Universidade Federal de Minas Gerais
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# ON THE CLASSICAL AND KULKARNI LIMIT SETS OF DISCRETE SUBGROUPS OF *PU(n,1)*

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**ON THE CLASSICAL AND KULKARNI LIMIT SETS OF DISCRETE SUBGROUPS OF *PU(n,1)***

**Final Version**

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FOLHA DE APROVAÇÃO

*On the classical e Kulkarni limit sets of discrete subgroups of PU(n,1)*

**ANTÔNIO AUGUSTO PEREIRA DOS SANTOS**

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ATA DA DEFESA DE DISSERTAÇÃO DE MESTRADO DO ALUNO ANTÔNIO AUGUSTO PEREIRA DOS SANTOS, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DO INSTITUTO DE CIÊNCIAS EXATAS DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 26 DE FEVEREIRO DE 2021.

Aos vinte e seis dias do mês de fevereiro de 2021, às 10h00, em reunião pública virtual na Plataforma Google Meet pelo link https://meet.google.com/uhm (conforme mensagem eletrônica da Pró-Reitoria de Pós-Graduação de 26/03/2020, com orientações para a atividade de defesa de dissertação durante a vigência da Portaria nº 1819), reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de dissertação do aluno **Antônio Augusto Pereira dos Santos**, intitulada: *“On the classical e Kulkarni limit sets of discrete subgroups of PU(n,1)"*, requisito final para obtenção do Grau de mestre em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Heleno da Silva Cunha, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 26 de fevereiro de 2021.

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**Resumo**

O grupo *PU(n,1)* mais a operação de conjugação complexa formam o grupo completo de isometrias do espaço hiperbólico complexo. O presente trabalho busca investigar as relações entre os conjuntos limites de subgrupos discretos de *PU(n,1)* conforme definidos por *Chen* e *Greenberg* e *Kulkarni*. Os conjuntos limites são importantes ferramentas no estudos desses subgrupos, no entanto não existe uma definição única de conjunto limite. Nesta dissertação vamos mostrar que pelo menos estas duas definições estão intimamente relacionadas, veremos que o conjunto limite conforme definido por *Chen* e *Greenberg* nada mais é que a intercessão entre o conjunto limite conforme definido por *Kulkarni* e a fronteira do espaço hiperbólico. Para mostrar isto utilizaremos como base um artigo publicado por *Navarrete* em que ele mostra essa igualdade em dimensão dois estendendo alguns dos resultados por ele encontrados para dimensão qualquer. Demonstraremos uma série de propriedades do conjunto limite no sentido de *Chen* e *Greenberg*, passando por dois importante resultdos relacionados a convergência de grupos compactos sob a ação de sequências de elementos discretos e uma relação de equivalência para pontos no conjunto limite, para ao final concluir com o resultado principal.

Palavras-chave: Espaço Hiperbólico Complexo. Subgrupos Discretos de *PU(n,1)*. Conjuntos Limites.

**Abstract**

The group *PU(n,1)* and complex conjugation form the complete group of isometries of the complex hyperbolic space. The present work aims to investigate how the limit sets of discrete subgroups of *PU(n,1)* as defined by *Chen* and *Greenberg* and as defined by *Kulkarni* are related. Limit sets are important tools in the study of these subgroups, however there is not an unique definition of what a limit set is. In this thesis we will show that the definition of limit set as given by *Chen* and *Greenberg* and as given by *Kulkarni* are intimately related, for the former definition is nothing more than the intersection of the latter definition and the boundary of the complex hyperbolic space. In order to show this we will rely on a paper by *Navarrete* in which it is shown that the above equality is valid in dimension two. We will generalize some of the results of *Navarrete* for any positive dimension. We will also show a series of properties of the limit set as defined by *Chen* and *Greenberg*, with two important results relating to the convergence of compact sets under the action of sequences of discrete elements and a equivalence relation for points in the limit set. We will then conclude with the main result of the work.

Keywords: Complex Hyperbolic Space. Discrete Subgroups of *PU(n,1)*. Limit Sets.

# Contents

# 1 INTRODUCTION

The set of all the unitary transformations of $\mathbb{C}^{n,1}$ form a group with the operation of composition. This group can be given a matrix representation as the group $PU(n,1)$ of $GL(n+1)$. When we consider the actions of subgroups of $PU(n,1)$ on $\mathbb{H}^n_{\mathbb{C}}$ it becomes apparent that, given a group, the orbits of any point $p \in \overline{\mathbb{H}^n_{\mathbb{C}}}$ tend to accumulate at certain positions. We define the set consisting of these accumulation points as the limit set of the group. This natural definition of limit set was first presented by Chen and Greenberg in [4]. In their paper they were able to prove that this definition does not depend on the orbit we choose to observe and that this definition is the smallest invariant set under the action of a subgroup of $PU(n,1)$ in a certain sense.

A lot of work has been done regarding this definition of limit set since its initial presentation by Chen and Greenberg. However, other authors have discovered different and equally natural definitions for limit sets in regard to subgroups of $PU(n,1)$. Kulkarni, for instance, in his study of the actions of groups on Hausdorff spaces came up with a different definition of limit set given by the union of infinite isotropy groups of points and the closure of accumulation points of compact subsets of the space. This definition can be applied to the same groups studied by Chen and Greenberg if we regard $\overline{\mathbb{H}^n_{\mathbb{C}}}$ as a part of $\mathbb{P}^n_{\mathbb{C}}$. Other definitions still exist for different purposes, but in the present work we will be concerned with only those two.

Among the subgroups of $PU(n,1)$ the most important are the discrete ones, as the actions of these subgroups in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ provide a multitude of interesting structures and phenomena. A point of particular interest in the study of discrete subgroups is the region where they act properly discontinuous. The limit set as defined by Chen and Greenberg allows us to answer this question in regard to $\overline{\mathbb{H}^n_{\mathbb{C}}}$ and the limit set as defined by Kulkarni extends the answer to the whole of $\mathbb{P}^n_{\mathbb{C}}$. Thus a question spontaneously arises: "Are these definitions related in some way?".

In his paper [10] Navarrete was able to prove that in the two dimensional case the Chen and Greenberg limit set is the intersection of the Kulkarni limit set and the boundary of the complex hyperbolic plane. Furthermore he was able to show that the Kulkarni limit set of a discrete subgroup of $PU(2,1)$ is given by the union of tangent complex lines of $\mathbb{P}^n_{\mathbb{C}}$ at points of the Chen and Greenberg limit set. Navarrete continued to work over these results in [3] with Cano and Seade, where they were able to generalize some of Navarrete previous conclusions and prove that the complement of the Chen and Greenberg limit set in $\mathbb{P}^n_{\mathbb{C}}$ is the subgroup's region of equicontinuity. Afterwards, Cano with Liu and Lopes in [2] was finally able to prove that in higher dimensions the Kulkarni limit set is the union of tangent hyperplanes at points of the Chen and Greenberg limit set.

We notice that in both [2] and [3] the theory used to achieve these results is noticeably different than the one used in [10]. Both [2] and [3] mainly use pseudo-projective transformations and the Cartan decomposition of elements of $PU(n,1)$, objects not used by [10]. Hence in the present work we wanted to focus in the theory presented in [10] and show that it can be used at least to show that for a $n$-dimensional complex hyperbolic space, the Chen and Greenberg limit set of a discrete subgroup of $PU(n,1)$ is the intersection of its Kulkarni limit set and the boundary of $\mathbb{H}^n_{\mathbb{C}}$.

In order to do this, in section 2 we will present the models of $\overline{\mathbb{H}^n_{\mathbb{C}}}$ that will be used throughout the thesis, nominally the projective model and the unitary ball model, and their relation. In section 3 we will present the group $PU(n,1)$ and discuss its action in the complex hyperbolic space. In section 4 we will give the definition of the Kulkarni limit set, discuss its relevant properties and give some examples. In section 5 we will do the bulk of the work in the thesis, giving the definition of the Chen and Greenberg limit set, presenting its relevant properties and presenting two important results, one about the convergence of compact set of $\overline{\mathbb{H}^n_{\mathbb{C}}}$ under the action of elements of $PU(n,1)$ and the other about the finite subgroups of $PU(n,1)$. Finally in section 6 we will present some considerations about how the results of sections 4 and 5 and the main result in this work.

# 2 THE COMPLEX HYPERBOLIC SPACE

## 2.1 THE PROJECTIVE MODEL

We first present a brief description of the projective model for a complex hyperbolic space of dimension $n$.

Let $\mathbb{C}$ denote the complex numbers and $\mathbb{C}^{n,1}$ the vector space of dimension $n+1$ with the following hermitian form associated to it $\langle , \rangle : \mathbb{C}^{n,1} \times \mathbb{C}^{n,1} \to \mathbb{R}$ defined as

$$\langle (z_0, z_1, ..., z_n), (w_0, w_1, ..., w_n) \rangle = -\overline{z_0} w_0 + \langle \langle (z_1, ..., z_n), (w_1, ..., w_n) \rangle \rangle,$$

Where

$$\langle \langle (z_1, ..., z_n), (w_1, ..., w_n) \rangle \rangle := \overline{z_1} w_1 + ... + \overline{z_n} w_n$$

Is the canonical hermitian form. This hermitian form divides our vector space in three distinct sets that we define as follows $V_0 := \{z \in \mathbb{C}^{n,1} : \langle z, z \rangle = 0\}$, $V_- := \{z \in \mathbb{C}^{n,1} : \langle z, z \rangle < 0\}$ and $V_+ := \{z \in \mathbb{C}^{n,1} : \langle z, z \rangle > 0\}$. We shall refer to the points of each of these sets as isotropic, negative or positive respectively.[4]

The set $\beta := \{e_0, e_1, ..., e_n\}$, where $e_i$ denotes the vector that is 0 in every coordinate except for the i-th, is an orthogonal basis of $\mathbb{C}^{n,1}$. We also have that $\langle e_0, e_0 \rangle = -1$ and $\langle e_i, e_i \rangle = 1$ for all $i \neq 0$, thus the hermitian form has a $(n, 1)$ signature.

Now we consider the projectivization of $\mathbb{C}^{n,1}$. We say that two elements $z, w \in \mathbb{C}^{n,1}$ are similar, if $\exists\ \lambda \in \mathbb{C} - \{0\}$ such that $z = \lambda w$, and we note $z \sim w$. The $n$-dimensional projective complex space can now be defined as $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^{n,1}) := \mathbb{C}^{n,1}/\sim$ and we obtain[11]:

**Definition 2.1.** *[11] $\mathbb{P}(V_-) := \mathbb{H}^n_{\mathbb{C}}$, the complex hyperbolic space of dimension $n$ and $\mathbb{P}(V_0) := \partial \mathbb{H}^n_{\mathbb{C}}$ the boundary of the complex hyperbolic space of dimension $n$.*

In the projective model we will assign a metric called the *Bergman metric* given indirectly by the following distance formula $\rho(\cdot, \cdot)$ [11]:

$$cosh^2 \left( \frac{\rho(z, w)}{2} \right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}$$

This formula does not depend on the representatives chosen, if $rz$ and $sw$ are different representatives of the equivalence classes of $z$ and $w$, then:

$$cosh^2 \left( \frac{\rho(rz, sw)}{2} \right) = \frac{\langle rz, sw \rangle \langle sw, rz \rangle}{\langle rz, rz \rangle \langle sw, sw \rangle}$$

$$= \frac{\overline{r} s \langle z, w \rangle \overline{s} r \langle w, z \rangle}{|r|^2 \langle z, z \rangle |s|^2 \langle w, w \rangle}$$

$$= \frac{(|r||s|)^2 \langle z, w \rangle \langle w, z \rangle}{(|r||s|)^2 \langle z, z \rangle \langle w, w \rangle}$$

$$= \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle} = cosh^2 \left( \frac{\rho(z, w)}{2} \right)$$

## 2.2 THE UNITARY BALL MODEL

The projective model will be the standard model for our calculations in this work, however, on occasion we shall use a different (but related) model, the unitary ball model. This model will prove useful in understanding some calculations by Navarrete and Chen and Greenberg.

Consider the hermitian form with signature $(n, 1)$ and the projectivization map defined above, a non-null element $z = (z_0, ..., z_n)$ may be isotropic or negative if, and only if, $z_0 \neq 0$. If $z_0 = 0$ then there exists one $z_i \neq 0$ ($z$ is a non-null vector) and so $\langle z, z \rangle = \langle \langle (z_1, ..., z_n), (z_1, ..., z_n) \rangle \rangle = \sum_{i=1}^{n} |z_i|^2 > 0$ which implies $z$ is positive.

With this result we may define a standard representation for equivalence classes in the projective model of $\overline{\mathbb{H}_{\mathbb{C}}^n}$, for every element of $\overline{\mathbb{H}_{\mathbb{C}}^n}$ is an equivalence class in $\mathbb{C}^{n,1}$. Let $[z] \in \overline{\mathbb{H}_{\mathbb{C}}^n}$ be one such class, and $z = (z_0, ..., z_n) \in [z]$ an element of this class, we choose the standard representative as being $z' = \frac{1}{z_0} z = (1, \frac{z_1}{z_0}, ..., \frac{z_n}{z_0})$. If $w = (w_0, ..., w_n)$ is a different element of $[z]$ then $w = \lambda z$ for some $\lambda \in \mathbb{C} - \{0\}$, so $\frac{1}{w_0} w = (1, \frac{w_1}{w_0}, ..., \frac{w_n}{w_0}) = (1, \frac{\lambda z_1}{\lambda z_0}, ..., \frac{\lambda z_n}{\lambda z_0}) = (1, \frac{z_1}{z_0}, ..., \frac{z_n}{z_0}) = z'$ and the standard representative is well defined. The standard representative is also called the standard lift of the class $[z]$. From now on the class $[z]$ we will be referred simply as $z$, and we will assume that $z$ is the standard representative of $[z]$.

With the previous definitions we have now that $z$ is negative, if, and only if

$$\langle z, z \rangle < 0$$

$$\Leftrightarrow -1 + |z_1|^2 + ... + |z_n|^2 < 0$$

$$\Leftrightarrow |z_1|^2 + ... + |z_n|^2 < 1$$

$$\Leftrightarrow (z_1, ..., z_n) \in B^n \subset \mathbb{C}^n$$

Where $B^n$ is the unitary ball of $\mathbb{C}^n$. An analogous calculation show us that an element is isotropic if, and only if, $(z_1, z_n) \in \partial B^n = S^{2n-1} \subset \mathbb{C}^n$. Thus we can identify $\overline{\mathbb{H}_{\mathbb{C}}^n}$ with $\overline{B^n}$ by taking the final n coordinates of the standard representation of an element of $\overline{\mathbb{H}_{\mathbb{C}}^n}$. This representation of the complex hyperbolic space is called the unitary ball model.

Similarly to the projective model, we can assign a metric in this model by defining the distance between two points as the distance between their standard lifts associated to them in the projective model [11].

# 3 THE GROUP PU(n,1)

## 3.1 PRESENTATION OF $PU(n,1)$

**Definition 3.1.** *[4] Let $g$ be an automorphism of $\mathbb{H}^n_{\mathbb{C}}$ such that $\langle g(z), g(w)\rangle = \langle z, w\rangle$ for all $z, w \in \mathbb{C}^{n,1}$, $g$ is called an unitary transformation, and $U(n,1)$ is the group given by all unitary transformations in $\mathbb{C}^{n,1}$*

As $g \in U(n,1)$ is an automorphism, $g$ is $\mathbb{C}$-linear, so it can be represented by an element of $GL(n+1,\mathbb{C})$ the group of $(n+1)\times(n+1)$ invertible matrices. Given $g \in U(n,1)$ and $\beta$ the canonical basis given in the previous section, then for each $i = 0, 1, ..., n$, we define $g(e_i)^T$ as the i-th column of the matrix representation $A$ of $g$. As $g$ is unitary, the $g(e_i)^T$ are all linearly independent, so $A$ is invertible, then $A \in GL(n+1,\mathbb{C})$. From now on, whenever we mention an element of $U(n,1)$ we will assume its matrix representation. [4]

**Proposition 3.2.** *[4] $U(n,1)$ acts isometrically on $\mathbb{H}^n_{\mathbb{C}}$.*

*Proof.* Let $A \in U(n,1)$ and $z, w \in \mathbb{H}^n_{\mathbb{C}}$ then we have that

$$cosh^2\left(\frac{\rho(Az,Aw)}{2}\right) = \frac{\langle Az, Aw\rangle\langle Aw, Az\rangle}{\langle Az, Az\rangle\langle Aw, Aw\rangle}$$

$$= \frac{\langle z, w\rangle\langle w, z\rangle}{\langle z, z\rangle\langle w, w\rangle} = cosh^2\left(\frac{\rho(z,w)}{2}\right)$$

$$\implies \rho(Az, Aw) = \rho(z, w)$$

□

Another notable feature of $U(n,1)$ is that its elements act transitively in $\mathbb{H}^n_{\mathbb{C}}$ and doubly transitively in $\partial\mathbb{H}^n_{\mathbb{C}}$. We present this fact as proved by .

**Proposition 3.3.** *[4] $U(n,1)$ acts transitively in $\mathbb{H}^n_{\mathbb{C}}$ and doubly transitively on the boundary $\partial\mathbb{H}^n_{\mathbb{C}}$.*

**Proposition 3.4.** *Let $A \in U(n,1)$ be a matrix and let $A' = \lambda A$ with $\lambda \in \mathbb{C}$, then $A' \in U(n,1)$ if, and only if $|\lambda| = 1$.*

*Proof.* $\langle A'z, A'w\rangle = \langle \lambda Az, \lambda Aw\rangle = \lambda\overline{\lambda}\langle Az, Aw\rangle = |\lambda|^2\langle z, w\rangle$. Now, if $A' \in U(n,1)$, then $\langle A'z, A'w\rangle = |\lambda|^2\langle z, w\rangle = \langle z, w\rangle \implies |\lambda| = 1$, and if $|\lambda| = 1$ then $\langle A'z, A'w\rangle = |\lambda|^2\langle z, w\rangle = \langle z, w\rangle \implies A' \in U(n,1)$. □

**Proposition 3.5.** *Let $A, A'$ as described in the previous proposition, then they act the same in $\overline{\mathbb{H}^n_{\mathbb{C}}}$.*

*Proof.* Let $z \in \overline{\mathbb{H}^n_{\mathbb{C}}}$, then $A'z = \lambda Az = A(\lambda z) = Az$. □

The two previous propositions make it clear that when we consider the actions of elements of $U(n,1)$ in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ we will have a lot of elements with identical actions. To solve this problem, we define an equivalence relation in $U(n,1)$ given by $A \sim B \Leftrightarrow \exists \lambda \in \mathbb{C}, |\lambda| = 1$ such that $A = \lambda B$.[4]

**Definition 3.6.** $PU(n,1) = U(n,1)/\sim$

**Remark.** *[4] $PU(n,1)$ can also be seen as the image of $U(n,1)$ in $PGL(n+1,\mathbb{C})$.*

**Remark.** *$PU(n,1)$ plus the operation of complex conjugation gives us the complete group of isometries of $\mathbb{H}^n_{\mathbb{C}}$, though we will not give a proof of this fact here.*

## 3.2 TOTALLY GEODESIC SUBMANIFOLDS

With $PU(n,1)$ well established, we can now present some results about totally geodesic submanifolds that will be used throughout our work. All of the results in this subsection were originally presented in [4], we will only give an abridged version of then.

**Definition 3.7.** *[4] A submanifold $M \subset \mathbb{H}^n_{\mathbb{C}}$ is totally geodesic if it contains every geodesic which is tangent to it.*

**Remark.** *[5] If $M$ is a subspace of $\mathbb{H}^n_{\mathbb{C}}$ and $\langle z, w\rangle \in \mathbb{R}$ for all $z, w \in M$, we say that $M$ is a totally real totally geodesic subspace.*

**Definition 3.8.** *[4] We call the element $s_0 = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in U(n,1)$, where $I_n$ is the $n \times n$ identity matrix, the symmetry at 0. If $\zeta \in B^n$, then $s_\zeta \in U(n,1)$ is called the symmetry at $\zeta$ and it is given by $gs_0g^{-1}$, where $g \in U(n,1)$ and $g(0) = \zeta$.*

**Proposition 3.9.** *[4] $M$ is a totally geodesic submanifold if and only if $s_\zeta(M) = M$ for all $\zeta \in M$.*

**Proposition 3.10.** *[4] The following are true:*

*(a) $\mathbb{H}^1_{\mathbb{R}}$ is a geodesic in $\mathbb{H}^n_{\mathbb{C}}$. Every geodesic is equivalent under $U(n,1)$ to $\mathbb{H}^1_{\mathbb{R}}$;*

*(b) The geodesics at 0 are precisely the $\mathbb{R}$-lines through 0. These are all equivalent under the isotropy group $U(1) \times U(n)$;*

*(c) Let $p, q \in \overline{\mathbb{H}^n_{\mathbb{C}}}$. Then there is a unique geodesic which connects $p$ to $q$.*

## 3.3 CONJUGACY CLASSES OF $U(n,1)$

The elements of $U(n,1)$ leave $\overline{\mathbb{H}^n_{\mathbb{C}}}$ invariant, thus, we can apply *Brower's fixed-point theorem* to obtain that any element of $U(n,1)$ fixes at least one point of $\overline{\mathbb{H}^n_{\mathbb{C}}}$[4].

This fact allows us to divide the elements of $U(n,1)$ in three categories:

**Definition 3.11.** *[4] Let $g$ be an element in $U(n,1)$. We shall call $g$ elliptic if it has a fixed point in $\mathbb{H}^n_{\mathbb{C}}$. We shall call $g$ parabolic if it has exactly one fixed point in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ and it lies on $\partial\mathbb{H}^n_{\mathbb{C}}$. $g$ will be called loxodromic if it has exactly two fixed points in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ and these belong to $\partial\mathbb{H}^n_{\mathbb{C}}$.*

We will not present a proof of the fact that the above definition covers all possibilities because the calculations involved would distract from the results being presented. If the reader is interested in seeing a proof, one can be found in [4].

# 4 THE KULKARNI LIMIT SET

The Kulkarni limit set was defined in [8] with the intention of studying higher dimensional Kleinian groups. In this section we will present its definition and some results pertaining to its relation with discrete groups of $PU(n,1)$ analogous to those presented in [10].

## 4.1 DEFINITIONS

We begin with a few basic definitions concerning the subject.

**Definition 4.1.** *[8] Let $X$ be a locally compact Hausdorff space and let $\Gamma$ be a group acting on $X$. The action of $\Gamma$ is said to be properly discontinuous on a $\Gamma$-invariant subset $\Omega$ of $X$ if for any two compact subsets $C$ and $D$ of $\Omega$, $\gamma(C) \cap D \neq \varnothing$ only for finitely many $\gamma \in \Gamma$.*

**Definition 4.2.** *[8] Let $\{A_\beta\}$ be a family of subsets of $X$ where $\beta$ runs over some infinite indexing set $B$. A point $p \in X$ is said to be a cluster point of $\{A_\beta\}$ if every neighborhood of $p$ intersects $A_\beta$ for infinitely many $\beta \in B$.*

**Definition 4.3.** *[8] Let $p \in X$, we define $\Gamma_p := \{\gamma \in \Gamma \text{ such that } \gamma(p) = p\}$ as the isotropy group of $p$ with respect to $\Gamma$.*

**Definition 4.4.** *[8] Let:*

$L_0(\Gamma) = \{$*the closure of the set of points in $X$ with infinite isotropy group*$\}$

$L_1(\Gamma) = \{$*the closure of the set of cluster points of $\{\gamma z\}_{\gamma\in\Gamma}$ where $z$ runs over $X - L_0(\Gamma)$*$\}$

$L_2(\Gamma) = \{$ *the closure set of cluster points of $\{\gamma K\}_{\gamma\in\Gamma}$ where $K$ runs over compact subsets of $X - \{L_0(\Gamma) \cup L_1(\Gamma)\}$*$\}$

**Definition 4.5** (The Kulkarni limit set)**.** *[8] The set $\Lambda(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)$ is called the limit set of $\Gamma$. The set $\Omega(\Gamma) = X - \Lambda(\Gamma)$ is called the domain of discontinuity of $\Gamma$.*

Let us consider a simple example for the sake of fixing ideas:

**Example.** *Let $g \in U(n,1)$ be an element of finite order and consider the action of $G = \langle g \rangle$ on $\mathbb{P}^n_{\mathbb{C}}$. As $G$ is a finite group, then no point of $\mathbb{P}^n_{\mathbb{C}}$ has infinite isotropy group with respect to $G$, thus $L_0(G) = \varnothing$. Similarly, the orbit of any compact subset of $\mathbb{P}^n_{\mathbb{C}}$ has a finite amount of elements, thus $L_1(G) = \varnothing = L_2(G)$. So $\Lambda(G) = \varnothing$.*

We present now an example given in [10], as a way of motivating the interest in the limit set of subgroups of $PU(n,1)$:

**Example.** *[10] Let $g$ be a loxodromic element of $PU(2,1)$ and consider the action of $G = \langle g \rangle$ on $\mathbb{P}^2_{\mathbb{C}}$. Through the analysis of the dynamic of $g$ on $\mathbb{P}^2_{\mathbb{C}}$ we can find that $G$ fixes three points in $\mathbb{P}^2_{\mathbb{C}}$, $\{a, r, s\}$, $a$ an attractor in $\mathbb{P}^2_{\mathbb{C}} - [r, s]$ (where $[r, s]$ represents the complex projective line that passes through these two points), $r$ a repeller in $\mathbb{P}^2_{\mathbb{C}} - [a, s]$ and $s$ a saddle point.*

*It becomes clear then that $L_0(G) = \{a, r, s\}$ for those are the only elements with infinite isotropy groups with respect to $G$. We also have that $L_1(G) = L_0(G)$, for if we take any point $p$ in $\mathbb{P}^2_{\mathbb{C}} - L_0(G)$, then $\lim_{n\to\infty} g^n(p) \in \{r, s\}$ and then $\lim_{n\to-\infty} g^n(p) \in \{a, s\}$.*

*Finally, we have that $L_2(G) = [a, s] \cup [r, s]$. We will not give a detailed proof of this fact, as it would involve too many calculations that would distract us from the main parts of interest, but we will give the general idea of the proof. Consider $K$, a 3-sphere in $\mathbb{P}^2_{\mathbb{C}}$ which bounds a ball around $s$, notice that $K \cap [a, s] \neq \varnothing \neq K \cap [r, s]$ for any such 3-sphere. This means that as $n \to \infty$, $K^n \to [a, s]$, so $[a, s] \subset L_2(G)$. Analogously as $n \to -\infty$, $K^n \to [r, s]$, so $[r, s] \subset L_2(G)$. For any point outside of $[a, s] \cup [r, s]$, we have that any compact subset that intersects either $[a, s]$ or $[r, s]$ can only have cluster points in $[a, s] \cup [r, s]$, and if it does not intersect either of those lines then it can only have $\{a, r, s\}$ as cluster points. Thus $L_2(G) = [a, s] \cup [r, s]$, and $\Lambda(G) = [a, s] \cup [r, s]$.*

## 4.2 PROPERTIES OF THE LIMIT SET

We shall discuss now a few properties of the Kulkarni limit set and its applications regarding discrete groups. Whenever there is no risk of confusion we will write $L_0, L_1, L_2, \Lambda$ and $\Omega$ instead of $L_0(\Gamma), L_1(\Gamma), L_2(\Gamma), \Lambda(\Gamma)$ and $\Omega(\Gamma)$

**Definition 4.6.** *[8] $\Gamma$ is said to have the Kleinian property if $\Omega \neq \varnothing$.*

**Proposition 4.7.** *[8] Let $X$ and $\Gamma$ be as above where $\Gamma$ is equipped with the compact open topology. Then $L_0, L_1, L_2, \Lambda$ and $\Omega$ are $\Gamma$-invariant and $\Gamma$ acts properly discontinuous on $\Omega$. If $\Gamma$ has the Kleinian property then it is discrete. If $X$ has a countable base for its topology then $\Gamma$ is countable.*

*Proof.* If $\Gamma_p$ is infinite and $\sigma \in \Gamma$, then $\sigma\Gamma_p\sigma^{-1}$ is infinite and it is also the isotropy group of $\sigma(p)$, thus $L_0$ is invariant under $\Gamma$. An analogous argument shows that $L_1, L_2, \Lambda$ and $\Omega$ are also $\Gamma$-invariant.

Given compact subsets $C$ and $D$ of $\Omega$ then define $S = \{\gamma \in \Gamma \text{ such that } \gamma(C) \cap D \neq \varnothing\}$. Suppose that $|S| = \infty$ then that means there are infinitely many $\gamma$ such that $\gamma(C) \cap D \neq \varnothing$. Choose some point $d_\gamma \in \gamma(C) \cap D$ for each $\gamma$. As $D$ is a compact subset, there exists a subsequence of $d_\gamma$ that converges to some point $d \in D$. This implies that $d$ is a cluster point of $\{\gamma(C)\}_{\gamma\in\Gamma}$, which implies that $d \in L_2$, which is absurd, for $D \subset X - \{L_0, L_1, L_2\}$. So we can conclude that $\Gamma$ acts properly discontinuous in $X$.

Now consider $K$ a compact neighborhood of a point $p \in \Omega$ and $\Gamma$ a group with the Kleinian property. If $T = \{\gamma \in \Gamma$ such that $\gamma(p) \in \text{int}(C)\}$ then $T \subset T' := \{\gamma \in \Gamma$ such that $\{\gamma(p)\} \cap C \neq \varnothing\}$. From the previous result we have that $\Gamma$ acts properly discontinuous on $\Omega$ and we conclude that $T'$ is finite. This also means that $T$ is finite, therefore open in $\Gamma$. Since $X$ is Hausdorff then the topology of $\Gamma$ is also Hausdorff. As $T$ is nonempty $\Gamma$ must be discrete.

Finally if $X$ has a countable base for its topology so does $\Omega$. Let $\{u_n\}$ $n \in \mathbb{N}$ be a countable base of relatively compact neighborhoods on $\Omega$ and let $\Gamma_n = \{\gamma \in \Gamma$ such that $\gamma(\overline{U_1}) \cap \overline{U_n} \neq \varnothing\}$. By the previous results each $\Gamma_n$ is finite and $\bigcup_{n\in\mathbb{N}} \Gamma_n = \Gamma$. So $\Gamma$ is countable. $\square$

With this result the usefulness of the Kulkarni limit set in considering the regions where a groups acts properly discontinuous becomes clear. We finish this section by presenting an example contained in [8] of a non-hyperbolic space where the traditional definition of the limit set (the cluster points of $\Gamma$-orbits) does not agree with the Kulkarni definition.

**Example.** *[8] Consider $X = \mathbb{R}^2$ and $\Gamma = \langle \gamma \rangle$ given by $\gamma(x, y) = (2x, \frac{y}{2})$. The origin is the only point fixed by $\gamma$. Furthermore it is the only cluster point of $\Gamma$-orbits of points as $\lim_{n\to\infty} \gamma^n(0, y) = (0, 0)$ for all $y \in \mathbb{R}$, $|\lim_{n\to\infty} \gamma^n(x, y)| = \infty$ for all $x \in \mathbb{R} - \{0\}$. Thus in the classical sense the limit of $\Gamma$ is the set $\{(0, 0)\}$.*

*In the Kulkarni sense, however, the previous reasoning only tells us that $L_1 \cup L_2 = \{(0, 0)\}$, we still need to consider the action of $\Gamma$ in the orbits of compact sets of $\mathbb{R}^2 - \{(0, 0)\}$. It is not hard to see that given any circle $S$ with the origin as its center and radius $r$ that $\gamma^n(S) \cap S \neq \varnothing$ and as $n \to \infty$ the intersection points tend to $(\pm r, 0)$ and as $n \to -\infty$ the intersection points tend to $(0, \pm r)$ thus $L_2 = \{(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$ such that either $x = 0$ or $y = 0\}$. We have then that $\Lambda(\Gamma) = \{(x, y)$ such that either $x = 0$ or $y = 0\}$*

# 5 THE CHEN-GREENBERG LIMIT SET

In this section we will present the classical notion of limit set, which we also call the Chen-Greenberg limit set, and some of its important properties. We will begin by giving its definition as presented in [4], then we will prove an important propositions about Chen-Greenberg limit sets in a similar fashion to the work done in [10].

Then we will make a few considerations about the isotropy group of a boundary point and present the notion of $G$-duality for limit points. This notion will allow us to conclude that any discrete subgroup of $PU(n,1)$ with more than one element in its Chen-Greenberg limit set has a loxodromic element. We will finish the section by stating a proposition of [3] that shall be important in the next section.

## 5.1 DEFINITION AND BASIC PROPERTIES

With the preliminary results presented so far we can now define one of the main points of interest in our work, the limit sets of discrete subgroups of $PU(n,1)$. Before we define the Chen-Greenberg limit set we need a proposition that guaranties that the limit set is well defined.

**Proposition 5.1.** *[4] Let $p$ be a point of $\mathbb{H}^n_{\mathbb{C}}$ and let $\{g_m\}$ be a sequence in $U(n,1)$ such that $\lim_{m\to\infty} g_m(p) = q \in \partial\mathbb{H}^n_{\mathbb{C}}$. Then $\lim_{m\to\infty} g_m(p') = q$ for all $p' \in \mathbb{H}^n_{\mathbb{C}}$.*

*Proof.* It is important to notice that the limit in the above definition is being considered in the ball model with respect to the euclidean metric of $B^n$, not the Bergman metric.

Let $p' \in \mathbb{H}^n_{\mathbb{C}}$ and suppose that $\lim_{m\to\infty} g_m(p') \neq q$. As $\overline{\mathbb{H}^n_{\mathbb{C}}}$ can be seen as $\overline{B^n}$, it is compact, thus there exist a subsequence of $\{g_{m_k}(p')\}$ that converges to a point $q' \in \overline{\mathbb{H}^n_{\mathbb{C}}}$. Without loss of generality we can assume that the convergent subsequence is $\{g_m(p')\}$ itself, so that $\lim_{m\to\infty} g_m(p') = q'$.

As stated in proposition 3.10 (c), there are unique geodesics that connects $p$ to $p'$ and $q$ to $q'$. Denote $l(r,s)$ as the length of the geodesic segment connecting the points $r$ and $s$, then we have that $\rho(p,p') = l(p,p') = l(g_m(p), g_m(p')) \to l(q,q') = \infty$, which is absurd. Therefore $\lim_{m\to\infty} g_m(p) = q = \lim_{m\to\infty} g_m(p')$ for all $p' \in \mathbb{H}^n_{\mathbb{C}}$ . $\square$

**Definition 5.2** (Chen-Greenberg limit set)**.** *[4] Let $G$ be a subgroup of $U(n,1)$ and let $p \in \mathbb{H}^n_{\mathbb{C}}$. The limit set of $G$ is defined to be the set $L(G) = \overline{G(p)} \cap \partial\mathbb{H}^n_{\mathbb{C}}$.*

Proposition 5.1 makes it clear that the definition of the Chen-Greenberg limit set is well defined, for any choice of $p$ in $\mathbb{H}^n_{\mathbb{C}}$ we will give us the same set $L(G)$. The following result will show us that the Chen-Greenberg limit set is the smallest invariant set under the action of a group $G$ [10].

**Proposition 5.3.** *[4] Let $G$ be a subgroup of $U(n,1)$, then $L(G)$ is invariant under $G$. Furthermore, if $X$ is a closed subset of $\partial\mathbb{H}^n_{\mathbb{C}}$ which contains more than one point and is invariant under $G$, then $L(G) \subset X$.*

*Proof.* Let $q \in L(G)$, and $h \in G$, then, by proposition 5.1, $q = \lim_{m\to\infty} g_m(p)$ for a sequence of elements $\{g_m\} \subset G$. As $h \in G$ then, $\{hg_m\} \subset G$ and the sequence $\{hg_m(p)\} \subset \mathbb{H}^n_{\mathbb{C}}$ is convergent in $\overline{\mathbb{H}^n_{\mathbb{C}}}$, because $h$ acts isometrically in $\mathbb{H}^n_{\mathbb{C}}$ and the sequence $\{g_m(p)\}$ is convergent in $\overline{\mathbb{H}^n_{\mathbb{C}}}$. Therefore, $h(q) = \lim_{m\to\infty} hg_m(p)$ implies that $h(q) \in L(G)$, proving the first part of our proposition.

In order to prove the second part, consider a pair of distinct elements $x, y \in X$ and assume, taking a subsequence if needed, that $\lim_{m\to\infty} g_m(x) = x'$, $\lim_{m\to\infty} g_m(y) = y'$ and $x' \neq q \neq y'$. Take now a point $p'$ in the geodesic given by $x$ and $y$. As for all $m$, $g_m$ acts isometrically in $\mathbb{H}^n_{\mathbb{C}}$, then $g_m(p')$ lies in the geodesic given by $g_m(x)$ and $g_m(y)$. Taking the limit as $m \to \infty$ we have that $\lim_{m\to\infty} g_m(p')$ is a point of the geodesic given by $x'$ and $y'$, which implies that $\lim_{m\to\infty} g_m(p') \neq q$ a contradiction with proposition 5.1. If $x' = y'$, then it is clear that $\lim_{m\to\infty} g_m(p') = x' = y' \neq q$, which also contradicts proposition 5.1. Thus, either $\lim_{m\to\infty} g_m(x) = q$ or $\lim_{m\to\infty} g_m(y) = q$. As $X$ is closed and invariant under $G$, we have that in either case $q \in X$, which implies $L(G) \subset X$. □

**Corollary 5.4.** *[10] Let $K \subset \mathbb{H}^n_{\mathbb{C}}$ be a compact subset and $\{g_m\} \subset U(n,1)$ a sequence such that $\lim_{m\to\infty} g_m(p) = q \in \partial\mathbb{H}^n_{\mathbb{C}}$ for some $p \in \mathbb{H}^n_{\mathbb{C}}$, then the sequence of functions $g_m|_K$ converges uniformly to the constant function with value equal to $q$.*

*Proof.* By proposition 5.1, for every $k \in K$ we have that $\lim_{m\to\infty} g_m(k) = q$, furthermore $\overline{B^n}$ is compact with respect to the euclidean metric, thus the functions $g_m$ converge uniformly to the constant function with value equal to $q$. □

We now have a basic understanding of the Chen-Greenberg limit set so we can start to work in proving the important propositions alluded in the beginning of this section.

## 5.2 SEQUENCES OF ELEMENTS IN $PU(n,1)$ AND THE CHEN-GREENBERG LIMIT

We state:

**Proposition 5.5.** *[10] If $\{g_m\}$ is a sequence of distinct elements of a discrete subgroup $G$ of $PU(n,1)$, then there exists a subsequence, still denoted $\{g_m\}$, and elements $x, y \in L(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$, such that $g_m(z) \to x$ uniformly on compact subsets of $\overline{\mathbb{H}^n_{\mathbb{C}}} - \{y\}$.*

This proposition is present in [10] and [9] for $\mathbb{H}^2_{\mathbb{C}}$ and $PU(2,1)$. In the form stated above it is generalized for dimension $n$. This proposition will be fundamental in proving the main theorem of this work, that $\Lambda(G) \cap \overline{\mathbb{H}^n_{\mathbb{C}}} = L(G)$ for all discrete $G \subset PU(n,1)$. Before we show a proof for it we will need some auxiliary results regarding bisectors, an important structure of hyperbolic geometry.

**Definition 5.6.** *[5] Given two distinct points $z_1, z_2 \in \mathbb{H}^n_{\mathbb{C}}$ we define the bisector equidistant from $z_1$ and $z_2$ (or the bisector of $\{z_1, z_2\}$) as*

$$\mathfrak{E}\{z_1, z_2\} = \{z \in \mathbb{H}^n_{\mathbb{C}} \text{ such that } \rho(z_1, z) = \rho(z_2, z)\}.$$

*Bisectors are also called equidistant hypersurfaces. The boundary of a bisector is defined as a spinal sphere in $\partial\mathbb{H}^n_{\mathbb{C}}$.*

Bisectors have a natural decomposition called the slice (Mostow) decomposition. We will give a brief explanation of how this decomposition works.

**Definition 5.7.** *[5] Given two distinct points $z_1, z_2 \in \mathbb{H}^n_{\mathbb{C}}$, let $\Sigma \subset \mathbb{H}^n_{\mathbb{C}}$ be the complex geodesic spanned by these two points. $\Sigma$ is called the complex spine (or $\mathbb{C}$-spine) of $\mathfrak{E}$ (with respect to $\{z_1, z_2\}$). The (real) spine $\sigma$ of $\mathfrak{E}$ (with respect to $\{z_1, z_2\}$) is defined as*

$$\sigma\{z_1, z_2\} = \mathfrak{E}\{z_1, z_2\} \cap \Sigma = \{z \in \Sigma \text{ such that } \rho(z_1, z) = \rho(z_2, z)\}.$$

*Notice that $\sigma$ is the orthogonal bisector of the geodesic segment joining $z_1$ and $z_2$ in $\Sigma$.*

**Proposition 5.8.** *[5] Let $L \subset B^n$ be a complex linear subspace with orthogonal projection $\Pi$. Then for all $u \in B^n - L$ and $s \in L$, the geodesic from $\Pi(u)$ to $u$ and to $s$ are orthogonal and span a totally real totally geodesic 2-plane. Furthermore*

$$\cosh\left(\frac{\rho(u, s)}{2}\right) = \cosh\left(\frac{\rho(u, \Pi(u))}{2}\right)\cosh\left(\frac{\rho(\Pi(u), s)}{2}\right).$$

We will not present a proof of this proposition, though one is found in [5]. We are most interested in the second assertion of the proposition, that

$$\cosh\left(\frac{\rho(u, s)}{2}\right) = \cosh\left(\frac{\rho(u, \Pi(u))}{2}\right)\cosh\left(\frac{\rho(\Pi(u), s)}{2}\right),$$

As this equality is fundamental in proving the Slice Decomposition Theorem. It suffices to say that given the first condition of the previous proposition, the equality is a direct consequence of the Pythagorean theorem.[5]

**Theorem 5.9** (Slice Decomposition Theorem)**.** *[5] Let $\mathfrak{E}, \Sigma$ and $\sigma$ be as above. Define $\Pi_\Sigma : \mathbb{H}^n_{\mathbb{C}} \to \Sigma$ the orthogonal projection onto $\Sigma$. Then $\mathfrak{E} = \Pi_\Sigma^{-1}(\sigma) = \bigcup_{s\in\sigma} \Pi_\Sigma^{-1}(s)$*

*Proof.* Let $z \in \mathbb{H}^n_{\mathbb{C}}$, by the previous proposition we have that

$$\cosh\left(\frac{\rho(z, z_i)}{2}\right) = \cosh\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right)\cosh\left(\frac{\rho(\Pi_\Sigma(z), z_i)}{2}\right)$$

So

$$\cosh\left(\frac{\rho(\Pi_\Sigma(z), z_i)}{2}\right) = \cosh\left(\frac{\rho(z, z_i)}{2}\right)\left(\cosh\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right)\right)^{-1} \tag{1}$$

for $i = 1, 2$ so

$$z \in \mathfrak{E}\{z_1, z_2\} \Leftrightarrow \rho(z, z_1) = \rho(z, z_2) \tag{2}$$

By the definition of bisector

$$(2) \Leftrightarrow \rho(\Pi_\Sigma(z), z_1) = \rho(\Pi_\Sigma(z), z_2)$$

Because of the equations (1) and (2). And finally

$$\Leftrightarrow \Pi_\Sigma(z) \in \sigma\{z_1, z_2\}$$

By the definition of $\sigma\{z_1, z_2\}$. □

**Definition 5.10.** *[5] The complex hyperplanes $\Pi_\Sigma^{-1}(s)$, for $s \in \sigma$, are called the slices of $\mathfrak{E}$ (with respect to $\{z_1, z_2\}$).*

The above results give us enough information about bisectors and their decompositions so that we may prove the results we need in this section. Though we will use it, we remark that bisectors, slices, spines and complex spines do not depend on the choice of $\{z_1, z_2\}$, in other words they are intrinsic. A proof of this fact and further discussion of its consequences can be found in [5].

A final lemma is needed before we may prove the main result of this subsection. Navarrete gives the original proof in dimension 2 in [10], we present a generalization of the result for dimension $n$.

**Lemma 5.11.** *[10] Let $\{x_t\}$ be a sequence of elements of $\mathbb{H}^n_\mathbb{C}$ such that $x_t \to q \in \partial\mathbb{H}^n_\mathbb{C}$. Consider the ball model and write $0$ as the origin in $B^n$ then:*

*(i) If $S_t$ denotes the closed half-space $\{z \in \mathbb{H}^n_\mathbb{C}$ such that $\rho(z, 0) \geq \rho(z, x_t)\}$, and $\partial S_t \subset \partial\mathbb{H}^n_\mathbb{C}$ denotes its ideal boundary, then the Euclidean diameter of $S_t \cup \partial S_t$ goes to 0 as $t \to \infty$;*

*(ii) If $(z_t)$ is a sequence such that $z_t \in S_t \cup \partial S_t$ for all $t \in \mathbb{N}$, then $z_t \to q$.*

*Proof.* Before we begin the proof proper we remark on the fact that $\{z \in \mathbb{H}^n_\mathbb{C} | \rho(z_1, z) \geq \rho(z_2, z)\} = \Pi_\Sigma^{-1}(\{z \in \Sigma | \rho(z_1, z) \geq \rho(z_2, z)\})$ which is an immediate consequence of the Slice decomposition theorem, for

$$\cosh\left(\frac{\rho(z, \Pi_\Sigma(z))}{2}\right) = \cosh\left(\frac{\rho(z, z_i)}{2}\right)\left(\cosh\left(\frac{\rho(\Pi_\Sigma(z), z_i)}{2}\right)\right)^{-1}$$

For $i = 1, 2$, so that

$$\cosh\left(\frac{\rho(z, z_1)}{2}\right)\cosh\left(\frac{\rho(\Pi_\Sigma(z), z_2)}{2}\right) = \cosh\left(\frac{\rho(z, z_2)}{2}\right)\cosh\left(\frac{\rho(\Pi_\Sigma(z), z_1)}{2}\right)$$

Then

$$\rho(z_1, z) \geq \rho(z_2, z)$$
$$\Updownarrow$$
$$\rho(z_1, \Pi_\Sigma(z)) \geq \rho(z_2, \Pi_\Sigma(z))$$

Now if we fix a point $x_t$, we can assume without loss of generality that it has $(r_t, 0, ..., 0)$ with $0 < r_t < 1$ as its coordinates. With these coordinates we have that the complex spine $\Sigma_t$ of the bisector $\mathfrak{E}\{0, x_t\}$ is equal to the disc $\mathbb{H}^1_{\mathbb{C}} \times \{0\}$, and the orthogonal projection $\Pi_{\Sigma_t} : \mathbb{H}^n_{\mathbb{C}} \to \Sigma_t$ is given by $\Pi_{\Sigma_t}((z_1, ..., z_n)) = (z_1, 0, ..., 0)$. Define $m_t$ as the intersection between $\mathbb{H}^1_{\mathbb{C}} \times \{0\}$ and the real spine $\sigma_t$ of $\mathfrak{E}\{0, x_t\}$ and $M_t = \{(z_1, 0, ..., 0) \in \mathbb{H}^1_{\mathbb{C}} \times \{0\} | Re(z_1) \geq m_t\}$.

As per stated in the initial remark $S_t = \Pi^{-1}_{\Sigma_t}(\{z \in \mathbb{H}^1_{\mathbb{C}} \times \{0\} | \rho(z, 0) \geq \rho(z, x_t)\})$, notice now that

$$\{z \in \mathbb{H}^1_{\mathbb{C}} \times \{0\} | \rho(z, 0) \geq \rho(z, x_t)\} \subset M_t$$
$$\implies S_t \subset \Pi^{-1}_{\Sigma_t}(M_t) = \{(z_1, ..., z_n) \in \mathbb{H}^n_{\mathbb{C}} | Re(z_1) \geq m_t\}$$

This implies that $S_t \cup \partial S_t \subset \{(z_1, ..., z_n) \in \overline{\mathbb{H}^n_{\mathbb{C}}} | Re(z_1) \geq m_t\}$. As $m_t \to 1$ when $t \to \infty$ then the Euclidean diameter of the set $\{(z_1, ..., z_n) \in \overline{\mathbb{H}^n_{\mathbb{C}}} | Re(z_1) \geq m_t\}$ goes to zero. This finishes the proof of statement $(i)$.

For statement $(ii)$ we have that $x_t \to q$ as $t \to \infty$ and that the Euclidean diameter of $S_t \cup \partial S_t$ goes to 0 as $t \to \infty$. As $x_t \in S_t \cup \partial S_t$ for all $t$ this implies that $\lim_{t\to\infty}(S_t \cup \partial S_t) = \{q\}$ proving statement $(ii)$. □

Finally we have all the tools needed for proving proposition 5.5.

**Proposition 5.5.** *[10] If $\{g_m\}$ is a sequence of distinct elements of a discrete subgroup $G$ of $PU(n, 1)$, then there exists a subsequence, still denoted $\{g_m\}$, and elements $x, y \in L(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$, such that $g_m(z) \to x$ uniformly on compact subsets of $\overline{\mathbb{H}^n_{\mathbb{C}}} - \{y\}$.*

*Proof.* We begin by assuming that $g_m(0) \neq 0$ for all $m$. In order to do this we may discard a finite number of elements that fixes the origin. If our sequence fixes the origin for an infinite number of elements then we can find an isometry $A$ of $\mathbb{H}^n_{\mathbb{C}}$ such that $A^{-1}(0)$ is fixed by at most a finite amount of elements of $\{g_m\}$. In this case we would have that $A(0) \neq 0$ and that the behavior of the sequence $\{A^{-1}g_mA\}$ would be identical to the behavior of $\{g_m\}$ in regards to the proposition.

As $\{g_m(0), g_m^{-1}(0) | m \in \mathbb{N}\} \subset \overline{B^n}$ a compact set, then there exists a subsequence of $\{g_m\}$ that we will still denote by $\{g_m\}$ such that $g_m(0) \to x$ and $g_m^{-1}(0) \to y$ as $m \to \infty$.

Define

$$S_{g_m} := \{z \in \mathbb{H}^n_{\mathbb{C}} | \rho(z, 0) \geq \rho(z, g_m^{-1}(0))\}$$

and

$$S_{g_m^{-1}} := \{z \in \mathbb{H}^n_{\mathbb{C}} | \rho(z, 0) \geq \rho(z, g_m(0))\}$$

Then we find ourselves in the same situation of the previous lemma with the Euclidean diameter of the sets $S_{g_m} \cup \partial S_{g_m}$ and $S_{g_m^{-1}} \cup \partial S_{g_m^{-1}}$ going to 0 as $m \to \infty$.

We also have that if $z \in S_{g_m} \cup \partial S_{g_m}$ then $\rho(z, 0) \geq \rho(z, g_m^{-1}(0))$ which implies that $\rho(g_m(z), g_m(0)) \geq \rho(g_m(z), 0)$ so that we have that $g_m(\overline{\mathbb{H}^n_{\mathbb{C}}} - S_{g_m} \cup \partial S_{g_m}) \subset S_{g_m^{-1}} \cup \partial S_{g_m^{-1}}$. Given any $K \subset \overline{\mathbb{H}^n_{\mathbb{C}}} - \{y\}$ compact, there exists $m_K$ such that $K \subset \overline{\mathbb{H}^n_{\mathbb{C}}} - S_{g_m} \cup \partial S_{g_m}$ for all $m \geq m_K$, which means that

$$g_m(K) \subset g_m(\overline{\mathbb{H}^n_{\mathbb{C}}} - S_{g_m} \cup \partial S_{g_m}) \subset S_{g_m^{-1}} \cup \partial S_{g_m^{-1}}$$

For all $m \geq m_K$. The result then follows from statement $(ii)$ of the previous proposition.

$\square$

**Corollary 5.12.** *[10] If $\{g_m\}$ is a sequence of distinct elements of a discrete subgroup $G$ of $PU(n, 1)$, then there exists a subsequence, still denoted $\{g_m\}$, and elements $x, y \in L(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$, such that $g_m(z) \to x$ uniformly on compact subsets of $\overline{\mathbb{H}^n_{\mathbb{C}}} - \{y\}$ and $g_m^{-1}(z) \to y$ uniformly on compact subsets of $\overline{\mathbb{H}^n_{\mathbb{C}}} - \{x\}$*

*Proof.* It suffices to apply proposition 5.5 twice, once for the sequence $\{g_m\}$ and once for the sequence $\{g_m^{-1}\}$ $\square$

**Corollary 5.13.** *If $G$ is a discrete subgroup of $PU(n, 1)$ then $L(G) = \varnothing$ if and only if $G$ is finite.*

*Proof.* By proposition 5.5 if $G$ is infinite then $L(G) \neq \varnothing$. This implies that if $L(G) = \varnothing$ then $G$ is finite.

If $G$ is finite then $|G(p)| < \infty$ so $\overline{G(p)} = G(p) \subset \mathbb{H}^n_{\mathbb{C}}$. Then by definition $L(G) = \overline{G(p)} \cap \partial\mathbb{H}^n_{\mathbb{C}} = \varnothing$. $\square$

**Corollary 5.14.** *If $g$ is an elliptic element of a discrete group of $PU(n, 1)$ then $g$ has finite order.*

*Proof.* Suppose that $g$ was an elliptic element of infinite order of the discrete group $G$ of $PU(n, 1)$ then there exists $z_0 \in \mathbb{H}^n_{\mathbb{C}}$ such that $g(z_0) = z_0$. Consider now the sequence $g_m := g^m$. We have that $g_m(z_0) = z_0$ for all $m \in \mathbb{N}$. As $\{g_m\}$ is a sequence of distinct elements of $G$ by proposition 5.5 there is $x \in \partial\mathbb{H}^n_{\mathbb{C}}$ such that $g_m(z) \to x$ for all $z \in \mathbb{H}^n_{\mathbb{C}}$ as $m \to \infty$, this is absurd as $g_m(z_0) \to z_0$ as $m \to \infty$. Thus $g$ must have finite order. $\square$

## 5.3 DISCRETE FINITE SUBGROUPS OF $PU(n,1)$

In this subsection we will develop an interesting set of tools that will culminate in a proof that any discrete subgroup that has at least two elements in its Chen-Greenberg limit set has a loxodromic element.

### 5.3.1 THE ISOTROPY GROUP OF A BOUNDARY POINT

The second to last corollary of the previous subsection gave us an indication about the possibilities of subgroups of $PU(n,1)$ with empty Chen-Greenberg limit sets. An interesting question that arises is if the size of nonempty Chen-Greenberg limit sets can give more information about its respective groups. First we will present a convenient transformation of the hyperbolic space that will allow us to better study groups that fix at least one point in the boundary of $\mathbb{H}^n_{\mathbb{C}}$, then we will show that if $|L(G)| \geq 2$ then $G$ has a loxodromic element in it, finally we will have all the theory needed to prove the main result of the section.

We begin by focusing on automorphisms of $\mathbb{H}^n_{\mathbb{C}}$ that fixes one point in $\partial\mathbb{H}^n_{\mathbb{C}}$. As the action of $U(n,1)$ is doubly transitive in the boundary the choice of coordinates for the fixed point does not matter as any choice can be made through conjugation by elements of $U(n,1)$. A convenient choice for this is the element $f_1 = (1,0,...,0) \in \partial B^n$. As $f_1$ is the standard lift of $e_0 + e_1$ we have that for $g \in U(n,1)$, $g$ fixes $f_1$ if and only if $g(e_0+e_1) = \lambda(e_0+e_1)$ for some $\lambda \in \mathbb{C}$.[4]

We consider now $\hat{\beta}$ a different basis for $\mathbb{C}^{n,1}$ given by $\hat{e}_0 = \frac{e_0-e_1}{2}, \hat{e}_1 = \frac{e_0+e_1}{2}$ and $\hat{e}_m = e_m$ for $2 \leq m \leq n$. The fact that all the $\hat{e}_m$ are linearly independent is immediate.[4] The matrix that corresponds to this change in basis is given by:

$$D = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}$$

Where $I_{n-1}$ is the $n-1 \times n-1$ identity matrix. We wish to consider now the actions of automorphisms in the hyperbolic space with the basis $\hat{\beta}$, we will accomplish this by considering the group $\hat{U}(n,1) := \{D^{-1}AD | A \in U(n,1)\}$. We consider now the following hermitian form $\Phi(z,w) = -(\overline{z_0}w_1 + \overline{z_1}w_0) + \langle\langle(z_2,...,z_n),(w_2,...,w_n)\rangle\rangle$[4]. We have that

$$\Phi(Dz, Dw) = \Phi\Bigg(\left(\frac{z_0+z_1}{\sqrt{2}}, \frac{-z_0+z_1}{\sqrt{2}}, z_2, ..., z_n\right), \left(\frac{w_0+w_1}{\sqrt{2}}, \frac{-w_0+w_1}{\sqrt{2}}, w_2, ..., w_n\right)\Bigg)$$

$$= -\left(\overline{\left(\frac{z_0+z_1}{\sqrt{2}}\right)}\left(\frac{-w_0+w_1}{\sqrt{2}}\right)+\overline{\left(\frac{-z_0+z_1}{\sqrt{2}}\right)}\left(\frac{w_0+w_1}{\sqrt{2}}\right)\right)$$

$$+ \langle\langle(z_2,...,z_n),(w_2,...,w_n)\rangle\rangle$$

$$= -\frac{1}{2}(-\overline{z_0}w_0+\overline{z_0}w_1-\overline{z_1}w_0+\overline{z_1}w_1-\overline{z_0}w_0-\overline{z_0}w_1+\overline{z_1}w_0+\overline{z_1}w_1)$$

$$+ \langle\langle(z_2,...,z_n),(w_2,...,w_n)\rangle\rangle$$

$$= -\overline{z_0}w_0+\overline{z_1}w_1+\langle\langle(z_2,...,z_n),(w_2,...,w_n)\rangle\rangle = \langle z,w\rangle$$

Thus if $D^{-1}AD \in \hat{U}(n,1)$ then

$$\Phi(D^{-1}AD(z), D^{-1}AD(w)) = \langle D^{-1}A(z), D^{-1}A(w)\rangle$$

$$= \langle D^{-1}(z), D^{-1}(w)\rangle = \Phi(z,w)$$

We can see then that $\Phi$ is the hermitian form preserved by $\hat{U}(n,1)$. The previous calculations also show us that $\hat{U}(n,1)$ is the group of linear transformations which leaves $D^{-1}(V_-)$ invariant. If $C = D^{-1}$ then the image of $C$ in $PSL(n+1)$ is called the Cayley transforms, it maps $B^n$ to the Siegel domain $C(B^n) := \{\zeta \in \mathbb{C}^n | Re(\zeta_1) > \frac{1}{2}\sum_{i=2}^n |\zeta_i|^2\}$. In this way, the action of $\hat{U}(n,1)$ in $C(B^n)$ is identical to the action of $U(n,1)$ in $B^n$ and an element $A \in U(n,1)$ fixes $f_1$ if and only if $D^{-1}AD(\hat{e}_1) = t\hat{e}_1$ for some $t \in \mathbb{C}$[4].

**Lemma 5.15.** *[4] [7] Let $A$ be an element in $\hat{U}(n,1)$ such that $\hat{A}(\hat{e}_1) := D^{-1}AD(\hat{e}_1) = t\hat{e}_1$ for some $t \in \mathbb{C}$. Then in the $\hat{\beta}$ basis*

$$\hat{A} = \begin{pmatrix} \mu & 0 & 0 \\ s & \lambda & b \\ a & 0 & A' \end{pmatrix}$$

*Where $\mu, \lambda, s \in \mathbb{C}$, $a$ is a $(n-1)\times 1$ matrix, $b$ is a $1\times(n-1)$ matrix and $A' \in U(n-1)$. Furthermore $\overline{\mu}\lambda = 1$, $Re(\overline{\mu}s) = \frac{1}{2}|a|^2$ (where $|a|$ is the Euclidean norm of $a$) and $b = \lambda\overline{a}^T A$ (where $T$ denotes the transpose).*

*Proof.* The proof will follow from the fact that $\hat{A}$ preserves the hermitian form $\Phi$ and that $\hat{A}(\hat{e}_1) = t\hat{e}_1$. Let $\hat{A}$ be

$$\hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1(n+1)} \\ \vdots & \ddots & \vdots \\ a_{(n+1)1} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}$$

In this notation we have that $a_{11} = \mu, a_{21} = s, a_{22} = \lambda, a^T = (a_{31} \quad \cdots \quad a_{(n+1)1})$, $b = (a_{13} \quad \cdots \quad a_{1(n+1)})$ . As $\hat{A}$ preserves the hermitian form $\Phi$, we can derive the following equations for $i = 1, 2$ and $j = 3, ..., n+1$:

$$\Phi(\hat{A}^T(\hat{e}_i), \hat{A}^T(\hat{e}_i)) = \Phi(\hat{e}_i, \hat{e}_i) = 0 \implies -2Re(\overline{a}_{i1}a_{i2}) + \sum_{k=3}^{n+1} |a_{ik}|^2 = 0 \tag{3}$$

$$\Phi(\hat{A}^T(\hat{e}_j), \hat{A}^T(\hat{e}_j)) = \Phi(\hat{e}_j, \hat{e}_j) = 1 \implies -2Re(\overline{a}_{j1}a_{j2}) + \sum_{k=3}^{n+1} |a_{jk}|^2 = 1 \tag{4}$$

$$\Phi(\hat{A}(\hat{e}_i), \hat{A}(\hat{e}_i)) = \Phi(\hat{e}_i, \hat{e}_i) = 0 \implies -2Re(\overline{a}_{1i}a_{2i}) + \sum_{k=3}^{n+1} |a_{ki}|^2 = 0 \tag{5}$$

$$\Phi(\hat{A}(\hat{e}_j), \hat{A}(\hat{e}_j)) = \Phi(\hat{e}_j, \hat{e}_j) = 1 \implies -2Re(\overline{a}_{j1}a_{j2}) + \sum_{k=3}^{n+1} |a_{kj}|^2 = 1 \tag{6}$$

$$\Phi(\hat{A}(\hat{e}_1), \hat{A}(\hat{e}_2)) = \Phi(\hat{e}_1, \hat{e}_2) = -1 \implies -(\overline{a}_{11}a_{22} + \overline{a}_{21}a_{12}) + \sum_{k=3}^{n+1} \overline{a}_{k1}a_{k2} = -1 \tag{7}$$

$$\Phi(\hat{A}(\hat{e}_1), \hat{A}(\hat{e}_j)) = \Phi(\hat{e}_1, \hat{e}_j) = 0 \implies -(\overline{a}_{11}a_{2j} + \overline{a}_{21}a_{1j}) + \sum_{k=3}^{n+1} \overline{a}_{k1}a_{kj} = 0 \tag{8}$$

Immediately from equation (5) we obtain that for $i = 1$ we have that $Re(\overline{a}_{11}a_{21}) = \frac{1}{2}\sum_{k=3}^{n+1} |a_{k1}|^2$ so that $Re(\overline{\mu}s) = \frac{1}{2}|a|^2$ holds.

As $\hat{A}(\hat{e}_1) = t\hat{e}_1$ then $\hat{A}(\hat{e}_1) = (a_{12} \quad \cdots \quad a_{(n+1)2}) = (0 \quad t \quad 0 \quad \cdots \quad 0)$ so we conclude that $a_{k2} = 0$ for all $k = 1, 3, 4, ..., n+1$, let us call this result $(*)$. Applying $(*)$ to equation (3) for $i = 1$ we obtain that $a_{12} = 0$ so that $\sum_{k=3}^{n+1} |a_{ik}|^2 = 0$ which implies that $a_{1k} = 0$ for all $k = 2, 3, ..., n+1$ let us call this result $(**)$. Applying $(*)$ to equation (7) we obtain that $\overline{a}_{11}a_{22} = 1$ let us call this result $(***)$.

In order to show that $b = \lambda\overline{a}^T A$ notice that

$$b = \lambda\overline{a}^T A$$

$$\Updownarrow$$

$$\begin{pmatrix} a_{13} & \cdots & a_{1(n+1)} \end{pmatrix} = \begin{pmatrix} a_{22}\overline{a}_{31} & \cdots & a_{22}\overline{a}_{(n+1)1} \end{pmatrix} \begin{pmatrix} a_{33} & \cdots & a_{3(n+1)} \\ \vdots & \ddots & \vdots \\ a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a_{13} & \cdots & a_{1(n+1)} \end{pmatrix} = \begin{pmatrix} a_{22}\sum_{k=3}^{n+1}\overline{a}_{k1}a_{k3} & \cdots & a_{22}\sum_{k=3}^{n+1}\overline{a}_{k1}a_{k(n+1)} \end{pmatrix}$$

$$\Leftrightarrow a_{2j} = a_{22}\sum_{k=3}^{n+1}\overline{a}_{k1}a_{kj}$$

By applying $(**)$ to equation (8) we can see that $a_{2j} = (\overline{a}_{11})^{-1}\sum_{k=3}^{n+1}\overline{a}_{k1}a_{kj}$. Applying $(***)$ to this last equality we obtain that $a_{2j} = a_{22}\sum_{k=3}^{n+1}\overline{a}_{k1}a_{kj}$, then $b = \lambda\overline{a}^T A$ holds.

Finally, to show that $A' \in U(n-1)$ take an element $r = (r_2, ..., r_n) \in \mathbb{C}^{n-1}$ and consider the element $\hat{r} = (0, 0, r_2, ..., r_n)$ in $\mathbb{C}^{n,1}$. We have that $\Phi(\hat{A}(\hat{r}), \hat{A}(\hat{r})) = \langle\langle A'r, A'r\rangle\rangle = \Phi(\hat{r}, \hat{r}) = \langle\langle r, r\rangle\rangle$ so $A' \in U(n-1)$. $\square$

By giving us such a rigid shape for our matrices this lemma will facilitate tremendously the calculations in the rest of the section. We take immediate advantage of this with the following lemma.

**Lemma 5.16.** *[4] Let $A$ be an elliptic element in $U(n,1)$ of finite order, and let $p, q \in \partial\mathbb{H}^n_{\mathbb{C}}$. If $A$ leaves $p$ and $q$ fixed, then it fixes every point on the geodesic $[p,q]$.*

*Proof.* As the action of $U(n,1)$ is doubly transitive in the boundary we can assume without loss of generality that $p = f_1$ and $q = -f_1 = (-1, 0, ..., 0)$. Applying lemma 5.15 we have that with respect to the basis $\hat{\beta}$ our transformation has the following matrix form

$$\hat{A} = \begin{pmatrix} \mu & 0 & 0 \\ s & \lambda & b \\ a & 0 & A' \end{pmatrix}$$

As $q = \hat{e}_0$ in the $\hat{\beta}$ basis and $\hat{A}$ fixes it, we obtain that $s = 0$ and $a = 0$, so that our matrix can be written as follows

$$\hat{A} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & A' \end{pmatrix}$$

Because $\hat{A}$ has finite order then $|\mu| = 1 = |\lambda|$ so $\mu = \lambda$. We can define now $W = \hat{e}_0\mathbb{R} + \hat{e}_1\mathbb{R}$. Given $w \in W$, we have $w = r_0\hat{e}_0 + r_1\hat{e}_1$, for some $r_0, r_1 \in \mathbb{R}$, so $\hat{A}(w) = \hat{A}(r_0\hat{e}_0 + r_1\hat{e}_1) = r_0\hat{A}(\hat{e}_0) + r_1\hat{A}(\hat{e}_1) = r_0 t\hat{e}_0 + r_1 t\hat{e}_1$. This means that when we consider the action in the projective space $\hat{A}(\mathbb{P}(w)) = \mathbb{P}(w)$, so that the whole of $\mathbb{P}(W)$ is pointwise fixed by $\hat{A}$. Finally notice that $[p,q] = [f_1, f_{-1}] = \mathbb{P}(W \cap V_-)$ so the lemma holds. $\square$

### 5.3.2 LOXODROMIC ELEMENTS IN DISCRETE GROUPS

After the considerations in the previous part we wish now to prove that if $G$ is a discrete subgroup of $PU(n,1)$ and $L(G) > 1$ then $G$ has at least one loxodromic element. First we state the following

**Proposition 5.17.** *If $A$ is a parabolic or loxodromic element of $PU(n,1)$ then $A$ has infinite order.*

Before we proceed we need a definition:

**Definition 5.18.** *[7] Let $x$ and $y$ be any two not necessarily distinct points in $\partial\mathbb{H}^n_{\mathbb{C}}$. If there exists a sequence $\{g_m\}$ of elements of $G$, a discrete subgroup of $PU(n,1)$, such that $\lim_{m\to\infty} g_m(p) = x$ and $\lim_{m\to\infty} g_m^{-1}(p) = y$ for any point $p \in B^n$, then we say that $x$ and $y$ are $G$-dual and denote this duality by $x \sim^d y$.*

This definition is closely related to proposition 5.5 which can be restated as "Given any sequence of elements in $G$ a discrete subgroup of $PU(n,1)$ there is a subsequence for which $x \sim^d y$.".

**Proposition 5.19.** *[7][10] Suppose that $G$ is a discrete subgroup of $PU(n,1)$ such that $x \sim^d y$ with respect to $G$ and $x \neq y$. Then there exists a loxodromic element of $G$.*

*Proof.* Let $U$ be an open neighborhood of $x$ and $V$ an open neighborhood of $y$ such that $\overline{U} \cap \overline{V} = \varnothing$. This means that the geodesic $[x,y]$ is not entirely contained in $\overline{U} \cup \overline{V}$. As $x \sim^d y$ there exists a sequence such that $\lim_{m\to\infty} g_m(p) = x$ and $\lim_{m\to\infty} g_m^{-1}(p) = y$ for any point $p \in B^n$. Applying corollary 5.12 we can see that there is a subsequence still denoted $\{g_m\}$ such that $g_m(z) \to x$ uniformly on compact subsets of $\overline{B^n} - \{y\}$ and $g_m^{-1}(z) \to y$ uniformly on compact subsets of $\overline{B^n} - \{x\}$. As $\overline{B^n} - V \subset \overline{B^n} - \{y\}$ is a compact subset we have that $\overline{B^n} - V$ converges uniformly to $\{x\}$ by $\{g_m\}$ so , there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, $g_m(\overline{B^n} - V) \subset U$. Thus for any $m \geq m_0$, $g_m(\overline{U} \cap \partial B^n) \subset U \cap \partial B^n$. By the Brouwer fixed point theorem, we have then that for any $m \geq m_0$, $g_m$ has a fixed point in $U \cap \partial B^n$. An analogous argument shows that there exists a number $m_1 \in \mathbb{N}$ such that for any $m \geq m_1$ then $g_m^{-1}$ has a fixed point in $V \cap \partial B^n$. This means that if $g_M$ is an element of the subsequence such that $M > m_0, m_1$ then $g_M$ fixes two points in the boundary. This means that $g_M$ must be either elliptic or loxodromic, but by lemma 5.16 if $g_M$ was elliptic, it would fix the geodesic $[x,y]$. As $[x,y] \cap (\overline{B^n} - \overline{U} \cup \overline{V}) \neq \varnothing$ this contradicts the fact that $g_M(\overline{B^n} - V) \subset U$ for any point $p \in [x,y] \cap (\overline{B^n} - \overline{U} \cup \overline{V})$ is fixed by $G_M$ and thus does not belong to $U$. So $g_M$ must be loxodromic. □

**Theorem 5.20.** *[6] [7] Let $G$ be a discrete subgroup of $U(n,1)$, then:*

*$(i)$ $G$-dual point $x$ and $y$ belong to the limit set $L(G)$;*

*$(ii)$ If $x \in L(G)$, then there is some point $y \in L(G)$ such that $x \sim^d y$;*

*$(iii)$ Denote $D(x) := \{y \in L(G) | x \sim^d y\}$. $D(X)$ is closed and $G$-invariant. If $|D(x)| \geq 2$ then $D(x) = L(G)$;*

*$(iv)$ If $|L(G) = 1$, then the point in $L(G)$ is $G$-dual to itself. If $|L(G)| \geq 2$, then any two points in $L(G)$ are $G$-dual.*

*Proof.* $(i)$ Is an immediate consequence of the definition of $L(G)$;

$(ii)$ Is an immediate consequence of corollary 5.12;

$(iii)$ Suppose that there is a sequence $\{y_j\} \subset D(x)$ such that $y_j \to y$ as $j \to \infty$. Since $L(G)$ is closed, $y \in L(G)$. Now for each $j$ there exists a sequence $\{g_m^j\} \subset G$ such that $\lim_{m\to\infty} g_m^j(p) = x$ and $\lim_{m\to\infty}(g_m^j)^{-1}(p) = y_j$ for any point $p \in B^n$. We take the sequence $\{g_m := g_m^m\}$ and then $\lim_{m\to\infty} g_m(p) = \lim_{m\to\infty} g_m^m(p) = x$ and $\lim_{m\to\infty} g_m^{-1}(p) = \lim_{m\to\infty}(g_m^m)^{-1}(p) = \lim_{m\to\infty} y_m = y$, so $x \sim^d y$ and we conclude $D(x)$ is closed.

Now let us see that it is $G$-invariant. Take $y \in D(x)$ and let $\{g_m\} \subset G$ be a sequence that satisfy the condition for duality between $x$ and $y$ and let $g \in G$. We have that for some $p \in B^n$, $g_m(p) \to x$ and $g_m^{-1} \to y$ as $m \to \infty$. Consider now the point $g^{-1}(p)$ and the sequence $\{g_m g^{-1}\} \subset G$, then $(g_m g^{-1})^{-1}(p) = g(g_m^{-1}(p)) \to g(y)$ and $g_m(g^{-1}(p)) \to x$ by proposition 5.1, which implies that $x \sim^d g(y)$ so that $g(y) \in D(x)$. Thus $D(x)$ is $G$-invariant and we conclude that is closed and $G$-invariant.

From statement $(i)$ we already have that $D(x) \subset L(G)$, now if $|D(x)| > 1$, by proposition 5.3 we have that $L(G) \subset D(x)$, so $L(G) = D(x)$.

$(iv)$ We divide the proof of this statement in three cases, $L(G) = 1, L(G) = 2$ and $L(G) > 2$.

If $L(G) = 1$ the result is immediate from corollary 5.12.

If $L(G) = 2$ assume $L(G) = \{x, y\}$. As $L(G)$ is $G$-invariant, $G$ cannot contain a parabolic element because for all $g \in G$, either $g(x) = y$ or $g(y) = x$ and in this case if $g$ was parabolic it would fix some other point $z \in \partial B^n$ and thus we would have $|L(G)| > 2$ a contradiction, or $g(x) = x$ and $g(y) = y$ and in this case $g$ could not be parabolic as all parabolic elements fix a single boundary point.

If $h \in G$ is a loxodromic element then of course $\{x, y\}$ must be fixed by it, otherwise $|L(G)| > 2$, and in this case the sequence $\{g^m\}$ would satisfy the condition for $x \sim^d y$. Thus to prove the statement, it suffices to show that there cannot be a discrete subgroup of $U(n, 1)$ composed only of the identity and elliptic elements.

Assume then that $G$ was such a subgroup and define $G_{x,y} := \{g \in G | g(x) = x \text{ and } g(y) = y\}$. If $G_{x,y}$ was infinite, then on one hand, by corollary 5.12 we would have a sequence $\{g_m\}$ such that $g_m(z) \to x$ uniformly on compact subsets of $\overline{B^n} - \{y\}$ and $g_m^{-1}(z) \to y$ uniformly on compact subsets of $\overline{B^n} - \{x\}$, on the other hand by lemma 5.16 we would have that $g_m$ would fix every point of the geodesic $[x, y]$, a contradiction. Thus $G_{x,y}$ is finite. Let us show now that $G - G_{x,y}$ is finite, suppose not, then there is a sequence of distinct elements $\{h_j\} \subset G - G_{x,y}$ such that $h_j(x) = y$ and $h_j(y) = x$ for all $j \in \mathbb{N}$. Consider now the sequence given by $\{f_j := h_1 h_j\}$, we have that $f_j(x) = h_1 h_j(x) = h_1(y) = x$ for all $j \in \mathbb{N}$ which implies $\{f_j\} \subset G_{x,y}$ and thus that $\{h_1, h_2, ...\}$ is a finite set, a contradiction. Hence $G - G_{x,y}$ is finite. As $G = G_{x,y} \cup (G - G_{x,y})$ we conclude $G$ is finite, a contradiction, as all finite sets have empty Chen-Greenberg limit sets by corollary 5.13. Hence if $L(G) = \{x, y\}$, then $G$ has at least one loxodromic element that satisfy the

$G$-duality definition for $x, y$ so that $x \sim^d y$.

Finally we assume $|L(G)| > 2$. Notice that if $|L(G)| > 2$ then there can is at most one element in $L(G)$ that is fixed by all elements of $G$. Suppose that this did not happen, then we would have at least two points $x, y$ fixed by all elements of $G$. Define $D := \{x, y\}$, by proposition 5.3 $L(G) \subset D$ a contradiction for $|L(G)| > 2$. Hence we can choose $\zeta$ a point in $L(G)$ which is not fixed by all elements of $G$, let us call $f$ one such element. Using statement $(ii)$ there is a point $\eta$ such that $\eta \sim^d \zeta$ (notice that we may have $\eta = \zeta$). By statement $(iii)$ we have that $D(\eta)$ is $G$-invariant so $f(\zeta) \in D(\eta)$ and $f(\zeta) \neq \zeta$. Then $|D(\eta)| > 1$ and we conclude that $D(\eta) = L(G)$, that is $\eta$ is $G$-dual to every point in $L(G)$.

Now choose $x, y \in L(G)$ such that $\eta, x$ and $y$ are all distinct and $W, U$ and $V$ are disjoint open neighborhoods of $\eta, x$ and $y$ respectively. By proposition 5.19 there are loxodromic elements $g, f \in G$ such that $g$ has fixed points in $U$ and $W$ and $h$ has fixed points in $V$ and $W$. If we show that $g$ and $f$ cannot have a fixed point in common in $W$ then either $g$ does not fix $\eta$ or $f$ does not fix $\eta$, so that (assume without loss of generality that $g(\eta) \neq \eta$) $\eta, g(\eta) \in D(x) \implies D(x) = L(G) = D(\eta)$ and we conclude any two points in $L(G)$ are $G$-dual.[7] Thus all we have to do now is show that $g$ and $f$ cannot have a fixed point in common in $W$.

Let us show then that if $g$ and $f$ have a common fixed point then $G$ is not discrete. Assume that $g$ and $f$ fix the same point. By lemmas 5.15 and 5.16 we can write both elements in matrix form with respect to the $\hat{\beta}$ basis as

$$g = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & A \end{pmatrix}, f = \begin{pmatrix} \psi & 0 & 0 \\ s & \phi & b \\ a & 0 & B \end{pmatrix}$$

And the commutator of $f$ and $g^m$ as

$$fg^m f^{-1} g^{-m} := h_m = \begin{pmatrix} 1 & 0 & 0 \\ \alpha^m & 1 & A_m \\ B_m & 0 & C_m \end{pmatrix}$$

Where

$$\alpha^{(m)} = s\psi^{-1} - \lambda^m s\psi^{-1}\mu^{-m} + \lambda^m bB^{-1}a\psi^{-1}\mu^{-m} - bA^m B^{-1}a\phi^{-1}\mu^{-m}$$

$$A_m = -\lambda^m bB^{-1}A^{-m} + bA^m B^{-1}A^{-m}$$

$$B_m = a\psi^{-1} - BA^m B^{-1}a\psi^{-1}\mu^{-m}$$

$$C_m = BA^m B^{-1}A^{-1}$$

First let us see that all the $h_m$ are pairwise distinct. Suppose that $h_k = h_{k'}$ with $k \neq k'$.

Then if $k'' = k - k'$, we have that $fg^{k''}f^{-1}g^{-k''} = I$ (where $I$ is the identity matrix). So, from the equations above we have that

$$B_{k''} = a\psi^{-1} - BA^{k''}B^{-1}a\psi^{-1}\mu^{-k''} = 0$$

$$\implies B^{-1}a\psi^{-1} = A^{k''}B^{-1}a\psi^{-1}\mu^{-k''}$$

Substituting this relation in our equation for $\alpha_{21}^{(k'')}$, and remembering that $\overline{\mu}\lambda = 1$, we can express it in terms of $\mu, \psi, a, b$ and $B$ such that

$$\alpha_{21}^{(k'')} = s\psi^{-1} - \overline{\mu}^{-k''}s\psi^{-1}\mu^{-k''} + \overline{\mu}^{-k''}bB^{-1}a\psi^{-1}\mu^{-k''} - bB^{-1}a\phi^{-1} = 0$$

Comparing real parts in the equation above we have that

$$Re(\alpha_{21}^{(k'')}) = Re(s\psi)|\psi|^{-2} - Re(s\psi)|\psi|^{-2}|\mu|^{-2k''} - (1 - |\mu|^{-2k''})Re(\phi\overline{a}^T BB^{-1}a\psi^{-1})$$

$$\left(\frac{1}{2}\right)|a|^2|\psi|^{-2}(1 - |\mu|^{-2m}) - |a|^2|\psi|^{-2}(1 - |\mu|^{-2m})$$

$$= \left(-\frac{1}{2}\right)|a|^2|\psi|^{-2}(1 - |\mu|^{-2m}) = 0$$

Thus we conclude that either $|\mu| = 1$ or $a = 0$. However if $|\mu| = 1$ then $g$ would not be loxodromic for we would have $\mu = \lambda$ and so $g$ would fix the entire geodesic connecting the points $[\hat{e}_0, \hat{e}_1]$ meaning it would be elliptic. It follows that $a = 0$ and then $\alpha_{21}^{(k'')} = s\psi^{-1} - Re(s\psi)|\psi|^{-2}|\mu|^{-2k''} = 0$. As $|\mu| \neq 1$ we conclude that $s = 0$ and therefore we would have that $f$ would fix $\hat{e}_0$ so $g$ and $f$ would have two fixed point in common which contradicts our hypothesis. Thus all the $h_m$ are distinct.

We can assume, taking a subsequence if necessary, that there exists $\alpha \in \mathbb{C}$ and $\alpha' \in \mathbb{C}$ such that $\alpha_{11}^{(m)} \to \alpha$ and $\alpha_{22}^{(m)} \to \alpha'$ as $m \to \infty$ (if $|\mu| > 1$) or as $m \to -\infty$ (if $|\mu| < 1$).

Assume $|\lambda| < 1$, as $U(n-1)$ is compact, we can assume that $C_m \to C \in U(n-1)$ as $m \to \infty$.And with this we can conclude that $A_m \to bB^{-1}C$ because $-\lambda^m bB^{-1}A^{-m} \to \overline{0}$ and $BA^mB^{-1}A^{-m} \to C \implies A^mB^{-1}A^{-m} \to B^{-1}C$ as $m \to \infty$. Similarly we also have that $\alpha^m \to s\psi^{-1}$ and $B_m \to a\psi^{-1}$ as $m \to \infty$, hence as $m \to \infty$:

$$h_m \longrightarrow h = \begin{pmatrix} 1 & 0 & 0 \\ s\psi^{-1} & 1 & bB^{-1}C \\ a\psi^{-1} & 0 & C \end{pmatrix}$$

Notice now that $h \in \hat{U}(n,1)$ as $\Phi(hz, hw) = (z, w)$ for all $z, w$. Let us show this by calculating the hermitian form of each line of $h$:

$$\Phi((1,0,...,0),(1,0,...,0)) = 0$$

$$\begin{aligned}\Phi((s\psi^{-1},1,bB^{-1}C),(s\psi^{-1},1,bB^{-1}C)) &= -(\overline{s\psi^{-1}}+s\psi^{-1}) + \langle\langle bB^{-1}C, bB^{-1}C\rangle\rangle \\ &= -2Re(s\psi^{-1}) + \langle\langle b,b\rangle\rangle \\ &= -2Re(s\psi^{-1}) + \langle\langle \overline{\psi a^T}, \overline{\psi a^T}\rangle\rangle \\ &= -2Re(s\psi^{-1}) + |\psi a|^2 = 0\end{aligned}$$

Write $a^T = (a_1 \quad \cdots \quad a_{n-1})$ and $c_j$ for line $j$ of $C$, $j = 1,...,n-1$ then

$$\Phi((a_j,0,c_j),(a_j,0,c_j)) = \langle\langle c_j, c_j\rangle\rangle > 0$$

for $C \in U(n-1)$ Hence $h$ preserves each of the elements of $\hat{\beta}$ so $h \in \hat{U}(n,1)$ and $G$ is not discrete. The calculation are identical if $|\lambda| > 1$, except that we take our sequence with $m \to \infty$.

Therefore we conclude that $g$ and $f$ do not have a common fixed point in $W$, otherwise $G$ would not be a discrete group and the proof of the theorem is finished. □

**Corollary 5.21.** *[10] If $|L(G)| > 1$ then $G$ has a loxodromic element.*

*Proof.* By theorem 5.20 there are points $x, y \in L(G)$ such that $x \sim^d y$ and $x \neq y$. By proposition 5.19 then $G$ has a loxodromic element. □

**Remark.** *[7] The condition that $|D(x)| > 1$ in statement $(iii)$ of theorem 5.20 is necessary as if $G = \langle g\rangle$, $g$ a loxodromic element, then we have that $D(x) = \{y\}$, $D(y) = x$ and $L(G) = \{x, y\}$.*

We may now present the final result of this chapter.

**Theorem 5.22.** *[3] If $G$ is a discrete subgroup of $PU(n,1)$ then $G$ is finite if and only if all of its elements have finite order.*

We will not present a proof for this theorem, but one can be found in [3]. The authors in [3] were able to prove the theorem for any discrete subgroup of $PSL(n+1,\mathbb{C})$, however the proof involves theory outside the scope of the present thesis. We remark however that given the results presented so far, we are able to show that if a subgroup $G$ is such that $|L(G)| > 1$ then $G$ has a loxodromic element and thus cannot be finite or have only elements of finite order. We were also able to prove that if $L(G) = \varnothing$ then $G$ is finite. Hence the only case that needs further verification in order to prove the theorem is the case $|L(G)| = 1$. In [10] he was able to show that in $PU(2,1)$ if $|L(G)| = 1$ then $G$ has a parabolic element. So far we were unable to generalize his proof to the higher dimensional case, but we still believe that with the theory developed so far it is possible.

# 6 EQUALITY OF LIMITS

We open this section by making a few observations about discrete subgroups of $PU(n,1)$ and discontinuous action.

**Definition 6.1.** *[9] We say that a subgroup $G$ of $PU(n,1)$ acts discontinuously at a point $x \in \mathbb{H}^n_{\mathbb{C}}$ if there is a neighborhood $U$ of $x$, so that $g(U) \cap U = \varnothing$ for all but finitely many $g \in G$.*

**Proposition 6.2.** *[9] Let $x$ be a point of $B^n$, and let $G$ be a subgroup of $PU(n,1)$. Then $G$ acts discontinuously at $x$ if and only if $G$ is a discrete subgroup of $PU(n,1)$.*

*Proof.* Assume $G$ not discrete and let $\{g_m\} \subset G$ be a sequence such that $g_m \to g \in G$ as $m \to \infty$. Consider now the sequence given by $\{f_m := g_m g^{-1}\}$. It is clear that $g_m g^{-1}(x) \to x$ as $m \to \infty$ so that given any neighborhood $U$ of $x$ there is some $M \in \mathbb{N}$ such that if $m > M$ then $g_m g^{-1}(x) \in U$. This means that $G$ does not act discontinuously at $x$.

Let $G$ now be a discrete subgroup of $PU(n,1)$ and $\{g_m\}$ any sequence of distinct elements of $G$. Assume now that there was some sequence of $G$ and a neighborhood $U$ of $x$ such that $g_m(U) \cap U \neq \varnothing$ for a infinite amount of $g_m$. We take the subsequence consisting only of the elements $g_m$ such that $g_m(U) \cap U \neq \varnothing$ (we will call this subsequence $\{g_k\}$). As $\overline{U}$ is a compact set, by corollary 5.12 we have that there exists a subsequence of $\{g_k\}$ such that $g_k(z) \to y$ uniformly for some $y \in \partial B^n$ a contradiction. Let $2\epsilon > 0$ be the Euclidean distance between $y$ and $U$, $g_m(U) \cap U \neq \varnothing$ implies that for all $m$ there exists a point $u_m \in U$ such that the Euclidean distance of $g_m(u)$ to $y$ is greater than $\epsilon$ a contradiction with the uniform convergence of the sequence $\{g_k\}$. $\square$

**Corollary 6.3.** *[9] Let $G$ be a subgroup of $PU(n,1)$. Then $G$ acts discontinuously at some point of $B^n$ if and only if $G$ acts discontinuously at every point of $B^n$*

*Proof.* If $G$ acts discontinuously at some point of $B^n$ then $G$ is not discrete so $G$ acts discontinuously at every point of $B^n$ by proposition 6.2. The converse is immediate. $\square$

**Lemma 6.4.** *[10] If $G$ is a discrete subgroup of $PU(n,1)$ then $\mathbb{H}^n_{\mathbb{C}} \subset \Omega(G)$*

*Proof.* We begin by noticing that $L_0(G) \cap \mathbb{H}^n_{\mathbb{C}} = \varnothing$ and $L_1(G) \cap \mathbb{H}^n_{\mathbb{C}} = \varnothing$ as a consequence of proposition 6.2, for if any point $x \in B^n$ was a cluster point with respect to $G$ then $G$ would not act discontinuously at $x$ and so would not be discrete.

Hence it suffices to prove that $L_2(G) \cap \mathbb{H}^n_{\mathbb{C}} = \varnothing$ to show that $\mathbb{H}^n_{\mathbb{C}} \subset \Omega(G)$. Assume there exists $K \subset \mathbb{P}^n_{\mathbb{C}} - (L_0(G) \cup L_0(G))$ discrete such that $x \in \mathbb{H}^n_{\mathbb{C}}$ is a cluster point of $\{g_m(K)\}$ for some sequence of distinct elements of $G$. As all elements of $PU(n,1)$ leave $\overline{\mathbb{H}^n_{\mathbb{C}}}$ invariant we may assume that $K \subset \mathbb{H}^n_{\mathbb{C}}$. Let $U$ be any neighborhood of $x$, then $U$ has nonempty intersection with an infinite amount of $\{g_m(K)\}$. Define the subsequence $\{g_{m_U}\}$

of all the elements such that $g_m(K) \cap U \neq \varnothing$. By proposition 5.5 there is a subsequence of $\{g_{m_U}\}$ that converges uniformly to a boundary point a contradiction with the fact that $g_m(K) \cap U \neq \varnothing$ for all elements of the subsequence. Thus $L_2(G) \cap \mathbb{H}^n_{\mathbb{C}} = \varnothing$ and so $\mathbb{H}^n_{\mathbb{C}} \cap \Lambda(G) = \mathbb{H}^n_{\mathbb{C}} \cap (L_0(G) \cup L_1(G) \cup L_2(G)) = (\mathbb{H}^n_{\mathbb{C}} \cup L_0(G)) \cap (\mathbb{H}^n_{\mathbb{C}} \cup L_1(G)) \cap (\mathbb{H}^n_{\mathbb{C}} \cup L_2(G)) = \varnothing \implies \mathbb{H}^n_{\mathbb{C}} \subset \Omega(G)$. □

**Corollary 6.5.** *If $G$ is a discrete subgroup of $PU(n,1)$ then $G$ has the Kleinian property.*

*Proof.* By lemma 6.4 $\mathbb{H}^n_{\mathbb{C}} \subset \Omega(G) \implies \Omega(G) \neq \varnothing$ so $G$ has the Kleinian property. □

These results already point in the direction of the main result of this work, as they indicate that cluster points cannot exist in $\mathbb{H}^n_{\mathbb{C}}$ for discrete subgroups of $PU(n,1)$. Let us show now a proposition regarding the convergence of points in $\mathbb{P}^n_{\mathbb{C}} - \overline{\mathbb{H}^n_{\mathbb{C}}}$.

**Definition 6.6.** *[5] Let $x \in \mathbb{C}^{n,1} - V_0$ then we define $\mathbb{P}(\langle x \rangle^\perp)$ as the polar hyperplane to $\mathbb{P}(x)$. If $y \in V_0$ we call $\mathbb{P}(\langle y \rangle^\perp)$ the tangent polar hyperplane at $\mathbb{P}(y)$.*

**Proposition 6.7.** *If $y \in V_0$ then $\mathbb{P}(\langle y \rangle^\perp) \cap \overline{\mathbb{H}^n_{\mathbb{C}}} = \mathbb{P}(y)$.*

This statement will be familiar to anybody that has already studied hyperbolic geometry, it suffices to say that the orthogonal subspace given by an isotropic point must be degenerate.

**Proposition 6.8.** *[10] Let $\{w_m\} \subset \mathbb{P}^n_{\mathbb{C}} - \overline{\mathbb{H}^n_{\mathbb{C}}}$ be a sequence such that $w_m \to w$ as $m \to \infty$. Define $W_m$ as the intersection between $\overline{\mathbb{H}^n_{\mathbb{C}}}$ and the polar hyperplane to $w_m$. Let $\{v_m\}$ be a sequence such that $v_m \in W_m$ for all $m$ and $v_m \to v$ as $m \to \infty$. Then:*

*$(i)$ If $w \in \partial\mathbb{H}^n_{\mathbb{C}}$ then $w = v$;*

*$(ii)$ If $w \in \mathbb{P}^n_{\mathbb{C}} - \overline{\mathbb{H}^n_{\mathbb{C}}}$ then $v \in W \cup \partial W$, where $W$ denotes the polar hyperplane to $w$. In particular, if $v \in \partial\mathbb{H}^n_{\mathbb{C}}$ then $w \in V$ where $V$ is the polar hyperplane tangent to $v$.*

*Proof.* Notice first that if we take the standard lift of $w_m$ and $v_m$ we have that $\langle v_m, w_m \rangle = 0$ for all $m \in \mathbb{N}$ so that $\langle v, w \rangle = 0$.

Let us prove now $(i)$. If $w \in \partial\mathbb{H}^n_{\mathbb{C}}$ then we have that $v$ belongs to the polar hyperplane tangent at $w$. As the intersection of the polar hyperplane tangent at $w$ with $\overline{\mathbb{H}^n_{\mathbb{C}}}$ is $\{w\}$ by proposition 6.7 and all the $W_m$ are contained in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ then $v \in \partial\mathbb{H}^n_{\mathbb{C}}$ and so $v = w$.

Now let us prove statement $(ii)$. Then, by the initial observation, we have that $v \in W \cup \partial W$. If $v \in \partial\mathbb{H}^n_{\mathbb{C}}$ then $v \in \partial W$ hence $w \in V$. □

We now have all the necessary tools needed to prove the main result of this work, so we begin:

**Proposition 6.9.** *[10] If $G$ is a discrete subgroup of $PU(n,1)$ then $L(G) = L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$.*

*Proof.* Let us divide the proof in three different cases.

First assume that $L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} = \varnothing$. Then no point in $\partial\mathbb{H}^n_{\mathbb{C}}$ has infinite isotropy group with respect to $G$. In particular this means that all $g \in G$ have finite order, otherwise there would be a point in the boundary fixed by an infinite amount of elements of $g$ (all infinite order elements are either parabolic or loxodromic and both of these cases fixes points in the boundary). By theorem 5.22 this means that $G$ is finite, hence $L(G) = \varnothing$. Thus we conclude $L(G) = \varnothing = L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$.

The second case is if $|L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}| = 1$. Let $L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} = \{x\}$, by corollary 5.12 there is a sequence of elements of $G$ such that $g_m(z) \to x$ uniformly in compact subsets of $\overline{\mathbb{H}^n_{\mathbb{C}}} - \{y\}$ for some $y \in L(G)$ then $x \in L(G)$ and so $L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} \subseteq L(G)$. Assume now that $|L(G)| > 1$ then, by corollary 5.21 $G$ has a loxodromic element. Let $z, w$ $(z \neq w)$ be the points fixed by this loxodromic element. As any loxodromic element has infinite order, both $z$ and $w$ must have infinite isotropy groups with respect to $G$. This means that $|L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}| > 1$ a contradiction with our initial assumption that $|L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}| = 1$. Thus we conclude $|L(G)| = 1$ and $L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} = L(G)$.

For the final case, assume that $|L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}| > 1$. Let $x \in L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ and $\{g_m\}$ a sequence of elements in $G_x$. By corollary 5.12 we have that there exists a subsequence of $\{g_m\}$ and points $x_1, x_2 \in L(G)$ such that the subsequence converges uniformly to $x_1$ in compacts of $\overline{\mathbb{H}^n_{\mathbb{C}}} - \{x_2\}$. As all $g_m \in G_x$ then $g_m(x) = x$ implies that either $x = x_1$ or $x = x_2$ hence $x \in L(G) \implies L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} \subset L(G)$. We notice now that $L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ is $G$-invariant for if $x$ and $\{g_m\}$ are as stated above and $h \in G$ is an arbitrary element of $G$ then the group generated by $\{f_m := hg_mh^{-1}\}$ is infinite and $f_m(h(x)) = hg_mh^{-1}(h(x)) = hg_m(x) = h(x)$ so $h(x) \in L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$. This means that $L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ is a closed (by definition), $G$-invariant set with more than two elements so $L(G) \subseteq L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ and we conclude $L(G) = L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ □

**Proposition 6.10.** *[10] If $G$ is a discrete subgroup of $PU(n,1)$ then $L(G) = L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$.*

*Proof.* If $G$ is finite then $L(G) = \varnothing$ and $L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} = \varnothing$ trivially. So we have $L(G) = L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ in this case.

Let us show that $L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} \subset L(G)$. Assume $G$ infinite and let $z \in \partial\mathbb{H}^n_{\mathbb{C}}$, $\zeta \in \mathbb{P}^n_{\mathbb{C}} - L_0(G)$ and $\{g_m\}$ a sequence of distinct elements in $G$ such that $g_m(\zeta) \to z$. In this case we must consider three situations.

For the first case, consider $\zeta \in \mathbb{H}^n_{\mathbb{C}} - L_0(G)$ then, by definition, $z \in L(G)$ and there is nothing more to show.

For the second case, consider $\zeta \in \partial\mathbb{H}^n_{\mathbb{C}} - L_0(G)$. By proposition 6.9 $L(G) = L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ so $\zeta \in \partial\mathbb{H}^n_{\mathbb{C}} - L_0(G) \implies \zeta \in \partial\mathbb{H}^n_{\mathbb{C}} - L(G)$. But by corollary 5.12 the sequence $g_m(\zeta)$ converges uniformly to some point in $L(G)$. As $g_m(\zeta) \to z$ then $z \in L(G)$.

For the final case, consider $\zeta \in \mathbb{P}^n_{\mathbb{C}} - (\overline{\mathbb{H}^n_{\mathbb{C}}} \cup L_0(G))$. Let $W$ be the intersection of the polar hyperplane to $\zeta$ and $\mathbb{H}^n_{\mathbb{C}}$. As all $g_m$ are unitary then $g_m(W)$ is the intersection of the polar hyperplane to $g_m(\zeta)$ with $\mathbb{H}^n_{\mathbb{C}}$. Now, let $w \in W$ and consider the sequence $\{g_m(w)\}$. By corollary 5.12 some subsequence must converge to a boundary point in $L(G)$, define $q$ as the convergence point of this subsequence. As $g_m(\zeta) \to z$ and $g_m(w) \to q$, by proposition 6.8 we must have $z = q$ and thus $z \in L(G)$.

Hence $L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} \subseteq L(G)$. Now notice that $L(G) = \overline{G(p)} \cap \partial\mathbb{H}^n_{\mathbb{C}}$ so any point of $G$ is a cluster point of $\{g(p)\}_{g \in G}$ thus $L(G) \subseteq L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ and $L(G) = L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ $\square$

**Proposition 6.11.** *[10] If $G$ is a discrete subgroup of $PU(n,1)$ then $L(G) = L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$.*

*Proof.* If $G$ is finite then both sets are empty and the equality holds. Let us assume that $G$ is infinite. By lemma 6.4 we have that $(L_0(G) \cup L_1(G)) \cap \mathbb{H}^n_{\mathbb{C}} = \varnothing$ so $L(G) \subseteq L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$ as for any point $x \in L(G)$, $x$ is a cluster point of the compact set $\{p\} \subset \mathbb{H}^n_{\mathbb{C}}$ by definition, so $x \in L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$.

All we have to do now is see that $L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} \subseteq L(G)$. Let $K$ be a compact subset of $\mathbb{P}^n_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ and let $z \in \partial\mathbb{H}^n_{\mathbb{C}}$ be a cluster point for $K$ with respect to $G$. Now we can choose sequences $\{k_m\} \subset K$ and $\{g_m\} \subset G$ such that $k_m \to k \in K$ and $g_m(k_m) \to z$ as $m \to \infty$. With this setup we need to consider four distinct situations.

The first is if $k \in \mathbb{H}^n_{\mathbb{C}} - (L_0(G) \cup L_1(G))$. In this case we may assume without loss of generality that all the $k_m \in \mathbb{H}^n_{\mathbb{C}}$. So, by corollary 5.12, there are points $x, y \in L(G)$ such that $g_m(k_m) \to x$ uniformly, so $z = x \in L(G)$.

The second case is if $k \in \partial\mathbb{H}^n_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ e some subsequence of $\{g_m\}$, still denoted by $\{g_m\}$, is entirely contained in $\overline{\mathbb{H}^n_{\mathbb{C}}}$. Then by propositions 6.9 and 6.10 $k \in \partial\mathbb{H}^n_{\mathbb{C}} - L(G)$ and we can apply corollary 5.12 to the compact set given by $\{k, k_1, ..\} \cap \overline{\mathbb{H}^n_{\mathbb{C}}}$, so that there are points $x, y \in L(G)$ such that $g_m(k_m) \to x$ uniformly. This implies that $z = x \in L(G)$.

The third case is if $k \in \partial\mathbb{H}^n_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ and only a finite amount of elements of $\{k_m\}$ belongs to $\overline{\mathbb{H}^n_{\mathbb{C}}}$. Then we denote by $W_m$ the intersection between the polar hyperplane to $k_m$ and $\mathbb{H}^n_{\mathbb{C}}$ and by $w_m$ a convergent sequence such that $w_m \in W_m$ for all $m$ (we may need to take a subsequence). Let $w_m \to w$ as $m \to \infty$. Applying proposition 6.8 we see that $w = k$. We consider now the action of $\{g_m\}$ on the sequence $\{w_m\}$, and by corollary 5.12 we see that for some $g_m(w_m)$ converges uniformly to some $x \in L(G)$. As all $g_m$ are unitary we have that if $w_m \in W_m$ then $g_m(W_m)$ is the intersection of the polar hyperplane to $g_m(k_m)$ and $\overline{\mathbb{H}^n_{\mathbb{C}}}$. Hence, as $g_m(k_m) \to z$ and $g_m(w_m) \to x$, by proposition 6.8 $z = x \in L(G)$ (remember that by definition $z \in \partial\mathbb{H}^n_{\mathbb{C}}$).

The final case is if $k \in \mathbb{P}^n_{\mathbb{C}} - (L_0(G) \cup L_1(G))$. In this situation we may assume without loss of generality that all the $k_m$ are in $\mathbb{P}^n_{\mathbb{C}} - \overline{\mathbb{H}^n_{\mathbb{C}}}$. Let $W_m$ be the intersection of the polar hyperplane of $k_m$ and $\overline{\mathbb{H}^n_{\mathbb{C}}}$, $W$ be the intersection of the polar hyperplane of $k$ and $\overline{\mathbb{H}^n_{\mathbb{C}}}$ and let $w_m$ be a sequence such that $w_m \in W_m$ for all $m$ and $w_m \to w \in W \cap \mathbb{H}^n_{\mathbb{C}}$. By corollary 5.12, there is a point $x \in L(G)$ such that $g_m(w_m) \to x$ uniformly. As all $g_m$ are unitary we have that if $w_m \in W_m$ then $g_m(W_m)$ is the intersection of the polar hyperplane to $g_m(k_m)$ and $\overline{\mathbb{H}^n_{\mathbb{C}}}$. Thus $g_m(k_m) \to z$ and $g_m(w_m) \to x$ implies that $z = x \in L(G)$ by proposition 6.8 (remember that by definition $z \in \partial\mathbb{H}^n_{\mathbb{C}}$).

Hence $L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} \subseteq L(G)$ and $L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}} = L(G)$. □

**Corollary 6.12.** *[10] If $G$ is a discrete subgroup of $PU(n,1)$ then $L(G) = \Lambda(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}$.*

*Proof.* The result follows from the three previous propositions as $\Lambda(G)\partial\mathbb{H}^n_{\mathbb{C}} = (L_0(G) \cup L_1(G) \cup L_2(G)) \cap \partial\mathbb{H}^n_{\mathbb{C}} = (L_0(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}) \cup (L_1(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}) \cup (L_2(G) \cap \partial\mathbb{H}^n_{\mathbb{C}}) = L(G)$. □

We finish by proving that :

In [2] they were able to prove the equality of the limit sets and further elaborate that that $\Lambda(G)$ is the union of all hyperplanes tangent to some point of $L(G)$.

# 7 CONCLUSION

With the previous section we finally conclude the thesis with the proof that the Chen-Greenberg limit set is the intersection of the Kulkarni limit set and the boundary of $\mathbb{H}^n_{\mathbb{C}}$. We believe, though we were still unable to show it conclusively that we can use the tools developed in this work to show that the Kulkarni limit set is the union of all tangent hyperplanes at a point of the Chen-Greenberg limit set.

Finally we would like to point out that for any non-elementary discrete subgroup of $PU(n, 1)$ the complement of the Kulkarni limit set is the maximal set where the subgroup acts discontinuously.[2]

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