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Some results on a pseudo-relativistic Hartree equation and on a magnetic Choquard equation

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Belo Horizonte-Minas Gerais

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Orientador: Prof. Dr. Hamilton Prado Bueno.
Coorientador: Prof. Dr. Gilberto de Assis Pereira.

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Some results on a pseudo-relativistic Hartree equation and on a magnetic Choquard equation

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Abstract

In the first part of this work, we consider the asymptotically linear, strongly coupled nonlinear system

$$\begin{cases} \sqrt{-\Delta + m^2}\, u = \dfrac{u^2+v^2}{1+s(u^2+v^2)} u + \lambda v, \\ \sqrt{-\Delta + m^2}\, v = \dfrac{u^2+v^2}{1+s(u^2+v^2)} v + \lambda u, \end{cases}$$

where $m > 0$, $0 < \lambda < m$ and $0 < s < 1/(\lambda + m)$ are constants.

By applying the Nehari-Pohozaev manifold, we prove that our system has a ground state solution.

We also prove that solutions of this system are radially symmetric and belong to $C^{0,\mu}(\mathbb{R}^N)$ for some $0 < \mu < 1$ and each $N > 1$.

In the final part of the work, we consider the stationary magnetic nonlinear Choquard equation

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right) . \frac{f(|u|)}{|u|} u,$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ is a vector potential, V is a scalar potential, $f : \mathbb{R} \to \mathbb{R}$ and F is the primitive of f . Under mild hypotheses, we prove the existence of a ground state solution for this problem. We also prove a simple multiplicity result by applying Ljusternik–Schnirelmann methods.

Contents

Introduction

In this work we deal with two types of differential equations. The first one is a pseudo-relativistic Hartree system and the second a magnetic Choquard equation. In both problems we consider the existence of a ground state solution, *i. e.*, a minimum energy solution.

In the first chapter we consider the strongly coupled system in $\mathbb{R}^N$

$$\begin{cases} \sqrt{-\Delta + m^2}\, u = \dfrac{u^2+v^2}{1+s(u^2+v^2)} u + \lambda v, \\ \sqrt{-\Delta + m^2}\, v = \dfrac{u^2+v^2}{1+s(u^2+v^2)} v + \lambda u, \end{cases} \tag{0.1}$$

where $0 < s < 1$ and $0 < \lambda < 1$ are constants. Changing $-\Delta u + u$ to $\sqrt{-\Delta + m^2}$, the same system was considered by Lehrer [9]:

$$\begin{cases} -\Delta u + u = \dfrac{u^2+v^2}{1+s(u^2+v^2)} u + \lambda v \\ -\Delta v + v = \dfrac{u^2+v^2}{1+s(u^2+v^2)} v + \lambda u \end{cases} \tag{0.2}$$

However, the pseudo-relativistic operator brings great complexity to this system. The right-hand side of (0.1) still satisfies the nonquadraticity condition formulated by Costa and Magalhães in [5], but the result obtained by Busca and Sirakov [4] - that guarantees that a positive solution (u, v) of (0.2) is necessarily radial and was crucial in Lehrer's paper - can not be applied to the operator $\sqrt{-\Delta + m^2}$.

In the first chapter, after defining the appropriate setting $E = H^{1/2}(\mathbb{R}^N) \times H^{1/2}(\mathbb{R}^N)$ for our problem and the meaning of a solution for the system, we consider the Euler-Lagrange functional Φ and the Pohozaev functional P attached to the problem. The functional Φ does not satisfy the Mountain Pass Geometry.

Therefore, a change of variables (as in [11]) was used to show that each point $z \in E \setminus \{0\}$ has a unique projection on the Nehari-Pohozaev manifold defined by $\mathcal{M} = \{z \in E \setminus \{0\}, J(z) = 0\}$, where $J(z) = 2P(z) - \Phi'(z) \cdot z$. That is, we denote $z_t(x) = \big(tu(t^{-2}x), tv(t^{-2}x)\big) \in \mathbb{R}^2$ and $h_z(t) = \Phi(z_t)$. We prove that, if $t \in (0, t_z]$, then $h_z(t)$ is positive and h_z increasing, whereas h_z is decreasing if $t > t_z$. So, we obtain a kind of Mountain Pass geometry after the change of variable. Finally, we show that $z \to t_z$ is continuous, $z_{t_z} \in \mathcal{M}$ and

$$\Phi(z_{t_z}) = \max_{t>0} \Phi(z_t) > 0.$$

We prove that $\mathcal{M}$ is a C^1 manifold, closed in E, bounded away from 0 and $\Phi > 0$ on $\mathcal{M}$.

As usual, we relate the minimax level of the geometry after the change of variables with the infimum in $\mathcal{M}$.

Since problem (0.1) is asymptotically linear, we consider Cerami sequences, which are shown to be bounded by applying concentration-compactness methods. The existence of a minimizing Cerami sequence is achieved by applying Ghoussoub-Preiss theorem and its convergence results from concentration-compactness methods.

This approach suffices to guarantee the existence of a solution to the problem (0.1). However, we are looking for a ground state solution. Thus, we had to prove that $\mathcal{M}$ is away from $0 \in E$ and that $\mathcal{M}$ is a natural constraint to (0.1). The proof of the last result is tricky and was obtained just before we complete this work, since some terms of the equation are not easy to handle in its natural setting $\mathbb{R}^N$. Thus, we consider the extension setting $\mathbb{R}^{N+1}_+$ and translate some results from there to $\mathbb{R}^N$. By applying Lagrange multipliers, we consider the equation $\Phi'(z) + \mu J'(z) = 0$ and proof the $\mu = 0$. The Pohozaev identity related to $\Phi'(z) + \mu J'(z)$ is obtained in 1.7 where we briefly recall some results about the extension problem.

By applying some results about a modified Bessel kernel and the moving plane method in integral form, we could prove that solutions of the system (0.1) are radially symmetric. The modified Bessel kernel has a key role in the proof that $u, v \in L^2(\mathbb{R}^N)$. In order to handle the immersions attached to the problem, we apply a result stated in D. Adams and L. Hedberg [1].

Considering the same modified Bessel kernel, the Hölder regularity of the solutions of (0.1) was obtained by applying some classical results. The proof is surprisingly simple.

The second chapter of this work deals with the stationary magnetic nonlinear Choquard equation

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\frac{f(|u|)}{|u|}u, \tag{0.3}$$

where $\nabla + iA(x)$ is the covariant derivative with respect to the C^1 vector potential $A : \mathbb{R}^N \to \mathbb{R}^N$, V is a scalar potential, $f\colon \mathbb{R} \to \mathbb{R}$ and F is the primitive of f.

The constant α belongs to the intervals $(0, N)$ and

$$\lim_{|x|\to\infty} A(x) = A_\infty \in \mathbb{R}^N.$$

The scalar potential $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous, bounded function satisfying

$(V1)$ $\inf_{\mathbb{R}^N} V > 0$;

$(V2)$ $V_\infty = \lim_{|y|\to\infty} V(y)$;

$(V3)$ $V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$.

We also suppose that

(AV) $|A(y)|^2 + V(y) < |A_\infty|^2 + V_\infty$.

The function F is the primitive of the nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$, which is non-negative in $(0, \infty)$ and satisfies, for any $r \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$,

$(f1)$ $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 0$,

$(f2)$ $\displaystyle\lim_{t\to\infty} \frac{f(t)}{t^{r-1}} = 0$,

$(f3)$ $\dfrac{f(t)}{t}$ is increasing if $t > 0$ and decreasing if $t < 0$.

Denoting

$$\tilde{f}(t) = \begin{cases} \dfrac{f(t)}{t}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0, \end{cases}$$

our hypotheses imply that $\tilde{f}$ is continuous. Therefore, problem (1.1) can be written in the form

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\tilde{f}(|u|)u. \tag{0.4}$$

The composition of f and F with $|u|$ gives a variational structure to the problem, allowing the application of the Mountain Pass Theorem.

The right-hand side of problem 0.4 can be viewed as a generalization of a problem studied by Cingolani, Clapp and Secchi in [9]. There, they consider the right-hand side of the equation was

$$\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u,$$

but the main part of that paper is devoted to the existence of multiple solutions of equation with the right-hand term (0.4) under the action of a closed subgroup G of the orthogonal group $O(N)$ of linear isometries of $\mathbb{R}^N$ if $A(gx) = gA(x)$ and $V(gx) = V(x)$ for all $g \in G$ and $x \in \mathbb{R}^N$. The authors look for solutions satisfying

$$u(gx) = \tau(g)u(x), \quad \text{for all } g \in G \text{ and } x \in \mathbb{R}^N,$$

where $\tau\colon G \to S^1$ is a given continuous group homomorphism into the unit complex numbers S^1.

In the second chapter we also address the multiplicity of solutions in a particular case of that treated in [9] and the existence of multiple solutions is much simpler in our simplified setting, resting on some classical results.

Denoting

$$\nabla_A u = \nabla u + iA(x)u$$

the natural setting for the problem (0.3) is and consider the space

$$H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) = \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) \,:\, \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C})\right\}$$

endowed with scalar product

$$\langle u, v\rangle_{A,V} = \mathfrak{Re}\int_{\mathbb{R}^N} \left(\nabla_A u \cdot \overline{\nabla_A v} + V(x)u\bar{v}\right)$$

and, therefore

$$\|u\|_{A,V}^2 = \int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2.$$

Observe that the norm generated by this scalar product is equivalent to the norm obtained by considering $V \equiv 1$, see [15, Definition 7.20].

As a consequence of the diamagnetic inequality, we have the continuous immersion

$$H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^q(\mathbb{R}^N, \mathbb{C}) \tag{0.5}$$

for any $q \in [2, \frac{2N}{N-2}]$. We denote $2^* = \frac{2N}{N-2}$.

The energy functional associated to problem (1.1) is given by

$$J_{A,V}(u) = \frac{1}{2}\|u\|_{A,V}^2 - D(u), \tag{0.6}$$

where

$$D(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * F(|u|)\right) F(|u|).$$

The functional $J_{A,V}$ is well-defined as a consequence of the Hardy-Littlewood-Sobolev (see [15, Theorem 4.3], since

$$\left|\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * F(|u|)\right) F(|u|)\right| \leq C\left(\|u\|^4 + \|u\|^{2r}\right). \tag{0.7}$$

We prove that $J_{A,V}$ satisfies the Mountain Pass geometry and we conclude the existence of a minimizing sequence $(u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$.

By considering the Nehari manifold $\mathcal{N}_{A,V}$ naturally attached to the problem, we show the existence of $\beta > 0$ such that $\|u\|_{A,V} \geq \beta$ for all $u \in \mathcal{N}_{A,V}$. As in the first chapter, $\max_{t\geq 0} \Phi(t)$ is achieved at a unique $t_u = t(u) > 0$ and $\Phi'(tu) > 0$ for $t < t_u$ and $\Phi'(tu) < 0$ for $t > t_u$. Furthermore, $\Phi'(t_u u) = 0$ implies that $t_u u \in \mathcal{N}_{A,V}$. Also as in the first chapter, the infimum in the Nehari manifold is proved to be equal the minimax level c.

By considering the limit problem

$$-(\nabla + iA_\infty)^2 u + V_\infty u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\frac{f(|u|)}{|u|}u,$$

we relate the minimax level c_∞ of the limit problem and the minimax level c of the problem (0.3) by showing that $0 < c < c_i nfty$. By applying Struwe's splitting lemma, we show that $J_{A,V}$ satisfies the Palais-Smale condition $(PS)_c$ for any $0 \leq c < c_\infty$. Thus, we obtain a ground state solution.

By applying a compactness result obtained by P.L. Lions and a result of Szulkin and Weth [20, Theorem 2] that uses Ljusternik-Schnirelmann methods, we obtain multiplicity of solutions for problem (0.3).

Chapter 1

Results on a strongly coupled, asymptotically linear pseudo-relativistic Hartree equation: ground state, radial symmetry and Hölder regularity

Abstract

In this chapter we consider the asymptotically linear, strongly coupled nonlinear system

$$\begin{cases} \sqrt{-\Delta + m^2}\, u = \dfrac{u^2+v^2}{1+s(u^2+v^2)}u + \lambda v, \\ \sqrt{-\Delta + m^2}\, v = \dfrac{u^2+v^2}{1+s(u^2+v^2)}v + \lambda u, \end{cases}$$

where $0 < s < 1$ and $0 < \lambda < 1$ are constants.

By applying the Nehari-Pohozaev manifold, we prove that our system has a ground state solution.

We also prove that solutions of this system are radially symmetric and belong to $C^{0,\mu}(\mathbb{R}^{\mathbb{N}})$ for some $0 < \mu < 1$.

1.1 Introduction

The Schrödinger equation represents well the propagation of a light beam. However, the interaction with different media can produce nonlinear effects. To study this phenomena, a weakly coupled system of Schrödinger equations was proposed by Weilnau, Ahles, Petter, Träger, Schröder and Denz [13].

The nonlinear weakly coupled system of Schrödinger equations

$$\begin{cases} i\varphi_t + \Delta\varphi + \dfrac{\alpha\left(|\varphi|^2+|\psi|^2\right)}{1+\left(|\varphi|^2+|\psi|^2\right)/I_0}\varphi = 0 \\ i\psi_t + \Delta\psi + \dfrac{\alpha\left(|\varphi|^2+|\psi|^2\right)}{1+\left(|\varphi|^2+|\psi|^2\right)/I_0}\psi = 0 \end{cases}$$

describes the propagation of a beam with two mutually incoherent components in a saturable medium. In the system above, φ and ψ stand for the amplitude of the components of the beam, α is a parameter, $|\varphi|^2 + |\psi|^2$ describes the intensity generated by the incoherent components of the beam, and I_0 is the saturation parameter.

Solitary waves with multiple coupled components are expected to arise. Considering solitary waves solutions $\varphi(x,t) = \sqrt{\alpha}u(x)e^{i\lambda_1 t}$ and $\psi(x,t) = \sqrt{\alpha}v(x)e^{i\lambda_2 t}$ for real functions u, v and propagation constants λ_1, λ_2, we obtain the weakly coupled elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{u^2+v^2}{1+s(u^2+v^2)}u \\ -\Delta v + \lambda_1 v = \dfrac{u^2+v^2}{1+s(u^2+v^2)}v, \end{cases}$$

where $s = \alpha/I_0$.

In Lehrer [9], the strongly coupled autonomous system

$$\begin{cases} -\Delta u + u = \dfrac{u^2+v^2}{1+s(u^2+v^2)}u + \lambda v \\ -\Delta v + v = \dfrac{u^2+v^2}{1+s(u^2+v^2)}v + \lambda u \end{cases} \tag{1.1}$$

was studied. Its solution depends heavily on a result obtained by Busca and Sirakov [4], that guarantees that a positive solution (u, v) of (1.1) is necessarily radial. So, applying the Mountain Pass Theorem in $H^1_{rad}(\mathbb{R}^N)$, the nonquadraticity condition (see Lemma 1.6 in the sequel), concentration-compactness methods and the principle of symmetric criticality, a ground state solution was obtained.

In this chapter we consider the system in $\mathbb{R}^N$

$$\begin{cases} \sqrt{-\Delta + m^2}\, u = \dfrac{u^2+v^2}{1+s(u^2+v^2)}u + \lambda v, \\ \sqrt{-\Delta + m^2}\, v = \dfrac{u^2+v^2}{1+s(u^2+v^2)}v + \lambda u, \end{cases} \tag{1.2}$$

where $m > 0$ $0 < s < 1$ and $0 < \lambda < 1$ are constants. Changing $-\Delta u + u$ to $\sqrt{-\Delta + m^2}$ brings great complexity to this system. The right-hand side of (1.2) still satisfies the nonquadraticity condition, but the result obtained by Busca and Sirakov [4] can not be applied to the operator $\sqrt{-\Delta + m^2}$.

Fractional Laplacian operators are the infinitesimal generators of Lévy stable diffusion processes, which have application in several areas, e.g., anomalous diffusion of plasmas, probability, finances and populations dynamics.

Our main result is the following. By a ground state we mean a minimal energy solution.

Theorem 1.1 *Suppose that* $N > 1$ *and* $(N+1)\lambda/N < m < \frac{1}{s} - \lambda$. *Then problem* (1.2) *has a positive ground state solution, for any* $0 < s < 1$ *and* $0 < \lambda < 1$.

In this chapter, after defining the appropriate setting for our problem, that is, $E = H^{1/2}(\mathbb{R}^N) \times H^{1/2}(\mathbb{R}^N)$, we define a solution for the system and consider the Euler-Lagrange functional Φ and the Pohozaev functional P attached to the problem. Changing variables, we show that each point $z \in E \setminus \{0\}$ has a unique projection on the Nehari-Pohozaev manifold defined by $\mathcal{M} = \{z \in E \setminus \{0\}, J(z) = 0\}$, where $J(z) = 2P(z) + \Phi'(z) \cdot z$.

The main step it then to prove that $\mathcal{M}$ has all the good properties and that is a natural constraint to our problem. To do this, we adapt some ideas of D. Qin, J. Chen and X. Tang [11]. To prove that any critical point of $\Phi|_{\mathcal{M}}$ is a critical point of Φ in E, we apply Lagrange multipliers and obtain the Nehari-Pohozaev functional attached to $\mathcal{M}$ by considering its expression in the extension setting $\mathbb{R}^{N+1}_+$, since some terms in $\mathbb{R}^{\mathbb{N}}$ are not easily handled. After some tricky calculations we conclude our proof. Some results about the extension problem are presented in an Appendix, see Section 1.7.

Then, relating the minimax value with the infimum of Φ in $\mathcal{M}$, the proof is nowadays standard. Since our problem is asymptotically linear at infinity, Cerami sequence are shown to be bounded. The existence of Cerami sequence is obtained by Ghoussoub-Preiss theorem and its convergence by concentration-compactness methods.

We also prove that solutions of the system (1.2) are radially symmetric. This is achieved by applying some results about a modified Bessel kernel and the moving plane method in integral form. The key step in this proof applies a result stated in D. Adams and L. Hedberg [1].

Theorem 1.2 *Any solution of* (1.2) *is radially symmetric and decreasing with respect to some point.*

Finally, we also prove the Hölder regularity of the solutions of (1.2) by applying some classical results and the same modified Bessel kernel used before.

Theorem 1.3 *If* $N \geq 1$*, then any solution of* (1.2) *belongs to* $C^{0,\mu}(\mathbb{R}^N)$ *for* $0 < \mu < \alpha - \frac{N}{2}$.

1.2 Functional setting

A usual definition of the operator $\sqrt{-\Delta + m^2}$ via Bessel potential is given by (see [2, 7])

$$\sqrt{-\Delta + m^2}u = C_N\, m^{\frac{N+1}{2}} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{\frac{N+1}{2}}} K_{\frac{N+1}{2}}(m|x-y|)\mathrm{d}y + mu(x). \tag{2.1}$$

Denoting by $\hat{u} = \mathcal{F}(u)$ the Fourier transform, the natural setting for problem (1.2) is the Hilbert space $E = H^{1/2}(\mathbb{R}^N) \times H^{1/2}(\mathbb{R}^N)$. That is, denoting $z(\cdot) = (u(\cdot), v(\cdot))$

$$E = \left\{ z \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \;:\; \int_{\mathbb{R}^N} 2\pi|\xi| \left(|\hat{u}(\xi)|^2 + |\hat{v}(\xi)|^2 \right) \mathrm{d}\xi < \infty \right\}$$

endowed with the norm

$$\|z\|_E^2 = \int_{\mathbb{R}^N} (1 + 2\pi|\xi|)|\hat{z}(\xi)|^2 \mathrm{d}\xi,$$

where $|z(\cdot)|^2 = |u(\cdot)|^2 + |v(\cdot)|^2$. (In other words, $|\cdot|$ denotes the norm generated by the natural inner product of $\mathbb{R}^2$).

If $\mathcal{F}(z)$ stands for $(\mathcal{F}(u), \mathcal{F}(v))$, by defining

$$(-\Delta + m^2)^{1/2} z = \mathcal{F}^{-1}\left((m^2 + 4\pi^2|\xi|^2)^{1/2}\mathcal{F}(z)(\xi)\right)$$

an equivalent norm in the space E is given by

$$\begin{aligned}
\|z\|^2 &= \int_{\mathbb{R}^N}\left[\left|(-\Delta + m^2)^{1/4}u(x)\right|^2 + \left|(-\Delta + m^2)^{1/4}v(x)\right|^2\right]\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\left[(m^2 + 4\pi^2|\xi|^2)^{1/2}|\hat{u}(\xi)|^2 + (m^2 + 4\pi^2|\xi|^2)^{1/2}|\hat{v}(\xi)|^2\right]\mathrm{d}\xi \\
&= \int_{\mathbb{R}^N}(m^2 + 4\pi^2|\xi|^2)^{1/2}|\hat{z}(\xi)|^2\mathrm{d}\xi = \int_{\mathbb{R}^N}\left|(-\Delta + m^2)^{1/4}z(x)\right|^2\mathrm{d}x. \qquad (2.2)
\end{aligned}$$

By applying (2.2), it follows from (2.1) that

$$\|z\|^2 = C_N\, m^{\frac{N+1}{2}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x) - z(y)|^2}{|x - y|^{\frac{N+1}{2}}}K_{\frac{N+1}{2}}(m|x - y|)\mathrm{d}y\mathrm{d}x + m\int_{\mathbb{R}^N}|z(x)|^2\mathrm{d}x.$$

Let us denote $|z| = (|u|, |v|)$. Then $|z| \in E$ and $|\,|z(x)| - |z(y)|\,| \leq |z(x) - z(y)|$. We conclude that

$$\|\,|z|\,\|^2 \leq \|z\|^2. \qquad (2.3)$$

,

an inequality that will be useful when addressing positive solutions.

We consider the space E endowed with the norm (2.2). If $\varphi = (\varphi_1, \varphi_2) \in E$, this equivalent norm derives from the inner product

$$\langle z, \varphi\rangle = \int_{\mathbb{R}^N}(m^2 + 4\pi^2|\xi|^2)^{1/2}\,\hat{z}(\xi)\cdot\hat{\varphi}(\xi)\,\mathrm{d}\xi,$$

where $\hat{z}(\xi)\cdot\hat{\varphi}(\xi) = \hat{u}(\xi)\hat{\varphi}_1(\xi) + \hat{v}(\xi)\hat{\varphi}_2(\xi)$ is the natural inner product in $\mathbb{R}^2$.

We define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F(\zeta, \eta) = \frac{\zeta^2 + \eta^2}{2s} - \frac{1}{2s^2}\ln\left(1 + s(\zeta^2 + \eta^2)\right) + \lambda\zeta\eta.$$

Using the symmetry of the operator $(-\Delta + m^2)^{1/2}$ in the space $\mathcal{S}(\mathbb{R}^N)$, we achieve the natural definition that $z = (u, v)$ is a weak solution of the problem (1.2),

Definition 1.4 *We say that $z \in E$ is a weak solution of* (1.2) *if*

$$\int_{\mathbb{R}^N} (-\Delta + m^2)^{1/4} z(x)\ (-\Delta + m^2)^{1/4} \varphi(x) \mathrm{d}x - \int_{\mathbb{R}^N} \nabla F(z(x)) \cdot \varphi(x) \mathrm{d}x = 0,$$

for all $\varphi \in E$.

The Euler-Lagrange functional attached to (1.2) is given by

$$\begin{aligned} \Phi(z) &= \frac{1}{2} \int_{\mathbb{R}^N} (m^2 + 4\pi^2 |\xi|^2)^{1/2} |\hat{z}(\xi)|^2 \,\mathrm{d}\xi - \int_{\mathbb{R}^N} F(z(x)) \mathrm{d}x \\ &= \frac{1}{2} \|z\|^2 - \int_{\mathbb{R}^N} F(z(x)) \mathrm{d}x. \end{aligned} \tag{2.4}$$

Since the derivative of the energy functional is

$$\Phi'(z) \cdot \varphi = \langle z, \varphi \rangle - \int_{\mathbb{R}^N} \nabla F(z(x)) \cdot \varphi(x) \,\mathrm{d}x.$$

we see that critical points of Φ are weak solution to (1.2). We look for critical points $z = (u, v)$ of Φ such that $u \geq 0$, $v \geq 0$ and $u(x) \to 0,\ v(x) \to 0$ as $|x| \to \infty$.

Since system (1.2) is asymptotically linear at infinity, not all functions in E can be projected on the Nehari manifold naturally attached to (1.2), see [6]. To overcome this problem, we deal with the Pohozaev functional attached to (1.2), namely

$$P(z) = \frac{N-1}{2} \|z\|^2 + \frac{m^2}{2} \int_{\mathbb{R}^N} \frac{|\hat{z}(\xi)|^2 \,\mathrm{d}\xi}{\sqrt{m^2 + 4\pi^2 |\xi|^2}} - N \int_{\mathbb{R}^N} F(z(x)) \,\mathrm{d}x. \tag{2.5}$$

Solutions to (1.2) satisfy $P(z) = 0$ (see [6]).

1.3 Preliminaries

Let us denote $|\cdot|_p$ the norm in $L^p(\mathbb{R}^N)$. We recall that there exist a constant γ_p such that, for all $2 < p < 2^* = \frac{2N}{N-1}$, it is true

$$\int_{\mathbb{R}^N} |z(x)|^p \mathrm{d}x \leq \gamma_p^p \|z\|^p. \tag{3.1}$$

We also note that

$$m\int_{\mathbb{R}^N}|z(x)|^2\mathrm{d}x \le \int_{\mathbb{R}^N}(m^2+4\pi^2|\xi|^2)^{1/2}|z(\xi)|^2\mathrm{d}\xi = \|z\|^2. \tag{3.2}$$

The next result will be helpful in the sequence.

Lemma 1.5 *We have*

$$\nabla F(\xi,\eta)=\left(\frac{\xi^2+\eta^2}{1+s(\xi^2+\eta^2)}\xi+\lambda\eta,\frac{\xi^2+\eta^2}{1+s(\xi^2+\eta^2)}\eta+\lambda\xi\right):=(f(\xi,\eta),g(\xi,\eta)),$$

$$K(\xi,\eta):=\nabla F(\xi,\eta)\cdot(\xi,\eta)=\frac{(\xi^2+\eta^2)^2}{1+s(\xi^2+\eta^2)}+2\lambda\xi\eta,$$

$$\nabla K(\xi,\eta)=\left(\frac{2\xi(\xi^2+\eta^2)(2+s(\xi^2+\eta^2))}{(1+s(\xi^2+\eta^2))^2}+2\lambda\eta,\frac{2\eta(\xi^2+\eta^2)(2+s(\xi^2+\eta^2))}{(1+s(\xi^2+\eta^2))^2}+2\lambda\xi\right)$$

and

$$\nabla K(\xi,\eta)\cdot(\xi,\eta)=\frac{2(\xi^2+\eta^2)^2(2+s(\xi^2+\eta^2))}{(1+s(\xi^2+\eta^2))^2}+4\lambda\xi\eta.$$

Lemma 1.6 *The function $F\colon\mathbb{R}^2\to\mathbb{R}$ satisfies the nonquadraticity condition (NQ), that is,*

(i) $\forall\,(\zeta,\eta)\in\mathbb{R}^2,\ \frac{1}{2}\nabla F(\zeta,\eta)\cdot(\zeta,\eta)-F(\zeta,\eta)\ge 0;$

(ii) $\lim_{|(\zeta,\eta)|\to\infty}\left(\frac{1}{2}\nabla F(\zeta,\eta)\cdot(\zeta,\eta)-F(\zeta,\eta)\right)=\infty.$

The nonquadraticity condition was formulated by Costa and Magalhães in [5] and the proof of Lemma 1.6 can be found in [10, Lemma 2.1].

The simple proof of the next result will be omitted.

Proposition 1.7 *The function $Q(t)=\frac{1}{2}\nabla F(t\zeta,t\eta))\cdot(t\zeta,t\eta)-F(t\zeta,t\eta)$ is increasing for any $t>0$, where $(\xi,\eta)\neq(0,0)$.*

We define $H\colon\mathbb{R}^2\to\mathbb{R}$ by

$$H(\zeta,\eta)=\frac{|\zeta|^2+|\eta|^2}{2s}-\frac{1}{2s^2}\ln\left(1+s(|\zeta|^2+|\eta|^2)\right)$$

The proof of the next results is straightforward.

Lemma 1.8 *The function H satisfies, for all* $2 < p < 2^*$ *and* $w = (\zeta, \eta) \in \mathbb{R}^2$

$$|H(\zeta,\eta)| < \frac{\epsilon}{2}\left(|\zeta|^2 + |\eta|^2\right) + C_\epsilon \left(|\zeta|^2 + |\eta|^2\right)^{p/2},$$
$$|\nabla H(\zeta,\eta)\cdot w| \le \frac{\epsilon}{2}\left(|\zeta|^2 + |\eta|^2\right) + 2C_\epsilon \left(|\zeta|^2 + |\eta|^2\right)^{p/2},$$

where the constant C_ϵ *depends on* ϵ.

1.4 Ground state solution

It is easy to see that the functional Φ does not satisfy the geometry of the Mountain Pass Theorem. Thus, we change variables and, for any $z = (u,v) \in E$, $t > 0$ and $x \in \mathbb{R}^N$, we denote

$$z_t(x) = \left(tu(t^{-2}x), tv(t^{-2}x)\right) \in \mathbb{R}^2 \qquad \text{and} \qquad h_z(t) = \Phi(z_t).$$

Changing variables, we obtain

$$\begin{aligned}
h_z(t) &= \frac{1}{2}\|z_t\|^2 - \int_{\mathbb{R}^N} F(z_t(x))\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(m^2 + 4\pi^2|\xi|^2\right)^{1/2}|\hat{z_t}(\xi)|^2\mathrm{d}\xi - \int_{\mathbb{R}^N} F(z_t(x))\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(m^2 + 4\pi^2|\xi|^2\right)^{1/2}|tt^{2N}\hat{z}(t^2\xi)|^2\mathrm{d}\xi - t^{2N}\int_{\mathbb{R}^N} F(tz(x))\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(m^2 + 4\pi^2\frac{|t^2\xi|^2}{t^4}\right)^{1/2} t^{2N+2}|\hat{z}(t^2\xi)|^2\mathrm{d}(t^2\xi) - t^{2N}\int_{\mathbb{R}^N} F(tz(x))\mathrm{d}x \\
&= t^{2N}\left(\int_{\mathbb{R}^N}\frac{\sqrt{t^4m^2 + 4\pi^2|\xi|^2}|\hat{z}(\xi)|^2}{2}\mathrm{d}\xi - \int_{\mathbb{R}^N} F(tz(x))\mathrm{d}x\right) && \text{(4.1a)} \\
&= t^{2N+2}\left[\int_{\mathbb{R}^N}\frac{1}{2}\sqrt{m^2 + \frac{4\pi^2|\xi|^2}{t^4}}|\hat{z}(\xi)|^2\mathrm{d}\xi - \int_{\mathbb{R}^N}\frac{F(tz(x))}{t^2|z(x)|^2}|z(x)|^2\mathrm{d}x\right] && \text{(4.1b)}
\end{aligned}$$

Lemma 1.9 *Let us suppose that* $0 < m < \frac{1}{s} - \lambda$. *Then, for any* $z = (u,v) \neq (0,0)$, h_z *satisfies*

(*i*) $\lim_{t\to\infty} h_z(t) = -\infty$;

(*ii*) $h_z(t) > 0$ *for any* $t > 0$ *small enough.*

Therefore, h_z *attains its maximum value in* $(0,\infty)$.

Proof. Since
$$\frac{F(tz(x))}{t^2|z(x)|^2} = \frac{1}{2s} - \frac{1}{2s^2}\frac{\ln(1+st^2|z(x)|^2)}{t^2|z(x)|^2} + \frac{\lambda u(x)v(x)}{|z(x)|^2},$$
making $t \to \infty$ and applying the inequality $uv \geq -(u^2+v^2)/2$ we obtain

$$\lim_{t\to\infty} \frac{F(tz(x))}{t^2|z(x)|^2} \geq \frac{1}{2s} - \frac{\lambda}{2}. \tag{4.2}$$

Thus, by applying the Dominated Convergence Theorem to (4.1b) we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\left(\int_{\mathbb{R}^N}\frac{1}{2}\sqrt{m^2+\frac{4\pi^2|\xi|^2}{t^4}}|\hat{z}(\xi)|^2 d\xi - \int_{\mathbb{R}^N}\frac{F(tz(x))}{t^2|z(x)|^2}|z(x)|^2\mathrm{d}x\right)\\
&\leq \int_{\mathbb{R}^N}\frac{1}{2}m|\hat{z}(\xi)|^2\mathrm{d}\xi - \int_{\mathbb{R}^N}\left(\frac{1}{2s}-\frac{\lambda}{2}\right)|z(x)|^2\mathrm{d}x\\
&= \frac{1}{2}\left(m-\frac{1}{s}+\lambda\right)\int_{\mathbb{R}^N}|z(x)|^2\mathrm{d}x < 0,
\end{aligned}$$

since $m < \frac{1}{s} - \lambda$. This proves (*i*).

It follows from (4.1a) that

$$\frac{h_z(t)}{t^{2N+2}} = \frac{1}{2t^2}\int_{\mathbb{R}^N}(t^4m^4+4\pi^2|\xi|^2)^{1/2}|\hat{z}(\xi)|^2\,\mathrm{d}\xi - \int_{\mathbb{R}^N}\frac{F(tz(x))}{t^2|z(x)|^2}|z(x)|^2\,\mathrm{d}x$$

diverges to $+\infty$ as $t \to 0^+$, since

$$\begin{aligned}
\lim_{t\to 0^+}\frac{F(tz(x))}{t^2|z(x)|^2} &= \lim_{t\to 0^+}\left(\frac{1}{2s}-\frac{1}{2s^2}\frac{\ln(1+st^2|z(x)|^2)}{t^2|z(x)|^2}+\frac{\lambda u(x)v(x)}{|z(x)|^2}\right)\\
&= \lim_{t\to 0^+}\left(\frac{1}{2s}-\frac{1}{2s^2}\frac{s}{(1+st^2|z(x)|^2)}+\frac{\lambda u(x)v(x)}{|z(x)|^2}\right)\\
&= \frac{\lambda u(x)v(x)}{|z(x)|^2} \leq \frac{\lambda}{2},
\end{aligned}$$

because $uv \leq (u^2+v^2)/2$. This proves (*ii*).

Since we can extend $h_z(t)$ continuously by defining $h_z(0) = 0$, we conclude that h_z attains a maximum in $t(z) = t_z > 0$ for any $0 \neq z \in E$. □

We will now show that the maximum point t_z is unique. For this, we observe that

(4.1a) implies

$$h_z'(t) = \int_{\mathbb{R}^N} Nt^{2N-1}\sqrt{t^4m^2+4\pi^2|\xi|^2}|\hat{z}(\xi)|^2 \mathrm{d}\xi + \int_{\mathbb{R}^N} \frac{t^{2N}t^3m^2|\hat{z}(\xi)|^2}{\sqrt{t^4m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi$$
$$- \int_{\mathbb{R}^N}\left[t^{2N}\nabla F(tz(x))\cdot z(x) + 2Nt^{2N-1}F(tz(x))\right]\mathrm{d}x. \tag{4.3}$$

Lemma 1.10 *For each $z \in E \setminus \{0\}$, there exists a unique $t_z = t(z) > 0$ where $h_z\colon (0,\infty) \to \mathbb{R}$ attains its maximum value. If $t \in (0, t_z]$, then $h_z(t)$ is positive and h_z increasing, whereas h_z is decreasing if $t > t_z$.*

Proof. It follows from (4.3) that $h_z'(t) = 0$ if, and only if,

$$t^{2N+1}\left[\int_{\mathbb{R}^N} \frac{m^2t^2|\hat{z}(\xi)|^2}{\sqrt{t^4m^2+4\pi^2|\xi|^2}}d\xi + \int_{\mathbb{R}^N} \frac{N\sqrt{t^4m^2+4\pi^2|\xi|^2}|\hat{z}(\xi)|^2}{t^2}d\xi\right.$$
$$\left. - \int_{\mathbb{R}^N} \frac{\nabla F(tz(x))\cdot tz(x) + 2NF(tz(x))}{t^2}\mathrm{d}x\right] = 0. \tag{4.4}$$

Denoting

$$I_1(t) = 2N\int_{\mathbb{R}^N} \frac{F(tz(x))}{t^2}\mathrm{d}x,$$
$$I_2(t) = \int_{\mathbb{R}^N} \frac{\nabla F(tz(x))\cdot tz(x)}{t^2}\mathrm{d}x$$
$$g(t) = \int_{\mathbb{R}^N} \frac{m^2t^2|\hat{z}(\xi)|^2}{\sqrt{t^4m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi + \int_{\mathbb{R}^N} \frac{N\sqrt{t^4m^2+4\pi^2|\xi|^2}|\hat{z}(\xi)|^2}{t^2}\mathrm{d}\xi,$$

we will show that the function $(g(t) - I_1(t) - I_2(t))$ is (strictly) decreasing by showing that I_1 and I_2 are increasing and g is decreasing.

In fact,

$$I_1'(t) = \int_{\mathbb{R}^N} 2N\left(\frac{t^2\nabla F(tz(x))\cdot z(x) - 2tF(tz(x))}{t^4}\right)\mathrm{d}x$$
$$= 2N\int_{\mathbb{R}^N} \frac{\nabla F(tz(x))\cdot tz(x) - 2F(tz(x))}{t^3}\mathrm{d}x > 0,$$

according to Lemma 1.6.

For all $t > 0$, by lemma (1.5) we also have

$$I_2'(t) = \int_{\mathbb{R}^N} \frac{2t|z(x)|^4}{(1+st^2|z(x)|^2)^2}\mathrm{d}x > 0.$$

Denoting $A(t) = \sqrt{t^4m^2 + 4\pi^2|\xi|^2}$, we have

$$g(t) = \int_{\mathbb{R}^N} |\hat{z}(\xi)|^2 \left[m^2 \left(\frac{t^2}{A(t)} \right) + N \left(\frac{t^2}{A(t)} \right)^{-1} \right] \mathrm{d}\xi.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{t^2}{A(t)} \right) = \frac{8t\pi^2|\xi|^2}{A^3(t)},$$

it follows

$$\begin{aligned} g'(t) &= \int_{\mathbb{R}^N} \frac{8\pi^2|\xi|^2|\hat{z}(\xi)|^2}{A(t)} \left(\frac{tm^2}{A^2(t)} - \frac{N}{t^3} \right) \mathrm{d}\xi \\ &= -\frac{2}{t^3} \int_{\mathbb{R}^N} \frac{4\pi^2|\xi|^2|\hat{z}(\xi)|^2}{\sqrt{t^4m^2 + 4\pi^2|\xi|^2}} \left(N - \frac{t^4m^2}{t^4m^2 + 4\pi^2|\xi|^2} \right) \mathrm{d}\xi < 0, \end{aligned}$$

because the term between parenthesis in the integral is positive, since $N > 1$.

The proof is complete. □

Evaluating $h_z'(t)$ at $t = 1$, by (4.3) and (2.5) we obtain

$$\begin{aligned} h_z'(t)\Big|_{t=1} &= N\|z\|^2 + \int_{\mathbb{R}^N} \frac{m^2|\hat{z}(\xi)|^2}{\sqrt{m^2 + 4\pi^2|\xi|^2}} \mathrm{d}\xi - 2N \int_{\mathbb{R}^N} F(z(x)) \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \nabla F(z(x)) \cdot z(x)\, \mathrm{d}x. \end{aligned} \tag{4.5}$$

Therefore

$$h_z'(t)\Big|_{t=1} = 2P(z) + \Phi'(z) \cdot z = J(z) \tag{4.6}$$

is an expression that motivates the definition of the Nehari-Pohozaev manifold

$$\mathcal{M} = \{ z \in E \setminus \{0\} \ : \ J(z) = 0 \}.$$

Observe that any solution z to (1.2) belongs to $\mathcal{M}$, since $\Phi'(z) \cdot z = 0$ and $P(z) = 0$.

Corollary 1.11 *If $z \in E \setminus \{0\}$, then*

$$z_t \in \mathcal{M} \ \textit{ if and only if } \ J(z_t) = 0 \ \textit{ if and only if } \ h_{z_t}'(1) = 0 \ \textit{ if and only if } \ h_z'(t) = 0.$$

Proof. An immediately consequence of $h_{z_t}'(1) = t h_z'(t)$, an equality that follows from (4.3). □

Lemma 1.12 *For $z \in E \setminus \{0\}$, let t_z be the unique maximum point of h_z. Suppose that $m < \frac{1}{s} - \lambda$. Then the function $z \to t_z$ is continuous, $z_{t_z} \in \mathcal{M}$ and*

$$\Phi(z_{t_z}) = \max_{t>0} \Phi(z_t) > 0.$$

Proof. We only need to prove that the function $z \to t_z$ is continuous. For this, suppose that $z_n \to z$ in E and denote $t_n = t_{z_n}$. We claim that (t_n) is bounded. If not, we can suppose that $t_n \to \infty$.

It follows from (4.4) that $h'_{z_n}(t_n) = 0$ if, and only if,

$$\begin{aligned}
0 = h'_{z_n}(t_n) &= \int_{\mathbb{R}^N} N t_n^{2N-1} \sqrt{t_n^4 m^2 + 4\pi^2|\xi|^2}|\hat{z_n}(\xi)|^2 \mathrm{d}\xi + \int_{\mathbb{R}^N} \frac{t_n^{2N} t_n^3 m^2 |\hat{z_n}(\xi)|^2}{\sqrt{t_n^4 m^2 + 4\pi^2|\xi|^2}} \mathrm{d}\xi \\
&\quad - \int_{\mathbb{R}^N} \left[t_n^{2N} \nabla F(t_n z_n(x)) \cdot z_n(x) + 2N t_n^{2N-1} F(t_n z_n(x)) \right] \mathrm{d}x. \\
&= t^{2N+1} \left(t_n^2 \int_{\mathbb{R}^N} \frac{m^2 |\hat{z_n}(\xi)|^2}{\sqrt{t_n^4 m^2 + 4\pi^2|\xi|^2}} \mathrm{d}\xi + \int_{\mathbb{R}^N} N t_n^{-2} \sqrt{t_n^4 m^2 + 4\pi^2|\xi|^2}|\hat{z_n}(\xi)|^2 \mathrm{d}\xi \right. \\
&\quad \left. - \int_{\mathbb{R}^N} \left[t_n^{-1} \nabla F(t_n z_n(x)) \cdot z_n(x) + 2N t_n^{-2} F(t_n z_n(x)) \right] \mathrm{d}x \right)
\end{aligned}$$

thus implying

$$\begin{aligned}
0 &= t_n^2 \int_{\mathbb{R}^N} \frac{m^2 |\hat{z_n}(\xi)|^2}{\sqrt{t_n^4 m^2 + 4\pi^2|\xi|^2}} \mathrm{d}\xi + \int_{\mathbb{R}^N} N t_n^{-2} \sqrt{t_n^4 m^2 + 4\pi^2|\xi|^2}|\hat{z_n}(\xi)|^2 \mathrm{d}\xi \\
&\quad - \int_{\mathbb{R}^N} \left[t_n^{-2} \nabla F(t_n z_n(x)) \cdot z_n(x) + 2N t_n^{-2} F(t_n z_n(x)) \right] \mathrm{d}x \\
&= I_{11} + I_{12} - I_{13},
\end{aligned} \tag{4.7}$$

with I_{1j} denoting the respective integral.

Let us consider I_{13}. From lemma (1.5), since

$$t_n^{-2} \nabla F(t_n z_n(x)) \cdot t_n z_n(x) + 2N t_n^{-2} F(t_n z_n(x)) =$$

$$\begin{aligned}
&= 2N \left(\frac{|z_n(x)|^2}{2s} - \frac{\ln(1 + s t_n^2 |z_n(x)|^2)}{2s^2 t_n^2} + \lambda u_n(x) v_n(x) \right) \\
&\quad + \left(\frac{t_n^2 |z_n(x)|^4}{1 + s t_n^2 |z_n(x)|^2} + 2\lambda u_n(x) v_n(x) \right)
\end{aligned}$$

converges to

$$\frac{N}{s}|z(x)|^2 + 2N\lambda u(x)v(x) + \frac{1}{s}|z(x)|^2 + 2\lambda u(x)v(x) \geq \left(\frac{N}{s} - N\lambda + \frac{1}{s} - \lambda\right)|z(x)|^2$$

the Dominated Convergence Theorem guarantees that

$$I_{13} \geq \int_{\mathbb{R}^N} (N+1)\left(\frac{1}{s} - \lambda\right)|z(x)|^2\, \mathrm{d}x \tag{4.8}$$

Since Plancharel's identity shows that

$$I_{11} + I_{12} \to \int_{\mathbb{R}^N} (N+1)m|\hat{z}(\xi)|^2\, \mathrm{d}\xi = (N+1)m\int_{\mathbb{R}^N} |z(x)|^2\, \mathrm{d}x,$$

it follows from (4.7) and (4.8) that

$$(N+1)\left(m - \frac{1}{s} + \lambda\right)\int_{\mathbb{R}^N} |z(x)|^2\, \mathrm{d}x = 0$$

and we have reached a contradiction, since $m - \frac{1}{s} + \lambda < 0$. Thus, (t_n) is bounded.

So, we suppose that $t_n \to t_0 \in (0, \infty)$. Therefore, as $n \to \infty$, it follows from (4.7) that $h_z'(t_0) = 0$. The uniqueness of t_z shows that $t_z = t_0$, proving that $t_n \to t_z$. □

Lemma 1.13 *The following affirmatives are true:*

(i) *Suppose that* $(N+1)\lambda/N) < m$. *Then, there exists* $\rho > 0$ *such that* $J(z) > 0$ *for all* $0 < \|z\| < \rho$ *and* $\mathcal{M}$ *is closed in E;*

(ii) *It holds* $\Phi(z) > 0$ *for all* $z \in \mathcal{M}$*;*

(iii) $\mathcal{M}$ *is a* C^1 *manifold;*

(iv) $\mathcal{M}$ *is a natural constraint of problem* (1.2), i.e., *any critical point of* $\Phi|_{\mathcal{M}}$ *is a critical point of* Φ *in E.*

Proof. It follows from Lemma 1.8 that

$$|F(z(x))| \leq |H(z(x))| + |\lambda u(x)v(x)| \leq \frac{1}{2}\epsilon|z(x)|^2 + C_\epsilon|z(x)|^p + \frac{\lambda}{2}|z(x)|^2$$

and

$$|\nabla F(z(x)) \cdot z(x)| \leq \frac{\epsilon}{2}|z(x)|^2 + 2C_\epsilon|z(x)|^p + \lambda|z(x)|^2,$$

for all $2 < p < 2^*$.

(i) From the definition of J, (3.2) and the recent stated inequalities we obtain

$$
\begin{aligned}
J(z) \geq N\|z\|^2 &- 2N\int_{\mathbb{R}^N}\left(\frac{1}{2}\epsilon|z(x)|^2 + C_\epsilon|z(x)|^p + \frac{\lambda}{2}|z(x)|^2\right)\mathrm{d}x \\
&- \int_{\mathbb{R}^N}\left(\frac{\epsilon}{2}|z(x)|^2 + 2C_\epsilon|z(x)|^p + \lambda|z(x)|^2\right)\mathrm{d}x \\
&= N\|z\|^2 - \left(\frac{(2N+1)}{2}\epsilon + (N+1)\lambda\right)\int_{\mathbb{R}^N}|z(x)|^2\mathrm{d}x \\
&- (2N+2)C_\epsilon\int_{\mathbb{R}^N}|z(x)|^p\mathrm{d}x \\
&= \left(\frac{2mN - 2(N+1)\lambda - (2N+1)\epsilon}{2m}\right)\|z\|^2 - (2N+2)C_\epsilon\int_{\mathbb{R}^N}|z(x)|^p\mathrm{d}x
\end{aligned}
$$

Now, define $\kappa = \dfrac{2mN - 2(N+1)}{2m+2N+1}$. For $0 < \epsilon < \kappa$ we have

$$
\begin{aligned}
\frac{2mN - 2(N+1)\lambda - (2N+1)\epsilon}{2m} &> \frac{2mN - 2(N+1)\lambda - (2N+1)\kappa}{2m} \\
&= \frac{2mN - 2(N+1)\lambda}{2m}\left(1 - \frac{2N+1}{2m+2N+1}\right) \\
&= \frac{2mN - 2(N+1)\lambda}{2m+2N+1} = \kappa.
\end{aligned}
$$

It follows that

$$J(z) \geq \kappa\|z\|^2 - (2N+2)\gamma_p^p C_\epsilon\|z\|^p.$$

Let $r = \|z\| > 0$ be, take $0 < \rho < 1$ small enough such that $\rho = \left(\dfrac{\kappa}{4(N+1)C_\epsilon\gamma_p^p}\right)^{\frac{1}{p-2}}$. Thus, if $0 < r \leq \rho$,

$$J(z) \geq \frac{N}{8}r^2 > 0$$

This proves that $z = 0$ is an isolated point of $J^{-1}(0) = \mathcal{M} \cup \{0\}$ and $\mathcal{M}$ is a closed set.

(ii) It follows from the definition of Φ and the (NQ)-condition - Lemma 1.6 (i) - that

$$\Phi(z) - \frac{1}{2N+2}J(z)$$

$$
\begin{aligned}
&= \frac{1}{2N+2}\|z\|^2 - \frac{1}{2N+2}\int_{\mathbb{R}^N} \frac{m^2|\hat{z}(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}} \mathrm{d}\xi \\
&\quad - \frac{2}{2N+2}\int_{\mathbb{R}^N} F(z(x))\mathrm{d}x + \frac{1}{2N+2}\int_{\mathbb{R}^N} \nabla F(z(x))\cdot z(x)\mathrm{d}x \\
&= \frac{1}{2N+2}\int_{\mathbb{R}^N}\left(\sqrt{m^2+4\pi^2|\xi|^2}|\hat{z}(\xi)|^2 - \frac{m^2|\hat{z}(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\right)\mathrm{d}\xi \\
&\quad + \frac{1}{2N+2}\int_{\mathbb{R}^N} \left(\nabla F(z(x))\cdot z(x) - 2F(z(x))\right)\mathrm{d}x \qquad (4.9)\\
&= \frac{1}{2N+2}\int_{\mathbb{R}^N} |\hat{z}(\xi)|^2\left(\frac{4\pi|\xi|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\right)\mathrm{d}\xi \\
&\quad + \frac{1}{2N+2}\int_{\mathbb{R}^N} \left[\nabla F(z(x))\cdot z(x) - 2F(z(x))\right]\mathrm{d}x > 0. \qquad (4.10)
\end{aligned}
$$

Since, for all $z \in \mathcal{M}$ it is true that $J(z) = 0$, it follows immediately from the last inequality that $\Phi(z) > 0$.

(*iii*) Recalling the notation $K(z(x)) = \nabla F(z(x)) \cdot z(x)$, the definition of J yields

$$
\begin{aligned}
J'(z)\cdot z = 2N\|z\|^2 + 2\int_{\mathbb{R}^N} \frac{m^2|\hat{z}(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi \\
- \int_{\mathbb{R}^N} \left[2N\nabla F(z(x))\cdot z(x) + \nabla K(z(x))\cdot z(x)\right]\mathrm{d}x.
\end{aligned}
$$

Since $J(z) = 0$, for all $z \in \mathcal{M}$,

$$
\begin{aligned}
J'(z)\cdot z = 2N\int_{\mathbb{R}^N} \left[2F(z(x)) - \nabla F(z(x))\cdot z(x)\right]\mathrm{d}x \\
+ \int_{\mathbb{R}^N} \left[2\nabla F(z(x))\cdot z(x) - \nabla K(z(x))\cdot z(x)\right]\mathrm{d}x.
\end{aligned}
$$

The (NQ)-condition (that is, Lemma 1.6) guarantees that $2F(z(x)) - \nabla F(z(x)) \cdot z(x) < 0$. Moreover from Lemma 1.5 follows

$$
2\nabla F(z(x))\cdot z(x) - \nabla K(z(x))\cdot z(x) = \frac{-2|z(x)|^4}{(1+s|z(x)|^2)^2} < 0,
$$

completing the proof that $J'(z(x)) \cdot z(x) < 0$.

(*iv*) Let z be a critical point of $\Phi|_{\mathcal{M}}$. By Lagrange multipliers there exists $\mu \in \mathbb{R}$ such that

$$
\Phi'(z) + \mu J'(z) = 0 \qquad (4.11)
$$

and therefore

$$\Phi'(z)\cdot z+\mu J'(z)\cdot z=0.$$

We must show that $\mu=0$.

According to the Appendix, the Nehari-Pohozaev functional attached to $\mathcal{M}$ is given by

$$\widetilde{J}(z)=J'(z)\cdot z+2\widetilde{P}(z),$$

where $\widetilde{P}(z)$ is the Pohozaev functional related to $\mathcal{M}$, that is, if $H=2NF+K$, then

$$\begin{aligned}\widetilde{P}(z)=2N(N-1)\|z\|^2+2(2N+1)m^2\int_{\mathbb{R}^N}\frac{|\mathcal{F}z(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi\\ -4N^2\int_{\mathbb{R}^N}H(z(x))\mathrm{d}x.\end{aligned}$$

If $z\in\mathcal{M}$, then $J(z)=0$. Thus,

$$\begin{aligned}\mu\widetilde{J}(z)&=J(z)+\mu\widetilde{J}(z)=\Phi'(z)\cdot z+2P(z)+\mu(J'(z)\cdot z+2\widetilde{P}(z))\\&=\Phi'(z)\cdot z+\mu J'(z)\cdot z+2(P(z)+\mu\widetilde{P}(z))\\&=0,\end{aligned}$$

since the Pohozaev identity attached to (4.11) is given by

$$P(z)+\mu\tilde{P}(z)=0.$$

To simplify the calculations, we define

$$T(z)=m^2\int_{\mathbb{R}^N}\frac{|\mathcal{F}z(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi.$$

We have

$$
\begin{aligned}
\widetilde{J}(z) &= 2N\|z\|^2 + 2T(z) \\
&\quad - \int_{\mathbb{R}^N} [2N\nabla F(z(x)) \cdot z(x) + \nabla K(z(x)) \cdot z(x)]\mathrm{d}x \\
&\quad + 4N(N-1)\|z\|^2 + 4(2N+1)T(z) - 8N^2 \int_{\mathbb{R}^N} [2NF(z) + K(z)]\,\mathrm{d}x \\
&= (4N^2 - 2N)\|z\|^2 + (8N+6)T(z) \\
&\quad - \int_{\mathbb{R}^N} \left[16N^3 F(z(x)) + (8N^2+2N)\nabla F(z(x)) \cdot z(x) + \nabla K(z(x)) \cdot z(x)\right] \mathrm{d}x.
\end{aligned}
$$

Observe that

$$
\mu\widetilde{J}(z) - (8N^2+2)\mu J(z) = 0. \tag{4.12}
$$

an identity that yields

$$
\begin{aligned}
0 &= \mu(4N^2 - 8N^3 - 4N)\|z\|^2 + \mu(8N - 8N^2 + 4)T(z) \\
&\quad + \mu \int_{\mathbb{R}^N} [2N(2F(z(x)) - \nabla F(z(x)) \cdot z(x))]\,\mathrm{d}x \\
&\quad + \mu \int_{\mathbb{R}^N} [2\nabla F(z(x)) \cdot z(x) - \nabla K(z(x)) \cdot z(x)]\,\mathrm{d}x.
\end{aligned} \tag{4.13}
$$

We observe:

- For each $N \geq 2$ we have $4N^2 - 8N^3 - 4N < 0$ and $8N - 8N^2 + 4 < 0$;
- the (NQ)-conditions guarantees that $2F(z(x)) - \nabla F(z(x)) \cdot z(x) \leq 0$;
- the last integrand satisfies

$$
2\nabla F(\xi,\eta) \cdot (\xi,\eta) - \nabla K(\xi,\eta) \cdot (\xi,\eta)
$$

$$
\begin{aligned}
&= 2\frac{(\xi^2+\eta^2)^2}{1+s(\xi^2+\eta^2)} + 4\lambda\xi\eta - \left[\frac{2(\xi^2+\eta^2)^2(2+s(\xi^2+\eta^2))}{(1+s(\xi^2+\eta^2))^2} + 4\lambda\xi\eta\right] \\
&= 2(\xi^2+\eta^2)^2\left[\frac{1}{1+s(\xi^2+\eta^2)} - \frac{(2+s(\xi^2+\eta^2))}{(1+s(\xi^2+\eta^2))^2}\right] \\
&= -\frac{2(\xi^2+\eta^2)^2}{(1+s(\xi^2+\eta^2))^2} < 0.
\end{aligned}
$$

Since all terms in the right-hand side of (4.13) are negative, we conclude that $\mu = 0$, concluding the proof. □

We now consider the minimax value

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0 \ \text{ and } \ \Phi(\gamma(1)) < 0\}$ and the infimum in the Nehari-Pohozaev manifold

$$\tilde{c} = \inf_{z \in \mathcal{M}} \Phi(z) = \inf_{z \in E} \max_{t > 0} \Phi(z_t) \geq 0$$

Lemma 1.14 *The level c is well-defined and $c = \tilde{c}$.*

Proof. Maintaining the notation $z_t(x) = tz(t^{-2}x)$, Lemma 1.9 guarantees the existence of $t_1 = t_1(z)$ such that $\Phi(z_{t_1}) < 0$. Changing scales, define $\gamma_0(t) = z_{tt_1}$, if $t > 0$, and $\gamma_0(0) = 0$. Then $\Phi(\gamma_0(1)) = \Phi(z_{t_1}) < 0$, and we conclude that $\gamma_0 \in \Gamma$.

Thus

$$\max_{t \geq 0} \Phi(z_{tt_1}) \geq \max_{0 \leq t \leq 1} \Phi(z_{tt_1}) \geq \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\gamma(t)) = c,$$

proving that $\hat{c} \geq c$.

According to Lemma 1.13, there exists $\rho > 0$ such that $J(z) \geq 0$, if $z \in E$ satisfies $\|z\| \leq \rho$. The proof of Lemma 1.13 shows that

$$\Phi(z) - \frac{1}{2N+2} J(z) \geq 0,$$

allowing us to conclude that $\Phi(z) \geq 0$, if $\|z\| \leq \rho$.

If $\gamma \in \Gamma$, because $\Phi(\gamma(1)) < 0$, a new application of the proof of Lemma 1.13 gives

$$J(\gamma(1)) \leq (2N+2)\Phi(\gamma(1)) < 0.$$

Since we also have $J(\gamma(0)) = 0$ and $J(\gamma(t)) > 0$ if $0 < \|\gamma(t)\| < \rho$, we conclude the existence of $\hat{t} \in (0,1)$ such that $J(\gamma(\hat{t})) = 0$, proving that γ intercepts $\mathcal{M}$. Therefore

$$\max_{0\leq t\leq 1} \Phi(\gamma(t)) \geq \inf_{z\in\mathcal{M}} \Phi(z) = \hat{c},$$

showing that $c \geq \hat{c}$ and completing the proof. □

Definition 1.15 *A sequence $(z_n) \in E$ is a Cerami sequence for Φ at the level θ if*

$$\Phi(z_n) \to \theta \quad \text{and} \quad \|\Phi'(z_n)\|_* (1 + \|z_n\|) \to 0,$$

where $\|\cdot\|_*$ stands for the norm in E^*.

Lemma 1.16 *Let (z_n) be a Cerami sequence for Φ at the level $\theta > 0$. Then, passing to a subsequence if necessary, (z_n) is bounded in E.*

Proof. Since (z_n) is a Cerami sequence, we have

$$\|\Phi'(z_n)\|_*(1 + \|z_n\|) \leq 1/n$$

for n large enough. Thus,

$$-\frac{1}{n} < \Phi'(z_n) \cdot z_n = \|z_n\|^2 - \int_{\mathbb{R}^N} \nabla F(z_n(x)) \cdot z_n(x)\, \mathrm{d}x < \frac{1}{n}. \tag{4.14}$$

Claim 1: For each $0 \neq z \in E$, the function $\ell_z \colon [0,\infty) \to \mathbb{R}$ given by

$$\ell_z(t) = \frac{t^2}{2}\nabla F(z(x)) \cdot z(x) - F(tz(x)) \tag{4.15}$$

attains its maximum at $t = 1$.

In fact, by lemma (1.5),

$$\frac{t^2}{2}\nabla F(z(x)) \cdot z(x) - F(tz) = \frac{t^2|z|^4}{2(1+s|z|^2)} - \frac{t^2|z|^2}{2s} + \frac{1}{2s^2}\ln\left(1 + st^2|z|^2\right)$$

satisfies

$$\ell_z'(t) = 0 \quad \text{if and only if} \quad t = 1.$$ [1]

[1] The proof of Lemma 1.6(i) is a consequence of this result.

Claim 2: For all $t > 0$ and $n \in \mathbb{N}$ we have

$$\Phi(tz_n) \leq \frac{t^2}{2n} + \int_{\mathbb{R}^N} \left(\frac{1}{2} \nabla F(z_n(x)) \cdot z_n(x) - F(z_n(x)) \right) \mathrm{d}x.$$

In fact, (4.14) guarantees that

$$\|z_n\|^2 < \frac{1}{n} + \int_{\mathbb{R}^N} \nabla F(z_n(x)) \cdot z_n(x) \, \mathrm{d}x.$$

Now, recalling that $h_z(t) = \Phi(z_t)$ and $h(t) \leq h(1)$, the definition of Φ yields

$$\begin{aligned}\Phi(tz_n) &= \frac{t^2}{2}\|z_n\|^2 - \int_{\mathbb{R}^N} F(tz_n(x))\mathrm{d}x \\ &\leq \frac{t^2}{2n} + \int_{\mathbb{R}^N} \left(\frac{t^2}{2} \nabla F(z_n(x)) \cdot z_n(x) - F(tz_n(x)) \right) \mathrm{d}x = \frac{t^2}{2n} + \int_{\mathbb{R}^N} \ell_z(t)\mathrm{d}x,\end{aligned}$$

where $\ell_z(t)$ is defined by the expression between parenthesis, that is, it is given by (4.15). Since, for all $0 \neq z \in E$, the maximum of $\ell_z(t)$ is attained at $t = 1$, we have

$$\left(\frac{t^2}{2} \nabla F(z_n(x)) \cdot z_n(x) - F(tz_n(x)) \right) \leq \left(\frac{1}{2} \nabla F(z_n(x)) \cdot z_n(x) - F(z_n(x)) \right),$$

proving Claim 2.

It also follows from (4.14) that $\Phi'(z_n) \cdot z_n > -1/n$. Therefore,

$$\|z_n\|^2 > -\frac{1}{n} + \int_{\mathbb{R}^N} \nabla F(z_n(x)) \cdot z_n(x) \, \mathrm{d}x,$$

showing that

$$\begin{aligned}\Phi(z_n) &= \frac{1}{2}\|z_n\|^2 - \int_{\mathbb{R}^N} F(z_n(x))\mathrm{d}x \\ &\geq -\frac{1}{2n} + \frac{1}{2} \int_{\mathbb{R}^N} \nabla F(z_n(x)) \cdot z_n(x) \, \mathrm{d}x - \int_{\mathbb{R}^N} F(z_n(x))\mathrm{d}x.\end{aligned}$$

Thus, we obtain

$$\int_{\mathbb{R}^N} \left(\frac{1}{2} \nabla F(z_n(x)) \cdot z_n(x) - F(z_n(x)) \right) \mathrm{d}x < \frac{1}{2n} + \Phi(z_n).$$

Since (z_n) satisfies the Cerami condition, Claim 2 and the last inequality imply that, for all $t > 0$ and $n \in \mathbb{N}$, we have

$$\frac{t^2}{2}\|z_n\|^2 - \int_{\mathbb{R}^N} F(tz_n(x))\mathrm{d}x = \Phi(tz_n) \leq \frac{t^2}{2n} + \left(\frac{1}{2n} + \Phi(z_n)\right) = \frac{t^2+1}{2n} + \theta + o_n(1).$$

Thus, accordingly to Lemma 1.8,

$$\begin{aligned}\frac{t^2}{2}\|z_n\|^2 &\leq \frac{t^2+1}{2n} + \theta + o_n(1) + \int_{\mathbb{R}^N} F(tz_n(x))\mathrm{d}x \\ &\leq \frac{t^2+1}{2n} + \theta + o_n(1) + \int_{\mathbb{R}^N} \left(\frac{t^2}{2}|z_n(x)|^2(\epsilon+\lambda) + C_\epsilon t^p |z_n(x)|^p\right)\mathrm{d}x.\end{aligned}$$

We now choose $0 < \epsilon < m - \lambda$ and define $k = \frac{m-\lambda-\epsilon}{m}$ and, for n large enough, $t_n := \frac{\sqrt{3\theta}}{\sqrt{k}\|z_n\|}$. Suppose, by contradiction, that $\|z_n\| \to \infty$, therefore, we have $t_n \to 0$ as $n \to \infty$.

Substituting t_n into the last inequality, yields

$$\frac{3\theta}{2k} \leq \theta + o_n(1) + \frac{3\theta}{2k}(\epsilon+\lambda)\int_{\mathbb{R}^N} \frac{|z_n(x)|^2\mathrm{d}x}{\|z_n\|^2} + \left(\frac{C_\epsilon\sqrt{3\theta}}{\sqrt{\lambda}}\right)^p \int_{\mathbb{R}^N} \frac{|z_n(x)|^p\mathrm{d}x}{\|z_n\|^p}.$$

Now, consider the bounded sequence $\tilde{z}_n = \frac{z_n}{\|z_n\|}$.

Since E is reflexive, passing to a subsequence, we can suppose $\tilde{z}_n \rightharpoonup \tilde{z}$ for some $\tilde{z} \in E$. There are two possible cases:

(i) $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{z}_n(x)|^2\mathrm{d}x = 0;$

(ii) $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{z}_n(x)|^2\mathrm{d}x > 0.$

If (i) occurs, Lion's lemma yields $\tilde{z}_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $2 < p < 2^*$. So,

$$\int_{\mathbb{R}^N} |\tilde{z}_n(x)|^p\mathrm{d}x \to 0.$$

Furthermore, from (3.2), $\int_{\mathbb{R}^N} |\tilde{z}_n(x)|^2\mathrm{d}x \leq \frac{\|\tilde{z}_n\|^2}{m} \leq \frac{1}{m}$, we have

$$\frac{3\theta}{2k} \leq o_n(1) + \theta + \frac{3\theta}{2k}\frac{(\epsilon+\lambda)}{m}, \text{ for } n \text{ large enough,}$$

and therefore we have reached the contradiction

$$\frac{\theta}{2} \leq o_n(1).$$

Suppose now that (ii) occurs. If $\delta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{z}_n(x)|^2 \mathrm{d}x > 0$, passing to a subsequence if necessary, for $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$ we have

$$\int_{B_1(y)} |\tilde{u}_n(x)|^2 \mathrm{d}x > \delta/2 \quad \text{or} \quad \int_{B_1(y)} |\tilde{v}_n(x)|^2 \mathrm{d}x > \delta/2$$

for some $y \in \mathbb{R}^N$.

Let us suppose that the first inequality is true, the other case being analogous. Then,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n(x)|^2 \mathrm{d}x > \frac{\delta}{2} > 0.$$

Thus, there exists a sequence (y_n) for all $n \in \mathbb{N}$

$$\int_{B_1(y_n)} |\tilde{u}_n(x)|^2 \mathrm{d}x > \delta/2 > 0.$$

By defining $U_n(w) = \tilde{u}_n(w + y_n)$, U_n is a translation of $\tilde{u}$ and therefore $\|U_n\| = 1$.

Thus, we can suppose that $U_n \rightharpoonup U$ in $H^{1/2}(\mathbb{R}^N)$, $U_n \to U$ in $L^2_{loc}(\mathbb{R}^N)$ and $U_n(x) \to U(x)$ a.e. in $\mathbb{R}^N$.

Since

$$\begin{aligned} \frac{\delta}{2} < \int_{B_1(y_n)} |\tilde{u}_n(x)|^2 \mathrm{d}x = \int_{B_1(0)} |\tilde{u}_n(w + y_n)|^2 \mathrm{d}w = \int_{B_1(0)} |U_n(w)|^2 \text{w} \\ \to \int_{B_1(0)} |U(w)|^2 \mathrm{d}w, \end{aligned}$$

there exists $\Omega \subset B_1(0)$ with $|\Omega| > 0$, such that $|U(w)| > 0$ for a.e. $w \in \Omega$.

Therefore,

$$0 < |U(w)| = \lim_{n\to\infty} |\tilde{u}_n(w + y_n)| = \lim_{n\to\infty} \frac{|u_n(w + y_n)|}{\|z_n\|}$$

for almost $w \in \Omega$ and, since $\|z_n\| \to \infty$, for such w we have

$$\lim_{n\to\infty} |u_n(w + y_n)| = \infty.$$

Thus, by applying Fatou Lemma and lemma (1.6)(ii),

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{2}\nabla F(z_n(w+y_n)) \cdot z_n(w+y_n) - F(z_n(w+y_n))\right) \mathrm{d}w \\
&\geq \int_{\mathbb{R}^N} \liminf_{n\to\infty} \left(\frac{1}{2}\nabla F(z_n(w+y_n)) \cdot z_n(w+y_n) - F(z_n(w+y_n))\right) \mathrm{d}w \\
&= \infty.
\end{aligned}$$

However, since $\Phi'(z_n) \cdot z_n \to 0$ and $\Phi(z_n) \to \theta$, we have both

$$\Phi(z_n) - \frac{1}{2}\Phi'(z_n) \cdot z_n = \theta + o_n(1)$$

and

$$\Phi(z_n) - \frac{1}{2}\Phi'(z_n) \cdot z_n = \int_{\mathbb{R}^N} \left(\frac{1}{2}\nabla F(z_n(x)) \cdot z_n(x) - F(z_n(x))\right) \mathrm{d}x \to +\infty.$$

We are done. □

The existence of Cerami sequence for Φ at the level c is a consequence of the Ghoussoub-Preiss Theorem(see [8]).

Theorem. (Ghoussoub-Preiss) Let X, X^* be a Banach spaces and $\Phi : X \to \mathbb{R}$ a continuous Gateaux-differentiable function, such that $\Phi' : X \to X^*$ is continuous from the norm topology of X to the weak* topology of X^*. Take two points $z_0,\ z_1$ in X and the set Γ of all continuous paths from z_0 to z_1 given by:

$$\Gamma = \{\gamma \in C([0,1]; X)\ : \gamma(0) = z_0,\ \gamma(1) = z_1\}.$$

Define the number c by:

$$c := \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \Phi(\gamma(t)).$$

Assume that there is a closed subset $\mathcal{M}$ of X such that

$$\mathcal{M} \cap \Phi_c \ \textit{separates}\ \ z_0 \ \text{ and } \ z_1,$$

with $\Phi_c = \{x \in X \, : \, \Phi(x) \geq c\}$.

Then, there exists a sequence (x_n) in X such that

(i) $\text{dist}\,(x_n, \mathcal{M}) \to 0$;

(ii) $\Phi(x_n) \to c$;

(iii) $(1 + \|x_n\|)\|\Phi'(x_n)\|_* \to 0$, as $n \to \infty$.

A closed subset $\mathcal{F} \subset X$ *separates* two points z_0 and z_1 in X if z_0 and z_1 belong to disjoint connected components in $X \setminus \mathcal{F}$.

In our case we consider $X = E$. By taking $z_0 = 0$ and $z_1 \in E$ such that $\Phi(z_1) < 0$, then

$$E \setminus \mathcal{M} = \{0\} \cup \{u \in H^{1/2}(\mathbb{R}^N) \, : J(u) > 0\} \cup \{u \in E \, : J(u) < 0\}.$$

We know that $0 \notin \mathcal{M}$ e $J(0) = 0$. According to Lemma 1.13, $B_\rho(0)$ belongs to a connected component of $\{0\} \cup \{u \in H^{1/2}(\mathbb{R}^N) \, : \, J(u) > 0\}$. Since $\Phi(z_1) < 0$, it follows from equation (4.10) that $J(z_1) < 0$. Thus, $\mathcal{M}$ separates $z_0 = 0$ and z_1. But we also have $\mathcal{M} \cap \Phi_c = \mathcal{M}$, since $\inf_{u \in \mathcal{M}} \Phi(u) = c$, as a consequence of Lemma 1.14. So, the hypotheses of the Ghoussoub-Preiss theorem are satisfied.

Theorem 1.17 *If* $(N+1)\lambda/N < m < \frac{1}{s} - \lambda$, *then the problem* (1.2) *has a ground state solution* $w \in E$. *In other words, there exists* w *satisfying* $\Phi(w) = c$ *and* $\Phi'(w) = 0$, *which has minimal energy among all solutions of* (1.2)

Proof. The Ghoussoub-Preiss theorem guarantees the existence of Cerami sequence $\{z_n\} \subset E$, which is bounded by Lemma 2.7. Thus, we can suppose that $z_n \rightharpoonup z$ in E, $z_n(x) \to z(x)$ almost everywhere $x \in \mathbb{R}^N$ and $z_n \to z$ in $L^p_{Loc}(\mathbb{R}^N) \times L^p_{Loc}(\mathbb{R}^N)$ for $p \in [2, 2^*)$.

We define

$$\delta = \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |z_n(x)|^2 \mathrm{d}x.$$

If $\delta = 0$, the concentration-compactness principle implies that $z_n \to 0$ in $L^p_{Loc}(\mathbb{R}^N) \times L^p_{Loc}(\mathbb{R}^N)$ for any $p \in (2, 2^*)$. Since z_n is bounded in $L^2_{Loc}(\mathbb{R}^N) \times L^2_{Loc}(\mathbb{R}^N)$, there exists $M_0 > 0$ such that $|z_n|_2 < M_0$, for all n.

Thus, by lemma (1.8) taking ϵ small enough, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} H(z_n(x))\mathrm{d}x &\leq \int_{\mathbb{R}^N} \left[\frac{\epsilon}{2}|z_n(x)|^2 + C_\epsilon |z_n(x)|^p\right] \mathrm{d}x = \frac{\epsilon}{2}|z_n|_2^2 + C_\epsilon |z_n|_p^p \\ &\leq \frac{\epsilon}{2}M_0 + C_\epsilon |z_n|_p^p \to 0,\end{aligned}$$

showing that $\int_{\mathbb{R}^N} H(z_n(x))\mathrm{d}x \to 0$ as $n \to \infty$.

Similarly,

$$\int_{\mathbb{R}^N} [\nabla H(z_n(x)) \cdot z_n(x)]\, \mathrm{d}x \to 0.$$

Comparing the expressions of $\Phi(z)$ and $(\Phi'(z) \cdot z)/2$ we obtain

$$\begin{aligned}\Phi(z_n) &= \frac{1}{2}\left(\Phi'(z_n) \cdot z_n + \int_{\mathbb{R}^N} [\nabla F(z_n(x)) \cdot z_n(x)]\, \mathrm{d}x\right) - \int_{\mathbb{R}^N} F(z_n(x))\mathrm{d}x \\ &= \frac{1}{2}\left(\Phi'(z_n) \cdot z_n + \int_{\mathbb{R}^N} [\nabla H(z_n(x)) \cdot z_n(x)]\, \mathrm{d}x\right) - \int_{\mathbb{R}^N} H(z_n(x))\mathrm{d}x\end{aligned}$$

Thus, we conclude that $\Phi(z_n) \to 0$, as $n \to \infty$, but

$$0 < c = \lim_{n\to\infty} \Phi(z_n) = 0,$$

a contradiction.

If $\delta > 0$, there exists a sequence (y_n) such that, for all $n \in \mathbb{N}$,

$$\int_{B_1(y_n)} |\tilde{z}_n(x)|^2 \mathrm{d}x > \frac{\delta}{2} > 0. \tag{4.16}$$

We define $w_n = z_n(x + y_n)$. Then $\|w_n\| = \|z_n\|$, $J(w_n) = J(z_n)$, $\Phi(w_n) = \Phi(z_n)$ and $\Phi'(w_n) \to 0$ as $n \to \infty$. Passing to a subsequence, we can suppose that, for $p \in [2, 2^*)$, we have $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^p_{Loc}(\mathbb{R}^N) \times L^p_{Loc}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ almost everywhere for $x \in \mathbb{R}^N$.

It follows from (4.16) that $w \neq 0$. Furthermore, for all $\varphi \in E$, we have

$$\begin{aligned}\Phi'(w) \cdot \varphi &= \lim_{n\to\infty}\left(\langle w_n, \varphi\rangle - \int_{\mathbb{R}^N} [\nabla F(w_n(x)) \cdot \varphi(x)]\, \mathrm{d}x\right) \\ &= \lim_{n\to\infty} \Phi'(w_n) \cdot \varphi = 0,\end{aligned}$$

thus allowing us to conclude that $\Phi'(w) = 0$, that is, w is a weak solution of (1.2) so that, satisfies the Pohozaev identity, but

$$J(w) = \Phi'(w)w + 2P(w) = 0$$

so that $w \in \mathcal{M}$ and by lemma (1.14), we get

$$\tilde{c} = \inf_{z \in \mathcal{M}} \Phi(z) = c$$

thus $\Phi(z) \geq c, \forall z \in \mathcal{M}$ and consequently $\Phi(w) \geq c$.

By applying Fatou's lemma to (4.10) with w instead to z, we obtain:

$$\begin{aligned}
\Phi(w) &= \Phi(w) - \frac{1}{2N+2}J(w) \\
&= \frac{1}{2N+2}\int_{\mathbb{R}^N} |\hat{w}(\xi)|^2\Big(\frac{4\pi|\xi|^2}{(m^2+4\pi^2|\xi|^2)^{1/2}}\Big) + \frac{1}{2N+2}\int_{\mathbb{R}^N}(\nabla F(w)\cdot w - 2F(w)) \\
&\leq \liminf_{n\to\infty}\frac{1}{2N+2}\Big[\int_{\mathbb{R}^N} |\hat{w}_n(\xi)|^2\Big(\frac{4\pi|\xi|^2}{(m^2+4\pi^2|\xi|^2)^{1/2}}\Big) + \int_{\mathbb{R}^N}(\nabla F(w_n)\cdot w_n - 2F(w_n))\Big] \\
&= \liminf_{n\to\infty}\Big[\Phi(w_n) - \frac{1}{2N+2}J(w_n)\Big] = \liminf_{n\to\infty}\Phi(w_n) = c.
\end{aligned}$$

Therefore $\Phi(w) = c$, and lemma (1.13) guarantees that w is a groun state solution.

Now we prove that the solution is positive. Clearly $-F(u,v) \geq -F(|u|,|v|)$. Consider $\bar{z} = (\bar{u}, \bar{v}) \in H^{\frac{1}{2}}(\mathbb{R}^N) \times H^{\frac{1}{2}}(\mathbb{R}^N)$ such that $\bar{z} \neq 0$ and

$$\Phi(\bar{z}) = \inf_{z=(u,v)\in\mathcal{M}} \Phi(z).$$

Since $|\bar{z}| = (|\bar{u}|, |\bar{v}|) \in H^{\frac{1}{2}}(\mathbb{R}^N) \times H^{\frac{1}{2}}(\mathbb{R}^N)$, Lemma 1.12 guarantees the existence of $t_0 > 0$ such that $|\bar{z}|_{t_0} \in \mathcal{M}$. Observe that $|z_t| = |z|_t$ for all $t > 0$. Thus, by applying (2.3), it follows

$$\begin{aligned}
\Phi(|\bar{z}|_{t_0}) &= \frac{1}{2}\|\, |\bar{z}|_{t_0}\|^2 - \int_{\mathbb{R}^{\mathbb{N}}} F(|\bar{z}|_{t_0})\mathrm{d}x \leq \frac{1}{2}\|\bar{z}_{t_0}\|^2 - \int_{\mathbb{R}^{\mathbb{N}}} F(\bar{z}_{t_0})\mathrm{d}x \\
&\leq \frac{1}{2}\|\bar{z}\|^2 - \int_{\mathbb{R}^{\mathbb{N}}} F(\bar{z})\mathrm{d}x = \Phi(\bar{z}),
\end{aligned}$$

since $\bar{z} \in \mathcal{M}$ and $\Phi(\bar{z}) = \max_{t>0}\Phi(\bar{z}_t)$. Therefore,

$$\Phi(|\bar{z}|_{t_0}) = \inf_{z\in\mathcal{M}} \Phi(z).$$

□

1.5 Radial solution

In this section we prove that solutions of system (1.2), that is,

$$\begin{cases} \sqrt{-\Delta + m^2}u = \frac{u^2+v^2}{1+s(u^2+v^2)}u + \lambda v & \text{in } \mathbb{R}^N \\ \sqrt{-\Delta + m^2}v = \frac{u^2+v^2}{1+s(v^2+u^2)}v + \lambda u & \text{in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

are radially symmetric.

In order to do that, we recall the definition of a modified Bessel kernel, defined for any $\alpha > 0$ by

$$g_\alpha(x) = \frac{1}{(4\pi)^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-m^2\delta/(4\pi)} \delta^{(2\alpha-N)/2} \frac{d\delta}{\delta}.$$

The proof of the result can be found in [6].

Proposition 1.18 *For every $\alpha > 0$ we have*

i) $g_\alpha \in L^1(\mathbb{R}^N)$;

ii) $g_\alpha(\xi) = (m^2 + 4\pi^2|\xi|^2)^{-\alpha}$.

Denoting, as before, $z(x) = (u(x), v(x))$, since

$$\mathcal{F}((-\Delta + m^2)^{1/2}z)(\xi) = (m^2 + 4\pi^2|\xi|^2)^{1/2}\hat{z}(\xi)$$

it follows from proposition 1.18 that

$$\begin{aligned} \hat{z}(\xi) &= (m^2 + 4\pi^2|\xi|^2)^{-1/2}\mathcal{F}((-\Delta + m^2)^{1/2})z)(\xi) = \hat{g}_{1/2}(\xi)\mathcal{F}((-\Delta + m^2)^{1/2}z)(\xi) \\ &= \mathcal{F}(g_{1/2} * (-\Delta + m^2)^{1/2}z)(\xi), \end{aligned} \tag{5.2}$$

thus

$$z = g_{1/2} * (-\Delta + m^2)^{1/2}z.$$

By defining $f(\xi,\eta) = \frac{\xi^2+\eta^2}{1+s(\xi^2+\eta^2)}\xi + \lambda\eta$, any solution of (5.1) satisfies

$$\begin{cases} u(x) = (g_{1/2} * f(u,v))\,(x) \\ v(x) = (g_{1/2} * f(v,u))\,(x). \end{cases} \tag{5.3}$$

We know that $g_{1/2}$ is radially symmetric and decreasing.

We have

$$\begin{aligned} \frac{\partial}{\partial\xi} f(\xi,\eta) &= \frac{\xi^2+\eta^2}{1+s(\xi^2+\eta^2)} + \xi\left[\frac{2\xi(1+s(\xi^2+\eta^2)) - (\xi^2+\eta^2))2s\xi}{(1+s(\xi^2+\eta^2)^2}\right] \\ &= \frac{\xi^2+\eta^2}{1+s(\xi^2+\eta^2)} + \frac{2\xi^2}{(1+s(\xi^2+\eta^2))^2} > 0,\ \forall(\xi,\eta) \neq (0,0). \end{aligned}$$

Observe that $\left|\frac{\partial}{\partial\xi} f(\xi,\eta)\right| \leq M$ and $\left|\frac{\partial}{\partial\xi} f(\xi,\eta)\right| \leq 3\xi^2 + \eta^2$. It follows that, if $u, v \in L^2(\mathbb{R}^N)$, then $\left|\frac{\partial}{\partial\xi} f(u,v)\right| \leq 3u^2 + v^2$ and we conclude that $\frac{\partial}{\partial\xi} f(u,v) \in L^1(\mathbb{R}^N)$. Note also that $\left|\frac{\partial}{\partial\xi} f(u,v)\right| \leq M$ implies $\frac{\partial}{\partial\xi} f(u,v) \in L^\infty(\mathbb{R}^N)$ and, by interpolation, we conclude that $\frac{\partial}{\partial\xi} f(u,v) \in L^q(\mathbb{R}^N)$ for all $q \in [1,+\infty]$.

For any $\lambda \in \mathbb{R}$ we set

$$\begin{aligned} \Sigma_\lambda &:= \left\{x = (x_1,\cdots,x_n) \in \mathbb{R}^N \,:\, x_1 \geq \lambda\right\}, \\ \Sigma_\lambda^u &:= \{x \in \Sigma_\lambda \,:\, u_\lambda(x) > u(x)\}, \\ \Sigma_\lambda^v &:= \{x \in \Sigma_\lambda \,:\, v_\lambda(x) > v(x)\}, \\ T_\lambda &:= \{x \in \Sigma_\lambda \,:\, x_1 = \lambda\} \end{aligned}$$

and denote

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n), \quad u_\lambda(x) = u(x^\lambda) \quad \text{e} \quad v_\lambda(x) = v(x^\lambda)$$

Lemma 1.19 *If $u, v \in H^{\frac{1}{2}}(\mathbb{R}^N)$ satisfy* (5.3), *then we have*

$$\begin{aligned} u_\lambda(x) - u(x) &= \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] \left[f(u_\lambda(y), v_\lambda(y)) - f(u(y), v_\lambda(y))\right] \mathrm{d}y \\ &+ \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] \left[f(u(y), v_\lambda(y)) - f(u(y), v(y))\right] \mathrm{d}y \end{aligned}$$

and

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] [f(v_\lambda(y), u_\lambda(y)) - f(v(y), u_\lambda(y))] \,\mathrm{d}y$$
$$+ \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] [f(v(y), u_\lambda(y)) - f(v(y), u(y))] \,\mathrm{d}y$$

for all $x \in \mathbb{R}^N$.

Proof. Since u satisfies (5.3), we have

$$u(x) = (g_{1/2} * f(u,v))(x) = \int_{\mathbb{R}^N} g_{1/2}(x-y) f(u(y), v(y)) \mathrm{d}y.$$

The change of variables $y \mapsto y^\lambda$ in the second integral below yields

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} g_{1/2}(x-y) f(u(y), v(y)) \mathrm{d}y + \int_{(\Sigma_\lambda)^c} g_{1/2}(x-y) f(u(y), v(y)) \mathrm{d}y \\ &= \int_{\Sigma_\lambda} g_{1/2}(x-y) f(u(y), v(y)) \mathrm{d}y + \int_{(\Sigma_\lambda)^c} g_{1/2}(x-y^\lambda) f(u(y^\lambda), v(y^\lambda)) \mathrm{d}y \\ &= \int_{\Sigma_\lambda} \Big(g_{1/2}(x-y) f(u(y), v(y)) + g_{1/2}(x^\lambda - y) f(u_\lambda(y), v_\lambda(y)) \Big) \,\mathrm{d}y \end{aligned}$$

since $g_{1/2}$ is radially symmetric and $|x^\lambda - y| = |x - y^\lambda|$.

Changing x into x^λ in the last inequality we obtain, since $g_{1/2}$ is radial and we have both $|x^\lambda - y| = |x - y^\lambda|$ and $|x - y| = |x^\lambda - y^\lambda|$,

$$u(x^\lambda) = \int_{\Sigma_\lambda} \Big(g_{1/2}(x^\lambda - y) f(u(y), v(y)) + g_{1/2}(x-y) f(u_\lambda(y), v_\lambda(y)) \Big) \,\mathrm{d}y,$$

and we have $u, v \in H^{\frac{1}{2}}(\mathbb{R}^N)$ satisfy (5.3) and $x \in \mathbb{R}^N$, then we have

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] [f(u_\lambda(y), v_\lambda(y)) - f(u(y), v(y))] \,\mathrm{d}y.$$

Since the analogous claim holds for v instead of u, we are done.

Lemma 1.20 *Denoting*

$$h_1(\lambda, y) = \frac{\partial f}{\partial \xi}(u(y) + \theta_1(y) u_\lambda(y), v_\lambda(y)), \quad h_2(\lambda, y) = \frac{\partial f}{\partial \eta}(u(y), v(y) + \theta_2(y) v_\lambda(y)),$$
$$h_3(\lambda, y) = \frac{\partial f}{\partial \xi}(v(y) + \theta_3(y) v_\lambda(y), u_\lambda(y)), \quad h_4(\lambda, y) = \frac{\partial f}{\partial \eta}(v(y), u(y) + \theta_4(y) u_\lambda(y)),$$

then

$$u_\lambda(x) - u(x) \le \int_{\Sigma_\lambda^u} g_{1/2}(x-y)h_1(\lambda,y)\,[u_\lambda(y)-u(y)]\,\mathrm{d}y + \int_{\Sigma_\lambda^v} g_{1/2}(x-y)h_2(\lambda,y)\,[v_\lambda(y)-v(y)]\,\mathrm{d}y \tag{5.4a}$$

and

$$v_\lambda(x) - v(x) \le \int_{\Sigma_\lambda^v} g_{1/2}(x-y)h_3(\lambda,y)\,[v_\lambda(y)-v(y)]\,\mathrm{d}y + \int_{\Sigma_\lambda^u} g_{1/2}(x-y)h_4(\lambda,y)\,[u_\lambda(y)-u(y)]\,\mathrm{d}y. \tag{5.4b}$$

Proof. Since $|x-y| \le |x^\lambda - y|$ for all $x, y \in \Sigma_\lambda$, we conclude that

$$g_{1/2}(x-y) - g_{1/2}(x^\lambda - y) \ge 0.$$

Therefore, it follows from Lemma 1.19

$$\begin{aligned}
u_\lambda(x) - u(x) &= \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] [f(u_\lambda(y), v_\lambda(y)) - f(u(y), v_\lambda(y))]\,\mathrm{d}y \\
&\quad + \int_{\Sigma_\lambda} \left[g_{1/2}(x-y) - g_{1/2}(x^\lambda - y)\right] [f(u(y), v_\lambda(y)) - f(u(y), v(y))]\,\mathrm{d}y \\
&\le \int_{\Sigma_\lambda^u} g_{1/2}(x-y)\,[f(u_\lambda(y), v_\lambda(y)) - f(u(y), v_\lambda(y))]\,\mathrm{d}y \\
&\quad + \int_{\Sigma_\lambda^v} g_{1/2}(x-y)\,[f(u(y), v_\lambda(y)) - f(u(y), v(y))]\,\mathrm{d}y \\
&= \int_{\Sigma_\lambda^u} g_{1/2}(x-y)\frac{\partial f}{\partial \xi}(u(y) + \theta_1(y)u_\lambda(y), v_\lambda(y))\,[u_\lambda(y) - u(y)]\,\mathrm{d}y \\
&\quad + \int_{\Sigma_\lambda^v} g_{1/2}(x-y)\frac{\partial f}{\partial \eta}(u(y), v(y) + \theta_2(y)v_\lambda(y))\,[v_\lambda(y) - v(y)]\,\mathrm{d}y,
\end{aligned}$$

where the last equality follows from the Mean Value Theorem. Analogously, we have

$$\begin{aligned}
v_\lambda(x) - v(x) &= \int_{\Sigma_\lambda^v} g_{1/2}(x-y)\frac{\partial f}{\partial \xi}(v(y) + \theta_3(y)v_\lambda(y), u_\lambda(y))\,[v_\lambda(y) - v(y)]\,\mathrm{d}y \\
&\quad + \int_{\Sigma_\lambda^u} g_{1/2}(x-y)\frac{\partial f}{\partial \eta}(v(y), u(y) + \theta_4(y)u_\lambda(y))\,[u_\lambda(y) - u(y)]\,\mathrm{d}y.
\end{aligned}$$

Thus, we have obtained (5.4a) and (5.4b). □

The proof of the next result can be found in D. Adams and Hedberg [1, Corollary 3.1.5].

Corollary 1.21 *Let $f \in L^p(\mathbb{R}^N)$, $0 < \alpha \le N$, $1 < p < \frac{N}{\alpha}$, and set $p^* = \frac{Np}{N-\alpha p}$. Then there is A such that for $p \le q \le p^*$*

$$||(g_\alpha * f)||_{L^q(\mathbb{R}^N)} \le A||f||_{L^p(\mathbb{R}^N)}.$$

We denote by χ_B the characteristic function of the set B.

Theorem 1.22 *Suppose that $u, v \in H^{\frac{1}{2}}(\mathbb{R}^N)$ satisfy* (5.3)*. Then, if $q \in (2, \frac{2N}{N-1}]$, we have*

$$\|u_\lambda - u\|_{L^q(\Sigma_\lambda^u)} \le A\|h_1(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma_\lambda^u)}\|u_\lambda - u\|_{L^q(\Sigma_\lambda^u)} + A\|h_2(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma_\lambda^v)}\|v_\lambda - v\|_{L^q(\Sigma_\lambda^v)}$$

and

$$\|v_\lambda - v\|_{L^q(\Sigma_\lambda^v)} \le A\|h_3(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma_\lambda^v)}\|v_\lambda - v\|_{L^q(\Sigma_\lambda^v)} + A\|h_4(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma_\lambda^u)}\|u_\lambda - u\|_{L^q(\Sigma_\lambda^u)},$$

Proof. Applying (5.4a), Corollary 1.21 and Hölder's inequality, we obtain

$$\begin{aligned}
\|u_\lambda - u\|_{L^q(\Sigma_\lambda^u)} &= \left(\int_{\mathbb{R}^N} |u_\lambda(x) - u(x)|^q \chi_{\Sigma_\lambda^u}(x)\mathrm{d}x\right)^{\frac{1}{q}} \\
&\le \left(\int_{\mathbb{R}^N} \chi_{\Sigma_\lambda^u}(x)\left[\int_{\mathbb{R}^N} g_{1/2}(x-y)h_1(\lambda,y)\left[u_\lambda(y) - u(y)\right]\chi_{\Sigma_\lambda^u}(y)\mathrm{d}y\right.\right. \\
&\quad \left.\left. + \int_{\mathbb{R}^N} g_{1/2}(x-y)h_2(\lambda,y)\left[v_\lambda(y) - v(y)\right]\chi_{\Sigma_\lambda^v}(y)\mathrm{d}y\right]^q \mathrm{d}x\right)^{\frac{1}{q}} \\
&\le \left(\int_{\mathbb{R}^N}\left[\int_{\mathbb{R}^N} g_{1/2}(x-y)h_1(\lambda,y)\left[u_\lambda(y) - u(y)\right]\chi_{\Sigma_\lambda^u}(y)\mathrm{d}y\right.\right. \\
&\quad \left.\left. + \int_{\mathbb{R}^N} g_{1/2}(x-y)h_2(\lambda,y)\left[v_\lambda(y) - v(y)\right]\chi_{\Sigma_\lambda^v}(y)\mathrm{d}y\right]^q \mathrm{d}x\right)^{\frac{1}{q}} \\
&\le \|(g_{1/2} * h_1(\lambda,\cdot)[u_\lambda - u]\chi_{\Sigma_\lambda^u})\|_{L^q(\mathbb{R}^N)} + \|(g_{1/2} * h_2(\lambda,\cdot)[v_\lambda - v]\chi_{\Sigma_\lambda^v})\|_{L^q(\mathbb{R}^N)} \\
&\le A\|h_1(\lambda,\cdot)[u_\lambda - u]\chi_{\Sigma_\lambda^u}\|_{L^2(\mathbb{R}^N)} + A\|h_2(\lambda,\cdot)[v_\lambda - v]\chi_{\Sigma_\lambda^v}\|_{L^2(\mathbb{R}^N)} \\
&= A\|h_1(\lambda,\cdot)[u_\lambda - u]\|_{L^2(\Sigma_\lambda^u)} + A\|h_2(\lambda,\cdot)[v_\lambda - v]\|_{L^2(\Sigma_\lambda^v)} \\
&\le A\|h_1(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma_\lambda^u)}\|u_\lambda - u\|_{L^q(\Sigma_\lambda^u)} + A\|h_2(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma_\lambda^v)}\|v_\lambda - v\|_{L^q(\Sigma_\lambda^v)}.
\end{aligned}$$

Analogously we obtain the statement about v. □

Corollary 1.23 *For any $q \in (2, \frac{2N}{N-1}]$ fixed we have*

$$\|u_\lambda - u\|_{L^q(\Sigma^u_\lambda)} = 0 \qquad \textit{and} \qquad \|v_\lambda - v\|_{L^q(\Sigma^v_\lambda)} = 0,$$

if λ is large enough.

Proof. Since $h_i(\lambda, \cdot) \in L^q(\mathbb{R}^N)$ for $i \in \{1,2,3,4\}$ and q as above, choosing $\lambda > 0$ large enough we obtain

$$\begin{aligned} A\|h_1(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma^u_\lambda)} < \tfrac{1}{4}, \;\; A\|h_2(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma^v_\lambda)} < \tfrac{1}{4}, \\ A\|h_3(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma^v_\lambda)} < \tfrac{1}{4}, \;\; A\|h_4(\lambda,\cdot)\|_{L^{\frac{2q}{q-2}}(\Sigma^u_\lambda)} < \tfrac{1}{4}, \end{aligned} \tag{5.5}$$

and it follows from Theorem 1.22 that

$$\begin{aligned} \|u_\lambda - u\|_{L^q(\Sigma^u_\lambda)} \le \frac{1}{4}\|u_\lambda - u\|_{L^q(\Sigma^u_\lambda)} + \frac{1}{4}\|v_\lambda - v\|_{L^q(\Sigma^v_\lambda)} \\ \|v_\lambda - v\|_{L^q(\Sigma^v_\lambda)} \le \frac{1}{4}\|v_\lambda - v\|_{L^q(\Sigma^v_\lambda)} + \frac{1}{4}\|u_\lambda - u\|_{L^q(\Sigma^u_\lambda)}. \end{aligned}$$

Adding the inequalities, we obtain

$$\|u_\lambda - u\|_{L^q(\Sigma^u_\lambda)} + \|v_\lambda - v\|_{L^q(\Sigma^v_\lambda)} \le \frac{1}{2}\left(\|u_\lambda - u\|_{L^q(\Sigma^u_\lambda)} + \|v_\lambda - v\|_{L^q(\Sigma^v_\lambda)}\right). \tag{5.6}$$

Thus,

$$||u_\lambda - u\|_{L^q(\Sigma^u_\lambda)} + ||v_\lambda - v\|_{L^q(\Sigma^v_\lambda)} = 0$$

and we are done. □

Theorem 1.24 *If $z(x)$ is a solution of* (5.1)*, then z is radially symmetric and decreasing with respect to some point.*

Proof. We claim that, for λ large enough, we have

$$u_\lambda(x) \le u(x) \quad \text{and} \quad v_\lambda(x) \le v(x) \;\; \text{a.e. in } \Sigma_\lambda. \tag{5.7}$$

In fact, combining Theorem 1.22 and Corollary 1.23 yields, for λ large enough,

$$\|u_\lambda - u\|_{L^q(\Sigma_\lambda^u)} = 0 \quad \text{and} \quad \|v_\lambda - v\|_{L^q(\Sigma_\lambda^v)} = 0$$

and we conclude our claim.

Considering only the first inequality in (5.7), we should look for $x \in \Sigma_\lambda$ where $u_\lambda(x) > u(x)$ if λ is not so big. In other words, we can move the plane T_λ in the x_1-axis from a neighborhood of $+\infty$ to the left whilst that inequality remains valid. We suppose we have to stop when $\lambda = \lambda_0 > 0$.

If there exists $x_* \in \Sigma_{\lambda_0}$ such that $u(x_*) = u_{\lambda_0}(x_*)$, then it follows from Lemma 1.19 that

$$\begin{aligned} 0 &= u_{\lambda_0}(x_*) - u(x_*) \\ &= \int_{\Sigma_{\lambda_0}} \left[g_{1/2}(x_* - y) - g_{1/2}(x_*^{\lambda_0} - y) \right] \left[f(u_{\lambda_0}(y), v_{\lambda_0}(y)) - f(u(y), v_\lambda(y)) \right] \mathrm{d}y \\ &\quad + \int_{\Sigma_{\lambda_0}} \left[g_{1/2}(x_* - y) - g_{1/2}(x_*^{\lambda_0} - y) \right] \left[f(u(y), v_{\lambda_0}(y)) - f(u(y), v(y)) \right] \mathrm{d}y. \end{aligned} \tag{5.8}$$

Since $g_{1/2}$ is radially decreasing and $|x_* - y| > |x_*^{\lambda_0} - y|$ in Σ_{λ_0}, we have

$$g_{1/2}(x^* - y) < g_{1/2}(x^*_{\lambda_0} - y),$$

from what follows

$$f(u_{\lambda_0}(y), v_{\lambda_0}(y)) = f(u(y), v_\lambda(y)) \quad \text{and} \quad f(u(y), v_{\lambda_0}(y)) = f(u(y), v(y)). \tag{5.9}$$

This contradicts the fact the f is strictly increasing in both variables. Observe that the same argument also applies to v instead of u. Therefore, we conclude that

$$u_{\lambda_0}(x) < u(x) \quad \text{and} \quad v_{\lambda_0}(x) < v(x), \;\; \forall\, x \in \Sigma_{\lambda_0}.$$

We now claim that

$$\lambda_0 = \inf \{\lambda \geq 0 \,:\, u_\lambda(x) \leq u(x), \, \forall x \in \Sigma_\lambda\} = 0. \tag{5.10}$$

Observe that $\lambda_0 = 0$ implies $u_0 = u$ and u is symmetrical with respect to x_1.

Supposing the contrary, that is $\lambda_0 > 0$, we show that the plane T_{λ_0} can be moved to the left, contradicting the definition of λ_0.

Since $h_i(\lambda, \cdot) \in L^q(\mathbb{R}^N)$ for all $q \in [1, \infty]$, it follows that for any $\epsilon > 0$ small enough, there exists $R > 0$ large enough so that

$$\int_{\mathbb{R}^N \setminus B_R(0)} \|h_1(\lambda, \cdot)\|^{\frac{2q}{q-2}} \mathrm{d}x < \epsilon \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R(0)} \|h_2(\lambda, \cdot)\|^{\frac{2q}{q-2}} \mathrm{d}x < \epsilon.$$

Applying Lusin's theorem, for any $\delta > 0$ there exists a closed set $F_\delta \subset B_R(0) \cap \Sigma_{\lambda_0} = E$ such that $(u_{\lambda_0} - u)\big|_{F_\delta}$ is continuous and $\mu(E - F_\delta) < \delta$.

Since we have $u_{\lambda_0}(x) < u(x)$ in the interior of Σ_{λ_0}, we obtain that $u_{\lambda_0}(x) < u(x)$ in F_δ.

Choose $\epsilon_1 > 0$ small enough so that, for any $\lambda \in (\lambda_0 - \epsilon_1, \lambda_0]$,

$$u_\lambda - u < 0, \quad \forall\, x \in F_\delta.$$

It follows that

$$\Sigma_\lambda^u \subset M := \left(\mathbb{R}^N \setminus B_R(0)\right) \cup (E \setminus F_\delta) \cup \left[(\Sigma_\lambda \setminus \Sigma_\lambda^u) \cap B_R(0)\right].$$

Now take ϵ, δ and ϵ_1 small enough such that the inequalities (5.5) are true. Then, proceeding as in the proof of Corollary 1.23, we conclude that Σ_λ^u and Σ_λ^v have null measure, contradicting the definition of λ_0. Thus, claim (5.10) is proved. Since the same argument applies to v instead of u, u and v are symmetrical with respect to axis x_1. Since the direction x_1 was chosen arbitrarily, the same argument applies to any axis. Considering our hypotheses, we conclude that our solution is radial with respect to some point $x_0 \in \mathbb{R}^N$ (see [4] and, for details, particularly [12]). □

1.6 Hölder regularity

We define, for each $\alpha > 0$,

$$L^{\alpha,q}(\mathbb{R}^N) = \{u = g_\alpha * f \,:\, f \in L^q(\mathbb{R}^N)\}.$$

We recall a regularity result:

Lemma 1.25 *If* $0 < \mu < \alpha - \frac{N}{q}$, *then* $L^{\alpha,q}(\mathbb{R}^N) \subset C^{0,\mu}(\mathbb{R}^N)$.

In the case of $\alpha = 1/2$, we will apply a bootstrap argument. Since

$$|f(\xi,\eta)| \leq \frac{1}{s}|\xi| + \lambda|\eta| \qquad \text{and} \qquad |f(\eta,\xi)| \leq \frac{1}{s}|\eta| + \lambda|\xi|,$$

we conclude that, if $u, v \in L^q(\mathbb{R}^N)$, then

$$f(u,v), f(v,u) \in L^q(\mathbb{R}^N). \tag{6.1}$$

Proposition 1.26 *If* $N \geq 1$, *then any solution of* (1.2) *belongs to* $C^{0,\mu}(\mathbb{R}^N)$ *for* $0 < \mu < \alpha - \frac{N}{2}$.

Proof. We apply a bootstrap argument, combining Corollary 1.21 and Lemma 1.25. We start taking $p = 2$, since it follows from Corollary 1.21 that $u, v \in L^{2^*}(\mathbb{R}^N)$ and therefore $f(u,v), f(v,u) \in L^{2^*}(\mathbb{R}^N)$.

Let us suppose that $(1/2)2^* < N$. Then, Corollary 1.21 guarantees that $u, v \in L^{2^*}(\mathbb{R}^N)$ and it follows that $f(u,v), f(v,u) \in L^{2^*}(\mathbb{R}^N)$. We define recursively

$$\begin{aligned}
2^*_0 &= 2,\\
2^*_1 &= 2^* = \frac{2^*_0 N}{N - \frac{1}{2}2^*_0} = \frac{2N}{N-1},\\
2^*_2 &= \frac{2^*_1 N}{N - \frac{1}{2}2^*_1} = \frac{2^* N}{N - \frac{1}{2}2^*},\\
\vdots &= \vdots\\
2^*_k &= \frac{2^*_{k-1} N}{N - \frac{1}{2}2^*_{k-1}} > 2^*_{k-1}.
\end{aligned}$$

We claim that there exists k_0 such that $\frac{1}{2}2^*_{k_0-1} \geq N$. In fact, if $\frac{1}{2}2^*_k < N$ for all k, then there exists $L = \lim_{k\to\infty} 2^*_k$.

But this implies

$$L = \lim_{k\to\infty} 2^*_k = \lim_{k\to\infty} \frac{2^*_{k-1} N}{N - \frac{1}{2}2^*_{k-1}} = \frac{LN}{N - \frac{1}{2}L} > L,$$

and we have reached a contradiction.

If $\frac{1}{2}2^*_{k_0-1} > N$, our result follows from Corollary 1.21. If, however, $\frac{1}{2}2^*_{k_0-1} = N$, then $2^*_{k_0}$ does not make sense, but we can take $r \in (2^*_{k_0-2}, 2^*_{k_0-1})$ tal que $(1/2)r^* > N$ and once again our result follows Corollary 1.21, since $u, v \in L^r(\mathbb{R}^N)$

The same argument applies to the case $(1/2)2^* = N$, since p^* does not make sense for $p = (1/2)2^*$, but we can take $r \in (2, 2^*)$ such that $(1/2)r^* > N$.

Finally, the case $(1/2)2^* > N$ implies $N = 1$ and follows immediately from Corollary 1.21.

□

1.7 Appendix

We start recalling the the Pohozaev identity for the problem $\sqrt{-\Delta + m^2}\, u = a(x)f(u)$ in $\mathbb{R}^N$.

Lemma 1.27 *For an appropriate f, let $v \in H^{\frac{1}{2}}(\mathbb{R}^N)$ be a weak solution to*

$$\sqrt{-\Delta + m^2}\, v = a(x)f(v) \quad in \ \mathbb{R}^N. \tag{7.1}$$

Then it satisfies the Pohozaev identity

$$\frac{(N-1)}{2}\int_{\mathbb{R}^N} |(-\Delta + m^2)^{\frac{1}{4}}u|^2 \mathrm{d}x + \frac{m^2}{2}\int_{\mathbb{R}^N} \frac{|\mathcal{F}u(\xi)|^2}{\sqrt{m^2 + 4\pi^2|\xi|^2}}\mathrm{d}\xi$$

$$= \int_{\mathbb{R}^N} \left[Na(x) + \nabla a(x)\cdot x\right] F(u(x))\mathrm{d}x \tag{7.2}$$

On its turn, the Pohozaev identity for the extension problem is given by

Lemma 1.28 *For appropriate f and a, let $u \in H^1(\mathbb{R}^{N+1}_+)$ be a weak solution to*

$$\begin{cases} -\Delta u + m^2 u &= 0, & in \ \mathbb{R}^{N+1}_+ \\ -\frac{\partial u(0,y)}{\partial x} &= a(y)f(u(0,y)), & on \ \mathbb{R}^N. \end{cases} \tag{7.3}$$

Then it satisfies the Pohozaev identity

$$\frac{(N-1)}{2}\iint_{\mathbb{R}^{N+1}_+}|\nabla u|^2\mathrm{d}x\mathrm{d}y+\frac{m^2}{2}(N+1)\iint_{\mathbb{R}^{N+1}_+}u^2\mathrm{d}x\mathrm{d}y$$

$$=\iint_{\mathbb{R}^N}[Na(y)+\nabla a(y)\cdot y]F(u(0,y))\mathrm{d}y \tag{7.4}$$

Since

$$\begin{aligned}J'(z)\cdot\varphi &= 2N\langle z,\varphi\rangle+m^2\int_{\mathbb{R}^N}\frac{\mathcal{F}z(\xi)\mathcal{F}\overline{\varphi}(\xi)+\mathcal{F}\varphi(\xi)\mathcal{F}\overline{z}(\xi)}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi\\ &-\int_{\mathbb{R}^N}[2N\nabla F(z(x))\cdot\varphi(x)+\nabla K(z(x))\cdot\varphi(x)]\mathrm{d}x,\end{aligned}$$

it follows

$$\begin{aligned}J'(z)\cdot z &= 2N||z||^2+2m^2\int_{\mathbb{R}^N}\frac{|\mathcal{F}z(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi\\ &-\int_{\mathbb{R}^N}[2N\nabla F(z(x))\cdot z(x)+\nabla K(z(x))\cdot z(x)]\mathrm{d}x.\end{aligned} \tag{7.5}$$

In this section we introduce the notation and results used in the proof that any critical point of J is a solution of the system

$$\begin{cases}2N(-\Delta\overline{u}+m^2\overline{u})+4m^2\overline{u} &= 0, \quad \text{in } \mathbb{R}^{N+1}_+\\ -\frac{\partial\overline{u}(0,y)}{\partial x} &= f_1(\overline{u}(0,y),\overline{v}(0,y)), \quad \text{on } \mathbb{R}^N,\\ 2N(-\Delta\overline{v}+m^2\overline{v})+4m^2\overline{v} &= 0, \quad \text{in } \mathbb{R}^{N+1}_+,\\ -\frac{\partial\overline{v}(0,y)}{\partial x} &= f_2(\overline{u}(0,y),\overline{v}(0,y)), \quad \text{on } \mathbb{R}^N,\end{cases} \tag{7.6}$$

where $\nabla F(\xi,\eta)=(f_1(\xi,\eta),f_2(\xi,\eta))$. Writing (7.6) in the form

$$\begin{cases}-\Delta\overline{u}+(m^2+\frac{4m^2}{2N})\overline{u} &= 0, \quad \text{in } \mathbb{R}^{N+1}_+\\ -\frac{\partial\overline{u}(0,y)}{\partial x} &= f_1(\overline{u}(0,y),\overline{v}(0,y)), \quad \text{on } \mathbb{R}^N,\\ -\Delta\overline{v}+(m^2+\frac{4m^2}{2N})\overline{v} &= 0, \quad \text{in } \mathbb{R}^{N+1}_+,\\ -\frac{\partial\overline{v}(0,y)}{\partial x} &= f_2(\overline{u}(0,y),\overline{v}(0,y)), \quad \text{on } \mathbb{R}^N.\end{cases} \tag{7.7}$$

According to the Pohozaev identity (1.28), defining $Z=(\overline{u},\overline{v})$ and $\nabla H=(f_1,f_2)$, system (7.7) satisfies

$$(N-1)\iint_{\mathbb{R}^{N+1}_+}|\nabla Z|^2\mathrm{d}x\mathrm{d}y+\left(m^2+\frac{4m^2}{2N}\right)(N+1)\iint_{\mathbb{R}^{N+1}_+}|Z|^2\mathrm{d}x\mathrm{d}y$$

$$=2N\int_{\mathbb{R}^N}H(\overline{u}(0,y),\overline{v}(0,y))\mathrm{d}y,$$

that is,

$$2N(N-1)\iint_{\mathbb{R}^{N+1}_+}\left[|\nabla Z|^2+m^2|Z|^2\right]\mathrm{d}x\mathrm{d}y+4(2N+1)m^2\iint_{\mathbb{R}^{N+1}_+}|Z|^2\mathrm{d}x\mathrm{d}y$$

$$=4N^2\int_{\mathbb{R}^N}H(\overline{u}(0,y),\overline{v}(0,y))\mathrm{d}y.$$

Projecting into $H^{\frac{1}{2}}(\mathbb{R}^N)$ (compare with Lemma 1.27), we obtain the Pohozaev identity attached to $\mathcal{M}$.

$$2N(N-1)||z||^2+2(2N+1)m^2\int_{\mathbb{R}^N}\frac{|\mathcal{F}z(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi=$$

$$=4N^2\int_{\mathbb{R}^N}H(z)\mathrm{d}x. \tag{7.8}$$

Therefore, we define the Pohozaev funtional $\widetilde{P}(z)$ by

$$\begin{aligned}\widetilde{P}(z)&=2N(N-1)\|z\|^2+2(2N+1)m^2\int_{\mathbb{R}^N}\frac{|\mathcal{F}z(\xi)|^2}{\sqrt{m^2+4\pi^2|\xi|^2}}\mathrm{d}\xi\\&-4N^2\int_{\mathbb{R}^N}H(z)\mathrm{d}x,\end{aligned} \tag{7.9}$$

and the Nehari-Pohozaev functional $\widetilde{J}(z)$ for $\mathcal{M}$:

$$\widetilde{J}(z)=J'(z)\cdot z+2\widetilde{P}(z).$$

where H satisfies $\nabla H=2N\nabla F+\nabla K$, that is, $H=2NF+K$.

The Pohozaev identity related to $\Phi'(z) + \mu J'(z) = 0$ is given by

$$P(z) + \mu \widetilde{P}(z) = 0$$

and we have

$$\Phi'(z) \cdot z + \mu J'(z) \cdot z = 0.$$

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Chapter 2

Ground state of a magnetic nonlinear Choquard equation

Abstract

We consider the stationary magnetic nonlinear Choquard equation

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\frac{f(|u|)}{|u|}u,$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ is a vector potential, V is a scalar potential, $f : \mathbb{R} \to \mathbb{R}$ and F is the primitive of f. Under mild hypotheses, we prove the existence of a ground state solution for this problem. We also prove a simple multiplicity result by applying Ljusternik-Schnirelmann methods.

2.1 Introduction

We consider the problem

$$-(\nabla + iA(x))^2u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\frac{f(|u|)}{|u|}u \tag{1.1}$$

where $\nabla + iA(x)$ is the covariant derivative with respect to the C^1 vector potential $A : \mathbb{R}^N \to \mathbb{R}^N$. (After stating our hypotheses, the form of equation (1.1) will be changed to (1.2).) The constant α belongs to the interval $(0, N)$ and

$$\lim_{|x|\to\infty} A(x) = A_\infty \in \mathbb{R}^N.$$

The scalar potential $V\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous, bounded function satisfying

$(V1)$ $\inf_{\mathbb{R}^N} V > 0$;

$(V2)$ $V_\infty = \lim_{|y|\to\infty} V(y)$;

$(V3)$ $V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$.

We also suppose that

(AV) $|A(y)|^2 + V(y) < |A_\infty|^2 + V_\infty$.

The function F is the primitive of the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$, which is non-negative in $(0, \infty)$ and satisfies, for any $r \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$,

$(f1)$ $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 0$,

$(f2)$ $\displaystyle\lim_{t\to\infty} \frac{f(t)}{t^{r-1}} = 0$,

$(f3)$ $\displaystyle\frac{f(t)}{t}$ is (strictly) increasing if $t > 0$ and (strictly) decreasing if $t < 0$.

For example, if $t \in \mathbb{R}$, the functions $t\ln(1 + |t|)$ and $|t|^{q_1-2}t + |t|^{q_2-2}t$ (where $2 < q_1, q_2 < r$) satisfy hypothesis $(f1)$, $(f2)$ and $(f3)$.

The case $A \equiv 0$ conduces to the Choquard equation. There is a huge collection of articles on the subject and a good review of the Choquard equation can be found in [17]. Recent advances in the study of this equation, including also the case of critical growth of the nonlinearity $f(u)$ can be found, e.g., in [1, 2, 3, 4, 12, 13, 16]

We denote

$$\tilde{f}(t) = \begin{cases} \dfrac{f(t)}{t}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Our hypotheses imply that $\tilde{f}$ is continuous. Therefore, problem (1.1) can be written in the form

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * F(|u|)\right)\tilde{f}(|u|)u. \tag{1.2}$$

The composition of f and F with $|u|$ gives a variational structure to the problem, allowing the application of the Mountain Pass Theorem. So, the right-hand side of problem (1.2) generalizes the term

$$\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \tag{1.3}$$

which was studied by Cingolani, Clapp and Secchi in [9]. Similar forms of problem (1.2) were studied in [7], [8] and [23].

Our aim in this chapter is to prove the existence of a ground state for problem (1.2), that is, a non-trivial solution with minimal energy. This is accomplished by showing that the mountain pass geometry is satisfied and then considering the asymptotic form of problem (1.2) and applying Struwe's splitting lemma.

The main part of the interesting paper by Cingolani, Clapp and Secchi [9] is devoted to the existence of multiple solutions of equation (1.2) - with (1.3) as the right-hand side - under the action of a closed subgroup G of the orthogonal group $O(N)$ of linear isometries of $\mathbb{R}^N$ if $A(gx) = gA(x)$ and $V(gx) = V(x)$ for all $g \in G$ and $x \in \mathbb{R}^N$. The authors look for solutions satisfying

$$u(gx) = \tau(g)u(x), \quad \text{for all } g \in G \text{ and } x \in \mathbb{R}^N,$$

where $\tau\colon G \to S^1$ is a given continuous group homomorphism into the unit complex numbers S^1. In this chapter we also address the multiplicity of solutions in a particular case of that treated in [9].

We define

$$\nabla_A u = \nabla u + iA(x)u$$

and consider the space

$$H^1_{A,V}(\mathbb{R}^N,\mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^N,\mathbb{C}) \;:\; \nabla_A u \in L^2(\mathbb{R}^N,\mathbb{C}) \right\}$$

endowed with scalar product

$$\langle u,v\rangle_{A,V} = \mathfrak{Re}\int_{\mathbb{R}^N} \left(\nabla_A u \cdot \overline{\nabla_A v} + V(x)u\bar{v}\right)$$

and, therefore

$$\|u\|^2_{A,V} = \int_{\mathbb{R}^N} |\nabla_A u|^2 + V|u|^2.$$

Observe that the norm generated by this scalar product is equivalent to the norm obtained by considering $V \equiv 1$, see [15, Definition 7.20].

If $u \in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N)$ and the *diamagnetic inequality* is valid (see [15, Theorem 7.21],[9])

$$|\nabla|u|(x)| \leq |\nabla u(x) + iA(x)u(x)|, \text{ a.e. } x \in \mathbb{R}^N.$$

As a consequence of the diamagnetic inequality, we have the continuous immersion

$$H^1_{A,V}(\mathbb{R}^N,\mathbb{C}) \hookrightarrow L^q(\mathbb{R}^N,\mathbb{C}) \tag{1.4}$$

for any $q \in [2, \frac{2N}{N-2}]$. We denote $2^* = \frac{2N}{N-2}$.

It is well-known that $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ is dense in $H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$, see [15, Theorem 7.22].

Remark 2.1 *It follows from $(f1)$-$(f2)$ that, for any fixed $\xi > 0$, there exists a constant C_ξ such that*

$$|f(t)| \leq \xi t + C_\xi t^{r-1}, \quad \forall\, t \geq 0. \tag{1.5}$$

Similarly, there exists $D_\xi > 0$ such that

$$|F(t)| \leq \xi t^2 + D_\xi t^r, \quad \forall\, t \geq 0.$$

Furthermore, $(f3)$ implies that f satisfies the Ambrosetti-Rabinowitz inequality

$$2F(t) < f(t)t, \quad \forall\, t > 0. \tag{1.6}$$

Observe that the function $f(t) = t\ln(1+|t|)$ satisfies the last inequality, but does not satisfy $\theta F(t) \leq tf(t)$ for any $\theta > 2$.

We state our results:

Theorem 2.2 *Suppose that $\alpha \in (0,N)$ and that conditions $(V1)$-$(V3)$, (AV) and $(f1)$-$(f3)$ are valid. Then, problem (1.1) has a ground state solution.*

In order to obtain our multiplicity result, we define the space

$$H^1_A(\mathbb{R}^N,\mathbb{C})^\tau = \left\{ u \in H^1_A(\mathbb{R}^N,\mathbb{C}) \,:\, u(gx) = \tau(g)u(x),\ \forall\, g \in G,\ \forall\, x \in \mathbb{R}^N \right\}$$

and suppose that the closed subgroup $G \subset O(N)$ satisfies the decomposition

$$G = O(N_1) \times O(N_2) \times \cdots \times O(N_k), \tag{1.7}$$

where $\sum_{j=1}^k N_j = N$, $N_j \geq 2$ for all $j \in \{1,\ldots,k\}$. Then we have

Theorem 2.3 *Let G be a closed subgroup of $O(N)$ satisfying the decomposition (1.7). Assume that $A(gx) = gA(x)$ and $V(gx) = V(x)$ for all $g \in G$ and $x \in \mathbb{R}^N$. Then problem (1.2) has a sequence $(u_n) \subset H^1_A(\mathbb{R}^N,\mathbb{C})^\tau$ such that $\lim\limits_{n\to\infty} \|u_n\|^2_{A,V} = \infty$.*

The chapter is organized as follows: Section 2.2 shows the mountain pass geometry and some basic results concerning the right-hand side of equation (1.2). Theorem 2.2 is proved in Section 2.3 and our multiplicity result in Section 2.4.

2.2 Variational Formulation

The energy functional associated to problem (1.1) is given by

$$J_{A,V}(u) = \frac{1}{2}\|u\|^2_{A,V} - D(u), \tag{2.1}$$

where

$$D(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * F(|u|)\right) F(|u|).$$

The energy functional is well-defined as a consequence of the Hardy-Littlewood-Sobolev (see [15, Theorem 4.3], since

$$\left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*F(|u|)\right)F(|u|)\right|\leq C\left(\|u\|^4+\|u\|^{2r}\right). \tag{2.2}$$

Remark 2.4 *Let us consider the case $F(t)=|t|^r$. By applying the Hardy-Littlewood-Sobolev inequality we have that*

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*F(|u|)\right)F(|u|)$$

is well-defined if $F(|u|)\in L^p(\mathbb{R}^N)$ for $p>1$ defined by

$$\frac{2}{p}+\frac{\alpha}{N}=2 \quad\Rightarrow\quad \frac{1}{p}=\frac{1}{2}\left(2-\frac{\alpha}{N}\right).$$

Consequently, in order to apply the immersion (1.4), *we must have*

$$pr\in[2,2^*]\Rightarrow\quad \frac{2N-\alpha}{N}\leq r\leq\frac{N}{N-2}\left(2-\frac{\alpha}{N}\right)=\frac{2N-\alpha}{N-2}.$$

This condition (taking the open interval satisfied by r) justifies hypothesis $(f2)$.

Since the derivative of the energy functional $J_{A,V}(u)$ is given by

$$\begin{aligned}J'_{A,V}(u)\cdot\psi &= \langle u,\psi\rangle_{A,V}-D'(u)\cdot\psi\\ &= \langle u,\psi\rangle_{A,V}-\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*F(|u|)\right)\tilde{f}(|u|)u\bar{\psi},\end{aligned}$$

we see that critical points of $J'_{A,V}(u)$ are weak solutions of (1.2). Note that, if $\psi=u$ we obtain

$$J'_{A,V}(u)\cdot u:=\|u\|^2_{A,V}-\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*F(|u|)\right)f(|u|)|u|. \tag{2.3}$$

Lemma 2.5 *The functional $J_{A,V}$ satisfies the Mountain Pass geometry. Precisely,*

(i) *there exist $\rho,\delta>0$ such that $J_{A,V}\big|_S\geq\delta>0$ for any $u\in S$, where*

$$S=\{u\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\,:\,\|u\|_{A,V}=\rho\};$$

(ii) *for any $u_0\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}$ there exists $\tau\in(0,\infty)$ such that $\|\tau u_0\|>\rho$ e $J_{A,V}(\tau u_0)<0$.*

Proof. Inequality (2.2) yields

$$J_{A,V}(u) \geq \frac{1}{2}\|u\|_{A,V}^2 - C\left(\|u\|_{A,V}^4 + \|u\|_{A,V}^{2r}\right),$$

thus implying (*i*) when we take $\|u\|_{A,V} = \rho > 0$ small enough.

In order to prove (*ii*), fix $u_0 \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ and consider the function $g_{u_0} \colon (0,\infty) \to \mathbb{R}$ given by

$$g_{u_0}(t) = D\left(\frac{tu_0}{\|u_0\|_{A,V}}\right) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^\alpha} * F\left(\frac{t|u_0|}{\|u_0\|_{A,V}}\right)\right] F\left(\frac{t|u_0|}{\|u_0\|_{A,V}}\right).$$

We have

$$\begin{aligned} g'_{u_0}(t) &= \int_{\mathbb{R}^N}\left[\frac{1}{|x|^\alpha} * F\left(\frac{t|u_0|}{\|u_0\|_{A,V}}\right)\right] f\left(\frac{t|u_0|}{\|u_0\|_{A,V}}\right)\frac{|u_0|}{\|u_0\|_{A,V}} \\ &= \frac{4}{t}\int_{\mathbb{R}^N}\frac{1}{2}\left[\frac{1}{|x|^\alpha} * F\left(\frac{t|u_0|}{\|u_0\|_{A,V}}\right)\right]\frac{1}{2} f\left(\frac{t|u_0|}{\|u_0\|_{A,V}}\right)\frac{t|u_0|}{\|u_0\|_{A,V}} \\ &\geq \frac{4}{t} g_{u_0}(t) \end{aligned}$$

as a consequence of the Ambrosetti-Rabinowitz condition (1.6). Observe that $g'_{u_0}(t) > 0$ for $t > 0$.

Thus,

$$\ln g_{u_0}(t)\Big|_1^{\tau\|u_0\|_{A,V}} \geq 4\ln t\Big|_1^{\tau\|u_0\|_{A,V}} \quad\Rightarrow\quad \frac{g_{u_0}(\tau\|u_0\|_{A,V})}{g_{u_0}(1)} \geq (\tau\|u_0\|_{A,V})^4,$$

proving that

$$D(\tau u_0) = g_{u_0}(\tau\|u_0\|_{A,V}) \geq M\,(\tau\|u_0\|_{A,V})^4 \tag{2.4}$$

for a constant $M > 0$. So,

$$J_{A,V}(\tau u_0) = \frac{\tau^2}{2}\|u_0\|_{A,V}^2 - D\,(\tau u_0) \leq C_1\tau^2 - C_2\tau^4$$

yields that $J_{A,V}(\tau u_0) < 0$ when τ is large enough. □

The mountain pass theorem without the PS condition (see [22, Teorema. 1.15]) yields a Palais-Smale sequence $(u_n) \subset H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that

$$J'_{A,V}(u_n) \to 0 \qquad \text{and} \qquad J_{A,V}(u_n) \to c,$$

where

$$c = \inf_{\alpha\in\Gamma} \max_{t\in[0,1]} J_{A,V}(\alpha(t)),$$

and $\Gamma = \Big\{\alpha \in C^1\Big([0,1], H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\Big) \,:\, \alpha(0) = 0,\, \alpha(1) < 0\Big\}$.

We now consider the Nehari manifold

$$\begin{aligned}\mathcal{N}_{A,V} &= \Big\{u \in H^1_{A,V}(\mathbb{R}^N,\mathbb{C}) \setminus \{0\} \,:\, J'_{A,V}(u)\cdot u = 0\Big\}\\ &= \Big\{u \in H^1_{A,V}(\mathbb{R}^N,\mathbb{C}) \setminus \{0\} \,:\, \|u\|^2_{A,V} = \int_{\mathbb{R}^N}\Big(\frac{1}{|x|^\alpha} * F(|u|)\Big) f(|u|)|u|\Big\}.\end{aligned}$$

It is not difficult to see that $\mathcal{N}_{A,V}$ is a manifold in $H^1_{A,V}(\mathbb{R}^N,\mathbb{C}) \setminus \{0\}$. The next result shows that $\mathcal{N}_{A,V}$ is a closed manifold in $H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$.

Lemma 2.6 *There exists $\beta > 0$ such that $\|u\|_{A,V} \geq \beta$ for all $u \in \mathcal{N}_{A,V}$.*

Another characterization of c in terms of the Nehari manifold is now standard: for $u \neq 0$, consider the function $\Phi(t) = (1/2)\|tu\|^2_{A,V} - D(tu)$, preserving the notation of Lemma 2.5. The proof of Lemma 2.5 assures that $\Phi(tu) > 0$ for t small enough, $\Phi(tu) < 0$ for t large enough and $g'_u(t) > 0$ if $t > 0$. Therefore, $\max_{t\geq 0}\Phi(t)$ is achieved at a unique $t_u = t(u) > 0$ and $\Phi'(tu) > 0$ for $t < t_u$ and $\Phi'(tu) < 0$ for $t > t_u$. Furthermore, $\Phi'(t_u u) = 0$ implies that $t_u u \in \mathcal{N}_{A,V}$.

The map $u \mapsto t_u$ ($u \neq 0$) is continuous and $c = c^*$, where

$$c^* = \inf_{u\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \max_{t\geq 0} J_{A,V}(tu).$$

For details, see [18, Section 3] or [10].

Standard arguments prove the next affirmative:

Lemma 2.7 *Let $(u_n) \subset H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$ be a sequence such that $J_{A,V}(u_n) \to c$ and $J'_{A,V}(u_n) \to 0$, where*

$$c = \inf_{u\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \max_{t\geq 0} J_{A,V}(tu).$$

Then (u_n) is bounded and (for a subsequence) $u_n \rightharpoonup u_0$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$.

Lemma 2.8 *Let $U \subseteq \mathbb{R}^N$ be any open set. For $1 < p < \infty$, let (f_n) be a bounded sequence in $L^p(U, \mathbb{C})$ such that $f_n(x) \to f(x)$ a.e. Then $f_n \rightharpoonup f$.*

The proof of Lemma 2.8 follows by adapting the arguments given for the real case, as in [14, Lemme 4.8, Chapitre 1].

Lemma 2.9 *Suppose that $u_n \rightharpoonup u_0$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ and $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$. Then*

$$\frac{1}{|x|^\alpha} * F(|u_n(x)|) \rightharpoonup \frac{1}{|x|^\alpha} * F(|u_0(x)| \quad in \;\; L^{2N/\alpha}(\mathbb{R}^N). \tag{2.5}$$

Proof. In this proof we adapt some ideas of [2].

The growth condition implies that $F(|u_n|)$ is bounded in $L^{\frac{2N-\alpha}{N-2}}(\mathbb{R}^N)$. Since we can suppose that $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$, it follows from the continuity of F that $F(|u_n(x)|) \to F(|u_0(x)|)$. From Lemma 2.8 follows

$$F(|u_n(x)|) \rightharpoonup F(|u_0(x)|).$$

As a consequence of the Hardy-Littlewood-Sobolev inequality, we have that

$$\frac{1}{|x|^\alpha} * w(x) \in L^{2N/\alpha}(\mathbb{R}^N)$$

for all $w \in L^{\frac{2N-\alpha}{N-2}}(\mathbb{R}^N)$; this is a bounded linear operator from $L^{\frac{2N-\alpha}{N-2}}(\mathbb{R}^N)$ to $L^{2N/\alpha}(\mathbb{R}^N)$. A new application of Lemma 2.8 yields (2.5). □

Corollary 2.10 *Suppose that $u_n \rightharpoonup u_0$ and consider*

$$D'(u_n) \cdot \psi = \mathfrak{Re} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|u_n|) \right] \tilde{f}(|u_n|)(u_n)\overline{\psi},$$

for $\psi \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$. Then $D'(u_n) \cdot \psi \to D'(u_0) \cdot \psi$.

Proof. It follows from the growth condition on f that $\tilde{f}(|u_n|)$ is bounded in $L^p(\mathbb{R}^N)$. Since $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$ and $\tilde{f}$ is continuous, by applying Lemma 2.8 we conclude that

$$\tilde{f}(|u_n|)u_n \rightharpoonup \tilde{f}(|u_0|)u_0 \quad \text{in} \;\; L^q(\mathbb{R}^N, \mathbb{C}). \tag{2.6}$$

Thus,

$$\left|\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{\alpha}}*F(|u_n|)\right]\tilde{f}(|u_n|)u_n\overline{\psi}-\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{\alpha}}*F(|u_0|)\right]\tilde{f}(|u_0|)v\overline{\psi}\right|$$

$$\leq\left|\int_{\mathbb{R}^N}\frac{1}{|x|^{\alpha}}*F(|u_n|)\left(\tilde{f}(|u_n|)u_n-\tilde{f}(|u_0|)u_0\right)\overline{\psi}\right| + \left|\int_{\mathbb{R}^N}\frac{1}{|x|^{\alpha}}*[F(|u_n|)-F(|u_0|)]\,\tilde{f}(|u_0|)u_0\overline{\psi}\right|.$$

The claim follows from Lemma 2.9 and (2.6). □

2.3 Ground state

In order to consider the general case of the potential $V(y)$, we adapt a well-known result due to M. Struwe [19].

Let (u_n) be the minimizing sequence given as consequence of Lemma 2.5, that is, $(u_n)\subset H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that

$$J'_{A,V}(u_n)\to 0 \qquad \text{and} \qquad J_{A,V}(u_n)\to c,$$

where

$$c=\inf_{u\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}}\max_{t\geq 0}J_{A,V}(tu).$$

We assume that $u_n\rightharpoonup u_0\in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$. We define $u_n^1=u_n-u_0$ and consider the limit problem

$$-(\nabla+iA_\infty)^2u+V_\infty u=\left(\frac{1}{|x|^{\alpha}}*F(|u|)\right)\frac{f(|u|)}{|u|}u, \tag{3.1}$$

where $A_\infty=\lim_{|x|\to\infty}A(x)$ and $V_\infty=\lim_{|x|\to\infty}V(y)$. The energy functional attached to this problem is, of course,

$$J_\infty(u)=\frac{1}{2}\|u\|^2_{A_\infty,V_\infty}-D(u).$$

Lemma 2.11 (Splitting Lemma) *Let* $(u_n) \subset H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$ *be such that*

$$J_{A,V}(u_n) \to c, \qquad J'_{A,V}(u_n) \to 0$$

and $u_n \rightharpoonup u_0$ *weakly on* $H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$. *Then* $J'_{A,V}(u_0) = 0$ *and we have* either

(i) $u_n \to u_0$ *strongly on* $H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$;

(ii) or there exist $k \in \mathbb{N}$, $(y^j_n) \in \mathbb{R}^N$ *such that* $|y^j_n| \to \infty$ *for* $j \in \{1,\dots,k\}$ *and nontrivial solutions* $u^1,\dots,u^k$ *of problem* (3.1) *so that*

$$J_{A,V}(u_n) \to J_{A,V}(u_0) + \sum_{j=1}^{k} J_\infty(u_j)$$

and

$$\left\| u_n - u_0 - \sum_{j=1}^{k} u^j(\cdot - y^j_n) \right\| \to 0.$$

Proof. (Sketch) We simply adapt the arguments presented in [11, Lemma 2.3] and [22, Theorem 8.4]. Since Corollary 2.10 guarantees that $D'(u_n)\cdot\phi \to D'(u_0)\cdot\phi$, it follows that $J'_{A,V}(u_0)\cdot\phi = 0$.

By setting $u^1_n = u_n - u_0$, we have

(a_1) $\|u^1_n\|^2_{A,V} = \|u_n\|^2_{A,V} - \|u_0\|^2_{A,V} + o_n(1)$;

(b_1) $J_\infty(u^1_n) \to c - J_{A,V}(u_0)$;

(c_1) $J'_\infty(u^1_n) \to 0$.

Let us define

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u^1_n|^2 \mathrm{d}x.$$

If $\delta = 0$, it follows that $u^1_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2^*)$. It follows that $u^1_n \to 0$ in $H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$, since $J'_\infty(v^1_n) \to 0$. In this case, the proof of Lemma 2.11 is complete.

So, let us suppose that $\delta > 0$. Then, we obtain a sequence $(y^1_n) \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u^1_n|^2 \mathrm{d}x \geq \frac{\delta}{2}.$$

By setting $v_n^1 = u_n^1(\cdot + y_n^1)$, we obtain a new bounded sequence (v_n^1). Therefore, we assume that $v_n^1 \rightharpoonup v_1$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ and $v_n^1 \to v$ a.e. in $\mathbb{R}^N$. Since

$$\int_{B_1(0)} |v_n^1|^2 \mathrm{d}x > \frac{\delta}{2},$$

we conclude that $u^1 \neq 0$ as consequence of Sobolev's immersion. We also conclude that (y_n) is unbounded, since $u_n^1 \rightharpoonup 0$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$. Therefore, we may assume that $|y_n^1| \to \infty$. Then, it is easy to see that $J'_\infty(u^1) = 0$.

We now define $u_n^2 = u_n^1 - u^1(\cdot - y_n)$. We then have

(a_2) $\|u_n^2\|^2_{A,V} = \|u_n\|^2_{A,V} - \|u_0\|^2_{A,V} - \|u^1\|^2_{A,V} + o_n(1)$;

(b_2) $J_\infty(u_n^2) \to c - J_{A,V}(u_0) - J_\infty(u_1)$;

(c_2) $J'_\infty(u_n^2) \to 0$.

Proceeding by iteration, we observe that, if u is a nontrivial critical point of J_∞ and $\bar{u}$ a ground state of problem (3.1), then the Ambrosetti-Rabinowitz condition implies that

$$J_\infty(u) \geq J_\infty(\bar{u}) = \int_{\mathbb{R}^N} \left(\frac{1}{2} f(|\bar{u}|)|\bar{u}| - F(|\bar{u}|) \right) =: \beta > 0.$$

Thus, it follows from (b_2) that the iteration process must end at some index $k \in \mathbb{N}$.
□

Remark 2.12 *Observe that, in particular, the proof shows that the sequence u_n^k converges to $\bar{u}$ and we have a solution of problem* (3.1).

The next result also follows [11, Corollary 2.3], see also [6]. We present the proof for the convenience of the reader.

Lemma 2.13 *The functional $J_{A,V}$ satisfies $(PS)_c$ for any $0 \leq c < c_\infty$.*

Proof. Let us suppose that (u_n) satisfies

$$J_{A,V}(u_n) \to c < c_\infty \qquad \text{and} \qquad J'_{A,V}(u_n) \to 0.$$

According to Lemma 2.7, we can suppose that the sequence (u_n) is bounded. Therefore, for a subsequence, we have $u_n \rightharpoonup u_0$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$. It follows from the Splitting

Lemma 2.11 that $J'_{A,V}(u_0) = 0$. Since

$$J'_{A,V}(u_0) \cdot u_0 = \|u_0\|^2_{A,V} - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * F(|u_0|) \right) f(|u_0|)|u_0|$$

we conclude that

$$\begin{aligned} J_{A,V}(u_0) &= \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|u_0|) \right] (f(|u_0|)|u_0| - 2F(|u_0|)) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|u_0|) \right] F(|u_0|) \\ &> \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|u_0|) \right] (f(|u_0|)|u_0| - 2F(|u_0|)) > 0 \end{aligned} \tag{3.2}$$

as a consequence of the Ambrosetti-Rabinowitz condition.

If $u_n \not\to u_0$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$, by applying again the Splitting Lemma we guarantee the existence of $k \in \mathbb{N}$ and nontrivial solutions $u^1, \ldots, u^k$ of problem (3.1) satisfying

$$\lim_{n\to\infty} J_{A,V}(u_n) = c = J_{A,V}(u_0) + \sum_{j=1}^{k} J_\infty(u^j) \geq kc_\infty \geq c_\infty$$

contradicting our hypothesis. We are done. □

We prove the next result by adapting the proof given in Furtado, Maia e Medeiros [11, Proposition 3.1], see also [6]:

Lemma 2.14 *Suppose that $V(y)$ satisfies (V_3). Then*

$$0 < c < c_\infty,$$

where c is characterized in Lemma 2.7.

Proof. Let $\bar{u}$ be the weak solution of (3.1) obtained in the proof of the Splitting Lemma (see Remark 2.12) and $t_{\bar{u}} > 0$ the unique number such that $t_{\bar{u}}\bar{u} \in \mathcal{N}_{A,V}$. We claim that $t_{\bar{u}} < 1$. Indeed, it follows from the condition (AV) that

$$\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|t_{\bar{u}}\bar{u}|) \right] f(|t_{\bar{u}}\bar{u}|)|t_{\bar{u}}\bar{u}| = t^2_{\bar{u}}\|\bar{u}\|_{A,V}$$

$$
\begin{aligned}
&< t_{\bar{u}}^2 \|\bar{u}\|_{A_\infty, V_\infty} \\
&= t_{\bar{u}}^2 \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|\bar{u}|)\right] f(|\bar{u}|)|\bar{u}| \\
&= t_{\bar{u}}^2 \bigg(\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|\bar{u}|)\right] f(|\bar{u}|)|\bar{u}| + \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|t_{\bar{u}}\bar{u}|)\right] f(|\bar{u}|)|\bar{u}| \\
&\qquad - \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|t_{\bar{u}}\bar{u}|)\right] f(|\bar{u}|)|\bar{u}|\bigg)
\end{aligned}
$$

thus yielding

$$
\begin{aligned}
0 > &\int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|t_{\bar{u}}\bar{u})|\right] (\tilde{f}(|t_{\bar{u}}\bar{u}|) - \tilde{f}(|\bar{u}|)) \\
&+ t_{\bar{u}}^2 \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * (F(|t_{\bar{u}}\bar{u}|) - F(|\bar{u}|))\right] f(|\bar{u}|)|\bar{u}|.
\end{aligned}
$$

If $t_{\bar{u}} \geq 1$, since $\tilde{f}$ is increasing, the first integral is non-negative and the second as well, since F is also increasing. We conclude that $t_{\bar{u}} < 1$.

Lemma 2.7 and its previous comments show that

$$
\begin{aligned}
c \leq \max_{t \geq 0} J_{A,V}(t\bar{u}) &= J_{A,V}(t_{\bar{u}}\bar{u}) \\
&= \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|t_{\bar{u}}\bar{u}|)\right] \left(\frac{1}{2} f(|t_{\bar{u}}\bar{u}|)|t_{\bar{u}}\bar{u}| - F(|t_{\bar{u}}\bar{u}|)\right).
\end{aligned}
$$

Since

$$
g(t) = \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|t\bar{u}|)\right] \left(\frac{1}{2} f(|t\bar{u}|)|t\bar{u}| - F(|t\bar{u}|)\right)
$$

is a strictly increasing function, we conclude that

$$
c = g(t_{\bar{u}}) < g(1) = \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\alpha} * F(|\bar{u}|)\right] \left(\frac{1}{2} f(|\bar{u}|)|\bar{u}| - F(|\bar{u}|)\right) = c_\infty,
$$

proving our result. □

Proof of Theorem 2.2. Let (u_n) be the minimizing sequence given by Lemma 2.5. It follows from Lemmas 2.13 and 2.14 that u_n converges to $u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ satisfying $J_{A,V}(u) = c$ and $J'_{A,V}(u) = 0$. □

2.4 On the multiplicity of solutions

In order to obtain multiplicity of solutions, we consider in this section a particular case of that considered by Cingolani, Clapp and Secchi in [9]. We think that the direct proof we present is interesting.

So, let G be a closed subgroup of $O(n)$, the group of orthogonal transformations in $\mathbb{R}^N$. As in [9], we suppose that $A(gx) = gA(x)$ and $V(gx) = V(x)$ for all $g \in G$ and $x \in \mathbb{R}^N$ and take a continuous group homomorphism $\tau\colon G \to S^1$ into the unit complex numbers S^1.

We consider the space

$$H^1_A(\mathbb{R}^N, \mathbb{C})^\tau = \left\{ u \in H^1_A(\mathbb{R}^N, \mathbb{C}) \,:\, u(gx) = \tau(g)u(x),\ \forall\, g \in G,\ \forall\, x \in \mathbb{R}^N \right\}.$$

We apply the following compactness result due to P.L. Lions:

Lemma 2.15 (Lions) *Let G be a closed subgroup of $O(N)$ and denote*

$$H^1_G = \left\{ u \in H^1(\mathbb{R}^N) \,:\, gu = u,\ \forall\, g \in G \right\}.$$

Suppose that $\sum_{j=1}^k N_j = N$, $N_j \geq 2$ for all $j \in \{1, \ldots, k\}$, and

$$G = O(N_1) \times O(N_2) \times \cdots \times O(N_k).$$

Then, the immersion $H^1_G(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is compact for $2 < p < 2^*$.

Observe that, if $u \in H^1_A(\mathbb{R}^N, \mathbb{C})^\tau$, then $|u| \in H^1_G(\mathbb{R}^N)$.

Proof of Theorem 2.3. For any $u \in H^1_{A,V}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_{A,V}$. It is easy to show that the application $0 \neq u \to t_u \in (0, \infty)$ is continuous. As a consequence, $u \mapsto t_u u$ is a homeomorphism between $\mathcal{N}_{A,V}$ and $\mathcal{S}$, the unit sphere with center at the origin in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$, see also [18]. Observe that $J_{A,V}$ is even. Since $J_{A,V} \in C^1(\mathcal{S}, \mathbb{R})$ is bounded below and satisfies (PS) for any level c (as a consequence of Lemma 2.15), the result follows from Theorem 2 in Szulkin and Weth [20], see also [21]. □

2.5 Apendix

In this section we are to show some results.

Proof of remark (2.1)

Proof. From $(f1)$-$(f2)$, given $\xi > 0$ there exist $\delta > 0$ such that, for each $|t| < \delta$ we have $f(t) < \xi|t|$.

From hypothesis **(f2)** ensured that $f(t) < t^{r-1}$ for all $t > M$.

How the f function is positive and $[\delta, M]$ is compact, so there is $c > 0$ such that $f(t) < c$ for all $t \in [\delta, M]$. Therefore:

$$f(t) \leq \begin{cases} \xi t, & \text{se } t \in (0,\delta) \\ c, & \text{se } t \in [\delta, M] \\ t^{r-1}, & \text{se } t \in [M, \infty) \end{cases} \tag{P}$$

Taking $C_\xi = \max\{1, c\}$, it follows that

$$f(t) < \xi t + C_\xi t^{r-1},$$

for all $t \geq 0$.

The proof for F is analogous.

From **(f3)**, notice that if $\left(\frac{f(t)}{t}\right)' > 0$ then $\dfrac{tf'(t) - f(t)}{t^2} > 0$, $t > 0$ implies $\frac{tf'(t)-f(t)}{t^2} > 0$ and using integration by part, we have

$$\begin{aligned} \int_0^{|u|} tf'(t) &> \int_0^{|u|} f(t)dt \\ tf(t)\Big|_0^{|u|} - \int_0^{|u|} f(t)dt &> \int_0^{|u|} f(t)dt \\ |u|f(|u|) &> 2F(|u|). \end{aligned} \tag{5.1}$$

proof of lemma (2.7)

Proof. From the definition of convergence to $I'_{A,V}(u_n) \to 0$:

Let $\epsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$

$$\sup_{v \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \frac{I'_{A,V}(u_n)}{\|v\|_{A,V}} = |I'_{A,V}(u_n)| < 1$$

for $v = u_n$ yields $|I'_{A,V}(u_n) \cdot u_n| < \|u_n\|_{A,V}$,

and $I_{A,V}(u_n) \to c$:

Let $\epsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then $|I_{A,V}(u_n) - c| < 1$, ou $c - 1 < I_{A,V}(u_n) < c + 1$.

We obtain $4I_{A,V}(u_n) - I'_{A,V}(u_n) \cdot u_n \leq 4(c+1) + \|u_n\|^2_{A,V}$.

Now using the definitions to $I_{A,V}(u_n)$ and $I_{A,V}(u_n) \cdot u_n$ we have

$$4I_{A,V}(u_n) - I'_{A,V}(u_n) \cdot u_n = 4\left(\frac{1}{2}\|u\|^2_{A,V} - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * F(|u_n|)\right)F(|u_n|)\right) - \tag{5.2}$$

$$-\|u_n\|^2_{A,V} + \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * F(|u_n|)\right)f(|u_n|)|u_n| \tag{5.3}$$

$$= \|u_n\|^2_{A,V} + \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * F(|u_n|)\right)(f(|u_n|)|u_n - 2|F(|u_n|)),$$

from Ambrosetti Rabinowitz inequality we have $4I_{A,V}(u_n) - I'_{A,V}(u_n) \cdot u_n \geq \|u_n\|^2_{A,V}$.

Therefore from previous estimates yields $4(c+1) + \|u_n\|^2_{A,V} \geq \|u_n\|^2_{A,V}$.

The last inequality would be false if $\|u_n\|$ it could be taken large enough. This conclude that $\{u_n\}$ is limited.

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