UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE PÓS-GRADUACÃO EM MATEMÁTICA

The finitistic dimension conjecture

Júlio César Magalhães Marques

Belo Horizonte - MG
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The finitistic dimension conjecture

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"The beauty of mathematics only shows itself to more patient followers."
Maryam Mirzakhani

Resumo expandido

No estudo de algebra homológica, uma importante ferramenta de caracterização são as dimensões homológicas. A ideia geral por trás de introduzir as dimensões homológicas, se baseia em encontrar a medida de desvio (ou do quão longe está) uma categoria de módulos da categoria "ideal" apresentada por Artin-Wedderburn, a qual se caracteriza pela projetividade de todos os objetos. Uma dessas dimensões homológicas é a dimensão global de uma algebra Λ

$$\mathrm{gldim}(\Lambda) = \sup \{\mathrm{pd}M \mid M \in \text{Mod-}\Lambda\}$$

onde $\mathrm{pd}M$ é a dimensão projetiva de um Λ-módulo M e Mod-Λ é a categoria dos Λ-módulos. Se restringirmos tal definição, considerando módulos finitamente gerados (ou não f.g.) com dimensão projetiva finita, obtemos duas novas dimensões homológicas, chamadas dimensões finitistas, as quais definimos a seguir

$$\mathrm{findim}(\Lambda) = \sup \{\mathrm{pd}M \mid M \in \text{mod-}\Lambda \text{ and } \mathrm{pd}(M) < \infty\}$$

onde mod-Λ é a categoria dos Λ-módulos finitamente gerados, e

$$\mathrm{Findim}(\Lambda) = \sup \{\mathrm{pd}M \mid M \in \text{Mod-}\Lambda \text{ and } \mathrm{pd}(M) < \infty\}$$

Essas dimensões são chamadas *pequena* dimensão finitista e *grande* dimensão finitista, respectivamente. Tais dimensões vem acompanhadas das conjecturas finitistas, apresentadas por H. Bass no artigo "Finitistic dimension and a homological generalization of semi-primary rings" [4], como segue:

- Se Λ é uma algebra de Artin, então $\mathrm{findim}(\Lambda) = \mathrm{Findim}(\Lambda)$.
- Se Λ é uma algebra de Artin, então $\mathrm{findim}(\Lambda) < \infty$ (pequena conjectura finitista).
- Se Λ é uma algebra de Artin, então $\mathrm{Findim}(\Lambda) < \infty$ (grande conjectura finitista).

No artigo supracitado, H. Bass dá uma prova parcial da primeira conjectura.

Theorem 1 (Bass, 1960)**.** *Se R é um anel, então são equivalentes:*

- $\mathrm{Findim}(\Lambda) = 0$.
- *R é perfeito a esquerda e* $\mathrm{findim}(\Lambda) = 0$.

Mais tarde, em 1992, B. Huisgen-Zimmermann apresenta um exemplo de algebras monomiais que não satisfazem a conjectura [12]. Até hoje apenas casos particulares da conjectura foram comprovados (veja [32], [28]). Sabemos que se a conjectura finitista for demonstrada outras importantes conjecturas também serão comprovadas, incluindo a conjectura generalizada de Nakayama que estabelece para todo Λ-módulo S, $\mathrm{Ext}^i_\Lambda(D\Lambda, S) \neq 0$ para algum $i \geq 0$, onde $D\Lambda$ é o dual do módulo regular.

Em 2005, K. Igusa and G. Todorov em seu trabalho entitulado "On the finitistic global dimension conjecture for Artin algebras" [14], introduzem as funções ϕ e ψ com a inteção de demonstrar a segunda conjectura. As funções de Igusa-Todorov determinam uma nova medida homológica, a qual generaliza a noção de dimensão projetiva e se tornou uma poderosa ferramenta para compreensão da conjectura da dimensão finitista. Novas ideias foram desenvolvidas com o uso das funcões de Igusa-Todorov ([7], [8], [28]). Em 2009, J. Wei [28] introduz a noção das álgebras de Igusa-Todorov, as quais, em particular, possuem dimensão finitista finita.

K. Igusa e G. Todorov, em seu trabalho acima citado, obtém sucesso ao demonstrar a conjectura para álgebras Artinianas com radical ao cubo zero e também para aquelas que possuem dimensão de representação no máximo três, estas últimas constituem uma ampla classe de álgebras. Por um tempo, pensou-se que todas as álgebras possuiam dimensão de representação no máximo três, contudo, Rouquier [25] apresentou o exemplo de uma álgebra com dimensão de representação quatro.

As conjecturas homológicas, em particular a conjectura da dimensão finitista, tem inspirado muitos pesquisadores nas últimas décadas. Nos anos 80 as categorias derivadas foram introduzidas na teoria de representações e por suas propriedades se tornaram uma forte ferramenta para interpretar tais conjecturas homológicas (veja [10]). Mais recentemente, J. Rickard em [22] considera que é mais natural e conveniente trabalhar com categorias derivadas ilimitadas formadas por complexos de módulos arbitrários, já que estás tem boas propriedades, em especial coprodutos arbitrários. Neste sentido, Rickard considera se há alguma conexão entre a conjectura da dimensão finitista e os anéis para os quais a categoria derivada ilimitada é gerada, como uma categoria triangulada com coprodutos arbitrários, pelos módulos injetivos sobre este anel (se este é o caso, dizemos que os "injetivos geram"). No trabalho citado, J. Rickard conclui que se os injetivos geram para uma algebra de dimensão finita então a grande conjectura da dimensão finitista vale para tais algebras (é claro que neste caso também vale a pequena conjectura da dimensão finitista).

Neste trabalho estudaremos a conjectura da dimensão finitista em dois contextos distintos: primeiro de acordo com o artigo de Igusa-Todorov [14] e em seguida segundo o artigo publicado por J. Rickard [22].

ABSTRACT

MARQUES, Júlio César Magalhães, Universidade Federal de Minas Gerais, August 2019. **The finitistic dimension conjecture**. Adviser: John William MacQuarrie.

We study two conditions under which the finitistic dimension conjecture holds. First, we study an article of K. Igusa and G. Todorov [14], which gives a simple condition that implies the finiteness of the little finitistic dimension for Artin algebras. We present their short proof of the finitistic dimension conjecture for radical cubed zero algebras and for algebras with representation dimension smaller than or equal to three. Secondly, following a recent article of J. Rickard [22], we considered the question of whether the injective modules generate the unbounded derived category of a ring as a triangulated category with arbitrary coproducts. We present Rickard's proof that if injectives generate for such algebra, then the big finitistic dimension conjecture holds for that algebra.

Keywords: Finitistic dimension conjecture, Derived Category, Artin algebra, Injectives generate, Representation dimension.

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Introduction

Homological algebra began in the early 1950's, with Cartan, Eilenberg, Nakayama, Auslander, Buchsbaum, Serre, and Nagata being the principal researchers. But there a result of 19-th century which is homological: "Uber die Theorie der algebraischen Formen", published by Hilbert in 1890 (see [11]). So we could consider this work as the begining of the homological algebra. In this paper Hilbert introduce the definition of syzygy as we know it these days and he prove the following theorem, know as Hilbert's Syzygy Theorem:

Theorem 2. *If k is a field, then* $\mathrm{gldim}(k[X_1, \ldots, X_n]) = n$.

Category theory, introduced by MacLane and Eilenberg in 1940s (see [26]) served to boost the develop of homological algebra. Another motivation for developing and study of homological algebra is their interaction with related areas, as for example Algebraic Geometry.

Theorem 3 (Auslander-Buchsbaum-Serre Theorem (1955/56))**.** *If V is an algebraic variety over an algebraically closed field and R its coordinate ring, then the global dimension of R is finite if and only if V is smooth. Moreover, in the smooth case,* $\mathrm{gldim}R = dimV$.

When we want consider the homological properties of an algebra a important tool is the homological dimensions, given by homological measures. The general idea behind introducing homological dimensions, was to find a measure for the deviation of a given module category from the "ideal" categories arising in the Artin-Wedderburn situation which is characterized by the projectivity of all objects. One of these homological dimensions is the global dimension of an algebra Λ,

$$\mathrm{gldim}(\Lambda) = \sup \{\mathrm{pd}M \mid M \in \text{Mod-}\Lambda\}$$

where $\mathrm{pd}M$ is the projective dimension of a Λ-module M and Mod-Λ is the category of Λ-modules. If we restrict this definition considering finitely generated (or not) modules of finite porjective dimension this gives rise to an another two homological dimensions, called finitistic dimensions, which are defined below

$$\mathrm{findim}(\Lambda) = \sup \{\mathrm{pd}M \mid M \in \text{mod-}\Lambda \text{ and } \mathrm{pd}(M) < \infty\}$$

where mod-Λ is the category of finitely generated Λ-modules, and

$$\mathrm{Findim}(\Lambda) = \sup \{\mathrm{pd}M \mid M \in \text{Mod-}\Lambda \text{ and } \mathrm{pd}(M) < \infty\}$$

These dimensions are called the *little* finitistic dimension and *big* finitistic dimension, respectively. They come with the finitistic conjectures, presented by H. Bass in the paper "Finitistic dimension and a homological generalization of semi-primary rings" [4], as follows:

- If Λ is an Artin algebra, then $\mathrm{findim}(\Lambda) = \mathrm{Findim}(\Lambda)$.
- If Λ is an Artin algebra, then $\mathrm{findim}(\Lambda) < \infty$ (called little finitistic dimension conjecture).
- If Λ is an Artin algebra, then $\mathrm{Findim}(\Lambda) < \infty$ (called big finitistic dimension conjecture).

In the article, cited before, H. Bass gives an partial proof of the first conjecture.

Theorem 4 (Bass, 1960). *If R is a ring, then the following are equivalent*

- $\mathrm{Findim}(\Lambda) = 0$.
- *R is left perfect and* $\mathrm{findim}(\Lambda) = 0$.

Afterward, in 1992, B. Huisgen-Zimmermann gave an example of monomial algebras that do not satisfy the first conjecture [12]. The conjecture is open until today, being proved only in particular cases(see [32], [28]). If the finitistic conjecture holds, then so do many other highly studied conjectures in the representation theory of algebras including the generalized Nakayama conjecture that for every simple Λ-module S, $\mathrm{Ext}^i_\Lambda(D\Lambda, S) \neq 0$ for some $i \geq 0$, where $D\Lambda$ is the dual of the regular module.

In 2005, K. Igusa and G. Todorov in their work "On the finitistic global dimension conjecture for Artin algebras" [14], introduce the functions ϕ and ψ with an intention to prove the second conjecture. These Igusa-Todorov functions determine new homological measures, generalising the notion of projective dimension, and have become a powerful tool to understand better the finitistic dimension conjecture. A lot of new ideas have been developed around the use of Igusa-Todorov functions ([7], [8], [28]). In 2009, J. Wei introduced in [28] the notion of Igusa-Todorov algebras, which in particular, have finite finitistic dimension.

K. Igusa and G. Todorov in cited paper have succeeded to show the conjecture for Artin algebras with radical cubed zero and for Artin algebra with representation dimension at most three, which constitue a very large class of algebras. For a while it was thought that all algebras have representation dimension at most three, however, there is an example by Rouquier of an algebra of representation dimension four [25], which make that conjecture be still open for this algebras.

The homological conjectures, in particular the finitistic dimension conjecture, have inspired a lot of work over the last few decades. In the 1980's derived categories were introduced into representation theory, and because of their good properties they became a powerful tool to interpret these homological results (see [10]). More recently, J. Rickard in [22] considers that it is more natural and convenient to work with the unbounded derived categories of complexes of arbitrary modules, since this has better properties, especially arbitrary coproducts. In that sense, Rickard considers the question of whether there is a connection to finitistic dimension conjecture and the rings for which the unbounded derived category is generated, as a triangulated category with infinite coproducts, by the injective modules (if this is the case, then we say that ”injectives generate”). In the cited work, he concludes that if injectives generate for a finite dimensional algebra then the big finitistic dimension conjecture holds for this algebra (of course, in this case, so does the little finitistic conjecture).

In this work we study the finitistic dimension conjecture in two distinct contexts, first according to the paper of Igusa-Todorov [14] and after that Rickard's paper [22].

The first chapter is dedicated to give some premilinaries results about category theory, homological algebra and, in particular, the category of modules.

In Chapter two, we introduce Igusa-Todorov functions as well their properties. These properties allow us to construct a condition, presented in the main theorem of the chapter, which implies that the little finitistic dimension conjecture holds for Artin algebras with radical cubed zero. The second part of this chapter is dedicated to the representation dimension of an algebra. We will consider some preliminary results which give us a base to show that the little finitistic conjecture holds for Artin algebras with representation dimension at most three.

In Chapter three we give a brief presentation of the category of complexes, the homotopy category, triangulated categories and derived categories, which includes the most important properties and equivalences of these categories that will ground the results presented in chapter four.

In Chapter four we present some properties of localizing subcategories of derived categories to show that the projective modules generate the derived category over the category of modules and show that injectives generate for any ring with finite global dimension. Lastly we show that if injectives generate for a finite dimensional algebra then the big finitistic dimension conjecture holds for this algebra (hence the little conjecture holds too).

Chapter 1

Preliminaries

This chapter is dedicated to introducing some basic concepts related to category theory and homological algebra, as well as establish the notation that will be used throughout this work. The category of modules is central in this work and will be presented with all the properties we will need to ground the results. Most of that will be presented here not will be proved, however, such proofs could be found in respective citations. For a complete course about theory presented in this chapter see [17], [13] and [31].

1.1 Categories and Functors

In this section, we introduce some of the basic language of category theory involving the notions of category, functor and natural transformations. We are interested to develop this language to deal with modules in a more general sense.

1.1.1 Categories

Definition 1.1. *We shall say that we have a* ***category*** $\mathcal{C}$ *if there are defined:*

1) *a class $Ob\mathcal{C}$, whose elements are called the* ***objects*** *of the category $\mathcal{C}$;*

2) *a class $\mathrm{Hom}_{\mathcal{C}}$, whose elements are called the* ***morphisms*** *of the category $\mathcal{C}$;*

3) *for any ordered triple $X,\ Y,\ Z \in Ob\mathcal{C}$ and any pair of morphisms $f : X \to Y$ and $g : Y \to Z$ there is a uniquely defined morphism $g \circ f : X \to Z$, which is called the* ***composition*** *of morphisms f and g.*

These objects, morphisms and compositions are required to satisfy the following conditions:

- *composition of morphisms is associative, i.e., for any triple of morphisms f, g, h one has $h \circ (g \circ f) = (h \circ g) \circ f$ whenever these compositions are defined;*
- *if $X \neq X'$ or $Y \neq Y'$, then $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X', Y')$ are disjoint;*
- *for any object $X \in Ob\mathcal{C}$ there exists a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $f \circ 1_X = f$ and $1_X \circ g = g$ for any morphisms $f : X \to Y$ and $g : Z \to X$.*

If the class $Ob\mathcal{C}$ is actually a set, then that category is called **small**. In sense to simplify, sometimes we denote

1. $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ by (X, Y);
2. $X \in Ob\mathcal{C}$ by $X \in \mathcal{C}$, of X is in $\mathcal{C}$;
3. The composition $g \circ f$ by gf.

Examples of Categories

Sets - category of sets. Whose objects are the class of all sets and morphisms are the set of all maps between sets.

Ab - category of Abelian groups. Whose objects are the class of all Abelian groups and morphism are the set of all Abelian group homomorphisms.

Mod-A **-** category of modules. Whose objects are the class of all right A-modules over an algebra A and morphisms are the set of homomorphisms of modules. If the modules are finitely generated denote it by mod-A. For left modules we denote by A-Mod and A-mod.

Morph$(\mathcal{C})$ **-** category of morphisms of a category $\mathcal{C}$. The objects of Morph$(\mathcal{C})$ are all triples (X, Y, f) where X and Y are in $Ob\mathcal{C}$ and f is in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and morphisms is the set of all pairs (α, β) where $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd}$$

where $f : X \to Y$ and $f' : X' \to Y'$ in $\mathcal{C}$.

Definition 1.2. *Let $\mathcal{C}$ be a category. A* ***subcategory*** *$\mathcal{C}'$ of $\mathcal{C}$ is a category such that*

(a) *$Ob\mathcal{C}' \subseteq Ob\mathcal{C}$;*

(b) *for all X and Y in $\mathcal{C}'$ we have that $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$.*

(c) *The composition of morphisms in $\mathcal{C}'$ is the same of the composition in $\mathcal{C}$ and the identity is the same in both for each object.*

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be a **full subcategory** of $\mathcal{C}$ if and only if $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all X, Y in $\mathcal{C}'$.

Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism in $\mathcal{C}$, f is an:

(a) **isomorphism** if and only if there is $g : Y \to X$ in $\mathcal{C}$ such that $gf = 1_X$ and $fg = 1_Y$;

(b) **epimorphism** if given $g, h : Y \to Z$ such that $gf = hf$, then $g = h$;

(c) **monomorphism** if given $g, h : U \to X$ such that $fg = fh$, then $g = h$.

In Mod-Λ the definition of epimorphism coincide with surjective homomorphism and monomoprhism with injective homomorphism.

1.1.2 Functors and natural transformations

Definition 1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **covariant functor** (resp. **contravariant**) F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a map that associate each object X in $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$ and each morphism $f : X \to Y$ in $\mathcal{C}$ corresponds a morphism $F(f) : F(X) \to F(Y)$ (resp. $F(f) : F(Y) \to F(X)$) in $\mathcal{D}$, such that the follwing conditions are satisfied:*

1) *$F(1_X) = 1_{F(X)}$ for all $X \in \mathcal{C}$;*

2) *if the composition of morphisms gf is defined in $\mathcal{C}$, then $F(gf) = F(g)F(f)$ (resp. $F(gf) = F(f)F(g)$).*

A functor in two variables is often called a **bifunctor**. Here follows some important examples of functors will be often used on this work.

Example 1.4. *Let $\mathcal{C}$ be a category and X an object in $\mathcal{C}$. We define the covariant functor $\mathrm{Hom}_{\mathcal{C}}(X,-)$ as follow:*

$$\begin{array}{rcllcl} \mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} & \longrightarrow & \mathit{Sets} & & & \\ A & \mapsto & \mathrm{Hom}_{\mathcal{C}}(X,A) & & & \\ A \xrightarrow{f} B & \mapsto & \mathrm{Hom}_{\mathcal{C}}(X,f) : & \mathrm{Hom}_{\mathcal{C}}(X,A) & \to & \mathrm{Hom}_{\mathcal{C}}(X,B) \\ & & & g & \mapsto & f \circ g \end{array}$$

Anagously, define the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,X)$

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C} & \longrightarrow & Sets \\ A & \mapsto & \mathrm{Hom}_{\mathcal{C}}(A,X) \\ A \xrightarrow{f} B & \mapsto & \mathrm{Hom}_{\mathcal{C}}(f,X) : \begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(A,X) & \to & \mathrm{Hom}_{\mathcal{C}}(A,X) \\ g & \mapsto & g \circ f \end{array} \end{array}$$

Example 1.5. *Given A in* Mod-R*, there is an additive functor defined by*

$$\begin{array}{rcl} A \otimes_R - : R\text{-Mod} & \longrightarrow & \mathbf{Ab} \\ B & \mapsto & A \otimes_R B \\ B \xrightarrow{g} B' & \mapsto & 1_A \otimes g \end{array}$$

*Similarly, given B in R-*Mod*, there is an additive functor $-\otimes_R B$. For more details about properties of these functors see [24].*

Example 1.6. *Let A be a finite dimensional K-algebra. We define the functor*

$$\begin{array}{rcl} D : \text{mod-}A & \longrightarrow & \text{mod-}A^{op} \\ M & \mapsto & DM := \mathrm{Hom}_K(M,K) \end{array}$$

Note that $\mathrm{Hom}_K(M,K)$ is a left A-module by the formula $(a\varphi)(m) = \varphi(ma)$ for $\varphi \in \mathrm{Hom}_K(M,K)$, $a \in A$ and $m \in M$. Note that D is a duality of categories, called the ***standard K-duality****, see chapter one in [13].*

Definition 1.7. *Let F and G be two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A* ***morphism****, or a* ***natural transformation****, from the functor F to the functor G is a map φ which assigns to each object $X \in \mathcal{C}$ a morphism $\varphi(X) : F(X) \to G(X)$ of the category $\mathcal{D}$ with the following property: for any pair of objects X, $Y \in \mathcal{C}$ and any any morphism $f : X \to Y$ of the category $\mathcal{C}$ we have $G(f)\varphi(X) = \varphi(Y)F(f)$, i.e., the following diagram commutes:*

$$\begin{tikzcd} F(X) \arrow[r, "\varphi(X)"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\varphi(Y)"'] & G(Y) \end{tikzcd}$$

A morphism of functors will be simply denoted by $\varphi : F \to G$, and we shall denote the collection of all morphisms from F to G by (F,G), which in general (F,G) is not a set. If $\mathcal{C}$ is small category then (F,G) is always a set. More generally if $\mathcal{C}$ is skeletally small (definition follows) then (F,G) is a set for all functor $F, G : \mathcal{C} \to \mathcal{D}$, see [3].

Definition 1.8. *Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be* ***dense*** *if and only if given any C in $\mathcal{C}$ there is a C' in $\mathcal{C}'$ such that $C \simeq C'$ in $\mathcal{C}$. The category $\mathcal{C}$ is said to be* ***skeletally***

small *if there is a small dense subcategory $\mathcal{C}'$ of $\mathcal{C}$.*

Example 1.9. *The category $Fun(\mathcal{C},\mathcal{D})$ which consists of all functors from $\mathcal{C}$ to $\mathcal{D}$ together with all (T_1,T_2), the natural transformations from T_1 to T_2 and the maps $(T_1,T_2)\times(T_2,T_3)\to(T_1,T_3)$ given by composition of morphisms of functors, so is a category provided each (T_1,T_2) is a set. Hence if $\mathcal{C}$ is skeletally small then $Fun(\mathcal{C},\mathcal{D})$ is a category for each category $\mathcal{D}$, which is called the* ***category of functors from** $\mathcal{C}$ **to** $\mathcal{D}$. The full subcategory of $Fun(\mathcal{C},\mathcal{D})$ consisting of all additive functors is denote by $(\mathcal{C},\mathcal{D})$.*

Definition 1.10. *Consider a functor $F:\mathcal{C}\to\mathcal{D}$ and*

$$F:(X,Y)_{\mathcal{C}}\to(F(X),F(Y))_{\mathcal{D}} \tag{1.11}$$

The functor F is said to be a:

- ***full** functor if (1.11) is a surjection for all X, Y in $\mathcal{C}$.*
- ***faithful** functor if (1.11) is a injection for all X, Y in $\mathcal{C}$.*
- ***fully faithful** functor if (1.11) is a isomorphism (of sets) for all X, Y in $\mathcal{C}$.*

The functor F is said to be a ***dense*** *functor if the full subcategory with objects $F(Ob\mathcal{C})$ is a dense subcategory of $\mathcal{D}$.*

The notion of an equivalence of categories is a important concept of relation between categories, because equivalent categories have essentially the same properties.

Definition 1.12. *The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there is a functor $F:\mathcal{C}\to\mathcal{D}$ and a functor $G:\mathcal{D}\to\mathcal{C}$ as well as a natural isomorphism $\beta: id_{\mathcal{D}}\to F\circ G$ and a natural isomorphism $\alpha: id_{\mathcal{C}}\to G\circ F$. The functor F is called a quasi-inverse of G, and G is a quasi-inverse of F.*

Proposition 1.13 ([3], Proposition 4.2)**.** *A functor $T:\mathcal{C}\to\mathcal{D}$ is an equivalence of categories if and only if T is a dense fully faithful functor.*

Let $\mathcal{C}$ be a category and $F:\mathcal{C}\to Sets$ a functor. With the intention to describe the morphisms from $\mathrm{Hom}_{\mathcal{C}}(C,-)$ to F for each C in $\mathcal{C}$ we state the following Theorem, know as **Yoneda's Lemma** which is central to all category theory.

Theorem 1.14 ([3], page 11)**.** *Let $\mathcal{C}$ be a category and $F:\mathcal{C}\to Sets$ a functor. If C is an object in $\mathcal{C}$, then the collection $((C,-),F)$ of all morphisms from $(C,-)$ to F is a set which is isomorphic to $F(C)$ under the map which sends a morphism $\varphi:(C,-)\to F$ to the element $\varphi_{\mathcal{C}}(1_C)$ in $F(C)$.*

Definition 1.15. *Let $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ be two functors between the categories $\mathcal{C}$ and $\mathcal{D}$. We say that F is* ***left adjoint*** *to G, and G is* ***right adjoint*** *to F, if we have the following isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(F-,-)\simeq\mathrm{Hom}_{\mathcal{C}}(-,G-)$$

1.1.3 Additive and abelian category

Let $\mathcal{C}$ be a category. A zero object in $\mathcal{C}$, denoted by 0 is such that for for all objects X of $\mathcal{C}$ the sets $\mathrm{Hom}_{\mathcal{C}}(X,0)$ and $\mathrm{Hom}_{\mathcal{C}}(0,X)$ has precisely one element.

Definition 1.16. *A category $\mathcal{C}$ is called an **additive category** if the following conditions hold:*

(A1) *For every X, Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group and the composition of morphisms is bilinear over the integers;*

(A2) *The category $\mathcal{C}$ contains a zero object 0;*

(A3) *For every pair of objects X, $Y \in \mathcal{C}$ there exists a coproduct $X \oplus Y$ in $\mathcal{C}$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and $F : \mathcal{A} \to \mathcal{B}$ be a functor. We said that F is an additive functor if, and only if, $F(A) \oplus F(B) \cong F(A \oplus B)$, for all A, $B \in \mathcal{A}$ and $F(f+g) = F(f) + F(g)$ for all $f, g \in \mathrm{Hom}_{\mathcal{A}}(A,B)$ (see [21], section 12.7). Note that the functors Hom and tensor product are additive.

Definition 1.17. *Let $f : X \to Y$ a morphism in $\mathcal{C}$, the kernel (resp. cokernel) of f is a morphism i (resp. p) together with the object $\mathrm{Ker}f$ (resp. $\mathrm{Coker}f$), such such $if = 0$ (resp. $fp = 0$), wich has the property that for every morphism g_1 (resp. g_2) such that $g_1 f = 0$ (resp. $fg_2 = 0$),*

$$\begin{tikzcd}
\mathrm{Ker}f \arrow[r, "i"] & X \arrow[r, "f"] & Y \arrow[r, "p"] \arrow[d, "g_2"] & \mathrm{Coker}f \arrow[dl, dotted, "h'"] \\
& A \arrow[u, "g_1"] \arrow[ul, dotted, "h"] & B &
\end{tikzcd}$$

there exists a unique morphism h (resp. h') such that the above diagram commute.

In category of module this definition reduces to the familiar notions of kernel and cokernel of modules.

Definition 1.18. *A category $\mathcal{C}$ is an **abelian category** if it is an additive category, have kernel and cokernel, every monomorphism is the kernel of some morphism in $\mathcal{C}$, every epimorphism is the cokernel of some morphism in $\mathcal{C}$ and every morphism $A \xrightarrow{\alpha} B$ can be written as $A \xrightarrow{u} I \xrightarrow{v} B$ such that u is an epimorphism and v is a monomorphism.*

1.2 The category of modules

In this section we develop some preliminary results about the category of modules. We are mainly interested in the category of modules (Mod-Λ and mod-Λ) over an Artin algebra and sometimes more strictly, finite dimensional algebras, both conditions over the algebra allows us consider the category mod-Λ as an abelian category (see Abelian category in [1]). For a more general view of the properties of modules and rings theory we recommend [17] and [9].

Let K be a field. Let Λ be an associative K-algebra with unity, that is, the ring Λ with a K-vector space structure compatible with the multiplication of the ring. A K-algebra Λ is said to be **finite dimensional** if the dimension $dim_K\Lambda$ of the K-vector space Λ is finite, otherwise is infinite dimensional. The (Jacobson) radical of the algebra Λ is the intersection of all it maximal right ideals, and is denoted by radΛ.

A right module over an algebra Λ is a vector space M with a right multiplication $M \times \Lambda \to M$ such that every pair (m, λ) corresponds to an element $m\lambda$ in M, this multiplication is associative and distributive on the operation in M. The definition of left module is analogous, but in this work we deal essentialy with right modules.

A Λ-module M is **finitely generated** if there is a finite number of elementes $m_1, m_2, \dots, m_n$ of M such that every element $m \in M$ can be written as $m = \sum_{i=1}^{n} m_i a_i$, where $a_i \in \Lambda$.

We will denote by Mod-Λ the category of the right Λ-modules and by mod-Λ the full subcategory of finitely generated Λ-modules. In the same way Λ-Mod and Λ-mod for left Λ-modules.

Definition 1.19. *A module M is called* ***Artin*** *or* ***Artinian*** *if satisfies the descending chain condition (d.c.c.), that is, if for every descending chain of submodules of M*

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

there exists an integer n such that $M_n = M_{n+1} = M_{n+2} = \dots$.

Definition 1.20. *A module M is called* ***Noetherian*** *if satisfies the ascending chain condition (a.c.c.), that is, if for every ascending chain of submodules of M*

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$$

there exists an integer n such that $M_n = M_{n+1} = M_{n+2} = \dots$.

A ring Λ is called a right **Artin** ring (resp. **Noetherian**) if the right regular module Λ_Λ is Artin (resp. Noetherian). From now Λ always denote an Artin algebra over a field K, except in cases we specify.

Proposition 1.21 ([17], Proposition 3.1.12)**.** *If Λ is a right Noetherian (resp. Artinian) ring, then any finitely generated right Λ-module M is Noetherian (resp. Artinian).*

Proposition 1.22 ([17], Corollary 3.1.4)**.** *If Λ is a right Noetherian ring, then any submodule of a finitely generated right Λ-module M is finitely generated.*

Let $\{M_i\}_{i\in I}$ be a family of modules in Mod-Λ, where I is an index set. The direct sum $\bigoplus_{i\in I} M_i$ is the set of infinite tuples $(m_i)_{i\in I}$ with $m_i \in M_i$ for all $i \in I$ and for almost all $i \in I$ m_i is equal to zero. If there is no assumption on the number of nonzero components then we obtain the direct product $\prod_{i\in I} M_i$ of the modules M_i. The direct sum coincides with the direct product of modules if the set I is finite. Sometimes we denote a sum of copies of the same module as a product, that is, $nM := \bigoplus_{i=1}^{n} M$.

Now, we summarize some properties of the abelian category Mod-Λ.

(AB3) For every set $\{A_i\}$ of objects of Mod-Λ, the coproduct, often called the direct sum, $\bigoplus A_i$, exists in Mod-Λ. Rather than say that Mod-Λ satisfies (AB3), we often say that Mod-Λ is cocomplete;

(AB3*) For every set $\{A_i\}$ of objects of Mod-Λ, the product $\prod A_i$ exists in Mod-Λ; Rather than say that Mod-Λ satisfies (AB3*), we usually say that Mod-Λ is complete

(AB4) Mod-Λ is cocomplete, and the direct sum of monomorphism is a monomorphism.

(AB4*) Mod-Λ is complete, and the product of epimorphism is an epimorphism.

From this follows that Mod-Λ satisfies (AB4*), then the product is an exact functor, see Appendix A.4 in [29].

Definition 1.23. *A module, which is isomorphic to a direct sum $M_1 \oplus M_2$, where M_1 and M_2 are nonzero modules, is said to be* ***decomposable****, otherwise it is called* ***indecomposable****.*

Theorem 1.24 ([13], Theorem 4.5)**.** *Let Λ a finite dimensional K-algebra. Every module M in* mod-Λ *has a decomposition $M \simeq M_1 \oplus \cdots \oplus M_m$, where $M_1, \ldots, M_m$ are indecomposable modules. This decomposition is unique up to isomorphism and order of summands.*

Definition 1.25. *An Λ-module M is called* ***free*** *if it is isomorphic to a direct sum of regular modules, that is, $M \simeq \bigoplus_{i\in I} M_i$, where $M_i \simeq \Lambda_\Lambda$ for all $i \in I$.*

Proposition 1.26 ([17], Proposition 1.5.4)**.** *Any module is isomorphic to a quotient module of a free module.*

Definition 1.27. *A nonzero module M is called* ***simple*** *if it has exactly two submodules (the two trivial submodules M and the zero module). A module M is called* ***semisimple*** *if it can be decomposed into a direct sum of simple modules.*

A ring Λ is called a right (resp. left) semisimple if it is semisimple as a right (resp. left) module over itself. Since Λ has an identity and any right submodule of Λ is just a right ideal, Λ is right semisimple if Λ is a direct sum of a finite number of simple right ideals. The following Proposition, well-know as the **Schur's Lemma**, give a property of simple modules.

Proposition 1.28 ([17], Proposition 2.2.1). *Any nonzero homomorphism between simple modules is an isomorphism. In particular, the endomorphism ring of a simple module is a division ring.*

In sense to characterize the semisimple modules we present the well-know **Wedderburn-Artin** Theorem.

Theorem 1.29 ([17], Theorem 2.2.2). *The following conditions are equivalent:*

(a) *The right Λ-module Λ_Λ is semisimple;*

(b) *Every right Λ-module is semisimple;*

(c) *The left Λ-module ${}_\Lambda\Lambda$ is semisimple;*

(d) *Every left Λ-module is semisimple;*

(e) *There exist positive integers $m_1, \dots, m_s$ and a K-algebra isomorphism*

$$\Lambda \cong \mathbb{M}_{m_1}(K) \times \cdots \times \mathbb{M}_{m_s}(K)$$

where $\mathbb{M}_{m_i}(K)$ are matrices over K.

The algebra Λ is called semisimple if one of the equivalent conditions in Theorem 1.29 are satisfied.

Proposition 1.30 ([24], Proposition 4.1). *A left Λ-module M is semisimple if and only if every submodule is a direct summand.*

Corollary 1.31 ([24], Corollary 4.2). *Every submodule and every quotient module of a semisimple module M is semisimple.*

Proposition 1.32 ([24], Proposition 4.5). *If Λ is a semisimple ring, then every right Λ-module M is a semisimple module.*

Proof. Since Λ is semisimple, by definition is semisimple as a module over itself; hence, every free left Λ-module is a semisimple module. Note that every module is a quotient of a free module (Proposition 1.26) and since quotient of semisimple is semisimple (Corollary 1.31), then a right Λ-module M is semisimple. $\square$

Now, a important concept that will be introduce is the radical of a module. First note that a submodule N of a module M is said to be maximal if $N \neq M$ and there is no submodule L, different from M and N, such that $N \subset L \subset M$.

Let M be an arbitrary Λ-module. Denote by $\mathrm{rad} M$ the intersection of all its maximal submodules. By convention, if M does not have maximal submodules we define $\mathrm{rad} M = M$. This submodule is called the **radical of the module** M.

For any nonzero homomorphism $\phi : M \to U$, where U is a simple Λ-module, we have $\mathrm{Im}\phi = U$. Hence $M/\mathrm{Ker}\phi \cong U$ is a simple module. Then $\mathrm{Ker}\phi$ is a maximal submodule of M.

Conversely, for any maximal submodule $M_1 \subset M$ we can build a projection $\pi : M \to M/M_1$ for which $\mathrm{Ker}\phi = M_1$ and M/M_1 is a simple module. Thus, we can give an equivalent definition of the radical of the module M:

Proposition 1.33 ([17], Proposition 3.4.1)**.**
$\mathrm{rad}M = \{ \bigcap \mathrm{Ker}\phi \mid \phi$ runs through all homomorphisms of M to all simple modules$\}$

Proposition 1.34 ([17], Proposition 3.4.2)**.** *Let $f : M \to N$ be a homomorphism of Λ-modules. Then $f(\mathrm{rad}M) \subseteq \mathrm{rad}N$.*

Proposition 1.35 ([13], Proposition 3.7)**.** *If M is in* mod-Λ*, then $M\mathrm{rad}\Lambda = \mathrm{rad}M$.*

Proposition 1.36 ([17], Proposition 3.4.3)**.** $\mathrm{rad}\left(\bigoplus_{\alpha\in I} M_\alpha\right) = \bigoplus_{\alpha\in I} \mathrm{rad}(M_\alpha)$.

Proposition 1.37 ([6], Proposition 2.3.6)**.** *If Λ is a finite dimensional algebra and M is in* mod-Λ*, then* $\mathrm{rad}M$ *is the smallest submodule of M such that $M/\mathrm{rad}M$ is semisimple. This quotient is denoted by* $\mathrm{top}M$*.*

Definition 1.38. *For M in* mod-Λ*, consider the decreasing sequence of submodules*

$$M \supset \mathrm{rad}M \supset \mathrm{rad}^2M \supset \cdots \supset \mathrm{rad}^nM \supset \ldots$$

This sequence is called the ***radical series****, or the descending Loewy series of M. There exists a smallest integer m such that $\mathrm{rad}^mM = 0$, this number m is called the* ***Loewy length*** *of M and is denoted by $ll(M)$.*

Proposition 1.39 ([16], Proposition 3.1)**.** *For an algebra Λ we have the following.*

(a) *The radical of Λ is nilpotent;*

(b) $\Lambda/\mathrm{rad}\Lambda$ *is a semisimple ring;*

(c) *A Λ-module M is semisimple if and only if* $\mathrm{rad}M = 0$*;*

(d) *There is only a finite number of nonisomorphic simple Λ-modules;*

(e) *Λ is left Noetherian;*

Projective and injective modules

We start with some definitions. A sequence $\ldots \longrightarrow X_{n-1} \xrightarrow{h_{n-1}} X_n \xrightarrow{h_n} X_{n+1} \xrightarrow{h_{n+1}} \ldots$ (infinite or finite) of right Λ-modules connected by Λ-homomorphisms is called **exact** if $\mathrm{Ker}h_n =$

$\mathrm{Im}h_{n-1}$ for any n. In particular

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a **short exact sequence** if f is a monomorphism, g is an epimorphism and $\mathrm{Ker}g = \mathrm{Im}f$. If there is a direct sum decomposition $M = \mathrm{Ker}g \oplus \mathrm{Im}f$, we say that the short exact sequence **splits**.

Definition 1.40.

(a) *A module P is called **projective** if for any epimorphism $\varphi : M \to N$ and for any homomorphism $\psi : P \to N$ there is a homomorphism $h : P \to M$ such that $\psi = \varphi h$, that is, the following diagram is commutative*

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "\psi"] & \\ M \arrow[r, "\varphi"'] & N \arrow[r] & 0 \end{tikzcd}$$

(b) *A module Q is called **injective** if for any monomorphism ϕ: $M \to N$ and for any homomorphism $\psi : M \to Q$ there exists a homomorphism $h : N \to Q$ such that $\psi = h\varphi$, that is, the following diagram is commutative*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\psi"'] & N \arrow[dl, dashed, "h"] \\ & Q & \end{tikzcd}$$

Let F be a free Λ-module with basis $\{f_i\}_{i \in I}$, so for $f \in F$ we have $f = \sum_{i \in I} f_i \lambda_i$, $\lambda_i \in \Lambda$ (in a unique way). In definition of projective module change P by F and define $\psi(f_i) = n_i \in N$. Since φ is an epimorhism, consider $\psi(m_i) = n_i$. Then, defining $h : F \to M$ by $h(f) = \sum m_i \lambda_i$ we have that the diagram commutes. Then a free module F is projective. Hence, since any module is isomorphic to a quotient module of a free module, we have that every module is isomorphic to a factor module of a projective module.

Note that the "duality" between the definitions of projective and injective modules implies that many statements for injective modules can be simply obtained by "inverting the arrows" in the theorems on projective modules. In this way we obtain immediately the following result

Proposition 1.41 ([17], Proposition 5.1.1. and 5.2.1.)**.**

(a) *An Λ-module P is projective if and only if* $\mathrm{Hom}_\Lambda(P, -)$ *is an exact functor.*

(b) *An Λ-module Q is injective if and only if* $\operatorname{Hom}_\Lambda(-,Q)$ *is an exact functor.*

Next, we will see some results about projective and injective modules that will be nedeed to ground the proofs of next chapter.

Proposition 1.42 ([17], Proposition 5.1.4. and 5.2.2.)**.**
Consider the following modules in Mod-Λ*, then:*

(a) *A direct sum $P = \bigoplus_{\alpha\in I} P_\alpha$ is a projective module if and only if each module P_α is projective.*

(b) *A direct product $Q = \prod_{\alpha\in I} Q_\alpha$ of injective modules Q_α is injective if and only if each Q_α is injective. Moreover, the direct sum of injectives is injective if, and only if, the algebra is Noetherian.*

For the product of projectives we need more conditions over the algebra, so this result will be given later. For now we follow with other properties of projectives and injectives modules.

Proposition 1.43 ([17], Proposition 5.1.6)**.**
For an Λ-module P the following statements are equivalent:

(a) *P is projective;*

(b) *every short exact sequence $0 \to N \to M \to P \to 0$ splits;*

(c) *P is a direct summand of a free Λ-module F.*

Definition 1.44.

(a) *An Λ-submodule L of M is* ***superfluous*** *or* ***small*** *if for every submodule X of M the equality $L + X = M$ implies $X = M$.*

(b) *An Λ-epimorphism $h : M \to N$ in* mod-Λ *is* ***minimal*** *if* Ker$h$ *is superfluous in M. An epimorphism $h : P \to M$ in* mod-Λ *is called a* ***projective cover*** *of M if P is a projective module and h is a minimal epimorphism.*

Theorem 1.45 ([13], Theorem 5.8)**.** *For any module M in* mod-Λ *there exists a projective cover $P(M) \xrightarrow{h} M \longrightarrow 0$. The projective cover $P(M)$ of a module M in* mod-Λ *is unique in the sense that if $h' : P' \to M$ is another projective cover of M, then there exists an isomorphism $g : P' \to P(M)$ such that $hg = h'$.*

Definition 1.46. *An exact sequence*

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

in mod-Λ *is called a* ***minimal projective presentation*** *of an* Λ*-module* M *if the* Λ*-module homomorphisms* $P_0 \xrightarrow{p_0} M$ *and* $P_1 \xrightarrow{p_1} \mathrm{Ker}(p_0)$ *are projective covers.*

Then, from Theorem 1.45, we have that any module M in mod-Λ admits a minimal projective presentation. The following proposition is a characterazation of projective covers.

Proposition 1.47 ([13], Lemma 5.6)**.** *An epimorphism* $h : P \to M$ *is a projective cover of an* Λ*-module* M *if and only if* P *is projective and for any* Λ*-homomorphism* $g : N \to P$ *the surjectivity of* hg *implies the surjectivity of* g*. In these case* h *is called an* ***essential epimorphism****.*

Proposition 1.48 ([16], Proposition 3.6)**.** *The following are equivalent for an epimorphism* $f : A \to B$*, where* A *and* B *are in* mod-Λ*.*

(a) f *is an essential epimorphism;*

(b) $\mathrm{Ker}(f) \subset \mathrm{rad}A$*;*

(c) *The induced epimorphism* $A/\mathrm{rad}A \to B/\mathrm{rad}B$ *is an isomorphism.*

Proposition 1.49 ([24], Proposition 3.12)**.** *Given exact sequences*

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\pi} M \longrightarrow 0$$

and

$$0 \longrightarrow K' \xrightarrow{i'} P' \xrightarrow{\pi'} M \longrightarrow 0$$

where P *and* P' *are projective, then there is an isomorphism*

$$K \oplus P' \cong K' \oplus P$$

Now we will give a definition of two types of rings, which are nedeed to understand the Propostition 1.52. A module M over a ring R will be called **finitely presented** if there exists an exact sequence

$$0 \longrightarrow \mathrm{Ker}f \longrightarrow F \xrightarrow{f} M \longrightarrow 0$$

of R-modules, where F is free and both F and $\mathrm{Ker}f$ are finitely generated.

Definition 1.50.

- *A ring* R *is* ***right coherent*** *if every finitely generated right ideal is finitely presented.*
- *A ring* R *is called* ***left perfect*** *if every left* R*-module has a projective cover.*

Theorem 1.51 ([9], Theorem 28.4)**.** *The following conditions are equivalent for a ring* R*.*

(a) *R is left perfect.*

(b) *R has the d.c.c. on principal right ideals.*

(c) *Every flat left R-module is projective.*

For now is necessary consider just the items (a) and (b) of Theorem 1.51, the item (c) will be used in another context after we define flat modules.

Proposition 1.52 ([5], Theorem 3.3.)**.** *Every direct product of projective left R-modules is projective if, and only if, R is left perfect and right coherent.*

Note that, by Theorem 1.51, a finite dimensional algebra verify the definitions of coherent and perfect ring, so the product of projectives is projective for this algebras.

Next, we present the dual notion of projective cover, called injective envelope. If N is a submodule of a module M, we shall say that M is an extension of N. A submodule N of M is called essential (or large) in M if it has nonzero intersection with every nonzero submodule of M. In this case we also say that M is an **essential extension** of N.

A module $E(M)$ is called an **injective envelope** of a module M if it is both an essential extension of M and an injective module. This could be represented by a monomorphism $u : M \to E(M)$. Every module M has an injective envelope, which is unique up to an isomorphism extending the identity of M.[[17], Theorem 5.3.4]

It is possible study the injective modules in mod-Λ by means of the projective modules in mod-Λ^{op} using the functor $D- = \mathrm{Hom}_K(-, K)$(Example 1.6) which defines two dualities

$$\text{mod-}\Lambda \xrightarrow{D} \text{mod-}\Lambda^{op} \xrightarrow{D} \text{mod-}\Lambda$$

The following Theorem present some relations give by these dualities on projectives and injectives modules. For details about the above definition see chapter I - 2.9 in [13].

Theorem 1.53 ([13], Theorem 5.13)**.** *Let $D-$ be the standard duality. Then the following hold.*

(a) *A sequence $0 \longrightarrow L \xrightarrow{u} N \xrightarrow{h} M \longrightarrow 0$ in* mod-Λ *is exact if and only if the induced sequence $0 \longrightarrow DM \xrightarrow{D(h)} DN \xrightarrow{D(u)} DL \longrightarrow 0$ is exact in* mod-Λ^{op}.

(b) *A module E in* mod-Λ *is injective if and only if the module DE is projective in* mod-Λ^{op}*. A module P in* mod-Λ *is projective if and only if the module DP is injective in* mod-Λ^{op}.

(c) *A module S in* mod-Λ *is simple if and only if the module DS is simple in* mod-Λ^{op}.

(d) *A monomorphism $u : M \to E$ in* mod-Λ *is an injective envelope if and only if the epimorphism $D(u) : DE \to DM$ is a projective cover in* mod-Λ^{op}*. An epimorphism $h : P \to M$ in* mod-Λ *is a projective cover if and only if the $D(h) : DM \to DP$ is an injective envelope in* mod-Λ^{op}.

1.3 Homological Algebra

This section brings basic notions and elementary facts from homological algebra needed in this work. In particular, we define the functors Ext and Tor, the projective and injective dimensions of a module, the global dimension of an algebra and the finitistic dimensions. Let's start with some definitions

A **chain complex** in the category Mod-Λ is a sequence

$$C_\bullet : \dots \longrightarrow C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \dots$$

of right Λ-modules connected by Λ-homomorphisms such that $d_n d_{n+1} = 0$ for all $n \geq 0$. A **cochain complex** in the category Mod-Λ is a sequence

$$C^\bullet : \dots \longrightarrow C^{n-2} \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \dots$$

of right Λ-modules connected by Λ-homomorphisms such that $d^{n-1}d^n = 0$ for all $n \geq 0$. The morphism d^n and d_n are called diferentials and the object C^n (resp. C_n) are called the n-th homogeneous component of $C^\bullet$ (resp. $C_\bullet$).

A moprhism $f^\bullet$ between complexes $X^\bullet$ and $Y^\bullet$ in Mod-Λ, is a family of morphisms $f^\bullet = (f^n : X^n \to Y^n)_{n\in\mathbb{Z}}$ such that $d_Y^n \circ f^n = f^{n+1} \circ d_X^n$, for each $n \in \mathbb{Z}$. That is, the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \dots \\
\dots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

The definition of morphism between chain complexes is analogous, sometimes we refer to this as a chain maps or cochain maps. Note that a module M in Mod-Λ can be viewed as a complex concentrated in 0-th homogeneous component. Sometimes, when there is no confusion, we omit the $\bullet$ in notation of complexes and morphisms of complexes. In Chapter 3 we retake the complexes to consider the category constitute by them.

For each $n \geq 0$, the n-th **homology** Λ-module of the chain complex $C_\bullet$ and the n-th **cohomology** Λ-module of the cochain complex $C^\bullet$ are the quotient Λ-modules

$$H_n(C_\bullet) = \mathrm{Ker} d_n / \mathrm{Im} d_{n+1} \quad \text{and} \quad H^n(C^\bullet) = \mathrm{Ker} d^n / \mathrm{Im} d^{n-1},$$

respectively. If $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism of complexes, we have $f^n(\mathrm{Ker} d_X^n) \subset \mathrm{Ker} d_Y^n$ and $f^n(\mathrm{Im} d_X^{n-1}) \subset \mathrm{Im} d_Y^{n-1}$ for each $n \in \mathbb{Z}$, then $f^\bullet$ induces a morphism $H^n(f^\bullet) : H^n(X^\bullet) \to H^n(Y^\bullet)$, analagously for homology (see page 95 in [18]). Therefore, H^n is an additive functor between the category of complexes over Mod-Λ and the category Mod-Λ for each n, called cohomological functor.

Lemma 1.54 (Fives Lemma). *Consider a commutative diagram with exact rows.*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "h_1"] & A_2 \arrow[r] \arrow[d, "h_2"] & A_3 \arrow[r] \arrow[d, "h_3"] & A_4 \arrow[r] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

(i) *If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective;*

(ii) *If h_2 and h_4 are injective and h_1 is surjective, then h_3 is injective;*

(iii) *If h_1, h_2, h_4, and h_5 are isomorphisms, then h_3 is an isomorphism.*

Proof. See Proposition 21.1 (page 35) in [19]. $\square$

Proposition 1.55 ([29], Exercise 1.2.1). *Let $\{M_\alpha\}_{\alpha\in I}$ be a family of complexes of right Λ-modules, then $\bigoplus_{\alpha\in I} H_n(M_\alpha) \simeq H_n\left(\bigoplus_{\alpha\in I} M_\alpha\right)$ and $\prod_{\alpha\in I} H_n(M_\alpha) \simeq H_n\left(\prod_{\alpha\in I} M_\alpha\right)$ for every n.*

Proof. For each complex consider the short exact sequence

$$0 \longrightarrow \mathrm{Im} d^\alpha_{n+1} \longrightarrow \mathrm{Ker} d^\alpha_n \longrightarrow H_n(M_\alpha) \longrightarrow 0$$

Since Mod-Λ is complete the product is an exact functor, then we have the follwing diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \prod \mathrm{Im} d^\alpha_{n+1} \arrow[r] \arrow[d, equal] & \prod \mathrm{Ker} d^\alpha_n \arrow[r] \arrow[d, equal] & \prod H_n(M_\alpha) \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & \mathrm{Im}(\prod d^\alpha_{n+1}) \arrow[r] & \mathrm{Ker}(\prod d^\alpha_n) \arrow[r] & H_n(\prod M_\alpha) \arrow[r] & 0 \end{tikzcd}$$

Then, by Lemma 1.54 we have the isomorphism. Analagously for sum and cohomology. $\square$

Theorem 1.56. *Let $\mathcal{A}$ be an abelian category. If*

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 0$$

is an exact sequence of complexes in $\mathcal{A}$, then, for each $n \in \mathbb{Z}$, there is a morphism in $\mathcal{A}$

$$\delta_n : H_n(C'') \longrightarrow H_{n-1}(C')$$

called the ***connecting homomorphism****, defined by*

$$\delta_n : z''_n + \mathrm{Im} d_{n+1}^{C''} \longmapsto i_{n-1}^{-1} d_n p_n^{-1} z''_n + \mathrm{Im} d_n^{C'}$$

Moreover, there exists the following ***long exact sequence*** *in* $\mathcal{A}$*.*

$$\dots \longrightarrow H_{n+1}(C'') \xrightarrow{\delta_{n+1}} H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(p)} H_n(C'') \xrightarrow{\delta_n} H_{n-1}(C') \longrightarrow \dots$$

in the same way, we have this result for homology.

Proof. See Proposition 6.9 and Theorem 6.10 (page 332) in [24]. □

Projective and injective dimensions

Let M be a module in Mod-Λ and each P_i a projective Λ-module. The following exact sequence

$$\dots \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{1.57}$$

is called the **projective resolution** of the Λ-module M. Sometimes we consider the projective resolution of a module M as the complex

$$P_\bullet = \quad \dots \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

Proposition 1.58 ([13], Lemma 5.4)**.** *Every module in* Mod-Λ *has a projective resolution. In particular, if such a module is in* mod-Λ*, so is their projective resolution.*

In the same way, we define the **injective resolution**Module!injective resolution of a Λ-module N to be the exact sequence

$$0 \longrightarrow N \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \dots \longrightarrow I_m \xrightarrow{i_{m+1}} \dots \tag{1.59}$$

where each I_i is a injective Λ-module.

An exact sequence (1.57) in mod-Λ is called a **minimal projective resolution** of M if P_0 is a projective cover of M and P_i is a projective cover of $\mathrm{Ker}(d_{i-1})$, for all $i \geq 1$. We called $\Omega^n M$ of the n-th **syzygy** of M and define as follows

$$\Omega^n M = \begin{cases} M & \text{if } n = 0 \\ \mathrm{Ker}(d_{n-1}) & \text{if } n \geq 1 \end{cases}$$

Definition 1.60. *The* ***projective dimension*** *of a right* Λ*-module* M *is the nonnegative integer* $\mathrm{pd}M = m$ *such that there exists a minimal projective resolution of* M *of length* m *and* M *has no* *projective resolution of length $m-1$, if such a number m exists. If M admits no projective resolution of finite length, we define the projective dimension* pd$M$ *of M to be infinity. Analagously, we define the* ***injective dimension****,* id$N$*, of an Λ-module N.*

It follows from the previous definitions that pd$M = 0$ if and only if M is projective and id$N = 0$ if and only if N is injective. Next we define the flat modules and the flat dimension (or weak dimension).

Definition 1.61. *A right Λ-module M is* ***flat*** *if $M \otimes_\Lambda -$ is an exact functor, that is, the functor preserves exactness of a sequence.*

The **flat resolution** of a module M is an analogous of projective and injective resolution, where we consider an exact sequence of the form

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

whith F_i flat modules for all $i \in \mathbb{N}$. The length of a flat resolution is the first subscript n such that F_n is nonzero and $F_i = 0$ for $i > n$. Then the flat dimension is the length of a flat resolution and is denoted by fd(M). If M does not admit a finite flat resolution, then the flat dimension is said to be infinite.

Lemma 1.62 ([17], Lemma 6.2.5)**.** *Suppose*

$$0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

be an exact sequence of modules. Then there is an exact sequence

$$0 \longrightarrow P'_\bullet \xrightarrow{f} P_\bullet \xrightarrow{g} P''_\bullet \longrightarrow 0$$

of projective resolutions, in which f extends φ and g extends ψ.

When we want consider the homological properties of an algebra a imporant tool is the homological dimensions. The **right global dimension** and the **left global dimension** of Λ are defined to be the numbers

$$\text{r.gl.dim}\Lambda = \max\{\text{pd}M \mid M \in \text{mod-}\Lambda\}$$

and

$$\text{l.gl.dim}\Lambda = \max\{\text{pd}L \mid L \in \Lambda\text{-mod}\}$$

respectively, if these numbers exists; otherwise, we say that the right global dimension of Λ (or the left global dimension of Λ, respectively) is infinity. This definition is may also given in terms of injective dimension. For a finite dimensional algebra the left and right global dimension coincide, so it is just called global dimension. If we restrict the definition of global dimension considering finitely generated (or not) modules of finite projective dimension this gives arrises an another two homological dimensions, called finitistic dimensions, which are defined below

$$\operatorname{findim}(\Lambda) = \sup \{\operatorname{pd}(M) \mid M \in \text{mod-}\Lambda \text{ and } \operatorname{pd}(M) < \infty\}$$

and

$$\operatorname{Findim}(\Lambda) = \sup \{\operatorname{pd}(M) \mid M \in \text{Mod-}\Lambda \text{ and } \operatorname{pd}(M) < \infty\}$$

This dimensions are called *little* finitistic dimension and *big* finitistic dimension, respectively.

Functors Tor and Ext

Definition 1.63. *For each $m \geq 0$, the m-th **extension functor** is*

$$\begin{array}{rcl} \operatorname{Ext}^m_\Lambda(-,N) : \text{Mod-}\Lambda & \longrightarrow & \text{Mod-}K \\ M & \mapsto & H^m(\operatorname{Hom}_\Lambda(P_\bullet, N)) \end{array}$$

where $P_\bullet$ is the projective resolution of M and $\operatorname{Ext}^m_\Lambda(f,N)$ is induced on the projective resolution by f, that is, $\operatorname{Hom}_\Lambda(f,N) = \operatorname{Hom}_\Lambda(f_m,N)_{m\in\mathbb{N}}$. The functor is additive and contravariant. Analagously $\operatorname{Ext}^m_\Lambda(M,-) : \text{Mod-}\Lambda \to \text{Mod-}K$ is a covariant additive functor.

Definition 1.64. *For each $m \geq 0$, the m-th **torsion functor** is*

$$\begin{array}{rcl} \operatorname{Tor}^\Lambda_m(-,N) : \text{Mod-}\Lambda & \longrightarrow & \text{Mod-}K \\ M & \mapsto & H_m(P_\bullet \otimes_\Lambda N) \end{array}$$

where $P_\bullet$ is the projective resolution of M and $\operatorname{Tor}^\Lambda_m(f,N)$ is the morphism induced on the projective resolution by f, that is, $f_\bullet \otimes 1_N : P_\bullet \otimes_\Lambda N \to P'_\bullet \otimes_\Lambda N$. The functor is additive and covariant. Analagously $\operatorname{Tor}^\Lambda_m(M,-) : \text{Mod-}\Lambda \to \text{Mod-}K$ is a covariant additive functor.

Remark 1.65. *Note that $Ext^0_\Lambda(M,N) \simeq \operatorname{Hom}_\Lambda(M,N)$ and $\operatorname{Tor}^\Lambda_0(M,N) \simeq M \otimes_\Lambda N$ (see [13], A.4.Theorem 4.5 and 4.10). Since the Theorem 1.56 gives us a long exact sequence defined in terms of the cohomological functor and the functors* Tor *and* Ext *are defined in terms of homology and cohomology, in particular we have long exact sequences defined with these functors, see the above cited Theorems.*

Theorem 1.66 ([2], Theorem 1.2 - chap X)**.** *Let M be a Λ-module. Then the following conditions are equivalent:*

(a) $\operatorname{pd}(M) \leq n$;

(b) $\operatorname{Ext}^k_\Lambda(M,-) = 0, \ \forall k > n$;

(c) $\operatorname{Ext}^{n+1}_\Lambda(M,-) = 0$;

(d) *For every exact sequence, with P_i projective for every i,*

$$0 \longrightarrow \Omega^n(M) \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

$\Omega^n(M)$ *is projective.*

Remark 1.67. *Follow from Theorem 1.66 that* $\mathrm{pd}M = \sup\{n \mid \mathrm{Ext}^n_\Lambda(M,-) \neq 0\}$*, so on that, for a family* $\{M_i\}_{i\in I}$ *in* Mod-Λ *we have* $\mathrm{pd}\left(\bigoplus_{i\in I} M_i\right) = \sup\{\mathrm{pd}M_i \mid i \in I\}$.

Lemma 1.68 ([29], Lemma 4.1.10)**.** *The following are equivalent for a right Λ-module M:*

1. $fd(M) \leq d$
2. $\mathrm{Tor}^\Lambda_n(M,N) = 0$ *for all $n > d$ and all left Λ-modules N.*
3. $\mathrm{Tor}^\Lambda_{d+1}(M,N) = 0$ *for all left Λ-modules N.*
4. *If* $0 \longrightarrow B_d \longrightarrow F_{d-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ *is a resolution with the F_i all flat, then B_d is also a flat Λ-module.*

Proposition 1.69 ([13], A.4.-Proposition 4.11)**.** *Let B be a finite dimensional K-algebra. For all modules Y and Z in* mod-B*, there exist functorial isomorphisms of K-vector spaces* $\mathrm{Hom}_B(Y,DZ) \cong D(Y \otimes_B Z)$ *and* $D\mathrm{Ext}^1_B(Y,DZ) \cong Tor^B_1(Y,Z)$.

Chapter 2

Igusa-Todorov Functions

Let Λ be an Artin algebra and mod-Λ the category of finitely generated (f.g.) right Λ-modules. Recall that the little finitistic dimension of Λ is

$$\text{findim}(\Lambda) = \sup \{\text{pd}(M) \mid M \in \text{mod-}\Lambda \text{ and } \text{pd}(M) < \infty\}$$

The finitistic dimension conjecture (for Artin algebra) is the conjecture that findim$(\Lambda) < \infty$ for every Artin algebra. The first version of this conjecture was publicized as a question by H. Bass ([4], 1960) and still open. The corresponding question for the big finitistic dimension is also open and will be present in Chapter 4. If the finitistic conjecture holds, then so do many other highly studied conjectures in the respresentation theory of algebras. However, there are a few cases for which this conjecture is verified to be true. One of the most impressive results in this direction is due to K. Igusa and G. Todorov [14]. In this chapter we will present the results proved in this paper, which consist of a condition which implies the finiteness of the finitistic dimension of Artin algebras with radical cubed zero and for algebras of representation dimension less then or equal to three.

Throughout this chapter we give some examples using quivers, however, we will not introduce the basic concepts related to this theory, for this and the notation that will be used, we recommend see [13] and [27].

It can be noted that the following Lemma differs from the slightly misstated Fitting's Lemma presented on the paper, more precisely we consider M a Λ-module not necessarily finitely generated. We made this correction for a suitable application on the results that will be proved.

Lemma 2.1 (**Fitting's Lemma**). *Let Λ be a Noetherian algebra and M be a Λ-module. For $f \in \text{End}_\Lambda(M)$, then:*

(a) *For any finitely generated submodule X of M, there is an integer n so that f sends $f^m(X)$*

isomorphically onto $f^{m+1}(X)$, for all $m \geq n$. Let $\eta_f(X)$ denote the smallest value of n with this property;

(b) *If Y is a submodule of X then $\eta_f(Y) \leq \eta_f(X)$;*

(c) *If in addition Λ is an Artin algebra and $X = M$ is in* mod-Λ, *then there is a direct sum decomposition $X = Y \oplus Z$ so that $Z = \mathrm{Ker}(f^m)$ and $Y = \mathrm{Im}(f^m)$, for all $m \geq \eta_f(X)$.*

Proof. **(a)** Restrict f to X and consider the following sequence:

$$X \xrightarrow{f} f(X) \xrightarrow{f} f^2(X) \longrightarrow \dots \longrightarrow f^m(X) \xrightarrow{f} f^{m+1}(X) \longrightarrow \dots \tag{2.2}$$

Take $a \in \mathrm{Ker}(f^n|_X)$, so $f^{n+1}(a) = f(f^n(a)) = f(0) = 0$ then $a \in \mathrm{Ker}(f^{n+1}|_X)$. We thus obtain an ascending chain of submodules

$$\mathrm{Ker}(f|_X) \subseteq \mathrm{Ker}(f^2|_X) \subseteq \dots \subseteq \mathrm{Ker}(f^m|_X) \subseteq \dots$$

Since Λ is Noetherian and X is finitely generated, X is Noetherian (Proposition 1.21). Then any ascending chain of submodules of X stabilizes, hence there is a integer n_f such that $\mathrm{Ker}(f^m|_X) = \mathrm{Ker}(f^{m+1}|_X)$, for all $m \geq n_f$.

Now, consider the map $f : f^m(X) \to f^{m+1}(X)$, for $m \geq n_f$. Let $b \in f^{m+1}(X)$, then $b = f^{m+1}(a)$ for some $a \in X$. Thus, $f(f^m(a)) = b$, that implies $f : f^m(X) \to f^{m+1}(X)$ is surjective, because there is $f^m(a) \in f^m(X)$ such that $f(f^m(a)) = b$.

Let $a \in \mathrm{Ker}(f : f^m(X) \to f^{m+1}(X))$. Hence $a = f^m(y)$, for some y in X, therefore,

$$f^{m+1}(y) = f(f^m(y)) = f(a) = 0$$

thus, $y \in \mathrm{Ker}(f^{m+1}|_X) = \mathrm{Ker}(f^m|_X)$. Then, $f^m(y) = 0 = a$ and consequently $f : f^m(X) \to f^{m+1}(X)$ is injective. So $f : f^m(X) \to f^{m+1}(X)$ sends $f^m(X)$ isomorphically onto $f^{m+1}(X)$.

(b) Note that a submodule of a finitely generated Noetherian module is finitely generated and Noetherian (Proposition 1.22). We have that $f^m(X) \cong f^{m+1}(X)$, for all $m \geq n_f(X)$, so $f^m(Y) \cong f^{m+1}(Y)$, for all $m \geq n_f(X)$. Since $n_f(Y)$ is the smallest value that this is true, we have $n_f(Y) \leq n_f(X)$.

(c) Consider the sequence (2.2) and $X = M$. Note that for $a \in \mathrm{Im}(f^{n+1})$, there is $b \in X$ such that $a = f^{n+1}(b) = f^n(f(b))$, then $a \in \mathrm{Im}(f^n)$. Since X is a finitely generated Λ-module and Λ is an Artin algebra, X is Artinian (Proposition 1.21). So, for descending chain below there is a integer for which it stabilizes.

$$\mathrm{Im}(f) \supseteq \mathrm{Im}(f^2) \supseteq \dots \supseteq \mathrm{Im}(f^k) \supseteq \dots$$

that is, $\mathrm{Im}(f^k) = \mathrm{Im}(f^{k+1})$, for all $k \geq n_0$. From part (a) X is Noetherian, so $\mathrm{Ker}(f^k) = \mathrm{Ker}(f^{k+1})$, for all $k \geq n_1$. Let $n_f = \max\{n_0, n_1\}$.

Now we want show that $\mathrm{Ker}(f^k)\bigcap \mathrm{Im}(f^k) = 0$, for all $k \geq n_f$. Let $x \in \mathrm{Ker}(f^k)\bigcap \mathrm{Im}(f^k)$, then $f^k(x) = 0$ and $x = f^k(y)$, for some $y \in X$. Note that $f^{2k}(y) = f^k(f^k(y)) = f^k(x) = 0$, and consequently $y \in \mathrm{Ker}(f^{2k}) = \mathrm{Ker}(f^k)$, then $x = f^k(y) = 0$.

To show that $X = \mathrm{Ker}(f^m) \oplus \mathrm{Im}(f^m)$ consider $c \in X$. Then, applying f^k on c we have that $f^k(c) \in \mathrm{Im}(f^k) = \mathrm{Im}(f^{2k})$, thus $f^k(c) = f^{2k}(a)$, for some $a \in X$. Note that

$$f^k(c - f^k(a)) = f^k(c) - f^{2k}(a) = 0$$

then, $c - f^k(a) \in \mathrm{Ker} f^k$. Thus, $c = c - f^k(a) + f^k(a)$, where $c - f^k(a) \in Z = \mathrm{Ker}(f^k)$ and $f^k(a) \in Y = \mathrm{Im}(f^k)$. Therefore given c in X we have $c = a + b$, where $a \in \mathrm{Ker}(f^k)$ and $b \in \mathrm{Im}(f^k)$, for all $k \geq n_f$. $\square$

2.1 The functions ϕ and ψ

In the this section we will define the Igusa-Todorov functions, which determine new homological measures, generalising the notion of projective dimension, and which have become a powerful tool to understand better the finitistic dimension conjecture.

Let K_0 be the abelian group generated by all symbols $[M]$, where M is in mod-Λ, modulo the relations,

$$\begin{cases} [C] = [A] + [B] & \text{, if } C \cong A \oplus B \\ [P] = 0 & \text{, if } P \text{ is projective} \end{cases}$$

Then K_0 is the free abelian group generated by the isomorphism classes of indecomposable finitely generated nonprojective Λ-modules. So, if we take some element of K_0 it will have the following form:

$$a_1[M_1] + \cdots + a_t[M_t]$$

where, for $i = 1, \ldots, t$, $a_i \in \mathbb{Z}$ and M_i is an indecomposable finitely generated Λ-module, with $M_i \not\cong M_j$, $i \neq j$. Recall that we define the syzygy of a module M as the kernel of maps in a minimal projective resolution.

$$\begin{tikzcd}
\ldots \arrow[r] & P_n \arrow[r, "d_n"] \arrow[dr, two heads] & P_{n-1} \arrow[r] & \ldots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[dr, two heads] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \\
\ldots & & \Omega^n M \arrow[u, hook] & \ldots & & \Omega M \arrow[u, hook] & &
\end{tikzcd}$$

Note that Ω commutes with direct sums. Consider the following projetive covers,

$$0 \longrightarrow \Omega M_1 \longrightarrow P_0^1 \xrightarrow{d_0^1} M_1 \longrightarrow 0$$

$$0 \longrightarrow \Omega M_2 \longrightarrow P_0^2 \xrightarrow{d_0^2} M_2 \longrightarrow 0$$

then, $0 \longrightarrow \Omega(M_1 \oplus M_2) \longrightarrow P_0^1 \oplus P_0^2 \longrightarrow M_1 \oplus M_2 \longrightarrow 0$ is exact. Because it is minimal $\Omega(M_1 \oplus M_2) = \Omega(M_1) \oplus \Omega(M_2)$. For projective modules the projective cover is the module itself, so the syzygy is zero, i.e., Ω takes projective modules to zero. This gives us that the map $L : K_0 \to K_0$ that sends $[M]$ to $[\Omega M]$ is a homomorphism of abelian groups. Note that the definition does not depend of the choice of the projective resolution of M (Proposition 1.49).

Let M in mod-Λ. We will denote by $\langle addM \rangle$ the subgroup of K_0 generated by all indecomposable summands of M. Note that K_0 is a free $\mathbb{Z}$-module, hence $\langle addM \rangle$ is a f.g. submodule of K_0 and since L is in $\mathrm{End}_{\mathbb{Z}}(K_0)$, by Fitting's Lemma, we define

$$\begin{aligned} \phi : \mathrm{mod}(\Lambda) &\longrightarrow \mathbb{N} \\ M &\longmapsto \phi(M) = \eta_L \langle addM \rangle \end{aligned}$$

that is, $L^n(\langle addM \rangle) \simeq L^{n+1}(\langle addM \rangle)$, $\forall n \geq \phi(M)$. On the other hand, consider the homomorphism L restrict to $\langle addM \rangle$, which gives rise to the following sequence when iterated,

$$\langle addM \rangle \xrightarrow{L} L(\langle addM \rangle) \xrightarrow{L} \dots \longrightarrow L^n(\langle addM \rangle) \xrightarrow{L} \dots$$

Note that the rank of $\langle addM \rangle$, denoted by $rk(\langle addM \rangle)$, is finite since M is in mod-Λ. Then,

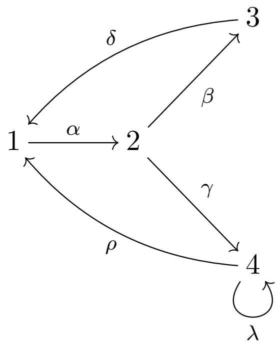
$$rk(\langle addM \rangle) \geq rk(L\langle addM \rangle) \geq \dots \geq rk(L^n\langle addM \rangle) \geq \dots \tag{2.3}$$

The well-ordering principle states that every non-empty set of positive integers contains a least element, thus there exists N such that,

$$rk(L^m\langle addM \rangle) = rk(L^{m+1}\langle addM \rangle), \forall m \geq N.$$

Note that this integer N coincides with the η_L, so alternatively, define $\phi(M)$ to be the smallest integer for which the rank in (2.3) stabilizes. This definition is very helpful since it is easier look at the rank than the isomorphisms of L in some cases.

Example 2.4. *Let Q be the quiver:*



Let $I = \langle \alpha\beta, \rho\alpha, \lambda^2, \gamma\lambda \rangle$ be an admissible ideal. Then define the algebra $\Lambda := KQ/I$ and consider right Λ-modules. So we have the indecomposable projective representations.

$$
\begin{array}{cccc}
P(1) & P(2) & P(3) & P(4) \\
\begin{smallmatrix}1\\|\\2\\|\\4\\|\\1\end{smallmatrix} &
\begin{array}{ccc} & 2 & \\ \swarrow & & \searrow \\ \begin{smallmatrix}3\\|\\1\\|\\2\\|\\4\\|\\1\end{smallmatrix} & & \begin{smallmatrix}4\\|\\1\end{smallmatrix}\end{array} &
\begin{smallmatrix}3\\|\\1\\|\\2\\|\\4\\|\\1\end{smallmatrix} &
\begin{array}{ccc} & 4 & \\ \swarrow & & \searrow \\ 1 & & \begin{smallmatrix}4\\|\\1\end{smallmatrix}\end{array}
\end{array}
$$

Let $X_1 := \begin{smallmatrix}4\\|\\1\end{smallmatrix}$ and $X_2 := \begin{smallmatrix}2\\|\\4\\|\\1\end{smallmatrix}$. In sense to give an example of how ϕ works in a decomposable module let $M_1 = \begin{smallmatrix}2\\|\\3\\|\\1\end{smallmatrix}$, $M_2 = \begin{smallmatrix}4\\|\\4\end{smallmatrix}$, $M_3 = S(4)$ be an indecomposable Λ-modules and $M = M_1 \oplus M_2 \oplus M_3$. Therefore $\langle addM \rangle = \langle [M_1], [M_2], [M_3] \rangle$ and $\mathrm{L}(\langle addM \rangle) = \langle [\Omega M_1], [\Omega M_2], [\Omega M_3] \rangle$, where

$$
\begin{aligned}
[\Omega M_1] &= [X_1 \oplus X_2] = [X_1] + [X_2] \\
[\Omega M_2] &= [S(1) \oplus S(1)] = [S(1)] + [S(1)] \\
[\Omega M_3] &= [X_1 \oplus S(1)] = [X_1] + [S(1)]
\end{aligned}
$$

Note that there is no linear combination between this simbols and hence $rk(\mathrm{L}(\langle addM \rangle)) = 3$. Also note that $\mathrm{L}(\langle addM \rangle) \leqslant \langle [X_1], [X_2], [S(1)] \rangle$, which is a subgroup with the same rank of the group.

However we have that

$$\Omega M = X_1 \oplus X_2 \oplus S(1) \oplus S(1) \oplus X_1 \oplus S(1) \oplus X_2.$$

Now,

$$\begin{aligned}
[\Omega^2 M_1] &= [X_1] \\
[\Omega^2 M_2] &= [X_2] + [X_2] \\
[\Omega^2 M_3] &= [X_1] + [X_2]
\end{aligned}$$

hence $L^2(\langle addM \rangle) = \langle [X_1], [X_1 \oplus X_2] \rangle$, *because* $[\Omega^2 M_3] - [\Omega^2 M_1] = [X_2]$ *which generate* $[\Omega^2 M_2]$. *And note that* $\Omega^2 M = X_1 \oplus P(3) \oplus X_2 \oplus X_2 \oplus X_1 \oplus X_2$. *To finish,*

$$\begin{aligned}
[\Omega^3 M_1] &= [X_1] \\
[\Omega^3 M_2] &= 0 \\
[\Omega^3 M_3] &= [X_1]
\end{aligned}$$

hence $L^3(\langle addM \rangle) = \langle [X_1] \rangle$ *and note that* $\Omega^3 M = X_1 \oplus P(3) \oplus P(3) \oplus X_1 \oplus P(3)$. *Since* L *sends* $[X_1]$ *isomorphically to* $[X_1]$, *then*

$$rk(\langle addM \rangle) = 3 \geq rk(L\langle addM \rangle) = 3 \geq rk(L^2\langle addM \rangle) = 2 \geq rk(L^3\langle addM \rangle) = 1 = \dots$$

Is not dificult to see that $\phi(M) = \mathrm{pd}(M)$ *when the projective dimension is finite. Then the function* ϕ *is a more thin measure than the projective dimension. Now, if we take the indecomposable module* $S(4)$ *we have the following projective resolution.*

$$\begin{tikzcd}
\dots \arrow[r] & P(4) \arrow[r] \arrow[dr, two heads] & P(3) \oplus P(4) \arrow[r] \arrow[dr, two heads] & P(2) \oplus P(4) \arrow[r] \arrow[dr, two heads] & P(1) \oplus P(4) \arrow[r] \arrow[dr, two heads] & P(4) \arrow[r] & S(4) \arrow[r] & 0 \\
 & & X_1 \arrow[u] & P(3) \oplus X_1 \arrow[u] & X_2 \oplus X_1 \arrow[u] & S(1) \oplus X_1 \arrow[u] & &
\end{tikzcd}$$

Note that $\mathrm{pd}S(4) = \infty$, *but the rank stabilize since the first term:*

$$\begin{aligned}
[\Omega S(4)] = [S(1) \oplus X_1] &\Longrightarrow rk(L\langle addM \rangle) = 1 \\
[\Omega^2 S(4)] = [X_2 \oplus X_1] &\Longrightarrow rk(L^2\langle addM \rangle) = 1 \\
[\Omega^3 S(4)] = [P(3) \oplus X_1] &\Longrightarrow rk(L^3\langle addM \rangle) = 1 \\
[\Omega^4 S(4)] = [X_1] &\Longrightarrow rk(L^4\langle addM \rangle) = 1 \\
&\vdots
\end{aligned}$$

So $\phi(S(4)) = 0$.

Lemma 2.5. **(a)** *If M has finite projective dimension then $\phi(M) = pdM$.*

(b) *If M is indecomposable with $pdM = \infty$ then $\phi(M) = 0$.*

(c) $\phi(A) \leq \phi(A \oplus B))$

(d) $\phi(kM) = \phi(M)$*, if* $k \geq 1$.

Proof. **(a)** Let M be a Λ-module with $pdM = m$, hence $\Omega^m(M)$ is projective (Theorem 1.66). So $L^n(\langle addM \rangle) = 0$ for $n \geq m$, then $\phi(M) \leq m$. Suppose now $\phi(M) = t < m$ and note that $\Omega^m(M)$ is projective, so

$$rk(\langle add\Omega^t M \rangle) = rk(\langle add\Omega^{t+1} M \rangle) = \cdots = rk(\langle add\Omega^m M \rangle) = 0$$

then $\Omega^t(M)$ is projective too, consenquently $pdM = t$ which is a contradiction with the value of projective dimension of M. Then $\phi(M) = \mathrm{pd}M$.

(b) We have that M is indecomposable, so $\langle addM \rangle = \langle [M] \rangle$ and $rk(\langle addM \rangle) = 1$. Therefore, $rk(\langle add\Omega^n M \rangle) \leq 1$, $\forall n \geq 0$. If $rk(\langle add\Omega^n M \rangle) = 0$ for some n, then $pdM = n$, which contradicts the fact that $pdM = \infty$. Then $rk(\Omega^n(M)) = 1$ $\forall n \geq 0$, hence $\phi(M) = 0$.

(c) This follows directly from part (b) of Fitting's Lemma, because M is a submodule of $M \oplus N$ and $\langle addM \rangle$ is a subgroup of $\langle add(M \oplus N) \rangle$.

(d) Recall that $kM = \displaystyle\bigoplus_{i=1}^{k} M$, then $\langle add(M) \rangle = \langle add(kM) \rangle$, hence $\phi(M) = \phi(kM)$, $k \geq 1$. $\square$

Definition 2.6. *For any finitely generated Λ-module M, let*

$$\psi(M) := \phi(M) + \sup\,\{\mathrm{pd}X \mid \mathrm{pd}X < \infty,\ X \text{ direct summand of } \Omega^{\phi(M)}M\}$$

Example 2.7. *Consider the same quiver of Example 2.4. Let* $M = \begin{array}{c}4\\ |\\ 1\end{array} \oplus \begin{array}{c}1\\ |\\ 2\\ |\\ 4\end{array} \oplus \begin{array}{c}3\\ |\\ 1\end{array} \oplus \begin{array}{c}4\\ |\\ 4\end{array}$ *and it projective resolution:*

$$\begin{tikzcd}
{} \arrow[r] & P(4)\oplus P(1)\oplus P(2)\oplus(P(1)\oplus P(1)) \arrow[r] & P(4)\oplus P(1)\oplus P(3)\oplus P(4) \arrow[r] & M \arrow[r] & 0 \\
& X_1\oplus X_2\oplus P(3)\oplus(X_2\oplus X_2) \arrow[u] & X_1\oplus S(1)\oplus X_2\oplus(S(1)\oplus S(1)) \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & P(4) \arrow[r] & P(4)\oplus P(3)\oplus(P(3)\oplus P(3)) \arrow[r] & P(4)\oplus P(2)\oplus P(3)\oplus(P(2)\oplus P(2)) \arrow[r] & {} \\
& X_1 \arrow[u] & X_1 \arrow[u] & X_1\oplus P(3)\oplus(P(3)\oplus P(3)) \arrow[u] &
\end{tikzcd}$$

Note that $\psi(M) = 3$. *If we take* $Z = X_2$ *a summand of* $\Omega^2(M)$ *(note that* $\mathrm{pd}Z < \infty$*), then* $\mathrm{pd}Z+2 \leq \psi(M)$. *The same happen if we take a summand of* $\Omega(M)$ *with finite projective dimension. The item* (d) *of the next Lemma generalize this fact.*

Lemma 2.8. **(a)** $\psi(M) = \phi(M) = \mathrm{pd}M$ *whenever* $\mathrm{pd}M < \infty$.

(b) $\psi(kM) = \psi(M)$, *if* $k \geq 1$.

(c) $\psi(A) \leq \psi(A \oplus B)$.

(d) *If* Z *is a summand of* $\Omega^n M$ *where* $n \leq \phi(M)$ *and* $\mathrm{pd}Z < \infty$ *then* $\mathrm{pd}Z + n \leq \psi(M)$.

Proof. **(a)** If $\mathrm{pd}M < \infty$, then $\Omega^{\phi(M)}(M) = \Omega^{\mathrm{pd}(M)}(M)$ and is projective. Therefore X, direct summand of $\Omega^{\phi(M)}(M)$, is projective too, hence $\mathrm{pd}X = 0$. So

$$\sup\,\{\mathrm{pd}X \mid \mathrm{pd}X < \infty,\ X \text{ direct summand of } \Omega^{\phi(M)}M\} = 0$$

and $\psi(M) = \phi(M) = \mathrm{pd}M$, by Lemma 2.5.

(b) We have that,

$$\psi(kM) = \phi(kM) + \sup\,\{\mathrm{pd}X \mid \mathrm{pd}X < \infty,\ X \text{ direct summand of } \Omega^{\phi(kM)}(kM)\}$$

and $\phi(kM) = \phi(M)$, by Lemma 2.5. Since Ω commute with direct sums, then $\Omega^{\phi(M)}(kM) = k\Omega^{\phi(M)}(M)$. If X is a summand of $\Omega^{\phi(M)}(kM)$, then X is isomorphic to a sum of copies of summands of $\Omega^{\phi(M)}(M)$, and because $\mathrm{pd}(kB) = \mathrm{pd}B$ for any B in mod-Λ, we have $\psi(kM) = \psi(M)$, if $k \geq 1$.

(c) Using definition of ψ, we have that

$$\psi(A \oplus B) = \phi(A \oplus B) + \ \sup\,\{\mathrm{pd}X \mid \mathrm{pd}X < \infty,\ X \text{ direct summand of } \Omega^{\phi(A\oplus B)}(A \oplus B)\}$$

and

$$\psi(A) = \phi(A) + \ \sup\,\{\mathrm{pd}Y \mid \mathrm{pd}Y < \infty,\ Y \text{ direct summand of } \Omega^{\phi(A)}(A)\}$$

Let $m = \ \sup\,\{\mathrm{pd}Y \mid \mathrm{pd}Y < \infty,\ Y \text{ direct summand of } \Omega^{\phi(A)}(A)\}$, consider $\Omega^{\phi(A)}(A) = M \oplus N$ such that $\mathrm{pd}M < \infty$ and $\mathrm{pd}N = \infty$. Hence $\mathrm{pd}M = m$, which implies that $\mathrm{pd}(\Omega^{\phi(A\oplus B)-\phi(A)}(M)) =$

$m-(\phi(A\oplus B)-\phi(A))$, as follow in the diagram below.

$$\begin{tikzcd}
\dots \arrow[r] & P_{\phi(A\oplus B)-1} \arrow[r] & \dots \arrow[r] & P_{\phi(A)-1} \arrow[r] & \dots \arrow[r] & P_0 \arrow[r] & A\oplus B \arrow[r] & 0 \\
& \Omega^{\phi(A\oplus B)}(A\oplus B) \arrow[u, hook] \arrow[rr, leftrightarrow, "\phi(A\oplus B)-\phi(A)"] & & \Omega^{\phi(A)}(A\oplus B) \arrow[u, hook] & & \Omega(A\oplus B) \arrow[u, hook] & &
\end{tikzcd}$$

Since $\Omega^{\phi(A\oplus B)-\phi(A)}(M)$ is a summand of $\Omega^{\phi(A\oplus B)}(A\oplus B)$, then

$$\sup\ \{\mathrm{pd}X \mid \mathrm{pd}X<\infty,\ X \text{ direct summand of } \Omega^{\phi(A\oplus B)}(A\oplus B)\}\geq m-(\phi(A\oplus B)-\phi(A))$$

therefore, $\psi(A\oplus B)\geq\psi(A)$.

(d) Let Z be a direct summand of $\Omega^n(M)$ i.e. $\Omega^n(M)=Z\oplus Y$. Note that

$$\Omega^k(\Omega^n(M))=\Omega^k(Z\oplus Y)\Longrightarrow\Omega^{k+n}(M)=\Omega^k(Z)\oplus\Omega^k(Y)$$

Let $k=\phi(M)-n$. Since $\mathrm{pd}Z<\infty$, then $\mathrm{pd}(\Omega^k(Z))<\infty$. Note that $\Omega^k(Z)$ is a summand of $\Omega^{\phi(M)}(M)$ so, by definiton of ψ, follows that $\phi(M)+\mathrm{pd}(\Omega^k(Z))\leq\psi(M)$. Hence,

$$\begin{aligned}
\mathrm{pd}(Z)+n&=\mathrm{pd}(Z)+\phi(M)-k\\
&=(\mathrm{pd}(Z)-k)+\phi(M)\\
&=\mathrm{pd}(\Omega^k(Z))+\phi(M)\leq\psi(M)
\end{aligned}$$

□

2.2 A simple condition

Theorem 2.9. *Suppose that* $0\longrightarrow A\xrightarrow{\chi}B\xrightarrow{\sigma}C\longrightarrow 0$ *is a short exact sequence of f.g.* Λ*-modules and* C *has finite projective dimension. Then* $\mathrm{pd}C\leqslant\psi(A\oplus B)+1$.

Proof. Suppose $\mathrm{pd}C=r$. From the Lemma 1.62 it follows that we have the followings short exact sequence for each $i\in\mathbb{N}$:

$$0\longrightarrow\Omega^iA\longrightarrow\Omega^iB\longrightarrow\Omega^iC\longrightarrow 0\tag{2.10}$$

We have that Ω^rC is projective, since $\mathrm{pd}C=r$, therefore these sequence splits for $i=r$. Hence $\Omega^rB\cong\Omega^rA\oplus\Omega^rC$, which implies $[\Omega^rB]=[\Omega^rA]$. Let $n=\min\ \{l:[\Omega^lB]=[\Omega^lA]\}$, then $\mathrm{pd}C\geq n$ and $\phi(A\oplus B)\geq n$. In fact, if $\phi(A\oplus B)<n$, then $\phi(A)<n$ and $\phi(B)<n$, this implies $[\Omega^kB]=[\Omega^kA]$, for $k<n$, and that contradicts the minimality of n.

Note that $[\Omega^n B] - [\Omega^n A] = 0$, hence the difference is sum a of projectives, which implies that $\Omega^n A \cong X \oplus P$ and $\Omega^n B \cong X \oplus Q$, where P and Q are projective and X has no projective summands. So the short exact sequence (2.10) for $i = n$ becomes to,

$$0 \longrightarrow X \oplus P \xrightarrow{t} X \oplus Q \xrightarrow{g} \Omega^n C \longrightarrow 0$$

where, $t = \left(\begin{smallmatrix} f & \gamma \\ \delta & \beta \end{smallmatrix}\right)$. Since $f : X \to X$ is an endomorphism and Λ Noetherian, by item (c) of Lemma 2.1 there is a direct sum decomposition $X = Y \oplus Z$, such that $Z = \operatorname{Ker}(f^m)$ and $Y = \operatorname{Im}(f^m)$, $\forall m \geq \eta_f(X)$. Note that, $f|_Y : Y \to Y$ is an automorphism and $f|_Z : Z \to Z$ is nilpotent. In fact, given $a \in Y$ there exist $b \in X$, such that $f^m(b) = a$, so $a = f^m(b) = f^{m+1}(b) = f(a)$, and for $f|_Z$ we have $f^m(Z) = f^m(\operatorname{Ker}(f^m)) = 0$. This gives a matricial representation for f

$$f = \begin{bmatrix} f|_Y & 0 \\ 0 & f|_Z \end{bmatrix}$$

Let S be a Λ-module and consider the functor $\operatorname{Hom}(-, S)$. So applying the long exact sequence Theorem (Remark 1.65) to the short exact sequence of n-th syzygies we have that

$$\ldots \longrightarrow \operatorname{Ext}^k(\Omega^n C, S) \longrightarrow \operatorname{Ext}^k(X \oplus Q, S) \longrightarrow \operatorname{Ext}^k(X \oplus P, S) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^{k+1}(\Omega^n C, S) \longrightarrow \operatorname{Ext}^{k+1}(X \oplus Q, S) \longrightarrow \operatorname{Ext}^{k+1}(X \oplus P, S) \longrightarrow \ldots$$

The functor Ext is additive and vanishes on projective module, so

$$\ldots \longrightarrow \operatorname{Ext}^k(\Omega^n C, S) \xrightarrow{\lambda_k} \operatorname{Ext}^k(X, S) \xrightarrow{\gamma_k} \operatorname{Ext}^k(X, S) \xrightarrow{\delta_{k+1}}$$

$$\xrightarrow{\delta_{k+1}} \operatorname{Ext}^{k+1}(\Omega^n C, S) \xrightarrow{\lambda_{k+1}} \operatorname{Ext}^{k+1}(X, S) \xrightarrow{\gamma_{k+1}} \operatorname{Ext}^{k+1}(X, S) \longrightarrow \ldots$$

where, $\gamma_k = \begin{bmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{bmatrix}$, $\beta_k = \operatorname{Ext}^k(f|_Z, S)$, $\alpha_k = \operatorname{Ext}^k(f|_Y, S)$, $\lambda_k = \operatorname{Ext}^k(g, S)$ and δ_{k+1} is the connecting homomorphism (Proposition 1.56). From definition $\operatorname{Ext}^k(f|_Z, S)$ is nilpotent when $f|_Z$ is nilpotent.

Note that, $\operatorname{pd}\Omega^n C = r - n$ so, by Theorem 1.66, $\operatorname{Ext}^j(\Omega^n C, S) = 0$, for all $j > r - n$. So, by exactness of the long exact sequence, γ_k is epimorphism for all $k > r - n - 1$, thus so is β_k. Then, because β_k is nilpotent it follows that $\operatorname{Ext}^k(Z, S) = 0$ for $k > r - n - 1$ and for all S in mod-Λ, consequently $\operatorname{pd}Z < \infty$.

Now suppose $\operatorname{Ext}^{k+1}(\Omega^n C, S) \neq 0$.

$$\ldots \xrightarrow{\gamma_k} \operatorname{Ext}^k(X, S) \xrightarrow{\delta_{k+1}} \operatorname{Ext}^{k+1}(\Omega^n C, S) \xrightarrow{\lambda_{k+1}} \operatorname{Ext}^{k+1}(X, S) \xrightarrow{\gamma_{k+1}} \ldots$$

Then $\delta_{k+1} \neq 0$ or $\lambda_{k+1} \neq 0$, not both.

- If $\delta_{k+1} \neq 0$, then γ_k is not an epimorphism, so isn't β_k. Then $\mathrm{Ext}^k(Z, S) \neq 0$.
- If $\lambda_{k+1} \neq 0$, then γ_{k+1} is not a monomorphism, so isn't β_{k+1}. Then $\mathrm{Ext}^{k+1}(Z, S) \neq 0$.

So we conclude that $\mathrm{Ext}^{k+1}(\Omega^n C, S) \neq 0$ implies either $\mathrm{Ext}^k(Z, S) \neq 0$ or $\mathrm{Ext}^{k+1}(Z, S) \neq 0$. Then, by Theorem 1.66, $\mathrm{pd}\Omega^n C \leq \mathrm{pd}Z + 1$. Since $\mathrm{pd}Z$ is finite, using item (d) of Lemma 2.8 for the last inequality, we have that

$$\mathrm{pd}C = n + \mathrm{pd}\Omega^n C \leq n + (\mathrm{pd}Z + 1) = 1 + (\mathrm{pd}Z + n) \leq 1 + \psi(A \oplus B)$$

. □

Example 2.11. *Consider the same quiver of Example 2.4. Let* $A = 2S(1) \oplus X_1$*,* $B = X_1 \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ 1 & & 1 \end{smallmatrix} \oplus X_2$ *and* $C = \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus X_2$*. Then we have the following short exact sequence:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Note that the projective resolution of C *is*

$$\begin{tikzcd}
0 \arrow[r] & P(3) \arrow[r] & P(2) \arrow[r] & 2P(1)\oplus P(3) \arrow[r] & 2P(2) \arrow[r] & C \arrow[r] & 0 \\
 & & P(3) \arrow[u] & X_2 \arrow[u] & (P(1)\oplus S(1))\oplus P(3) \arrow[u] & &
\end{tikzcd}$$

and,

$$\begin{tikzcd}
\dots \arrow[r] & P(4)\oplus P(4) \arrow[r] & \begin{smallmatrix} P(3)\oplus P(3)\oplus P(4) \\ \oplus \\ P(4)\oplus P(3) \end{smallmatrix} \arrow[r] & \begin{smallmatrix} P(2)\oplus P(2)\oplus P(4) \\ \oplus \\ P(4)\oplus P(2)\oplus P(3) \end{smallmatrix} \arrow[r] & \begin{smallmatrix} P(1)\oplus P(1)\oplus P(4) \\ \oplus \\ P(4)\oplus P(2)\oplus P(2) \end{smallmatrix} \arrow[r, two heads] & A\oplus B \\
 & X_1\oplus X_1 \arrow[u] & X_1\oplus X_1 \arrow[u] & \begin{smallmatrix} P(3)\oplus P(3)\oplus X_1 \\ \oplus \\ X_1\oplus P(3) \end{smallmatrix} \arrow[u] & \begin{smallmatrix} X_2\oplus X_2\oplus X_1 \\ \oplus \\ X_1\oplus X_2\oplus P(3) \end{smallmatrix} \arrow[u] &
\end{tikzcd}$$

that is $\psi(A \oplus B) = 2$*. Since* $\mathrm{pd}C = 3 < \infty$*, we have*

$$\mathrm{pd}C \leq \psi(A \oplus B) + 1$$

Corollary 2.12. *If* M *is a finitely generated* Λ*-module with Loewy length 2 and finite projective dimension then*

$$\mathrm{pd}M \leq \psi(\Lambda/\mathrm{rad}\Lambda \oplus \Lambda/(\mathrm{rad}^2\Lambda)) + 1$$

Proof. Let P be a projective cover of M. So

$$0 \longrightarrow \Omega M \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

is exact. Note that $f(\mathrm{rad}^2 P) \subseteq \mathrm{rad}^2 M$ (Proposition 1.34), and by definition of Loewy length of M we have $\mathrm{rad}^2 M = 0$, hence $f(\mathrm{rad}^2 P) = 0$. From this we have the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r] \arrow[d] & P \arrow[r, "f"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & P/(\mathrm{rad}P)^2 \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

where $g(p) = f(p)$ for all p in P and $A = \mathrm{Ker}(g)$, that is, $A = \Omega M + \mathrm{rad}^2 P$. So by Theorem 2.9

$$\mathrm{pd}M \leqslant \psi(A \oplus P/\mathrm{rad}^2)P + 1 \tag{2.13}$$

To conclude the proof we need to show that $\psi(A \oplus P/(\mathrm{rad}^2 P)) \leq \psi(\Lambda/\mathrm{rad}\Lambda \oplus \Lambda/(\mathrm{rad}^2\Lambda))$. Because $\Omega M \subseteq \mathrm{rad}P$ (Proposition 1.48), A is a submodule of the semisimple module $\mathrm{rad}P/\mathrm{rad}^2 P$, hence A is semisimple too. The module $\Lambda/\mathrm{rad}\Lambda$ is the largest semisimple quotient of possible, so the simple modules summands of A are in $add(\Lambda/\mathrm{rad}\Lambda)$, i.e., A is a summand of $\Lambda/\mathrm{rad}\Lambda$.

Now, we want to show that $P/\mathrm{rad}^2 P$ is a summand of $\Lambda/\mathrm{rad}^2\Lambda$. Since P is projective, so $P \oplus Q = \bigoplus \Lambda_\Lambda$, for some Λ-module Q. Note that

$$(P \oplus Q)/\mathrm{rad}^2(P \oplus Q) = (P \oplus Q)/(\mathrm{rad}^2 P \oplus \mathrm{rad}^2 Q) \simeq P/\mathrm{rad}^2 P \oplus Q/\mathrm{rad}^2 Q$$

Because $(\bigoplus \Lambda)/\mathrm{rad}^2(\bigoplus \Lambda) = \bigoplus(\Lambda/\mathrm{rad}^2\Lambda)$, we have that

$$P/\mathrm{rad}^2 P \oplus Q/\mathrm{rad}^2 Q \simeq \oplus(\Lambda/\mathrm{rad}^2\Lambda)$$

Then, $A \oplus P/\mathrm{rad}^2 P$ is a summand of $\Lambda/\mathrm{rad}\Lambda \oplus \Lambda/(\mathrm{rad}^2\Lambda)$. Therefore, by Lemma 2.8 (c) and inequality (2.13) we conclude that

$$\mathrm{pd}M \leqslant \psi(\Lambda/\mathrm{rad}\Lambda \oplus \Lambda/(\mathrm{rad}^2\Lambda)) + 1$$

□

Example 2.14. *Consider the same quiver of Example 2.4. Let* $M = \begin{smallmatrix} 2 \\ | \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ | \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ | \\ 4 \end{smallmatrix}$. *Note that* $\mathrm{rad}^2 M = 0$, *and we have the following projective resolution of* M*:*

$$\begin{tikzcd}
{} \arrow[r] & P(3)\oplus P(1)\oplus P(2)\oplus(P(1)\oplus P(1)) \arrow[r] & P(2)\oplus P(3)\oplus P(4) \arrow[r] & M \arrow[r] & 0 \\
& X_2\oplus P(3)\oplus(X_2\oplus X_2) \arrow[u] & (P(3)\oplus S(1))\oplus X_2\oplus(S(1)\oplus S(1)) \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & P(3)\oplus(P(3)\oplus P(3)) \arrow[r] & P(2)\oplus P(3)\oplus(P(2)\oplus P(2)) \arrow[r] & {} \\
& & P(3)\oplus(P(3)\oplus P(3)) \arrow[u] &
\end{tikzcd}$$

Hence, $\mathrm{pd}M = 3$. *Note that* $\Lambda/\mathrm{rad}\Lambda \simeq S(1)\oplus S(2)\oplus S(3)\oplus S(4)$ *and* $\Lambda/\mathrm{rad}^2\Lambda \simeq P(1)/\mathrm{rad}^2P(1)\oplus P(2)/\mathrm{rad}^2P(2)\oplus P(3)/\mathrm{rad}^2P(3)\oplus P(4)/\mathrm{rad}^2P(4)$, *since* $\Lambda/\mathrm{rad}^2\Lambda \simeq KQ_0\oplus KQ_1$. *Then we have the projective resolution:*

$$\begin{tikzcd}
{} \arrow[r] & \begin{matrix} P(2)\oplus(P(3)\oplus P(4))\oplus P(1)\oplus(P(1)\oplus P(4)) \\ \oplus \\ P(4)\oplus(P(1)\oplus P(1))\oplus P(2)\oplus P(1) \end{matrix} \arrow[r] & \begin{matrix} P(1)\oplus P(2)\oplus P(3)\oplus P(4) \\ \oplus \\ P(1)\oplus P(2)\oplus P(3)\oplus P(4) \end{matrix} \arrow[r] & \begin{matrix} \Lambda/\mathrm{rad}\Lambda \\ \oplus \\ \Lambda/\mathrm{rad}^2\Lambda \end{matrix} \arrow[r] & 0 \\
& \begin{matrix} P(3)\oplus X_1\oplus(X_2\oplus X_1) \\ \oplus \\ X_1\oplus X_2\oplus P(3)\oplus X_2 \end{matrix} \arrow[u] & \begin{matrix} X_2\oplus(P(3)\oplus X_1)\oplus P(1)\oplus(S(1)\oplus X_1) \\ \oplus \\ X_1\oplus(P(1)\oplus S(1))\oplus X_2\oplus S(1) \end{matrix} \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \begin{matrix} P(4)\oplus P(4) \\ \oplus \\ P(4) \end{matrix} \arrow[r] & \begin{matrix} P(4)\oplus P(4) \\ \oplus \\ P(4) \end{matrix} \arrow[r] & \begin{matrix} P(4)\oplus(P(3)\oplus P(4)) \\ \oplus \\ P(4)\oplus P(3)\oplus P(3) \end{matrix} \arrow[r] & \begin{matrix} P(3)\oplus P(4)\oplus(P(2)\oplus P(4)) \\ \oplus \\ P(4)\oplus P(2)\oplus P(3)\oplus P(2) \end{matrix} \arrow[r] & {} \\
& & \begin{matrix} X_1\oplus X_1 \\ \oplus \\ X_1 \end{matrix} \arrow[u] & \begin{matrix} X_1\oplus X_1 \\ \oplus \\ X_1 \end{matrix} \arrow[u] & \begin{matrix} X_1\oplus(P(3)\oplus X_1) \\ \oplus \\ X_1\oplus P(3)\oplus P(3) \end{matrix} \arrow[u] &
\end{tikzcd}$$

Of course $\mathrm{pd}(\Lambda/\mathrm{rad}\Lambda\oplus\Lambda/\mathrm{rad}^2\Lambda)$ *is infinite, but we have that* $\psi(\Lambda/\mathrm{rad}\Lambda\oplus\Lambda/\mathrm{rad}^2\Lambda) = 3$. *Therefore*

$$\mathrm{pd}(M) \leq \psi(\Lambda/\mathrm{rad}\Lambda\oplus\Lambda/\mathrm{rad}^2\Lambda) + 1$$

Corollary 2.15. *Suppose that* $\mathrm{rad}^3\Lambda = 0$, *then*

$$\mathrm{findim}(\Lambda) \leq \psi(\Lambda/\mathrm{rad}\Lambda\oplus\Lambda/(\mathrm{rad}^2\Lambda)) + 2$$

in particular $\mathrm{findim}(\Lambda)$ *is finite.*

Proof. Let M be a f.g. Λ-module with $\mathrm{pd}M < \infty$ and P a projective cover of M. Since $M\mathrm{rad}\Lambda = \mathrm{rad}M$ (Proposition 1.35) and $\Omega M \subseteq \mathrm{rad}P$ (Proposition 1.48), we have

$$\mathrm{rad}^2\Omega M = \mathrm{rad}(\mathrm{rad}(\Omega M)) = \mathrm{rad}(\Omega M\mathrm{rad}\Lambda) = \Omega M\mathrm{rad}^2\Lambda \leq \mathrm{rad}P\mathrm{rad}^2\Lambda = P\mathrm{rad}^3\Lambda = 0$$

where the inequality means submodule. Then, the Loewy length of ΩM is less or equal 2. Hence, by Corollary 2.12, we have that

$$\mathrm{pd}M = \mathrm{pd}\Omega M + 1 \leqslant \psi(\Lambda/\mathrm{rad}\Lambda \oplus \Lambda/\mathrm{rad}^2\Lambda) + 2$$

□

2.3 Representation dimension

In 1971, Auslander [3] has introduced the notion of representation dimension of an Artin algebra. We will work with a different but equivalent definition. We consider that $\mathrm{repdim}(\Lambda) \leq n$, if there is a finitely generated Λ-module X such that $\mathrm{gldim}(\mathrm{End}_\Lambda(X)^{op}) \leq n$ and $addX$ contains all projective and all injective Λ-modules.

Our aim in this subsection is to prove the little finitistic dimension conjecture for Artin algebras with repdim≤ 3, which constitutes a very large class of Artin algebras. For a while, it was thought that all algebras had representation dimension at most three, however, there is an example by Rouquier (2006) of an algebra of dimension representation four [25], and the conjecture be still is still open for this algebras. We start this subsection with some results that will ground our claims.

Suppose P is a projective Λ-module. Then we denote by mod-P the full subcategory of mod-Λ whose objects are those X in mod-Λ which have projective presentation $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$, with P_i in $addP$ for $i = 0, 1$.

Proposition 2.16 ([16], Proposition 2.5)**.** *Let P be a projective Γ-module and let $\Lambda = \mathrm{End}_\Gamma(P)^{op}$. Then the restriction $\mathrm{Hom}_\Gamma(P, -)|_{\text{mod-}P}$: mod-$P \to$ mod-Λ of the evaluation functor $\mathrm{Hom}_\Gamma(P, -)$ is an equivalence of categories.*

Corollary 2.17. *Let $\Lambda = \mathrm{End}_\Gamma(P)^{op}$, where P is a projective module over an artin algebra Γ with* $\mathrm{gldim}\Gamma \leq 3$. *Then*

$$\mathrm{findim}(\Lambda) \leq \psi((P, \Gamma)) + 3$$

where $(P, \Gamma) = \mathrm{Hom}_\Gamma(P, \Gamma)$ is considered as a Λ-module.

Proof. Consider the equivalence of categories presented in Proposition 2.16.

$$e_P := \mathrm{Hom}_\Gamma(P, -)|_{\text{mod-}P} : \text{mod-}P \to \text{mod-}\Lambda$$

Let X be a Λ-module. By density of e_P there is $C \in$ mod-P such that $(P, C) \cong X$. Note that gldim$\Gamma \leq 3$, so C has projective dimension at most three, since is a Γ-module. Consider the minimal projective resolution of C:

$$0 \longrightarrow P_3 \xrightarrow{p_3} P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} C \longrightarrow 0$$

applying functor e_P we have

$$0 \longrightarrow (P, P_3) \xrightarrow{p_3^*} (P, P_2) \xrightarrow{p_2^*} (P, P_1) \xrightarrow{p_1^*} (P, P_0) \xrightarrow{p_0^*} X \longrightarrow 0$$

Because C is in mod-P, (P, P_0) and (P, P_1) are projective. Now, consider the short exact sequence

$$0 \longrightarrow (P, P_3) \xrightarrow{p_3^*} (P, P_2) \xrightarrow{p_2^*} \mathrm{coker}(p_3^*) \longrightarrow 0$$

Since $\Omega^2 X = \mathrm{Ker} p_1^*$, then $\mathrm{pd}(\Omega^2 X) = \mathrm{pd}(\mathrm{coker}(p_3^*))$. Remark that $\mathrm{pd}X = \mathrm{pd}(\Omega^n X) + n$ when we consider the minimal projective resolution, so

$$\mathrm{pd}X = \mathrm{pd}(\Omega^2 X) + 2 = \mathrm{pd}(\mathrm{coker}(p_3^*)) + 2$$

By Theorem 2.9,

$$\mathrm{pd}(\mathrm{coker}(p_3^*)) + 2 \leq \psi((P, P_3) \oplus (P, P_2)) + 3 = \psi((P, P_3 \oplus P_2)) + 3$$

Since $P_3 \oplus P_2$ is a projective Γ-module, so is a summand of $k\Gamma$ for some k, then $(P, P_3 \oplus P_2)$ is summand of $(P, k\Gamma) \simeq \bigoplus_{i=1}^{k} (P, \Gamma)$. Therefore by Lemma 2.8 (c) we have that

$$\mathrm{pd}X \leq \psi((P, P_3 \oplus P_2)) + 3 \leq \psi((P, \Gamma)) + 3$$

□

Now consider that repdim$(\Lambda) \leq 3$, then there is a finitely generated Λ-module X such that gldim$(\mathrm{End}_\Lambda(X)^{op}) \leq 3$ and *addX* contains all projective and all injective Λ-modules. Let $\mathcal{A}$ be the additive category of all Λ-modules which are summands of arbitrary sums of copies of X. We will fix this notation for $\mathcal{A}$ and X until the end of this chapter.

Now consider the category Morph$(\mathcal{A})$, as in Example 2.18. Let (g_1, g_2) be a morphism in

Morph($\mathcal{A}$), i.e. a commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "g_1"'] & A_2 \arrow[d, "g_2"] \\ A_1' \arrow[r, "f'"'] & A_2' \end{tikzcd} \tag{2.18}$$

We say that (g_1, g_2) is projectively trivial if there is a morphism $h : A_2 \to A_1'$ such that $f'h = g_2$. Consider the following additive equivalence relation on Morph($\mathcal{A}$),

$$(g_1, g_2)\mathbb{P}(g_1', g_2') \text{ if, and only if, } (g_1 - g_1', g_2 - g_2') \text{ is projectively trivial.}$$

For notation in follow definition recall Example 1.9.

Definition 2.19. *We say that a functor $F : \mathcal{A}^{op} \to \mathrm{Ab}$ is **coherent** if there is an exact sequence $(-, A_1) \longrightarrow (-, A_2) \longrightarrow F \longrightarrow 0$, for some A_1 and A_2 in $\mathcal{A}$. We denote the full subcategory of $(\mathcal{A}^{op}, \mathrm{Ab})$ consisting of all coherent functors by $\hat{\mathcal{A}}$.*

Proposition 2.20. *The category $\mathrm{Morph}(\mathcal{A})/\mathbb{P}$ is equivalent to $\hat{\mathcal{A}}$.*

Proof. First define the functor $\mathrm{Morph}(\mathcal{A}) \to (\mathcal{A}^{op}, \mathrm{Ab})$ that send an object $A_1 \to A_2$ to the functor $F : \mathcal{A}^{op} \to \mathrm{Ab}$ which is given by $F(A) = \mathrm{Coker}((A, A_1) \to (A, A_2))$ for all A in $\mathcal{A}$ and also the exact sequence $(-, A_1) \longrightarrow (-, A_2) \longrightarrow F \longrightarrow 0$ of functors in $(\mathcal{A}^{op}, \mathrm{Ab})$.

A morphism in Morph($\mathcal{A}$) is a commutative diagram (2.18) which send to the following commutative diagram

$$\begin{tikzcd} (-, A_1) \arrow[r] \arrow[d] & (-, A_2) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ (-, A_1') \arrow[r] & (-, A_2') \arrow[r] & F' \arrow[r] & 0 \end{tikzcd}$$

Note that the morphism $F \to F'$ is the unique natural transformation which makes the diagram commute and a morphism in Morph($\mathcal{A}$) goes to zero if and only if it is projectively trivial. Therefore our functor $\mathrm{Morph}(\mathcal{A}) \to (\mathcal{A}^{op}, \mathrm{Ab})$ induces the fully faithful functor $\mathrm{Morph}(\mathcal{A})/\mathbb{P} \to (\mathcal{A}^{op}, \mathrm{Ab})$.

Dense: Let $F \in \hat{\mathcal{D}}$, so $(-, A_1) \longrightarrow (-, A_2) \longrightarrow F \longrightarrow 0$ for $A_1,\ A_2 \in \mathcal{D}$. Using Yoneda's Lemma (1.14) we have that $(-, A_1) \to (-, A_2)$ becomes to (A_1, A_2). Then by Five's Lemma (1.54) we are done.

Full: By Yoneda's Lemma we have that $(-, A)$ are projective objects in $(\mathcal{D}^{op}, \mathrm{Ab})$, so we can complete the diagram with γ_2. Because the right square commutes and since $(-, A_1)$ is a projective object we have γ_1.

$$\begin{tikzcd}
(-,A_1) \arrow[r] \arrow[d, dotted, "\gamma_1"] & (-,A_2) \arrow[r] \arrow[d, dotted, "\gamma_2"] & F \arrow[r] \arrow[d, "\phi"] & 0 \\
(-,A_1') \arrow[r] & (-,A_2') \arrow[r] & F' \arrow[r] & 0
\end{tikzcd}$$

Again by Yoneda's Lemma we have a morphism going to ϕ.

Faithful: The morphism is unique by since F is a cokernel.

□

Proposition 2.21. *Let Λ be an Artin algebra such that $\mathrm{rad}^n\Lambda = 0$ and $\Gamma = \mathrm{End}(X)$. The restriction $\hat{\mathcal{A}} \to \text{Mod-}\Gamma^{op}$ of the exact functor $(\mathcal{A}^{op}, \mathrm{Ab}) \to \text{Mod-}\Gamma^{op}$ given by $F \to F(X)$ has the following properties:*

- **(a)** *If A in $\mathcal{A}$ is a finite sum of copies of X then (X, A) is a finitely generated free Γ^{op}-module. Hence $\mathcal{P}$, the full subcategory of $\text{Mod-}\Gamma^{op}$ whose objects are the Γ^{op}-modules (X, A) for all A in $\mathcal{A}$, is an additive subcategory of $\text{Mod-}\Gamma^{op}$ consisting of finitely generated projective Γ^{op}-modules. Further, $\mathcal{P}$ contains all finitely generated free Γ^{op}-modules.*
- **(b)** *If F is in $\hat{\mathcal{A}}$, then $F(X)$ is a finitely presented Γ^{op}-module. Hence the image of $\hat{\mathcal{A}}$ in $\text{Mod-}\Gamma^{op}$ is contained in $\text{mod-}\Gamma^{op}$.*
- **(c)** *The induced functor $\hat{\mathcal{A}} \to \text{mod-}\Gamma^{op}$ is an equivalence of categories.*

Proof. **(a)** In fact (X, A) is a finitely generated free Γ^{op}-module. Note that every morphism $\alpha : X \to \bigoplus X$ can ben seen as a column vector, then the set of canonical vectors with identity morphism on i-th position, e_i, with $i = 1, \ldots, n$ form a basis for (X, A) as a Γ^{op}-module, therefore is free and finitely generated.

Because $\mathrm{Hom}(-,-)$ is an additive bifunctor and $\mathcal{A}$ is an additive category, then $\mathcal{P}$ is an additive category. So (X, A), for any A in $\mathcal{A}$, is finitely generated and projective, since is free. Now we want to show that $\mathcal{P}$ contains all f.g. free Γ^{op}-modules. Let N be a f.g. free module, then there is an isomorphism $N \simeq \bigoplus_{i\in I} \mathrm{End}X^{op}$, so

$$N \simeq \bigoplus_{i=1}^{n} \mathrm{End}X^{op} \simeq \bigoplus_{i=1}^{n} (X, X) \simeq \left(X, \bigoplus_{i=1}^{n} X \right)$$

then N is in $\mathcal{P}$.

(b) If F is in $\hat{A}$ then there exist an exact sequence $(-, A_1) \longrightarrow (-, A_2) \longrightarrow F \longrightarrow 0$ for some A_1, A_2 in $\mathcal{A}$. That implies $(X, A_1) \longrightarrow (X, A_2) \longrightarrow F(X) \longrightarrow 0$ is exact, where by (a),

$(X, A_1), (X, A_2)$ are finitely generated free Γ^{op}-modules, then $F(X)$ is finitely presented which is equivalent to be finitely generated as a Γ^{op}-module.

(c) Let $\Phi : \hat{\mathcal{A}} \to \text{mod-}\Gamma^{op}$ be the functor between this categories, such that $\Phi(F) = F(X)$ and $\Phi(\phi) = \phi_X$. To show that Φ is a dense functor, we need show that given N in mod-Γ^{op} there exists F in $\hat{\mathcal{A}}$ such that $F(X) \simeq N$. To do this, we will show that there exists $\alpha \in \text{Morph}(\mathcal{A})$ such that $F = \text{Coker}(\text{Hom}(-, \alpha))$ is our desired functor.

Let N in mod-Γ^{op}, and $\{n_1, \ \ldots, \ n_m\}$ a set of generators of N. So there is a surjective map $(X, mX) \to N$. Hence we have a minimal projective presentation of N:

$$\begin{tikzcd} (X,kX) \arrow[r, "\beta"] \arrow[d, two heads] & (X,mX) \arrow[r, "\alpha"] & N \arrow[r] & 0 \\ Ker(\alpha) \arrow[ur, hook, "i"'] & & & \end{tikzcd}$$

consequently,

$$N \simeq (X, mX)/\text{Ker}(\alpha) = (X, mX)/\text{Im}(\beta) = \text{Coker}(\beta) \tag{2.22}$$

conversely, we have $\beta \in \text{Hom}_\Gamma((X, kX)_\mathcal{A}, (X, mX)_\mathcal{A})$. Since the functor $(X, -)$ commutes with arbitrary sums, then

$$\text{Hom}_\Gamma((X, kX)_\mathcal{A}, (X, mX)_\mathcal{A}) = \bigoplus_{i=1}^{k} \bigoplus_{i=1}^{m} \text{Hom}_\Gamma((X, X)_\mathcal{A}, (X, X)_\mathcal{A})$$

and $\text{Hom}_\Gamma((X, X), (X, X)) = (\Gamma, \Gamma)_\Gamma \simeq \Gamma_\Gamma \simeq (X, X)_\mathcal{A}$, hence

$$\bigoplus_{i=1}^{k} \bigoplus_{i=1}^{m} \text{Hom}_\Gamma((X, X)_\mathcal{A}, (X, X)_\mathcal{A}) \simeq \bigoplus_{i=1}^{k} \bigoplus_{i=1}^{m} \text{Hom}_\mathcal{A}(X, X)$$

that is,

$$\text{Hom}_\Gamma((X, kX)_\mathcal{A}, (X, mX)_\mathcal{A}) \xrightarrow{\sim} \text{Hom}_\mathcal{A}(kX, mX)$$

so there is $\alpha : kX \to mX$ in $\text{Morph}(\mathcal{A})$ such that $\beta \mapsto \alpha$. By equivalence of categories $\text{Morph}(\mathcal{A})$ and $\hat{\mathcal{A}}$ (Proposition 2.20), there is a coherent funtor F such that $F(X) = \text{Coker}((X, kX) \to (X, mX))$ then $F(X) \simeq N$ (2.22) and Φ is dense.

We shall prove that the functor is full and faithful. Let $g, h : F(X) \to G(X)$ be different homomorphisms in mod-Γ^{op}. So by coherentness there exist exact rows:

$$\begin{array}{c} (-, A_1) \longrightarrow (-, A_2) \longrightarrow F \longrightarrow 0 \\ (-, A'_1) \longrightarrow (-, A'_2) \longrightarrow G \longrightarrow 0 \end{array}$$

Since A_1, A_2, A'_1 and A_2 are in $\mathcal{A}$ they are summands of aX for some a, that is, for some integers

m, n, k, p and applying X on the sequences, we have that:

$$\begin{tikzcd}
(X, mX) \arrow[r, "\beta_1"] & (X, nX) \arrow[r] & F(X) \arrow[r] \arrow[d, shift right, "h"'] \arrow[d, shift left, "g"] & 0 \\
(X, pX) \arrow[r, "\beta_2"] & (X, kX) \arrow[r] & G(X) \arrow[r] & 0
\end{tikzcd}$$

From the item (a) of this proposition (X, A) is projective for all $A \in \mathcal{A}$, then there exist γ_2 such that the right square commutes.

$$\begin{tikzcd}
(X, mX) \arrow[r, "\beta_1"] \arrow[d, "\gamma_1"'] & (X, nX) \arrow[r] \arrow[d, "\gamma_2"] & F(X) \arrow[r] \arrow[d, shift right, "h"'] \arrow[d, shift left, "g"] & 0 \\
(X, pX) \arrow[r, "\beta_2"'] & (X, kX) \arrow[r] & G(X) \arrow[r] & 0
\end{tikzcd}$$

Because (X, mX) is projective and the right square commutes there exist γ_1 making the left square commutes. Using the same argument of the first part of this proof, we can find α_1 and α_2 such that the following diagram is commutative

$$\begin{tikzcd}
mX \arrow[r, "\alpha_1"] \arrow[d] & nX \arrow[d] \\
pX \arrow[r, "\alpha_2"'] & kX
\end{tikzcd}$$

then, we have a morphism in $\mathrm{Morph}(\mathcal{A})$ which are associated to the following commutative diagram

$$\begin{tikzcd}
(-, mX) \arrow[r] \arrow[d, "g_1^*"] & (-, nX) \arrow[r] \arrow[d, "g_2^*"] & F \arrow[r] \arrow[d, "\phi"] & 0 \\
(-, pX) \arrow[r] & (-, kX) \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Note that F is a cokernel and the natural transformation $\phi : F \to G$ is unique, then $\phi_X = h = g$. The existence of ϕ guarantee that functor is full and uniqueness that is faithful. □

Proposition 2.23. *Let Λ be a left Artin ring such that $\mathrm{rad}^n\Lambda = 0$ and $\Gamma = \mathrm{End}(X)$. Then there is a finitely generated projective Γ-module P such that $\mathrm{End}_\Gamma(P)^{op} \simeq \Lambda$.*

Proof. Recall that $\mathcal{A}$ is the additive category of all Λ-modules which are summands of arbitrary sums of copies of X. Note that, by Yoneda's Lemma (1.14), we have that

$$\mathrm{End}_{\hat{\mathcal{A}}}((-, \Lambda)) \simeq \mathrm{End}(\Lambda) \simeq \Lambda^{op}.$$

Since $\hat{\mathcal{A}} \to \text{mod-}\Gamma^{op}$ is an equivalence of categories (Proposition 2.21) and $(-, \Lambda)$ is a projective object in $\hat{\mathcal{A}}$ it follows that $P = (X, \Lambda)$ in $\text{mod-}\Gamma^{op}$ is projective and $\mathrm{End}_\Gamma(P) \simeq \Lambda^{op}$. Hence

$\mathrm{End}_\Gamma(P)^{op} \simeq \Lambda$. □

Theorem 2.24. *If* $\mathrm{repdim}(\Lambda) \leq 3$ *then* $\mathrm{findim}(\Lambda) < \infty$.

Proof. The Proposition 2.23 gives us that $\Lambda \simeq \mathrm{End}_\Gamma(P)^{op}$, where $\Gamma = \mathrm{End}_\Lambda(X)^{op}$ and $P = (X, \Lambda)$ is projective in mod-Γ^{op}. Then we are under the hypothesis of Corollary 2.17, so $\mathrm{findim}(\Lambda) < \infty$. □

Chapter 3

Derived and triangulated categories

This section is devoted to introduce triangulated and derived categories of an abelian category. For this we define the category of complexes and homotopy category of complexes. The definitions are given in a general way, but our main point is work with these concepts in the category of modules. The results and properties given here are fundamental to study the last chapter. For more details see [18], [31] and [30].

3.1 Triangulated category

In modern representation theory the notion of a triangulated categories is crucial. They provide an abstract framework for derived and for stable module categories in which most of the modern theories and correspondences are built. Moreover, they are even sufficiently flexible to allow equivalences of intermediate degree.

Historically triangulated categories were introduced by Verdier in order to obtain a nice framework for the various derived functors occurring in algebraic geometry. After Verdier's pioneering work triangulated categories became very useful in algebraic geometry of course, but also in complex analysis, representation theory as well as in algebraic topology.

Let $\mathcal{C}$ be an additive category and let T be an additive functor which is an automorphism of the category $\mathcal{C}$. We call T the **translation functor** on $\mathcal{C}$ and use the notation $T^n(X) = X[n]$ for any $n \in \mathbb{Z}$. A **triangle** is given by three objects X, Y, Z of $\mathcal{C}$ and three morphisms $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\beta \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $\gamma \in \mathrm{Hom}_{\mathcal{C}}(Z, TX)$. A morphism from a triangle $(X, Y, Z, \alpha, \beta, \gamma)$ to a triangle $(X', Y', Z', \alpha', \beta', \gamma')$ is a triple $f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, Z)$ so that the squares in the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[r, "\gamma"] \arrow[d, "h"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "\alpha'"'] & Y' \arrow[r, "\beta'"'] & Z' \arrow[r, "\gamma'"'] & TX' \end{tikzcd}$$

are commutative. Two triangles are isomorphic if there is a morphism of triangles which is formed by a triple which consists of three isomorphisms in $\mathcal{C}$.

Definition 3.1 (Verdier)**.** *An additive category $\mathcal{T}$ furnished with a self-equivalence $T : \mathcal{T} \to \mathcal{T}$ (called shift functor or suspension functor) and a class of "distinguished triangles", is a **triangulated category** if it satisfies the following axioms.*

(TR1) *A triangle which is isomorphic to a distinguished triangle is itself distinguished. The triangle*

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow TX$$

is distinguished. Every morphism $\alpha : X \to Y$ can be completed into a distinguished triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX$$

called the triangle above α.

(TR2) *If*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX$$

is a distinguished triangle, then

$$Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX \xrightarrow{-T\alpha} TY \quad \text{and} \quad T^{-1}Z \xrightarrow{-T^{-1}} X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

are distinguished triangles.

(TR3) *If*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} TX$$

is a distinguished triangle, and if

$$X' \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z' \xrightarrow{\gamma'} TX'$$

is a distinguished triangle, $f : X \to X'$ and $g : Y \to Y'$ then for any pair with $\alpha' \circ f = g \circ \alpha$

there is a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[r, "\gamma"] \arrow[d, dotted, "h"] & TX \arrow[d, "Tf"] \\
X' \arrow[r, "\alpha'"'] & Y' \arrow[r, "\beta'"'] & Z' \arrow[r, "\gamma'"'] & TX'
\end{tikzcd}$$

(TR4) *Given three distinguished triangles $X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow TX$, $Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow TY$ and $X \xrightarrow{vu} Z \longrightarrow Y' \longrightarrow TX$, there exists a distinguished triangle $Z' \xrightarrow{u} Y' \longrightarrow X' \longrightarrow TZ'$ for which the following diagram commute*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "1_X"'] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d] & TX \arrow[d, "1_{TX}"] \\
X \arrow[r, "vu"] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, "1_Z"] & Y' \arrow[r] \arrow[d] & TX \arrow[d, "Tu"] \\
Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "1_{X'}"] & TY \arrow[d] \\
Z' \arrow[r, "u"] & Y' \arrow[r] & X' \arrow[r] & TZ'
\end{tikzcd}$$

Given a morphism u in a triangulated category $\mathcal{T}$, we have then the triangle $A \xrightarrow{u} B \longrightarrow C \longrightarrow TA$. We call C the **cone of** u.

Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories. The additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called **triangulated functor** if:

- $T_{\mathcal{C}'} \circ F \cong F \circ T_{\mathcal{C}}$, where $T_{\mathcal{C}'}$ and $T_{\mathcal{C}}$ are the shift functors of $\mathcal{C}'$ and $\mathcal{C}$ respectively.
- F sends distinguished triangles of $\mathcal{C}$ to distinguished triangles in $\mathcal{C}'$.

Definition 3.2. *Let $\mathcal{T}$ be a triangulated category. A full additive subcategory $\mathcal{S}$ in $\mathcal{T}$ is called a **triangulated subcategory** if every object isomorphic to an object of S is in $\mathcal{S}$, if $T\mathcal{S} = \mathcal{S}$, and if for any distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow TX$$

such that the objects X and Y are in $\mathcal{S}$, the object Z is also in $\mathcal{S}$.

Clearly, a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ leads to an infinite diagram

$$\dots \xrightarrow{T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{T(f)} \dots \tag{3.3}$$

Lemma 3.4. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *be a distinguished triangle. Then the composition of any two consecutive morphisms in the triangle is equal to zero.*

Proof. See Lemma 1.3.1 (page 57) in [18]. □

Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ an abelian category. Let $F : \mathcal{C} \to \mathcal{A}$ be an additive functor. For any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow TX$ we have $F(g) \circ F(f) = 0$ (Lemma 3.4). Moreover, from (3.3), we have the following complex

$$\dots \xrightarrow{F(T^{-1}(h))} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(TX) \xrightarrow{F(T(f))} \dots \tag{3.5}$$

of objects in $\mathcal{A}$.

Definition 3.6. *Let* $\mathcal{C}$ *be a triangulated category and* $\mathcal{A}$ *an abelian category. The additive functor* $F : \mathcal{C} \to \mathcal{A}$ *is a* ***cohomological functor*** *if for any distinguished triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *, we have an exact sequence*

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

in $\mathcal{A}$*. Therefore, the complex (3.5) is exact, when* F *is cohomological.*

Proposition 3.7. *Let* $\mathcal{C}$ *be a triangulated category and* U *an object in* $\mathcal{C}$*. Then* $\mathrm{Hom}_{\mathcal{C}}(U, -)$ *and* $\mathrm{Hom}_{\mathcal{C}}(-, U)$ *are cohomological functors.*

Proof. See Proposition 1.4.1 (page 58) in [18]. □

3.1.1 The category of complexes

Recall from Section 1.3 that a complex on an additive category $\mathcal{A}$ is a family $X^\bullet = (X^n, d_X^n)_{n\in\mathbb{Z}}$ of objects X^n of $\mathcal{A}$ and morphisms $d_X^n : X^n \to X^{n+1}$ such that $d_X^n \circ d_X^{n-1} = 0$, can be write as a sequence:

$$X^\bullet : \quad \dots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \dots$$

A moprhism $f^\bullet$ between complexes $X^\bullet$ and $Y^\bullet$ in $\mathcal{A}$, is a family of morphisms $f^\bullet = (f^n : X^n \to Y^n)_{n\in\mathbb{Z}}$ such that $d_Y^n \circ f^n = f^{n+1} \circ d_X^n$, for each $n \in \mathbb{Z}$. The set of complexes on $\mathcal{A}$ with their morphisms give the additive category of complexes over $\mathcal{A}$, which we denote by $\mathcal{C}(\mathcal{A})$.

A complex $X^\bullet$ on $\mathcal{A}$ is said to be **bounded below** (**bounded above**) if there exists $n_0 \in \mathbb{Z}$ such that $X^n = 0$, for each $n < n_0$ ($n > n_0$). A complex is said to be bounded when it is bounded above and below . Denote by $\mathcal{C}^+(\mathcal{A})$, $\mathcal{C}^-(\mathcal{A})$ and $\mathcal{C}^b(\mathcal{A})$ the category of complexes bounded below, bounded above and bounded, respectively. The notation $\mathcal{C}^*(\mathcal{A})$ refers to one of the categories above.

We define a translation functor $T : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ as the functor which attaches to a complex $X^\bullet$ the complex $TX^\bullet$ such that

$$(TX^\bullet)^n = X[1]^n \quad \text{and} \quad d^n_{TX} = -d^{n+1}_X$$

for any $n \in \mathbb{Z}$. To any morphism $f^\bullet : X^\bullet \to Y^\bullet$, define $T(f^\bullet) : TX^\bullet \to TY^\bullet$ as

$$(Tf^\bullet)^n = f[1]^n$$

for any $n \in \mathbb{Z}$. Clearly, T is an automorphism of the category $\mathcal{C}(\mathcal{A})$. Often we are going to use the notation $T^p(X^\bullet) = X^\bullet[p]$ where $X^\bullet[p]$ is the complex $X^\bullet$ shifted to left p times.

Definition 3.8. *Let $\mathcal{A}$ be an additive category and $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of complex in $\mathcal{C}(\mathcal{A})$. We define the graded object $C^\bullet_f$ by ${C_f}^n = X^{n+1} \oplus Y^n$ for all $n \in \mathbb{Z}$. We also define $d^n_{C_f} : {C_f}^n \to {C_f}^{n+1}$ by*

$$d^n_{C_f} = \begin{bmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{bmatrix}$$

Note that $(C^\bullet_f, d_{C_f})$ is a complex in $\mathcal{C}(\mathcal{A})$, since $d^{n+1}_{C_f} \circ d^n_{C_f} = 0$. This complex is called the **mapping cone of** f.

3.1.2 The homotopy category

Definition 3.9. *Let $\mathcal{A}$ be an additive category. Two morphisms $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ in the category $\mathcal{C}(\mathcal{A})$ of complexes are called **homotopic**, denoted by $f^\bullet \sim g^\bullet$, if there exists a family $(s^n)_{n\in\mathbb{Z}}$ of morphisms $s^n : X^n \to Y^{n-1}$ in $\mathcal{A}$ satisfying*

$$f^n - g^n = d^{n-1}_Y s^n + s^{n+1} d^n_X$$

*In particular, setting g to be zero morphism, we can speak of morphisms being **homotopic to zero** and write $f^\bullet \sim 0$. Note that homotopy is an equivalence relation.*

Definition 3.10. *Let $\mathcal{A}$ be an additive category. The homotopy category $\mathcal{K}(\mathcal{A})$ has the same objects as the category $\mathcal{C}(A)$ of complexes over $\mathcal{A}$. The morphisms in the homotopy category are the equivalence classes of morphisms in $\mathcal{C}(\mathcal{A})$ modulo homotopy.*

Proposition 3.11 ([18], Lemma 1.3.3., Theorem 2.1.1.)**.** *Let $\mathcal{A}$ be an additive category. Then the homotopy category $\mathcal{K}(\mathcal{A})$ is again an additive category. Moreover, the additive category $\mathcal{K}(\mathcal{A})$ equipped with the translantion functor and the class of distinguished triangles is a triangulated category.*

As before we define the full subcategories $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ $\mathcal{K}^b(\mathcal{A})$ of the bounded below complexes, bounded above complexes and bounded complexes.

Now recall the definition of cohomogical functors (see Chapter 1 - Section 1.3). We may consider these functors over an abelian category $\mathcal{A}$, that is, $H^p : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ is a functor for each $p \in \mathbb{Z}$. These functors have the following property:

$$H^p(T(X^\bullet)) = H^{p+1}(X^\bullet)$$

and

$$H^p(T(f)) = H^{p+1}(f)$$

Therefore,

$$H^p = H^0 \circ T^p$$

for any $p \in \mathbb{Z}$.

Lemma 3.12 ([18], Lemma 1.4.1)**.** *If $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ two homotopic morphisms of complexes, then $H^p(f^\bullet) = H^p(g^\bullet)$, for all $p \in \mathbb{Z}$.*

Definition 3.13.

- *A morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathcal{K}(\mathcal{A})$ is called a* ***quasi-isomorphism*** *if the morphisms $H^p(f^\bullet) : H^p(X^\bullet) \to H^p(Y^\bullet)$ are isomorphisms for all p in $\mathbb{Z}$.*
- *An object $X^\bullet$ in $\mathcal{K}(\mathcal{A})$ is called* ***acyclic*** *if $H^p(X^\bullet) = 0$ for all $p \in \mathbb{Z}$.*

As a consequence of Lemma 3.12 we have that $g^\bullet$ is a quasi-isomorphism when $f^\bullet$ it is and $f^\bullet \sim g^\bullet$. Because of that, if a morphism is a quasi-isomorphism in $\mathcal{K}(\mathcal{A})$ so is the elements of the class which is belong.

Example 3.14. *Let X in* Mod-Λ*, and let*

$$\dots \longrightarrow P_n \longrightarrow \dots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be a projective resolution of X. Then the complex with X concentrated in degree zero and the complex

$$\dots \longrightarrow P_n \longrightarrow \dots \longrightarrow P_0 \longrightarrow 0 \longrightarrow \dots$$

are quasi-isomorphic. The same happens if we take injective resolutions of X.

Proposition 3.15. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathcal{K}(\mathcal{A})$. Then the following conditions are equivalent:*

(a) *The morphism f is a quasi-isomorphism.*

(b) *The cone of f is acyclic.*

Proof. See Chapter 3, sec. 3 - Lemma 3.1.1 in [18]. □

Definition 3.16.

- *Two complexes $X^\bullet$ and $Y^\bullet$ are* ***homotopy equivalent*** *if there exists chain maps $f^\bullet : X^\bullet \to Y^\bullet$ and $g^\bullet : Y^\bullet \to X^\bullet$ such that $f^\bullet \circ g^\bullet \sim id_{X^\bullet}$ and $g^\bullet \circ f^\bullet \sim id_{Y^\bullet}$. In this case we call $f^\bullet$ and $g^\bullet$* ***homotopy equivalence****.*
- *A chain complex $X^\bullet$ is said to be* ***contractible*** *if it is homotopy equivalent to the zero complex.*

As a consequence of the definition, a chain complex $X^\bullet$ is said to be contractible if, and only if, $id_{X^\bullet}$ is null-homotopic. In the same way, if $f^\bullet$ is a homotopy equivalence, then by Lemma 3.12, $f^\bullet$ is a quasi-isomorphism.

3.1.3 Derived Category

The derived category of an abelian category is constructed from the homotopy category of complexes. This construction is done by localizing that category with respect to the class of quasi-isomorphisms.

Let $\mathcal{A}$ be an abelian category and let $\mathcal{K}(\mathcal{A})$ be the corresponding homotopy category of complexes with the triangulated structure (Proposition 3.11). Let S^* be the class of all quasi-isomoprhism in $\mathcal{K}(\mathcal{A})$, we know that this class of objects is compatible with the triangulation in $\mathcal{K}(\mathcal{A})$ (Proprosition 3.1.2 in [18]).

Let $Q : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ a additive functor, such that:

- $Q(s)$ is an isomorphism for all $s \in S^*$;
- $Q(Ob\mathcal{K}(\mathcal{A})) = Ob\mathcal{K}(\mathcal{A})$;

with the following universal property: *If $\mathcal{B}$ is an additive category and $F : \mathcal{K}(\mathcal{A}) \to \mathcal{B}$ is an additive functor such that $F(s)$ is an isomorphism for all $s \in S^*$, then there exists an unique additive functor $G : A[(S^*)^{-1}] \to \mathcal{B}$ such that $F = G \circ Q$, that is, making the following diagram commutative*

$$\begin{tikzcd}
\mathcal{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{B} \\
\mathcal{K}(\mathcal{A})[(S^*)^{-1}] \arrow[ur, "G"'] &
\end{tikzcd}$$

The functor Q is called the localization functor. We call attention to the fact that the class of quasi-isomorphisms in $\mathcal{C}(\mathcal{A})$ usually does not satisfy the conditions for being a localizing class, because of that is necessary to consider the homotopy category.

The localization of $\mathcal{K}(\mathcal{A})$ with respect to S^* is denoted by $\mathcal{D}(\mathcal{A}) := \mathcal{K}(\mathcal{A})[(S^*)^{-1}]$ and called the **derived category** of $\mathcal{A}$.

So, the category $\mathcal{D}(\mathcal{A})$ is defined in the following way:

- $Obj\ \mathcal{D}(\mathcal{A}) = Obj\ \mathcal{C}(\mathcal{A})$;
- $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$ is the class of left fractions $$\begin{tikzcd} & X' \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$, where f and s are morphisms in $\mathcal{K}(\mathcal{A})$.

Denote by $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ the localization of categories $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ with respect to S^*. The notation $\mathcal{D}^*(\mathcal{A})$ refers to one of above categories.

Now we will present some results associated to the derived category which are necessary in next chapter. First, note that $\mathcal{D}(\mathcal{A})$ has triangulated structure, inherited of $\mathcal{K}(\mathcal{A})$. Denote by $\underline{f}$ a morphism from $X^\bullet$ to $Y^\bullet$ in $\mathcal{D}(\mathcal{A})$.

Proposition 3.17. *Let* $0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$ *a short exact sequence in* $\mathcal{C}(\mathcal{A})$. *Then it determines a distinguished triangle* $X^\bullet \xrightarrow{\underline{f}} Y^\bullet \xrightarrow{\underline{g}} Z^\bullet \longrightarrow TX^\bullet$ *in* $\mathcal{D}(\mathcal{A})$.

Proof. See Chapter 3, sec. 3 Proposition 3.5.2 in [18]. □

Let $\mathcal{A}$ be an abelian category. Denote by $\mathcal{I}$ the full subcategory of $\mathcal{A}$ consisting of all injective objects in $\mathcal{A}$. Since the sum of two injective objects is injective, $\mathcal{I}$ is a full additive subcategory of $\mathcal{A}$. Let $\mathcal{K}^+(\mathcal{I})$ the homotopy category of $\mathcal{I}$-complexes. We can view it as a full subcategory of $\mathcal{K}^+(\mathcal{A})$. Since the direct sum of injective objects is injective, for any two complexes $I^\bullet$ and $J^\bullet$ in $\mathcal{K}^+(\mathcal{I})$, the cone of a morphism $f : I^\bullet \to J^\bullet$ in $\mathcal{C}^*(\mathcal{A})$ is in $\mathcal{K}^+(\mathcal{I})$. This implies that $\mathcal{K}^+(\mathcal{I})$ is a full triangulated subcategory of $\mathcal{K}^+(\mathcal{A})$.

Theorem 3.18 ([18], Theorem 2.1.1.)**.** *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ its full subcategory which contains 0 and such that for any X in $\mathcal{A}$ there exists M in $\mathcal{B}$ and a monomorphism $i : X \to M$.*

Let $X^\bullet$ be a complex in $\mathcal{C}^+(\mathcal{A})$. Then there exists a complex $M^\bullet$ in $\mathcal{C}^+(\mathcal{B})$ and a quasi-isomorphism $s : X^\bullet \to M^\bullet$.

This theorem gives a general view of the fact that an injective resolution of an object is quasi-isomorphic to the complex concentrated in zero degree. Since in the module category all objects have projective covers and injective envelopes, this result, and it dual, always hold in the category of modules (see Example 3.14). In the case describe on Theorem we denote $M^\bullet$ by $\iota X^\bullet$, and for the dual case we denote by $pX^\bullet$.

Lemma 3.19. *Let $I^\bullet$ be a complex in $\mathcal{K}^+(\mathcal{I})$ and $X^\bullet$ a complex in $\mathcal{K}^+(\mathcal{A})$. Let $s : I^\bullet \to X^\bullet$ be a quasi-isomorphism. Then there exists a morphism $t : X^\bullet \to I^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ such that $t \circ s = id_I$, i.e., $t \circ s$ is homotopic to the identity on $I^\bullet$.*

Proof. See Lemma 2.2.1 (page 222 and 223) in [18]. □

Lemma 3.20. *Let $I^\bullet$ be a complex in $\mathcal{K}^+(\mathcal{I})$ and $X^\bullet$ a complex in $\mathcal{K}^+(\mathcal{A})$. Assume that $X^\bullet$ is acyclic. Then any morphism $f : X^\bullet \to I^\bullet$ is homotopic to zero.*

Proof. See Lemma 2.2.2 (page 222) in [18]. □

Let $Q : \mathcal{K}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$ be the quotient functor (from localization). Then, by restricting to $\mathcal{K}^+(\mathcal{I})$ it defines an exact functor $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$.

Theorem 3.21. *The natural functor $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$ is fully faithful.*

Proof. See Theorem 2.2.4 (page 223) in [18]. □

Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough injectives if for any object M in $\mathcal{A}$ there exists an injective object I and a monomorphism $s : M \to I$.

Corollary 3.22. *Let $\mathcal{A}$ be an abelian category which has enough injectives. Then the natural morphism $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$ is an equivalence of categories.*

Proof. See Corollary 2.2.5 (page 224) in [18]. □

We summarize the dual of this fact in the following proposition.

Proposition 3.23. *Let A be a finite dimensional k-algebra, then the localization functor induces equivalences*

- $\mathcal{D}^-(A\text{-Mod}) \simeq \mathcal{K}^-(A\text{-Proj})$;
- $\mathcal{D}^b(A\text{-Mod}) \simeq \mathcal{K}^-(A\text{-Proj})$ *with bounded homology;*
- $\mathcal{D}^-(A\text{-mod}) \simeq \mathcal{K}^-(A\text{-proj})$;
- $\mathcal{D}^b(A\text{-mod}) \simeq \mathcal{K}^-(A\text{-proj})$ *with bounded homology.*

Proof. See Proposition 3.5.43 (page 332) in [31]. □

Corollary 3.24. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $pX^\bullet \to X^\bullet$ be a quasi-isomorphism in the homotopy category $\mathcal{K}^-(\mathcal{A})$ so that $pX^\bullet$ is an object in the homotopy category $\mathcal{K}^-(\mathcal{P})$ of right bounded complexes of projective objects. Then there is a natural isomorphism $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathcal{K}^-(\mathcal{P})}(pX^\bullet, Y^\bullet)$ for every right bounded complex Y of objects in $\mathcal{A}$.*

Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $Y^\bullet \to \iota Y^\bullet$ be a quasi-isomorphism in the homotopy category $\mathcal{K}^+(\mathcal{A})$ so that $\iota Y^\bullet$ is an object in $\mathcal{K}^+(\mathcal{I})$ of left bounded complex of injective objects. Then there is a natural isomorphism $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{\mathcal{K}^+(\mathcal{I})}(X^\bullet, \iota Y^\bullet)$ for every left bounded complex $X^\bullet$ of objects in $\mathcal{A}$.

Proof. See Corollary 3.5.47 (page 338) in [31]. □

Chapter 4

Derived categories and the finitistic dimension conjecture

In the 1980's derived categories were introduced into representation theory, and because of their good properties they became a powerful tool to interpret homological results (see [10]). More recently, J. Rickard in [22] considers that it is more natural and convenient to work with the unbounded derived categories of complexes of arbitrary modules, since this has better properties, especially arbitrary coproducts. In that sense, Rickard considers the question of whether there is a connection between the finitistic dimension conjecture and the rings for which the unbounded derived category is generated, as a triangulated category with infinite coproducts, by the injective modules (if this is the case, then we say that "injectives generate"). In cited work, Rickard also considers what conditions a ring must have for injectives to generate.

In this chapter we present some properties of localizing subcategories of derived categories to show that the projective modules generate the derived category over the category of modules and show that injectives generate for any ring with finite global dimension. For this is need to introduce some properties of a localizing subcategory of a derived category. Lastly, and most importantly, we show that if injectives generate for a finite dimensional algebra then the big finitistic dimension conjecture holds for this algebra.

Although this proof does not put an end point on the conjecture, this result proved by Rickard gives a strong tool to analyse when an algebra has finite finitistic dimension. For this we need to introduce the projective dimension of a infinite number of modules (by definition of Findim), the result allows us just verify, for a finite dimensional algebra, if (for example) the simple modules are in the localizing subcategory generated by injectives.

4.1 Localizing subcategory

In this section we shall fix some notation and discuss some generalities on localizing subcategories.

- A is a finite dimensional K-algebra;
- We shall be considering right A-modules;
- By a complex we mean a cochain complex, unless we specify otherwise.

We impose no finiteness conditions on the modules in Mod-A and no boundness conditions on the complexes in $\mathcal{K}(A)$ and $\mathcal{D}(A)$. We shall regard Mod-A as a full subcategory of both $\mathcal{K}(A)$ and $\mathcal{D}(A)$, in the usual way, identifying a module M with the complex which has M in degree zero and zero in all other degrees.

Because is generated on Mod-A the derived category $\mathcal{D}(A)$ is a triangulated category with small coproducts, so we consider the following definition.

Definition 4.1 ([20], Definition 1.12.)**.** *Let $\mathcal{D}(A)$ be the derived category of cochain complexes of A-modules, and let $\mathcal{S}$ a particular class of objects in $\mathcal{D}(A)$. The **localizing subcategory** generate by $\mathcal{S}$ is the smallest triangulated subcategory of $\mathcal{D}(A)$ that contains $\mathcal{S}$ and is closed under coproducts. Denote it by $\langle\mathcal{S}\rangle$.*

Given a complex $X^\bullet = \ \ldots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \ldots$, we call the brutal truncation of $X^\bullet$ in degree n the complex

$$\sigma^{<n}X^\bullet = \ \ldots \longrightarrow X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{0} 0 \longrightarrow \ldots$$

and the good truncation of $X^\bullet$ in degree n the complex

$$\mathcal{T}^{\leq n}X^\bullet = \ \ldots \longrightarrow X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} \mathrm{Ker}(d^n) \xrightarrow{0} 0 \longrightarrow \ldots$$

We shall start by considering some easy concrete properties of localizing subcategories.

Proposition 4.2. *Let $\mathcal{C}$ be a localizing subcategory of $\mathcal{D}(A)$.*

(a) *If $0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$ is a short exact sequence of complexes, and two of the three objects $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ are in $\mathcal{C}$, then so is the third.*

(b) *If a complex $X^\bullet$ is in $\mathcal{C}$ then so is the shifted complex $X^\bullet[t]$ for every $t \in \mathbb{Z}$.*

(c) *If $X^\bullet$ and $Y^\bullet$ are quasi-isomorphic complexes and $X^\bullet$ is in $\mathcal{C}$, then so is $Y^\bullet$.*

(d) *If $\{X^\bullet{}_i \mid i \in I\}$ is a set of objects of $\mathcal{C}$, then $\bigoplus\limits_{i\in I} X^\bullet{}_i$ is in $\mathcal{C}$.*

(e) *If $X^\bullet \oplus Y^\bullet$ is in $\mathcal{C}$ then so are $X^\bullet$ and $Y^\bullet$.*

(f) *If $X^\bullet$ is a bounded complex, where the module X^i is in $\mathcal{C}$ for every i, then $X^\bullet$ is in $\mathcal{C}$.*

(g) *If $X^\bullet{}_0 \xrightarrow{\alpha_0} X^\bullet{}_1 \xrightarrow{\alpha_1} X^\bullet{}_2 \xrightarrow{\alpha_2} \dots$ is a sequence of cochain maps between complexes, with $X^\bullet{}_i$ in $\mathcal{C}$ for all i, then $\varinjlim X^\bullet{}_i$ is in $\mathcal{C}$.*

(h) *If $X^\bullet$ is a bounded above complex, where the module X^i is in $\mathcal{C}$ for every i, then $X^\bullet$ is in $\mathcal{C}$.*

Proof. **(a)** Consider the short exact sequence $0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0\,.$ By Proposition 3.17 it determines the following triangle in $\mathcal{D}(\mathcal{A})$.

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow TX^\bullet \tag{4.3}$$

Suppose $X^\bullet$ and $Y^\bullet$ in $\mathcal{C}$, then by definition of triangulated subcategory $Z^\bullet$ is in $\mathcal{C}$. Since 4.3 is a distinguished triangle, by (TR2), so is

$$Y^\bullet \longrightarrow Z^\bullet \longrightarrow TX^\bullet \longrightarrow TY^\bullet$$

and

$$Z^\bullet \longrightarrow TX^\bullet \longrightarrow TY^\bullet \longrightarrow TZ^\bullet$$

Now if we suppose $Y^\bullet$ and $Z^\bullet$ in $\mathcal{C}$, then by definition of triangulated subcategory $X^\bullet$ is in $\mathcal{C}$. Analagously for the last case.

(b) By definition of triangulated subcategory we have $T\mathcal{C} = \mathcal{C}$, so the result is an iterated application of the definition.

(c) Since a quasi-isomorphism becomes to an isomorphism in derived category we have the desired results.

(d) Note that a localizing subcategory is closed under coproducts, so the result is by definition.

(e) Consider the short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{i} (X^\bullet \oplus Y^\bullet \oplus X^\bullet \oplus Y^\bullet \oplus \dots) \xrightarrow{p} (Y^\bullet \oplus X^\bullet \oplus Y^\bullet \oplus \dots) \longrightarrow 0$$

where i is the natural inclusion and p kills the first term of the sum. Note that $X^\bullet \oplus Y^\bullet \oplus X^\bullet \oplus Y^\bullet \oplus \dots \cong \bigoplus (X^\bullet \oplus Y^\bullet) \cong Y^\bullet \oplus X^\bullet \oplus Y^\bullet \oplus \dots$, then by item **(a)** and **(d)** of this propostition $X^\bullet$ is in $\mathcal{C}$. Using the same argument we conclude that $Y^\bullet$ is in $\mathcal{C}$.

(f) Given a complex $X^\bullet = \ \ldots \longrightarrow 0 \longrightarrow X^{-n} \longrightarrow \ldots \longrightarrow X^0 \longrightarrow \ldots \longrightarrow X^n \longrightarrow 0 \longrightarrow \ldots$, we say that $X^\bullet$ is bounded in degree n. The proof will be made by induction on n, since $X^\bullet = X^*$ is a bounded complex. First consider that $X^\bullet$ is bounded in degree $n = 0$, then we have the following exact sequence:

$$\begin{tikzcd}
0 \arrow[d] & & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
X^0[0] \arrow[d] & = & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
X^\bullet \arrow[d] & = & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
\sigma^{<0}X^\bullet \arrow[d] & = & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
0 & & \ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

We have X^i in $\mathcal{C}$ for every i, so the complexes $X^0[0]$ and $\sigma^{<0}X^\bullet$ are in $\mathcal{C}$. Then, by item **(a)** of this Proposition, $X^\bullet$ is in $\mathcal{C}$. Suppose it is true for $X^\bullet$ bounded in degree $n = k$, that is:

$$X^\bullet|_k := X^\bullet \ = \ \ldots \longrightarrow 0 \longrightarrow X^{-k} \longrightarrow \ldots \longrightarrow X^0 \longrightarrow \ldots \longrightarrow X^k \longrightarrow 0 \longrightarrow \ldots$$

is in $\mathcal{C}$, since X^i is in $\mathcal{C}$ for every i. Now, suppose $X^\bullet$ is bounded in degree $k+1$ and X^i is in $\mathcal{C}$ for every i. Consider the short exact sequence:

$$0 \longrightarrow X^{k+1}[-(k+1)] \longrightarrow X^\bullet \longrightarrow \sigma^{<k+1}X^\bullet \longrightarrow 0$$

Then we have that $X^{k+1}[-(k+1)] = \ \ldots \longrightarrow 0 \longrightarrow X^{k+1} \longrightarrow 0 \longrightarrow \ldots$ is in $\mathcal{C}$ since $X^i \in \mathcal{C}$ for every i. Note that $\sigma^{<k+1}X^\bullet = X^{k+1}[-(k+1)] \oplus X^\bullet|_k$ and we have that if the summands is in $\mathcal{C}$ so is the sum (item **(d)**). Then, since $X^{k+1}[-(k+1)] \in \mathcal{C}$ and by hypothesis $X^\bullet|_k \in \mathcal{C}$ so $\sigma^{<k+1}X^\bullet$ is in $\mathcal{C}$. Therefore by item **(a)** we conclude that $X^\bullet$ is in $\mathcal{C}$.

(g) First we define direct limit. Let $I = (I, \leq)$ be a partially ordered set. A direct or inductive system of A-modules over I consists of a collection $\{X_i\}$ of A-modules indexed by I and a collection of homomorphisms $\varphi_{ij} : X_i \to X_j$, defined whenever $i \leq j$, such that the diagrams of the form

$$\begin{tikzcd}
X_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & X_k \\
& X_j \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

commute whenever $i \leq j \leq k$. We shall denote such a system by $\{X_i, \varphi_{ij}, I\}$. For an A-module X assume that $\psi_i : X_i \to X$ is a homomorphism for each $i \in I$. These mappings ψ_i are said to be compatible if $\psi_j\varphi_{ij} = \psi_i$ whenever $i \leq j$. One says that a A-module X together with compatible homomorphisms $\varphi_i : X_i \to X$ $(i \in I)$ is a **direct limit** or an inductive limit of the direct system $\{X_i, \varphi_{ij}, I\}$, if the following universal property is satisfied:

$$\begin{tikzcd} X \arrow[r, dotted, "\psi"] & B \\ X_i \arrow[u, "\varphi_i"] \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

whenever B is a A-module and $\psi_i : X_i \to B$ $(i \in I)$ is a set of compatible homomorphisms, then there exists a unique homomorphism ψ such that $\varphi_i\psi = \psi_i$. The direct limit of a direct system always exist (see Proposition 1.2.1 page 15 in [15]), we denote it by $(X_i, \psi_i)_{i\in I} = \varinjlim X_i$. The direct limit can be constructed explicitly as

$$\varinjlim X_i = \bigoplus_{i\in I} X_i / \sim$$

where $\sim$ is the following equivalence relation: for $x \in X_i$ and $y \in X_j$, $x \sim y$ if there exists $k \in I$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$.

Note that the complexes $X^\bullet{}_i$ with the compositions of α_i form a direct system indexed by $\mathbb{N}$. Define a relation, $x \in X_i$, $y \in X_j$, for $j \geq i$, so $x \sim y$ to be the smallest equivalence relation such that $\alpha_{ij}(x) = y$, where $\alpha_{ij} = \alpha_i \circ \cdots \circ \alpha_{j-1}$. We will show that the following sequence is an exact sequence

$$0 \longrightarrow \bigoplus_{i\in\mathbb{N}} X^\bullet{}_i \xrightarrow{\psi} \bigoplus_{i\in\mathbb{N}} X^\bullet{}_i \xrightarrow{\rho} \varinjlim X^\bullet{}_i \longrightarrow 0$$

The function ψ sends the vector $(x_0, x_1, x_2, \dots)$ in $(-x_0, \alpha_0(x_0) - x_1, \alpha_1(x_1) - x_2, \dots)$. So, since the sum has a finite number of nonzero entries, ψ is injective.

Then $\varinjlim X^\bullet{}_i = \bigoplus_{i\in\mathbb{N}} X^\bullet{}_i / \sim$ and ρ is the canonical projection for this relation, hence surjective.

Now, we want show that $\mathrm{Ker}(\rho) = \mathrm{Im}(\psi)$. Let $b \in \mathrm{Ker}(\rho)$, so $b = (\dots, 0, -x_i, \alpha_i(x_i), 0, \dots)$ (or a sum of elements of this form) because $x_i \sim \alpha_i(x_i)$, i.e. $\alpha_i(x_i) - x_i \sim 0$. Of course if we take $x = (\dots, 0, x_i, 0, \dots)$ in $\bigoplus_{i\in\mathbb{N}} X^\bullet{}_i$, we have $\psi(x) = b$, so $b \in \mathrm{Im}(\psi)$. Conversely, if we take $a \in \mathrm{Im}(\psi)$, then $a = (\dots, \alpha_i(x_i) - x_{i+1}, \alpha_{i+1}(x_{i+1}) - x_{i+2}, \dots)$, which we can write as

$$a = (\dots, -x_i, \alpha_i(x_i), \dots) + (\dots, -x_{i+1}, \alpha_{i+1}(x_{i+1}), \dots) + \dots$$

hence we have a sum of elements of $\mathrm{Ker}(\rho)$, then a is in $\mathrm{Ker}(\rho)$. So we have the short exact sequence. Since $\mathcal{C}$ is closed under coproducts (item **(d)**), we have $\bigoplus_{i\in\mathbb{N}} X^\bullet{}_i \in \mathcal{C}$, then by item **(a)** $\varinjlim X_i$ is in $\mathcal{C}$.

(h) Suppose, without loss of generality, X^* is bounded above in degree zero.

$$X^* = \ \dots \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow 0 \longrightarrow \dots$$

Consider the brutal truncation of X^*

$$\sigma^{\geq i} X^* = \ \dots \longrightarrow 0 \longrightarrow X^i \longrightarrow \dots \longrightarrow X^0 \longrightarrow 0 \longrightarrow \dots$$

We claim that $\varinjlim \sigma^{\geq i} X^* = X^*$. Note that if $i > 0$ then $\sigma^{\geq i} X^*$ is the zero complex. So we have the following of cochain maps between complexes

$$\sigma^{\geq 0} X^* \xrightarrow{\alpha_0} \sigma^{\geq -1} X^* \xrightarrow{\alpha_{-1}} \sigma^{\geq -2} X^* \longrightarrow \dots \tag{4.4}$$

where $\alpha_n : \sigma^{\geq n} X^* \to \sigma^{\geq n-1} X^*$, $n \in \mathbb{Z}^-$, is identity on degrees which has nonzero objects of $\sigma^{\geq n} X^*$ and zero for the others. The elements of (4.4) with the composition of the α_n's form a direct system. Consider the map $\psi : \bigoplus_{i \in \mathbb{Z}^-} \sigma^{\geq i} X^* \longrightarrow \bigoplus_{i \in \mathbb{Z}^-} \sigma^{\geq i} X^*$ that apply α_n on $\sigma^{\geq n} X^*$. Then,

$$\bigoplus_{i \in \mathbb{Z}^-} \sigma^{\geq i} X^* / \mathrm{Im}\psi = \varinjlim \sigma^{\geq i} X^* = X^*$$

So by item **(f)** we have $\sigma^{\geq n} X^*$ in $\mathcal{C}$ since X^i is in $\mathcal{C}$ for every i, and by item **(g)** we have that $\varinjlim \sigma^{\geq i} X^*$ is in $\mathcal{C}$, then X^* is in $\mathcal{C}$. $\square$

4.2 Projectives and Injectives generate

On this section, we will show that projectives generate the unbounded derived category as a triangulated category with arbitrary coproducts. We will see too that if the regular right module A_A has finite injective dimension so injectives generate A. For this lest result there is a particular case that this holds for any ring with finite global dimension.

Proposition 4.5. *The projective A-modules generate $\mathcal{D}(A)$.*

Proof. Let $\mathcal{P}$ be the collection of projective A-modules. Denote by $\langle \mathcal{P} \rangle$ the localizing subcategory of $\mathcal{D}(A)$. We want to show that any complex in $\mathcal{D}(A)$ is in $\langle \mathcal{P} \rangle$.

Let $X^\bullet$ be a complex in $\mathcal{D}(A)$

$$X^\bullet = \ \dots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \dots$$

and consider the good truncation of $X^\bullet$

$$\mathcal{T}^{\leq n} X^\bullet = \ \dots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} \mathrm{Ker}(d^n) \xrightarrow{0} 0 \longrightarrow \dots$$

By the same ideia used to prove that $\varinjlim \sigma^{\geq i} X^* = X^*$ (see Proposition 4.2 item **(g)**), we have that $\varinjlim \mathcal{T}^{\leq n} X^\bullet = X^\bullet$. Now, note that every module in a homogeneous component of $X^\bullet$ has a projective resolution. Each of this modules, viewed as a complex concentrated in degree zero, is quasi-isomorphic to it projective resolution (recall Example 3.14), that is

$$P_\bullet^n = \ \ldots \longrightarrow P^n \longrightarrow \ldots \longrightarrow P^0 \longrightarrow 0 \longrightarrow \ldots$$

is quasi-isomorphic to the complex with X^n concentrated in degree zero, denoted by $(X^n)^\bullet$. By definition $P_\bullet^n$ is in $\langle \mathcal{P} \rangle$, the localizing subcategory generated by the projectives modules of Mod-A. Since $P_\bullet^n$ is in $\langle \mathcal{P} \rangle$ and is quasi-isomorphic $(X^n)^\bullet$, then $(X^n)^\bullet \in \langle \mathcal{P} \rangle$ (Proposition 4.2 item **(c)**). We can do this for every $n \in \mathbb{Z}$.

Note that $\mathcal{T}^{\leq n} X^\bullet$ is a bounded above complex and every X^n is in $\langle \mathcal{P} \rangle$, by the previous argument. Then, by Proposition 4.2 item **(h)**, $\mathcal{T}^{\leq n} X^\bullet$ is in $\langle \mathcal{P} \rangle$. To finish, note that $\varinjlim \mathcal{T}^{\leq n} X^\bullet$ is in $\langle \mathcal{P} \rangle$ (Proposition 4.2 item **(g)**), hence $X^\bullet \in \langle \mathcal{P} \rangle$. □

Remark 4.6. *We have that a projective module is a direct summand of a free module, that is, a direct sum of copies of the regular right A-module A_A. Therefore, this shows that a single object A_A generates $\mathcal{D}(A)$.*

A similar proof to that of Proposition 4.5 shows that the injective A-modules generate $\mathcal{D}(A)$ as a triangulated category with products (i.e., the colocalizing subcategory generated by the injectives is the whole of $\mathcal{D}(A)$), but we shall consider the localizing subcategory generated by injectives, which is less obvious. We will denote by $\mathcal{I}$ the category of injective A-modules, so this localizing subcategory will be denoted by $\langle \mathcal{I} \rangle$.

Definition 4.7. *If $\langle \mathcal{I} \rangle = \mathcal{D}(A)$ then we say that **injectives generate** for A.*

Theorem 4.8. *If the regular right module A_A has finite injective dimension, then injectives generate for A. In particular, injectives generate for any ring with finite global dimension.*

Proof. Using the same idea of Example 3.14, we can show that A_A is quasi-isomorphic to it's injective resolution. Because the injective dimension is finite, the injective resolution is a bounded complex of injectives. Then, by Proposition 4.2 **(c)**, A_A is in $\langle \mathcal{I} \rangle$. Therefore, by Remark 4.6 A_A generates $\mathcal{D}(A)$, so injectives generate.

Since the global dimension can be taken as $\sup \{\mathrm{id} M \mid M \in \text{Mod-}A\}$, it finitness implies, in particular, that A_A has finite injective dimension, so we repeat the first part of the proof. Then injectives generate for a ring in this condition. □

4.3 The finitistic dimension conjecture

Recall the definition of the big finitistic dimension of a finite dimensional algebra A.

$$\text{Findim}(A) = \sup \{\text{pd}(M) \mid M \in \text{Mod-}A \text{ and } \text{pd}(M) < \infty\}$$

We say that A satisfies the big finitistic dimension conjecture if $\text{Findim}(A) < \infty$. The main point on this section is show that if injectives generate for A then A satisfies the big finitistic dimension conjecture. For this we give some preliminary results that will be useful throughout the proof of the main Theorem of this section.

Recall the notation of Example 1.6, where the functor duality $D-$ is given by $DM = \text{Hom}_K(M, K)$ and recall too that mod-A is the full subcategory of Mod-A consisting of finitely generated modules.

One simplifying factor is that if A is a finite dimensional algebra then every injective A-module is a direct summand of a direct sum of copies of DA, so $\langle \mathcal{I} \rangle$ is generated by the single object DA.

Theorem 4.9 ([23], Theorem 1.7.7.). *A chain map between chain complexes of A-modules is a homotopy equivalence if, and only if, it's mapping cone is contractible.*

Proof. ($\Longrightarrow$) Suppose the cochain map $f^\bullet : K^\bullet \to L^\bullet$ is a homotopy equivalence, so there is $g^\bullet : L^\bullet \to K^\bullet$, such that

$$\begin{aligned} id_{K^{n+1}} - g^{n+1} \circ f^{n+1} &= d_K^n \circ h_K^{n+1} + h_K^{n+2} \circ d_K^{n+1} \\ id_{L^n} - f^n \circ g^n &= d_L^{n-1} \circ h_L^n + h_L^{n+1} \circ d_L^n \end{aligned}$$

where $h_K^n : K^n \to K^{n-1}$ and $h_L^n : L^n \to L^{n-1}$ are the homotopy maps. By definition the mapping cone of f is the complex $C_f^\bullet$:

$$(C_f)^n = L^n \oplus K^{n+1}, \quad d^n = \begin{pmatrix} d_L^n & f^{n+1} \\ 0 & -d_K^{n+1} \end{pmatrix}$$

.

We want to construct a certain homotopy $h^n : C_f^n \to C_f^{n-1}$, such that

$$d^{n-1} \circ h^n + h^{n+1} \circ d^n = id_{C_f^n} = \begin{pmatrix} id_{L^n} & 0 \\ 0 & id_{K^{n+1}} \end{pmatrix}$$

Try a rather obvious candidate,

$$h^n = \begin{pmatrix} h_L^n & 0 \\ g^n & -h_K^{n+1} \end{pmatrix}$$

Replacing in the homotopy equation, we have

$$d^{n-1} \circ h^n + h^{n+1} \circ d^n = \begin{pmatrix} id_{L^n} & -f^n \circ h_K^{n+1} + h_L^{n+1} \circ f^{n+1} \\ 0 & id_{K^{n+1}} \end{pmatrix} =: \phi^n$$

Let $k^n := -f^n \circ h_K^{n+1} + h_L^{n+1} \circ f^{n+1}$ which is not necessarily zero. Note that the matrix ϕ^n has an inverse, which is $\psi^n = \begin{pmatrix} id_{L^n} & -k^n \\ 0 & id_{K^{n+1}} \end{pmatrix}$. Taking the composition with h^n, we have the desired:

$$d^{n-1} \circ (h^n \circ \psi^n) + (h^{n+1} \circ \psi^{n+1}) \circ d^n = \begin{pmatrix} id_{L^n} & 0 \\ 0 & id_{K^{n+1}} \end{pmatrix}$$

so $C_f^\bullet$ is contractible.

$(\Longleftarrow)$ Now suppose $C_f^\bullet$ is contractible, so there is $h^n : C_f^n \to C_f^{n-1}$ such that

$$d^{n-1} \circ h^n + h^{n+1} \circ d^n = id_{C_f^n} = \begin{pmatrix} id_{L^n} & 0 \\ 0 & id_{K^{n+1}} \end{pmatrix}.$$

Writing $h^n = \begin{pmatrix} h_1^n & h_2^n \\ h_3^n & h_4^n \end{pmatrix}$ and replacing in the equation we have:

$$\begin{pmatrix} d_L^{n-1} & f^n \\ 0 & -d_K^n \end{pmatrix} \begin{pmatrix} h_1^n & h_2^n \\ h_3^n & h_4^n \end{pmatrix} + \begin{pmatrix} h_1^{n+1} & h_2^{n+1} \\ h_3^{n+1} & h_4^{n+1} \end{pmatrix} \begin{pmatrix} d_L^n & f^{n+1} \\ 0 & -d_K^{n+1} \end{pmatrix} = \begin{pmatrix} id_{L^n} & 0 \\ 0 & id_{K^{n+1}} \end{pmatrix}$$

which implies that,

$$\begin{pmatrix} d_L^{n-1} \circ h_1^n + f^n \circ h_3^n + h_1^{n+1} \circ d_L^n & T \\ -d_K^n \circ h_3^n + h_3^{n+1} \circ d_L^n & -d_K^n \circ h_4^n + h_3^{n+1} \circ f^{n+1} - d_K^{n+1} \circ h_4^{n+1} \end{pmatrix} = \begin{pmatrix} id_{L^n} & 0 \\ 0 & id_{K^{n+1}} \end{pmatrix}$$

where $T = d_L^{n-1} \circ h_2^n + f^n \circ h_4^n + h_1^{n+1} \circ f^{n+1} + h_2^{n+1} \circ (-d_K^{n+1})$. Without loss of generality suppose $h_2^n = h_2^{n+1} = 0$,

$$\begin{pmatrix} d_L^{n-1} \circ h_1^n + f^n \circ h_3^n + h_1^{n+1} \circ d_L^n & f^n \circ h_4^n + h_1^{n+1} \circ f^{n+1} \\ -d_K^n \circ h_3^n + h_3^{n+1} \circ d_L^n & -d_K^n \circ h_4^n + h_3^{n+1} \circ f^{n+1} - d_K^{n+1} \circ h_4^{n+1} \end{pmatrix} = \begin{pmatrix} id_{L^n} & 0 \\ 0 & id_{K^{n+1}} \end{pmatrix}$$

which means that $id_{L^n} \sim f^n \circ h_3^n$, $id_{K^{n+1}} \sim h_3^{n+1} \circ f^{n+1}$, h_3 is a chain map between $L^\bullet$ and $K^\bullet$, and $f^n \circ h_4^n + h_1^{n+1} \circ f^{n+1} = 0$. Then f is a homotopy equivalence. ☐

Lemma 4.10. *Every module over a finite dimensional algebra is an iterated extension of coproducts of simple modules.*

Proof. Let A be a finite dimensional k-algebra and M a right A-module. Note that $M/M\mathrm{rad}A$ is an $A/\mathrm{rad}A$-module by the following law:

$$\begin{array}{rcl} (M/M\mathrm{rad}A,\ A/\mathrm{rad}A) & \to & M/M\mathrm{rad}A \\ (m+M\mathrm{rad}A,\ a+\mathrm{rad}A) & \mapsto & ma+M\mathrm{rad}A \end{array}$$

Suppose m_1, m_2 are elements of the same class in $M/M\mathrm{rad}A$ and a_1, a_2 elements of the same class in $A/\mathrm{rad}A$. Then $m_1 = m_2 + ma$, $m \in M$ and $a \in \mathrm{rad}A$, and $a_1 = a_2 + a'$, $a' \in \mathrm{rad}A$, hence

$$\begin{array}{rcl} m_1a_1 - m_2a_2 & = & (m_2+ma)(a_2+a') - m_2a_2 \\ & = & m_2a_2 + ma - m_2a_2 \\ & = & ma \end{array}$$

and it is well defined. Is easy to check that is in fact a module. Since $A/\mathrm{rad}A$ is semisimple, then $M/M\mathrm{rad}A$ is a semisimple module, Proposition 1.32. Then, by Theorem 1.29 $\mathrm{rad}(M/M\mathrm{rad}A) = 0$. Let $\{I_i\}_i$ be the maximal submodules of M, then

$$\frac{\mathrm{rad}M}{M\mathrm{rad}A} = \frac{\bigcap_i I_i}{M\mathrm{rad}A} \subseteq \bigcap_i \left(\frac{I_i}{M\mathrm{rad}A}\right) = \mathrm{rad}\left(\frac{M}{M\mathrm{rad}A}\right)$$

therefore $\frac{\mathrm{rad}M}{M\mathrm{rad}A} = 0$, so $\mathrm{rad}M \subseteq M\mathrm{rad}A$. Since A is a finite dimensional algebra the radical of A is nilpotent i.e. $\mathrm{rad}^n A = 0$, hence $\mathrm{rad}^n M = 0$.

Note that $\mathrm{rad}^{n-1}M/\mathrm{rad}^n M$ is semisimple, then $\mathrm{rad}^{n-1}M$ is semisimple (because $\mathrm{rad}^n M = 0$). And since $\mathrm{rad}^{n-2}M/\mathrm{rad}^{n-1}M$ is semisimple too, we have the following extension

$$0 \longrightarrow \mathrm{rad}^{n-1}M \longrightarrow \mathrm{rad}^{n-2}M \longrightarrow \mathrm{rad}^{n-2}M/\mathrm{rad}^{n-1}M \longrightarrow 0$$

hence $\mathrm{rad}^{n-2}M$ is an extension of $\mathrm{rad}^{n-2}M/\mathrm{rad}^{n-1}M$ by $\mathrm{rad}^{n-1}M$. Continuing, we have the extensions

$$\begin{array}{c} 0 \longrightarrow \mathrm{rad}^{n-1}M \longrightarrow \mathrm{rad}^{n-2}M \longrightarrow \mathrm{rad}^{n-2}M/\mathrm{rad}^{n-1}M \longrightarrow 0 \\ 0 \longrightarrow \mathrm{rad}^{n-2}M \longrightarrow \mathrm{rad}^{n-3}M \longrightarrow \mathrm{rad}^{n-3}M/\mathrm{rad}^{n-2}M \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \mathrm{rad}^{n-(n-1)}M \longrightarrow \mathrm{rad}^{0}M \longrightarrow \mathrm{rad}^{0}M/\mathrm{rad}^{n-(n-1)}M \longrightarrow 0 \end{array}$$

the last is equal to $0 \longrightarrow \mathrm{rad}M \longrightarrow M \longrightarrow M/\mathrm{rad}M \longrightarrow 0$. Then M is an iterated extension of coproducts of simple modules. $\square$

Lemma 4.11. *The tensor product with a finitely presented module preserves both products and coproducts.*

Proof. Let M be a finitely presented A-module, then

$$0 \longrightarrow mA \longrightarrow nA \longrightarrow M \longrightarrow 0$$

is exact for some m and n integers. Then, applying tensor, we have the follow diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (\prod N_i) \otimes_A mA \arrow[r] \arrow[d, "f"] & (\prod N_i) \otimes_A nA \arrow[r] \arrow[d, "g"] & (\prod N_i) \otimes_A M \arrow[r] \arrow[d, dotted, "h"] & 0 \\
0 \arrow[r] & \prod(N_i \otimes_A mA) \arrow[r] & \prod(N_i \otimes_A nA) \arrow[r] & \prod(N_i \otimes_A M) \arrow[r] & 0
\end{tikzcd}$$

Note that f and g are isomorphisms,

$$\left(\prod N_i\right) \otimes_A mA \simeq \bigoplus^m \left(\left(\prod N_i\right) \otimes_A A\right) \simeq \bigoplus^m \left(\prod (N_i \otimes_A A)\right) \simeq \prod (N_i \otimes_A mA)$$

so by Lemma 1.54, h is an isomorphism too. The proof is the same for coproducts. $\square$

Definition 4.12. *The* ***Nakayama functor*** *of* mod-A *is defined to be the endofunctor* $D\mathrm{Hom}_A(-, A) : \text{mod-}A \to \text{mod-}A$.

Lemma 4.13. *The Nakayama functor defined in 4.12 is right exact and is functorially isomorphic to* $- \otimes_A DA$*.*

Proof. See Chapter III - Lemma 2.9 in [13]. $\square$

Proposition 4.14. *The restriction of the Nakayama functor to the full subcategory* proj-A *of* mod-A *whose objects are the projective modules induces an equivalence between* proj-A *and the full subcategory* inj-A *of* mod-A *whose objects are the injective modules. The quasi-inverse of this restriction is given by* $\mathrm{Hom}_A(D(_AA), -) : \text{inj-}A \to \text{proj-}A$.

Proof. See Chapter III - Proposition 2.10 in [13]. $\square$

Theorem 4.15. *Let A be a finite dimensional K-algebra for which injectives generate, then A satisfies the big finitistic dimension conjecture, that is* $\mathrm{Findim}(A) < \infty$ *(hence also* $\mathrm{findim}(A) < \infty$*).*

Proof. Suppose A does not satisfy the big finitistic dimension conjecture, so there exists an inifinite family of nonzero A-modules $\{M_i : i \in I\}$ with $\mathrm{pd}M_i < \infty$ and $\mathrm{pd}M_i \neq \mathrm{pd}M_j$ for $i \neq j$. Let $\mathrm{pd}M_i = d_i$ and P_i be a minimal projective resolution of M_i,

$$P_i = \; \ldots \longrightarrow 0 \longrightarrow {P_i}^{d_i} \xrightarrow{\partial^{d_i}} {P_i}^{d_i - 1} \xrightarrow{\partial^{d^i - 1}} \ldots \longrightarrow {P_i}^{0} \longrightarrow 0 \longrightarrow \ldots$$

Then $P_i[-d_i]$ considered as a cochain complex has cohomology concentrated in degree d_i, in fact $H^n(P_i[-d_i]) = 0$, for $n \neq d_i$ and $H^{d_i}(P_i[-d_i]) = M_i$.

Note that both $\bigoplus_i P_i[-d_i]$ and $\prod_i P_i[-d_i]$ have homology isomorphic to M_i in degree d_i.

$$H_{d_j}(\bigoplus_i P_i[-d_i]) \cong \bigoplus_i H_{d_j}(P_i[-d_i]) \cong M_j$$

where the first isomorphism follows from Proposition 1.55 and $\mathrm{pd}M_j = d_j$, the same happens for product since it commutes with homology (Proposition 1.55). Because each $P_i[-d_i]$ has cohomology concentrated in degree d_i and there are no two complexes with the same length (because the projective dimensions are distinct) then just one $P_i[-d_i]$, of the sum for each i, has nonzero cohomology. So the natural inclusion

$$\iota : \bigoplus_i P_i[-d_i] \longrightarrow \prod_i P_i[-d_i]$$

is a quasi-isomorphism. In fact applying homology functor on ι, we have

$$\begin{array}{rcl} H_n(\iota) : H_n(\bigoplus_i P_i[-d_i]) & \longrightarrow & H_n(\prod_i P_i[-d_i]) \\ p + \mathrm{Ker}(\oplus \partial_0) & \mapsto & \iota(p) + \mathrm{Ker}(\prod \partial_0) \end{array}$$

Since $\iota(p) = p$ (natural inclusion) this is an isomorphism, because the homology is concentrated in a degree.

If ι is a homotopy equivalence, then there exists a map γ such that $\iota \circ \gamma \sim id$ and $\gamma \circ \iota \sim id$, hence for every additive functor F we have $F(\iota \circ \gamma) \sim F(id)$ and $F(\gamma \circ \iota) \sim F(id)$, that is, $F(\iota) \circ F(\gamma) \sim F(id)$ and $F(\gamma) \circ F(\iota) \sim F(id)$, hence by Lemma 3.12 $F(\iota)$ would be a quasi-isomorphism. We want to show that ι is not a homotopy equivalence, so it suffices to find an additive functor for which $F(\iota)$ is not a quasi-isomorphism. For this we choose the additive functor $- \otimes_A A/\mathrm{rad}A$, whose cohomology is $\mathrm{Tor}_n^A(-, A/\mathrm{rad}A)$, $n \in \mathbb{Z}$.

For finite dimensional algebras, flat modules are projective (Theorem 1.51), so the projective dimension of a module is the same as it's the flat dimension, which is

$$\mathrm{pd}M_i = \sup \{n \mid \mathrm{Tor}_n^A(M_i, X) \neq 0,\ \text{X left } A\text{-module}\}$$

or alternatively using Proposition 1.69, that is, consider the dual fucntor $D- = \mathrm{Hom}_k(-, k)$ (Example 1.6), for any left module X we have $\mathrm{Hom}_A(M_i, DX) \cong D(M \otimes_A X)$ and so taking derived functors $\mathrm{Ext}_A^n(M_i, DX) \cong D\mathrm{Tor}_n^A(M, X)$, then by Theorem 1.66 we have the same result.

Note that the class of the left modules such that $\mathrm{Tor}_n^A(M_i, X) = 0$ is closed by coproducts because homology and tensor are additive functors, and if we take the long exact sequence for the functor Tor (Remark 1.65) we will see that this class is closed by extensions too. It follows from this and Lemma 4.10 that,

$$\mathrm{pd}M_i = \sup \{n \mid \mathrm{Tor}_n^A(M_i, S) \neq 0,\ \text{S simple left } A\text{-module}\}$$

Since $\mathrm{pd}M_i = d_i \neq 0$, there is a simple module for which $\mathrm{Tor}^A_{d_i}(M_i, S) \neq 0$. Recall that the tensor product with a finitely presented module preserves both products and coproducts (Lemma 4.11) and $A/\mathrm{rad}A$ contains all simple modules as a summands, so the map on degree zero cohomology induced by $\iota \otimes_A (A/\mathrm{rad}A)$ is the natural map

$$\bigoplus_i \mathrm{Tor}^A_{d_i}(M_i, A/\mathrm{rad}A) \longrightarrow \prod_i \mathrm{Tor}^A_{d_i}(M_i, A/\mathrm{rad}A)$$

which is not an isomorphism, because elements of the left side have a finite number of nonzero entries while elements of the right side has a infinite number of nonzero entries, so the map is not surjective.

Let C_ι be the mapping cone of ι, so the n-th homogeneous component of this complex is

$$(C_\iota)^n = \left(\bigoplus_i P_i[-d_i]\right)^{n+1} \oplus \left(\prod_i P_i[-d_i]\right)^n$$

Since A is a finite dimensional algebra, the product of projectives is projective (Proposition 1.52), hence C_ι is a bounded below complex of projectives A-modules. Because ι is a quasi-isomorphism C_ι is acyclic (Lemma 3.15), but not contractible provided that ι is not a homotopy equivalence (see Theorem 4.9).

Recall that the functor $- \otimes_A DA$ is an equivalence from the category of projective A-modules to the category of injective A-modules (Proposition 4.14). So, applying this functor to C_ι, we get a bounded below complex of injective A-modules that is not contractible (since $- \otimes_A DA$ is an equivalence). Note that, if $C_\iota \otimes_A DA$ is acyclic then $id : C_\iota \otimes_A DA \to C_\iota \otimes_A DA$ is homotopic to zero (Lemma 3.20), contradiction since this complex is not contractible. Then $C_\iota \otimes_A DA$ is not acyclic.

Since C_ι is acyclic, $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, C_\iota[t]) = 0$, and so $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(DA, C_\iota \otimes_A DA[t]) = 0$ for all $t \in \mathbb{Z}$. In fact, let $f \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, C_\iota[t])$,

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f^n"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & C_\iota{}^{n-1} \arrow[r, "\partial^{n-1}"] & C_\iota^n \arrow[r, "\partial^n"] & C_\iota{}^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

Since $\partial^n \circ f^n = 0$ then $\mathrm{Im}f^n \subseteq \mathrm{Ker}(\partial_n) = \mathrm{Im}(\partial_{n-1})$ (because is acyclic). Since A is projective as a right A-module there is the following diagram,

$$\begin{tikzcd} & A \arrow[dl, dotted, "\exists!"'] \arrow[d] \\ C_\iota^{n-1} \arrow[r] & \mathrm{Im}\partial^{n-1} \end{tikzcd}$$

Let $h_n : A \to C_\iota^{n-1}$ be this morphism, such that $\partial_{n-1} \circ h_n = f^n$. Hence $f \sim 0$.

Let $g \in \mathrm{Hom}_{K(\mathcal{A})}(DA, C_\iota \otimes_A DA[t])$ and suppose $g \neq 0$ (in homotopic sense). Since $- \otimes_A DA$ is an equivalence, consider the quasi-inverse $\psi := \mathrm{Hom}_A(D({}_AA), -)$. Then we have the following diagram

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \psi(DA) \arrow[r] \arrow[d, "\psi(g^n)"] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \psi(C_\iota{}^{n-1} \otimes_A DA) \arrow[r, "\psi(\partial^{n-1})"] & \psi(C_\iota^n \otimes_A DA) \arrow[r, "\psi(\partial^n)"] & \psi(C_\iota{}^{n+1} \otimes_A DA) \arrow[r] & \dots \end{tikzcd}$$

Thus, $\psi(g) \neq 0$ since $g \neq 0$, which contradicts that $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, C_\iota[t]) = 0$. Then $g \sim 0$, hence $\mathrm{Hom}_{K(\mathcal{A})}(DA, C_\iota \otimes_A DA[t]) = 0$.

Since $C_\iota \otimes_A DA$ is a bounded below complex of injectives by Corollary 3.24 we have:

$$\mathrm{Hom}_{D(\mathcal{A})}(DA, C_\iota \otimes_A DA[t]) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(DA, C_\iota \otimes_A DA[t]) = 0 \tag{4.16}$$

Note that $\mathbf{S} = \{X^\bullet \in \mathcal{D}(\mathcal{A}) \mid \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, C_\iota \otimes_A DA[t]) = 0,\ \forall t \in \mathbb{Z}\}$ form a localizing subcategory of $\mathcal{D}(\mathcal{A})$, in fact:

- <u>$X^\bullet \in \mathbf{S}$ and $Y^\bullet \simeq X^\bullet$, then $Y^\bullet$ is in $\mathbf{S}$.</u>

  Let $f : X^\bullet \to Y^\bullet$ be an isomorphism, then

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d, "f"'] & \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, C_\iota \otimes_A DA) \\ Y^\bullet \arrow[r] & \mathrm{Hom}_{D(\mathcal{A})}(Y^\bullet, C_\iota \otimes_A DA) \arrow[u] \end{tikzcd}$$

  Hence $\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, C_\iota \otimes_A DA) = 0$ implies that $\mathrm{Hom}_{D(\mathcal{A})}(Y^\bullet, C_\iota \otimes_A DA) = 0$.

- <u>$T(X)$ is in $\mathbf{S}$, $\forall X \in \mathbf{S}$.</u>

  Note that a morphism $TX^\bullet = X^\bullet[1] \to C_\iota \otimes_A DA[t]$ is equivalent to a morphism $X^\bullet \to C_\iota \otimes_A DA[t+1]$ for every t, so we are done.

- <u>Given a triangle $X \to Y \to Z \to T(X)$ with X, Y in $\mathbf{S}$, then so is Z.</u>

Using Proposition 3.7 with the functor $\operatorname{Hom}(-, C_\iota \otimes_A DA)$, we have the long exact sequence

$$\ldots \longrightarrow \operatorname{Hom}(TX, C_\iota \otimes_A DA) \longrightarrow \operatorname{Hom}(Z, C_\iota \otimes_A DA) \longrightarrow \operatorname{Hom}(Y, C_\iota \otimes_A DA) \longrightarrow \ldots$$

Note that X is in $\mathbf{S}$ so is TX, therefore, since $Y \in \mathbf{S}$ and by exactness of the sequence $\operatorname{Hom}(Z, C_\iota \otimes_A DA) = 0$.

- <u>$\mathbf{S}$ is closed under coproducts</u>.

  This comes from the fact that $\operatorname{Hom}(-, C_\iota \otimes_A DA)$ is an additive functor.

Since A is a finite dimensional algebra, then every injective A-module is a direct summand of a direct sum of copies of DA, so $\langle \mathcal{I} \rangle$ is generated by the single object DA. Hence $\langle \mathcal{I} \rangle \subset \mathbf{S}$, because $DA \in \mathbf{S}$ (4.16). But, $C_\iota \otimes_A DA$ is not in $\langle \mathcal{I} \rangle$ because it does not belong to $\mathbf{S}$. In fact, if $C_\iota \otimes_A DA \in \mathbf{S}$ then,

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(C_\iota \otimes_A DA, C_\iota \otimes_A DA[t]) = 0, \ \forall t \in \mathbb{Z}$$

and, in particular, $C_\iota \otimes_A DA$ is contractible (contradiction). Then, injectives do not generate for A. □

Lemma 4.17. *If every simple A-module is in $\langle \mathcal{I} \rangle$, then injectives generate for A.*

Proof. Every semisimple module is a coproduct of simple modules, and therefore is in the localizing subcategory generated by simple modules. So, since every module is an interetated extension of coproduct of simple modules (Lemma 4.10) using item (a) of Proposition 4.2, we have that every module is in the localizing subcategory generated by simple modules. □

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