NOVEL STABILITY AND STABILIZATION CONDITIONS FOR TIME-DELAYED LPV SYSTEMS

A Linear Matrix Inequality-based Approach

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Master Thesis
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"Novel Stability and Stabilization Conditions for Time-delayed LPV Systems - A Linear Matrix Inequality-based Approach"

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Dissertação de Mestrado submetida à Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em Engenharia Elétrica da Escola de Engenharia da Universidade Federal de Minas Gerais, como requisito para obtenção do grau de Mestre em Engenharia Elétrica.

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ABSTRACT

This work investigates the problem of stability, state-feedback and static output-feedback control design for linear parameter-varying systems with time-varying delays. The uncertain parameters are assumed to belong to a polytope with bounded known variation rates. The new conditions are based on the Lyapunov theory and are expressed through Linear Matrix Inequalities. An alternative parameter-dependent Lyapunov-Krasovskii functional is employed and its time-derivative is handled using recent integral inequalities for quadratic functions proposed in the literature. As main results, a novel sufficient stability condition for delay-dependent systems as well as new sufficient conditions are stated to design gain-scheduling state-feedback and also gain-scheduling static output-feedback control. In the new proposed methodology, the Lyapunov matrices and the system matrices are put separated, making it suitable for supporting in a new way the design of the stabilization controllers. Some examples, including some based on models of real-world problems, are provided to illustrate the effectiveness of the proposed methods.

Keywords: Linear parameter-varying (LPV) systems, Time-delay systems, Gain-scheduled control, Delay-dependent stability criterion.

RESUMO

Este trabalho apresenta condições suficientes para análise de estabilidade e projeto de controladores de ganho escalonado para sistemas lineares com parâmetro variante no tempo sujeitos a atraso. Os parâmetros incertos são considerados pertencentes a um politopo com taxas de variação conhecidas e limitadas. As novas condições são baseadas na teoria de Lyapunov e são expressas por meio de Desigualdades Matriciais Lineares. Um funcional alternativo de Lyapunov-Krasovskii dependente do parâmetro é utilizado e sua derivada no tempo é tratada através de desigualdades integrais para funções quadráticas recentemente propostas na literatura. Como resultados principais, uma nova condição suficiente para análise de estabilidade de sistemas dependentes do atraso é obtida, bem como novas condições suficientes para projeto de controladores de ganho escalonado por realimentação de estado e também por realimentação estática de saída. Na nova metodologia proposta, as matrizes de Lyapunov e as matrizes do sistema são separadas, tornando-as adequadas para suportar de uma nova forma o projeto dos controladores. Alguns exemplos, incluindo alguns baseados em problemas do mundo real, são fornecidos para ilustrar a eficácia dos métodos propostos.

Palavras-chave: Sistemas lineares com parâmetros variantes no tempo (LPV), Sistemas com atraso, Controle de ganho escalonado, Condições de estabilidade dependentes do atraso.

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NOTATION

$\mathbb{R}^n$	denotes the n-dimensional Euclidean space;
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real matrices;
$\mathbb{S}_+^n$	refers to the set of symmetric positive definite matrices;
$X > (<) \ 0$	indicates that X is a symmetric positive (or negative) definite matrix;
I	identity matrix of appropriate dimension;
0	null matrix of appropriate dimension;
X^T	transpose of matrix X;
$\text{He}(X)$	is denoted by $(X + X^T)$;
$\text{diag}(A, B)$	block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$;
$\text{col}\{a, b\}$	denotes a column vector whose elements are a, b;
$\star$	indicates symmetric block in a symmetric matrix;
Λ_N	unit simplex of dimension N;
$A \otimes B$	indicates the Kronecker product of matrices A and B;
C_n	denotes the Banach space of continuous functions $\phi : [-h_2, 0] \to \mathbb{R}^n$, where h_2 is a real constant scalar;

LIST OF ACRONYMS

IQC Integral Quadratic Constraint

LKF Lyapunov-Krasovskii functional
LMI Linear Matrix Inequality
LPV system Linear Parameter-varying system
LTI Linear Time-Invariant system

SOF Static output-feedback

1

INTRODUCTION

Most dynamical systems in real-world applications present nonlinear and even time-varying behavior. As a matter of fact, some classes of those systems can be represented as linear systems subjected to time-varying parameters, so-called Linear Parameter-varying systems (LPV systems). The dynamics of these systems, which are widely studied in recent years, vary as a function of a scheduling parameter vector, unknown a priori, but measurable in real-time with known bounds. This representation allows the analysis of gain-scheduled control design, where the scheduling parameters that describe the current operating point are used to adjust the gain controller automatically [1]. Likewise, systems with time delays have also received attention in the literature [2–9]. As the delay is usually a source of instability, to investigate stability and also gain-scheduling control for time-varying delayed LPV systems have been a topic of increasing interest.

Over the past few decades, LPV systems have emerged as a useful modeling tool in many applications, such as manufacturing processes [10], fault-tolerant control [11, 12], induction motor [13], power systems [14, 15], and robotic systems [16]. Regarding the stability and stabilization problem, most of the current approaches are based on the Lyapunov theory, which carries the dependence on the time-varying parameters to derive sufficient conditions, usually in terms of Linear Matrix Inequalities (LMIs) [17–19]. One can list different types of approaches that, basically, use the dependency on the parameter [20, 21] or parameter-independent [22] Lyapunov functional. It is known that less conservative results can be obtained by selecting parameter-dependent Lyapunov functional, but the problem is that in the case the parameter is time-varying its time-derivative has to be taken into consideration.

On the other hand, processes that require the transmission of information, measurement of data, or transportation of fluids induce intrinsic time-delays in their dynamics systems. Thus, as the delay is usually a source of poor performance, it should not be neglected when analyzing system stability or synthesizing control laws [3, 5]. In the case of Linear Time-Invariant systems (LTIs) and constant time-delay, some theoretical tools such as direct eigenvalue analysis [23] are well established and allow to obtain efficient conditions to certify stability with relatively low numerical complexity. Nevertheless, for uncertain and LPV systems with time-varying delay, this type of criterion generally fails to assure stability. For time-varying delay, stability conditions can be derived in the frequency domain employing the Integral Quadratic Constraint (IQC) framework [24, 25]. In the time-domain, there exist two main Lyapunov theory based approaches: the first one is based on the so-called Lyapunov-Krasovskii functional (LKF) and the second one is the Razumikhin method of Lyapunov functions [26, 27]. Usually, the Lyapunov-Krasovskii based approaches, which is used in this Thesis, lead to less conservative results than the Razumikhin technique, and it is more frequently used.

For time-delay systems, to develop sufficient stability conditions in terms of LMIs, it is crucial to choose an appropriate LKF and a method to estimate the bound of the derivative of the LKF [28]. Many bound techniques have been developed to reduce the conservativeness of the stability conditions, as the Jensen inequality [3], Wirtinger-based inequality [29], Bessel–Legendre inequality [30], and a more general auxiliary function-based integral inequalities [31, 32]. The analyses of the application of the Moon's et al. inequality [33] merged with the convex analysis was performed in [34].

As time-delay is naturally present in several processes that can be modeled as LPV systems, the study of time-varying delayed LPV systems is a problem that has attracted interest [35, 36]. It is known that in the case that the presence of time-delay is not considered when synthesizing control laws, it can cause deterioration in the performance or instability of the resulting closed-loop system. The state-feedback stabilization of LPV systems with time-varying delays has been performed on [37] using an LKF combined with the Jensen inequality. In [38], stability and stabilization conditions of polytopic LPV systems with parameter-varying time delays have been addressed. Studies have also been done demonstrating that adding useful new terms in the LKF can lead to less conservative results, for example, the addition of triple integrals as seen in [39, 40]. [41] uses a single integral in the LKF and addresses the stability analysis and also the state-feedback controller problem that guarantees the desired $\mathscr{L}_2$ gain performance for the LPV time-delayed systems. In [42, 43], the state-feedback control design technique is employed using double integrals in the LKF.

The control of LPV systems with time-varying delay has been examined in several recent works [44–48]. In [49] the affine quadratic stability condition is stated as well as both state-feedback and dynamic output-feedback controllers are derived. In [50] a synthesis conditions of delay-scheduled state-feedback controllers for LPV systems is stated. The design of reduced order observer for time-delayed LPV systems has been addressed in [51]. Delay-dependent stability analysis and $\mathscr{H}_\infty$ state-feedback control for LPV systems was investigated in [52], where the LMIs have been solved with the aid of interactive algorithms. In [53] is considered the robust dissipative state-feedback control for LPV systems with multiple input delays.

In addition, the bounded variation of the scheduling parameter in LPV systems can be directly treated with robust control as an uncertainty problem [54]. However, gain-scheduled controllers have been designed to avoid conservatism of the robust control approach [55]. Gain-scheduled controllers in most cases take advantage of the parameter dependence of system variable to achieve less conservative results with enhanced performance in comparison to linear time-invariant (LTI) robust controllers.

In [46] a gain-scheduled state-feedback has been designed to achieve finite-time boundedness for the closed-loop LPV system with a parameter-dependent state delay. A transformation based on the maximum value of the delay has been used in [56] to allow the design of a gain-scheduled state-feedback controller. On the other hand, it is required an observer design whenever full state information is not accessible. In [57] is proposed the introduction of an additional decision matrix to decouple the parameter-dependent Lyapunov matrix from the system matrix to design a gain-scheduled state-feedback controller. However, as pointed out by the authors, the resulting condition is non-convex and it is necessary to convert the original problem to a nonlinear minimization problem to obtain the desired controller. The work [58] addresses the gain-scheduled stabilization problem of LPV systems with time-delays and a transformation is adopted to obtain a memory-type controller design.

Most of the conditions previously discussed assume that the states of the systems are available, but it is well known that in practical situations the state vector may not be fully available. Based on that, state-feedback control can not be employed. Thus, a possible solution to deal with the problem of missing measurements to control the system is to use the Static output-feedback (SOF) control (if it exists). In comparison with the dynamic output-feedback control and observer-based control, the SOF is much simpler to be applied in practice [59]. For this reason, several researchers have been committed to exploit the problem of designing SOF controller type. However, despite some literature about this topic, the design of SOF controllers is still a challenging issue in control theory due to its non-convex characterization [60]. Nonetheless, due to recent improvements in LMIs, several works have been addressed to provide SOF LMI-based conditions. Some methods which require the output matrix to be constant, or iterative convex optimization scheme, or additional equality constrains, or specific conditions in the system or in the Lyapunov matrix, have emerged in the literature [60–67]. Nevertheless, the existing sufficient conditions based on SOF formulations may be often too restrictive. Despite having a number of results for SOF control that can be found in the literature, it is worth mentioning that, to the best of the candidate's knowledge, most of the results for LPV systems demand the imposition of some additional procedure to obtain numerical tractable results. Notice also that most of the current results for SOF control for LPV systems are not usually prepared to handle time-delays in an easy way.

1.1 OBJECTIVES

This Master Thesis investigates the problem of stability, state-feedback and static output-feedback stabilization of time-delayed LPV systems with polytopic time-varying parameters. The main objective is to propose sufficient delay-dependent conditions for stability analysis and gain-scheduled stabilization of time-delayed LPV systems via the use of parameter-dependent Lyapunov-Krasovskii functionals.

1.2 CONTRIBUTIONS

Motivated by the previously discussion, this Thesis provides novel stability and stabilization conditions for state time-delayed LPV systems. The main contributions of this work are summarized as follows:

- a novel sufficient stability result for LPV time-delayed systems is stated. The selected parameter-dependent Lyapunov-Krasovskii functional candidate encompasses triple integral and, based on bound techniques as well a reciprocally convex method combined with Moon's inequality, new LMI conditions are derived;

- a novel gain-scheduled state-feedback stabilization result for LPV time-delayed systems is also derived. It is worth mentioning there is no need to establish a particular structure to the Lyapunov matrices;
- a novel gain-scheduled static output-feedback stabilization condition for LPV time-delayed systems is also presented. As in the state-feedback case, there is no need to establish a particular structure to the Lyapunov matrices;
- as a subproduct, the selection of a proper augmented vector in a new fashion contributes to decouple, via Finsler's Lemma [68], the Lyapunov-Krasovskii matrices from key matrices of the system aiming to less conservative results. This kind of strategy facilitates to separate important matrices variables used to obtain the state-feedback and static output-feedback control gains. This makes easier to obtain the new LMI conditions for stabilization.

1.3 OUTLINE

The remainder of this thesis is structured as follows:

- Chapter 2: The concepts on the stability of time-delay systems and its main advances in recent years is reviewed. The modeling representation of time-delayed LPV systems with polytopic time-varying parameters used in this Master Thesis is presented.
- Chapter 3: The stability analysis of LPV time-delayed systems is investigated and a novel sufficient LMI condition is derived. Examples are used to illustrate the performance of the proposed method.
- Chapter 4: A novel gain-scheduling state-feedback control condition for LPV time-delayed systems is obtained based on the novel condition presented in Chapter 3. An example is used to illustrate the performance of the proposed method.
- Chapter 5: A novel gain-scheduled static output-feedback control for LPV time-delayed systems is derived based on the novel condition presented in Chapter 3. An example is used to illustrate the performance of the proposed method.
- Chapter 6: The conclusions and possible future directions are presented.

2

BACKGROUND

In this section, the fundamental theoretical background used in this Master Thesis is presented. Firstly, it is introduced concepts on the stability of time-delay systems and its main advances in recent years. Finally, it is presented the modeling representation of LPV systems subject to time-delay employed in this Thesis.

2.1 STABILITY OF LINEAR TIME-DELAY SYSTEMS

To introduce essential concepts about the stability of time-delay systems, consider a linear time-delay system as follows:

$$\begin{aligned} \dot{x}(t) = & \; f(t, x_t), \quad \forall t \geq t_0, \\ x(t) = & \; \phi(t), \quad \forall t \in [-h_2, 0], \end{aligned} \tag{2.1}$$

where $f : \mathbb{R} \times C_n \to \mathbb{R}^n$ is continuous and is Lipschitzian in x_t, $f(t, 0) = 0$, $x_t = x(t - h(t))$, and $\phi(t)$ is the initial condition. The time-varying delay $h(t)$ is continuous and satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \quad h_{12} \triangleq h_2 - h_1, \tag{2.2}$$

where h_1 and h_2 are, respectively, the lower and upper bounds.

In this Master Thesis, the technique used to obtain stability conditions for time-delay systems is the following Lyapunov-Krasovskii method.

Theorem 2.1: [69, 70] Lyapunov-Krasovskii Stability Theorem

The system (2.1) is asymptotically stable if there exists a continuous differentiable functional $V(t, \phi)$ such that

$$\begin{aligned} & u\left(\|\phi(0)\|\right) \leq V(t, \phi) \leq v\left(\|\phi\|_c\right), \\ & \dot{V}(t, \phi) \leq -w\left(\|\phi(0)\|\right), \end{aligned}$$

where u, v and w are continuous strictly increasing functions satisfying $u(0) = v(0) = w(0) = 0$ and $\lim_{s \to \infty} u(s) = +\infty$. The function $\| \cdot \|_c$ is defined as

$$\|\phi\|_c = \max_{-h_2 \leq h(t) \leq 0} \|\phi(h(t))\|.$$

Hence, for the sake of simplicity, along this work the Lyapunov-Krasovskii functional $V(t, \phi)$ is denoted by $V(t)$.

The system (2.1) is called a time-delay system because the future evolution of this system depends not only on its present state but also on its past history [71].

Stability conditions for time-delay systems can be classified into two categories. One is delay-independent stability conditions and the other is delay-dependent stability conditions. Generally, delay-dependent stability conditions are less conservative than delay-independent ones mainly when the time-delay is small. Thus, this Master Thesis deals with delay-dependent case.

Definition 2.1: Categories of stability conditions

- Delay-independent stability: when a system is stable for all $h(t) \geq 0$.
- Delay-dependent stability: when a system is stable for finite intervals of the delay value.

2.1.1 *Integral Inequalities*

To develop a stability condition for a system as in (2.1) three issues are crucial. One is the choice of an appropriate LKF, the second is the calculation of the time-derivative of this LKF candidate, and the latter is to estimate a bound for the derivative of the LKF.

Among the LKFs used to delay-dependent stability analysis of time-delayed systems, one of the most relevant terms, which was introduced in [72] is a double integral quadratic term given by

$$V(t) = h_{12} \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta,$$

where $R \in \mathbb{S}_n^+$. This class of LKF terms has been widely used in the literature mainly because it leads to conditions which depend on the explicit value of the delay h_1 and h_2. In fact when time-differentiating this term yields

$$\dot{V}(t) = h_{12}^2 \dot{x}^T(t) R \dot{x}(t) - h_{12} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) ds. \tag{2.3}$$

Notice that the integral term in (2.3) can not be expressed in terms of an LMI in the form it is. To transform (2.3) into a suitable LMI setup, the integral term should be expressed through bound techniques. Hence, to obtain a more accurate bound for this integral term, and thus, to reduce the conservatism of the resulting stability conditions, different bounding techniques have been employed in the literature.

In Lemma 2.1 is described a bound technique used to analyze the stability of time-delay systems based on the Jensen inequality [3].

Lemma 2.1: [3] Jensen's inequality

For any matrix $R \in \mathbb{S}_n^+$ and any differentiable function $\{x(u) | u \in [a, b]\}$, the following inequality holds:

$$\int_a^b \dot{x}^T(\delta) R \dot{x}(\delta) d\delta \geq \frac{1}{b-a} \Omega_1^T R \Omega_1,$$

where

$$\Omega_1 = x(b) - x(a).$$

Proof. The proof is omitted and can be found in [3]. □

The conservativeness of the Jensen inequality has been analyzed in [73] using the Gruss inequality. In order to obtain less conservative results, the following Lemma 2.2 proposed in [29] provides an inequality called Wirtinger-based inequality, which encompasses Jensen inequality as a particular case.

Lemma 2.2: [29] Wirtinger-based inequality

For any matrix $R \in \mathbb{S}_n^+$ and any differentiable function $\{x(u) | u \in [a, b]\}$, the following inequality holds:

$$\int_a^b \dot{x}^T(\delta) R \dot{x}(\delta) d\delta \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2,$$

where

$$\Omega_1 = x(b) - x(a), \quad \Omega_2 = x(b) + x(a) - \frac{2}{b-a} \int_a^b x(u) du.$$

Proof. The proof is omitted and can be found in [29]. □

In addition to the LKF functional containing double integral, it is shown by simulation results [28] that employing Lyapunov functional containing triple integral terms is quite effective to reduce the conservatism of the stability conditions. Consider the following LKF term

$$V(t) = \int_{-h_2}^{-h_1} \int_{\gamma}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta d\gamma.$$

The time-derivative of the previous functional is given by

$$\begin{aligned}\dot{V}(t) =& \frac{h_{12}^2}{2}\dot{x}^T(t)R\dot{x}(t) - \int_{-h(t)}^{-h_1}\int_{t+\theta}^{t-h_1} \dot{x}^T(s)R\dot{x}(s)dsd\theta \\ &- \int_{-h_2}^{-h(t)}\int_{t+\theta}^{t-h(t)} \dot{x}^T(s)R\dot{x}(s)dsd\theta \\ &- (h_2 - h(t))\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)R\dot{x}(s)ds.\end{aligned} \tag{2.4}$$

Although the Wirtinger inequality provides less conservative results than Jensen's inequality, Wirtinger only deal with single integral terms of quadratic functions while upper bounds of double integral terms should also be estimated if triple integral terms are introduced in the Lyapunov–Krasovskii functional to reduce the conservatism. Recently, an integral inequality called Bessel–Legendre (B–L) inequality has been developed in [30] which encompasses the Jensen inequality and the Wirtinger-based integral inequality. However, this inequality only deals with single integral terms of quadratic functions as well.

A more general integral inequality for quadratic functions, described in Lemma 2.3, which encompasses Jensen inequality, Wirtinger-based inequality and Bessel-Legendre has been developed in [31] and can be applied to deal with the double integral terms of Equation (2.4).

Lemma 2.3: [31] Auxiliary function-based integral inequalities

For a matrix $R \in \mathbb{S}_n^+$ and a differentiable function $\left\{x(u)|u \in [a,b]\right\}$, the following inequalities holds:

$$\begin{aligned}&\int_a^b \dot{x}^T(\delta)R\dot{x}(\delta)d\delta \geq \frac{1}{b-a}\Omega_1^T R\Omega_1 + \frac{3}{b-a}\Omega_2^T R\Omega_2 + \frac{5}{b-a}\Omega_3^T R\Omega_3, \\ &\int_a^b\int_\beta^b \dot{x}^T(\delta)R\dot{x}(\delta)d\delta d\beta \geq 2\Omega_4^T R\Omega_5 + 4\Omega_5^T R\Omega_5, \\ &\int_a^b\int_a^\beta \dot{x}^T(\delta)R\dot{x}(\delta)d\delta d\beta \geq 2\Omega_6^T R\Omega_6 + 4\Omega_7^T R\Omega_7,\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 =& x(b) - x(a), \\
\Omega_2 =& x(b) + x(a) - \frac{2}{b-a}\int_a^b x(\delta)d\delta, \\
\Omega_3 =& x(b) - x(a) + \frac{6}{b-a}\int_a^b x(\delta)d\delta - \frac{12}{(b-a)^2}\int_a^b\int_\beta^b x(\delta)d\delta d\beta, \\
\Omega_4 =& x(b) - \frac{1}{b-a}\int_a^b x(\delta)d\delta, \\
\Omega_5 =& x(b) + \frac{2}{b-a}\int_a^b x(\delta)d\delta - \frac{6}{(b-a)^2}\int_a^b\int_\beta^b x(\delta)d\delta d\beta, \\
\Omega_6 =& x(a) - \frac{1}{b-a}\int_a^b x(\delta)d\delta, \\
\Omega_7 =& x(a) - \frac{4}{b-a}\int_a^b x(\delta)d\delta + \frac{6}{(b-a)^2}\int_a^b\int_\beta^b x(\delta)d\delta d\beta.
\end{aligned}$$

Proof. The proof is omitted and can be found in [31]. □

2.1.2 *Convex approaches*

When employing the Lemmas discussed in Section 2.1.1, a matrix $\begin{bmatrix} \frac{1}{\lambda}R & 0 \\ \star & \frac{1}{1-\lambda}R \end{bmatrix}$ appears and it has to be bounded to derive a stability condition due to the time-varying scalar $\lambda \in (0,1)$. Based on the Jensen inequality, a reciprocally convex combination approach was introduced by [74] for deriving delay-dependent stability conditions.

Lemma 2.4: [74] RCCL - Reciprocally convex combination lemma inequality

For any matrices $R \in \mathbb{S}_n^+$ and $X \in \mathbb{R}^{n\times n}$ the following inequality holds

$$\begin{bmatrix} \frac{1}{\lambda}R & 0 \\ 0 & \frac{1}{1-\lambda}R \end{bmatrix} \geq \begin{bmatrix} R & X \\ \star & R \end{bmatrix}, \quad \forall \lambda \in (0,1),$$

subject to

$$\begin{bmatrix} R & X \\ \star & R \end{bmatrix} > 0.$$

Proof. The proof is omitted and can be found in [74]. □

Notice that after the convex analysis, a stability condition can be expressed in terms of an LMI. In order to obtain less conservatives results, the Moon's et al. inequality [33] merged with the convex analysis was performed in [34] and can be seen in Lemma 2.5.

Lemma 2.5: [34] Reciprocally convex method combined with Moon's et al. inequality

For any matrices $R_1 \in \mathbb{S}_n^+, R_2 \in \mathbb{S}_n^+, Y_1 \in \mathbb{R}^{2n \times n}$ and $Y_2 \in \mathbb{R}^{2n \times n}$ the following inequality holds

$$\begin{bmatrix} \frac{1}{\lambda}R_1 & 0 \\ 0 & \frac{1}{1-\lambda}R_2 \end{bmatrix} \geq \Theta_M(\lambda), \quad \forall \lambda \in (0,1),$$

where

$$\Theta_M(\lambda) = \text{He}(Y_1[I_n \ 0_{n\times n}] + Y_2[0_{n\times n} \ I_n]) - \lambda Y_1 R_1^{-1} Y_1^T - (1-\lambda) Y_2 R_2^{-1} Y_2^T.$$

Proof. The proof is omitted and can be found in [34]. □

Indeed, in [34] it has been proved that RCCL is a particular case of the reciprocally convex method combined with the Moon's et al. inequality. Thus, it is expected that applying Lemma 2.5 provides a tighter lower bound than applying Lemma 2.4.

2.2 POLYTOPIC LPV TIME-DELAYED SYSTEMS MODELING

The polytopic modeling representation of Linear Parameter-varying systems (LPV systems) has been broadly used for robust analysis and robust control, and will be used in this Master Thesis. Consider the following LPV system with time-varying delay described by:

$$\begin{aligned} \dot{x}(t) &= A(\varrho(t))x(t) + A_d(\varrho(t))x(t-h(t)) + B(\varrho(t))u(t), \quad \forall t \geq 0, \\ x(t) &= \phi(t), \quad \forall t \in [-h_2, 0], \end{aligned} \tag{2.5}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and $\phi(t)$ is the initial condition. The time-varying delay $h(t)$ is continuous and satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \quad h_{12} \triangleq h_2 - h_1, \tag{2.6}$$

where h_1 and h_2 are, respectively, the lower and upper bounds. Furthermore, the system matrices $A(\varrho(t)) \in \mathbb{R}^{n\times n}$, $A_d(\varrho(t)) \in \mathbb{R}^{n\times n}$, and $B(\varrho(t)) \in \mathbb{R}^{n\times m}$ belong to a polytopic domain parameterized in terms of a time-varying parameter $\varrho(t)$, being defined by:

$$X(\varrho(t)) = \sum_{i=1}^{N} \varrho_i(t) X_i, \quad \varrho(t) \in \Lambda_N, \tag{2.7}$$

where $X(\varrho(t))$ represents any matrix of the system in (2.5), N denotes the number of vertices of the polytope and Λ_N is the unit simplex given by

$$\Lambda_N = \left\{ \varrho(t) \in \mathbb{R}^N : \sum_{i=1}^{N} \varrho_i(t) = 1, \varrho_i(t) \geq 0,\ i = 1, \ldots, N \right\}. \tag{2.8}$$

Notice that as $X(\varrho(t)) = \sum_{i=1}^{N} \varrho_i(t) X_i$, then taking its time-derivative on both sides yields

$$\frac{\mathrm{d}}{\mathrm{d}t} X(\varrho(t)) = \sum_{i=1}^{N} \dot{\varrho}_i(t) X_i. \tag{2.9}$$

Hence, along this work, the time-derivative of a generic matrix $X(\varrho(t))$ that depends polynomially on $\varrho(t)$ is denoted by $\dot{X}(\varrho(t)) = \sum_{i=1}^{N} \dot{\varrho}_i(t) X_i$ (in shorthand notation: $\dot{X}(\varrho(t))$).

From the constraint $\sum_{i=1}^{N} \varrho_i(t) = 1$ in (2.8), it is easy to notice that:

$$\sum_{i=1}^{N} \dot{\varrho}_i(t) = \dot{\varrho}_1(t) + \ldots + \dot{\varrho}_N(t) = 0. \tag{2.10}$$

In this work, the following assumption is considered:

Assumption 2.1

The time-varying parameter $\varrho(t)$ is assumed to be measured or estimated on-line. Besides that, the variation rate of the time-varying parameter is assumed to be bounded and known.

Based on Assumption 2.1, consider the following known bounds of the parameter variation rates:

$$\underline{b}_i \leq \dot{\varrho}_i(t) \leq \bar{b}_i, \quad \bar{b}_i > \underline{b}_i, \quad 0 \in [\underline{b}_i, \bar{b}_i], \quad i = 1, \ldots, N, \tag{2.11}$$

and notice that (2.10) and (2.11) can be expressed, respectively, in the form $A_e x = b_e$ and $\bar{A} x \leq b$ with:

$$x = \begin{bmatrix} \dot{\varrho}_1(t) \\ \vdots \\ \dot{\varrho}_N(t) \end{bmatrix}, \ A_e = \underbrace{\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}}_{N \text{ times}}, \ b_e = 0, \ \bar{A} = I_N \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ b = \begin{bmatrix} \bar{b}_1 \\ -\underline{b}_1 \\ \vdots \\ \bar{b}_N \\ -\underline{b}_N \end{bmatrix}. \tag{2.12}$$

Then the space where $\dot{\varrho}_i(t)$ assumes values can be determined via the region defined by the intersection between the linear constraints (2.10) and (2.11). As shown in [75], this region is convex and can be represented by the set

$$\mathscr{D} = \left\{ \varphi \in \mathbb{R}^N : \varphi = \sum_{\ell=1}^{M} \beta_\ell h^\ell, \quad \sum_{i=1}^{N} h_i^\ell = 0, \quad \forall \ell = 1, \ldots, M, \quad \beta \in \Lambda_M \right\}, \tag{2.13}$$

being the vectors h^ℓ the vertices of the polytope. As pointed out in [75], the computation of h^l can be efficiently performed by vertex enumeration algorithms [76], having the matrices in (2.12) as inputs.

3 STABILITY ANALYSIS

This chapter investigates the problem of stability for time-delayed Linear Parameter-varying system (LPV system) with polytopic time-varying parameters. The selected parameter-dependent Lyapunov-Krasovskii functional (LKF) candidate encompasses triple integral and, based on bound techniques as well a reciprocally convex method combined with Moon's inequality, a new Linear Matrix Inequality (LMI) condition is derived. In Section 3.1 a sufficient condition is derived. Some examples that illustrate the performance of the proposed method are presented in Section 3.2.

3.1 STABILITY OF TIME-DELAYED LPV SYSTEMS

To analyze the stability problem consider the closed-loop time-delayed LPV system:

$$\dot{x}(t) = A(\varrho(t))x(t) + A_d(\varrho(t))x(t-h(t)), \tag{3.1}$$

with initial condition as in (2.5). The following result provides a new sufficient condition for ensuring the asymptotic stability of the system (3.1).

Theorem 3.1

If there exist matrices $P(\varrho(t)) \in \mathbb{S}_+^{4n}$ defined as in (2.7), Q_1, Q_2, R_1, R_2, Z_1, Z_2, Z_3 and $Z_4 \in \mathbb{S}_+^n$, $M_{\hat{k}} \in \mathbb{R}^{11n\times 3n}$, $\hat{k} = 1, 2$, $W_{\tilde{k}}(\varrho(t)) \in \mathbb{R}^{n\times n}$ defined as in (2.7), $\tilde{k} = 1, 2, 3$, and given scalars $0 \le h_1 \le h_2$, such that the following LMIs hold

$$\begin{bmatrix} \Omega_0(h_2) & h_{12}M_1 \\ \star & -\widetilde{R}_{21} \end{bmatrix} < 0, \quad \begin{bmatrix} \Omega_0(h_1) & h_{12}M_2 \\ \star & -\widetilde{R}_{22} \end{bmatrix} < 0, \tag{3.2}$$

with

$$\begin{aligned} \Omega_0(h) = {} & \Gamma(h)P(\varrho(t))\Upsilon^T + \Upsilon P(\varrho(t))\Gamma^T(h) + \Gamma(h)\dot{P}(\varrho(t))\Gamma^T(h) + \widetilde{Q} \\ & + \Upsilon\left(h_1^2\hat{R}_1 + h_{12}^2\hat{R}_2 + \frac{h_1^2}{2}\hat{Z}_1 + \frac{h_1^2}{2}\hat{Z}_2 + \frac{h_{12}^2}{2}\hat{Z}_3 + \frac{h_{12}^2}{2}\hat{Z}_4\right)\Upsilon^T \\ & - G_0^T\widetilde{R}_1G_0 - G_3^T\widetilde{Z}_1G_3 - G_4^T\widetilde{Z}_2G_4 - G_5^T\widetilde{Z}_3G_5 - G_6^T\widetilde{Z}_3G_6 \\ & - G_7^T\widetilde{Z}_4G_7 - G_8^T\widetilde{Z}_4G_8 + G_1^T\check{Z}_3G_1 + G_2^T\check{Z}_4G_2 \\ & - h_{12}\mathrm{He}(M_1G_1 + M_2G_2) + \mathscr{X}\mathscr{B} + \mathscr{B}^T\mathscr{X}^T, \end{aligned} \tag{3.3}$$

$$\widetilde{Q} = \mathrm{diag}(0, Q_1, Q_2 - Q_1, 0, -Q_2, 0, 0, 0, 0, 0, 0),$$
$$\hat{R}_1 = \mathrm{diag}(R_1, 0, 0, 0), \quad \hat{R}_2 = \mathrm{diag}(R_2, 0, 0, 0), \quad \hat{Z}_1 = \mathrm{diag}(Z_1, 0, 0, 0),$$
$$\hat{Z}_2 = \mathrm{diag}(Z_2, 0, 0, 0), \quad \hat{Z}_3 = \mathrm{diag}(Z_3, 0, 0, 0), \quad \hat{Z}_4 = \mathrm{diag}(Z_4, 0, 0, 0),$$
$$\widetilde{R}_1 = \mathrm{diag}(R_1, 3R_1, 5R_1), \quad \widetilde{R}_2 = \mathrm{diag}(R_2, 3R_2, 5R_2), \quad \widetilde{Z}_1 = \mathrm{diag}(2Z_1, 4Z_1),$$
$$\widetilde{Z}_2 = \mathrm{diag}(2Z_2, 4Z_2), \quad \widetilde{Z}_3 = \mathrm{diag}(2Z_3, 4Z_3), \quad \widetilde{Z}_4 = \mathrm{diag}(2Z_4, 4Z_4),$$
$$\check{Z}_3 = \mathrm{diag}(Z_3, 3Z_3, 5Z_3), \quad \check{Z}_4 = \mathrm{diag}(Z_4, 3Z_4, 5Z_4),$$
$$\widetilde{R}_{21} = \widetilde{R}_2 + \check{Z}_3, \quad \widetilde{R}_{22} = \widetilde{R}_2 + \check{Z}_4,$$

$$\mathscr{X} = \begin{bmatrix} W_1(\varrho(t)) & W_2(\varrho(t)) & 0 & W_3(\varrho(t)) & 0_{n\times 7n} \end{bmatrix}^T,$$
$$\mathscr{B} = \begin{bmatrix} -I & A(\varrho(t)) & 0 & A_d(\varrho(t)) & 0_{n\times 7n} \end{bmatrix},$$
$$\Gamma(h) = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_1 I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (h(t)-h_1)I & (h_2 - h(t))I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 I & 0 & 0 \end{bmatrix}^T,$$
$$\Upsilon = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2I & 0 & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$H_1 = \begin{bmatrix} I & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 2I & 0 & 0 & -3I \end{bmatrix}, \; H_2 = \begin{bmatrix} -I & 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & -4I & 0 & 0 & 3I \end{bmatrix},$$
$$H_3 = \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & -2I & 0 & 0 & 0 \\ I & -I & 0 & 0 & 6I & 0 & 0 & -6I \end{bmatrix}, \; G_0 = \left[\begin{array}{c|c|cc} 0 & & 0 & 0 \\ 0 & H_3 & 0 & 0 \\ 0 & & 0 & 0 \end{array}\right],$$
$$G_1 = \left[\begin{array}{cc|c|c} 0 & 0 & & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & & 0 \end{array}\right], \; G_2 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & 0 & 0 & H_3 \\ 0 & 0 & 0 & \end{array}\right],$$
$$G_3 = \left[\begin{array}{c|c|cc} 0 & H_1 & 0 & 0 \\ 0 & & 0 & 0 \end{array}\right], \; G_4 = \left[\begin{array}{cc|c|cc} 0 & 0 & H_2 & 0 & 0 \\ 0 & 0 & & 0 & 0 \end{array}\right], G_5 = \left[\begin{array}{cc|c|c} 0 & 0 & H_1 & 0 \\ 0 & 0 & & 0 \end{array}\right],$$
$$G_6 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & H_1 \\ 0 & 0 & 0 & \end{array}\right], G_7 = \left[\begin{array}{ccc|c|c} 0 & 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & & 0 \end{array}\right], G_8 = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & H_2 \\ 0 & 0 & 0 & 0 & \end{array}\right],$$

then the system (3.1) is asymptotically stable for any time-varying delay $h(t)$ satisfying (2.6).

Proof. Consider a parameter-dependent Lyapunov-Krasovskii functional selected as:

$$V(t) = \sum_{i=0}^{5} V_i(t),$$

with

$$\begin{aligned}
V_0(t) &= \eta^T(t)P(\varrho(t))\eta(t),\\
V_1(t) &= \int_{t-h_1}^{t} x^T(\delta)Q_1x(\delta)d\delta + \int_{t-h_2}^{t-h_1} x^T(\delta)Q_2(t)x(\delta)d\delta,\\
V_2(t) &= h_1\int_{-h_1}^{0}\int_{t+\beta}^{t} \dot{x}^T(\delta)R_1\dot{x}(\delta)d\delta d\beta + h_{12}\int_{-h_2}^{-h_1}\int_{t+\beta}^{t} \dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta d\beta,\\
V_3(t) &= \int_{-h_1}^{0}\int_{\gamma}^{0}\int_{t+\beta}^{t} \dot{x}^T(\delta)Z_1\dot{x}(\delta)d\delta d\beta d\gamma + \int_{-h_1}^{0}\int_{-h_1}^{\gamma}\int_{t+\beta}^{t} \dot{x}^T(\delta)Z_2\dot{x}(\delta)d\delta d\beta d\gamma,\\
V_4(t) &= \int_{-h_2}^{-h_1}\int_{\gamma}^{-h_1}\int_{t+\beta}^{t} \dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta d\gamma,\\
V_5(t) &= \int_{-h_2}^{-h_1}\int_{-h_2}^{\gamma}\int_{t+\beta}^{t} \dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta d\gamma,
\end{aligned}$$

where

$$\eta(t) = \operatorname{col}\left\{x(t), \int_{t-h_1}^{t} x(\delta)d\delta, \int_{t-h_2}^{t-h_1} x(\delta)d\delta, (2/h_1)\int_{-h_1}^{0}\int_{t+\beta}^{t} x(\delta)d\delta d\beta\right\}.$$

Furthermore, consider an augmented vector $\xi(t)$ given by:

$$\xi(t) = \operatorname{col}\left\{\begin{bmatrix}\dot{x}(t)\\ x(t)\\ x(t-h_1)\\ x(t-h(t))\\ x(t-h_2)\end{bmatrix}, \begin{bmatrix}\frac{1}{h_1}\int_{t-h_1}^{t} x(\delta)d\delta\\ \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(\delta)d\delta\\ \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(\delta)d\delta\\ \frac{2}{h_1^2}\int_{-h_1}^{0}\int_{t+\beta}^{t} x(\delta)d\delta d\beta\\ \frac{2}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1} x(\delta)d\delta d\beta\\ \frac{2}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)} x(\delta)d\delta d\beta\end{bmatrix}\right\}.$$

As $\int_{t-h_2}^{t-h_1} x(\delta)d\delta = \int_{t-h(t)}^{t-h_1} x(\delta)d\delta + \int_{t-h_2}^{t-h(t)} x(\delta)d\delta$ the following relation is obtained

$$\eta^T(t) = \xi(t)^T\Gamma(h) = \begin{bmatrix} x(t) \\ \int_{t-h_1}^{t} x(\delta)d\delta \\ \int_{t-h_2}^{t-h_1} x(\delta)d\delta \\ \frac{2}{h_1}\int_{-h_1}^{0}\int_{t+\beta}^{t} x(\delta)d\delta d\beta \end{bmatrix}^T,$$

$$\dot{\eta}(t) = \Upsilon^T\xi(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h_1) \\ x(t-h_1) - x(t-h_2) \\ 2(x(t) - \frac{1}{h_1}\int_{t-h_1}^{t} x(\delta)d\delta) \end{bmatrix}.$$

Therefore, according to the augmented vector, differentiating $V_0(t)$ along the trajectories of the system (3.1) yields

$$\dot{V}_0(t) = \xi^T(t)(\Gamma(h)P(\varrho(t))\Upsilon^T + \Upsilon P(\varrho(t))\Gamma^T(h) + \Gamma(h)\dot{P}(\varrho(t))\Gamma^T(h))\xi(t).$$

On the other hand, from the definition of the matrix $\widetilde{Q}$, the time-derivative of $V_1(t)$ along the trajectories of the system leads to

$$\begin{aligned}\dot{V}_1(t) &= x^T(t)Q_1x(t) - x^T(t-h_1)Q_1x(t-h_1) \\ &\quad + x^T(t-h_1)Q_2x(t-h_1) - x^T(t-h_2)Q_2x(t-h_2) \\ &= \xi^T(t)\widetilde{Q}\xi(t).\end{aligned}$$

The time-derivative of $V_2(t)$ along the trajectories of the system yields

$$\begin{aligned}\dot{V}_2(t) =& \dot{x}^T(t)(h_1^2R_1 + h_{12}^2R_2)\dot{x}(t) - h_1\int_{t-h_1}^{t} \dot{x}^T(\delta)R_1\dot{x}(\delta)d\delta \\ & - h_{12}\int_{t-h(t)}^{t-h_1} \dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta - h_{12}\int_{t-h_2}^{t-h(t)} \dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta,\end{aligned}$$

and according to the definition of matrices Υ, $\hat{R}_1$ and $\hat{R}_2$, the previous equation can be rewritten as

$$\begin{aligned}\dot{V}_2(t) =& \xi^T(t)\Upsilon(h_1^2\hat{R}_1 + h_{12}^2\hat{R}_2)\Upsilon^T\xi(t) - h_1\int_{t-h_1}^{t} \dot{x}^T(\delta)R_1\dot{x}(\delta)d\delta \\ & - h_{12}\int_{t-h(t)}^{t-h_1} \dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta - h_{12}\int_{t-h_2}^{t-h(t)} \dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta.\end{aligned}$$

Based on the matrix $\widetilde{R}_1$, applying Lemma 2.3 in the first integral of the previous equation yields the following inequality relation:

$$
\begin{aligned}
&-h_1\int_{t-h_1}^{t}\dot{x}^T(\delta)R_1\dot{x}(\delta)d\delta \leqslant \\
&-\begin{bmatrix} x(t)-x(t-h_1) \\ x(t)+x(t-h_1)-\frac{2}{h_1}\int_{t-h_1}^{t}x(\delta)d\delta \\ x(t)-x(t-h_1)+\frac{6}{h_1}\int_{t-h_1}^{t}x(\delta)d\delta-\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+\beta}^{t}x(\delta)d\delta d\beta \end{bmatrix}^T \times \\
&\widetilde{R}_1\times\begin{bmatrix} x(t)-x(t-h_1) \\ x(t)+x(t-h_1)-\frac{2}{h_1}\int_{t-h_1}^{t}x(\delta)d\delta \\ x(t)-x(t-h_1)+\frac{6}{h_1}\int_{t-h_1}^{t}x(\delta)d\delta-\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+\beta}^{t}x(\delta)d\delta d\beta \end{bmatrix}
\end{aligned}
\tag{3.4}
$$

and from the definition of G_0, Equation (3.4) can be rewritten as

$$
-h_1\int_{t-h_1}^{t}\dot{x}^T(\delta)R_1\dot{x}(\delta)d\delta \leqslant -\xi^T(t)G_0^T\widetilde{R}_1G_0\xi(t),
$$

by noting that

$$
G_0\xi(t)=\begin{bmatrix} x(t)-x(t-h_1) \\ x(t)+x(t-h_1)-\frac{2}{h_1}\int_{t-h_1}^{t}x(\delta)d\delta \\ x(t)-x(t-h_1)+\frac{6}{h_1}\int_{t-h_1}^{t}x(\delta)d\delta-\frac{12}{h_1^2}\int_{-h_1}^{0}\int_{t+\beta}^{t}x(\delta)d\delta d\beta \end{bmatrix}.
$$

Applying Lemma 2.3 and the same procedure as before to the other two integrals of $\dot{V}_2(t)$ yields

$$
\begin{aligned}
-h_{12}\int_{t-h(t)}^{t-h_1}\dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta &\leqslant -\frac{h_{12}}{h(t)-h_1}\xi^T(t)G_1^T\widetilde{R}_2G_1\xi(t), \\
-h_{12}\int_{t-h_2}^{t-h(t)}\dot{x}^T(\delta)R_2\dot{x}(\delta)d\delta &\leqslant -\frac{h_{12}}{h_2-h(t)}\xi^T(t)G_2^T\widetilde{R}_2G_2\xi(t).
\end{aligned}
$$

The time-derivative of $V_3(t)$ along the trajectories of the system leads to

$$
\begin{aligned}
\dot{V}_3(t)=&\frac{h_1^2}{2}\dot{x}^T(t)Z_1\dot{x}(t)-\int_{-h_1}^{0}\int_{t+\beta}^{t}\dot{x}^T(\delta)Z_1\dot{x}(\delta)d\delta d\beta+\frac{h_1^2}{2}\dot{x}^T(t)Z_2\dot{x}(t) \\
&-\int_{-h_1}^{0}\int_{t-h_1}^{t+\beta}\dot{x}^T(\delta)Z_2\dot{x}(\delta)d\delta d\beta.
\end{aligned}
$$

The previous equation can be rewritten as

$$
\begin{aligned}
\dot{V}_3(t)=\xi^T(t)\Upsilon\left(\frac{h_1^2}{2}\hat{Z}_1+\frac{h_1^2}{2}\hat{Z}_2\right)\Upsilon^T\xi(t)-\int_{-h_1}^{0}\int_{t+\beta}^{t}\dot{x}^T(\delta)Z_1\dot{x}(\delta)d\delta d\beta \\
-\int_{-h_1}^{0}\int_{t-h_1}^{t+\beta}\dot{x}^T(\delta)Z_2\dot{x}(\delta)d\delta d\beta,
\end{aligned}
$$

and applying again Lemma 2.3, the following upper bounds are obtained

$$
\begin{aligned}
-\int_{-h_1}^{0}\int_{t+\beta}^{t}\dot{x}^T(\delta)Z_1\dot{x}(\delta)d\delta d\beta &\leqslant -\xi^T(t)G_3^T\widetilde{Z}_1G_3\xi(t),\\
-\int_{-h_1}^{0}\int_{t-h_1}^{t+\beta}\dot{x}^T(\delta)Z_2\dot{x}(\delta)d\delta d\beta &\leqslant -\xi^T(t)G_4^T\widetilde{Z}_2G_4\xi(t).
\end{aligned}
$$

The time-derivative of $V_4(t)$ along the trajectories of the system yields

$$
\begin{aligned}
\dot{V}_4(t) =& \frac{h_{12}^2}{2}\dot{x}^T(t)Z_3\dot{x}(t) - \int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta\\
&- \int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta\\
&- (h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta.
\end{aligned}
$$

The previous equation can be rewritten as

$$
\begin{aligned}
\dot{V}_4(t) =& \ \xi^T(t)\Upsilon\left(\frac{h_{12}^2}{2}\hat{Z}_3\right)\Upsilon^T\xi(t) - \int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta\\
&- \int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta\\
&- (h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta,
\end{aligned}
$$

and applying Lemma 2.3, the following upper bounds are obtained

$$
\begin{aligned}
&-\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta \leqslant -\xi^T(t)G_5^T\widetilde{Z}_3G_5\xi(t),\\
&-\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta d\beta \leqslant -\xi^T(t)G_6^T\widetilde{Z}_3G_6\xi(t),\\
&-(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta \leqslant -\left(\frac{h_{12}}{h(t)-h_1}-1\right)\xi^T(t)G_1^T\check{Z}_3G_1\xi(t).
\end{aligned}
$$

The latter inequality can be rewritten as

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{x}^T(\delta)Z_3\dot{x}(\delta)d\delta \leqslant& -\frac{h_{12}}{h(t)-h_1}\xi^T(t)G_1^T\check{Z}_3G_1\xi(t)\\
&+\xi^T(t)G_1^T\check{Z}_3G_1\xi(t).
\end{aligned}
$$

Finally, the time-derivative of $V_5(t)$ along the trajectories of the system yields

$$
\begin{aligned}
\dot{V}_5(t) =& \frac{h_{12}^2}{2}\dot{x}^T(t)Z_4\dot{x}(t) - \int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta\\
&- \int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta\\
&- (h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta.
\end{aligned}
$$

The previous equation can be rewritten as

$$\begin{aligned}\dot{V}_5(t) =&\xi^T(t)\Upsilon\left(\frac{h_{12}^2}{2}\hat{Z}_4\right)\Upsilon^T\xi(t) - \int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta\\ &- \int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta\\ &- (h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta,\end{aligned}$$

applying again Lemma 2.3, the following upper bounds are obtained

$$\begin{aligned}&-\int_{-h(t)}^{-h_1}\int_{t-h(t)}^{t+\beta}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta \leqslant -\xi^T(t)G_7^T\widetilde{Z}_4G_7\xi(t),\\ &-\int_{-h_2}^{-h(t)}\int_{t-h_2}^{t+\beta}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta d\beta \leqslant -\xi^T(t)G_8^T\widetilde{Z}_4G_8\xi(t),\\ &-(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta \leqslant -\left(\frac{h_{12}}{h_2-h(t)}-1\right)\xi^T(t)G_2^T\check{Z}_4G_2\xi(t).\end{aligned}$$

Notice that the latter inequality can be rewritten as:

$$\begin{aligned}-(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(\delta)Z_4\dot{x}(\delta)d\delta \leqslant &-\frac{h_{12}}{h_2-h(t)}\xi^T(t)G_2^T\check{Z}_4G_2\xi(t)\\ &+\xi^T(t)G_2^T\check{Z}_4G_2\xi(t).\end{aligned}$$

Now, defining $\lambda = \frac{h(t)-h_1}{h_{12}}$, the terms in $\dot{V}_2(t)$, $\dot{V}_4(t)$ and $\dot{V}_5(t)$ that depend on $\frac{h(t)-h_1}{h_{12}}$ can be rewritten as:

$$\begin{aligned}&-\frac{1}{\lambda}\xi^T(t)G_1^T\widetilde{R}_2G_1\xi(t) - \frac{1}{1-\lambda}\xi^T(t)G_2^T\widetilde{R}_2G_2\xi(t)\\ &-\frac{1}{\lambda}\xi^T(t)G_1^T\check{Z}_3G_1\xi(t) - \frac{1}{1-\lambda}\xi^T(t)G_2^T\check{Z}_4G_2\xi(t),\end{aligned} \tag{3.5}$$

then from the definition of $\widetilde{R}_{21}$ and $\widetilde{R}_{22}$, Equation (3.5) can be rewritten as

$$-\xi^T(t)\begin{bmatrix}G_1\\ G_2\end{bmatrix}^T\begin{bmatrix}\frac{1}{\lambda}\widetilde{R}_{21} & 0\\ 0 & \frac{1}{1-\lambda}\widetilde{R}_{22}\end{bmatrix}\begin{bmatrix}G_1\\ G_2\end{bmatrix}\xi(t).$$

Applying Lemma 2.5, with $Y_1 = h_{12}[\hat{M}_1{}^T 0]^T$ and $Y_2 = h_{12}[0\ \hat{M}_2{}^T]^T$, it follows that

$$
\begin{aligned}
&-\xi^T(t)\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\lambda}\widetilde{R}_{21} & 0 \\ 0 & \frac{1}{1-\lambda}\widetilde{R}_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\xi(t) \\
&\qquad \leqslant -\xi(t)\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}^T \left(\mathrm{He}\left(\begin{bmatrix} h_{12}\hat{M}_1 \\ 0 \end{bmatrix}\begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ h_{12}\hat{M}_2 \end{bmatrix}\begin{bmatrix} 0 & I \end{bmatrix}\right)\right. \\
&\qquad\quad - h_{12}(h(t)-h_1)\begin{bmatrix} \hat{M}_1 \\ 0 \end{bmatrix}\widetilde{R}_{21}^{-1}\begin{bmatrix} \hat{M}_1{}^T & 0 \end{bmatrix} \\
&\qquad\quad \left. - h_{12}(h_2-h(t))\begin{bmatrix} 0 \\ \hat{M}_2 \end{bmatrix}\widetilde{R}_{22}^{-1}\begin{bmatrix} 0 & \hat{M}_2{}^T \end{bmatrix}\right)\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\xi(t).
\end{aligned}
\tag{3.6}
$$

Let $M_1 = G_1^T\hat{M}_1$ and $M_2 = G_2^T\hat{M}_2$, then the following inequality holds

$$
\begin{aligned}
&-\xi^T(t)\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\lambda}\widetilde{R}_{21} & 0 \\ 0 & \frac{1}{1-\lambda}\widetilde{R}_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\xi(t) \\
&\qquad \leqslant -\xi(t)(h_{12}\mathrm{He}(M_1G_1 + M_2G_2) - h_{12}(h(t)-h_1)M_1\widetilde{R}_{21}^{-1}M_1^T \\
&\qquad\quad - h_{12}(h_2-h(t))M_2\widetilde{R}_{22}^{-1}M_2^T)\xi(t).
\end{aligned}
$$

To introduce extra slack variables, consider the Finsler's Lemma [68]. Let $\xi^T(t)\mathscr{Q}\xi(t) < 0$ be the sum of the terms generated by the differentiation of $V(t)$ and application of Lemmas 2.3 and 2.5, where $\mathscr{Q}$ is given by

$$
\begin{aligned}
\mathscr{Q} = {} & \Gamma(h)P(\varrho(t))\Upsilon^T + \Upsilon P(\varrho(t))\Gamma^T(h) + \Gamma(h)\dot{P}(\varrho(t))\Gamma^T(h) + \widetilde{Q} - G_0^T\widetilde{R}_1G_0 \\
& + \Upsilon\left(h_1^2\hat{R}_1 + h_{12}^2\hat{R}_2 + \frac{h_1^2}{2}\hat{Z}_1 + \frac{h_1^2}{2}\hat{Z}_2 + \frac{h_{12}^2}{2}\hat{Z}_3 + \frac{h_{12}^2}{2}\hat{Z}_4\right)\Upsilon^T \\
& - G_3^T\widetilde{Z}_1G_3 - G_4^T\widetilde{Z}_2G_4 - G_5^T\widetilde{Z}_3G_5 - G_6^T\widetilde{Z}_3G_6 - G_7^T\widetilde{Z}_4G_7 \\
& - G_8^T\widetilde{Z}_4G_8 + G_1^T\check{Z}_3G_1 + G_2^T\check{Z}_4G_2 - h_{12}\mathrm{He}(M_1G_1 + M_2G_2) \\
& + h_{12}(h(t)-h_1)M_1\widetilde{R}_{21}^{-1}M_1^T + h_{12}(h_2-h(t))M_2\widetilde{R}_{22}^{-1}M_2^T.
\end{aligned}
\tag{3.7}
$$

Based on the system (3.1), let $\mathscr{B} = \begin{bmatrix} -I & A(\varrho(t)) & 0 & A_d(\varrho(t)) & 0_{n\times 7n} \end{bmatrix}$ and note that $\mathscr{B}\xi(t) = 0$. Then, according to Finsler's Lemma, $\xi^T(t)\mathscr{Q}\xi(t) < 0$ is equivalent to

$$
\widetilde{\Omega}(h) \triangleq \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}^T\mathscr{X}^T < 0,
\tag{3.8}
$$

with

$$
\mathscr{X} = \begin{bmatrix} W_1(\varrho(t)) & W_2(\varrho(t)) & 0 & W_3(\varrho(t)) & 0_{n\times 7n} \end{bmatrix}^T.
$$

Notice also that $\widetilde{\Omega}(h)$ (defined in (3.8)) can be rewritten as $\widetilde{\Omega}(h) = \Omega_0(h) + J$ with

$$J = h_{12}(h(t) - h_1)M_1\widetilde{R}_{21}^{-1}M_1^T + h_{12}(h_2 - h(t))M_2\widetilde{R}_{22}^{-1}M_2^T,$$

and $\Omega_0(h)$ is given in (3.3).

Notice that applying the Schur Complement in Equation (3.8) one has Equation (3.2). Thus, if Equation (3.2) is satisfied, one can conclude that Equation (3.8) holds and consequently $\dot{V}(t) < 0$ which, by the Lyapunov-Krasovskii theory, ensures the asymptotic stability of the system (3.1). This concludes the proof. □

Remark 3.1

It is worth highlighting that $\dot{P}(\varrho(t))$ in (3.3) can be described as

$$\dot{P}(\varrho(t)) = \sum_{i=1}^{N} \dot{\varrho}_i(t)P_i.$$

Moreover, using the representation given in (2.10)-(2.13), one can compute the expression $\dot{P}(\varrho(t)$ in a polytopic way in the following way

$$\dot{P}(\varrho(t)) = \sum_{i=1}^{N}\sum_{\ell=1}^{M} \beta_\ell h_i^\ell P_i, \quad \beta \in \Lambda_M.$$

For more details on how to obtain finite conditions from the presented parameter-dependent LMIs see Appendix A.

Remark 3.2

Differently from existing conditions in the literature (see, e.g. [43, 77] and the references therein), the chosen strategy to deal with the various integrals, composing the LKF, and also to obtain proper upper bounds to them, has been based on Lemma 2.3. This alternative is more general than the results proposed in [3, 29, 30]. Therefore is expected to ensure tighter bounds [31, 32]. Besides that, it has been applied Lemma 2.5 in Equation (3.6) since, in general, it may produce less conservative results [34]. The strategy, as proposed to bound the various integrals in the LKF, to the best of the candidate's knowledge, is new in the context of time-delayed LPV systems.

3.2 EXAMPLES

The next examples illustrates the effectiveness of the proposed Theorem 3.1. The first example is motivated by the model of a milling process, the second is from [71, Example 3.4.1] and the third one has been randomly generated. The simulations are implemented in Matlab, a finite set of LMIs is automatically obtained by employing the ROLMIP parser [75] that works jointly with Yalmip and the solver MOSEK.

Example 3.1. Consider a chatter during the milling process [78, 79]. Milling is the process of machining using rotary cutters to remove material by advancing a cutter into a workpiece. The simplified geometry of a milling process is depicted in Figure 3.1.

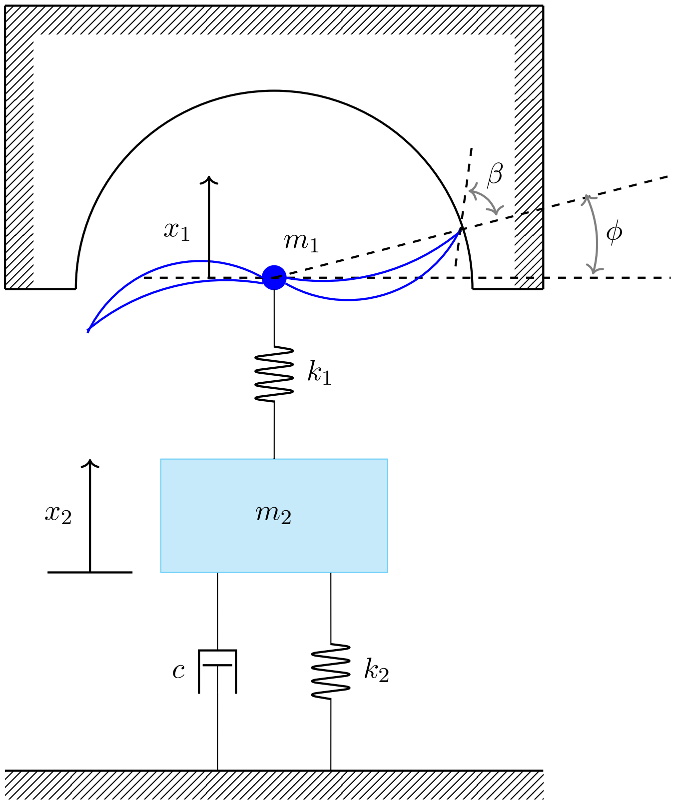

Figure 3.1: Simplified geometry of a milling process.

The cutter has two blades that are used to remove material from the clamped workpiece. The force acting on the tool is a function of the current displacement of the tool and the workpiece surface characteristics, and hence the displacement at the previous tool pass. This phenomenon induces a speed-dependent delay into the system and the equations that describe the motion are given by

$$\begin{aligned} m_1\ddot{x}_1(t) + k_1\left(x_1(t) - x_2(t)\right) &= f(t), \\ m_2\ddot{x}_2(t) + c\dot{x}_2(t) + k_1\left(x_2(t) - x_1(t)\right) + k_2 x_2(t) &= u(t), \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} f(t) &= k\sin(\phi+\beta)l(t) - w(t), \\ l(t) &= \sin(\phi)\left[x_1(t-h(t)) - x_1(t)\right], \end{aligned}$$

where k_1 and k_2 are the stiffness of the two springs, c is the damping coefficient, m_1 and m_2 are, respectively, the masses of the blade and the tool. Further, $x_1(t)$ and $x_2(t)$ are, respectively, the displacements of the blade and the tool. The angle β depends on the particular material and the tool used. The angle ϕ denotes the angular position of the blade, k is the cutting force coefficient, $w(t)$ denotes the disturbance, $u(t)$ denotes the control input and $h(t)$ denotes the time-varying delay that is approximated to be $\pi/\omega(t)$, where $\omega(t)$ is the rotation speed of the blade. Equation (3.9) can be rewritten as:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \ddot{x}_1(t)\\ \ddot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k_1+k\sin(\phi)\sin(\phi+\beta)}{m_1} & \frac{k_1}{m_1} & 0 & 0\\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & 0 & -\frac{c}{m_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac{k\sin(\theta+\beta)\sin(\phi)}{m_1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t-\pi/\omega(t))\\ x_2(t-\pi/\omega(t))\\ \dot{x}_1(t-\pi/\omega(t))\\ \dot{x}_2(t-\pi/\omega(t))\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -1\\ 0\end{bmatrix}w(t) + \begin{bmatrix}0\\ 0\\ 0\\ \frac{1}{m_2}\end{bmatrix}u(t).\end{aligned} \tag{3.10}$$

Consider the parameters: $m_1 = 1$, $m_2 = 2$, $k_1 = 10$, $k_2 = 20$, $k = 2$, $c = 0.5$ and $\beta = 70°$. Notice also that:

$$\begin{aligned}\sin(\phi+\beta)\sin(\phi) &= 0.5[\cos(\beta) - \cos(2\phi+\beta)]\\ &= 0.1710 - 0.5\cos(2\phi+\beta).\end{aligned}$$

Considering $\alpha_1(t) = \cos(2\phi+\beta)$ notice that $\alpha_1(t) \in [\underline{\alpha}_1\ \overline{\alpha}_1]$. The rotate of the speed blade is assumed to be between 200 rpm and 2000 rpm, and 1000 rpm is the maximum variation rate. Hence $\alpha_1(t) = \cos(2\phi+\beta) \in [-1\ 1]$ and $|\mathrm{d}\alpha_1(t)/\mathrm{d}t| = |-2\sin(2\phi+\beta)\omega| \leqslant 2\times 2000 \times 2\pi/60 = 418.9$ rad/s.

Substituting the given model parameter values in Equation (3.10), it follows the time-delayed LPV system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -10.34+\alpha_1(t) & 10 & 0 & 0\\ 5 & -15 & 0 & -0.25\end{bmatrix}x(t)\\ &+ \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0.34-\alpha_1(t) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}x(t-\pi/\omega(t)) + \begin{bmatrix}0\\ 0\\ -1\\ 0\end{bmatrix}w(t) + \begin{bmatrix}0\\ 0\\ 0\\ 0.5\end{bmatrix}u(t).\end{aligned} \tag{3.11}$$

Considering $\varrho_1(t) = \frac{\alpha_1(t)-\underline{\alpha}_1}{\overline{\alpha}_1-\underline{\alpha}_1}$ and $\varrho_2(t) = 1-\varrho_1(t)$, an LPV system with 2 vertices can be obtained

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11.34 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9.34 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix},$$

$$A_{d_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_{d_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

The time-derivative of $\varrho_1(t) = \frac{\alpha_1(t)-\underline{\alpha}_1}{\overline{\alpha}_1-\underline{\alpha}_1}$ and $\varrho_2(t) = 1-\varrho_1(t)$ are given by

$$\dot{\varrho}_1(t) = \frac{\dot{\alpha}_1(t)(\overline{\alpha}_1 - \underline{\alpha}_1)}{(\overline{\alpha}_1 - \underline{\alpha}_1)^2} = \frac{\dot{\alpha}_1(t)}{2},$$
$$\dot{\varrho}_2(t) = -\dot{\varrho}_1(t).$$

Then, $|\mathrm{d}\varrho_1(t)/\mathrm{d}t| \leqslant 209.45$ rad/s. To verify the effectiveness of Theorem 3.1 concerning stability, a comparison with other methods in the literature is performed considering $w(t) = u(t) = 0$.

The problem to be solved is to find, for a given cutting force coefficient k, the maximum allowed delay $\bar{\tau} \in [0, h_2]$ such that system (3.11) still remains stable. It is performed tests employing Theorem 3.1, and the parameter-dependent proposed approaches in [71, Corollary 3.4.1] and [79, Corollary 4], considering the interval $k \in [0.1 \; 0.5]$. The results of this analysis are depicted in Figure 3.2.

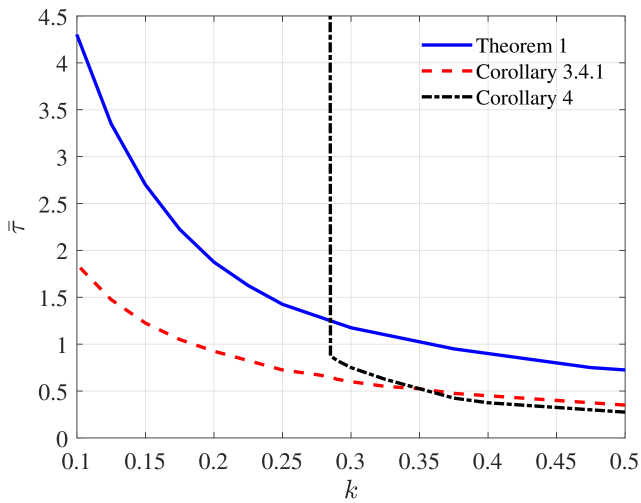

Figure 3.2: Stability Analysis - Maximum allowable time delay.

As can be seen in Figure 3.2, Theorem 3.1 provides less conservative results than the one in [71] and [79]. This result is somehow expected since the proposed approach in Theorem 3.1 included several new relaxations as described before.

These results illustrate the effectiveness of the proposed Theorem 3.1 for stability analysis for time-delay LPV systems.

Example 3.2. [71, Example 3.4.1] Consider the system (3.1) with polytopic uncertainties and with

$$A_1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \; A_3 = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix},$$
$$A_{d_1} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \; A_{d_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; A_{d_3} = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}.$$

The objective is to find the maximum upper bound of $\bar{\tau} \in [0, h_2]$. The information about the time-derivative of the parameter is not given in this example, then it is considered to be null. The upper bound on the delay is 0.7963 in [2, 80], and 0.9090 in [71]. However, Theorem 3.1 in this section shows that the system is asymptotically stable for $h_2 = 1.1098$. Theorem 3.1 yields a larger maximum upper bound on the allowable size of the delay than [2, 71, 80] do. Moreover, Table 3.1 compares the upper bounds obtained. It is consider the lower bound of the delay $h_1 = 0$ to compare with the existing results, although Theorem 3.1 allows to analyze the stability when h_1 is greater than 0.

Table 3.1: Allowable upper bound, h_2, Example 3.2.

Method	Maximum h_2
[2]	0.7963
[80]	0.7963
[71]	0.9090
Theorem 3.1	1.1098

Example 3.3. Consider the following LPV system with time delay

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 + 0.2\alpha_1(t) \\ -10 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} x(t - h(t)), \tag{3.12}$$

where $\alpha_1(t) \in [-1, 1]$. Consider $\varrho_1(t) = \frac{\alpha_1(t) - \underline{\alpha}_1}{\bar{\alpha}_1 - \underline{\alpha}_1}$ and $\varrho_2(t) = 1 - \varrho_1(t)$, and let the variation rate of each parameter bounded by

$$-1 \leq \dot{\varrho}_1(t) \leq 1, \quad 1 \leq \dot{\varrho}_2(t) \leq 1.$$

The vectors h^ℓ that describe the polytopic region where ϱ_i assume values are given by

$$\begin{bmatrix} h^1 & h^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and a graphical representation is depicted in Figure 3.3.

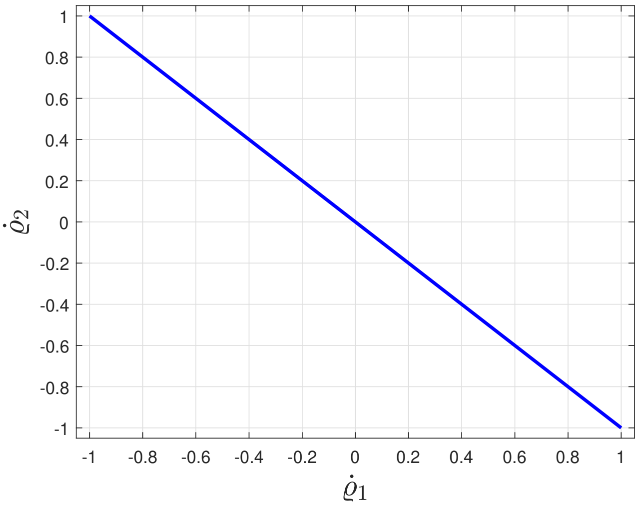

Figure 3.3: Feasible region subject to $-1 \leq \dot{\varrho}_i \leq 1$ and $\dot{\varrho}_1 + \dot{\varrho}_2 = 0$.

Substituting the given model parameter values in Equation (3.12), the time-delayed LPV system with 2 vertices can be obtained

$$A_1 = \begin{bmatrix} 0 & 0.8 \\ -10 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1.2 \\ -10 & -1 \end{bmatrix},$$
$$A_{d_1} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \; A_{d_2} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

The objective is to find the maximum upper bound of h_2. The comparison is performed in relation to [71, Corollary 3.4.1]. The upper bound on the delay is 0.155 in [71] when the lower bound $h_1 = 0$. However, the proposed Theorem 3.1 shows that the system is asymptotically stable for $h_2 = 1.264$ when $h_1 = 0$. Thus, Theorem 3.1 yields a larger maximum upper bound on the allowable size of the delay. Furthermore, Table 3.2 compares the upper bounds obtained. Due to the strategy used in condition [71, Corollary 3.4.1], the allowable upper bound only can be evaluated when $h_1 = 0$, although Theorem 3.1 allows to analyze the stability when h_1 is greater than 0.

Table 3.2: Allowable upper bound, h_2, Example 3.3.

h_1	0	0.1	0.2	0.4	0.5
[71]	0.155	–	–	–	–
Theorem 3.1	1.264	1.411	1.605	2.034	2.205

3.3 CHAPTER CONCLUSIONS

This chapter investigated the stability analyses of time-delayed LPV systems. First a novel stability result was stated, based on a parameter-dependent Lyapunov-Krasovskii functional, and bound techniques as well a reciprocally convex method combined with Moon's inequality. Finally, the results in the examples illustrate the effectiveness of the proposed Theorem 3.1.

4

STATE-FEEDBACK STABILIZATION

This chapter investigates the problem of state-feedback control for time-delayed Linear Parameter-varying system (LPV system) with polytopic time-varying parameters. A novel gain-scheduled state-feedback stabilization result for LPV time-delayed systems is derived based on Theorem 3.1. It is worth mentioning there is no need to establish a particular structure to the Lyapunov matrices. In Section 4.1 a sufficient condition is derived. An example that illustrates the performance of the proposed method is presented in Section 4.2.

4.1 GAIN-SCHEDULED STATE-FEEDBACK CONTROL

Considering Assumption 2.1, the following scheduling state-feedback control is adopted

$$u(t) = K(\varrho(t))x(t). \tag{4.1}$$

Closing the loop with Equation (4.1) and the system (2.5), then the closed-loop system is given by

$$\dot{x}(t) = \widetilde{A}(\varrho(t))x(t) + A_d(\varrho(t))x(t-h(t)), \tag{4.2}$$

where $\widetilde{A}(\varrho(t)) = A(\varrho(t)) + B(\varrho(t))K(\varrho(t))$.

Now, the problem is to design a control law, as in Equation (4.1), that stabilizes the delayed LPV closed-loop system described in (4.2). The LMI based condition to obtain the gain-scheduled control gain is stated in the following theorem.

Theorem 4.1

If there exist matrices $P(\varrho(t)) \in \mathbb{S}^{+}_{4n}$ defined as in (2.7), Q_1, Q_2, R_1, R_2, Z_1, Z_2, Z_3 and $Z_4 \in \mathbb{S}^{+}_{n}$, $M_{\hat{k}} \in \mathbb{R}^{11n\times 3n}$, $\hat{k} = 1, 2$, $W(\varrho(t)) \in \mathbb{R}^{n\times n}$ defined as in (2.7), $L(\varrho(t)) \in \mathbb{R}^{m\times n}$ also defined as in (2.7), and given scalars $0 \le h_1 \le h_2$, such that the following LMIs hold

$$\begin{bmatrix} \Omega_0(h_2) & h_{12}M_1 \\ \star & -\widetilde{R}_{21} \end{bmatrix} < 0, \quad \begin{bmatrix} \Omega_0(h_1) & h_{12}M_2 \\ \star & -\widetilde{R}_{22} \end{bmatrix} < 0,$$

where $\Omega_0(h)$, $\Gamma(h)$, Υ, $\widetilde{Q}$, $\hat{R}_1$, $\hat{R}_2$, $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Z}_3$, $\hat{Z}_4$, $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{Z}_1$, $\widetilde{Z}_2$, $\widetilde{Z}_3$, $\widetilde{Z}_4$, $\check{Z}_3$, $\check{Z}_4$, $\widetilde{R}_{21}$, $\widetilde{R}_{22}$, H_1, H_2, H_3, G_1, G_2, G_3, G_4, G_5, G_6, G_7 and G_8 are defined in Theorem 3.1 and

$$\mathscr{B}^T\mathscr{X}^T = \left[\begin{array}{cccc:c} -W(\varrho(t)) & -W(\varrho(t)) & 0 & -W(\varrho(t)) & \\ \mathscr{A} & \mathscr{A} & 0 & \mathscr{A} & 0_{4n\times 7n} \\ 0 & 0 & 0 & 0 & \\ A_d(\varrho(t))W(\varrho(t)) & A_d(\varrho(t))W(\varrho(t)) & 0 & A_d(\varrho(t))W(\varrho(t)) & \\ \hdashline \multicolumn{4}{c:}{0_{7n\times 4n}} & 0_{7n\times 7n} \end{array}\right],$$

with $\mathscr{A} = A(\varrho(t))W(\varrho(t)) + B(\varrho(t))L(\varrho(t))$.

Then $K(\varrho(t)) = L(\varrho(t))W(\varrho(t))^{-1}$ is a scheduling state-feedback gain control ensuring that system (4.2) is asymptotically stable for any time-varying delay $h(t)$ satisfying (2.6).

Proof. Taking Theorem 3.1 as the starting point. Consider the closed-loop system (4.2) and let

$$\mathscr{B} = \begin{bmatrix} -I & \widetilde{A}^T(\varrho(t)) & 0 & A_d^T(\varrho(t)) & 0_{n\times 7n} \end{bmatrix},$$

such that $\xi^T(t)\mathscr{B}^T = 0$.

Then, according to Finsler's Lemma, $\xi^T(t)\mathscr{Q}\xi(t) < 0$ (with $\mathscr{Q}$, given in (3.7)) is equivalent to:

$$\mathscr{Q} + \mathscr{B}^T\mathscr{X}^T + \mathscr{X}\mathscr{B} < 0,$$

where,

$$\mathscr{X}^T = \begin{bmatrix} W(\varrho(t)) & W(\varrho(t)) & 0 & W(\varrho(t)) & 0_{n\times 7n} \end{bmatrix}.$$

Substituting $L(\varrho(t)) = K(\varrho(t))W(\varrho(t))$, the scheduled state-feedback gain that guarantees the asymptotic stability of the closed-loop system (4.2) follows in a straightforward way: $K(\varrho(t)) = L(\varrho(t))W(\varrho(t))^{-1}$. This concludes the proof. □

Remark 4.1

Notice that matrix Υ multiplies matrix $P(\varrho(t))$ in Equation (3.3) but Υ does not contain any of the system matrices. This is due to the new proposed approach used for selecting the augmented vector. Then, there is no need to establish a particular structure to the Lyapunov matrices in the control design, since the Lyapunov matrices and the system matrices are decoupled. This is also a new strategy for designing gain-scheduled state-feedback controllers for delayed LPV systems.

4.2 EXAMPLE

The next example illustrates the effectiveness of the proposed Theorem 4.1. The example is motivated by the same model of a milling process seen in Section 3.2. The simulations are implemented in Matlab, a finite set of LMIs is automatically obtained by employing the ROLMIP parser [75] that works jointly with Yalmip and the solver MOSEK.

Consider the model of a milling process as in Example 3.1.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k\sin(\phi)\sin(\phi+\beta)}{m_1} & \frac{k_1}{m_1} & 0 & 0 \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k\sin(\theta+\beta)\sin(\phi)}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\pi/\omega(t)) \\ x_2(t-\pi/\omega(t)) \\ \dot{x}_1(t-\pi/\omega(t)) \\ \dot{x}_2(t-\pi/\omega(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u(t).$$

Consider the parameters as in Example 3.1. Based on Assumption 2.1, consider $\alpha_1(t) = \cos(2\phi + \beta)$ such that $\alpha_1(t) \in [\underline{\alpha}_1 \ \overline{\alpha}_1]$ as scheduling parameters, which is measurable in real-time and can be used to obtain a gain-scheduled controller.

Substituting the given model parameter values, it follows the time-delayed LPV system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10.34 + \alpha_1(t) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix} x(t)$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.34 - \alpha_1(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t-\pi/\omega(t)) + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u(t). \tag{4.3}$$

In the sequel, some experiments are discussed for the design of gain-scheduling controllers for time-varying delayed LPV systems. The control design is based on the new result proposed in Theorem 4.1. In the first experiment, Case I, the objective is to find the controller that ensures stability for a maximum time-delay for a given cutting force coefficient, k. In the second one, Case II, it is investigated the performance of the gain-scheduled controller if a disturbance force, $w(t)$, occurs. In the third experiment, Case III, it is considered the situation in which the friction of the blades varies during the milling process (modeled as an uncertain parameter) and how the proposed control design strategy performs in this case even if it is also affected by disturbance.

- Case I – Maximum allowed time-delay

In this case, the purpose is to find, for a given cutting force coefficient k, the maximum allowed delay $\bar{\tau} \in [0, h_2]$ such that a controller designed via Theorem 4.1 stabilizes the system (4.3). Several tests have been performed considering W and L fixed in order to obtain a fixed-gain controller. Also, it is performed tests considering $W(\varrho(t))$ and $L(\varrho(t))$ dependent on the parameter $\varrho(t)$ to obtain a scheduled-gain controller. The result of this analysis is depicted in Figure 4.1. It can be seen that the scheduled-gain controller presents better results than a fixed-gain controller.

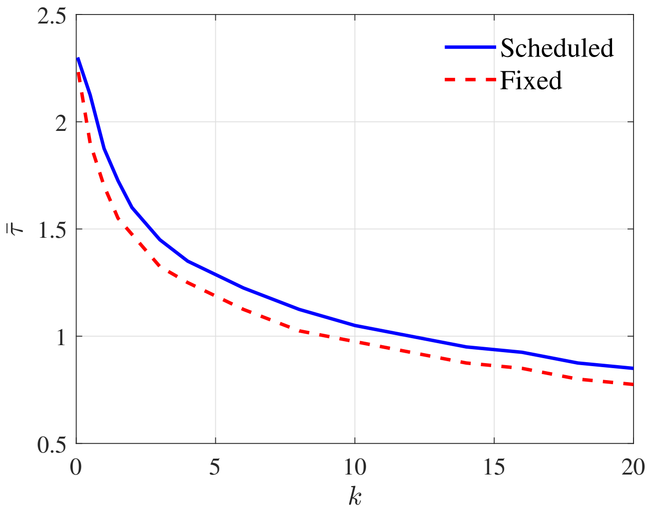

Figure 4.1: Case I - Maximal allowable time delay obtained via Theorem 4.1.

- Case II – Presence of disturbance

In this case, the goal is to investigate the performance of the gain-scheduled controller in the presence of a disturbance force $w(t)$.

Consider the disturbance $w(t)$ given by a rectangular signal:

$$w(t) = \begin{cases} 1, & 0 \le t \le 4 \\ 0, & t > 4. \end{cases}$$

The blade rotating speed $\omega(t)$ varies as depicted in Figure 4.2 and, therefore, the time-varying delay $h(t)$ is according to Figure 4.3. The initial condition of the states, in this case, are null.

In Figure 4.4 it is possible to notice the system behavior without control. In Figure 4.5, it is possible to notice the performance of the controller in correcting the disturbance that has been presented up to the instant of 4 seconds. It is possible to see that the controller effectively stabilizes the system.

- Case III – Uncertain friction of the blades

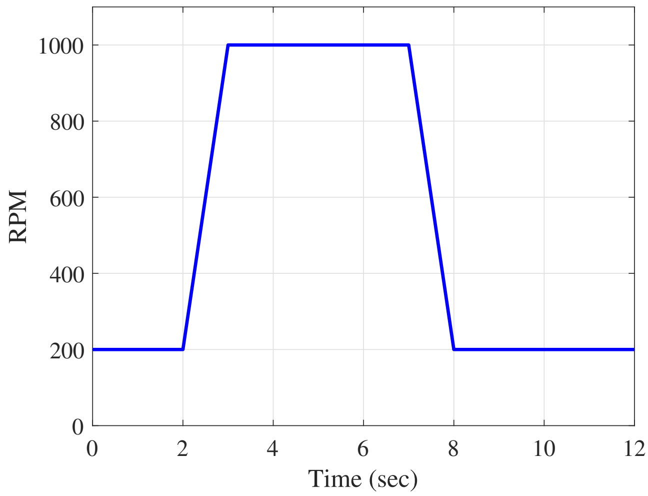

Figure 4.2: Blade rotating speed.

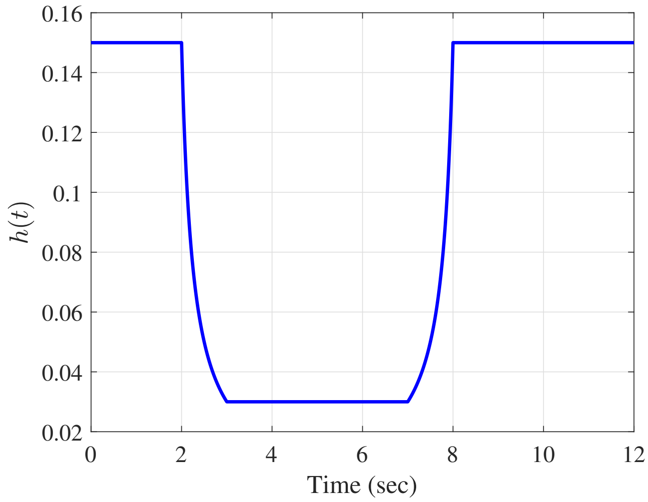

Figure 4.3: Time-varying delay.

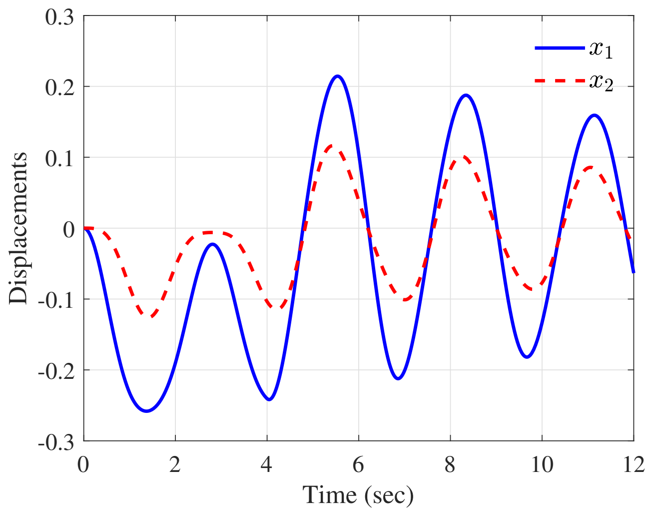

Figure 4.4: Case II – Displacement of the masses without control.

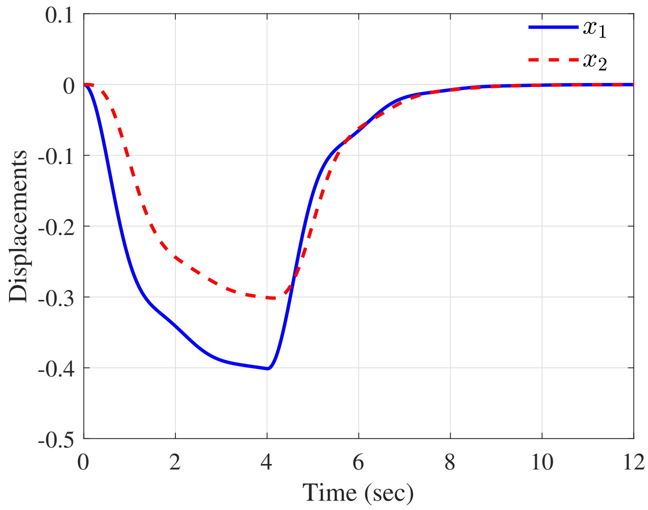

Figure 4.5: Case II – Displacement of the masses with the controller via Theorem 4.1.

Now, suppose that the friction of the blades during the milling process leads the spring temperature to rise and there is a variation in the stiffness described as $k_1 \in [5, 10]$. In this case, matrix $A(\varrho(t))$ is given by

$$A(\varrho(t)) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.34 + \alpha_1(t) - \alpha_2(t) & \alpha_2(t) & 0 & 0 \\ \frac{\alpha_2(t)}{2} & -\frac{\alpha_2(t)}{2} - 10 & 0 & -0.25 \end{bmatrix},$$

and the other system matrices do not modify since only $A(\varrho(t))$ depends on k_1. In this case, the scheduling parameters are $\alpha_1(t) \in [-1 \ 1]$ and $\alpha_2(t) \ \in \ [5 \ 10]$, and the combination of these parameters belongs to a unit simplex as in Equation (2.8). Thus, an LPV system with 4 vertices can be obtained

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6.34 & 5 & 0 & 0 \\ 2.5 & -12.5 & 0 & -0.25 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11.34 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4.34 & 5 & 0 & 0 \\ 2.5 & -12.5 & 0 & -0.25 \end{bmatrix}, \; A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9.34 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix},$$

$$A_{d_1} = A_{d_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_{d_3} = A_{d_4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

Furthermore, the time-derivative of $\varrho_1(t)$ and $\varrho_2(t)$ are given in Example 3.1, and it is considered $|\dot{\varrho}_3(t)| = |\dot{\varrho}_4(t)| = 0$.

The simulation of this second time-delayed LPV plant has been performed with the same rotating speed $\omega(t)$. The disturbance $w(t)$ is given by:

$$w(t) = \begin{cases} 1, & 8 \leq t \leq 10 \\ 0, & 0 \leq t < 8 \text{ and } t > 10. \end{cases}$$

For this simulation, the initial conditions is $x(0) = \begin{bmatrix} 0.1 & 0.2 & 0.5 & 0.5 \end{bmatrix}^T$. The uncertain k_1 assumes values between 5 to 10.

Figure 4.6 depicts the trajectories of the system with k_1 assuming values between 5 to 10. The trajectories of the states x_1 and x_2 are represented in blue and red color, respectively. Unlike the previous case, the system begins with a non-null initial condition. In approximately 4 seconds the system stabilizes, and a disturbance occurs in 8 seconds. Once more, a gain-scheduled controller $K(\varrho(t))$ is obtained using Theorem 4.1 and its effectiveness is depicted in Figure 4.6.

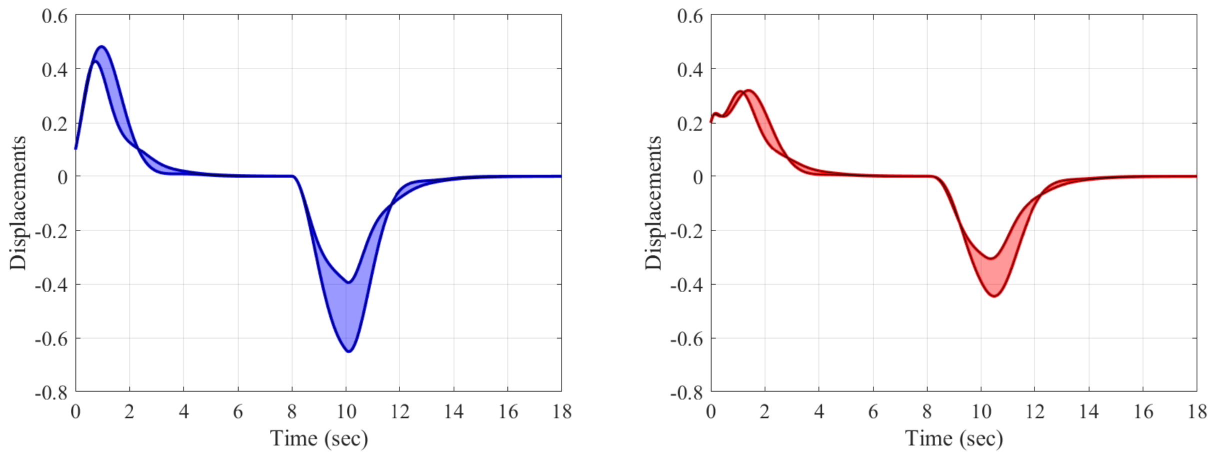

Figure 4.6: Case III – Trajectories of x_1 with $k_1 \in [5, 10]$ are depicted in blue; Trajectories of x_2 with $k_1 \in [5, 10]$ are depicted in red.

4.3 CHAPTER CONCLUSIONS

This chapter investigated the state-feedback stabilization analyses of time-delayed LPV systems. First a novel gain-scheduled state-feedback stabilization result for LPV time-delayed systems is stated. It is worth mentioning there is no need to establish a specific structure to the Lyapunov matrices. Finally, the results presented in the example illustrate the effectiveness of the proposed Theorem 4.1.

5

STATIC OUTPUT-FEEDBACK STABILIZATION

This chapter investigates the problem of static output-feedback control for time-delayed Linear Parameter-varying system (LPV system) with polytopic time-varying parameters. A novel gain-scheduled static output-feedback stabilization result for LPV time-delayed systems is derived based on Theorem 3.1. As in Theorem 4.1, it is worth mentioning there is no need to establish a particular structure to the Lyapunov matrices. In Section 5.1 a sufficient condition is derived. An example that illustrates the performance of the proposed method is presented in Section 5.2.

5.1 GAIN-SCHEDULED STATIC OUTPUT-FEEDBACK CONTROL

Considering Assumption 2.1, the following scheduling static output-feedback control is adopted

$$u(t) = L(\varrho(t))y(t), \tag{5.1}$$

where $y(t) = C(\varrho(t))x(t)$ is the output vector, with $C(\varrho(t)) \in \mathbb{R}^{n_y \times n}$.

Closing the loop with Equation (5.1) and the system (2.5), leads to:

$$\dot{x}(t) = \bar{A}(\varrho(t))x(t) + A_d(\varrho(t))x(t - h(t)), \tag{5.2}$$

where $\bar{A}(\varrho(t)) = A(\varrho(t)) + B(\varrho(t))L(\varrho(t))C(\varrho(t))$.

The problem can be stated as to design a control law, as in Equation (5.1), that stabilizes the delayed LPV closed-loop system described in (5.2). The LMI based condition to obtain the gain-scheduled control gain is presented in the following theorem.

Theorem 5.1

If there exist matrices $P(\varrho(t)) \in \mathbb{S}^+_{4n}$ defined as in (2.7), Q_1, Q_2, R_1, R_2, Z_1, Z_2, Z_3 and $Z_4 \in \mathbb{S}^+_n$, $M_{\hat{k}} \in \mathbb{R}^{(11n+m)\times 3n}$, $\hat{k} = 1, 2$, $E(\varrho(t))$, $X(\varrho(t))$ and $Z(\varrho(t)) \in \mathbb{R}^{n\times n}$ defined as in (2.7), $J(\varrho(t)) \in \mathbb{R}^{m\times n}$ defined as in (2.7), $G(\varrho(t)) \in \mathbb{R}^{m\times m}$ defined as in (2.7), $K(\varrho(t)) \in \mathbb{R}^{m\times n_y}$ also defined as in (2.7), a given matrix $\mathscr{T} \in \mathbb{R}^{n\times m}$, and given scalars ζ and $0 \le h_1 \le h_2$, such that the following LMIs hold

$$\begin{bmatrix} \Omega_0(h_2) & h_{12}M_1 \\ \star & -\widetilde{R}_{21} \end{bmatrix} < 0, \quad \begin{bmatrix} \Omega_0(h_1) & h_{12}M_2 \\ \star & -\widetilde{R}_{22} \end{bmatrix} < 0,$$

where $\Omega_0(h)$, $\hat{R}_1$, $\hat{R}_2$, $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Z}_3$, $\hat{Z}_4$, $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{Z}_1$, $\widetilde{Z}_2$, $\widetilde{Z}_3$, $\widetilde{Z}_4$, $\breve{Z}_3$, $\breve{Z}_4$, $\widetilde{R}_{21}$, $\widetilde{R}_{22}$, H_1, H_2 and H_3 are defined in Theorem 3.1 and

$$
\Gamma(h)=\begin{bmatrix}
0_{n\times m} & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0_{n\times m} & 0 & 0 & 0 & 0 & 0 & h_1 I & 0 & 0 & 0 & 0 & 0\\
0_{n\times m} & 0 & 0 & 0 & 0 & 0 & 0 & (h(t)-h_1)I & (h_2-h(t))I & 0 & 0 & 0\\
0_{n\times m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 I & 0 & 0
\end{bmatrix}^T,
$$

$$
\Upsilon=\begin{bmatrix}
0_{n\times m} & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0_{n\times m} & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0_{n\times m} & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0\\
0_{n\times m} & 0 & 2I & 0 & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T,
$$

$$
\widetilde{Q}=\mathrm{diag}(0_m, 0, Q_1, Q_2-Q_1, 0, -Q_2, 0, 0, 0, 0, 0, 0),
$$

$$
G_0=\left[\begin{array}{cc:c:cc}
0_{n\times m} & 0 & & 0 & 0\\
0_{n\times m} & 0 & H_3 & 0 & 0\\
0_{n\times m} & 0 & & 0 & 0
\end{array}\right],\;
G_1=\left[\begin{array}{ccc:c:c}
0_{n\times m} & 0 & 0 & & 0\\
0_{n\times m} & 0 & 0 & H_3 & 0\\
0_{n\times m} & 0 & 0 & & 0
\end{array}\right],
$$

$$
G_2=\left[\begin{array}{cccc:c}
0_{n\times m} & 0 & 0 & 0 & \\
0_{n\times m} & 0 & 0 & 0 & H_3\\
0_{n\times m} & 0 & 0 & 0 &
\end{array}\right],\;
G_3=\left[\begin{array}{cc:c:cc}
0_{n\times m} & 0 & \multirow{2}{*}{H_1} & 0 & 0\\
0_{n\times m} & 0 & & 0 & 0
\end{array}\right],
$$

$$
G_4=\left[\begin{array}{ccc:c:cc}
0_{n\times m} & 0 & 0 & \multirow{2}{*}{H_2} & 0 & 0\\
0_{n\times m} & 0 & 0 & & 0 & 0
\end{array}\right],\;
G_5=\left[\begin{array}{ccc:c:c}
0_{n\times m} & 0 & 0 & \multirow{2}{*}{H_1} & 0\\
0_{n\times m} & 0 & 0 & & 0
\end{array}\right],
$$

$$
G_6=\left[\begin{array}{cccc:c}
0_{n\times m} & 0 & 0 & 0 & \multirow{2}{*}{H_1}\\
0_{n\times m} & 0 & 0 & 0 &
\end{array}\right],\;
G_7=\left[\begin{array}{cccc:c:c}
0_{n\times m} & 0 & 0 & 0 & \multirow{2}{*}{H_2} & 0\\
0_{n\times m} & 0 & 0 & 0 & & 0
\end{array}\right],
$$

$$
G_8=\left[\begin{array}{ccccc:c}
0_{n\times m} & 0 & 0 & 0 & 0 & \multirow{2}{*}{H_2}\\
0_{n\times m} & 0 & 0 & 0 & 0 &
\end{array}\right],
$$

$$
\mathscr{X}\mathscr{B}=\left[\begin{array}{ccccc:c}
F_{11} & -\zeta J(\varrho(t)) & F_{13} & 0_{m\times n} & F_{15} & \\
F_{21} & -E(\varrho(t)) & F_{23} & 0 & F_{25} & \\
F_{31} & -X(\varrho(t)) & F_{33} & 0 & F_{35} & 0_{(4n+m)\times 7n}\\
0_{n\times m} & 0 & 0 & 0 & 0 & \\
F_{51} & -Z(\varrho(t)) & F_{53} & 0 & F_{55} & \\
\hdashline
\multicolumn{5}{c:}{0_{7n\times(4n+m)}} & 0_{7n\times 7n}
\end{array}\right],
$$

with

$$
\begin{aligned}
&F_{11} = \zeta J(\varrho(t))B(\varrho(t)) - G(\varrho(t)), \quad F_{21} = E(\varrho(t))B(\varrho(t)) - \zeta \mathscr{T} G(\varrho(t)),\\
&F_{31} = X(\varrho(t))B(\varrho(t)) - \zeta \mathscr{T} G(\varrho(t)), \quad F_{51} = Z(\varrho(t))B(\varrho(t)) - \zeta \mathscr{T} G(\varrho(t)),\\
&F_{13} = \zeta J(\varrho(t))A(\varrho(t)) + K(\varrho(t))C(\varrho(t)),\\
&F_{23} = E(\varrho(t))A(\varrho(t)) + \zeta \mathscr{T} K(\varrho(t))C(\varrho(t)),\\
&F_{33} = X(\varrho(t))A(\varrho(t)) + \zeta \mathscr{T} K(\varrho(t))C(\varrho(t)),\\
&F_{53} = Z(\varrho(t))A(\varrho(t)) + \zeta \mathscr{T} K(\varrho(t))C(\varrho(t)),\\
&F_{15} = \zeta J(\varrho(t))A_d(\varrho(t)), \quad F_{25} = E(\varrho(t))A_d(\varrho(t)),\\
&F_{35} = X(\varrho(t))A_d(\varrho(t)), \quad F_{55} = Z(\varrho(t))A_d(\varrho(t)).
\end{aligned}
$$

Then $L(\varrho(t)) = G(\varrho(t))^{-1}K(\varrho(t))$ is a scheduling static output-feedback gain control ensuring that system (5.2) is asymptotically stable for any time-varying delay $h(t)$ satisfying (2.6).

Proof. Use Theorem 3.1 as starting point and consider the augmented vector $\xi(t)$ given by:

$$
\xi(t) = \operatorname{col}\left\{ \begin{bmatrix} u(t) \\ \dot{x}(t) \\ x(t) \\ x(t-h_1) \\ x(t-h(t)) \\ x(t-h_2) \end{bmatrix}, \begin{bmatrix} \frac{1}{h_1}\int_{t-h_1}^{t} x(\delta)d\delta \\ \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x(\delta)d\delta \\ \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x(\delta)d\delta \\ \frac{2}{h_1^2}\int_{-h_1}^{0}\int_{t+\beta}^{t} x(\delta)d\delta d\beta \\ \frac{2}{(h(t)-h_1)^2}\int_{-h(t)}^{-h_1}\int_{t+\beta}^{t-h_1} x(\delta)d\delta d\beta \\ \frac{2}{(h_2-h(t))^2}\int_{-h_2}^{-h(t)}\int_{t+\beta}^{t-h(t)} x(\delta)d\delta d\beta \end{bmatrix} \right\}.
$$

Taking $\xi(t)$ given above, the matrices $\Gamma(h)$, Y, $\widetilde{Q}$, G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7, and G_8 are redefined following the same reasoning applied along the proof of Theorem 3.1.

Consider the closed-loop system (5.2) and let

$$
\mathscr{B} = \begin{bmatrix} B(\varrho(t)) & -I & A(\varrho(t)) & 0 & A_d(\varrho(t)) & 0_{n\times 7n} \\ -I & 0 & L(\varrho(t))C(\varrho(t)) & 0 & 0 & 0_{m\times 7n} \end{bmatrix},
$$

such that $\mathscr{B}\xi(t) = 0$.

Then, according to Finsler's Lemma, $\xi^T(t)\mathscr{Q}\xi(t) < 0$ (with $\mathscr{Q}$, given in (3.7)) is equivalent to:

$$
\mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}^T\mathscr{X}^T < 0,
$$

with

$$\mathscr{X} = \begin{bmatrix} \zeta J(\varrho(t)) & G(\varrho(t)) \\ E(\varrho(t)) & \zeta \mathscr{T} G(\varrho(t)) \\ X(\varrho(t)) & \zeta \mathscr{T} G(\varrho(t)) \\ 0 & 0 \\ Z(\varrho(t)) & \zeta \mathscr{T} G(\varrho(t)) \\ 0_{7n\times n} & 0_{7n\times m} \end{bmatrix}.$$

Substituting $K(\varrho(t)) = G(\varrho(t))L(\varrho(t))$, the scheduled static output-feedback gain that guarantees the asymptotic stability of the closed-loop system (5.2) follows in a straightforward way: $L(\varrho(t)) = G(\varrho(t))^{-1}K(\varrho(t))$. This concludes the proof. □

Remark 5.1

Notice that matrix $\mathscr{T}$ is used only to achieve less conservative results, namely, the given matrix $\mathscr{T}$ is not used to make the static output-feedback problem convex.

Remark 5.2

As in Theorem 4.1, the matrix Y multiplies matrix $P(\varrho(t))$ in Equation (3.3) but Y does not contain any of the system matrices. This is due to the new proposed approach used for selecting the augmented vector. Then, there is no need to establish a particular structure to the Lyapunov matrices in the control design, as done in the current literature [62], since the Lyapunov matrices and the system matrices are decoupled. This is also a new strategy for designing gain-scheduled static output-feedback controllers for delayed LPV systems.

5.2 EXAMPLE

The next example illustrates the effectiveness of the proposed Theorem 5.1. The example is motivated by the model of a magnetic suspension system investigated in [81, Example 1]. The simulations are implemented in Matlab, a finite set of LMIs is automatically obtained by employing the ROLMIP parser [75] that works jointly with Yalmip and the solver MOSEK.

Consider a magnetic suspension system [81]. Magnetic suspension is a system where an object is suspended without a physical support, using only magnetic fields. In this systems, a magnetic force is used to counteract the effects of forces caused by gravitational acceleration and any other accelerations. Magnetic suspensions systems are commonly found in high-speed trains, magnetic bearings, gyroscopes, and accelerometers. The simplified geometry of a magnetic suspension system is depicted in Figure 5.1.

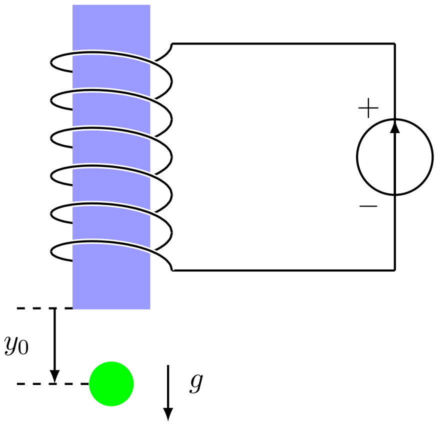

Figure 5.1: Simplified geometry of magnetic suspension system.

The magnetic suspension system is composed by a ball and a variable electromagnet. The force acting on the ball is function of the gravitational force and the attraction force of the electromagnet. The equations that describe the system are given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{g\mu\left(\mu x_1(t) + 2\mu y_0 + 2\right)x_1(t)}{\left(1+\mu\left(x_1(t)+y_0\right)\right)^2}x_1(t) - \frac{K_m}{m}x_2(t) + \frac{\lambda\mu}{2m\left(1+\mu\left(x_1(t)+y_0\right)\right)^2}u(t), \end{aligned} \tag{5.3}$$

where K_m is the viscous friction coefficient, λ is the inductance, m is the mass of the suspended ball. Further, $x_1(t)$ and $x_2(t)$ are, respectively, the ball deviation around its desired position and vertical velocity. The position y_0 is the desired vertical position. The acceleration g denotes the gravity acceleration, μ is the coefficient of inductance variation, and $u(t)$ denotes the control input. Assuming a delay in the state $x_2(t)$ due to practical sensor dynamics in the measured velocity, Equation (5.3) can be rewritten as:

$$\begin{aligned} \dot{x}_1(t) =& cx_2(t) + (1-c)x_2(t-h(t)) \\ \dot{x}_2(t) =& \frac{g\mu\left(\mu x_1(t) + 2\mu y_0 + 2\right)x_1(t)}{\left(1+\mu\left(x_1(t)+y_0\right)\right)^2}x_1(t) - c\frac{K_m}{m}x_2(t) - (1-c)\frac{K_m}{m}x_2(t-h(t)) \\ &+ \frac{\lambda\mu}{2m\left(1+\mu\left(x_1(t)+y_0\right)\right)^2}u(t), \end{aligned} \tag{5.4}$$

where $c = 1$ means there is no delay as in Equation (5.3), and as c decreases the dynamics of the sensor induces delay in the system.

In addition, consider that the measured output is:

$$y(t) = Cx(t), \tag{5.5}$$

where $y(t) \in \mathbb{R}^{n_y \times n}$ is the output vector.

Equations (5.4) and (5.5) can be rewritten as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & c \\ \frac{g\mu(\mu x_1(t)+2\mu y_0+2)x_1(t)}{(1+\mu(x_1(t)+y_0))^2} & -c\frac{K_m}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 1-c \\ 0 & -(1-c)\frac{K_m}{m} \end{bmatrix} \begin{bmatrix} x_1(t-h(t)) \\ x_2(t-h(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\lambda\mu}{2m(1+\mu(x_1(t)+y_0))^2} \end{bmatrix} u(t), \tag{5.6}$$
$$y(t) = C \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Consider the parameters: $m = 0.068$, $K_m = 0.001$, $\lambda = 0.46$, $g = 9.8$, $\mu = 2$, $c = 0.7$ and $y_0 = 0.05$. The time varying delay $h(t)$ satisfies (2.6). The physical structure of the assembling imposes that $0 \leq \bar{x}_1(t) \leq 0.1$, where $\bar{x}_1(t) = x_1(t) + y_0$ and, thus, we have $-0.05 \leq x_1(t) \leq 0.05$. In order to obtain an LPV system, consider the schedules parameters

$$\alpha_1(t) = \frac{g\mu\left(\mu x_1(t) + 2\mu y_0 + 2\right) x_1(t)}{\left(1 + \mu\left(x_1(t) + y_0\right)\right)^2}, \ \alpha_2(t) = \frac{\lambda\mu}{2m\left(1 + \mu\left(x_1(t) + y_0\right)\right)^2},$$

such that $\alpha_1(t) \in [\underline{\alpha}_1 \ \overline{\alpha}_1]$ and $\alpha_2(t) \in [\underline{\alpha}_2 \ \overline{\alpha}_2]$ are scheduling parameters, which are measurable in real-time and can be used to obtain a gain-scheduled controller. Substituting the scheduling parameters in Equation (5.6), it follows the time-delayed LPV system

$$\dot{x}(t) = \begin{bmatrix} 0 & c \\ \alpha_1(t) & -c\frac{K_m}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1-c \\ 0 & -(1-c)\frac{K_m}{m} \end{bmatrix} x(t-h(t)) + \begin{bmatrix} 0 \\ \alpha_2(t) \end{bmatrix} u(t)$$
$$y(t) = Cx(t).$$

Substituting the given parameters values the scheduling parameters are $\alpha_1(t) \in [-2.058 \ 1.5653]$, where $|\mathrm{d}\alpha_1(t)/\mathrm{d}t| = \left|\frac{47.43}{(2x_1(t)+1.1)^3}\right| \leqslant 47.43$ and $\alpha_2(t) \in [4.6977 \ 6.7647]$, where $|\mathrm{d}\alpha_2(t)/\mathrm{d}t| = \left|\frac{-27.05}{(2x_1(t)+1.1)^3}\right| \leqslant 27.05$. The combination of these parameters belongs to a unit simplex as in (2.8). Thus, an LPV system with 4 vertices can be obtained

$$A_1 = A_2 = \begin{bmatrix} 0 & 0.7000 \\ -2.0580 & -0.0103 \end{bmatrix}, \ A_3 = A_4 = \begin{bmatrix} 0 & 0.7000 \\ 1.5653 & -0.0103 \end{bmatrix},$$
$$A_{d_1} = A_{d_2} = A_{d_3} = A_{d_4} = \begin{bmatrix} 0 & 0.3000 \\ 0 & -0.0044 \end{bmatrix}, \tag{5.7}$$
$$B_1 = B_3 = \begin{bmatrix} 0 \\ 4.6988 \end{bmatrix}, \ B_2 = B_4 = \begin{bmatrix} 0 \\ 6.7647 \end{bmatrix}.$$

The time-derivative of $\varrho_1(t)$, $\varrho_2(t)$, $\varrho_3(t)$, and $\varrho_4(t)$ are given by

$$\begin{aligned}
\dot{\varrho}_1(t) &= \frac{\dot{\alpha}_1(t)(\overline{\alpha}_1 - \underline{\alpha}_1)}{(\overline{\alpha}_1 - \underline{\alpha}_1)^2} = \frac{\dot{\alpha}_1(t)}{3.6233},\\
\dot{\varrho}_2(t) &= -\dot{\varrho}_1(t),\\
\dot{\varrho}_3(t) &= \frac{\dot{\alpha}_2(t)(\overline{\alpha}_2 - \underline{\alpha}_2)}{(\overline{\alpha}_2 - \underline{\alpha}_2)^2} = \frac{\dot{\alpha}_2(t)}{2.067},\\
\dot{\varrho}_4(t) &= -\dot{\varrho}_3(t).
\end{aligned}$$

In this example, consider that the measured output is:

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Furthermore, consider also that:

$$\zeta = 1.6, \;\; \mathscr{T} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In the sequel, some experiments are discussed for the design of gain-scheduling controllers of the time-varying delayed LPV system describing the magnetic suspension system. The control design is based on the new result proposed in Theorem 5.1. In the first experiment, Case I, it is investigated the performance of the gain-scheduled static output-feedback controller to stabilize the magnetic suspension system given an initial condition. In the second one, Case II, the objective is to find the controller that ensures stability for a maximum time-delay for a given sensor dynamics in the measured velocity coefficient, c.

- Case I – Stabilization of (5.4)-(5.5) for a given initial condition

In this case, the goal is to investigate the performance of the gain-scheduled controller proposed in Theorem 5.1 to stabilize the time-delayed LPV system which vertices are depicted in (5.7). Consider a time-varying delay depicted in Figure 5.2.

At 0s to 2s and 8s to 12s of the simulation the delay is constant and its value is 1 second. At 2s to 3s and 7s to 8s, the delay varies in time, as depicted in Figure 5.2. For this simulation, the initial condition is $x(0) = \begin{bmatrix} -0.05 & 0.15 \end{bmatrix}^T$.

Clearly if the system is in open-loop it is unstable (the ball falls) and it is not necessary to depict its state trajectories. In Figure 5.3, it is possible to notice the performance of the controller stabilizing the system (5.4)-(5.5), when closing the loop for the given initial condition, which shows the controller effective.

- Case II – Maximum allowable time-delay

In this case, the purpose is to find, for a given sensor dynamics in the measured velocity coefficient c, the maximum allowed delay $\bar{\tau} \in [0, h_2]$ such that a controller designed via Theorem 5.1 stabilizes the system (5.4)-(5.5). Several tests have been performed considering G and K fixed in order to obtain a fixed-gain controller. Also, it is performed tests considering $G(\varrho(t))$ and

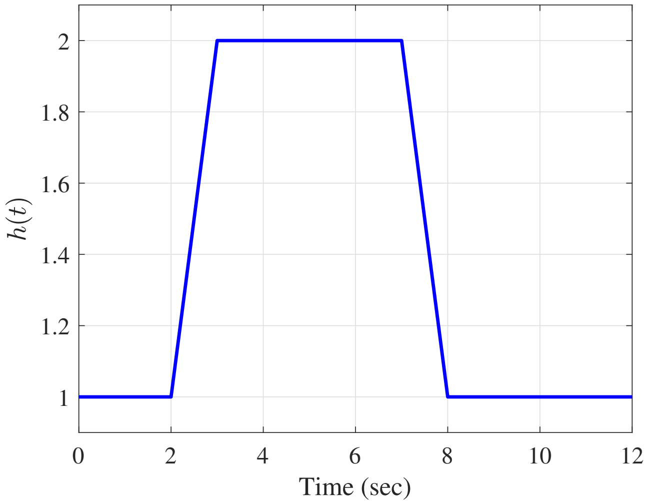

Figure 5.2: Case I – Time-varying delay.

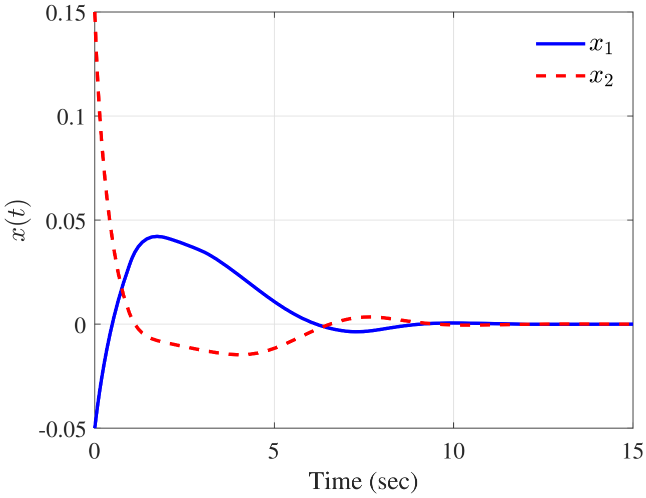

Figure 5.3: Case I – Trajectories of x_1 and x_2 - closed-loop magnetic suspension system.

$K(\varrho(t))$ depending on the parameter $\varrho(t)$ to obtain a scheduled-gain controller. The result of this analysis is depicted in Figure 5.4. It can be seen that the scheduled-gain controller presents better results than a fixed-gain controller.

Remark 5.3

Notice that in this example, the results may be better or, at least achieve less conservative results, if one searches for another combination for the controller matrix $\mathscr{T}$ and scalar ζ.

5.3 CHAPTER CONCLUSIONS

This chapter investigated the static output-feedback stabilization analyses of time-delayed LPV systems. Firstly, a novel gain-scheduled static output-feedback stabilization result for LPV time-delayed systems is stated. It is worth mentioning there is no need to establish a specific structure

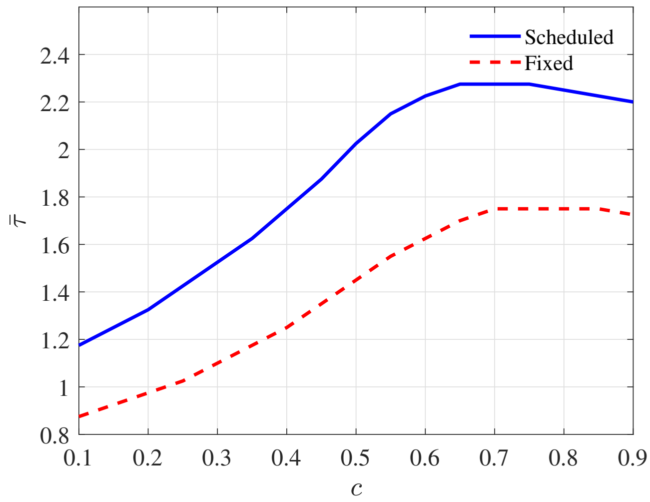

Figure 5.4: Case II - Maximal allowable time delay obtained via Theorem 5.1

to the Lyapunov matrices. Finally, the results presented in the example illustrate the effectiveness of the proposed SOF control design in Theorem 5.1.

6 CONCLUSIONS AND POSSIBLE FUTURE DIRECTIONS

In this Master Thesis new results for stability and stabilization of Linear Parameter-varying systems (LPV systems) with polytopic time-varying parameters subject to time-varying delay have been proposed. The considered parameter-dependent Lyapunov-Krasovskii functional has been combined with recent relaxations proposed in the literature for time-delayed systems as, for instance, the Auxiliary function-based integral inequalities for quadratic functions, to estimate the best upper bound to the functional derivative. The new obtained conditions have the advantage of separating the Lyapunov matrices and the system matrices, making it very suitable for describing the new stabilization conditions.

Chapter 3 has provided a novel sufficient stability result for state time-delayed LPV systems employing a parameter-dependent Lyapunov-Krasovskii functional. The use of new relaxations proposed in the literature for time-delay systems, as the reciprocally convex method combined with Moon's inequality, have contributed to achieve less conservative results as discussed in the examples.

Considering as starting point the novel stability conditions presented in Chapter 3, new LMI-based conditions to calculate gain-scheduling state-feedback control gains for state time-delayed LPV systems have been introduced in Chapter 4. The selected augmented vector, which is used in a new fashion, has contributed to decouple the Lyapunov matrices from the system matrices, making it easier to obtain the conditions for stabilization.

On the other hand, Chapter 5 has proposed new LMI conditions to compute gain-scheduling Static output-feedback (SOF) for state time-delayed LPV systems. It is worth mentioning that, unlike most results found in the literature, the derived LMI conditions do not require additional procedure to provide numerical tractable solutions.

Finally, this Thesis has presented a few examples to illustrate the effectiveness of the proposed conditions.

6.1 POSSIBLE FUTURE DIRECTIONS

Following the same strategy proposed in this Master Thesis, new LMI conditions can be derived to stabilization of similar classes of systems and problems. More specifically, the future directions of this work can be summarized as follows:

- to employ different bound techniques to upper bound the derivative of the Lyapunov-Krasovskii functional (LKF). As discussed in Chapter 2, to reduce the conservatism of stability conditions for time-delayed systems, a few different bounding techniques have been proposed in the literature in the recent years. These techniques, usually applied in time-delayed Linear Time-Invariant systems (LTIs), can be extended to deal with time-delayed LPV systems. The stabilization conditions may also be extended to design $\mathscr{H}_\infty$ like controllers;

- as pointed out in Chapter 1, to explore alternative Lyapunov–Krasovskii functionals can lead to less conservative results, for example, the addition of triple integrals. Based on that, less conservative results can be obtained adding higher order integrals, or exploring other classes of LKF;
- as the issue of time-delay is of fundamental importance in Networked Control Systems (NCS), a natural discussion is to extend the proposed approaches to NCS for LPV systems with time-delays induced along the network.

6.2 SUBMITTED JOURNAL PAPER

- L. T. F. de Souza, M. L. C. Peixoto, and R. M. Palhares. "New Gain-Scheduling Control Conditions for Time-Varying Delayed LPV Systems." Currently in revision: *Journal of the Franklin Institute* (2020).

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A
FINITE-DIMENSIONAL LMI RELAXATIONS

The parameter-dependent LMI conditions proposed in Theorem 3.1, Theorem 4.1 and Theorem 5.1 are assumed to be polynomially parameter-dependent, that is, they are of infinite dimension. To illustrate how those conditions can be numerically solved, a general scheme to build finite set of LMIs is given in the sequel.

Consider the multiple summation given by:

$$\Phi(\varrho(t), \dot{\varrho}(t)) = \sum_{i=1}^{N}\sum_{j=1}^{N} \varrho_i(t)\varrho_j(t)\Phi_{ij} + \sum_{k=1}^{N}\sum_{l=1}^{M} \beta_l h_k^l \Phi_k < 0,$$
$$\forall \varrho(t) \in \Lambda_N, \ \forall \dot{\varrho}(t) \in \mathscr{D}. \quad \text{(A.1)}$$

Equation (A.1) is negative definite if

$$\Phi_{ii} + \sum_{k=1}^{N} h_k^l \Phi_k < 0, \quad i = j = 1, \ldots, N, \quad l = 1, \ldots, M,$$
$$\Phi_{ij} + \Phi_{ji} + 2\sum_{k=1}^{N} h_k^l \Phi_k < 0, \ i = 1, \ldots, N-1, \ j = 1+1, \ldots, N, \ l = 1, \ldots, M,$$

where the vectors h^l are the vertices of the polytope $\mathscr{D}$ given in (2.13). Notice that, the second part of (A.1), $(\sum_{k=1}^{N}\sum_{l=1}^{M} \beta_l h_k^l \Phi_k)$, corresponds to the polytopic representation of the matrices that depend on the derivative of the time-varying parameter $\dot{\varrho}(t)$. For more details on how to solve the proposed LMI conditions, see [75].