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Existence and non-existence of solutions to problems involving conformal operators on sphere and hemisphere

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Abstract

In this work, we study the existence and nonexistence of nonconstant solutions for the following equation

$$\begin{aligned} \mathcal{A}_{2s}u &= f(u) \qquad \text{in } M,\\ \tfrac{\partial u}{\partial \nu} &= 0 \qquad \text{on } \partial M, \end{aligned}$$

and system

$$\begin{aligned} \mathcal{A}_{2s}u_1 &= f_1(u_1,u_2) \qquad \text{in } M,\\ \mathcal{A}_{2s}u_2 &= f_2(u_1,u_2) \qquad \text{in } M,\\ \tfrac{\partial u_1}{\partial \nu} &= \tfrac{\partial u_2}{\partial \nu} = 0 \qquad \text{on } \partial M, \end{aligned}$$

where M is the n-dimensional standard unit sphere or hemisphere, $n > 2$ and $\mathcal{A}_{2s}$ is the fractional conformal or *intertwining* operator for $s \in (0,1]$ or $s = 2$. Under some conditions on f, f_1 and f_2, we will prove that the only positive solutions to the above problems are constants. The main techniques used are the moving plane method in an integral form and the geometry of M. In addition, we will show that the equation has infinitely many sign-changing solutions for any $s \in (0,1)$.

Key words: Fractional conformal operator, moving plane, sign-changing solution.

Resumo

Neste trabalho, estudamos a existência e não existência de soluções não constantes para a seguinte equação

$$\begin{aligned} \mathcal{A}_{2s}u = f(u) \qquad &\text{in } M, \\ \tfrac{\partial u}{\partial \nu} = 0 \qquad &\text{on } \partial M, \end{aligned}$$

e o sistema

$$\begin{aligned} \mathcal{A}_{2s}u_1 = f_1(u_1, u_2) \qquad &\text{in } M, \\ \mathcal{A}_{2s}u_2 = f_2(u_1, u_2) \qquad &\text{in } M, \\ \tfrac{\partial u_1}{\partial \nu} = \tfrac{\partial u_2}{\partial \nu} = 0 \qquad &\text{on } \partial M, \end{aligned}$$

onde M é a esfera unitaria ou semi-esfera canônica de dimensão $n > 2$ e $\mathcal{A}_{2s}$ é o operador conforme fracionário ou *intertwining* para $s \in (0, 1]$ ou $s = 2$. Sob certas condições de f, f_1 e f_2, vamos provar que as únicas soluções positivas dos problemas acima são constantes. As principais técnicas usadas são o método *moving plane* na forma integral e a geometria de M. Além disso, mostraremos que a equação possui infinitas soluções que mudam de sinal para qualquer $s \in (0, 1)$.

Palavras chaves: Operador conforme fracionário, moving plane, solução mudando de sinal.

Contents

Introduction

In recent years, there has been independent study of fractional order operators by two different group of mathematicians. On one hand, there are extensive works that study properties of fractional Laplacian operators as non-local operators together with its applications [18], and many others (see the related articles [20, 24, 34, 37, 69]); on the other hand, there is the work of Graham and Zworski [40] (see also [2, 3, 19, 38, 57, 45, 46], for instance), that study a general class of conformal operators P_γ, parameterized by a real number γ and defined on the boundary of a conformally compact Einstein manifold, and which includes the fractional Laplacian operators as a special case when the boundary is the Euclidean space setting as boundary of the hyperbolic space. Thus, the study about the existence and nonexistence of solutions for problems involving conformal operators is closely related with the study of problems involving the fractional Laplace operator.

This Thesis is organized in five chapters.

Chapter 1:

In [39], Graham et al. constructed a sequence of conformally covariant elliptic operators P_k^g, on Riemannian manifolds (M^n, g) for all positive integers k if n is odd, and for $k \in \{1, .., n/2\}$ if n is even. Moreover, P_1^g is the well known conformal Laplacian $-\Delta_g + c(n)R_g$, where Δ_g is the Laplace-Beltrami operator, $c(n) = (n-2)/4(n-1)$, $P_1^g(1) = R_g$ is the scalar curvature of M, $n \geq 3$; P_2^g is the Paneitz operator and $P_2^g(1)$ is the Q-curvature with $n \geq 5$.

Making use of a generalized Dirichlet to Neumann map, Graham and Zworski [40] introduced a meromorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds (see Mazzeo and Melrose [56]). Recently, Chang and González [19] reconciled the way of Graham and Zworski to define conformally invariant operators P_s^g of non-integer order $s \in (0, n/2)$ and the localization method of Caffarelli and Silvestre [18] for fractional Laplacian on the Euclidean space $\mathbb{R}^n$. These lead naturally to a fractional order curvature $R_s^g = P_s^g(1)$, which is called s-curvature.

There are several works on these conformally invariant equations of fractional order and prescribing s-curvature problems (fractional Yamabe problem and fractional Nirenberg problem), see e. g. [37, 38, 45, 46, 47] and references therein.

If $M = \mathbb{S}^n$ is the unit sphere, $n > 2$, provided with the standard metric $g = g_{\mathbb{S}^n}$, then the operator $\mathcal{A}_{2s} =: P_s^{g_{\mathbb{S}^n}}$ has the formula (see [15])

$$\mathcal{A}_{2s} = \frac{\Gamma\left(B + \frac{1}{2} + s\right)}{\Gamma\left(B + \frac{1}{2} - s\right)}, \ B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2},$$

where Γ is the Gamma function and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace-Beltrami operator on $\mathbb{S}^n$. Then our goal will be to discuss the relationship between the class of conformal operators $\mathcal{A}_{2s}$ defined on the unit sphere $\mathbb{S}^n$ and the fractional Laplace or polyharmonic operator $(-\Delta)^s$ defined on the Euclidean space $\mathbb{R}^n$, namely,

$$(\mathcal{A}_{2s}u) \circ \mathcal{F} = |J_{\mathcal{F}}|^{-\frac{n+2s}{2n}}(-\Delta)^s(|J_{\mathcal{F}}|^{\frac{n-2s}{2n}}(u \circ \mathcal{F})), \quad \text{for all } u \in C^\infty(\mathbb{S}^n),$$

where $\mathcal{F}^{-1}$ is the stereographic projection and $J_{\mathcal{F}}$ is the Jacobian of $\mathcal{F}$. Moreover, we will see that the existence and nonexistence of solutions for some problems involving $\mathcal{A}_{2s}$ implies the existence and nonexistence of solutions for some problems involving $(-\Delta)^s$ and vice versa.

Chapter 2:

The study of semilinear elliptic equations and systems involving critical growth with Neumann or nonlinear boundary conditions has received considerable attention in last years, see e.g., [4, 17, 18, 52, 58, 67]. We consider the following problem

$$(P) \quad \begin{cases} -L_g u = f(u) & in\ M, \\ u > 0 & in\ M, \\ \frac{\partial u}{\partial \nu} = 0 & on\ \partial M \end{cases}$$

where (M^n, g), $n \geq 3$, is a compact Riemannian manifold (possibly with non-empty boundary), L_g is a second order partial differential operator on M^n with respect to the metric g, and $\frac{\partial u}{\partial \nu}$ is the normal derivative of u with respect to the unit exterior normal vector field ν of the boundary ∂M, and $f : (0, \infty) \to \mathbb{R}$ is a function. If the boundary of M is empty, we do not assume $\frac{\partial u}{\partial \nu} = 0$ on ∂M in the problem (P). Our main interest here is to find conditions on f and on the geometry or topology of M which imply that the problem (P) admits only positive constant solutions. A particular case of the problem

(P) is the following one:

$$(Q)\begin{cases} -\Delta_g u + \lambda u - F(u)u^{\frac{n+2}{n-2}} = 0 & in\ M, \\ u > 0 & in\ M, \\ \frac{\partial u}{\partial \nu} = 0 & on\ \partial M, \end{cases}$$

where Δ_g is the Laplace-Beltrami operator on M^n with respect to the metric g, λ is a real smooth function on M and $F : (0, \infty) \to \mathbb{R}$ is a real smooth function. Note that when $\lambda > 0$ is a constant and $F(t)t^{\frac{n+2}{n-2}} = t^p$, $p > 1$, then $u = \lambda^{\frac{1}{p-1}}$ is a solution of the problem (Q). In the case where F is a constant and $\lambda = \frac{(n-2)}{4(n-1)} R_g$, where R_g denotes the scalar curvature of the Riemannian manifold (M, g), the problem (Q) is just the Yamabe problem in the conformal geometry for the closed case or, if ∂M is not empty, for the case of minimal boundary. See Escobar's work [32]. If $M^n = \mathbb{S}^n$ is the standard unit n-sphere and g is the standard metric, there are infinitely many solutions for the Yamabe problem with respect to the metric g since the conformal group of the standard unit n-sphere is also infinite. In a more specific situation, the problem (Q) was studied by Lin, Ni and Takagi for the case when $F(t)t^{\frac{n+2}{n-2}} = t^p$, $p > 1$, $\lambda > 0$ is a constant function and M is a bounded convex domain with smooth boundary in the Euclidean space $\mathbb{R}^n$ (see [53], [58] and the references therein). When p is a subcritical exponent, that is, $p < (n+2)/(n-2)$, Lin, Ni and Takagi [53] showed that problem has a unique solution if λ is sufficiently small. Such kind of uniqueness results about radially symmetric solution of (P) were also obtained by Lin and Ni in [52] when Ω is an annulus and $p > 1$ or when Ω is a ball and $p > (n+2)/(n-2)$. Based on this, Lin and Ni [52] made the following conjecture.

Conjecture 0.1. *(Lin - Ni [52]) Assume $F(t) = 1$. Then the problem (Q) admits only the constant solution for $0 < \lambda$ small.*

When Ω is the unit ball and $n = 4, 5, 6$, it was shown by Adimurthi and Yadava [5] that Lin-Ni's conjecture is not true, namely, the problem (P) has at least two radial solutions if $\lambda > 0$ and is close to 0 (see also [17]). In the case where Ω is a non-convex domain, this conjecture has negative answer [64]. For other cases, the conjecture is open. But, we will show this conjecture is true for case $M = \mathbb{S}^n_+$.

In [16], Brezis and Li studied the problem (P) for the case of the standard unit sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and, using results due to Gigas, Ni and Nirenberg [35], they showed that the problem (P) admits only constant solutions provided that $L_g = \Delta_{g_{\mathbb{S}^n}}$ and f is such that the function $h(t) = t^{-\frac{n+2}{n-2}}(f(t) + n(n-2)t/4)$ is a decreasing function on $(0, +\infty)$. Hence, considering the particular problem (Q) on the standard sphere, they showed that if $0 < \lambda < \frac{n(n-2)}{4}$ and F is a decreasing function on $(0, +\infty)$, then the only positive solution to (Q) is the constant one.

Motivated by the results in [16], and by the technique applied in that work, we study first the following nonlinear elliptic equations and systems:

$$\begin{cases} -\Delta_{g_0} u = f(u), \ u > 0 & \text{in } \mathbb{S}^n_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{S}^n_+, \end{cases} \tag{1}$$

and

$$\begin{cases} -\Delta_{g_0} u_1 = f_1(u_1, u_2) & \text{in } \mathbb{S}^n_+, \\ -\Delta_{g_0} u_2 = f_2(u_1, u_2) & \text{in } \mathbb{S}^n_+ \\ u_1, \ u_2 > 0 & \text{in } \mathbb{S}^n_+, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\mathbb{S}^n_+, \end{cases} \tag{2}$$

where g_0 is the standard metric on the hemisphere $\mathbb{S}^n_+$, $n \geq 3$, $\frac{\partial}{\partial \nu}$ is the derivate with respect to the outward normal vector field ν, and $f : (0, +\infty) \to \mathbb{R}$, $f_1, f_2 : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ are continuous functions.

Our goals are to show the nonexistence of nonconstant positive solutions of (1) and (2). This will be a consequence of the following results.

Theorem 0.1. *Assume that*

$$h_1(t) := t^{-\frac{n+2}{n-2}} \left(f(t) + \frac{n(n-2)}{4} t \right) \quad \textit{is decreasing in} \quad (0, +\infty).$$

Then the problem (1) admits only constant solutions.

Theorem 0.2. *Let $h_{i1} : (0, \infty) \times (0, \infty) \to \mathbb{R}$, $i = 1, 2$, be two functions defined by*

$$h_{i1}(t_1, t_2) := t_i^{-\frac{n+2}{n-2}} \left(f_i(t_1, t_2) + \frac{n(n-2)}{4} t_i \right), \ t_1 > 0, \ t_2 > 0.$$

Assume that

$$\begin{aligned} &h_{i1}(t_1, t_2) \textit{ is nondecreasing in } t_j > 0, \ \textit{with } i \neq j, \\ &h_{i1}(t_1, t_2) t_i^{\frac{n+2}{n-2}} \textit{ is nondecreasing in } t_i > 0, \\ &h_{i1}(a_1 t, a_2 t) \textit{ is decreasing in } t > 0 \textit{ for any } a_i > 0, \end{aligned}$$

for $i, j = 1, 2$. Then the problem (2) admits only constant solutions.

To prove the theorems above we will use the moving planes method and technique based on identities of integrals on $\mathbb{R}^n$.

Chapter 3:

Considerable attention has been given to the study of the problem (P) on smooth compact Riemannian (M^n, g) when the operator L_g is replaced by a fourth order partial differential operator, see e.g. [9, 31, 52]. A particular case is the Paneitz operator [60] defined by

$$P_2^g u = \Delta_g^2 u - div_g(a_n R_g g + b_n Ric_g)du + \frac{n-4}{2} Q_g u,$$

where $n > 4$, R_g denotes the scalar curvature of (M^n, g), Ric_g denotes the Ricci curvature of (M^n, g), a_n and b_n are constants dependent of n, and Q_g is called Q-curvature. See [30] for details about the properties of P_2^g. Problems such as prescribing scalar curvature and Paneitz curvature on $\mathbb{S}^n$ were studied extensively in last years, see e.g., [7, 55, 21, 50, 62] and [1, 30, 66].

On the unit sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$, $n \geq 5$, with the standard metric $g_{\mathbb{S}^n}$, the operator $P_2^{g_{\mathbb{S}^n}}$ has the expression

$$P_2^{g_{\mathbb{S}^n}} u = \Delta_{g_{\mathbb{S}^n}}^2 u - c_n \Delta_{g_{\mathbb{S}^n}} u + d_{n,2} u,$$

where $c_n = (n^2 - 2n - 4)/2$ and $d_{n,2} = (n-4)n(n^2-4)/16$.

We consider the following nonlinear equations related with the Paneitz operator:

$$\begin{cases} \Delta_g^2 u - c_n \Delta_g u = f(u) & \text{in } \mathbb{S}^n, \\ u > 0 & \text{in } \mathbb{S}^n; \end{cases} \tag{3}$$

and

$$\begin{cases} \Delta_g^2 u_1 - c_n \Delta_g u_1 = f_1(u_1, u_2) & \text{in } \mathbb{S}^n, \\ \Delta_g^2 u_2 - c_n \Delta_g u_2 = f_2(u_1, u_2) & \text{in } \mathbb{S}^n \\ u_1,\ u_2 > 0 & \text{in } \mathbb{S}^n, \end{cases} \tag{4}$$

where $f : (0, +\infty) \to \mathbb{R}$, f_1, $f_2 : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ are continuous functions.

Our goal is to show that under conditions on f, f_1 and f_2, the problems above have only constant solutions. We use the same arguments used in the proof of Theorems 0.1 and 0.2 to show the following results.

Theorem 0.3. *Assume that*

$$h_2(t) := t^{-\frac{n+4}{n-4}}(f(t) + d_{n,2}t) \text{ is decreasing non-negative in } (0, +\infty) \text{ and}$$
$$h_2(t)t^{\frac{n+4}{n-4}} \text{ is nondecreasing in } (0, +\infty).$$

Then the problem (3) admits constant solutions.

Theorem 0.4. *Let $h_{i2} : (0,\infty)\times(0,\infty) \to \mathbb{R}$, $i = 1,2$, be two functions defined by*

$$h_{i2}(t_1,t_2) := t_i^{-\frac{n+4}{n-4}} \left(f_i(t_1,t_2) + d_{n,2}t_i\right),\ t_i > 0.$$

Assume that for $i,j = 1,2$: $h_{i,2}$ are non-negative,

$$\begin{aligned}
&h_{i2}(t_1,t_2) \text{ is nondecreasing in } t_j > 0, \text{ with } i \neq j,\\
&h_{i2}(t_1,t_2)t_i^{\frac{n+4}{n-4}} \text{ is nondecreasing in } t_i > 0,\\
&h_{i2}(a_1t,a_2t) \text{ is decreasing in } t > 0 \text{ for any } a_i > 0.
\end{aligned}$$

Then the problem (4) admits only constant solutions.

The results above determine the nonexistence of nonconstant solutions of (3) and (4).

Chapter 4:

When $s \in (0,1)$, Pavlov and Samko [61] showed that

$$\mathcal{A}_{2s}(u)(\zeta) = C_{n,-s}\int_{\mathbb{S}^n} \frac{u(\zeta)-u(z)}{|\zeta - z|^{n+2s}}dz + \mathcal{A}_{2s}(1)u(\zeta),\ u \in C^2(\mathbb{S}^n),\ \zeta \in \mathbb{S}^n,$$

where $C_{n,-s} = \frac{2^{2s}s\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-s)}$, $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$ and $\int_{\mathbb{S}^n}$ is understood as $\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}$.

We denote

$$\mathcal{D}_s u(q) =: C_{n,-s}\int_{\mathbb{S}^n} \frac{u(\zeta)-u(z)}{|\zeta - z|^{n+2s}}dz,\ u \in C^2(\mathbb{S}^n) \text{ and } d_{n,s} =: \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)} = \mathcal{A}_{2s}(1).$$

We will study the existence of constant solutions of the following problems:

$$\begin{cases} \mathcal{D}_s u = f(u) & \text{in } \mathbb{S}^n,\\ \quad u > 0 & \text{in } \mathbb{S}^n; \end{cases} \tag{5}$$

and

$$\begin{cases} \mathcal{D}_s u_1 = f_1(u_1,u_2) & \text{in } \mathbb{S}^n,\\ \mathcal{D}_s u_2 = f_2(u_1,u_2) & \text{in } \mathbb{S}^n,\\ u_1,u_2 > 0 & \text{in } \mathbb{S}^n. \end{cases} \tag{6}$$

Motivated by the previous results we will study the nonexistence of nonconstant positive solutions for problems (5) and (6) on some conditions of f, f_1 and $f4_2$. Our main results are:

Theorem 0.5. *Let $s \in (0,1)$. Assume that*

$$h_s(t) := t^{-\frac{n+2s}{n-2s}} \left(f(t) + d_{n,s} t\right) \quad \textit{is decreasing in} \quad (0, +\infty).$$

Then the problem (5) admits only constant solutions.

Theorem 0.6. *Let $s \in (0,1)$ and let $h_{is} : (0,\infty) \times (0,\infty) \to \mathbb{R}$, $i = 1, 2$, be two functions defined by*

$$h_{is}(t_1, t_2) := t_i^{-\frac{n+2s}{n-2s}} \left(f_i(t_1, t_2) + d_{n,s} t_i\right), \ t_i > 0.$$

Assume that for $i, j = 1, 2$: $h_{i,s}$ are non-negative,

$$\begin{aligned} &h_{is}(t_1, t_2) \textit{ is nondecreasing in } t_j > 0, \ \textit{with } i \neq j, \\ &h_{is}(t_1, t_2) t_i^{\frac{n+2s}{n-2s}} \textit{ is nondecreasing in } t_i > 0, \\ &h_{is}(a_1 t, a_2 t) \textit{ is decreasing in } t > 0 \textit{ for any } a_i > 0. \end{aligned}$$

Then the problem (6) admits only constant solutions.

Chapter 5:

Looking at the previous results we can conclude that if the problem (1), (3) or (5) has nonconstant solution then the solution is negative or its sign changes. Thus, a question arises: given the following problem

$$\mathcal{A}_{2s} u = f(u) \text{ in } \mathbb{S}^n, \tag{7}$$

where $0 < 2s < n$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, are there nonconstant solutions of (7)?

If $f(t) = |t|^{\frac{4s}{n-2s}} t$, we will see that the problem (7) is closely related to the following problem

$$(-\Delta)^s v = |v|^{\frac{4s}{n-2s}} v \text{ in } \mathbb{R}^n. \tag{8}$$

If $s = 1$, we have $(-\Delta)^s = -\Delta$ the classical Laplacian operator. In [35], Gidas, Ni and Nirenberg proved that any positive solution of

$$-\Delta v = |v|^{\frac{4}{n-2}} v, \ \text{ in } \mathbb{R}^n, \tag{9}$$

which has finite energy is necessarily of the form

$$v(x) = \frac{[n(n-2)a^2]^{\frac{n-2}{4}}}{(a^2 + |x - x_0|^2)^{\frac{n-2}{2}}}, \tag{10}$$

where $a > 0$, $x_0 \in \mathbb{R}^n$. Damascelli and Gladiali [26] studied the problem of nonexistence of positive solutions for more general elliptic equations. Many people tried to show, without success, that all the solutions which are positive somewhere, are given by (10). However, Ding [29] showed that (9) has an unbounded sequence of solutions that are different from those given by (10).

For $n > 2m$, $m \in \mathbb{N}$, Lin [51], Wei and Xu [65] used the moving plane method to showed that all positive solutions of the polyharmonic problem

$$(-\Delta)^m v = |v|^{\frac{4m}{n-2m}} v, \text{ in } \mathbb{R}^n, v \in D^{2,m}(\mathbb{R}^n), \tag{11}$$

take the form

$$v(x) = \frac{C_{n,m}}{(a^2 + |x - x_0|^2)^{\frac{n-2m}{2}}},$$

where $C_{n,m}$ is a constant depending of n and m. $D^{2,m}(\mathbb{R}^n)$ denotes the set of real-functions v on $\mathbb{R}^n$ such that $v \in L^{2n/(n-2m)}(\mathbb{R}^n)$ and

$$\begin{aligned} \Delta^{m/2} v \in L^2(\mathbb{R}^n) &\qquad \text{if } m \text{ is even,} \\ \nabla \Delta^{(m-1)/2} v \in L^2(\mathbb{R}^n) &\qquad \text{if } m \text{ is odd.} \end{aligned}$$

Later, Guo and Liu [43] generalized Wei and Xu's results. Following the same idea of [29], Bartsch, Schneider and Weth [9] proved the existence of infinitely many sign-changing weak solutions of (11).

When $s \in (0, 2n)$, Chen, Li and Ou [24] showed that the positive solutions of (9) are the form

$$v(x) = \frac{C_{n,s}}{(a^2 + |x - x_0|^2)^{\frac{n-2s}{2}}}, \ a \in \mathbb{R}, \ x_0 \in \mathbb{R}^n, \tag{12}$$

where $C_{n,s}$ is a positive constant depending of n and s. In [22], they generalized their results showing the nonexistence of positive solutions for a class of nonlocal equations. There is a paper [33] where the author constructs sign-changing solutions to (8). Nevertheless, we believe that the construction and the computations are not clear to us, but it was shown in [27] for $1/2 < s < 1$.

Motivated by those results we will answer the question posed above. Some conditions on f will be required to show the existence of infinitely many solutions of (7). In particular, the result will lead to the following.

Theorem 0.7. *For $0 < s < 1$, there exists an unbounded sequence $\{v_l\}_{l \in \mathbb{N}}$ in $D^{s,2}(\mathbb{R}^n)$ of sign-changing solutions of (8).*

All the chapters are related to the preprints [2, 3].

Chapter 1

A conformal operator on the unit sphere

Let $(\mathbb{S}^n, g_{\mathbb{S}^n})$ be the standard unit sphere equipped with standard metric $g_{\mathbb{S}^n}$ of dimension $n > 2$. Denote by $[g_{\mathbb{S}^n}] = \{\tilde{\rho} g_{\mathbb{S}^n}; 0 < \tilde{\rho} \in C^\infty(\mathbb{S}^n)\}$ the conformal class of $g_{\mathbb{S}^n}$. Graham and Zworski [40] showed the existence of conformally covariant (pseudo) differential operators P_s^g for $g \in [g_{\mathbb{S}^n}]$, where $0 < 2s < n$ and these satisfy the conformal transformation relation (see also [19, 38])

$$P_s^g(\phi) = \rho^{-\frac{n+2s}{n-2s}} P_s^{g_{\mathbb{S}^n}}(\rho\phi), \text{ for all } \rho, \phi \in C^\infty(\mathbb{S}^n) \text{ with } g = \rho^{\frac{4}{n-2s}} g_{\mathbb{S}^n}.$$

Branson showed [15, Theorem 2.8] that the operator $\mathcal{A}_{2s} := P_s^{g_{\mathbb{S}^n}}$ is the unique operator on $\mathbb{S}^n$ satisfying the following properties:

(i) $\mathcal{A}_{2s}$ is positive:

$$\int_{\mathbb{S}^n} u \mathcal{A}_{2s} u \, d\eta > 0, \text{ for } u \in C^\infty(\mathbb{S}^n), \ u \neq 0.$$

(ii) $\mathcal{A}_{2s}$ is self-adjoint:

$$\int_{\mathbb{S}^n} u \mathcal{A}_{2s} w \, d\eta \ = \ \int_{\mathbb{S}^n} w \mathcal{A}_{2s} u \, d\eta, \text{ for } u, w \in C^\infty(\mathbb{S}^n).$$

(iii) $\mathcal{A}_{2s}$ is an *intertwining operator*:

$$\tau^* \mathcal{A}_{2s} (\tau^{-1})^* = |\mathcal{J}_\tau|^{-\frac{n+2s}{2n}} \mathcal{A}_{2s} |\mathcal{J}_\tau|^{\frac{n-2s}{2n}},$$

where τ is a conformal transformations of $\mathbb{S}^n$, $\mathcal{J}_\tau$ is the Jacobian of τ and τ^* denotes the natural pullback

$$\tau^* g_{\mathbb{S}^n} = |\mathcal{J}_\tau|^{\frac{2}{n}} g_{\mathbb{S}^n} \text{ and } \tau^* u = u \circ \tau \text{ for each } u \in C^\infty(\mathbb{S}^n).$$

The operator $\mathcal{A}_{2s}$ can be written as

$$\mathcal{A}_{2s} = \frac{\Gamma\left(B + \frac{1}{2} + s\right)}{\Gamma\left(B + \frac{1}{2} - s\right)}, \ B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2}, \tag{1.1}$$

where Γ is the Gamma function and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace-Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. In particular, $\mathcal{A}_2$ is the conformal Laplacian operator

$$\mathcal{A}_{2s} u = -\Delta_{g_{\mathbb{S}^n}} u + \frac{n(n-2)}{4} u,$$

and $\mathcal{A}_4$ is the Paneitz operator [60]

$$\mathcal{A}_{2s} u = \Delta_g^2 u - c_n \Delta_g u + d_n u,$$

where $c_n = (n^2 - 2n - 4)/2$ and $d_n = (n-4)n(n^2-4)/16$.

Denote by $H^s(\mathbb{S}^n)$ the set of functions $u : \mathbb{S}^n \to \mathbb{R}$ such that $u \in L^2(\mathbb{S}^n)$ and

$$\int_{\mathbb{S}^n} u \mathcal{A}_{2s} u \; d\eta < +\infty.$$

The set $H^s(\mathbb{S}^n)$ will be called *Sobolev space on* $\mathbb{S}^n$. We will see later that this name is given by the relationship between $H^s(\mathbb{S}^n)$ and the Sobolev space $H^s(\mathbb{R}^n)$.

The sharp Sobolev inequality on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ was established by Becker [11, Theorem 6] as follows

$$\frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)} \omega_n^{\frac{2s}{n}} \left(\int_{\mathbb{S}^n} |u|^{2_s^*} d\eta \right)^{\frac{2}{2_s^*}} \leq \int_{\mathbb{S}^n} u \mathcal{A}_{2s} u \; d\eta, \text{ for all } u \in H^s(\mathbb{S}^n), \tag{1.2}$$

where ω_n is the volume of $\mathbb{S}^n$ and $2_s^* = 2n/(n-2s)$. Equality holds only for functions of the form

$$u(\zeta) = c|1 - \langle a, \zeta \rangle|, \ c \in \mathbb{R}, \ a \in \mathbb{B}^{n+1}, \ \zeta \in \mathbb{S}^n.$$

The operator $\mathcal{A}_{2s}$ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection [57]. Let S be the south pole of $\mathbb{S}^n$, and $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ the stereographic projection, which is the inverse of

$$\mathcal{F}:\mathbb{R}^n\to\mathbb{S}^n\backslash\{S\},\ y\mapsto\left(\frac{2y}{1+|y|^2},\frac{1-|y|^2}{1+|y|^2}\right).$$

We recall that $\mathcal{F}$ is a conformal diffeomorphism. More precisely, if $g_{\mathbb{R}^n}$ denotes the flat euclidean metric on $\mathbb{R}^n$, then the pullback of $g_{\mathbb{S}^n}$ to $g_{\mathbb{R}^n}$ satisfies

$$\mathcal{F}^* g_{\mathbb{S}^n}=\frac{4}{(1+|\cdot|^2)^2}g_{\mathbb{R}^n}.$$

Moreover, the corresponding volume element is given by

$$d\eta=\left(\frac{2}{1+|y|^2}\right)^n\ dy. \tag{1.3}$$

For a function $u:\mathbb{S}^n\to\mathbb{R}$, we may define

$$Pu:\mathbb{R}^n\to\mathbb{R},\ (Pu)(y):=\xi_s(y)u(\mathcal{F}(y)),$$

where

$$\xi_s(y)=\left(\frac{2}{1+|y|^2}\right)^{\frac{n-2s}{2}}. \tag{1.4}$$

Sometimes ξ_s is called *conformal factor*. From (1.3), it is easy to see that P defines an isometric isomorphism between $L^{2^*_s}(\mathbb{S}^n)$ and $L^{2^*_s}(\mathbb{R}^n)$.

Let $(\dot{H}^s(\mathbb{R}^n),\|\cdot\|_s)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ smooth functions with compact support in $\mathbb{R}^n$ under the norm $\|\cdot\|_s$ induce by scalar product $\langle\cdot,\cdot\rangle_s$, where:

(i) if $s\in\mathbb{N}$ and s even,

$$\langle v,w\rangle_s=\int_{\mathbb{R}^n}\Delta^{\frac{s}{2}}(v)\Delta^{\frac{s}{2}}(w)\ dy; \tag{1.5}$$

(ii) if $s\in\mathbb{N}$ and s odd,

$$\langle v,w\rangle_s=\int_{\mathbb{R}^n}\nabla\Delta^{\frac{s-1}{2}}(v)\nabla\Delta^{\frac{s-1}{2}}(w)\ dy; \tag{1.6}$$

(iii) if $s\in(0,1)$,

$$\langle v,w\rangle_s=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+2s}}dxdy, \tag{1.7}$$

where Δ, ∇ are the laplacian and gradient operator on $(\mathbb{R}^n,g_{\mathbb{R}^n})$.

The norm $\|\cdot\|_s$ is well defined for the following reason. *The fractional Laplace operador on* $\mathbb{R}^n$ is defined by

$$\widehat{(-\Delta)^s u} := |\cdot|^{2s}\widehat{u}, \;\; u \in C_c^\infty(\mathbb{R}^n),$$

where $s > 0$ and $\widehat{\ }$ denotes the Fourier transform. When $s \in \mathbb{N}$, $(-\Delta)^s$ is the polyharmonic operator. From Appendix and (1.15) we have that

$$\langle v, v\rangle_s = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dy = \int_{\mathbb{R}^n} v(-\Delta)^s v \; dy \geq C \left(\int_{\mathbb{R}^n} v^{2_s^*} dy \right)^{\frac{2}{2_s^*}} \text{ for all } v \in C_c^\infty(\mathbb{R}^n). \tag{1.8}$$

So, $\|\cdot\|_s$ is a norm and the Sobolev space is given by

$$\dot{H}^s(\mathbb{R}^n) = \{v \in L^{2_s^*}(\mathbb{R}^n); \; \|v\|_s < +\infty\}.$$

Let $\dot{H}^s(\mathbb{S}^n)$ be the completion of the space of smooth functions $C^\infty(\mathbb{S}^n)$ under the norm $\|\cdot\|_*$ induced by scalar product $\langle \cdot, \cdot\rangle_*$ in $C^\infty(\mathbb{S}^n)$:

$$\langle u, w\rangle_* := \langle Pu, Pw\rangle_s, \;\; u, w \in C^\infty(\mathbb{S}^n). \tag{1.9}$$

Then, by construction,

P is also an isometric isomorphism between$(\dot{H}^s(\mathbb{S}^n), \|\cdot\|_*)$ and $(\dot{H}^s(\mathbb{R}^n), \|\cdot\|_s)$.

Next we note that $\langle \cdot, \cdot\rangle_*$ is the quadratic form of a unique positive self adjoint operator in $L^2(\mathbb{S}^n)$ denoted by $\dot{\mathcal{A}}_{2s}$, and

$$\|u\|_*^2 = \int_{\mathbb{S}^n} u\dot{\mathcal{A}}_{2s} u \; d\eta, \text{ for all } u \in C^\infty(\mathbb{S}^n). \tag{1.10}$$

Then, from (1.8)-(1.10) gets

$$(\dot{\mathcal{A}}_{2s} u) \circ \mathcal{F} = \xi_s^{-\frac{n+2s}{n-2s}} (-\Delta)^s (\xi_s (u \circ \mathcal{F})), \quad \text{for all } u \in C^\infty(\mathbb{S}^n).$$

From [57, Preposition 2.2], $\dot{\mathcal{A}}_{2s}$ is an intertwining operator, and from the uniqueness of the fractional conformal operator on $\mathbb{S}^n$, we have that $\dot{\mathcal{A}}_2 s = \mathcal{A}_{2s}$ and $\dot{H}^s(\mathbb{S}^n) = H^s(\mathbb{S}^n)$.

The operator $\mathcal{A}_{2s}$ is called sometimes conformal fractional operator because of the relationship between $\mathcal{A}_{2s}$ and $(-\Delta)^s$. There are many problems in conformal geometry that involve this operator, see e.g. [2, 3, 19, 38, 40, 45, 46, 47] and references therein.

For $n > 2$ and $0 < 2s < n$, we consider the following problems:

$$\begin{cases} \mathcal{A}_{2s} u = f(u) \text{ in } \mathbb{S}^n, \\ u \in H^s(\mathbb{S}^n), \end{cases} \tag{1.11}$$

where $\mathcal{A}_{2s}$ is the conformal fractional operator defined by (1.1), and

$$\begin{cases} (-\Delta)^s v = \xi_s^{\frac{n+2s}{n-2s}} f\left(\frac{v}{\xi_s}\right) \text{ in } \mathbb{R}^n, \\ v \in D^{s,2}(\mathbb{R}^n), \end{cases} \tag{1.12}$$

where $(-\Delta)^s$ is the polyharmonic operator if $s \in \mathbb{N}$, or is the fractional Laplace operator if $s \in (1, 0)$, $D^{s,2}(\mathbb{R}^n)$ denotes the space of real-valued functions $v \in L^{2^*_s}(\mathbb{R}^n)$ whose energy associated to $(-\Delta)^s$ is finite, i.e.,

$$\|v\|_s^2 := \langle v, v \rangle_s < +\infty. \tag{1.13}$$

We use the scalar products $\langle \cdot, \cdot \rangle_*$ and $\langle \cdot, \cdot \rangle_s$ to define solutions of (1.11) and (1.12).

Definition 1.1. *We say that*

(i) $u \in H^s(\mathbb{S}^n)$ is a weak solution of (1.11) if

$$\langle u, \varphi \rangle_* = \int_{\mathbb{S}^n} f(u)\varphi \, d\eta, \quad \textit{for all } \varphi \in H^s(\mathbb{S}^n);$$

(ii) $v \in D^{s,2}(\mathbb{R}^n)$ is a weak solution of (1.12) if

$$\langle v, \psi \rangle_s = \int_{\mathbb{R}^n} \xi_s^{\frac{n+2s}{n-2s}} f\left(\frac{v}{\xi_s}\right) \psi \, dy \quad \textit{for all } \psi \in D^{s,2}(\mathbb{R}^n).$$

The following lemma constitutes the bridge between (1.11) and (1.12). From now on, solution means a solution in the weak sense.

Lemma 1.1. *Every solution u of (1.11) corresponds to a solution v of (1.12) and*

$$\|u\|_* = \|v\|_s. \tag{1.14}$$

Proof. Let $u \in H^s(\mathbb{S}^n)$ be a solution of (1.11). Considere $v(y) = Pu(y)$. We shall prove first that $v \in D^{s,2}(\mathbb{R}^n)$. We have from (1.9) and (1.10) that

$$\|v\|_s^2 = \int_{\mathbb{S}^n} u\mathcal{A}_{2s}u \; d\eta < +\infty.$$

By the isometry of P and from Sobolev inequality in $\mathbb{S}^n$ (1.2),

$$\begin{aligned}\int_{\mathbb{R}^n} |v|^{2_s^*} dy &= \int_{\mathbb{S}^n} |u|^{2_s^*} \; d\eta \\ &\le C \int_{\mathbb{S}^n} u\mathcal{A}_{2s}u \; d\eta < +\infty.\end{aligned}$$

Thus $v \in D^{s,2}(\mathbb{R}^n)$ and (1.14) clearly follows.

Now, we will prove that v is solution of (1.12). Let $\varphi \in D^{s,2}(\mathbb{R}^n)$. Then $(\varphi/\xi_s) \circ \mathcal{F}^{-1} \in H^s(\mathbb{S}^n)$ and

$$\begin{aligned}\int_{\mathbb{R}^n} \xi_s^{\frac{n+2s}{n-2s}} f\left(\frac{v}{\xi_s}\right) \varphi \; dy &= \int_{\mathbb{R}^n} f(u \circ \mathcal{F}) \frac{\varphi}{\xi_s} \left(\frac{2}{1+|y|^2}\right)^n dy \\ &= \int_{\mathbb{S}^n} f(u) \left(\frac{\varphi}{\xi_s} \circ \mathcal{F}^{-1}\right) d\eta \\ &= \int_{\mathbb{S}^n} \mathcal{A}_{2s} v P^{-1}\varphi \; d\eta \\ &= \langle u, P^{-1}\varphi \rangle_* = \langle Pu, \varphi \rangle_s \\ &= \langle v, \varphi \rangle_s\end{aligned}$$

□

The Lemma 1.1 also implies that $\dot{H}^s(\mathbb{R}^n)$ coincides with $D^{s,2}(\mathbb{R}^n)$ and there is a relation between the sharp Sobolev inequality on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $(\mathbb{R}^n, g_{\mathbb{R}^n})$, which is (see [25])

$$\frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)} \omega_n^{\frac{2s}{n}} \left(\int_{\mathbb{R}^n} |v|^{2_s^*} d\eta\right)^{\frac{2}{2_s^*}} \le \int_{\mathbb{R}^n} v(-\Delta)^s v \; dy, \text{ for all } v \in D^{s,2}(\mathbb{R}^n). \tag{1.15}$$

Equality holds only for functions of the form

$$v(y) := \frac{c}{(c^2 + |y-b|^2)^{\frac{n-2s}{2}}}, \; c \in \mathbb{R}, \; b \in \mathbb{R}^n.$$

Now we analyze the case for systems. We consider

$$\begin{cases} \mathcal{A}_{2s}u_1 = f_1(u_1,u_2) & \text{in } \mathbb{S}^n, \\ \mathcal{A}_{2s}u_2 = f_2(u_1,u_2) & \text{in } \mathbb{S}^n, \\ u_1,u_2 \in H^s(\mathbb{S}^n), \end{cases} \tag{1.16}$$

and

$$\begin{cases} (-\Delta)^s v_1 = v_1^{\frac{n+2s}{n-2s}} f_1(v_1,v_2) & \text{in } \mathbb{R}^n, \\ (-\Delta)^s v_2 = v_2^{\frac{n+2s}{n-2s}} f_2(v_1,v_2) & \text{in } \mathbb{R}^n, \\ \quad v_1,v_2 \in D^{s,2}(\mathbb{R}^n). \end{cases} \tag{1.17}$$

Based on the case for one equation, we can define a solution pair of (1.16) and (1.17), and establish the relation bewteen them.

Definition 1.2. *We say that*

(i) $(u_1,u_2) \in H^s(\mathbb{S}^n) \times H^s(\mathbb{S}^n)$ *is a weak solution pair of (1.16) if*

$$\langle u_1, \varphi_1\rangle_* = \int_{\mathbb{S}^n} f_1(u_1,u_2)\varphi_1 \, d\eta \quad \text{and} \quad \langle u_2, \varphi_2\rangle_* = \int_{\mathbb{S}^n} f_2(u_1,u_2)\varphi_2 \, d\eta,$$

for all $(\varphi_1,\varphi_2) \in H^s(\mathbb{S}^n) \times H^s(\mathbb{S}^n)$;

(ii) $(v_1,v_2) \in D^{s,2}(\mathbb{R}^n) \times D^{s,2}(\mathbb{R}^n)$ *is a weak solution pair of (1.17) if*

$$\langle v_1, \psi_1\rangle_s = \int_{\mathbb{R}^n} \xi_s^{\frac{n+2s}{n-2s}} f_1\left(\frac{v_1}{\xi_s}, \frac{v_2}{\xi_s}\right) \psi_1 \, dy \quad \text{and} \quad \langle v_2, \psi_2\rangle_s = \int_{\mathbb{R}^n} \xi_s^{\frac{n+2s}{n-2s}} f_2\left(\frac{v_1}{\xi_s}, \frac{v_2}{\xi_s}\right) \psi_2 \, dy\,,$$

for all $(\psi_1,\psi_2) \in D^{s,2}(\mathbb{R}^n) \times D^{s,2}(\mathbb{R}^n)$.

The following result is analogous to Lemma 1.1

Lemma 1.2. *Every solution pair* (u_1,u_2) *of (1.16) corresponds to a solution pair of (1.17) and*

$$\|u_1\|_* = \|v_1\|_s \quad \text{and} \quad \|u_2\|_* = \|v_2\|_s.$$

Proof. Consider $v_1(y) = Pu_1(y)$ and $v_2(y) = Pu_2(y)$, where $y \in \mathbb{R}^n$. The rest of the proof is very similar to the proof of Lemma 1.1. ☐

The Lemmas 1.1 and 1.2 will be of great help in the study of the next chapters.

Chapter 2

Nonexistence of nonconstant positive solutions for Neumann problems

2.1 Case for an equation

In this section, we will study the following problem

$$\begin{cases} -\Delta_{g_0} u = f(u), \ u > 0 & \text{in } \mathbb{S}^n_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathbb{S}^n_+, \end{cases} \tag{2.1}$$

where $\mathbb{S}^n_+$ is the hemisphere provided with the standard metric g_0, $n > 2$, Δ_{g_0} is the Laplace-Beltrami operator on $(\mathbb{S}^n_+, g_0)$, $\frac{\partial u}{\partial \nu}$ is the normal derivative of u with respect to the unit exterior normal vector field ν of the boundary $\partial \mathbb{S}^n_+$, and $f : (0, \infty) \to \mathbb{R}$ is a continuous function.

Let $H^1(\mathbb{S}^n_+)$ be the completion of the space of smooth functions $C^\infty(\overline{\mathbb{S}^n_+})$ under the norm $\|\cdot\|_{*,+}$ induced for by scalar product

$$\langle u, w\rangle_{*,+} := \frac{n(n-2)}{4} \int_{\mathbb{S}^n_+} uw \, d\eta + \int_{\mathbb{S}^n_+} g_0(\nabla_{\mathbb{S}^n_+} u, \nabla_{\mathbb{S}^n_+} w) \, d\eta, \ u, w \in C^\infty(\overline{\mathbb{S}^n_+}),$$

where $\nabla_{\mathbb{S}^n_+}$ is the gradient on $\mathbb{S}^n_+$.

Definition 2.1. *We say that $u \in C(\overline{\mathbb{S}^n_+}) \cap H^1(\mathbb{S}^n_+)$ is a solution of (2.1) if*

$$\langle u, \varphi\rangle_{*,+} = \int_{\mathbb{S}^n_+} \left[f(u)\varphi + \frac{n(n-2)}{4} u\varphi \right] d\eta, \quad \textit{for all } \varphi \in H^1(\mathbb{S}^n_+). \tag{2.2}$$

If the solution u of (2.1) is smooth, then we can write (2.2) as

$$\int_{\mathbb{S}^n_+} \varphi \mathcal{A}_2 u \, d\eta = \int_{\mathbb{S}^n_+} \left[f(u)\varphi + \frac{n(n-2)}{4} u\varphi \right] d\eta, \quad \text{for all } \varphi \in H^1(\mathbb{S}^n_+),$$

where $\mathcal{A}_2$ is defined by (1.1).

Some conditions on f for (2.1) are motived by the study of the following problem

$$-\Delta_{g_{\mathbb{S}^n}} v = f(v), \;\; v > 0 \text{ in } \mathbb{S}^n. \tag{2.3}$$

For $f(t) = t^p - \lambda t$ with $1 < p \leq (n+2)/(n-2)$, Gidas and Spruck [36] showed that only solutions of (2.3) are $v \equiv \lambda^{1/(p-1)}$ provided that $0 < \lambda < n(n-2)/4$. For $p = (n+2)/(n-2)$ and $\lambda = n(n-2)/4$, the solutions of (2.3) are

$$u(\zeta) = \left(\frac{n(n-2)}{4}(\beta^2 - 1) \right)^{\frac{n-2}{4}} (\beta - \cos |\zeta_0 - \zeta|_{\mathbb{S}^n})^{\frac{2-n}{2}}, \;\; \zeta \in \mathbb{S}^n$$

where $\beta > 1$, $|\cdot|_{\mathbb{S}^n}$ is the distance in $\mathbb{S}^n$, $\zeta_0 \in \mathbb{S}^n$. On the other hand, Bandle and Wei [8] showed that there are nonconstant solution of (2.3) as $\lambda \to +\infty$. All these authors used some remarkable identities on $\mathbb{S}^n$ to conclude their results while Brezis and Li [16] used the moving planes method to deduce the following result.

Theorem 2.1. *(Brezis-Li [16]) Assume that f is smooth and*

$$h_1(t) := t^{-\frac{n+2}{n-2}} \left(f(t) + \frac{n(n-2)}{4} t \right) \text{ is decreasing on } (0, +\infty).$$

Then any solution of (2.3) is constant.

In the proof of Theorem 2.1 was also used some results of symmetry and related properties via the maximum principle [35]. So, the smoothness of f and the solution are necessary. However, we will use the method of moving planes and some techniques based on inequalities of integrals to show the following main result without needing the smoothness of f.

Theorem 2.2. *Assume that*

$$h_1(t) := t^{-\frac{n+2}{n-2}} \left(f(t) + \frac{n(n-2)}{4} t \right) \;\; \text{is decreasing in} \;\; (0, +\infty).$$

Then the problem (2.1) admits only constant solutions.

Example 2.1. *Consider the function*

$$f(t) = t^p - \lambda t, \ p > 1, \ \lambda > 0.$$

Then (2.1) becomes

$$\begin{cases} -\Delta_g u = u^p - \lambda u, \ u > 0 & \text{in } \mathbb{S}^n_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathbb{S}^n_+. \end{cases} \tag{2.4}$$

The following result guarantees that the Li-Ni's conjecture is true for the case hemisphere $\mathbb{S}^n_+$.

Corollary 2.1. *Assume that $p \leq (n+2)/(n-2)$ and $\lambda \leq n(n-2)/4$, and at least one of these inequalities is strict. Then the only solution of (2.4) is the constant $u \equiv \lambda^{1/(p-1)}$.*

In order to prove the Theorem 2.2 we will study a problem in $\mathbb{R}^n_+$ and the symmetry of its solutions. We use the stereographic projection to formulate an equivalent problem with (2.1).

Let ζ be an arbitrary point on $\partial \mathbb{S}^n_+$, which we will rename the south pole S. Let $\mathcal{F}^{-1} : \overline{\mathbb{S}^n_+} \backslash \{S\} \to \overline{\mathbb{R}^n_+}$ be the stereographic projection, which is the inverse of

$$\mathcal{F}(y) = \left(\frac{2y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2} \right), \ y \in \mathbb{R}^n.$$

We have that $\mathcal{F}$ is a conformal diffeomorphism and

$$\mathcal{F}^* g_0 = \frac{4}{(1+|\cdot|^2)^2} g_{\mathbb{R}^n}.$$

Let u be a solution of (2.1). Consider $v(y) := \xi_1(y) u(\mathcal{F}(y))$, $y \in \mathbb{R}^n_+$. Then

$$v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+). \tag{2.5}$$

From Appendix we have that if u is smooth, then v satisfies the problem

$$\begin{cases} -\Delta v = h_1 \left(\dfrac{v}{\xi_1} \right) v^{\frac{n+2}{n-2}}, \ v > 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial y_n} = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases} \tag{2.6}$$

We use the moving plane method to prove symmetry with respect to the axis y_n of the solutions of problem (2.6). We will also use some techniques based on inequalities of integrals. These techniques are used in works concerning Liouville type theorems for elliptic equation and system with general nonlinearity (see e.g. [6, 12, 26, 42, 43, 49, 68, 67] and references therein), but those techniques were originally based on the ideas of Terracini [63]. On the other hand, this method was also widely used in integral equation and system that are closely related to the fractional differential equation and system, see e.g., [22, 24, 23, 34, 69] and references therein. In those works, the authors use the Kelvin transform to show results about the nonexistence of solutions. However, the Kelvin transform does not contribute anything to our purposes. So, we will use the geometry of $\mathbb{S}^n_+$ or $\mathbb{S}^n$.

Lemma 2.1. *Let u be a smooth solution of (2.1). Under the assumptions of Theorem 2.2, $v = \xi_1(u \circ \mathcal{F})$ is symmetric with respect to the axis y_n.*

Proof. Given $t \in \mathbb{R}$ we set

$$Q_t = \{y \in \mathbb{R}^n_+;\ y_1 < t\}; \quad U_t = \{y \in \mathbb{R}^n_+;\ y_1 = t\},$$

where $y_t := I_t(y) := (2t - y_1, y')$ is the image of a point $y = (y_1, y')$ under the reflection through the hyperplane U_t. We define the reflected function by $v^t(y) := v(y_t)$. The proof is carried out in three steps. In the first step we show that

$$\Lambda := \inf\{t > 0;\ v \geq v^\mu \text{ in } Q_\mu,\ \forall \mu \geq t\}$$

is a number, that is $\Lambda < +\infty$, and in fact $v \geq v^\mu$ in $Q_\mu, \forall \mu > \Lambda$. The second step consists in proving that if $\Lambda > 0$ then $v \equiv v^\Lambda$ in Q_Λ. In the third step we conclude that $\Lambda = 0$, which implies the symmetry of v.

Step 1. $\Lambda < +\infty$.

Assume by an argument of contradiction that exists a $t > 0$, such that $v^t(y) > v(y)$ for some $y \in Q_t$. Since $|y| < |y_t|$, we have

$$\frac{v(y)}{\xi_1(y)} < \frac{v^t(y)}{\xi_1^t(y)},$$

and

$$
\begin{aligned}
-\Delta(v^t - v) &= h_1\left(\frac{v^t}{\xi_1^t}\right)(v^t)^{\frac{n+2}{n-2}} - h_1\left(\frac{v}{\xi_1}\right)v^{\frac{n+2}{n-2}} \\
&\leq h_1\left(\frac{v^t}{\xi_1^t}\right)(v^t)^{\frac{n+2}{n-2}} - h_1\left(\frac{v^t}{\xi_1^t}\right)v^{\frac{n+2}{n-2}} \\
&= h_1\left(\frac{v^t}{\xi_1^t}\right)((v^t)^{\frac{n+2}{n-2}} - v^{\frac{n+2}{n-2}}) \\
&\leq \frac{n+2}{n-2}\max\left\{h_1\left(\frac{v^t}{\xi_1^t}\right), 0\right\}(v^t)^{\frac{4}{n-2}}(v^t - v) \\
&= C(v^t)^{\frac{4}{n-2}}(v^t - v), \qquad (2.7)
\end{aligned}
$$

where the last inequality is a consequence of $h_1(\frac{v}{\xi}) \in L^\infty(\mathbb{R}^n_+)$ and C be a constant. By the fact of that $v(y) \to 0$ as $|y| \to \infty$, we can take $(v^t - v - \varepsilon)^+$, where $^+$ denotes the positive part of function, as a test function with compact support in Q_t, for $\varepsilon > 0$ and small. From (2.7) we obtain

$$
\int_{Q_t} |\nabla(v^t - v - \varepsilon)^+|^2 dy \leq C\int_{Q_t}(v^t)^{\frac{4}{n-2}}(v^t - v)(v^t - v - \varepsilon)^+ dy.
$$

By (2.5), the right hand side of the above inequality is limited by the integral of a function that does not dependent of ε. Indeed, if $(v^t(y) - v(y) - \varepsilon)^+ > 0$ for some $y \in Q_t$, then $v^t(y) > v(y)$ and

$$
(v^t)^{\frac{4}{n-2}}(v^t - v)(v^t - v - \varepsilon)^+ \leq (v^t)^{\frac{2n}{n-2}} \in L^1(\mathbb{R}^n_+).
$$

Using Fatou's lemma, Hölder's and Sobolev's inequalities, and Dominate Convergence

Theorem, we have

$$
\begin{aligned}
\int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2}} dy &\le \liminf_{\varepsilon \to 0} \int_{Q_t} [(v^t - v - \varepsilon)^+]^{\frac{2n}{n-2}} dy \\
&\le \liminf_{\varepsilon \to 0} C \left(\int_{Q_t} |\nabla (v^t - v - \varepsilon)^+|^2 dy \right)^{\frac{n}{n-2}} \\
&\le C \liminf_{\varepsilon \to 0} \left(\int_{Q_t} (v^t)^{\frac{4}{n-2}} (v^t - v)(v^t - v - \varepsilon)^+ dy \right)^{\frac{n}{n-2}} \\
&= C \left(\int_{Q_t} (v^t)^{\frac{4}{n-2}} [(v^t - v)^+]^2 dy \right)^{\frac{n}{n-2}} \\
&\le C \left[\left(\int_{Q_t} (v^t)^{\frac{2n}{n-2}} dy \right)^{\frac{2}{n}} \left(\int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} \right]^{\frac{n}{n-2}} \\
&\le \varphi(t) \left(\int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2}} dy \right), \qquad (2.8)
\end{aligned}
$$

where

$$
\varphi(t) = C \left(\int_{Q_t} (v^t)^{\frac{2n}{n-2}} dy \right)^{\frac{2}{n-2}}.
$$

Since $v^{\frac{2n}{n-2}} \in L^1(\mathbb{R}^n_+)$, then $\lim_{t \to +\infty} \varphi(t) = 0$. Thus, choosing $t_1 > 0$ large enough, in a such way that $\varphi(t_1) < 1$, we obtain from (2.8) that

$$
\int_{Q_t} [(v^t - v)^+]^{\frac{2n}{n-2}} dy = 0, \qquad \text{for all } t > t_1.
$$

Therefore, $(v^t - v)^+ \equiv 0$, for $t > t_1$. Which is a contradiction with our assumption.

Step 2. If $\Lambda > 0$ then $v \equiv v_\Lambda$ in Q_Λ.

Since the solution is continuous, by the definition of Λ we get $v \ge v^\Lambda$ and $\xi_1 > \xi_1^\Lambda$ in Q_Λ. Then suppose that exists a point $y_0 \in Q_\Lambda$ such that $v(y_0) = v^\Lambda(y_0)$. Then, there is $r > 0$ sufficiently small, so that

$$
\frac{v}{\xi_1} < \frac{v^\Lambda}{\xi_1^\Lambda} \quad \text{in} \quad B(y_0, r).
$$

Hence, for $y \in B(y_0, r)$,

$$
\begin{aligned}
-\Delta(v(y) - v^{\Lambda}(y)) &= h_1\left(\frac{v}{\xi_1}\right) v^{\frac{n+2}{n-2}}(y) - h_1\left(\frac{v^{\Lambda}}{\xi_1^{\Lambda}}\right) (v^{\Lambda})^{\frac{n+2}{n-2}} \\
&\geq h_1\left(\frac{v^{\Lambda}}{\xi_1^{\Lambda}}\right) (v^{\frac{n+2}{n-2}}(y) - (v^{\Lambda})^{\frac{n+2}{n-2}}(y)) \\
&\geq -C(v(y) - v^{\Lambda}(y)), \qquad (2.9)
\end{aligned}
$$

where C is a non-negative constant. The last inequality is a consequence of the limitation in $\mathbb{R}^n$ of v, v^{Λ} and $h_1(\frac{v^{\Lambda}}{\xi^{\Lambda}})$. From Maximum Principle [42, Proposition 3.7], we obtain $v \equiv v^{\Lambda}$ in $B(y_0, r)$. As the set $\{y \in Q_{\Lambda};\ v(y) = v^{\Lambda}(y)\}$ is open and closed in Q_{Λ}, then $v \equiv v^{\Lambda}$ in Q_{Λ}.

Now, assume that $v > v^{\Lambda}$ in Q_{Λ}. We can choose a compact $K \subset Q_{\Lambda}$ and a number $\delta > 0$ satisfying $\forall t \in (\Lambda - \delta, \Lambda)$ that $K \subset Q_t$ and

$$
\varphi(t) = C\left(\int_{Q_t \backslash K} (v^t)^{\frac{2n}{n-2}} dy\right)^{\frac{2}{n-2}} < \frac{1}{2}. \qquad (2.10)
$$

Moreover, there exists $0 < \delta_1 < \delta$, such that

$$
v > v^t, \text{ in } K\ \forall t \in (\Lambda - \delta_1, \Lambda). \qquad (2.11)
$$

Thus, we get that $(v^t - v)^+ \equiv 0$ on K. Using (2.10) and following as in Step 1, since the integrals are over $Q_t \backslash K$, we see that $(v^t - v)^+ \equiv 0$ in $Q_t \backslash K$. By (2.11) we get $v > v^t$ in Q_t for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of Λ.

Step 3. Symmetry.

If $\Lambda > 0$, then

$$
h_1\left(\frac{v}{\xi_1}\right) = -\frac{\Delta v}{v^{\frac{n+2}{n-2}}} = -\frac{\Delta v^{\Lambda}}{(v^{\Lambda})^{\frac{n+2}{n-2}}} = h_1\left(\frac{v^{\Lambda}}{\xi_1^{\Lambda}}\right) = h_1\left(\frac{v}{\xi_1^{\Lambda}}\right) < h_1\left(\frac{v}{\xi_1}\right) \quad \text{in } Q_{\Lambda}.
$$

This is a contradiction. Thus $\Lambda = 0$. By continuity of v, we have $v^0(y) \leq v(y)$ for all $y \in Q_0$. We can also perform the moving plane procedure from the left and find a corresponding Λ'. An analogue to Step 1 and Step 2 implies that $\Lambda' = 0$. Then we get $v^0(y) \geq v(y)$ for $y \in Q_0$. These inequalities say us that v is symmetric with respect to U_0. Therefore, if $\Lambda = \Lambda' = 0$ for all directions that are vertical to the y_n direction, then v is symmetric with respect to the axis y_n.

□

Now, we will study the symmetry of the solution in the hemisphere. In others words, we go to analyse the symmetry in the meridians. In that context, the next result is a first crucial step.

Lemma 2.2. *Let u be a smooth solution of (2.1). Under the assumptions of Theorem 2.2, we have that for each $r \in [0, \pi/2]$, u is constant in*

$$A_r = \{\zeta \in \mathbb{S}^n_+;\ r = \inf\{|\zeta - \zeta_0|_{\mathbb{S}^n};\ \zeta_0 \in \partial\mathbb{S}^n_+\}\}, \tag{2.12}$$

where $|\cdot|_{\mathbb{S}^n}$ is the distance in $\mathbb{S}^n$.

Proof. Let ζ_1, $\zeta_2 \in A_r$, $r > 0$. Then exists a $\zeta_0 \in \partial\mathbb{S}^n_+$ such that $|\zeta_1 - \zeta_0|_{\mathbb{S}^n} = |\zeta_2 - \zeta_0|_{\mathbb{S}^n}$. Let $\mathcal{F}^{-1} : \overline{\mathbb{S}^n_+}\backslash\{\zeta_0\} \to \overline{\mathbb{R}^n_+}$ be the stereographic projection. Then $\mathcal{F}^{-1}(\zeta_1)$ and $\mathcal{F}^{-1}(\zeta_2)$ are symmetrical points with respect to the axis y_n. We define $v = \xi_1(u \circ \mathcal{F})$ in $\mathbb{R}^n_+$. From Lemma 2.1 we obtain that v is symmetric with respect to the axis y_n. By the definition of v and the symmetry of ξ_1, we have $u(\zeta_1) = u(\zeta_2)$. Therefore u is constant in A_r for each $r \in (0, \pi/2]$. By continuity of u, we have u is constant in A_0.

□

Proof of Theorem 2.2

Let u be the solution of problem 2.1. We take an arbitrary point $p \in \partial\mathbb{S}^n_+$, and let $\mathcal{F}^{-1} : \overline{\mathbb{S}^n_+}\backslash\{p\} \to \overline{\mathbb{R}^n_+}$ be the stereographic projection. We consider the equation (2.1), where $v = \xi_1(u \circ \mathcal{F})$ in $\mathbb{R}^n_+$.

Define

$$v^*(y) = \begin{cases} v(y', y_n), & \text{if } y_n \geq 0, \\ v(y', -y_n), & \text{if } y_n < 0, \end{cases}$$

where $y = (y', y_n) \in \mathbb{R}^n_+$. Then $v^* \in C^1(\mathbb{R}^n)$ is a weak solution of problem

$$-\Delta v^* = h\left(\frac{v^*}{\xi_1}\right) v^{*\frac{n+2}{n-2}},\ v^* \geq \ \text{in } \mathbb{R}^n. \tag{2.13}$$

We can apply Lemma 2.1 for v^* in whole space $\mathbb{R}^n$ to get radial symmetry in $\mathbb{R}^n_+$. Then we obtain that v^* is radially symmetric, which implies

$$v^*(y) = v(y) = C \text{ for all } y \in \mathbb{R}^n_+ \text{ such that } |y| = 1, \tag{2.14}$$

where C is a constant.

On the other hand, the set $\mathcal{F}(\{y \in \mathbb{R}^n_+;\ |y| = 1\})$ intersects perpendicularly to A_r for any $r \in (0, \pi/2]$. Therefore, from (2.14) we have that u is constant in $\mathcal{F}(\{y \in \mathbb{R}^n_+; |y| = 1\})$, and from Lemma 2.2, we have that u is constant in $\mathbb{S}^n_+$. □

Remark 2.1. *We can remove the condition on the smoothness of f in the Theorem 2.1 and have the same conclution.*

In fact, from Lemma 1.1 gets that for each solution u of (2.3), the function $v = \xi_1(u \circ \mathcal{F})$ is the solution of

$$-\Delta v = h_1 \left(\frac{v}{\xi_1} \right) v^{\frac{n+2}{n-2}}, v > 0 \ \text{ in } \mathbb{R}^n,$$

From Lemma 2.1 we have that v is radially symmetric with respect to origin. Then u is constant in each parallel $\mathcal{F}(\{y \in \mathbb{R}^n;\ |y| = r\})$, $r > 0$. If we consider the stereographic projection $\tilde{\mathcal{F}}$ in relation to a point on the equator, then we can use the same argument to show that u is constant in each meridian of sphere $\mathbb{S}^n$. Thus, u is constant in $\mathbb{S}^n$.

Example 2.2. *Consider the function*

$$f(t) = \begin{cases} -\beta t, & \textit{if } 0 < t < (\lambda - \beta)^{\frac{1}{p-1}}, \\ t^p - \lambda t, & \textit{if} (\lambda - \beta)^{\frac{1}{p-1}} \leq t, \end{cases}$$

where $p > 1$, $\lambda, \beta > 0$. The function f is not smooth in $(\lambda - \beta)^{1/(p-1)}$ if $\lambda - \beta > 0$. Then (2.3) becomes

$$-\Delta_{g_{\mathbb{S}^n}} u = \begin{cases} -\beta u, & \textit{if } 0 < u < (\lambda - \beta)^{\frac{1}{p-1}}, \\ u^p - \lambda u, & \textit{if } (\lambda - \beta)^{\frac{1}{p-1}} \leq u. \end{cases} \tag{2.15}$$

Corollary 2.2. *Assume that $p \leq (n+2)/(n-2)$ and $\lambda \leq n(n-2)/4$, and at least one of these inequalities is strict. If $\lambda - \beta > 0$ then the only positive solution of (2.15) is the constant $u \equiv \lambda^{1/(p-1)}$.*

2.2 Case for systems

The study of systems arise naturally as a generalization of the study of an equation, see e.g, [44, 42, 23, 67].

In this section we study the following system

$$\begin{cases} -\Delta_{g_0} u_1 = f_1(u_1, u_2) & \text{in } \mathbb{S}^n_+, \\ -\Delta_{g_0} u_2 = f_2(u_1, u_2) & \text{in } \mathbb{S}^n_+ \\ \quad u_1,\ u_2 > 0 & \text{in } \mathbb{S}^n_+, \\ \quad \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial \mathbb{S}^n_+, \end{cases} \tag{2.16}$$

where g_0 is the standard metric on the hemisphere $\mathbb{S}^n_+$, $n \geq 3$, Δ_{g_0} is the Laplace-Beltrami operator on $(\mathbb{S}^n_+, g_0)$, $\frac{\partial}{\partial \nu}$ is the derivate with respect to the outward normal vector field ν, and f_1, $f_2 : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ are continuous functions.

Analogously to Theorem 2.2, under some conditions on f_1 and f_2 we have our next main result.

Theorem 2.3. *Let $h_{i1} : (0,\infty) \times (0,\infty) \to \mathbb{R}$, $i = 1,2$, be two functions defined by*

$$h_{i1}(t_1,t_2) := t_i^{-\frac{n+2}{n-2}} \left(f_i(t_1,t_2) + \frac{n(n-2)}{4} t_i \right), \ t_1 > 0, \ t_2 > 0.$$

Assume that

$$\begin{aligned} &h_{i1}(t_1,t_2) \text{ is nondecreasing in } t_j > 0, \text{ with } i \neq j, \\ &h_{i1}(t_1,t_2) t_i^{\frac{n+2}{n-2}} \text{ is nondecreasing in } t_i > 0, \\ &h_{i1}(a_1 t, a_2 t) \text{ is decreasing in } t > 0 \text{ for any } a_i > 0, \end{aligned}$$

for $i,j = 1,2$. Then the problem (2.16) admits only constant solutions.

Example 2.3. *An example for system is the case*

$$f_1(t_1,t_2) = t_1^{\alpha_1} t_2^{\alpha_2} - \lambda_1 t_1, \ f_2(t_1,t_2) = t_1^{\beta_1} t_2^{\beta_2} - \lambda_2 t_2,$$

where $\alpha_i, \beta_i, \lambda_i$ are positive constants for $i = 1,2$ with $\alpha_1 > 1$ and $\beta_2 > 1$. So that (2.16) becomes

$$\begin{cases} -\Delta_{g_0} u_1 = u_1^{\alpha_1} u_2^{\alpha_2} - \lambda_1 u_1 & \text{in } \mathbb{S}^n_+, \\ -\Delta_{g_0} u_2 = u_1^{\beta_1} u_2^{\beta_2} - \lambda_2 u_2 & \text{in } \mathbb{S}^n_+, \\ u_1, \ u_2 > 0 & \text{in } \mathbb{S}^n_+, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\mathbb{S}^n_+, \end{cases} \tag{2.17}$$

Corollary 2.3. *Assume that $\alpha_1 + \alpha_2 \leq (n+2)/(n-2)$, $\beta_1 + \beta_2 \leq (n+2)/(n-2)$, $\lambda_1 \leq n(n-2)/4$ and $\lambda_2 \leq n(n-2)/4$, and at least two of these inequalities are strict. Then*

a) if $(\alpha_1 - 1)(\beta_2 - 1) \neq \alpha_2 \beta_1$, then the problem (2.17) has a unique positive solution pair (u_1, u_2),

b) if $(\alpha_1 - 1)(\beta_2 - 1) = \alpha_2 \beta_1$, then the problem (2.17) has no positive solution pair (u_1, u_2) provided that $\lambda_1^{\beta_2 - 1} \neq \lambda_2^{\alpha_2}$.

Similarly to Section 2.1, we consider the functions v_1, v_2 defined on $\mathbb{R}^n_+$ by

$$v_1(y) = \xi_1(y) u_1(\mathcal{F}(y)), \ v_2(y) = \xi_1(y) u_2(\mathcal{F}(y)),$$

where ξ_1 is defined in (1.4), (u_1, u_2) is the solution of (2.16) and $\mathcal{F}$ is the stereographic projection. Then we have that

$$v_1, v_2 \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+). \tag{2.18}$$

From Appendix we have

$$\begin{cases} -\Delta v_1 = h_{11}\left(\dfrac{v_1}{\xi_1}, \dfrac{v_2}{\xi_1}\right) v_1^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ -\Delta v_2 = h_{21}\left(\dfrac{v_1}{\xi_1}, \dfrac{v_2}{\xi_1}\right) v_2^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ v_1, \ v_2 > 0 & \text{in } \mathbb{R}^n_+, \\ \quad \dfrac{\partial v_1}{\partial y_n} = \dfrac{\partial v_2}{\partial y_n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.19}$$

To show the Theorem 2.2, we will use the same arguments that were used in the proof of the Theorem 2.2.

Lemma 2.3. *Let (u_1, u_2) be a solution of (2.16). Then $v_1 = \xi_1(u_1 \circ \mathcal{F})$ and $v_2 = \xi_1(u_2 \circ \mathcal{F})$ are symmetric with respect to the axis y_n.*

Proof. For any $t \in \mathbb{R}$, let Q_t, y_t and U_t as in the proof of Theorem 2.2. Consider the reflected functions $v_1^t(y) := v_1(y_t)$, and $v_2^t := v_2(y_t)$. Finally, let

$$\Lambda := \inf\{t > 0; \ v_1 \geq v_1^\mu, v_2 \geq v_2^\mu \text{ in } Q_\mu, \ \forall \mu \geq t\}.$$

As before, the proof is carried out in three steps.

Step 1. $\Lambda < +\infty$.

Assume, again by an argument of contradiction that exists $t > 0$ such that $v_1^t(y) > v_1(y)$ for some $y \in Q_t$. Since $|y| < |y_t|$, we have, $\frac{v_1}{\xi_1} < \frac{v_1^t}{\xi_1^t}$.

If $v_2 > v_2^t$, then by conditions on h_{i1} we have

$$
\begin{aligned}
-\Delta(v_1^t - v_1) &= h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) v_1^{\frac{n+2}{n-2}} \\
&\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2^t}{\xi_1}\right) v_1^{\frac{n+2}{n-2}} \\
&= h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2^t v_1}{\xi_1 v_1}\right) v_1^{\frac{n+2}{n-2}} \\
&\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t v_1^t}{\xi_1^t v_1}\right) v_1^{\frac{n+2}{n-2}} \\
&\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right) v_1^{\frac{n+2}{n-2}}.
\end{aligned}
$$

That is

$$
\begin{aligned}
-\Delta(v_1^t - v_1) &\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)\left((v_1^t)^{\frac{n+2}{n-2}} - v_1^{\frac{n+2}{n-2}}\right) \\
&\le C(v_1^t)^{\frac{4}{n-2}} \max\left\{h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right), 0\right\}(v_1^t - v_1) \\
&\le C(v_1^t)^{\frac{4}{n-2}}(v_1^t - v_1),
\end{aligned} \tag{2.20}
$$

where C is a non-negative constant. If $v_2^t > v_2$, then $\frac{v_1^t v_2^t}{\xi_1^t} > \frac{v_1 v_2}{\xi_1}$ and

$$
\begin{aligned}
-\Delta(v_1^t - v_1) &= h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) v_1^{\frac{n+2}{n-2}} \\
&= h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1 v_2}{\xi_1 v_2}, \frac{v_2 v_1}{\xi_1 v_1}\right) v_1^{\frac{n+2}{n-2}} \\
&\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1^t v_2^t}{\xi_1^t v_2}, \frac{v_2^t v_1^t}{\xi_1^t v_1}\right) v_1^{\frac{n+2}{n-2}} \\
&\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1^t v_2^t}{\xi_1^t v_2}, \frac{v_2^t}{\xi_1^t}\right) v_1^{\frac{n+2}{n-2}} \\
&\le h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)(v_1^t)^{\frac{n+2}{n-2}} - h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)\left(\frac{v_2}{v_2^t}\right)^{\frac{n+2}{n-2}} v_1^{\frac{n+2}{n-2}} \\
&= (v_2^t)^{-\frac{n+2}{n-2}} h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right)\left((v_1^t v_2^t)^{\frac{n+2}{n-2}} - (v_1 v_2)^{\frac{n+2}{n-2}}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&-\Delta(v_1^t - v_1) \\
&\quad \le C(v_1^t)^{\frac{4}{n-2}} \max\left\{h_{11}\left(\frac{v_1^t}{\xi_1^t}, \frac{v_2^t}{\xi_1^t}\right), 0\right\}[(v_1^t - v_1) + v_1^t(v_2^t)^{-1}(v_2^t - v_2)] \\
&\quad \le C(v_1^t)^{\frac{4}{n-2}}[(v_1^t - v_1) + (u_1 \circ \mathcal{F})^t((u_2 \circ \mathcal{F})^t)^{-1}(v_2^t - v_2)] \\
&\quad \le C(v_1^t)^{\frac{4}{n-2}}[(v_1^t - v_1) + (v_2^t - v_2)],
\end{aligned} \tag{2.21}
$$

where the last inequality is a consequence of $(u_1 \circ \mathcal{F})^t((u_2 \circ \mathcal{F})^t)^{-1} \in L^\infty(\mathbb{R}^n)$ and C is a non-negative constant. Since $v_1(y) \to 0$ as $|y| \to \infty$, then for $\varepsilon > 0$, we can take $(v_1^t - v_1 - \varepsilon)^+$ as a test function with compact support in Q_t for (2.20) and (2.21). Then we obtain

$$
\int_{Q_t} |\nabla(v_1^t - v_1 - \varepsilon)^+|^2 dy \le C \int_{Q_t} (v_1^t)^{\frac{4}{n-2}}[(v_1^t - v_1) + (v_2^t - v_2)^+](v_1^t - v_1 - \varepsilon)^+ dy
$$

By (2.18), we obtain that the right hand side of the above inequality is limited by the integral of a function independent of ε. In fact, if

$$
(v_1^t(y) - v_1(y) - \varepsilon)^+ > 0 \quad \text{and} \quad (v_2^t(y) - v_2(y))^+ > 0 \quad \text{for some} \quad y \in Q_t,
$$

then $v_1^t(y) > v_1(y)$, $v_2^t(y) > v_2(y)$ and

$$
(v_1^t)^{\frac{4}{n-2}}[(v_1^t - v_1) + (v_2^t - v_2)^+](v_1^t - v_1 - \varepsilon)^+ \le 4(v_1^t)^{\frac{n+2}{n-2}}(v_1^t + v_2^t) \in L^1(\mathbb{R}^n_+).
$$

Using Fatou's lemma, Hölder's and Sobolev's inequalities, and dominate convergence theorem, we have

$$
\begin{aligned}
&\left(\int_{Q_t}[(v_1^t-v_1)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\\
&\le \liminf_{\varepsilon\to 0}\left(\int_{Q_t}[(v_1^t-v_1-\varepsilon)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\\
&\le \liminf_{\varepsilon\to 0} C\int_{Q_t}|\nabla(v_1^t-v_1-\varepsilon)^+|^2dy\\
&\le C\liminf_{\varepsilon\to 0}\int_{Q_t}(v_1^t)^{\frac{4}{n-2}}[(v_1^t-v_1)+(v_2^t-v_2)^+](v_1^t-v_1-\varepsilon)^+dy\\
&= C\int_{Q_t}(v_1^t)^{\frac{4}{n-2}}[(v_1^t-v_1)+(v_2^t-v_2)^+](v_1^t-v_1)^+dy\\
&\le C\int_{Q_t}(v_1^t)^{\frac{4}{n-2}}\{[(v_1^t-v_1)^+]^2+(v_2^t-v_2)^+(v_1^t-v_1)^+\}dy\\
&\le C\psi_1(t)\left[\left(\int_{Q_t}[(v_1^t-v_1)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\right.\\
&\left.+\left(\int_{Q_t}[(v_2^t-v_2)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}[(v_1^t-v_1)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\right],
\end{aligned}
$$

where

$$\psi_1(t)^{\frac{n}{2}}=\int_{Q_t}(v_1^t)^{\frac{2n}{n-2}}dy.$$

This implies that

$$(1-C\psi_1(t))\left(\int_{Q_t}[(v_1^t-v_1)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\le C\psi_1(t)\left(\int_{Q_t}[(v_2^t-v_2)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}. \quad (2.22)$$

Similarly, we have

$$(1-C\psi_2(t))\left(\int_{Q_t}[(v_2^t-v_2)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\le C\psi_2(t)\left(\int_{Q_t}[(v_1^t-v_1)^+]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}, \quad (2.23)$$

where

$$\psi_2(t)^{\frac{n}{2}}=\int_{Q_t}(v_2^t)^{\frac{2n}{n-2}}dy.$$

Since $v_1,\, v_2\in L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$, we obtain

$$\lim_{t\to\infty}\psi_1(t)=\lim_{t\to\infty}\psi_2(t)=0.$$

Then, we can choose a $t_1 \in \mathbb{R}$, such that

$$C\psi_1(t) < \frac{1}{2} \ \text{ and } \ C\psi_2(t) < \frac{1}{2}, \ \text{ for all } t > t_1, \tag{2.24}$$

and from (2.22) - (2.24), we have

$$\int_{Q_t} [(v_1^t - v_1)^+]^{\frac{2n}{n-2}} dy = 0, \ \int_{Q_t} [(v_2^t - v_2)^+]^{\frac{2n}{n-2}} dy = 0 \ \text{ for all } t > t_1.$$

Therefore, $(v_1^t - v_1)^+ \equiv 0$ and $(v_2^t - v_2)^+ \equiv 0$ in Q_t for all $t > t_1$. Which is a contradiction with our assumption. Therefore, $\Lambda < +\infty$.

Step 2. If $\Lambda > 0$ then $v_1 \equiv v_1^\Lambda$ or $v_2 \equiv v_2^\Lambda$ in Q_Λ.

By definition of Λ and the continuity of solutions, we get $v_1 \geq v_1^\Lambda$ and $v_2 \geq v_2^\Lambda$ in Q_Λ. Then

$$\begin{cases} -\Delta(v_1 - v_1^\Lambda) = h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) v_1^{\frac{n-2}{n+2}} - h_{11}\left(\frac{v_1^\Lambda}{\xi_1^\Lambda}, \frac{v_2^\Lambda}{\xi_1^\Lambda}\right) (v_1^\Lambda)^{\frac{n-2}{n+2}} \geq 0 & \text{in } Q_\Lambda, \\ -\Delta(v_2 - v_2^\Lambda) = h_{21}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) v_2^{\frac{n-2}{n+2}} - h_{11}\left(\frac{v_1^\Lambda}{\xi_1^\Lambda}, \frac{v_2^\Lambda}{\xi_1^\Lambda}\right) (v_2^\Lambda)^{\frac{n-2}{n+2}} \geq 0 & \text{in } Q_\Lambda, \\ v_1 - v_1^\Lambda \geq 0, v_2 - v_2^\Lambda \geq 0 & \text{in } Q_\Lambda. \end{cases}$$

From Maximum Principle, we obtain either $v_i \equiv v_i^\Lambda$ in Q_Λ for some $i = 1, 2$ or $v_i > v_i^\Lambda$ in Q_Λ for all $i = 1, 2$. Suppose $v_1 > v_1^\Lambda$ and $v_2 > v_2^\Lambda$ in Q_Λ. We can choose a compact $K \subset Q_\Lambda$ and a number $\delta > 0$ such that $\forall t \in (\Lambda - \delta, \Lambda)$ we have $K \subset Q_t$ and

$$C\psi_i(t) = \left(\int_{Q_t \setminus K} (v_i^t)^{\frac{2n}{n-2}} dy\right) < \frac{1}{4} \ \text{ for } i = 1, 2. \tag{2.25}$$

On the other hand, there exists $0 < \delta_1 < \delta$, such that

$$v_1 > v_1^t \text{ and } v_2 > v_2^t \text{ in } K, \ \forall t \in (\Lambda - \delta_1, \Lambda). \tag{2.26}$$

Using (2.25) and following as in Step 1, since the integrals are over $Q_t \setminus K$, we see that $(v_i^t - v_i)^+ \equiv 0$ in $Q_t \setminus K$ for $i = 1, 2$. By (2.26) we get $v_1 > v_1^t$ and $v_2 > v_2^t$ in Q_t for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of Λ.

Step 3. Symmetry.

Suppose $\Lambda > 0$. From Step 2 we can assume $v_1 = v_1^\Lambda$. Then

$$h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) = -\frac{\Delta v_1}{v_1^{\frac{n+2}{n-2}}} = -\frac{\Delta v_1^\Lambda}{(v_1^\Lambda)^{\frac{n+2}{n-2}}} = h_{11}\left(\frac{v_1^\Lambda}{\xi_1^\Lambda}, \frac{v_2^\Lambda}{\xi_1^\Lambda}\right) \leq h_{11}\left(\frac{v_1}{\xi_1^\Lambda}, \frac{v_2}{\xi_1^\Lambda}\right) < h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right).$$

This is a contradiction. Thus $\Lambda = 0$. By continuity of v_1 and v_2, we have $v_1^0(y) \leq v_1(y)$ and $v_2^0(y) \leq v_2(y)$ for all $y \in Q_0$. We can also perform the moving plane procedure from the left and find a corresponding Λ'. An analogue of Step 1 and Step 2 we can show that $\Lambda' = 0$. Then we get $v_1^0(y) \geq v_1(y)$ and $v_2^0(y) \geq v_1(y)$ for $y \in Q_0$. This fact and the above inequality imply that v_1 and v_2 are symmetric with respect to U_0. Therefore, if $\Lambda = \Lambda' = 0$ for all directions that are vertical to the y_n direction, then v_1 and v_2 are symmetric with respect to the axis y_n. □

Lemma 2.4. *Let (u_1, u_2) be a solution of (2.16). Then for each $r \in [0, \pi/2]$, we have that $u_1 = u_1(r)$ and $u_2 = u_2(r)$ in A_r, where A_r is defined by (2.12).*

Proof. The arguments for the proof are the same as the Lemma 2.2. □

Proof of Theorem 2.2.

Let (u_1, u_2) be the solution of problem (2.16). We take a $p \in \partial\mathbb{S}^n_+$ and let $\mathcal{F}^{-1} : \overline{\mathbb{S}^n_+}\backslash\{p\} \to \overline{\mathbb{R}^n_+}$ be the stereographic projection. We consider the problema (2.19), where $v_1 = \xi_1(u_1 \circ \mathcal{F})$ and $v_2 = \xi_1(u_2 \circ \mathcal{F})$ in $\mathbb{R}^n_+$. Denote $y = (y', y_n) \in \mathbb{R}^n_+$, and define

$$v_1^*(y) = \begin{cases} v_1(y', y_n) & \text{if } y_n \geq 0, \\ v_1(y', -y_n) & \text{if } y_n < 0, \end{cases} \quad \text{and} \quad v_2^*(y) = \begin{cases} v_2(y', y_n) & \text{if } y_n \geq 0, \\ v_2(y', -y_n) & \text{if } y_n < 0. \end{cases}$$

Then, $(v_1^*, v_2^*) \in C^1(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ are weak solutions of the problem

$$\begin{cases} -\Delta v_1^* = h_{11}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) & \text{in } \mathbb{R}^n, \\ -\Delta v_2^* = h_{21}\left(\frac{v_1}{\xi_1}, \frac{v_2}{\xi_1}\right) & \text{in } \mathbb{R}^n, \\ v_1^*, v_2^* > 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Applying Lemma 2.3 for (v_1^*, v_2^*) in the whole space $\mathbb{R}^n$, we obtain that v_1^* and v_2^* are radially symmetrical. This implies

$$v_1^*(y) = v_1(y) = C_1 \text{ and } v_2^*(y) = v_2(y) = C_2, \text{ for all } y \in \mathbb{R}^n_+ \text{ such that } |y| = 1, \tag{2.27}$$

where C_1 and C_2 are constant.

On the other hand $\mathcal{F}(\{y \in \mathbb{R}^n_+; |y| = 1\})$ intersects perpendicularly to A_r for all $r \in (0, \pi/2)$. Therefore, from (2.27) we have that u_1 and u_2 are constant in $\mathcal{F}(\{y \in \mathbb{R}^n_+; |y| = 1\})$ and from Lemma 2.4, we have that u_1 and u_2 are constant in $\mathbb{S}^n_+$. □

Chapter

3

Problems on the sphere involving the Paneitz operator

3.1 Constant solutions for an equation

In this section we study the following problem

$$\Delta^2_{g_{\mathbb{S}^n}} u - c_n \Delta_{g_{\mathbb{S}^n}} u = f(u), u > 0 \ \text{ in } \mathbb{S}^n, \tag{3.1}$$

where $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$, $n \geq 5$, $c_n = (n^2 - 2n - 4)$ and $f : (0, \infty) \to \mathbb{R}$ is a continuous function.

We recall that

$$\mathcal{A}_4 = \Delta^2_{g_{\mathbb{S}^n}} - c_n \Delta_{g_{\mathbb{S}^n}} + d_{n,2} \text{ in } C^\infty(\mathbb{S}^n),$$

where $d_{n,2} = (n-4)(n^2-4)/16$.

Following the results of Chapter 1 we have our main result for this section.

Theorem 3.1. *Assume that*

$$h_2(t) := t^{-\frac{n+4}{n-4}}(f(t) + d_{n,2}t) \text{ is decreasing non-negative in } (0, +\infty) \text{ and}$$
$$h_2(t)t^{\frac{n+4}{n-4}} \text{ is nondecreasing in } (0, +\infty).$$

Then the problem (3.1) admits constant solutions.

Example 3.1. *A typical example is the case*

$$f(t) = t^p - \lambda t, \ p > 1, \ \lambda > 0.$$

So that (3.1) becomes

$$\Delta^2 a_{g_{\mathbb{S}^n}} u - c_n \Delta_{g_{\mathbb{S}^n}} u = u^p - \lambda u, \;\; u > 0 \text{ in } \mathbb{S}^n. \tag{3.2}$$

Corollary 3.1. *Assume that $p \le (n+4)/(n-4)$ and $\lambda \le d_{n,2}$, and at least one of these inequalities is strict. Then the only solution of (3.2) is the constant $u \equiv \lambda^{1/(p-1)}$.*

In order to prove Theorem 3.1 we will use the moving planes method and thechniques based on inequalities of integrals. First we will show the symmetry of solutions for a problem on $\mathbb{R}^n$ equivalent to (3.1).

Let ζ be an arbitrary point on $\mathbb{S}^n$, which we will rename the south pole S. Let $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ be the stereographic projection.

Let u be a solution of (3.1). We define

$$v(y) = \xi_2(y) u(\mathcal{F}(y)), \;\; y \in \mathbb{R}^n,$$

where ξ_2 is defined by (1.4). Then we have

$$\begin{aligned} |y|^{-2} v &\in L^2(\mathbb{R}^n \backslash B_r) \cap L^\infty(\mathbb{R}^n \backslash B_r), \\ |y|^{-2} v \Delta v &\in L^1(\mathbb{R}^n \backslash B_r) \cap L^\infty(\mathbb{R}^n \backslash B_r), \end{aligned} \tag{3.3}$$

where B_r is any ball with center zero and radius $r > 0$. By Lemma 1.1, v is solution of

$$\Delta^2 v = h_2 \left(\frac{v}{\xi_2} \right) v^{\frac{n+4}{n-4}}, \;\; v > 0 \text{ in } \mathbb{R}^n, \tag{3.4}$$

where

$$h_2(t) = t^{-\frac{n+4}{n-4}} \left(f(t) + d_{n,2} t \right), \;\; t > 0$$

and $d_{n,2} = n(n-4)(n^2-4)/16$.

Denote $w_1 = v$ and $w_2 = -\Delta w_1$, we have

$$\begin{cases} -\Delta w_1 = w_2 & \text{in } \mathbb{R}^n, \\ -\Delta w_2 = h_2 \left(\frac{w_1}{\xi_2} \right) w_1^{\frac{n+4}{n-4}} & \text{in } \mathbb{R}^n. \end{cases} \tag{3.5}$$

We use the moving plane method to prove radial symmetry of solution of the problem (3.4). The next lemma shows the non-negativity of w_2, which is necessary for the proof of Theorem 3.1.

Lemma 3.1. *$w_2 = -\Delta w_1$ is non-negative in $\mathbb{R}^n$.*

Proof. Suppose that there exists $y_0 \in \mathbb{R}^n$ such that $w_2(y_0) < 0$. We can assume that $y_0 = 0$. We introduce the spherical average of a function

$$\overline{w}(r) = \frac{1}{|S_r|}\int_{S_r} w \, d\sigma,$$

where $|S_r|$ is the measure of the sphere of radius r. By definition of w_1, we have $\overline{w}_1 \in L^\infty(\mathbb{R}^n)$ and

$$\begin{cases} -\Delta\overline{w}_1 &= \overline{w}_2, \\ -\Delta\overline{w}_2 &= \overline{h_2(\frac{w_1}{\xi_2})w_1^{\frac{n+4}{n-4}}}. \end{cases}$$

Since $\overline{w}_2(0) = w_2(0) < 0$ and $-\Delta\overline{w}_2 = \overline{h_2(\frac{w_1}{\xi_2})w_1^{\frac{n+4}{n-4}}} \geq 0$, from Maximum Principle, we obtain for all $r > 0$,

$$\begin{cases} \overline{w}_2(r) &\leq \overline{w}_2(0) &< 0, \\ -\Delta\overline{w}_1(r) &= \overline{w}_2(r) &\leq \overline{w}_2(0), \end{cases}$$

or

$$-\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}\overline{w}_1) \leq \overline{w}_2(0). \tag{3.6}$$

Integrating (3.6), we have

$$\overline{w}_1(r) \geq \overline{w}_1(0) - \frac{\overline{w}_2(0)}{4}r^2, \quad \text{for all } r > 0.$$

Since $w_2(0) < 0$, we have $\overline{w}_1(r) \to +\infty$ as $r \to +\infty$. This leads to a contradiction.

□

Proof. Theorem 3.1.

Let u be the solution of problem 3.1. We take an arbitrary point $\zeta \in \mathbb{S}^n$ as the south pole S, and let $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ be the stereographic projection. We define $v = \xi_2(u \circ \mathcal{F})$ in $\mathbb{R}^n$. For given $t \in \mathbb{R}$ we define as before the elements Q_t, U_t, and y_t. Thus, the reflected function is $v^t(y) := v(y_t)$. Also, we denote $w_1 := v$ and $w_2 := -\Delta w_1$. Following the same argument, the proof is carried out in three steps. Let

$$\Lambda := \inf\{t > 0;\ w_1 \geq w_1^\mu, w_2 \geq w_2^\mu \text{ in } Q_\mu, \forall \mu \geq t\}.$$

Step 1. $\Lambda < +\infty$.

Again, we follows an argument of contradiction. For $\varepsilon > 0$ and $t > 0$, we denote

$W^t_{i,\varepsilon} = w^t_i - w_i - \varepsilon$ and $W^t_i = w^t_i - w_i$ for $i = 1, 2$. Assume that $W^t_i > 0$. Then

$$
\begin{aligned}
&\int_{Q_t} |\nabla\{(W^t_{1,\varepsilon})^+ |y_t|^{-1}\}|^2 dy \\
&= \int_{Q_t} |\nabla(W^t_{1,\varepsilon})^+|^2 |y_t|^{-2} dy + \int_{Q_t} 2|y_t|^{-1}\nabla(W^t_{1,\varepsilon})^+ . \nabla\{|y_t|^{-1}\} dy \\
&+ \int_{Q_t} [(W^t_{1,\varepsilon})^+]^2 (\nabla\{|y_t|^{-1}\})^2 dy \\
&= \int_{Q_t} \nabla(W^t_{1,\varepsilon})^+ . \nabla\{(W^t_{1,\varepsilon})^+ |y_t|^{-2}\} dy + \int_{Q_t} [(W^t_{1,\varepsilon})^+]^2 (\nabla\{|y_t|^{-1}\})^2 dy \\
&= \int_{Q_t} \nabla(W^t_{1,\varepsilon}) . \nabla\{(W^t_{1,\varepsilon})^+ |y_t|^{-2}\} dy + \int_{Q_t} [(W^t_{1,\varepsilon})^+]^2 (\nabla\{|y_t|^{-1}\})^2 dy \\
&= \int_{Q_t} \nabla W^t_1 . \nabla\{(W^t_{1,\varepsilon})^+ |y_t|^{-2}\} dy + \int_{Q_t} [(W^t_{1,\varepsilon})^+]^2 (\nabla\{|y_t|^{-1}\})^2 dy. \qquad (3.7)
\end{aligned}
$$

Since $w_1(y) \to 0$ as $|y| \to \infty$, then for $\varepsilon > 0$, we can take $(W^t_{1,\varepsilon})^+ |y_t|^{-2}$ as test function with compact support in Q_t for the problem (3.5). Then, from (3.7) we obtain

$$
\begin{aligned}
&\int_{Q_t} |\nabla\{(W^t_{1,\varepsilon})^+ |y_t|^{-1}\}|^2 dy \\
&= \int_{Q_t} (w^t_2 - w_2)(W^t_{1,\varepsilon})^+ |y_t|^{-2} dy + \int_{Q_t} [(W^t_{1,\varepsilon})^+]^2 (\nabla\{|y_t|^{-1}\})^2 dy \\
&\le \int_{Q_t} (W^t_2)^+ (W^t_{1,\varepsilon})^+ |y_t|^{-2} dy \int_{Q_t} [(W^t_{1,\varepsilon})^+]^2 |y_t|^{-4} dy \\
&= I_\varepsilon + II_\varepsilon. \qquad (3.8)
\end{aligned}
$$

From Lemma 3.1 and by (3.3) we can see that if $(W^t_{1,\varepsilon})^+(y) > 0$ and $(W^t_2)^+(y) \ge 0$ for some $y \in Q_t$, then $w^t_1(y) > w_1(y)$, $w^t_2(y) \ge w_2(y)$ and

$$
\begin{aligned}
(W^t_2)^+ (W^t_{1,\varepsilon})^+ |y_t|^{-2} &\le 4 w^t_1 w^t_2 |y_t|^{-2} \in L^1(\mathbb{R}^n) \\
[(W^t_{1,\varepsilon})^+]^2 |y_t|^{-4} &\le 4 (w^t_1)^2 |y_t|^{-4} \in L^1(\mathbb{R}^n).
\end{aligned}
$$

Thus, by Fatou's lemma, Sobolev's inequality and dominate convergence theorem we get

$$
\begin{aligned}
\left(\int_{Q_t} [(W^t_1)^+ |y_t|^{-1}]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} &\le \liminf_{\varepsilon \to 0} \left(\int_{Q_t} [(W^t_{1,\varepsilon})^+ |y_t|^{-1}]^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} \\
&\le C \liminf_{\varepsilon \to 0} \int_{Q_t} |\nabla\{(W^t_1)^+ |y_t|^{-1}\}|^2 dy \\
&\le \liminf_{\varepsilon \to 0} (I_\varepsilon + II_\varepsilon) < +\infty. \qquad (3.9)
\end{aligned}
$$

By Hardy's inequality, we know that

$$\left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \le \int_{\mathbb{R}^n} |\nabla u|^2 dx,\ \ u \in H^1(\mathbb{R}^n).$$

Which together with Holder's inequality we obtain

$$\begin{aligned} I_\varepsilon &\le \left(\int_{Q_t} [(W_2^t)^+]^2 dy\right)^{\frac{1}{2}} \left(\int_{Q_t} \left[\frac{(W_{1,\varepsilon}^t)^+}{|y_t|^2}\right]^2 dy\right)^{\frac{1}{2}} \\ &\le \left(\frac{2}{n-2}\right) \left(\int_{Q_t} [(W_2^t)^+]^2 dy\right)^{\frac{1}{2}} \left(\int_{Q_t} |\nabla\{(W_{1,\varepsilon}^t)^+|y_t|^{-1}\}|^2 dy\right)^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

Moreover,

$$II_\varepsilon = \int_{Q_t} \left[\frac{(W_{1,\varepsilon}^t)^+}{|y_t|^2}\right]^2 dy \le \left(\frac{2}{n-2}\right)^2 \int_{Q_t} |\nabla\{(W_{1,\varepsilon}^t)^+|y_t|^{-1}\}|^2 dy. \tag{3.11}$$

By (3.8), (3.10) and (3.11), we have

$$\left(1-\left(\frac{2}{n-2}\right)^2\right)^2 \int_{Q_t} |\nabla\{(W_{1,\varepsilon}^t)^+|y_t|^{-1}\}|^2 dy \le \int_{Q_t} [(W_2^t)^+]^2 dy \tag{3.12}$$

On the other hand, for $t > 0$, we get $\xi_2 > \xi_2^t$ in Q_t. By conditions on h_2 we have: if $w_1^t > w_1$, then

$$\begin{aligned} -\Delta W_2^t &= h_2\left(\frac{w_1^t}{\xi_2^t}\right)(w_1^t)^{\frac{n+4}{n-4}} - h_2\left(\frac{w_1}{\xi_2}\right) w_1^{\frac{n+4}{n-4}} \\ &\le h_2\left(\frac{w_1^t}{\xi_2^t}\right)\left((w_1^t)^{\frac{n+4}{n-4}} - (w_1)^{\frac{n+4}{n-4}}\right) \\ &\le \frac{n+4}{n-4}(w_1^t)^{\frac{8}{n-4}} h_2\left(\frac{w_1^t}{\xi_2^t}\right)(w_1^t - w_1); \end{aligned}$$

if $w_1^t < w_1$, then

$$-\Delta W_2^t \le h_2\left(\frac{w_1^t}{\xi_2^t}\right)(w_1^t)^{\frac{n+4}{n-4}} - h_2\left(\frac{w_1}{\xi_2^t}\right) w_1^{\frac{n+4}{n-4}} \le 0.$$

Thus,

$$-\Delta W_2^t \le C(w_1^t)^{\frac{8}{n-4}}(w_1^t - w_1)^+, \tag{3.13}$$

where the last inequality is a consequence of $h_2(\frac{w_1}{\xi_2}) \in L^\infty(\mathbb{R}^n)$, and C is a positive constant. Since $w_2(y) \to 0$ as $|y| \to \infty$, then for $\varepsilon > 0$, we can take $(W_{2,\varepsilon}^t)^+|y_t|^2$ as a test function with compact support in Q_t for (3.5), and one gets

$$\begin{aligned}\int_{Q_t}|\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}|^2dy &= \int_{Q_t}\nabla W^t_2\nabla\{(W^t_{2,\varepsilon})^+|y_t|^2\}dy+\int_{Q_t}[(W^t_{2,\varepsilon})^+]^2dy\\ &\le C\int_{Q_t}(w^t_1)^{\frac{8}{n-4}}(W^t_1)^+(W^t_{2,\varepsilon})^+|y_t|^2dy+\int_{Q_t}[(W^t_{2,\varepsilon})^+]^2dy\\ &= CIII_\varepsilon+IV_\varepsilon.\end{aligned}\tag{3.14}$$

From Hölder's, Sobolev's and Hardy's inequalities, and (3.9) we have

$$\begin{aligned}III_\varepsilon &\le \left(\int_{Q_t}[(w^t_1)^{\frac{8}{n-4}}|y_t|^2]^{\frac{n}{2}}dy\right)^{\frac{2}{n}}\left(\int_{Q_t}\left[\frac{(W^t_1)^+}{|y_t|}\right]^{\frac{2n}{n-2}}dy\int_{Q_t}\left[\frac{(W^t_{2,\varepsilon})^+}{|y_t|^{-1}}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\\ &\le \varphi(t)\left(\int_{Q_t}\left[\frac{(W^t_1)^+}{|y_t|}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}^2dy\right)^{\frac{1}{2}},\end{aligned}\tag{3.15}$$

where

$$\varphi(t)=\left(\int_{Q_t}(w^t_1)^{\frac{4n}{n-4}}|y_t|^n dy\right)^{\frac{2}{n}}\quad\text{and}\quad\lim_{t\to0}\varphi(t)=0,\tag{3.16}$$

because $(w^t_1)^{\frac{4n}{n-4}}|y_t|^n\in L^1(\mathbb{R}^n)$; and from Hardy's inequality,

$$IV_\varepsilon\le\left(\frac{2}{n-2}\right)^2\int_{Q_t}|\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}|^2dy.\tag{3.17}$$

Then, by (3.14), (3.15) and (3.17), we get

$$\begin{aligned}\int_{Q_t}|\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}|^2dy\le\varphi(t)\left(\int_{Q_t}[(W^t_1)^+|y_t|^{-1}]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}|\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}|^2dy\right)^{\frac{1}{2}}\\ +\left(\frac{2}{n-2}\right)^2\int_{Q_t}|\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}|^2dy.\end{aligned}$$

Hence, for all $\varepsilon>0$,

$$\left(1-\left(\frac{2}{n-2}\right)^2\right)^2\left(\int_{Q_t}|\nabla\{(W^t_{2,\varepsilon})^+|y_t|\}|^2dy\right)\le\varphi(t)^2\left(\int_{Q_t}[(W^t_1)^+|y_t|^{-1}]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}.\tag{3.18}$$

From Fatou's lemma, Hardy's inequality and (3.18), we obtain

$$\begin{aligned}\int_{Q_t}[(W_2^t)^+]^2dy &\le \liminf_{\varepsilon\to 0}\int_{Q_t}[(W_{2,\varepsilon}^t)^+]^2dy\\ &\le \left(\frac{2}{n-2}\right)^2\liminf_{\varepsilon\to 0}\int_{Q_t}|\nabla\{(W_{2,\varepsilon}^t)^+|y_t|\}|^2dy\\ &\le C\varphi(t)^2\left(\int_{Q_t}\left[(W_1^t)^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}},\end{aligned}\tag{3.19}$$

where C is a positive constant depending of n. From Sobolev's inequality, (3.12), (3.19) and letting $\varepsilon\to 0$, we have

$$\left(\int_{Q_t}\left[(W_1^t)^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\le C\varphi(t)^2\left(\int_{Q_t}\left[(W_1^t)^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}.$$

Thus, choosing t_1 sufficiently large such that $\varphi(t)^2<\frac{1}{C}$ for all $t>t_1$, we have

$$\int_{Q_t}\left[(W_1^t)^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\equiv 0,\quad \text{in } Q_t\quad \text{for all } t>t_1.$$

Then, $(W_1^t)^+\equiv 0$ in Q_t for all $t>t_1$, and from (3.19), one gets $(W_2^t)^+\equiv 0$ in Q_t for all $t>t_1$.

Which implies a contradiction with our assumption. Therefore, $\Lambda<+\infty$.

Step 2. If $\Lambda>0$, then $w_1\equiv w_1^\Lambda$ or $w_2\equiv w_2^\Lambda$ in Q_Λ.

By definition of Λ and the continuity of the solutions, we get $w_1\ge w_1^\Lambda$ and $w_2\ge w_2^\Lambda$ in Q_Λ. From conditions of h_2 and (3.5) we have

$$\begin{cases}-\Delta(w_1-w_1^\Lambda)=w_2-w_2^\Lambda\ge 0 & \text{in } Q_\Lambda,\\ -\Delta(w_2-w_2^\Lambda)=h_2\left(\frac{w_1}{\xi_2}\right)w_1^{\frac{n+4}{n-2}}-h_2\left(\frac{w_1^\Lambda}{\xi_2^\Lambda}\right)(w_1^\Lambda)^{\frac{n+4}{n-2}}\ge 0 & \text{in } Q_\Lambda,\\ w_1-w_1^\Lambda\ge 0,\ w_2-w_2\ge 0 & \text{in } Q_\Lambda.\end{cases}$$

Then, from Maximum Principle we have either $w_i\equiv w_i^\Lambda$ in Q_Λ for some $i=1,2$ or $w_i>w_i^\Lambda$ in Q_Λ for all $i=1,2$. Suppose $w_1>w_1^\Lambda$ and $w_2>w_2^\Lambda$ in Q_Λ. We can choose a compact $K\subset Q_\Lambda$ and a number $\delta>0$ such that $\forall t\in(\Lambda-\delta,\Lambda)$ we have $K\subset Q_t$ and

$$C\varphi(t)^2=C\left(\int_{Q_t\setminus K}(w_1^t)^{\frac{8}{n-4}}|y_t|^2dy\right)^{\frac{4}{n}}<\frac{1}{2}.\tag{3.20}$$

On the other hand, there exists $0 < \delta_1 < \delta$, such that

$$w_1 > w_1^t,\ w_2 > w_2^t \text{ in } K \text{ for all } t \in (\Lambda - \delta_1, \Lambda). \tag{3.21}$$

Using (3.20) and following as in Step 1, considering the integrals are over $Q_t \backslash K$, we see that $(w_1^t - w_1)^+ \equiv 0$ in $Q_t \backslash K$. By (3.21) we get $w_1 > w_1^t$ in Q_t for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of Λ.

Step 3. Symmetry

Suppose $\Lambda > 0$. From Step 2 we can assume $w_2 \equiv w_2^\Lambda$ in Q_Λ. Then

$$\begin{aligned} h_2\left(\frac{w_1}{\xi_2}\right) &= -\frac{\Delta w_2}{w_1^{\frac{n+4}{n-4}}} = -\frac{\Delta w_2^\Lambda}{w_1^{\frac{n+4}{n-4}}} = h_2\left(\frac{w_1^\Lambda}{\xi_2^\Lambda}\right)\left(\frac{w_1^\Lambda}{w_1}\right)^{\frac{n+4}{n-4}} \\ &\leq h_2\left(\frac{w_1}{\xi_2^\Lambda}\right) < h_2\left(\frac{w_1}{\xi_2}\right). \end{aligned}$$

Which clearly is a contradiction.

Therefore, $\Lambda = 0$ for all directions. This implies that w_1 is radially symmetric in $\mathbb{R}^n$. By definition of w_1, we obtain that u is constant on every $(n-1)$-sphere whose elements $q \in \mathbb{S}^n$ satisfy $|q - S| =$ constant. Since $\zeta \in \mathbb{S}^n$ is arbitrary on $\mathbb{S}^n$, u is constant. $\square$

3.2 Constant solutions for systems

In this section we generalize the Theorem 3.1 for systems.

We consider the following problem

$$\begin{cases} \Delta^2_{g_{\mathbb{S}^n}} u_1 - c_n \Delta_{g_{\mathbb{S}^n}} u_1 = f_1(u_1, u_2) & \text{in } \mathbb{S}^n, \\ \Delta^2_{g_{\mathbb{S}^n}} u_2 - c_n \Delta_{g_{\mathbb{S}^n}} u_2 = f_2(u_1, u_2) & \text{in } \mathbb{S}^n \\ u_1,\ u_2 > 0 & \text{in } \mathbb{S}^n, \end{cases} \tag{3.22}$$

where $f_1,\ f_2 : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ are continuous functions.

Based on the Theorem 2.3 we have our main result for system.

Theorem 3.2. *Let $h_{i1} : (0, \infty) \times (0, \infty) \to \mathbb{R}$, $i = 1, 2$, be two functions defined by*

$$h_{i1}(t_1, t_2) := t_i^{-\frac{n+2}{n-2}} \left(f_i(t_1, t_2) + d_{n,2} t_i\right),\ t_1 > 0,\ t_2 > 0.$$

Assume that

$$h_{i1}(t_1,t_2) \text{ is nondecreasing in } t_j > 0, \text{ with } i \neq j,$$
$$h_{i1}(t_1,t_2)t_i^{\frac{n+2}{n-2}} \text{ is nondecreasing in } t_i > 0,$$
$$h_{i1}(a_1t,a_2t) \text{ is decreasing in } t > 0 \text{ for any } a_i > 0,$$

for $i,j = 1,2$. *Then the problem (3.22) admits only constant solutions.*

The result above means that the problem (3.22) has no nonconstan solutions. To prove the Theorem 3.2, we will use the same arguments that were used in the proof of the Theorem 3.1. Let (u_1,u_2) be a solution of (3.22). We define the functions

$$v_1(y) = \xi_2(y)u_1(\mathcal{F}(y)), \ v_2(y) = \xi_2(y)u_2(\mathcal{F}(y)), \ y \in \mathbb{R}^n,$$

where ξ_2 is defined in (1.4). Then we have that

$$\begin{aligned} &|y|^{-2}v_1, \ |y|^{-2}v_2 \in L^2(\mathbb{R}^n\backslash B_r) \cap L^\infty(\mathbb{R}^n\backslash B_r), \\ &|y|^{-2}v_1\Delta v_1, \ |y|^{-2}v_2\Delta v_2 \in L^1(\mathbb{R}^n\backslash B_r) \cap L^\infty(\mathbb{R}^n\backslash B_r), \end{aligned} \tag{3.23}$$

where B_r is any ball with center zero and radius $r > 0$. From Lemma 1.2 we obtain

$$\begin{cases} \Delta^2 v_1 = h_{12}\left(\dfrac{v_1}{\xi_2}, \dfrac{v_2}{\xi_2}\right) v_1^{\frac{n+4}{n-4}}, \ v_1 > 0 \text{ in } \mathbb{R}^n, \\ \Delta^2 v_2 = h_{22}\left(\dfrac{v_1}{\xi_2}, \dfrac{v_2}{\xi_2}\right) v_2^{\frac{n+4}{n-4}}, \ v_2 > 0 \text{ in } \mathbb{R}^n, \end{cases} \tag{3.24}$$

where

$$h_{i2}(t_1,t_2) = t_i^{-\frac{n+4}{n-4}}(f_i(t_1,t_2) + d_n t_i), \ t_1 > 0, \ t_2 > 0 \text{ for } \ i = 1,2,$$

and $d_n = n(n-4)(n^2-4)/16$.

Denote $w_{11} = v_1$, $w_{12} = -\Delta w_{11}$, $w_{21} = v_2$ and $w_{22} = -\Delta w_{21}$. Then we have

$$\begin{cases} -\Delta w_{11} &= w_{12}, & \text{in } \mathbb{R}^n \\ -\Delta w_{12} &= h_{12}\left(\frac{w_{11}}{\xi_2}, \frac{w_{12}}{\xi_2}\right) w_{11}^{\frac{n+4}{n-4}} & \text{in } \mathbb{R}^n \\ -\Delta w_{21} &= w_{22}, & \text{in } \mathbb{R}^n \\ -\Delta w_{22} &= h_{22}\left(\frac{w_{11}}{\xi_2}, \frac{w_{12}}{\xi_2}\right) w_{21}^{\frac{n+4}{n-4}} & \text{in } \mathbb{R}^n. \end{cases} \tag{3.25}$$

The moving plane method is the main ingredient to prove radial symmetry of solution of the problem (3.24). We start with

Lemma 3.2. *For* $i = 1,2$, *we have* $-\Delta w_{i1}$ *are non-negative in* $\mathbb{R}^n$.

Proof. The arguments for the proof are the same as the Lemma 3.1. $\square$

Proof of Theorem 3.2. Since the arguments to prove Theorem are similar to those that were used in the prove of Theorem 2.3 and 3.1, then we will simplify some calculations. Let (u_1, u_2) be the solution of problem (3.22). We take a $\zeta \in \mathbb{S}^n$ as the south pole S, and let $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ be the stereographic projection. We define

$$v_1 = \xi_2(u_1 \circ \mathcal{F}), \quad \text{and} \quad v_2 = \xi_2(u_2 \circ \mathcal{F}) \ \text{ in } \ \mathbb{R}^n.$$

Given $t \in \mathbb{R}$, let Q_t, U_t as before, and y_t the reflection through the hyperplane U_t. Define the reflected function by $v_i^t(y) := v_i(y_t)$, $i = 1, 2$, and $w_{i1} := v_i$ and $w_{i2} := -\Delta w_{i1}$, $i = 1, 2$. The prove is carried out in three steps. Define

$$\Lambda := \inf\{t > 0; \ w_{ij} \geq w_{ij}^{\mu}, \text{ in } Q_\mu, \forall \mu \geq t, \ i, j = 1, 2.\}.$$

Step 1. $\Lambda < +\infty$.
Again, we follows an argument of contradiction. For $\varepsilon > 0$ and $t > 0$, we denote $W_{ij,\varepsilon}^t = w_{ij}^t - w_{ij} - \varepsilon$ and $W_{ij}^t = w_{ij}^t - w_{ij}$ for $i, j = 1, 2$. We can take $(W_{i1,\varepsilon}^t)^+ |y_t|^{-2}$ as test function with compact support in Q_t for the problem (3.25). Then

$$\begin{aligned} \int_{Q_t} |\nabla\{(W_{i1,\varepsilon}^t)^+ |y_t|^{-1}\}|^2 dy &\leq \int_{Q_t} (W_{i2}^t)^+ (W_{i1,\varepsilon}^t)^+ |y_t|^{-2} dy + \int_{Q_t} [(W_{i1,\varepsilon}^t)^+]^2 |y_t|^{-4} dy \\ &= I_{i,\varepsilon} + II_{i,\varepsilon} \end{aligned} \tag{3.26}$$

From the Hölder and Hardy inequalities we obtain

$$I_{i,\varepsilon} \leq \left(\frac{2}{n-2}\right) \left(\int_{Q_t} [(W_{i2}^t)^+]^2 dy\right)^{\frac{1}{2}} \left(\int_{Q_t} |\nabla\{(W_{i1,\varepsilon}^t)^+ |y_t|^{-1}\}|^2 dy\right)^{\frac{1}{2}}$$

and

$$II_{i,\varepsilon} \leq \left(\frac{2}{n-2}\right)^2 \int_{Q_t} |\nabla\{(W_{i1,\varepsilon}^t)^+ |y_t|^{-1}\}|^2 dy.$$

Thus,

$$\left(1 - \left(\frac{2}{n-2}\right)^2\right)^2 \int_{Q_t} |\nabla\{(W_{i1,\varepsilon}^t)^+ |y_t|^{-1}\}|^2 dy \leq \int_{Q_t} [(W_{i2}^t)^+]^2 dx \ \text{ for } \ i = 1, 2. \tag{3.27}$$

On the other hand, using the same arguments as in (2.20) and (2.21), we have

$$-\Delta W_{i2}^t \leq C(w_{i1}^t)^{\frac{8}{n-4}} [(w_{11}^t - w_{11})^+ + (w_{21}^t - w_{21})^+], \tag{3.28}$$

where C is a positive constant. We take $(W^t_{i2,\varepsilon})^+|y_t|^2$ as test function in (3.28). Then

$$\begin{aligned}
\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy &= \int_{Q_t}\nabla W^t_{i2}\nabla\{(W^t_{i2,\varepsilon})^+|y_t|^2\}dy+\int_{Q_t}[(W^t_{i2,\varepsilon})^+]^2dy\\
&\le C\int_{Q_t}w_{i1}^{\frac{8}{n-4}}[(W^t_{11})^++(W^t_{21})^+](W^t_{i2,\varepsilon})^+|y_t|^2dy\\
&\quad+\int_{Q_t}[(W^t_{i2,\varepsilon})^+]^2dy\\
&= C\int_{Q_t}w_{i1}^{\frac{8}{n-4}}(W^t_{11})^+(W^t_{i2,\varepsilon})^+|y_t|^2dy\\
&\quad+C\int_{Q_t}w_{i1}^{\frac{8}{n-4}}(W^t_{21})^+(W^t_{i2,\varepsilon})^+|y_t|^2dy+\int_{Q_t}[(W^t_{i2,\varepsilon})^+]^2dy\\
&= III_{i,\varepsilon}+IV_{i,\varepsilon}+V_{i,\varepsilon}.
\end{aligned}\tag{3.29}$$

From Hölder, Sobolev and Hardy inequalities, we have

$$III_{i,\varepsilon}\le\varphi_i(t)\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}^2dy\right)^{\frac{1}{2}},\tag{3.30}$$

where

$$\varphi_i(t)=C\left(\int_{Q_t}(w^t_{i1})^{\frac{4n}{n-4}}|y_t|^ndy\right)^{\frac{2}{n}}\quad\text{and}\quad\lim_{t\to0}\varphi_i(t)=0,\tag{3.31}$$

$$IV_{i,\varepsilon}\le\varphi_i(t)\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}^2dy\right)^{\frac{1}{2}},\tag{3.32}$$

and

$$V_{i,\varepsilon}\le\left(\frac{2}{n-2}\right)^2\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy.\tag{3.33}$$

From (3.29)-(3.32) and (3.33), gets

$$\begin{aligned}
&\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy\\
&\le\varphi_i(t)\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy\right)^{\frac{1}{2}}\\
&+\varphi_i(t)\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\left(\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy\right)^{\frac{1}{2}}\\
&+\left(\frac{2}{n-2}\right)^2\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy.
\end{aligned}$$

Hence, for all $\varepsilon > 0$, $i = 1, 2$, we obtain

$$\begin{aligned}&\left(1-\left(\frac{2}{n-2}\right)^2\right)\left(\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy\right)^{\frac{1}{2}}\\&\le \varphi_i(t)\left[\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}+\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{2n}}\right].\end{aligned} \tag{3.34}$$

From Fatou's lemma and Hardy's inequality, we obtain

$$\begin{aligned}&\int_{Q_t}[(W^t_{i2})^+]^2dy \le \left(\frac{2}{n-2}\right)^2\liminf_{\varepsilon\to 0}\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy\\&\le C\varphi_i(t)^2\left[\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}+\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\right]\end{aligned} \tag{3.35}$$

where C is a positive constant. From Fatou's lemma, Sobolev's inequality, (3.27), (3.35) and letting $\varepsilon \to 0$, we have

$$\begin{aligned}&\left(\int_{Q_t}\left[(W^t_{i1})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}} \le C\liminf_{\varepsilon\to 0}\int_{Q_t}|\nabla\{(W^t_{i2,\varepsilon})^+|y_t|\}|^2dy\\&\le C\varphi_i(t)^2\left[\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}+\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\right].\end{aligned}$$

So,

$$\begin{aligned}&\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}+\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\\&\le \varphi(t)^2\left[\left(\int_{Q_t}\left[(W^t_{11})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}+\left(\int_{Q_t}\left[(W^t_{21})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy\right)^{\frac{n-2}{n}}\right],\end{aligned}$$

where $\varphi(t)^2 = C(\varphi_1(t)^2 + \varphi_2(t)^2)$. Thus, choosing t_1 sufficiently large such that $\varphi(t)^2 < 1$ for all $t > t_1$, we have

$$\int_{Q_t}\left[(W^t_{i1})^+|y_t|^{-1}\right]^{\frac{2n}{n-2}}dy \equiv 0, \quad \text{in } Q_t \quad \text{for all } t > t_1, \;\; i = 1, 2.$$

Then, $(W^t_{i1})^+ \equiv 0$ in Q_t for all $t > t_1$, and from (3.35), gets $(W^t_{i2})^+ \equiv 0$ in Q_t for all $t > t_1$, $i = 1, 2$. Which implies a contradiction with our assumption. Therefore, $\Lambda < +\infty$.

Step 2. If $\Lambda > 0$ then $w_{ij} \equiv w^{\Lambda}_{ij}$ in Q_Λ for some $i, j = 1, 2$.

By definition of Λ and continuity of solutions, we get $w_{ij} \geq w_{ij}^{\Lambda}$, $i, j = 1, 2$ in Q_{Λ}, and from (3.25) and the conditions on $h_{i,j}$, $i, j = 1, 2$, we have:

$$\begin{cases} -\Delta(w_{i1} - w_{i1}^{\Lambda}) = w_{i2} - w_{i2}^{\Lambda} \geq 0 & \text{in } Q_{\Lambda}, \\ -\Delta(w_{i2} - w_{i2}^{\Lambda}) = h_{i2}\left(\frac{w_{11}}{\xi_2}, \frac{w_{12}}{\xi_2}\right) w_{i1}^{\frac{n+4}{n-2}} - h_{i2}\left(\frac{w_{11}^{\Lambda}}{\xi_2^{\Lambda}}, \frac{w_{12}^{\Lambda}}{\xi_2^{\Lambda}}\right) (w_{i1}^{\Lambda})^{\frac{n+4}{n-2}} \geq 0 & \text{in } Q_{\Lambda}, \\ w_{ij} - w_{ij}^{\Lambda} \geq 0 & \text{in } Q_{\Lambda}. \end{cases}$$

Then, from Maximum Principle we have either $w_{ij} \equiv w_{ij}^{\Lambda}$ in Q_{Λ} for some $i, j = 1, 2$, or $w_{ij} > w_{ij}^{\Lambda}$ in Q_{Λ} for all $i, j = 1, 2$. Suppose $w_{ij} > w_{ij}^{\Lambda}$ in Q_{Λ}. We can choose a compact $K \subset Q_{\Lambda}$ and a number $\delta > 0$ such that $\forall t \in (\Lambda - \delta, \Lambda)$ we have $K \subset Q_t$ and

$$C\varphi_i(t)^2 = C\left(\int_{Q_t \setminus K} (w_{i1}^t)^{\frac{8}{n-4}} |y_t|^2 dy\right)^{\frac{4}{n}} < \frac{1}{2}. \tag{3.36}$$

On the other hand, there exists $0 < \delta_1 < \delta$, such that

$$w_{ij} > w_{ij}^t \text{ in } K \quad \forall t \in (\Lambda - \delta_1, \Lambda),\ \forall i, j = 1, 2. \tag{3.37}$$

Using (3.36) and following as in Step 1, since the integrals are over $Q_t \setminus K$, we see that $(w_{ij}^t - w_{ij})^+ \equiv 0$ in $Q_t \setminus K$ for all $i, j = 1, 2$. By (3.37) we get $w_{ij} > w_{ij}^t$ in Q_t for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of Λ.

Step 3. Symmetry

Supose $\Lambda > 0$. From Step 2 we can assume that $w_{12} \equiv w_{12}^{\Lambda}$ in Q_{Λ}. Then

$$\begin{aligned} h_{12}\left(\frac{w_{11}}{\xi_2}, \frac{w_{21}}{\xi_2}\right) &= -\frac{\Delta w_{12}}{w_{11}^{\frac{n+4}{n-4}}} = -\frac{\Delta w_{12}^{\Lambda}}{w_{11}^{\frac{n+4}{n-4}}} = h_{12}\left(\frac{w_{11}^{\Lambda}}{\xi_2^{\Lambda}}, \frac{w_{21}^{\Lambda}}{\xi_2^{\Lambda}}\right)\left(\frac{w_{11}^{\Lambda}}{w_{11}}\right)^{\frac{n+4}{n-4}} \\ &\leq h_{12}\left(\frac{w_{11}}{\xi_2^{\Lambda}}, \frac{w_{21}}{\xi_2^{\Lambda}}\right) < h_{12}\left(\frac{w_{11}}{\xi_2}, \frac{w_{21}}{\xi_2}\right). \end{aligned}$$

This is a contradiction.

Therefore, $\Lambda = 0$ for all directions. Which implies that w_{ij} is radially symmetrical in $\mathbb{R}^n$ for $i, j = 1, 2$. By definition of $v_1 = w_{11}$ and $v_2 = w_{21}$, we obtain that u_1 and u_2 are constant on every $(n-1)$-sphere whose elements $q \in \mathbb{S}^n$ satisfy $|q - S| =$ constant. Since $\zeta \in \mathbb{S}^n$ is arbitrary on $\mathbb{S}^n$, u_1 and u_2 are constant. $\square$

Chapter 4

Problems on the sphere involving a conformal fractional operator

4.1 Nonexitence of solutions for an equation

In this section we study the following problem

$$\begin{cases} \mathcal{D}_s u = f(u) & \text{in } \mathbb{S}^n, \\ \quad u > 0 & \text{in } \mathbb{S}^n, \end{cases} \tag{4.1}$$

where $0 < s < 1$, $n > 2$, $f : (0, \infty) \to \mathbb{R}$ is a continuous function and

$$\mathcal{D}_s u(\zeta) = (\mathcal{A}_{2s} u)(\zeta) - \mathcal{A}_{2s}(1) u(\zeta), \ \zeta \in \mathbb{S}^n, \ u \in C^\infty(\mathbb{S}^n).$$

The nonexistence of nonconstant solutions of problem (4.1) is characterized by the following result.

Theorem 4.1. *Let $s \in (0,1)$. Assume that*

$$h_s(t) := t^{-\frac{n+2s}{n-2s}} (f(t) + d_{n,s} t) \quad \textit{is decreasing in} \quad (0, +\infty).$$

Then the problem (4.1) admits only constant solutions.

In order to prove Theorem 4.1 we will use the same arguments that were used in the proof of previous theorems.

Let ζ be an arbitrary point on $\mathbb{S}^n$, which we will rename the north pole N. Let $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection.

Let $s \in (0,1)$. Let u be a solution of (4.1). We define

$$v(y) = \xi_s(y)u(\mathcal{F}(y)),\ y \in \mathbb{R}^n.$$

Then we have that

$$v \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{4.2}$$

From Lemma 1.1 we get the following equation

$$(-\Delta)^s v = h_s\left(\frac{v}{\xi_s}\right) v^{\frac{n+2s}{n-2s}},\ v > 0 \text{ in } \mathbb{R}^n, \tag{4.3}$$

where

$$h_s(t) = t^{-\frac{n+2s}{n-2s}}\left(f(t) + d_{n,s}t\right),\ t > 0 \ \text{ and } \ d_{n,s} = \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)}.$$

In our arguments, we follows the same strategy as before to show the symmetry of the solution. So, for given $t \in \mathbb{R}$, we set Q_t, U_t, and y_t the same previous objects. We define the reflected function by $v^t(y) := v(y_t)$, and the following functions for $t \geq 0$ and $\varepsilon > 0$:

$$w_\varepsilon^t(y) = \begin{cases} (v^t(y) - v(y) - \varepsilon)^+,\ y \in Q_t, \\ (v^t(y) - v(y) + \varepsilon)^-,\ y \in Q_t^c, \end{cases} \tag{4.4}$$

where $^-$ denotes the negative part of function. Our first result to prove that the solutions are constants is:

Lemma 4.1. *Under the assumptions of Theorem 4.1, for $t > 0$ exists a constant $C > 0$ such that*

$$\left(\int_{Q_t} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} \leq C \int_{Q_t} (-\Delta)^s(v^t - v)(v_\varepsilon^t - v - \varepsilon)dy. \tag{4.5}$$

Proof. Given $t > 0$, we have that

$$w_\varepsilon^t(y) = \max\{v^t(y) - v(y) - \varepsilon, 0\} = -\min\{v^t(y_t) - v(y_t) + \varepsilon, 0\} = -w_\varepsilon^t(y_t),$$

for $y \in Q_t$. Similarly, $w_\varepsilon^t(y) = -w_\varepsilon^t(y_t)$ for $y \in Q_t^c$. So

$$w_\varepsilon^t(y) = -w_\varepsilon^t(y_t) \ \text{ for all } y \in \mathbb{R}^n. \tag{4.6}$$

This implies that

$$\int_{\mathbb{R}^n} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy = \int_{Q_t} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy + \int_{Q_t^c} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy = 2\int_{Q_t} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy. \tag{4.7}$$

Moreover, we see that for any $y \in Q_t \cap \operatorname{supp}(w_\varepsilon^t)$, we have

$$
\begin{aligned}
&(-\Delta)^s w_\varepsilon^t(y) - (-\Delta)^s(v^t - v - \varepsilon)(y) \\
&= \int_{\mathbb{R}^n} \frac{w_\varepsilon^t(y) - w_\varepsilon^t(z)}{|y-z|^{n+2s}} dz - \int_{\mathbb{R}^n} \frac{(v^t - v - \varepsilon)(y) - (v^t - v - \varepsilon)(z)}{|y-z|^{n+2s}} dz \\
&= \int_{\mathbb{R}^n} \frac{(v^t - v - \varepsilon)(z) - w_\varepsilon^t(z)}{|y-z|^{n+2s}} dz \\
&= \int_{Q_t \cap \operatorname{supp}(w_\varepsilon^t)^c} \frac{(v^t - v - \varepsilon)(z)}{|y-z|^{n+2s}} dz + \int_{Q_t^c \cap \operatorname{supp}(w_\varepsilon^t)^c} \frac{(v^t - v - \varepsilon)(z)}{|y-z|^{n+2s}} dz \\
&\quad - \int_{Q_t^c \cap \operatorname{supp}(w_\varepsilon^t)} \frac{2\varepsilon}{|y-z|^{n+2s}} dz \\
&= \int_{Q_t \cap \operatorname{supp}(w_\varepsilon^t)^c} \frac{(v^t - v - \varepsilon)(z)}{|y-z|^{n+2s}} dz + \int_{Q_t \cap \operatorname{supp}(w_\varepsilon^t)^c} \frac{(v^t - v - \varepsilon)(z_t)}{|y-z_t|^{n+2s}} dz \\
&\quad - \int_{Q_t^c \cap \operatorname{supp}(w_\varepsilon^t)} \frac{2\varepsilon}{|y-z|^{n+2s}} dz \\
&= \int_{Q_t \cap \operatorname{supp}(w_\varepsilon^t)^c} (v^t - v - \varepsilon)(z) \left[\frac{1}{|y-z|^{n+2s}} - \frac{1}{|y-z_t|^{n+2s}} \right] dz \\
&\quad - \int_{Q_t \cap \operatorname{supp}(w_\varepsilon^t)^c} \frac{2\varepsilon}{|y-z_t|^{n+2s}} dz - \int_{Q_t^c \cap \operatorname{supp}(w_\varepsilon^t)} \frac{2\varepsilon}{|y-z|^{n+2s}} dz \\
&\leq 0,
\end{aligned}
\tag{4.8}
$$

where the last two integrals are finite, $v^t - v - \varepsilon \leq 0$ in $Q_t \cap \operatorname{supp}(w_\varepsilon^t)^c$, and $|y-z| < |y-z_t|$ for $y, z \in Q_t$. Using the same arguments as in (2.7) and (2.8), we have

$$
(-\Delta)^s(v^t - v)(y) \leq C v^t(y)^{\frac{4s}{n-2s}} (v^t - v)(y) \;\; \text{for } v^t(y) \geq v(y), \;\, y \in Q_t. \tag{4.9}
$$

From (4.8) and (4.9), we get

$$
\begin{aligned}
\int_{Q_t} (-\Delta)^s w_\varepsilon^t w_\varepsilon^t dy &\leq \int_{Q_t} (-\Delta)^s (v^t - v - \varepsilon)(v^t - v - \varepsilon)^+ dy \\
&= \int_{Q_t} (-\Delta)^s (v^t - v)(v^t - v - \varepsilon)^+ dy \\
&\leq C \int_{Q_t} (v^t)^{\frac{4s}{n-2s}} (v^t - v)(v^t - v - \varepsilon)^+ dy \\
&\leq 4C \int_{Q_t} (v^t)^{\frac{4s}{n-2s}} (v^t)^2 dy < \infty,
\end{aligned}
\tag{4.10}
$$

where the last inequality is a consequence of (4.2). From (4.10) and (4.6) we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^t|^2 dy &= \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^t|^2 dy + \int_{Q_t^c} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^t|^2 dy \\ &= 2 \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^t|^2 dy. \end{aligned} \tag{4.11}$$

Using Sobolev's inequality, (4.7), (4.10) and (4.11) we obtain

$$\begin{aligned}\left(\int_{Q_t} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} &= \left(\frac{1}{2}\int_{\mathbb{R}^n} |w_\varepsilon^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} \le C \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^t|^2 dy \\ &= \frac{C}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w_\varepsilon^t|^2 dy = \frac{C}{2} \int_{\mathbb{R}^n} (-\Delta)^s w_\varepsilon^t w_\varepsilon^t dy \\ &= C \int_{Q_t} (-\Delta)^s w_\varepsilon^t w_\varepsilon^t dy \\ &\le C \int_{Q_t} (-\Delta)^s (v^t - v)(v^t - v - \varepsilon)^+ dy. \end{aligned}$$

This concludes the proof of the lemma. □

Proof of Theorem 4.1

Let u be the solution of problem 4.1. We take an arbitrary point $\zeta \in \mathbb{S}^n$ as the south pole, S and let $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ be the stereographic projection. We define $v = \xi_s(u \circ \mathcal{F})$ in $\mathbb{R}^n$. Let

$$\Lambda := \inf\{t > 0;\ v \ge v^\mu,\ \text{in } Q_\mu, \forall \mu \ge t\}.$$

Following our scheme, we start with

Step 1. $\Lambda < +\infty$.

For $\varepsilon > 0$ and $t > 0$, we consider the functions w_ε^t and w^t defined by (4.4). Using Fatou's lemma, Lemma 4.1, (4.9), Höölder's and Sobolev's inequalities, and dominate convergence theorem, we find that

$$
\begin{aligned}
\left(\int_{Q_t}|w^t|^{\frac{2n}{n-2s}}dy\right)^{\frac{n-2s}{n}} &\le \liminf_{\varepsilon\to 0}\left(\int_{Q_t}|w^t_\varepsilon|^{\frac{2n}{n-2s}}dy\right)^{\frac{n-2s}{n}}\\
&\le C\liminf_{\varepsilon\to 0}\int_{Q_t}(-\Delta)^s(v^t-v)(v^t-v-\varepsilon)^+dy\\
&\le C\liminf_{\varepsilon\to 0}\int_{Q_t}(v^t)^{\frac{4s}{n-2s}}(v^t-v)(v^t-v-\varepsilon)^+dy\\
&\le C\int_{Q_t}(v^t)^{\frac{4s}{n-2s}}[(v^t-v)^+]^2dy\\
&\le C\left(\int_{Q_t}(v^t)^{\frac{2n}{n-2s}}dy\right)^{\frac{2s}{n}}\left(\int_{Q_t}[(v^t-v)^+]^{\frac{2n}{n-2s}}dy\right)^{\frac{n-2s}{n}}\\
&\le \phi(t)\left(\int_{Q_t}|w^t|^{\frac{2n}{n-2s}}dy\right)^{\frac{n-2s}{n}}, \qquad (4.12)
\end{aligned}
$$

where $\phi(t)=C(\int_{Q_t}(v^t)^{\frac{2n}{n-2s}}dy)^{\frac{2s}{n}}$. Since $v^{\frac{2n}{n-2s}}\in L^1(\mathbb{R}^n)$, we obtain that $\lim_{t\to+\infty}\phi(t)=0$. Thus, choosing $t_1>0$ large sufficiently such that $\varphi(t_1)<1$, we have from (4.12)

$$\int_{Q_t}|w^t|^{\frac{2n}{n-2s}}dy=0, \qquad \text{for all } t>t_1.$$

This implies $(v^t-v)^+\equiv 0$ in Q_t for $t>t_1$. Which complete this step.

Step 2. $\Lambda=0$.

Assume $\Lambda>0$. By definition of Λ and the continuity of the solution, we have $v\ge v^\Lambda$ and $\xi_s>\xi_s^\Lambda$ in Q_Λ.

Suppose that exists a point $y_0\in Q_\Lambda$ such that $v(y_0)=v^\Lambda(y_0)$. Using the fact of h is decreasing, we have

$$(-\Delta)^s v(y_0)-(-\Delta)^s v^\Lambda(y_0)=\left[h_s\left(\frac{v(y_0)}{\xi_s(y_0)}\right)-h_s\left(\frac{v(y_0)}{\xi_s^\Lambda(y_0)}\right)\right]v(y_0)^{\frac{n+2s}{n-2s}}>0. \qquad (4.13)$$

On the other hand,

$$
\begin{aligned}
&(-\Delta)^s v(y_0) - (-\Delta)^s v^\Lambda(y_0)\\
&= -\int_{\mathbb{R}^n} \frac{v(z) - v(z_\Lambda)}{|y_0 - z|^{n+2s}} dz\\
&= -\int_{Q_\Lambda} \frac{v(z) - v(z_\Lambda)}{|y_0 - z|^{n+2s}} dz - \int_{Q_\Lambda^c} \frac{v(z) - v(z_\Lambda)}{|y_0 - z|^{n+2s}} dz\\
&= -\int_{Q_\Lambda} (v(z) - v(z_\Lambda)) \left(\frac{1}{|y_0 - z|^{n+2s}} - \frac{1}{|y_0 - z_\Lambda|^{n+2s}} \right) dz \leq 0.
\end{aligned}
$$

Which contradicts (4.13). As a consequence, $v > v^\Lambda$ in Q_Λ. So, we can choose a compact $K \subset Q_\Lambda$ and a number $\delta > 0$ such that $\forall t \in (\Lambda - \delta, \Lambda)$ we have $K \subset Q_t$ and

$$
\phi(t) = C \left(\int_{Q_t \backslash K} (v^t)^{\frac{2n}{n-2s}} dy \right)^{\frac{2s}{n}} < \frac{1}{2}. \tag{4.14}
$$

We also have that exists $0 < \delta_1 < \delta$, such that

$$
v > v^t, \text{ in } K \; \forall t \in (\Lambda - \delta_1, \Lambda). \tag{4.15}
$$

Using (4.14), (4.15) and following as in (4.12), by noticing that the integrals are over $Q_t \backslash K$, we see that $(v^t - v)^+ \equiv 0$ in $Q_t \backslash K$. By (4.15) we obtain $v > v^t$ in Q_t for all $t \in (\Lambda - \delta_1, \Lambda)$, contradicting the definition of Λ.

Step 3. Symmetry.

By Step 2 we have $\Lambda = 0$ for all directions. This implies that v is radially symmetrical in $\mathbb{R}^n$. By definition of v, we obtain that u is constant on every $(n-1)$-sphere whose elements $q \in \mathbb{S}^n$ satisfy $|q - S| =$ constant. Since $\zeta_0 \in \mathbb{S}^n$ is arbitrary on $\mathbb{S}^n$, we conclude that u is constant. $\square$

4.2 Nonexitence of solutions for systems

Following the results of Chapters 1, 2 and 3, we will study the following system

$$
\begin{cases}
\mathcal{D}_s u_1 = f_1(u_1, u_2) & \text{in } \mathbb{S}^n,\\
\mathcal{D}_s u_2 = f_2(u_1, u_2) & \text{in } \mathbb{S}^n,\\
u_1, u_2 > 0 & \text{in } \mathbb{S}^n.
\end{cases} \tag{4.16}
$$

Similarly to Theorem 4.1, the nonexistence of nonconstant solutions of problem (4.16) is characterized by the following result.

Theorem 4.2. *Let $s \in (0,1)$ and let $h_{is} : (0,\infty)\times(0,\infty) \to \mathbb{R}$, $i = 1,2$, be two functions defined by*

$$h_{is}(t_1,t_2) := t_i^{-\frac{n+2s}{n-2s}} \left(f_i(t_1,t_2) + d_{n,s}t_i\right),\ t_i > 0.$$

Assume that for $i, j = 1, 2$: $h_{i,s}$ are non-negative,

$$\begin{aligned}
&h_{is}(t_1,t_2) \text{ is nondecreasing in } t_j > 0, \text{ with } i \neq j,\\
&h_{is}(t_1,t_2)t_i^{\frac{n+2s}{n-2s}} \text{ is nondecreasing in } t_i > 0,\\
&h_{is}(a_1t, a_2t) \text{ is decreasing in } t > 0 \text{ for any } a_i > 0.
\end{aligned}$$

Then the problem (4.16) admits only constant solutions.

The arguments to be used in the proof of Theorem 4.2 are similar to those used in the proof of Theorem 4.2. Then we will simplify some calculations.

Let (u_1,u_2) be a solution of (4.16). We define

$$v_1(y) = \xi_s(y)u_1(\mathcal{F}(y)),\ v_2(y) = \xi_s(y)u_2(\mathcal{F}(y)),$$

where ξ_s is defined in (1.4) and $\mathcal{F}$ is the inverse of stereographic projection. Then we have that

$$v_1,\ v_2 \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{4.17}$$

From Lemma 1.2 gets the following equation

$$\begin{cases}
(-\Delta)^s v_1 = h_{1s}\left(\dfrac{v_1}{\xi_s(y)}, \dfrac{v_2}{\xi_s(y)}\right) v_1^{\frac{n+2s}{n-2s}},\ v_1 > 0 \text{ in } \mathbb{R}^n,\\
(-\Delta)^s v_2 = h_{2s}\left(\dfrac{v_1}{\xi_s(y)}, \dfrac{v_2}{\xi_s(y)}\right) v_2^{\frac{n+2s}{n-2s}},\ v_2 > 0 \text{ in } \mathbb{R}^n,
\end{cases} \tag{4.18}$$

where

$$h_{is}(t) = t^{-\frac{n+2s}{n-2s}}(f(t) + d_{n,s}t),\ t > 0,\ i = 1,2, \text{ and } d_{n,s} = \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)}.$$

We define the following functions for $t \geq 0$, $i = 1, 2$, and $\varepsilon \geq 0$:

$$w_{i,\varepsilon}^t(y) = \begin{cases}
(v_i^t(y) - v_i(y) - \varepsilon)^+,\ y \in Q_t,\\
(v_i^t(y) - v_i(y) + \varepsilon)^-,\ y \in Q_t^c.
\end{cases} \tag{4.19}$$

Then, we have that these functions satisfy the following inequality.

Lemma 4.2. *Under the assumptions of Theorem 4.2, there exists a constant $C > 0$ such that, for $t > 0$ and $\varepsilon > 0$, gets*

$$\left(\int_{Q_t} |w_{i,\varepsilon}^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} \leq C \int_{Q_t} (-\Delta)^s (v_i^t - v_i)(v_i^t - v_i - \varepsilon) dy, \;\; i = 1, 2. \tag{4.20}$$

Proof. Given $t > 0$,$\varepsilon > 0$ and arguing as (4.6)-(4.8) we have

$$w_{i,\varepsilon}(y) = -w_{i,\varepsilon}^t(y_t) \;\text{ for all } y \in \mathbb{R}^n, \text{ and } \int_{\mathbb{R}^n} |w_{i,\varepsilon}^t|^{\frac{2n}{n-2s}} dy = 2 \int_{Q_t} |w_{i,\varepsilon}^t|^{\frac{2n}{n-2s}} dy, \tag{4.21}$$

and

$$(-\Delta)^s w_{i,\varepsilon}^t(y) \leq (-\Delta)^s (v_i^t - v_i - \varepsilon)(y), \;\; y \in Q_t \cap \operatorname{supp}(w_{i,\varepsilon}^t). \tag{4.22}$$

Using the same arguments as in (2.20) and (2.21), we have

$$(-\Delta)^s (v_i^t - v_i)(y) \leq C v_i^t(y)^{\frac{4s}{n-2s}} [(v_1^t - v_1) + (v_2^t - v_2 - \varepsilon)](v_i^t - v_i)(y), \tag{4.23}$$

for $v_i^t(y) \geq v_i(y)$, $y \in Q_t$, $i = 1, 2$. From (4.22) and (4.23), we get

$$\begin{aligned}
\int_{Q_t} (-\Delta)^s w_{i,\varepsilon}^t w_{i,\varepsilon}^t dy &\leq \int_{Q_t} (-\Delta)^s (v_i^t - v_i)(v_i^t - v_i - \varepsilon)^+ dy && (4.24)\\
&\leq C \int_{Q_t} (v_i^t)^{\frac{4s}{n-2s}} [(v_1^t - v_1)^+ + (v_2^t - v_2)^+](v_i^t - v_i - \varepsilon)^+ dy \\
&\leq C \int_{Q_t} (v_i^t)^{\frac{n+2s}{n-2s}} (v_1^t + v_2^t) dy < +\infty. && (4.25)
\end{aligned}$$

Moreover,

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} w_{i,\varepsilon}^t|^2 dy = 2 \int_{Q_t} |(-\Delta)^{\frac{s}{2}} w_{i,\varepsilon}^t|^2 dy, \;\; i = 1, 2. \tag{4.26}$$

Using Sobolev's inequality, (4.21), (4.24) and (4.26) we obtain

$$\left(\int_{Q_t} |w_{i,\varepsilon}^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} \leq C \int_{Q_t} (-\Delta)^s (v_i^t - v_i)(v_i^t - v_i - \varepsilon)^+ dy, \;\; \text{for } i = 1, 2.$$

This completes the proof of Lemma. □

Proof of Theorem 4.2

Let (u_1, u_2) be the solution of problem (4.16). We take a $\zeta \in \mathbb{S}^n$ as the south pole S, and let $\mathcal{F}^{-1} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ be the stereographic projection. Define $v_1 = \xi_s(u_1 \circ \pi)$ and

$v_2 = \xi_s(u_2 \circ \mathcal{F})$ in $\mathbb{R}^n$. As in the case of one equation, given $t \in \mathbb{R}$ we set Q_t, U_t and y_t. Let

$$\Lambda := \inf\{t > 0;\ v_i \geq v_i^\mu,\ \text{in } Q_\mu, \forall \mu \geq t,\ i = 1,2\}.$$

The proof is carried out in three steps.

Step 1. $\Lambda < +\infty$.

For $\varepsilon > 0$ and $t > 0$ we consider the functions $w_{i,\varepsilon}^t$ and w_i^t defined by (4.19). Using Fatou's lemma, Lemma 4.2, (4.23), Hölder's and Sobolev's inequalities, and Dominate Convergence, we get

$$\begin{aligned}
\left(\int_{Q_t} |w_1^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} &\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} (-\Delta)^s (v_1^t - v_1)(v_1^t - v_1 - \varepsilon)^+ dy \\
&\leq C \liminf_{\varepsilon \to 0} \int_{Q_t} (v_1^t)^{\frac{4s}{n-2s}} (v_1^t - v_1)(v_1^t - v_1 - \varepsilon)^+ dy \\
&+ C \liminf_{\varepsilon \to 0} \int_{Q_t} (v_1^t)^{\frac{4s}{n-2s}} (v_2^t - v_2)(v_1^t - v_1 - \varepsilon)^+ dy \\
&\leq C \left(\int_{Q_t} (v_1^t)^{\frac{2n}{n-2s}} dy\right)^{\frac{2s}{n}} \left[\left(\int_{Q_t} [(v_1^t - v_1)^+]^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}}\right. \\
&\left.\left(\int_{Q_t} [(v_2^t - v_2)^+]^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}}\right] \\
&\leq \phi_1(t) \left[\left(\int_{Q_t} |w_1^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} + \left(\int_{Q_t} |w_2^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}}\right], \qquad (4.27)
\end{aligned}$$

where $\phi_1(t) = C(\int_{Q_t} (v_1^t)^{\frac{2n}{n-2s}} dy)^{\frac{2s}{n}}$. Since $v_1^{\frac{2n}{n-2s}} \in L^1(\mathbb{R}^n)$, then $\lim_{t \to +\infty} \phi_1(t) = 0$. Similarly, we have

$$\left(\int_{Q_t} |w_2^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} \leq \phi_2(t) \left[\left(\int_{Q_t} |w_1^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}} + \left(\int_{Q_t} |w_2^t|^{\frac{2n}{n-2s}} dy\right)^{\frac{n-2s}{n}}\right], \qquad (4.28)$$

where $\phi_2(t) = C(\int_{Q_t} (v_2^t)^{\frac{2n}{n-2s}} dy)^{\frac{2s}{n}}$. Since $v_2^{\frac{2n}{n-2s}} \in L^1(\mathbb{R}^n)$, then $\lim_{t \to +\infty} \phi_2(t) = 0$.

Thus, choosing $t_1 > 0$ large sufficiently such that $\phi_i(t_1) < 1/4$ for $i = 1,2$, we have from (4.27) and (4.28)

$$\int_{Q_t} |w_i^t|^{\frac{2n}{n-2s}} dy = 0, \qquad \text{for all } t > t_1.$$

This implies that $(v_i^t - v_i)^+ \equiv 0$ in Q_t for $t > t_1$ and $i = 1,2$. Therefore Λ is well defined, i.e. $\Lambda < +\infty$.

Step 2. $\Lambda = 0$.

Assume $\Lambda > 0$. By definition of Λ and continuity of the solution, we get $v_i \geq v_i^\Lambda$ in Q_Λ for $i = 1, 2$. Suppose there is a point $y_0 \in Q_\Lambda$ such that $v_1(y_0) = v_1^\Lambda(y_0)$. By the conditions on h_{is}, we have

$$\begin{aligned}
&(-\Delta)^s v_1(y_0) - (-\Delta)^s v_1^\Lambda(y_0) \\
&= h_{1s}\left(\frac{v_1(y_0)}{\xi_s(y_0)}, \frac{v_2(y_0)}{\xi_s(y_0)}\right) v_1(y_0)^{\frac{n-2s}{n+2s}} - h_{1s}\left(\frac{v_1^\Lambda(y_0)}{\xi_s^\Lambda(y_0)}, \frac{v_2^\Lambda(y_0)}{\xi_s^\Lambda(y_0)}\right) v_1^\Lambda(y_0)^{\frac{n-2s}{n+2s}} \\
&= \left[h_{1s}\left(\frac{v_1(y_0)}{\xi_s(y_0)}, \frac{v_2(y_0)}{\xi_s(y_0)}\right) - h_{1s}\left(\frac{v_1(y_0)}{\xi_s^\Lambda(y_0)}, \frac{v_2^\Lambda(y_0)}{\xi_s^\Lambda(y_0)}\right)\right] v_1(y_0)^{\frac{n-2s}{n+2s}} \\
&\geq \left[h_{1s}\left(\frac{v_1(y_0)}{\xi_s(y_0)}, \frac{v_2(y_0)}{\xi_s(y_0)}\right) - h_{1s}\left(\frac{v_1(y_0)}{\xi_s^\Lambda(y_0)}, \frac{v_2(y_0)}{\xi_s^\Lambda(y_0)}\right)\right] v_1(y_0)^{\frac{n-2s}{n+2s}} > 0.
\end{aligned} \tag{4.29}$$

On the other hand,

$$\begin{aligned}
&(-\Delta)^s v_1(y_0) - (-\Delta)^s v_1^\Lambda(y_0) \\
&= -\int_{Q_\Lambda} (v_1(z) - v_1(z_\Lambda))\left(\frac{1}{|y_0 - z|^{n+2s}} - \frac{1}{|y_0 - z_\Lambda|^{n+2s}}\right) dz \leq 0,
\end{aligned}$$

which contradicts (4.29). Thus, $v_1 > v_1^\Lambda$ in Q_Λ.

Similarly, we have that $v_2 > v_2^\Lambda$ in Q_Λ. We can choose a compact $K \subset Q_\Lambda$ and a number $\delta > 0$ such that $\forall t \in (\Lambda - \delta, \Lambda)$ we have $K \subset Q_t$ and

$$\phi_i(t) = C\left(\int_{Q_t \setminus K} (v_i^t)^{\frac{2n}{n-2s}} dy\right)^{\frac{2s}{n}} < \frac{1}{2}, \; i = 1, 2. \tag{4.30}$$

On the other hand, there exists $0 < \delta_1 < \delta$, such that

$$v_1 > v_1^t \text{ and } v_2 > v_2^t, \text{ in } K \; \forall t \in (\Lambda - \delta_1, \Lambda). \tag{4.31}$$

Using (4.30), (4.31) and proceeding as in Step 1, in (4.27), since the integrals are over $Q_t \backslash K$, we see that $(v^t - v)^+ \equiv 0$ in $Q_t \backslash K$. By (4.31) we get $v_i > v_i^t$ in Q_t for all $t \in (\Lambda - \delta_1, \Lambda)$ and $i = 1, 2$, contradicting the definition of Λ.

Step 3. Symmetry.

By Step 2, the functions v_i are radially symmetrical in $\mathbb{R}^n$. By the definition of v_i, we obtain that u_i is constant on every $(n-1)$-sphere whose elements $q \in \mathbb{S}^n$ satisfy $|q - S| = $ constant. Since $p \in \mathbb{S}^n$ is arbitrary on $\mathbb{S}^n$, u_i is constant. $\square$

Chapter

5

Existence of solutions for a conformally invariant fractional equation on the sphere

5.1 Main results

Consider then the following problem

$$\mathcal{A}_{2s}u = f(u) \text{ in } \mathbb{S}^n, \;\; u \in H^s(\mathbb{S}^n), \tag{5.1}$$

where $s \in (0,1)$, $n > 2$, $\mathcal{A}_{2s}$ is the conformal fractional operator given by (1.1) and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying the following conditions:

(h_1) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$;

(h_2) There exist a positive constant C and $p \in (2, 2n/(n-2s)]$ such that

$$|f(t)| \leq C(1 + |t|^{p-1}) \;\text{ for all } t \in \mathbb{R};$$

(h_3) There are two constants $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(t) \leq t f(t) \;\text{ for all } |t| \geq R,$$

where $F(t) := \int_0^t f(\tau)d\tau$ for all $t \in \mathbb{R}$.

The operator $\mathcal{A}_{2s}$ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection:

$$(\mathcal{A}_{2s}u) \circ \mathcal{F} = \xi_s{}^{-\frac{n+2s}{n-2s}}(-\Delta)^s(\xi_s(u \circ \mathcal{F})), \quad \text{for all } u \in C^\infty(\mathbb{S}^n),$$

where $\mathcal{F}$ is the inverse of the stereographic projection, ξ_s is defined by (1.4) and $(-\Delta)^s$ is the fractional Laplacian operator.

The study of the existence of sign-changing solutions for some problems involving the fractional Laplace and conformal operatos have been studied with great interest in recent years, see e. g., [10, 20, 41, 48, 54] and the references therein. Motivated by these results we can state our main result about the existence of solutions as follows.

Theorem 5.1. *Assume that function f satisfies (h_1), (h_2), (h_3). Then there exists an unbounded sequence of solutions $\{u_l\}_{l\in\mathbb{N}}$ in $H^s(\mathbb{S}^n)$ of (5.1).*

Our result also guarantes the existence of solutions for some problems on $\mathbb{R}^n$ involving the fractional Laplacian operator. These problems arise from (5.1) using the stereographic projection (see Chapter 1).

Consider the equation

$$\begin{cases} (-\Delta)^s v = |v|^{\frac{4s}{n-2s}} v \text{ in } \mathbb{R}^n, \\ v \in D^{s,2}(\mathbb{R}^n), 0 < s < 1. \end{cases} \tag{5.2}$$

Then all positive solutions are the form

$$v(x) = \frac{C_{n,s}}{(a^2 + |x - x_0|^2)^{\frac{n-2s}{2}}}, \; a \in \mathbb{R}, \; x_0 \in \mathbb{R}^n, \tag{5.3}$$

where $C_{n,s}$ is a positive constant depending of n and s. We remark that the equation (5.2) is invariant under the conformal transformations on $\mathbb{R}^n$. Thus, if $v(x)$ is solution, then for each $a > 0$ and $x_0 \in \mathbb{R}^n$, $a^{(n-2s)/2}v\left((x - x_0)/a\right)$ is also solution. Moreover, for each equation, all solutions obtained in this way have the same energy, and we will say that these solutions of each equation are equivalent. In particular, the solutions (5.3) are equivalent. Therefore, our main theorem implies the existence of infinitely many inequivalent solutions to the problem (5.2).

Corollary 5.1. *The problem (5.2) has an unbounded sequence $\{v_l\}_{l\in\mathbb{N}}$ of sign-changing solutions.*

Proof. By Theorem 5.1, the equation

$$\mathcal{A}_{2s}u = |u|^{\frac{4s}{n-2s}} u$$

has an unbounded sequence $\{u_l\}_{l\in\mathbb{N}}$ in $H^s(\mathbb{S}^n)$ of solutions. From Lemma 1.1, each function $v_l = Pu_l$, $l \in \mathbb{N}$, is a solution of (5.2) and

$$\|v_l\|_s \to +\infty \text{ as } l \to +\infty.$$

Follow from [24] that the solutions v_l are not equivalent. Therefore, v_l changes sign. $\square$

We recall that the Theorem 4.1 shows that the only positive solutions to problem (5.1) are constants provided that the function $\tilde{h}(t) = t^{-\frac{n+2s}{n-2s}} f(t)$ is decreasing on $(0, +\infty)$. The following result shows the existence of solutions no constants.

Corollary 5.2. *Assume that function f satisfies (h_1), (h_2), (h_3) and $\tilde{h}(t) = t^{-\frac{n+2s}{n-2s}} f(t)$ is decreasing on $(0, +\infty)$. Then there exists an unbounded sequence $\{u_l\}_{l\in\mathbb{N}}$ in $H^s(\mathbb{S}^n)$ of sign-changing solutions of (5.1).*

Proof. By Theorem 5.1, (5.1) has an unbound sequence $\{u_l\}_{l\in\mathbb{N}}$ in $H^s(\mathbb{S}^n)$. Unless of subsequence, assume that $\{u_l\}_{l\in\mathbb{N}}$ is a positive sequence. Then, from Theorem 4.1, each u_l is costanst and

$$f(u_l)u_l = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} f(u_l)u_l \, d\eta = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} u_l \mathcal{A}_{2s} u_l \, d\eta = \frac{1}{|\mathbb{S}^n|}\|u_l\|_*^2 \to +\infty \text{ as } l \to +\infty.$$

Moreover,

$$f(u_l)u_l = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} u_l \mathcal{A}_{2s} u_l \, d\eta = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} u_l \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)} u_l d\eta = \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)}(u_l)^2. \tag{5.4}$$

Thus, $u_l \to +\infty$ as $l \to +\infty$.

On the other hand, from (h_3), there are positive constans a_1, a_2 such that

$$tf(t) \geq \mu F(t) \geq a_1 t^\mu - a_2, \text{ for all } t > 0 \text{ large}.$$

From (5.4) and as $\mu > 2$, we have

$$u_l f(u_l) \geq \mu F(u_l) > \frac{\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{n}{2}-s)}(u_l)^2 = u_l f(u_l) \text{ for } l \text{ large}.$$

Thus, we obtain a contradiction. $\square$

Example 5.1. *Consider the following function:*

$$f(t) = |t|^{p-2}t + \lambda t, \;\; p > 2, \;\; 0 \leq \lambda \leq d_{n,s},$$

where $d_{n,s} = \Gamma(n/2+s)/\Gamma(n/2-s)$. *So that (5.1) becomes*

$$\mathcal{A}_{2s}u = |u|^{p-2}u + \lambda u \quad in\ \mathbb{S}^n. \tag{5.5}$$

Corollary 5.3. *Assume that* $p \leq (n+2)/(n-2s)$ *and* $\lambda \leq d_{n,s}$, *and at least one of these inequalities is strict. Then there exists an unbounded sequence* $\{v_l\}_{l\in\mathbb{N}}$ *in* $H^s(\mathbb{S}^n)$ *of sign changing solutions of (5.5).*

The proof of our main theorem consists in use some standard variational techniques. As always, we note that the symmetries of $\mathbb{S}^n$ will play an important role.

5.2 Existence of infinitely many solutions

We note that solutions of (5.1) are related with the critical points of the functional

$$\begin{aligned} J(u) &:= \frac{1}{2}\int_{\mathbb{S}^n} u\mathcal{A}_{2s}u \ d\eta - \int_{\mathbb{S}^n} F(u)\ d\eta \\ &= \frac{1}{2}\|u\|_*^2 - \int_{\mathbb{S}^n} F(u) d\eta. \end{aligned} \tag{5.6}$$

Recall that $\mathcal{A}_{2s}$ is adjoint and $2_s^* = 2n/(n-2s)$ is just the critical exponent for the embedding $H^s(\mathbb{S}^n) \subset L^p(\mathbb{S}^n)$, $1 \leq p \leq 2_s^*$. Therefore, by conditions on f, the functional J is well defined and differentiable in $H^s(\mathbb{S}^n)$, but it fails to satisfy the Palais-Smale compactness condition in $H^s(\mathbb{S}^n)$. However, applying the fountain theorem [10], and the principle of symmetric criticality [59] we obtain the following result whose demonstration is found in the Appendix.

Lemma 5.1. *Let G be a compact topological group acting linearly and isometrically on closed subset* $X_G \subset H^s(\mathbb{S}^n)$ *such that*

(i) J *is G-invariant;*

(ii) the embedding $X_G \hookrightarrow L^p$ *is compact;*

(iii) X_G *has infinite dimension.*

Then J has a unbounded sequence of critical points $\{u_l\}_{l\in\mathbb{N}}$ *in* $H^s(\mathbb{S}^n)$.

Ding [29] considered an infinite dimensional closed subset $X_G \subset H^1(\mathbb{S}^n)$ and showed that the embedding $X_G \hookrightarrow L^p$ is compact. To show this compactness he used Sobolev's inequality for functions in H^1 on limited domains of $\mathbb{R}^k$, $k < n$, and some integral inequalities. In order to show the item (ii) of Lemma 5.1 we will use the Sobolev's inequality in fractional spaces H^s (see [28]), but this inequality involves an double integral similar to the expression (5.31). Hence we need to give an expression of double integral in $\mathbb{S}^n$ instead of the first integral of (5.6). This expression is given by Proposition 5.2 and we will give a brief proof. When $s \in (0,1)$, Pavlov and Samko [61] showed that

$$\mathcal{A}_{2s}u(\zeta) = \mathcal{A}_{2s}(1)u(\zeta) + c_{n,-s}P.V.\int_{\mathbb{S}^n} \frac{u(\zeta)-u(\omega)}{|\zeta-\omega|^{n+2s}}d\eta_\omega, \quad \text{for all } u \in C^2(\mathbb{S}^n), \tag{5.7}$$

where

$$\mathcal{A}_{2s}(1) = \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma\left(\frac{n}{2}-s\right)}, \; c_{n,-s} = \frac{2^{2s}\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)},$$

and $P.V. \int_{\mathbb{S}^n}$ is understood as $\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}$. In order to remove the singularity of the integral we will use the following lemma.

Lemma 5.2. *There is a positive constant $C = C(n,s)$ such that*

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(\omega)-u(\zeta)|^2}{|\omega-\zeta|^{n+2s}}\, d\eta_\omega d\eta_\zeta \leq C\|u\|^2_{H^1(\mathbb{S}^n)}, \quad \textit{for all } u \in C^1(\mathbb{S}^n). \tag{5.8}$$

Proof. First, let us estimate (5.8) on $V_\zeta = \mathbb{S}^n \cap \{\omega \in \mathbb{R}^{n+1};\; |\omega-\zeta| < 1\}$, $\zeta \in \mathbb{S}^n$. Consider the diffeomorphism

$$h : B_r \subset \mathbb{R}^n \to V_S, \;\; h(y) = (y, -\sqrt{1-|y|^2}).$$

where $B_r(0)$ is the ball of center 0 and radius r, $0 < r < 1$, and $S = (0,-1)$ is the south pole of $\mathbb{S}^n$. Then $R_\zeta \circ h$ maps V_ζ diffeomorphically onto B_r, where R_ζ is an orthogonal rotation in $\mathbb{R}^{n+1}$. Thus, there is a positive constant c_1 that is independent of ζ such that

$$c_1^{-1}|z| \leq |R_\zeta z| \leq c_1|z| \;\; \text{for all } z \in \mathbb{R}^n,$$

and the metric matrices in the charts $(V_\zeta, (R_\zeta \circ h)^{-1})$ satisfies

$$g^\zeta_{ij} \leq c_1 I \;\; \text{for each } \zeta \in \mathbb{S}^n,$$

where I is the $n \times n$ identity matrix. Moreover,

$$\frac{|h'(ty)|}{|h(y)-h(0)|} \leq \frac{1}{(1-t^2|y|^2)^{\frac{1}{2}}} < c_, \text{ for all } t \in (0,1) \text{ and } y \in B_r,$$

where c_2 is a positive constant. Then

$$
\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\cap\{|\omega-\zeta|<1\}}\frac{|u(\omega)-u(\zeta)|^2}{|\omega-\zeta|^{n+2s}}\,d\eta_\omega d\eta_\zeta\\
&=\int_{\mathbb{S}^n}\int_{B_r}\frac{|u(R_\zeta\circ h)(y)-u(R_\zeta\circ h)(0)|^2}{|(R_\zeta\circ h)(y)-(R_\zeta\circ h)(0)|^{n+2s}}\sqrt{det(g_{ij}^{\zeta})}\,dyd\eta_\zeta\\
&\leq C\int_{\mathbb{S}^n}\int_{B_r}\left[\int_0^1\frac{|(u\circ R_\zeta\circ h)'(ty)|}{|R_\zeta(h(y)-h(0))|}\,dt\right]^2\frac{1}{|R_\zeta(h(y)-h(0))|^{n+2(s-1)}}dyd\eta_\zeta\\
&\leq C\int_{\mathbb{S}^n}\int_{B_r}\left[\int_0^1\frac{|\nabla_{\mathbb{S}^n}u[R_\zeta(h(ty))]|\,|h'(ty)|}{|h(y)-h(0)|}\,dt\right]^2\frac{1}{|h(y)-h(0)|^{n+2(s-1)}}dyd\eta_\zeta\\
&\leq C\int_{\mathbb{S}^n}\int_{B_r}\left[\int_0^1\frac{|\nabla_{\mathbb{S}^n}u[R_\zeta(h(ty))]|}{|h(y)-h(0)|^{\frac{n+2(s-1)}{2}}}\,dt\right]^2\,dyd\eta_\zeta\\
&\leq C\int_{\mathbb{S}^n}\int_{B_r}\int_0^1\frac{|\nabla_{\mathbb{S}^n}u[R_\zeta(h(ty))]|^2}{|h(y)-h(0)|^{n+2(s-1)}}\,dtdyd\eta_\zeta\\
&\leq C\int_{B_r}\int_0^1\int_{\mathbb{S}^n}\frac{|\nabla_{\mathbb{S}^n}u[R_\zeta(h(ty))]|^2}{|y|^{n+2(s-1)}}d\eta_\zeta dtdy\\
&\leq C\int_{B_r}\int_0^1\frac{\|\nabla_{\mathbb{S}^n}u\|^2_{L^2(\mathbb{S}^n)}}{|y|^{n+2(s-1)}}dtdy\\
&\leq C\|u\|^2_{H^1(\mathbb{S}^n)},
\end{aligned}
\tag{5.9}
$$

where $C=C(n,s)>0$ and $\nabla_{\mathbb{S}^n}$ is the gradient on $\mathbb{S}^n$.

Now, let us estimate (5.8) on $\mathbb{S}^n\cap\{\omega\in\mathbb{R}^{n+1};\ |\omega-\zeta|>1\}$, $\zeta\in\mathbb{S}^n$.

$$
\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\cap\{|\omega-\zeta|>1\}}\frac{|u(\omega)-u(\zeta)|^2}{|\omega-\zeta|^{n+2s}}\,d\eta_\omega d\eta_\zeta\\
&\qquad\leq C\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\cap\{|\omega-\zeta|>1\}}\left[|u(\omega)|^2+|u(\zeta)|^2\right]d\eta_\omega d\eta_\zeta\leq C\|u\|^2_{L^2(\mathbb{S}^n)}.
\end{aligned}
\tag{5.10}
$$

From (5.9) and (5.10) we get (5.8). □

Proposition 5.1. *If $s\in(0,1)$, then*

$$
P.V.\int_{\mathbb{S}^n}\frac{u(\zeta)-u(\omega)}{|\zeta-\omega|^{n+2s}}d\eta^{(\omega)}=\frac{C_{n,-s}}{2}\int_{\mathbb{S}^n}\frac{2u(\zeta)-u(\omega)-u(\theta(\zeta,\omega))}{|\zeta-\omega|^{n+2s}}dv^{(\omega)}_{g_{\mathbb{S}^n}},\ \zeta\in\mathbb{S}^n, u\in C^2(\mathbb{S}^n),
$$

where $\theta(\zeta,\omega)$ denotes the symmetric value of ω on $\mathbb{S}^n$ with respect to ζ.

Proof. Given $u\in C^2(\mathbb{S}^n)$ and $z\in\mathbb{S}^n$, we use the stereographic coordinates with ζ being the south pole to obtain the following equality

$$P.V. \int_{\mathbb{S}^n} \frac{u(\zeta)-u(\omega)}{|\zeta-\omega|^{n+2s}} d\eta^{(\omega)} = P.V. \int_{\mathbb{R}^n} \frac{u(\mathcal{F}(0))-u(\mathcal{F}(y))}{|\mathcal{F}(0)-\mathcal{F}(y)|^{n+2s}} \left(\frac{2}{1+|y|^2}\right)^n dy, \tag{5.11}$$

where $\mathcal{F}$ is the inverse of the stereographic projection. For simplicity, we assume that ζ is the canonical vector $-e_{n+1} \in \mathbb{R}^{n+1}$, since we can apply a orthogonal rotation on the right side of (5.1). Substituting $\tilde{y} = -y$ on the left side of the above equality, we have

$$\begin{aligned} P.V. \int_{\mathbb{R}^n} & \frac{u(\mathcal{F}(0))-u(\mathcal{F}(y))}{|\mathcal{F}(0)-\mathcal{F}(y)|^{n+2s}} \left(\frac{2}{1+|y|^2}\right)^n dy \\ &= P.V. \int_{\mathbb{R}^n} \frac{u(\mathcal{F}(0))-u(\mathcal{F}(-\tilde{y}))}{|\mathcal{F}(0)-\mathcal{F}(-\tilde{y})|^{n+2s}} \left(\frac{2}{1+|\tilde{y}|^2}\right)^n d\tilde{y}, \end{aligned}$$

and so,

$$\begin{aligned} 2P.V. & \int_{\mathbb{S}^n} \frac{u(\zeta)-u(\omega)}{|\zeta-\omega|^{n+2s}} d\eta^{(\omega)} \\ &= P.V. \int_{\mathbb{R}^n} \frac{2u(\mathcal{F}(0))-u(\mathcal{F}(y))-u(\mathcal{F}(-y))}{|\mathcal{F}(0)-\mathcal{F}(y)|^{n+2s}} \left(\frac{2}{1+|y|^2}\right)^n dy. \end{aligned} \tag{5.12}$$

The above representation is useful to remove the singularity of the integral at the origin. Indeed, we use a second order Taylor expansion to obtain

$$\frac{2u(\mathcal{F}(0))-u(\mathcal{F}(y))-u(\mathcal{F}(-y))}{|\mathcal{F}(0)-\mathcal{F}(y)|^{n+2s}} \left(\frac{2}{1+|y|^2}\right)^n \leq C \frac{\|D^2(u\circ\mathcal{F})\|_{L^\infty}}{|y|^{n+2s-2}},$$

which is integrable near 0, and thus one can get rid of the P.V. □

The following result expresses the norm $\|\cdot\|_*$ in terms of a singular integral operator.

Proposition 5.2. *If $s \in (0,1)$, then*

$$\|u\|_{s,g_{\mathbb{S}^n}} = \frac{C_{n,-s}}{2} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} + P_s(1) \int_{\mathbb{S}^n} u^2 dv_{g_{\mathbb{S}^n}}$$

for all $u \in H^s(\mathbb{S}, g_{\mathbb{S}^n})$.

Proof. Denote by $\theta(\zeta,\omega)$ the symmetric value of ω on $\mathbb{S}^n$ with respect to $\zeta \in \mathbb{S}^n$. Given

$u \in C^2(\mathbb{S}^n)$, from Proposition 5.1 and the Fubini theorem, we have

$$\begin{aligned}
&\int_{\mathbb{S}^n} P.V. \int_{\mathbb{S}^n} \frac{u(\zeta)^2 - u(\zeta)u(\omega)}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\omega)} dv_{g_{\mathbb{S}^n}}^{(\zeta)} \\
&= \frac{1}{2}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{2u(\zeta)^2 - u(\zeta)u(\omega) - u(\zeta)u(\theta(\zeta,\omega))}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\omega)} dv_{g_{\mathbb{S}^n}}^{(\zeta)} \\
&= \frac{1}{2}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} + \frac{1}{2}\frac{[u(\zeta)-u(\theta(\zeta,\omega))]^2}{|\zeta-\theta(\zeta,\omega)|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} \\
&- \frac{1}{2}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{u(\omega)^2 - u(\omega)u(\zeta) + u(\theta(\zeta,\omega))^2 - u(\theta(\zeta,\omega))u(\zeta)}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} \\
&= \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} - \frac{1}{2}\int_{\mathbb{S}^n} P.V. \int_{\mathbb{S}^n} \frac{u(\omega)^2 - u(\omega)u(\zeta)}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} \\
&- \frac{1}{2}\int_{\mathbb{S}^n} P.V. \int_{\mathbb{S}^n} \frac{u(\theta(\zeta,\omega))^2 - u(\theta(\zeta,\omega))u(\zeta)}{|\zeta-\theta(\zeta,\omega)|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} \\
&= \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)} - \int_{\mathbb{S}^n} P.V. \int_{\mathbb{S}^n} \frac{u(\omega)^2 - u(\omega)u(\zeta)}{|\zeta-\omega|^{n+2s}} dv_{g_{\mathbb{S}^n}}^{(\zeta)} dv_{g_{\mathbb{S}^n}}^{(\omega)}
\end{aligned}$$

Therefore, the proposition follows immediately from the above equality and the above proposition. $\square$

Now, we consider as in [29] the set

$$X_G = \{u \in H^1(\mathbb{S}^n);\ u(g\zeta) = u(\zeta), \forall g \in G \text{ and a.e } \zeta \in \mathbb{S}^n\}, \tag{5.13}$$

where $G = O(k) \times O(m)$ is the compact subgroup of the compact Lie group $O(n+1)$, $k + m = n + 1$ and $k \geq m \geq 2$. For $g = (g_1, g_2) \in G$, where $g_1 \in O(k)$ and $g_2 \in O(m)$, the action of G on $\mathbb{S}^n$ is defined by $g(\zeta_1, \zeta_2) = (g_1\zeta_1, g_2\zeta_2)$, where $\zeta_1 \in \mathbb{R}^k$ and $\zeta_2 \in \mathbb{R}^m$. With this choice of G and by the continuity of $H^1(\mathbb{S}^n) \hookrightarrow H^s(\mathbb{S}^n)$ we obtain that $X_G^s = (X_G, \|.\|_*)$ is an infinite-dimensional closed subspace of $H^s(\mathbb{S}^n)$.

Lemma 5.3. *For $1 \leq p \leq 2k/(k-2s)$, we have the continuous embeddings $X_G^s \hookrightarrow L^p(\mathbb{S}^n)$. Moreover, the embeddings are compact if $1 \leq p < 2k/(k-2s)$.*

Proof. Notice first that if $v \in X_G^s$, then $v(\zeta) = v(|\zeta_1|, |\zeta_2|)$ i.e. v depends only on $|\zeta_1|$, or equivalently, v depends only on $|\zeta_2|$, since $|\zeta_1|^2 + |\zeta_2|^2 = 1$. Now, for any $\overline{\zeta} = (\overline{\zeta}_1, \overline{\zeta}_2) \in \mathbb{S}^n$, assume first that $\overline{\zeta}_2 \neq 0$. Then $\overline{\zeta}_2^i \neq 0$ for some $1 \leq i \leq m$. Set

$$h^{-1}(\zeta_1, \zeta_2) = (\zeta_1^1, ..., \zeta_1^k, \zeta_2^1, ..., \zeta_2^{i-1}, \zeta_2^{i+1}, ..., \zeta_2^m) \in \mathbb{R}^n.$$

Then there exists a neighborhood U of $\overline{\zeta}$ in $\mathbb{S}^n$ and $\delta > 0$ such that h^{-1} maps U diffeomorphically onto the open set $B^k_\delta(\overline{\zeta}_1) \times B^{m-1}_{6\delta}(\overline{\zeta}'_2)$ in $\mathbb{R}^n$, where

$$\overline{\zeta}'_2 = (\overline{\zeta}^1_2, ..., \overline{\zeta}^{i-1}_2, \overline{\zeta}^{i+1}_2, ..., \overline{\zeta}^m_2) \in \mathbb{R}^{m-1}.$$

Note that in the chart (U, h^{-1}), if $v \in X^s_G$ then v depends only on $|\zeta_1|$ where $\zeta_1 \in B^k_\delta(\overline{\zeta}_1)$. Next, if $\overline{\zeta}_2 = 0$, then $\overline{\zeta}^i_1 \neq 0$ for some $1 \leq i \leq k$. We can likewise take a chart in which $v \in X^s_G$ depends only on $|\zeta_2|$, where $\zeta_2 \in B^m_\delta(\overline{\zeta}_2)$.

(Continuity) We may assume that $\mathbb{S}^n$ is covered by a finite number of such charts, say $(U_\alpha, h^{-1}_\alpha)$, $1 \leq \alpha \leq N$, and that the metric matrices in these charts satisfies

$$c^{-1} I \leq (g^\alpha_{ij}) \leq cI, \ 1 \leq \alpha \leq N,$$

where $c > 0$ is a constant and I is the $n \times n$ identity matrix. Moreover h_α is Lipschitz in $h^{-1}_\alpha(U_\alpha)$ and its Lipschitz constant no depending of α. Since the functions in X^s_G behave locally like functions of k or m independent variables in these charts, we have for $r = k$ or m,

$$\begin{aligned}
&\int_U \int_U \frac{[u(\zeta) - u(\omega)]^2}{|\zeta - \omega|^{n+2s}} d\eta_\zeta d\eta_\omega \\
&= \int_{B^r_\delta \times B^{n-r}_{6\delta}} \int_{B^r_\delta \times B^{n-r}_{6\delta}} \frac{[u(h(\zeta_1, \zeta'_2)) - u(h(\omega_1, \omega'_2))]^2}{|h(\zeta_1, \zeta'_2) - h(\omega_1, \omega'_2)|^{n+2s}} \sqrt{det(g_{ij_\zeta}) det(g_{ij_\omega})} d\zeta_1 d\zeta'_2 d\omega_1 d\omega'_2 \\
&\geq C \int_{B^r_\delta} \int_{B^r_\delta} \int_{B^{n-r}_{6\delta}} \int_{B^{n-r}_{6\delta}} \frac{[u(\zeta_1) - u(\omega_1)]^2}{|(\zeta_1, \zeta'_2) - (\omega_1, \omega'_2)|^{n+2s}} d\zeta'_2 d\omega'_2 d\zeta_1 d\omega_1. \qquad (5.14)
\end{aligned}$$

Making change vaviable we have

$$
\begin{aligned}
&\int_{B_{6\delta}^{n-r}}\int_{B_{6\delta}^{n-r}}\frac{1}{|(\zeta_1,\zeta_2')-(\omega_1,\omega_2')|^{n+2s}}d\zeta_2'd\omega_2'\\
&=\int_{B_{6\delta}^{n-r}(0)}\int_{B_{6\delta}^{n-r}(0)}\frac{1}{[|\zeta_1-\omega_1|^2+|\zeta_2'-\omega_2'|^2]^{\frac{n+2s}{2}}}d\zeta_2'd\omega_2'\\
&=\int_{B_{6\delta}^{n-r}(0)}\int_{B_{6\delta}^{n-r}(\zeta_2')}\frac{1}{[|\zeta_1-\omega_1|^2+|\omega_2'|^2]^{\frac{n+2s}{2}}}d\omega_2'd\zeta_2'\\
&\geq\int_{B_{3\delta}^{n-r}(0)\setminus B_{2\delta}^{n-r}(0)}\int_{B_{6\delta}^{n-r}(\zeta_2')}\frac{1}{[|\zeta_1-\omega_1|^2+|\omega_2'|^2]^{\frac{n+2s}{2}}}d\omega_2'd\zeta_2'\\
&\geq\int_{B_{3\delta}^{n-r}(0)\setminus B_{2\delta}^{n-r}(0)}\int_{B_{\delta}^{n-r}(0)}\frac{1}{[|\zeta_1-\omega_1|^2+|\omega_2'|^2]^{\frac{n+2s}{2}}}d\omega_2'd\zeta_2'\\
&\geq\int_{B_{3\delta}^{n-r}(0)\setminus B_{2\delta}^{n-r}(0)}\int_{B_{\frac{|\zeta_1-\omega_1|}{2}}^{n-r}(0)}\frac{1}{[|\zeta_1-\omega_1|^2+|\omega_2'|^2]^{\frac{n+2s}{2}}}d\omega_2'd\zeta_2'\\
&\geq C\int_{B_{3\delta}^{n-r}(0)\setminus B_{2\delta}^{n-r}(0)}\frac{1}{|\zeta_1-\omega_1|^{n+2s}}\int_{B_{\frac{|\zeta_1-\omega_1|}{2}}^{n-r}(0)}d\omega_2'd\zeta_2'\\
&\geq C\int_{B_{3\delta}^{n-r}(0)\setminus B_{2\delta}^{n-r}(0)}\frac{1}{|\zeta_1-\omega_1|^{r+2s}}d\zeta_2\\
&\geq C\frac{1}{|\zeta_1-\omega_1|^{r+2s}},
\end{aligned}
\tag{5.15}
$$

where $C=C(\delta,n,r)>0$. From (5.14), (5.15) we obtain

$$\int_U\int_U\frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}}d\eta_\zeta d\eta_\omega\geq C\int_{B_\delta^r}\int_{B_\delta^r}\frac{[u(\zeta_1)-u(\omega_1)]^2}{|\zeta_1-\omega_1|^{r+2s}}d\zeta_1d\omega_1.$$

Hence, there exists a constant $C_1=C_1(\alpha,\delta,n,r)>0$ such that

$$\int_{U_\alpha}\int_{U_\alpha}\frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}}d\eta_\zeta d\eta_\omega\geq C_1\int_{B_{\alpha,\delta}^r}\int_{B_{\alpha,\delta}^r}\frac{[u(\zeta_1)-u(\omega_1)]^2}{|\zeta_1-\omega_1|^{r+2s}}d\zeta_1d\omega_1.\tag{5.16}$$

Similarly, we can prove that

$$\int_{U_\alpha}|u|^pd\eta\leq C_2\int_{B_{\alpha,\delta}^r}|u|^pd\zeta_1,\tag{5.17}$$

for some $C_2=C_2(\alpha,\delta,n,r)>0$ and all $v\in X_G^s$. Combining (5.16), (5.17) and the Sobolev inequality on $H^s(B^r_{\alpha,\delta})$ yields that

$$\int_{U_\alpha} |u|^p d\eta \leq C_3 \int_{U_\alpha}\int_{U_\alpha} \frac{[u(\zeta)-u(\omega)]^2}{|\zeta-\omega|^{n+2s}} d\eta_\zeta d\eta_\omega,\ 1\leq p\leq \frac{2r}{r-2s}, \tag{5.18}$$

for some $C_3 = C_3(\alpha,\delta,n,r) > 0$ and for all $u \in X^s_G$. The global inequality now follows easily:

$$\|u\|_{L^p(\mathbb{S}^n)} \leq C\|u\|_{H^s(\mathbb{S}^n)},\ 1\leq p\leq \frac{2k}{k-2s},$$

for all $u \in X^s_G$. This proves that $X^s_G \hookrightarrow L^p(\mathbb{S}^n)$ is continuous for $1 \leq p \leq 2k/(k-2s)$.

(Compactnes) Let $\{u_l\}_{l\in\mathbb{N}} \subset X^s_G$ be a limited sequence in the norm $\|\cdot\|_{H^s(\mathbb{S}^n)}$. We will show that unless subsequence $\{u_l\}$ converges in $L^p(\mathbb{S}^n)$, $1 \leq p < 2k/(k-2s)$. By (5.16) with $r = k$ or m, there is a $M > 0$ such that

$$\|u_l\|_{H^s(B^r_{\alpha,\delta})} \leq M,\ \text{for all } l,\ 1\leq\alpha\leq N.$$

From (5.17) and since $H^s(B^r_{\alpha,\delta}) \hookrightarrow L^p(B^r_{\alpha,\delta})$ is compact for $1 \leq p < 2r/(r-2s)$, then unless subsequence $\{u_l\}$ is a Cauchy sequence in U_α, $1 \leq \alpha \leq N$. Thus for each $\alpha = 1, ..., N$ there exists $u_\alpha \in L^p(U_\alpha)$ such that $v_l \to u_\alpha$ in $L^p(U_\alpha)$ as $l \to +\infty$. Define $u \in L^p(\mathbb{S}^n)$ as

$$u(\zeta) := u_\alpha(\zeta),\ \zeta\in U_\alpha.$$

Then

$$\int_{\mathbb{S}^n} |u-u_l|^p d\eta \leq \sum_{\alpha=1}^N \int_{U_\alpha} |u_\alpha - u_l|^p d\eta \to 0,\ \text{as } l\to+\infty.$$

This proves that $X^s_G \hookrightarrow L^p(\mathbb{S}^n)$ is compact for $1 \leq p < 2k/(k-2s)$.

□

Proof of Theorem 5.1. Since $2k/(k-2s) > 2^*_s$ and by Lemma 5.3, the embedding $X^s_G \hookrightarrow L^{2^*_s}$ is compact. Therefore, we may apply Lemma 5.1 to complete the proof of Theorem 5.1. □

Conclusion

The fractional conformal operator $\mathcal{A}_{2s}$ ($s \in (0,1) \cup \mathbb{N}$, defined by (1.1)) is related to the fractional Laplace operator through the following identity

$$\mathcal{A}_{2s} u \circ \mathcal{F} = \xi_s^{-\frac{n+2s}{n-2s}} (-\Delta)^s (\xi_s (u \circ \mathcal{F})), \; u \in C^\infty(\mathbb{S}^n),$$

where $\mathcal{F}$ is the stereographic projection and ξ_s is the conformal factor. Thus, the study of some problems on $\mathbb{S}^n$ involving these operators $\mathcal{A}_{2s}$ is equivalent to the study of problems on $\mathbb{R}^n$ involving the fractional Laplace operator $(-\Delta)^s$.

A class of problems that we studied was the nonexistence of nonconstant positive solutions of the problem

$$\begin{cases} \mathcal{A}_{2s} u = f(u) \quad \text{in } \mathbb{S}^n, \\ \quad u \in H^s(\mathbb{S}^n), \end{cases} \tag{5.19}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Using method of moving plane, it was concluded that if $h_s(t) := t^{-\frac{n+2s}{n-2s}} f(t)$ is decreasing on $(0, +\infty)$, then it was shown that any positive solution of (5.19) is constant. In addition, the result was generalized for systems.

A natural question that arose was about the existence of nonconstant solutions of (5.19), which implies that the solution changes sign. For $0 < s < 1$, it was shown that, under certain conditions on f, the problem (5.19) has a unbounded sequence of sing-changing solutions (Theorem 5.1 and Corollary 5.2).

An interesting work would be to study these conformal operators in the Hyperbolic space $\mathbb{H}^n$ or on the Cylinder $\mathbb{R} \times \mathbb{S}^n$ as well as the existence and nonexistence of solutions for problems involving these operators.

Also, using method of moving plane, we studied the nonexistence of positive solutions of

$$\begin{cases} -\Delta_{g_{\mathbb{S}^n_+}} u = f(u) & \text{in } \mathbb{S}^n_+; \\ \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathbb{S}^n_+. \end{cases} \tag{5.20}$$

The main result was that any positive smooth solution of (5.20) is constant provided that $h(t) := t^{\frac{n+2}{n-2}}(f(t) + n(n-2)/4t)$ is decreasing. As a particular case it was that Ni-Li's conjecture is true in the case of the hemisphere, i.e., for $f(t) = t^{-\frac{n+2}{n-2}} - \lambda t$ and $0 < \lambda$ small, the only positive solution of (5.20) is $u \equiv \lambda^{\frac{n-2}{4}}$. Moreover, this result was also generalized for systems.

An interesting line of research is to know and study what class of fractional conformal operators are defined in the hemisphere since we are not able to define these operators by the method of Chang and Gonzales [19]. Likewise, it is possible to study some problems in the space $\mathbb{R}^n_+$ through the study of problems in the hemisphere $\mathbb{S}^n_+$, see e.g., [29, 32].

Appendix

In this chapter, we gather some elementary results used in this thesis, see [14, 13, 28].

Conformal Laplacian in coordinates

Let u be a smooth function defined on Riemannian manifold (M^n, g), where g is a metric. Then, the Laplace-Beltrami operator in coordinates is given by

$$\Delta_g u(q) = \sum_{i,j=1}^{n} (g^{ji}(q)\partial_i\partial_j u(q) - g^{ij}(q)\sum_{k=1}^{n}\Gamma_{ij}^k(q)\partial_k u(q)),\ q \in M^n, \tag{5.21}$$

where $\psi : U \subset \mathbb{R}^n \to M^n$ is a parameterization, U is neighborhood of $y = (y^1, y^2, ..., y^n)$, $\psi(y) = q$, $\partial_i u(q) = \frac{\partial}{\partial y^i}(u \circ \psi)(y)$, $g = (g_{ij})$, $g^{-1} = (g^{ij})$ and Γ_{ij}^k are Christoffel symbols,

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{n}(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij})g^{km}, \tag{5.22}$$

for $i, j, k = 1, ..., n$.

Let $(\mathbb{S}^n, g)$ be the unit sphere provided with the standard metric g, $n > 2$. Let p be an point on $\mathbb{S}^n$, which we will rename the south pole S. Let $\mathcal{F} : \mathbb{R}^n \to \mathbb{S}^n \backslash \{S\}$ be defined by the stereographic projection. Then

$$\mathcal{F}(y) = \left(\frac{4y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2}\right),\ y \in \mathbb{R}^n,$$

and

$$g_{ij}(q) = \frac{4}{(1+|y|^2)^2}\delta_i^j, \quad \text{and} \quad g^{ij} = \frac{(1+|y|^2)^2}{4}\delta_i^j, \quad y \in \mathbb{R}^n,\ q = \mathcal{F}(y), \tag{5.23}$$

where g is called standard metric and δ_i^j is the Kronecker delta.

Theorem A 5.1. *If u is a smooth function in $(\mathbb{S}^n, g)$, $n > 2$, where g is the standard metric, then*

$$\Delta_g u(p) = \frac{(1+|y|^2)^2}{4}\Delta(u\circ\mathcal{F})(y) - (n-2)\frac{1+|y|^2}{2}y.\nabla(u\circ\mathcal{F})(y),\ y\in\mathbb{R}^n, \tag{5.24}$$

where Δ and ∇ are the Laplacian and gradient operators on $\mathbb{R}^n$.

Proof. From (5.23) we have

$$\partial_m g_{ij} = \frac{\partial}{\partial y^m}\left(\frac{4}{(1+|y|^2)^2}\right)\delta_i^j = -2\left(\frac{2}{1+|y|^2}\right)^3 y^m\delta_i^j, \quad \text{for } i,j,m = 1,...,n; \tag{5.25}$$

and from (5.22) and (5.25)

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})g^{kk}\\ &= \left(\frac{2}{1+|y|^2}\right)(-y^i\delta_j^k - y^j\delta_i^k + y^k\delta_i^j)\end{aligned} \tag{5.26}$$

Therefore, by (5.21), (5.23) and (5.26), we have

$$\begin{aligned}\Delta_g u(q) &= \sum_{i=1}^n (g^{ii}\partial_i\partial_i u(q) - g^{ii}\sum_{k=1}^n \Gamma_{ii}^k\partial_k u(q))\\ &= \left(\frac{1+|y|^2}{2}\right)^2\Delta u(\mathcal{F}(y)) - \left(\frac{1+|y|^2}{2}\right)^2\sum_{i,k=1}^n\Gamma_{ii}^k\partial_k u(q))\\ &= \left(\frac{1+|y|^2}{2}\right)^2\Delta u(\mathcal{F}(y)) - \left(\frac{1+|y|^2}{2}\right)(n-2)y.\nabla u(\mathcal{F}(y)).\end{aligned}$$

□

Let us denote from here $\partial_i = \frac{\partial}{\partial y^i}$. Let

$$\xi(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad y\in\mathbb{R}^n.$$

Then, for $i, j = 1, ..., n$,

$$\partial_i \xi(y) = -\frac{n-2}{2} \left(\frac{2}{1+|y|^2} \right)^{\frac{n}{2}} y^i, \tag{5.27}$$

$$\partial_i \partial_j \xi(y) = -\frac{n-2}{2} \left(\frac{2}{1+|y|^2} \right)^{\frac{n}{2}} \left\{ -\frac{n y^i y^j}{1+|y|^2} + \delta_i^j \right\}, \tag{5.28}$$

$$\Delta \xi(y) = -\frac{n(n-2)}{4} \left(\frac{2}{1+|y|^2} \right)^{\frac{n+2}{2}} = -\frac{n(n-2)}{4} \xi(y)^{\frac{n+2}{n-2}}. \tag{5.29}$$

Theorem A 5.2. *Let u be a smooth function in $(\mathbb{S}^n, g)$, $n > 2$, where g is the standar metric. Denote*

$$v(y) = \xi(y) u(\mathcal{F}(y)), \ y \in \mathbb{R}^n.$$

Then

$$-\Delta v(y) = \xi(y)^{\frac{n+2}{n-2}} \left(-\Delta_g u(q) + \frac{n(n-2)}{4} u(q) \right), \ q = \mathcal{F}(y), \ y \in \mathbb{R}^n. \tag{5.30}$$

Proof. From Theorem A5.1, (5.27)-(5.29), we have

$$\begin{aligned}
-\Delta v(y) &= -\xi(y) \Delta u(\mathcal{F}(y)) - 2 \nabla \xi(y) . \nabla u(\mathcal{F}(y)) - \Delta \xi(y) u(\mathcal{F}(y)) \\
&= -\xi(y) \left((\frac{2}{1+|y|^2})^2 \Delta_g u(q) + (n-2) \frac{2}{1+|y|^2} y . \nabla (u(\mathcal{F}(y)) \right) \\
&\quad - 2 \left(-\frac{n-2}{2} \left(\frac{2}{1+|y|^2} \right)^{\frac{n}{2}} y . \nabla u(\mathcal{F}(y)) \right) + \frac{n(n-2)}{4} \xi(y)^{\frac{n+2}{n-2}} u(\mathcal{F}(y)) \\
&= -\xi(y)^{\frac{n+2}{n-2}} \Delta u(q) + \frac{n(n-2)}{4} \xi(y)^{\frac{n+2}{n-2}} u(q).
\end{aligned}$$

□

Similarly, we have:

Theorem A 5.3. *Let $y, \eta \in \mathbb{R}^n$ and $\nu := d\mathcal{F}_y(\eta)$. If $u \in C^1(\mathbb{S}^n)$, then*

$$\frac{\partial u}{\partial \nu}(p) = \frac{(1+|y|^2)^2}{4} \left((u \circ \mathcal{F})(y) \frac{\partial \xi}{\partial \eta}(y) + \xi(y) \frac{\partial (u \circ \mathcal{F})}{\partial \eta}(y) \right), \quad p = \mathcal{F}(y),$$

where $\frac{\partial}{\partial \nu}$ means the covariant derivative in the direction of ν.

Fractional Laplace Operator

In this section we will briefly comment on some results involving the fractional Laplacian operator. The fractional Laplace operator $(-\Delta)^s$ on $\mathbb{R}^n$, $s > 1$ is defined using the Fourier transform by

$$\widehat{(-\Delta)^s} v(y) := |y|^{2s}\widehat{v}(y), \text{ for all } v \in C_c^\infty(\mathbb{R}^n),$$

where

$$\widehat{v}(y) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi i x.y} v(x)\ dx.$$

From [28] we have the following result.

Theorem A 5.4. *Let* $0 < s < 1$*. Up to a positive constant we have*

$$\int_{\mathbb{R}} v(-\Delta)^s v\ dy = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dy. \tag{5.31}$$

$$(-\Delta)^s u(x) = P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy,$$

where P.V. means $\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}$.

By the Sobolev inequality, we have that

$$\|v\|_s := \int_{\mathbb{R}^n} v(-\Delta)^s v\ dy$$

is a norm. Let $D^{s,2}(\mathbb{R}^n)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ in the norm $\|\cdot\|_s$. The set $D^{s,2}(\mathbb{R}^n)$ is called *the Sobolev space*. Then we have a characterization for the Sobolev space.

Theorem A 5.5. *For* $s \in (0,1) \cup \mathbb{N}$*, we have*

$$D^{s,2}(\mathbb{R}^n) = \{v \in L^{\frac{2n}{n-2s}}, \|v\|_s < +\infty\}. \tag{5.32}$$

Proof. Denotes by H the right side of (5.32). For $s \in \mathbb{N}$, see [13, Section 11.8]. Let $0 < s < 1$. Select $\rho \in C_c^1(\mathbb{R}^n)$ so that

$$\begin{cases} 0 \le \rho \le 1, & \rho \equiv 1 \text{ on } B(0,1), \\ spt(\rho) \subset B(0,2), & |\nabla\rho| \le 2. \end{cases}$$

For each $k = 1, 2, ...,$ set $\rho_k(y) = \rho(x/k)$. Given $v \in H$, we write $v_k = \rho_k v$. Then

$v_k \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2n/(n-2s)$ and $(-\Delta)^{s/2}(v_k) \in L^2(\mathbb{R}^n)$, and

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)-v_k(x)+v_k(y)|^2}{|x-y|^{n+2s}}dxdy \\
&\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\rho_k(x)|^2|v(x)-v(y)|^2}{|x-y|^{n+2s}}dxdy + C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)|^2|\rho_k(x)-\rho_k(y)|^2}{|x-y|^{n+2s}}dxdy \\
&\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n-B(0,k)} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}dxdy + \frac{C}{k^2}\int_{\mathbb{R}^n}|v(x)|^2\int_{B(0,2k)} \frac{|\nabla\rho(\frac{x}{k})|^2}{|z|^{n+2s-2}}dydx \\
&\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n-B(0,k)} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}dxdy + \frac{C}{k^2}\int_{B(0,2k)-B(0,k)}|v(x)|^2dx \\
&\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n-B(0,k)} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}dxdy + C\int_{\mathbb{R}^n-B(0,k)}|v(x)|^2dx \to 0
\end{aligned}
$$

Then, $\|v-v_k\|_* \to 0$ as $k \to +\infty$. From [28, Theorem 2.4] gets that for each k, exists a sequence $\tilde{v}_{k_l}$ in $C_c^\infty(\mathbb{R}^n)$ such that $\|v_k - \tilde{v}_{k_l}\|_* \to 0$ as $l \to +\infty$.

Therefore, unless the subsequence, $\|v-\tilde{v}_{k_l}\|_* \to 0$ as $l \to +\infty$. □

Fountain theorem

In this section we are going to show Lemma 3.1, which is a consequence of the principle of symmetric criticality and the fountain theorem.

Theorem A 5.6. *(Principle of symmetric criticality [59]) Let G be a group of isometries of a Riemannian manifold M and let $J : M \to \mathbb{R}$ be a C^1 function invariant under G. Then the set X of stationary points of M under the action of G is a totally geodesic smooth submanifold M, and if $u \in X$ is a critical point of $J|_X$ then u is in fact a critical point of J.*

The Principle states that in order for a symmetric point u to be a critical point it suffices that it be a critical point of $J|_X$, the restriction of J to X.

Theorem A 5.7. *(Fountain theorem [10]) Let E be a Hilbert space, $(e_j : j \in \mathbb{N})$ an orthonormal sequence, and set $E_k := span(e_1, e_2, ..., e_k)$. Consider a C^1-functional $J : E \to \mathbb{R}$ that satisfies the following hypotheses.*

(g_1) J is even: $J(-v) = J(v)$ for all $v \in E$;

(g_2) $b_k := \sup_{\rho\geq 0}\inf_{v\in E_k^\perp, \|v\|=\rho} J(v) \to +\infty$ as $k \to +\infty$;

(g_3) $\inf_{r>0}\sup_{v\in E_k, \|v\|\geq r} J(v) < 0$ for every $k \in \mathbb{N}$;

(g_4) *The Palais- Smale condition hold above* 0*, i.e., any sequence* $\{v_m\}$ *in* E *which satisfies* $J(v_m) \to c > 0$ *and* $J'(v_k) \to 0$ *contains a convergent subsequence.*

Then J *posseses an unbouded sequence of critical values* c_k*.*

Proof of Lemma 5.1. Let $E := X_G$. By the principle of symmetric criticality [59], it sufficies to find an unbouded sequence of critical values of the restriction $J : E \to \mathbb{R}$.

We claim that $J : E \to \mathbb{R}$ satisfies the assumptions of the Fountain Theorem 3.1. Let $(e_j; j \in \mathbb{N})$ be a Hilbert basis of E and set $E_k = span(e_1, ..., e_k)$. Clearly, J satisfies (g_1) because of (h_1). For the proof of (g_2) we defined

$$\mu_k := \sup_{v \in E_{k-1}^{\perp}, v \neq 0} \frac{\|v\|_{L^p(\mathbb{S}^n)}}{\|v\|_*}.$$

From [59, Lemma 3.3] we have $\mu_k \to 0$ as $k \to +\infty$. Using (h_2), there is a constant $C > 0$ such that

$$|F(t)| \leq C(1 + |t|^p), \quad \text{for all } t \in \mathbb{R}.$$

Thus, for $v \in E_{k-1}^{\perp}$ we have

$$\begin{aligned} J(v) &\geq \frac{1}{2}\|v\|_*^2 - C\|v\|_{L^p(\mathbb{S}^n)}^p - C|\mathbb{S}^n| \\ &\geq \frac{1}{2}\|v\|_*^2 - C\mu_k^p\|v\|_*^p - C|\mathbb{S}^n|, \end{aligned}$$

where $|\mathbb{S}^n|$ is the measure of $\mathbb{S}^n$. Set $r_k := (C\mu_k^p p)^{1/(2-p)}$. Then

$$J(v) \geq \left(\frac{1}{2} - \frac{1}{p}\right)(C\mu_k^p p)^{2/(2-p)}$$

for every $v \in E_{k-1}^{\perp}$ with $\|v\|_* = r_k$. Therefore,

$$b_k \geq \inf_{v \in E_{k-1}^{\perp}, \|v\|_* = r_k} J(v) \to +\infty \quad \text{as } k \to +\infty,$$

because $\mu_k \to 0$ and $p > 2$. This prove (g_2).

To show (g_3), from (h_3) gets

$$F(t) \geq a_1 t^\mu - a_2,$$

for some positive constants a_1, a_2. Then

$$J(v) \leq \frac{1}{2}\|v\|_*^2 - a_1\|v\|_{L^\mu(\mathbb{S}^n)}^\mu - a_2|\mathbb{S}^n|.$$

Since E_k is finite dimensional, all norms are equivalent on E_k. Therefore, $\mu > 2$ implies

$$\sup_{v \in E_k, \|v\|_* \geq r} J(v) \to -\infty, \quad \text{as } r \to +\infty.$$

It remains to prove the condition (g_4). Consider a Palais-Smale sequence $\{v_m\}$ in E, so that $J(v_m) \to c$ and $J'(v_m) \to 0$. we will check that $\{v_m\}$ is bounded. For m largue enough, using (h_3) we have

$$\begin{aligned}
1 + c + \|v_m\|_* &\geq J(v_m) - \frac{1}{\mu} J'(v_m) v_m \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|v_m\|_*^2 - \int_{\mathbb{S}^n \cap \{|u_m| \leq R\}} \left(F(v_m) - \frac{1}{\mu} f(v_m) v_m\right) d\eta \\
&- \int_{\mathbb{S}^n \cap \{|u_m| > R\}} \left(F(v_m) - \frac{1}{\mu} f(v_m) v_m\right) d\eta \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|v_m\|_*^2 - \max_{0 \leq t \leq R} \left\{F(t) - \frac{1}{\mu} f(t) t\right\} |\mathbb{S}^n|.
\end{aligned}$$

It follows that $\{v_m\}$ is bounded in E. Unless of subsequences, we have $v_m \rightharpoonup v$ in E for some $v \in E$. By (f_2) we have that $v_m \to v$ in $L^p(\mathbb{S}^n)$ and $v_m \to v$ a.e. in $\mathbb{S}^n$. By (h_2) and for standard arguments,

$$\int_{\mathbb{S}^n} (f(v_m) - f(v))(v_m - v) d\eta \to 0 \quad \text{as } m \to +\infty.$$

Since

$$\|v_m - v\|_*^2 = (J'(v_m) - J'(v))(v_m - v) + \int_{\mathbb{S}^n} (f(v_m) - f(v))(v_m - v) d\eta$$

and

$$(J'(v_m) - J'(v))(v_m - v) \to 0 \quad \text{as } m \to +\infty,$$

we have

$$\|v_m - v\|_* \to 0 \quad \text{as } m \to +\infty.$$

Therefore, we can apply the Fountain Theorem 3.1 as clameid.

□

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