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Nonlinear perturbations of a periodic magnetic nonlinear Choquard equation with Hardy-Littlewood-Sobolev critical exponent

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ATA DA CENTÉSIMA QUINQUASÉSIMA DEFESA DE TESE DO ALUNO LEANDRO DA LUZ VIEIRA, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 04 DE AGOSTO DE 2020.

Aos quatro dias do mês de agosto de 2020, às 09h30, em reunião pública virtual na Plataforma Google Meet pelo link https://meet.google.com/qqv-otjv-dqd (conforme mensagem eletrônica da Pró-Reitoria de Pós-Graduação de 26/03/2020, com orientações para a atividade de defesa de tese durante a vigência da Portaria nº 1819), reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Leandro da Luz Vieira**, intitulada: "*Nonlinear perturbations of a periodic magnetic nonlinear Choquard equation with Hardy-Littlewood-Sobolev critical exponent*", requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Hamilton Prado Bueno, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente, sem a presença do aluno, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 04 de agosto de 2020.

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ABSTRACT

In this work we consider the following magnetic nonlinear Choquard equations

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{2^*_{\alpha}}\right)|u|^{2^*_{\alpha}-2}u + \lambda f(u) \ \text{ in } \ \mathbb{R}^N (N \geq 3)$$

and

$$(-\Delta)^s_A u + V(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda g(u) \ \text{ in } \ \mathbb{R}^N (N = 3),$$

where $s \in (0,1)$, $2^*_{\alpha} = \frac{2N-\alpha}{N-2}$ and $2^*_{\alpha,s} = \frac{6-\alpha}{3-2s}$ are critical exponents in the sense of the Hardy-Littlewood-Sobolev inequality. Moreover, in both problems $0 < \alpha < N$, $\lambda > 0$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is an C^1, $\mathbb{Z}^N$-periodic vector potential and V is a continuous scalar potential given as a perturbation of a periodic potential. Considering different types of nonlinearities f and g, namely, $f(x,u) = \left(\frac{1}{|x|^{\alpha}} * |u|^p\right)|u|^{p-2}u$ for $(2N-\alpha)/N < p < 2^*_{\alpha}$, then $f(u) = |u|^{p-1}u$ for $1 < p < 2^* - 1$ and $f(u) = |u|^{2^*-2}u$ (where $2^* = 2N/(N-2)$), $g(x,u) = \left(\frac{1}{|x|^{\alpha}} * |u|^p\right)|u|^{p-2}u$ for $(6-\alpha)/3 < p < 2^*_{\alpha,s}$, then $g(u) = |u|^{p-1}u$ for $1 < p < 2^*_s - 1$ and $g(u) = |u|^{2^*_s-2}u$ (where $2^*_s = 6/(3-2s)$), we prove the existence of at least one ground state solution for these equations by variational methods if p belongs to some intervals depending on N, λ and also on s in the second problem.

Key Words: Variational methods, magnetic Choquard equation, fractional magnetic Choquard equation, Hardy-Littlewood-Sobolev critical exponent.

RESUMO

Neste trabalho nós consideramos as seguintes equações de Choquard magnéticas não lineares

$$-(\nabla + iA(x))^2u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha - 2}u + \lambda f(u) \quad \text{em} \quad \mathbb{R}^N (N \geq 3)$$

e

$$(-\Delta)^s_A u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s} - 2}u + \lambda g(u) \quad \text{em} \quad \mathbb{R}^N (N = 3),$$

em que $s \in (0,1)$, $2^*_\alpha = \frac{2N-\alpha}{N-2}$ e $2^*_{\alpha,s} = \frac{6-\alpha}{3-2s}$ são os expoentes críticos no sentido da desigualdade de Hardy-Littlewood-Sobolev. Além disso, em ambos os problemas $0 < \alpha < N$, $\lambda > 0$, $A : \mathbb{R}^N \to \mathbb{R}^N$ é um potencial vetorial de classe C^1, $\mathbb{Z}^N$-periódico e V é potencial escalar contínuo dado como uma perturbação de um potencial periódico. Considerando diferentes tipos de não linearidades f e g, a saber, $f(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$ para $(2N-\alpha)/N < p < 2^*_\alpha$, depois $f(u) = |u|^{p-1}u$ para $1 < p < 2^* - 1$ e $f(u) = |u|^{2^*-2}u$ (em que $2^* = 2N/(N-2)$), $g(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$ para $(6-\alpha)/3 < p < 2^*_{\alpha,s}$, depois $g(u) = |u|^{p-1}u$ para $1 < p < 2^*_s - 1$ e $g(u) = |u|^{2^*_s-2}u$ (em que $2^*_s = 6/(3-2s)$), nós provamos a existência de ao menos uma solução de estado fundamental para estas equações por métodos variacionais se p pertence a alguns intervalos dependendo de N, λ e também de s no segundo problema.

Palavras-chave: Métodos variacionais, equação de Choquard magnética, equação de Choquard magnética fracionária, expoente crítico de Hardy-Littlewood-Sobolev.

Summary

Introduction

In this thesis we consider the problems

$$-(\nabla + iA(x))^2u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha-2}u + \lambda f(u) \ \text{ in } \ \mathbb{R}^N (N \geq 3) \tag{1}$$

and

$$(-\Delta)^s_A u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda g(u) \ \text{ in } \ \mathbb{R}^N (N = 3), \tag{2}$$

where $\nabla + iA(x)$ is the covariant derivative with respect to the C^1, $\mathbb{Z}^N$-periodic vector potential $A\colon \mathbb{R}^N \to \mathbb{R}^N$, i.e,

$$A(x+y) = A(x),\ \forall\, x \in \mathbb{R}^N,\ \forall\, y \in \mathbb{Z}^N,$$

and, for $u \in C^\infty_c(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, $(-\Delta)^s_A u$ is the fractional magnetic operator defined by

$$(-\Delta)^s_A u(x) = c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}} \mathrm{d}y,$$

where $c_{N,s}$ is a normalizing constant.

The second equation of this thesis was suggested by Prof. G. M. Figueiredo after his reading of our article [16] - which handles (1) - and advising us to obtain similar results in the framework of the nonlocal fractional magnetic operator, bringing into our notice references [11, 12].

In both problems we consider $0 < \alpha < N$ and the exponents $2^*_\alpha = \frac{2N-\alpha}{N-2}$ and $2^*_{\alpha,s} = \frac{6-\alpha}{3-2s}$, $s \in (0,1)$, which are critical in the sense of the Hardy-Littlewood-Sobolev inequality. Moreover $\lambda > 0$, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous scalar potential and f and g stand for different types of nonlinearities. Namely, in the first chapter we consider $f(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$ for $(2N-\alpha)/N < p < 2^*_\alpha$, then $f(u) = |u|^{p-1}u$ for $1 < p < 2^* - 1$ (where 2^* is the critical exponent of the immersion $H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{2^*}(\mathbb{R}^N, \mathbb{C})$), and finally we examine $f(u) = |u|^{2^*-2}u$.

In the second chapter, we initially consider $g(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$ for $(6-\alpha)/3 < p < 2^*_\alpha$, then $g(u) = |u|^{p-1}u$ for $1 < p < 2^*_s - 1$ (where 2^*_s is the critical exponent of the immersion $H^s_{A,V}(\mathbb{R}^3, \mathbb{C}) \hookrightarrow L^{2^*_s}(\mathbb{R}^3, \mathbb{C})$), and finally we examine $g(u) = |u|^{2^*_s-2}u$.

Inspired by the seminal work of Coti Zelati and Rabinowitz [19], but also by Alves, Carrião and Miyagaki [2] and by Alves and Figueiredo [3], we assume that there is a continuous, $\mathbb{Z}^N$-periodic potential $V_{\mathcal{P}} : \mathbb{R}^N \to \mathbb{R}$, constants $V_0, W_0 > 0$ and $W \in L^{\frac{N}{2}}(\mathbb{R}^N, \mathbb{R})$ with $W(x) \geq 0$ such that

(V_1) $V_{\mathcal{P}}(x) \geq V_0, \quad \forall\, x \in \mathbb{R}^N$;

(V_2) $V(x) = V_{\mathcal{P}}(x) - W(x) \geq W_0, \quad \forall\, x \in \mathbb{R}^N$,

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^N$.

For technical reasons, in the problem (2) we consider only the case $N = 3$. However, since both problems (1) and (2) are considered in the whole space $\mathbb{R}^N$ and have a critical nonlinearity in the Hardy-Littlewood-Sobolev sense, the verification of any compactness condition is not easy.

The first equation of this work is motivated by Gao and Yang in [29], where a classical Choquard equation is considered in a bounded domain, i.e., the case $A \equiv 0$ and $V \equiv 0$ is studied in a bounded domain Ω. There is a huge literature about the Choquard equation and we cite only Moroz and Van Schaftingen [41] for a good review of results on this important subject. In [29], Gao and Yang proved the existence of a ground state solution (*i.e, a least energy nontrivial solution*) under restriction on N and λ. Other recent advances in the study of the Choquard equation can be found, e.g., in [8, 25, 26, 31, 39, 46] for critical exponents, in [6] for multi-bump solutions, and in [4, 7, 40] for the concentration behavior of solutions.

In Mukherjee and Sreenadh [43], the magnetic problem

$$-(\nabla + iA(x))^2 u + \mu g(x) u = \lambda u + \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2} u \quad \text{in} \quad \mathbb{R}^N$$

was examined. In this equation $\mu > 0$ is also a parameter that interacts with the linear term in the right-hand side of the equation. Existence of a ground state solution was proved supposing that g satisfies the assumptions

(g_1) $g \in C(\mathbb{R}^N, \mathbb{R})$, $g \geq 0$ and $\Omega :=$ interior of $g^{-1}(0)$ is a nonempty bounded set with smooth boundary and $g^{-1}(0) = \overline{\Omega}$;

(g_2) There exists $M > 0$ such that the set $\{x \in \mathbb{R}^N \, : \, g(x) \leq M\}$ has finite Lebesgue measure in $\mathbb{R}^N$.

The concentration of solutions as $\mu \to \infty$ was also studied.

Changing the right-hand side of (1) to

$$\left(\frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2} u, \tag{3}$$

the problem was studied by Cingolani, Clapp and Secchi in [18]. In that paper the authors proved existence and multiplicity of solutions. In [15], the right-hand side (3) was generalized and a ground state solution was obtained, but the multiplicity result depend on more restrictive hypotheses than in [18].

Recent years have witnessed a growth of interest in the study of magnetic equations. By using variational methods, penalization techniques and Ljusternik–Schnirelmann theory, Alves, Figueiredo and Yang [5] proved existence of multiple solutions to the magnetic equation

$$\left(\frac{\varepsilon}{i} \nabla - A(x) \right)^2 u + V(x) u = \varepsilon^{\mu - N} \left(\frac{1}{|x|^\alpha} * F(|u|^2) \right) f(|u|^2) u, \quad x \in \mathbb{R}^N, \tag{4}$$

where $\varepsilon > 0$ is a parameter, $N \geq 2$, $0 < \mu < 2$ and $F(s) = \int_0^s f(t)\mathrm{d}t$.

The same type of techniques were used by d'Avenia and Ji [22] to obtain multiplicity and concentration of solutions of (4) with the right-hand side of that equation changed to $f(|u|^2)u$, with f having critical exponential growth.

On its turn, the class of magnetic fractional equations is an object of increasing interest since the pioneering works of Fiscella, Pinamonti and Vecchi [28] and d'Avenia and Squassina [21].

In [28], for $s \in (0,1)$ and a parameter λ, the problem

$$(-\Delta)^s_A u = \lambda f(|u|)u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega$$

was studied in a bounded domain Ω. Considering different types of nonlinearities f, variational techniques were used to prove the existence of at least two solutions.

In [21] d'Avenia and Squassina considered the minimization problems

$$m_A := \inf_{u \in L} \left(\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x + \frac{c_s}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|e^{-i(x-y)\cdot A(\frac{x+y}{2})}u(x) - u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y \right) \tag{5}$$

and

$$m^c_A := \inf_{u \in L^c} \frac{c_s}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|e^{-i(x-y)\cdot A(\frac{x+y}{2})}u(x) - u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y, \tag{6}$$

where $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous magnetic potential whith locally bounded gradient, $2 < p < \frac{6}{3-2s}$, $L := \left\{ u \in H^s_A(\mathbb{R}^3, \mathbb{C}) \, : \, \int_{\mathbb{R}^3} |u(x)|^p \mathrm{d}x = 1 \right\}$ and $L^c := \{u \in D^s_A(\mathbb{R}^3, \mathbb{C}) \, : \, \int_{\mathbb{R}^3} |u(x)|^{6/(3-2s)} \mathrm{d}x\}$, where $H^s_A(\mathbb{R}^3, \mathbb{C})$ and $D^s_A(\mathbb{R}^3, \mathbb{C})$ are suitable Hilbert spaces defined in that article.

By applying concentration-compactness arguments, the existence of a solution to (5) for a class of potentials A yields a solution to the problem

$$(-\Delta)^s_A u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3.$$

as a consequence of Lagrange multipliers. On its turn, if there exists a solution u to (6), then a representation formula for u is obtained.

In [37], Mingqi, Pucci, Squassina and Zhang proved existence and multiplicity of nontrivial solutions for the problem

$$M([u]^2_{s,A})(-\Delta)^s_A u + V(x)u = f(x, |u|)u \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$,

$$[u]^2_{s,A} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right),$$

$M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is a Kirchhoff function, $V : \mathbb{R}^N \to \mathbb{R}^+$ is a scalar potential, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential, $(-\Delta)^s_A$ is the associated fractional magnetic operator, and $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is a sublinear or superlinear nonlinearity. In the sublinear case a solution is obtained by the direct methods, whereas the mountain pass theorem and Nehari manifold are applied in the superlinear case. Multiplicity of solutions is handled by the symmetric mountain pass theorem.

Also by applying variational methods and Ljusternick–Schnirelmann theory, Ambrosio and d'Avenia [9] proved existence and multiplicity of solutions for the equation

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon} u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a parameter and $N \geq 3$. As usual, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ (for $\alpha \in (0,1]$) are the electric and magnetic potentials respectively, and $f : \mathbb{R}^N \to \mathbb{R}$ is a subcritical nonlinearity. The same equation with the term $\varepsilon^{-2t}(|x|^{2t-3} * |u|^2)u$ added to the left-hand side of the equation was considered by Ambrosio [12], where $t \in (0,1)$ is a parameter and $N = 3$. Existence, multiplicity and concentration of solutions was obtained for $\varepsilon > 0$ small enough also by applying Ljusternick–Schnirelmann theory.

In [10] Ambrosio investigated existence and concentration of nontrivial solutions to the fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^2_{A/\varepsilon}u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\alpha} * F(|u|^2)\right) f(|u|^2)u \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $0 < \mu < 2s$ and $N \geq 3$. It is supposed that the potential V is positive and has a local minimum and f is a continuous nonlinearity with subcritical growth.

By applying concentration-compactness, a fractional Kato type inequality, Ljusternik–Schnirelmann and minimax methods, Ambrosio [11] proved existence, multiplicity and concentration of nontrivial solutions for the following fractional magnetic Kirchhoff equation with critical growth

$$\left(a\varepsilon^{2s} + b\varepsilon^{4s-3}[u]^2_{s,A/\varepsilon}\right)(-\Delta)^s_{A/\varepsilon}u + V(x)u = f(|u|^2)u + |u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^3.$$

In this equation, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $A \in C^{0,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$, ($\alpha \in (0,1]$) are the electric and magnetic potentials, respectively; $\varepsilon > 0$ is a small parameter, a and b are positive constants, $s \in (\frac{3}{4}, 1)$, $2^*_s = \frac{6}{3-2s}$ is the fractional critical exponent and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 subcritical nonlinearity.

The main results of this thesis are the following theorems.

Theorem 1 *For $\frac{2N-\alpha}{N} < p < 2^*_\alpha$, under the hypotheses already stated on A, V and α, problem*

$$-(\nabla + iA(x))^2u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha-2}u + \lambda\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{7}$$

has at least one ground state solution if either

(i) $\frac{N+2-\alpha}{N-2} < p < 2^*_\alpha$, $N = 3, 4$ *and* $\lambda > 0$;

(ii) $\frac{2N-\alpha}{N} < p \leq \frac{N+2-\alpha}{N-2}$, $N = 3, 4$ *and* λ *sufficiently large;*

(iii) $\frac{2N-2-\alpha}{N-2} < p < 2^*_\alpha$, $N \geq 5$ *and* $\lambda > 0$;

(iv) $\frac{2N-\alpha}{N} < p \leq \frac{2N-2-\alpha}{N-2}$, $N \geq 5$ *and* λ *sufficiently large.*

Theorem 2 *For $1 < p < 2^* - 1$, under the hypotheses already stated on A, V and α, problem*

$$-(\nabla + iA(x))^2u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha-2}u + \lambda|u|^{p-1}u \quad \text{in } \mathbb{R}^N.$$

has at least one ground state solution if either

(i) $3 < p < 5$, $N = 3$ *and* $\lambda > 0$;

(ii) $p > 1$, $N \geq 4$ *and* $\lambda > 0$;

(iii) $1 < p \leq 3$, $N = 3$ *and* λ *sufficiently large.*

Theorem 3 *Under the hypotheses already stated on* A, V *and* α, *the problem*

$$-(\nabla + iA(x))^2u + V(x)u = \lambda\left(\frac{1}{|x|^{\alpha}} * |u|^p\right)|u|^{p-2}u + |u|^{2^*-2}u \quad in \quad \mathbb{R}^N,$$

has at least one ground state solution in the intervals already described in Theorem 1.

Theorem 4 *For* $2 - \frac{\alpha}{3} < p < 2^*_{\alpha,s}$, *under the hypotheses already stated on* A, V *and* α, *problem*

$$(-\Delta)^s_A u + V(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda\left(\frac{1}{|x|^{\alpha}} * |u|^p\right)|u|^{p-2}u \quad in \quad \mathbb{R}^3 \tag{8}$$

has at least one ground state solution if either

(i) $s \in (\frac{3}{4}, 1)$, $\frac{7-2s-\alpha}{3-2s} < p < 2^*_{\alpha,s}$ *and* $\lambda > 0$;

(ii) $s \in (0, 1)$, $\frac{6-\alpha}{3} < p \leq \frac{7-2s-\alpha}{3-2s}$ *and* λ *sufficiently large.*

Theorem 5 *For* $s \in (\frac{3}{4}, 1)$ *and* $1 < p < 2^*_s - 1$, *under the hypotheses already stated on* A, V *and* α, *problem*

$$(-\Delta)^s_A u + V(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda|u|^{p-1}u \quad in \quad \mathbb{R}^3.$$

has at least one ground state solution if either

(i) $\frac{6s-3}{3-2s} < p < 2^*_s - 1$ *and* $\lambda > 0$;

(ii) $1 < p \leq \frac{6s-3}{3-2s}$ *and* λ *sufficiently large.*

Theorem 6 *Under the hypotheses already stated on* A, V *and* α, *the problem*

$$(-\Delta)^s_A u + V(x)u = \lambda\left(\frac{1}{|x|^{\alpha}} * |u|^p\right)|u|^{p-2}u + |u|^{2^*_s-2}u \quad in \quad \mathbb{R}^3,$$

has at least one ground state solution in the intervals already described in Theorem 4.

Problems (1) and (2) will be considered in Chapter 1 and 2, respectively. In both chapters, we start proving the existence of a ground state solution for problems (1) and (2), respectively, considering the potential $V = V_{\mathcal{P}}$, that is, we consider the problems

$$-(\nabla + iA(x))^2u + V_{\mathcal{P}}(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{2^*_{\alpha}}\right)|u|^{2^*_{\alpha}-2}u + \lambda f(u) \quad \text{in} \quad \mathbb{R}^N \tag{9}$$

and

$$(-\Delta)^s_A u + V_{\mathcal{P}}(x)u = \left(\frac{1}{|x|^{\alpha}} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda g(u) \quad \text{in} \quad \mathbb{R}^N \tag{10}$$

and f and g as in Theorems 1, 2, 3, 4, 5 and 6, maintaining the notation introduced before and supposing that (V_1) is valid.

As in Gao and Yang in [29], the key step to proof the existence of a ground state solution of problems (9) and (10) is the use of cut-off techniques on the extreme function that attains the best constants $\mathcal{S}_{H,L}$, and $\mathcal{S}_0^s$ defined in the sequence. This allows us to estimate the mountain pass values c_λ and c_{λ_s} associated to the respective "energy" functionals attached to (9) and (10). We prove that the (PS)-condition holds for these levels. In a demanding proof, this lead us to establish intervals for p (depending on N and λ in the first case and N, λ and s in the second one) where the (PS)-condition is satisfied, as in the seminal work of Brézis and Nirenberg [14]. After that, the proof is completed by showing the mountain pass geometry, introducing the Nehari manifold associated with (9) and (10) and applying concentration-compactness arguments. In the sequel, we consider (1) and (2) for the different nonlinearities f and g and prove that each problem has at least one ground state solution.

We observe that the conclusion of Theorem 2 is similar to that of Theorem 1.1 in Alves, Carrião and Miyagaki [2] and Theorem 1.1 in Miyagaki [38]. Precisely, in [2] the authors have discussed the existence of a positive solution to the semilinear elliptic problem involving critical exponents

$$-\Delta u + V(x)u = \lambda u^q + u^p \quad \text{in} \quad \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, $1 < q < p = 2^* - 1$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function. On its turn, Miyagaki [38] has studied the existence of nontrivial solution for the following class of semilinear elliptic equation in $\mathbb{R}^N$ $(N \geq 3)$ involving critical Sobolev exponents

$$-\Delta u + a(x)u = \lambda |u|^{q-1} + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N,$$

where $1 < q < p \leq 2^* - 1 = \frac{N+2}{N-2}$ and $\lambda > 0$ are constants and $a : \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that $a(x) \geq a_0$ for all $x \in \mathbb{R}^N$, where $a_0 > 0$ is a constant.

Problems (9) and (1) - and (10) and (2) as well - are then related by showing that the minimax value d_λ of (1) (or d_{λ_s} of (2)) satisfies $d_\lambda < c_\lambda$ (respectively, $d_{\lambda_s} < c_{\lambda_s}$). Once more, concentration-compactness arguments are applied to show the existence of a ground state solution for each problem.

In a nutshell, by adopting the same techniques applied to study problem (1), we have proved our results for problem (2).

This work is organized as follows.

In each chapter, initially, some preliminary results will be established (see Sections 1.1 and 2.1). Then, Sections 1.2, 1.3 and 1.4 are then devoted to the proofs of Theorems 1, 2 and 3, respectively, and Sections 1.3, 2.3 and 2.4 proves Theorems 4, 5 and 6, respectively.

The text in Chapter 1 is very similar to that of the submitted paper [16].

In the appendices A and B we gather some of the main results used in this work and we justify some of the facts used in the proofs of our results, respectively.

Chapter 1

Magnetic Choquard equation with Hardy-Littlewood-Sobolev critical exponent

In this chapter we deal with problem (1)

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha - 2}u + \lambda f(u) \ \text{ in } \ \mathbb{R}^N (N \geq 3)$$

and prove Theorems 1, 2 and 3.

1.1 Preliminary results

We denote

$$\nabla_A u = \nabla u + iA(x)u.$$

We handle problem (1) in the space

$$H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) \, : \, \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}), \quad \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, \mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_{A,V} = \left(\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V(x)|u|^2) \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Observe that the norm generated by this scalar product is equivalent to the norm obtained by considering $V \equiv 1$, see [35, Definition 7.20].

If $u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N)$ and the *diamagnetic inequality* is valid (see [18] or [35, Theorem 7.21])

$$|\nabla |u|(x)| \leq |\nabla u(x) + iA(x)u(x)|, \quad \text{a.e. } x \in \mathbb{R}^N.$$

As a consequence of the diamagnetic inequality, we have the continuous immersion

$$H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^s(\mathbb{R}^N, \mathbb{C}) \tag{1.1}$$

for any $s \in [2, \frac{2N}{N-2}]$. We denote $2^* = \frac{2N}{N-2}$ and $\| \cdot \|_s$ the norm in $L^s(\mathbb{R}^N, \mathbb{C})$.

It is well-known that $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$, see [35, Theorem 7.22].

Following Gao and Yang [30], we denote by $S_{H,L}$

$$S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N,\mathbb{R})\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}\, |u(y)|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\alpha}}} \tag{1.2}$$

$$= \inf_{u \in D^{1,2}_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla_A u|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}\, |u(y)|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\alpha}}} =: S_A,$$

where $D^{1,2}_A(\mathbb{R}^N, \mathbb{C}) = \{u \in L^{2^*}(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C})\}$. The equality between $S_{H,L}$ and S_A was proved in Mukherjee and Sreenadh [43]. We remark that S_A is attained if and only if $\operatorname{rot} A = 0$ [43, Theorem 4.1]. See also [13, Theorem 1.1].

We state a result proved in [30].

Proposition 7 (Gao and Yang [30]) *The constant $S_{H,L}$ defined in (1.2) is achieved if and only if*

$$u(x) = C \left(\frac{b}{b^2 + |x-a|^2}\right)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. Furthermore,

$$S_{H,L} = \frac{S}{C(N,\alpha)^{\frac{N-2}{2N-\alpha}}},$$

where S is the best Sobolev constant of the immersion $D^{1,2}(\mathbb{R}^N, \mathbb{R}) \hookrightarrow L^{2^*}(\mathbb{R}^N, \mathbb{R})$ and $C(N, \alpha)$ depends on N and α.

If we consider the minimizer for S given by $U(x) := \frac{[N(N-2)]^{\frac{N-2}{4}}}{(|1+|x|^2)^{\frac{N-2}{2}}}$ (see [48, Theorem 1.42]), then

$$\bar{U}(x) = S^{\frac{(N-\alpha)(2-\alpha)}{4(N+2-\alpha)}} C(N,\alpha)^{\frac{2-N}{2(N+2-\alpha)}} \frac{[N(N-2)]^{\frac{N-2}{4}}}{(|1+|x|^2)^{\frac{N-2}{2}}}$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$-\triangle u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{2^*_\alpha - 2} u \quad \text{in } \mathbb{R}^N$$

with

$$\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 \mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{U}(x)|^{2^*_\alpha} |\bar{U}(y)|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

Lemma 8 *Let $\Omega \subsetneqq \mathbb{R}^N$ be any open set. For $1 < t < \infty$, let (f_n) be a bounded sequence in $L^t(\Omega, \mathbb{C})$ such that $f_n(x) \to f(x)$ a.e. Then $f_n \rightharpoonup f$ in $L^t(\Omega, \mathbb{C})$.*

The proof of Lemma 8 only adapts the arguments given for the real case, as in [33, Lemme 4.8, Chapitre 1].

1.2 The case $f(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$

1.2.1 The periodic problem

In this subsection we deal with the case $V = V_{\mathcal{P}}$ and prove the existence of a ground state solution for problem (9) and $f(x,u)$ as above, that is, we consider the problem

$$-(\nabla + iA(x))^2 u + V_{\mathcal{P}}(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha - 2}u + \lambda\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \tag{1.3}$$

where $\frac{2N-\alpha}{N} < p < 2^*_\alpha$.

To deal with this problem, we consider the space

$$H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C}) = \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) \,:\, \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C})\right\}$$

endowed with scalar product

$$\langle u, v\rangle_{A,V_{\mathcal{P}}} = \mathfrak{Re}\int_{\mathbb{R}^N}\left(\nabla_A u \cdot \overline{\nabla_A v} + V_{\mathcal{P}}(x)u\bar{v}\right)\mathrm{d}x$$

and, therefore

$$\|u\|^2_{A,V_{\mathcal{P}}} = \int_{\mathbb{R}^N}\left(|\nabla_A u|^2 + V_{\mathcal{P}}|u|^2\right)\mathrm{d}x.$$

We observe that the energy functional $J_{A,V_{\mathcal{P}}}$ on $H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$ associated with (1.3) is given by

$$J_{A,V_{\mathcal{P}}}(u) := \frac{1}{2}\|u\|^2_{A,V_{\mathcal{P}}} - \frac{1}{2\cdot 2^*_\alpha}D(u) - \frac{\lambda}{2p}B(u),$$

where

$$B(u) = \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)^p||u(y)|^p}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y$$

and

$$D(u) = \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha}\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)^{2^*_\alpha}||u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y.$$

Remark 1.2.1 If $t \in [(2N-\alpha)/N, (2N-\alpha)/(N-2)]$ and $r = 2N/(2N-\alpha)$, then $2 \le tr \le 2^*$. So, for $u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$, it follows from (1.1) that $u \in L^{tr}(\mathbb{R}^N, \mathbb{C})$, that is, $|u|^t \in L^r(\mathbb{R}^N, \mathbb{C})$. Since $\frac{2}{r} + \frac{\alpha}{N} = 2$, the Hardy-Littlewood-Sobolev inequality (see Appendix A, Proposition 49) yields

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^t|u(y)|^t}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \le C(N,\alpha)\|u\|^{2t}_{tr}.$$

Therefore,

$$B(u) \le C_1(N,\alpha)\|u\|_{pr}^{2p} \tag{1.4}$$

and

$$D(u) \le C_2(N,\alpha)\|u\|_{2^*}^{2\cdot 2^*_\alpha} \tag{1.5}$$

for constants $C_1(N,\alpha)$ and $C_2(N,\alpha)$. Therefore, $J_{A,V_{\mathcal{P}}}$ is well-defined for $u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$.

Here, as also in [4], $\frac{2N-\alpha}{N}$ is called the lower critical exponent and $2^*_\alpha = \frac{2N-\alpha}{N-2}$ the upper critical exponent. This lead us to say that (1) is a critical nonlocal elliptic equation. Moreover, by Lemma 2.5 of [47], $J_{A,V_{\mathcal{P}}} \in C^1(H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C});\mathbb{R})$.

Observe that

$$S_A = \inf_{u\,\in\, D^{1,2}_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla_A u|^2 \mathrm{d}x}{D(u)^{\frac{N-2}{2N-\alpha}}}. \tag{1.6}$$

Definition 1.2.1 *A function $u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$ is a weak solution of (1.3) if*

$$\langle u,\psi\rangle_{A,V_{\mathcal{P}}} - \mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha-2}u\bar{\psi}\,\mathrm{d}x - \lambda\,\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^{p}\right)|u|^{p-2}u\bar{\psi}\,\mathrm{d}x = 0$$

for all $\psi \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$.

Since the derivative of the energy functional $J_{A,V_{\mathcal{P}}}$ is given by

$$J'_{A,V_{\mathcal{P}}}(u)\cdot\psi = \langle u,\psi\rangle_{A,V_{\mathcal{P}}} - \mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha-2}u\bar{\psi}\,\mathrm{d}x - \lambda\,\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^{p}\right)|u|^{p-2}u\bar{\psi}\,\mathrm{d}x,$$

we see that critical points of $J_{A,V_{\mathcal{P}}}$ are weak solutions of (1.3).

Note that, if $\psi = u$ we obtain

$$J'_{A,V_{\mathcal{P}}}(u)\cdot u := \|u\|^2_{A,V_{\mathcal{P}}} - D(u) - \lambda B(u). \tag{1.7}$$

Lemma 9 *The functional $J_{A,V_{\mathcal{P}}}$ satisfies the mountain pass geometry. Precisely,*

(*i*) *there exist $\rho,\delta > 0$ such that $J_{A,V_{\mathcal{P}}}\big|_S \ge \delta > 0$ for any $u \in \mathcal{S}$, where*

$$\mathcal{S} = \{u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})\,:\,\|u\|_{A,V_{\mathcal{P}}} = \rho\};$$

(*ii*) *for any $u_0 \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}$ there exists $\tau \in (0,\infty)$ such that $\|\tau u_0\|_{V_{\mathcal{P}}} > \rho$ and $J_{A,V_{\mathcal{P}}}(\tau u_0) < 0$.*

Proof. Inequalities (1.4) and (1.5) yields

$$J_{A,V_{\mathcal{P}}}(u) \ge \frac{1}{2}\|u\|^2_{A,V_{\mathcal{P}}} - \frac{C_2(\alpha,N)}{2\cdot 2^*_\alpha}\|u\|^{2\cdot 2^*_\alpha}_{A,V_{\mathcal{P}}} - \frac{\lambda C_1(\alpha,N)}{2p}\|u\|^{2p}_{A,V_{\mathcal{P}}},$$

thus implying (*i*) if we take $\|u\|_{A,V_{\mathcal{P}}} = \rho > 0$ sufficiently small.

In order to prove (*ii*), fix $u_0 \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}$ and consider the function $g_{u_0}\colon(0,\infty)\to\mathbb{R}$ given by

$$g_{u_0}(t) := J_{A,V_{\mathcal{P}}}(tu_0) = \frac{1}{2}\|tu_0\|^2_{A,V_{\mathcal{P}}} - \frac{1}{2\cdot 2^*_\alpha}D(tu_0) - \frac{\lambda}{2p}B(tu_0).$$

We have

$$B(tu_0) = t^{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)^p||u_0(y)|^p}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = t^{2p} B(u_0)$$

and

$$D(tu_0) = t^{2\cdot 2^*_\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^{2^*_\alpha}|u_0(y)|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = t^{2\cdot 2^*_\alpha} D(u_0).$$

Thus,

$$\begin{aligned} g_{u_0}(t) &= \frac{1}{2} t^2 \|u_0\|^2_{A,V_{\mathcal{P}}} - \frac{1}{2\cdot 2^*_\alpha} t^{2\cdot 2^*_\alpha} D(u_0) - \frac{\lambda}{2p} t^{2p} B(u_0) \\ &= \frac{1}{2} t^{2\cdot 2^*_\alpha} \left(\frac{\|u_0\|^2_{A,V_{\mathcal{P}}}}{t^{(2(2^*_\alpha - 1))}} - \frac{1}{2^*_\alpha} D(u_0) - \frac{\lambda}{p} \frac{B(u_0)}{t^{(2(2^*_\alpha - p))}} \right) \end{aligned}$$

Since $1 < \frac{2N-\alpha}{N} < p < 2^*_\alpha$, we have

$$\lim_{t\to+\infty} J_{A,V_{\mathcal{P}}}(tu_0) = -\infty$$

completing the proof of (ii). □

The mountain pass theorem without the PS condition (see [48, Theorem 1.15]) yields a Palais-Smale sequence $(u_n) \subset H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$ such that

$$J'_{A,V_{\mathcal{P}}}(u_n) \to 0 \qquad \text{and} \qquad J_{A,V_{\mathcal{P}}}(u_n) \to c_\lambda, \tag{1.8}$$

where

$$c_\lambda = \inf_{\alpha\in\Gamma} \max_{t\in[0,1]} J_{A,V_{\mathcal{P}}}(\gamma(t)),$$

and $\Gamma = \left\{ \gamma \in C^1\left([0,1], H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})\right) \,:\, \gamma(0) = 0,\ J_{A,V_{\mathcal{P}}}(\gamma(1)) < 0 \right\}$.

Lemma 10 *Suppose that $u_n \rightharpoonup u_0$ in $H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$. Then*

$$\frac{1}{|x|^\alpha} * |u_n|^t \rightharpoonup \frac{1}{|x|^\alpha} * |u_0|^t \;\; in \;\; L^{\frac{2N}{\alpha}}(\mathbb{R}^N), \tag{1.9}$$

for all $\frac{2N-\alpha}{N} \le t \le 2^*_\alpha$.

Proof. In this proof we adapt some ideas of [7]. We can suppose that $|u_n(x)|^t \to |u_0(x)|^t$ a.e. in $\mathbb{R}^N$ and, as consequence of the immersion (1.1), $|u_n|^t$ is bounded in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$. Thus, Lemma 8 allows us to conclude that

$$|u_n(x)|^t \rightharpoonup |u_0(x)|^t \;\text{ in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N, \mathbb{C})$$

as $n \to \infty$.

By the Hardy–Littlewood–Sobolev inequality (see Appendix A, Proposition 49), the map $T : L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N, \mathbb{C}) \to L^{\frac{2N}{\alpha}}(\mathbb{R}^N, \mathbb{C})$ defined by $T(w) = |x|^{-\alpha} * w$ is well-defined (see Appendix A, Lemma 54), moreover it is linear and continuous. Hence, the result follows by applying Proposition 56, Appendix A. □

Corollary 11 *Suppose that $u_n \rightharpoonup u_0$ and consider*

$$B'(u_n)\cdot\psi = \mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_n|^p\right)|u_n|^{p-2}u_n\bar{\psi}\,\mathrm{d}x$$

and

$$D'(u_n)\cdot\psi = \mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha-2}u_n\bar{\psi}\,\mathrm{d}x,$$

for $\psi\in C_c^\infty(\mathbb{R}^N,\mathbb{C})$. Then $B'(u_n)\cdot\psi\to B'(u_0)\cdot\psi$ and $D'(u_n)\cdot\psi\to D'(u_0)\cdot\psi$.

Proof. The immersion (1.1) guarantees that $|u_n|^{p-2}u_n$ is bounded in $L^{\frac{2N}{N+2-\alpha}}(\mathbb{R}^N,\mathbb{C})$. Since we can suppose that $|u_n(x)|^p\to|u_0(x)|^p$ a.e. in $\mathbb{R}^N$, by applying Lemma 8, we conclude that

$$|u_n|^{p-2}u_n \rightharpoonup |u_0|^{p-2}u \quad\text{in }\ L^{\frac{2N}{N+2-\alpha}}(\mathbb{R}^N,\mathbb{C}) \tag{1.10}$$

for all $\frac{2N-\alpha}{N}\le p\le 2^*_\alpha$, as $n\to+\infty$.

Combining (1.9) with (1.10) yields

$$\left(\frac{1}{|x|^\alpha}*|u_n|^p\right)|u_n|^{p-2}u_n \rightharpoonup \left(\frac{1}{|x|^\alpha}*|u_0|^p\right)|u_0|^{p-2}u_0 \ \text{ in }\ L^{\frac{2N}{N+2}}(\mathbb{R}^N)$$

as $n\to+\infty$, for all $\frac{2N-\alpha}{N}\le p\le 2^*_\alpha$. Consequently, for $\psi\in C_c^\infty(\mathbb{R}^N,\mathbb{C})$, it follows that

$$\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_n|^p\right)|u_n|^{p-2}u_n\bar{\psi}\,\mathrm{d}x\to\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_0|^p\right)|u_0|^{p-2}u_0\bar{\psi}\,\mathrm{d}x$$

and

$$\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha-2}u_n\bar{\psi}\,\mathrm{d}x\to\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_0|^{2^*_\alpha}\right)|u_0|^{2^*_\alpha-2}u_0\bar{\psi}\,\mathrm{d}x,$$

that is,

$$B'(u_n)\cdot\psi\to B'(u_0)\cdot\psi \qquad\text{and}\qquad D'(u_n)\cdot\psi\to D'(u_0)\cdot\psi.$$

□

Lemma 12 *If $(u_n)\subset H^1_{A,V_\mathcal{P}}(\mathbb{R}^N,\mathbb{C})$ is a $(PS)_b$ sequence for $J_{A,V_\mathcal{P}}$, then (u_n) is bounded. In addition, if $u_n\rightharpoonup u$ weakly in $H^1_{A,V_\mathcal{P}}(\mathbb{R}^N,\mathbb{C})$ as $n\to\infty$, then u is a weak solution to problem* (1.3)*.*

Proof. Standard arguments prove that (u_n) is bounded in $H^1_{A,V_\mathcal{P}}(\mathbb{R}^N,\mathbb{C})$. Then, up to a subsequence, we have $u_n\rightharpoonup u$ weakly in $H^1_{A,V_\mathcal{P}}(\mathbb{R}^N,\mathbb{C})$ as $n\to\infty$.

From Corollary 11 it follows that, for all $\psi\in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N,\mathbb{C})$, we have

$$\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u_n|^t\right)|u|^{t-2}u_n\bar{\psi}\,\mathrm{d}x=\mathfrak{Re}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*|u|^t\right)|u|^{t-2}u\bar{\psi}\,\mathrm{d}x+o_n(1),\quad\text{as }\ n\to\infty,$$

where $t=p$ or $t=2^*_\alpha$.

Thus, since for all $\psi \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ we have $J'_{A,V_\mathcal{P}}(u_n) \cdot \psi = o_n(1)$, we obtain

$$J'_{A,V_\mathcal{P}}(u) \cdot \psi = 0, \ \ \forall\, \psi \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C}),$$

that is, u is a weak solution to (1.3). □

We now consider the Nehari manifold associated with the $J_{A,V_\mathcal{P}}$.

$$\mathcal{M}_{A,V_\mathcal{P}} = \left\{ u \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \,:\, \|u\|^2_{A,V_\mathcal{P}} = D(u) + \lambda B(u) \right\}.$$

Lemma 13 *There exists a unique $t_u = t_u(u) > 0$ such that $t_u u \in \mathcal{M}_{A,V_\mathcal{P}}$ for all $u \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ and $J_{A,V_\mathcal{P}}(t_u u) = \max_{t \geq 0} J_{A,V_\mathcal{P}}(tu)$. Moreover $c_\lambda = c^*_\lambda = c^{**}_\lambda$, where*

$$c^*_\lambda = \inf_{u \,\in\, \mathcal{M}_{A,V}} J_{A,V_\mathcal{P}}(u) \quad \text{and} \quad c^{**}_\lambda = \inf_{u \,\in\, H^1_{A,V_\mathcal{P}}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \max_{t \geq 0} J_{A,V_\mathcal{P}}(tu).$$

Proof. Let $u \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ and g_u defined on $(0, +\infty)$ given by

$$g_u(t) = J_{A,V_\mathcal{P}}(tu).$$

By the mountain pass geometry (Lemma 9), there exists $t_u > 0$ such that

$$g_u(t_u) = \max_{t \geq 0} g_u(t) = \max_{t \geq 0} J_{A,V_\mathcal{P}}(t_u u).$$

Hence

$$0 = g'_u(t_u) = J'_{A,V_\mathcal{P}}(t_u u) \cdot u = J'_{A,V_\mathcal{P}}(t_u u) \cdot t_u u,$$

implying that $t_u u \in \mathcal{M}_{A,V_\mathcal{P}}$, as consequence of (1.7). We now show that t_u is unique. To this end, we suppose that there exists $s_u > 0$ such that $s_u u \in \mathcal{M}_{A,V_\mathcal{P}}$. Thus, we have both

$$\|u\|^2_{A,V_\mathcal{P}} = t_u^{2(2^*_\alpha - 1)} D(u) + \lambda t_u^{2(p-1)} B(u) \qquad \text{and} \qquad \|u\|^2_{A,V_\mathcal{P}} = s_u^{2(2^*_\alpha - 1)} D(u) + \lambda s_u^{2(p-1)} B(u).$$

Hence

$$0 = \left(t_u^{2(2^*_\alpha - 1)} - s_u^{2(2^*_\alpha - 1)} \right) D(u) + \lambda \left(t_u^{2(p-1)} - s_u^{2(p-1)} \right) B(u).$$

Since both terms in parentheses have the same sign if $t_u \neq s_u$ and we also have $B(u) > 0$, $D(u) > 0$ and $\lambda > 0$, it follows that $t_u = s_u$.

Now, the rest of the proof follows arguments similar to that found in [2, 27, 44, 48]. (See Appendix B, Lemma 57.) □

Taking into account Lemma 13, we can now redefine a ground state solution.

Definition 1.2.2 *We say that $u \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C})$ is a ground state for problem* (1.3) *if $J'_{A,V_\mathcal{P}}(u) = 0$ and $J_{A,V_\mathcal{P}}(u) = c_\lambda$, that is, if u is a solution to the equation $J'_{A,V_\mathcal{P}}(u) = 0$ which has minimal energy in the set of all nontrivial solutions.*

The following result controls the level c_λ of a Palais-Smale sequence of $J_{A,V_\mathcal{P}}$.

Lemma 14 *Let $(u_n) \subset H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$ be a $(PS)_b$ sequence for $J_{A,V_{\mathcal{P}}}$ such that*

$$u_n \rightharpoonup 0 \quad \text{weakly in } H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C}), \quad \text{as } n \to \infty,$$

with

$$b < \frac{N+2-\alpha}{2(2N-\alpha)} S_A^{\frac{2N-\alpha}{N-\alpha+2}}.$$

Then the sequence (u_n) verifies either

(i) $u_n \to 0$ strongly in $H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$, as $n \to \infty$,

or

(ii) There exists a sequence $(y_n) \subset \mathbb{R}^N$ and constants $r, \theta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \, \mathrm{d}x \geq \theta$$

where $B_r(y)$ denotes the ball in $\mathbb{R}^N$ of center at y and radius $r > 0$.

Proof. Suppose that *(ii)* does not hold. Applying a result by Lions [48, Lemma 1.21], it follows from inequality (1.4) that

$$B(u_n) \to 0, \quad \text{as} \quad n \to \infty.$$

Since $J'_{A,V_{\mathcal{P}}}(u_n)u_n = o_n(1)$ as $n \to \infty$, we obtain

$$\|u_n\|^2_{A,V_{\mathcal{P}}} = D(u_n) + o_n(1) \text{ as } n \to \infty. \tag{1.11}$$

Let us suppose that

$$\|u_n\|^2_{A,V_{\mathcal{P}}} \to \ell \ \ (\ell > 0) \quad \text{as} \quad n \to \infty.$$

Thus, as consequence of (1.11), we have

$$D(u_n) \to \ell, \quad \text{as} \quad n \to \infty.$$

Since

$$J_{A,V_{\mathcal{P}}}(u_n) = \frac{1}{2}\|u_n\|^2_{A,V_{\mathcal{P}}} - \frac{\lambda}{2p}B(u_n) - \frac{1}{2\cdot 2^*_\alpha}D(u_n),$$

making $n \to \infty$ yields

$$b = \frac{\ell}{2}\left(1 - \frac{1}{2^*_\alpha}\right) = \ell\left(\frac{N+2-\alpha}{2(2N-\alpha)}\right). \tag{1.12}$$

On the other hand, it follows from (1.6) that

$$\|u_n\|^2_{A,V_{\mathcal{P}}} \geq \int_{\mathbb{R}^N} |\nabla_A u_n|^2 \, \mathrm{d}x \geq S_A(D(u_n))^{\frac{N-2}{2N-\alpha}}, \quad \forall\, u \in D^{1,2}_A(\mathbb{R}^N,\mathbb{C}).$$

Thus,

$$\ell \geq (S_A)^{\frac{2N-\alpha}{N+2-\alpha}} \tag{1.13}$$

and from (1.12) and (1.13) we conclude that $b \geq \frac{N+2-\alpha}{2(2N-\alpha)} S_A^{\frac{2N-\alpha}{N+2-\alpha}}$, which is a contradiction. Therefore, *(i)* is valid and the proof is complete. □

We now state our result about the periodic problem (1.3).

Theorem 15 *Under the hypotheses already stated on A and α, suppose that (V_1) is valid. Then problem (1.3) has at least one ground state solution if either*

(i) $\frac{N+2-\alpha}{N-2} < p < 2^*_\alpha$, $N = 3, 4$ *and* $\lambda > 0$*;*

(ii) $\frac{2N-\alpha}{N} < p \leq \frac{N+2-\alpha}{N-2}$, $N = 3, 4$ *and* λ *sufficiently large;*

(iii) $\frac{2N-\alpha-2}{N-2} < p < 2^*_\alpha$, $N \geq 5$ *and* $\lambda > 0$*;*

(iv) $\frac{2N-\alpha}{N} < p \leq \frac{2N-\alpha-2}{N-2}$, $N \geq 5$ *and* λ *sufficiently large.*

Proof. Let c_λ be the mountain pass level and consider a sequence $(u_n) \subset H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C})$ such that

$$J'_{A,V_\mathcal{P}}(u_n) \to 0 \qquad \text{and} \qquad J_{A,V_\mathcal{P}}(u_n) \to c_\lambda.$$

Claim. We affirm that $c_\lambda < \frac{N+2-\alpha}{2(2N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}$, a result that will be shown after completing our proof, since it is very technical.

Lemma 12 guarantees that (u_n) is bounded. So, passing to a subsequence if necessary, there is $u \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C})$ such that

$$u_n \rightharpoonup u \text{ in } H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C}), \qquad u_n \to u \text{ in } L^2_{loc}(\mathbb{R}^N, \mathbb{C}) \qquad \text{and} \qquad u_n \to u \text{ a.e. } x \in \mathbb{R}^N. \tag{1.14}$$

If $u = 0$, it follows from Lemma 14 the existence of $\theta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \,\mathrm{d}x \geq \theta. \tag{1.15}$$

A direct computation shows that we can assume that $(y_n) \subset \mathbb{Z}^N$. In fact, if $y_n = (y_n^1, y_n^2, \ldots, y_n^N)$ there exists $z_n^i \in \mathbb{Z}$, $1 \leq i \leq N$, such that $|y_n^i - z_n^i| \leq \frac{1}{2}$. Considering $z_n = (z_n^1, z_n^2, \ldots z_n^N)$, we have that $|z_n - y_n| \leq \sqrt{\sum_{i=1}^N |z_n^i - y_n^i|^2} \leq \frac{\sqrt{N}}{2}$. Thus, $B_r(y_n) \subset B_{\frac{\sqrt{N}}{2}+r}(z_n)$, since if $x \in B_r(y_n)$ then $|z_n - x| \leq |z_n - y_n| + |y_n - x| < \frac{\sqrt{N}}{2} + r$. Therefore

$$\limsup_{n\to\infty} \int_{B_{\frac{\sqrt{N}}{2}+r}(z_n)} |u_n|^2 \,\mathrm{d}x \geq \limsup_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \,\mathrm{d}x \geq \theta > 0.$$

Let

$$v_n(x) := u_n(x + y_n).$$

Since both $V_\mathcal{P}$ and A are $\mathbb{Z}^N$-periodic, we have

$$\|v_n\|_{A,V_\mathcal{P}} = \|u_n\|_{A,V_\mathcal{P}} \quad J_{A,V_\mathcal{P}}(v_n) = J_{A,V_\mathcal{P}}(u_n) \quad \text{and} \quad J'_{A,V_\mathcal{P}}(v_n) \to 0, \quad \text{as } n \to \infty.$$

Therefore there exists $v \in H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C})$ such that $v_n \rightharpoonup v$ weakly in $H^1_{A,V_\mathcal{P}}(\mathbb{R}^N, \mathbb{C})$ and $v_n \to v$ in $L^2_{loc}(\mathbb{R}^N, \mathbb{C})$.

We claim that $v \neq 0$. In fact, it follows from (1.15)

$$0 < \theta \leq \|v_n\|_{L^2(B_r(0))} \leq \|v_n - v\|_{L^2(B_r(0))} + \|v\|_{L^2(B_r(0))}.$$

Since $v_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$, we have $\|v_n - v\|_{L^2(B_r(0))} \to 0$ as $n \to \infty$, proving our claim.

But Corollary 11 guarantees that $J'_{A,V_\mathcal{P}}(v_n)\cdot\psi \to J'_{A,V_\mathcal{P}}(v)\cdot\psi$ and it follows that $J'_{A,V_\mathcal{P}}(v)\cdot\psi = 0$. Consequently, v is a weak solution of (1.3).

Since $v \in \mathcal{M}_{A,V_\mathcal{P}}$, of course we have $c^*_\lambda \le J_{A,V_\mathcal{P}}(v)$. But

$$\begin{aligned} c^*_\lambda = c_\lambda &= J_{A,V_\mathcal{P}}(v_n) - \frac{1}{2}J'_{A,V_\mathcal{P}}(v_n)\cdot v_n + o_n(1) \\ &= \lambda\left(\frac{1}{2} - \frac{1}{2p}\right)B(v_n) - \frac{N+2-\alpha}{2(2N-\alpha)}D(v_n) + o_n(1). \end{aligned}$$

Fatou's Lemma then guarantees that, as $n \to \infty$, we have

$$c^*_\lambda \ge \lambda\left(\frac{1}{2} - \frac{1}{2p}\right)B(v) - \frac{N+2-\alpha}{2(2N-\alpha)}D(v) = J_{A,V_\mathcal{P}}(v)$$

that is, $J_{A,V_\mathcal{P}}(v) = c_\lambda$, and we are done. The same argument applies to the case $u \neq 0$ in (1.14). $\square$

We now prove the postponed Claim, that is, we show that $c_\lambda < \frac{N+2-\alpha}{2(2N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}$. Observe that, once proved the existence of u_ϵ as in our next result, then

$$0 < c_\lambda = \inf_{\alpha\in\Gamma}\max_{t\in[0,1]} J_{A,V_\mathcal{P}}(\gamma(t)) \le \sup_{t\ge 0} J_{A,V_\mathcal{P}}(tu_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}.$$

Lemma 16 *There exists u_ε such that*

$$\sup_{t\ge 0} J_{A,V_\mathcal{P}}(tu_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}.$$

provided that either

(i) $\frac{N+2-\alpha}{N-2} < p < 2^*_\alpha$, $N = 3, 4$ *and* $\lambda > 0$*;*

(ii) $\frac{2N-\alpha}{N} < p \le \frac{N+2-\alpha}{N-2}$, $N = 3, 4$ *and* λ *sufficiently large;*

(iii) $\frac{2N-2-\alpha}{N-2} < p < 2^*_\alpha$, $N \ge 5$ *and* $\lambda > 0$*;*

(iv) $\frac{2N-\alpha}{N} < p \le \frac{2N-2-\alpha}{N-2}$, $N \ge 5$ *and* λ *sufficiently large.*

The arguments of this proof were adapted from the articles [29, 38]. Observe that the conditions stated in this result are exactly the same of Theorem 1 and Theorem 15.

Proof. We know that $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ is a minimizer for S, the best Sobolev constant of the immersion $D^{1,2}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{R})$ (see [48, Theorem 1.42] or [13, Section 3]) and also a minimizer for $S_{H,L}$, according to Proposition 7.

If B_r denotes the ball in $\mathbb{R}^N$ of center at origin and radius r, consider the balls B_δ and $B_{2\delta}$ and take $\psi \in C^\infty_0(\mathbb{R}^N)$ such that, for a constant $C > 0$,

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_\delta, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus B_{2\delta}, \end{cases} \qquad 0 \le |\psi(x)| \le 1, \quad |D\psi(x)| \le C, \quad \forall\, x \in \mathbb{R}^N.$$

We define, for $\varepsilon > 0$,

$$U_\varepsilon(x) := \varepsilon^{(2-N)/2} U\left(\frac{x}{\varepsilon}\right) \quad \text{and} \qquad u_\varepsilon(x) := \psi(x) U_\varepsilon(x) \tag{1.16}$$

In the proof we apply the estimates

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x = C(N,\alpha)^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}} S_A^{\frac{N}{2}} + O(\varepsilon^{N-2}) \tag{1.17}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2^*_\alpha}|u_\varepsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \geq C(N,\alpha)^{\frac{N}{2}} S_A^{\frac{2N-\alpha}{2}} - O(\varepsilon^{N-\frac{\alpha}{2}}), \tag{1.18}$$

which were obtained by Gao and Yang [30].

Case 1. $\frac{N+2-\alpha}{N-2} < p < 2^*_\alpha$ and $N = 3, 4$ or $\frac{2N-2-\alpha}{N-2} < p < 2^*_\alpha$ and $N \geq 5$.
Proof of Case 1. Consider the function $f : [0, +\infty) \to \mathbb{R}$ defined by

$$f(t) = J_{A,V_\mathcal{P}}(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2_{A,V_\mathcal{P}} - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha} D(u_\varepsilon) - \frac{\lambda t^{2p}}{2p} B(u_\varepsilon).$$

The mountain pass geometry (Lemma 9) implies the existence of $t_\varepsilon > 0$ such that $\sup_{t\geq 0} J_{A,V_\mathcal{P}}(tu_\varepsilon) = J_{A,V_\mathcal{P}}(t_\varepsilon u_\varepsilon)$. Since $t_\varepsilon > 0$, $B(u_\varepsilon) > 0$ and $f'(t_\varepsilon) = 0$, we obtain

$$0 < t_\varepsilon < \left(\frac{\|u_\varepsilon\|^2_{A,V_\mathcal{P}}}{D(u_\varepsilon)}\right)^{\frac{1}{2(2^*_\alpha - 1)}} := S_A(\varepsilon),$$

thus implying

$$\|u_\varepsilon\|^2_{A,V_\mathcal{P}} = D(u_\varepsilon)\left(S_A(\varepsilon)\right)^{2(2^*_\alpha-1)}. \tag{1.19}$$

Now define $g : [0, S_A(\varepsilon)] \to \mathbb{R}$ by

$$g(t) = \frac{t^2}{2}\|u_\varepsilon\|^2_{A,V_\mathcal{P}} - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha} D(u_\varepsilon).$$

So,

$$g(t) = \frac{t^2}{2} D(u_\varepsilon)\left(S_A(\varepsilon)\right)^{2(2^*_\alpha-1)} - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha} D(u_\varepsilon).$$

Since $t > 0$ and $D(u_\varepsilon) > 0$, it follows that $g'(t) > 0$, and, consequently, g is increasing in this interval. Thus,

$$0 < g(t_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)} D(u_\varepsilon)(S_A(\varepsilon))^{2\cdot 2^*_\alpha}.$$

We conclude that

$$D(u_\varepsilon)(S_A(\varepsilon))^{2\cdot 2^*_\alpha} = \frac{(\|u_\varepsilon\|^2_{A,V_\mathcal{P}})^{\frac{2N-\alpha}{N+2-\alpha}}}{D(u_\varepsilon)^{\frac{N-2}{N+2-\alpha}}}$$

and therefore

$$0 < g(t_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)} \cdot \frac{(\|u_\varepsilon\|^2_{A,V_{\mathcal{P}}})^{\frac{2N-\alpha}{N+2-\alpha}}}{D(u_\varepsilon)^{\frac{N-2}{N+2-\alpha}}}.$$

Since $J_{A,V_{\mathcal{P}}}(tu_\varepsilon) = g(t) - \frac{\lambda}{2p}t^{2p}B(u_\varepsilon)$, we have

$$J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)} \left(\frac{\|u_\varepsilon\|^2_{A,V_{\mathcal{P}}}}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}} \right)^{\frac{2N-\alpha}{N+2-\alpha}} - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon).$$

But $\|u_\varepsilon\|^2_{A,V_{\mathcal{P}}} = \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x + \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x)|u_\varepsilon|^2)\mathrm{d}x$ implies

$$\frac{\|u_\varepsilon\|^2_{A,V_{\mathcal{P}}}}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}} = \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x + \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x)|u_\varepsilon|^2)\mathrm{d}x.$$

Therefore, we conclude that

$$\begin{aligned} J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)} \Bigg(&\frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x \\ &+ \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x))|u_\varepsilon|^2 \mathrm{d}x \Bigg)^{\frac{2N-\alpha}{N+2-\alpha}} - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon). \end{aligned}$$

Since, for all $\beta \geq 1$ and any $a, b \geq 0$ we have $(a+b)^\beta \leq a^\beta + \beta(a+b)^{\beta-1}b$, considering

$$a = \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x, \quad b = \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x)|u_\varepsilon|^2)\mathrm{d}x$$

and

$$\beta = \frac{2N-\alpha}{N+2-\alpha},$$

it follows

$$\begin{aligned} J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)} \Bigg[&\left(\frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x \right)^{\frac{2N-\alpha}{N+2-\alpha}} + \\ &\frac{2N-\alpha}{N+2-\alpha} \left(\frac{1}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x + \frac{1}{(D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x)|u_\varepsilon|^2)\mathrm{d}x \right)^{\frac{N-2}{N+2-\alpha}} \\ &\cdot \frac{1}{((D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x)|u_\varepsilon|^2)\mathrm{d}x \Bigg] - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon). \end{aligned} \tag{1.20}$$

Taking into account (1.17) and (1.18), we conclude that

$$\left(\frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x \right)^{\frac{2N-\alpha}{N+2-\alpha}} \leq \left(\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}} \cdot S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{\left(C(N,\alpha)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\alpha}{2}} - O(\varepsilon^{\frac{2N-\alpha}{2}}) \right)^{\frac{N-2}{2N-\alpha}}} \right)^{\frac{2N-\alpha}{N+2-\alpha}}.$$

We also have

$$\left(\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}+O(\varepsilon^{N-2})}{\left(C(N,\alpha)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\alpha}{2}}-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}} = (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}}\cdot\left(\frac{1+O(\varepsilon^{N-2})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}} \tag{1.21}$$

and

$$\left(\frac{1+O(\varepsilon^{N-2})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}} < 1+C(N,\alpha)\cdot\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{\left(1-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}. \tag{1.22}$$

(See proof of (1.21) and (1.22) in Appendix B.1)

We observe that, for $\varepsilon>0$ sufficiently small, it holds

$$(1-O(\varepsilon^{\frac{N-2}{2N-\alpha}}))^{\frac{N-2}{2N-\alpha}}\geq\frac{1}{2}.$$

So,

$$\left(\frac{1+O(\varepsilon^{N-2})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}} < 1+2C(N,\alpha)\left(O\left(\varepsilon^{N-2}\right)+O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right) < 1+O\left(\varepsilon^{\min\{N-2,\frac{2N-\alpha}{2}\}}\right).$$

Therefore, we conclude that, for any $\varepsilon>0$ sufficiently small, we have

$$\left(\frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^{\frac{2N-\alpha}{N+2-\alpha}} < (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}}+O\left(\varepsilon^{\min\{N-2,\frac{2N-\alpha}{2}\}}\right). \tag{1.23}$$

Combining (1.20) with (1.23), for ε sufficiently small, we have

$$\begin{aligned} J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < {} & \frac{N+2-\alpha}{2(2N-\alpha)}(S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}}+O\left(\varepsilon^{\min\{N-2,\frac{2N-\alpha}{2}\}}\right) \qquad (1.24)\\ & +\frac{1}{2}\left(\frac{1}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x+\frac{1}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}}\int_{\mathbb{R}^N}(|A(x)|^2+V_{\mathcal{P}}(x))|u_\varepsilon|^2)\mathrm{d}x\right)^{\frac{N-2}{N+2-\alpha}}\\ & \cdot\frac{1}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}}\int_{\mathbb{R}^N}(|A(x)|^2+V_{\mathcal{P}}(x))|u_\varepsilon|^2\mathrm{d}x-\frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon). \end{aligned}$$

We claim that there is a positive constant C_0 such that, for all $\varepsilon>0$

$$t_\varepsilon^{2p}\geq C_0. \tag{1.25}$$

In fact, suppose that there is a sequence $(\varepsilon_n)\subset\mathbb{R}$, $\varepsilon_n\to 0$ as $n\to\infty$, such that $t_{\varepsilon_n}\to 0$ as $n\to\infty$. Thus,

$$0<c_\lambda\leq\sup_{t\geq 0}J_{A,V}(tu_{\varepsilon_n})=J_{A,V_{\mathcal{P}}}(t_{\varepsilon_n}u_{\varepsilon_n}).$$

Since $u_{\varepsilon_n} \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$ is bounded and $t_{\varepsilon_n} \to 0$, as $n \to \infty$, we have $t_{\varepsilon_n} u_{\varepsilon_n} \to 0$ as $n \to \infty$, in $H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$.

The continuity of $J_{A,V_{\mathcal{P}}}$ implies that $J_{A,V_{\mathcal{P}}}(t_{\varepsilon_n} u_{\varepsilon_n}) \to J_{A,V_{\mathcal{P}}}(0) = 0$. Therefore,

$$0 < c_\lambda \le \lim_{n\to\infty} J_{A,V_{\mathcal{P}}}(t_{\varepsilon_n} u_{\varepsilon_n}) = 0,$$

a contradiction that proves the claim.

From (1.19), (1.24) and (1.25) we conclude that, for some constant $C_0 > 0$ and $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned}
J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) &< \frac{N+2-\alpha}{2(2N-\alpha)} (S_A)^{\frac{2N-\alpha}{N+2-\alpha}} + O\left(\varepsilon^{\min\{N-2,\frac{2N-\alpha}{2}\}}\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}}\|u_\varepsilon\|^2_{AV_{\mathcal{P}}}\right)^{\frac{N-2}{N+2-\alpha}} \cdot \frac{1}{(D(u_\varepsilon)^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x))|u_\varepsilon|^2 \mathrm{d}x - C_0 B(u_\varepsilon) \\
&< \frac{N+2-\alpha}{2(2N-\alpha)} (S_A)^{\frac{2N-\alpha}{N+2-\alpha}} \qquad (1.26) \\
&\quad + O\left(\varepsilon^{\min\{N-2,\frac{2N-\alpha}{2}\}}\right) + \frac{S_A(\varepsilon)^2}{2} \cdot \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x))|u_\varepsilon|^2 \mathrm{d}x - C_0 B(u_\varepsilon) \\
&= \frac{N+2-\alpha}{2(2N-\alpha)} (S_A)^{\frac{2N-\alpha}{N+2-\alpha}} + O(\varepsilon^\eta) + C_1 \int_{\mathbb{R}^N} a(x)|u_\varepsilon|^2 \mathrm{d}x - C_0 B(u_\varepsilon),
\end{aligned}$$

where $C_1 = \frac{S_A(\varepsilon)^2}{2}$, $a(x) = |A(x)|^2 + V_p(x)$ and $\eta = \min\{N-2, \frac{2N-\alpha}{2}\}$.

By direct computation we know that, for $\varepsilon < 1$, since $\psi(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_{2\delta}$ and $\psi \equiv 1$ in B_δ, we have

$$\begin{aligned}
B(u_\varepsilon) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^p |u_\varepsilon(y)|^p}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(x)U_\varepsilon(x)|^p |\psi(y)U_\varepsilon(y)|^p}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \\
&= \int_{B_{2\delta}}\int_{B_{2\delta}} \frac{|\psi(x)U_\varepsilon(x)|^p |\psi(y)U_\varepsilon(y)|^p}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \ge \int_{B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x)|^p |U_\varepsilon(y)|^p}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \\
&= \int_{B_\delta}\int_{B_\delta} \frac{\varepsilon^{\frac{(2-N)p}{2}}[N(N-2)]^{\frac{(N-2)p}{4}} \varepsilon^{\frac{(2-N)p}{2}}[N(N-2)]^{\frac{(N-2)p}{4}}}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)p}{2}} |x-y|^\alpha (1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)p}{2}}} \mathrm{d}x\mathrm{d}y \\
&\ge [N(N-2)]^{\frac{(N-2)p}{2}} \varepsilon^{2N-\alpha-(N-2)p} \int_{B_\delta}\int_{B_\delta} \frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}} |x-y|^\alpha (1+|y|^2)^{\frac{(N-2)p}{2}}} \mathrm{d}x\mathrm{d}y \\
&= C_3 \varepsilon^{2N-\alpha-(N-2)p}.
\end{aligned}$$

Since $a(x)$ is bounded, (1.26) and the last inequality imply that

$$J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{N+2-\alpha}{2(N-\alpha)} (S_A)^{\frac{2N-\alpha}{N+2-\alpha}} + O(\varepsilon^\eta) + C_2 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{2N-\alpha-(N-2)p}. \qquad (1.27)$$

We are going to see that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\eta}\left(C_2 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{2N-\alpha-(N-2)p}\right) = -\infty. \qquad (1.28)$$

In order to do that, it suffices to show that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\eta}\left(C_2\int_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right) = -\infty \tag{1.29}$$

and

$$C_2\int_{B_{2\delta}\setminus B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p} = O(\varepsilon^\eta). \tag{1.30}$$

Assuming (1.29) and (1.30), let us proceed with our proof. Since

$$O(\varepsilon^\eta)+C_2\int_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x-C_3\varepsilon^{2N-\alpha-(N-2)p} = \varepsilon^\eta\left[\frac{O(\varepsilon^\eta)}{\varepsilon^\eta}+\varepsilon^{-\eta}\left(C_2\int_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right)\right],$$

from (1.28) follows

$$O(\varepsilon^\eta)+C_2\int_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p} < 0 \tag{1.31}$$

for $\varepsilon > 0$ sufficiently small.

Thus, (1.27) and (1.31) imply

$$\sup_{t\geq 0} J_{A,V_{\mathcal{P}}}(tu_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)}\left(S_A\right)^{\frac{2N-\alpha}{N+2-\alpha}}$$

for $\varepsilon > 0$ sufficiently small and fixed. Once (1.29) and (1.30) are verified, the proof of Case 1 is complete. □

We now prove (1.29).

Lemma 17 *If* $\frac{N+2-\alpha}{N-2} < p < 2^*_\alpha$ *and* $N = 3, 4$ *or* $\frac{2N-2-\alpha}{N-2} < p < 2^*_\alpha$ *and* $N \geq 5$ *it follows that*

$$\lim_{\varepsilon\to 0}\varepsilon^{-\eta}\left(C_2\int_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right) = -\infty$$

Proof. This limit is evaluated considering the cases $N = 3$, $N = 4$ and $N \geq 5$ as follows. We initially observe that direct computation allows us to conclude that

$$\int_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x = N\omega_N[N(N-2)]^{\frac{N-2}{2}}\varepsilon^2\int_0^{\frac{\delta}{\varepsilon}}\frac{r^{N-1}}{(1+r^2)^{N-2}}\mathrm{d}r, \tag{1.32}$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.

Now, define

$$I_\varepsilon := \varepsilon^{-\eta}\left(C_2\int_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right) = \varepsilon^{-\eta}\left(C_4\varepsilon^2\int_0^{\frac{\delta}{\varepsilon}}\frac{r^{N-1}}{(1+r^2)^{N-2}}\mathrm{d}r - C_3\varepsilon^{2N-\alpha-(N-2)p}\right),$$

the second equality being a consequence of (1.32).

• **The case N = 3.** In this case we have $5-\alpha < p < 2^*_\alpha$ and therefore $5-\alpha-p < 0$. We also observe that $0 < \alpha < N$ implies $\min\{N-2, \frac{2N-\alpha}{2}\} = N-2 = 1$.

It is easy to show that

$$\varepsilon^2 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^2}{1+r^2} \mathrm{d}r = \varepsilon \left(\delta - \varepsilon \arctan \left(\frac{\delta}{\varepsilon} \right) \right).$$

Thus,

$$I_\varepsilon = C_4 \left(\delta - \varepsilon \arctan \left(\frac{\delta}{\varepsilon} \right) \right) - C_3 \varepsilon^{5-\alpha-p}.$$

Our claim follows.

• **The case N = 4.** In this case, $\frac{6-\alpha}{2} < p < 2_\alpha^*$ implies $6-\alpha-2p < 0$ and $\min\{N-2, \frac{2N-\alpha}{2}\} = N-2 = 2$, since $0 < \alpha < 4$.

We have

$$\varepsilon^2 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^3}{(1+r^2)^2} \mathrm{d}r = \frac{\varepsilon^2}{2} \left[\ln \left(1 + \frac{\delta^2}{\varepsilon^2} \right) + \frac{\varepsilon^2}{\varepsilon^2+\delta^2} - 1 \right].$$

So,

$$I_\varepsilon = \frac{C_4}{2} \left(\ln \left(1 + \frac{\delta^2}{\varepsilon^2} \right) + \frac{\varepsilon^2}{\varepsilon^2+\delta^2} - 1 \right) - C_3 \varepsilon^{6-\alpha-2p}.$$

Our claim follows.

• **The case N ≥ 5**. We have

$$I_\varepsilon = \varepsilon^{2-\min\{N-2, \frac{2N-\alpha}{2}\}} \left(C_4 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{N-2}} \mathrm{d}r - C_3 \varepsilon^{2N-\alpha-(N-2)p-2} \right).$$

It is easy to show that, if $N \geq 5$, then the integral

$$\lim_{\varepsilon \to 0} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{N-2}} \mathrm{d}r$$

converges.

There are two cases to be considered:

- $0 < \alpha < 4$ and $N \geq 5$;
- $\alpha \geq 4$ and $N \geq 5$.

Let us suppose $0 < \alpha < 4$ and $N \geq 5$. Since $0 < \alpha < 4$ we have

$$2 - \eta = 2 - \min\{N-2, \frac{2N-\alpha}{2}\} = -N + 4 < 0.$$

Also $\frac{2N-\alpha-2}{N-2} < p < \frac{2N-\alpha}{N-2}$ implies $2N - \alpha - (N-2)p - 2 < 0$. Therefore, $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$. Now we consider the case $\alpha \geq 4$ and $N \geq 5$. We have $N - 2 \geq \frac{2N-\alpha}{2}$ and therefore

$$2 - \eta = 2 - \min\left\{ N-2, \frac{2N-\alpha}{2} \right\} = 2 - N + \frac{\alpha}{2} < 0.$$

Since

$$I_\varepsilon = \varepsilon^{2-N+\frac{\alpha}{2}}\left[C_4\int_0^{\frac{\delta}{\varepsilon}}\frac{r^{N-1}}{(1+r^2)^{N-2}}\mathrm{d}r - C_3\varepsilon^{2N-\alpha-(N-2)p-2}\right],$$

we conclude that $I_\varepsilon \to -\infty$. We are done. $\square$

We now prove (1.30).

Lemma 18 *It holds*

$$C_2\int_{B_{2\delta}\setminus B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p} = O(\varepsilon^\eta).$$

Proof. Fix $\delta>0$ sufficiently large so that $U_\varepsilon^2(x)\le\varepsilon^{1+\eta}$ if $|x|\ge\delta$. Since

$$\begin{aligned}\frac{1}{\varepsilon^\eta}\left[C_2\int_{B_{2\delta}\setminus B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right] &< \frac{C_2}{\varepsilon^\eta}\int_{B_{2\delta}\setminus B_\delta}\psi^2(x)U_\varepsilon^2(x)\mathrm{d}x \le C_2\varepsilon\|\psi\|_2\\ &\le C_1\varepsilon\|\psi\|_{A,V_{\mathcal{P}}},\end{aligned}$$

our proof is complete. $\square$

Case 2. For λ sufficiently large, $\frac{2N-\alpha}{N}<p\le\frac{N+2-\alpha}{N-2}$ and $N=3,4$ or $\frac{2N-\alpha}{N}<p\le\frac{2N-2-\alpha}{N-2}$ and $N\ge5$.

Proof of Case 2. Define $g_\lambda:[0,+\infty)\to\mathbb{R}$ by

$$g_\lambda(t)=J_{A,V_{\mathcal{P}}}(tu_\varepsilon)=\frac{t^2}{2}\int_{\mathbb{R}^N}\left[|\nabla u_\varepsilon|^2+\left(|A(x)|^2+V_{\mathcal{P}}(x)\right)|u_\varepsilon|^2\right]\mathrm{d}x-\frac{\lambda}{2p}t^{2p}B(u_\varepsilon)-\frac{1}{2\cdot2_\alpha^*}t^{2\cdot2_\alpha^*}D(u_\varepsilon).$$

We already know that $\max_{t\ge0}g_\lambda(t)$ is attained at some $t_\lambda>0$. Since $g'_\lambda(t_\lambda)=0$ we have

$$\int_{\mathbb{R}^N}\left[|\nabla u_\varepsilon|^2+\left(|A(x)|^2+V_{\mathcal{P}}(x)\right)|u_\varepsilon|^2\right]\mathrm{d}x=\lambda t_\lambda^{2(p-1)}B(u_\varepsilon)+t_\lambda^{2(2_\alpha^*-1)}D(u_\varepsilon).$$

Thus $t_\lambda\to0$ as $\lambda\to+\infty$ and

$$\begin{aligned}\max_{t\ge0}J_{A,V_{\mathcal{P}}}(tu_\varepsilon)&=\frac{{t_\lambda}^2}{2}\int_{\mathbb{R}^N}\left[|\nabla u_\varepsilon(x)|^2+\left(|A(x)|^2+V_{\mathcal{P}}(x)\right)|u_\varepsilon(x)|^2\right]\mathrm{d}x-\frac{\lambda}{2p}{t_\lambda}^{2p}B(u_\varepsilon)-\frac{1}{2\cdot2_\alpha^*}t^{2\cdot2_\alpha^*}D(u_\varepsilon)\\&<\frac{{t_\lambda}^2}{2}\int_{\mathbb{R}^N}\left[|\nabla u_\varepsilon|^2+\left(|A(x)|^2+V_{\mathcal{P}}(x)\right)|u_\varepsilon(x)|^2\right]\mathrm{d}x.\end{aligned}$$

Since $t_\lambda\to0$ as $\lambda\to+\infty$ and $\frac{N+2-\alpha}{2(N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}>0$, we conclude that

$$\frac{{t_\lambda}^2}{2}\int_{\mathbb{R}^N}\left[|\nabla u_\varepsilon|^2+\left(|A(x)|^2+V_{\mathcal{P}}(x)\right)|u_\varepsilon(x)|^2\right]\mathrm{d}x<\frac{N+2-\alpha}{2(2N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}},$$

for $\lambda>0$ sufficiently large.

Therefore,

$$\sup_{t\ge0}J_{A,V_{\mathcal{P}}}(tu_\varepsilon)<\frac{N+2-\alpha}{2(2N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}$$

for $\lambda>0$ sufficiently large. $\square$

1.2.2 The proof of Theorem 1

Some arguments of this proof were adapted from the articles by C.O. Alves and G.M. Figueiredo [3] and by O.H. Miyagaki [38].

Maintaining the notation introduced in subsection 1.2.1, consider the energy functional $I_{A,V} : H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ given by

$$I_{A,V}(u) = \frac{1}{2}\|u\|^2{}_{A,V} - \frac{1}{2 \cdot 2^*_\alpha} D(u) - \frac{\lambda}{2p} B(u).$$

We denote by $\mathcal{N}_{A,V}$ the Nehari manifold related to $I_{A,V}$, that is,

$$\mathcal{N}_{A,V} = \left\{ u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \; : \; \|u\|^2_{A,V} = D(u) + \lambda B(u) \right\},$$

which is non-empty as a consequence of Theorem 15. As before, the functional $I_{A,V}$ satisfies the mountain pass geometry. Thus, there exists a sequence $(u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that

$$I'_{A,V}(u_n) \to 0 \qquad \text{and} \qquad I_{A,V}(u_n) \to d_\lambda,$$

where d_λ is the minimax level, also characterized by

$$d_\lambda = \inf_{u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} I_{A,V}(tu) = \inf_{\mathcal{N}_{A,V}} I_{A,V}(u) > 0.$$

We stress that, as a consequence of (V_2), we have $I_{A,V}(u) < J_{A,V_{\mathcal{P}}}(u)$ for all $u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$.

The next lemma compares the levels d_λ and c_λ.

Lemma 19 *The levels d_λ and c_λ verify the inequality*

$$d_\lambda < c_\lambda < \frac{N + 2 - \alpha}{2(2N - \alpha)} (S_A)^{\frac{2N-\alpha}{N+2-\alpha}}$$

for all $\lambda > 0$.

Proof. Let u be the ground state solution of problem (1.3) and consider $\bar{t}_u > 0$ such that $\bar{t}_u u \in \mathcal{N}_{A,V}$, that is

$$0 < d_\lambda \leq \sup_{t \geq 0} I_{A,V}(tu) = I_{A,V}(\bar{t}_u u).$$

It follows from (V_2) that

$$0 < d_\lambda \leq I_{A,V}(\bar{t}_u u) < J_{A,V_{\mathcal{P}}}(\bar{t}_u u) \leq \sup_{t \geq 0} J_{A,V_{\mathcal{P}}}(tu) = J_{A,V_{\mathcal{P}}}(u) = c_\lambda.$$

Therefore,

$$d_\lambda < c_\lambda.$$

The second inequality was already known. □

Proof of Theorem 1. Let (u_n) be a $(PS)_{d_\lambda}$ sequence for $I_{A,V}$. As before, (u_n) is bounded in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$. Thus, there exists $u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that

$$u_n \rightharpoonup u \;\text{ in }\; H^1_{A,V}(\mathbb{R}^N, \mathbb{C}).$$

By the same arguments given in the proof of Theorem 15, u is a ground state solution of problem (7), if $u \neq 0$.

Following closely [3], we will show that $u = 0$ cannot occur. Indeed, Lemma 8 yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W|u_n|^2 \, \mathrm{d}x = 0 \tag{1.33}$$

since $W \in L^{\frac{N}{2}}(\mathbb{R}^N, \mathbb{C})$ and $u_n \rightharpoonup 0$ in $H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ (see Appendix B.2). So,

$$|J_{A,V_{\mathcal{P}}}(u_n) - I_{A,V}(u_n)| = o_n(1)$$

showing that

$$J_{A,V_{\mathcal{P}}}(u_n) \to d_\lambda.$$

But, for $\varphi \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that $\|\varphi\|_{A,V} \leq 1$, we have

$$|(J'_{A,V_{\mathcal{P}}}(u_n) - I'_{A,V}(u_n)) \cdot \varphi| \leq \left(\int_{\mathbb{R}^N} W|u_n|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = o_n(1).$$

Thus,

$$J'_{A,V_{\mathcal{P}}}(u_n) = o_n(1)$$

Let $t_n > 0$ such that $t_n u_n \in \mathcal{M}_{A,V_{\mathcal{P}}}$. Mimicking the argument found in [2, 27, 44, 48], it follows that

$$t_n \to 1 \ \text{ as } \ n \to \infty \tag{1.34}$$

(see Appendix B.3). Therefore,

$$c_\lambda \leq J_{A,V_{\mathcal{P}}}(t_n u_n) = J_{A,V_{\mathcal{P}}}(u_n) + o_n(1) = d_\lambda + o_n(1).$$

Letting $n \to +\infty$, we get

$$c_\lambda \leq d_\lambda$$

obtaining a contradiction with Lemma 19. This completes the proof of Theorem 1. □

1.3 The case $f(u) = |u|^{p-1}u$

1.3.1 The periodic problem

In this subsection we deal with problem (9) for $f(u)$ as above, that is,

$$-(\nabla + iA(x))^2 u + V_{\mathcal{P}}(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2}u + \lambda |u|^{p-1}u, \tag{1.35}$$

where $1 < p < 2^* - 1$.

We observe that in this case the energy functional $J_{A,V_{\mathcal{P}}}$ is given by

$$J_{A,V_{\mathcal{P}}}(u) := \frac{1}{2}\|u\|^2_{A,V_{\mathcal{P}}} - \frac{1}{2 \cdot 2^*_\alpha} D(u) - \frac{\lambda}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \mathrm{d}x,$$

where, as before

$$D(u) = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * |u|^{2_\alpha^*} \right) |u|^{2_\alpha^*} \, \mathrm{d}x = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)^{2_\alpha^*}||u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y.$$

By the Sobolev immersion (1.1) and the Hardy-Littlewood-Sobolev inequality (see Appendix A, Proposition 49), we have that $J_{A,V_{\mathcal{P}}}$ is well defined.

Definition 1.3.1 *A function $u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$ is a weak solution of* (1.35) *if*

$$\langle u, \varphi \rangle_{A,V_{\mathcal{P}}} - \mathfrak{Re} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * |u|^{2_\alpha^*} \right) |u|^{2_\alpha^*-2} u \bar{\psi} \, \mathrm{d}x - \lambda \, \mathfrak{Re} \int_{\mathbb{R}^N} |u|^{p-1} u \bar{\psi} \, \mathrm{d}x = 0$$

for all $\psi \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$.

As before, we see that critical points of $J_{A,V_{\mathcal{P}}}$ are weak solutions of (1.35) and

$$J'_{A,V_{\mathcal{P}}}(u) \cdot u := \|u\|^2_{A,V_{\mathcal{P}}} - D(u) - \lambda \|u\|^{p+1}_{p+1}.$$

We obtain that $J_{A,V_{\mathcal{P}}}$ satisfies the geometry of the mountain pass (see the proof of Lemma 9).

As in Section 1.2, the mountain pass theorem without the PS condition yields a sequence $(u_n) \subset H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C})$ such that

$$J'_{A,V_{\mathcal{P}}}(u_n) \to 0 \qquad \text{and} \qquad J_{A,V_{\mathcal{P}}}(u_n) \to c_\lambda,$$

where

$$c_\lambda = \inf_{\alpha \in \Gamma} \max_{t \in [0,1]} J_{A,V_{\mathcal{P}}}(\gamma(t))$$

and

$$\Gamma = \left\{ \gamma \in C^1 \left([0,1], H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C}) \right) : \gamma(0) = 0, \; J_{A,V_{\mathcal{P}}}(\gamma(1)) < 0 \right\}.$$

Considering the Nehari manifold $J_{A,V_{\mathcal{P}}}$

$$\mathcal{M}_{A,V_{\mathcal{P}}} = \left\{ u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \; : \; \|u\|^2_{A,V_{\mathcal{P}}} = D(u) + \lambda \|u\|^{p+1}_{p+1} \right\},$$

by proceeding as in the proof of Lemma 13 we obtain

Lemma 20 *There exists a unique $t_u = t_u(u) > 0$ such that $t_u u \in \mathcal{M}_{A,V_{\mathcal{P}}}$ for all $u \in H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ and $J_{A,V_{\mathcal{P}}}(t_u u) = \max_{t \geq 0} J_{A,V_{\mathcal{P}}}(tu)$. Moreover $c_\lambda = c^*_\lambda = c^{**}_\lambda$, where*

$$c^*_\lambda = \inf_{u \,\in\, \mathcal{M}_{A,V_{\mathcal{P}}}} J_{A,V_{\mathcal{P}}}(u) \quad \textit{and} \quad c^{**}_\lambda = \inf_{u \,\in\, H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} J_{A,V_{\mathcal{P}}}(tu).$$

Lemma 21 *Suppose that $u_n \rightharpoonup u_0$ and consider*

$$B'(u_n) \cdot \psi = \mathfrak{Re} \int_{\mathbb{R}^N} |u|^{p-1} u \bar{\psi} \, \mathrm{d}x$$

and

$$D'(u_n) \cdot \psi = \mathfrak{Re} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * |u_n|^{2_\alpha^*} \right) |u_n|^{2_\alpha^*-2} u_n \bar{\psi} \, \mathrm{d}x$$

for $\psi \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$. Then $B'(u_n) \cdot \psi \to B'(u_0) \cdot \psi$ and $D'(u_n) \cdot \psi \to D'(u_0) \cdot \psi$ as $n \to \infty$.

Lemma 22 *If* $(u_n) \subset H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$ *is a* $(PS)_b$ *sequence for* $J_{A,V_{\mathcal{P}}}$, *then* (u_n) *is bounded. In addition, if* $u_n \rightharpoonup u$ *weakly in* $H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$, *as* $n\to\infty$, *then* u *is a weak solution of* (1.35).

Lemma 23 *If* $(u_n) \subset H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})$ *is a sequence* $(PS)_b$ *for* $J_{A,V_{\mathcal{P}}}$ *such that*

$$u_n \rightharpoonup 0 \quad \text{weakly in } H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C}) \ \text{ as } n\to\infty,$$

with

$$b < \frac{N+2-\alpha}{2(2N-\alpha)} S_A^{\frac{2N-\alpha}{N+2-\alpha}},$$

then there exists a sequence $(y_n) \in \mathbb{R}^N$ *and constants* $R, \theta > 0$ *such that*

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \, \mathrm{d}x \geq \theta,$$

where $B_r(y)$ *denotes the ball in* $\mathbb{R}^N$ *of center at* y *and radius* $r>0$.

The proof of Lemmas 21, 22 and 23 is similar to that of Corollary 11, Lemmas 12 and 14, respectively.

Lemma 24 *Let* $1 < p < 2^* - 1$ *and* u_ε *as defined in* (1.16). *Then, there exists* ε *such that*

$$\sup_{t\geq 0} J_{A,V_{\mathcal{P}}}(tu_\varepsilon) < \frac{N+2-\alpha}{2(2N-\alpha)} (S_A)^{\frac{2N-\alpha}{N+2-\alpha}}.$$

provided that either

(i) $3 < p < 5$, $N = 3$ *and* $\lambda > 0$;

(ii) $p > 1$, $N \geq 4$ *and* $\lambda > 0$;

(iii) $1 < p \leq 3$, $N = 3$ *and* λ *sufficiently large.*

Proof. Consider, for the cases (i) and (ii) the function $f : [0,+\infty) \to \mathbb{R}$ defined by

$$f(t) = J_{A,V_{\mathcal{P}}}(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2_{A,V_{\mathcal{P}}} - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha} D(u_\varepsilon) - \frac{\lambda t^{p+1}}{p+1}\|u_\varepsilon\|^{p+1}_{p+1}$$

and proceed as in the proof of Case 1, Lemma 16.

In the case of $1 < p \leq 3$, $N = 3$ and λ sufficiently large, consider $g_\lambda : [0,+\infty) \to \mathbb{R}$ defined by

$$g_\lambda(t) = J_{A,V_{\mathcal{P}}}(tu_\varepsilon) = \frac{t^2}{2}\int_{\mathbb{R}^N} \left[|\nabla u_\varepsilon|^2 + \left(|A(x)|^2 + V_{\mathcal{P}}(x)\right)|u_\varepsilon|^2\right] \mathrm{d}x - \frac{1}{2\cdot 2^*_\alpha} t^{2\cdot 2^*_\alpha} D(u_\varepsilon) - \frac{\lambda t^{p+1}}{p+1}\|u_\varepsilon\|^{p+1}_{p+1}$$

and proceed as in the proof of Case 2, Lemma 16. □

Similar to the proof of Theorem 15, we now state our result about the periodic problem (1.35).

Theorem 25 *Under the hypotheses already stated on* A *and* α, *suppose that* (V_1) *is valid. Then problem* (1.35) *has at least one ground state solution if either*

(i) $3 < p < 5$, $N = 3$ *and* $\lambda > 0$;

(ii) $p > 1$, $N \geq 4$ *and* $\lambda > 0$;

(iii) $1 < p \leq 3$, $N = 3$ *and* λ *sufficiently large.*

1.3.2 Proof of Theorem 2

Some arguments of this proof were adapted from the proof of Theorem 1 below , that in turn were adapted from articles [3, 38].

Maintaining the notation already introduced, consider the functional $I_{A,V} : H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ defined by

$$I_{A,V}(u) := \frac{1}{2}\|u\|^2{}_{A,V} - \frac{1}{2 \cdot 2^*_\alpha} D(u) - \frac{\lambda}{p+1}\|u\|_{p+1}^{p+1}$$

for all $u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$.

We denote by $\mathcal{N}_{A,V}$ the Nehari Manifold related to $I_{A,V}$, that is,

$$\mathcal{N}_{A,V} = \left\{ u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \,:\, \|u\|^2_{A,V} = D(u) + \lambda \|u\|_{p+1}^{p+1} \right\},$$

which is non-empty as a consequence of Theorem 48. As before, the functional $I_{A,V}$ satisfies the mountain pass geometry. Thus, there exists a $(PS)_{d_\lambda}$ sequence $(u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$, that is, a sequence satisfying

$$I'_{A,V}(u_n) \to 0 \qquad \text{and} \qquad I_{A,V}(u_n) \to d_\lambda,$$

where d_λ is the minimax level, also characterized by

$$d_\lambda = \inf_{u \in H^1_{A,V}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \max_{t \geq 0} I_{A,V}(tu) = \inf_{\mathcal{N}_{A,V}} I_{A,V}(u) > 0.$$

As in the Section 1.2, we have $I_{A,V}(u) < J_{A,V_{\mathcal{P}}}(u)$ for all $u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})$ as a consequence of (V_2).

Similar to the proof of Lemma 19 we have the following conclusion that shows an important inequality involving the levels d_λ and c_λ, which completes the proof of Theorem 2.

Lemma 26 *The levels d_λ and c_λ verify the inequality*

$$d_\lambda < c_\lambda < \frac{N+2-\alpha}{2(N-\alpha)}(S_A)^{\frac{2N-\alpha}{N+2-\alpha}}$$

for all $\lambda > 0$.

1.4 The case $f(u) = |u|^{2^*-2}u$

1.4.1 Proof of Theorem 3

As observed by Gao and Yang [29], the proof of Theorem 3 is analogous to the proof of Theorem 1. The principal distinction is that the $(PS)_{c_\lambda}$ condition holds true below the level $\frac{1}{N}S^{\frac{N}{2}}$. It follows from [48, Lemma 1.46] that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x = S^{\frac{N}{2}} + O(\varepsilon^{N-2})$$

and

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*}\mathrm{d}x = S^{\frac{N}{2}} + O(\varepsilon^N).$$

So, we have

$$\sup_{t\geq 0} J_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} + O(\varepsilon^{N-2}) + C_2\int_{\mathbb{R}^N} |u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p} < \frac{1}{N}S^{\frac{N}{2}},$$

since

$$\lim_{\varepsilon\to 0}\varepsilon^{-(N-2)}\left(C_2\int_{\mathbb{R}^N} |u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right) = -\infty.$$

Observe that the last result is a consequence of

$$\lim_{\varepsilon\to 0}\varepsilon^{-(N-2)}\left(C_2\int_{B_\delta} |u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p}\right) = -\infty$$

and

$$C_2\int_{B_{2\delta}\setminus B_\delta} |u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\alpha-(N-2)p} = O(\varepsilon^{N-2}).$$

The rest of the proof is omitted here. □

Chapter 2

Fractional Magnetic Choquard equation with Hardy-Littlewood-Sobolev critical exponent

In this chapter we deal with problem (2)

$$(-\Delta)^s_A u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda g(u) \ \text{ in } \ \mathbb{R}^N(N=3)$$

and prove Theorems 4, 5 and 6.

2.1 Preliminary results

In this section we first provide some basic functional setting and then we give some results that will be used in the next sections. The critical exponent 2^*_s is defined as $2^*_s = \frac{6}{3-2s}$.

Following [37], we introduce the space of Lebesgue functions $L^2(\mathbb{R}^3, V)$.

$$L^2_V(\mathbb{R}^3,\mathbb{R}) = \left\{u : \mathbb{R}^3 \to \mathbb{R} \,:\, \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x < \infty\right\}$$

equipped with norm

$$\|u\|_{2,V} := \left(\int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

The fractional Sobolev space $H^s_V(\mathbb{R}^3,\mathbb{R})$ is then defined as

$$H^s_V(\mathbb{R}^3,\mathbb{R}) = \left\{u \in L^2_V(\mathbb{R}^3,\mathbb{R}) \,:\, [u]_s < \infty\right\}$$

where $[\cdot]_s$ is the Gagliardo semi-norm given by

$$[u]_s = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\, dxdy\right)^{\frac{1}{2}}.$$

The space $H^s_V(\mathbb{R}^3;\mathbb{R})$ is endowed with the norm

$$\|u\|_{s,V} := \left(\|u\|^2_{2,V} + [u]^2_s\right)^{\frac{1}{2}}.$$

We observe that if $V \equiv 1$ then we recover the space

$$H^s(\mathbb{R}^3,\mathbb{R}) := \{u \in L^2(\mathbb{R}^3,\mathbb{R}) : [u]_s < \infty\}.$$

The embedding $H^s_V(\mathbb{R}^3;\mathbb{R}) \hookrightarrow L^q(\mathbb{R}^3)$ is continuous for any $q \in [2, 2^*_s]$, see [23, Theorem 6.7]. Namely, there exists a positive constant C such that

$$\|u\|_{L^q(\mathbb{R}^3)} \le C\|u\|_{s,V} \text{ for all } u \in H^s_V(\mathbb{R}^3;\mathbb{R}).$$

Considering a compact subset $K \subset \mathbb{R}^3$, we also define the localized norm on the space $H^s_V(K;\mathbb{R})$ by

$$\|u\|_{s,K,V} := \left(\int_K V(x)|u|^2 \mathrm{d}x + \int_K\int_K \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}. \tag{2.1}$$

Now we introduce the spaces of complex functions. Let $L^2_V(\mathbb{R}^3,\mathbb{C})$ be the Lebesgue space of functions $u : \mathbb{R}^3 \to \mathbb{C}$ such that $\int_{\mathbb{R}^3} V(x)|u|^2\mathrm{d}x < \infty$ endowed with the (real) scalar product

$$\langle u, v\rangle_{L^2_V(\mathbb{R}^3,\mathbb{C})} = \mathfrak{Re}\int_{\mathbb{R}^3} V(x)u\bar{v}\,dx \text{ for all } u, v \in L^2_V(\mathbb{R}^3,\mathbb{C}),$$

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Following [21], we also introduce the magnetic Gagliardo semi-norm given by

$$[u]_{s,A} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$$

and define the space

$$H^s_{A,V}(\mathbb{R}^3,\mathbb{C}) = \{u \in L^2(\mathbb{R}^3,\mathbb{C}) : u \in L^2_V(\mathbb{R}^3,\mathbb{C}),\ [u]^2_{s,A} < \infty\}$$

endowed with the inner product

$$\langle u, v\rangle_{s,A,V} = \langle u, v\rangle_{L^2_V} + \mathfrak{Re}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left(u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)\right)\overline{\left(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)\right)}}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\right).$$

Therefore

$$\|u\|_{s,A,V} := \langle u, u\rangle^{1/2}_{s,A,V} = \left(\|u\|^2_{L^2_V} + [u]^2_{s,A}\right)^{\frac{1}{2}}.$$

We deal with problem (2) in the space $H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$ endowed with the norm $\|\cdot\|_{s,A,V}$.

The next result is proved in [9, Lemma 2.2].

Lemma 27 *The space $H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$ is complete and $C^\infty_c(\mathbb{R}^3,\mathbb{C})$ is dense in $H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$*

A simple adaptation of [21, Lemma 3.1] proves the next result.

Lemma 28 (Diamagnetic inequality) *For each* $u \in H^s_{A,V}(\mathbb{R}^3, \mathbb{C})$

$$|u| \in H^s(\mathbb{R}^3, \mathbb{R}) \quad \text{and} \quad \||u|\|_{s,1} \leq \|u\|_{s,A,V.}$$

Remark 2.1.1 (Pointwise diamagnetic inequality) *There holds*

$$\big||u(x)| - |u(y)|\big| \leq \left|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)\right| \tag{2.2}$$

for a.e. $x, y \in \mathbb{R}^3$. *(see [21, Remark 3.2].)*

Arguing as in [21, Lemma 3.5] and applying Lemma 28 we obtain

Lemma 29 *The embedding*

$$H^s_{A,V}(\mathbb{R}^3, \mathbb{C}) \hookrightarrow L^q(\mathbb{R}^3, \mathbb{C}) \tag{2.3}$$

is continuous for $q \in [2, 2^*_s]$. *Furthermore, for any compact subset* $K \subset \mathbb{R}^3$ *and all* $q \in [1, 2^*_s)$ *the embeddings*

$$H^s_{A,V}(\mathbb{R}^3; \mathbb{C}) \hookrightarrow H^s_{0,V}(K, \mathbb{C}) \hookrightarrow L^q(K, \mathbb{C})$$

are continuous and the latter is compact, where $H^s_V(K, \mathbb{C})$ *is endowed with* (2.1).

The proof of the next result can be found in [27, Lemma 2.2].

Lemma 30 *Let* $q \in [2, 2^*_s]$. *If* (u_n) *is a bounded sequence in* $H^s(\mathbb{R}^3, \mathbb{R})$ *and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n|^q dx = 0$$

for some $r > 0$, *then* $u_n \to 0$ *em* $L^t(\mathbb{R}^3, \mathbb{R})$ *for all* $t \in (2, 2^*_s)$.

The proof of the next result only adapts arguments given for the real case, as in [33, Lemme 4.8, Chapitre 1].

Lemma 31 *Let* $U \subseteqq \mathbb{R}^3$ *be any open set. For* $1 < t < \infty$, *let* (f_n) *be a bounded sequence in* $L^t(U, \mathbb{C})$ *such that* $f_n(x) \to f(x)$ *a.e. Then* $f_n \rightharpoonup f$ *in* $L^t(U, \mathbb{C})$.

Now, we consider the minimization problem

$$\mathcal{S}^s_A := \inf_{u \,\in\, H^s_{A,V}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}} \frac{[u]^2_{s,A}}{\left(\displaystyle\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\alpha,s}}\ |u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y\right)^{\frac{3-2s}{6-\alpha}}} = \inf_{u \,\in\, D^s_{A,V}(\mathbb{R}^3,\mathbb{C})} [u]^2_{s,A}.$$

where

$$D^s_{A,V}(\mathbb{R}^3, \mathbb{C}) = \left\{ u \in H^s_{A,V}(\mathbb{R}^3, \mathbb{C}) \,:\, \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\alpha,s}}\ |u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = 1 \right\}.$$

By density, we have

$$\mathcal{S}^s_A := \inf_{u \,\in\, D^s_{A,V}(\mathbb{R}^3,\mathbb{C})\cap C^\infty_c(\mathbb{R}^3,\mathbb{C})} [u]^2_{s,A}$$

$$\mathcal{S}^s_0 := \inf_{u \,\in\, D^s_{0,V}(\mathbb{R}^3,\mathbb{C})\cap C^\infty_c(\mathbb{R}^3,\mathbb{C})} [u]^2_{s,0}$$

where

$$D^s_{0,V}(\mathbb{R}^3,\mathbb{C}) = \left\{ u \in H^s_{0,V}(\mathbb{R}^3,\mathbb{C}) \; : \; \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\alpha,s}}\;|u(y)|^{2^*_{\alpha,s}}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y = 1 \right\}.$$

We claim that

$$\mathcal{S}^s_0 = \inf_{u\,\in\, D^s_{0,V}(\mathbb{R}^3,\mathbb{R})\cap C^\infty_c(\mathbb{R}^3;\mathbb{R})} [u]^2_{s,0}. \tag{2.4}$$

In fact, since the set $D^s_{0,V}(\mathbb{R}^3,\mathbb{C})\cap C^\infty_c(\mathbb{R}^3,\mathbb{C})$ has less restrictions than $D^s_{0,V}(\mathbb{R}^3,\mathbb{R})\cap C^\infty_c(\mathbb{R}^3,\mathbb{R})$, we have

$$\inf_{u\,\in\, D^s_{0,V}(\mathbb{R}^3,\mathbb{C})\cap C^\infty_c(\mathbb{R}^3,\mathbb{C})} [u]^2_{s,0} \le \inf_{u\,\in\, D^s_{0,V}(\mathbb{R}^3,\mathbb{R})\cap C^\infty_c(\mathbb{R}^3;\mathbb{R})} [u]^2_{s,0},$$

that is

$$\mathcal{S}^s_0 \le \inf_{u\,\in\, D^s_{0,V}(\mathbb{R}^3,\mathbb{R})\cap C^\infty_c(\mathbb{R}^3;\mathbb{R})} [u]^2_{s,0}.$$

On the other hand, since $[|u|]_{s,0} \le [u]_{s,0}$ (see Appendix B, Lemma 58) we have

$$\inf_{u\,\in\, D^s_{0,V}(\mathbb{R}^3,\mathbb{R})\cap C^\infty_c(\mathbb{R}^3;\mathbb{R})} [u]^2_{s,0} \le \mathcal{S}^s_0$$

and (2.4) is proved.

The next result can be found in Gao and Yang [30, Lemma 1.2]. See also [17, 42, 45]. Lemma 1.2 in [30] deals with another Sobolev constant, which is also a multiple of $\mathcal{S}_s$.

Lemma 32 *$\mathcal{S}^s_0$ is a multiple of $\mathcal{S}_s$, the best constant of the Sobolev embedding $H^s(\mathbb{R}^3,\mathbb{R}) \hookrightarrow L^{2^*_s}(\mathbb{R}^3,\mathbb{R})$, more specifically*

$$\mathcal{S}^s_0\, C(3,\alpha)^{\frac{1}{2^*_{\alpha,s}}} = \mathcal{S}_s. \tag{2.5}$$

Therefore, $\mathcal{S}^s_0$ is achieved if, and only if, u has the form

$$C\left(\frac{t}{t^2+|x-x_0|^2}\right)^{\frac{3-2s}{2}},\; x \in \mathbb{R}^3,$$

for some $x_0 \in \mathbb{R}^3$, $C>0$ and $t>0$. Furthermore, u satisfies

$$(-\Delta)^s u = \left(\int_{\mathbb{R}^3} \frac{|u|^{2^*_{\alpha,s}}}{|x-y|^{\alpha}}dy\right)|u|^{2^*_{\alpha,s}-2}u \text{ in } \mathbb{R}^3. \tag{2.6}$$

for $\alpha = 4s$.

Remark 2.1.2 *We emphasize that the constant $C(3,\alpha)$ above, which will also appear repeatedly below, depends only on α, although we prefer to denote it thus to make explicity its dependence on the dimension of Euclidean space $\mathbb{R}^3$.*

Now, for fixed constants $\kappa \in \mathbb{R}\setminus\{0\}$ and $\mu>0$, we consider

$$u(x) = \kappa(\mu^2+|x|^2)^{-\frac{3-2s}{2}}, \qquad u^*(x) = \frac{1}{\|u\|_{2^*_s}} u\left(\frac{x}{S_s^{\frac{1}{2s}}}\right),$$

and the family of functions $\{U_\varepsilon\}$ defined for $x \in \mathbb{R}^3$ as

$$U_\varepsilon(x) = \varepsilon^{-\frac{(3-2s)}{2}} u^* \left(\frac{x}{\varepsilon}\right).$$

Then, according to [45, Claim 7], for each $\varepsilon > 0$, U_ε satisfies

$$(-\Delta)^s u = |u|^{2_s^*-2} u \text{ in } \mathbb{R}^3$$

and verifies the equality

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} |U_\varepsilon(x)|^{2_s^*}\,\mathrm{d}x = \mathcal{S}_s^{\frac{3}{2s}}. \tag{2.7}$$

Then

$$\bar{U}_\varepsilon(x) = \mathcal{S}_s^{\frac{(3-\alpha)(2s-3)}{4s(3-\alpha+2s)}} C(3,\alpha)^{\frac{2s-3}{2(3-\alpha+2s)}} U_\varepsilon(x)$$

gives a family of minimizers for $\mathcal{S}_0^s$ which satisfies (2.6) and

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\bar{U}_\varepsilon(x) - \bar{U}_\varepsilon(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\bar{U}_\varepsilon(x)|^{2^*_{\alpha,s}}|\bar{U}_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = (\mathcal{S}_0^s)^{\frac{6-\alpha}{3-\alpha+2s}}.$$

The arguments in the proof of the next result were adapted from [21, Lemma 4.6].

Lemma 33 *It holds*

$$\mathcal{S}_A^s = \mathcal{S}_0^s$$

Proof.

Let $\varepsilon > 0$ and $u \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ be such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 1, \qquad [u]_s^2 < \mathcal{S}_0^s + \varepsilon.$$

For $\sigma > 0$, considering the scaling

$$u_\sigma(x) = \sigma^{-\frac{3-2s}{2}} u\left(\frac{x}{\sigma}\right),$$

we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\sigma(x)|^{2^*_{\alpha,s}}|u_\sigma(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 1$$

and

$$[u_\sigma]_{s,0} = [u]_{s,0}.$$

(see Appendix B, Lemma 59).

Changing variables, it follows

$$[u_\sigma]_{s,A}^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - e^{i\sigma(x-y)\cdot A(\sigma\frac{x+y}{2})}u(y)|^2}{|x-y|^{3+2s}} dxdy.$$

Then, we have

$$[u_\sigma]^2_{s,A} - [u]^2_{s,0} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - e^{i\sigma(x-y)\cdot A(\sigma\frac{x+y}{2})}u(y)|^2 - |u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_\sigma(x,y)\, dxdy = \int_K\int_K \Phi_\sigma(x,y)\, dxdy,$$

where K is the compact support of u and

$$\Phi_\sigma(x,y) := \frac{2\mathfrak{Re}\left(\left(1 - e^{i\sigma(x-y)\cdot A(\sigma\frac{x+y}{2})}\right)u(x)u(y)\right)}{|x-y|^{3+2s}}$$
$$= \frac{2\left(1-\cos\left(\sigma(x-y)\cdot A(\sigma\frac{x+y}{2})\right)\right)u(x)u(y)}{|x-y|^{3+2s}}$$

a.e. in $\mathbb{R}^3\times\mathbb{R}^3$ (see Appendix B, Lemma 60). (Observe that $\Phi_\sigma(x,y)\to 0$ a.e $(x,y)\in\mathbb{R}^3\times\mathbb{R}^3$.)

Since $A\colon \mathbb{R}^3\to\mathbb{R}^3$ is a C^1, $\mathbb{Z}^3$-periodic vector potential, there exists $C>0$ such that

$$1-\cos\left(\sigma(x-y)\cdot A\left(\sigma\frac{x+y}{2}\right)\right) \le C|x-y|^2, \tag{2.8}$$

for all $x,y\in K$. In fact, since $1-\cos x \le \frac{1}{2}|x|^2$ for all $x\in\mathbb{R}$ (see Apêndice A, Lemma 55), we have

$$1-\cos\left(\sigma(x-y)\cdot A\left(\sigma\frac{x+y}{2}\right)\right) \le \frac{\sigma^2}{2}\left|(x-y)\cdot A\left(\sigma\frac{x+y}{2}\right)\right|^2 \le \frac{\sigma^2}{2}|x-y|^2\cdot\left|A\left(\sigma\frac{x+y}{2}\right)\right|^2$$
$$< C|x-y|^2$$

for some $C>0$ and for all $x,y\in K$.

Therefore, since u is bounded, for $x,y\in K$ and an adequate constant C we have

$$|\Phi_\sigma(x,y)| \le \frac{C}{|x-y|^{1+2s}}, \quad \text{if } \ |x-y|<1$$

and

$$|\Phi_\sigma(x,y)| \le \frac{C}{|x-y|^{3+2s}}, \quad \text{if } \ |x-y|\ge 1.$$

(The first inequality is a consequence of the boundeness of u and (2.8), whereas the second is a consequence of the boundeness of u and the inequality $0\le 1-\cos\left(\sigma(x-y)\cdot A\left(\sigma\frac{x+y}{2}\right)\right)\le 2$.)

Consider

$$w(x,y) := C\min\left\{\frac{1}{|x-y|^{1+2s}}, \frac{1}{|x-y|^{3+2s}}\right\}$$

for all $x,y\in K$.

It follows straightforwardly that

$$|\Phi_\sigma(x,y)| \le w(x,y) \quad \text{and} \quad w\in L^1(K\times K).$$

Summarizing, we have

a) $\Phi_\sigma(x,y) \to 0$ a.e $(x,y) \in K \times K$ as $\sigma \to 0$;

b) $|\Phi_\sigma(x,y)| \leq w(x,y)$ for all $x, y \in K$;

c) $w \in L^1(K \times K)$.

So, the Dominated Convergence Theorem yields

$$\lim_{\varepsilon \to 0} [u_\sigma]^2_{s,A} = [u]^2_{s,0}.$$

Of course we have

$$\mathcal{S}^s_A \leq [u_\sigma]^2_{s,A}.$$

So,

$$\mathcal{S}^s_A \leq \lim_{\sigma \to 0} [u_\sigma]^2_{s,A} = [u]^2_{s,0} < \mathcal{S}^s_0 + \varepsilon,$$

proving that

$$\mathcal{S}^s_A \leq \mathcal{S}_0$$

by the arbitrariness of ε.

The opposite inequality is trivial as a consequence of the Pointwise Diamagnetic inequality (Remark 2.1.1). In fact, by (2.2) we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\big||u(x)| - |u(y)|\big|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\mathrm{d}y \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)\right|}{|x-y|^{3-2s}}\,\mathrm{d}x\mathrm{d}y,$$

which immediately yields

$$\mathcal{S}_0 \leq \mathcal{S}^s_A.$$

□

2.2 The case $g(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$

2.2.1 The periodic problem

In this subsection we deal with problem (10) considering $g(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$, that is,

$$(-\Delta)^s_A u + V_{\mathcal{P}}(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u + \lambda\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \tag{2.9}$$

where $\frac{6-\alpha}{3} < p < 2^*_{\alpha,s}$.

We consider the space

$$H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C}) = \left\{u \in L^2(\mathbb{R}^3,\mathbb{C}) \,:\, [u]_{s,A} < \infty\right\}$$

endowed with scalar product

$$\langle u,v\rangle_{s,A,V_{\mathcal{P}}} = \langle u,v\rangle_{L^2_{V_{\mathcal{P}}}} + \mathfrak{Re}\left(\int_{\mathbb{R}3}\int_{\mathbb{R}^3} \frac{\left(u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)\right)\overline{\left(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y)\right)}}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y\right).$$

and, therefore corresponding norm

$$\|u\|_{s,A,V_{\mathcal{P}}} = \left(\|u\|^2_{L^2_{V_{\mathcal{P}}}} + [u]^2_{s,A}\right)^{\frac{1}{2}}.$$

We observe that the energy functional $J^s_{A,V_{\mathcal{P}}}$ on $H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ associated to (2.9) is given by

$$J^s_{A,V_{\mathcal{P}}}(u) := \frac{1}{2}\|u\|^2_{s,A,V_{\mathcal{P}}} - \frac{1}{2\cdot 2^*_{\alpha,s}}D_s(u) - \frac{\lambda}{2p}B(u),$$

where

$$B(u) = \int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p \mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y$$

and

$$D_s(u) = \int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}} \mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y.$$

We affirm that $J^s_{A,V_{\mathcal{P}}}$ is well-defined for $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$. In fact if $t \in [(6-\alpha)/3, (6-\alpha)/(3-2s)]$ and $r = 6/(6-\alpha)$, then $2 \leq tr \leq 2^*_s$. So, for $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$, it follows from the immersion (2.3) that $u \in L^{tr}(\mathbb{R}^3,\mathbb{C})$, that is, $|u|^t \in L^r(\mathbb{R}^3,\mathbb{C})$. Since $\frac{2}{r} + \frac{\alpha}{3} = 2$, the Hardy-Littlewood-Sobolev inequality (see Appendix A, Proposition 49) yields

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^t|u(y)|^t}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \leq C(3,\alpha)\|u\|^{2t}_{tr}.$$

Therefore,

$$B(u) \leq C_1(3,\alpha)\|u\|^{2p}_{pr}$$

and

$$D_s(u) \leq C_2(3,\alpha)\|u\|^{2\cdot 2^*_{\alpha,s}}_{2^*_s}$$

for constants $C_1(3,\alpha)$ and $C_2(3,\alpha)$, and the affirmation follows. Moreover, by Lemma 2.5 of [47], $J_{A,V_{\mathcal{P}}} \in C^1(H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C});\mathbb{R})$.

Here $\frac{6-\alpha}{3}$ is called the lower critical exponent and $2^*_{\alpha,s} = \frac{6-\alpha}{3-2s}$ the upper critical exponent. This lead us to say that (2) is a critical nonlocal elliptic equation.

Observe that

$$\mathcal{S}^s_A = \inf_{u\,\in\, H^s_{A,V}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}}\frac{[u]^2_{s,A}}{D_s(u)^{\frac{3-2s}{6-\alpha}}}.$$

Definition 2.2.1 *A function $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ is a weak solution of (2.9) if*

$$\langle u,\psi\rangle_{s,A,V_{\mathcal{P}}} - \mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u\bar{\psi}\,dx - \lambda\,\mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u\bar{\psi}\,dx = 0$$

for all $\psi \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$.

Since the derivative of the energy functional $J^s_{A,V_{\mathcal{P}}}$ is given by

$$J^{s\prime}_{A,V_{\mathcal{P}}}(u)\cdot\varphi = \langle u,\varphi\rangle_{s,A,V_{\mathcal{P}}} - \mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha}*|u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u\bar{\varphi}\,\mathrm{d}x - \lambda\,\mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha}*|u|^{p)}\right)|u|^{p-2}u\bar{\varphi}\,\mathrm{d}x,$$

we see that critical points of $J^s_{s,A,V_{\mathcal{P}}}$ are weak solutions of (2.9).

Note that, if $\varphi = u$ we obtain

$$J^{s\prime}_{A,V_{\mathcal{P}}}(u)\cdot u := \|u\|^2_{s,A,V_{\mathcal{P}}} - D_s(u) - \lambda B(u).$$

Similarly to Lemma 9, Corollary 11, Lemma 12 and Lemma 13, respectively, we have the following results.

Lemma 34 *The functional $J^s_{A,V_{\mathcal{P}}}$ satisfies the mountain pass geometry. Precisely,*

(i) there exist $\rho,\delta>0$ such that $J^s_{A,V_{\mathcal{P}}}\big|_S \geq \delta > 0$ for any $u\in\mathcal{S}$, where

$$\mathcal{S} = \{u\in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\,:\,\|u\|_{A,s,V_{\mathcal{P}}} = \rho\};$$

(ii) for any $u_0\in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}$ there exists $\tau\in(0,\infty)$ such that $\|\tau u_0\|_{V_{\mathcal{P}}} > \rho$ and $J^s_{A,V_{\mathcal{P}}}(\tau u_0) < 0$.

The mountain pass theorem without the (PS)-condition (see [48, Theorem 1.15]) yields a Palais-Smale sequence $(u_n)\subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ such that

$$J^{s\prime}_{A,V_{\mathcal{P}}}(u_n)\to 0 \qquad\text{and}\qquad J^s_{A,V_{\mathcal{P}}}(u_n)\to c_{\lambda_s},$$

where

$$c_{\lambda_s} = \inf_{\alpha\in\Gamma}\max_{t\in[0,1]} J_{A,V_{\mathcal{P}}}(\gamma(t)),$$

and $\Gamma = \left\{\gamma\in C^1\left([0,1],H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\right)\,:\,\gamma(0)=0,\ J^s_{A,V_{\mathcal{P}}}(\gamma(1))<0\right\}$.

Lemma 35 *Suppose that $u_n\rightharpoonup u_0$ and consider*

$$B'(u_n)\cdot\psi = \mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha}*|u_n|^p\right)|u_n|^{p-2}u_n\bar{\psi}\,\mathrm{d}x$$

and

$$D'_s(u_n)\cdot\psi = \mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha}*|u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}-2}u_n\bar{\psi}\,\mathrm{d}x,$$

for $\psi\in C^\infty_c(\mathbb{R}^3,\mathbb{C})$. Then $B'(u_n)\cdot\psi\to B'(u_0)\cdot\psi$ and $D'_s(u_n)\cdot\psi\to D'_s(u_0)\cdot\psi$.

Lemma 36 *If $(u_n)\subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ is a $(PS)_b$ sequence for $J^s_{A,V_{\mathcal{P}}}$, then (u_n) is bounded. In addition, if $u_n\rightharpoonup u$ weakly in $H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ as $n\to\infty$, then u is a weak solution to problem (2.9).*

We now consider the Nehari manifold associated with the $J_{A,V_{\mathcal{P}}}$.

$$\mathcal{M}_{s,A,V_{\mathcal{P}}} = \left\{u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C}) \setminus \{0\} \,:\, \|u\|^2_{s,A,V_{\mathcal{P}}} = D_s(u) + \lambda B(u)\right\}.$$

Lemma 37 *There exists a unique $t_u = t_u(u) > 0$ such that $t_u u \in \mathcal{M}_{s,A,V_{\mathcal{P}}}$ for all $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\setminus \{0\}$ and $J^s_{A,V_{\mathcal{P}}}(t_u u) = \max_{t\geq 0} J^s_{A,V_{\mathcal{P}}}(tu)$. Moreover $c_{\lambda_s} = c^*_{\lambda_s} = c^{**}_{\lambda_s}$, where*

$$c^*_{\lambda_s} = \inf_{u\, \in\, \mathcal{M}_{s,A,V_{\mathcal{P}}}} J^s_{A,V_{\mathcal{P}}}(u) \quad \textit{and} \quad c^{**}_{\lambda_s} = \inf_{u\, \in\, H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}} \max_{t\geq 0} J^s_{A,V_{\mathcal{P}}}(tu).$$

Taking into account Lemma 37, we can now redefine a ground state solution.

Definition 2.2.2 *We say that $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ is a ground state solution for problem (2.9) if ${J^s}'_{A,V_{\mathcal{P}}}(u) = 0$ and $J^s_{A,V_{\mathcal{P}}}(u) = c_{\lambda_s}$, that is, if u is a solution to the equation ${J^s}'_{A,V_{\mathcal{P}}}(u) = 0$ which has minimal energy in the set of all nontrivial solutions.*

The following result controls the level c of a Palais-Smale sequence of $J^s_{A,V_{\mathcal{P}}}$.

Lemma 38 *Let $(u_n) \subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ a $(PS)_b$ sequence for ${J^s}_{A,V_{\mathcal{P}}}$ such that*

$$u_n \rightharpoonup 0 \quad \textit{weakly in } H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C}), \quad \textit{as } n\to\infty,$$

with

$$b < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}^s_A)^{\frac{6-\alpha}{3+2s-\alpha}}.$$

Then the sequence (u_n) verifies either

(i) $u_n \to 0$ strongly in $H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$, as $n\to\infty$,

or

(ii) There exists a sequence $(y_n) \subset \mathbb{R}^3$ and constants $r,\theta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \,\mathrm{d}x \geq \theta$$

where $B_r(y)$ denotes the ball in $\mathbb{R}^3$ of center at y and radius $r > 0$.

We now state our result about the periodic problem (2.9).

Theorem 39 *Under the hypotheses already stated on A and α, suppose that (V_1) is valid. Then problem (2.9) has at least one ground state solution if either*

(i) $s \in (\frac{3}{4},1)$, $\frac{7-2s-\alpha}{3-2s} < p < 2^*_{\alpha,s}$ and $\lambda > 0$;

(ii) $s \in (0,1)$, $\frac{6-\alpha}{3} < p \leq \frac{7-2s-\alpha}{3-2s}$ and λ sufficiently large.

Proof. Let c_{λ_s} be the mountain pass level and consider a sequence $(u_n) \subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3, \mathbb{C})$ such that

$$J^{s\prime}{}_{A,V_{\mathcal{P}}}(u_n) \to 0 \qquad \text{and} \qquad J^{s}{}_{A,V_{\mathcal{P}}}(u_n) \to c_{\lambda_s}.$$

Claim. We affirm that $c_{\lambda_s} < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}^s_A)^{\frac{6-\alpha}{3+2s-\alpha}}$, a result that will be shown after completing our proof, since it is very technical.

Lemma 36 guarantees that (u_n) is bounded. So, passing to a subsequence if necessary, there is $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3, \mathbb{C})$ such that

$$u_n \rightharpoonup u \ \text{ in } \ H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3, \mathbb{C}), \qquad u_n \to u \ \text{ in } \ L^2_{loc}(\mathbb{R}^3, \mathbb{C}) \qquad \text{and} \qquad u_n \to u \ \text{ a.e. } \ x \in \mathbb{R}^3.$$

If $u \neq 0$ we are done. If $u = 0$, it follows from Lemma 38 the existence of $r, \theta > 0$ and $(y_n) \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \, \mathrm{d}x \geq \theta. \tag{2.10}$$

We already know from the proof of Theorem 15 that we can assume that $(y_n) \subset \mathbb{Z}^3$.

Let

$$v_n(x) := u_n(x + y_n).$$

Since both $V_{\mathcal{P}}$ and A are $\mathbb{Z}^3$-periodic, we have

$$\|v_n\|_{s,A,V_{\mathcal{P}}} = \|u_n\|_{s,A,V_{\mathcal{P}}} \quad J^s_{A,V_{\mathcal{P}}}(v_n) = J^s_{A,V_{\mathcal{P}}}(u_n) \quad \text{and} \quad J^{s\prime}{}_{s,A,V_{\mathcal{P}}}(v_n) \to 0, \quad \text{as } \ n \to \infty.$$

Therefore there exists $v \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3, \mathbb{C})$ such that $v_n \rightharpoonup v$ weakly in $H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3, \mathbb{C})$ and $v_n \to v$ in $L^2_{loc}(\mathbb{R}^3, \mathbb{C})$.

We claim that $v \neq 0$. In fact, it follows from (2.10) that

$$0 < \theta \leq \|v_n\|_{L^2(B_r(0))} \leq \|v_n - v\|_{L^2(B_r(0))} + \|v\|_{L^2(B_r(0))}.$$

Since $v_n \to v$ in $L^2_{loc}(\mathbb{R}^3)$, we have $\|v_n - v\|_{L^2(B_r(0))} \to 0$ as $n \to \infty$, proving our claim.

But Lemma 35 guarantees that $J^{s\prime}{}_{A,V_{\mathcal{P}}}(v_n)\cdot\psi \to J^{s\prime}{}_{A,V_{\mathcal{P}}}(v)\cdot\psi$ and it follows that $J^{s\prime}{}_{A,V_{\mathcal{P}}}(v)\cdot\psi = 0$. Consequently, v is a weak solution of (2.9).

Since $v \in \mathcal{M}_{s,A,V_{\mathcal{P}}}$, of course we have $c^*_{\lambda_s} \leq J^s_{A,V_{\mathcal{P}}}(v)$, being $c^*_{\lambda_s}$ as in Lemma 37 . But, from Lemma 37, it follows that

$$\begin{aligned} c^*_{\lambda_s} = c_{\lambda_s} &= J^s_{A,V_{\mathcal{P}}}(v_n) - \frac{1}{2} J^{s\prime}{}_{A,V_{\mathcal{P}}}(v_n) \cdot v_n + o_n(1) \\ &= \lambda \left(\frac{1}{2} - \frac{1}{2p}\right) B(v_n) + \frac{3+2s-\alpha}{2(6-\alpha)} D_s(v_n) + o_n(1). \end{aligned}$$

Fatou's Lemma then guarantees that, as $n \to \infty$, we have

$$c^*_{\lambda_s} = c_{\lambda_s} \geq \lambda \left(\frac{1}{2} - \frac{1}{2p}\right) B(v) + \frac{3+2s-\alpha}{2(6-\alpha)} D_s(v) = J^s_{A,V_{\mathcal{P}}}(v)$$

that is, $J^s_{A,V_{\mathcal{P}}}(v) = c_{\lambda_s}$, and we are done. The same argument applies to the case $u \neq 0$ in (1.14).

□

We now prove the postponed Claim, that is, we show that $c_{\lambda_s} < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}_A^s)^{\frac{6-\alpha}{3+2s-\alpha}}$. Observe that, once proved the existence of u_ϵ as in our next result, then

$$0 < c_{\lambda_s} = \inf_{\alpha\in\Gamma}\max_{t\in[0,1]} J^s_{A,V_{\mathcal{P}}}(\gamma(t)) \le \sup_{t\ge 0} J^s_{A,V_{\mathcal{P}}}(tu_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}_A^s)^{\frac{6-\alpha}{3+2s-\alpha}}.$$

Lemma 40 *There exists u_ε such that*

$$\sup_{t\ge 0} J^s{}_{A,V_{\mathcal{P}}}(tu_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}_A^s)^{\frac{6-\alpha}{3+2s-\alpha}}$$

provided that

(*i*) $s \in (\frac{3}{4}, 1)$, $\frac{7-2s-\alpha}{3-2s} < p < 2^*_{\alpha,s}$ *and* $\lambda > 0$;

(*ii*) $s \in (0,1)$, $\frac{6-\alpha}{3} < p \le \frac{7-2s-\alpha}{3-2s}$ *and λ sufficiently large.*

Observe that the condition stated in this result is exactly the same of Theorem 4 and Theorem 39.

Proof. Following [30, Lemma 14] we see that $u(x) = 1/(1+|x|^2)^{(3-2s)/2}$ is a minimizer for $\mathcal{S}_0^s$. From [20, Theorem 1.1] it is known that it is also a minimizer for $\mathcal{S}_s$, where $\mathcal{S}_s = \inf_{u\in H^s(\mathbb{R}^3,\mathbb{R})\setminus\{0\}}[u]^2_{s,0}/\|u\|^2_{2^*_s}$.

If B_r denotes the ball in $\mathbb{R}^3$ of center at origin and radius r, consider the balls B_δ and $B_{2\delta}$ and let $\psi \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ be such that $0 \le \psi \le 1$ in $\mathbb{R}^3$ and

$$\psi \equiv 1 \quad \text{in} \quad B_\delta \quad \text{and} \quad \psi \equiv 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B_{2\delta}. \tag{2.11}$$

We consider

$$U_\varepsilon(x) := \varepsilon^{-(3-2s)/2} u^*\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad u_\varepsilon(x) := \psi(x)U_\varepsilon(x) \tag{2.12}$$

for $\varepsilon > 0$ and $x \in \mathbb{R}^3$, where $u^*(x) = \frac{u\left(\frac{x}{\mathcal{S}_s^{1/(2s)}}\right)}{\|u\|_{2^*_s}}$.

In the proof we apply the estimate

$$\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^{2^*_{\alpha,s}}|u_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{3-2s}{6-\alpha}} \ge \left(C(3,\alpha)^{\frac{3}{2s}}(\mathcal{S}_0^s)^{\frac{6-\alpha}{2s}} - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}} \tag{2.13}$$

which can be proved following [30, Section 3] or [42, Sections 2 and 4].

Following closely Guo and Melgaard [32, Lemmas 4.2 and 4.3] and Servadei and Valdinoci [45, Proposition 21] we have the following inequalities (See Apendix B, Lemma 61).

$$[u_\varepsilon]^2_{s,A} \le [U_\varepsilon]^2_{s,A} + O(\varepsilon^{3-2s})$$

and

$$[U_\varepsilon]^2_{s,A} \le [u^*]^2_{s,0} + O(\varepsilon^2).$$

that imply that

$$[u_\varepsilon]_{s,A}^2 \leq \left(C(3,\alpha)^{\frac{3-2s}{6-\alpha}}\mathcal{S}_0^s\right)^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + O(\varepsilon^2) \tag{2.14}$$

since $[u^*]_{s,0}^2 = \left(C(3,\alpha)^{\frac{3-2s}{6-\alpha}}\mathcal{S}_0^s\right)^{\frac{3}{2s}}$ (see (2.5) and (2.7)).

Case 1. $s \in (\frac{3}{4}, 1)$, $\frac{7-2s-\alpha}{3-2s} < p < 2^*_{\alpha,s}$ and $\lambda > 0$.

Proof of Case 1. Consider the function $f : [0,+\infty) \to \mathbb{R}$ defined by

$$f(t) = J^s_{A,V_\mathcal{P}}(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2_{s,A,V_\mathcal{P}} - \frac{t^{2\cdot 2^*_{\alpha,s}}}{2\cdot 2^*_{\alpha,s}}D_s(u_\varepsilon) - \frac{\lambda t^{2p}}{2p}B(u_\varepsilon).$$

The mountain pass geometry (Lemma 9) implies the existence of $t_\varepsilon > 0$ such that $\sup_{t\geq 0} J^s_{A,V_\mathcal{P}}(tu_\varepsilon) = J^s_{A,V_\mathcal{P}}(t_\varepsilon u_\varepsilon)$. Since $t_\varepsilon > 0$, $B(u_\varepsilon) > 0$ and $f'(t_\varepsilon) = 0$, we obtain

$$0 < t_\varepsilon < \left(\frac{\|u_\varepsilon\|^2_{s,A,V_\mathcal{P}}}{D_s(u_\varepsilon)}\right)^{\frac{1}{2(2^*_{\alpha,s}-1)}} := \mathcal{S}_A(\varepsilon),$$

thus implying

$$\|u_\varepsilon\|^2_{s,A,V_\mathcal{P}} = D_s(u_\varepsilon)\left(\mathcal{S}_A(\varepsilon)\right)^{2(2^*_{\alpha,s}-1)}. \tag{2.15}$$

Now define $g : [0, S_A(\varepsilon)] \to \mathbb{R}$ by

$$g(t) = \frac{t^2}{2}\|u_\varepsilon\|^2_{s,A,V_\mathcal{P}} - \frac{t^{2\cdot 2^*_{\alpha,s}}}{2\cdot 2^*_{\alpha,s}}D_s(u_\varepsilon).$$

So,

$$g(t) = \frac{t^2}{2}D_s(u_\varepsilon)\left(\mathcal{S}_A(\varepsilon)\right)^{2(2^*_{\alpha,s}-1)} - \frac{t^{2\cdot 2^*_{\alpha,s}}}{2\cdot 2^*_{\alpha,s}}D_s(u_\varepsilon).$$

Considering $t \in (0, S_A(\varepsilon))$, since $t > 0$ and $D_s(u_\varepsilon) > 0$, it follows that $g'(t) > 0$, and, consequently, g is increasing in the interval $[0, S_A(\varepsilon)]$. Thus,

$$0 < g(t_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}D_s(u_\varepsilon)(S_A(\varepsilon))^{2\cdot 2^*_{\alpha,s}}.$$

We conclude that

$$D_s(u_\varepsilon)(S_A(\varepsilon))^{2\cdot 2^*_{\alpha,s}} = \frac{\left(\|u_\varepsilon\|^2_{s,A,V_\mathcal{P}}\right)^{\frac{6-\alpha}{3+2s-\alpha}}}{D_s(u_\varepsilon)^{\frac{3-2s}{3+2s-\alpha}}}$$

and therefore

$$0 < g(t_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}\cdot\frac{\left(\|u_\varepsilon\|^2_{s,A,V_\mathcal{P}}\right)^{\frac{6-\alpha}{3+2s-\alpha}}}{D_s(u_\varepsilon)^{\frac{3-2s}{3+2s-\alpha}}}.$$

Since $J^s_{A,V_{\mathcal{P}}}(tu_\varepsilon) = g(t) - \frac{\lambda}{2p}t^{2p}B(u_\varepsilon)$, we have

$$J^s_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}\left(\frac{\|u_\varepsilon\|^2_{s,A,V_{\mathcal{P}}}}{D_s(u_\varepsilon)^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon).$$

But $\|u_\varepsilon\|^2_{s,A,V_{\mathcal{P}}} = \|u_\varepsilon\|^2_{L^2_{V_{\mathcal{P}}}} + [u_\varepsilon]^2_{s,A}$ implies

$$\frac{\|u_\varepsilon\|^2_{s,A,V_{\mathcal{P}}}}{D_s(u_\varepsilon)^{\frac{3-2s}{6-\alpha}}} = \frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x + \frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy$$

Therefore, we conclude that

$$J^s_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}\left(\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x))|^2\mathrm{d}x + \frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{6-\alpha}{3+2s-\alpha}} - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon).$$

Since, for all $\beta \geq 1$ and any $a, b \geq 0$ we have $(a+b)^\beta \leq a^\beta + \beta(a+b)^{\beta-1}b$, considering

$$a = \frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy,$$

$$b = \frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x$$

and $\beta = \frac{6-\alpha}{3+2s-\alpha}$, it follows

$$\begin{aligned} J^s_{A,V_{\mathcal{P}}}(t_\varepsilon v_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}\Bigg[&\left(\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{\frac{6-\alpha}{3+2s-\alpha}} \\ &+ \frac{6-\alpha}{3+2s-\alpha}\left(\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy\right. \\ &\left.+\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x\right)^{\frac{3-2s}{3+2s-\alpha}} \\ &\cdot\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x\Bigg] - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon). \end{aligned} \tag{2.16}$$

Taking into account (2.13) and (2.14), since $s \in (\frac{3}{4}, 1)$ we conclude that

$$
\begin{aligned}
\left(\frac{[u_\varepsilon]_{s,A}^2}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} &\leq \left(\frac{[u_\varepsilon]_{s,0}^2 + O(\varepsilon^{3-2s}) + O(\varepsilon^2)}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} \\
&= \left(\frac{[u_\varepsilon]_{s,0}^2 + O(\varepsilon^{3-2s})}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} \\
&\leq \left(\frac{(C(3,\alpha))^{\frac{3-2s}{6-\alpha}\cdot\frac{3}{2s}} \cdot (\mathcal{S}_0^s)^{\frac{3}{2s}} + O(\varepsilon^{3-2s})}{\left(C(3,\alpha)^{\frac{3}{2s}}(\mathcal{S}_0^s)^{\frac{6-\alpha}{2s}} - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}}.
\end{aligned}
$$

Similar to (1.21) and (1.22) we have

$$
\frac{(C(3,\alpha))^{\frac{3-2s}{6-\alpha}\cdot\frac{3}{2s}} \cdot (\mathcal{S}_0^s)^{\frac{3}{2s}} + O(\varepsilon^{3-2s})}{\left(C(3,\alpha)^{\frac{3}{2s}}(\mathcal{S}_0^s)^{\frac{6-\alpha}{2s}} - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}}} = \mathcal{S}_0^s \cdot \frac{1 + O(\varepsilon^{3-2s})}{\left(1 - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}}}
$$

and

$$
\left(\frac{1 + O(\varepsilon^{3-2s})}{\left(1 - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} < 1 + C(3,\alpha) \cdot \frac{O(\varepsilon^{3-2s}) + O(\varepsilon^{\frac{6-\alpha}{2}})}{\left(1 - O(\varepsilon^{\frac{6-\alpha}{2}})\right)^{\frac{3-2s}{6-\alpha}}}.
$$

Moreover, since for $\varepsilon > 0$ sufficiently small it holds

$$
\left(1 - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}} \geq \frac{1}{2},
$$

we get, for $\varepsilon > 0$ sufficiently small

$$
\begin{aligned}
\left(\frac{1 + O(\varepsilon^{3-2s})}{\left(1 - O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right)^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} &< 1 + 2C(3,\alpha)\left(O\left(\varepsilon^{3-2s}\right) + O\left(\varepsilon^{\frac{6-\alpha}{2}}\right)\right) \\
&< 1 + O\left(\varepsilon^{\min\{3-2s, \frac{6-\alpha}{2}\}}\right) = 1 + O\left(\varepsilon^{3-2s}\right),
\end{aligned}
$$

where the least equality follows from $0 < \alpha < 3$ and $s \in (\frac{3}{4}, 1)$.

Therefore, we conclude that, for any $\varepsilon > 0$ sufficiently small, we have

$$
\left(\frac{[u_\varepsilon]_{s,A}^2}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\right)^{\frac{6-\alpha}{3+2s-\alpha}} < (\mathcal{S}_0^s)^{\frac{6-\alpha}{3+2s-\alpha}} + O\left(\varepsilon^{3-2s}\right). \tag{2.17}
$$

Combining (2.16) with (2.17), for ε sufficiently small, we have

$$J^s_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}\left(\mathcal{S}^s_0\right)^{\frac{6-\alpha}{3+2s-\alpha}} + O\left(\varepsilon^{3-2s}\right) \tag{2.18}$$
$$+\frac{1}{2}\left(\frac{1}{D_s(u_\varepsilon)^{\frac{3-2s}{6-\alpha}}}[u_\varepsilon]^2_{s,A} + \frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x\right)^{\frac{3-2s}{3+2s-\alpha}}$$
$$\cdot\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x - \frac{\lambda}{2p}t_\varepsilon^{2p}B(u_\varepsilon).$$

We claim that there is a positive constant C_0 such that, for all $\varepsilon > 0$

$$t_\varepsilon^{2p} \geq C_0. \tag{2.19}$$

So, from Lemma 33, (2.15), (2.18) and (2.19) we conclude that, for some constant $C_0 > 0$ and $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned}
J^s_{A,V_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) &< \frac{3+2s-\alpha}{2(6-\alpha)}\left(\mathcal{S}^A_s\right)^{\frac{6-\alpha}{3+2s-\alpha}} + O\left(\varepsilon^{3-2s}\right)\\
&\quad+\frac{1}{2}\left(\frac{1}{D_s(u_\varepsilon)^{\frac{3-2s}{6-\alpha}}}\|u_\varepsilon\|^2_{s,A,V_{\mathcal{P}}}\right)^{\frac{3-2s}{3+2s-\alpha}}\cdot\frac{1}{(D_s(u_\varepsilon))^{\frac{3-2s}{6-\alpha}}}\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x - C_0B(u_\varepsilon)\\
&< \frac{3+2s-\alpha}{2(6-\alpha)}\left(\mathcal{S}^A_s\right)^{\frac{6-\alpha}{3+2s-\alpha}} \qquad (2.20)\\
&\quad+O\left(\varepsilon^{3-2s}\right) + \frac{(\mathcal{S}_A(\varepsilon))^2}{2}\cdot\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x - C_0B(u_\varepsilon).\\
&= \frac{3+2s-\alpha}{2(6-\alpha)}\left(\mathcal{S}^A_s\right)^{\frac{6-\alpha}{3+2s-\alpha}} + O(\varepsilon^{3-2s}) + C_1\int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|u_\varepsilon(x)|^2\mathrm{d}x - C_0B(u_\varepsilon),
\end{aligned}$$

where $C_1 = \frac{\mathcal{S}_A(\varepsilon)^2}{2}$.

By direct computation we know that, for $\varepsilon < 1$, since $\psi(x) = 0$ for all $x \in \mathbb{R}^3 \setminus B_{2\delta}$ and $\psi \equiv 1$ in B_δ, we have

$$B(u_\varepsilon) \geq C_3\varepsilon^{6-\alpha-(3-2s)p}.$$

where $C_3 := C(3, s, p)$.

Since $V_{\mathcal{P}}(x)$ is bounded, (2.20) and the last inequality imply that

$$J^s_{A,u_{\mathcal{P}}}(t_\varepsilon u_\varepsilon) < \frac{3+2s-\alpha}{2(3-\alpha)}\left(\mathcal{S}^s_A\right)^{\frac{6-\alpha}{3+2s-\alpha}} + O(\varepsilon^{3-2s}) + C_2\int_{\mathbb{R}^3}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p}. \tag{2.21}$$

We are going to see that

$$\lim_{\varepsilon\to 0}\varepsilon^{-3+2s}\left(C_2\int_{\mathbb{R}^3}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p}\right) = -\infty. \tag{2.22}$$

In order to do that, it suffices to show that

$$\lim_{\varepsilon\to 0}\varepsilon^{-3+2s}\left(C_2\int_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p}\right) = -\infty \tag{2.23}$$

and

$$C_2 \int_{B_{2\delta}\setminus B_\delta} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{6-\alpha-(3-2s)p} = O(\varepsilon^{3-2s}). \tag{2.24}$$

Since

$$O(\varepsilon^{3-2s}) + C_2 \int_{\mathbb{R}^3} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{6-\alpha-(3-2s)p} = \varepsilon^{3-2s} \left[\frac{O(\varepsilon^{3-2s})}{\varepsilon^{3-2s}} \right.$$
$$\left. + \varepsilon^{-3+2s} \left(C_2 \int_{\mathbb{R}^3} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{6-\alpha-(3-2s)p} \right) \right],$$

from (2.22) follows

$$O(\varepsilon^{3-2s}) + C_2 \int_{\mathbb{R}^3} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{6-\alpha-(3-2s)p} < 0 \tag{2.25}$$

for $\varepsilon > 0$ sufficiently small.

Thus, (2.21) and (2.25) imply

$$\sup_{t \geq 0} J^s_{A,V_{\mathcal{P}}}(tu_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)} \left(\mathcal{S}^A_s\right)^{\frac{6-\alpha}{3+2s-\alpha}}$$

for $\varepsilon > 0$ sufficiently small and fixed. Once (2.23) and (2.24) are verified, the proof of Case 1 is complete. □

We now prove (2.23).

Lemma 41 *If $s \in (\frac{3}{4}, 1)$, $\frac{7-2s-\alpha}{3-2s} < p < 2^*_{\alpha,s}$ and $\lambda > 0$, it follows that*

$$\lim_{\varepsilon \to 0} \varepsilon^{-3+2s} \left(C_2 \int_{B_\delta} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{6-\alpha-(3-2s)p} \right) = -\infty$$

Proof. We initially observe that direct computation allows us to conclude that

$$\int_{B_\delta} |u_\varepsilon(x)|^2 \mathrm{d}x = 3\omega_3 \mathcal{S}^{\frac{3(3-3s)}{4s}} \varepsilon^{2s} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^2}{(1+r^2)^{3-2s}} \mathrm{d}r, \tag{2.26}$$

where ω_3 denotes the volume of the unit ball in $\mathbb{R}^3$.

Now, define

$$\begin{aligned} I_\varepsilon := &\ \varepsilon^{-3+2s} \left(C_2 \int_{B_\delta} |u_\varepsilon(x)|^2 \mathrm{d}x - C_3 \varepsilon^{6-\alpha-(3-2s)p} \right) \\ = &\ \varepsilon^{-3+2s} \left(C_4 \varepsilon^{2s} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^2}{(1+r^2)^{3-2s}} \mathrm{d}r - C_3 \varepsilon^{6-\alpha-(3-2s)p} \right), \end{aligned}$$

the equality between integrals being a consequence of (2.26).

It is easy to show that

$$\varepsilon^{2s} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^2}{(1+r^2)^{3-2s}} \mathrm{d}r < \varepsilon^{2s} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^2}{1+r^2} \mathrm{d}r = \varepsilon^{2s-1} \left(\delta - \varepsilon \arctan\left(\frac{\delta}{\varepsilon} \right) \right).$$

Thus,

$$I_\varepsilon < C_4\varepsilon^{-4+4s}\left(\delta - \varepsilon \arctan\left(\frac{\delta}{\varepsilon}\right)\right) - C_3\varepsilon^{3+2s-\alpha-3p+2sp}.$$
$$= \varepsilon^{-4+4s}\left[C_4\left(\delta - \varepsilon \arctan\left(\frac{\delta}{\varepsilon}\right)\right) - C_3\varepsilon^{7-2s-\alpha-3p+2sp}\right].$$

We have $\frac{7-2s-\alpha}{3-2s} < p < 2^*_{\alpha,s}$ and then $7-2s-\alpha-3p+2sp<0$. Therefore, we conclude that $I_\varepsilon \to -\infty$. We are done. □

The proof of (2.24) follows in similar fashion to that of [16, Lemma 16].

Case 2. For λ sufficiently large, $s \in (0,1)$ and $\frac{6-\alpha}{3} < p \leq \frac{7-2s-\alpha}{3-2s}$.

Proof of Case 2. The proof follows in similar fashion to that of Lemma 16, case 2. □

2.2.2 The proof of Theorem 4

Maintaining the notation introduced in subsection 2.2.1, consider the energy functional $I^s_{A,V} : H^s_{A,V}(\mathbb{R}^3,\mathbb{C}) \to \mathbb{R}$ given by

$$I^s_{A,V}(u) = \frac{1}{2}\|u\|^2{}_{s,A,V} - \frac{1}{2\cdot 2^*_{\alpha,s}}D_s(u) - \frac{\lambda}{2p}B(u).$$

We denote by $\mathcal{N}_{s,A,V}$ the Nehari Manifold related to $I^s_{A,V}$, that is,

$$\mathcal{N}_{s,A,V} = \left\{u \in H^s_{A,V}(\mathbb{R}^3,\mathbb{C}) \setminus \{0\} \,:\, \|u\|^2_{s,A,V} = D_s(u) + \lambda B(u)\right\},$$

which is non-empty as a consequence of Theorem 39. As before, the functional $I^s_{A,V}$ satisfies the mountain pass geometry. Thus, there exists a sequence $(u_n) \subset H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$ such that

$$I^{s'}{}_{A,V}(u_n) \to 0 \qquad \text{and} \qquad I^s_{A,V}(u_n) \to d_{\lambda_s},$$

where d_{λ_s} is the minimax level, also characterized by

$$d_{\lambda_s} = \inf_{u\,\in H^s_{A,V}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}} \max_{t\geq 0} I^s_{A,V}(tu) = \inf_{\mathcal{N}^s_{A,V}} I^s_{A,V}(u) > 0.$$

The next lemma compares the levels d_{λ_s} and c_{λ_s}.

Lemma 42 *The levels d_{λ_s} and c_{λ_s} verify the inequality*

$$d_{\lambda_s} < c_{\lambda_s} < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}_A)^{\frac{6-\alpha}{3+2s-\alpha}}$$

for all $\lambda > 0$.

Proof. Firstly, we stress that, as a consequence of (V_2), we have $I^s_{A,V}(u) < J^s_{A,V_{\mathcal{P}}}(u)$ for all $u \in H^s_{A,V}(\mathbb{R}^3, \mathbb{C})$. In fact, we have

$$\|u\|^2_{s,A,V} = \|u\|^2_{L^2_V} + [u]^2_{s,A} = \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx + [u]^2_{s,A} = \int_{\mathbb{R}^3} [V_p(x) - W(x)]|u(x)|^2 dx + [u]^2_{s,A}$$

that implies

$$\int_{\mathbb{R}^3} W(x)|u(x)|^2 dx = \int_{\mathbb{R}^3} V_p(x)|u(x)|^2 dx - \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx < \infty.$$

Since $\int_{\mathbb{R}^3} W(x)|u(x)|^2 dx \geq 0$, we have

$$\int_{\mathbb{R}^3} V(x)|u(x)|^2 dx = \int_{\mathbb{R}^3} V_p(x)|u(x)|^2 dx - \int_{\mathbb{R}^3} W(x)|u(x)|^2 dx < \int_{\mathbb{R}^3} V_p(x)|u(x)|^2 dx.$$

So,

$$\begin{aligned} \|u\|^2_{s,A,V} &:= \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx + [u]^2_{s,A} \\ &< \int_{\mathbb{R}^3} V_p(x)|u|^2 dx + [u]^2_{s,A} \\ &= \|u\|^2_{s,A,V_{\mathcal{P}}}, \end{aligned}$$

and the affirmative follows.

Let u be the ground state solution of problem (2.9) and consider $\bar{t}_u > 0$ such that $\bar{t}_u u \in \mathcal{N}_{s,A,V}$, that is

$$0 < d_{\lambda_s} \leq \sup_{t \geq 0} I^s_{A,V}(tu) = I^s_{A,V}(\bar{t}_u u).$$

It follows from (V_2) that

$$0 < d_{\lambda_s} \leq I^s_{A,V}(\bar{t}_u u) < J^s_{A,V_{\mathcal{P}}}(\bar{t}_u u) \leq \sup_{t \geq 0} J^s_{A,V_{\mathcal{P}}}(tu) = J^s_{A,V_{\mathcal{P}}}(u) = c_{\lambda_s}.$$

Therefore,

$$d_{\lambda_s} < c_{\lambda_s}.$$

The second inequality was already known. □

Proof of Theorem 4. Let (u_n) be a $(PS)_{d_{\lambda_s}}$ sequence for $I^s_{A,V}$. As before, (u_n) is bounded in $H^s_{A,V}(\mathbb{R}^3, \mathbb{C})$. Thus, there exists $u \in H^s_{A,V}(\mathbb{R}^3, \mathbb{C})$ such that

$$u_n \rightharpoonup u \ \text{ in } \ H^s_{A,V}(\mathbb{R}^3, \mathbb{C}).$$

By the same arguments given in the proof of Theorem 39, u is a ground state solution of problem (8), if $u \neq 0$.

Following closely [3], we will show that $u = 0$ cannot occur. Indeed, Lemma 31 yields

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} W|u_n|^2 \, \mathrm{d}x = 0,$$

since $W \in L^{\frac{3}{2}}(\mathbb{R}^3, \mathbb{C})$ and $u_n \rightharpoonup 0$ in $H^s_{A,V}(\mathbb{R}^3, \mathbb{C})$. So,

$$|J^s_{A,V_\mathcal{P}}(u_n) - I^s_{A,V}(u_n)| = o_n(1)$$

showing that

$$J^s_{A,V_\mathcal{P}}(u_n) \to d_{\lambda_s}.$$

But, for $\varphi \in H^s_{A,V}(\mathbb{R}^3, \mathbb{C})$ such that $\|\varphi\|_{A,V} \leq 1$, we have

$$|({J^s}'_{A,V_\mathcal{P}}(u_n) - {I^s}'_{A,V}(u_n)) \cdot \varphi| \leq \left(\int_{\mathbb{R}^3} W|u_n|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = o_n(1).$$

Thus,

$${J^s}'_{A,V_\mathcal{P}}(u_n) = o_n(1)$$

Let $t_n > 0$ such that $t_n u_n \in \mathcal{M}_{s,A,V_\mathcal{P}}$. Mimicking the argument found in [2, 27, 44, 48], it follows that $t_n \to 1$ as $n \to \infty$. Therefore,

$$c_{\lambda_s} \leq J^s_{A,V_\mathcal{P}}(t_n u_n) = J^s_{A,V_\mathcal{P}}(u_n) + o_n(1) = d_{\lambda_s} + o_n(1).$$

Letting $n \to +\infty$, we get

$$c_{\lambda_s} \leq d_{\lambda_s}$$

obtaining a contradiction with Lemma 42. This completes the proof of Theorem 4.

2.3 The case $g(u) = |u|^{p-1}u$

2.3.1 The periodic problem

In this subsection we deal with problem (10) for $g(u)$ as above, that is,

$$(-\Delta)^s_A u + V_\mathcal{P}(x)u = \left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}} \right) |u|^{2^*_{\alpha,s}-2}u + \lambda|u|^{p-1}u, \tag{2.27}$$

where $1 < p < 2^*_s - 1$.

We observe that in this case the energy functional $J^s_{A,V_\mathcal{P}}$ is given by

$$J^s_{A,V_\mathcal{P}}(u) := \frac{1}{2}\|u\|^2_{s,A,V_\mathcal{P}} - \frac{1}{2 \cdot 2^*_\alpha} D_s(u) - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \mathrm{d}x,$$

where, as before

$$D_s(u) = \int_{\mathbb{R}^3} \left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}} \right) |u|^{2^*_{\alpha,s}} \, \mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)^{2^*_{\alpha,s}}||u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y.$$

By the Sobolev immersion (2.3) and the Hardy-Littlewood-Sobolev inequality (see Appendix A, Proposition 49), we have that $J^s_{A,V_\mathcal{P}}$ is well defined.

Definition 2.3.1 *A function $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ is a weak solution of (2.27) if*

$$\langle u,\varphi\rangle_{s,A,V_{\mathcal{P}}} - \mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha} * |u|^{2^*_{\alpha,s}}\right)|u|^{2^*_{\alpha,s}-2}u\bar{\psi}\,\mathrm{d}x - \lambda\,\mathfrak{Re}\int_{\mathbb{R}^3}|u|^{p-1}u\bar{\psi}\,\mathrm{d}x = 0$$

for all $\psi \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$.

As before, we see that critical points of $J^s_{A,V_{\mathcal{P}}}$ are weak solutions of (2.27) and

$$J^{s\prime}_{A,V_{\mathcal{P}}}(u)\cdot u := \|u\|^2_{s,A,V_{\mathcal{P}}} - D_s(u) - \lambda\|u\|^{p+1}_{p+1}.$$

We obtain that $J^s_{A,V_{\mathcal{P}}}$ satisfies the geometry of the mountain pass (see the proof of Lemma 34).

As in Section 2.2, the mountain pass theorem without the (PS)-condition yields a sequence $(u_n) \subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ such that

$$J^{s\prime}_{A,V_{\mathcal{P}}}(u_n) \to 0 \qquad \text{and} \qquad J^s_{A,V_{\mathcal{P}}}(u_n) \to c_{\lambda_s},$$

where

$$c_{\lambda_s} = \inf_{\alpha\in\Gamma}\max_{t\in[0,1]} J_{A,V_{\mathcal{P}}}(\gamma(t))$$

and

$$\Gamma = \left\{\gamma \in C^1\left([0,1], H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\right) \,:\, \gamma(0)=0,\ J^s_{A,V_{\mathcal{P}}}(\gamma(1))<0\right\}.$$

Considering the Nehari manifold associated with $J^s_{A,V_{\mathcal{P}}}$

$$\mathcal{M}_{s,A,V_{\mathcal{P}}} = \left\{u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\setminus\{0\} \,:\, \|u\|^2_{A,V_{\mathcal{P}}} = D_s(u) + \lambda\|u\|^{p+1}_{p+1}\right\},$$

by proceeding as in the proof of Lemma 13 we obtain

Lemma 43 *There exists a unique $t_u = t_u(u) > 0$ such that $t_u u \in \mathcal{M}_{s,A,V_{\mathcal{P}}}$ for all $u \in H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}$ and $J^s_{A,V_{\mathcal{P}}}(t_u u) = \max_{t\ge 0} J^s_{A,V_{\mathcal{P}}}(tu)$. Moreover $c_{\lambda_s} = c^*_{\lambda_s} = c^{**}_{\lambda_s}$, where*

$$c^*_{\lambda_s} = \inf_{u\,\in\,\mathcal{M}_{s,A,V_{\mathcal{P}}}} J^s_{A,V_{\mathcal{P}}}(u) \quad \textit{and} \quad c^{**}_{\lambda_s} = \inf_{u\,\in\,H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}} \max_{t\ge 0} J^s_{A,V_{\mathcal{P}}}(tu).$$

Lemma 44 *Suppose that $u_n \rightharpoonup u_0$ and consider*

$$B'(u_n)\cdot\psi = \mathfrak{Re}\int_{\mathbb{R}^3}|u|^{p-1}u\bar{\psi}\,\mathrm{d}x$$

and

$$D'(u_n)\cdot\psi = \mathfrak{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha} * |u_n|^{2^*_{\alpha,s}}\right)|u_n|^{2^*_{\alpha,s}-2}u_n\bar{\psi}\,\mathrm{d}x$$

for $\psi \in C^\infty_c(\mathbb{R}^3,\mathbb{C})$. Then $B'(u_n)\cdot\psi \to B'(u_0)\cdot\psi$ and $D'(u_n)\cdot\psi \to D'(u_0)\cdot\psi$ as $n\to\infty$.

Lemma 45 *If $(u_n) \subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ is a $(PS)_b$ sequence for $J^s_{A,V_{\mathcal{P}}}$, then (u_n) is bounded. In addition, if $u_n \rightharpoonup u$ weakly in $H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$, as $n\to\infty$, then u is ground state solution for problem (2.27).*

Lemma 46 *If* $(u_n) \subset H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C})$ *is a sequence* $(PS)_b$ *for* $J^s_{A,V_{\mathcal{P}}}$ *such that*

$$u_n \rightharpoonup 0 \quad \text{weakly in } H^s_{A,V_{\mathcal{P}}}(\mathbb{R}^3,\mathbb{C}) \quad \text{as } n\to\infty,$$

with

$$b < \frac{3+2s-\alpha}{2(6-\alpha)}\mathcal{S}_A^{\frac{6-\alpha}{3+2s-\alpha}},$$

then there exists a sequence $(y_n) \in \mathbb{R}^3$ *and constants* $R,\theta > 0$ *such that*

$$\limsup_{n\to\infty}\int_{B_r(y_n)} |u_n|^2 \,\mathrm{d}x \geq \theta,$$

where $B_r(y)$ *denotes the ball in* $\mathbb{R}^3$ *of center at* y *and radius* $r>0$.

The proof of Lemmas 44, 45 and 46 is similar to that of Lemma 35, Lemmas 12 and 38, respectively.

Lemma 47 *Let* $1 < p < 2^*_s - 1$ *and* u_ε *as defined in* (40). *Then, for* $s \in (\frac{3}{4}, 1)$, *there exists* ε *such that*

$$\sup_{t\geq 0} J^s_{A,V_{\mathcal{P}}}(tv_\varepsilon) < \frac{3+2s-\alpha}{2(6-\alpha)}(\mathcal{S}_A)^{\frac{6-\alpha}{3+2s-\alpha}}.$$

provided that either

(*i*) $\frac{6s-3}{3-2s} < p < 2^*_s$ *and* $\lambda > 0$;

(*ii*) $1 < p \leq \frac{6s-3}{3-2s}$ *and* λ *sufficiently large.*

Proof. Consider, for the cases (i) the function $f : [0,+\infty) \to \mathbb{R}$ defined by

$$f(t) = J^s_{A,V_{\mathcal{P}}}(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2_{s,A,V_{\mathcal{P}}} - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha}D_s(u_\varepsilon) - \frac{\lambda t^{p+1}}{p+1}\|u_\varepsilon\|^{p+1}_{p+1}$$

and proceed as in the proof of Case 1, Lemma 40.

In the case of $s \in (0,1)$, $1 < p \leq \frac{6s-3}{3-2s}$ and λ sufficiently large, consider $g_\lambda : [0,+\infty) \to \mathbb{R}$ defined by

$$g_\lambda(t) = J^s_{A,V_{\mathcal{P}}}(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2_{s,A,V_{\mathcal{P}}} - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha}D_s(u_\varepsilon) - \frac{\lambda t^{p+1}}{p+1}\|u_\varepsilon\|^{p+1}_{p+1}$$

and proceed as in the proof of Case 2, Lemma 40. □

Similar to the proof of Theorem 39, we now state our result about the periodic problem (2.27).

Theorem 48 *Under the hypotheses already stated on* A *and* α, *suppose that* (V_1) *is valid. Then, for* $s \in (\frac{3}{4}, 1)$, *problem* (39) *has at least one ground state solution if either*

(*i*) $\frac{6s-3}{3-2s} < p < 2^*_s - 1$ *and* $\lambda > 0$;

(*ii*) $1 < p \leq \frac{6s-3}{3-2s}$ *and* λ *sufficiently large.*

2.3.2 Proof of Theorem 5

Some arguments of this proof were adapted from the proof of Theorem 1.

Maintaining the notation already introduced, consider the functional $I^s_{A,V}: H^s_{A,V}(\mathbb{R}^3,\mathbb{C}) \to \mathbb{R}$ defined by

$$I^s_{A,V}(u) := \frac{1}{2}\|u\|^2{}_{s,A,V} - \frac{1}{2\cdot 2^*_{\alpha,s}} D_s(u) - \frac{\lambda}{p+1}\|u\|^{p+1}_{p+1}$$

for all $u \in H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$.

We denote by $\mathcal{N}_{s,A,V}$ the Nehari Manifold related to $I^s_{A,V}$, that is,

$$\mathcal{N}_{s,A,V} = \left\{ u \in H^s_{A,V}(\mathbb{R}^3,\mathbb{C}) \setminus \{0\} \,:\, \|u\|^2_{s,A,V} = D_s(u) + \lambda\|u\|^{p+1}_{p+1} \right\},$$

which is non-empty as a consequence of Theorem 48. As before, the functional $I^s_{A,V}$ satisfies the mountain pass geometry. Thus, there exists a $(PS)_{d_{\lambda_s}}$ sequence $(u_n) \subset H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$, that is, a sequence satisfying

$$I^{s\prime}{}_{A,V}(u_n) \to 0 \qquad \text{and} \qquad I^s_{A,V}(u_n) \to d_{\lambda_s},$$

where d_λ is the minimax level, also characterized by

$$d_{\lambda_s} = \inf_{u\, \in H^s_{A,V}(\mathbb{R}^3,\mathbb{C})\setminus\{0\}} \max_{t\geq 0} I^s_{A,V}(tu) = \inf_{\mathcal{N}_{s,A,V}} I^s_{A,V}(u) > 0.$$

As in the Section 2.2, we have $I^s_{A,V}(u) < J^s_{A,V_\mathcal{P}}(u)$ for all $u \in H^s_{A,V}(\mathbb{R}^3,\mathbb{C})$ as a consequence of (V_2).

Similar to the proof of Lemma 19 we have the following conclusion that shows an important inequality involving the levels d_{λ_s} and c_{λ_s}, which completes the proof of Theorem 4.

2.4 The case $g(u) = |u|^{2^*_s-2}u$

2.4.1 Proof of Theorem 6

The proof of Theorem 6 is analogous to the proof of Theorem 4.The principal distinction is that the $(PS)_{c_{\lambda_s}}$ condition holds true below the level $\frac{s}{3}\mathcal{S}_s^{\frac{3}{2}}$. It follows from [42, Proposition 4.1] that

$$[u_\varepsilon]^2_{s,0} = \mathcal{S}_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s})$$

and

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{2^*_s}\mathrm{d}x = \mathcal{S}_s^{\frac{3}{2s}} + O(\varepsilon^3).$$

So, we have

$$\sup_{t\geq 0} J^s_{A,V_\mathcal{P}}(t_\varepsilon u_\varepsilon) < \frac{s}{3}\mathcal{S}_s^{\frac{3}{2}} + O(\varepsilon^{3-2s}) + C_2\int_{\mathbb{R}^3}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p} < \frac{s}{3}\mathcal{S}_s^{\frac{3}{2}},$$

since

$$\lim_{\varepsilon\to 0}\varepsilon^{-(3-2s)}\left(C_2\int_{\mathbb{R}^3}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p}\right) = -\infty.$$

Observe that the last result is a consequence of

$$\lim_{\varepsilon\to 0}\varepsilon^{-(3-2s)}\left(C_2\int_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p}\right) = -\infty$$

and

$$C_2\int_{B_{2\delta}\setminus B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{6-\alpha-(3-2s)p} = O(\varepsilon^{3-2s}).$$

The rest of the proof is omitted here. □

Appendix A

Main results used in the work

In this appendix we gather some of the main results used in this work.

Proposition 49 (Hardy-Littlewood-Sobolev inequality) *Suppose that* $f \in L^t(\mathbb{R}^N)$ *and* $h \in L^r(\mathbb{R}^N)$ *for* $t, r > 1$ *and* $0 < \alpha < N$ *satisfying* $\frac{1}{t} + \frac{\alpha}{N} + \frac{1}{r} = 2$*. Then, there exists a sharp constant* $C(t, N, \alpha)$*, independent of* f *and* h*, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \leq C(t, N, \alpha)\|f\|_t \|h\|_r. \tag{A.1}$$

If $t = r = \frac{2N}{2N-\alpha}$*, then*

$$C(t, N, \alpha) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\alpha}{2})}{\Gamma(N - \frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{-1+\frac{\alpha}{N}}.$$

In this case there is equality in (A.1) *if and only if* $h = cf$ *for a constant* c *and*

$$f(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\alpha)/2}$$

for some $A \in \mathbb{C}$*,* $0 \neq \gamma \in \mathbb{R}$ *and* $a \in \mathbb{R}^N$*.*

Proof. See [35, Proposition 4.3]. □

We emphasize that inequality (A.1) is sometimes referred to as the weak Young inequality. Note that (A.1) looks almost like Young's inequality (see [35, Theorem 4.2]) with $g(x)$ replaced by $|x|^{-\alpha}$ in that inequality. This function is, however, not in any L^t-space but nevertheless we have an inequality analogous to Young's inequality.

Folowing [35], we introduce the weak L^q - space $L^q_w(\mathbb{R}^N, \mathbb{R})$.

Definition A.0.1 *For* $q > 1$*, we define the weak* L^q *- space* $L^q_w(\mathbb{R}^N, \mathbb{R})$ *as the space of all mensurable functions* u *such that*

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^N \ ; \ u(x) > \lambda\}|^{\frac{1}{q}} < \infty,$$

where $|\cdot|$ *denotes the Lebesgue measure on* $\mathbb{R}^N$*.*

We consider on $L^q_w(\mathbb{R}^N,\mathbb{R})$ the norm

$$\|u\|_{q,w} := \sup_A |A|^{-\frac{1}{q'}} \int_A |u(x)|\, dx,$$

where $\frac{1}{q}+\frac{1}{q'}=1$ and A denotes an arbitrary measurable set of measure $|A|<\infty$.

The following results hold.

Lemma 50 $L^q(\mathbb{R}^N) \subset L^q_w(\mathbb{R}^N)$.

Proof. For all $u \in L^q(\mathbb{R}^N)$ and $\lambda > 0$ we have

$$\begin{aligned}\|u\|_q^q \geq \int_{\{|u|>\lambda\}} |u(x)|^q\, dx \geq \int_{\{|u|>\lambda\}} \lambda^q\, dx =& \lambda^q |\{x\in\mathbb{R}^N\ ;\ |u(x)|>\lambda\}| \\ =& \big(\lambda |\{x\in\mathbb{R}^N\ ;\ |u(x)|>\lambda\}|^{\frac{1}{q}}\big)^q\end{aligned}$$

which implies

$$\lambda |\{x\in\mathbb{R}^N\ ;\ |u(x)|>\lambda\}|^{\frac{1}{q}} \leq \|u\|_q.$$

Consequently,

$$\sup_{\lambda>0} \lambda |\{x\in\mathbb{R}^N\ ;\ |u(x)|>\lambda\}|^{\frac{1}{q}} \leq \|u\|_q.$$

□

Lemma 51 *If* $u(x)=|x|^{-\alpha}$, $0<\alpha<N$, *then* $u\in L^q_w(\mathbb{R}^N)$ *with* $q=\frac{N}{\alpha}$ *and*

$$\|u\|_{\frac{N}{\alpha},w} = \frac{N}{N-\alpha}\left(\frac{|S^{N-1}|}{N}\right)^{\frac{\alpha}{N}} \tag{A.2}$$

where $|S^{N-1}|$, *which is equal to* $2\pi^{N/2}/\Gamma(N/2))$, *is the area of the unit sphere* $S^{N-1}\subset\mathbb{R}^N$.

Proof. Since the Lebesgue measure $|\cdot|$ of a ball $B_r(0)$ is

$$|B_r(0)| = \frac{1}{N}|S^{N-1}| r^N$$

(see [35, Section 1.2, eq.(8)]), we have

$$\begin{aligned}\lambda |\{x\in\mathbb{R}^N\ ;\ u(x)>\lambda\}|^{\frac{1}{q}} &= \lambda \left|\left\{x\in\mathbb{R}^N\ ;\ |x|^{-\alpha}>\lambda\right\}\right|^{\frac{\alpha}{N}} \\ &= \lambda \left|\left\{x\in\mathbb{R}^N\ ;\ |x|<\lambda^{-\frac{1}{\alpha}}\right\}\right|^{\frac{\alpha}{N}} \\ &= \lambda |B_{\lambda^{-\frac{1}{\alpha}}}(0)|^{\frac{\alpha}{N}} = \lambda\left(\frac{1}{N}|S^{N-1}|\lambda^{-\frac{N}{\alpha}}\right)^{\frac{\alpha}{N}} = \left(\frac{1}{N}|S^{N-1}|\right)^{\frac{\alpha}{N}},\end{aligned}$$

which implies that

$$\sup_{\lambda>0} \lambda |\{x\in\mathbb{R}^N\ ;\ u(x)>\lambda\}|^{\frac{1}{q}} = \left(\frac{1}{N}|S^{N-1}|\right)^{\frac{\alpha}{N}} < \infty.$$

This together with the measurability of u implies that $u \in L^q_w(\mathbb{R}^N)$, with proves the first part of lemma.

By defintion, we have

$$\|u\|_{q,w} = \sup_A |A|^{-\frac{1}{q'}} \int_A |x|^{-\alpha}\, dx = \sup_A |A|^{\frac{\alpha}{N}-1} \int_A |x|^{-\alpha}\, dx.$$

Considering $A = B_1(0)$ it hold

$$|A| = \frac{|S^{N-1}|}{N} \tag{A.3}$$

and

$$\int_A |x|^{-\alpha}\, dx = |S^{N-1}| \int_0^1 r^{N-\alpha-1}\, dr = \frac{|S^{N-1}|}{N-\alpha}.$$

So,

$$|A|^{\frac{\alpha}{N}-1} \int_A |x|^{\alpha}\, dx = \left(\frac{|S^{N-1}|}{N}\right)^{\frac{N}{\alpha}-1} \frac{|S^{N-1}|}{N-\alpha} = \frac{N}{N-\alpha}\left(\frac{|S^{N-1}|}{N}\right)^{\frac{\alpha}{N}}.$$

Now, we are going to show that for arbitrary measurable set of measure $|A| < \infty$ it holds

$$|A|^{\frac{\alpha}{N}-1} \int_A |x|^{-\alpha}\, dx \le \frac{N}{N-\alpha}\left(\frac{|S^{N-1}|}{N}\right)^{\frac{\alpha}{N}}. \tag{A.4}$$

Let A be a measurable set of measure $|A| < \infty$ and $R > 0$ such that $|A| = |B_R(0)|$, that is,

$$R = \left(\frac{N|A|}{|S^{N-1}|}\right)^{\frac{1}{N}}. \tag{A.5}$$

Since

$$|B_R(0)| = |A \cap B_R(0)| + |A^c \cap B_R(0)|$$

and

$$|A| = |A \cap B_R(0)| + |A \cap B_R(0)^c|$$

it follows that

$$|A^c \cap B_R(0)| = |A \cap B_R(0)^c|. \tag{A.6}$$

So, from (A.6), (A.5) and (A.3) we have

$$\begin{aligned}
\int_A |x|^{-\alpha}\, dx &= \int_{A\cap B_R(0)} |x|^{-\alpha}\, dx + \int_{A\cap B_R(0)^c} |x|^{-\alpha}\, dx \le \int_{A\cap B_R(0)} |x|^{-\alpha}\, dx + \int_{A\cap B_R(0)^c} R^{-\alpha}\, dx \\
&= \int_{A\cap B_R(0)} |x|^{-\alpha}\, dx + \int_{A^c\cap B_R(0)} R^{-\alpha}\, dx \le \int_{A\cap B_R(0)} |x|^{-\alpha}\, dx + \int_{A^c\cap B_R(0)} |x|^{-\alpha}\, dx \\
&= \int_{B_R(0)} |x|^{-\alpha}\, dx = |S^{N-1}| \int_0^R t^{N-\alpha-1}\, dt = \frac{|S^{N-1}|}{N-\alpha} R^{N-\alpha} \\
&= \frac{|S^{N-1}|}{N-\alpha}\left(\frac{N|A|}{|S^{N-1}|}\right)^{1-\frac{\alpha}{N}} = \frac{N}{N-\alpha}|A|^{1-\frac{\alpha}{N}}\left(\frac{|S^{N-1}|}{N}\right)^{\frac{\alpha}{N}}
\end{aligned}$$

which clearly proves (A.4). Therefore, lemma is proved. □

Similar to Young's Inequality (see [35, Theorem 4.2]) we have the following Weak Young Inequality, whose proof can be found in [36].

Proposition 52 *For $f \in L^t(\mathbb{R}^N)$, $g \in L^q_w(\mathbb{R}^N)$, $h \in L^r(\mathbb{R}^N)$ with $1 < t, q, r < \infty$ with $\frac{1}{t}+\frac{1}{q}+\frac{1}{r} = 2$ it holds*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)g(x-y)h(y)dxdy \right| \leq K_{t,q,r,N} \|f\|_t \|g\|_{q,w} \|h\|_r \tag{A.7}$$

for some number $K_{t,q,r,N}$.

It's possible to show that the sharp constant is given by

$$K_{t,q,r,N} = (1/q')(N/|S^{N-1}|)^{1/q} C(N, N/q, t) \tag{A.8}$$

(see [35, Pag. 107]). Moreover, using Holder's Inequality, it can be shown that the best choice for h (up to a constant) when f and g are given in Proposition 52 is

$$h(x) = e^{-i\theta(x)} |(g * f)(x)|^{r'/r}, \tag{A.9}$$

where $\theta(x)$ is defined by $g * f = e^{i\theta}|g * f|$.

The following result tell us we can also view the HLS inequality as the statement that convolution is a bounded map from $L^t(\mathbb{R}^N) \times L^q_w(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$.

Proposition 53 *Let $f \in L^t(\mathbb{R}^N)$ and $g \in L^q_w(\mathbb{R}^N)$ be. Then*

$$\|g * f\|_s \leq \frac{1}{q'} \left(\frac{N}{|S^{N-1}|} \right)^{\frac{1}{q}} C\left(N, \frac{N}{q}, t\right) \|g\|_{q,w} \|f\|_t \tag{A.10}$$

with $\frac{1}{t} + \frac{1}{q} = 1 + \frac{1}{s}$.

Proof. Replacing function h given in (A.9) into (A.7) ans using (A.8) we have

$$\int_{\mathbb{R}^N} |g * f|^{r'} \leq (1/q')(N/|S^{N-1}|)^{1/q} C(N, N/q, t) \|f\|_t \|g\|_{q,w} \|g * f\|_{r'}^{r'/r},$$

which implies that

$$\|g * f\|_{r'} \leq (1/q')(N/|S^{N-1}|)^{1/q} C(N, N/q, t) \|f\|_t \|g\|_{q,w}.$$

Making $r' = s$ we have

$$\|g * f\|_s \leq (1/q')(N/|S^{N-1}|)^{1/q} C(N, N/q, t) \|f\|_t \|g\|_{q,w}$$

whit $\frac{1}{t} + \frac{1}{q} = 1 + \frac{1}{s}$ and lemma is proved. □

Lemma 54 *For all $w \in L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$*

$$\frac{1}{|x|^\alpha} * w \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N).$$

Proof.

Considering $t = \frac{2N}{2N-\alpha}$, $q = \frac{N}{\alpha}$, $g(x) = |x|^{-\alpha}$ and $f = w \in L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ into (A.10), and taking into account that

$$\frac{1}{t} + \frac{1}{q} = 1 + \frac{1}{r} \Leftrightarrow \frac{2N-\alpha}{2N} + \frac{2\alpha}{2N} - \frac{2N}{2N} = \frac{1}{r} \Leftrightarrow \frac{\alpha}{2N} = \frac{1}{r} \Leftrightarrow r = \frac{2N}{\alpha},$$

$$\frac{1}{q} + \frac{1}{q'} = 1 \Leftrightarrow \frac{\alpha}{N} + \frac{1}{q'} = \frac{N}{N} \Leftrightarrow \frac{1}{q'} = \frac{N-\alpha}{N} \Leftrightarrow q' = \frac{N}{N-\alpha},$$

and

$$C(N, \frac{N}{q}, t) = C(N, \alpha, 2N/(2N-\alpha)) = C(N, \alpha)$$

we have

$$\begin{aligned}\left\| \frac{1}{|x|^\alpha} * w \right\|_{\frac{2N}{\alpha}} &\le \frac{N-\alpha}{N} \left(\frac{N}{|S^{N-1}|} \right)^{\frac{\alpha}{N}} C(N,\alpha) \| |x|^{-\alpha} \|_{\frac{N}{\alpha}, w} \|w\|_{\frac{2N}{2N-\alpha}} \\ &= \frac{N-\alpha}{N} \left(\frac{N}{|S^{N-1}|} \right)^{\frac{\alpha}{N}} C(N,\alpha) \frac{N}{N-\alpha} \left(\frac{|S^{n-1}|}{N} \right)^{\frac{\alpha}{N}} \|w\|_{\frac{2N}{2N-\alpha}}\end{aligned}$$

where , in the last equality, we used (A.2).
Therefore, we have

$$\left\| \frac{1}{|x|^\alpha} * w \right\|_{\frac{2N}{\alpha}} \le C(N,\alpha) \|w\|_{\frac{2N}{2N-\alpha}},$$

which proves the lemma. □

Lemma 55 *For all $x \in \mathbb{R}$ it holds*

$$1 - \cos x \le \frac{1}{2}|x|^2.$$

Proof. If $x = 0$ it is immediate.

If $x > 0$ we know that $\sin x < x$ and $|x| = x$. In this case, since obviously $\frac{x}{2} > 0$ we have

$$1 - \cos\, x = 2\sin^2 \left(\frac{x}{2}\right) < 2\left(\frac{x}{2}\right)^2 = \frac{1}{2}x^2 = \frac{1}{2}|x|^2$$

and the inequality holds.

If $x < 0$ we have obviously $|x| = -x$ and $-x > 0$, consequently, $\frac{-x}{2} > 0$. So, by using previous case and the cosine function parity, we have

$$1 - \cos\, x = 1 - \cos(-x) = 2\sin^2 \left(\frac{-x}{2}\right) < 2\left(\frac{-x}{2}\right)^2 = \frac{1}{2}(-x)^2 = \frac{1}{2}|x|^2$$

Therefore, the lemma is proved.] □

Proposition 56 *Le X, Y be normed linear spaces and let A be a linear operator from X into Y. Then*

$$x_n \rightharpoonup x \Rightarrow Ax_n \rightharpoonup Ax.$$

Proof. See [24, Proposition 2.1.27]. □

Appendix B

Statements used in proofs of the main results of work

In this appendix we justify some of the facts used in this work.

B.1 Proof of the statements (1.21) and (1.22)

We have that

$$
\begin{aligned}
&\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}+O(\varepsilon^{N-2})}{\left(C(N,\alpha)^{\frac{N}{2}}(S_{H,L})^{\frac{2N-\alpha}{2}}-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\\
&=\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}+(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}O(\varepsilon^{N-2})}{\left(C(N,\alpha)^{\frac{N}{2}}(S_{H,L})^{\frac{2N-\alpha}{2}}-(C(N,\alpha))^{\frac{N}{2}}(S_{H,L})^{\frac{2N-\alpha}{2}}O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\\
&=\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}\left[1+O(\varepsilon^{N-2})\right]}{\left(C(N,\alpha)^{\frac{N}{2}}(S_{H,L})^{\frac{2N-\alpha}{2}}\left(1-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)\right)^{\frac{N-2}{2N-\alpha}}}\\
&=\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}\left[1+O(\varepsilon^{N-2})\right]}{(C(N,\alpha))^{\frac{N}{2}\cdot\frac{N-2}{2N-\alpha}}(S_{H,L})^{\frac{2N-\alpha}{2}\cdot\frac{N-2}{2N-\alpha}}\left[1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right]^{\frac{N-2}{2N-\alpha}}}\\
&=S_{H,L}\cdot\frac{1+O(\varepsilon^{N-2})}{\left[1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right]^{\frac{N-2}{2N-\alpha}}},
\end{aligned}
$$

that is,,

$$
\left(\frac{(C(N,\alpha))^{\frac{N-2}{2N-\alpha}\cdot\frac{N}{2}}(S_{H,L})^{\frac{N}{2}}+O(\varepsilon^{N-2})}{\left(C(N,\alpha)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\alpha}{2}}-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}}=(S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}}\cdot\left(\frac{1+O(\varepsilon^{N-2})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}}.
$$

On the other hand,

$$\begin{aligned}\frac{1+O(\varepsilon^{N-2})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}} &= \frac{(1-O(\varepsilon^{\frac{2N-\alpha}{2}}))+O\left(\varepsilon^{N-2}\right)+O(\varepsilon^{\frac{2N-\alpha}{2}})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}}\\ &= (1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{1-\frac{N-2}{2N-\alpha}}+\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{(1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{\frac{N-2}{2N-\alpha}}}\\ &= (1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{\frac{N+2-\alpha}{2N-\alpha}}+\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{(1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{\frac{N-2}{2N-\alpha}}}\end{aligned}$$

which implies that

$$\begin{aligned}&\left(\frac{1+O(\varepsilon^{N-2})}{\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2N-\alpha}{N+2-\alpha}}\\ &= \left[(1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{\frac{N+2-\alpha}{2N-\alpha}}+\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{(1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{\frac{N-2}{2N-\alpha}}}\right]^{\frac{2N-\alpha}{N+2-\alpha}}\\ &\le 1-O(\varepsilon^{\frac{2N-\alpha}{2}})\\ &+\frac{2N-\alpha}{N+2-\alpha}\left((1-O(\varepsilon^{\frac{2N-\alpha}{2}}))^{\frac{N+2-\alpha}{2N-\alpha}}+\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{\left(1-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{N-2}{N+2-\alpha}}\cdot\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{\left(1-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\\ &< 1+C(N,\alpha)\cdot\frac{O(\varepsilon^{N-2})+O(\varepsilon^{\frac{2N-\alpha}{2}})}{\left(1-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\end{aligned}$$

where in the penultimate inequality we use the Mean Value Theorem considering

$$a=\left(1-O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{N+2-\alpha}{2N-\alpha}},\qquad b=\frac{O(\varepsilon^{N-2})+O\left(\varepsilon^{\frac{2N-\alpha}{2}}\right)}{\left(1-O(\varepsilon^{\frac{2N-\alpha}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\qquad\text{and}\qquad\beta=\frac{2N-\alpha}{N+2-\alpha}.$$

Therefore, (1.21) and (1.22) are proved.

B.2 Proof of the statement (1.33)

If $u=0$, then $u_n\rightharpoonup 0$ on $H^1_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $u_n\to 0$ a.e $x\in\mathbb{R}^N$, which implies that

$$|u_n|^2\to 0\ \text{ a.e }\ x\in\mathbb{R}^N.$$

Thanks to continuous immersion $H^1_{A,V}(\mathbb{R}^N,\mathbb{C}) \hookrightarrow L^{2^*}(\mathbb{R}^N,\mathbb{C})$ we have $(u_n) \in L^{2^*}(\mathbb{R}^N,\mathbb{C})$, what implies that $|u_n|^2 \in L^{\frac{N}{3-2}}(\mathbb{R}^N,\mathbb{C})$. Moreover, we also have that $|u_n|^2$ is bounded on $L^{\frac{N}{N-2}}(\mathbb{R}^N,\mathbb{C})$. So, Lemma 8 it follows that

$$|u_n|^2 u_n \rightharpoonup 0 \ \text{ in } \ L^{\frac{N}{N-2}}(\mathbb{R}^N,\mathbb{C}),$$

what implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \Phi |u_n|^2 \, dx = 0$$

for all $\Phi \in L^{\left(\frac{N}{N-2}\right)'}(\mathbb{R}^N,\mathbb{C}) = L^{\frac{N}{2}}(\mathbb{R}^N,\mathbb{C})$. In particular, since $W \in L^{\frac{N}{2}}(\mathbb{R}^N,\mathbb{C})$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W |u_n|^2 \, dx = 0.$$

□

B.3 Proof of the statement (1.34)

Consider $t_n > 0$ such that $t_n u_n \in \mathcal{M}_{A,V}$. So

$$t_n^2 \|u_n\|^2_{A,V} = \lambda t_n^{2p} B(u_n) + t_n^{2\cdot 2^*_\alpha} D(u_n)$$

equivalently,

$$\|u_n\|^2_{A,V} = \lambda t_n^{2(p-1)} B(u_n) + t_n^{2(2^*_\alpha - 1)} D(u_n), \ \ \forall\, n \in \mathbb{N}. \tag{B.1}$$

we also have

$$o_n(1) = J'_{A,V}(u_n) \cdot u_n = \|u_n\|^2_{A,V} - \lambda B(u_n) - D(u_n),$$

that is,

$$\|u_n\|^2_{A,V} = \lambda B(u_n) + D(u_n) + o_n(1), \ \forall\, n \in \mathbb{N}. \tag{B.2}$$

Firstly, we are going to show that
a) $t_n \not\to 0$ as $n \to \infty$.
b) $\limsup\limits_{n\to\infty} t_n \leq 1$.

In fact, suppose that $t_n \to 0$ as $n \to \infty$. Since the sequences $(B(u_n))$ and $(D(u_n))$ are bounded, it follows from (B.1) that $\|u_n\|_{A,V} \to 0$ as $n \to \infty$. Since $J_{A,V}(u_n) \leq \frac{1}{2}\|u_n\|^2_{A,V}$, we deduce that

$$0 < c_\lambda \leq 0,$$

which is a contradiction. Therefore,

$$t_n \not\to 0 \ \text{ as } \ n \to \infty,$$

which proves a).

Now, suppose that there exists a subsequence of (t_n), still denoted by (t_n), such that $t_n \geq 1+\delta$, for all $n \in \mathbb{N}$ and for some $\delta > 0$. From (B.1) and (B.2) we have

$$0 = \lambda(t_n^{2(p-1)} - 1) B(u_n) + (t_n^{2(2^*_\alpha - 1)} - 1) D(u_n) + o_n(1),$$

that is,

$$o_n(1) = \lambda(t_n^{2(p-1)} - 1)B(u_n) + (t_n^{2(2_\alpha*-1)} - 1)D(u_n). \tag{B.3}$$

Consider $f : [0, +\infty] \to \mathbb{R}$ defined by

$$f(t) = t^\mu, \mu > 0$$

Since f is increasing , then

$$t_n \geq 1 + \delta > 1 \Rightarrow f(t_n) > f(1).$$

Taking $\mu = 2(p-1)$, we have

$$t_n^{2(p-1)} \geq (1+\delta)^{2(p-1)} > 1,$$

that is,

$$t_n^{2(p-1)} - 1 \geq (1+\delta)^{2(p-1)} - 1 > 0.$$

Similarly,

$$t_n^{2(2_\alpha^*-1)} - 1 \geq (1+\delta)^{2(2_\alpha^*-1)} - 1 > 0.$$

From (B.3) and $t_n \geq 1 + \delta$, we obtain

$$o_n(1) > \lambda[(1+\delta)^{2(p-1)} - 1]B(u_n) + [(1+\delta)^{2(2_\alpha^*-1)} - 1]D(u_n), \tag{B.4}$$

for all $n \in \mathbb{N}$ e $u_n \in H^1_{A,V}(R^N, \mathbb{C}) \setminus \{0\}$. From (B.4) we see that

$$0 < \lambda[(1+\delta)^{2(p-1)} - 1]B(u_n) < o_n(1)$$

and

$$0 < \lambda[(1+\delta)^{2(2_\alpha^*-1)} - 1]D(u_n) < o_n(1),$$

So, $B(u_n) \to 0$ e $D(u_n) \to 0$ as $n \to \infty$.

By using this last statement and (B.2), we see that $\|u_n\|_{A,V} \to 0$; this implies that

$$c_\lambda \leq 0,$$

what does not hold. Therefore, b) turns out. From a) and b), we conclude that (t_n) is bounded and, going on a subsequence if necessary, $t_n \to t_0$, with $t_0 \in (0, 1]$.

Now we will prove that $0 < t_0 < 1$ cannot occur. In fact, suppose $0 < t_0 < 1$. Since (u_n) is bounded, from (1.4) and (1.5) we have that $(B(u_n))$ and $(D(u_n))$ are bounded on $\mathbb{R}$. So, going to a subsequence if necessary, $B(u_n) \to l_1$ and $D(u_n) \to l_2$, with $l_1 \geq 0$ and $l_2 \geq 0$. Notice that l_1 e l_2 are not both null, otherwise we would have $\|u_n\|_{A,V} \to 0$ and, consequently, $c_\lambda \leq 0$, what is a contradiction. Taking the limit on (B.3), as $n \to \infty$, we obtain

$$0 = \lambda(t_0^{2(p-1)} - 1)l_1 + \left(t_0^{2(2_\alpha^*-1)} - 1\right)l_2 < 0,$$

which is a contradiction. Therefore, $t_0 = 1$, that is,

$$t_n \to 1.$$

□

Lemma 57 *We have*

$$c_\lambda = c_\lambda^* = c_\lambda^{**}$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{A,V_{\mathcal{P}}}(\gamma(t)),$$

$$\Gamma = \left\{\gamma \in C^1\left([0,1], H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})\right) \,:\, \gamma(0)=0,\, \gamma(1)<0\right\},$$

$$c_\lambda^* := \inf_{u\,\in\,\mathcal{M}_{A,V_{\mathcal{P}}}} J_{A,V_{\mathcal{P}}}(u),$$

and

$$c_\lambda^{**} := \inf_{u\,\in\,H^1_{A,V_{\mathcal{P}}}(\mathbb{R}^N,\mathbb{C})\backslash\{0\}} \max_{t\geq 0} J_{A,V_{\mathcal{P}}}(tu).$$

Proof. From the first part of the lemma 13 we have

$$\max_{t\geq 0} J_{A,V_{\mathcal{P}}}(tu) = J_{A,V_{\mathcal{P}}}(t_u u).$$

In this way, it follows that

$$c_\lambda^{**} = c_\lambda^*$$

since $t_u u \in \mathcal{M}_{A,V}$.

Consider $u \in \mathcal{M}_{A,V}$ and

$$g(t) := J_{A,V_{\mathcal{P}}}(tu).$$

We have

$$\begin{aligned} g(t) &= \frac{1}{2}\|tu\|^2_{A,V_{\mathcal{P}}} - \frac{\lambda}{2p}B(tu) - \frac{1}{2\cdot 2^*_\alpha}D(tu) \\ &= \frac{t^2}{2}\|u\|^2_{A,V_{\mathcal{P}}} - \frac{\lambda t^{2p}}{2p}B(u) - \frac{t^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha}D(u). \end{aligned}$$

So, since $u \in \mathcal{M}_{A,V_{\mathcal{P}}}$, we have

$$\begin{aligned} g'(t) &= t\|u\|^2_{A,V_{\mathcal{P}}} - \lambda t^{2p-1}B(u) - t^{2\cdot 2^*_\alpha - 1}D(u) \\ &= t[\lambda B(u) + D(u)] - \lambda t^{2p-1}B(u) - t^{2\cdot 2^*_\alpha - 1}D(u) \\ &= t\lambda B(u) + tD(u) - \lambda t^{2p-1}B(u) - t^{2\cdot 2^*_\alpha - 1}D(u) \\ &= \left(t\lambda - \lambda t^{2p-1}\right)B(u) + \left(t - t^{2\cdot 2^*_\alpha - 1}\right)D(u). \end{aligned}$$

Moreover, since $u \in \mathcal{M}_{A,V_{\mathcal{P}}}$, it follows that $u \neq 0$ and, consequently, $B(u) \neq 0$ and $D(u) \neq 0$.

We also have

- $g'(1) = 0$
- $g'(t) > 0$ if $t \in (0,1)$ and $g'(t) < 0$ if $t > 1$.

So, the function g has a single maximum point that is reached in $t = 1$. Then

$$\max_{t\geq 0} J_{A,V_{\mathcal{P}}}(tu) = J_{A,V_{\mathcal{P}}}(u). \tag{B.5}$$

Choose $t_0 \in \mathbb{R}$ and $\bar{u} = t_0 u$ such that $J_{A,V_{\mathcal{P}}}(\bar{u}) < 0$. (Observe that Lemma (9) guarantees that $J_{A,V}(tu) < 0$ for t sufficiently large). So, $\bar{\gamma}(t) = t\bar{u} \in \Gamma$, what implies, from (B.5), that

$$\begin{aligned} c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\,\in\,[0,1]} J_{A,V_{\mathcal{P}}}(\gamma(t)) &\leq \max_{t\,\in\,[0,1]} J_{A,V_{\mathcal{P}}}(\bar{\gamma}(t)) \\ &= \max_{t\,\in\,[0,1]} J_{A,V_{\mathcal{P}}}(t\bar{u}) \\ &= \max_{t\,\in\,[0,1]} J_{A,V_{\mathcal{P}}}(tt_0 u) \\ &\leq \max_{t\geq 0} J_{A,V_{\mathcal{P}}}(tu) \\ &= J_{A,V_{\mathcal{P}}}(u). \end{aligned}$$

consequently,

$$c_\lambda \leq c_\lambda^*.$$

In the following we are going to show the reverse inequality.

Let (u_n) a sequence a $(PS)_{c_\lambda}$ satisfying (1.8). Since (u_n) is bounded (see Lemma 12) we conclude that $J'_{A,V_{\mathcal{P}}}(u_n)u_n \to 0$, as $n \to \infty$, moreover, from (B.5), for each $n \in \mathbb{N}$ there exists a single $t_n \in \mathbb{R}^+$ such that $J'_{A,V_{\mathcal{P}}}(t_n u_n)t_n u_n = 0, \forall\, n$, that is, $t_n u_n \in \mathcal{M}_{A,V}$. Consequently, we have

$$\|u_n\|^2_{A,V_{\mathcal{P}}} = \lambda t_n^{2(p-1)} B(u_n) + t_n^{2(2^*_\alpha - 1)} D(u_n),\ \forall\, n. \tag{B.6}$$

We claim that t_n does not converge to 0; otherwise, from boundedness of $B(u_n)$ and $D(u_n)$, by using (B.6), we would have $\|u_n\|_{A,V_{\mathcal{P}}} \to 0$, as $n \to \infty$, which is impossible since $c_\lambda > 0$.

Also, t_n doens't converge to infinity due to boundedness of (u_n). So, the sequence (t_n) is bounded and, going to a subsequence if necessary, there exists $t_0 \in (0, \infty)$such that $t_n \to t_0$, as $n \to \infty$.

Since $J'_{A,V_{\mathcal{P}}}(u_n)u_n \to 0$, as $n \to \infty$, we obtain

$$\|u_n\|^2_{A,V_{\mathcal{P}}} = \lambda B(u_n) + D(u_n) + o(1),\ \text{as } n \to \infty. \tag{B.7}$$

subtracting (B.6) from (B.7) we have

$$o(1) = \lambda(t_n^{2(p-1)} - 1)B(u_n) + (t_n^{2(2^*_\alpha - 1)} - 1)D(u_n),\ \text{as } n \to \infty. \tag{B.8}$$

Passing to the limit into (B.8) we obtain

$$\lambda(t_0^{2(p-1)} - 1)l_1 + (t_0^{2(2^*_\alpha - 1)} - 1)l_2 = 0$$

where

$$\lim_{n\to\infty} B(u_n) = l_1 > 0 \ \text{ and } \ \lim_{n\to\infty} D(u_n) = l_2 > 0.$$

Therefore, $t_0 = 1$, that is,

$$t_n \to 1,\ \text{as } n \to \infty. \tag{B.9}$$

Notice that by (B.9) and recalling that $t_n u_n \in \mathcal{M}_{A,V}$ we have

$$\begin{aligned} c_\lambda^* &\le J_{A,V_{\mathcal{P}}}(t_n u_n) \\ &= \frac{t_n^2}{2}\|u_n\|^2_{A,V_{\mathcal{P}}} - \frac{\lambda}{2p} t_n^{2p} B(u_n) - \frac{1}{2\cdot 2_\alpha^*} t_n^{2\cdot 2_\alpha^*} D(u_n) \\ &= t_n^2 \left[\frac{1}{2}\|u_n\|^2_{A,V_{\mathcal{P}}} - \frac{\lambda}{2p} t_n^{2(p-1)} B(u_n) - \frac{1}{2\cdot 2_\alpha^*} t_n^{2(2_\alpha^*-1)} D(u_n)\right] \\ &= t_n^2 \left[J_{A,V_{\mathcal{P}}}(u_n) + \frac{\lambda}{2p}\left(1 - t_n^{2(p-1)}\right) B(u_n) + \frac{1}{2\cdot 2_\alpha^*}\left(1 - t_n^{2(2_\alpha^*-1)}\right) D(u_n)\right] \\ &= t_n^2 J_{A,V_{\mathcal{P}}}(u_n) + o(1) \\ &= (t_n^2 - 1) J_{A,V_{\mathcal{P}}}(u_n) + J_{A,V_{\mathcal{P}}}(u_n) + o(1). \end{aligned}$$

Passing to the limit we obtain $c_\lambda^* \le c_\lambda$.
This concludes the verification of lemma. □

Lemma 58

$$[|u|]_{s,0} \le [u]_{s,0}$$

for all $u \in H^s_{0,V}(\mathbb{R}^3, \mathbb{C})$.

Proof. We have

$$||u(x)| - |u(y)||^2 = |u(x)|^2 - 2|u(x)\overline{u(y)}| + |u(y)|^2,$$

$$|u(x) - u(y)|^2 = |u(x)|^2 - 2\mathfrak{Re}(u(x)\overline{u(y)}) + |u(y)|^2$$

and

$$\mathfrak{Re}(u(x)\overline{u(y)}) \le |u(x)\overline{u(y)}|$$

which implies that

$$|u(x) - u(y)|^2 \ge ||u(x)| - |u(y)||^2.$$

So,

$$[|u|]^2_{s,0} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{||u(x)| - |u(y)||^2}{|x-y|^{3+2s}}\, dxdy \le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}}\, dxdy = [u]^2_{s,0}$$

that is,

$$[|u|]_{s,0} \le [u]_{s,0}.$$

□

Lemma 59 *It holds*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x)|^{2^*_{\alpha,s}} |u_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = 1$$

and

$$[u_\varepsilon]_{s,0} = [u]_{s,0}$$

for $\varepsilon > 0$.

Proof. Since $\displaystyle\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 1$ and $2^*_{\alpha,s} = \frac{6-\alpha}{3-2s}$, changing the variables we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)|^{2^*_{\alpha,s}}|u_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{dxdy} &= \varepsilon^{-(6-\alpha)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u\left(\frac{x}{\varepsilon}\right)|^{2^*_{\alpha,s}}|u\left(\frac{y}{\varepsilon}\right)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}dxdy\\ &= \varepsilon^{-(6-\alpha)}\cdot\varepsilon^{6-\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}dxdy\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}dxdy\\ &= 1\end{aligned}$$

and

$$\begin{aligned}[u_\varepsilon]^2_{s,0} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{3+2s}}\mathrm{dxdy} &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\varepsilon^{-\frac{3-2s}{2}}u\left(\frac{x}{\varepsilon}\right)-\varepsilon^{-\frac{3-2s}{2}}u\left(\frac{y}{\varepsilon}\right)|^2}{|x-y|^{3+2s}}dxdy\\ &= \varepsilon^{-3+2s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u\left(\frac{x}{\varepsilon}\right)-u\left(\frac{y}{\varepsilon}\right)|^2}{|x-y|^{3+2s}}dxdy\\ &= \varepsilon^{3+2s}\cdot\varepsilon^{-3-2s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\ &= [u]^2_{s,0}\end{aligned}$$

and, so, the lemma is proved.

□

Lemma 60 *It holds*

$$\begin{aligned}\Phi_\varepsilon(x,y) &= \frac{2\mathfrak{Re}\left(\left(1-e^{i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}\right)u(x)u(y)\right)}{|x-y|^{3+2s}}\\ &= \frac{2\left(1-\cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right)\right)u(x)u(y)}{|x-y|^{3+2s}}\end{aligned}$$

Proof. We have

$$\begin{aligned}|u(x)-e^{i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}u(y)|^2 &= |e^{i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}u(y)-u(x)|^2\\ &= |u(y)|^2-2\mathfrak{Re}\left(e^{i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}u(y)\overline{u(x)}\right)+|u(x)|^2\end{aligned}$$

and

$$|u(x)-u(y)|^2 = |u(y)-u(x)|^2 = |u(y)|^2-2\mathfrak{Re}\left(u(y)\overline{u(x)}\right)+|u(x)|^2$$

So, since $u(x) \in \mathbb{R}$, it follows that

$$
\begin{aligned}
|u(x) - e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}u(y)|^2 - |u(x)-u(y)|^2 &= 2\left[\mathfrak{Re}\left(u(y)\overline{u(x)}\right)\right. \\
&\left. - \mathfrak{Re}\left(e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}u(y)\overline{u(x)}\right)\right] \\
&= 2\mathfrak{Re}\left(u(y)\overline{u(x)} - e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}u(y)\overline{u(x)}\right) \\
&= 2\mathfrak{Re}\left(\left(1 - e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}\right)u(y)\overline{u(x)}\right) \\
&= 2\mathfrak{Re}\left(\left(1 - e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}\right)u(x)u(y)\right).
\end{aligned}
$$

Moreover, since

$$
1 - e^{i\varepsilon(x-y)\cdot A\left(\varepsilon \frac{x+y}{2}\right)} = 1 - \cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right) - i\sin\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right),
$$

follows that

$$
\begin{aligned}
\left(1 - e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}\right)u(x)u(y) &= \left[1 - \cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right)\right]u(x)u(y) \\
&- i\left[\sin\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right)\right]u(x)u(y)
\end{aligned}
$$

which implies that

$$
|u(x) - e^{i\varepsilon(x-y)\cdot A(\varepsilon \frac{x+y}{2})}u(y)|^2 - |u(x)-u(y)|^2 = 2\left[1 - \cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right)\right]u(x)u(y)
$$

and prove the lemma.

Lemma 61 *It holds*

$$
[u_\varepsilon]^2_{s,A} \leq [U_\varepsilon]^2_{s,A} + O(\varepsilon^{3-2s}) \tag{B.10}
$$

and

$$
[U_\varepsilon]^2_{s,A} \leq [u^*]^2_{s,0} + O(\varepsilon^2). \tag{B.11}
$$

Proof. Initially, we are going to prove (B.10). For this, condider the assertions below.

Claim 1. Let $\varrho > 0$ be. If $x \in B^c_\varrho$, then

$$
|u_\varepsilon(x)| \leq |U_\varepsilon(x)| \leq C\varepsilon^{(3-2s)/2}
$$

for any $\varepsilon > 0$ and for some positive constant $C = C(\varrho, s)$.

In fact, if $x \in B^c_\varrho$ we get

$$
|U_\varepsilon(x)| \leq C\varepsilon^{-(3-2s)/2}\left(1 + \left(\frac{\varrho}{\varepsilon\mathcal{S}_s^{1/(2s)}}\right)^2\right)^{-(3-2s)/2} \leq C\varepsilon^{(3-2s)/2}.
$$

Since $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^3$ so Claim 1 follows.

Claim 2. Let $\varrho > 0$ be. If $x \in B_\varrho^c$ then

$$|\nabla u_\varepsilon(x)| \le C\varepsilon^{(3-2s)/2}$$

for any $\varepsilon > 0$ and for some positive constant $C = C(\varrho, s)$.

In fact, first of all we observe that, for any $|z| \ge \varrho$, we have that

$$\begin{aligned}
\left(1+\left|\frac{z}{\varepsilon}\right|^2\right)^{-(3-2s)/2} + \frac{|z|}{\varepsilon^2}\left(1+\left|\frac{z}{\varepsilon}\right|^2\right)^{-1-(3-2s)/2} &\le \left(1+\left|\frac{z}{\varepsilon}\right|^2\right)^{-(3-2s)/2} \\
&\quad + \frac{1}{\varrho}\left|\frac{z}{\varepsilon}\right|^2\left(1+\left|\frac{z}{\varepsilon}\right|^2\right)^{-1-(3-2s)/2} \\
&\le \left(1+\frac{1}{\varrho}\right)\left(1+\left|\frac{z}{\varepsilon}\right|^2\right)^{-(3-2s)/2} \\
&\le \left(1+\frac{1}{\varrho}\right)\left(1+\left|\frac{\varrho}{\varepsilon}\right|^2\right)^{-(3-2s)/2} \qquad \text{(B.12)} \\
&\le \left(1+\frac{1}{\varrho}\right)\left(\frac{\varepsilon}{\varrho}\right)^{3-2s}
\end{aligned}$$

From (2.12) and considering $\varepsilon\mathcal{S}_s^{1/(2s)}$ in (B.12), we have that, for any $x \in B_\varrho^c$,

$$\begin{aligned}
|\nabla u_\varepsilon(x)| \le & C\varepsilon^{-(3-2s)/2}\left[\left(1+\left|\frac{x}{\varepsilon\mathcal{S}_s^{1/(2s)}}\right|^2\right)^{-(3-2s)/2}\right. \\
& \left. + \frac{1}{\varepsilon\mathcal{S}_s^{1/(2s)}}\left|\frac{x}{\varepsilon\mathcal{S}_s^{1/(2s)}}\right|\left(1+\left|\frac{x}{\varepsilon\mathcal{S}_s^{1/(2s)}}\right|^2\right)^{-1-(3-2s)/2}\right] \\
& \le C\varepsilon^{-(3-2s)/2}\cdot\varepsilon^{3-2s} = C\varepsilon^{(3-2s)/2},
\end{aligned}$$

which proves Claim 2.

Claim 3. Let δ be as in (2.11). Then for any $\varepsilon > 0$ and for some constant $C = C(\delta, s)$ it follow

a) For any $x \in \mathbb{R}^3$ and $y \in B_\delta^c$ with $|x-y| \le \frac{\delta}{2}$,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\varepsilon^{\frac{3-2s}{2}}|x-y|. \qquad \text{(B.13)}$$

b) For any $x, y \in B_\delta^c$

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\varepsilon^{\frac{3-2s}{2}}\min\{1, |x-y|\}, \qquad \text{(B.14)}$$

and

$$|u_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)| \le C\varepsilon^{\frac{3-2s}{2}}\min\{1, |x-y|\}. \qquad \text{(B.15)}$$

Proof of assertion a): Let $x \in \mathbb{R}^3$ and $y \in B_\delta^c$ with $|x-y| \leq \frac{\delta}{2}$, and let z be any point on the segment joining x and y, that is, $z = (1-t)x + ty$ for some $t \in [0,1]$. Then

$$|z| = |y + t(x-y)| \geq |y| - t|x-y| \geq \delta - t\,(\delta/2) \geq \delta/2.$$

This and Claim 2 (considering $\varrho = \delta/2$) imply that $|\nabla u_\varepsilon(z)| \leq C\varepsilon^{(3-2s)/2}$; so, by mean value inequality, see e.g [34, Theorem 5.1],

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C\varepsilon^{\frac{3-2s}{2}}|x-y|,$$

which proves (B.13).

Proof of assertion b): Let $x, y \in B_\delta^c$. If $|x-y| \leq \delta/2$, then $b)$ follows from $a)$.

Suppose $|x-y| > \delta/2$. Then, by Claim 1 (considering $\varrho = \delta$) we have

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq |u_\varepsilon(x)| + |u_\varepsilon(y)| \leq C\varepsilon^{(3-2s)/2},$$

which proves (B.14).

Now, we show (B.15).

Since A is bounded, there exists $C > 0$ such that

$$\left| e^{i(x-y)\cdot A(\frac{x+y}{2})} - 1 \right| \leq C \min\{1, |x-y|\}. \tag{B.16}$$

Then, by claim 1, claim 3 (here, in particular, we use (B.14)) and (B.16), we obtain

$$\begin{aligned}\left| u_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y) \right| &\leq \left| e^{i(x-y)\cdot A(\frac{x+y}{2})} - 1 \right| |u_\varepsilon(x)| + |u_\varepsilon(x) - u_\varepsilon(y)| \\ &\leq C\varepsilon^{\frac{3-2s}{2}} \min\{1, |x-y|\},\end{aligned}$$

which proves (B.15). So, Claim 3 is proved.

Set

$$L = \left\{ (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in B_\delta,\ y \in B_\delta^c \text{ and } |x-y| \leq \frac{\delta}{2} \right\},$$

and

$$G = \left\{ (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in B_\delta,\ y \in B_\delta^c \text{ and } |x-y| > \frac{\delta}{2} \right\},$$

where δ is as in (2.11).

By (2.12) we have that

$$\begin{aligned}[u_\varepsilon]_{s,A}^2 &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y)|^2}{|x-y|^{3+2s}} dxdy \\ &= \int_{B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} U_\varepsilon(y)|^2}{|x-y|^{3+2s}} dxdy \\ &+ \int_{B_\delta}\int_{B_\delta} \frac{|u_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y)|^2}{|x-y|^{3+2s}} dxdy \\ &+ 2\int_{L} \frac{|u_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y)|^2}{|x-y|^{3+2s}} dxdy \\ &+ 2\int_{G} \frac{|u_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_\varepsilon(y)|^2}{|x-y|^{3+2s}} dxdy\end{aligned} \tag{B.17}$$

By (2.11) ($\eta|_{B^c_{2\delta}}=0$) and Claim 3 (here, in particular, we use (B.15) we have

$$\int_{B^c_\delta}\int_{B^c_\delta}\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy \le C\varepsilon^{3-2s}\int_{B_{2\delta}}\int_{B_{2\delta}}\frac{\min\{1,|x-y|^2\}}{|x-y|^{3+2s}}dxdy$$
$$\le C\varepsilon^{3-2s}\left(\int_{\substack{|x|<2\delta\\|x-y|<1}}\frac{|x-y|^2}{|x-y|^{3+2s}}dxdy+\right. \tag{B.18}$$
$$\left.\int_{\substack{|x|<2\delta\\|x-y|\ge1}}\frac{1}{|x-y|^{3+2s}}dxdy\right)=O(\varepsilon^{3-2s}).$$

For $(x,y)\in L$, by claim 1, claim 3 (Here, in particular, we use (B.13)) and (B.16), we have

$$\left|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)\right|\le|u_\varepsilon(x)-u_\varepsilon(y)|+\left|e^{i(x-y)\cdot A(\frac{x+y}{2})}-1\right||u_\varepsilon(y)|$$
$$\le|u_\varepsilon(x)-u_\varepsilon(y)|+C|x-y|\varepsilon^{\frac{3-2s}{2}} \tag{B.19}$$
$$\le C\varepsilon^{\frac{3-2s}{2}}|x-y|.$$

Then, by (B.19), we obtain that

$$\int_L\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy\le C\varepsilon^{3-2s}\int_{\substack{|x|<\delta\\|x-y|\le\frac{\delta}{2}}}\frac{|x-y|^2}{|x-y|^{3+2s}}dxdy \tag{B.20}$$
$$=O(\varepsilon^{3-2s}),$$

as $\varepsilon\to0$.

Now, in (B.17) it remains to estimate the integral on G, that is,

$$\int_G\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy. \tag{B.21}$$

For this, recalling that $u_\varepsilon(x)=U_\varepsilon(x)$ for any $x\in B_\delta$ thanks to (2.12), we note that, for any $(x,y)\in G$,

$$|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2=|U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2$$
$$=|(U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y))+(U_\varepsilon(y)-u_\varepsilon(y))|^2$$
$$\le|U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)|^2+|U_\varepsilon(y)-u_\varepsilon(y))|^2$$
$$+2|U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)||U_\varepsilon(y)-u_\varepsilon(y))|,$$

so that

$$\int_G\frac{|u_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy\le\int_G\frac{|U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy$$
$$+\int_G\frac{|U_\varepsilon(y)-u_\varepsilon(y))|^2}{|x-y|^{3+2s}}dxdy \tag{B.22}$$
$$+2\int_G\frac{|U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)||U_\varepsilon(y)-u_\varepsilon(y))|}{|x-y|^{3+2s}}dxdy$$

Hence, in order to estimate (B.21), we bound the last two terms in the right-hand side of (B.22).

By claim 1 (here used with $\varrho = \delta$), we obtain

$$\begin{aligned}\int_G \frac{|U_\varepsilon(y)-u_\varepsilon(y))|^2}{|x-y|^{3+2s}}dxdy &\le \int_G \frac{(|U_\varepsilon(y)|+|u_\varepsilon(y)|)^2}{|x-y|^{3+2s}}dxdy \\ &\le 4\int_G \frac{U_\varepsilon^2(y)}{|x-y|^{3+2s}}dxdy \\ &\le C\varepsilon^{3-2s}\int_{\substack{x\in B_\delta \\ |x-y|>\frac{\delta}{2}}} \frac{1}{|x-y|^{3+2s}}dxdy \\ &= C\varepsilon^{3-2s}\int_{|\zeta|<\delta} d\zeta \int_{|\xi|>\frac{\delta}{2}} |\xi|^{-3-2s}d\xi \\ &= O(\varepsilon^{3-2s}).\end{aligned} \tag{B.23}$$

as $\varepsilon \to 0$.

Now, we are going to estimate the last term in the right-hand side of (B.22).

Recalling (2.12) and claim 1 it follows

$$\int_G \frac{|U_\varepsilon(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)||U_\varepsilon(y)-u_\varepsilon(y))|}{|x-y|^{3+2s}}dxdy \le 2\int_G \frac{U_\varepsilon(x)U_\varepsilon(y)+U_\varepsilon^2(y)}{|x-y|^{3+2s}}dxdy.$$

By (2.12) (which is valid for any $x \in \mathbb{R}^3$) and Claim 1, we have

$$|U_\varepsilon(x)||U_\varepsilon(y)| \le C\left(1+\left|\frac{x}{\varepsilon\mathcal{S}_s^{1/(2s)}}\right|^2\right)^{-(3-2s)/2} \tag{B.24}$$

for any $(x,y) \in G$. So, by using the shorthand notation

$$\delta_\varepsilon := \delta/(\varepsilon\mathcal{S}_s^{1/2s})$$

and the change of variable $z := x/(\varepsilon\mathcal{S}_s^{1/2s})$ and $w := x-y$, observing that since $B_1 \subset B_{\delta_\varepsilon}$ as

$\varepsilon \to 0$ and recalling that $s \in (\frac{3}{4}, 1)$, up to remaning C, by (B.24), it follows that

$$\begin{aligned}\int_G \frac{|U_\varepsilon(x)||U_\varepsilon(y)|}{|x-y|^{3+2s}}dxdy &\le C\int_G \left(1+\left|\frac{x}{\varepsilon \mathcal{S}_s^{1/(2s)}}\right|^2\right)^{-(3-2s)/2} |x-y|^{-(3+2s)}dxdy \\ &= C\varepsilon^3 \int_{\substack{z\in B_{\delta_\varepsilon} \\ |w|>\frac{\delta}{2}}} (1+|z|^2)^{-(3-2s)/2}|w|^{-(3+2s)}dzdw \\ &= C\varepsilon^3 \int_{z\in B_{\delta_\varepsilon}} (1+|z|^2)^{-(3-2s)/2}dz \\ &= C\varepsilon^3 \left[\int_{z\in B_1} (1+|z|^2)^{-(3-2s)/2}dz + \int_{z\in B_{\delta_\varepsilon}\setminus B_1} (1+|z|^2)^{-(3-2s)/2}dz\right] \qquad \text{(B.25)} \\ &\le C\varepsilon^3 \left[\int_{z\in B_1} |z|^{-(3-2s)}dz + \int_{z\in B_{\delta_\varepsilon}\setminus B_1} |z|^{-(3-2s)}dz\right] \\ &= C\varepsilon^3 \left[1/(2s) + \int_1^{\delta/(\varepsilon\mathcal{S}_s)^{1/(2s)}} r^{-(3-2s)+2}dr\right] \\ &= C\varepsilon^3 \cdot (\varepsilon^{-2s}) = O(\varepsilon^{3-2s}),\end{aligned}$$

as $\varepsilon \to 0$.

On other hand, by claim 1, we have

$$\begin{aligned}\int_G \frac{U_\varepsilon^2(y)}{|x-y|^{3+2s}}dxdy &\le C\varepsilon^{3-2s}\int_{\substack{x\in B_\delta, y\in B_\delta \\ |x-y|>\frac{\delta}{2}}} \frac{1}{|x-y|^{3+2s}}dxdy \qquad \text{(B.26)} \\ &= O(\varepsilon^{3-2s}).\end{aligned}$$

as $\varepsilon \to 0$.

So, by (B.25) and (B.26) we have

$$\int_G \frac{|U_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)||U_\varepsilon(y)-u_\varepsilon(y))|}{|x-y|^{3+2s}}dxdy \le O(\varepsilon^{3-2s}) \qquad \text{(B.27)}$$

Finally by (B.17), (B.18), (B.20), (B.22), (B.23) and (B.27) we have

$$\begin{aligned}[u_\varepsilon]_{s,A}^2 &\le \int_{B_\delta}\int_{B_\delta} \frac{|U_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy \\ &\quad + 2\int_G \frac{|U_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy + O(\varepsilon^{3-2s}) \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|U_\varepsilon(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}U_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy + O(\varepsilon^{3-2s})\end{aligned}$$

as $\varepsilon \to 0$, which proves (B.10)

Now, we are going to prove (B.11).
We have

$$
\begin{aligned}
[U_\varepsilon]_{s,A}^2 &= \varepsilon^{-(3-2s)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u^*(\frac{x}{\varepsilon}) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u^*(\frac{y}{\varepsilon})|^2}{|x-y|^{3+2s}}dxdy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u^*(x) - e^{i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}u^*(y)|^2}{|x-y|^{3+2s}}dxdy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^{*2}(x) + u^{*2}(y)}{|x-y|^{3+2s}}dxdy \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2u^*(x)u^*(y)\cos\left(\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})\right)}{|x-y|^{3+2s}}dxdy.
\end{aligned}
$$

Let $(u_n) \subset C_c^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$
u_n \to u^* \quad \text{as} \quad n \to \infty.
$$

(The sequence (u_n) exists since $C_c^\infty(\mathbb{R}^3, \mathbb{R})$ is dense in $H^s(\mathbb{R}^N, \mathbb{R})$ - See [1, Theorem 7.38]).
Since

$$
[u_n]_{s,0}^2 = \int_{\mathbb{R}^6} \frac{u_n(x)^2 + u_n(y)^2 - 2u_n(x)u_n(y)}{|x-y|^{3+2s}}dxdy
$$

we have

$$
\begin{aligned}
&[U_\varepsilon]_{s,A}^2 - [u_n]_{s,0}^2 \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2\left(1 - \cos\left(\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})\right)\right)u_n(x)u_n(y)}{|x-y|^{3+2s}}dxdy \\
&:= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi_\varepsilon(x,y)\,dxdy = \int_{K_n\times K_n} \Phi_\varepsilon(x,y)\,dxdy,
\end{aligned}
\tag{B.28}
$$

where K_n is the compact support of u_n. For ε small and $x, y \in K_n$, it follows from the boundedness of A that

$$
1 - \cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right) \le \varepsilon^2|x-y|^2.
$$

Moreover, noticing that $|x-y|$ is bounded for $x, y \in K_n$, we have

$$
1 - \cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right) \le C\varepsilon^2.
$$

Therefore, since (u_n) is bounded, there exists $C > 0$ such that

$$
|\Phi_{\varepsilon,n}(x,y)| \le \begin{cases} \dfrac{C\varepsilon^2}{|x-y|^{1+2s}} & \text{if } |x-y| < 1 \\ \dfrac{C\varepsilon^2}{|x-y|^{3+2s}} & \text{if } |x-y| \ge 1 \end{cases}
$$

So,

$$\int_{K_n}\int_{K_n} \Phi_{\varepsilon,n}(x,y)\,dxdy = \int_{(K_n\times K_n)\cap\{|x-y|<1\}} \Phi_{\varepsilon,n}(x,y)\,dxdy + \int_{(K_n\times K_n)\cap\{|x-y|\geq 1\}} \Phi_{\varepsilon,n}(x,y)\,dxdy$$
$$\leq C\varepsilon^2 \int_{K_n} dw \int_{\{|z|<1\}} \frac{1}{|z|^{1+2s}}\,dz + C\varepsilon^2 \int_{K_n} dw \int_{\{|z|\geq 1\}} \frac{1}{|x-y|^{3+2s}}\,dz \tag{B.29}$$
$$= O(\varepsilon^2).$$

Hence, by (B.28) and (B.29) we have

$$[U_\varepsilon]^2_{s,A} \leq [u_n]^2_{s,0} + O(\varepsilon^2)$$

for all n, and taking the limit as $n \to \infty$ we have (B.11). Therefore the lemma is proved. □

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