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Generalized entropy of wandering dynamics

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Abstract

The notion of generalized entropy (introduced by Correa and Pujals, [3]) extends the classical notion of entropy and it is a useful tool to study dynamical systems with zero topological entropy. On the other hand, Hauseux and Le Roux ([5]) studied the polynomial entropy of the wandering part of an invertible dynamical system on a compact metric space. In this work, we study wandering dynamics from the perspective of generalized entropy. We construct examples with arbitrarily generalized entropy (in its proper context). And we also classify the generalized entropy of maps on the sphere with finite non-wandering points.

Keywords: Generalized entropy. Wandering dynamics. Brouwer homeomorphisms.

Resumo

A noção de entropia generalizada (introduzida por Correa e Pujals, [3]) estende a noção clássica de entropia e é uma ferramenta útil para estudar sistemas dinâmicos com entropia topológica nula. Por outro lado, Hauseux e Le Roux ([5]) estudaram a entropia polinomial da parte errante de um sistema dinâmico invertível em um espaço métrico compacto. Neste trabalho, estudamos a dinâmica errante sob a perspectiva da entropia generalizada. Construímos exemplos com entropia generalizada arbitrária (em um contexto específico). E também classificamos a entropia generalizada de mapas na esfera com finitos pontos não-errantes.

Palavras-chave: Entropia generalizada. Dinâmicas errantes. Homeomorfismos de Brouwer.

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Contents

Introduction

It is a classical question to estimate the complexity of a dynamical system, although it could be not simple in many cases. A traditional tool to do this is the topological entropy. In a sense, the topological entropy describes in a crude but suggestive way the total exponential complexity of the orbit structure with a single number.

One often distinguishes between systems with zero topological entropy and systems with positive topological entropy and there are several criteria to prove that a given system has a positive topological entropy. Even if there is no criterion for a system to have zero entropy (unless being an isometry or contracting), there are several well-known zero entropy continuous systems: for instance the harmonic and anharmonic oscillators, the simple pendulum, homeomorphisms of the circle, elliptic billiards, and cylindrical cascades.

Although all these systems have the same topological entropy, it seems obvious that the harmonic oscillator is simpler than the anharmonic one which is simpler than the simple pendulum. In the same way, a rotation on the circle is simpler than a homeomorphism that possesses both periodic and wandering points, and finally circular billiard looks simpler than any other elliptic billiards. It is therefore a natural question to estimate the complexity of such systems more precisely. And it is in this context that became necessary to consider an invariant that detects a complexity of systems that are not distinguished by topological entropy.

To do this, we start to consider no longer an exponential measure of the complexity, as the topological entropy, but a polynomial measure of the complexity. In this direction, Marco introduced the concept of polynomial entropy in the framework of integrable Hamiltonian systems (see [11], [12]). The polynomial entropy evaluates the polynomial growth rate of the number of orbits that one needs to know to understand the entire set of orbits within a given precision. As an application, one sees that the polynomial entropy of the harmonic oscillator is smaller than that of the anharmonic oscillator which is smaller than that of the simple pendulum.

In this direction, Labrousse studied the polynomial entropy of circle homeomorphisms and torus flows and showed that circle homeomorphisms have polynomial entropy 0 or 1 and that the value 0 characterizes the conjugacy classes of rotations (see [8], [9], [10]). She also studied the polynomial entropy of geodesic flows for Riemannian metrics on the two-torus: in a work with Patrick Bernard, they showed that the geodesic flow has polynomial entropy 1 if and only if the torus is isometric to a flat torus (see [2]).

Also, Artigue, Carrasco-Olivera, and Monteverde in [1] construct a homeomorphism on a compact metric space with vanishing polynomial entropy that it is not equicontinuous. And, they give examples with arbitrarily small polynomial entropy.

Finally, Hauseux and Le Roux, in [5], proposed to study the polynomial entropy of the wandering part of dynamical systems. They show that the polynomial entropy localizes near certain finite sets, and that it may be computed by a simple dynamical coding. In this thesis, we extend the approach presented by Hauseux and Le Roux and it will be very useful for our purposes. In their work, they also compute the polynomial entropy of Brouwer homeomorphisms (fixed point free orientation preserving homeomorphisms of the plane) and showed that a Brouwer homeomorphism has wandering polynomial entropy 1 if and only if it is conjugated to a translation. None Brouwer homeomorphism has wandering polynomial entropy in the interval $(1, 2)$. And, for every $\alpha \in [2, +\infty]$ there exists a Brouwer homeomorphism f_α with wandering polynomial entropy α.

Katić and Perić, in [6], adapted the construction from Hauseux and Le Roux ([5]) to obtain a method for computing the polynomial entropy for a continuous map with finitely many non-wandering points. They compute the maximal cardinality of a singular set of Morse negative gradient systems and apply this method to compute the polynomial entropy for Morse gradient systems. It is crucial to observe that they impose an important restriction on the dynamics which is the singularity index $0, [n/2]$ and n. The problem of polynomial entropy, even for Morse-Smale gradients, has not been solved yet. In this thesis, we also adapt the approach from Hauseux and Le Roux for non-wandering sets with finite points. However, we choose a different method from Katić and Perić, which allows us to consider more general cases.

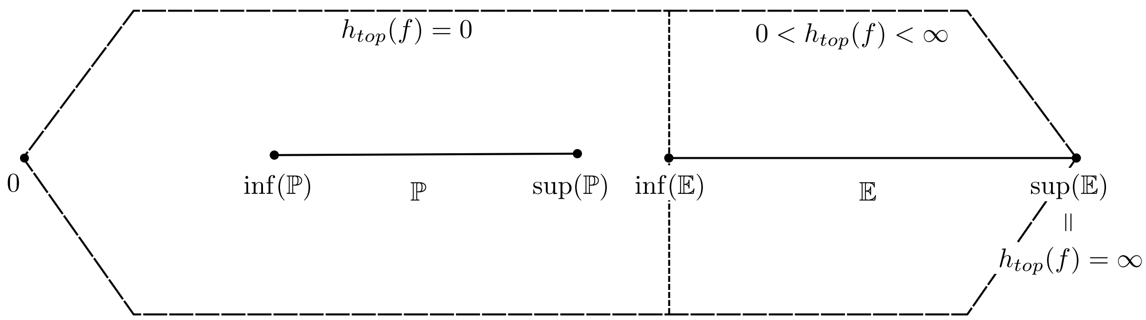
In another direction, Correa and Pujals, in [3], constructed the complete set of orders of growth and define on it the generalized entropy of a dynamical system. With this object, they provide a framework where it is possible to study the separation of orbits of a map beyond the scope of exponential or polynomial growth.

Consider the space of non-decreasing sequences $\mathcal{O}$. We say that two sequences in $\mathcal{O}$ are related if they both have the same order of growth. Then, we consider the quotient space $\mathbb{O}$, which is called the space of orders of growth. There is in $\mathbb{O}$ a notion of a order of growth being faster than another. This concept defines a partial order in $\mathbb{O}$. For the purposes this space was developed, it would be useful that "limits" could be taken and therefore, it needs to be completed. Then, we consider now $\overline{\mathbb{O}}$ the Dedekind - MacNeille completion of $\mathbb{O}$. This is the smallest complete lattice which contains $\mathbb{O}$. The space $\overline{\mathbb{O}}$ is the complete set of orders of growth. (This construction was introduced in [3] and will be presented in details in section 1.1).

Let us present now the generalized entropy of a map. Consider a continuous map f, defined in a compact metric space. As in the case of topological entropy or polynomial entropy, we consider the cardinality of the minimal (n, ε)-generator sets, $g(f, \varepsilon, n)$. If we fix $\varepsilon > 0$, then $g(f, \varepsilon, n)$ is an increasing sequence of natural numbers, so $g_{f,\varepsilon}(n) := g(f, \varepsilon, n) \in \mathcal{O}$. We consider the elements $[g_{f,\varepsilon}(n)]$ in $\overline{\mathbb{O}}$, and the generalized entropy of f, $o(f)$, is defined as the supremum of the set $\{[g_{f,\varepsilon}(n)] \in \mathbb{O} : \varepsilon > 0\}$, that is $o(f) \in \overline{\mathbb{O}}$.

The generalized entropy is a topological invariant, and coincides with the topological entropy of a map in the following sense: the exponential orders of growth are the set $\mathbb{E} = \{[\exp(tn)]; t \in (0, \infty)\} \subset \mathbb{O}$. The classical notion of topological entropy is the projection of the generalized entropy into the family of exponential orders of growth. Another important family that we consider in $\overline{\mathbb{O}}$ is the polynomial orders of growth, that are the set $\mathbb{P} = \{[n^t]; t \in (0, \infty)\} \subset \mathbb{O}$. The projection of the generalized entropy of a map over the family $\mathbb{P}$ provides us the polynomial entropy of such map.

We present here a simple representation of the set of the generalized entropies of continuous maps:



The definition of generalized entropy, some properties, and how it is related to the classical topological entropy and polynomial entropy will be presented in details in section 1.2.

In this context of generalized entropy, Correa and Pujals studied, in [3], the space of homeomorphisms of the circle, and presented a method to distinguishes such homeomorphisms. With this, they showed that generalized entropy can give us information about maps that are indistinguishable by classical topological entropy. Furthermore, they constructed examples of cylindrical cascades with arbitrarily slow generalized entropy, namely for every $o \in \overline{\mathbb{O}}$ they present a cylindrical cascade f such that $0 < o(f) \leq o$. They showed that for the family of cylindrical cascades, polynomial entropy is not sufficient.

In this thesis, we translate and extend to the context of orders of growth and generalized entropy the results of Hauseux and Le Roux. We define wandering generalized entropy, $o_w(f)$, and show that we can calculate it using the approach of coding and singular sets as presented in [5]. Furthermore, we show that we can use such techniques in the context where the non-wandering set is not just a fixed point, but a finite set.

We extend the construction of Hauseux and Le Roux to obtain a Brouwer homeomorphism whose wandering polynomial entropy coincides with the entropy of the homeomorphisms in their context, but whose wandering generalized entropy is different. It is important to point out that in order to calculate the entropy of Brouwer homeomorphisms we compactify the plane by adding the point at infinity to obtain a compact metric space ($\mathbb{S}^2$). We show that:

Theorem A. *For every $L \geq 2$, there exists a Brouwer homeomorphism f with*

$$o_w(f) = [n^L \cdot \log n].$$

Such Brouwer homeomorphism has wandering polynomial entropy equals to $h_{pol}(f) = L$, for any integer $L \geq 2$, which is equivalent to $[n^L]$, not $[n^L \cdot \log n]$. With this, we can see that polynomial entropy is not enough to characterize Brouwer homeomorphisms.

Then, we use this to construct Brouwer homeomorphisms with, in some way, arbitrary generalized entropy. In, [3], Correa and Pujals propose a question about the realization of orders of growth. They ask: if given $\mathcal{H}$ a family of dynamical systems such that $o_i = \inf\{o(f); f \in \mathcal{H}\}$ and $o_s = \sup\{o(f); f \in \mathcal{H}\}$, does for every $o_i < o < o_s$ exists $f \in \mathcal{H}$ such that $o(f) = o$?

In our case, we consider the family of homeomorphisms $\mathcal{H} = \{f : S^2 \to S^2; \Omega(f) = \{\infty\}\}$. In particular, this family contains the compactification of Brouwer homeomorphisms. And for such family, we give an answer to their question.

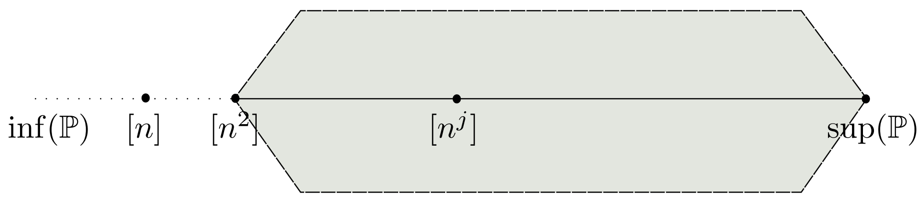
Let us consider $\underline{\mathbb{O}}$ the set of orders of growth that are supremum of countable sets in $\mathbb{O}$. It is enough consider $\underline{\mathbb{O}}$, because if f is a continuous map, then its generalized entropy satisfies $o(f) \in \underline{\mathbb{O}}$. From Hauseux and Le Roux work, we infer that the wandering generalized entropy of a translation is the linear order, $[n]$. And for every

$f \in \mathcal{H}$ that is not conjugated to the translation, we have $[n^2] \leq o(f) \leq \sup \mathbb{P}$.

Theorem B. *Let* $o \in \underline{\mathbb{O}}$ *with* $[n^2] \leq o \leq \sup \mathbb{P}$. *Then there exists* $f \in \mathcal{H}$ *such that*

$$o(f) = o.$$

We can represent the set of the generalized entropies of the family $\mathcal{H}$ in the following figure. The horizontal line represents the family of polynomial orders of growth, $\mathbb{P}$.



Although the compactification of the translations belongs to the family $\mathcal{H}$, theorem A answer positively the question proposed by Correa and Pujals. Theorem A is also a generalization in the context of orders of growth and generalized entropy of the results of Hauseux and Le Roux, in [5].

Generalized entropy is a very sensitive tool. There exists many examples where $o(f, \Omega(f)) < o(f)$. For instance, if $f \in \mathcal{H}$, then we have $o(f, \Omega(f)) = 0$ with $o(f) > 0$. Also, in [3], Correa and Pujals presented a map such that $o(f, Rec(f)) \leq o(f, \Omega(f)) \leq o(f)$. This does not happen with topological entropy, since we have $h_{top}(f, \Omega(f)) = h_{top}(f)$, as a consequence from the variational principle. This means that $o(f)$ can detect the separation of orbits in places where topological entropy can not, like in the wandering set. One natural question is why does this leap happen? And moreover, how can we mensurate it?

A classical way to compute the topological entropy of a map is through the coding of the itinerary of the orbits. For instance, if the map is an Axiom A, then we use Markov partitions. Now, for the context where $h_{top}(f) = 0$, we believe that Hauseux and Le Roux ([5]) approach is the right way to codify the dynamics in the wandering set. To prove theorems A and B, the techniques of coding and singular sets are used in a context where f is such that the non-wandering set contains only one fixed point. However, we show that such techniques allow us to compute the generalized entropy in the context where the non-wandering set is finite. For the family of homeomorphisms $\mathcal{H}' = \{f : S^2 \to S^2; \Omega(f) = \{y_1, y_2, \cdots, y_k\}\}$ we have the following theorem:

Theorem C. *The set of the generalized entropies of maps in the family* $\mathcal{H}'$ *is the interval from* $[n^2]$ *to* $\sup \mathbb{P}$ *(in* $\underline{\mathbb{O}}$*) and* $[n]$.

Another reason why it is interesting to calculate the entropy when the non-wandering set is finite is that we believe this approach can be used to precisely calculate the entropy of Morse-Smale diffeomorphisms.

This work is structured as follows:

In chapter 1, we present in details the construction of the space of orders of growth introduced in [3] by Correa and Pujals. We show some of the many interesting properties of this space. We also present the definition of generalized entropy, how it is related to the classical topological entropy and the polynomial entropy, and some of its properties.

In chapter 2, we define the wandering generalized entropy and show that it can be calculated in terms of coding and singular sets, as a generalization of the approach of Hauseux and Le Roux, in [5]. In this chapter we consider a homeomorphism defined in a compact metric space and such that the non-wandering set contains only one fixed point.

In chapter 3, we generalize the construction presented by Hauseux and Le Roux, in [5], to obtain Brouwer homeomorphisms with wandering generalized entropy that can not be detected by the polynomial entropy. This is the proof of Theorem A. We extend even more the construction to achieve Brouwer homeomorphisms with, in some way, arbitrary wandering generalized entropy. This is the proof of Theorem B.

Finally, in chapter 4, we show that if f is a homeomorphism defined in a compact metric space and such that the non-wandering set is finite, we also can calculate the generalized entropy of f in terms of coding and singular sets. And with this we conclude the proof of Theorem C.

Chapter 1

Generalized entropy

1.1 Orders of growth

Consider the space of non-decreasing sequences in $[0, \infty)$,

$$\mathcal{O} = \{a : \mathbb{N} \to [0, \infty) : a(n) \leq a(n+1), \forall n \in \mathbb{N}\}.$$

We define a relation $\approx$ in this space and we say that $a \approx b$, for $a, b \in \mathcal{O}$, if and only if there exists $c_1, c_2 \in (0, \infty)$ such that $c_1 a(n) \leq b(n) \leq c_2 a(n)$ for all $n \in \mathbb{N}$, or equivalently $\limsup_{n\to\infty} \frac{b(n)}{a(n)} < \infty$ and $\liminf_{n\to\infty} \frac{a(n)}{b(n)} > 0$.

It is easy to see that $\approx$ is an equivalence relation, that is, it satisfies:

1. (reflexive property) For $c = 1$ we have $a(n) \leq a(n)$, then $a \approx a$.
2. (symmetric property) If $a \approx b$ then $\dfrac{1}{c_1}, \dfrac{1}{c_2}$ satisifes $\dfrac{1}{c_2} b(n) \leq a(n) \leq \dfrac{1}{c_1} b(n)$, for all $n \in \mathbb{N}$, then $b \approx a$.
3. (transitive property) If $a \approx b$ and $b \approx c$, let $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that $c_1 a(n) \leq b(n) \leq c_2 a(n)$ and $c_3 b(n) \leq c(n) \leq c_4 b(n)$, for all $n \in \mathbb{N}$. Then $c_1 c_3 a(n) \leq c(n) \leq c_2 c_4 a(n)$, thus $a \approx c$.

The meaning for two sequences to be related is that both of them have the same order of growth. Because of this, the quotient space $\mathbb{O} = \mathcal{O}/_{\approx}$ is called the space of orders of growth. If a belongs to $\mathcal{O}$ we are going to note $[a(n)]$ the class associated to a, which is an element of $\mathbb{O}$. If a sequence is defined by one formula (for example n^2), then the order of growth associated with it will be represented by the formula between brackets ($[n^2] \in \mathbb{O}$).

Since $\mathbb{O}$ is the space of orders of growth, there is a clear notion of a order of growth being faster than another. This concept defines a partial order in $\mathbb{O}$ which is formalized through the following construction: given $[a(n)], [b(n)] \in \mathbb{O}$ we say that $[a(n)] \leq [b(n)]$ if there exists $C > 0$ such that $a(n) \leq Cb(n)$, for all $n \in \mathbb{N}$, or equivalently $\liminf_{n\to\infty} \frac{b(n)}{a(n)} > 0$.

This partial order is well defined because it does not depend on the choices of the representants a and b. In fact, let $\hat{a} \approx a$ and $\hat{b} \approx b$ sequences in $\mathcal{O}$, and $a(n) \leq Cb(n)$, for some $C > 0$ and all $n \in \mathbb{N}$. Let $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that $c_1 a(n) \leq \hat{a}(n) \leq c_2 a(n)$ and $c_3 b(n) \leq \hat{b}(n) \leq c_4 b(n)$, then $\hat{C} = \frac{Cc_3}{c_2} > 0$ satisfies $\hat{C}\hat{a}(n) \leq \hat{b}(n)$, for all $n \in \mathbb{N}$.

We have that $(\mathbb{O}, \leq)$ is a partial order, that means that it satisfies:

1. (Reflexivity) If $C = 1$, $a(n) \leq a(n)$, for all $n \in \mathbb{N}$, then $[a(n)] \leq [a(n)]$, for all $[a(n)] \in \mathbb{O}$.

2. (Antisymmetry) If $[a(n)] \leq [b(n)]$ and $[b(n)] \leq [a(n)]$, then there exists $C_1, C_2 > 0$ such that $a(n) \leq C_1 b(n)$ and $b(n) \leq C_2 a(n)$, then $\frac{1}{C_1} a(n) \leq b(n) \leq C_2 a(n)$, thus $a \approx b$ and $[a] = [b]$, for $[a(n)], [b(n)] \in \mathbb{O}$.

3. (Transitivity) If $[a(n)] \leq [b(n)]$ and $[b(n)] \leq [c(n)]$ then there exists $C_1, C_2 > 0$ such that $a(n) \leq C_1 b(n)$ and $b(n) \leq C_2 c(n)$, for all $n \in \mathbb{N}$, then $a(n) \leq C_1 C_2 c(n)$, thus $[a(n)] \leq [c(n)]$, for all $[a(n)], [b(n)], [c(n)] \in \mathbb{O}$.

For the purposes this space was developed, it would be useful that "limits" could be taken and therefore, it needs to be complete. We say that a set L with a partial order is a complete lattice if every subset $A \subset L$ has both an infimum and a supremum.

We consider now $\overline{\mathbb{O}}$ the Dedekind - MacNeille completion of $\mathbb{O}$. This is the smallest complete lattice which contains $\mathbb{O}$. In particular, it is uniquely defined and we will always consider that $\mathbb{O} \subset \overline{\mathbb{O}}$. We will also call $\overline{\mathbb{O}}$ the complete set of orders of growth. Another way to define $\overline{\mathbb{O}}$ is to consider in $\mathbb{O}$ the order topology and then consider the compactification of $\mathbb{O}$ respecting the partial order.

Since $\overline{\mathbb{O}}$ is not a complete order, just a partial order, the elements $o \in \overline{\mathbb{O}}$ are not represented in a line. They are going to be represented on the plane. Given $o, u \in \overline{\mathbb{O}}$ if o is to the right of u, then o and u may or may not be comparable but if they are, $u < o$. However, if they are on the same horizontal line and o is to the right of u, then $u < o$ holds. In the figure 1.1, we have $u < o_1$ and, if u and o_2 are comparable, then $u < o_2$. The orders o_1 and o_2 are not comparable.

Figure 1.1: representation of elements of $\overline{\mathbb{O}}$.

Let us see now a few properties about the space $\overline{\mathbb{O}}$. These properties do not depend on the choice of representants.

1. (Sum) If $[a(n)], [b(n)] \in \mathbb{O}$, then $[a(n)] + [b(n)] := [a(n) + b(n)] \in \mathbb{O}$, where $[a(n) + b(n)]$ is the class of the sequence $a(n) + b(n)$, for $n \in \mathbb{N}$.

2. (Product) If $[a(n)], [b(n)] \in \mathbb{O}$, then $[a(n)] \cdot [b(n)] := [a(n) \cdot b(n)] \in \mathbb{O}$, where $[a(n) \cdot b(n)]$ is the class of the sequence $a(n) \cdot b(n)$, for $n \in \mathbb{N}$.

3. (Supremum) If $[a(n)], [b(n)] \in \mathbb{O}$, then $\sup\{[a(n)], [b(n)]\} = [\max\{a(n), b(n)\}] \in \mathbb{O}$.

   In fact, we know that $[a(n)] \leq [\max\{a(n), b(n)\}]$ and $[b(n)] \leq [\max\{a(n), b(n)\}]$, then $\sup\{[a(n)], [b(n)]\} \leq [\max\{a(n), b(n)\}]$. But, if we suppose

$$\sup\{[a(n)], [b(n)]\} < [\max\{a(n), b(n)\}],$$

   then there exists $[c(n)] \in \mathbb{O}$ such that

$$\sup\{[a(n)], [b(n)]\} < [c(n)] < [\max\{a(n), b(n)\}].$$

   Since $\sup\{[a(n)], [b(n)]\} < [c(n)]$ and by definition we have

$$[a(n)] \leq \sup\{[a(n)], [b(n)]\},$$

   then $[a(n)] \leq [c(n)]$, and by the same argument $[b(n)] \leq [c(n)]$. Thus must there exists constants $k_1, k_2 > 0$ such that $a(n) \leq k_1 c(n)$ and $b(n) \leq k_2 c(n)$. Let $k = \max\{k_1, k_2\}$, then we have $\max\{a(n), b(n)\} \leq k c(n)$, which implies $[\max\{a(n), b(n)\}] \leq [c(n)]$, what is a contradiction. Therefore,

$$\sup\{[a(n)], [b(n)]\} = [\max\{a(n), b(n)\}],$$

   as we wanted.

4. (Supremum of suprema) If $[a(n)]$, $[b(n)]$, $[c(n)]$, $[d(n)] \in \mathbb{O}$, then

$$\sup\{\sup\{[a(n)],\ [b(n)]\},\ \sup\{[c(n)],\ [d(n)]\}\} = \sup\{[a(n)],\ [b(n)],\ [c(n)],\ [d(n)]\}.$$

In fact, we show that $\sup\{\cup_{i\in\Lambda} A_i\} = \sup\{\sup A_i;\ i \in \Lambda\}$. It is easy to see that we have $\sup\{\cup_{i\in\Lambda} A_i\} \geq a_i$, for all $a_i \in A_i$, for all $i \in \Lambda$. Then, $\sup A_i \leq \sup\{\cup_{i\in\Lambda} A_i\}$, for all $i \in \Lambda$, and then $\sup\{\sup A_i;\ i \in \Lambda\} \leq \sup\{\cup_{i\in\Lambda} A_i\}$. On the other hand, $\sup\{\sup A_i;\ i \in \Lambda\} \geq \sup A_i$, for all $i \in \Lambda$, and since $\sup A_i \geq a_i$, for all $a_i \in A_i$, for all $i \in \Lambda$, we have $\sup\{\sup A_i;\ i \in \Lambda\} \geq a_i$, for all $a_i \in A_i$, for all $i \in \Lambda$, and thus $\sup\{\cup_{i\in\Lambda} A_i\} \leq \sup\{\sup A_i;\ i \in \Lambda\}$.

Remark 1.1.1. The property 3, of the supremum, is only true when we consider the supremum of a finite set of orders of growth, in otherwise we can have that the supremum is an element of $\overline{\mathbb{O}}$, but not an element of $\mathbb{O}$.

There are many other properties of the space of orders of growth that we will not approach here, but there is another important property that we want to show in the following lemma:

Lemma 1.1.1. *Let $B \subset \mathbb{O}$ a countable subset of orders of growth, then there exists a countable and ordered subset $A \subset \mathbb{O}$ such that*

$$\sup(A) = \sup(B) \in \overline{\mathbb{O}}.$$

Proof. Let $B = \{[b_1(n)], [b_2(n)], \cdots, [b_k(n)], \cdots\} \subset \mathbb{O}$, we construct the subset A as follows:

$$\begin{aligned}
[a_1(n)] &= \sup\{[b_1(n)], [b_2(n)]\} = [\max\{b_1(n), b_2(n)\}],\\
[a_2(n)] &= \sup\{[a_1(n)], [b_3(n)]\} = \sup\{\sup\{[b_1(n)], [b_2(n)]\}, [b_3(n)]\}\\
&= \sup\{[b_1(n)], [b_2(n)], [b_3(n)]\} = [\max\{b_1(n), b_2(n), b_3(n)\}],\\
&\vdots\\
[a_k(n)] &= \sup\{[a_{k-1}(n)], [b_{k+1}(n)]\} = \cdots = [\max\{b_1(n), b_2(n), \cdots, b_{k+1}(n)\}],\\
&\vdots
\end{aligned}$$

Thus, $A = \{[a_1(n)], [a_2(n)], \cdots, [a_k(n)], \cdots\}$ is a subset of $\mathbb{O}$, by property 3 above. A is clearly a countable subset and it is easy see that it is a ordered set, in fact

$$\begin{aligned}
[a_1(n)] &= \sup\{[b_1(n)], [b_2(n)]\} \leq \sup\{[b_1(n)], [b_2(n)], [b_3(n)]\}\\
&= [a_2(n)] \leq \cdots [a_k(n)] \leq [a_{k+1}(n)] \leq \cdots
\end{aligned}$$

Let see now that $\sup(B) = \sup(A)$. Since $\sup(A) \geq [a_k(n)]$, for all $k \in \mathbb{N}$, and $[a_j(n)] \geq [b_j(n)]$, for all $j \leq k+1$, we have $\sup(A) \geq [b_j(n)]$, for all $j \in \mathbb{N}$, then $\sup(B) \leq \sup(A)$. On the other hand, $\sup(B) \geq [b_k(n)]$, for all $k \in \mathbb{N}$, but $[b_k(n)] \leq [a_k(n)]$, for all k, then $\sup(B) \geq [a_k(n)]$, for all $k \in \mathbb{N}$, and we have $\sup(A) \leq \sup(B)$. Hence, we conclude that A is a countable and ordered subset of $\mathbb{O}$ such that

$$\sup(A) = \sup(B),$$

as we wanted. $\square$

Let us consider a special subset in $\overline{\mathbb{O}}$; the subset of all orders of growth in $\overline{\mathbb{O}}$ that are supremum of countable sets and we denote it by

$$\underline{\mathbb{O}} = \{o \in \overline{\mathbb{O}}; o = \sup(B), \text{ where } B \subset \mathbb{O} \text{ is countable}\}.$$

We have $\mathbb{O} \subset \underline{\mathbb{O}} \subset \overline{\mathbb{O}}$, and we know that the set $\overline{\mathbb{O}}$ is really big and have many properties that we still do not understand completely.

The subset $\underline{\mathbb{O}}$ is special because the generalized entropy of continuous maps is always an element in $\underline{\mathbb{O}}$, which we will show later.

1.2 Generalized entropy

We want now to define the entropy of a dynamical system in the complete space of orders of growth. We will consider that the notion of topological entropy is a well-known subject. For more details see, for instance, [7], [13] and [14].

Given (M, d) a compact metric space and $f : M \to M$ a continuous map. We define in M the dynamical metric $d_n^f(x, y) = \max\{d(f^k(x), f^k(y)); 0 \leq k \leq n-1\}$, and we denote the dynamical ball as $B(x, n, \varepsilon) = \{y \in M; d_n^f(x, y) < \varepsilon\}$.

A set $G \subset M$ is (n, ε)-generator if $M = \cup_{x \in G} B(x, n, \varepsilon)$. By compactness of M there always exists a finite (n, ε)-generator set. We define then $g(f, \varepsilon, n)$ as the smallest possible cardinality of a finite (n, ε)-generator. If we fix $\varepsilon > 0$, then $g(f, \varepsilon, n)$ is an increasing sequence of natural numbers, in fact, $d_n^f(x, y) \leq d_{n+1}^f(x, y)$ implies $B(x, n+1, \varepsilon) \subset B(x, n, \varepsilon)$, then $g(f, \varepsilon, n) \leq g(f, \varepsilon, n+1)$, for all natural n.

The sequence $g_{f,\varepsilon} \in \mathcal{O}$ is defined by $g_{f,\varepsilon}(n) = g(f, \varepsilon, n)$. For a fixed n, if $\varepsilon_1 < \varepsilon_2$, then $g_{f,\varepsilon_1}(n) \geq g_{f,\varepsilon_2}(n)$ and we have $[g_{f,\varepsilon_1}(n)] \geq [g_{f,\varepsilon_2}(n)]$ in $\mathbb{O}$. We consider the set $G_f = \{[g_{f,\varepsilon}(n)] \in \mathbb{O} : \varepsilon > 0\}$, and the generalized entropy of f is defined by

$$o(f) = \text{“}\lim_{\varepsilon \to 0}\text{”}[g_{f,\varepsilon}(n)] = \sup G_f \in \overline{\mathbb{O}}.$$

We can also develop the generalized entropy through the point of view of (n,ε)-separated sets. We say that $E \subset M$ is (n,ε)-separated if $B(x,n,\varepsilon) \cap E = \{x\}$, for all $x \in E$. We define $s(f,\varepsilon,n)$ the maximal cardinality of an (n,ε)-separated set. Analogously, as with $g(f,\varepsilon,n)$, if we fix $\varepsilon > 0$, then we know that $s(f,\varepsilon,n)$ is a non-decreasing sequence of natural numbers. Then, we define the sequence $s_{f,\varepsilon} \in \mathcal{O}$ by $s_{f,\varepsilon}(n) = s(f,\varepsilon,n)$. Again, for a fixed n, if $\varepsilon_1 < \varepsilon_2$, thus $s_{f,\varepsilon_1}(n) \geq s_{f,\varepsilon_2}(n)$ and we have $[s_{f,\varepsilon_1}(n)] \geq [s_{f,\varepsilon_2}(n)]$ in $\mathbb{O}$.

The numbers $g_{f,\varepsilon}(n)$ and $s_{f,\varepsilon}(n)$ are relationated by the following lemma.

Lemma 1.2.1. *Let M a compact metric space and $f : M \to M$ a continous map. Then $g_{f,\varepsilon}(n) \leq s_{f,\varepsilon}(n) \leq g_{f,\frac{\varepsilon}{2}}(n)$, for all n and all $\varepsilon > 0$.*

Proof. A maximal (n,ε)-separated set $E \subset M$ is a (n,ε)-generator set, in fact, suppose that there exists $y \in M$ such that $d_n^f(x,y) \geq \varepsilon$ for all $x \in E$, then $E \cup \{y\}$ is a (n,ε)-separated, which is a contradiction with maximality of E. Then, $g_{f,\varepsilon}(n) \leq s_{f,\varepsilon}(n)$, for all n and all $\varepsilon > 0$.

Now, let E be a (n,ε)-separated set and G be a $(n,\frac{\varepsilon}{2})$-generator set. For each $x \in M$ we can associate a $g(x) \in G$ such that $x \in B(g(x),n,\frac{\varepsilon}{2})$. If $g(x) = g(y)$, then $d_n^f(x,y) < \varepsilon$, since E is (n,ε)-separated, it follows that g is injective on E. Hence $\#E \leq \#G$, for all E (n,ε)-separated and G $(n,\frac{\varepsilon}{2})$-generator, thus $s_{f,\varepsilon}(n) \leq g_{f,\frac{\varepsilon}{2}}(n)$, for all n and all $\varepsilon > 0$. ☐

If we consider the set $S_f = \{[s_{f,\varepsilon}(n)] \in \mathbb{O} : \varepsilon > 0\}$, then the generalized entropy of f can also be defined by

$$o(f) = \text{“}\lim_{\varepsilon \to 0}\text{”}[s_{f,\varepsilon}(n)] = \sup S_f \in \overline{\mathbb{O}}.$$

Theorem 1.2.1. *Let M and N be two compact metric spaces and $f : M \to M$, $g : N \to N$ two continuous map. Suppose there exists $h : M \to N$ a homeomorphism such that $h \circ f = g \circ h$. Then $o(f) = o(g)$.*

Proof. Given $\varepsilon > 0$, consider $\delta > 0$, from the uniform continuity of h, such that $d(x,y) < \delta$ implies $d_n^f(x,y) < \varepsilon$. Let $E \subset N$ be an (n,ε)-separated set of g. We will show that $h^{-1}(E) \subset M$ is an (n,δ)-separated set of f. In fact, suppose that there exists $x, y \in h^{-1}(E)$ such that $d_n^f(x,y) < \delta$, this is $d(f^k(x), f^k(y)) < \delta$, for all $0 \leq k \leq n-1$. Then $d(h(f^k(x)), h(f^k(y)) < \varepsilon$, since h conjugates f and g, we have $d(g^k(h(x)), g^k(h(y)) < \varepsilon$, for all $0 \leq k \leq n-1$, what is a contradiction, since E is (n,ε)-separated and $h(x), h(y) \in E$.

If E is maximal we have $s_{g,\varepsilon}(n) = \#E = \#h^{-1}E \leq s_{f,\delta}(n)$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$, and $\delta \to 0$ when $\varepsilon \to 0$, such that $[s_{g,\varepsilon}(n)] \leq [s_{f,\delta}(n)]$, and thus $o(g) \leq o(f)$.

Since h is a homeomorphism we analogously prove that $o(f) \leq o(g)$ and then we conclude that $o(f) = o(g)$. $\square$

Example 1.2.1. Let $\Sigma_k = \{0, \cdots, k-1\}^{\mathbb{Z}}$ be the space of infinite sequences of symbols in $\{0, \cdots, k-1\}$, equipped with the metric d defined as

$$d(\underline{a}, \underline{b}) = \sum_{n=-\infty}^{\infty} \frac{|a_n - b_n|}{(10k)^{|n|}},$$

for $\underline{a}, \underline{b} \in \Sigma_k$. Then the symetric cylinder $C_m^{\alpha} = \{\underline{a} \in \Sigma_k; a_i = \alpha_i, |i| \leq m\}$, for a finite sequence of symbols $\alpha = (\alpha_{-m}, \cdots, \alpha_0, \cdots, \alpha_m)$, coincides with the ball centered in each $\underline{a}$ in the cylinder C_m^{α} with radius $\varepsilon_m = \frac{1}{2(10k)^m}$.

Consider the shift $\sigma : \Sigma_k \to \Sigma_k$ and fix the symbols $\alpha_{-m}, \cdots, \alpha_0, \cdots, \alpha_{m+n}$, then the cylinder $C_{(-m,\cdots,m+n)}^{(\alpha_{-m},\cdots,\alpha_{m+n})}$ coincides with the dynamical ball $B(\underline{a}, n, \varepsilon_m)$ centered in each element $\underline{a}$ in the cylinder, associated with the dynamical metric d_n^{σ}.

Thus, any two dynamic balls of radius ε_m are either equal or disjunct, and there are exactly k^{n+2m+1} cylinders of type $C_{(-m,\cdots,m+n)}^{(\alpha_{-m},\cdots,\alpha_{m+n})}$ in Σ_k. If we consider a cover of Σ_k by such dynamical balls, we find a (n, ε)-generator set that is minimal and has cardinality equal to k^{n+2m+1}.

Then,

$$g_{\sigma,\varepsilon_m}(n) = k^n \cdot k^{2m+1} = \exp(\log(k^n \cdot k^{2m+1})) = \exp(\log k^n) \cdot c(m),$$

where $c(m)$ is a constant which does not depend on n. When we consider the order of growth associated to such sequence, we have $[g_{\sigma,\varepsilon_m}(n)] = [\exp(\log k \cdot n)]$, for all $\varepsilon_m > 0$. And we conclude that $o(\sigma) = [\exp(\log k \cdot n)]$.

Recalling that the topological entropy of a map is defined as

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(g_{f,\varepsilon}(n)).$$

The natural question is how the generalized entropy is related to topological entropy. The answer to this question is that the classical notion of topological entropy is the projection of the generalized entropy into the family of exponential orders of growth. The exponential orders of growth are the classes of the sequences $\{\exp(tn)\}_{n\in\mathbb{N}}$ where t is a number between 0 and ∞. Then, the family of exponential orders of growth is the set $\mathbb{E} = \{[\exp(tn)]; t \in (0, \infty)\} \subset \mathbb{O}$.

Although it is not necessary for now, we take the opportunity to remark that the elements $\inf(\mathbb{E})$ and $\sup(\mathbb{E})$ belong to $\overline{\mathbb{O}}$ and are both abstract orders of growth which are not realizable by any sequence.

Once we have established the family of exponential growths $\mathbb{E}$, we say how to compare an element $o \in \overline{\mathbb{O}}$ with $\mathbb{E}$. Given $o \in \overline{\mathbb{O}}$ we consider the interval $I_{\mathbb{E}}(o) = \{t \in (0,\infty); o \leq [\exp(tn)]\} \subset \mathbb{R}$. We would like to observe that the order of growth o might not be comparable to any element of $\mathbb{E}$ and therefore the set $I_{\mathbb{E}}(o)$ might be the empty set. In any case, we define the projection $\pi_{\mathbb{E}} : \overline{\mathbb{O}} \to [0,\infty]$ as following:

$$\pi_{\mathbb{E}}(o) = \begin{cases} \inf(I_{\mathbb{E}}(o)), & \text{if } I_{\mathbb{E}}(o) \neq \emptyset \\ \infty, & \text{if } I_{\mathbb{E}}(o) = \emptyset \end{cases}.$$

Now that we know how to project a order of growth into the family of exponential orders of growth, we can relate generalized entropy and the classical topological entropy. This theorem is due to Correa and Pujals, see [3].

Theorem 1.2.2. *Let M be a compact metric space and $f : M \to M$ a continuous map. Then, $\pi_{\mathbb{E}}(o(f)) = h_{top}(f)$. And, $o(f) \leq \sup(\mathbb{E})$.*

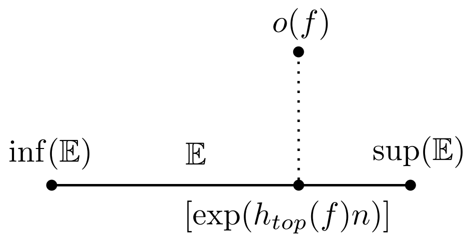


Figure 1.2: theorem 1.2.2: projection over $\mathbb{E}$.

Looking back to the example 1.2.1 we can see that $\pi_{\mathbb{E}}(o(\sigma)) = h_{top}(\sigma) = \log k$.

It is important to point out that the projection is into the closure of the set of indexes that define $\mathbb{E}$ and not into $\mathbb{E}$ itself. The reason for this is that the set $\mathbb{E}$ is not a closed set. In fact, for every $\varepsilon > 0$, we have

$$\exp((t-\varepsilon)\cdot n) < \frac{\exp(tn)}{n} < \exp(tn) < \exp(tn)\cdot n < \exp((t+\varepsilon)\cdot n),$$

and when we consider the classes, we see that set $\mathbb{E}$ is a discrete set.

Another important family that we consider in $\overline{\mathbb{O}}$ is the polynomial orders of growth, which are the classes of the sequences $\{n^t\}_{n\in\mathbb{N}}$ where t is a number between 0 and ∞. We denote

$$\mathbb{P} = \{[n^t]; t \in (0,\infty)\} \subset \mathbb{O}.$$

Just as happens to the family $\mathbb{E}$, the elements $\inf \mathbb{P}$ and $\sup(\mathbb{P})$ are not elements in $\mathbb{O}$. They are both abstract orders of growth which belongs to $\overline{\mathbb{O}}$ and are not realizable by sequences in $\mathbb{O}$. We also have how to compare an element $o \in \overline{\mathbb{O}}$ with

$\mathbb{P}$. Given $o \in \overline{\mathbb{O}}$, we consider the interval $I_{\mathbb{P}} = \{t \in (0, \infty); o \leq [n^t]\} \subset \mathbb{R}$. And we define a similar projection to the exponential case by $\pi_{\mathbb{P}} : \overline{\mathbb{O}} \to [0, \infty]$ as

$$\pi_{\mathbb{P}}(o) = \begin{cases} \inf(I_{\mathbb{P}}(o)), & \text{if } I_{\mathbb{P}}(o) \neq \emptyset \\ \infty, & \text{if } I_{\mathbb{P}}(o) = \emptyset \end{cases} .$$

And, with this tool to project orders of growth into the family of polynomial orders of growth, we can relate generalized entropy with polynomial entropy, as in Theorem 1.2.2. Before, let us remember that the polynomial entropy of a continuous map is defined as

$$h_{pol}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log(g_{f,\varepsilon}(n))}{\log n}.$$

For the definition in details, see [12]. The projection of the generalized entropy of a map f over the family $\mathbb{P}$ provides us the polynomial entropy of such map, that is, $\pi_{\mathbb{P}}(o(f)) = h_{pol}(f)$.

There is an important element of $\overline{\mathbb{O}}$ that deserves our attention. Since $\overline{\mathbb{O}}$ is a complete lattice, it has a minimum. The interesting is that his minimum is in fact an element of $\mathbb{O}$ and it is the equivalence class of the constant sequence. To simplify the notation we denote such element by 0.

The maps such that satisfies $o(f) = 0$ are those where there is none separation of orbits. Correa and Pujals, [3], in the following theorem tells us which is the class of such maps. Furthermore, it gives us a condition to obtain at least linear growth.

Theorem 1.2.3. *Let M be a compact metric space and $f : M \to M$ a continuous map. Then, $o(f) = 0$ if and only if f is Lyapunov stable. In addition, if f is a homeomorphism, and there exists $x \in M$ such that $x \notin \alpha(x)$, then $o(f) \geq [n]$.*

The first part of theorem 1.2.3, has been proved by Blanchard, Host, and Massin in [4], where the property $o(f) = 0$ is called bounded complexity and Lyapunov stable maps are called equicontinuous. However, Correa and Pujals gave an alternate proof in [3].

As a corollary of the second part of the theorem 1.2.3, we have:

Corollary 1.2.1. *Let $f : M \to M$ be a continuous map on a compact metric space. If $o(f) < [n]$, then every point is recurrent and therefore $Rec(f) = \Omega(f) = M$. In particular, when M is connected, f has a point x whose w-limit is not a periodic orbit.*

Let us consider again the subset $\underline{\mathbb{O}}$ of all orders of growth in $\overline{\mathbb{O}}$ that are supremum of countable sets,

$$\underline{\mathbb{O}} = \{o \in \overline{\mathbb{O}}; o = \sup(B), B \subset \mathbb{O} \text{ is countable}\}.$$

As we said before, this set is special, because it contains the generalized entropy of continuous maps.

Proposition 1.2.1. $\{o(f); f$ *is a continuous map*$\} \subset \underline{\mathbb{O}}$.

Proof. By the definition of generalized entropy, we have

$$o(f) = \sup\{[g_{f,\varepsilon}(n)] \in \mathbb{O} : \varepsilon > 0\} = \sup\{[g_{f,1/m}(n)]; m \in \mathbb{N}\} \in \underline{\mathbb{O}}.$$

□

With all the discussion until here, the set $\{o(f); f$ is a continuous map$\}$ can be represented as follows:

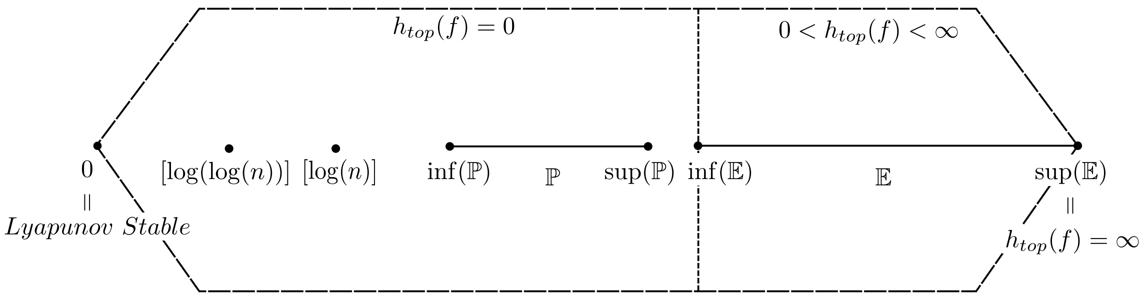


Figure 1.3: $\{o(f); f$ is a continuous map$\}$.

We end this section by presenting some important properties of classical entropy which also are verified by generalized entropy. The topological entropy of a map f is related to the topological entropy of f^m, $m \geq 1$, by the formula $h_{top}(f^m) = m \cdot h_{top}(f)$. In the polynomial case we have $h_{pol}(f^m) = h_{pol}(f)$. For the generalized entropy, we have:

Proposition 1.2.2. *Let M be a compact metric space and $f : M \to M$ a continuous map. The following inequalities hold:*

$$o(f) \leq o(f^2) \leq \cdots \leq o(f^m) \leq \cdots$$

Proof. Observe that $d_n^{f^m}(x,y) \leq \max\{d(f^i(x), f^i(y)); 0 \leq i \leq mn-1\} = d_{mn}^f(x,y)$, for every $m \geq 1$. Then, we have $g_{f^m,\varepsilon}(n) \leq g_{f,\varepsilon}(m \cdot n)$. On the other hand, by uniform continuity, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $B(x,\delta) \subset B(x,m,\varepsilon)$, for every $x \in M$. Then,

$$B_{f^m}(x,n,\delta) = \bigcap_{k=0}^{n-1} f^{-km}\left(B(f^{km}(x),\delta)\right) \subset \bigcap_{k=0}^{n-1} f^{-km}\left(B(f^{km}(x),m,\varepsilon)\right) = B_f(x,m{\cdot}n,\varepsilon),$$

where $B_{f^m}(x,n,\delta)$ represents the dynamical ball with respect the metric $d_n^{f^m}$ and $B_f(x,m\cdot n,\varepsilon)$ represents the dynamical ball with respect the metric d^f_{mn}. Thus, $g_{f,\varepsilon}(m{\cdot}n) \leq g_{f^m,\delta}(n)$. And we see that $g_{f^m,\varepsilon}(n) \leq g_{f,\varepsilon}(m{\cdot}n) \leq g_{f^m,\delta}(n)$. Since when $\varepsilon \to 0$ we have $\delta \to 0$, we see $o(f^m) \leq o(f,m) \leq o(f^m)$, that is $o(f^m) = o(f,m)$, where $o(f,m) = \sup\{[g_{f,\varepsilon}(m \cdot n)] \in \mathbb{O} : \varepsilon > 0\}$.

Since $g_{f,\varepsilon}$ is non-decreasing, we have $g_{f,\varepsilon}(n) \leq g_{f,\varepsilon}(2{\cdot}n) \leq \cdots \leq g_{f,\varepsilon}(m{\cdot}n) \leq \cdots$. Therefore,

$$o(f) \leq o(f^2) \leq \cdots \leq o(f^m) \leq \cdots,$$

as we wanted. □

When f is a homeomorphism, we know that $h_{top}(f) = h_{top}(f^{-1})$, the same for the polinomial case. And this property is also true for $o(f)$.

Proposition 1.2.3. *Let M be a compact metric space and $f : M \to M$ a homeomorphism, then $o(f) = o(f^{-1})$.*

Proof. If E is an (n,ε)-separated set for f, then $f^{n-1}(E)$ is an (n,ε)-separated set for f^{-1}. Thus, $s_{f,\varepsilon}(n) = s_{f^{-1},\varepsilon}(n)$, and we conclude

$$o(f) = o(f^{-1}).$$

□

Chapter 2

Wandering generalized entropy

2.1 Context and definition

Let (W, d) be a metric space, and $g : W \to W$ be a homeomorphism. Remember that a set Y is wandering if $g^n(Y) \cap Y = \emptyset$ for every $n \neq 0$, and a point is wandering if it admits a wandering neighborhood. We denote the set of non-wandering points of g by $\Omega(g)$, which is invariant under g.

Let $\sim$ denote the relation that identifies every point of $\Omega(g)$ at a single point, which we are going to denote by ∞. Let $\tilde{W} = W/_\sim$ denotes the quotient obtained by such identification and $\tilde{g} : \tilde{W} \to \tilde{W}$ the induced homeomorphism. Observe that every point of $\tilde{W}$ is wandering under $\tilde{g}$, except for the point ∞. The Poincaré recurrence theorem implies that the only invariant measure for $\tilde{g}$ is the Dirac measure at the point at ∞, and thus the topological entropy of $\tilde{g}$ vanishes. This motivates the following definition.

Definition 2.1.1. *The wandering generalized entropy of a homeomorphism $g : W \to W$ in a metric space is the generalized entropy of $\tilde{g} : \tilde{W} \to \tilde{W}$. We denote $o_w(g) = o(\tilde{g})$.*

The topological space $\tilde{W}$ is metrizable and a natural metric is given by $\tilde{d}(x, y) = \min\{d(x, y), d(x, \Omega(g)) + d(y, \Omega(g))\}$, if $x, y \in \tilde{W} \setminus \{\infty\}$, and $\tilde{d}(x, \infty) = d(x, \Omega(g)) = \inf\{d(x, y); y \in \Omega(g)\}$ otherwise. Thus we are led to study the generalized entropy of a homeomorphism of a compact metric space whose non-wandering set is reduced to one single point. This is the setting where we are going to develop the general theory of the wandering generalized entropy.

Throughout this chapter, X denotes a compact metric space, and ∞ denotes some given point of X. We consider a homeomorphism $f : X \to X$ that fixes ∞

and satisfies that every point except ∞ is a wandering point.

2.2 Coding

Let $\mathcal{F}$ be a finite family of non empty subsets of $X \setminus \{\infty\}$. We denote by $\cup\mathcal{F}$ the union of all the elements of $\mathcal{F}$, and by $\infty_{\mathcal{F}}$ the complement of $\cup\mathcal{F}$ (when there is no risk of confusion with the point ∞ we will denote it just by ∞). We fix a positive integer n. Let $\underline{x} = (x_0, \ldots, x_{n-1})$ be a finite sequence of points in X, and $\underline{w} = (w_0, \ldots, w_{n-1})$ be a finite sequence of elements of $\mathcal{F} \cup \{\infty_{\mathcal{F}}\}$. We say that $\underline{w}$ is a coding of $\underline{x}$, relative to $\mathcal{F}$, if for every $k = 0, \ldots, n-1$ we have $x_k \in w_k$. Observe that if the sets of $\mathcal{F}$ are not disjoint we can have several codings for a sequence. We denote by $\mathcal{A}_n(f, \mathcal{F})$ the set of all codings of all orbits $(x, f(x), \ldots, f^{n-1}(x))$ of length n. We define the sequence $c_{f,\mathcal{F}}(n) = \#\mathcal{A}_n(f, \mathcal{F})$ and is easy to see that $c_{f,\mathcal{F}} \in \mathcal{O}$, this is, $c_{f,\mathcal{F}}(n)$ is a non-decreasing sequence. If $\mathcal{F} = \{Y\}$ we denote $c_{f,\mathcal{F}}(n) = c_{f,Y}(n)$.

Example 2.2.1: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation $(x, y) \mapsto (x + 1, y)$. To fit our setting we first compactify the plane by adding the point at infinity to get a set X and a map $f : X \to X$ such that $\Omega(f) = \{\infty\}$; but since we work with subsets of $X \setminus \{\infty\}$ we may identify them with subsets of $\mathbb{R}^2$. Let Y be a compact subset of $\mathbb{R}^2$ and let suppose that its diameter is less than 1, then we have $c_{f,Y}(n) = n$. Indeed, the elements of $\mathcal{A}_n(f, Y)$ are exactly all the words of the form $(\infty, \ldots, \infty, Y, \infty, \ldots, \infty)$, thus it contains n elements.

Example 2.2.2: Let A be the linear map $(x, y) \mapsto (\lambda x, y/\lambda)$, where $\lambda > 1$, in the plane. Again, we need to compactify the plane by adding the point at infinity, and then identify the point at infinity and the fixed point 0 to get the set X and the map f; and since we work with subsets of $X \setminus \{\infty\}$ we also may identify them with subsets of $\mathbb{R}^2 \setminus \{0\}$. Let Y_1, Y_2 be two disks, not containing the origin, whose interiors met respectively the 'y' and the 'x' axes. To simplify the computation we assume the disks are small in order to each one does not meet its image under A.

First, we have $c_{f,Y_1}(n) = c_{f,Y_2}(n) = n$, as in the example 2.2.1. Next, we have $c_{f,\{Y_1,Y_2\}}(n) = n(n-1)/2$. Indeed, the elements of $\mathcal{A}_n(f, \{Y_1, Y_2\})$ are exactly all the words of the form

$$(\infty, \ldots, \infty, Y_1, \infty, \ldots, \infty, Y_2, \infty, \ldots, \infty),$$

and thus it has $n(n-1)/2$ elements. Then, we see that $[c_{f,\{Y_1,Y_2\}}] = [n^2]$.

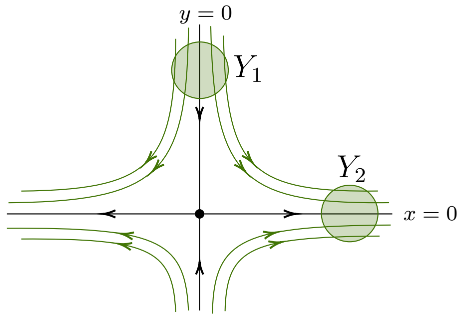


Figure 2.1: example 2.2.2.

Example 2.2.3 (Reeb's flow/Brouwer's counter-example): Consider in the plane the regions $R_1 = \{(x,y) \in \mathbb{R}^2; y \geq 1\}$, $R_2 = \{(x,y) \in \mathbb{R}^2; -1 < y < 1\}$ and $R_3 = \{(x,y) \in \mathbb{R}^2; y \leq -1\}$ and the map $H : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\begin{array}{rcl} H|_{R_1} : R_1 & \to & R_1 \\ (x,y) & \mapsto & (x+1,y) \end{array} \quad , \quad \begin{array}{rcl} H|_{R_3} : R_3 & \to & R_3 \\ (x,y) & \mapsto & (x-1,y) \end{array} .$$

In R_2, let us consider the curves $\gamma_c = \dfrac{1}{(y+1)(y-1)} + c$, $c \in \mathbb{R}$, parametrized by the arc-lenght, and we define $H|_{R_2}$ such that for every $c \in \mathbb{R}$ and every pair $(x,y) \in \gamma_c$ we have $d_c((x,y), H(x,y)) = 1$, where $d_c(\cdot,\cdot)$ is the distance in γ_c. The map H is the time-one of Reeb's flow (see figure 2.2) and it is an example of a Brouwer homeomorphism that is not conjugated to a translation. In fact, for a translation f, given any compact subset $K \subset \mathbb{R}^2$, there exists $n_0 \in \mathbb{N}$ such that $f^n(K) \cap K = \emptyset$, for every $n \geq n_0$. For H, let K be a line segment joining a point in the line $y = 1$ to a point in the line $y = -1$, then we have $H^n(K) \cap K \neq \emptyset$, for every $n \in \mathbb{Z}$.

Let Y_1, Y_2 again be two disks, not containing the origin, whose interiors now meet respectively the lines $y = 1$ and $y = -1$. Again, to simplify the computation, we assume the disks are small in order to each one does not meet its image under H. By the same argument in the example 2.2.2 we conclude that $[c_{f,\{Y_1,Y_2\}}] = [n^2]$, where the map f is the induced map in the compactification of the plane.

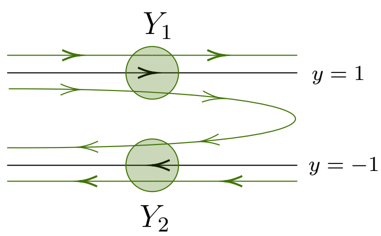


Figure 2.2: example 2.2.3.

The idea of these examples will be generalized with the notion of singular sets in section 2.4 below.

We have $[c_{f,\mathcal{F}}(n)] \in \mathbb{O}$ and we want to show that the generalized entropy of f can be calculated as the supremum of $[c_{f,Y}(n)]$ taken among all compact sets Y of $X \setminus \{\infty\}$, formulated in the following proposition.

Proposition 2.2.1. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism such that $\Omega(f) = \{\infty\}$. Then*

$$o(f) = \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \textit{ is compact}\} \in \overline{\mathbb{O}}.$$

Remark 2.2.1. Looking back to the examples 2.2.1 and 2.2.3, the proposition 2.2.1 provides us a tool to prove that Brouwer's counter-example is not conjugated to the translation, since their generalized entropy are not equal.

Before proving the proposition we will show a few properties of $[c_{f,\mathcal{F}}(n)]$. For every subset Y of $X \setminus \{\infty\}$ we denote by $M(Y)$ the maximum number of terms of an orbit that belongs to Y, $M(Y) = \sup_{x\in X} \#\{n; f^n(x) \in Y\}$.

Observe that if Y is compact, it may be covered by a finite number of wandering open sets, and every orbit intersects a wandering set at most once, thus in this case $M(Y)$ is finite. Also observe that if n is large compared to $M(\cup\mathcal{F})$, then most of the letters of a word in $\mathcal{A}_{f,\mathcal{F}}(n)$ are equal to $\infty_{\mathcal{F}}$. This remark leads to the following lemma.

Lemma 2.2.1. *Let $\mathcal{F}$ be a finite family of subsets of X such that $M(\cup\mathcal{F}) < +\infty$.*

1. *(monotonocity) Let $\mathcal{F}'$ be another finite family of subsets of X. If each element of $\mathcal{F}'$ is included in an element of $\mathcal{F}$, that we will denote $\mathcal{F}' \subset \mathcal{F}$, then*

$$[c_{f,\mathcal{F}'}(n)] \leq [c_{f,\mathcal{F}}(n)].$$

2. *(aditivity)*

$$[c_{f,\cup\mathcal{F}}(n)] = [c_{f,\mathcal{F}}(n)].$$

3. *(wandering aditivity) If $\mathcal{F} = \{Y_1, \ldots, Y_L\}$ is such that $Y_1 \cup Y_2$ is wandering, then*

$$[c_f, \mathcal{F}(n)] = \sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\},$$

where $\mathcal{F}_1 = \{Y_1, Y_3, \ldots, Y_L\}$ and $\mathcal{F}_2 = \{Y_2, Y_3, \ldots, Y_L\}$.

Proof. To prove the first point, we fix an integer n and define a map Φ from $\mathcal{A}_n(f,\mathcal{F}')$ to $\mathcal{A}_n(f,\mathcal{F})$ in the following way. Let w' be a word in $\mathcal{A}_n(f,\mathcal{F}')$, take $x \in X$ such that w' is the coding of $\{x, f(x), \ldots, f^{n-1}(x)\}$ relative to $\mathcal{F}'$, and we choose some coding $\Phi(w')$ of $\{x, f(x), \ldots, f^{n-1}(x)\}$ relative to $\mathcal{F}$. Let us evaluate the number of inverse images w' of some word w in $\mathcal{A}_n(f,\mathcal{F})$. The word w codes the orbit of some point x relative to $\mathcal{F}$. We have that the ith letter of w is $'\infty'$ exactly when $f^i(x) \notin \cup\mathcal{F}$, in which case $f^i(x) \notin \cup\mathcal{F}'$ and thus the ith letter in w' has to be $'\infty'$ also. On the other hand there are at most $M(\cup\mathcal{F})$ letters in w which are distinct from $'\infty'$, and since w' is a word on an alphabet consisting of $\#\mathcal{F}'+1$ letters, this gives at most

$$k = (\#\mathcal{F}'+1)^{M(\cup\mathcal{F})},$$

possibilities for w'. We deduce that

$$\#A_n(f,\mathcal{F}') \leq k \cdot \#\mathcal{A}_n(f,\mathcal{F}),$$

whence

$$c_{f,\mathcal{F}'}(n) \leq k \cdot c_{f,\mathcal{F}}(n),$$

and, since k does not depends on n this gives the inequality $[c_{f,\mathcal{F}'}(n)] \leq [c_{f,\mathcal{F}}(n)]$, as wanted.

Let us turn to the second point. The first point applies to the families $\mathcal{F}$ and $\{\cup\mathcal{F}\}$ and provides the inequality $[c_{f,\mathcal{F}}(n)] \leq [c_{f,\cup\mathcal{F}}(n)]$. On the other hand, every word in $\mathcal{A}_n(f,\cup\mathcal{F})$ is one word in $\mathcal{A}_n(f,\mathcal{F})$, then $c_{f,\cup\mathcal{F}}(n) \leq c_{f,\mathcal{F}}(n)$ and thus $[c_{f,\cup\mathcal{F}}(n)] \leq [c_{f,\mathcal{F}}(n)]$. Then,

$$[c_{f,\cup\mathcal{F}}(n)] = [c_{f,\mathcal{F}}(n)].$$

Finally we prove the third point. Since $Y_1 \cup Y_2$ is wandering, no word in $\mathcal{A}_n(f,\mathcal{F})$ contains both letters $'Y_1'$ and $'Y_2'$, as a consequence

$$\mathcal{A}_n(f,\mathcal{F}) = \mathcal{A}_n(f,\mathcal{F}_1) \cup A_n(f,\mathcal{F}_2),$$

and this is a disjoint union. Thus

$$\begin{array}{rcl} c_{f,\mathcal{F}}(n) & = & c_{f,\mathcal{F}_1}(n) + c_{f,\mathcal{F}_2}(n) \\ & \leq & 2 \cdot \max\{c_{f,\mathcal{F}_1}(n), c_{f,\mathcal{F}_2}(n)\}, \end{array}$$

then

$$[c_{f,\mathcal{F}}(n)] \leq [\max\{c_{f,\mathcal{F}_1}(n), c_{f,\mathcal{F}_2}(n)\}] = \sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\}.$$

To the reverse inequality, applying the first point, we deduce $c_{f,\mathcal{F}_i}(n) \leq c_{f,\mathcal{F}}(n)$, for $i = 1, 2$, thus

$$\max\{c_{f,\mathcal{F}_1}(n), c_{f,\mathcal{F}_2}(n)\} \leq c_{f,\mathcal{F}}(n),$$

then,

$$\sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\} = [\max\{c_{f,\mathcal{F}_1}(n), c_{f,\mathcal{F}_2}(n)\}] \leq [c_{f,\mathcal{F}}(n)],$$

which entails the wanted equality.

□

The following lemma says that we can estimate the order of the growth of the cardinality of the set of codings relative to a compact subset Y of $X \setminus \{\infty\}$ in terms of a family of subsets of Y with controlled diameters. We recall that $\mathcal{F}' \subset \mathcal{F}$ means that each element of the family $\mathcal{F}'$ is included in an element of the family $\mathcal{F}$, and saying $\mathcal{F}'$ is disjoint we mean that its elements are pairwise disjoint.

Lemma 2.2.2. *For every compact subset Y of $X \setminus \{\infty\}$, and every $\varepsilon > 0$ there exists a finite family $\mathcal{F} = \{Y_1, \ldots, Y_L\}$ of wandering compact subsets of Y with diameters less than ε, such that*

$$[c_{f,Y}(n)] \leq \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \textit{ is disjoint}\}.$$

Proof. Since $Y \subset X \setminus \{\infty\}$, every point of Y admits a wandering compact neighborhood, and by compactness, up to disminishing ε, every subset of Y of diameter less than 2ε is wandering. Again by compactness there is a finite cover $\mathcal{F} = \{Y_1, \ldots, Y_k\}$ of Y by wandering compact subsets of Y with diameter less than ε. Then, for every $i, j \in \{1, \ldots, k\}$, if Y_i meets Y_j, then $Y_i \cup Y_j$ is a wandering set. And, we have $[c_{f,Y}(n)] \leq [c_{f,\mathcal{F}}(n)]$.

If $\mathcal{F}$ is disjoint, since $[c_{f,\mathcal{F}'}(n)] \leq [c_{f,\mathcal{F}}(n)]$ for every $\mathcal{F}' \subset \mathcal{F}$ and as $[c_{f,\mathcal{F}}(n)] \in \{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\}$ we have $[c_{f,\mathcal{F}}(n)] = \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\}$ and it is proved.

Otherwise we select two distinct elements which intersects, and to simplify the notation we can name it Y_1 and Y_2. Then by wandering aditivity

$$[c_{f,\mathcal{F}}(n)] = \sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\},$$

where $\mathcal{F}_1 = \{Y_1, Y_3, \ldots, Y_k\}$ and $\mathcal{F}_2 = \{Y_2, Y_3, \ldots, Y_k\}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both disjoint, we have

$$\sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}](n)\} \leq \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\},$$

and any disjoint family $\mathcal{F}' \subset \mathcal{F}$ satisfies $\mathcal{F}' \subset \mathcal{F}_1$ or $\mathcal{F}' \subset \mathcal{F}_2$, thus $[c_{f,\mathcal{F}'}(n)] \leq [c_{f,\mathcal{F}_1}(n)]$ or $[c_{f,\mathcal{F}'}(n)] \leq [c_{f,\mathcal{F}_2}](n)$, then $[c_{f,\mathcal{F}'}(n)] \leq \sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\}$, that implies

$$\sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\} \leq \sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\}.$$

Thus

$$[c_{f,\mathcal{F}}(n)] = \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\},$$

and it is proved.

If $\mathcal{F}_i$ are not disjoint, for $i = 1, 2$, we select two distinct elements which intersects, and we can name it Y_{i1} and Y_{i2}, and again, by wandering aditivity

$$[c_{f,\mathcal{F}_i}(n)] = \sup\{[c_{f,\mathcal{F}_{i1}}(n)], [c_{f,\mathcal{F}_{i2}}(n)]\},$$

and if $\mathcal{F}_{i1}, \mathcal{F}_{i2}$, for $i =, 2$, are all disjoint we have

$$[c_{f,\mathcal{F}}(n)] = \sup\{\sup\{[c_{f,\mathcal{F}_{11}}(n)], [c_{f,\mathcal{F}_{12}}(n)]\}, \sup\{[c_{f,\mathcal{F}_{21}}(n)], [c_{f,\mathcal{F}_{22}}(n)]\}\},$$

that implies, by property of Supremum of suprema,

$$[c_{f,\mathcal{F}}(n)] = \sup\{[c_{f,\mathcal{F}_{11}}(n)], [c_{f,\mathcal{F}_{12}}(n)], [c_{f,\mathcal{F}_{21}}(n)], [c_{f,\mathcal{F}_{22}}(n)]\},$$

where $\mathcal{F}_{i1} = \{Y_{i1}, Y_{i3}, \dots, Y_{ik}\}$, $\mathcal{F}_{i2} = \{Y_{i2}, Y_{i3}, \dots, Y_{ik}\}$, for $i = 1, 2$. And, again, we have

$$\sup\{[c_{f,\mathcal{F}_{11}}(n)], [c_{f,\mathcal{F}_{12}}(n)], [c_{f,\mathcal{F}_{21}}(n)], [c_{f,\mathcal{F}_{22}}(n)]\} = \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\},$$

since any disjoint family $\mathcal{F}' \subset \mathcal{F}$ satisfies $\mathcal{F}'$ is a subset of one of the families $\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{21}, \mathcal{F}_{22}$. Then, we deduce

$$[c_{f,\mathcal{F}}(n)] = \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\},$$

and it is proved.

If $\mathcal{F}_{i1}$ or $\mathcal{F}_{i2}$ are not all disjoint, for $i = 1, 2$, we repeat the algorithm until produce disjoint families of subsets of $\mathcal{F}$ and conclude

$$[c_{f,\mathcal{F}}(n)] = \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\},$$

as we wanted.

□

The next lemma provides the lower bound for generalized entropy.

Lemma 2.2.3. *For every compact subset Y of $X \setminus \{\infty\}$ we have*

$$[c_{f,Y}(n)] \le o(f).$$

Proof. By the preceding lemma there exists a finite family $\mathcal{F}$ of compact subsets of Y whose elements are wandering and such that

$$[c_{f,Y}(n)] \le \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\}.$$

Let $\mathcal{F}' = \{Y_1, \ldots, Y_L\}$ be a disjoint family of subsets of $\mathcal{F}$ and choose some compact disjoint wandering respective neighborhoods $U_1, \ldots, U_L$ of the elements $Y_1, \ldots, Y_L$. Let $\varepsilon_0 > 0$ be smaller, for every $i = 1, \ldots, L$, than the distance from Y_i to the complement of U_i.

Fix some positive integer n. For every $\mathcal{G} \subset \mathcal{F}'$ let $\mathcal{A}_n(\mathcal{F}', \mathcal{G})$ denote the set of elements of $\mathcal{A}_n(f, \mathcal{F}')$ whose set of the letters is exactly $\mathcal{G} \cup \{\infty_{\mathcal{F}'}\}$. We fix some $\mathcal{G} \subset \mathcal{F}'$ and we consider two points x, y in X and two words $\underline{w} = (w_0, \ldots, w_{n-1})$, $\underline{z} = (z_0, \ldots, z_{n-1})$ in $\mathcal{A}_n(\mathcal{F}, \mathcal{G})$ which represent respectively the orbits $(x, \ldots, f^{n-1}(x))$ and $(y, \ldots, f^{n-1}(y))$. Then, we claim that if the symbols $\underline{w}$ and $\underline{z}$ are distinct the points x and y are (n, ε_0)-separated. Indeed, let $i \in \{0, \ldots, n-1\}$ be such that $w_i \ne z_i$. If both $w_i \ne \infty$, $z_i \ne \infty$, then $f^i(x)$ and $f^i(y)$ belongs to distinct sets Y_i's, and these are more than ε_0 apart. If, say, $w_i = \infty$, then $f^i(y) \in Y_{z_i}$ and $f^i(x) \notin Y_{z_i}$. By definition of $\mathcal{A}_n(\mathcal{F}', \mathcal{G})$ there exists some $j \ne i$ in $\{0, \ldots, n-1\}$ such that $f^j(x) \in Y_{z_i} \subset U_{z_i}$. Since U_{z_i} is wandering, we see that $f^i(x) \notin U_{z_i}$, thus again $f^i(x)$ and $f^i(y)$ are more than ε_0 apart, and the claim is proved.

As an immediate consequence we have that $\#\mathcal{A}_n(\mathcal{F}', \mathcal{G}) = c_{f,\mathcal{G}}(n) \le s_{f,\varepsilon_0}(n)$. Since the $\mathcal{A}_n(\mathcal{F}', \mathcal{G})$'s form a partition of $\mathcal{A}_n(f, \mathcal{F}')$ into $2^{\#\mathcal{F}'} = 2^L$ elements we have

$$c_{f,\mathcal{F}'}(n) \le 2^L \cdot c_{f,\mathcal{G}}(n) \le 2^L \cdot s_{f,\varepsilon_0}(n),$$

which implies

$$[c_{f,\mathcal{F}'}(n)] \le [s_{f,\varepsilon_0}(n)] \le o(f),$$

and thus,

$$[c_{f,Y}(n)] \le \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\} \le o(f).$$

□

Now, we can prove the proposition 2.2.1 as we want.

Proof of Proposition 2.2.1. The previous lemma entails that

$$\sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \text{ is compact}\} \leq o(f).$$

To prove the reverse inequality, we will show that for every $\varepsilon > 0$ there exists a compact Y subset of $X \setminus \{\infty\}$ such that $s_{f,\varepsilon}(n) \leq c_{f,Y}(n)$, for every n. Given $\varepsilon > 0$ let $\mathcal{F} = \{Y_1, \ldots, Y_L\}$ be a family of (a priori non disjoint) subsets of $X \setminus \{\infty\}$ with diameters less than ε and such that $Y_\infty = X \setminus (Y_1 \cup \cdots \cup Y_L)$ also has diameter less than ε. We fix a positive integer n, and consider some (n, ε)-separated set E. For every point x in E, choose some coding $\underline{w}(x)$ of the sequence $(x, f(x), \ldots, f^{n-1}(x))$ with respect to the family $\mathcal{F}$. Since E is (n, ε)-separated and the sets $Y_1, \ldots, Y_L, Y_\infty$ have diameters less than ε, the map who associate points in E to words of the set $\mathcal{A}_n(f, \mathcal{F})$ is one to one. Thus

$$s_{f,\varepsilon}(n) \leq c_{f,\mathcal{F}}(n) = c_{f,Y}(n),$$

where $Y = Y_1 \cup \cdots \cup Y_L$.

Since this works for every $\varepsilon > 0$, we see that

$$o(f) \leq \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \text{ is compact}\},$$

and we conclude

$$o(f) = \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \text{ is compact}\}.$$

□

2.3 Localization

Let $S = \{x_1, \ldots, x_L\}$ be a finite set of points in $X \setminus \{\infty\}$. Consider for every x_i, $i = 1, \ldots, L$, a decreasing family of compacts $\{U_i^m\}_{m \geq 0}$ which forms a basis of neighborhoods of each x_i. We denote all of these families as $\mathcal{U}^m = \{U_1^m, \ldots, U_L^m\}_{m \geq 0}$, with $x_i \in U_i^m$, for $i = 1, \ldots, L$ and every $m \geq 0$. For a fixed m, we have $c_{f,\mathcal{U}^m}(n)$ is a non-decreasing sequence of natural numbers, where $c_{f,\mathcal{U}^m}(n) = \#\mathcal{A}_n(f, \mathcal{U}^m)$, then $[c_{f,\mathcal{U}^m}(n)] \in \mathbb{O}$. Since $\mathcal{U}^{m+1} \subset \mathcal{U}^m$, by monotonicity we deduce $[c_{f,\mathcal{U}^{m+1}}(n)] \leq [c_{f,\mathcal{U}^m}(n)]$ and we can define

$$o(f, S, \mathcal{U}) = \inf\{[c_{f,\mathcal{U}^m}(n)]; m \geq 0\} = \text{“}\lim_{m\to\infty}\text{”}[c_{f,\mathcal{U}^m}(n)] \in \overline{\mathbb{O}}.$$

Let $\mathcal{V}^m = \{V_1^m, \dots, V_L^m\}_{m\geq 0}$ be another decreasing family of compacts which forms a basis of neighborhoods of each x_i, $i = 1, \dots, L$. We gona see that $o(f, S, \mathcal{U}) = o(f, S, \mathcal{V})$. Indeed, we know that $U_i^m \cap V_i^m \neq \emptyset$, for every $m \geq 0$ and $i = 1, \dots, L$, and both diameters of U_i^m and V_i^m tends to zero when m tends to infinity. Choose positive integers m_1 and m_2 such that $U_i^{m_1} \subset V_i^{m_2}$ for every $i = 1, \dots, L$, then $[c_{f,\mathcal{U}^{m_1}}(n)] \leq [c_{f,\mathcal{V}^{m_2}}(n)]$, and taking the infimum we have $o(f, S, \mathcal{U}) \leq o(f, S, \mathcal{V})$. The same holds choosing positive integers n_1 and n_2 such that $V_i^{n_1} \subset U_i^{n_2}$ for every $i = 1, \dots, L$, and we conclude $[c_{f,\mathcal{V}^{n_1}}(n)] \leq [c_{f,\mathcal{U}^{n_2}}(n)]$, thus $o(f, S, \mathcal{V}) \leq o(f, S, \mathcal{U})$.

Therefore, this definition does not depend on the choice of the sequence of neighborhoods, but only on the x_i's. And we can define the generalized entropy of f at $S = \{x_1, \dots, x_L\}$, and we denote as $o(f, S)$.

Our goal now is to show that we can calculate the generalized entropy of f as the supremum of the generalized entropy of f on finite sets of $X \setminus \{\infty\}$.

Proposition 2.3.1. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism such that $\Omega(f) = \{\infty\}$. Then,*

$$o(f) = \sup\{o(f, S); S \subset X \setminus \{\infty\} \textit{ is finite}\} \in \overline{\mathbb{O}}.$$

To prove the proposition, we need the following lemma.

Lemma 2.3.1. *For every compact subset Y of $X \setminus \{\infty\}$ holds*

$$[c_{f,Y}(n)] \leq \sup\{o(f, S); S \subset Y \textit{ is finite}\}.$$

Proof. By the lemma 2.2.2 we have, for every $\varepsilon_0 > 0$, a finite family $\mathcal{F}$ of wandering compact subsets of Y with diameters less than ε_0 such that

$$[c_{f,Y}(n)] = \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\}.$$

Let $\mathcal{F}' = \{Y_1, \dots, Y_L\}$ be a disjoint family of subsets of $\mathcal{F}$. We will see that $[c_{f,\mathcal{F}'}] \leq o(f, S)$, where $S = \{x_1, \dots, x_L\}$ with $x_i \in Y_i$, for $i = 1, \dots, L$, stepwise as follows:

Step 1. Consider a partition of each Y_i of $\mathcal{F}'$ in k_i^1 compact subsets with diameters less than $\varepsilon_1 < \varepsilon_0$ which we denote by $Y_{ij_i}^1$ and we name the family of these $k_1^1 + \cdots + k_L^1$ compact sets as $\mathcal{F}^1 = \{Y_{ij_i}^1; i = 1, \dots, L,\ j_i = 1, \dots, k_i^1\}$. We denote $\mathsf{F}^1 = \{\mathcal{G} \subset \mathcal{F}^1; \mathcal{G} = \{Y_{1j_1}^1, Y_{2j_2}^1, \dots, Y_{Lj_L}^1\}\}$ as the set of all families $\mathcal{G}$

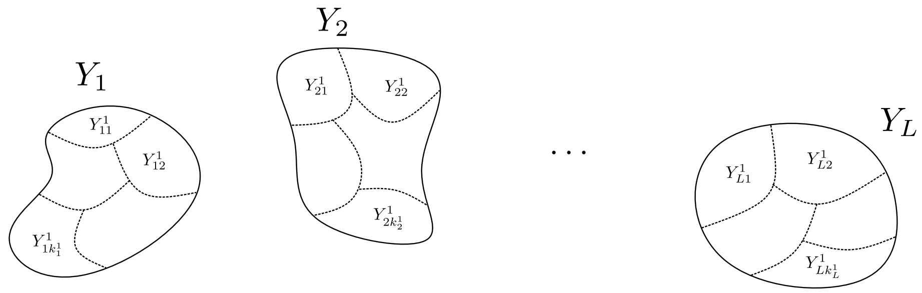


Figure 2.3: step 1.

by the $k_1^1 \cdot k_2^1 \cdots k_L^1$ possible combinations of subsets of $\mathcal{F}^1$ which take only one subset of each Y_i. Since $\mathcal{F}^1$ is a compact cover of $\cup\mathcal{F}'$ by subsets with diameters less then ε_1, we see that

$$[c_{f,\mathcal{F}'}(n)] = [c_{f,\cup\mathcal{F}'}(n)] = \sup\{[c_{f,\mathcal{H}}(n)]; \mathcal{H} \subset \mathcal{F}^1 \text{ is disjoint}\}.$$

Let $\mathcal{H}$ be a disjoint family of subsets of $\mathcal{F}^1$. If $\mathcal{H}$ contains sets $Y^1_{kl_1}$ and $Y^1_{kl_2}$, for $k \in \{1,\dots,L\}$ and $l_1 \neq l_2 \in \{1,\dots,k^1_k\}$, since Y_k is a wandering set, for every $k \in \{1,\dots,L\}$, then $Y^1_{kl_1} \cup Y^1_{kl_2}$ is wandering, and by wandering aditivity we have

$$[c_{f,\mathcal{H}}(n)] = \sup\{[c_{f,\mathcal{G}_{l_1}}(n)], [c_{f,\mathcal{G}_{l_2}}(n)]\},$$

where $\mathcal{G}_{l_1} = \{Y^1_{1j_1},\dots,Y^1_{kl_1},\dots,Y^1_{Lj_L}\}$ and $\mathcal{G}_{l_2} = \{Y^1_{1j_1},\dots,Y^1_{kl_2},\dots,Y^1_{Lj_L}\}$, with $\mathcal{G}_{l_1},\mathcal{G}_{l_2} \in \mathsf{F}^1$, thus

$$[c_{f,\mathcal{H}}(n)] = \sup\{[c_{f,\mathcal{G}}(n)]; \mathcal{G} \in \mathsf{F}^1\},$$

and we have

$$[c_{f,\mathcal{F}'}(n)] = \sup\{[c_{f,\mathcal{G}}(n)]; \mathcal{G} \in \mathsf{F}^1\}.$$

Setp 2. Consider now a partition of each $Y^1_{ij_i}$ of $\mathcal{F}^1$ in k^2_i compact subsets with diameters less than $\varepsilon_2 < \varepsilon_1$ which we denote by $Y^2_{ij_i}$ and we name the family of these $k^2_1 + \cdots + k^2_L$ compact sets as $\mathcal{F}^2 = \{Y^2_{ij_i}; i = 1,\dots,L, j_i = 1,\dots,k^2_i\}$. We denote now $\mathsf{F}^2 = \{\mathcal{G} \subset \mathcal{F}^2; \mathcal{G} = \{Y^2_{1j_1},\dots,Y^2_{Lj_L}\}\}$ as the set of all families $\mathcal{G}$ by the $k^2_1 \cdot k^2_2 \cdots k^2_L$ possible combinations of subsets of $\mathcal{F}^2$ which take only one subset of each Y_i. And we have $[c_{f,\mathcal{F}'}(n)] = \sup\{[c_{f,\mathcal{G}}(n)]; \mathcal{G} \in \mathsf{F}^2\}$.

Proceeding inductively we produce a decreasing sequence of families $\mathcal{G}^m = \{Y^m_{1j_1},\dots,Y^m_{Lj_L}\}$ of compact sets such that $Y^{m+1}_{ij_i} \subset Y^m_{ij_i}$, for $i = 1,\dots,L$ and $j_i = 1,\dots,k^m_i$, with the diameter of $Y^m_{ij_i}$ tending to 0 when m tends to ∞, and $[c_{f,\mathcal{G}^m}(n)]$ does not depending on m, this is

Figure 2.4: step 2.

$$[c_{f,\mathcal{F}'}(n)] = \sup\{[c_{f,\mathcal{G}^m}(n)]\} \text{ for every } m \geq 0.$$

Which implies

$$[c_{f,Y}(n)] = sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \text{ is disjoint}\} = \sup\{[c_{f,\mathcal{G}^m}(n)]; \mathcal{G}^m \in \mathsf{F}^m\},$$

for every $m \geq 0$.

Let now $S = \{x_1, \ldots, x_L\}$ be a set of limit points of the sequences in $\mathcal{G}^m$,

$$\bigcap_{m \geq 0} Y^m_{ij_i} = \{x_i\} \in Y_i.$$

For any decreasing family of compacts $\mathcal{U}^m$ which forms a basis of neighborhoods of each x_i, we can choose m large enough such that $Y^m_{ij_i} \subset U^m_i$, for $i = 1, \ldots, L$, and we have

$$[c_{f,\mathcal{G}^m}(n)] \leq [c_{f,\mathcal{U}^m}(n)].$$

Thus

$$[c_{f,Y}(n)] = \sup\{[c_{f,\mathcal{G}^m}(n)]; \mathcal{G}^m \in \mathsf{F}^m\} \leq \sup\{[c_{f,\mathcal{U}^m}(n)]; \mathcal{U}^m \text{ is a neighborhood of } S\},$$

for every $m \geq 0$, where the supremum is taken over every family of neighborhoods $\mathcal{U}^m$ of S. This implies

$$[c_{f,Y}(n)] \leq \sup\{\inf_{m \geq 0} [c_{f,\mathcal{U}^m}(n)]; \mathcal{U}^m \text{ is a neighborhood of } S\},$$

and we conclude

$$[c_{f,Y}(n)] \leq \sup\{o(f,S); S \subset Y \text{ is finite}\}.$$

$\square$

Now we can prove the proposition 2.3.1.

Proof of Proposition 2.3.1. By the Proposition 2.2.1 we know that

$$o(f) = \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \text{ is compact}\},$$

and by definition we have

$$\begin{aligned} o(f,S) &= \inf_{m\geq 0}\{[c_{f,\mathcal{U}^m}(n)]\} \\ &\leq [c_{f,\mathcal{U}^1}(n)] = [c_{f,\cup\mathcal{U}^1}(n)] \\ &\leq \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \text{ is compact}\}, \end{aligned}$$

since $\cup\mathcal{U}^1 = \cup_{i=1}^{L} U_L^1$ is a compact subset of $X \setminus \{\infty\}$. Thus, for every finite subset S of $X \setminus \{\infty\}$ it holds $o(f,S) \leq o(f)$, and then,

$$\sup\{o(f,S); S \subset X \setminus \{\infty\} \text{ is finite}\} \leq o(f).$$

By the previous lemma, for every compact subset Y of $X \setminus \{\infty\}$ we have $[c_{f,Y}(n)] \leq \sup\{o(f,S); S \subset Y \text{ is finite}\}$. Thus,

$$[c_{f,Y}(n)] \leq \sup\{o(f,S); S \subset X \setminus \{\infty\} \text{ is finite}\},$$

then, by proposition 2.2.1,

$$\begin{aligned} o(f) &= \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \{\infty\} \text{ is compact}\} \\ &\leq \sup\{o(f,S); S \subset X \setminus \{\infty\} \text{ is finite}\}. \end{aligned}$$

Therefore,

$$o(f) = \{o(f,S); S \subset X \setminus \{\infty\} \text{ is finite}\},$$

as we want. □

We end the section with the following useful lemma that establishes a way to calculate the generalized entropy of f at a finite set $S = \{x_1, \ldots, x_L\}$ taking any point in the orbit of the x_i's.

Lemma 2.3.2. *For every finite set $S = \{x_1, \ldots, x_L\}$ of $X \setminus \{\infty\}$ holds*

$$o(f,S) = o(f,S'),$$

where $S' = \{x_1, f^{n_1}(x_1), x_2 \ldots, x_L)\}$.

Proof. Given any decreasing family of compacts $\mathcal{U}^m = \{U_1^m, U_{n_1}^m, U_2^m, \ldots, U_L^m\}$ which forms a basis of neighborhoods of S', and also a family $\underline{\mathcal{U}}^m = \{U_1^m, U_2^m, \ldots, U_L^m\}$

a basis of neighborhoods of S. For each fixed $m \geq 0$, by monotonicity, we have $[c_{f,\underline{\mathcal{U}}^m}(n)] \leq [c_{f,\mathcal{U}^m}(n)]$, thus $o(f,S) \leq o(f,S')$.

On the other hand, we may assume that $f^{n_1}(x_1) \neq x_i$, for every $i = 1,\ldots,L$. Let's consider a family $\mathcal{U}^m = \{U_1^m, f^{n_1}(U_1^m), U_2^m, \ldots, U_L^m\}$ of neighborhoods of S' such that $\mathcal{U}^m$ is disjoint for every $m \geq 0$, and let $\underline{\mathcal{U}}^m = \{U_1^m, U_2^m, \ldots, U_L^m\}$ be a family of neighborhoods of S. For each $n \geq n_1$, we define a map

$$\Phi : \mathcal{A}_n(f,\underline{\mathcal{U}}^m) \to \mathcal{A}_n(f,\mathcal{U}^m),$$

as follow: given a word $w \in \mathcal{A}_n(f,\underline{\mathcal{U}}^m)$, we choose x such that w is the coding of the n-orbit $(x, f(x), \ldots, f^{n-1}(x))$, and then we define $\Phi(w)$ to be the coding relative to $\mathcal{A}_n(f,\mathcal{U}^m)$ of $(x, f(x), \ldots, f^{n-1}(x))$. This map is onte-to-one: indeed, let $w, w' \in \mathcal{A}_n(f,\underline{\mathcal{U}}^m)$ be words such that $\Phi(w) = \Phi(w) \in \mathcal{A}_n(f,\mathcal{U}^m)$. Let x, y the points in X such that w is the coding of the n-orbit $(x, f(x), \ldots, f^{n-1}(x))$ and w' is the coding of the n-orbit $(y, f(y), \ldots, f^{n-1}(y))$. Since $(\Phi(w))_i = (\Phi(w'))_i$ for every $i = 0, \ldots, n-1$, we have two cases:

1. $(\Phi(w))_i = U_j^m$, for $j \in \{1, \ldots, L\}$, then $f^i(x) \in U_j^m$ and $f^i(y) \in U_j^m$, thus $w_i = w_i^{'}$.

2. $(\Phi(w))_i = \infty$, then $f^i(x) \notin \cup\mathcal{U}^m$ and $f^i(y) \notin \cup\mathcal{U}^m$, thus

$$w_i = \begin{cases} \infty, \text{ if } f^i(x) \notin \cup\underline{\mathcal{U}}^m \\ f^{n_1}(U_1^m), \text{ if } f^i(x) \in f^{n_1}(U_1^m) \end{cases},$$

and

$$w_i^{'} = \begin{cases} \infty, \text{ if } f^i(y) \notin \cup\underline{\mathcal{U}}^m \\ f^{n_1}(U_1^m), \text{ if } f^i(y) \in f^{n_1}(U_1^m) \end{cases}.$$

If we suppose, without loss of generality, that $w_i = f^{n_1}(U_1^m)$, then

$$f^i(x) \in f^{n_1}(U_1^m) \Rightarrow f^{i-n_1}(x) \in U_1^m \Rightarrow (\Phi(w))_{i-n_1} = U_1^m,$$

since $(\Phi(w))_{i-n_1} = (\Phi(w'))_{i-n_1}$, we see that $(\Phi(w'))_{i-n_1} = U_1^m$, thus

$$f^{i-n_1}(y) \in U_1^m \Rightarrow f^i(y) \in f^{n_1}(U_1^m) \Rightarrow w_i^{'} = f^{n_1}(U_1^m),$$

and we conclude $w_i = {w'}_i$.

If $w_i = \infty$ the same argument shows that ${w'}_i = \infty$, which proves the injectivity.

Therefore, we have $c_{f,\underline{\mathcal{U}}^m}(n) \leq c_{f,\mathcal{U}^m}(n)$, for every $n \geq n_1$, that is $[c_{f,\underline{\mathcal{U}}^m}(n)] \leq$

$[c_{f,\mathcal{U}^m}(n)]$, for every $m \geq 0$, which implies $o(f, S') \leq o(f, S)$, and we conclude the proof of the lemma. $\square$

If we consider $S = \{x_1, \ldots, x_L\}$ and $S' = \{x_1, f^{n_1}(x_1), \ldots, x_L, f^{n_L}(x_L)\}$, we also get $o(f, S) = o(f, S')$, it is just apply the lemma repeatedly. And if we consider $S' = \{f^{n_1}(x_1), f^{n_2}(x_2), \ldots, f^{n_L}(x_L)\}$ we can use the same argument on the lemma choosing $S = \{f^{n_1}(x_1), x_1, f^{n_2}(x_2), \ldots, f^{n_L}(x_L)\}$, and, again, we have $o(f, S) = o(f, S')$.

Thus, we conclude that in order to calculate the generalized entropy of f at a finite set $S = \{x_1, \ldots, x_L\}$ of $X \setminus \{\infty\}$ we can replace a point x_i, for $i = 1, \ldots, L$, by any of its iterates, which means that $o(f, S)$ is unchanged when we consider this replacement.

2.4 Singular sets

We say that the subsets $U_1, \ldots, U_L$ of $X \setminus \{\infty\}$ are *mutually singular* if for every $M > 0$ there exists a point x and times $n_1, \ldots, n_L$ such that $f^{n_i}(x) \in U_i$ for every $i = 1, \ldots, L$, and $|n_i - n_j| > M$ for every $i \neq j$. We say that a finite subset $S = \{x_1, \ldots, x_L\}$ of $X \setminus \{\infty\}$ is *singular* if every family of respective neighborhoods $U_1, \ldots, U_L$ of $x_1, \ldots, x_L$ are mutually singular.

Example 2.4.1: In both examples 2.2.2 and 2.2.3, the sets Y_1, Y_2 are mutually singular. In the example 2.2.2, every pair of points $\{x_1, x_2\}$, with x_1 in the 'y' axis and x_2 in the 'x' axis, is a singular set. In the example 2.2.3, every pair of points $\{x_1, x_2\}$, with x_1 in the line $y = 1$ and x_2 in the line $y = -1$, is a singular set.

If $U_1, \ldots, U_L$ are compact subsets of $X \setminus \{\infty\}$ which are mutually singular, then there exists a singular set $\{x_1, \ldots, x_L\}$ such that $x_i \in U_i$, for $i = 1, \ldots, L$. Indeed, given $M > 0$, let $x^M \in X$ be the point such that $f^{n_1}(x^M) \in U_i$, for $i = 1, \ldots, L$, we define $y_i^M = f^{n_i}(x^M)$, then $\{y_i^M\}_{M>0}$ is a sequence in U_i, which admits a convergente subsequence. The $y_i's$, which are the limit points of the subsequences of y_i^M, are singular points. Also note that a unitary set, $S = \{x\}$, is always a singular set.

The following proposition says that the generalized entropy at a finite set always comes from singular sets.

Proposition 2.4.1. *Let S be a finite set of $X \setminus \{\infty\}$. Then*

$$o(f, S) = \sup\{o(f, S'); S' \subset S \text{ is singular}\}.$$

Proof. Let S be a finite subset of $X \setminus \{\infty\}$. Assuming S is not singular, we will show that there exists a subset S' of S which is singular and satisfies $o(f,S) = o(f,S')$. The proposition follows by a finite backward induction, since a singleton is always singular.

We assume that $S = \{x_1, \dots, x_L\}$ is not singular, and we consider a sequence of wandering families of respective neighborhoods $\mathcal{U}^m = \{U_1^m, \dots, U_L^m\}$ of $x_1, \dots, x_L$ which are not mutually singular. Fix $m > 0$, to simplify notation we will denote $\mathcal{U}^m = \mathcal{U}$ and $U_i^m = U_i$. Let n be a positive integer. Like in the proof of the lemma 2.2.3, for every $\mathcal{U}' \subset \mathcal{U}$ we denote $\mathcal{A}_n(\mathcal{U},\mathcal{U}')$ the set of elements of $\mathcal{A}_n(f,\mathcal{U})$ whose set of letters is exactly $\mathcal{U}' \cup \{\infty\}$. In particular, the elements of $\mathcal{A}_n(\mathcal{U},\mathcal{U})$ uses exactly the letters $U_1, \dots, U_L$ and ∞. Since the U_i's are wandering, each letter but ∞ appears at most once, thus

$$\mathcal{A}_n(f,\mathcal{U}) = \bigcup_{\mathcal{U}' \subset \mathcal{U}} \mathcal{A}_n(\mathcal{U},\mathcal{U}').$$

Since the U_i's are not mutually singular, there exists a number M such that if a point $x \in X$ satisfies $f^{n_i}(x) \in U_i$ and $f^{n_j}(x) \in U_j$, then $|n_i - n_j| \leq M$. For every $i \neq j \in \{1, \dots, L\}$, denote by $\mathcal{A}_n(\{i,j\})$ the set of elements w of $\mathcal{A}_n(\mathcal{U},\mathcal{U})$ such that the letters U_i and U_j appear at places at most M apart. We have

$$\mathcal{A}_n(\mathcal{U},\mathcal{U}) = \bigcup_{(i,j)} \mathcal{A}_n(\{i,j\}),$$

and

$$\mathcal{A}_n(f,\mathcal{U}) = \bigcup_{\mathcal{U}' \subsetneq \mathcal{U}} \mathcal{A}_n(\mathcal{U},\mathcal{U}') \cup \bigcup_{(i,j)} \mathcal{A}_n(\{i,j\}).$$

Then,

$$c_{f,\mathcal{U}}(n) = \sum_{\mathcal{U}' \subsetneq \mathcal{U}} c_{\mathcal{U},\mathcal{U}'}(n) + \sum_{(i,j)} c_{\{i,j\}}(n).$$

Thus,

$$
\begin{aligned}
[c_{f,\mathcal{U}}(n)] &= \left[\sum_{\mathcal{U}'\subsetneq\mathcal{U}} c_{\mathcal{U},\mathcal{U}'}(n) + \sum_{(i,j)} c_{\{i,j\}}(n)\right] \\
&\leq \left[\max\{\sum_{\mathcal{U}'\subsetneq\mathcal{U}} c_{\mathcal{U},\mathcal{U}'}(n), \sum_{(i,j)} c_{\{i,j\}}(n)\}\right] \\
&= \sup\{[\sum_{\mathcal{U}'\subsetneq\mathcal{U}} c_{\mathcal{U},\mathcal{U}'}(n)], [\sum_{(i,j)} c_{\{i,j\}}(n)]\} \\
&\leq \sup\{\sup_{\mathcal{U}'\subsetneq\mathcal{U}}\{[c_{\mathcal{U},\mathcal{U}'}(n)]\}, \sup_{(i,j)}\{[c_{\{i,j\}}(n)]\}\} \\
&= \sup\{[c_{\mathcal{U},\mathcal{U}'}(n)], [c_{\{i,j\}}(n)]; \mathcal{U}'\subsetneq\mathcal{U}, (i,j)\}.
\end{aligned}
$$

On the other hand, for each positive integer n, we have $\mathcal{A}_n(\mathcal{U},\mathcal{U}') \subset \mathcal{A}_n(f,\mathcal{U})$, for every $\mathcal{U}'\subsetneq\mathcal{U}$, thus $[c_{\mathcal{U},\mathcal{U}'}(n)] \leq [c_{f,\mathcal{U}}(n)]$, and then

$$\sup\{[c_{\mathcal{U},\mathcal{U}'}(n)]; \mathcal{U}'\subsetneq\mathcal{U}\} \leq [c_{f,\mathcal{U}}(n)].$$

Also, for each positive integer n, if w is an element of $\mathcal{A}_n(\{i,j\})$, let w' be obtained from w changing the letter 'U_i', that appears exactly once in w, into '∞' and w' is uniquely determined. Since $\mathcal{U}$ is a disjoint family, we have $w' \in \mathcal{A}_n(\mathcal{U},\mathcal{U}')$, where $\mathcal{U}' = \{U_1,\dots,U_{i-1},U_{i+1},\dots,U_L\}$. The word w also contains the letter 'U_j', and the letter 'U_i' is at most M places appart: thus w' has at most $2M$ inverse images under the map $w \mapsto w'$. We have

$$\#\mathcal{A}_n(\{i,j\}) \leq 2M\#\mathcal{A}_n(\mathcal{U},\mathcal{U}').$$

Then

$$[c_{\{i,j\}}(n)] \leq [c_{\mathcal{U},\mathcal{U}'}(n)] \leq [c_{f,\mathcal{U}}(n)] \text{ for every } (i,j),$$

thus,

$$\sup\{[c_{\{i,j\}}(n)]; (i,j)\} \leq [c_{f,\mathcal{U}}(n)],$$

and we have

$$[c_{f,\mathcal{U}}(n)] = \sup\{[c_{\mathcal{U},\mathcal{U}'}(n)]; \mathcal{U}'\subsetneq\mathcal{U}\}.$$

Returning to the notation, we deduce

$$[c_{f,\mathcal{U}^m}(n)] = \sup\{[c_{\mathcal{U}^m,\mathcal{U}^{m'}}(n)]; \mathcal{U}^{m'}\subsetneq\mathcal{U}^m\},$$

and taking $m\to\infty$, we find

$$o(f,S) = \sup\{o(f,S'); S'\subset S \text{ is singular}\}.$$

Since $o(f) = \sup\{o(f,S); S \subset X \setminus \{\infty\} \text{ is finite}\}$, we conclude

$$o(f) = \sup\{o(f,S'); S' \subset S \text{ is singular }\}.$$

$\square$

With this proposition, we can reformulate the proposition 2.3.1 as follows:

Proposition 2.3.1'. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism such that $\Omega(f) = \{\infty\}$. Then*

$$o(f) = \sup\{o(f,S); S \subset X \setminus \{\infty\} \textit{ is singular}\} \in \overline{\mathbb{O}}.$$

All over this chapter, we consider X to be a compact metric space and f a homeomorphism with a single non-wandering fixed point, $\Omega(f) = \{\infty\}$. Is it possible to calculate the generalized entropy using such techniques when the non-wandering set is more general? Whether it has finite points? These questions are going to be answered in the chapter 4.

Chapter 3

Generalized entropy of Brouwer homeomorphisms

Hauseux and Le Roux, in [5], showed that plane translations have wandering polynomial entropy equal to 1. And, they showed that a Brouwer homeomorphism has polynomial entropy equal to 1 if and only if it is conjugated to a translation. If a Brouwer homeomorphism is not conjugate to the translation, its wandering polynomial entropy is greater or equal to 2, and then they construct, for every $\alpha \in [2, +\infty]$, a Brouwer homeomorphism f_α whose wandering polynomial entropy is α. Recall that a Brouwer homeomorphism is a homeomorphism in the plane that is free of fixed points and preserves the orientation.

In this chapter, we generalize their construction to obtain Brouwer homeomorphisms with wandering generalized entropy of any order of growth between $[n^2]$ and $\sup(\mathbb{P})$. We are going to construct a Brouwer homeomorphism with wandering generalized entropy equal to $[n^{L-1} \cdot \log n]$, for $L \geq 3$, whose wandering polynomial entropy coincides with $\alpha = \{2, 3, \cdots\}$. This shows that polynomial entropy is not sufficient to characterize the Brouwer homeomorphisms.

3.1 Brouwer homeomorphisms by gluing translations

In this section, we construct a Brouwer homeomorphism whose wandering polynomial entropy coincides with the entropy of the homeomorphisms in the context of results of Hauseux and Le Roux but whose generalized entropy is different. With this, we can see that polynomial entropy is not enough to characterize Brouwer homeomorphisms.

Theorem A. *For every $L \geq 2$, there exists a Brouwer homeomorphism f with*

$$o(f) = [n^L \cdot \log n].$$

Before to prove the theorem, we are going to presents a construction that provides us a way to achieve Brouwer homeomorphisms with several generalized entropies. This construction is going to be useful to prove theorem A but also will be used in the next section.

Let $L \geq 3$ be an integer. We consider L copies $P_1, \ldots, P_L$ of the plane $\mathbb{R}^2$, and denote by $O_k = \{(x, y) \in P_k; y > 0\}$ the open upper half plane. For each $k = 1, \ldots, L-1$, let $\Phi_k : O_k \to O_{k+1}$ be of the form

$$(x, y) \mapsto (x + \varphi_k(y), y),$$

where φ_k is a continuous map from $(0, +\infty)$ to $\mathbb{R}$ whose limit in 0 is $-\infty$. Let P be the quotient space

$$\cup P_k/_\sim,$$

where $\sim$ denotes the equivalence relation generated by the identification of every point (x, y) in O_k to the point $\Phi_k(x, y)$ in O_{k+1}. P is a Housdorff simply connected non-compact surface, and thus is homeomorphic to the plane.

Let $T : \cup P_k \to \cup P_k$ be defined as the translation $(x, y) \mapsto (x + 1, y)$ on each P_k. The map T commutes with each Φ_k, and thus it defines a Brouwer homeomorphism $F : P \to P$. We compactify the plane by adding the point at infinity, and we have a homeomorphism $f : S^2 \to S^2$ (remember $S^2 = \{(x, y, z) \in \mathbb{R}^3; \|(x, y, z)\| = 1\}$).

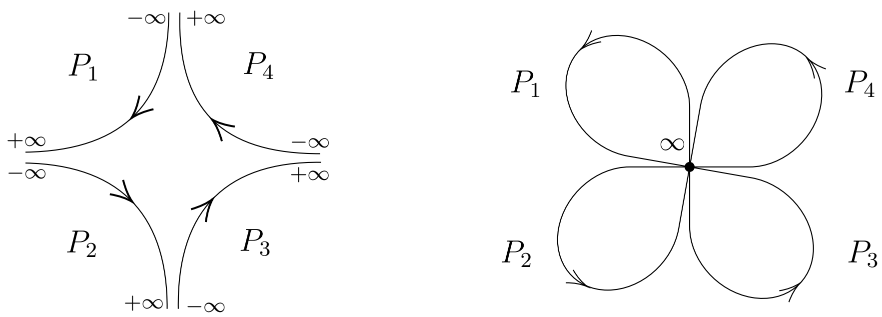


Figure 3.1: $F : P \to P$ and $f : S^2 \to S^2$ (case $L = 4$).

We note that the singular sets of f consists of all sets $\{M_1, \cdots, M_L\}$ with M_k on the boundary of the half plane O_k, ∂O_k, in P_k, and their subsets.

Remark 3.1.1. To construct Brouwer homeomorphisms by gluing translations, we can consider $L \geq 2$. However, the case $L = 2$ is a simpler case, which provides a Brouwer homeomorphism such as in the example 2.2.3 (Reeb's flow/Brouwer's counter-example), which we already know, does have generalized entropy equal to $[n^2]$.

3.1.1 Construction of f

We now fix $L \geq 3$. We are going to specify the gluing maps Φ_k in order to obtain generalized entropy $[n^{L-1} \cdot \log n]$. Let F be the resulting Brouwer homeomorphism and f the homeomorphism induced in the sphere, where the gluing maps have the following properties.

Assumption on the φ_k:

- φ_1 is negative, increasing, and, for each positive integer k_1, assumes the value $-k_1$ on a non trivial interval I_{k_1}. This collection of intervals tends to 0 when k_1 tends to $+\infty$. For convenience, we assume that all these intervals are included in the interval $(0, \frac{2}{3}]$.

- For each positive integer k_1, the restriction of φ_2 to I_{k_1} is increasing from $-2k_1$ to $-k_1$, and for each integer $-k_2$ between $-2k_1$ and $-k_1$, it assumes the value $-k_2$ on a non trivial sub-interval I_{k_1,k_2} of I_{k_1}. All these intervals are called the steps of order k_1 of φ_2. Between two sucessives steps φ_2 is monotonous.

- Likewise, φ_3 is incresing on each step I_{k_1,k_2} of order k_1 of φ_2, and assumes each integer values $-k_3$ between $-2k_1$ and $-k_1$ on a non trivial sub-interval I_{k_1,k_2,k_3} of I_{k_1,k_2}.

- Inductively, until φ_{L-1} : on each step of order k_1 of φ_{L-2}, this map is increasing and assumes each integer value $-k_{L-1}$ between $-k_1 - a(k_1)$ and $-k_1$ on a sub-interval $I_{k_1,\ldots,k_{L-1}}$ of $I_{k_1,\ldots,k_{L-2}}$, where the sequence $a(k)$ is chosen to satisfy

$$\sum_{k=1}^{n/2L} k^{L-2} \cdot a(k) = n^{L-1} \cdot \log n,$$

  and we will specify it later. All these maps are monotonous between two successive steps.

Graphs of the gluing maps φ_k's:

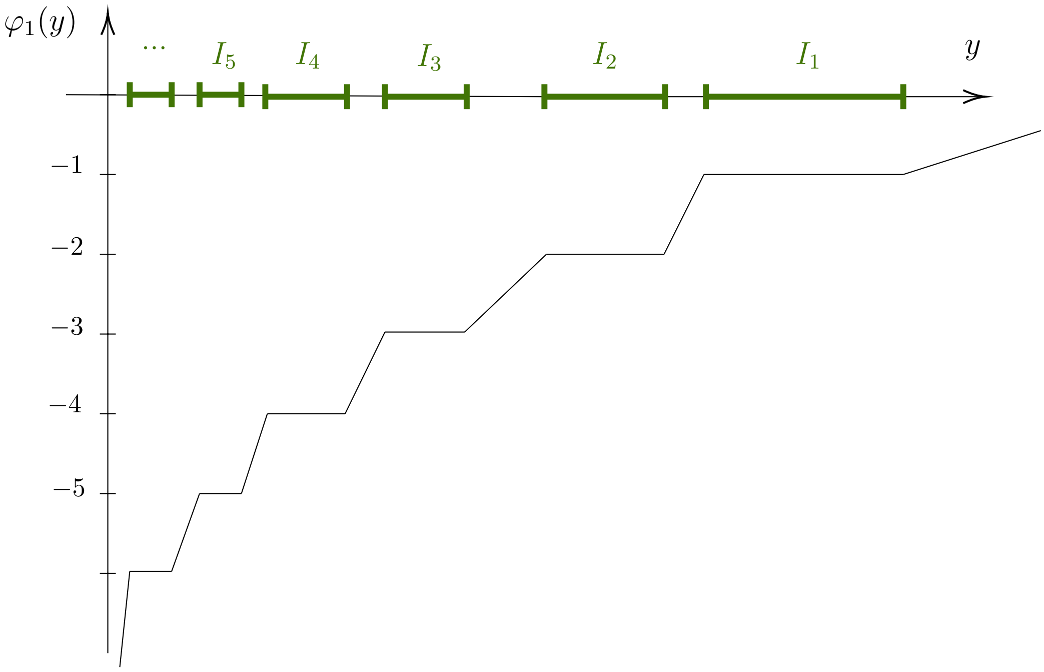


Figure 3.2: graph of φ_1.

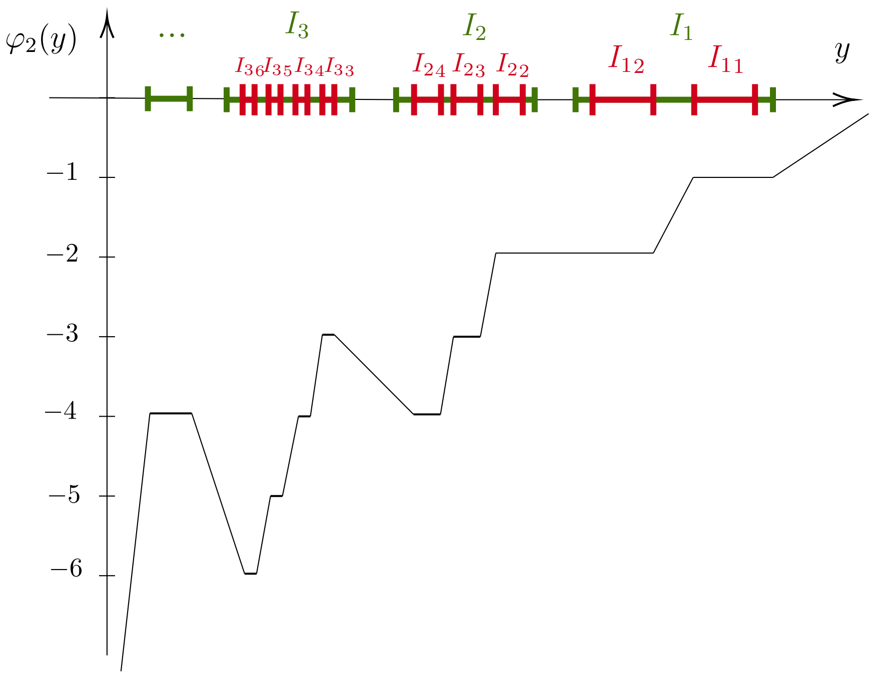


Figure 3.3: graph of φ_2.

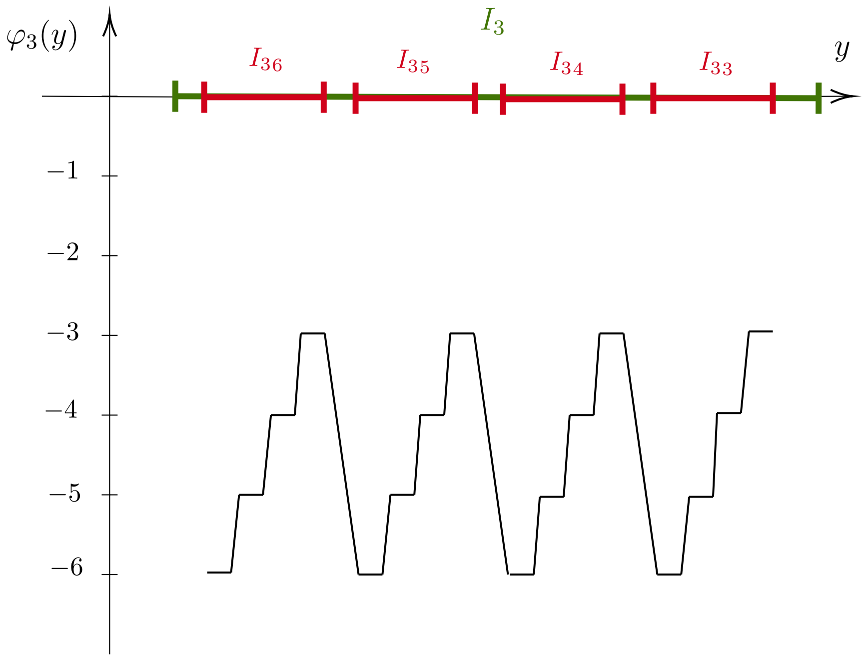


Figure 3.4: graph of φ_3 (zoom at I_3).

Construction of the sequence $a(k)$:

We want a sequence of positive integers $a(k)$ that satisfies $\sum_{k=1}^{n/2L} k^{L-2} \cdot a(k) = n^{L-1} \cdot \log n$, then, for each positive integer k we define $a(k)$ as follows:

- If $n = 2L$, we set $a(1) = (2L)^{L-1} \cdot \log(2L)$.
- If $n = 4L$, we set $a(2) = \dfrac{(4L)^{L-1} \cdot \log(4L) - (2L)^{L-1} \cdot \log(2L)}{2^{L-2}}$.
- If $n = 6L$, we set $a(3) = \dfrac{(6L)^{L-1} \cdot \log(6L) - (4L)^{L-1} \cdot \log(4L)}{3^{L-2}}$.

Proceeding inductively, we construct, for each positive integer k, the sequence

$$a(k) = \frac{(k2L)^{L-1} \cdot \log(k2L) - ((k-1)2L)^{L-1} \cdot \log((k-1)2L)}{k^{L-2}}.$$

3.1.2 Generalized entropy of f

The key to the computation of the generalized entropy is the following estimate.

Lemma 3.1.1. *Let U_i be the compact set $\left[-\frac{2}{3}, \frac{2}{3}\right]^2$ in the plane P_i, for $i = 1, \cdots, L$. Then*

$$[c_{f,\mathcal{U}}(n)] = [n^{L-1} \cdot \log n],$$

where $\mathcal{U} = \{U_1, \cdots, U_L\}$.

Proof. Let n be a positive integer, we want to estimate the number of elements of $\mathcal{A}_n(f,\mathcal{U})$. If we put aside the element which has only the letter ∞, every other element of this set has the form

$$(\infty,\cdots,\infty,U_i,\infty,\cdots,\infty,U_{i+1},\infty,\cdots,\infty,U_j,\infty,\cdots,\infty),$$

for some $i \le j$ in $\{1,\cdots,L\}$, or a similar form where some of the letters of $\mathcal{U}$ are doubled (since a point may have two successive iterates in some U_k). First, assume that $i > 1$ or $j < L$. Then in the above word, there are at most L maximal subwords with only the letter ∞, each of which has length less than n, and the length of the last one is determined by the length of the others since the total length is n. Taking into account the possibility of doubling the letters, the number of such words (for fixed values of i and j) is less than

$$2^{L-1}\cdot n^{L-1}.$$

Now, if we have $i = 1$ and $j = L$, what means the words $\underline{w}$ in $\mathcal{A}_n(f,\mathcal{U})$ are of the form

$$\underline{w} = (\infty,\cdots,\infty,U_1,\infty,\cdots,\infty,U_2,\infty,\cdots,\infty,U_{L-1},\infty,\cdots,\infty,U_L,\infty,\cdots,\infty).$$

In $\underline{w}$ there are L letters of $\mathcal{U}$, and $L+1$ maximal subwords with only the letter ∞, each has length at most n, furthermore the last maximal subword has the length determined by the other ones.

Let k_1 be a positive integer less than $\frac{n}{2L}$. Let $k_2,\cdots,k_{L-2}$ be integers between k_1+1 and $2k_1$. Let k_{L-1} be an integer between k_1+1 and $k_1+a(k_1)$. Finally, choose some y in the interval $I_{k_1,\cdots,k_{L-1}}$. Let z be the point of the plane whose coordinates in the plane P_1 are $(0,y)$. The point z is in U_1, once that $I_{k_1,\cdots,k_{L-1}} \subset (0,\frac{2}{3}]$. Since $\varphi_1(y) = -k_1$, the coordinates of z in the plane P_2 are $(-k_1,y)$, thus an iterate $f^k(z)$ is in U_2 if and only if $k = k_1$. Likewise $\varphi_2(y) = -k_2$, thus the coordinates of $f^{k_1}(z)$ in P_3 are $(-k_2,y)$, and an iterate $f^k(f^{k_1}(z))$ is in U_3 if and only if $k = k_2$, and so on. Let k_0 be an integer between 1 and k_1. Since k_1 is less than $\frac{n}{2L}$, the coding of the n first terms of the orbit of $f^{-k_0}(z)$ begins by

$$\underbrace{\infty\cdots\infty}_{(k_0-1)\text{ letters}}\ U_1\ \underbrace{\infty\cdots\infty}_{(k_1-1)\text{ letters}}\ U_2\ \underbrace{\infty\cdots\infty}_{(k_2-1)\text{ letters}}\ U_3\cdots\cdots U_{L-1}\ \underbrace{\infty\cdots\infty}_{(k_{L-1}-1)\text{ letters}}\ U_L\infty\cdots\infty.$$

Distinct values of k_1 provide distinct codings. Furthermore for a given value of k_1 we have approximately $k_1^{L-2}\cdot a(k_1)$ possibilities for the $(L-2)$-uplet

$(k_0, k_2, k_3, \cdots, k_{L-1})$. Thus, we find

$$\sum_{k=1}^{n/2L} k^{L-2} \cdot a(k) = n^{L-1} \cdot \log n,$$

words of the form $\underline{w}$ in $\mathcal{A}_n(f,\mathcal{U})$. When ounting the words of the form $\underline{w}$ in $\mathcal{A}_n(f,\mathcal{U})$ we can "ignore" the words where some of the letters of $\mathcal{U}$ are doubled, since this number does not increase on n. In any case, we are going to denote this number of such words with doubled letters by C.

Then, we have

$$n^{L-1} \cdot \log n \leq \#\mathcal{A}_n(f,\mathcal{U}) = c_{f,\mathcal{U}}(n),$$

and on the other hand

$$c_{f,\mathcal{U}}(n) = \#\mathcal{A}_n(f,\mathcal{U}) \leq n^{L-1} \cdot \log n + 2^{L-1} \cdot n^{L-1} + C.$$

Thus we conclude

$$[c_{f,\mathcal{U}}(n)] = [n^{L-1} \cdot \log n],$$

as we wanted. □

Now, we can prove Theorem A:

Proof of Theorem A. Using the lemma we deduce that $o(f) = [n^{L-1} \cdot \log n]$, for $L \geq 3$. Indeed, by aditivity we have $[c_{f,\mathcal{U}}(n)] = [c_{f,\cup\mathcal{U}}(n)]$ and since $\cup\mathcal{U}$ is a compact set we see that $o(f) \geq [n^{L-1} \cdot \log n]$. On the other hand, by the relation between generalized entropy and finite sets of mutually singular points, and by the description of all sets given above, we have

$$o(f) = \sup\{o(f,S); S = \{x_1, \cdots, x_L\}, x_k \in \partial O_k, \text{ for } k = 1, \cdots, L\}.$$

Furthermore, we know that the entropy is unchanged when we replace a point by one of its iterates. Since every point in ∂O_k has an iterate in the interior if U_k, in the last formula we can further restrict the supremum by demanding that each x_k belongs to $\partial O_k \cap \mathrm{Int}(U_k)$. By definition, $o(f,S) \leq [c_{f,\mathcal{U}}(n)]$, and then $o(f) \leq [n^{L-1} \cdot \log n]$. □

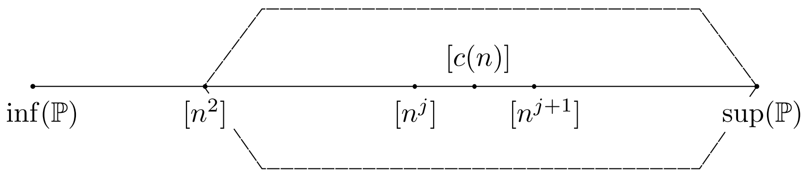
3.2 Wandering generalized entropy of Brouwer homeomorphisms

We want to show here that for any order of growth between $[n^2]$ and $\sup(\mathbb{P})$

we can construct a Brouwer homeomorphism such that the wandering generalized entropy has such order of growth.

We opted for a step by step construction in this section. This means that we are going to start with the case where $[c(n)] \in \mathbb{O}$ is an order of growth such that $[c(n)] = [n^j] \cdot [b(n)]$, for some integer $j \geq 2$ and $[b(n)] < [n]$ (that generalizes the construction on the previously section). Then we will consider the case where $[c(n)] \in \mathbb{O}$ is a order of growth such that $[n^j] < [c(n)] < [n^{j+2}]$, for some integer $j \geq 2$, but $[c(n)]$ is not comparable to $[n^{j+1}]$. And finally we will consider the general case, where $[c(n)] \in \mathbb{O}$ is such that $[n^j] \leq [c(n)] \leq [n^l]$, where $j \geq 2$ is the largest integer and l the smallest integer that satisfy the inequality. We believe that such step by step is the most didactic way to present the construction, but the reader who wants could go directly for the general case.

Let us suppose, first, that $[c(n)] \in \mathbb{O}$ is an order of growth such that $[c(n)] = [n^j] \cdot [b(n)]$, for some integer $j \geq 2$, where $[b(n)] < [n]$, that is $[n^j] < [c(n)] < [n^{j+1}]$. Using the gluing translations and the construction made previously, we can construct a Brouwer homeomorphism with wandering generalized entropy equal to $[c(n)]$ as follows.



Consider L copies of the plane, with $L = j+1$, and let $F : P \to P$ be the Brouwer homeomorphism given by the gluing of these L planes, and $f : S^2 \to S^2$ the induced homeomorphism, where $P = \cup_{k=1}^{L} P_k/_\sim$, as in the section 3.1. The dynamic of f depends on the choices of the φ_k's:

- φ_1 assumes the value $-k_1$ on a non trivial interval I_{k_1}, for each positive integer k_1.

- For each positive integer k_1, the restriction of φ_2 to I_{k_1} assumes the value $-k_2$ on a non trivial sub-interval I_{k_1,k_2} of I_{k_1} for each integer $-k_2$ between $-2k_1$ and $-k_1$.

- Inductively, until φ_{L-1} : on each step of order k_1 of φ_{L-2} this map assumes each integer value $-k_{L-1}$ between $-k_1 - a(k_1)$ and $-k_1$ on a sub-interval $I_{k_1,\ldots,k_{L-1}}$ of $I_{k_1,\ldots,k_{L-2}}$, where the sequence $a(k)$ satisfies

$$a(k) = \frac{(k2L)^{L-1} \cdot b(k2L) - ((k-1)2L)^{L-1} \cdot b((k-1)2L)}{k^{L-2}}, \text{ for } k \geq 1.$$

Chosing the family $\mathcal{U} = \{U_1, \cdots, U_L\}$, where U_i is the compact subset $\left[-\frac{2}{3}, \frac{2}{3}\right]^2$ in the plane P_i, for $i = 1, \cdots, L$. We count

$$\sum_{k=1}^{n/2L} k^{L-2} \cdot a(k) = n^j \cdot b(n),$$

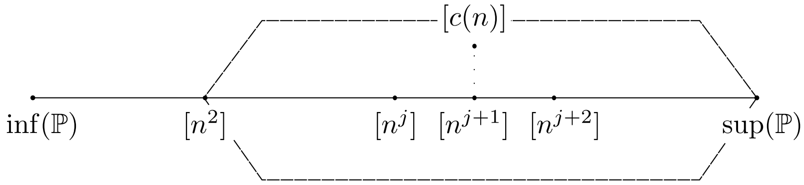
words of the form $\underline{w} = (\infty \cdots \infty\, U_1\, \infty \cdots \infty\, U_2\, \infty \cdots \infty\, U_{L-1}\, \infty \cdots \infty U_L\, \infty \cdots \infty)$ in the set $\mathcal{A}_n(f, \mathcal{U})$. Counting all the words in $\mathcal{A}_n(f, \mathcal{U})$, we see that

$$n^j \cdot b(n) \leq \#\mathcal{A}_n(f, \mathcal{U}) \leq n^j \cdot b(n) + 2^j \cdot n^j + C,$$

as in the proof of lemma 3.1.1. Then, $[c_{f,\mathcal{U}}(n)] = [n^j] \cdot [b(n)] = [c(n)]$, and we can conclude

$$o(f) = [c(n)].$$

Let suppose now that $[c(n)] \in \mathbb{O}$ is a order of growth such that $[n^j] < [c(n)] < [n^{j+2}]$, for some integer $j \geq 2$, but $[c(n)]$ is not comparable to $[n^{j+1}]$. We can write $[c(n)] = [n^j] \cdot [b(n)]$, where $[b(n)] < [n^2]$, but it is not comparable to $[n]$.



Again, we consider L copies of the plane, with $L = j+1$, and proceed analogously until the choice of the φ_k's:

- φ_1 assumes the value $-k_1$ on a non trivial interval I_{k_1}, for each positive integer k_1.

- For each positive integer k_1, the restriction of φ_2 to I_{k_1} assumes the value $-k_2$ on a non trivial sub-interval I_{k_1,k_2} of I_{k_1} for each integer $-k_2$ between $-2k_1$ and $-k_1$.

- Inductively, until φ_{L-2}: on each step of order k_1 of φ_{L-3} this map assumes each integer value $-k_{L-2}$ between $-k_1 - a(k_1)$ and $-k_1$ on a sub-interval $I_{k_1,\ldots,k_{L-2}}$ of $I_{k_1,\ldots,k_{L-3}}$.

- also φ_{L-1} assumes the value $-k_{L-1}$ for each integer $-k_{L-1}$ between $-k_1 - a(k_1)$ and $-k_1$, where the sequence $a(k)$ satisfies

$$a(k) = \sqrt{\frac{(k2L)^{L-1} \cdot b(k2L) - ((k-1)2L)^{L-1} \cdot b((k-1)2L)}{k^{L-3}}}, \text{ for } k \geq 1.$$

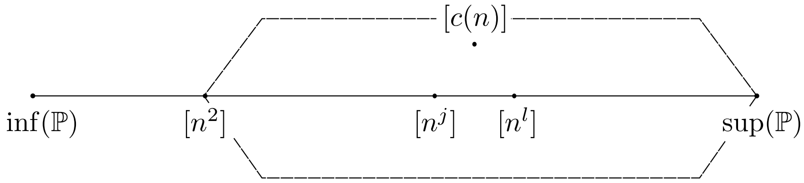
Again, setting the family $\mathcal{U} = \{U_1, \cdots, U_L\}$, where U_i is the compact subset $\left[-\frac{2}{3}, \frac{2}{3}\right]^2$ in the plane P_i, for $i = 1, \cdots, L$. We have

$$\sum_{k=1}^{n/2L} k^{L-3} \cdot a(k)^2 = n^j \cdot b(n),$$

words of the form $\underline{w}$ in the set $\mathcal{A}_n(f, \mathcal{U})$. Then, $[c_{f,\mathcal{U}}(n)] = [n^j] \cdot [b(n)] = [c(n)]$, and have

$$o(f) = [c(n)].$$

Finally, let $[c(n)] \in \mathbb{O}$ and let $j \geq 2$ the largest integer such that $[n^j] \leq [c(n)]$ and l the smallest integer such that $[c(n)] \leq [n^l]$. We can write $[c(n)] = [n^j] \cdot [b(n)]$, where $[b(n)] \leq [n^{l-j}]$.



Again, we consider L copies of the plane, now with $L = j + (l - j) = l$, and proceeds analogously until the choice of the φ_k's:

- φ_1 assumes the value $-k_1$ on a non trivial interval I_{k_1}, for each positive integer k_1.

- For each positive integer k_1, the restriction of φ_2 to I_{k_1} assumes the value $-k_2$ on a non trivial sub-interval I_{k_1,k_2} of I_{k_1} for each integer $-k_2$ between $-2k_1$ and $-k_1$.

- Inductively, until φ_j: on each step of order k_1 of φ_{j-1} this map assumes each integer value $-k_j$ between $-k_1 - a(k_1)$ and $-k_1$ on a sub-interval $I_{k_1,\ldots,k_j}$ of $I_{k_1,\ldots,k_{j-1}}$.

- until φ_{l-1}: it assumes the value $-k_{l-1}$ for each integer $-k_{l-1}$ between $-k_1 - a(k_1)$ and $-k_1$, where the sequence $a(k)$ satisfies

$$a(k) = \sqrt[l-j]{\frac{(k2l)^{l-1} \cdot b(k2l) - ((k-1)2l)^{l-1} \cdot b((k-1)2l)}{k^{j-1}}}, \text{ for } k \geq 1.$$

Setting the family $\mathcal{U} = \{U_1, \cdots, U_L\}$, where U_i is the compact subset $\left[-\frac{2}{3}, \frac{2}{3}\right]^2$

in the plane P_i, for $i = 1, \cdots, L$. We have

$$\sum_{k=1}^{n/2L} k^{j-1} \cdot a(k)^{l-j} = n^j \cdot b(n),$$

words of the form $\underline{w}$ in the set $\mathcal{A}_n(f, \mathcal{U})$. Then, $[c_{f,\mathcal{U}}(n)] = [n^j] \cdot [b(n)] = [c(n)]$, and have

$$o(f) = [c(n)].$$

Now, let $o \in \overline{\mathbb{O}}$ such that $o = \sup(A)$, where $A \subset \mathbb{O}$ is a countable ordered set, that is $A = \{[a_k(n)] \in \mathbb{O}; k \in \mathbb{N}, [a_k(n)] \leq [a_{k+1}(n)]\}$, and $[n^2] \leq [a_k(n)] \leq \sup(\mathbb{P})$, for every $k \in \mathbb{N}$. It is possible to construct a homeomorphism $f : S^2 \to S^2$ such that $o(f) = o$?

Let us consider the set $A = \{[n^{k+1}]; k \in \mathbb{N}\}$ and we will construct a homeomorphism that answers that question.

Consider three copies P_1, P_2, P_3 of the plane $\mathbb{R}^2$, and let O_1, O_2 and O_3 be their respectives open upper half plane, $O_k = \{(x, y) \in P_k; y > 0\}$, for $k = 1, 2, 3$. Let $\Phi_k : O_k \to O_{k+1}$, here $k = 1, 2$, defined as $(x, y) \mapsto \Phi_k(x, y) = (x + \varphi_k(y), y)$, where φ_k is a continuous map from $(0, +\infty)$ to $\mathbb{R}$, whose limit when y tends to 0 is $-\infty$. Let P be the quotiente space

$$P = \bigcup_{k=1}^{3} P_k/_\sim,$$

where $\sim$ denotes the equivalence relation generated by the identification of every point $(x, y) \in O_k$ to the point $\Phi(x, y) \in O_{k+1}$, that is $(x, y) \sim \Phi_1(x, y) \sim \Phi_2 \circ \Phi_1(x, y)$, where $(x, y) \in O_1$. We know that P is a Hausdorff simply connected non-compact surface, and it is homeomorphic to the plane.

Let $T : \cup P_k \to \cup P_k$ be defined as the translation $(x, y) \mapsto (x + 1, y)$ on each P_k, $k = 1, 2, 3$. The map T commutes with each Φ_k, $k = 1, 2$, and thus it defines a Brouwer homeomorphism $F_1 : P \to P$, as in the section 3.1. We compactify the plane by adding the point at infinity, and we have a homeomorphism $f_1 : S^2 \to S^2$.

We know that by a good choice, as we did before, of the gluing maps $\varphi_k' s$ we can obtain $o(f_1) = [n^2]$.

Let us consider now in P_1 the open lower half plane $A_1 = \{(x, y) \in P_1; y < 0\}$ and three other copies P_{11}, P_{12}, P_{13} of the plane $\mathbb{R}^2$, and let O_{11}, O_{12}, O_{13} be their respectives open upper half plane, $O_{1k} = \{(x, y) \in P_k; y > 0\}$, for $k = 1, 2, 3$. Let $\Phi : A_1 \to O_{11}$ and $\Phi_{1k} : O_{1k} \to O_{1(k+1)}$, here $k = 1, 2$, defined as $(x, y) \mapsto \Phi(x, y) = (x + \varphi(|y|), |y|)$ and $(x, y) \mapsto \Phi_{1k}(x, y) = (x + \varphi_{1k}(y), y)$, where φ and φ_{1k} are continuous maps from $(0, +\infty)$ to $\mathbb{R}$ whose limit is $-\infty$ when $|y|$ tends to 0.

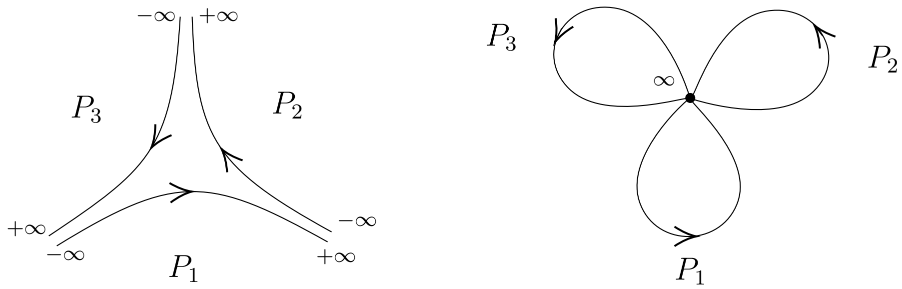


Figure 3.5: $F_1 : P \to P$ and $f_1 : S_2 \to S^2$.

Let $\approx$ denotes the equivalence relation generated by the identification of every point $(x, y) \in A_1$ to the point $\Phi(x, y) \in O_{11}$ and every point $(x, y) \in O_{1k}$ to the point $\Phi_{1k}(x, y) \in O_{1(k+1)}$, that is $(x, y) \approx \Phi(x, y) \approx \Phi_{11} \circ \Phi(x, y) \approx \Phi_{12} \circ \Phi_{11} \circ \Phi(x, y)$, where $(x, y) \in A_1$. Let $\overline{P_1}$ be the quotiente space

$$\overline{P_1} = \left(\left(\cup_{k=1}^{3} P_{1k}\right) \cup P_1\right) /_{\approx}.$$

We know that $\overline{P_1}$ is a Hausdorff simply connected non-compact surface, and it is homeomorphic to the plane $\mathbb{R}^2$, then it is homeomorphic to the plane P_1. And we replace the plane P_1 for the plane $\overline{P_1}$ to obtain

$$P = \left(\overline{P_1} \cup P_2 \cup P_3\right) /_{\sim}.$$

We can define a Brouwer homeomorphism $F_2 : P \to P$ as above and obtain the induced homeomorphism $f_2 : S^2 \to S^2$.

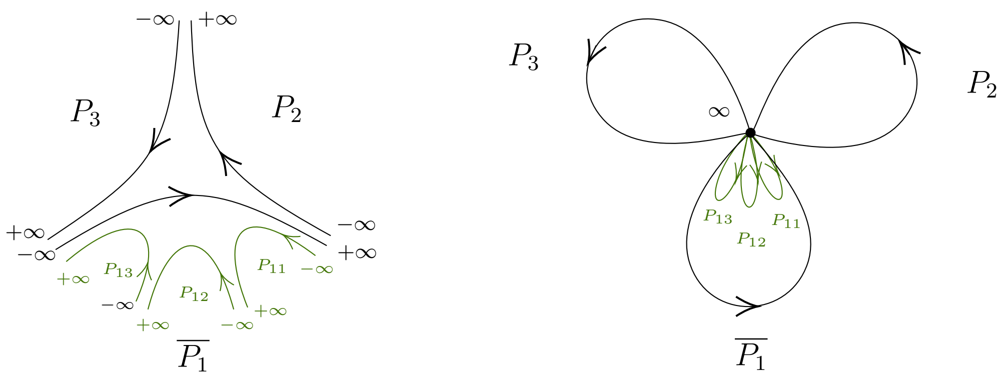


Figure 3.6: $F_2 : P \to P$ and $f_2 : S^2 \to S^2$.

Now, strategically choosing the gluing maps φ and $\varphi_k' s$, as we did before, we can obtain $o(f_2) = [n^3]$.

In the next step, by adding four copies $P_{111}, P_{112}, P_{113}, P_{114}$ of the plane $\mathbb{R}^2$ in

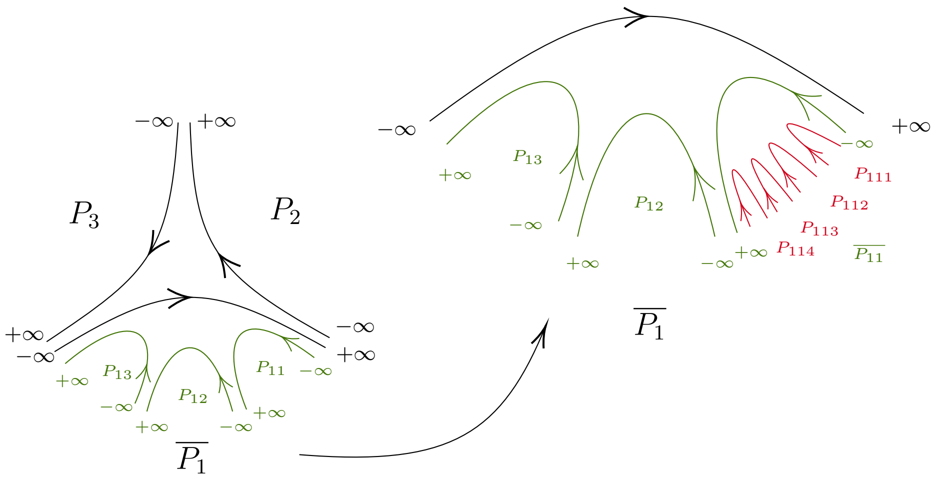


Figure 3.7: $F_3 : P \to P$.

the plane P_{11}, we construct a Brouwer homeomorphism $F_3 : P \to P$, and hence a homeomorphism $f_3 : S^2 \to S^2$ such that $o(f_3) = [n^4]$.

Inductively, in the k-th step we have a Brouwer homeomorphism $F_k : P \to P$, and hence a homeomorphism $f_k : S^2 \to S^2$ such that $o(f_k) = [n^{k+1}]$, for all $k \in \mathbb{N}$.

On the limit we have a Brouwer homeomorphism $F : P \to P$, and an induced homeomorphism $f : S^2 \to S^2$ who satisfies: for every $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that the dynamic of f is equal to the dynamic of f_{k_0}, that is, the dynamic of each f_k, if $k \geq k_0$, is contained in invariant subsets of diameters less than ε. And f is such that $o(f) = o$, where $o = \sup(A)$ for $A = \{[n^{k+1}]; k \in \mathbb{N}\}$, as we wanted. In particular, $o = \sup(\mathbb{P})$, the family $\mathbb{P}$ of the polynomial orders of growth.

By the same construction, with a good choice of the gluing maps, we construct a Brouwer homeomorphism $F : P \to P$ and the induced homeomorphism $f : S^2 \to S^2$ such that $o(f) = o$, for any $o = \sup(A)$, where

$$A = \{[a_k(n)] \in \mathbb{O}; k \in \mathbb{N},\ [a_k(n)] \leq [a_{k+1}(n)]\} \subset \mathbb{O}$$

is a countable ordered set of growth orders in $\mathbb{O}$, with $[n^2] \leq [a_k(n)] \leq \sup(\mathbb{P})$, for all $k \in \mathbb{N}$. And we also know that if $B \subset \mathbb{O}$ is a countable set of growth orders we can construct a countable and ordered set $A \subset \mathbb{O}$ such that $\sup(A) = \sup(B)$.

Let us consider the set of all orders of growth in $\overline{\mathbb{O}}$ that are supremum of countable sets, $\underline{\mathbb{O}} = \{o \in \overline{\mathbb{O}}; o = \sup(B),\ \text{where } B \subset \mathbb{O} \text{ is countable}\}$, and consider the family of homeomorphisms $\mathcal{H} = \{f : S^2 \to S^2;\ \Omega(f) = \{\infty\}\}$.

We have proved the following theorem:

Theorem B. *Let* $o \in \underline{\mathbb{O}}$ *with* $[n^2] \leq o \leq \sup(\mathbb{P})$. *Then there exists* $f \in \mathcal{H}$ *such that*

$$o(f) = o.$$

This theorem generalizes to the context of orders of growth and generalized entropy the result of Hauseux and Le Roux, in [5], who construct, for every $\alpha \in [2, +\infty]$, a Brouwer homeomorphism f_α whose wandering polynomial entropy is α.

Chapter 4

Entropy of wandering dynamics

Throughout Chapter 2, we considered X a compact metric space and f a homeomorphism with a single fixed non-wandering point, $\Omega(f) = \{\infty\}$. Now, we want to extend and use the techniques developed in that chapter in more general contexts. We considere here the case where the non-wandering set is finite: we will show that we can calculate the generalized entropy using the idea of coding and singular sets.

4.1 Finite points

In this section, we consider X a compact metric space and $f : X \to X$ a homeomorphism such that $\Omega(f) = \{y_1, y_2, \cdots, y_k\}$.

4.1.1 Coding

Let $\mathcal{F}$ be a finite family of non empty subsets of $X \setminus \Omega(f)$. We recall that $\cup\mathcal{F}$ denotes the union of all the elements of $\mathcal{F}$, and $\infty_{\mathcal{F}}$ denotes the complement of $\cup\mathcal{F}$. We fix a positive integer n. We say that $\underline{w}$ is a coding of $\underline{x}$, relative to $\mathcal{F}$, if for every $k = 0, \dots, n-1$ we have $x_k \in w_k$. Just like before we denote by $\mathcal{A}_n(f, \mathcal{F})$ the set of all codings of all orbits $(x, f(x), \dots, f^{n-1}(x))$ of length n. We define the sequence $c_{f,\mathcal{F}}(n) = \#\mathcal{A}_n(f, \mathcal{F}) \in \mathcal{O}$, and again we have $[c_{f,\mathcal{F}}(n)] \in \mathbb{O}$.

In comparison to coding in the case where $\Omega(f) = \{\infty\}$, now we possibly have the same word coding different orbits. However, since $\Omega(f)$ is finite, the number of orbits that are possibly coded by the same coding does not depend on the length of the segment of the orbit considered, that is, it does not depend on n.

Example 4.1.1. Let f be the time-one map of the flow whose orbits are given by the figure bellow. The non-wanderig set of f is given by the finite set of fixed points $\Omega(f) = Fix(f) = \{y_1, y_2, y_3\}$.

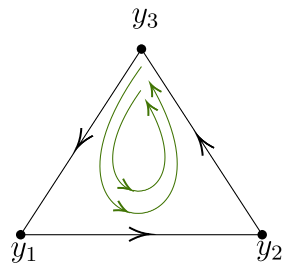


Figure 4.1: example 4.1.1.

If we consider the family $\mathcal{F} = \{U_1, U_2, U_3\}$ of compact subsets as in the figure 4.2 bellow:

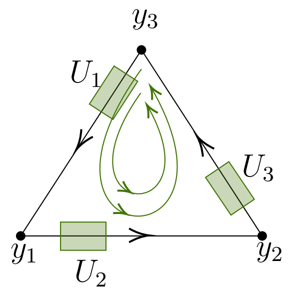


Figure 4.2: family $\mathcal{F}$.

The words $\underline{w} \in \mathcal{A}_n(f, \mathcal{F})$ have the form

$$(\infty, \cdots, \infty, U_1, \infty, \cdots, \infty, U_2, \infty, \cdots, \infty, U_3, \infty, \cdots, \infty),$$

and we conclude that $[n^2] \leq [c_{f,\mathcal{F}}(n)] \leq [n^3]$.

We will show that the generalized entropy of f also can be calculated as the supremum of $[c_{f,Y}(n)]$ taken among all compact sets Y of $X \setminus \Omega(f)$, as before. We will enunciate all the results again, making the necessary adaptations for the current context and we will omit the proofs that do not need any changes. We will demonstrate only the results whose proofs need to be adapted to the present case. Let us start with the properties of $[c_{f,\mathcal{F}}(n)]$.

Lemma 4.1.1. *Let $\mathcal{F}$ be a finite family of subsets of $X \setminus \Omega(f)$, such that $M(\cup\mathcal{F}) < +\infty$.*

1. *(monotonocity)Let $\mathcal{F}'$ be another finite family of subsets of $X \setminus \Omega(f)$. If $\mathcal{F}' \subset \mathcal{F}$, then*
$$[c_{f,\mathcal{F}'}(n)] \leq [c_{f,\mathcal{F}}(n)].$$

2. *(aditivity)*
$$[c_{f,\cup\mathcal{F}}(n)] = [c_{f,\mathcal{F}}(n)].$$

3. *(wandering aditivity) If $\mathcal{F} = \{Y_1, \ldots, Y_L\}$ is such that $Y_1 \cup Y_2$ is wandering, then*
$$[c_f, \mathcal{F}(n)] = \sup\{[c_{f,\mathcal{F}_1}(n)], [c_{f,\mathcal{F}_2}(n)]\},$$
where $\mathcal{F}_1 = \{Y_1, Y_3, \ldots, Y_L\}$ and $\mathcal{F}_2 = \{Y_2, Y_3, \ldots, Y_L\}$.

The proof of this lemma is the same for the lemma 2.2.1. We recall that $M(Y) = \sup_{x\in X} \#\{n; f^n(x) \in Y\}$, for any subset Y of $X \setminus \Omega(f)$, and that $\mathcal{F}' \subset \mathcal{F}$ means that each element of the family $\mathcal{F}'$ is included in an element of the family $\mathcal{F}$.

The following lemma provides us a way to estimate the order of the growth of the cardinality of the set of codings relative to a compact subset Y of $X \setminus \Omega(f)$, in terms of a family of subsets of Y with controlled diameters, as we did before.

Lemma 4.1.2. *For every compact subset Y of $X\setminus\Omega(f)$, and every $\varepsilon > 0$ there exists a finite family $\mathcal{F} = \{Y_1, \ldots, Y_L\}$ of wandering compact subsets of Y with diameters less than ε, such that*
$$[c_{f,Y}(n)] \leq \sup\{[c_{f,\mathcal{F}'}(n)]; \mathcal{F}' \subset \mathcal{F} \textit{ is disjoint}\}.$$

The proof of this lemma is the same for the lemma 2.2.2.

And again, we obtain a lower bound for generalized entropy, as before.

Lemma 4.1.3. *For every compact subset Y of $X \setminus \Omega(f)$ we have*
$$[c_{f,Y}(n)] \leq o(f).$$

The proof of this lemma is the same for the lemma 2.2.3.

Now we can enunciate the proposition which relates the generalized entropy of f with the order of the growth of the cardinality of the set of codings relative to compact subsets of $X \setminus \Omega(f)$:

Proposition 4.1.1. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism such that $\Omega(f) = \{y_1, y_2, \cdots, y_k\}$. Then*
$$o(f) = \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \Omega(f) \textit{ is compact}\} \in \overline{\mathbb{O}}.$$

Before we presents the proof of the proposition, we need some auxiliary results. At first, as in the proof of the proposition 2.2.1 we will show that for every $\varepsilon > 0$ there exists a compact Y subset of $X \setminus \Omega(f)$ such that $s_{f,\varepsilon}(n) \leq c_{f,Y}(n)$, for every n. Given $\varepsilon > 0$ let $\mathcal{F} = \{Y_1, \dots, Y_L\}$ be a family of wandering subsets of $X \setminus \Omega(f)$ with diameters less than ε and such that each connected component of $X \setminus \cup\mathcal{F}$ also has diameter less than ε, let denote such components as $V_1, V_2, \cdots, V_k$.

We choose a positive integer n, and consider a maximal (n, ε)-separated set E. Let $\Phi : E \to \mathcal{A}_n(f, \mathcal{F})$ be the map who associates for every point x in E some coding $\Phi(x) = w \in \mathcal{A}_n(f, \mathcal{F})$ of the sequence $(x, f(x), \dots, f^{n-1}(x))$ with respect the family $\mathcal{F}$. It is clear that the map Φ is not one-to-one (as in the case $\Omega(f) = \{\infty\}$), however we will show that the set $\Phi^{-1}(w)$ has its cardinality bounded by a constant that does not depend on n, for every word $w \in \mathcal{A}_n(f, \mathcal{F})$.

Let us then consider the graph G whose vertices are given by the set $V(G) = \{Y_1, \cdots, Y_L, V_1, \cdots, V_k\}$, and we say that there exists an edge of type (Y_i, Y_j) if $f(Y_i) \cap Y_j \neq \emptyset$, for $i, j \in \{1, \cdots, L\}$, and there exists an edge of type (Y_i, V_j) if $f(Y_i) \cap V_j \neq \emptyset$, for $i = 1, \cdots, L$ and $j = 1, \cdots, k$.

Observe that G satisfies the following properties:

1. There is no edge of the type (Y_i, Y_i), since Y_i is a wandering set for every $i = 1, \cdots, L$. More generally, there is no walk in the graph G with both initial and final vertices Y_i.

2. If there exists edge of the type (Y_i, V_j), then there is no edge of the type (Y_i, V_l), with $l \neq j$. For this, let $d = \min\{\frac{d(y_i, y_j)}{2}, i \neq j\}$. By uniform continuity of f, we have that there exists $\delta(d) > 0$, such that $d(x, y) < \delta(d)$ implies $d(f(x), f(y)) < d$, for every $x, y \in X$. If we choose $\varepsilon < \delta(d)$, then for every $x_j, x_l \in Y_i$, we have $d(f(x_j), f(x_l)) < d$.

3. If there exists edge of type (V_i, V_j), then there is no edge of the type (V_i, V_l), with $l \neq j$. That is, in edges of type (V_i, V_j), each V_i is in only one edge as initial vertex and in only one edge as final vertex.

Recall that a walk in graph G is a finite sequence of edges of the form

$$\{(b_0, b_1), (b_1, b_2), \cdots, (b_{j-1}, b_j)\},$$

where $b_i \in V(G)$, also denoted by $\{a_0, a_1, \cdots, a_j\}$, wher e each a_i is an edge of G, and which any two consecutive edges are adjacent or identical. The number of edges in a walk is called its length. A walk in which all the edges are distinct is a trail.

Let $P_n(G)$ be the set of all walks in G with length $n-1$, that is walks in G with n vertices and $n-1$ edges. Let us consider the map $P : E \to P_n(G)$ defined as follow: P associates every point $x \in E$, where E is the (n, ε)-separated set considered above, into a walk $P(x) \in P_n(G)$, where $P(x)$ is the walk whose vertices satisfy $f^i(x) \in P_i(x)$, for $i = 0, \cdots, n-1$.

Lemma 4.1.4. *The map P is injective.*

Proof. Given $x \neq y \in E$, let $P(x)$ and $P(y)$ in $P_n(G)$ be their respective images. Since x and y are (n, ε)-separated and we have chosen both Y_i, for $i = 1, \cdots, L$ and V_j, for $j = 1, \cdots, k$, with diameters less than ε, there exists $l \in \{0, \cdots, n-1\}$ such that $d(f^l(x), f^l(y)) \geq \varepsilon$, then $P_l(x) \neq P_l(y)$, and thus $P(x) \neq P(y)$. □

Consider now the map $Q : P_n(G) \to \mathcal{A}_n(f, \mathcal{F})$ defined as follows: for each walk $p = \{(b_0, b_1), (b_1, b_2), \cdots, (b_{n-2}, b_{n-1})\} \in P_n(G)$, where $b_i \in V(G)$, the map Q associates a word $Q(p) = w = (b_0^*, b_1^*, \cdots, b_{n-1}^*) \in \mathcal{A}_n(f, \mathcal{F})$ and each b_i^* is given by

$$b_i^* = \begin{cases} Y_{j_i}, \text{ if } b_i = Y_{j_i} \\ \infty, \text{ if } b_i = V_{j_i} \end{cases}.$$

Let us consider as well the map $\phi : P_n(G) \to T(G)$, where $T(G)$ is the set of all the trails in G, defined as follows: given $p = \{a_0, a_1, \cdots, a_{n-2}\} \in P_n(G)$, where a_i denotes the edge (b_i, b_{i+1}), for each $b_i \in V(G)$, the map ϕ associates the trail $\phi(p) \in T(G)$ where $\phi_i(p) = a_{k_i}$; $k_i = \min\{k > k_{i-1}; a_k \neq a_{k_j}, \text{ for } j \leq i-1\}$, $k_0 = 0$, that is

$$\begin{aligned} \phi_0(p) &= a_0, \\ \phi_1(p) &= a_{k_1}; \quad k_1 = \min\{k > k_0; a_k \neq a_0\}, \\ \phi_2(p) &= a_{k_2}; \quad k_2 = \min\{k > k_1; a_k \neq a_{k_1}, a_k \neq a_{k_0}\}, \\ &\vdots \end{aligned}$$

The map ϕ eliminates the repeated edges, transforming the walk p with $n-1$ edges into a trail $\phi(p)$ with the same edges. In particular, for our case, by properties 1, 2, and 3, the map ϕ eliminates only the edges of type (V_i, V_j).

Lemma 4.1.5. *For each word $w \in \mathcal{A}_n(f, \mathcal{F})$, the restriction $\phi : Q^{-1}(w) \to T(G)$ of the map ϕ is injective.*

Proof. Let p_1 and p_2 be two walks in $P_n(G)$ such that $Q(p_1) = Q(p_2) = w \in \mathcal{A}_n(f, \mathcal{F})$ and $\phi(p_1) = \phi(p_2) \in T(G)$, we want to show that $p_1 = p_2$. If we write

$$\begin{aligned} p_1 = & \;\{(b_0^1, b_1^1), (b_1^1, b_2^1), \cdots, (b_{n-2}^1, b_{n-1}^1)\} \\ p_2 = & \;\{(b_0^2, b_1^2), (b_1^2, b_2^2), \cdots, (b_{n-2}^2, b_{n-1}^2)\} \end{aligned},$$

let (b_i^1, b_{i+1}^1), for $i \in \{0, \cdots, n-2\}$, be an edge of the walk p_1, then, we have 3 possibilites.

1. (b_i^1, b_{i+1}^1) is an edge of type $(Y_{j_i}, Y_{j_{i+1}})$:

   Since $Q(p_1) = Q(p_2) = w = (b_0^*, b_1^*, \cdots, b_{n-1}^*)$, we have $b_i^1 = Y_{j_i}$ and $b_{i+1}^1 = Y_{j_{i+1}}$, thus $(b_i^2, b_{i+1}^2) = (Y_{j_i}, Y_{j_{i+1}}) = (b_i^1, b_{i+1}^1)$.

2. (b_i^1, b_{i+1}^1) is an edge of type $(Y_{j_i}, V_{j_{i+1}})$:

   This type of edge does not repeat, then it is not eliminated by the map ϕ. Thus, we have that $a_l = (Y_{j_i}, Y_{j_{i+1}})$, for some $l \leq i$, is an edge of $\phi(p_1) = \phi(p_2)$, and thus $(b_i^2, b_{i+1}^2) = a_l = (Y_{j_i}, V_{j_{i+1}}) = (b_i^1, b_{i+1}^1)$.

3. (b_i^1, b_{i+1}^1) is an edge of type $(V_{j_i}, V_{j_{i+1}})$:

   – If this edge is not eliminated by the map ϕ, then we have that $a_l = (V_{j_i}, V_{j_{i+1}})$, for some $l \leq i$, is an edge of $\phi(p_1) = \phi(p_2)$, and thus $(b_i^2, b_{i+1}^2) = a_l = (V_{j_i}, V_{j_{i+1}}) = (b_i^1, b_{i+1}^1)$.

   – If this edge is eliminated by the map ϕ, then we consider the previous edge

   $$(b_{i-1}^1, b_i^1) = \begin{cases} (Y_{j_{i-1}}, V_{j_i}) \\ (V_{j_{i-1}}, V_{j_i}) \end{cases} :$$

     - if $(b_{i-1}^1, b_i^1) = (Y_{j_{i-1}}, V_{j_i})$, then the edge (b_i^1, b_{i+1}^1) will not be eliminated by the map ϕ and we have that $a_l = (V_{j_i}, V_{j_{i+1}})$, for some $l \leq i$, is an edge of $\phi(p_1) = \phi(p_2)$, and thus $(b_i^2, b_{i+1}^2) = a_l = (V_{j_i}, V_{j_{i+1}}) = (b_i^1, b_{i+1}^1)$.
     - if $(b_{i-1}^1, b_i^1) = (V_{j_{i-1}}, V_{j_i})$, then it may or may not be eliminated by the map ϕ. If it is eliminated, then we repeat the argument until we obtain (b_{i-m}^1, b_{i-m+1}^1) for some $m \leq i$, which is not eliminated by the map ϕ. This must happen because the walks have finite length.

In all cases, we conclude that the edges of the walks p_1 and p_2 are necessarily the same, as we wanted. □

And now we can present the proof of the proposition 4.1.1.

Proof of Proposition 4.1.1. The lemma 4.1.3 entails that

$$\sup\{[c_{f,Y}(n)]; Y \subset X \setminus \Omega(f), \text{is compact}\} \leq o(f).$$

To the reverse inequality, considering the maps $\Phi : E \to \mathcal{A}_n(f,\mathcal{F})$, $P : E \to P_n(G)$, $Q : P_n(G) \to \mathcal{A}_n(f,\mathcal{F})$ and $\phi : P_n(G) \to T(G)$, we have the following diagram:

$$\begin{tikzcd} & E \arrow[dl, "P"'] \arrow[dr, "\Phi"] & \\ P_n(G) \arrow[rr, "Q"'] \arrow[d, "\phi"'] & & \mathcal{A}_n(f,\mathcal{F}) \\ T(G) & & \end{tikzcd}$$

Combining the lemmas 4.1.4 and 4.1.5, we have for every word $w \in \mathcal{A}_n(f,\mathcal{F})$:

$$\#(\Phi^{-1}(w)) = \#(P^{-1} \circ Q^{-1}(w)) \leq \#(Q^{-1}(w)) = \#(\phi \circ Q^{-1}(w)) \leq \#(T(G)).$$

Since the graph $(G, V(G))$ is finite, we have $\#(T(G))$ is a constant which does not depend on n, let denote it by $\#(T(G)) = B$. Thus,

$$\#(E) \leq \#(\Phi^{-1}(w)) \cdot \#(\mathcal{A}_n(f,\mathcal{F})) \leq B \cdot \#(\mathcal{A}_n(f,\mathcal{F})).$$

And then

$$s_{f,\varepsilon}(n) \leq B \cdot c_{f,Y}(n),$$

where $Y = \cup_{i=1}^{L} Y_i$ is a compact subset of $X \setminus \Omega(f)$. Since this holds for every $\varepsilon < \delta(d)$, we conclude

$$o(f) \leq [c_{f,Y}(n)] \leq \sup\{[c_{f,Y}(n)]; Y \subset X \setminus \Omega(f) \, \text{is compact}\}.$$

□

4.1.2 Localization

Let $S = \{x_1, \ldots, x_L\}$ be a finite set of points in $X \setminus \Omega(f)$. We consider for every x_i, $i = 1, \ldots, L$, a decreasing family of compacts $\{U_i^m\}_{m \geq 0}$ which forms a basis of neighborhoods of each x_i. As before, we denote all of these families as $\mathcal{U}^m = \{U_1^m, \ldots, U_L^m\}_{m \geq 0}$, with $x_i \in U_i^m$, for $i = 1, \ldots, L$ and every $m \geq 0$. For a fixed m, we have $c_{f,\mathcal{U}^m}(n)$ is a non-decreasing sequence of natural numbers, where $c_{f,\mathcal{U}^m}(n) = \#\mathcal{A}_n(f, \mathcal{U}^m)$, then $[c_{f,\mathcal{U}^m}(n)] \in \mathbb{O}$. Since $\mathcal{U}^{m+1} \subset \mathcal{U}^m$, by monotonicity we see that $[c_{f,\mathcal{U}^{m+1}}(n)] \leq [c_{f,\mathcal{U}^m}(n)]$ and we can define

$$o(f, S, \mathcal{U}) = \inf\{[c_{f,\mathcal{U}^m}(n)]; m \geq 0\} = \text{“} \lim_{m \to \infty} \text{”} [c_{f,\mathcal{U}^m}(n)] \in \overline{\mathbb{O}}.$$

Just like in chapter 2, this definition does not depend on the choice of the sequence of neighborhoods, but only on the x_i's, thus we can define the generalized entropy of f at $S = \{x_1, \ldots, x_L\}$, and we denote as $o(f, S)$.

Again, we can calculate the generalized entropy of f as the supremum of the generalized entropy of f at the finite sets of $X \setminus \Omega(f)$.

Proposition 4.1.2. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism such that $\Omega(f) = \{y_1, y_2, \cdots, y_k\}$. Then,*

$$o(f) = \sup\{o(f, S); S \subset X \setminus \Omega(f) \ \textit{is finite}\} \in \overline{\mathbb{O}}.$$

The proof follows just like the proof for the proposition 2.3.1.

We also have the following useful lemma that establishes a way to calculate the generalized entropy of f at a finite set $S = \{x_1, \ldots, x_L\}$ in $X \setminus \Omega(f)$ taking any point in the orbit of the x_i's.

Lemma 4.1.6. *For every finite set $S = \{x_1, \ldots, x_L\}$ of $X \setminus \Omega(f)$ holds*

$$o(f, S) = o(f, S'),$$

where $S' = \{f^{n_1}(x_1), \ldots, f^{n_L}(x_L)\}$.

The proof of this lemma is the same for the lemma 2.3.2 with the argument that proceeds in section 2.3.

4.1.3 Singular Sets

We say that the subsets $U_1, \ldots, U_L$ of $X \setminus \Omega(f)$ are *mutually singular* if for every $M > 0$ there exists a point x and times $n_1, \ldots, n_L$ such that $f^{n_i}(x) \in U_i$ for every $i = 1, \ldots, L$, and $|n_i - n_j| > M$ for every $i \neq j$. We say, as before, that a finite subset $S = \{x_1, \ldots, x_L\}$ of $X \setminus \{\infty\}$ is *singular* if every family of respective neighborhoods $U_1, \ldots, U_L$ of $x_1, \ldots, x_L$ are mutually singular.

The following proposition says that the generalized entropy at a finite set always comes from singular sets, as in the case where $\Omega(f) = \{\infty\}$.

Proposition 4.1.3. *Let S be a finite set of $X \setminus \Omega(f)$. Then*

$$o(f, S) = \sup\{o(f, S'); S' \subset S \ \textit{is singular}\}.$$

The proof of this proposition is the same for the proposition 2.4.1. And, with this proposition, we can also reformulate the proposition 4.1.2 in terms of singular sets:

Proposition 4.1.2'. *Let X be a compact metric space and $f : X \to X$ be a homeomorphism such that $\Omega(f) = \{y_1, y_2, \cdots, y_k\}$. Then*

$$o(f) = \sup\{o(f, S); S \subset X \setminus \Omega(f), \text{ is singular}\} \in \overline{\mathbb{O}}.$$

Finaly, if we consider the family of homeomorphisms $\mathcal{H}' = \{f : S^2 \to S^2; \Omega(f) = \{y_1, y_2, \cdots, y_k\}\}$ we have the following theorem:

Theorem C. *The set of the generalized entropies of maps in the family $\mathcal{H}'$ is the interval from $[n^2]$ to $\sup \mathbb{P}$ (in $\underline{\mathbb{O}}$) and $[n]$.*

Proof. For every $f \in \mathcal{H}'$ that is not conjugated to the compactification of the translation, by proposition 4.1.2', we have $[n^2] \leq o(f) \leq \sup \mathbb{P}$.

On the other hand, since $\mathcal{H} \subset \mathcal{H}'$, by theorem B, for every $o \in \underline{\mathbb{O}}$, there exists a homeomorphim $f \in \mathcal{H}'$ such that $o(f) = o$. □

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