarly, if $v \perp \text{span}\{W, \nabla W, W_1\}$ in $\dot{H}^1$, then u is orthogonal to u_0, $u_{1,i}$, $1 \leq i \leq N+1$, and thus

$$\int_{\mathbb{R}^N} |\nabla v|^2 - W^{p_c} v^2 dy \geq \frac{4\lambda_2}{N(N-2)} \int_{\mathbb{R}^N} W^{p_c} v^2 dy,$$

which is equivalent to

$$\int_{\mathbb{R}^N} |\nabla v|^2 - p_c W^{p_c} v^2 dy \geq \frac{4}{N+2} \int_{\mathbb{R}^N} |\nabla v|^2 dy.$$

Therefore, we proved that, for $h \in G^{\perp}$,

$$\Phi(h) \geq \frac{4}{N+2} \|h\|_{\dot{H}^1}.$$

Step 2. Coerciveness of Φ *in* $\tilde{G}^{\perp}$.

We first claim that $B(\mathcal{Y}_+, \mathcal{Y}_-) \neq 0$. If $B(\mathcal{Y}_+, \mathcal{Y}_-)$ was 0, then Φ would be identically 0 on $\text{span}\{\nabla W, iW, W_1, \mathcal{Y}_+, \mathcal{Y}_-\}$, a subspace of dimension $N+4$. But this cannot happen, given Φ is positive definite on $G^{\perp}$, which is of codimension $N+3$.

We now show that $\Phi(h) > 0$ on $\tilde{G}^{\perp} \backslash \{0\}$. Assume, by contradiction, that there exists $h \in \tilde{G}^{\perp} \backslash \{0\}$ such that $\Phi(h) \leq 0$. Recall that $\ker \mathcal{L} = \text{span}\{\nabla W, iW, W_1\}$, and that, by definition of $\tilde{G}^{\perp} \backslash \{0\}$, $B(\mathcal{Y}_+, h) = 0$. Hence, the vectors $\partial_k W$, $k \leq N$, iW, W_1, $\mathcal{Y}_+$ and h are mutually orthogonal under the symmetric form B. Since

$$\Phi(\partial_k W) = \Phi(iW) = \Phi(W_1) = \Phi(\mathcal{Y}_+) = 0,$$

we get

$$\Phi_{|\text{span}\{\nabla W, iW, W_1, \mathcal{Y}_+, h\}} \leq 0.$$

We claim that these vectors are independent. Indeed, if

$$\sum_k \alpha_k \partial_k W + \beta i W + \gamma W_1 + \delta \mathcal{Y}_+ + \epsilon h = 0,$$

then

$$\delta B(\mathcal{Y}_+, \mathcal{Y}_-) = 0$$

and since $B(\mathcal{Y}_+, \mathcal{Y}_-) \neq 0$, $\delta = 0$. Therefore, the claim is proven, since $\partial_k W$, iW, W_1 and h are orthogonal in the real Hilbert space $\dot{H}^1$.

To prove coercivity, we rely on a compactness argument. Suppose, by contradiction, that there exists $\{h_n\} \subset \tilde{G}^\perp$ such that

$$\lim \Phi(h_n) = 0, \quad \|h_n\|_{\dot{H}^1} = 1.$$

Up to a subsequence, we may assume $h_n \rightharpoonup h^*$ weakly in $\dot{H}^1$. This implies $h^* \in \tilde{G}^\perp$. Since the operator $\int W^{p_c - 1} |\cdot|^2$ is compact, we have $\int W^{p_c-1}|h_*|^2 > 0$ and

$$\Phi(h^*) \leq \liminf \Phi(h_n) = 0.$$

This contradicts the strict positivity of Φ on $\tilde{G}^\perp \backslash \{0\}$.

□

Proof of Lemma 4.3.5, intercritical case. Since the explicit formula for Q in the intercritical case is not available, we cannot proceed as in the energy-critical case. We follow here [87] and [25].

Step 1. Non-negativity on $G^\perp$. Define the functional

$$J(u) = \frac{\left(\int |\nabla u|^2\right)^a \left(\int |u|^2\right)^b}{\int |u|^{p+1}},$$

where

$$a = \frac{N(p-1)}{4}, \quad b = \frac{2p+2-N(p-1)}{4}.$$

By the sharp Gagliardo-Nirenberg inequality, we see that this functional achieves an absolute minimum at Q. Therefore, the minimization condition $\frac{d^2}{d\varepsilon^2}J(Q+\varepsilon h)_{|\varepsilon=0} \geq 0$ for all functions $h \in H^1$ gives

$$\Phi(h) \geq \frac{b}{\left(\int Q^2\right)}\left[\frac{1}{a}\left(\int \Delta Q h_1\right)^2 + \frac{1}{b}\left(\int Q h_1\right)^2 - \left(\int Q^p h_1\right)^2\right].$$

Since a and b are positive if $0 < s_c < 1$, we have that $\Phi(h) \geq 0$ if $\int Q^p h_1 = 0$. Therefore, Φ must be non-negative on $G^\perp$.

Step 2. Coercivity on $G^\perp$. We now employ compactness to show that, for every real function $h \in G^\perp$,

$$(L_+h, h)_{L^2} \gtrsim \|h\|_{L^2}, \text{ and } (L_-h, h)_{L^2} \gtrsim \|h\|_{L^2}$$

If we prove the last inequalities, then (again) by compactness, the coercivity follows. Suppose that there is a sequence of real H^1 functions h_n in $G^\perp$ such that

$$\lim_{n\to\infty}(L_+h_n, h_n)_{L^2} = \Phi(h_n) = 0, \text{ and } \|h_n\|_{L^2} = 1.$$

This implies

$$0 \leq \frac{1}{2}\int |\nabla h_n|^2 = -\frac{1}{2} + \frac{p}{2}\int Q^{p-1}h_n^2 + \Phi(h_n) \lesssim 1.$$

Therefore, $\|\nabla h_n\| \lesssim 1$ and, for large n, $\int Q^{p-1}h_n^2 \gtrsim 1$. Passing to a subsequence, and recalling that Q decays at infinity, we get that there exists $h_* \in G^\perp$ such that

$$h_n \rightharpoonup h_* \text{ weakly in } H^1, \text{ and } \int Q^{p-1}h_*^2 > 0.$$

In particular, $h_* \neq 0$. Moreover,

$$\Phi(h_*) \leq \frac{1}{2}\liminf_{n+\to\infty}\|h_n\|_{H^1} - \frac{p}{2}\lim_{n\to\infty}\int Q^{p-1}h_n^2 = \liminf_{n+\to\infty}\Phi(h_n) = 0.$$

Recall that $\Phi(h_*) \geq 0$ by Step 1. Therefore, $\Phi(h_*) = 0$ and h_* is the solution to the minimization problem

$$0 = (L_+h_*, h_*)_{L^2} = \min_{f\in E}(L_+h, h)_{L^2}, \text{ where}$$

$$E := \left\{ h \in H^1; \|h\|_{L^2} = \|h_*\|_{L^2} \text{ and } h \in G^{\perp} \right\}.$$

Thus, there exist Lagrange multipliers $\lambda_0, \cdots, \lambda_{N+1}$ such that

$$L_+ h_* = \lambda_0 Q^p + \sum_{j=1}^{N} \lambda_j \partial_j Q + \lambda_{N+1} h_*.$$

Since $h_* \in G^{\perp} \backslash \{0\}$ and $(L_+ h_*, h_*)_{L^2} = 0$, we have $\lambda_{N+1} = 0$. By testing the last equation against $\partial_j Q$ and using that $L_+(\partial_k Q) = 0$, for all $k \leq N$, we conclude that

$$L_+ h_* = \lambda_0 Q^p.$$

Recalling that that $L_+ Q = -\frac{p-1}{2} Q^p$ and that $\ker(L_+) = \text{span}\{\nabla Q\}$, we conclude that there exist $\mu, \cdots, \mu_N$ such that

$$h_* = -\frac{2\lambda_0}{p-1} Q + \sum_{j=1}^{N} \mu_j \partial_j Q.$$

Noting that $\int Q \partial_j Q = \frac{1}{2} \int \partial_j (Q^2) = 0$, and recalling that $h_* \in G^{\perp}$ gives $\mu_j = 0$ for all j. Therefore,

$$h_* = \frac{2\lambda_0}{p-1} Q.$$

And, by direct calculation,

$$(L_+ h_*, h_*)_{L^2} = -\left(\frac{2\lambda_0}{p-1} \right)^2 \int Q^{p+1} < 0.$$

This contradicts $(L_+ h_*, h_*)_{L^2} \geq 0$ and $h_* \neq 0$, and proves that

$$(L_+ h, h)_{L^2} \gtrsim \|h\|_{L^2}$$

for any real function $h \in G^{\perp}$. The proof for L_- is analogous. In particular, we have strict positivity of Φ on $G^{\perp} \backslash \{0\}$ and, by compactness, the coercivity follows on $G^{\perp}$.

Step 3. Coercivity on $\tilde{G}^{\perp}$. The proof relies on a (co)dimensional argument, together with compactness, as in the energy-critical case. □

4.10.2 Proof of modulation results

Proof of Lemma 4.5.1. The proof is already classical (see, for instance, [25, Section 7.1] and [70, Section 2]). We first show the lemma when u is close to W. Define the functionals $J = (J_0, \cdots, J_{N+1})$ on $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R} \times \dot{H}^1$ as

$$\begin{aligned} J_0 &: (\theta, x, \lambda, u) \mapsto (f_{[x,\lambda,\theta]}, iW)_{\dot{H}^1}, \\ J_k &: (\theta, x, \lambda, u) \mapsto (f_{[x,\lambda,\theta]}, \partial_k W)_{\dot{H}^1},\ 0 \le k \le N, \\ J_{N+1} &: (\theta, x, \lambda, u) \mapsto (f_{[x,\lambda,\theta]}, W_1)_{\dot{H}^1}. \end{aligned}$$

By direct calculation, one can check that

$$\det\left(\frac{\partial J}{\partial(\theta, x, \lambda)}\right) = \left(\int |W|^2\right)\left(\prod_k \int |\partial_k W|^2\right)\left(-\int |W_1|^2\right) \neq 0,$$

and that $J(0,1,0,W) = 0$. Hence, by the Implicit Function Theorem, there exist ϵ_0, η_0 such that, if $f \in \dot{H}^1$ and $\|f - W\|_{\dot{H}^1} < \epsilon_0$, then there exists a unique n-tuple (x, λ, θ) such that

$$|x| + |\lambda| + |\theta - 1| \le \eta_0, \text{ and } J(\theta, x, \lambda, f) = 0.$$

Now, if u is as in the lemma, by the variational characterization of W, if $d(u)$ is small, then there exists $(x_0, \lambda_0, \theta_0)$ such that $u_{[x_0,\lambda_0,\theta_0]} = W + f$, with $\|f\|_{\dot{H}^1} \le \epsilon(d(f))$. We are thus back to the preceding case. Existence, local uniqueness and regularity follow again from the Implicit Function Theorem. □

Proof of Lemma 4.5.2. For a fixed t, write $v = u_{[x(t),\lambda(t),\theta(t)]}(t) - W = \alpha(t)W + h(t)$ as in (4.5.1). Since $h \in G^\perp$, we have

$$\|v\|_{\dot{H}^1}^2 = \alpha^2 \|W\|_{\dot{H}^1}^2 + \|h\|_{\dot{H}^1}^2. \tag{4.10.3}$$

Since $h \in G^\perp$, and W satisfies the equation $\Delta W + W^{p_c} = 0$, we have

$$B(W, h) = \frac{1}{2}\int \nabla W \cdot \nabla h_1 + \frac{p}{2}\int \Delta W h_1 = 0.$$

Therefore, $\Phi(v) = \Phi(\alpha W + h) = \Phi(W)\alpha^2 + \Phi(h)$. Recalling that W is a critical point for the energy functional E, we have $E(W+v) = E(W) + \Phi(v) + O(\|v\|^3_{\dot{H}^1})$. Since $E(W+v) = E(W)$, and by the coercivity given by Lemma 4.3.5, we have $\Phi(h) \approx \|h\|_{\dot{H}^1}$. Thus, we have

$$\alpha^2 \lesssim \|h\|^2_{\dot{H}^1} + \|v\|^3_{\dot{H}^1} \tag{4.10.4}$$

and

$$\|h\|^2_{\dot{H}^1} \lesssim \alpha^2 + \|v\|^3_{\dot{H}^1}. \tag{4.10.5}$$

Since $\|v\|_{\dot{H}^1}$ is small when $d(u)$ is small, estimates (4.10.3), (4.10.4), and (4.10.5) give $|\alpha| \approx \|h\|_{\dot{H}^1} \approx \|v\|_{\dot{H}^1}$. Finally, since

$$d(u) = \left| \|W+v\|^2_{\dot{H}^1} - \|W\|^2_{\dot{H}^1} \right| = |\|v\|^2_{\dot{H}^1} + 2\alpha\|W\|^2_{\dot{H}^1}|,$$

we have $d(u) \approx |\alpha|$, and (4.5.2) is proved.

It remains to prove (4.5.3). Consider the variables y and s given by

$$y = \frac{x}{\mu(t)}, \text{ and } dt = \frac{1}{\mu^2(t)} ds.$$

In view of (4.5.2) and the decomposition (4.5.1), we can rewrite (1.0.2) as

$$i\partial_s h + \Delta h - i\alpha_s W + \theta_s W - ix_s \cdot \nabla W + i\frac{\lambda_s}{\lambda} W_1 = O\left(\epsilon(s)\right) \text{ in } \dot{H}^1, \tag{4.10.6}$$

where $\epsilon(s) := d\left(d + |\theta_s| + |x_s| + \left|\frac{\lambda_s}{\lambda}\right|\right)$. Since $h \in G^\perp$, projecting (4.10.6) in $\dot{H}^1$ onto W, iW, ∇W and W_1 and integrating by parts (possible due to a standard regularization argument) yields

$$|\alpha_s| + |\theta_s| + |x_s| + \left|\frac{\lambda_s}{\lambda}\right| = O(d + \epsilon(s)),$$

which is enough to conclude (4.5.3) and finishes the proof of Lemma 4.5.2. □

Proof of Lemma 4.5.3. The proof is analogous to the proof of Lemma 4.5.1 and will be omitted. □

Proof of Lemma 4.5.4. The orthogonality condition (4.3.5) implies $B(Q,h) = 0$. Since $E(u) = E(Q)$, and Q is a critical point for E, we have

$$\alpha^2\Phi(Q) + \Phi(h) = \Phi(\alpha Q + h) = O(|\alpha|^3 + \|h\|^3_{\dot{H}^1}).$$

By coercivity, $\Phi(h) \approx \|h\|_{H^1}$, and hence $|\alpha| \approx \|h\|_{H^1}$. The relation $M(u) = M(Q)$ gives

$$\left|\alpha \int Q^2 + \int Qh_1\right| = \frac{1}{2}\int |\alpha Q + h|^2 = O(|\alpha|^2), \tag{4.10.7}$$

and thus

$$|\alpha| \approx \left|\int Qh_1\right|.$$

Now, using (4.3.5),

$$d(u) = \left|\int |\nabla u|^2 - \int |\nabla Q|^2\right| = \left|2\left(\alpha \int |\nabla Q|^2 - \int Qh_1\right) + O(|\alpha|^2)\right|,$$

which, together with (4.10.7), gives

$$d(u) = \left|2\alpha\left(\int |\nabla Q|^2 - \int Q^2\right) + O(|\alpha|^2)\right|.$$

Since, by Pohozaev identities (2.6.2), $\|\nabla Q\|_{L^2} \neq \|Q\|_{L^2}$ for any N and any p in the intercritical range, we conclude $d(u) \approx |\alpha|$ and hence (4.5.6) holds. The rest of the proof goes along the same lines of the proof of Lemma 4.5.2 (without the need of self-similar variables), and will be omitted.

□

4.10.3 Proof of an incertainty principle

Proof of Lemma 4.6.3. Let $\delta(f) = \int |\nabla W|^2 - \int |\nabla f|^2$ and $\lambda \in \mathbb{R}$. By Sobolev inequality

$$\|\nabla(e^{i\lambda\varphi} f)\|_2 \geq \frac{\|\nabla W\|_2}{\|W\|_{2^*}}\|f\|_{2^*}.$$

Squaring the last inequality and expanding the term $\|\nabla(e^{i\lambda\varphi} f)\|_2$,

$$\lambda^2 \int |\nabla\varphi|^2|\nabla f|^2 + 2\lambda \operatorname{Im}\int (\nabla\varphi \cdot \nabla f)\bar{f} + \int |\nabla f|^2 - \frac{\|\nabla W\|_2^2}{\|W\|_{2^*}^2}\|f\|_{2^*}^2 \geq 0.$$

The discriminant of this quadratic form must be non-positive, and we have

$$\left(\operatorname{Im}\int (\nabla\varphi \cdot \nabla f)\bar{f}\right)^2 \leq \left(\int |\nabla f|^2 - \frac{\|\nabla W\|_2^2}{\|W\|_{2^*}^2}\|f\|_{2^*}^2\right)\left(\int |\nabla\varphi|^2|\nabla f|^2\right)$$

Since

$$\int |\nabla f|^2 = \int |\nabla W|^2 - \delta(f),$$

we have, by $E(f) = E(W)$,

$$0 < \int |f|^{2^*} = \int |W|^{2^*} - \frac{N}{N-2}\delta(f).$$

Therefore,

$$\begin{aligned}\int |\nabla f|^2 - \frac{\|\nabla W\|_2^2}{\|W\|_{2^*}^2}\|f\|_{2^*}^2 &= \int |\nabla W|^2 - \delta(f) - \frac{\|\nabla W\|_2^2}{\|W\|_{2^*}^2}\left(\int |W|^{2^*} - \frac{N}{N-2}\delta(f)\right)^{\frac{N-2}{N}} \\ &= \int |\nabla W|^2 - \delta(f) - \frac{\|\nabla W\|_2^2}{\|W\|_{2^*}^2}\left(\|W\|_{2^*}^2 - \delta(f) + O(\delta(f)^2)\right) \\ &= O(\delta(f)^2),\end{aligned}$$

and Lemma 4.6.3 is proved. □

Proof of Lemma 4.6.6. The proof is analogous to the proof of Lemma 4.6.3 and will be omitted. □

Proof of Lemma 4.8.1. By Lemma 4.3.12, we can assume that

$$\|h(t)\|_{S(L^2,\,[t,+\infty))} \lesssim e^{-c_0 t}. \tag{4.10.8}$$

We first normalize the eigenfunctions of $\mathcal{L}$. Define

$$f_0 := \frac{iW}{\|W\|_{\dot{H}^1}}, \quad f_k := \frac{\partial_k W}{\|\partial_k W\|_{\dot{H}^1}}, \quad f_{N+1} := \frac{W_1}{\|W_1\|_{\dot{H}^1}}.$$

We have

$$B(f_k, h) = 0, \quad \|f_j\|_{\dot{H}^1} = 1, \quad \forall k \leq N+1, \forall h \in \dot{H}^1.$$

Recall that $B(\mathcal{Y}_+, \mathcal{Y}_-) \neq 0$. Normalize $\mathcal{Y}_+$, $\mathcal{Y}_-$ such that $B(\mathcal{Y}_+, \mathcal{Y}_-) = 1$. Next, write

$$h(t) = \alpha_+(t)\mathcal{Y}_+ + \alpha_-(t)\mathcal{Y}_- + \sum_k \beta_k(t) f_k + g(t), \quad g(t) \in \tilde{G}^\perp, \tag{4.10.9}$$

where, recalling that $\mathcal{L}_{|\text{span}\{f_k,\, k\leq N+1\}} = 0$ and that $\Phi(\mathcal{Y}_+) = \Phi(\mathcal{Y}_-) = 0$,

$$\alpha_+(t) = B(h(t), \mathcal{Y}_-), \quad \alpha_-(t) = B(h(t), \mathcal{Y}_+), \tag{4.10.10}$$

$$\beta_k(t) = (h(t), f_k)_{\dot{H}^1} - \alpha_+(t)(\mathcal{Y}_+, f_k)_{\dot{H}^1} - \alpha_-(t)(\mathcal{Y}_-, f_k)_{\dot{H}^1}, \quad \forall k \leq N+1 \tag{4.10.11}$$

Step 1. Differential inequalities on the coefficients. We will show

$$\frac{d}{dt}\Big(e^{e_0 t}\alpha_+(t)\Big) = e^{e_0 t}B(\mathcal{Y}_-, \epsilon), \quad \frac{d}{dt}\Big(e^{-e_0 t}\alpha_-(t)\Big) = e^{-e_0 t}B(\mathcal{Y}_+, \epsilon), \tag{4.10.12}$$

$$\frac{d}{dt}\Big(e^{-e_0 t}\beta_k(t)\Big) = (f_k, \epsilon)_{\dot{H}^1} - (\mathcal{Y}_+, f_k)_{\dot{H}^1}B(\mathcal{Y}_-, \epsilon) - (\mathcal{Y}_-, f_k)_{\dot{H}^1}B(\mathcal{Y}_+, \epsilon) - (\mathcal{L}g, f_k)_{\dot{H}^1}, \tag{4.10.13}$$

$$\frac{d\Phi(h(t))}{dt} = 2B(h, \epsilon). \tag{4.10.14}$$

By equation (4.8.1),

$$\begin{aligned}\alpha_+'(t) &= B(\partial_t h, \mathcal{Y}_-) = B(-\mathcal{L}h + \epsilon, \mathcal{Y}_-)\\ &= B(h, \mathcal{L}\mathcal{Y}_-) + B(\epsilon, \mathcal{Y}_-) = -e_0\alpha_+(t) + B(\epsilon, \mathcal{Y}_-).\end{aligned}$$

This yields the first equation in (4.10.12). The second equation follows similarly.

Now, differentiating (4.10.11), we obtain

$$\beta_k' = (-\mathcal{L}h + \epsilon - \alpha_+'\mathcal{Y}_+ - \alpha_-'\mathcal{Y}_-, f_k)_{\dot{H}^1}.$$

Note that $\mathcal{L}h = e_0\alpha_+\mathcal{Y}_+ - e_0\alpha_-\mathcal{Y}_- + \mathcal{L}g$, by (4.10.9), which proves (4.10.13), in view of (4.10.12).

Finally, differentiating $\Phi(h(t))$,

$$\frac{d}{dt}\Phi(t) = 2B(h, \partial_t h) = -2B(h, \mathcal{L}h) + 2B(h, \epsilon) = 2B(h, \epsilon),$$

by the skew-symmetry of $\mathcal{L}$ in (4.3.3). Equation (4.10.14) is then proved.

Step 2. Estimates on $\alpha_\pm$. We claim

$$|\alpha_-(t)| \lesssim e^{-c_1 t} \tag{4.10.15}$$

$$|\alpha_+(t)| \lesssim \begin{cases} e^{-c_1 t} & \text{if } e_0 < c_0, \\ e^{-e_0 t} + e^{-c_1^- t} & \text{if } e_0 \geq c_0 \end{cases} \tag{4.10.16}$$

We will need the following claim, which is an immediate application of Hölder inequality.

Claim 4.10.1. *If I is a finite time interval, $f \in L_I^\infty L^{\frac{2N}{N-2}}$, $g \in L_I^\infty L_x^{\frac{2N}{N+2}}$ are such that $\nabla f \in L_I^2 L_x^{\frac{2N}{N-2}}$, $\nabla g \in L_I^2 L_x^{\frac{2N}{N+2}}$, then*

$$\int_I |B(f(t), g(t))| dt \lesssim \|\nabla f\|_{L_I^2 L_x^{\frac{2N}{N-2}}} \|\nabla g\|_{L_I^2 L_x^{\frac{2N}{N+2}}} + |I| \|f\|_{L_I^\infty L_x^{\frac{2N}{N-2}}} \|g\|_{L_I^\infty L_x^{\frac{2N}{N+2}}}.$$

The above claim, (4.10.8) and (4.10.12) yield

$$\int_t^{t+1} |e^{-e_0 s} B(\mathcal{Y}_+, \epsilon(s))| ds \leq e^{-e_0 t} \int_t^{t+1} |B(\mathcal{Y}_+, \epsilon(s))| ds \lesssim e^{-(e_0+c_1)t}.$$

By triangle inequality, integrating the second equation in (4.10.12) gives

$$|\alpha_-(t)| \lesssim e^{e_0 t} \int_t^{+\infty} |e^{-e_0 s} B(\mathcal{Y}_+, \epsilon(s))| ds \lesssim e^{-c_1 t},$$

which proves (4.10.15).

To prove (4.10.16), consider first the case $c_0 > e_0$. Then, by (4.10.8) and (4.10.10), $e^{e_0 t}\alpha_+(t)$ vanishes as $t \to +\infty$. By Claim 4.10.1

$$\int_t^{t+1} |e^{e_0 s} B(\mathcal{Y}_+, \epsilon(s))| ds \lesssim e^{e_0 t} \int_t^{t+1} |B(\mathcal{Y}_+, \epsilon(s))| ds \lesssim e^{(e_0-c_1)t},$$

integrating the equation on α_+ in (4.10.12), recalling that $c_1 > c_0$, and using triangle inequality, we get (4.10.16) if $c_0 > e_0$.

Assume now that $c_0 \leq e_0$. Integrating (4.10.12),

$$|\alpha_+(t) - e^{-e_0 t}\alpha_+(0)| \leq e^{-e_0 t} \int_0^t e^{e_0 s} |B(\mathcal{Y}_-, \epsilon(s))| ds \lesssim e^{-c_1^- t},$$

and the proof of (4.10.16) is finished.

Step 3. Bounds on g and β_k. We will prove

$$\|g(t)\|_{\dot{H}^1} + \sum_k |\beta_k(t)| \lesssim e^{-\frac{(c_0+c_1)}{2}t}. \tag{4.10.17}$$

Again by Claim 4.10.1, $\int_t^{t+1} |B(h(s),\epsilon(s))|ds \lesssim e^{-(c_0+c_1)t}$. By triangle inequality, integrating (4.10.14), we get

$$\Phi(h(t)) \lesssim e^{-(c_0+c_1)t}.$$

Therefore,

$$|2\alpha_+\alpha_- B(\mathcal{Y}_+,\mathcal{Y}_-) + \Phi(g)| = |\Phi(h)| \lesssim e^{-(c_0+c_1)t}.$$

By Step 2,

$$|\Phi(g)| \lesssim \begin{cases} e^{-(c_0+c_1)t} + e^{-2c_1 t} & \text{if } c_0 > e_0, \\ e^{-(c_0+c_1)t} + e^{-c_1 t}(e^{-e_0 t} + e^{-c_1^- t}) & \text{if } c_0 \le e_0. \end{cases}$$

In any case, $|\Phi(g)| \le e^{-(c_0+c_1)t}$. Using the coercivity of Φ, given by Lemma 4.3.5, estimate for g in (4.10.17) is proven.

Consider now (4.10.13). By (4.10.8),

$$\beta_k(t+1)-\beta_k(t) \lesssim e^{-c_1 t} + \int_t^{t+1} |(f_k,\mathcal{L}g(s))_{\dot{H}^1}|ds = e^{-c_1 t} + \int_t^{t+1} \left|\mathrm{Re}\int \mathcal{L}^*(\Delta f_k)\overline{g}(s)\right| d(s),$$

where $\mathcal{L}^* = \begin{pmatrix} 0 & L_+ \\ -L_- & 0 \end{pmatrix}$ is the L^2-adjoint of $\mathcal{L}$.

One can check explicitly that, for any $0 \le k \le N+1$, $|\mathcal{L}^*(\Delta f_k)| \lesssim \frac{1}{1+|x|^{N+4}}$. Therefore, $\mathcal{L}^*(\Delta f_k) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, so that, by the estimate on g in (4.10.17),

$$\left|\mathrm{Re}\int \mathcal{L}^*(\Delta f_k)\overline{g}(t)\right| \lesssim \|g(t)\|_{\frac{2N}{N-2}} \lesssim \|g\|_{\dot{H}^1} \lesssim e^{-\frac{(c_0+c_1)}{2}t}$$

Step 4. Closure

By the decomposition (4.10.9), as well as steps 2 and 3, so far we have

$$\|h(t)\|_{\dot{H}^1} \lesssim \begin{cases} e^{-\frac{(c_0+c_1)}{2}t} & \text{if } c_0 > e_0 \\ e^{-e_0 t} + e^{-\frac{(c_0+c_1)}{2}t} & \text{if } c_0 \le e_0. \end{cases}$$

Now, if $e_0 \notin [c_0, c_1)$, by iterating the argument, we obtain

$$\|h(t)\|_{\dot{H}^1} \lesssim e^{-c_1^- t},$$

which proves (4.8.3).

Assume now $e_0 \in [c_0, c_1)$. Then, estimate (4.10.12) on α_+ ensures the existence of a limit A to $e^{e_0 t}\alpha_+(t)$, as $t \to +\infty$. Integrating (4.10.12) from t to $+\infty$,

$$|A - e^{e_0 t}\alpha_+| \leq e^{e_0 t} \int_t^{+\infty} |B(\mathcal{Y}_+, \epsilon(s))| ds \lesssim e^{(e_0 - c_1)t}.$$

In view of decomposition (4.10.9) and estimates (4.10.15), (4.10.16) and (4.10.17), we get

$$\|h(t) - Ae^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} \lesssim e^{-\frac{(c_0+c_1)}{2}t}.$$

Since $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$, we see that $\tilde{h}(t) := h(t) - Ae^{-e_0 t}\mathcal{Y}_+$ satisfies the differential equation (4.8.1) with the same ϵ, and with c_0 replaced by $\frac{c_0+c_1}{2} > c_0$ in condition (4.8.2). By iterating the argument a finite number of times, we end up under condition (4.8.3), which implies condition (4.8.4) for the original h, and finishes the proof of Lemma 4.8.1. □

Proof of Lemma 4.8.7. We first normalize the eigenfunctions of $\mathcal{L}$. Define

$$f_0 := \frac{iQ}{\|Q\|_2}, \quad f_k := \frac{\partial_k Q}{\|\partial_k Q\|_2},$$

We have

$$B(f_k, h) = 0, \quad \|f_j\|_2 = 1, \quad \forall k \leq N, \, \forall h \in H^1.$$

Recall that $B(\mathcal{Y}_+, \mathcal{Y}_-) \neq 0$. Normalize $\mathcal{Y}_+$, $\mathcal{Y}_-$ such that $B(\mathcal{Y}_+, \mathcal{Y}_-) = 1$. Next, write

$$h(t) = \alpha_+(t)\mathcal{Y}_+ + \alpha_-(t)\mathcal{Y}_- + \sum_k \beta_k(t) f_k + g(t), \quad g(t) \in \tilde{G}^{\perp}. \tag{4.10.18}$$

where, recalling that $\mathcal{L}_{|\text{span}\{f_k, k \leq N\}} = 0$ and that $\Phi(\mathcal{Y}_+) = \Phi(\mathcal{Y}_-) = 0$,

$$\alpha_+(t) = B(h(t), \mathcal{Y}_-), \quad \alpha_-(t) = B(h(t), \mathcal{Y}_+), \tag{4.10.19}$$

$$\beta_k(t) = (h(t), f_k)_{H^1} - \alpha_+(t)(\mathcal{Y}_+, f_k)_{H^1} - \alpha_-(t)(\mathcal{Y}_-, f_k)_{H^1}, \quad \forall k \leq N. \tag{4.10.20}$$

Step 1. Differential inequalities on the coefficients. We will show

$$\frac{d}{dt}\left(e^{e_0 t}\alpha_+(t)\right) = e^{e_0 t}B(\mathcal{Y}_-, \epsilon), \quad \frac{d}{dt}\left(e^{-e_0 t}\alpha_-(t)\right) = e^{-e_0 t}B(\mathcal{Y}_+, \epsilon), \tag{4.10.21}$$

$$\frac{d}{dt}\left(e^{-e_0 t}\beta_k(t)\right) = (f_k, \epsilon)_{H^1} - (\mathcal{Y}_+, f_k)_{H^1}B(\mathcal{Y}_-, \epsilon) - (\mathcal{Y}_-, f_k)_{H^1}B(\mathcal{Y}_+, \epsilon) - (\mathcal{L}g, f_k)_{H^1}, \tag{4.10.22}$$

$$\frac{d\Phi(h(t))}{dt} = 2B(h, \epsilon). \tag{4.10.23}$$

By equation (4.8.14),

$$\begin{aligned} \alpha_+'(t) &= B(\partial_t h, \mathcal{Y}_-) = B(-\mathcal{L}h + \epsilon, \mathcal{Y}_-) \\ &= B(h, \mathcal{L}\mathcal{Y}_-) + B(\epsilon, \mathcal{Y}_-) = -e_0\alpha_+(t) + B(\epsilon, \mathcal{Y}_-). \end{aligned}$$

This yields the first equation in (4.10.21). The second equation follows similarly.

Now, differentiating (4.10.20), we obtain

$$\beta_k' = (\mathcal{L}h + \epsilon - \alpha_+'\mathcal{Y}_+ - \alpha_-'\mathcal{Y}_-, f_k)_{H^1}.$$

Note that $\mathcal{L}h = e_0\alpha_+\mathcal{Y}_+ - e_0\alpha_-\mathcal{Y}_- + \mathcal{L}g$, by (4.10.18), which proves (4.10.22), in view of (4.10.21).

Finally, differentiating $\Phi(h(t))$,

$$\frac{d}{dt}\Phi(t) = 2B(h, \partial_t h) = -2B(h, \mathcal{L}h) + 2B(h, \epsilon) = 2B(h, \epsilon),$$

by the skew-symmetry of $\mathcal{L}$ in (4.3.3). Equation (4.10.23) is then proved.

Step 2. Estimates on $\alpha_\pm$. We claim

$$|\alpha_-(t)| \lesssim e^{-c_1 t} \tag{4.10.24}$$

$$|\alpha_+(t)| \lesssim \begin{cases} e^{-c_1 t} & \text{if } c_0 \le e_0, \\ e^{-e_0 t} + e^{-c_1^- t} & \text{if } c_0 > e_0 \end{cases} \tag{4.10.25}$$

We will need the following inequality, which is an immediate application of Hölder inequality.

$$\int_I |B(f(t),g(t))|dt \lesssim \|\langle\nabla\rangle f\|_{S(L^2,I)}\|\langle\nabla\rangle g\|_{S'(L^2,I)} \tag{4.10.26}$$

The above inequality, assumption (4.8.15) and (4.10.21) yield

$$\int_t^{+\infty} |e^{-e_0 s} B(\mathcal{Y}_+,\epsilon(s))|ds \le e^{-e_0 t}\int_t^{+\infty} |B(\mathcal{Y}_+,\epsilon(s))|ds \lesssim e^{-(e_0+c_1)t}.$$

By integrating the second equation in (4.10.21) gives

$$|\alpha_-(t)| \lesssim e^{e_0 t}\int_t^{+\infty} |e^{-e_0 s} B(\mathcal{Y}_+,\epsilon(s))|ds \lesssim e^{-c_1 t},$$

which proves (4.10.24).

To prove (4.10.25), consider first the case $c_1 > c_0 > e_0$. Then, by assumption (4.8.15) and (4.10.19), $e^{e_0 t}\alpha_+(t)$ vanishes as $t \to +\infty$. By (4.10.26), integrating the equation on α_+ in (4.10.21),

$$|e^{e_0 t}\alpha_+(t)| \lesssim \int_t^{+\infty} |e^{e_0 s} B(\mathcal{Y}_+,\epsilon(s))|ds \lesssim e^{(e_0-c_1)t},$$

and we get (4.10.25) if $c_0 > e_0$.

Assume now that $c_0 \le e_0$. By (4.10.21), we have

$$|\alpha_+(t) - e^{-e_0 t}\alpha_+(0)| \le e^{-e_0 t}\int_0^t e^{e_0 s}|B(\mathcal{Y}_-,\epsilon(s))|ds \lesssim e^{-c_1^- t},$$

and the proof of (4.10.25) is finished.

Step 3. Bounds on g and β_k. We will prove

$$\|g(t)\|_{H^1} + \sum_k |\beta_k(t)| \lesssim e^{-\frac{(c_0+c_1)}{2}t}. \tag{4.10.27}$$

Again by (4.10.26), $\int_t^{+\infty} |B(h(s),\epsilon(s))|ds \lesssim e^{-(c_0+c_1)t}$. By integrating (4.10.23), we get

$$\Phi(h(t)) \lesssim e^{-(c_0+c_1)t}.$$

Therefore

$$|2\alpha_+\alpha_- B(\mathcal{Y}_+,\mathcal{Y}_-) + \Phi(g)| = |\Phi(h)| \lesssim e^{-(c_0+c_1)t}.$$

By Step 2,

$$|\Phi(g)| \lesssim \begin{cases} e^{-(c_0+c_1)t} + e^{-2c_1 t} & \text{if } c_0 > e_0, \\ e^{-(c_0+c_1)t} + e^{-c_1 t}(e^{-e_0 t} + e^{-c_1^- t}) & \text{if } c_0 \leq e_0. \end{cases}$$

In any case, $|\Phi(g)| \leq e^{-(c_0+c_1)t}$. Using the coercivity of Φ, given by Lemma 4.3.5, estimate for Φ in (4.10.27) is proven.

Consider now (4.10.22). By assumption (4.8.15),

$$\beta_k(t+1) - \beta_k(t) \lesssim e^{-c_1 t} + \int_t^{t+1} |(f_k, \mathcal{L}g(s))_{H^1}| ds = e^{-c_1 t} + \int_t^{t+1} \left| \mathrm{Re} \int \mathcal{L}^*(\Delta f_k)\overline{g}(s) \right| d(s),$$

where $\mathcal{L}^* = \begin{pmatrix} 0 & L_+ \\ -L_- & 0 \end{pmatrix}$ is the L^2-adjoint of $\mathcal{L}$.

One can check that, for any $0 \leq k \leq N$, $|\mathcal{L}^*(\Delta f_k)| \lesssim e^{-|x|}$. Therefore, $\mathcal{L}^*(\Delta f_k) \in L^2$ so that, by the estimate on g in (4.10.27),

$$\left| \mathrm{Re} \int \mathcal{L}^*(\Delta f_k)\overline{g}(t) \right| \lesssim \|g(t)\|_{L^2} \leq \|g\|_{H^1} \lesssim e^{-\frac{(c_0+c_1)}{2} t}$$

Step 4. Closure

By the decomposition (4.10.18), as well as steps 2 and 3, so far we have

$$\|h(t)\|_{H^1} \lesssim \begin{cases} e^{-\frac{(c_0+c_1)}{2} t} & \text{if } c_0 > e_0 \\ e^{-e_0 t} + e^{-\frac{(c_0+c_1)}{2} t} & \text{if } c_0 \leq e_0. \end{cases}$$

Now, if $e_0 \notin [c_0, c_1)$, by iterating the argument, we obtain

$$\|h(t)\|_{H^1} \lesssim e^{-c_1^- t},$$

which proves (4.8.16).

Assume now $e_0 \in [c_0, c_1)$. Then, estimate (4.10.21) on α_+ ensures the existence of a limit

A to $e^{e_0 t}\alpha_+(t)$, as $t \to +\infty$. Integrating (4.10.21) from t to $+\infty$,

$$|A - e^{e_0 t}\alpha_+| \leq e^{e_0 t} \int_t^{+\infty} |B(\mathcal{Y}_+, \epsilon(s))| ds \lesssim e^{(e_0 - c_1)t}.$$

In view of decomposition (4.10.18) and estimates (4.10.24), (4.10.25) and (4.10.27), we get

$$\|h(t) - Ae^{-e_0 t}\mathcal{Y}_+\|_{H^1} \lesssim e^{-\frac{(c_0+c_1)}{2}t}.$$

Since $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$, we see that $\tilde{h}(t) := h(t) - Ae^{-e_0 t}\mathcal{Y}_+$ satisfies the differential equation (4.8.14) with the same ϵ, and with c_0 replaced by $\frac{c_0+c_1}{2} > c_0$ in condition (4.8.15). By iterating the argument a finite number of times, we end up under condition (4.8.16), which implies condition (4.8.17) for the original h, and finishes the proof of Lemma 4.8.7. □

5 Scattering and blowup criteria for the INLS above the threshold

5.1 Introduction

We consider here the initial value problem associated to INLS (1.0.3), with initial data *above* the mass-energy threshold. This question was considered for the classical NLS by Duyckaerts and Roudenko [26].

For the sake of readability, we normalize here the scale-invariant quantities defined in the preliminaries chapter. Namely, we redefine the *mass-energy*

$$\mathcal{ME}[u] = \mathcal{ME}[u_0] \;= \frac{M[u_0]^{\frac{1-s_c}{s_c}} E[u_0]}{M[Q]^{\frac{1-s_c}{s_c}} E[Q]},$$

the *mass-kinetical-energy*

$$\mathcal{MK}[u(t)] = \frac{M[u_0]^{\frac{1-s_c}{s_c}} \displaystyle\int |\nabla u(t)|^2}{M[Q]^{\frac{1-s_c}{s_c}} \displaystyle\int |\nabla Q|^2},$$

and define here the *mass-potential-energy*

$$\mathcal{MP}[u(t)] \;= \frac{M[u_0]^{\frac{1-s_c}{s_c}} \displaystyle\int |x|^{-b}|u(t)|^{p+1}}{M[Q]^{\frac{1-s_c}{s_c}} \displaystyle\int |x|^{-b}|Q|^{p+1}}.$$

Also, if u is a solution to (1.0.3) and $u_0 \in \Sigma := \left\{f \in H^1(\mathbb{R}^N); |x|f \in L^2(\mathbb{R}^N)\right\}$, we define its variance at time t as

$$V(t) = \int |x|^2 |u(x,t)|^2 \, dx.$$

Recalling the *Virial identities* ((2.3.1) and (2.3.2)), we have

$$V_t(t) = 4 \text{ Im} \int x \cdot \nabla u(x,t)\overline{u}(x,t)\, dx \tag{5.1.1}$$

and

$$V_{tt}(t) = 4(N(p-1)+2b)E[u] - 2(N(p-1)+2b-4)\|\nabla u\|^2_{L^2(\mathbb{R}^N)}. \tag{5.1.2}$$

In previous works, Farah and Guzmán [31] and Dinh [21] studied the global behavior of solutions to (1.0.3) below the *mass-energy threshold*, i.e, in the case $\mathcal{ME}[u_0] < 1$. They proved a dichotomy between blow-up and scattering, depending on the quantity $\mathcal{MK}[u_0]$. We rewrite the global behavior of solutions to (1.0.3) with $\mathcal{ME}[u_0] < 1$ in the following way

Theorem 5.1.1. *Let u be a solution to INLS* (1.0.3) *and $0 < s_c < 1$. Assume $\mathcal{ME}[u_0] < 1$. Then*

(i) If $\mathcal{MP}[u_0] > 1$, and either $V(0) < \infty$, or u_0 is radial, or $N = 1$, then the solution blows up in finite time, in both time directions.

(ii) If $\mathcal{MP}[u_0] < 1$, $N \geq 2$, and u_0 is radial then the solution is global. Moreover, if $0 < b < \min\left\{\frac{N}{3}, 1\right\}$ and u is radial, then it scatters in H^1, in both time directions.

Remark 5.1.2. The case $\mathcal{MP}[u_0] = 1$ cannot occur if $\mathcal{ME}[u_0] < 1$ (see Farah and Guzmán [31, Lemma 4.2, item (ii)]).

Remark 5.1.3. In Farah and Guzmán [31] and Dinh [21], this theorem was proven using $\mathcal{MK}[u_0]$ instead of $\mathcal{MP}[u_0]$. We show the equivalence, if $\mathcal{ME}[u_0] \leq 1$ in Proposition 5.2.1. Therefore, since the equivalence does not hold in the case $\mathcal{ME}[u_0] > 1$, the quantity that governs the dichotomy between blow-up and scattering is, in fact, $\mathcal{MP}[u_0]$.

We are interested here in criteria that includes initial data *above* the threshold $\mathcal{ME}[u_0] = 1$. The first theorem we prove is a dichotomy

Theorem 5.1.4. *Let u be a solution to INLS* (1.0.3)*, where $0 < s_c < 1$. Assume $N \geq 2$,*

$V(0)<\infty$, $u_0\in H^1(\mathbb{R}^N)$ and

$$\mathcal{ME}[u_0]\left(1-\frac{(V_t(0))^2}{32E[u_0]V(0)}\right)\leq 1. \tag{5.1.3}$$

(i) (Blow-up) If

$$\mathcal{MP}[u_0]>1 \tag{5.1.4}$$

and

$$V_t(0)\leq 0, \tag{5.1.5}$$

then $u(t)$ blows-up in finite positive, $T_+<\infty$.

(ii) (Boundedness and scattering) If

$$\mathcal{MP}[u_0]<1 \tag{5.1.6}$$

and

$$V_t(0)\geq 0, \tag{5.1.7}$$

then

$$\limsup_{t\to T_+(u)} M[u_0]^{1-s_c}\left(\int |x|^{-b}|u(t)|^{p+1}\right)^{s_c} < M[Q]^{1-s_c}\left(\int |x|^{-b}|Q|^{p+1}\right)^{s_c}. \tag{5.1.8}$$

In particular, $T_+=+\infty$. Moreover, if $b<\min\left\{\frac{N}{3},1\right\}$ and u is radial, then it scatters forward in time in H^1.

Remark 5.1.5. If $\mathcal{ME}[u_0]<1$, the conclusion of Theorem 5.1.4 follows from Theorem 5.1.1. Thus, the conclusions of Theorem 5.1.4 is new in the case $\mathcal{ME}[u_0]\geq 1$.

Remark 5.1.6. The proof of Theorem 5.1.4 shows that there are two disjoint subsets (defined by (5.1.3), (5.1.4) and (5.1.5); and by (5.1.3), (5.1.6) and (5.1.7)) that are stable under the INLS flow and contain solutions with arbitrary mass and energy (see, for example, Remark 5.1.10 below).

Remark 5.1.7. We prove in Section 5.3 that any solution of (1.0.3) that satisfies (5.1.8) scatters for positive time. Replacing $\mathcal{MP}[u_0]$ by $\mathcal{MK}[u_0]$, this result is already known (see [31]). Due to the one-sided implication (5.2.1), our assumption is weaker. Therefore, Theorem 5.1.4 improves known results.

Remark 5.1.8. The scattering statement of Theorem 5.1.4 is optimal in the following sense: If $u_0 \in H^1(\mathbb{R}^N)$ has finite variance and scatters forward in time, then there exists $t_0 \geq 0$ such that (5.1.3), (5.1.6) and (5.1.7) are satisfied by $u(t)$ for all $t \geq t_0$. In fact, if $u(t)$ scatters forward in time, then $\int |x|^{-b}|u(t)|^{p+1} \to 0$. This implies $E[u_0] > 0$ and, by (5.1.2),

$$V_t(t) \approx 16E[u_0]t \qquad \text{and } V(t) \approx 8E[u_0]t^2$$

which implies

$$\mathcal{ME}[u_0]\left(1 - \frac{(V_t(t))^2}{32E[u_0]V(t)}\right) \to 0, \text{ as } t \to +\infty.$$

As a consequence of Theorem 5.1.4, we obtain

Corollary 5.1.9. *Let $\gamma \in \mathbb{R}\backslash\{0\}$, $v_0 \in H^1(\mathbb{R}^N)$ with finite variance be such that $\mathcal{ME}[v_0] < 1$, and u^γ be the solution to INLS* (1.0.3) *with initial data*

$$u_0^\gamma = e^{i\gamma|x|^2}v_0.$$

(i) If $\mathcal{MP}[v_0] > 1$, then for any $\gamma < 0$, u^γ blows up in finite positive time;

(ii) If $\mathcal{MP}[v_0] < 1$, then for any $\gamma > 0$, u^γ satisfies (5.1.8)*. Moreover, if $b < \min\left\{\frac{N}{3}, 1\right\}$ and v_0 is radial, then u^γ scatters forward in time in $H^1(\mathbb{R}^N)$.*

Remark 5.1.10. With the above corollary, we can predict the behavior of a class of solutions with arbitrarily large energy. If $\mathcal{ME}[v_0] < 1$, then

$$E[u_0^\gamma] = 4\gamma^2\|xv_0\|_{L^2}^2 + 4\gamma \text{ Im} \int x \cdot \nabla v_0 \bar{v}_0 + E[v_0]$$

and $E[u_0^\gamma] \to +\infty$ as $\gamma \to \pm\infty$.

Remark 5.1.11. Note that the statement of Theorem 5.1.4 is not symmetric in time as the statement of Theorem 5.1.1. Indeed, Corollary 5.1.12 below shows solutions with different behaviors in positive and negative times.

Corollary 5.1.12. *Let $\gamma \in \mathbb{R}$ and Q^γ be the solution to INLS* (1.0.3) *with initial data*

$$Q_0^\gamma = e^{i\gamma|x|^2}Q.$$

(i) If $\gamma > 0$, then Q^γ is globally defined on $[0,+\infty)$, scatters forward in time and blows up backwards in time.

(ii) If $\gamma < 0$, then Q^γ is globally defined on $(-\infty, 0]$, scatters backward in time and blows up forward in time.

Blow-up criteria

The blow up criterion of Glassey [44] for the NLS uses the second derivative of the the variance $V(t)$ to show that finite variance, negative energy solutions blow up in finite time. The second derivative of the variance is also used by Lushnikov [69], but with an approach based on classical mechanics, resulting in a finer blow-up criterion. This and another criteria were proven by Holmer, Platte and Roudenko [52] for the 3D cubic NLS. The argument was extended by Duyckaerts and Roudenko [26] to the focusing mass-supercritical NLS in any dimension. We extend these criteria for the intercritical INLS equation in any dimension.

Theorem 5.1.13. *Suppose that $u_0 \in H^1(\mathbb{R}^N)$, $N \geq 1$ and $V(0) < \infty$. The following inequality is a sufficient condition for blow-up in finite time for solutions to the INLS 1.0.3 with $0 < s_c < 1$ and $E[u_0] > 0$*

$$\frac{V_t(0)}{M[u_0]} < \sqrt{8Ns_c g\left(\frac{4}{Ns_c}\frac{E[u_0]V(0)}{M[u_0]^2}\right)},$$

where

$$g(x) = \begin{cases} \sqrt{\frac{1}{kx^k} + x - (1+\frac{1}{k})} & \text{if } 0 < x \leq 1 \\ -\sqrt{\frac{1}{kx^k} + x - (1+\frac{1}{k})} & \text{if } x \geq 1 \end{cases} \quad \text{with } k = \frac{(p-1)s_c}{2}. \tag{5.1.9}$$

Theorem 5.1.14. *Suppose that $u_0 \in H^1(\mathbb{R}^N)$ and $V(0) < \infty$. The following inequality is a sufficient condition for blow-up in finite time for solutions to INLS (1.0.3) with $0 < s_c < 1$ and $E[u_0] > 0$*

$$\frac{V_t(0)}{M[u_0]} < \frac{4\sqrt{2}M[u_0]^{\frac{1}{2}-\frac{p+1}{N(p-1)+2b}}E[u_0]^{\frac{s_c}{N}}}{C} g\left(C^2 \frac{E[u_0]^{\frac{4}{N(p-1)+2b}}V(0)}{M[u_0]^{1+\frac{2(p+1)}{N(p-1)+2b}}}\right),$$

where g is defined in (5.1.9),

$$C=\left(\frac{2(p+1)}{s_c(p-1)}(C_{p,N})^{\frac{N(p-1)+2b}{2}+(p+1)}\right)^{\frac{2}{N(p-1)+2b}}.$$

and $C_{p,N}$ is a sharp constant in the interpolation inequality (5.2.3).

Remark 5.1.15. For real-valued initial data, Theorem 5.1.14 is an improvement over Theorem 5.1.13 if

$$\mathcal{ME}[u_0]>\left(\frac{Ns_cC^2}{4}\right)^{\frac{N(p-1)+2b}{N(p-1)+2b-4}}.$$

Remark 5.1.16. In both theorems, the restriction $s_c<1$ is only needed to ensure the local well-posedness.

Remark 5.1.17. In both theorems, the restriction $s_c<1$ is only needed to ensure the local well-posedness of the INLS equation, proved by Genoud [38] and Guzmán [48].

This chapter is structured as follows: In section 5.2, we prove the boundedness and blow-up part of Theorem 5.1.4. The scattering part is proven in section 5.3. In section 5.4, we show two non-equivalent blow-up criteria for the INLS (Theorems 5.1.13 and 5.1.14).

5.2 Boundedness versus Blow-up

Coercivity and equivalence between criteria

We start this section with the proof of the equivalence between using $\mathcal{MK}[u_0]$ and $\mathcal{MP}[u_0]$ in the dichotomy when $\mathcal{ME}[u_0]\leq 1$.

Proposition 5.2.1. *If $f\in H^1(\mathbb{R}^N)$, then*

$$\mathcal{MK}[f]<1\Longrightarrow\mathcal{MP}[f]<1. \tag{5.2.1}$$

Furthermore, assume $\mathcal{ME}[f]\leq 1$. Then

$$\mathcal{MK}[f]<1\Longleftrightarrow\mathcal{MP}[f]<1. \tag{5.2.2}$$

Proof. We write the sharp Gagliardo-Nirenberg inequality (2.6.5) as

$$(\mathcal{MP}[f])^{\frac{4}{N(p-1)+2b}} \leq \mathcal{MK}[f],$$

and (5.2.1) follows. Now, if $\mathcal{MP}[f] < 1$ and $\mathcal{ME}[f] \leq 1$, then:

$$M[Q]^{\frac{1-s_c}{s_c}} E[Q] \geq M[f]^{\frac{1-s_c}{s_c}} E[f] > \frac{1}{2} M[f]^{\frac{1-s_c}{s_c}} \int |\nabla f|^2 \, dx - \frac{1}{p+1} M[Q]^{\frac{1-s_c}{s_c}} \int |x|^{-b} |Q|^{p+1} \, dx$$

taking the first and last member, we conclude $\mathcal{MK}[f] < 1$ □

We also point that inequalities in (5.2.2) can be replaced by equalities: we can scale f so that $M[f] = M[Q]$. By similar arguments as the ones used in proving (5.2.1) and (5.2.2), $\mathcal{MP}[f] = 1$ or $\mathcal{MK}[f] = 1$ in the case $\mathcal{ME}[f] \leq 1$, implies $\mathcal{MP}[f] = \mathcal{MK}[f] = \mathcal{ME}[f] = 1$. In this case, f is equal to Q up to scaling and phase.

We now turn to the proof of Theorem 5.1.4. Start rewriting the Gagliardo-Nirenberg inequality (2.6.5) as

$$\left(\int |x|^{-b} |f|^{p+1} \, dx\right)^{\frac{4}{N(p-1)+2b}} \leq C_Q M[f]^{\kappa} \int |\nabla u|^2 \, dx, \qquad \kappa = \frac{2(p+1)}{N(p-1)+2b} - 1, \quad (5.2.3)$$

where

$$C_Q := (C_{p,N})^{\frac{4}{N(p-1)+2b}} = \frac{2(p+1)}{N(p-1)+2b} \frac{\left(\int |x|^{-b} |Q|^{p+1} \, dx\right)^{\frac{4}{N(p-1)+2b} - 1}}{M[Q]^{\kappa}}$$

$$= \left(\frac{8(p+1)}{A}\right)^{\frac{4}{N(p-1)+2b}} \frac{s_c(p-1)}{N(p-1)+2b} \cdot \frac{E[Q]}{M[Q]^{\kappa}}^{\frac{4}{N(p-1)+2b} - 1}$$

and

$$A := 2(N(p-1) + 2b - 4) = 4(p-1)s_c.$$

We make use of the following Cauchy-Schwarz inequality proved by Banica [6]. We include the proof here for the sake of completeness.

Lemma 5.2.2. *Let $f \in H^1(\mathbb{R}^N)$ such that $|x|f \in L^2(\mathbb{R}^N)$. Then,*

$$\left(\operatorname{Im}\int x\cdot\nabla f\,\overline{f}\,dx\right)^2 \leq \int |x|^2|f|^2\,dx\left[\int |\nabla f|^2\,dx - \frac{1}{C_Q M^\kappa}\left(\int |x|^{-b}|f|^{p+1}\,dx\right)^{\frac{4}{N(p-1)+2b}}\right].$$

Proof. Given $f \in H^1(\mathbb{R}^N)$ and $\lambda > 0$, we have

$$\nabla\left(e^{i\lambda|x|^2}f\right) = 2i\lambda e^{i\lambda|x|^2}xf + e^{i\lambda|x|^2}\nabla f = e^{i\lambda|x|^2}(2i\lambda\, xf + \nabla f).$$

Thus,

$$\begin{aligned}\int\left|\nabla\left(e^{i\lambda|x|^2}f\right)\right|^2\,dx &= \int e^{i\lambda|x|^2}(2i\lambda\,xf+\nabla f)e^{-i\lambda|x|^2}(-2i\lambda\,x\overline{f}+\nabla\overline{f})\,dx\\ &= 4\lambda^2\int |x|^2|f|^2\,dx + 4\lambda\operatorname{Im}\int x\cdot\nabla f\,\overline{f}\,dx + \int|\nabla f|^2\,dx\end{aligned}$$

and of the Gagliardo-Niremberg inequality (5.2.3), for all $\lambda \in \mathbb{R}$ we get

$$\begin{aligned}C_Q M[f]^\kappa&\left[4\lambda^2\int|x|^2|f|^2\,dx + 4\lambda\operatorname{Im}\int x\cdot\nabla f\,\overline{f}\,dx + \int|\nabla f|^2\,dx\right]\\ &- \left(\int|x|^{-b}|f|^{p+1}\,dx\right)^{\frac{4}{N(p-1)+2b}} \geq 0.\end{aligned}$$

Note that the left-hand side of inequality above is a quadratic polynomial in λ . The discriminant of this polynomial is non-positive, which yields the conclusion of the lemma.

□

5.2.1 Dichotomy above the threshold

Proof of Theorem 5.1.4. We assume

$$\mathcal{ME}[u_0] \geq 1, \tag{5.2.4}$$

as the case $\mathcal{ME}[u_0] < 1$ has been proven by [31]. By (5.1.2), we have

$$\int|\nabla u|^2\,dx = \frac{4(N(p-1)+2b)E[u_0] - V_{tt}}{A}, \tag{5.2.5}$$

where we recall $A = 2(N(p-1)+2b-4) = 4(p-1)s_c$. Furthermore,

$$\begin{aligned}\int |x|^{-b}|u|^{p+1}\,dx &= (p+1)\frac{8\|\nabla u\|_2^2 - V_{tt}}{4(N(p-1)+2b)}\\ &= (p+1)\frac{16E[u_0]-V_{tt}}{4(N(p-1)+2b)} + \frac{16}{4(N(p-1)+2b)}\int |x|^{-b}|u|^{p+1}\,dx.\end{aligned}$$

Solving the equality above for $\int |x|^{-b}|u|^{p+1}\,dx$, we have

$$\int |x|^{-b}|u|^{p+1}\,dx = (p+1)\frac{16E[u_0]-V_{tt}}{2A}. \tag{5.2.6}$$

Note that the expression (5.2.6) implies that $V_{tt} \leq 16E[u_0]$ for all t. In view of the equation (5.1.1), the derivative of variance $V(t)$, and Lemma 5.2.2 we get,

$$\begin{aligned}(V_t(t))^2 &= 16\left(\operatorname{Im}\int x\cdot\nabla u(t)\,\overline{u}(t)\,dx\right)^2\\ &\leq 16\int V(t)\left[\int |\nabla u(t)|^2\,dx - \frac{1}{C_Q M[u_0]^\kappa}\left(\int |x|^{-b}|u(t)|^{p+1}\,dx\right)^{\frac{4}{N(p-1)+2b}}\right]\end{aligned} \tag{5.2.7}$$

If $z(t) = \sqrt{V(t)}$, then

$$z_t(t) = \frac{1}{2}\frac{V_t(t)}{\sqrt{V(t)}}.$$

Dividing (5.2.7) by $V(t)$, using (5.2.5), (5.2.6) and (5.2.7), we have

$$\begin{aligned}z_t^2(t) &= \frac{1}{4}\frac{(V_t(t))^2}{V(t)}\\ &\leq 4\left[\frac{4(N(p-1)+2b)E[u_0]-V_{tt}}{A} - \frac{1}{C_Q M[u_0]^\kappa}\left(\frac{(p+1)(16E[u_0]-V_{tt})}{2A}\right)^{\frac{4}{N(p-1)+2b}}\right],\end{aligned}$$

that is,

$$z_t^2(t) \leq 4\varphi(V_{tt}), \tag{5.2.8}$$

where

$$\varphi(\alpha) = \left[\frac{4(N(p-1)+2b)E[u_0]-\alpha}{A} - \frac{1}{C_Q M[u_0]^\kappa}\left(\frac{(p+1)(16E[u_0]-\alpha)}{2A}\right)^{\frac{4}{N(p-1)+2b}}\right]$$

is defined for $\alpha \in (-\infty, 16E[u_0]]$. We have

$$\varphi'(\alpha) = -\frac{1}{A} + \frac{4}{C_Q M[u_0]^\kappa (N(p-1)+2b)} \left(\frac{p+1}{2A}\right)^{\frac{4}{N(p-1)+2b}} (16E[u_0] - \alpha)^{\frac{4}{N(p-1)+2b}-1}.$$

Consider $\alpha_m \in (-\infty, 16E[u_0])$ such that $\varphi'(\alpha_m) = 0$, that is,

$$\frac{1}{A} = \frac{4}{C_Q M[u_0]^\kappa (N(p-1)+2b)} \left(\frac{p+1}{2A}\right)^{\frac{4}{N(p-1)+2b}} (16E[u_0] - \alpha_m)^{\frac{4}{N(p-1)+2b}-1}. \quad (5.2.9)$$

Since $s_c > 0$,

$$\frac{4}{N(p-1)+2b} - 1 = \frac{4 - N(p-1) - 2b}{N(p-1)+2b} = -\frac{2s_c}{(p-1)(N(p-1)+2b)} < 0,$$

therefore φ is decreasing on $(-\infty, \alpha_m)$ and increasing on $(\alpha_m, 16E[u_0]]$. Note that (5.2.9) implies

$$\frac{\alpha_m}{8} = \frac{(\alpha_m - 16E)(N(p-1)+2b)}{4A} + \frac{4(N(p-1)+2b)E}{A} - \frac{\alpha_m}{A} = \varphi(\alpha_m).$$

Using (5.2.9) and the fact that Q is associated to the sharp constant in the Gagliardo-Nirenberg (2.2.3), we have

$$\frac{E[Q]^{\frac{4}{N(p-1)+2b}-1}}{M[Q]^\kappa} = \frac{\left(E[u_0] - \frac{\alpha_m}{16}\right)^{\frac{4}{N(p-1)+2b}-1}}{M[u_0]^\kappa},$$

hence raising both sides to $\frac{2(p-1)}{N(p-1)+2b}$, we get

$$\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0] - \frac{\alpha_m}{16}}{E[Q]} = 1. \quad (5.2.10)$$

As a consequence of (5.2.4)

$$\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0] - \frac{\alpha_m}{16}}{E[Q]} = 1 \leq \mathcal{ME}[u_0] = \left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{E[u_0]}{E[Q]},$$

i.e.,

$$\alpha_m \geq 0,$$

and by (5.1.3) and (5.2.10),

$$\begin{aligned} z_t^2(0) &= -\left(1 - \frac{(V_t(0))^2}{32E[u_0]V(0)}\right)\frac{8E[u_0]\mathcal{ME}[u_0]}{\mathcal{ME}[u_0]} + 8E[u_0] \\ &\geq -\frac{8E[u_0]}{\mathcal{ME}[u_0]}\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}}\frac{E[u_0]-\frac{\alpha_m}{16}}{E[Q]} + 8E[u_0] \\ &= \frac{\alpha_m}{2} = 4\varphi(\alpha_m). \end{aligned} \tag{5.2.11}$$

We first prove case (i) of Theorem 5.1.4. Suppose that $u \in H^1(\mathbb{R}^N)$ satisfies (5.1.4) and (5.1.5). Note that (5.1.5) is equivalent to

$$z_t(0) = \frac{V_t(0)}{2\sqrt{V(0)}} \leq 0. \tag{5.2.12}$$

In view of (2.6.4), the assumption (5.1.4) means

$$\left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{A\int |x|^{-b}|u_0|^{p+1}\,dx}{(p+1)E[Q]} = \left(\frac{M[u_0]}{M[Q]}\right)^{\frac{1-s_c}{s_c}} \frac{\int |x|^{-b}|u_0|^{p+1}\,dx}{\int |x|^{-b}|Q|^{p+1}\,dx} > 1$$

and consequently, from (5.2.6)

$$V_{tt}(0) = -\frac{2A}{p+1}\int |x|^{-b}|u_0|^{p+1} + 16E[u_0] < \alpha_m. \tag{5.2.13}$$

Note that, for all $t > 0$

$$z_{tt}(t) = \frac{d}{dt}\left[\frac{V_t(t)}{2\sqrt{V(t)}}\right] = \frac{V_{tt}(t)}{2\sqrt{V(t)}} - \frac{(V_t(t))^2}{4\sqrt{V(t)^3}} = \frac{1}{z(t)}\left(\frac{V_{tt}(t)}{2} - z_t^2(t)\right). \tag{5.2.14}$$

Hence from (5.2.11) and (5.2.13), we have

$$z_{tt}(0) = \frac{1}{z(0)}\left(\frac{V_{tt}(0)}{2} - (z_t(0))^2\right) < \frac{1}{z(0)}\left(\frac{\alpha_m}{2} - \frac{\alpha_m}{2}\right) = 0.$$

Suppose that $z_{tt}(\tilde{t}) \geq 0$ for some $\tilde{t}$ belonging to $[0, T_+(u))$. Then, as z_{tt} is continuous on $[0, T_+(u))$, by the intermediate value theorem there exists $t_0 \in (0, T_+(u))$ such that

$$\forall \in [0, t_0),\ z_{tt}(0) < 0 \text{ and } z_{tt}(t_0) = 0.$$

Thus for (5.2.11) and (5.2.12)

$$\forall t \in (0, t_0],\ z_t(t) < z_t(0) \leq -\sqrt{4\varphi(\alpha_m)}.$$

We have, thus,

$$\forall t \in (0, t_0],\ z_t^2(t) > 4\varphi(\alpha_m).$$

Using the inequality above and (5.2.8),

$$\forall t \in (0, t_0],\ 4\varphi(V_{tt}(t)) \geq z_t^2(t) > 4\varphi(\alpha_m).$$

Therefore, $V_{tt}(t) \neq \alpha_m$ for $t \in (0, t_0]$. Since $V_{tt}(0) < \alpha_m$ and by the continuity of V_{tt},

$$\forall t \in [0, t_0],\ V_{tt}(t) < \alpha_m. \tag{5.2.15}$$

Since $V_{tt}(t) \neq \alpha_m$ and by (5.2.15), we get

$$z_{tt}(t_0) = \frac{1}{z(t_0)}\left(\frac{V_{tt}(t_0)}{2} - z_t^2(t_0)\right) < \frac{1}{z(t_0)}\left(\frac{\alpha_m}{2} - \frac{\alpha_m}{2}\right),$$

contradicting the definition of t_0. Therefore,

$$z_{tt} < 0 \text{ for all } t \in [0, T_+(u)). \tag{5.2.16}$$

By contradiction, suppose that $T_+(u) = +\infty$. From (5.2.12) and (5.2.16),

$$\forall t > 0,\ z_t(t) < z_t(0) \leq 0,$$

a contradiction with nonnegativity of $z(t)$.

We now prove case (ii) of Theorem 5.1.4. We assume, besides the conditions (5.1.3) and (5.2.4), that (5.1.6) and (5.1.7) hold. That implies, in the same way as we did in case (i),

$$z_t(0) \geq 0 \tag{5.2.17}$$

$$V_{tt}(0) > \alpha_m. \tag{5.2.18}$$

We affirm that there is $t_0 \geq 0$ such that

$$z_t(t_0) > 2\sqrt{\varphi(\alpha_m)}. \tag{5.2.19}$$

Indeed, by (5.2.11) and (5.2.17),

$$z_t(0) \geq 2\sqrt{\varphi(\alpha_m)}. \tag{5.2.20}$$

If $z_t(0) > 2\sqrt{\varphi(\alpha_m)}$, then choose $t_0 = 0$ and we have the result. If not,

$$z_{tt}(0) = \frac{1}{z(0)}\left(\frac{V_{tt}(0)}{2} - z_t^2(0)\right) > \frac{1}{z(0)}\left(\frac{\alpha_m}{2} - \frac{\alpha_m}{2}\right) = 0,$$

by (5.2.18) and (5.2.20). Hence, there is a small $t_0 > 0$ satisfying (5.2.19).

Let ε_0 be a positive small number and assume

$$z_t(t_0) \geq 2\sqrt{\varphi(\alpha_m)} + 2\varepsilon_0 \tag{5.2.21}$$

We will show that, for all $t \leq t_0$

$$z_t(t) > 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0. \tag{5.2.22}$$

Suppose (5.2.22) is false, and define

$$t_1 = \inf\{t \geq t_0;\ z_t(t) \leq 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0\}.$$

By (5.2.21) $t_1 > t_0$. By continuity of z_t,

$$z_t(t_1) = 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0 \tag{5.2.23}$$

and

$$\forall \in [t_0, t_1], \quad z_t(t) \geq 2\sqrt{\varphi(\alpha_m)} + \varepsilon_0. \tag{5.2.24}$$

In view of (5.2.8),

$$\forall t \in [t_0, t_1],\ (2\sqrt{\varphi(\alpha_m)} + \varepsilon_0)^2 \leq z_t^2(t) \leq 4\varphi(V_{tt}(t)). \tag{5.2.25}$$

Hence, $\varphi(V_{tt}(t)) > \varphi(\alpha_m)$ for all $t \in [t_0, t_1]$, so, $V_{tt}(t) \neq \alpha_m$ and by continuity $V_{tt}(t) > \alpha_m$ for $t \in [t_0, t_1]$. Using the Taylor expansion of φ around $\alpha = \alpha_m$, there exists $a > 0$ such that, if $|\alpha - \alpha_m| \leq 1$, then

$$\varphi(\alpha) \leq \varphi(\alpha_m) + a(\alpha - \alpha_m)^2. \tag{5.2.26}$$

We show that there exists a universal constant $D > 0$ such that

$$\forall\, t \in [t_0, t_1] \; V_{tt}(t) \geq \alpha_m + \frac{\sqrt{\varepsilon_0}}{D}. \tag{5.2.27}$$

Consider two cases:

a) If $V_{tt}(t) \geq \alpha_m + 1$, then for $D > 0$ large, we get (5.2.27)

b) If $\alpha_m < V_{tt}(t) \leq \alpha_m + 1$, then by (5.2.25) and (5.2.26), we obtain

$$(2\sqrt{\varphi(\alpha_m)} + \varepsilon_0)^2 \leq z_t^2(t) \leq 4\varphi(V_{tt}(t)) \leq 4\varphi(\alpha_m) + 4a(V_{tt}(t) - \alpha_m)^2.$$

Thus,

$$4\sqrt{\varphi(\alpha_m)}\varepsilon_0 < 4\sqrt{\varphi(\alpha_m)}\varepsilon_0 + \varepsilon_0^2 \leq 4a(V_{tt} - \alpha_m)^2,$$

and choosing $D = \sqrt{a}(\varphi(\alpha_m))^{-\frac{1}{4}}$, (5.2.27) holds.

Furthermore, by (5.2.14) and (5.2.24)

$$\begin{aligned} z_{tt}(t_1) &= \frac{1}{z(t_1)}\left(\frac{V_{tt}(t_1)}{2} - z_t^2(t_1)\right) \\ &\geq \frac{1}{z(t_1)}\left(\frac{\alpha_m}{2} + \frac{\sqrt{\varepsilon_0}}{2D} - (2\sqrt{\varphi(\alpha_m)} + \varepsilon_0)^2\right) \\ &\geq \frac{1}{z(t_1)}\left(\frac{\sqrt{\varepsilon_0}}{2D} - 4\varepsilon\sqrt{\varphi(\alpha_m)} - \varepsilon_0^2\right) > 0, \end{aligned}$$

if ε_0 is small enough. That is, z_t is increasing close to t_1, contradicting (5.2.23) and(5.2.24). This shows (5.2.22). Note that we have also shown that the inequality (5.2.27) holds for all $t \in [t_0, T_+(u))$. Hence, by (5.2.6), (2.6.4) and (5.2.10)

$$\begin{aligned} M[u_0]^{1-s_c}\left(\int |x|^{-b}|u(t)|^{p+1}\right)^{s_c} &= M[u_0]^{1-s_c}\left[\frac{p+1}{2A}(16E[u_0] - V_{tt}(t))\right]^{s_c} \\ &\leq M[u_0]^{1-s_c}\left[\frac{p+1}{2A}\left(16E[u_0] - \alpha_m - \frac{\sqrt{\varepsilon_0}}{D}\right)\right]^{s_c} \end{aligned}$$

$$
\begin{aligned}
&< M[u_0]^{1-s_c}\left[\frac{p+1}{2A}\left(16E[u_0]-\alpha_m\right)\right]^{s_c}\\
&= M[u_0]^{1-s_c}\left[\frac{8(p+1)}{A}E[Q]\right]^{s_c}\\
&= M[Q]^{1-s_c}\left[\int |x|^{-b}|Q|^{p+1}\right]^{s_c}.
\end{aligned}
$$

□

5.2.2 Quadratic-phase initial data

We now prove Corollary 5.1.9, except for the scattering statement, which will follow from the results in Section 5.3.

Proof of Corollary 5.1.9. Let v_0 satisfy $\mathcal{ME}[v_0] < 1$, $\gamma \in \mathbb{R}\backslash\{0\}$ and u be the solution with initial data $u_0 = e^{i\gamma|x|^2}v_0$. We assume

$$\mathcal{ME}[u_0] \geq 1$$

(otherwise the result follows from Theorem 5.1.1).

We will now show that u_0 satisfies the assumption of Theorem 5.1.4. We need to calculate

$$E[u_0] = E[v_0] + 2\gamma\,\mathrm{Im}\int x\cdot\nabla v_0\bar{v}_0\,dx + 2\gamma^2\int |x|^2|v_0|^2\,dx \tag{5.2.28}$$

and

$$\mathrm{Im}\int \bar{u}_0\,x\cdot\nabla u_0\,dx = \ \mathrm{Im}\int \bar{v}_0\,x\cdot\nabla v_0\,dx + 2\gamma\int |x|^2|v_0|^2\,dx.$$

Rewriting the above equations,

$$E[u_0] - \frac{\left(\mathrm{Im}\int \bar{u}_0\,x\cdot\nabla u_0\,dx\right)^2}{2\int |x|^2|u_0|^2\,dx} = E[v_0] - \frac{\left(\mathrm{Im}\int \bar{v}_0\,x\cdot\nabla v_0\,dx\right)^2}{2\int |x|^2|v_0|^2\,dx} \leq E[v_0], \tag{5.2.29}$$

or,

$$\mathcal{ME}[u_0]\left[1 - \frac{\left(\mathrm{Im}\int \bar{u}_0 x\cdot\nabla u_0\right)^2}{2E[u_0]\int |x|^2|u_0|^2}\right] = \mathcal{ME}[v_0] \leq 1 \tag{5.2.30}$$

Therefore, the assumption (5.1.3) follows from (5.1.1) and (5.2.30).

We will assume here $\gamma > 0$ and $\mathcal{MP}[v_0] < 1$, as the proof of the other case is analogous. First note that, since $\mathcal{ME}[v_0] < 1$ and $\int |x|^2|v_0|^2 > 0$, there is only one positive solution of

$$M[v_0]^{\frac{1-s_c}{s_c}} \left(E[v_0] + 2\gamma \operatorname{Im} \int x \cdot \nabla v_0 \bar{v}_0 \, dx + 2\gamma^2 \int |x|^2 |v_0|^2 \, dx \right) = M[Q]^{\frac{1-s_c}{s_c}} E[Q]. \tag{5.2.31}$$

Now, since $\mathcal{ME}[u_0] \geq 1$ and $\gamma > 0$, (5.2.28), we have $\gamma \geq \gamma_c^+$, where γ_c^+ is the positive solution of (5.2.31). Rewriting (5.2.31), we have

$$\gamma_c^+ \operatorname{Im} \int x \cdot \nabla v_0 \bar{v}_0 \, dx + (\gamma_c^+)^2 \int |x|^2 |v_0|^2 \, dx = \frac{M[Q]^{\frac{1-s_c}{s_c}} E[Q] - M[v_0]^{\frac{1-s_c}{s_c}} E[v_0]}{2M[v_0]^{\frac{1-s_c}{s_c}}} > 0,$$

which implies

$$\operatorname{Im} \int x \cdot \nabla v_0 \bar{v}_0 \, dx + \gamma_c^+ \int |x|^2 |v_0|^2 \, dx > 0.$$

Using that $\gamma \geq \gamma_c^+$, we see that

$$\operatorname{Im} \int x \cdot \nabla u_0 \bar{u}_0 \, dx = \operatorname{Im} \int x \cdot \nabla v_0 \bar{v}_0 \, dx + \gamma \int |x|^2 |v_0|^2 \, dx > 0.$$

which yields (5.1.7). Since Theorem 5.1.4 applies, we conclude the proof. □

We next prove Corollary 5.1.12, except for the scattering statement.

Proof of Corollary 5.1.12. Given that $\bar{u}(x,-t)$ is a solution of (1.0.3) if $u(x,t)$ is a solution, we can assume $\gamma > 0$. We only need to prove that

$$\operatorname{Im} \int x \cdot \nabla Q^\gamma(t_0) \overline{Q^\gamma}(t_0) \, dx \geq 0,$$

$$\mathcal{MP}[Q^\gamma(t_0)] < 1$$

and

$$ME[Q^\gamma(t_0)] \left(1 - \frac{(V_t(t_0))^2}{32 E[Q^\gamma(t_0)] V(t_0)} \right) \leq 1,$$

for some $t_0 > 0$, where $V(t) = \displaystyle\int |x|^2 |Q^\gamma(x,t)|^2 \, dx$. First note that, for $Q_0^\gamma = e^{i\gamma|x|^2} Q$, we have

$$\nabla Q_0^\gamma = (2i\gamma x Q + \nabla Q) e^{i\gamma|x|^2}, \text{ and}$$

$$\triangle Q_0^\gamma = e^{i\gamma|x|^2}(2iN\gamma Q + 4i\gamma x\cdot\nabla Q - 4\gamma^2|x|^2Q + \triangle Q). \tag{5.2.32}$$

Thus,

$$\begin{aligned}\operatorname{Im}\int x\cdot\nabla Q_0^\gamma Q_0^\gamma\,dx =&\ \operatorname{Im}\int x\cdot(2i\gamma xQ + \nabla Q)e^{i\gamma|x|^2}e^{-i\gamma|x|^2}Q\,dx \\ =&\ \operatorname{Im}\int x\cdot(2i\gamma xQ+\nabla Q)Q\,dx \\ =&\ 2\gamma\int |x|^2Q^2\,dx > 0.\end{aligned} \tag{5.2.33}$$

which shows $\operatorname{Im}\int x\cdot\nabla Q^\gamma(t_0)\overline{Q^\gamma}(t_0)\,dx > 0$ for sufficiently small t_0. Moreover, using the fact that Q^γ is a solution to (1.0.3), we have

$$\begin{aligned}\frac{d}{dt}\int |x|^{-b}|Q^\gamma|^{p+1}\,dx &= (p+1)\ \operatorname{Re}\int |x|^{-b}(\partial_t Q^\gamma\overline{Q^\gamma})|Q^\gamma|^{p-1}\,dx \\ &= (p+1)\ \operatorname{Re}\int |x|^{-b}(i\triangle Q^\gamma\overline{Q^\gamma})|Q^\gamma|^{p-1}\,dx \\ &= -(p+1)\ \operatorname{Im}\int |x|^{-b}|Q^\gamma|^{p-1}\triangle Q^\gamma\overline{Q^\gamma}\,dx.\end{aligned}$$

Consequently, from (5.2.32),

$$\begin{aligned}\left[\frac{d}{dt}\int |x|^{-b}|Q^\gamma|^{p+1}\,dx\right]\Bigg|_{t=0} &= \left[-(p+1)\ \operatorname{Im}\int |x|^{-b}|Q^\gamma|^{p-1}\triangle Q^\gamma\overline{Q^\gamma}\,dx\right]\Bigg|_{t=0} \\ &= -(p+1)\ \operatorname{Im}\int |x|^{-b}|Q_0^\gamma|^{p-1}Q(2iN\gamma Q + 4i\gamma x\cdot\nabla Q \\ &\qquad - 4\gamma^2|x|^2Q + \triangle Q)\,dx \\ &= -2N\gamma(p+1)\int |x|^{-b}Q^{p+1}\,dx - 4\gamma(p+1)\int Q^p x\cdot\nabla Q\,dx \\ &= -2N\gamma(p-1)\int |x|^{-b}Q^{p+1}\,dx < 0.\end{aligned}$$

Since

$$M[Q_0^\gamma]^{\frac{1-s_c}{s_c}}\int |x|^{-b}|Q_0^\gamma|^{p+1}\,dx = M[Q]^{\frac{1-s_c}{s_c}}\int |x|^{-b}|Q|^{p+1}\,dx,$$

we get, for sufficiently small t_0

$$\mathcal{MP}[Q^\gamma(t_0)] < 1.$$

Now, define the function F as

$$F(t) = M[Q^\gamma]^{\frac{1-s_c}{s_c}} \left[E[Q^\gamma] - \frac{\left(\operatorname{Im} \int x \cdot \nabla Q^\gamma(t) \overline{Q^\gamma}(t)\, dx \right)^2}{2 \int |x|^2 |Q^\gamma(t)|^2\, dx} \right] - M[Q]^{\frac{1-s_c}{s_c}} E[Q]. \quad (5.2.34)$$

In view of (5.2.29), with $v_0 = Q$, we conclude $F(0) = 0$. We just need to check that $F(t) \leq 0$ for small positive t. Let

$$V(t) = \int |x|^2 |Q^\gamma(x,t)|^2\, dx, \qquad z(t) = \sqrt{V(t)}.$$

We can rewrite (5.2.34) as

$$F(t) = M[Q^\gamma]^{\frac{1-s_c}{s_c}} \left(E[Q^\gamma] - \frac{1}{8} z_t^2(t) \right) - M[Q]^{\frac{1-s_c}{s_c}} E[Q],$$

and thus,

$$F_t(t) = -\frac{1}{4} M[Q^\gamma]^{\frac{1-s_c}{s_c}} z_t(t) z_{tt}(t).$$

Using (5.1.1), (5.1.2) and the fact that Gagliardo-Nirenberg inequality (2.6.5) is an equality for $f = Q = e^{-i\gamma|x|^2} Q_0^\gamma$, we conclude that $z_{tt}(0) = 0$. Therefore,

$$\begin{aligned} F_{tt}(0) &= -\frac{1}{4} M[Q^\gamma]^{\frac{1-s_c}{s_c}} \left(z_t(0) z_{ttt}(0) + z_{tt}^2(0) \right), \\ &= -\frac{1}{4} M[Q^\gamma]^{\frac{1-s_c}{s_c}} z_t(0) z_{ttt}(0). \end{aligned}$$

On the other hand,

$$V_{tt} = 2(z_t)^2 + 2 z z_{tt}, \qquad V_{ttt} = 6 z_t z_{tt} + 2 z z_{ttt}.$$

Thus, $V_{ttt}(0) = 2z(0) z_{ttt}(0)$. Hence, $F_{tt}(0)$ and $-V_{ttt}(0)$ have the same sign, but from (5.2.33) $z_t(0) > 0$. By (5.2.6), we get that this sign is the same as the one of

$$\left[\frac{d}{dt} \int |x|^{-b} |Q^\gamma|^{p+1}\, dx \right] \Bigg|_{t=0} = -\frac{(p+1)}{2A} V_{ttt}(0).$$

Therefore, $F_{tt}(0) < 0$, which shows that $F(t)$ is negative for small $t > 0$. This completes the proof. $\square$

5.3 Scattering

We now prove the scattering part of Theorem 5.1.4. We start with a lemma:

Lemma 5.3.1. *Let $0 < a < A < 1$. Then, there exists $\epsilon_0 = \epsilon_0(a, A)$ such that for all $f \in H^1(\mathbb{R}^N)$ with*

$$a \leq \mathcal{MP}[f] \leq A,$$

one has

$$\int |\nabla f|^2 \, dx - \frac{N(p-1)+2b}{2(p+1)} \int |x|^{-b}|f|^{p+1} \, dx \geq \epsilon_0 M[f]^{1-\frac{1}{s_c}} \tag{5.3.1}$$

and

$$E[f] \geq \frac{\epsilon_0}{2} M[f]^{1-\frac{1}{s_c}}. \tag{5.3.2}$$

Proof. Recalling the sharp Gagliardo-Nirenberg inequality, we have:

$$\begin{aligned}
&M[f]^{\frac{1}{s_c}-1}\left[\int |\nabla f|^2 \, dx - \frac{N(p-1)+2b}{2(p+1)} \int |x|^{-b}|f|^{p+1} \, dx\right] \\
&\geq \frac{1}{c_Q} M[f]^{\frac{1}{s_c}-1-\kappa} \left(\int |x|^{-b}|f|^{p+1} \, dx\right)^{\frac{4}{N(p-1)+2b}} - M[f]^{\frac{1}{s_c}-1} \frac{N(p-1)+2b}{2(p+1)} \int |x|^{-b}|f|^{p+1} \, dx \\
&= \frac{y^{\frac{4}{N(p-1)+2b}}}{c_Q} - \frac{N(p-1)+2b}{2(p+1)} y.
\end{aligned}$$

where $y = M[f]^{\frac{1}{s_c}-1} \int |x|^{-b}|f|^{p+1} \, dx$. The function $y \mapsto \frac{y^{\frac{4}{N(p-1)+2b}}}{c_Q} - \frac{N(p-1)+2b}{2(p+1)} y$ has only one zero y^* on $(0, +\infty)$ and is positive on $(0, y^*)$. Since the inequality (5.3.1) is an equality when $f = Q$, y^* is exactly $M[Q]^{\frac{1}{s_c}-1} \int |x|^{-b}|Q|^{p+1} \, dx$, and (5.3.1) follows. Noting that

$$E[f] \geq \frac{1}{2}\left(\int |\nabla f|^2 \, dx - \frac{N(p-1)+2b}{2(p+1)} \int |x|^{-b}|f|^{p+1} \, dx\right),$$

we get (5.3.2), because $\frac{N(p-1)+2b}{4} \geq 1$. □

It is already known that scattering follows from the finiteness of the $S(\dot{H}^{s_c})$ norm (see Farah and Guzmán [31, Proposition 1.4]).

Proposition 5.3.2. *Define $S(L, A)$ as the supremum of $\|u\|_{S(\dot{H}^{s_c})}$ such that u is a radial* *solution to (1.0.3) on* $[0, +\infty)$ *with*

$$\mathcal{ME}[u_0] \leq L$$

and

$$\sup_{t\in[0,+\infty)} \mathcal{MP}[u(t)] \leq A \tag{5.3.3}$$

If $A < 1$*, then* $S(L, A) < +\infty$.

Proof. The proof goes along the spirit of Duyckaerts and Roudenko [26], Farah and Guzmán [31] and (see also Guevara [47]). As this proof is already considered classical, and it is considerably long, we give an outline of the proof, highlighting the main differences.

First we note that, if $L > 0$ is small enough (i.e., $L < 1$), then $S(L, A) < +\infty$. Assume, by contradiction, that $S(L, A) = +\infty$ for some $L \in \mathbb{R}$. Note that, if $u \not\equiv 0$ satisfies (5.3.3), with $A < 1$, then by Lemma 5.3.1, $E[u] > 0$. Thus, the quantity L_c given by

$$L_c = L_c(A) := \inf \{L \in \mathbb{R} \text{ s.t. } S(L, A) = +\infty\}$$

is well-defined and positive.

Moreover, there exists a sequence $\{u_n\}$ of (global) radial solutions such that

$$M[u_n] = 1,$$

$$\|u_n\|_{S(\dot{H}^{s_c})} \to +\infty,$$

$$E[u_n] \searrow L_c$$

and

$$\sup_{t\in[0,+\infty)} \int |x|^{-b}|u|^{p+1}\, dx \leq A.$$

Therefore, using the radial linear profile decomposition ([31, Proposition 5.1]) for the initial conditions $u_{n,0}$ (note that $\{u_{n,0}\}$ is bounded in $H^1(\mathbb{R}^N)$) and the existence of wave operators for large times (see [31] and [47]), we obtain, for each $M \in \mathbb{N}$ (passing, if necessary, to a subsequence) a nonlinear profile decomposition of the form:

$$u_{n,0} = \sum_{j=1}^{M} \tilde{u}^j\left(-t_n^j\right) + \tilde{W}_n^M,$$

where, for each j, $\tilde{u}^j$ is a solution to (1.0.3) and:

1. for $k \neq j$, $|t_n^k - t_n^j| \to +\infty$;
2. for each j, there exists $T_j > 0$ such that, if $t_n^j \to +\infty$, then $\tilde{u}^j$ is defined on $(-\infty, -T_j]$, and if $t_n^j \to -\infty$, then $\tilde{u}^j$ is defined on $[T_j, +\infty)$;
3. for each j, there exists $v^j \in H^1$ such that $\|\tilde{u}^j\,(-t_n^j) - e^{-it_n^j\Delta}v^j\|_{H^1} \to 0$;
4. $\lim\limits_{M\to+\infty}\left[\lim\limits_{n\to+\infty}\left\|e^{it\Delta}\tilde{W}_n^M\right\|_{S\left(\dot{H}^{s_c}\right)}\right] = 0$;
5. for fixed $M \in \mathbb{N}$ and any $0 \leq s \leq 1$, the asymptotic Pythagorean expansion:

$$\|u_{n,0}\|_{\dot{H}^s}^2 = \sum_{j=1}^{M}\left\|\tilde{u}^j\left(-t_n^j\right)\right\|_{\dot{H}^s}^2 + \left\|\tilde{W}_n^j\right\|_{\dot{H}^s}^2 + o_n(1)$$

   and the energy Pythagorean decomposition:

$$E[u_{n,0}] = \sum_{j=1}^{M} E\left[\tilde{u}^j\right] + E\left[\tilde{W}_n^j\right] + o_n(1).$$

We denote the solution to (1.0.3) in time t, with initial data ψ by $INLS(t)\psi$. Note that, unlike in [31], we do not know whether the nonlinear profiles evolve into global solutions, because the quantity $E[\tilde{u}^j]^{s_c}M[\tilde{u}^j]^{1-s_c}$ may not be small. Thus, in order to prove that $INLS(t)\tilde{u}^j(-t_n^j)$ exists on $[0,+\infty)$, we need to track $\|\nabla INLS(t)\tilde{u}^j(-t_n^j)\|_{L^2}$. But the long-time perturbation theory (Farah and Guzmán [31, Proposition 4.14], see also Guevara [47, Lemma 3.9]), shows that the asymptotic orthogonality at $t = 0$ can be extended to the $INLS$ flow.

Lemma 5.3.3. *(Pythagorean decomposition along the bounded INLS flow). Suppose $u_{n,0}$ is a radial bounded sequence in $H^1(\mathbb{R}^N)$. Let $T \in (0,+\infty)$ be a fixed time. Assume that $u_n(t) = INLS(t)u_{n,0}$ exists up to time T for all n; and $\lim\limits_n \|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x} < +\infty$. Consider the nonlinear profile decomposition (5.3) and denote $W_n^M(t) = INLS(t)W_n^m$. Then for all j, the nonlinear profiles $\tilde{v}^j(t) = INLS(t)\tilde{u}^j(-t_n^j)$ exist up to time T and for* *all* $t \in [0, T]$,

$$\|\nabla u_n(t)\|_{L^2}^2 = \sum_{j=1}^{M} \left\|\nabla \tilde{v}^j\,(t)\right\|_{L^2}^2 + \left\|\tilde{W}_n^j(t)\right\|_{L^2}^2 + o_n(1),$$

where $o_n(1) \to 0$ *uniformly on* $0 \leq t \leq T$.

Invoking (5.3.2) and (5.3.3) and using this orthogonality along the INLS flow, one gets that $v^j(t)$ is defined on $[0, +\infty)$ as well, and satisfies, for every j,

$$M[v^j] \leq 1,$$

$$\mathcal{ME}[v^j] \leq L$$

and

$$\sup_{t\in[0,+\infty)} \mathcal{MP}[v^j] \leq A.$$

The rest of the proof follows the same lines as [26] and [31], using the criticality of L_c to show the existence of only one non-zero profile, say, $v^1(t)$, and letting $u_c(t) = v^1(t)$. This criticality also shows that $M[u_c] = 1$ and $E[u_c]^{s_c} M[u_c]^{1-s_c} = L_c^{s_c}$. Long-time perturbation theory yields $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$. At this point, we have the classical compactness lemma.

Lemma 5.3.4 (Compactness)**.** *Assume that there exists* $L_0 \in \mathbb{R}$ *and a positive number* $A < 1$ *such that* $S(L_0, A) = +\infty$. *Then there exists a radial global solution* u_c *of (1.0.3) such that the set*

$$K = \{u_c(x,t), t \in [0, +\infty)\}$$

has a compact closure in $H^1(\mathbb{R}^N)$.

Using this compactness lemma and the virial identity (5.1.2), we also have the classic rigidity lemma.

Lemma 5.3.5 (Rigidity)**.** *There is no solution* u_c *of (1.0.3) satisfying the conclusion of Lemma 5.3.4.*

The proof goes on the same lines as in Duyckaerts and Roudenko [26] and Farah and Guzmán [31]. We point here that the restriction $b < \min\left\{\frac{N}{3}, 1\right\}$ is technical and comes from the proof of long-time perturbation in [31]. □

5.4 Proof of the blowup criteria

In this section we prove two criteria for blow up in finite time. The first one is a generalization of Lushnikov's criterion in [69] and of Holmer-Platte-Roudenko criteria in [52] for the INLS, and the second one is the modification of the first approach, where the generalized uncertainty principle is replaced by the interpolation inequality (5.4.10). The two criteria are the INLS versions of the criteria proved by Duyckaerts and Roudenko in [26].

Proof of Theorem 5.1.13. Integrating by parts,

$$\begin{aligned}\|u\|_{L^2}^2 &= \int |u|^2\,dx = \frac{1}{N}\sum_{j=1}^{N}\int \partial_j x_j |u|^2\,dx = -\frac{1}{N}\sum_{j=1}^{N}\int x_j\partial_j(|u|^2)\,dx \\ &= -\frac{1}{N}\sum_{j=1}^{N}\int x_j(\partial_j u\overline{u} + u\partial_j\overline{u})\,dx = -\frac{2}{N}\sum_{j=1}^{N}\mathrm{Re}\int x_j\partial_j u\overline{u}\,dx \\ &= -\frac{2}{N}\mathrm{Re}\int (x\cdot\nabla u)\overline{u}\,dx.\end{aligned}$$

Since $|z|^2 = |\mathrm{Re}\,z|^2 + |\mathrm{Im}\,z|^2$, using Hölder's inequality

$$\begin{aligned}\|xu\|_{L_2}^2\|\nabla u\|_{L^2}^2 \geq \left|\int (x\cdot\nabla u)\overline{u}\,dx\right|^2 &= \left|\mathrm{Re}\int (x\cdot\nabla u)\overline{u}\,dx\right|^2 + \left|\mathrm{Im}\int (x\cdot\nabla u)\overline{u}\,dx\right|^2 \\ &= \frac{N^2}{4}\|u\|_{L^2}^4 + \left|\mathrm{Im}\int (x\cdot\nabla u)\overline{u}\,dx\right|^2.\end{aligned}$$

From the definition of variance and the identity for the first derivative of the variance (5.1.1), we get the uncertainty principle

$$\frac{N^2}{4}\|u_0\|_{L^2}^2 + \left|\frac{V_t}{4}\right|^2 \leq V(t)\|\nabla u(t)\|_{L^2}^2. \tag{5.4.1}$$

Using the equation (5.1.2) for the second derivative of the variance, we obtain

$$V_{tt}(t) = 4(N(p-1)+2b)E[u_0] - 4(p-1)s_c\|\nabla u(t)\|_{L^2}^2. \tag{5.4.2}$$

Substituting (5.4.2) in the uncertainty principle (5.4.1), we have

$$V_{tt}(t) \leq 4(N(p-1)+2b)E[u_0] - N^2(p-1)s_c\frac{(M[u_0])^2}{V(t)} - \frac{(p-1)s_c}{4}\frac{|V_t(t)|^2}{V(t)}. \quad (5.4.3)$$

Now, we rewrite equation (5.4.3) in order to cancel the term V_t^2. For this, define

$$V = B^{\frac{1}{\alpha+1}}, \qquad \alpha = \frac{(p-1)s_c}{4} = \frac{N(p-1)-4+2b}{8}. \quad (5.4.4)$$

Then,

$$V_t = \frac{1}{\alpha+1}B^{-\frac{\alpha}{\alpha+1}} \quad \text{and} \quad V_{tt} = -\frac{\alpha}{(\alpha+1)^2}B^{-\frac{2\alpha+1}{\alpha+1}}B_t^2 + \frac{1}{\alpha+1}B^{-\frac{\alpha}{\alpha+1}}B_{tt},$$

which gives

$$B_{tt} \leq 4(\alpha+1)N(p-1)E[u_0]B^{\frac{\alpha}{\alpha+1}} - (\alpha+1)N^2(p-1)s_c(M[u_0])^2B^{\frac{\alpha-1}{\alpha+1}},$$

that is, for all $t \in [0, T_+(u)$

$$B_{tt} \leq \frac{N(p-1)(N(p-1)+4+2b)}{2}\left(E[u_0]B^{\frac{N(p-1)-4+2b}{N(p-1)+4+2b}} - \frac{Ns_c}{4}(M[u_0])^2B^{\frac{N(p-1)-12+2b}{N(p-1)+4+2b}}\right).$$

In order to further simplify inequality, let us make a rescaling. Define $B(t) = \mu\Phi(\lambda t)$, with

$$\mu = \left(\frac{Ns_c(M[u_0])^2}{4E[u_0]}\right)^{\frac{N(p-1)+4+2b}{8}}, \qquad \lambda = \frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]}. \quad (5.4.5)$$

Then letting $s = \lambda t$, we obtain

$$\omega\Phi_{ss} \leq \Phi^\gamma - \Phi^\delta, \quad s \in [0, T_+/a), \quad (5.4.6)$$

where

$$\gamma = \frac{N(p-1)-4+2b}{N(p-1)+4+2b}, \qquad \delta = \frac{N(p-1)-12+2b}{N(p-1)+4+2b} = 2\gamma - 1,$$

$$\omega = \frac{64}{N(p-1)(N(p-1)+4+2b)}$$

and since $p > 1 + \frac{4}{N}$,

$$0 < \gamma < 1, \quad -1 < \delta < \gamma.$$

We rewrite (5.4.6) as

$$\omega \Phi_{ss} + \frac{\partial U}{\partial \Phi} \leq 0, \tag{5.4.7}$$

for $t \in [0, T_+/a)$, where $U(\Phi) = \frac{\Phi^{\delta+1}}{\delta+1} - \frac{\Phi^{\gamma+1}}{\gamma+1}$. Define the energy of the particle

$$\mathcal{E}(s) = \frac{\omega}{2}\Phi_s^2(s) + U(\Phi(s))$$

which is conserved for solutions of

$$\omega \Phi_{ss} + \frac{\partial U}{\partial \Phi} = 0.$$

Based on the ideas of Lushnikov [69], Duyckaerts and Roudenko [26] studied this model and showed the following proposition

Proposition 5.4.1. *Let Φ be a nonnegative solution of (5.4.7) such that one of the following holds:*

*(**A**)* $\mathcal{E}(0) < U_{max}$ *and* $\Phi(0) < 1$,

*(**B**)* $\mathcal{E}(0) > U_{max}$ *and* $\Phi_s(0) < 0$,

*(**C**)* $\mathcal{E}(0) = U_{max}, \Phi_s(0) < 0$ *and* $\Phi(0) < 1$.

Then $T_+ < \infty$.

Proof. For the sake of completeness of this work, we will give the proof of the proposition. Multiplying equation (5.4.7) by Φ_s, we get

$$\Phi_s(s) > 0 \Rightarrow \mathcal{E}_s(s) < 0, \quad \Phi_s(s) < 0 \Rightarrow \mathcal{E}_s(s) > 0. \tag{5.4.8}$$

We argue by contradiction, assuming $T_+ = T_+(u) = +\infty$.

We first assume (A). Let us prove by contradiction that

$$\exists\, s > 0, \;\; \Phi_s(s) < 0.$$

If not, $\Phi_s(s) \leq 0$ for all s, and (5.4.8) implies that the energy decays. By (A), $\mathcal{E}(s) \leq \mathcal{E}(0) < U_{max}$ for all s. Thus, $|\Phi(s) - 1| \geq \varepsilon_0$ (where $\varepsilon_0 > 0$ depends on $\mathcal{E}(0)$) for all s. Since by (A) $\Phi(0) < 1$, we obtain by continuity of Φ that $\Phi(s) \leq 1 - \varepsilon_0$ for all s. By equation (5.4.6), we deduce $\Phi_{ss} \leq -\varepsilon_1$ for all s, where $\varepsilon_1 > 0$ depends on ε_0. Thus, Φ is strictly concave, a contradiction with the fact that Φ is positive and $T_+ = +\infty$.

We have proved that there exists $s > 0$ such that $\Phi_s(s) < 0$. Letting

$$t_1 = \inf\{s > 0; \Phi_s(s) < 0\},$$

we get by (5.4.8) that the energy is nonincreasing on $[0, t_1]$. Thus, $\mathcal{E}(s) < \mathcal{E}(0) \leq U_{max}$ on $[0, t_1]$, which proves that $\Phi(s) \neq 1$ on $[0, t_1]$. Since $\Phi(0) < 1$, we deduce by the intermediate value theorem that $\Phi(t_1) < 1$ and by (5.4.6) that $\Phi_{ss}(t_1) < 0$. Since $\Phi_s(t_1) \leq 0$, an elementary bootstrap argument, together with equation (5.4.6) shows that $\Phi(s) \leq 1 - \varepsilon_0$, $\Phi_s(s) < 0$ and $\Phi_{ss}(s) \leq -\varepsilon_1$ for $s > t_1$, for some positive constants $\varepsilon_0, \varepsilon_1$. This is again a contradiction with the positivity of Φ.

We next assume (B). Let t_1 be such that $\Phi_s(s) < 0$ on $[0, t_1]$. By (5.4.8), $\mathcal{E}$ is nondecreasing on $[0, t_1]$, and thus, $\mathcal{E}(s) \geq \mathcal{E}(0) > U_{max}$ for all s on $[0, t_1]$. As a consequence, $\frac{1}{2}\Phi_s(s)^2 \geq \mathcal{E}(0) - U_{max} > 0$ for all s in $[0, t_1]$, which shows that the inequality $\Phi_s(s) \leq -\sqrt{\mathcal{E}(0) - U_{max}}$ holds on $[0, t_1]$. Finally, an elementary bootstrap argument shows that the inequality $\Phi_s(s) \leq -\sqrt{\mathcal{E}(0) - U_{max}}$ is valid for all $s \geq 0$, a contradiction with the positivity of Φ.

Finally, we assume (C). By bootstrap again, $\Phi_s(s) < 0$, $\Phi(s) < 1$ and $\Phi_{ss}(s) < 0$ for all positive s, proving again that Φ is a strictly concave function, a contradiction. □

Since

$$\alpha = \frac{(p-1)s_c}{4} = \frac{N(p-1) - 4 + 2b}{8},$$

we have

$$2\alpha + 1 = \frac{N(p-1) + 2b}{4}, \quad \alpha + 1 = \frac{N(p-1) + 4 + 2b}{8},$$

$$(\alpha+1)(\delta+1)=2\alpha, \quad (\alpha+1)(\gamma+1)=2\alpha+1 \text{ and } \omega=\frac{2}{(2\alpha+1)(\alpha+1)}.$$

By making $\Phi = v^{\alpha+1}$, then

$$\mathcal{E}=\frac{\omega}{2}\Phi_s^2(s)+U(\Phi(s))=\frac{\alpha+1}{2\alpha+1}(v')^2v^{2\alpha}+\frac{\alpha+1}{2\alpha}v^{2\alpha}-\frac{\alpha+1}{2\alpha+1}v^{2\alpha+1}$$

and

$$U_{max}=\frac{1}{2\alpha}\frac{\alpha+1}{2\alpha+1}.$$

Consider the function f given for

$$f(x)=\sqrt{\frac{1}{kx^k}+x-\left(1+\frac{1}{k}\right)}, \tag{5.4.9}$$

where $k=\frac{(p-1)s_c}{2}=2\alpha$. Hence, if $v_s(0)$ satisfies the condition

$$v_s(0)<\begin{cases} +f(v(0)), & \text{if } v(0)<1,\\ -f(v(0)), & \text{if } v(0)\geq 1,\end{cases}$$

then $\Phi = v^{\alpha+1}$ satisfies the conditions of Proposition 5.4.1. Indeed, the condition $\mathcal{E} < U_{max}$ is equivalent to

$$2\alpha(v')^2v^{2\alpha}+(2\alpha+1)v^{2\alpha}-2\alpha v^{2\alpha+1}<1$$

that is,

$$|v_s|<f(v).$$

Hence, the condition (A) means

$$v(0)<1 \quad \text{and} \quad -f(v(0))<v_s(0)<f(v(0))$$

and the condition (B) holds if and only if

$$|v_s(0)|>f(v(0)) \quad \text{and} \quad v_s(0)<0.$$

More precisely,

$$v_s(0)<-f(v(0))$$

and the condition (C) is equivalent to

$$v(0) < 1 \quad \text{and} \quad v_s(0) = -f(v(0)).$$

Therefore, from (5.4.4), (5.4.5) and from the definition of v, we have

$$\begin{aligned} V(0) = (\mu\Phi(\lambda t))^{\frac{1}{\alpha+1}}\Big|_{t=0} &= \mu^{\frac{8}{N(p-1)+4+2b}} v\left(\frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]}t\right)\Bigg|_{t=0} \\ &= \mu^{\frac{8}{N(p-1)+4+2b}} v(0) = \frac{Ns_c M^2}{4E[u_0]} v(0) \end{aligned}$$

and

$$V_t(0) = \mu^{\frac{8}{N(p-1)+4+2b}} \frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]} v_s(0) = \frac{Ns_c M^2}{4E[u_0]}\frac{8\sqrt{2}}{\sqrt{Ns_c}}\frac{E[u_0]}{M[u_0]} v_s(0) = M[u_0]\sqrt{8Ns_c} v_s(0).$$

Furthermore,

$$\frac{V_t(0)}{M[u_0]} = \sqrt{8Ns_c} v_s(0) < \sqrt{8Ns_c} g(v(0)) = \sqrt{8Ns_c} g\left(\frac{4}{Ns_c}\frac{V(0)E[u_0]}{M[u_0]^2}\right),$$

which completes the proof of Theorem 5.1.13. □

We now proceed to the proof of Theorem 5.1.14. For that, we consider the following proposition.

Proposition 5.4.2. *Let $p > 1$ and $N \geq 1$. Then, the following inequality*

$$\|u\|_{L^2}^2 \leq C_{p,N}\left(\|xu\|_{L^2}^{\frac{N(p-1)+2b}{2}} \||\cdot|^{\frac{-b}{p+1}}u\|_{L^{p+1}}^{p+1}\right)^{\frac{2}{N(p-1)+2(p+1)+2b}} \tag{5.4.10}$$

holds with the sharp constant $C_{p,N}$ (depending on the nonlinearity p and dimension N) given by (5.4.14). Moreover, the equality occurs if and only if there exists $\beta \geq 0$, $\alpha \leq 0$ such that $|u(x)| = \beta\phi(\alpha x)$, where

$$\phi(x) = \begin{cases} |x|^{\frac{b}{p-1}}(1-|x|^2)^{\frac{1}{p-1}} & \textit{if } 0 \leq |x| < 1, \\ 0 & \textit{if } |x| > 1. \end{cases}$$

The proof of Proposition 5.4.2 follows the ideas of [26].

Proof. Let $R > 0$ to be specified later. Split the mass of u as follows

$$\int |u(x)|^2\,dx = \frac{1}{R^2}\int_{|x|\le R}(R^2-|x|^2)|u(x)|^2\,dx+\frac{1}{R^2}\int_{|x|\le R}|x|^2|u(x)|^2\,dx+\int_{|x|\ge R}|u(x)|^2\,dx.$$

By Hölder inequality we have

$$\begin{aligned}\frac{1}{R^2}\int(R^2-|x|^2)|u(x)|^2\,dx &\le \frac{1}{R^2}\left(\int_{|x|\le R}|x|^{\frac{2b}{p-1}}(R^2-|x|^2)^{\frac{p+1}{p-1}}\,dx\right)^{\frac{p-1}{p+1}}\left(\int |x|^{-b}|u(x)|^{p+1}\,dx\right)^{\frac{2}{p+1}}\\ &\le \frac{1}{R^2}\left(\int_{|x|\le 1}R^{\frac{2b}{p-1}}|y|^{\frac{2b}{p-1}}(R^2-R^2|y|^2)^{\frac{p+1}{p-1}}R^N\,dy\right)^{\frac{p-1}{p+1}}\left(\int |x|^{-b}|u(x)|^{p+1}\,dx\right)^{\frac{2}{p+1}}\\ &= R^{\frac{N(p-1)+2b}{p+1}}D_{p,N}\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{p+1}^2, \qquad (5.4.11)\end{aligned}$$

where

$$D_{p,N}=\left(\int_{|y|\le 1}|y|^{\frac{2b}{p-1}}(1-|y|^2)^{\frac{p+1}{p-1}}\,dy\right)^{\frac{p-1}{p+1}}.$$

Furthermore,

$$\frac{1}{R^2}\int_{|x|\le R}|x|^2|u(x)|^2\,dx+\int_{|x|\ge R}|u(x)|^2\,dx \le \frac{1}{R^2}\int |x|^2|u(x)|^2\,dx. \qquad (5.4.12)$$

Combining (5.4.11) and (5.4.12), we get

$$\forall R>0,\quad \|u\|_{L^2}^2 \le D_{p,N}\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{L^{p+1}}^2 R^{\frac{N(p-1)+2b}{p+1}}+\frac{1}{R^2}\|xu\|_{L^2}^2. \qquad (5.4.13)$$

Let $F:(0,+\infty)\to\mathbb{R}$ given by $F(R)=AR^\alpha+BR^{-2}$, where $A,B>0$ and $\alpha>0$. The minimum value of F is reached at $R=\left(\frac{2B}{\alpha A}\right)^{\frac{1}{\alpha+2}}$ and

$$F\left(\left(\frac{2B}{\alpha A}\right)^{\frac{1}{\alpha+2}}\right)=A\left(\frac{2B}{\alpha A}\right)^{\frac{\alpha}{\alpha+2}}+B\left(\frac{\alpha A}{2B}\right)^{\frac{2}{\alpha+2}}=\frac{2+\alpha}{\alpha}(\alpha A)^{\frac{2}{\alpha+2}}(2B)^{\frac{\alpha}{\alpha+2}}.$$

Thus, by taking

$$R=\left(\frac{p+1}{N(p-1)+2b}\frac{2\|xu\|_{L^2}^2}{D_{p,N}\left\||\cdot|^{-\frac{b}{p+1}}u\right\|_{L^{p+1}}^2}\right)^{\frac{p+1}{N(p-1)+2(p+1)+2b}}$$

in (5.4.13), we have

$$\|u\|_{L^2}^2 \leq C_{p,N}^2 \left\| |\cdot|^{-\frac{b}{p+1}} u \right\|_{L^{p+1}}^{\frac{4(p+1)}{N(p-1)+2(p+1)+2b}} \|xu\|_{L^2}^{\frac{2N(p-1)+4b}{N(p-1)+2(p+1)+2b}},$$

where

$$C_{p,N} = \left(\frac{N(p-1)+2(p+1)+2b}{2N(p-1)+4b}\right)^{\frac{1}{2}} \left(\frac{N(p-1)+2b}{p+1} D_{p,N}\right)^{\frac{(p+1)}{N(p-1)+2(p+1)+2b}} 2^{\frac{N(p-1)+2b}{2N(p-1)+4(p+1)+4b}}. \tag{5.4.14}$$

Note that equality in (5.4.10) holds if and only if there exists $R > 0$ such that (5.4.13) is an equality. This is equivalent to the fact that for some $R > 0$, both (5.4.11) and (5.4.12) are equalities. The inequality (5.4.11) is an equality if and only if, for $|x| < R$, $|x|^{-b}|u(x)|^{p+1} = c|x|^{\frac{2b}{p-1}}(R^2 - |x|^2)^{\frac{p+1}{p-1}}$ for some constant $c \geq 0$, and inequality (5.4.12) is an equality if and only if $u(x) = 0$ for $|x| \geq R$. This completes the proof of Proposition 5.4.2. □

Proof of Theorem 5.1.14. Since the energy is

$$E[u_0] = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{1}{p+1}\left\| |\cdot|^{-\frac{b}{p+1}} u(t)\right\|_{L^{p+1}}^{p+1},$$

from (5.1.2), we obtain

$$\begin{aligned} V_{tt}(t) &= 4(N(p-1)+2b)E[u_0] - 2(N(p-1)+2b-4)\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^2 \\ &= 16E[u_0] - \frac{8(p-1)s_c}{p+1}\left\| |\cdot|^{-\frac{b}{p+1}} u(t)\right\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Using the sharp interpolation inequality (5.4.10)

$$V_{tt}(t) \leq 16E[u_0] - \frac{8(p-1)s_c}{(p+1)(C_{p,N})^{\frac{N(p-1)}{2}+(p+1)+b}} \frac{M[u_0]^{\frac{N(p-1)}{4}+\frac{(p+1)}{2}+\frac{b}{2}}}{V(t)^{\frac{N(p-1)+2b}{4}}}, \tag{5.4.15}$$

with $C_{p,N}$ from (5.4.10). As done in the proof of Proposition 5.1.13, take $v(s)$ with $s = at$ such that

$$V(t) = \mu v(\lambda t), \quad \lambda = \sqrt{\frac{32E[u_0]}{\mu}},$$

where
$$\mu = \left(\frac{s_c(p-1)}{2(p+1)}\right)^{\frac{4}{N(p-1)+2b}} \frac{M[u_0]^{1+(p+1)\left(\frac{2}{N(p-1)+2b}\right)}}{(C_{p,N})^{2+(p+1)\left(\frac{4}{N(p-1)+2b}\right)} E[u_0]^{\frac{4}{N(p-1)+2b}}}.$$

Hence, applying in the inequality (5.4.15), we have

$$v_{ss}(s) \leq \frac{1}{2}\left(1 - v^{-\frac{N(p-1)+2b}{4}}(s)\right).$$

If the inequality in the above expression is replaced by an equality, then we have that the following energy is conserved

$$\mathcal{E}(s) = \frac{k}{1+k}\left((v(s))^2 - v(s) - \frac{1}{kv(s)^k}\right),$$

where as before $k = \frac{(p-1)s_c}{2} = \frac{N(p-1)+2b}{4} - 1$. The maximum of the function

$$f(x) = \frac{k}{1+k}\left(x + \frac{1}{kx^k}\right),$$

attained at $x = 1$, is -1. As we did to (A), (B) and (C), we identify the three sufficient conditions for blow-up in finite time.

(A^*) $\mathcal{E}(0) < -1$ and $v(0) < 1$,

(B^*) $\mathcal{E}(0) > -1$ and $v_s(0) < 0$,

(C^*) $\mathcal{E}(0) = -1$, $v_s(0) < 0$ and $v(0) < 1$.

If $v_s(0)$ satisfies the condition

$$v_s(0) < \begin{cases} +f(v(0)), & \text{if } v(0) < 1 \\ -f(v(0)), & \text{if } v(0) \geq 1, \end{cases}$$

then v satisfies one of the conditions (A*), (B*) and (C*). Indeed, recalling the function f from (5.4.9) and using the definition of $\mathcal{E}$, we obtain

a) $\mathcal{E} < -1$ if and only if $|v_s| < f(v)$.

b) $\mathcal{E} \geq -1$ if and only if $|v_s| \geq f(v)$.

Then the previous conditions can be written in the following form:

$(A^*) \Leftrightarrow v(0) < 1$ and $-f(v(0)) < v_s(0) < f(v(0))$,

$(B^*) \Leftrightarrow v_s(0) < -f(v(0)),$

$(C^*) \Leftrightarrow v_s(0) = -f(v(0)), \;\; v(0) < 1.$

Substituting back $V(t)$, we obtain

$$\frac{V_t(0)}{\lambda\mu} < g\left(\frac{V(0)}{\mu}\right),$$

where g is defined in (5.1.9). Hence,

$$\frac{V_t(0)}{4\sqrt{2}} \cdot \left(\frac{2(p+1)}{s_c(p-1)}(C_{p,N})^{\frac{N(p-1)+2b}{2}+(p+1)}\right)^{\frac{2}{N(p-1)+2b}} \frac{(C_{p,N})^{1+(p+1)\left(\frac{2}{N(p-1)+2b}\right)}}{E[u_0]^{\frac{s_c}{N}} M[u_0]^{\frac{1}{2}+(p+1)\left(\frac{1}{N(p-1)+2b}\right)}} < g(\theta),$$

with

$$\theta = \left(\frac{2(p+1)}{s_c(p-1)}(C_{p,N})^{\frac{N(p-1)+2b}{2}+(p+1)}\right)^{\frac{4}{N(p-1)+2b}} \frac{E[u_0]^{\frac{4}{N(p-1)+2b}}}{M[u_0]^{1+(p+1)\left(\frac{2}{N(p-1)+2b}\right)}} V(0).$$

This completes the proof of Theorem 5.1.14. □

6 Future work

We list here some possibilities for future works on topics related to this thesis.

1. Give a classification of solutions at the mass-energy threshold for different nonlinearities, such as the INLS equation and the Hartree-type equations;

2. Answer to questions regarding stability/instability of standing waves for the INLS equation, extending, for example the results of [19];

3. Study the global well-posedness of solutions to INLS for initial data with less regularity than H^1. There are two possible approaches: the high-low decomposition method, introduced by Bourgain [9] and its improved version, the I-method, introduced by Colliander-Keel-Staffilani-Takaoka-Tao [17];

4. Perform numerical simulations on the INLS to investigate whether the restrictions on N, p and b in the theorems regarding long-time behavior are really necessary, or if they are technical.

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