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**Departamento de Matemática**

# Differential Geometry and Stability of Hypersurfaces in Minkowski Spaces

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*Differential Geometry and Stability of Hypersurfaces in Minkowski Spaces*

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# Abstract

The main topic of the thesis is the study of differential geometry in (normed or) Minkowski spaces. The thesis is divided into two Parts: (I) Some topics in differential geometry of normed spaces; and (II) Stability of hypersurfaces in Minkowski spaces. In part I, with a smooth norm instead of an inner product, one can define analogous concepts of principal curvatures, with such concepts in mind various questions appear. In this part we extend some results already known from the Euclidean case to the Minkowski case. In part II, we introduce the concept of Minkowski area and stability with respect to Birkhoff normal variations and compute the formula of the second variation of the area with respect to these variations. Finally, using the second variation formula, we extend to Minkowsky spaces the classical result of Barbosa and do Carmo [9] that characterizes the euclidean sphere as the unique compact stable CMC hypersurface of $\mathbb{R}^n$.

**Key-words:** geometry of normed spaces, Birkhoff-Gauss map, Minkowski mean curvature, stability.

# Resumo

O tema principal da tese é o estudo da geometria diferencial em espaços (normados ou) de Minkowski. A tese está dividida em duas partes: (I) Alguns tópicos em geometria diferencial de espaços normados; e (II) Estabilidade de hipersuperfícies em espaços de Minkowski. Na parte I, com uma norma suave em vez de um produto interno, podemos definir conceitos análogos de curvaturas principais, com esses conceitos em mente várias questões aparecem. Nesta parte estendemos alguns resultados já conhecidos do caso euclidiano para o caso Minkowski. Na parte II, apresentamos o conceito de área de Minkowski e estabilidade com relação a variações normais de Birkhoff e calculamos a fórmula da segunda variação da área com relação a essas variações. Finalmente, usando a fórmula da segunda variação, estendemos aos espaços de Minkowsky o resultado clássico de Barbosa e do Carmo [9] que caracteriza a esfera euclidiana como a única hipersuperfície compacta CMC estável de $\mathbb{R}^n$.

**Palavras-Chave:** geometria de espaços normados, aplicação de Birkhoff-Gaus, curvatura média de Minkowski, estabilidade.

# Contents

# Introduction

The thesis is divided into two Parts: (I) Some topics in differential geometry of normed spaces; and (II) Stability of hypersurfaces in Minkowski spaces. Each Part is divided into Chapters.

## 0.1 Some topics in differential geometry of normed spaces

This thesis deals with the study of differential geometry on normed vector spaces of finite dimension, that are called Minkowski spaces. According to Thompson [32], such a geometry is called Minkowski geometry. Our goal is to extend results from classical differential geometry of hypersurfaces to Minkowski spaces. Such a hypersurface immersed in such a space becomes a Finsler manifold with the induced metric of the norm of the ambient, so this work can be seen from the perspective of Finsler Geometry. Our construction, following Balestro, Martini and Teixeira [6, 5, 7], is a particular equiaffine immersion (as in [25]). Therefore, from the point of view of affine differential geometry, we can study whether our particular case has a special behavior. The affine differential geometry approach given in the book [25] refers to hypersurface immersed in an affine space $\mathbb{R}^{n+1}$ equipped with a transversal vector field that plays the role of a normal vector field. Following [6], in this work we provide such a hypersurface immersed in a Minkowski space with a transversal vector obtained through Birkhoff's orthogonality associated with the norm, such a transversal field gives rise to immersions that are equiaffine, which allows us to extend some concepts of geometry classic differential. To talk a little about the chronology of Minkowski spaces, we will use the text in [23], namely that the axioms of Minkowski spaces were introduced in 1896 by Minkowski [24], in connection with problems from number theory. However, it seems that the earliest reference to non-Euclidean geometry in the sense of Minkowski Geometry was made by Riemann, in 1868, in his Habilitationsvortrag [29], where he mentioned the $l_4$-norm. Hilbert [17] in his famous lecture delivered before the International Congress of Mathematicians in 1900 gives a description of Minkowski Geometry in his fourth problem. Two papers considering Minkowski Geometry from a geometric (as opposed to analytic) point of view are GoŁab [15] of 1932 and GoŁab and Härlen [16] of 1931. Minkowski Geometry was studied, especially by Busemann [10] in 1950, in order to throw more light on Finsler Geometry, introduced by Finsler [14] in 1918. For others developments in Finsler Geometry, see Álvarez [3].

Let us concretely explain the initial idea behind our constructions. The approach to Minkowski geometry given in the paper [6] regards surfaces immersed in an three-dimensional Minkowski space with the transversal vector field obtained via the Birkhoff orthogonality associated to the norm. Our idea in this thesis is to extend this theory to $(n-1)$-dimensional hypersurfaces immersed in Minkowski spaces

We begin by briefly describing the theory developed here. We work with an immersion $x : M \to (\mathbb{R}^n, ||\cdot||)$ of a hypersurface $M$ in the space $\mathbb{R}^n$ endowed with a norm $||\cdot||$, which is considered to be admissible. This means that the unit sphere $\partial\mathbb{B} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ of the normed or

Minkowski space $(\mathbb{R}^n, ||\cdot||)$ has strictly positive Gaussian curvature as a surface of the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Note that the unit sphere is the boundary of the unit ball $\mathbb{B} := \{x \in \mathbb{R}^n : ||x|| \leq 1\}$, which is a compact, convex set with interior points centered at the origin. Respective homothetical copies are called Minkowski spheres and Minkowski balls. We say that a vector $v \in \mathbb{R}^n$ is Birkhoff orthogonal to a hyperplane $H \subset \mathbb{R}^n$ if for each $w \in H$ we have $||v+w|| \geq ||v||$ (see [2]). Geometrically, a vector $v$ is Birkhoff orthogonal to a plane $H$ if $H$ supports the unit ball of $(\mathbb{R}^n, ||\cdot||)$ at $v/||v||$.

Due to the admissibility of the norm, it follows that Birkhoff orthogonality is unique both on left and on right. The Birkhoff-Gauss map of M is an analogue to the Gauss map defined in terms of Birkhoff orthogonality as follows: for each $p \in M$, the Birkhoff normal vector to $M$ at $p$ is a vector $\eta(p) \in \partial\mathbb{B}$ which is Birkhoff orthogonal to the tangent space to $M$ at $p$. Such a vector field can be globally defined if $M$ is orientable, and hence we will always assume this hypothesis.

At each point, the eigenvalues of the differential map $d\eta_p$ are called principal curvatures. Their product is the Minkowski Gaussian curvature, and their arithmetic mean is the Minkowski mean curvature. We also endow $M$ with an induced connection $\tilde{\nabla}$ by means of the Gauss equation

$$D_X Y = \tilde{\nabla}_X Y + h(X,Y)\eta, \tag{1}$$

where $X, Y$ are smooth vector fields in $M$, and $h(X,Y)$ is a symmetric bilinear form which can be regarded as the second fundamental form in our context. For this bilinear form, we have the formula

$$h(X,Y) = -\frac{\langle du^{-1}_{\eta(p)} Y, d\eta_p X\rangle}{\langle \eta, \xi\rangle}$$

where $\xi$ denotes the usual Euclidean Gauss map of $M$ , and $u^{-1}$ is the Euclidean Gauss map of the unit sphere $\partial\mathbb{B}$. Notice that we have $\eta = u \circ \xi$ (where $\circ$ denotes the usual composition of maps). We also define the normal curvature $k_{M,p}(X)$ of $M$ at a point $p$ in direction $X$ to be the circular curvature of the curve obtained by intersecting $M$ with the plane spanned by $\eta(p)$ and $X$ (translated to pass through p, of course). For the normal curvature we have the equality

$$k_{M,p}(X) = \frac{\langle du^{-1}_{\eta(p)}X, d\eta_p X\rangle}{\langle du^{-1}_{\eta(p)}X, X\rangle}.$$

Now we describe the structure of this part of the thesis. The first chapter contains the basic concepts and results that supported the development of the entire thesis. In chapter two we present extensions, for any dimension, of results proposed by Balestro, Martini, Teixeira in the papers [6], [7], [8]. In these articles, the authors generalize to three-dimensional Minkowski spaces, some known results of classic Differential Geometry. The key point is in the section 2.1, where we prove that the signs of the principal curvatures (positive and negative) are, in equal quantity, in any two Minkowski geometries and this allowed us to extend results, almost without modification, as Hadamard-type theorems, which we present in section 2.2, Global theorems.

Section 2.3 contemplates Weyl's tube formula. In [7] the authors observed that such a formula, in this case $n = 3$, could be obtained without making use of a particular parameterization. Here we present it for a general parametrization, for any dimension $n$. Lemma 2.15 is the key point for this extension.

In [8] the authors provide a formula for the first variation of the area for a particular variation, in dimension $n = 3$. In section 2.4 we present a formula for the first variation of the area for general variations. With our formula we can see that hypersurfaces with Minkowski mean curvature $H_m = 0$ are critical points of the functional area, as in the Euclidean case.

We close the chapter with section 2.5 where we present the definition of volume in analogy with the Euclidean case and use section 2.4 to verify that an immersion $x : M \to \mathbb{R}^n$ has Minkowski mean curvature $H_m$ constant if and only if it is critical points for the functional area for variations that preserve volume.

## 0.2 Stability of hypersurfaces in Minkowski spaces

A well-known result before 1900 says that if $M$ is a surface in $\mathbb{R}^3$, with a smaller area between all surfaces that limit the same volume, then the mean curvature of $M$ should be constant. At that time, the following fact was already known: The sphere is the surface, among all that limit the same volume, the one with the smallest area, and that its mean curvature is constant different from zero (H.A Schwarz, 1890, see [31]). The following question then arises: Is a surface at $\mathbb{R}^3$ of constant mean curvature, different from zero, necessarily a sphere? A physical counterpart, equipped with the properties of the soap film, which says that a soap bubble should have a constant mean curvature (Laplace-Young equation), would say: Can a soap bubble have another shape if not the round sphere?

Answers came up. 1900: If a surface, strictly convex, compact in $\mathbb{R}^3$ has constant mean curvature, then it must be a round sphere (Liebmann, see [21]). 1950: A compact surface simply connected with constant mean curvature immersed in $\mathbb{R}^3$ is a embedded round sphere (Heinz Hopf, see [18]). In 1962: The only closed hypersurfaces of constant mean curvature and embedded in Euclidean spaces are the round spheres (Aleksandrov, see [1]).

Since Hopf, Aleksandrov's theorem for immersed surfaces was believed to be true, rather than embedded. This became known as Hopf's conjecture. Aleksandrov's theorem further emphasized the idea that this conjecture was valid. Only 35 years after Hopf's theorem this conjecture was solved, and given as false, by the American mathematician Henry Wente (see [33]).

Following the chronology, in 1979, do Carmo and Peng proved that the plane is the only minimal and stable complete surface at $\mathbb{R}^3$. Intrigued to know what would happen, when trying to extend to surfaces of constant mean curvature, results involving minimal surfaces, do Carmo, in 1981, proposed to JL Barbosa to study the problem (Revista Matemática Universitária. Nº 16, July 1994. 1-18 ). João Lucas Barbosa became interested in the subject and then in 1984 they proved Theorem 0.1 below. First, let's look at some definitions.

Let $M$ be an $n-1$ oriented differentiable manifold and $x : M \to \mathbb{R}^n$ a smooth immersion. The area of the immersion $x$ is defined as

$$A(x) = \int_M dS$$

where $dS$ is the area element induced by $x$. We can define the volume of $x$ as

$$V(x) = \frac{1}{n}\int_M \langle x, \xi \rangle dS$$

where $\xi$ denotes the unit normal vector field determined by the orientation of $M$. The definition of $V$ is justified by the fact that when $x$ is an embedding and $M$ is closed, $V$ represents the volume of the interior of $M$.

A variation of $x : M \to \mathbb{R}^n$ is a smooth function $F : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$ such that for every $t \in (-\varepsilon, \varepsilon)$, the function $F^t = F(t, \cdot)$ is also an immersion.

For such a variation, the area and volume defined above give one-parameter functions $A(t) = A(F^t), V(t) = V(F^t)$. We say that a variation has compact support if $F(t,p) = x(p)$ for every $p$ outside a compact subset of $M$.

It is known that *minimal surfaces*, this is, surfaces with zero mean curvature, arise naturally as critical points of the area function, whereas CMC surfaces (surfaces with constant mean curvature) correspond to the critical points of the area with restricted volume. More precisely, we say that an immersion is *minimal* if for every variation with compact support we have $A'(0) = 0$. It is known that for any immersion and any variation (see Theorem 2.20),

$$A'(0) = \int_M (n-1)H_e(p)\langle F_t(0,p), \xi(p)\rangle dS$$

where $H_e$ denotes the mean curvature and $F_t(0,p) = \frac{\partial}{\partial t}F(0,p)$, so minimal immersions are characterized as immersions with zero mean curvature. Minimal surfaces arise naturally as solutions of the Plateau problem, that is finding a surface with prescribed boundary, minimizing the surface area measure. If we restrict to immersions having a fixed volume, we see from the well known formula

$$V'(0) = \int_M \langle F_t(0,p), \xi(p)\rangle dS$$

that the immersions such that $A'(0) = 0$ for every volume-preserving variation of compact support, are precisely those with constant mean curvature.

Surfaces minimizing the area with restricted volume are CMC but the converse is not true, meaning that a CMC surface may be deformed locally into nearby surfaces with less area. This is the case for example, of cylindrical and plane surfaces.

We say that an immersion is *stable* if it has constant mean curvature and if for every variation of compact support that preserves the volume, $A''(0) \geq 0$.

For a variation of any minimal immersion, it is known that the formula of second variation holds

$$A''(0) = \int_M f(\Delta f + B_e^2 f) dS$$

where $f(p) = \langle F_t(0,p), \xi(p) \rangle$ is the normal component of the variation, $\Delta$ is the Laplace-Beltrami operator and $B_e$ is the norm of the second fundamental form of $M$.

If $M = \partial\Omega$ is a closed smooth embedded surface that minimizes the surface area measure among all "nearby" surfaces enclosing the same volume as $M$, then $M$ must be a round sphere. The same holds true if $M$ is only closed and CMC. This fact is known since the classical work of Alexandrov [1] where he introduced the *moving planes* method. But for immersed surfaces this fact is false in general. In [18] Hopf showed that a CMC immersion $\mathbb{S}^2 \to \mathbb{R}^3$ must be a round sphere. Latter Hsiang constructed a CMC immersion $\mathbb{S}^3 \to \mathbb{R}^4$ that is not the round sphere, and generalized the result to higher dimensions in [19]. In 1986 Wente exhibited in [33] a closed immersed surface of genus one (the Wente torus) in $\mathbb{R}^3$ and Kapouleas [20] constructed surfaces in $\mathbb{R}^3$ of higher genus. In contrast, Barbosa and do Carmo in [9] showed that the stability property characterizes the round sphere. They proved the following.

**Theorem 0.1** (Theorem 1.3, [9])**.** *Let $M^{n-1}$ be compact, orientable, and let $x : M \to \mathbb{R}^n$ be an immersion with non-zero constant mean curvature. Then $x$ is stable if and only if $x(M) \subset \mathbb{R}^n$ is a (round) sphere $S^{n-1} \subset \mathbb{R}^n$.*

In parallel to CMC surfaces, the theory of surfaces immersed in general $n$-dimensional normed spaces (Minkowski spaces) was studied by Busemann [10], Petty [28]. The aspects of differential geometry were further developed more recently by Balestro et al. [6], [8] in a systematic way.

We briefly recall the basic definitions. Let $\mathbb{B}$ be a compact convex set with the origin in the interior (there is no need to assume that $\mathbb{B}$ is symmetric), and suppose it has smooth boundary with positive Gauss curvature. The Gauss map $\partial\mathbb{B} \to \mathbb{S}^{n-1}$ is a smooth diffeomorphism and we denote the inverse by $u : \mathbb{S}^{n-1} \to \partial\mathbb{B}$.

Given an immersed surface $M$ with Gauss map $\xi : M \to \mathbb{S}^{n-1}$ we define $\eta = u \circ \xi$ and observe that $\eta(p) \in \partial\mathbb{B}$ is the unique point such that $T_{\eta(p)}\partial\mathbb{B} = T_pM$. We call $\eta$ the Birkhoff-Gauss map of $M$ with respect to $\mathbb{B}$.

The differential $d_p\eta$ can thus be regarded as an endomorphism of $T_pM$ and can be shown to be diagonalizable with real eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. This is done considering a second Riemannian metric in $M$ called the *Dupin metric* $(\cdot,\cdot)_b$. For $v, w \in T_pM$ define $(v,w)_b = \langle d_p u^{-1}(v), w\rangle$ and notice that $d_p\eta$ is self-adjoint with respect to $(\cdot,\cdot)_b$. The Dupin metric is shown in [8], [5] to be a useful tool for translating properties that are valid in classical differential geometry, to the Minkowski case.

The Minkowskian mean curvature and Minkowskian Gauss curvature are defined by $H_m = \frac{\lambda_1+\cdots+\lambda_{n-1}}{n-1}$ and $K_m = \lambda_1 \cdots \lambda_{n-1}$ respectively.

The support function of $\mathbb{B}$ is defined by

$$h_{\mathbb{B}}(v) = \max_{x\in\mathbb{B}} \langle v, x\rangle \text{ for } v \in \mathbb{R}^n$$

and, for a unit vector $v$, it measures the distance from the origin to the supporting hyperplane of $\mathbb{B}$ perpendicular to $v$. Notice that $h_{\mathbb{B}}(\xi(p)) = \langle \eta(p), \xi(p)\rangle$, since by definition, $T_{\eta(p)}\partial\mathbb{B}$ is this supporting hyperplane.

The Minkowskian area measure $\omega = \omega_{\mathbb{B}}$ is defined by

$$A_m(x) = \int_M d\omega = \int_M \langle \eta, \xi\rangle dS \tag{2}$$

and if $F(t,p)$ is a variation of $M$ with compact support, the first variation formula for $A_m$ is given by

$$A'_m(0) = \int_M (n-1)H_m(p)N_\eta(F_t(0,p))d\omega(p) \tag{3}$$

where $N_\eta$ is the projection to the second coordinate in the direct-sum decomposition $\mathbb{R}^n = T_pM \oplus \langle \eta \rangle$. Formula (3) was proven in [8] for variations in a more restricted class than [9], but the formula extends easily to the general case. We prove this in Appendix B.

If $M$ is the (smooth) boundary of a bounded open set $\Omega$, this area measure already appears in the literature as the *mixed volume* of $\Omega$ and $\mathbb{B}$ (see [30]). The mixed volume of any two compact convex sets $K$ and $L$ with non-empty interior, is defined as

$$V(K,L) = \frac{1}{n}\lim_{\varepsilon\to 0}\frac{\mathrm{vol}(K+\varepsilon L)-\mathrm{vol}(K)}{\varepsilon}$$

where $K + L = \{x + y : x \in K, y \in L\}$ is the Minkowski sum of sets. The integral representation (see [30])

$$V(K,L) = \frac{1}{n}\int_{\partial K} h_L(\xi(p))dS(p)$$

shows that the Minkowskian area measure of $M$ is precisely $nV(\Omega,\mathbb{B})$. The mixed volume inequality (see Theorem 7.2.1, [30]) states that

$$V(K,L) \geq \mathrm{vol}(K)^{\frac{n-1}{n}}\,\mathrm{vol}(L)^{\frac{1}{n}}$$

with equality if and only if $K$ and $L$ are homothetic.

As a consequence of the mixed volume inequality we deduce that the least possible value of the Minkowskian area measure of an embedded surface $M = \partial\Omega$ with fixed volume $\mathrm{vol}(\Omega) = v$ is $A_m(M) = nv^{\frac{n-1}{n}}\,\mathrm{vol}(\mathbb{B})^{\frac{1}{n}}$, and this value is attained if and only if $\Omega$ is homothetic to $\mathbb{B}$ and thus $M = \partial(x_0+\lambda\mathbb{B})$ is a *Minkowskian sphere*.

The subject of Minkowskian differential geometry, although it was born more than half a century ago, remains vastly unexplored from the point of view of differential geometry. The above considerations imply that a Minkowskian version of the usual *isoperimetric inequality* holds for the area measure $A_m$. Also an immersed surface that minimizes the Minkowskian area with restricted volume must have constant Minkowskian mean curvature, as it is clear from the first variation formula (3).

The concept of Stability for the Minkowskian area measure was treated by Palmer in [26], and later [27] in the non-smooth case. In [26] the author established the corresponding stability theorem by following Wente's proof in [34], thus avoiding necesity of computing the Jacobi operator for $\mathbb{B}$, this is, the second order term appearing in the second variation formula. In the paper [35], the author calculated the second variation formula. These papers are developed from the viewpoint of Wulff shapes: a functional area is defined having the form

$$\mathcal{F}(x) := \int_M F(\xi) dS$$

where $x : M^{n-1} \to \mathbb{R}^n$ is a smooth immersion of closed hypersurfaces, $F$ is "weight" function, $\xi$ is the (usual) Gauss map and $dS$ is the area element. The function $F : \mathbb{S}^{n-1} \to \mathbb{R}_+$ is considered to be a smooth and positive which is extended to $\mathbb{R}^n \backslash \{0\}$ by $F(tu) = tF(u)$ for $u \in \mathbb{S}^{n-1}$ and $t > 0$. The authors translate the "convexity" property of $F$ by requiring that it is *elliptic*, in the sense that for each $u \in \mathbb{S}^{n-1}$, the restriction of the Hessian of $F$ to the tangent space $u^{\perp}$ is a positive definite endomorphism. Under this hypothesis, it follows that the set

$$\{v \in \mathbb{R}^n : \langle v, u \rangle \ \leq F(u) \text{ for all } u \in \mathbb{S}^{n-1}\}$$

is a convex body whose boundary, $W_F$, is called the Wulff shape of $F$. Under this viewpoint, the functional $\mathcal{F}$ can be seen as a sort of surface area which is coeherent with "the geometry of $F$". What we realized is that there is a stronger geometric reasoning for that. If one starts with a convex body $\mathbb{B} \subset \mathbb{R}^n$ having the origin as an interior point (not necessarily symmetric) and whose boundary is $C^2_+$, then the *gauge function* of $\mathbb{B}$ defined as

$$||v||_{\mathbb{B}} := \inf\{\lambda \geq 0 : v \in \lambda\mathbb{B}\}$$

yields a geometry in $\mathbb{R}^n$ (which is commonly called a Minkowski geometry). This geometry has a natural orthogonality concept between directions and hyperplanes called Birkhoff orthogonality. Instead of considering a "weight" on a given immersed surface $M$ (which may seem a little bit "artificial"), one can endow $M$ with its Birkhoff-Gauss map. By doing that, we have a structure which rely solely on the geometry given by $\mathbb{B}$. The "coherent" surface area measure is not obtained from a "weight" in the Euclidean Gauss map, but otherwise obtained via the $(n-1)$-form

$$d\omega(X_1, \cdots, X_{n-1}) := \det[X_1, \cdots, X_{n-1}, \eta(p)]$$

at each $p \in M$, where $\eta(p)$ denotes the Birkhoff-Gauss normal of $M$ at $p$. In the field of affine differential geometry, it is common to introduce analogously a surface area measure related to the affine normal field. If we are wiling to use an auxiliary Euclidean structure (which was not needed thus far!) one can show that

$$\int_M d\omega = \int_M \langle \eta, \xi \rangle dS$$

where $\eta$ and $\xi$ stands for the Birkhoff-Gauss and usual Gauss map, respectively, and $dS$ denotes the usual surface area element.

Because of these facts, we propose to study the concept of stability in Minkowski geometry, from the "Minkowskian differential geometry" viewpoint.

Our contributions are the following:

**Theorem 0.2.** *Let $x : M \to \mathbb{R}^n$ be an immersed surface with Birkhoff-Gauss map $\eta$ and constant Minkowskian mean curvature, and let $F : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$, be a volume-preserving variation of compact support given by*

$$F(t,p) = F^t(p) = x(p) + g(t,p)\eta(p). \tag{4}$$

*Denote $f(p) = \frac{\partial}{\partial t} g(t,p)\big|_{t=0}$ and $A_m(t) = A_m(F(t,\cdot))$ the area defined by (2). Then,*

$$\begin{aligned} A_m''(0) &= \int_M \left(-B_m^2 f^2 + \langle \eta, \xi \rangle (\nabla^b f, \nabla^b f)_b\right) d\omega \\ &= -\int_M f\left(B_m^2 f + \langle \eta, \xi \rangle^{-1} \operatorname{div}_M (\langle \eta, \xi \rangle^2 du(\nabla f)\right) d\omega \end{aligned} \tag{5}$$

*Here $\nabla^b f$ is the gradient of $f$ with respect to the Dupin metric and can be computed as $\nabla^b f = du(\nabla f)$ where $\nabla f$ is the gradient with respect to the usual metric. Also $B_m$ is the norm of the Minkowski second fundamental form, $B_m^2 = \sum_{i=1}^{n-1} \lambda_i^2$. Finally,* $\operatorname{div}_M X|_p = \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle|_p$, *where $\{e_1, \cdots, e_{n-1}\}$ is a ortonormal basis of $T_pM$.*

The novelty in this formula is the term

$$\Delta_m(f) = \langle \eta, \xi \rangle^{-1} \operatorname{div}_M \left(\langle \eta, \xi \rangle^2 du(\nabla f)\right),$$

that reduces to the usual Laplace-Beltrami operator when $\mathbb{B}$ is the unit Euclidean ball.

This formula is quite remarkable from the viewpoint of Minkowskian differential geometry. The Dupin and weighted Dupin metrics are new concepts, which seem to contain a lot of information regarding the geometry given by $\mathbb{B}$. So this formula shed some light into concepts which are characteristic of the field of Minkowski geometry.

The variational formulas obtained in [35], [26], [27] cannot, be straightforwardly "translated" to the "language" of Minkowski geometry. Moreover, the fact that the Minkowskian approach leads to the same results as the approach by Wulff shapes is not trivial at a first glance.

We also verify using formula (5), that the proof of Theorem 0.1 carries on to the Minkowski case. We say that an immersion is stable with respect to the Minkowskian structure if $A''_m(0) \geq 0$ for every volume-preserving, Birkhoff normal variation of compact support.

**Theorem 0.3.** *Let $x : M \to \mathbb{R}^n$ be a compact immersed surface without boundary, with constant Minkowskian mean curvature and stable with respect to the Minkowskian structure. Then $x(M)$ is an embedded Minkowski sphere, this is, $x(M)$ is homothetic to $\partial\mathbb{B}$.*

The proof of Theorem 0.3 follows the same lines of Theorem 0.1 with some adaptations. The main difficulty here is to compute $\Delta_m(f)$ when $f$ is a suitable test function. This is done in Lemma 3.5.

The rest of the thesis is organized as follows: In the Section 3.1 we recall some basic definitions and lemmas. In the Sections 3.2 and 3.3 we prove Theorems 0.2 and 0.3 respectively. The proof of Theorem 0.3 relies on a lengthy computation that we postpone to the Appendix A.

# Chapter 1

# Preliminaries

## 1.1 Basic concepts

We work with a normed space $(\mathbb{R}^n, ||\cdot||)$, whose *unit ball* is the set $\mathbb{B} := \{x \in \mathbb{R}^n : ||x|| \leq 1\}$, and its boundary $\partial\mathbb{B} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ is the *unit sphere*. Througout the text we will always assume that the norm is *admissible*, meaning that the unit sphere is a embedded hypersurface whose (Euclidean) Gaussian curvature is strictly positive everywhere, where we are assuming that $\mathbb{R}^n$ is equipped with the standard inner product

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}.$$

In particular, we get that $(\mathbb{R}^n, ||\cdot||)$ is a smooth and strictly convex normed space.

In an $n$-dimensional normed space, we define *Birkhoff orthogonality* between vectors and hyperplanes stating that a vector $v \in (\mathbb{R}^n, ||\cdot||)$ is *Birkhoff orthogonal* (or simply *orthogonal*) to a hyperplane $H \subseteq (\mathbb{R}^n, ||\cdot||)$ whenever $||v|| \leq ||v + w||$ for any $w \in H$. We denote this relation by $v \dashv_B H$. Since the normed space $(\mathbb{R}^n, ||\cdot||)$ is smooth and strictly convex, it follows that we have existence and uniqueness for the Birkhoff orthogonality both on left and on right. This means that for each non-zero vector $v \in \mathbb{R}^n$, there exists a unique hyperplane $H \subseteq \mathbb{R}^n$ such that $v \dashv_B H$, and for each hyperplane $H$ there exists a non-zero vector $v \in \mathbb{R}^n$ with $v \dashv_B H$, and such vector is unique up to scalar multiplication. Geometrically, when $v \dashv_B H$ and $v \neq 0$, the hyperplane $H$ supports the unit ball at $v/||v||$.

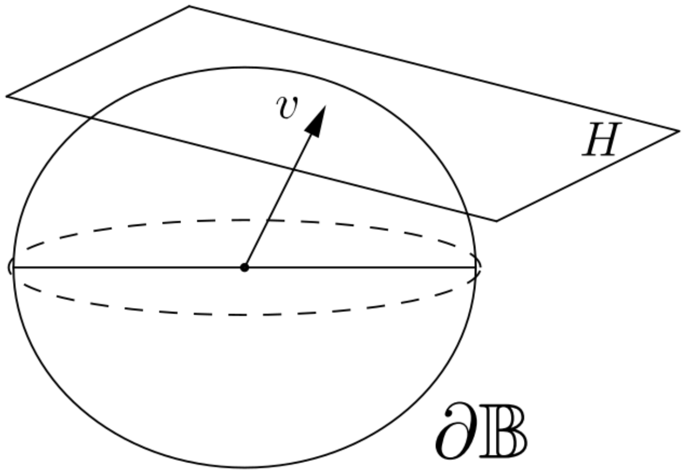


Figure 1.1: Birkhorff Orthogonality

As in [6], we shall use the concept of Birkhoff orthogonality to endow an immersed hypersurface with an analogue of the Gauss map.

Let $x : M \to (\mathbb{R}^n, ||\cdot||)$ be an immersed hypersurface. A *transversal vector field* on $M$ is a smooth section $\eta$ of the restriction $T\mathbb{R}^n|_M$ such that

$$\mathbb{R}^n \simeq T_pM \oplus \mathrm{span}\{\eta_p\}$$

for each $p \in M$, where we are naturally identifying $T_pM \simeq x_*(T_pM)$. A transversal vector field induces a connection $\tilde{\nabla}$ on $M$ by the decomposition

$$D_XY = \tilde{\nabla}_XY + h(X,Y)\eta, \tag{1.1}$$

where $D$ denotes the standard connection in $\mathbb{R}^n$, and $X, Y$ are smooth vector fields in $M$. Observe that $h$ is symmetric bilinear form and $\tilde{\nabla}$ is *torsion-free*, meaning that $\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = 0$. It is easy to see that $h$ is bilinear. We will prove that $h$ is symmetric and $\tilde{\nabla}$ is torsion-free. Of the previous equation, we have

$$[X,Y] - (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = (h(X,Y) - h(Y,X))\eta$$

hence

$$h(X,Y) - h(Y,X) = \frac{\langle [X,Y] - (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X), \xi \rangle}{\langle \eta, \xi \rangle}$$

as $[X,Y] - (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X)$ is tangent, the first result follows. As the connection $D$ is torsion-free we have

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X,Y] - (h(X,Y) - h(Y,X))\eta$$

follows that $\tilde{\nabla}$ is torsion-free from the $h$ symmetry.

The map $h$ is a bilinear form which is called *affine fundamental form*, and it is clear that if $\eta$ is the (Euclidean) unit normal map of $M$, then $h$ is the usual second fundamental form.

We denote by det the usual determinant in $\mathbb{R}^n$. Any transversal vector field $\eta$ on $M$ induces an $n$-dimensional volume element by

$$\omega(X_1, \ldots, X_{n-1}) = \det(X_1, \ldots, X_{n-1}, \eta).$$

For a very complete tract on transversal vector fields (oriented towards affine differential geometry) we refer the reader to the book [25].

## 1.2 The Birkhoff-Gauss map and curvature

Assume that $x : M \to (\mathbb{R}^n, ||\cdot||)$ is an immersed oriented hypersurface. Hence we have two globally defined smooth vector fields $\eta$ such that $||\eta(p)|| = 1$ and $\eta(p) \dashv_B T_pM$ for each $p \in M$. The choice of such a vector field is the

*Birkhoff-Gauss map* of $M$, and it can clearly be seen both as a transversal vector field and as a map $\eta : M \to \partial\mathbb{B}$. We shall prove that $\eta$ is *equiaffine*, that is, the derivative $D_X\eta$ is always tangential.

**Proposition 1.1.** *For any smooth vector field $X$ on $M$ and any $p \in M$, we have that $D_X\eta(p) \in T_pM$. Consequently, regarding the natural identification $T_pM \simeq T_{\eta(p)}\partial\mathbb{B}$.*

*Proof.* If $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = X$, then $\eta \circ \gamma$ is a smooth curve on the unit sphere $\partial\mathbb{B}$. Therefore,

$$D_X\eta(p) = \frac{d}{dt}(\eta \circ \gamma)\bigg|_{t=0} \in T_{\eta(p)}\partial\mathbb{B} \simeq T_pM.$$

Notice that the identification $T_pM \simeq T_{\eta(p)}\partial\mathbb{B}$ comes from the definition of $\eta$, and from the uniqueness of Birkhoff orthogonality.

$\square$

Recall that we consider that $(\mathbb{R}^n, ||\cdot||)$ is endowed with the standard inner product $\langle \cdot, \cdot \rangle$, which induces an Euclidean norm that we shall denote by $||\cdot||_e$. The corresponding Euclidean unit ball and unit sphere will be denoted by $\mathbb{B}_e$ and $\mathbb{S}^{n-1}$, respectively. The Minkowski unit sphere $\partial\mathbb{B}$ is an embedded hypersurface, and hence it has an outward pointing Euclidean Gauss map. We denote the inverse of this map by $u : \mathbb{S}^{n-1} \to \partial\mathbb{B}$. We also denote by $\xi : M \to \mathbb{S}^{n-1}$ the Euclidean Gauss map of $M$, note that $\eta = u \circ \xi$. For each $p \in M$, the linear maps $d\eta_p$, $du^{-1}_{\eta(p)}$ and $d\xi_p$ can be seen as endomorphisms of $T_pM$. Hence we have the natural identification $T_pM \simeq T_{\eta(p)}\partial\mathbb{B} \simeq T_{\xi(p)}\mathbb{S}^{n-1}$, see Figure 1.2.

**Proposition 1.2.** *For each $p \in M$, the map $d\eta_p : T_pM \to T_pM$ is self-adjoint with respect to the inner product $b = \langle du^{-1}_{\eta(p)}(\cdot), \cdot \rangle : T_pM \times T_pM \to \mathbb{R}$. In particular, $d\eta_p$ has a basis of orthonormal eigenvectors second $b$.*

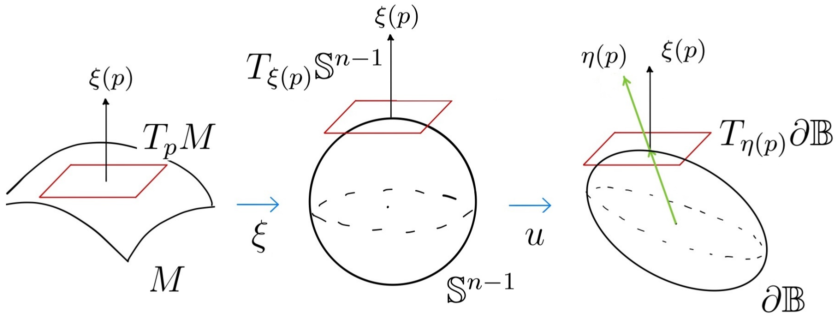


Figure 1.2: Birkhorff normal

*Proof.* This is a consequence of the fact that both maps $du^{-1}_{\eta(p)}$ and $d\xi_p$ are self-adjoint with respect to the standard inner product, since they are usual Gauss maps (actually, the fact that $b$ is symmetric comes from the self-adjointness of $du^{-1}$). We calculate

$$b(d\eta_p X, Y) = \langle du^{-1} \circ d\eta(X), Y \rangle = \langle d\xi(X), Y \rangle = \langle X, d\xi(Y) \rangle$$
$$= \langle X, du^{-1} \circ d\eta Y \rangle = b(X, d\eta_p Y),$$

where we omitted some subindices for the simplicity of the notation. This concludes the proof.

□

**Remark 1.3.** *The metric* $b = \langle du^{-1}_{\eta(p)}(\cdot), \cdot \rangle$ *is the* Dupin metric *of* $M$ *defined in [5].*

As a consequence of the last proposition, it follows that for each $p \in M$ the map $d\eta_p : T_pM \to T_pM$ has $n-1$ real eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ (possibly repeated). These numbers are called the *(Minkowski) principal curvatures* of $M$ at $p$. The respective eigenvectors are called the *(Minkowski) principal directions* of $M$ at $p$. The *(Minkowski) Gaussian curvature* and *(Minkowski)* *mean curvature* of $M$ at $p$ are defined as

$$K_m(p) := \det(d\eta_p) = \lambda_1 \cdot \ldots \cdot \lambda_{n-1} \quad \text{and}$$
$$H_m(p) := \frac{1}{n-1}\mathrm{tr}(d\eta_p) = \frac{\lambda_1 + \ldots + \lambda_{n-1}}{n-1}.$$

These are clearly analogues of the corresponding concepts in the Euclidean differential geometry, in the sense that they coincide with the Euclidean Gaussian and mean curvatures in the case where the norm is Euclidean.

We also have an useful formula for the affine fundamental form $h$ defined in (1.1), which was also derived for the three-dimensional case in [6].

**Lemma 1.4.** *If $\eta$ is the Birkhoff normal field of an immersed hypersurface $M$, $h$ stands for its associated affine fundamental form, and $\xi$ is the Euclidean Gauss map of $M$, then*

$$h(X,Y) = -\frac{\langle du^{-1}_{\eta(p)}Y, d\eta_p X\rangle}{\langle \eta, \xi\rangle}$$

*Proof.* For any $p \in M$ and any $X, Y \in T_pM$, we have that $D_XY = \tilde{\nabla}_XY + h(X,Y)\eta$. Thus,

$$\langle D_XY, \xi\rangle = \langle \nabla_XY, \xi\rangle + h(X,Y)\langle \eta, \xi\rangle = h(X,Y)\langle \eta, \xi\rangle.$$

On the other hand, $\langle D_XY, \xi\rangle = X\langle Y, \xi\rangle - \langle Y, D_X\xi\rangle = -\langle Y, d\xi_pX\rangle$. Now,

$$\begin{aligned} h(X,Y) &= \frac{\langle D_XY, \xi\rangle}{\langle \eta, \xi\rangle} = -\frac{\langle Y, d\xi_pX\rangle}{\langle \eta, \xi\rangle} = -\frac{\langle Y, du^{-1}_{\eta(p)} \circ d\eta_pX\rangle}{\langle \eta, \xi\rangle} \\ &= -\frac{\langle du^{-1}_{\eta(p)}Y, d\eta_pX\rangle}{\langle \eta, \xi\rangle}, \end{aligned}$$

where the last equality justifies since $u^{-1}$ is the Euclidean Gauss map of $\partial\mathbb{B}$, and hence $du^{-1}_{\eta(p)}$ is self-adjoint with respect to the usual inner product.

□

**Corollary.** *Let $\lambda, \sigma \in \mathbb{R}$ be distinct Minkowski principal curvatures of $M$ at a point $p$, and assume that $V_1, V_2 \in T_pM$ are respective Minkowski principal directions. Then, $b(V_1, V_2) = h(V_1, V_2) = 0$.*

*Proof.* First, notice that $b(V_1, V_2) = 0$, since $V_1$ and $V_2$ are eigenvectors of a linear map which is self-adjoint with respect to $b$. Therefore,

$$h(V_1, V_2) = -\frac{\langle du^{-1}_{\eta(p)} V_1, d\eta_p V_2 \rangle}{\langle \eta, \xi \rangle} = -\frac{\langle du^{-1}_{\eta(p)} V_1, \lambda_2 V_2 \rangle}{\langle \eta, \xi \rangle} = -\frac{\lambda_2}{\langle \eta, \xi \rangle} b(V_1, V_2) = 0.$$

□

We say that two non-zero vectors $X, Y \in T_pM$ are *conjugate* if $D_X Y$ is tangential (of course, this is independent of the considered extension of $Y$). From the Gauss equation (1.1) we have that two non-zero vectors $X, Y \in T_pM$ are conjugate if and only if $h(X, Y) = 0$. Hence the corollary above guarantees that Minkowski principal directions corresponding to distinct Minkowski principal curvatures are conjugate.

**Lemma 1.5.** *Let $x : M \to (\mathbb{R}^n, || \cdot ||)$ be an immersed hypersurface with Birkhoff-Gauss map $\eta$ and usual Euclidean Gauss map $\xi$. For any non-zero vectors $X, Y \in T_pM$ , the following statements are equivalent:*

*(a) the vectors $X$ and $Y$ are conjugate directions in the Euclidean sense,*

*(b) the derivative $D_X Y$ is tangential, and*

*(c) $h(X, Y) = 0$.*

*Proof.* From the equality $D_X Y = \tilde{\nabla}_X Y + h(X, Y)\eta$ we have that $D_X Y$ is tangential if and only if $h(X, Y) = 0$. Furthemore, we have

$$h(X, Y) = \frac{\langle D_X Y, \xi \rangle}{\langle \eta, \xi \rangle} = -\frac{\langle Y, D_X \xi \rangle}{\langle \eta, \xi \rangle}.$$

Since the derivative $D_X \xi$ at a point $p \in M$ is precisely $d\xi_p(X)$, the proof is complete. □

Let's define curvature line as in [6]. A regular connected curve $\gamma : J \subset \mathbb{R} \to M$ is said to be a curvature line if for each $t \in J$ the tangent vector $\gamma'(t)$ gives a principal direction at $\gamma(t)$. We will characterize the curvature lines of a hypersurface in a Minkowski space in a similar manner as it is done for the Euclidean subcase.

**Proposition 1.6.** *Let $\gamma : J \to M$ be a regular connected curve. Then $\gamma$ is a curvature line of $M$ if and only if there exists a function $\lambda : J \to \mathbb{R}$ such that*

$$(\eta \circ \gamma)'(t) = \lambda(t)\gamma'(t),$$

*for each $t \in J$.*

*Proof.* First suppose that $\gamma$ is a curvature line. Then, for each $t \in J$, we have that $\gamma'(t)$ is a principal direction, and therefore

$$(\eta \circ \gamma)'(t) = d\eta_{\gamma(t)}(\gamma'(t)) = \lambda(t)\gamma'(t),$$

where $\lambda(t)$ is an eigenvalue of $d\eta_{\gamma(t)}$.

Conversely, assume that $\gamma$ is a connected curve for which $(\eta \circ \gamma)'(t) = \lambda(t)\gamma'(t)$ holds for some function $\lambda : J \to \mathbb{R}$. For each $t \in J$ we have that $\gamma'(t)$ is an eigenvector of $d\eta_{\gamma(t)}$. Thus, $\gamma$ is a curvature line. □

## 1.3 An analogue of the normal curvature

Before we define normal curvature we will define circular curvature, following [4]. To define it, let $\gamma(s) : [0, l(\gamma)] \to (\mathbb{R}^2, ||\cdot||)$ be a curve parametrized by arc length, and assume that $\phi(t) : [0, l(S)] \to (\mathbb{R}^2, ||\cdot||)$ is a parametrization of the unit circle, $S$, by arc length. We let $t$ be the function which associates each $s \in [0, l(\gamma)]$ to the number $t(s) \in [0, l(S)]$ such that

$$\gamma'(s) = \frac{d\phi}{dt}(t(s))$$

In other words, $t(s)$ is the length traveled along the unit circle when the vector field $\gamma'(s)$ varies as its tangent field.

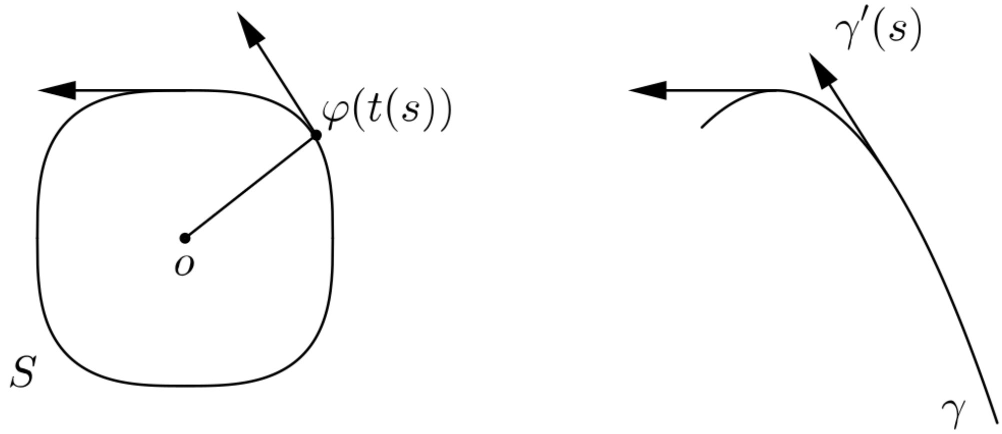


Figure 1.3: Circular curvature

We define the circular curvature of $\gamma$ measuring the variation of the length $t(s)$ with respect to $s$. Formally, the circular curvature of $\gamma$ at $\gamma(s)$ is given by

$$k_c(s) := t'(s).$$

Throughout this section we still always assume that the norm fixed in the space is admissible. As usual, we let $u : \mathbb{S}^{n-1} \to \partial\mathbb{B}$ be the inverse of the Euclidean Gauss map of $\partial\mathbb{B}$. Recall also that we are denoting by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^n$. Given an immersed hypersurface $x : M \to \mathbb{R}^n$, we still denote by $\eta$ and $\xi$ the Birkhoff-Gauss and usual Euclidean Gauss maps of $M$, respectively.

In Euclidean differential geometry, the normal curvature of a surface $M$ in a given point $p \in M$ and a given direction $X \in T_pM$ can be regarded as the (signed) length of the projection of the normal vector of a curve in $M$, passing through $p$ with tangent vector $X$, onto $\xi(p)$. In particular, the considered curve can be taken as the intersection of the plane spanned by

$\xi(p)$ and $X$, and therefore the normal curvature is the usual curvature of this (plane) curve at $p$ (see [13]). This observation allows us to extend this notion to our general case. Let $x : M \to (\mathbb{R}^n, ||\cdot||)$ be an immersed hypersurface, and fix $p \in M$ and $X \in S_p \subset T_pM$, where $S_p$ denotes the unit sphere of $T_pM$. Denote by $\pi$ the plane spanned by $\eta(p)$ and $X$. Let $\gamma : (-\epsilon, \epsilon) \to M$ be a local arc-length parametrization of the curve given by the intersection of the plane $p + \pi$ with $M$, and assume that $\gamma(0) = p$ and $\gamma'(0) = X$.

**Definition 1.7.** *The Minkowski normal curvature of $M$ at $p \in M$ in the direction $X \in S_p$ is the circular curvature of $\gamma$ at $p$ in the plane geometry endowed in $\pi$ by the norm $||\cdot||$ (in other words, the geometry in $\pi$ whose unit circle is the intersection of $\partial\mathbb{B}$ with $\pi$). We will denote this number by $k_{M,p}(X)$.*

We will give a formula for the Minkowski normal curvature in terms of the auxiliary Euclidean structure fixed in the plane. To do so, we first notice that this is essentially a problem in the plane $\pi$. Following [4], the circular curvature of $\gamma$ at $p$ is the ratio between its usual plane Euclidean curvature and the usual plane Euclidean curvature of the circle $\partial\mathbb{B}\cap\pi$ at a point whose tangent lies in the direction $X$.

**Theorem 1.8.** *For any $p \in M$ and $X \in T_pM$ we have*

$$k_{M,p}(X) = \frac{\langle du^{-1}_{\eta(p)}X, d\eta_p X\rangle}{\langle du^{-1}_{\eta(p)}X, X\rangle}, \tag{1.2}$$

*where we are considering the natural identification $T_pM \simeq T_{\eta(p)}\partial\mathbb{B} \simeq T_{\xi(p)}\mathbb{S}^{n-1}$.*

*Proof.* Let us first look at $\partial\mathbb{B}$ as an immersed hypersurface. The Euclidean normal curvature of $\partial\mathbb{B}$ at $\eta(p)$ in the direction $X$ is given by $\langle du^{-1}_{\eta(p)}X, X\rangle$

since $u^{-1}$ is the Euclidean Gauss map of $\partial\mathbb{B}$. Following [13], this normal curvature can be obtained from the curve $\phi := \partial\mathbb{B} \cap \pi$ as

$$-\langle du^{-1}_{\eta(p)}X, X\rangle = k_\phi(\eta(p))\langle \zeta, \xi(p)\rangle \tag{1.3}$$

where $k_\phi(\eta(p))$ is the (plane) Euclidean curvature of the curve $\phi$ at $\eta(p)$, and $\zeta$ is the unit Euclidean normal vector to $X$ at the plane $\pi$, which is also the Euclidean normal vector of the curve $\phi$ at $\eta(p)$. On the other hand, the Euclidean normal curvature of $M$ at $p$ in the direction $X$ is given by $\langle d\xi_p X, X\rangle$, since $\xi$ is the Euclidean Gauss map of $M$. As in the previous argument, this normal curvature can be obtained from the curve $\gamma$ as

$$-\langle d\xi_p X, X\rangle = k_\gamma(p)\langle \zeta, \xi(p)\rangle \tag{1.4}$$

where $k_\gamma(p)$ is the (plane) Euclidean curvature of $\gamma$ at $p$. Now, from (1.3) and (1.4) we have

$$k_{M,p}(X) = \frac{k_\gamma(p)}{k_\phi(\eta(p))} = \frac{\langle d\xi_p X, X\rangle}{\langle du^{-1}_{\eta(p)}X, X\rangle}$$

Since $\xi = u^{-1} \circ \eta$, and since the differential of the Euclidean Gauss map of any immersed hypersurface is self-adjoint at any point, it follows that $\langle d\xi_p X, X\rangle = \langle du^{-1}_{\eta(p)}X, d\eta_p X\rangle$. This gives equality (1.2). □

## 1.4 Area and Volume

Let $M$ be an $n$ oriented differentiable manifold and $x : M \to \mathbb{R}^n$ a smooth immersion. The usual determinant in $\mathbb{R}^n$ induces an area element in $x$ as

$$d\omega(X_1, \cdots, X_{n-1}) := \det(X_1, \cdots, X_{n-1}, \eta),$$

for each $X_1, \cdots, X_{n-1} \in TM$, where $\eta$ is the Birkhorff-Gauss map of $M$. With such an area element, that we will call Minkowski area measure, the

*Minkowski area* of an open bounded subset $D \subset M$ is, following [7], given as

$$A_{m,D}(x) := \int_D d\omega$$

Observe that if $\xi$ is the Gauss map of $M$ we can see

$$A_{m,D}(x) = \int_D \langle \eta, \xi \rangle dS$$

where $dS$ is the area element induced by immersion $x$.

We will define the volume of $x$ as in the Euclidean case

$$V_D(x) := \frac{1}{n} \int_D \langle x, \xi \rangle dS$$

that seen in another way, it is written as

$$V_D(x) := \frac{1}{n} \int_D \frac{\langle x, \xi \rangle}{\langle \eta, \xi \rangle} dw$$

where $\rho := \frac{\langle x,\xi \rangle}{\langle \eta,\xi \rangle}$ is the Birkhoff normal component of $x$, that is, $\rho$ is the projection to the second coordinate in the direct-sum decomposition $\mathbb{R}^n = T_pM \oplus \langle \eta \rangle$ of $x$. The definition of $V$ is justified by the fact that when $x$ is an embedding and $M$ is closed, $V$ represents the volume of the interior of $M$.

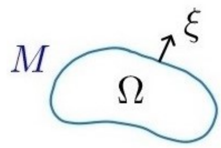


Figure 1.4: $n\,\mathrm{vol}(\Omega) = \int_\Omega \operatorname{div} \mathrm{id}(x)\, dx = \int_M \langle x, \xi \rangle dS.$

A variation of $x : M \to \mathbb{R}^n$ is a smooth function $F : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$ such that for every $t \in (-\varepsilon, \varepsilon)$, the function $F^t = F(t, \cdot)$ is also an immersion and $F(0, p) = x(p)$, $\forall p \in M$. For such a variation, the area and volume defined above give one-parameter functions

$$A_m(t) = A_m(F^t), \quad V(t) = V(F^t).$$

We say that a variation has compact support if $F(t,p) = x(p)$ for every $p$ outside a compact subset of $M$.

It is known that *minimal hypersurfaces*, this is, hypersurfaces with zero mean curvature, arise naturally as critical points of the area function, whereas CMC hypersurfaces (hypersurfaces with constant mean curvature) correspond to the critical points of the area with restricted volume. More precisely, we say that an immersion is *minimal* if for every variation with compact support we have $A'_m(0) = 0$. It is known that for any immersion and any variation (see Theorem 2.20),

$$A'_m(0) = \int_M (n-1)H_e(p)\langle F_t(0,p), \xi(p)\rangle dS$$

where $H_e$ denotes the mean curvature and $F_t(0,p) = \frac{\partial}{\partial t}F(0,p)$ the variational field, so minimal immersions are characterized as immersions with zero mean curvature. Minimal hypersurfaces arise naturally as solutions of the Plateau problem, that is finding a hypersurface with prescribed boundary, minimizing the hypersurface area measure. If we restrict to immersions having a fixed volume, we see from the well known formula

$$V'(0) = \int_M \langle F_t(0,p), \xi(p)\rangle dS$$

that the immersions such that $A'_m(0) = 0$ for every volume-preserving variation of compact support, are precisely those with constant mean curvature. We will see later, in the sections 2.5 and 2.4 that such a characterization also happens in the Minkowskian case.

# Chapter 2

# Some topics in differential geometry of normed spaces

In this chapter we present extensions for any dimension, from Differential Geometry results with $n = 3$ proposed by Balestro, Martini, Teixeira in the articles [6], [7], [8]. The key point is in the section 2.1, where we prove that the signs of the principal curvatures (positive and negative) are, in equal quantity, in any two Minkowski geometries and this allowed us to extend results, almost without modification, as Hadamard-type theorems, which we present in section 2.2, Global theorems.

Section 2.3 contemplates Weyl's tube formula. In [7] the authors observed that such a formula, in this case $n = 3$, could be obtained without making use of a particular parameterization. Here we present it for a general parametrization, for any dimension $n$. Lemma 2.15 is the key point for this extension.

In [8] the authors provide a formula for the first variation of the area for a particular variation, in dimension $n = 3$. In section 2.4 we present a formula for the first variation of the area for general variations. With our formula we can see that hypersurfaces with $H_m = 0$ are critical points of the functional area, as in the Euclidean case.

We close the chapter with section 2.5 where we present the definition of volume in analogy with the Euclidean case and use section 2.4 to verify that an immersion $x : M \to \mathbb{R}^n$ has Minkowski mean curvature $H_m$ constant if and only if it is critical points for the functional area for variations that preserve volume.

## 2.1 Principal curvatures: Relationship between two geometries

The Lemma 2.5 is the base result for this section, he says that vectors which are linear combinations of Minkowski principal directions associated with positive (negative) Minkowski principal curvatures are directions with Euclidean normal curvature positive (negative). With it we prove the Theorem 2.7, which relates the signs of the principal curvatures of any two Minkowski geometries.

**Lemma 2.1.** *There exists a base of Minkowski principal directions of $T_pM$, namely $\{V_1, ..., V_{n-1}\}$, such that, for all $i \neq j$ one has*

$$\langle du^{-1}_{\eta(p)} V_i, V_j \rangle = 0.$$

*In particular the affine fundamental form satisfies, $h(V_i, V_j) = 0$, but not $h(V_i, V_i) \neq 0$, due the existence of null eigenvalues.*

*Proof.* For each $i = 1, ..., n-1$ let $\lambda_i$ consider the Minkowski principal curvature associated the $V_i$, this is, $d\eta_p V_i = \lambda_i V_i$. First suppose $\lambda_i \neq \lambda_j$, in this case, assume $\lambda_i \neq 0$. Thus,

$$\begin{aligned}\langle du^{-1}_{\eta(p)} V_i, V_j\rangle &= \frac{1}{\lambda_i}\langle du^{-1}_{\eta(p)} d\eta_p V_i, V_j\rangle \\ &= \frac{1}{\lambda_i}\langle d\xi_p V_i, V_j\rangle \\ &= \frac{1}{\lambda_i}\langle V_i, d\xi_p V_j\rangle \\ &= \frac{1}{\lambda_i}\langle V_i, du^{-1}_{\eta(p)} d\eta_p V_j\rangle \\ &= \frac{\lambda_j}{\lambda_i}\langle V_i, du^{-1}_{\eta(p)} V_j\rangle \\ &= \frac{\lambda_j}{\lambda_i}\langle du^{-1}_{\eta(p)} V_i, V_j\rangle,\end{aligned}$$

since $\lambda_i \neq \lambda_j$ the result follows.

Otherwise, if $\lambda_i = \lambda_j$, with $i \neq j$, this is, $d\eta_p V_i = \lambda_i V_i$ and $d\eta_p V_j = \lambda_i V_j$, given any $V = \alpha V_i + \beta V_j$ we have $d\eta_p V = \lambda_i V$. So choose

$$\tilde{V}_j = V_j - \frac{\langle du^{-1}_{\eta(p)} V_j, V_i\rangle}{\langle du^{-1}_{\eta(p)} V_i, V_i\rangle} V_i.$$

In this case, we have $\langle du^{-1}_{\eta(p)} V_i, \tilde{V}_j\rangle = 0$ and $du^{-1}_{\eta(p)} \tilde{V}_j = \lambda_j \tilde{V}_j$, therefore just change $V_j$ for $\tilde{V}_j$ on the base given initially. $\square$

Another way to obtain such a result is via the spectral Theorem. Once the operator $d\eta$ is self-adjoint with respect to the metric Dupin, $b : X, Y \mapsto \langle du^{-1} X, Y\rangle$.

Remember that we denote $k_{M,p}(V)$ the Minkowski normal curvature of $M$ at $p$, in direction $V$. We will use the equation (1.2) here, which says

$$k_{M,p}(V) = \frac{\langle du^{-1}_{\eta(p)} V, d\eta_p V\rangle}{\langle du^{-1}_{\eta(p)} V, V\rangle}.$$

Observe that if $V \in T_pM$ is a Minkowski principal direction at $p \in M$, with associated Minkowski principal curvature $\lambda$, then $k_{M,p}(V) = \lambda$. Of course, we have $d\eta_p V = \lambda V$, thus

$$k_{M,p}(V) = \frac{\langle du^{-1}_{\eta(p)}V, \lambda V\rangle}{\langle du^{-1}_{\eta(p)}V, V\rangle} = \lambda.$$

For the Theorem below denote by $\lambda_i$ the Minkowki principal curvature of $M$ at $p$ associated with the direction $V_i$.

In the Euclidean subcase, the principal curvatures of $M$ at a point $p$ are characterized by maximum and the minimum values of the normal curvature in this point. This is also true in the general Minkowski case. More precisely, we have.

**Theorem 2.2.** *Consider $\lambda_1, \cdots, \lambda_{n-1}$ Minkowski principal curvatures of $M$ at $p$ ordered such that*

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{n-1},$$

*and $W_i$ the space generated by the principal directions $V_1, \cdots, V_i$, as in the Lemma 2.1. Then*

$$\lambda_{n-1} = \max_{V \in T_pM} k_{M,p}(V),$$

$$\lambda_1 = \min_{V \in T_pM} k_{M,p}(V)$$

$$\lambda_i = \max_{V \in W_i} k_{M,p}(V),$$

*Proof.* Choose a base for $T_pM$ as in Lemma 2.1. Consider $V \in T_pM$ and write $V = \alpha_1 V_1 + ... + \alpha_{n-1}V_{n-1}$. Thus,

$$\begin{aligned}
k_{M,p}(V) &= \frac{\langle du^{-1}_{\eta(p)}V, d\eta_p V\rangle}{\langle du^{-1}_{\eta(p)}V, V\rangle} \\
&= \frac{\langle du^{-1}_{\eta(p)}\left(\sum_{i=1}^{n-1}\alpha_i V_i\right), d\eta_p\left(\sum_{j=1}^{n-1}\alpha_j V_j\right)\rangle}{\langle du^{-1}_{\eta(p)}\left(\sum_{i=1}^{n-1}\alpha_i V_i\right), \sum_{j=1}^{n-1}\alpha_j V_j\rangle} \\
&= \frac{\sum_{i,j=1}^{n-1}\alpha_i\alpha_j\lambda_j\langle du^{-1}_{\eta(p)}V_i, V_j\rangle}{\sum_{i,j=1}^{n-1}\alpha_i\alpha_j\langle du^{-1}_{\eta(p)}V_i, V_j\rangle} \\
&= \frac{\sum_{i=1}^{n-1}\lambda_i\alpha_i^2\langle du^{-1}_{\eta(p)}V_i, V_i\rangle}{\sum_{i=1}^{n-1}\alpha_i^2\langle du^{-1}_{\eta(p)}V_i, V_i\rangle} \\
&\leq \lambda_{n-1} = k_{M,p}(V_{n-1}).
\end{aligned}$$

It shows that $\lambda_{n-1} = \max\limits_{V \in T_pM} k_{M,p}(V)$ and that $\lambda_1 = \min\limits_{V \in T_pM} k_{M,p}(V)$. Now take $V \in T_pM$ such that $\langle du^{-1}_{\eta(p)}V, V_{n-1}\rangle = 0$, this is, $V \in W_{n-2}$. Thus $V$ is of the form $V = \alpha_1 V_1 + ... + \alpha_{n-2}V_{n-2}$. Repeating the previous process we obtain $k_{M,p}(V) \leq \lambda_{n-2}$ for such a $V$ and therefore $\lambda_{n-2} = \max\limits_{V \in W_{n-2}} k_{M,p}(V)$. The theorem follows repeating this process until obtaining $\lambda_2$. □

**Corollary 2.3.** *The Minkowski normal curvature of an immersed connected hypersurface is constant if and only if this hypersurface is contained in a hyperplane or in a Minkowski sphere. The first case occurs if and only if* $k_{M,p} = 0$*, and in the second case the radius of the sphere is given by* $|k_{M,p}|^{-1}$

.

*Proof.* By the previous Theorem we have that all points of $M$ are umbilic, hence the result follows from Lemma 2.12, that we will see later. □

**Corollary 2.4.** *A point $p \in M$ is umbilic if and only if $k_{M,p}$ is constant in $T_pM\backslash\{0\}$. In this case, $k_{M,p}(V)$ equals the Minkowski principal curvature of $M$ at $p$, for all $V \in T_pM$.*

*Proof.* By Theorem 2.2, if $p$ is umbilic, the minimum and maximum of $k_{M,p}$ are equal, so the Minkowski principal curvatures are equal. Conversely, if the Minkowski principal curvatures are equal, the minimum and maximum of $k_{M,p}$ are equal and therefore $k_{M,p}$ is constant. □

An interesting result, which we will see next, is that Minkowski's geometry preserves the amount of positive, null and negative Euclidean principal curvatures of a hypersurface. We will be more precise in the Theorem 2.7, before, let's see the following lemma.

**Lemma 2.5.** *Let $V_1$, $V_2$, ... , $V_l$ be Minkowski's principal directions of $M$ at $p$ (as in Lemma 2.1) associated Minkowski's principal curvatures $\lambda_1$, ... , $\lambda_l$, all positive (respectively, all negative). If $V = \alpha_1 V_1 + \alpha_2 V_2 + \ ... + \alpha_l V_l$ then $k_{E,p}(V) > 0$ (respectively $k_{E,p}(V) < 0$), where $k_{E,p}(V)$ is the Euclidean's normal curvature of $M$ at $p$ in the direction $V$.*

*Proof.* Without loss of generality, consider $V$ unit with respect to the Euclidean metric.

$$\begin{aligned} K_{E,p}(V) &= \langle d\xi_p V, V\rangle \\ &= \langle du^{-1}_{\eta(p)} d\eta_p V, V\rangle \\ &= \big\langle du^{-1}_{\eta(p)}\big(\sum_i \lambda_i \alpha_i V_i\big), \sum_j \alpha_j V_j\big\rangle \\ &= \sum_{ij} \lambda_i \alpha_i \alpha_j \langle du^{-1}_{\eta(p)} V_i, V_j\rangle \\ &= \sum_i \lambda_i \alpha_i^2 \langle du^{-1}_{\eta(p)} V_i, V_i\rangle. \end{aligned}$$

since the bilinear form associated with $du^{-1}_{\eta(p)}$ is positively definited, from the above account we conclude that $K_{E,p}(V) > 0$ (respectively $K_{E,p}(V) < 0$) if $\lambda_i > 0$ (respectively $\lambda_i < 0$) for all $i = 1, ..., l$. □

**Corollary 2.6.** *Let $V_1$, $V_2$, ... , $V_l$ be Minkowski's principal directions of $M$ at $p$ (as in the Lemma 2.1) associated Minkowski's principal curvatures $\lambda_1$, ... , $\lambda_l$, all non negative (all non positive). If $V = \alpha_1 V_1 + \alpha_2 V_2 + ... + \alpha_l V_l$ then $K_{E,p}(V) \geq 0$ ($K_{E,p}(V) \leq 0$).*

As mentioned above, the following theorem presents a relationship between the signs of the Minkowskinian and Euclidean principal curvatures.

**Theorem 2.7.** *Let consider $M$ a hypersurface in $\mathbb{R}^n$, and a point $p$ in $M$. The signs of principal curvatures of $M$ at $p$ are equally distributed with respect to any two geometries.*

*Proof.* It is enough to show that the signs of the principal curvatures of $M$ in $p$ in any geometry are equally distributed with respect to Euclidean geometry. Suppose the signs of Minkowski's principal curvatures are divided as follows:

**m** positive; **q** negative and **z** null

Note that $\ker d\xi = \ker d\eta$, since $d\xi_p V = du_{\eta(p)}^{-1} d\eta_p V$, hence we will also have **z** zero Euclidean's principal curvatures. We claim that there are at most **q** negative Euclidean's principal curvatures. In fact, otherwise we would have a space $W_- \subset T_pM$ generated by the principal Euclidean directions associated with these negative Euclidean principal curvatures with $\dim W_- \geq q+1$ e $K_{E,p}(V) < 0$, whenever $V \in W_- \backslash \{0\}$. In addition, by Corollary 2.6 there is a space $W \subset T_pM$ with $K_{E,p}(V) \geq 0$, whenever $V \in W\backslash\{0\}$ and $\dim W \geq m + z$. Since $m + z + q + 1 > n - 1$ there would be $V \in W \cap W_-$, not null, which is absurd. The same argument tells us that we have at most $m$ positive Euclidean principal curvatures. This ends the demonstration. □

**Corollary 2.8.** *Assume, as usual, that $||\cdot||$ is an admissible norm, and let $x : M \to (\mathbb{R}^n, ||\cdot||)$ be an immersed hypersurface. Denote by $K_m$ and $K_e$ the Minkowski and the Euclidean Gaussian curvatures of $M$, respectively. For a point $p \in M$ , the following statements are equivalent:*

*(a)* $K_e(p) > 0$,

*(b)* $K_m(p) > 0$.

*Proof.* The proof that (a) $\iff$ (b) follows immediately from the previous theorem. □

With the previous result we are able to prove, as in [8], the versions of the Hadamard theorems for suitable hypotheses regarding the Minkowskian Gauss curvature.

## 2.2 Global theorems

With the proof of Theorem 2.7 the results of this section are obtained almost without modification of the Euclidean case and the Minkowski case to $n = 3$, presented in [8].

We assume that all the immersed hypersurfaces are complete in the Euclidean ambient geometry, meaning that all of the (Euclidean) geodesics are defined for all of the values of the parameter.

**Theorem 2.9.** *Let $x : M \to (\mathbb{R}^n, ||\cdot||)$ be a simply connected immersed hypersurface. If the Minkowskian Gauss curvature of $M$ is non-positive, then $M$ is diffeormorphic to a plane.*

*Proof.* We are assuming that $M$ is complete in the Euclidean geometry. From Theorem 2.7 it follows that the Euclidean Gauss curvature is non-positive. Hence the result comes as a consequence of the Hadamard theorem in Euclidean geometry. $\square$

**Theorem 2.10.** *Let $x : M \to (\mathbb{R}^n, ||\cdot||)$ be a compact, connected immersed hypersurface. If the Minkowskian Gauss curvature of $M$ is positive, then the Birkhoff-Gauss map $\eta : M \to \partial\mathbb{B}$ is a diffeomorphism.*

*Proof.* Again it follows from Theorem 2.7 that the Euclidean Gauss curvature of $M$ is positive. Therefore, the Euclidean Gauss map $\xi : M \to \mathbb{S}^{n-1}$ is a local diffeomorphism defined in a compact, consequently a covering map, whose image is simply connected, therefore a difeomorphism (see [[13], Section 5.6 B, Theorem 2]). Since the norm is admissible, the Minkowskian unit sphere $\partial B$ is itself a compact, connected immersed hypersurface with positive Euclidean Gauss curvature. It follows that $u^{-1} : \partial\mathbb{B} \to \mathbb{S}^{n-1}$ is a diffeomorphism. Hence also $\eta = u \circ \xi$ is a diffeomorphism. $\square$

We continue working with the area induced by Birkhoff-Gauss map (and the usual determinant in $\mathbb{R}^n$). In this case, the Minkowski area of the unit sphere $\partial\mathbb{B}$ is given by

$$A_m(\partial\mathbb{B}) := \int_{\partial\mathbb{B}} d\omega_{\partial\mathbb{B}}$$

In what follows, we say that a hypersurface is closed whenever it is compact and without boundary. For the sake of simplicity, all of the immersions are assumed to be embeddings.

**Theorem 2.11.** *Let $x : M \to (\mathbb{R}^n, ||\cdot||)$ be a closed hypersurface with positive Minkowski Gaussian curvature. Then*

$$\int_M K_m d\omega = A_m(\partial\mathbb{B})$$

*Proof.* Since $M$ is compact and its Minkowski Gaussian curvature is positive, it follows that the Birkhoff-Gauss map $\eta : M \to \partial\mathbb{B}$ is a diffeomorphism (see Theorem 2.10). Hence

$$\int_M K_m d\omega = \int_M \det(d\eta) d\omega = \int_{\partial\mathbb{B}} d\omega_{\partial\mathbb{B}} = A_m(\partial\mathbb{B}),$$

where the second equality comes from the standard changing of variables formula, and from the fact that, for each $p \in M$, the Birkhoff normal is the same for $M$ at $p$ and $\partial\mathbb{B}$ at $\eta(p)$. $\square$

Notice that the principal curvatures of a plane are 0 at any point. Also, any Minkowski sphere has constant, equal principal curvatures at each of its points. Indeed, the Birkhoff-Gauss map can be regarded as the map $\eta : M \to \partial\mathbb{B}$ given by $\eta(x) = \frac{1}{\rho}(x - p)$, where $p$ is the center $M$ and $\rho$ is the radius. Clearly, $d\eta_p = \frac{1}{\rho} Id_{T_pM}$, and hence the principal curvatures of $M$ at any point $p$ are $\frac{1}{\rho}$.

Remember that a point $p \in M$ is an umbilic point if the principal curvatures of $M$ at $p$ have the same value. Equivalently, a point $p \in M$ is umbilic when the differential of the Birkhoff normal vector field at $p$ is a multiple of the identity map. By the previous observation, all the points of a hyperplane or of a Minkowski sphere are umbilic. The next proposition states that, as in the Euclidean subcase, these are the only possible hypersurfaces with such property. Balestro, Martini and Teixeira proved it to dimension three, but it extends without modification to dimension $n$.

**Lemma 2.12** ([6], Proposition 4.5)**.** *A connected hypersurface immersed* $\mathbb{R}^n$*, all whose points are umbilic is contained in a plane or in a Minkowski sphere.*

*Proof.* For each $p \in M$ we have that $d\eta_p(X) = \lambda(p)X$ for any $X \in T_pM$, where $\lambda : M \to \mathbb{R}$ is a smooth function. Our first step is to prove that the function $\lambda$ is constant. For this sake, fix linearly independent vectors $X, Y \in T_pM$ and denote also by $X$ and $Y$ the parallel transport of $X$ through a curve tangent to $Y$ at $p$, and the parallel transport of $Y$ through a curve tangent to $X$ at $p$, both with respect to the induced connection $\tilde{\nabla}$, given by (1). We have then $\tilde{\nabla}_X Y|_p = \tilde{\nabla}_Y X|_p = 0$. Now, extending smoothly both vector fields to an open neighborhood of $p$, we may calculate at $p$

$$D_Y D_X \eta = D_Y(\lambda X) = Y(\lambda)X + \lambda h(Y, X)\eta \text{ and}$$

$$D_X D_Y \eta = X(\lambda)Y + \lambda h(X, Y)\eta.$$

Since $D$ is a flat connection, we write

$$0 = D_Y D_X \eta - D_X D_Y \eta - D_{[X,Y]}\eta = Y(\lambda)X - X(\lambda)Y - \lambda[X, Y].$$

Recalling that $\tilde{\nabla}$ is a torsion-free connection, it follows that $[X, Y] = 0$ at $p$. Hence we have $X(\lambda)Y - Y(\lambda)X = 0$, and this gives $X(\lambda) = Y(\lambda) = 0$ (since $X$ and $Y$ are linearly independent). This argument shows that the derivative of the function $\lambda$ at any point $p \in M$ and with respect to any direction $X \in T_pM$ equals 0. It follows that $\lambda$ is constant. $\lambda = 0$, then the Birkhoff-Gauss map is constant, and this means that the Birkhoff normal vector is the same for each point of $M$. In particular, the Euclidean normal vector is also the same for every point, and therefore $M$ is contained in a plane. If $\lambda \neq 0$, then the map $p \in M \mapsto p - \frac{1}{\lambda}\eta(p) \in \mathbb{R}^n$ is clearly a constant map. Indeed, for any point $p \in M$ and any direction $X \in T_pM$ , we have

$$D_X\Big(p - \frac{1}{\lambda}\eta(p)\Big) = X - \frac{1}{\lambda}(\lambda X) = 0.$$

Thus, $M$ is contained in the Minkowski sphere whose center is this constant point, and whose radius equals $\frac{1}{\lambda}$. □

## 2.3 Weyl's tube formula and intrisic volumes

In this section we extend some results from [7]. In that paper the authors make use of a parameterization whose coordinate curves are curvature lines. The key point here is the Lemma 2.15, that allows us to work with a general parameterization, without using continuity arguments to deal with umbilic points.

To begin the section we will give an interpretation, in analogy to the Euclidean case, of the Minkowski Gaussian curvature in terms of the Birkhoff Gauss map $\eta$. As in [13] we will make the convention that the area of a region contained in a connected neighborhood $D \subset M$, where $K_m \neq 0$, and the area of its image by $\eta$ have the same sign if $K_m > 0$ in $D$, and opposite signs if $K_m < 0$ in $D$ (since $D$ is connected, $K_m$ does not change sign in $D$).

**Theorem 2.13.** *Let $p \in M$ be a point where $K_m(p) \neq 0$, and let $U \subset M$ be a connected neighborhood of $p$ where the sign of $K_m$ does not change. Then*

$$K_m(p) = \lim_{D \to p} \frac{A_{\partial\mathbb{B}}(\eta(D))}{A_M(D)}$$

*where $D \subset U$ and $\eta(D) \subset \partial\mathbb{B}$ denotes the image of $D$ under the Birkhoff-Gauss map. $A_{\partial\mathbb{B}}(\eta(D)) = \int_{\eta(D)} d\omega_{\partial\mathbb{B}}$ and $A_M(D) = \int_D d\omega$.*

*Proof.* Let $\phi : V \to D$ be a local parametrization, where $V$ is an open of $\mathbb{R}^{n-1}$. The Minkowski area of $D$ writes

$$A_M(D) = \int_V d\omega(\phi_{u_1}, \cdots, \phi_{u_{n-1}}) du_1 \cdots du_{n-1},$$

where $\phi_{u_i}$ is a coordinate vector field. Since $K_m(p) \neq 0$, by the inverse function theorem one can take $D$ small enough such that the restriction $\eta|_D$ is a diffeomorphism onto its image. In that case, the map $\eta \circ \phi : V \to \partial\mathbb{B}$ becomes a local parametrization of $\eta(D)$. Using that $T_{\eta(p)}\partial\mathbb{B}$ is parallel to $T_pM$, the Minkowski area of this region is calculated, using the above convention, by

$$\begin{aligned} A_{\partial\mathbb{B}}(\eta(D)) &= \int_V d\omega_{\partial\mathbb{B}}(d\eta(\phi_{u_1}), \cdots, d\eta(\phi_{u_{n-1}})) du_1 \cdots du_{n-1} \qquad (2.1) \\ &= \int_V \det(d\eta) d\omega_{\partial\mathbb{B}}(\phi_{u_1}, \cdots, \phi_{u_{n-1}}) du_1 \cdots du_{n-1} \\ &= \int_V \det(d\eta) d\omega(\phi_{u_1}, \cdots, \phi_{u_{n-1}}) du_1 \cdots du_{n-1} \end{aligned}$$

Denoting by $A(V)$ the usual area of $V$, we get from the mean value theorem for integrals that for each region $D$ there exists a point $p_1 \in D$ such that

$$\begin{aligned} d\omega_{p_1}(\phi_{u_1}, \cdots, \phi_{u_{n-1}}) &= \frac{1}{A(V)} \int_V d\omega(\phi_{u_1}, \cdots, \phi_{u_{n-1}}) du_1 \cdots du_{n-1} \\ &= \frac{A_M(D)}{A(V)}, \end{aligned}$$

and the same holds for the other integral, for some point $p_2 \in D$. As $D \to p$, we get that $p_1, p_2 \to p$, and hence continuity yields

$$\begin{aligned} \lim_{D \to p} \frac{A_{\partial\mathbb{B}}(\eta(D))}{A_M(D)} &= \lim_{D \to p} \frac{\frac{A_{\partial\mathbb{B}}(\eta(D))}{A(V)}}{\frac{A_M(D)}{A(V)}} \\ &= \lim_{D \to p} \frac{\det(d\eta_{p_2}) d\omega_{p_2}(\phi_{u_1}, \cdots, \phi_{u_{n-1}})}{d\omega_{p_1}(\phi_{u_1}, \cdots, \phi_{u_{n-1}})} \\ &= K_m(p) \end{aligned}$$

completing the proof. $\square$

A homothety of the space is a map $F : \mathbb{R}^n \to \mathbb{R}^n$ given as $F(p) = cp$ for some constant $c > 0$. As a consequence of the previous theorem we will describe what happens with the Minkowski Gaussian curvature of an immersed hypersurface under a homothety.

**Corollary 2.14.** *Let $M$ be an immersed hypersurface in $(\mathbb{R}^n \ , ||\cdot||)$, and $c > 0$ be a constant. For each $p \in M$, the Minkowski Gaussian curvature $\overline{K_m}(F(p))$ of the image $\overline{M}$ of $M$ by the homothety $F(p) = cp$ at $F(p)$ is given as*

$$\overline{K_m}(F(p)) = \frac{K_m(p)}{c^{n-1}}$$

*where $K_m(p), \overline{K_m}(F(p)) \neq 0$.*

*Proof.* Let $p \in M$ and $\phi(u_1, \cdots, u_{n-1}) : V \to D$ be a local parametrization of a neighborhood $D \subset M$ of $p$. Then, $c\phi : V \to D$ is a local parametrization of $F(D) = \overline{D}$ around $F(p)$, and

$$\begin{aligned} A_{\overline{M}}(\overline{D}) &= \int_V d\omega_{\overline{M}}(c\phi_{u_1}, \cdots, c\phi_{u_{n-1}})du_1 \cdots du_{n-1} \\ &= c^{n-1}A_M(D). \end{aligned}$$

Where we use that the normal vector to $M$ at any $\tilde{p} \in D$ is the same as the normal vector to $\overline{M}$ at $F(\tilde{p}) \in \overline{D}$, thus $\eta(D) = \eta_{\overline{M}}(\overline{D})$, where symbol $\eta$ is the Birkhof normal to $M$ and $\eta_{\overline{M}}$ the Birkhoff normal to $\overline{M}$. Therefore,

$$A_{\partial B}(\eta_{\overline{M}}(\overline{D})) = A_{\partial B}(\eta(D)).$$

Now we just calculate, by the previous theorem

$$\begin{aligned}\overline{K_m}(F(p)) &= \lim_{\overline{D}\to F(p)} \frac{A_{\partial B}(\eta_{\overline{M}}(\overline{D}))}{A_{\overline{M}}(\overline{D})}\\ &= \lim_{D\to p} \frac{A_{\partial B}(\eta(D))}{c^{n-1}A_M(D)}\\ &= \frac{1}{c^{n-1}}\lim_{D\to p} \frac{A_{\partial B}(\eta(D))}{A_M(D)}\\ &= \frac{K_m(p)}{c^{n-1}}.\end{aligned}$$

where we use the clear fact that $\overline{D} \to F(p)$ if and only if $D \to p$. □

This can be also obtained, without restrictions on Minkowski Gaussian curvature in $p$ and $F(p)$, from the fact that the principal curvatures of $M$ are divided by $c$ under the homothety $p \mapsto cp$. In fact, consider $\gamma : I \to M$, with $\gamma(0) = p$, $\gamma'(0) = v$ and $d\eta_p v = \lambda v$, note that $\eta_{\overline{M}}(F(\gamma(t))) = \eta(\gamma(t))$, calculating the derivative at $t = 0$, we have $d(\eta_{\overline{M}})_{F(p)} cv = d\eta_p v = \lambda v$.

Given a hypersurface $M \subset \mathbb{R}^n$ (recall that we are identifying $x(M)$ with $M$), a parallel hypersurface of $M$ is a hypersurface $\overline{M} := \{p + c\eta(p) : p \in M\}$ for some constant $c \in \mathbb{R}$. As a further consequence of Theorem 2.13 we get a formula for the Minkowski Gaussian curvature of a parallel hypersurface in a regular point (a parallel hypersurface can have singular points). Before, let's look at the following Lemma.

**Lemma 2.15.** *Let $\overline{M}$ be a parallel hypersurface as defined above. Then the Minkowski area of an open bounded subset $\overline{D} \subset \overline{M}$, of the image $\overline{D}$ of $D$ by homothety $F(p) = p + c\eta(p)$ is given as*

$$A_{\overline{M}}(\overline{D}) = \int_D \prod_{i=1}^{n-1}(1 + c\lambda_i)d\omega$$

*where* $|c| < \frac{1}{\max\limits_{i=1,\cdots,n-1}|\lambda_i(p)|}, \forall p \in D$; *and $\lambda_i(p)$ denotes a Minkowski principal curvature of $M$ at $p$.*

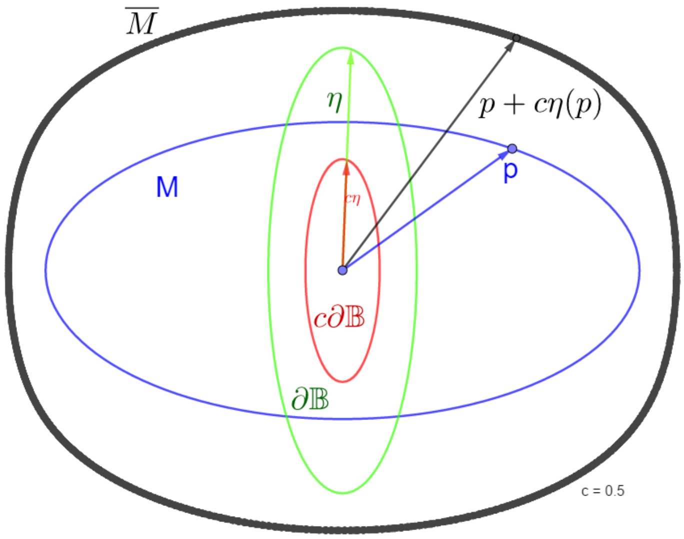


Figure 2.1: Parallel Hypersurface.

*Proof.* Let $\phi : V \to D$ be a local parametrization of neighborhood of $p \in M$. The map $\psi : V \to \overline{D}$ defined as $\psi(q) = \phi(q) + c\eta(\phi(q))$ is a local parametrization of the neighborhood $\overline{D}$ of $p + c\eta(p) \in \overline{M}$. We have

$$\psi_{u_i} = \phi_{u_i} + cd\eta\phi_{u_i},$$

for all $i = 1, \cdots, n-1$ and also, that the Birkhof normal to $M$ at $p$ is the same as the Birkhoff normal to $\overline{M}$ at $p + c\eta(p)$. If we define $G$ and $\overline{G}$ the matrix of $d\eta_p$ and $d\psi_{\phi^{-1}(p)}$ in the base $\{\phi_{u_1}, \cdots, \phi_{u_{n-1}}\}$, we have

$$\overline{G} = Id + cG$$

Thus

$$\det \overline{G} = \det(A^{-1}A + c\,A^{-1}\,D\,A) = \prod_{i=1}^{n-1}(1 + c\lambda_i(p))$$

where $A$ is invertible and $D$ is the diagonal matrix of the principal Minkowski curvature $\lambda_i$ of $M$. Therefore

$$d\omega_{F(p)}(\psi_{u_1}, \cdots, \psi_{u_{n-1}}) = \prod_{i=1}^{n-1}|1 + c\lambda_i(p)|d\omega_p(\phi_{u_1}, \cdots, \phi_{u_{n-1}}),$$

and using the restriction on $c$ the result follows. □

**Theorem 2.16.** *Let $M$ be an immersed surface with Minkowski Gaussian curvature $K_m$. Given a constant $c \in \mathbb{R}$ as in the previous theorem, let $\overline{M}$ be the parallel surface as defined above. Then its Minkowski Gaussian curvature is given by the formula*

$$\overline{K_m}(p + c\eta(p)) = \frac{K_m(p)}{\prod_{i=1}^{n-1}(1 + c\lambda_i(p))},$$

*where $K_m(p), \overline{K_m}(F(p)) \neq 0$.*

*Proof.* Using the notation of the previous Lemma we get that

$$A_{\overline{M}}(\overline{D}) = \int_V \prod_{i=1}^{n-1}(1 + c\lambda_i) d\omega(\phi_{u_1}, \cdots \phi_{u_{n-1}}) du_1 \cdots du_{n-1}$$

and from the mean value theorem for integrals we get that

$$\frac{A_{\overline{M}}(\overline{D})}{A(V)} = \prod_{i=1}^{n-1}(1 + c\lambda_i(p_1)) d\omega_{p_1}(\phi_{u_1}, \cdots \phi_{u_{n-1}})$$

for some $p_1 \in V$. From equality (2.1) and, using again the argument of the mean value theorem for the ratio $\frac{A_{\partial\mathbb{B}}(\eta(D))}{A(V)}$ we calculate the Minkowski Gaussian curvature of $\overline{M}$ at $p + c\eta(p)$ as

$$\overline{K_m}(p + c\eta(p)) = \lim_{D \to p} \frac{A_{\partial\mathbb{B}}(\eta(\overline{D}))}{A_{\overline{M}}(\overline{D})}$$

$$= \lim_{D \to p} \frac{\frac{A_{\partial\mathbb{B}}(\eta(D))}{A(V)}}{\frac{A_{\overline{M}}(\overline{D})}{A(V)}}$$

$$= \lim_{D \to p} \frac{K_m(p_2) d\omega_{p_2}(\phi_{u_1}, \cdots, \phi_{u_{n-1}})}{\prod_{i=1}^{n-1}(1 + c\lambda_i(p_1)) d\omega_{p_1}(\phi_{u_1}, \cdots, \phi_{u_{n-1}})}.$$

Follow that

$$\overline{K_m}(p + c\eta(p)) = \frac{K_m(p)}{\prod_{i=1}^{n-1}(1 + c\lambda_i(p))}$$

where we again use the fact that $p_1, p_2 \to p$ as $D \to p$. □

Again, this can be also obtained, without restrictions on Minkowski Gaussian curvature in $p$ and $F(p) = p + c\eta(p)$, from the fact that the principal curvatures $\lambda$ of $M$ are divided by $1 + c\lambda$ under the homothety $F$. In fact, consider $\gamma : I \to M$, with $\gamma(0) = p$, $\gamma'(0) = v$ and $d\eta_p v = \lambda v$, note that $\eta_{\overline{M}}(F(\gamma(t))) = \eta(\gamma(t))$, calculating the derivative at $t = 0$, we have $d(\eta_{\overline{M}})_{F(p)}(1 + c\lambda)v = d\eta_p v = \lambda v$.

Now let's get an analogue of Weyl's tube formula, which characterizes the volume of the set of the points $\epsilon$-next to a surface $M$ as a polynomial of degree $n$ in the variable $\epsilon$ ($> 0$, say). In what follows, the volume in $\mathbb{R}^n$ is given by the usual determinant.

**Theorem 2.17** (Weyl's tube formula). *Let $M$ be a surface in $(\mathbb{R}^n, || \cdot ||)$, and let $M_\epsilon$ be the $\epsilon$-tube defined as*

$$M_\epsilon := \{z \in \mathbb{R}^n : \text{dist.}(z, M) \leq \epsilon\},$$

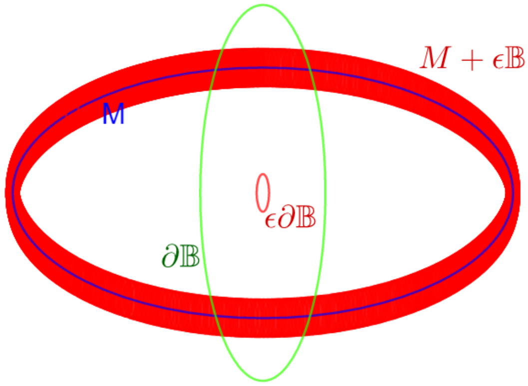


Figure 2.2: $\epsilon$-tube

*where* $\text{dist.}(z, M) = \inf\{||z - p|| : p \in M\}$*. For sufficiently small* $\epsilon > 0$*, the volume of* $M_\epsilon$ *is given by the formula*

$$\text{vol}(M_\epsilon) = 2\epsilon A_M(M) + \sum_{k=1}^{n-1} \frac{t^{k+1}}{k+1}\bigg|_{-\epsilon}^{\epsilon} \int_M \sum_{i_1<\cdots<i_k}^{n-1} \prod_{j=1}^{k} (\lambda_{i_j})\, d\omega dt$$

*Proof.* The idea of the proof is to see $M_\epsilon$ as a family of parallel hypersurfaces, as in Lemma 2.15. Given a parametrization $\phi : V \to D$ of a neighborhood $D \subset M$, for sufficiently small $\epsilon > 0$ one can parametrize $D_\epsilon := \{z \in \mathbb{R}^n : \text{dist.}(z, D) \leq \epsilon\}$ as

$$\psi(q, t) = \phi(q) + t\eta(\phi(q)),$$

for $(q, t) \in V \times (-\epsilon, \epsilon)$. Write $(q, t) = x$ for simplicity, hence the volume of $D_\epsilon$ is calculated as

$$\begin{aligned}
\text{vol}(D_\epsilon) &= \int_{V\times(-\epsilon,\epsilon)} \det(\psi_{u_1}, \cdots, \psi_{u_{n-1}}, \psi_t) dx \\
&= \int_{D\times(-\epsilon,\epsilon)} \prod_{i=1}^{n-1} (1 + t\lambda_i) d\omega dt \\
&= \int_{D\times(-\epsilon,\epsilon)} \left(1 + \sum_{k=1}^{n-1} \sum_{i_1<\cdots<i_k}^{n-1} \prod_{j=1}^{k} (t\lambda_{i_j})\right) d\omega dt \\
&= 2\epsilon A_M(D) + \sum_{k=1}^{n-1} \frac{t^{k+1}}{k+1}\bigg|_{-\epsilon}^{\epsilon} \int_D \sum_{i_1<\cdots<i_k}^{n-1} \prod_{j=1}^{k} (\lambda_{i_j})\, d\omega,
\end{aligned}$$

where we used Fubini's theorem. Now, using partitions of the unity we get equality for $M_\epsilon$. $\square$

**Corollary 2.18.** *Let $\Omega \subset \mathbb{R}^n$ be an open, convex set with smooth boundary $\partial\Omega = M$. Consider the set*

$$\Omega_\epsilon = \Omega + \epsilon\partial\mathbb{B} = \Omega \cup \psi\left([0,\epsilon], M\right)$$

*where $\partial\mathbb{B} = \{q \in \mathbb{R}^n : ||q|| = 1\}$ and $\psi : \mathbb{R} \times M \to \mathbb{R}^n$ is defined by*

$$\psi(t,p) = p + t\eta(p).$$

*Then*

$$\mathrm{vol}(\Omega_\epsilon) = \mathrm{vol}(\Omega) + \sum_{k=1}^{n} \frac{\epsilon^k}{k} \int_M \sum_{i_1<\cdots<i_{k-1}}^{n-1} \prod_{j=1}^{k-1} (\lambda_{i_j})\, d\omega.$$

*Proof.* Follow of the Theorem 2.17 that

$$\begin{aligned}
\mathrm{vol}(\Omega_\epsilon) &= \mathrm{vol}(\Omega) + \int_M \int_0^\epsilon \prod_{i=1}^{n-1}(1+t\lambda_i) d\omega dt \\
&= \mathrm{vol}(\Omega) + \epsilon A_M(M) + \sum_{k=1}^{n-1} \left.\frac{t^{k+1}}{k+1}\right|_0^\epsilon \int_M \sum_{i_1<\cdots<i_k}^{n-1} \prod_{j=1}^{k} (\lambda_{i_j})\, d\omega. \\
&= \mathrm{vol}(\Omega) + \epsilon A_M(M) + \sum_{k=1}^{n-1} \frac{\epsilon^{k+1}}{k+1} \int_M \sum_{i_1<\cdots<i_k}^{n-1} \prod_{j=1}^{k} (\lambda_{i_j})\, d\omega. \\
&= \mathrm{vol}(\Omega) + \sum_{k=0}^{n-1} \frac{\epsilon^{k+1}}{k+1} \int_M \sum_{i_1<\cdots<i_k}^{n-1} \prod_{j=1}^{k} (\lambda_{i_j})\, d\omega. \\
&= \mathrm{vol}(\Omega) + \sum_{k=1}^{n} \frac{\epsilon^{k}}{k} \int_M \sum_{i_1<\cdots<i_{k-1}}^{n-1} \prod_{j=1}^{k-1} (\lambda_{i_j})\, d\omega.
\end{aligned}$$

$\square$

**Remark 2.19.** *This is a nice proof of the Steiner formula.*

*Comparing to the Steiner formula*

$$\begin{aligned}\mathrm{vol}(\Omega+\epsilon\mathbb{B}) &= \sum_{k=0}^{n}\binom{n}{k}\epsilon^k V(\Omega[n-k],\mathbb{B}[k]) \\ &= \mathrm{vol}(\Omega)+\sum_{k=1}^{n}\binom{n}{k}\epsilon^k V(\Omega[n-k],\mathbb{B}[k]),\end{aligned}$$

*where $V(\Omega[n-k],\mathbb{B}[k])$ is the mixed volume called $k$-th quermassintegral of $\Omega$, we obtain*

$$\binom{n}{k}V(\Omega[n-k],\mathbb{B}[k]) = \frac{1}{k}\int_M \sum_{i_1<\cdots<i_{k-1}}^{n-1}\prod_{j=1}^{k-1}(\lambda_{i_j})\,d\omega.$$

## 2.4 First variation formula for the area

In [8] the authors provide a formula for the first variation of the area for a particular variation, in dimension $n=3$. In this section we prove the first variation formula (3) for general variations.

From this section onwards we will identify the usual Euclidean connection notation, $D$, previously used, with the $\nabla$. If $X$ is a vector field on $M$, then we let $X^\top$ and $X^{\tilde{\top}}$ denote the tangential and "Birkhoff tangential" components, respectively. More precisely, $X^{\tilde{\top}}$ is the projection to the first coordinate in the direct-sum decomposition $\mathbb{R}^n = T_pM \oplus \langle\eta\rangle$, where $\eta$ is the Birkhorff normal field on $M$. The covariant derivative $\nabla$ on $\mathbb{R}^n$ then induces a usual covariant derivative $\nabla_M$ on $M$. That is, the induced covariant derivative $\nabla_M$ is given by $\nabla_M = (\nabla)^\top$, to simplify the notation we will identify $\nabla_M$ with $\nabla$.

**Theorem 2.20.** *Assume $M$ is a closed manifold, $x: M\to\mathbb{R}^n$ a smooth immersion and $F:(-\varepsilon,\varepsilon)\times M\to\mathbb{R}^n$ a smooth variation of $x$. The first variation of the area is given by*

$$A_m'(0) = \int_M (n-1)H_m(p)N_\eta\left(\frac{\partial F}{\partial t}(0,p)\right)d\omega(p)$$

*where $N_\eta$ is the projection to the second coordinate in the direct-sum decomposition $\mathbb{R}^n = T_pM \oplus \langle \eta \rangle$.*

Before we need the following technical lemma.

**Lemma 2.21.** *Let $X^\top$ denote the orthogonal projection of $X$ in $T_pM$.*

a) $\left.\dfrac{d\xi}{dt}\right|_0 = -\nabla\langle F_t, \xi\rangle + \nabla_{F_t^\top}\xi$

b) $\nabla_{\eta^\top}\xi = \nabla\langle \eta, \xi\rangle$

*Proof.* For the first part let $\{e_i\}$ be an orthonormal basis for $T_pM$ and $e_i^t = dF_p^t(e_i)$.

$$\left\langle \left.\frac{\partial}{\partial t}\xi\right|_{t=0}, e_i \right\rangle = \left.\frac{\partial}{\partial t}\langle \xi^t, e_i^t\rangle\right|_{t=0} - \left\langle \xi, \left.\frac{\partial}{\partial t}e_i^t\right|_{t=0}\right\rangle$$

As $e_i^t$ is orthogonal to $\xi^t$ along $t$

$$\langle \left.\frac{\partial}{\partial t}\xi\right|_{t=0}, e_i\rangle = -\langle \xi, \nabla_{F_t} e_i^t\rangle$$

Using that $[F_t, e_i^t] = 0$ and $d\xi_p$ is a self-adjoint operator

$$\begin{aligned}
\left\langle \left.\frac{\partial}{\partial t}\xi\right|_{t=0}, e_i \right\rangle &= -\langle \xi, \nabla_{e_i} F_t\rangle \\
&= -(e_i\langle \xi, F_t\rangle - \langle \nabla_{e_i}\xi, F_t\rangle) \\
&= -(e_i\langle \xi, F_t\rangle - \langle e_i, \nabla_{F_t^\top}\xi\rangle) \\
&= -e_i\langle \xi, F_t\rangle + \langle e_i, \nabla_{F_t^\top}\xi\rangle
\end{aligned}$$

For the second part compute for any $v \in T_pM$,

$$\langle \nabla_{\eta^\top}\xi, v\rangle = \langle \eta^\top, \nabla_v \xi\rangle = v\langle \eta, \xi\rangle = \langle \nabla\langle \eta, \xi\rangle, v\rangle.$$

$\square$

*Proof of Theorem 2.20.* We have $A_m(t) = \int_M \langle \eta^t, \xi^t \rangle dS_t$. Choose the coordinate system such that at $p \in M$ it is orthonormal, as in the lemma above. Note that

$$\frac{\partial}{\partial t}\langle \eta^t, \xi^t\rangle\Big|_{t=0} = \left\langle \eta, \frac{\partial}{\partial t}\xi\Big|_{t=0}\right\rangle,$$

since $\frac{\partial}{\partial t}\eta\big|_{t=0} \in T_pM$. Furthemore $\frac{\partial}{\partial t}\sqrt{\det(\langle e_i^t, e_j^t\rangle)}\Big|_{t=0} = \operatorname{div}_M F_t$ (see eq. 1.44, [11]). Thus

$$A_m'(0) = \int_M \left\{ \left\langle \eta, \frac{\partial}{\partial t}\xi\Big|_{t=0}\right\rangle + \langle \eta, \xi\rangle \operatorname{div}_M F_t \right\} dS.$$

Writing $F_t = F_t^{\tilde{\top}} + \dfrac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\eta$, we have

$$\operatorname{div}_M F_t = \operatorname{div}_M F_t^{\tilde{\top}} + \operatorname{div}_M \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\eta.$$

On the other hand,

$$\begin{aligned} \operatorname{div}_M \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\eta &= \left\langle \nabla \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}, \eta \right\rangle + \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle} \operatorname{div}_M \eta \\ &= \left\langle \nabla \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}, \eta \right\rangle + \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle} (n-1) H_m. \end{aligned}$$

Then

$$\begin{aligned} A'(0) = \int_M &\left\{ \left\langle \eta, \frac{\partial}{\partial t}\xi\Big|_{t=0}\right\rangle + \langle \eta, \xi\rangle \operatorname{div}_M F_t^{\tilde{\top}} + \langle \eta, \xi\rangle \left\langle \nabla \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}, \eta \right\rangle \right\} dS \\ &+ \int_M \langle \eta, \xi\rangle \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle} (n-1) H_m \, dS. \end{aligned}$$

Using again the decomposition $F_t = F_t^{\tilde{\top}} + \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\eta$, we obtain

$$\begin{aligned} \tfrac{\partial}{\partial t}\xi\big|_{t=0} &= -\nabla\langle F_t, \xi\rangle + \nabla_{F_t^{\tilde{\top}}}\xi + \nabla_{\frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\eta^{\top}}\xi \\ &= -\nabla\langle F_t, \xi\rangle + \nabla_{F_t^{\tilde{\top}}}\xi + \tfrac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\nabla_{\eta^{\top}}\xi. \end{aligned}$$

Furthermore,

$$\nabla\langle F_t, \xi\rangle = \nabla\left(\frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\langle \eta, \xi\rangle\right) = \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\nabla\langle \eta, \xi\rangle + \langle \eta, \xi\rangle \nabla \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}. \tag{2.2}$$

Using (2.2) and the Lemma 4.2-b, the Lemma 2.21 yields

$$\frac{\partial}{\partial t}\xi\bigg|_{t=0} = -\langle \eta, \xi\rangle \nabla \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle} + \nabla_{F_t^{\tilde{\top}}}\xi.$$

Replacing in $A'(0)$, we have:

$$A'(0) = \int_M (n-1)H_m \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\, d\omega + \int_M \left\{ \langle \nabla_{F_t^{\tilde{\top}}}\xi, \eta\rangle + \langle \eta, \xi\rangle \operatorname{div}_M F_t^{\tilde{\top}} \right\} dS,$$

as

$$\operatorname{div}_M \langle \eta, \xi\rangle F_t^{\tilde{\top}} = F_t^{\tilde{\top}}\langle \eta, \xi\rangle + \langle \eta, \xi\rangle \operatorname{div}_M F_t^{\tilde{\top}} = \langle \nabla_{F_t^{\tilde{\top}}}\xi, \eta\rangle + \langle \eta, \xi\rangle \operatorname{div}_M F_t^{\tilde{\top}},$$

we have

$$A'(0) = \int_M (n-1)H_m \frac{\langle F_t, \xi\rangle}{\langle \eta, \xi\rangle}\, d\omega.$$

□

## 2.5 First variation for the volume

Let's consider $F : (-\epsilon, \epsilon) \times M \to (\mathbb{R}^n, ||\cdot||)$ an variation of an immersion $x : M \to \mathbb{R}^n$, we can calculate the volume $V(t)$ of each immersion $F_t : M \to (\mathbb{R}^n, ||\cdot||)$ given by $F_t(p) = F(t,p)$ as follows:

$$V(t) = \frac{1}{n}\int_M \frac{\langle F^t, \xi_t\rangle}{\langle \eta_t, \xi^t\rangle}\, \omega_t = \frac{1}{n}\int_M \langle F^t, \xi^t\rangle\, dS_t.$$

The first variation of volume (see Rafael López, [22], p. 263) is given by:

**Proposition 2.22.** $\displaystyle \frac{dV}{dt}\bigg|_{t=0} = \int_M \Big\langle F_t(0,p)\, ,\, \xi(p) \Big\rangle\, dS$

The Lemma below is an almost immediate adaptation to the Minkowski case of the Euclidean case (see [9], p. 341).

**Lemma 2.23.** *Let $g : M \to \mathbb{R}$ be a piecewise smooth function such that*

$$\int_M g\omega = 0.$$

*Then there exists a volume-preserving normal variation whose variation vector is $g\eta$. If, in addition, $g \equiv 0$, on $\partial D$, the variation can be so chosen that it fixes the boundary $\partial D$.*

*Proof.* Let consider $F(t, \bar{t}) = f + (tg + \bar{t}\tilde{g})$, where $\tilde{g} : M \to \mathbb{R}$ is a piecewise smooth function with $\tilde{g} = 0$ on $\partial D$ and

$$\int_M \tilde{g}\,\omega \neq 0.$$

Let $V(t, \bar{t})$ be the volume determined by $F(t, \bar{t})$ and consider the equation

$$V(t, \bar{t}) = constant \tag{2.3}$$

Follows from the first variation for the volume, that at $t = \bar{t} = 0$, we have

$$\frac{\partial V}{\partial \bar{t}} = \int_M \tilde{g}\omega \neq 0.$$

Thus we can apply the implicit function theorem to equation (2.3) and obtain that $\bar{t} = \phi(t)$, where $\phi$ is a smooth function of $t$ in a neighborhood of $t = 0$. It follows that the variation

$$y_t = F(t, \phi(t)) = f + (tg + \phi(t)\tilde{g})\eta$$

is volume preserving. Furthemore, the variation vector of $y_t$ is given by

$$\left.\frac{dy_t}{dt}\right|_{t=0} = (g + \phi'(0)\tilde{g})\eta = g\eta,$$

since

$$\phi'(0) = \left(\frac{\partial V}{dt}\right)_0 \left(\frac{\partial V}{\partial \bar{t}}\right)_0^{-1} = \left(\int_M g\omega\right)\left(\int_M \tilde{g}\omega\right)^{-1} = 0.$$

It is also clear from variation $y_t$ that, if $g = 0$ on $\partial D$, the variation $y_t$ fixes the boundary. □

Another result of the Euclidean case that extends without much difficulty to the Minkowskian case is the following proposition (see [9], p. 342).

**Proposition 2.24.** *Let* $x : M^{n-1} \to (\mathbb{R}^n, ||\cdot||)$. *Consider* $J : (-\epsilon, \epsilon) \to \mathbb{R}$ *defined by*

$$J(t) = A_m(t) + nH_0V(t),$$

*where* $H_0 = \dfrac{1}{A_m}\displaystyle\int_M H_m d\omega$. *The following statements are equivalent:*

*(i)* *$x$ has constant Minkowski mean curvature $H_0 \neq 0$.*

*(ii)* *For each relatively compact domain $D \subset M$ with smooth boundary, and each volume-preserving variation $F_t : D \to (\mathbb{R}^n, ||\cdot||)$, that fixes the boundary $\partial D$, $A'_{m,D}(0) = 0$.*

*(iii)* *For each $D \subset M$ as in (ii) and each (not necessarily volume-preserving) variation that fixes the boundary $\partial D$, $J'(0) = 0$.*

*Proof.* $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ follow from $J$ and $V'$ formules. We will prove that $(ii) \to (i)$. Suppose that exist $p \in D$ such that $(H_m - H_0)(p) \neq 0$. Assume $(H_m - H_0)(p) > 0$. Set

$$D^+ = \{q \in D; (H_m - H_0)(q) > 0\} \text{ and } D^- = \{q \in D; (H_m - H_0)(q) < 0\}.$$

Let $\phi$ and $\psi$ nonnegative real piecewise smooth functions on $\overline{D}$ such that

$$p \in \operatorname{supp}\phi \subset D^+, \ \operatorname{supp}\psi \subset D^-, \ \int_D (\phi + \psi)(H_m - H_0)d\omega = 0.$$

where supp $\phi$ denotes the support of $\phi$. Set $g = (\phi + \psi)(H_m - H_0)$. Then $g = 0$ on $\partial D$ and $\int g d\omega = 0$. By previous Lemma, there is a Birkhoff-normal variation that preserves volume whose variational field is $g\eta$. By hypothesis, for such a variation

$$0 = A'_{m,D}(0) = (n-1)\int_D gH_m d\omega.$$

Thus

$$0 = \int_D gH_m d\omega - H_0 \int_D g d\omega = \int_D g(H_m - H_0) d\omega = \int_D (\phi + \psi)(H_m - H_0)^2 d\omega > 0.$$

a contradiction. It follows that $H_m = H_0$ in $D$. Since (ii) holds for each $D \subset M$, $H_m = H_0$.

□

# Chapter 3

# Stability of hypersurfaces in Minkowski spaces

## 3.1 Preliminaries

In this section we recall some observations that are important to the development of Section 3.3.

The following lemma is proved in [9] in the Euclidean context. Its extension to the Minkowskian case is immediate, we present the proof for completeness.

**Lemma 3.1.** *For any immersion $x : M \to \mathbb{R}^n$ with Minkowskian mean curvature $H_m$ and $B_m^2 = \lambda_1^2 + \cdots + \lambda_{n-1}^2$ ($\lambda_1, \cdots, \lambda_{n-1}$ the Minkowski principal curvatures of $x$), we have $B_m^2 \geq (n-1)H_m^2$ with equality at a point $p$ if and only if $p$ is an umbilic point (meaning that $\lambda_1 = \cdots = \lambda_{n-1}$).*

*Proof.* Let $\lambda_1, \cdots, \lambda_{n-1}$ be the Minkowski principal curvatures of $x$ at $p \in M$.

Then $B_m^2 = \sum_{i=1}^{n-1} \lambda_i^2$, and

$$B_m^2 - (n-1)^2 H_m^2 = -2 \sum_{i<j} \lambda_i \lambda_j, \qquad i, j = 1, \cdots, n-1.$$

It can easily be shown, by induction, that

$$\sum_{i<j} (\lambda_i^2 + \lambda_j^2) = (n-2) \sum_{i=1}^{n-1} \lambda_i^2.$$

Thus

$$\sum_{i<j} (\lambda_i - \lambda_j)^2 = (n-2) \sum_{i=1}^{n-1} \lambda_i^2 - 2 \sum_{i<j} \lambda_i \lambda_j = (n-2) B_m^2 - 2 \sum_{i<j} \lambda_i \lambda_j,$$

hence

$$(n-1)(B_m^2 - (n-1) H_m^2) = \sum_{i<j} (\lambda_i - \lambda_j)^2.$$

□

The characterization of umbilical surfaces in Minkowskian geometry was done by Balestro, Martini, and Teixeira for dimension 3. The proof extends without modification to dimension $n$.

**Lemma** (2.12)**.** *A connected hypersurface immersed $\mathbb{R}^n$, all whose points are umbilic is contained in a plane or in a Minkowski sphere.*

*Proof.* For a demonstration, see Lemma 2.12. □

The next lemma allows us to work with variations whose initial velocity has zero average, but are not necessarily volume-preserving. The proof is a standard application of the implicit function theorem.

**Lemma 3.2.** *Let $x : M \to \mathbb{R}^n$ be an immersion with constant Minkowskian mean curvature and $F$ a variation of compact support of the form* (4) *and set $f(p) = \frac{\partial}{\partial t} g(t, p)\big|_{t=0}$.*

*Consider the functional* $J(t) = A_m(t) - (n-1)H_0V(t)$*, as in Proposition 2.24, then we have*

*(a) If* $\int_M f d\omega = 0$ *then there exists a volume-preserving variation*

$$\tilde{F}(t,p) = x(p) + \tilde{g}(t,p)\eta$$

*such that* $\frac{\partial}{\partial t}\tilde{g}(t,p)\big|_{t=0} = f(p)$*.*

*(b) For such a variation* $\tilde{F}$ *we have* $A''_m(0) = J''_m(0)$*.*

*Proof.* The proof of (a) follow of the Lemma 2.23. Proof of (b) is immediate. □

As mentioned in the introduction, the Minkowskian isoperimetric inequality is proven in [36] (with different notation) by extending the concept of mixed volume, and the mixed volume inequality to general domains.

**Lemma 3.3** ([36], Lemma 3.2)**.** *Let* $K \subseteq \mathbb{R}^n$ *be a compact domain with smooth boundary and* $\mathbb{B}$ *a compact convex set with the origin in the interior. Then*

$$\frac{1}{n}\int_{\partial K} d\omega_{\mathbb{B}} \geq \mathrm{vol}(K)^{\frac{n-1}{n}}\,\mathrm{vol}(\mathbb{B})^{\frac{1}{n}}$$

*with equality if and only if* $K$ *and* $\mathbb{B}$ *are homothetic.*

## 3.2 Second variation formula

In this section we prove Theorem 0.2. For the sake of convenience we enunciate it here again

**Theorem.** *Let* $x : M \to \mathbb{R}^n$ *be an immersed surface with Birkhoff-Gauss map* $\eta$ *and constant Minkowskian mean curvature, and let* $F : (-\varepsilon,\varepsilon)\times M \to \mathbb{R}^n$*, be a volume-preserving variation of compact support given by*

$$F(t,p) = F^t(p) = x(p) + g(t,p)\eta(p).$$

*Denote* $f(p) = \frac{\partial}{\partial t} g(t,p)\big|_{t=0}$ *and* $A_m(t) = A_m(F(t,\cdot))$ *the area defined by* (2). *Then,*

$$\begin{aligned} A_m''(0) &= \int_M \left(-B_m^2 f^2 + \langle \eta, \xi\rangle (\nabla^b f, \nabla^b f)_b\right) d\omega \\ &= -\int_M f\left(B_m^2 f + \langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f)\right) d\omega \end{aligned}$$

*Here* $\nabla^b f$ *is the gradient of* $f$ *with respect to the Dupin metric and can be computed as* $\nabla^b f = du(\nabla f)$ *where* $\nabla f$ *is the gradient with respect to the usual metric. Also* $B_m$ *is the norm of the Minkowski second fundamental form,* $B_m^2 = \sum_{i=1}^{n-1} \lambda_i^2$.

In view of Lemma 3.2-b it suffices to compute $J_m''$.

Let $p \in M$, there is a neighborhood of $p$ where the restriction of $x$ is an embedding. Without loss of generality we will assume for the local computations, that $M \subseteq \mathbb{R}^n$ is a submanifold and $x$ is the identity. Take an orthonormal basis $e_1, \ldots, e_{n-1}$, of $T_pM$ consisting of *euclidean* principal directions. For $t \in (-\varepsilon, \varepsilon)$ the vectors $e_i^t = d_pF^t(e_i)$ span the tangent space of $M^t = F^t(M)$ at $F(t,p)$. We denote by $g_{i,j}^t$ the coefficients of the metric of $M^t$, $(g^{i,j}(t))$ the inverse of the matrix $(g_{i,j}^t)$ and $H_m^t$ the Minkowskian mean curvature of $M^t$. Again without loss of generality the functions $\eta^t, g_{i,j}^t, H_m^t$ can be regarded as functions defined on (neighborhoods of) $M$ or $M^t$ via composition with $F^t$. Denote by $\nabla$ the usual connection in $\mathbb{R}^n$.

*Proof of Theorem 0.2.* Take coefficients $a_{k,i}^t$ such that

$$d\eta^t(e_i^t) = \sum_k a_{k,i}^t e_k^t$$

and note that

$$\langle e_j^t, \nabla_{e_i^t} \eta^t \rangle = \sum_k a_{k,i}^t g_{j,k}^t$$

$$\sum_j \langle e_j^t, \nabla_{e_i^t}\eta^t\rangle g^{i,j}(t) = \sum_{k,j} a_{k,i}^t g_{j,k}^t g^{i,j}(t) = a_{i,i}^t$$
$$(n-1)H_m^t = \sum_i a_{i,i}^t = \sum_{i,j} g^{i,j}(t)\langle e_j^t, \nabla_{e_i^t}\eta^t\rangle.$$

Now we compute the derivative

$$\frac{\partial}{\partial t}(n-1)H_m^t\bigg|_{t=0} = \sum_{i,j} \frac{\partial}{\partial t} g^{i,j}(t)\bigg|_{t=0} \langle e_j, \nabla_{e_i}\eta\rangle + \delta_{i,j}\,\frac{\partial}{\partial t}\langle e_j^t, \nabla_{e_i^t}\eta^t\rangle\bigg|_{t=0}$$

Using that

$$\frac{\partial}{\partial t} g^{i,j}(t)\bigg|_{t=0} = -\frac{\partial}{\partial t} g_{i,j}(t)\bigg|_{t=0},$$

we have

$$\begin{aligned}\frac{\partial}{\partial t}(n-1)H_m^t\bigg|_{t=0} &= \sum_{i,j} -\frac{\partial}{\partial t} g_{i,j}(t)\bigg|_{t=0} \langle e_j, \nabla_{e_i}\eta\rangle + \delta_{i,j}\,\frac{\partial}{\partial t}\langle e_j^t, \nabla_{e_i^t}\eta^t\rangle\bigg|_{t=0} \\ &= \sum_{i,j} -(\langle \nabla_{F_t} e_i, e_j\rangle + \langle \nabla_{F_t} e_j, e_i\rangle)\langle e_j, \nabla_{e_i}\eta\rangle \\ &+ \delta_{i,j}\,\frac{\partial}{\partial t}\langle e_j^t, \nabla_{e_i^t}\eta^t\rangle\bigg|_{t=0},\end{aligned}$$

where

$$F_t = \frac{\partial}{\partial t}\bigg|_{t=0} F = f\eta.$$

As

$$\nabla_{F_t} e_j - \nabla_{e_j} F_t = [F_t, e_j] = 0,$$

follow that

$$(n-1)\left.\frac{\partial}{\partial t}H_m^t\right|_{t=0}$$

$$= \sum_{i,j} -(\langle \nabla_{e_i} F_t, e_j\rangle + \langle \nabla_{e_j} F_t, e_i\rangle)\langle e_j, \nabla_{e_i}\eta\rangle + \delta_{i,j} \left.\frac{\partial}{\partial t}\langle e_j^t, \nabla_{e_i^t}\eta^t\rangle\right|_{t=0}$$

$$= \sum_{i,j} -(\langle \nabla_{e_i} f\eta, e_j\rangle + \langle \nabla_{e_j} f\eta, e_i\rangle)\langle e_j, \nabla_{e_i}\eta\rangle + \delta_{i,j} \left.\frac{\partial}{\partial t}\langle e_j^t, \nabla_{e_i^t}\eta^t\rangle\right|_{t=0}$$

$$= \sum_{i,j} -(\langle e_i(f)\eta + f\nabla_{e_i}\eta, e_j\rangle + \langle e_j(f)\eta + f\nabla_{e_j}\eta, e_i\rangle)\langle e_j, \nabla_{e_i}\eta\rangle$$

$$+ \sum_i \langle e_i(f)\eta + f\nabla_{e_i}\eta, \nabla_{e_i}\eta\rangle + \langle e_i, \nabla_{F_t}\nabla_{e_i^t}\eta^t\rangle.$$

As

$$\sum_{i,j}\langle \nabla_{e_i}\eta, e_j\rangle\langle e_j, \nabla_{e_i}\eta\rangle = \sum_i \|\nabla_{e_i}\eta\|^2,$$

we have

$$(n-1)\left.\frac{\partial}{\partial t}H_m^t\right|_{t=0}$$

$$= -f\left(\sum_i \|\nabla_{e_i}\eta\|^2 + \sum_{i,j}\langle e_i, \nabla_{e_j}\eta\rangle\langle e_j, \nabla_{e_i}\eta\rangle\right) + f\sum_i \|\nabla_{e_i}\eta\|^2$$

$$- \sum_{i,j}(\langle e_i(f)\eta, e_j\rangle + \langle e_j(f)\eta, e_i\rangle)\langle e_j, \nabla_{e_i}\eta\rangle + \sum_i \langle e_i(f)\eta, \nabla_{e_i}\eta\rangle$$

$$+ \sum_i \langle e_i, \nabla_{F_t}\nabla_{e_i^t}\eta^t\rangle$$

$$= -fB_m^2 - \sum_i e_i(f)\langle \sum_j \langle \eta, e_j\rangle e_j, \nabla_{e_i}\eta\rangle - \sum_{i,j} e_j(f)\langle \eta, e_i\rangle\langle e_j, \nabla_{e_i}\eta\rangle$$

$$+ \sum_i \langle e_i(f)\eta, \nabla_{e_i}\eta\rangle + \langle e_i, \nabla_{F_t}\nabla_{e_i^t}\eta^t\rangle.$$

Note that

$$\sum_i e_i(f)\langle \sum_j \langle \eta, e_j\rangle e_j, \nabla_{e_i}\eta\rangle = \langle \eta^T, \nabla_{\nabla f}\eta\rangle,$$

hence

$$\frac{\partial}{\partial t}(n-1)H_m^t\bigg|_{t=0} = -fB_m^2 - \langle \eta^T, \nabla_{\nabla f}\eta\rangle - \sum_i \langle \eta, e_i\rangle\langle \nabla f, \nabla_{e_i}\eta\rangle + \langle \eta^T, \nabla_{\nabla f}\eta\rangle$$

$$+\sum_i \langle e_i, \nabla_{F_t}\nabla_{e_i^t}\eta^t\rangle$$
$$= -fB_m^2 - \langle \nabla f, \nabla_{\eta^T}\eta\rangle + \sum_i \langle e_i, \nabla_{F_t}\nabla_{e_i^t}\eta^t\rangle.$$

Recording that $d\eta = du \circ d\xi$, we have

$$\frac{\partial}{\partial t}(n-1)H_m^t\bigg|_{t=0} = -fB_m^2 - \langle \nabla f, du(\nabla_{\eta^T}\xi)\rangle + \sum_i \langle e_i, \nabla_{e_i}\nabla_{F_t}\eta^t\rangle.$$

Using that $\nabla_{\eta^T}\xi = \nabla(\langle \eta, \xi\rangle)$ and $\frac{\partial}{\partial t}\eta\big|_{t=0} = -\langle \eta, \xi\rangle du(\nabla f)$ (see Proposition 4.3 in the Appendix), follow that

$$\frac{\partial}{\partial t}(n-1)H_m^t\bigg|_{t=0} = -fB_m^2 - \langle du(\nabla f), \nabla(\langle \eta, \xi\rangle)\rangle + \operatorname{div}_M\left(\frac{\partial}{\partial t}\eta\bigg|_{t=0}\right)$$
$$= -fB_m^2 - \langle du(\nabla f), \nabla(\langle \eta, \xi\rangle)\rangle - \operatorname{div}_M(\langle \eta, \xi\rangle du(\nabla f)).$$

Now observe that

$$\langle \eta, \xi\rangle^{-1}\operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f)) = \langle du(\nabla f), \nabla(\langle \eta, \xi\rangle)\rangle + \operatorname{div}_M(\langle \eta, \xi\rangle du(\nabla f)),$$

(see Proposition 4.3 in the Appendix). Therefore

$$\frac{\partial}{\partial t}(n-1)H_m^t\bigg|_{t=0} = -fB_m^2 - \langle \eta, \xi\rangle^{-1}\operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f)).$$

Finally we observe that

$$J'(t) = A'(t) - (n-1)H_m^0 V'(t)$$
$$= (n-1)\int_M g_t(H_m^t - H_m^0)d\omega.$$

Hence

$$J''(0) = (n-1)\int_M f\left(\frac{\partial}{\partial t}H_m^t\bigg|_{t=0}\right)d\omega,$$

and the result follows. □

## 3.3 Stability

We start by showing that the Minkowski sphere is stable.

**Theorem 3.4.** *Let $B = x_0 + \lambda(\mathbb{B})$ and $F_t$ a volume-preserving variation of $\partial B$, then $A'_m(0) = 0$ and $A''_m(0) \geq 0$.*

*Proof.* By Lemma 3.2 we may assume without loss of generality that $F(t,p) = p + g(t,p)\eta_p$.

Since $\eta$ is a transversal vector field we have that for small values of $t$, $F(t, \partial B)$ is the (smooth) boundary for some compact domain $B_t$. By Lemma 3.3,

$$A_m(t) \geq n \operatorname{vol}(B_t)^{\frac{n-1}{n}} \operatorname{vol}(\mathbb{B})^{\frac{1}{n}} = n \operatorname{vol}(B)^{\frac{n-1}{n}} \operatorname{vol}(\mathbb{B})^{\frac{1}{n}} = A_m(0)$$

and the result follows. □

The following lemma is the key component of our stability theorem. The proof is a lengthy computation and will be presented in Appendix A, to improve readability.

**Lemma 3.5.** *Let $x : M \to \mathbb{R}^n$ have constant Minkowskian mean curvature and let $\rho(x) = \langle \eta, \xi\rangle^{-1}\langle x, \xi\rangle$. Then*

$$\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla\rho)) = (n-1)H_m - \rho B_m^2 \tag{3.1}$$

We shall verify the Minkowski identity for the function $\rho$ in (3.1) (see equation (5.2), [7]). To this end, consider the variation $x(t,p) = (t+1)p$ and notice that

$$A_m(t) = (t+1)^{n-1}A_m(0)$$
$$A'_m(0) = (n-1)\int_M d\omega.$$

On the other hand, by formula (3) we have

$$A'_m(0) = (n-1)\int_M H_m(p)\rho(p)d\omega$$

then we obtain

$$\int_M \rho H_m d\omega = \int_M d\omega.$$

Thus taking $f(p) = 1 - \rho(p)H_m$ we have

$$\int_M f d\omega = 0. \tag{3.2}$$

Now we are in conditions to prove our main theorem, the Theorem 0.3, which we will state again here:

**Theorem.** *Let $x : M \to \mathbb{R}^n$ be a compact immersed surface without boundary, with constant Minkowskian mean curvature and stable with respect to the Minkowskian structure. Then $x(M)$ is an embedded Minkowski sphere, this is, $x(M)$ is homothetic to $\partial\mathbb{B}$.*

*Proof of Theorem 0.3.* Let $\rho$ be as in Lemma 3.5. By (3.2) and by Lemma 3.2-a there is a volume-preserving variation

$$F(t,p) = p + g(t,p)\eta$$

with $g_t(0,p) = f = 1 - \rho H_m$.

The second variation formula reads

$$0 \leq A''(0) = \int_M f\left(-B_m^2 f - \langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f))\right) d\omega.$$

Using that $H_m$ is constant and the Lemma 3.5, we compute

$$\begin{aligned}\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f)) &= \langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(-H_m \nabla\rho)) \\ &= -(n-1)H_m^2 + \rho B_m^2 H_m.\end{aligned}$$

Multiplying by $-f$ and subtracting $B_m^2 f^2$ in the previous equation, we have

$$\begin{aligned}
-B_m^2 f^2 - \langle \eta, \xi \rangle^{-1} \operatorname{div}_M(\langle \eta, \xi \rangle^2 du(\nabla f) f &= -B_m^2 f^2 + (n-1) H_m^2 f - \rho H_m B_m^2 f \\
&= -B_m^2 f^2 + (n-1) H_m^2 f - (1-f) B_m^2 f \\
&= ((n-1) H_m^2 - B_m^2) f.
\end{aligned}$$

And using that $\int_M f d\omega = 0$ we obtain

$$A_m''(0) = -\int_M B_m^2 f d\omega = -\int_M B_m^2 (1 - \rho H_m) d\omega. \tag{3.3}$$

For other side, as $M$ has no boundary, we have

$$\begin{aligned}
0 &= \int_M \operatorname{div}_M(\langle \eta, \xi \rangle^2 du(\nabla \rho)) dS \\
&= \int_M \langle \eta, \xi \rangle^{-1} \operatorname{div}_M(\langle \eta, \xi \rangle^2 du(\nabla \rho)) d\omega.
\end{aligned}$$

Using (3.1) and the fact that $H_m$ is constant we obtain

$$\begin{aligned}
0 &= \int_M \langle \eta, \xi \rangle^{-1} \operatorname{div}_M(\langle \eta, \xi \rangle^2 du(\nabla \rho)) d\omega = \int_M \left((n-1) H_m - \rho B_m^2\right) d\omega \\
0 &= \int_M \left((n-1) H_m^2 - \rho H_m B_m^2\right) d\omega.
\end{aligned}$$

Hence

$$\int_M (n-1) H_m^2 d\omega = \int_M B_m^2 \rho H_m d\omega.$$

Substituting in (3.3) we get

$$A_m''(0) = -\int_M \left(B_m^2 - (n-1) H_m^2\right) d\omega \leq 0$$

where the integrand is non-negative by Lemma 3.1, implying that $B_m^2 = (n-1) H_m^2$ for every $p \in M$, since the immersion $x$ is stable, hence all points of $M$ are umbilic. Since $M$ is compact, Lemma 2.12 implies that $x(M) \subset \mathbb{R}^n$ is a Minkowski sphere. $\square$

## 3.4 An alternative proof for the Stability Theorem

Consider $M$ a compact oriented $n$-manifold and $x : M \to \mathbb{R}^n$ an immersion. For a such immersion we compute the area $A_m(x)$,

$$A_m(x) = \int_M d\omega,$$

where $d\omega$ is the Minkowski area element of $M$ induced for the immersion $x$. Also we compute the "oriented" volume, $V(x)$, enclosed by $x(M)$. It is given by formula

$$V(x) = \frac{1}{n}\int_M \frac{\langle x\,,\,\xi\rangle}{\langle \eta, \xi\rangle} d\omega,$$

where $\xi$ is the normal Euclidean unit vector field determined by the orientation of $M$ and the imersion $x$ and $\eta$ the Birkhoff-normal field of $M$.

**Theorem 3.6.** *Let $M$ be a compact oriented $n$-manifold and $x : M \to \mathbb{R}^n$ an immersion with non-zero Minkowski mean constant curvature $H_m$. Then $x$ is stable if and only if $x(M) \subset \mathbb{R}^n$ is a Minkowski sphere $\partial B$ in $\mathbb{R}^n$.*

*Proof.* Let $x : M \to \mathbb{R}^n$ be a compact immersion, where we suppose that $x(M)$ has non-zero Minkowski mean constant curvature $H_m$. Let $F(t) = x + t\eta$ be one-parameter family of surfaces. Observe that $F(t)$ has the same Birkhoff-normal unit vetorial field as $x$, because

$$\frac{\partial F(t)}{\partial u} = \frac{\partial x}{\partial u} + t\frac{\partial \eta}{\partial u},$$

and then,

$$\left\langle \xi, \frac{\partial F(t)}{\partial u} \right\rangle = \left\langle \xi, \frac{\partial x}{\partial u} \right\rangle + t\left\langle \xi, \frac{\partial \eta}{\partial u} \right\rangle = 0.$$

Therefore $T_{F(t,p)}M$ is parallel to $T_pM$. Hence $\eta(t)$ is parallel to $\eta(0) = \eta$ at $p$.

Furthemorer, the area $A_m(t) = A_m(F(t))$ is given by:

$$A_m(t) = \int_M \prod_{i=1}^{n-1}(1+\lambda_i t)\ d\omega, \tag{3.4}$$

where $\lambda_1, \cdots, \lambda_{n-1}$ are Minkowski's principal curvatures of $x = F(0)$.

We shall give proofs of this in the appendix.

The right side of the equation (3.4) is a polynomial of degree $n-1$ in $t$ and maybe expanded in the form

$$\begin{aligned}
A_m(t) &= a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1} \\
a_0 &= \int d\omega = A_m(x) \\
a_1 &= \int (\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1}) d\omega = (n-1)H_m a_0 \\
a_2 &= \int H_{m,2}\, d\omega, \ \text{ where } H_{m,2} = \prod_{\lambda_i<\lambda_j} \lambda_i\lambda_j \\
\\
a_k &= \int H_{m,k}\, d\omega, \ \text{ where } H_{m,k} = \prod_{\lambda_{i_1}<\cdots<\lambda_{i_k}} \lambda_{i_1}\cdots\lambda_{i_k}
\end{aligned}$$

Notice that $a_{n-1} = \int K_m\, d\omega$ where $K_m$ is a Minkowski Gauss curvature of $M$.

Notice also that the volume function satisfies

$$\frac{dV(t)}{dt} = A_m(t)$$

For a proof see the appendix.

Thus, we have:

$$V(t) = v_0 + v_1 t + v_2 t^2 + \cdots + v_n t^n,$$

where

$$v_1 = a_0, \quad 2v_2 = a_1 = (n-1)H_m a_0, \quad \text{etc.} \tag{3.5}$$

The family $F(t)$ is not volume preserving. In order to obtain a volume-preserving family we apply the appropriate homothety. Namely, $y = sF(t) = s(x + t\eta)$ a two-parameter family of immersions. Thus:

$$A_m(sF(t)) = s^{n-1}A_m(t) = s^{n-1}(a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1}). \qquad (3.6)$$

$$V(sF(t)) = s^n V(t) = s^n(v_0 + v_1 t + v_2 t^2 + \cdots + v_n t^n). \qquad (3.7)$$

We will now determine $s = s(t)$ such that $V(sF(t)) = v_0$.

By use of formula (3.7), we have:

$$\begin{aligned} s^n(v_0 + v_1 t + v_2 t^2 + \cdots + v_n t^n) &= v_0 \\ s^{n-1}(t) &= \left(\frac{v_0}{v_0 + v_1 t + v_2 t^2 + \cdots + v_n t^n}\right)^{\frac{n-1}{n}} \\ s^{n-1}(t) &= \left(1 + \frac{v_1}{v_0}t + \frac{v_2}{v_0}t^2 + \cdots + \frac{v_n}{v_0}t^n\right)^{-\frac{n-1}{n}} \end{aligned}$$

Using the binomial theorem (see Theorem 4.5) we obtain the serie for $s^{n-1}$ (needing terms only through $t^2$),

$$\begin{aligned} & s^{n-1}(t) \\ &= 1 + \left(-\frac{n-1}{n}\right)\left(\frac{v_1}{v_0}t + \frac{v_2}{v_0}t^2 + \cdots + \frac{v_n}{v_0}t^n\right) + \\ &+ \frac{1}{2}\cdot\frac{-(n-1)}{n}\cdot\frac{-2(n-1)-1}{n}\left(\frac{v_1}{v_0}t + \frac{v_2}{v_0}t^2 + \cdots + \frac{v_n}{v_0}t^n\right)^2 + \cdots \\ &= 1 + t\left(\frac{-(n-1)}{n}\frac{v_1}{v_0}\right) + t^2\left(\frac{-(n-1)}{n}\frac{v_2}{v_0} + \frac{(n-1)(2(n-1)+1)}{2n^2}\frac{v_1^2}{v_0^2}\right) + \cdots \end{aligned}$$

Substituing $s^{n-1}(t)$ in (3.6), and calling $A_m(t) \equiv A_m[s(t)F(t)]$ we find (needing terms only through $t^2$)

$$\begin{aligned} A_m(t) =& a_0 + t\left[\frac{-(n-1)}{n}\frac{v_1}{v_0}a_0 + a_1\right] \\ &+t^2\left\{\left[\frac{-(n-1)}{n}\frac{v_2}{v_0} + \frac{(n-1)(2(n-1)+1)}{2n^2}\frac{v_1^2}{v_0^2}\right]a_0\right. \\ &+\left.\left[\frac{-(n-1)}{n}\frac{v_1}{v_0}\right]a_1 + a_2\right\} + \cdots . \end{aligned} \tag{3.8}$$

The fact that $A_m'(0) = 0$ (see appendix), (3.8) and (3.5) leads us to

$$\frac{-(n-1)}{n}\frac{v_1}{v_0}a_0 + a_1 = 0,$$

multiplying by $-\dfrac{n}{n-1}\dfrac{v_0}{a_0}$

$$\frac{-(n-1)}{n}\frac{a_0}{v_0}a_0 + (n-1)H_m a_0 = 0.$$

Then

$$v_0 = \frac{a_0}{nH_m} \tag{3.9}$$

Substituing the identities (3.5) and (3.9) into the coefficient of $t^2$ em (3.8) we obtain

$$\begin{aligned}
\frac{A''_m(0)}{2} &= \left[\frac{-(n-1)}{n}\frac{v_2}{v_0} + \frac{(n-1)(2(n-1)+1)}{2n^2}\frac{v_1^2}{v_0^2}\right] a_0 \\
&+ \left[\frac{-(n-1)}{n}\frac{v_1}{v_0}\right] a_1 + a_2 \\
&= \left[\frac{-(n-1)}{n}\frac{\frac{1}{2}(n-1)H_m a_0}{v_0} + \frac{(n-1)(2(n-1)+1)}{2n^2}\frac{a_0^2}{v_0^2}\right] a_0 \\
&+ \left[\frac{-(n-1)}{n}\frac{a_0}{v_0}\right](n-1)H_m a_0 + a_2 \\
&= \left[\frac{-(n-1)}{n}\frac{\frac{1}{2}(n-1)H_m a_0}{\frac{a_0}{nH_m}} + \frac{(n-1)(2(n-1)+1)}{2n^2}\frac{a_0^2}{(\frac{a_0}{nH_m})^2}\right] a_0 \\
&+ \left[\frac{-(n-1)}{n}\frac{a_0}{\frac{a_0}{nH_m}}\right](n-1)H_m a_0 + a_2 \\
&= \left[\frac{-(n-1)^2 H_m^2}{2} + \frac{(n-1)(2(n-1)+1)H_m^2}{2}\right] a_0 \\
&- \left[(n-1)H_m\right](n-1)H_m a_0 + a_2 \\
&= \left[\frac{(n-1)^2 H_m^2}{2} + \frac{(n-1)H_m^2}{2}\right] a_0 - (n-1)^2 H_m^2 a_0 + a_2 \\
&= -\frac{((n-1)^2-(n-1))H_m^2}{2} a_0 + a_2 \\
&= -\int \left[\frac{((n-1)^2-(n-1))H_m^2}{2} - H_{m,2}\right] d\omega.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{A''_m(0)}{2} &= -\int_M \left[\left(\frac{(n-1)^2-(n-1)}{2}\right)\left(\sum_{i=1}^{n-1}\frac{\lambda_i}{(n-1)}\right)^2 - \sum_{i<j}\lambda_i\lambda_j\right] d\omega \\
&= -\frac{1}{2(n-1)}\int_M \left(\sum_{i<j}(\lambda_i-\lambda_j)^2\right) d\omega.
\end{aligned}$$

The last equality above is seen as follows (multiplying member to member by $2(n-1)$.)

$$\begin{aligned}
&(n-2)(n-1)^2 H_m^2 - 2(n-1)H_{m,2}\\
&\qquad =(n-2)(n-1)^2\left(\sum_i \lambda_i\right)^2 \frac{1}{(n-1)^2} - 2(n-1)\left(\sum_{i<j}\lambda_i\lambda_j\right)\\
&\qquad =(n-2)\left(\sum_{i=1}^{n-1}\lambda_i^2\right) + 2(n-2)\sum_{i<j}\lambda_i\lambda_j - 2(n-1)\sum_{i<j}\lambda_i\lambda_j\\
&\qquad =(n-2)\left(\sum_i \lambda_i^2\right) - 2\sum_{i<j}\lambda_i\lambda_j\\
&\qquad =\sum_{i<j}(\lambda_i-\lambda_j)^2.
\end{aligned}$$

In the last account we use that, by induction, it is worth:

$$\left(\sum_{i=1}^{n-1}\lambda_i\right)^2 = \sum \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j.$$

We conclude then that:

$$A_m''(0) = -\frac{1}{2(n-1)}\int_M \left(\sum_{i<j}(\lambda_i-\lambda_j)^2\right) d\omega. \tag{3.10}$$

From (3.10) we see that if $x$ is not all umbilic then $A''(0)$ is negative and the immersion is unstable

With this we prove that $x(M)$ is the Minkowski sphere in $\mathbb{R}^n$ (see Lemma 2.12).

□

# Appendix

## 4.1 Appendix A: Proof of Lemma 3.5

The proof is divided in several lemmas. Let $p \in M$ and $e_1, \ldots, e_{n-1}$ an orthonormal basis of $T_pM$ consisting of *euclidean* principal directions. Translate the basis, by parallel transport along geodesics issuing from $p$, to all points in a geodesic neighborhood in $M$. Extend the vector fields $\{e_i\}$ to a neighbourhood of $p$ in $\mathbb{R}^n$ and notice that $\nabla_{e_i} e_j(p) = 0$, $[e_i, e_j](p) = 0$ and thus $\nabla_{e_i}\nabla_{e_j} X - \nabla_{e_j}\nabla_{e_i} X = \nabla_{[e_i,e_j]} X = 0$.

Recall that $u : \mathbb{S}^{n-1} \to \partial\mathbb{B}$ is the inverse of the *euclidean* Gauss map of $\partial\mathbb{B}$, $\eta = u \circ \xi$ and that $\rho = \langle \eta, \xi \rangle^{-1} \langle x, \xi \rangle$.

**Lemma 4.1.** *Consider $X \in T_pM$ and $\{e_i\}$ as above. Then*

a) $\langle e_k, \nabla_{e_j}\nabla_{e_i}\xi \rangle = \langle e_i, \nabla_{e_j}\nabla_{e_k}\xi \rangle$

b) $\sum_{i,j} \langle X, e_i \rangle \langle du(e_j), \nabla_{e_j}\nabla_{e_i}\xi \rangle = \sum_{i,j} \langle X, \nabla_{e_j}\nabla_{e_i}\xi \rangle \langle du(e_i), e_j \rangle$

*Proof.* For the first item, recall that $d\xi$ is self-adjoint

$$\begin{aligned}
\langle e_k, \nabla_{e_j}\nabla_{e_i}\xi \rangle &= e_j \langle e_k, \nabla_{e_i}\xi \rangle \\
&= e_j \langle e_i, \nabla_{e_k}\xi \rangle \\
&= \langle e_i, \nabla_{e_j}\nabla_{e_k}\xi \rangle.
\end{aligned}$$

For the second item, we will use the first item, which $du$ is self-adjoint

$$\begin{aligned}
\sum_i \langle X, \nabla_{e_j}\nabla_{e_i}\xi\rangle\langle du(e_i), e_j\rangle &= \sum_{i,k}\langle\langle X, e_k\rangle e_k, \nabla_{e_j}\nabla_{e_i}\xi\rangle\langle du(e_i), e_j\rangle \\
&= \sum_{i,k}\langle X, e_k\rangle\langle e_k, \nabla_{e_j}\nabla_{e_i}\xi\rangle\langle du(e_i), e_j\rangle \\
&= \sum_{i,k}\langle X, e_k\rangle\langle e_i, \nabla_{e_j}\nabla_{e_k}\xi\rangle\langle du(e_j), e_i\rangle \\
&= \sum_{i,k}\langle X, e_k\rangle\langle\langle du(e_j), e_i\rangle e_i, \nabla_{e_j}\nabla_{e_k}\xi\rangle \\
&= \sum_{k}\langle X, e_k\rangle\langle du(e_j), \nabla_{e_j}\nabla_{e_k}\xi\rangle \\
&= \sum_{i}\langle X, e_i\rangle\langle du(e_j), \nabla_{e_j}\nabla_{e_i}\xi\rangle.
\end{aligned}$$

and the result follows. □

**Lemma 4.2.** *Concerning $\rho$ we have the following properties:*

a) $e_i(\rho) = \langle \eta, \xi\rangle^{-1}\langle x - \rho\eta, \nabla_{e_i}\xi\rangle$

b) $\sum_j \langle \nabla_{e_j} du(e_i), e_j\rangle \nabla_{e_i}\xi = -\sum_j \langle du(e_j), \nabla_{e_j}\nabla_{e_i}\xi\rangle e_i$

c) $\sum_{i,j} e_i(\rho)\langle \eta, \xi\rangle\langle \nabla_{e_j} du(e_i), e_j\rangle = -\sum_{i,j}\langle x - \rho\eta, \nabla_{e_j}\nabla_{e_i}\xi\rangle\langle du(e_i), e_j\rangle$

*Proof.* The first assertion is a direct calculation:

$$\begin{aligned}
e_i(\rho) &= e_i(\langle \eta, \xi\rangle^{-1}\langle x, \xi\rangle) \\
&= \langle \eta, \xi\rangle^{-1} e_i(\langle x, \xi\rangle) + \langle x, \xi\rangle e_i(\langle \eta, \xi\rangle^{-1}) \\
&= (\langle e_i, \xi\rangle + \langle x, \nabla_{e_i}\xi\rangle)\langle \eta, \xi\rangle^{-1} + \langle x, \xi\rangle(-1)\langle \eta, \xi\rangle^{-2}\langle \eta, \nabla_{e_i}\xi\rangle \\
&= \langle \eta, \xi\rangle^{-1}\langle x - \rho\eta, \nabla_{e_i}\xi\rangle.
\end{aligned}$$

For the second assertion we use that, in $p \in M$, $\{e_i\}$ are eigenvalues of $d\xi$, that $\nabla_{e_i} e_j = 0$, $\nabla_{e_i}\nabla_{e_j} = \nabla_{e_j}\nabla_{e_i}$ and that $du$ is self-adjoint.

$$
\begin{aligned}
\langle \nabla_{e_j} du(e_i), e_j \rangle \nabla_{e_i}\xi &= e_j(\langle du(e_i), e_j \rangle)\nabla_{e_i}\xi \\
&= e_j(\langle du(e_j), e_i \rangle) k_i e_i \\
&= \langle \nabla_{e_j} du(e_j), \nabla_{e_i}\xi \rangle e_i \\
&= (-\langle du(e_j), \nabla_{e_j}\nabla_{e_i}\xi \rangle + e_j(\langle du(e_j), \nabla_{e_i}\xi \rangle))e_i \\
\sum_j e_j(\langle du(e_j), \nabla_{e_i}\xi \rangle) &= \sum_j e_j(\langle e_j, \nabla_{e_i}\eta \rangle) \\
&= \sum_j \langle e_j, \nabla_{e_j}\nabla_{e_i}\eta \rangle \\
&= \sum_j \langle e_j, \nabla_{e_i}\nabla_{e_j}\eta \rangle \\
&= e_i(\sum_j \langle e_j, \nabla_{e_j}\eta \rangle) \\
&= e_i((n-1)H_m) = 0,
\end{aligned}
$$

since $H_m$ is constant. Also note that from the above calculation and $H_m$ constant, it is worth:

$$
\sum_j \langle e_j, \nabla_{e_j}\nabla_{e_i}\eta \rangle = 0 \tag{4.1}
$$

The third one follows directly from the calculation below.

From the first statement, we have

$$
e_i(\rho)\langle \eta, \xi \rangle \langle \nabla_{e_j} du(e_i), e_j \rangle = \langle x - \rho\eta, \nabla_{e_i}\xi \rangle \langle \nabla_{e_j} du(e_i), e_j \rangle
$$

as $du$ is self-adjoint and $\nabla_{e_i} e_j = 0$ in $p$, we obtain

$$
\begin{aligned}
\langle \nabla_{e_j} du(e_i), e_j \rangle &= e_j\langle du(e_i), e_j \rangle - \langle du(e_i), \nabla_{e_j} e_j \rangle \\
&= e_j\langle e_i, du(e_j) \rangle \\
&= \langle e_i, \nabla_{e_j} du(e_j) \rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
& e_i(\rho)\langle\eta,\xi\rangle\langle\nabla_{e_j}du(e_i),e_j\rangle \\
& \qquad = \langle x-\rho\eta,\nabla_{e_i}\xi\rangle\langle\nabla_{e_j}du(e_j),e_i\rangle \\
& \qquad = \langle x-\rho\eta,k_ie_i\rangle\langle\nabla_{e_j}du(e_j),e_i\rangle \\
& \qquad = \langle x-\rho\eta,e_i\rangle\langle\nabla_{e_j}du(e_j),\nabla_{e_i}\xi\rangle \\
& \qquad = \langle x-\rho\eta,e_i\rangle\left(e_j\langle du(e_j),d\xi(e_i)\rangle-\langle du(e_j),\nabla_{e_j}\nabla_{e_i}\xi\rangle\right) \\
& \qquad = \langle x-\rho\eta,e_i\rangle\left(e_j\langle e_j,d\eta(e_i)\rangle-\langle du(e_j),\nabla_{e_j}\nabla_{e_i}\xi\rangle\right) \\
& \qquad = \langle x-\rho\eta,e_i\rangle\left(\langle e_j,\nabla_{e_j}\nabla_{e_i}\eta\rangle-\langle du(e_j),\nabla_{e_j}\nabla_{e_i}\xi\rangle\right)
\end{aligned}
$$

By equation (4.1) in statement two, we obtain

$$
\sum_j e_i(\rho)\langle\eta,\xi\rangle\langle\nabla_{e_j}du(e_i),e_j\rangle=-\sum_j\langle x-\rho\eta,e_i\rangle\langle du(e_j),\nabla_{e_j}\nabla_{e_i}\xi\rangle,
$$

and the result follow from the 4.1-b.

□

Remember the statement of the Lemma 3.5:

**Lemma** (3.5)**.** *Let* $x: M\to\mathbb{R}^n$ *have constant Minkowskian mean curvature and let* $\rho(x)=\langle\eta,\xi\rangle^{-1}\langle x,\xi\rangle$. *Then*

$$
\langle\eta,\xi\rangle^{-1}\operatorname{div}_M(\langle\eta,\xi\rangle^2du(\nabla\rho))=(n-1)H_m-\rho B_m^2.
$$

*Proof of Lemma 3.5.* Using Lemma 4.2-a compute

$$
\begin{aligned}
e_je_i(\rho) &= e_j(\langle\eta,\xi\rangle^{-1}\langle x-\rho\eta,\nabla_{e_i}\xi\rangle) \\
&= -\langle\eta,\xi\rangle^{-2}\langle\eta,\nabla_{e_j}\xi\rangle\langle x-\eta\rho,\nabla_{e_i}\xi\rangle \\
&+ \langle\eta,\xi\rangle^{-1}\langle e_j-\rho\nabla_{e_j}\eta-e_j(\rho)\eta,\nabla_{e_i}\xi\rangle \\
&+ \langle\eta,\xi\rangle^{-1}\langle x-\rho\eta,\nabla_{e_j}\nabla_{e_i}\xi\rangle.
\end{aligned}
$$

Again applying Lemma 4.2-a in the first term above, we have

$$e_j e_i(\rho) = \langle \eta, \xi\rangle^{-1}\left(-e_i(\rho)\langle \eta, \nabla_{e_j}\xi\rangle + \langle e_j - \rho\nabla_{e_j}\eta - e_j(\rho)\eta, \nabla_{e_i}\xi\rangle\right) \quad (4.2)$$
$$+ \langle \eta, \xi\rangle^{-1}\langle x - \rho\eta, \nabla_{e_j}\nabla_{e_i}\xi\rangle.$$

Since $\{e_i\}$ is orthonormal we compute the divergence as

$$\operatorname{div}_M(X) = \sum_i \langle \nabla_{e_i} X, e_i\rangle.$$

In the following we will omit the summation sings

$$\begin{aligned}
\operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla\rho)) &= \langle \nabla(\langle \eta, \xi\rangle^2), du(\nabla\rho)\rangle + \langle \eta, \xi\rangle^2 \operatorname{div}_M(du(\nabla\rho)) \\
&= 2\langle \langle \eta, \xi\rangle \nabla\langle \eta, \xi\rangle, du(\nabla\rho)\rangle + \langle \eta, \xi\rangle^2 \langle \nabla_{e_i}(du(\nabla\rho)), e_i\rangle \\
&= 2\langle \langle \eta, \xi\rangle \langle \eta, \nabla_{e_i}\xi\rangle e_i, e_j(\rho) du(e_j)\rangle \\
&+ \langle \eta, \xi\rangle^2 \langle \nabla_{e_i}(e_j(\rho) du(e_j)), e_i\rangle \\
&= 2\langle \eta, \xi\rangle \langle \eta, \nabla_{e_i}\xi\rangle e_j(\rho)\langle e_i, du(e_j)\rangle \\
&+ \langle \eta, \xi\rangle^2 e_i e_j(\rho)\langle du(e_j), e_i\rangle \\
&+ \langle \eta, \xi\rangle^2 e_j(\rho)\langle \nabla_{e_i}(du(e_j)), e_i\rangle.
\end{aligned}$$

Using the equation (4.2)

$$\begin{aligned}
&\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla\rho)) \\
&\qquad = 2\langle \eta, \nabla_{e_i}\xi\rangle e_j(\rho)\langle e_i, du(e_j)\rangle \\
&\qquad + \left(-e_i(\rho)\langle \eta, \nabla_{e_j}\xi\rangle + \langle e_j - \rho\nabla_{e_j}\eta - e_j(\rho)\eta, \nabla_{e_i}\xi\rangle\right)\langle du(e_j), e_i\rangle \\
&\qquad + \langle x - \rho\eta, \nabla_{e_j}\nabla_{e_i}\xi\rangle\langle du(e_j), e_i\rangle \\
&\qquad + \langle \eta, \xi\rangle e_j(\rho)\langle \nabla_{e_i}(du(e_j)), e_i\rangle.
\end{aligned}$$

Applying Lemma 4.2-c

$$
\begin{aligned}
&\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla\rho)) \\
&\qquad\qquad = 2\langle \eta, \nabla_{e_i}\xi\rangle e_j(\rho)\langle e_i, du(e_j)\rangle \\
&\qquad\qquad + \left(-e_i(\rho)\langle \eta, \nabla_{e_j}\xi\rangle + \langle e_j - \rho\nabla_{e_j}\eta - e_j(\rho)\eta, \nabla_{e_i}\xi\rangle\right)\langle du(e_j), e_i\rangle.
\end{aligned}
$$

Recalling that $du$ is self-adjoint and canceling the terms $e_i(\rho)\langle \eta, \nabla_{e_j}\xi\rangle\langle du(e_j), e_i\rangle$, follow that

$$\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla\rho)) = \langle e_j - \rho\nabla_{e_j}\eta, \nabla_{e_i}\xi\rangle\langle e_j, du(e_i)\rangle.$$

Finally, using that $e_i$ are eigenvectors of $d\xi$, we have

$$
\begin{aligned}
\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla\rho)) &= \langle e_j - \rho\nabla_{e_j}\eta, e_i\rangle\langle e_j, d\eta(e_i)\rangle \\
&= \langle e_i, d\eta(e_i)\rangle - \rho\langle d\eta(e_j), e_i\rangle\langle e_j, d\eta(e_i)\rangle.
\end{aligned}
$$

Since $d\eta$ is diagonalizable, $[d\eta] = C.D.C^{-1}$ where $[d\eta]$ is the matrix of $d\eta$ in the basis $\{e_i\}$, $D$ is diagonal and $C$ is invertible. Then

$$\sum_{i,j}\langle d\eta(e_j), e_i\rangle\langle e_j, d\eta(e_i)\rangle = \operatorname{tr}([d\eta]^2) = \operatorname{tr}(C.D^2.C^{-1}) = \sum_i \lambda_i^2$$

and the result follows. $\square$

## 4.2 Appendix B: Details of proof for Second variation of Area

**Proposition 4.3.** *Let $x : M \to \mathbb{R}^n$ be an immersed surface with Birkhoff-Gauss map $\eta$ and constant mean curvature, and let $F : (\epsilon, \epsilon) \times M \to \mathbb{R}^n$, be a volume-preserving variation of compact support given by*

$$F(t,p) = F^t(p) = x(p) + g(t,p)\eta(p).$$

*Denote $f(p) = \frac{\partial}{\partial t} g(t,p)\big|_{t=0}$. Then*

a) $\nabla_{\eta^\top}\xi = \nabla\langle \eta, \xi\rangle$, *where* $\eta^\top$ *is the orthogonal projection of* $\eta$ *on* $T_pM$

b) $\frac{\partial}{\partial t}\eta\Big|_{t=0} = -\langle \eta, \xi\rangle du\nabla f$

c) $\langle \eta, \xi\rangle^{-1} \operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f)) = \langle du(\nabla f), \nabla(\langle \eta, \xi\rangle)\rangle + \operatorname{div}_M(\langle \eta, \xi\rangle du(\nabla f))$, *where* $\operatorname{div}_M X = \sum_{i=1}^n \langle \nabla_{e_i} X, e_i\rangle$, *and* $\{e_1, \cdots, e_{n-1}\}$ *a ortonormal basis of* $T_pM$.

*Proof.* The first item is the letter (a) of the Lemma 2.21.

For the second item, we combine the letter (b) of the Lemma 2.21 with the present particular variation, and then

$$\begin{aligned}
\frac{\partial}{\partial t}\xi\Big|_{t=0} &= -\nabla\langle F_t, \xi\rangle + \nabla_{F_t^\top}\xi \\
&= -\nabla\langle f\eta, \xi\rangle + \nabla_{f\eta^\top}\xi \\
&= -\langle \eta, \xi\rangle \nabla f - f\nabla\langle \eta, \xi\rangle + f\nabla\langle \eta, \xi\rangle \\
&= -\langle \eta, \xi\rangle\nabla f.
\end{aligned}$$

Hence $\frac{\partial}{\partial t}\eta\Big|_{t=0} = du \circ \frac{\partial}{\partial t}\xi\Big|_{t=0} = -\langle \eta, \xi\rangle du\nabla f$.

The third item is a simple calculation.

$$\begin{aligned}
\operatorname{div}_M(\langle \eta, \xi\rangle^2 du(\nabla f)) =& \langle \nabla\langle \eta, \xi\rangle^2, du(\nabla f)\rangle + \langle \eta, \xi\rangle^2 \operatorname{div}_M(du(\nabla f)) \\
=& 2\langle \eta, \xi\rangle\langle \nabla\langle \eta, \xi\rangle, du(\nabla f)\rangle + \langle \eta, \xi\rangle^2 \operatorname{div}_M(du(\nabla f)) \\
=& 2\langle \eta, \xi\rangle\langle \nabla\langle \eta, \xi\rangle, du(\nabla f)\rangle + \langle \eta, \xi\rangle \operatorname{div}_M(\langle \eta, \xi\rangle du(\nabla f)) \\
& -\langle \eta, \xi\rangle\langle \nabla\langle \eta, \xi\rangle, du(\nabla f)\rangle \\
=& \langle \eta, \xi\rangle\langle \nabla\langle \eta, \xi\rangle, du(\nabla f)\rangle + \langle \eta, \xi\rangle \operatorname{div}_M(\langle \eta, \xi\rangle du(\nabla f)).
\end{aligned}$$

□

## 4.3 Appendix C: Details of alternative proof for Stability Theorem

We will show that, for the variation $F(t) = x + t\eta$, we have

$$A_m(t) = \int_M \prod_{i=1}^{n-1} (1 + \lambda_i t) \ d\omega.$$

*Proof.* In fact, let $\phi : V \to D$ be a local parametrization of neighborhood of $p \in M$. The map $\psi : V \to \overline{D}$ defined as $\psi(q) = \phi(q) + c\eta(\phi(q))$ is a local parametrization of the neighborhood $\overline{D}$ of $p + c\eta(p) \in \overline{M}$. We have

$$\psi_{u_i} = \phi_{u_i} + cd\eta\phi_{u_i},$$

for all $i = 1, \cdots, n-1$ and also, that the Birkhof normal to $M$ at $p$ is the same as the Birkhoff normal to $\overline{M}$ at $p + c\eta(p)$. If we define $G$ and $\overline{G}$ the matrix of $d\eta_p$ and $d\psi_p$ in the base $\{\phi_{u_1}, \cdots, \phi_{u_{n-1}}\}$, we have

$$\overline{G} = Id + cG$$

Thus

$$\det \overline{G} = \det(C^{-1}C + c\, C^{-1}\, D\, C) = \prod_{i=1}^{n-1} (1 + ck_i)$$

where $C$ is invertible and $D$ is the diagonal matrix of the principal Minkowski curvature $k_i$ of $M$. Therefore

$$\omega_t(\psi_{u_1}, \cdots, \psi_{u_{n-1}}) = \prod_{i=1}^{n-1} (1 + ck_i)\omega(\phi_{u_1}, \cdots, \phi_{u_{n-1}}).$$

$\square$

For the same variation given above, we see that

$$\frac{dV(t)}{dt} = A_m(t)$$

*Proof.* In fact, it is well known that

$$V'(t) = \int_M \langle F_t(t), \xi(t) \rangle \; dM_t$$

Using that $F_t(t) = \eta$ and that $\eta(t)$ is parallel the $\eta$, since fixed $p \in M$, for each $t$, $T_pM$ is parallel to $T_{F(t,p)}M_t$, follows that

$$V'(t) = \int_M d\omega_t = A_m(t).$$

$\square$

**Proposition 4.4.** *Consider the variation $y(t) = s(t)F(t) = s(t)(x + t\eta)$ presented in the Stability Theorem alternative statement. Given that*

$$A_m(t) = \int_M \langle \eta(t), \xi(t) \rangle dM_t.$$

*We affirm that*

$$A'_m(0) = 0.$$

*Proof.* In fact, as $\eta(t) = \eta$ e $\xi(t) = \xi$, just calculate $\left.\frac{d}{dt}\right|_0 dM_t$. We know, of the first variation of area, that

$$\left.\frac{d}{dt}\right|_0 dM_t = \mathrm{div}_M \left( \left.\frac{dy}{dt}\right|_0 \right) dM.$$

Now,

$$\left.\frac{dy}{dt}\right|_0 = s'(0)x + s(0)\eta = -H_m x + \eta.$$

For the calculation of $s'(0)$, we derive the equation (3.7), to obtain

$$\begin{aligned} 0 =& \left.\frac{d(V(sF_t))}{dt}\right|_{t=0} = \left.\frac{d(s^{n+1}V(F_t))}{dt}\right|_{t=0} \\ =& (n+1)s(0)^n s'(0) v_0 + s^{n+1}(0) \left.\frac{d(V(F_t))}{dt}\right|_{t=0} \\ =& (n+1)s'(0)v_0 + a_0, \end{aligned}$$

since $s(0) = 1$. Remember that $a_0$ and $v_0$ are Minkowksi area and volume of the immersion $x$, as in section 3.4. Thus, using that $\int_M (1 - \rho H)\, d\omega = 0$, we have

$$\frac{-1}{(n+1)} \frac{a_0}{v_0} = -H_m,$$

Using that $H_m$ is constant, $\mathrm{div}_M(x) = n$ and $\mathrm{div}_M(\eta) = nH_m$, we have:

$$\mathrm{div}_M \left( \left. \frac{dy}{dt} \right|_0 \right) = -nH_m + nH_m = 0.$$

□

To conclude the section, we present the binomial theorem, which allowed us to obtain the expression of $s^n$ by specifying the coefficients up to $t^2$.

**Theorem 4.5** (Binomial Theorem). *If $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$, with $|z| < 1$. Then:*

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,$$

*where* $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(k-1))}{k(k-1)(k-2)\cdots 1}$.

## 4.4 Basic tools of Riemannian Geometry

A good reference, which we indicate for this section, is the book by Manfredo Perdigão do Carmo (see [12])

**Tangent space**

**Definition 4.6.** *A smooth manifold of dimension $n$ is a Hausdorff topological space $M$ with an enumerable base equipped with a maximal atlas.*

**Definition 4.7.** *Let $\gamma : I \to M$ be a differentiable curve with $\gamma(t_0) = p$. Let*

$$D_p(M) = \{f : M \to \mathbb{R} : f \text{ is differentiable in} p\}$$

*be the vectorial space of the reals functions in $M$ differentiable in $p$. The tangent vector to curve $\gamma$ in $p$ is the function $\gamma'(t_0) : D_p(M) \to \mathbb{R}$ definided by $\gamma'(t_0)f = (f \circ \gamma)'(t_0)$.*

*A tangent vector in $p$ is the tangent vector in $t = 0$ of any curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. The set of the tangent vectors to $M$ in $p$ will be indicated by $T_pM$.*

A base for $T_pM$ can be given choosing a local chart $x : U \to x(U)$ in $p$ and considering the map $\frac{\partial(p)}{\partial x_i} : C^\infty(M) \to \mathbb{R}$ defined by,

$$\frac{\partial}{\partial x_i}(p)f = \frac{\partial}{\partial x_i} f \circ x(x^{-1}(p))$$

Thus, the functions $\dfrac{\partial}{\partial x_i}(p)$ according to definition, they are tangent vectors to $M$ in $p$ and the set $\left\{ \dfrac{\partial}{\partial x_1}(p), ..., \dfrac{\partial}{\partial x_m}(p) \right\}$ form a basis for $T_pM$.

We can see the tangent vectors to $M$ in $p$ in another way:

**Definition 4.8.** *Let $M$ be a smooth manifold and $p \in M$. The linear map $X_p : C^\infty(M) \to \mathbb{R}$, defined in the set of all the infinitely differentiable functions in a neighborhood of $p$ is called a derivation in $p$ if the product rule is satisfied,*

$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$$

*for all $f$, $g \in C^\infty(M)$.*

*$X_p$ is called a tangent vector to $M$ in $p$.*

The set of all derivations from $C^\infty(M)$ in $p$ has a vector space structure, called *tangent space* to $M$ in $p$, denoted by $T_pM$. An element from $T_pM$ is called *tangent vector* to $M$ in $p$.

The *tangent bundle* $TM$ is defined by $TM := \bigcup_{p \in M} T_pM$.

**Definition 4.9.** *A vectors field $X$ in a smooth manifold $M$ is a correspondece that at each point $p \in M$ associates a vector $X(p) \in T_pM$. The field $X$ is differentiable if the map $X : M \to TM$ is differentiable.*

**Definition 4.10.** *A point $p \in M$ is said to be a regular point of $f : M \to N$ when the derivative $f'(p) : T_pM \to T_{f(p)}N$ is injective. Otherwise, $p$ is said a singular or critical point of $f$.*

**Definition 4.11.** *A differentiable map $f : M \to N$ is said to be an immersion if every point $p \in M$ is a regular point for $f$, that is, the derivative $f'(p) : T_pM \to T_{f(p)}N$ is injective for each $p \in M$.*

**Definition 4.12.** *Given two fields $X, Y \in \mathcal{T}(M)$, vector field $[X, Y]$ defined by,*

$$[X, Y]_p f = (XY - YX)f = X_p(Y(f)) - Y_p(X(f))$$

*it's called a bracket.*

**Riemannian manifolds**

**Definition 4.13.** *A Riemannian metric in a smooth manifold $M$ is a correspondence that associates to each point of $p$ of $M$ an internal product $\langle , \rangle_p$ in the tangent space $T_pM$. Sometimes we use the notation $g( , ) = g_p( , )$ for the Riemannian metric.*

The above definition requires that the metric $\langle , \rangle_p$ is differentiable in the following sense: if $x : U \to x(U)$ is a system of local coordinate in $p \in x(U)$, for $q \in x(U)$ with $q = x(x_1, ..., x_m)$ we should have that the function of $U$ in $\mathbb{R}$, $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle_q : x(U) \to \mathbb{R}$ to be a differentiable function for all $i, j = \{1, \ldots, m\}$.

**Definition 4.14.** *The functions* $g_{ij}(x_1, ..., x_m) = \left\langle \frac{\partial(q)}{\partial x_i}, \frac{\partial(q)}{\partial x_j} \right\rangle_q$ *are called expression of the Riemannian metric in the coordinate system* $x$*. A differentiable manifold with a given Riemannian metric is called Riemannian manifold.*

**Affine connection**

**Definition 4.15.** *A affine connection* $\nabla$ *in a smooth manifold* $M$ *is a map*

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

*which is indicated by* $\nabla(X, Y) = \nabla_X Y$ *and which satisfies the following properties:*

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z,$

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z,$

(iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y,$

where $X, Y, Z \in \mathcal{T}(M)$, $f, g \in C^\infty(M)$. The symbol $\nabla_X Y$ reads: covariant derivative $Y$ in direction of $X$. When the affine connection satisfies the following properties:

(i) $X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$, (metric compatibility)

(ii) $\nabla_X Y - \nabla_Y X = [X, Y]$, (symmetry)

it is called *Levi-Civita connection* (or *Rimannian connection*).

**Curvature**

**Definition 4.16.** *The* $R$ *curvature of a* $M$ *Riemannian manifold corresponds to each pair* $X, Y \in \mathcal{T}(M)$ *a map* $R(X, Y) : \mathcal{T}(M) \to \mathcal{T}(M)$ *given by,*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad Z \in \mathcal{T}(M)$$

*where* $\nabla$ *is the Riemannian connection of* $M$*.*

**Proposition 4.17.** *The $R$ curvature in $M$ has the following properties*

*(a) $R$ is bilinear in $\mathcal{T}(M) \times \mathcal{T}(M)$, that is,*

$$R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$$

$$R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2)$$

*with $f, g \in C^\infty(M)$, and $\quad X_1, X_2, Y_1, Y_2 \in \mathcal{T}(M)$*

*(b) For all $X, Y \in \mathcal{T}(M)$, the curvature operator $R(X, Y) : \mathcal{T}(M) \to \mathcal{T}(M)$ is linear, that is,*

$$R(X, Y)(Z + W) = R(X, Y)Z + R(X, Y)W$$

$$R(X, Y)fZ = fR(X, Y)Z$$

*with $f \in C^\infty(M), \quad Z, W \in \mathcal{T}(M)$*

*(c) (Bianchi's first identity)*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

A demonstration of this result can be found at[12].

**Proposition 4.18.** *Given a Riemannian manifold $(M, g)$ with a curvature $R$, using the notation $R(X, Y, Z, T) = g(R(X, Y)Z, T)$ for any fields $X, Y, Z, T \in \mathcal{T}(M)$ we have the following properties:*

*(a) $R(X, Y, Z, T) + R(Y, Z, X, T) + R(Z, X, Y, T) = 0$*

*(b) $R(X, Y, Z, T) = -R(Y, X, Z, T)$*

*(c) $R(X, Y, Z, T) = -R(X, Y, T, Z)$*

*(d) $R(X, Y, Z, T) = R(Z, T, X, Y)$*

**Proposition 4.19.** *Let $\sigma \subset T_pM$ be a two-dimensional subspace of space $T_pM$ e sejam $X, Y \in \sigma$ two linearly independent vectors. So*

$$K(X,Y) = \frac{(X,Y,Y,X)}{|x \wedge Y|^2}$$

*onde $|X \wedge Y|^2 = \sqrt{|X|^2|Y|^2 - g(X,Y)^2}$, independent of the choice of vectors $X, Y \in \sigma$.*

**Definition 4.20.** *Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$ the real number $K(X,Y) = K(\sigma)$, where $\{X,Y\}$ is a any base of $\sigma$ is called* seccional curvature *of $\sigma$ in $p$*

**Definition 4.21.** *Let $M$ be a Riemannian manifold. The* Ricci's curvature *tensor of $M$ (or simply* Ricci tensor*) denoted* Ric*, is the covariant tensor field of order 2 defined as the trace of endomorfism curvature tensor in relation to its first covariant index and its only contravariant index or, equivalently, as the trace in relation to the metric of the curvature tensor in its first and last indexes. Therefore, the components of the Ricci curvature are given by*

$$R_{ij} = \sum_{k=1}^{n} = \sum_{k,m=1}^{n} g^{km} R_{kijm}$$

Due to the symmetries of the curvature endomorphism tensor, using different traces would not make a difference or would only imply a signal exchange.

**Definition 4.22.** *Let $M$ be a Riemannian manifold. The* scalar curvature *of $M$, denoted $S$, is the real function $S : M \to \mathbb{R}$ defined by the trace in relation to the metrics of the Ricci tensor:*

$$S = \mathrm{tr}_g \mathrm{Ric} = \sum_{i,j=1}^{n} g^{ij} R_{ij}.$$

**Definition 4.23.** *Let $M^n$ and $\overline{M}^{n+k}$ $(k \geq 1)$ be Riemannian manifolds. A immersion $\phi : M^n \to \overline{M}^{n+k}$ is called isometric if $\langle d\phi_p(v), d\phi_p(w)\rangle_{\overline{M}} = \langle v, w\rangle_M, \forall v, w \in T_pM$.*

Given an isometric immersion $\phi : M^n \to \overline{M}^{n+k}$, we can establish relationships between objects defined in both manifolds. Let us remember that if $\phi : M^n \to \overline{M}^{n+k}$ is an immersion, then $\phi$ is locally embedding. In these conditions, we can identify an open $U$ of $M$ with $\phi(U)$, and said that $\phi$ is the inclusion map locally. Moreover, we can consider $U$ as a submanifold of $M$. In particular, we are identifying $p \in U$ with $\phi(p) \in \phi(U)$.

Consequently, for each $p \in M$, the tangent space $T_pM$ is considered a vector subspace of $T_p\overline{M}$ of dimension $n$ (already considering the identification above).

Take now, local vector fields $X$ and $Y$ tangents to $M$. How $\phi|_U$ is a embedding, there are local extensions $\overline{X}$ and $\overline{Y}$ of $X$ and $Y$, respectively, in a neighborhood of $U$ in $\overline{M}$. Thus, if $\overline{\nabla}$ is the connection of Levi-Civita of $\overline{M}$, it makes sense to calculate $\overline{\nabla}_{\overline{X}}\overline{Y}$, or even $\overline{\nabla}_X\overline{Y}$.

It can be shown that $\overline{\nabla}_X\overline{Y}$ does not depend on the extension $\overline{Y}$ of $Y$ that we take, and therefore, for simplicity of notation, we will denote $\overline{\nabla}_X\overline{Y}$ by $\overline{\nabla}_XY$, remembering that this means taking an extension of $Y$ to calculate the covariant derivative.

We then have:

$$\overline{\nabla}_XY = (\overline{\nabla}_XY)^\top + (\overline{\nabla}_XY)^\perp.$$

However, it is possible to verify that $(\overline{\nabla}_{\cdot}\cdot)^\top$ is the $M$ Levi-Civita connection (which we will denote by $\nabla$), this is, $(\overline{\nabla}_XY)^\top = \nabla_XY$.

Let us denote by $\mathcal{T}(M^n)^\perp$ the space of the differentiable vector fields normal to $M^n$. The *second fundamental form* of immersion $x$ is the map

$\mathrm{I\!I} : \mathcal{T}(M^n) \times \mathcal{T}(M^n) \to \mathcal{T}(M^n)^{\perp}$, defined by

$$\mathrm{I\!I}(X, Y) = \overline{\nabla}_X Y - \nabla_X Y, \quad \forall X, Y \in \mathcal{T}(M^n).$$

Since, for every $p \in M^n$, $\mathrm{I\!I}$ is a symmetric bilinear map, for each unit vector $N$ normal to the $M^n$ in $p$, we can associate it with a self-adjoint linear mapping $S_N : T_pM^n \to T_pM^n$, given by

$$\langle S_N(X), Y\rangle = \langle \mathrm{I\!I}(X, Y), N\rangle, \forall X, Y \in T_pM^n.$$

Then, let's define the *mean curvature* $H$ of the immersion $x : M^n \to \mathbb{R}^{n+1}$ by

$$H = \frac{1}{n}\mathrm{tr}(S_N).$$

Here $\mathrm{tr}(S_N)$ means the trace of the matrix of $S_N$.

Let $N \in \mathcal{T}(M^n)^{\perp}$ and $X$, $Y \in \mathcal{T}(M^n)$ be vector fields, then $\langle N, Y\rangle = 0$. This implies that

$$\langle \overline{\nabla}_X Y, N\rangle = \langle -\overline{\nabla}_X N, Y\rangle.$$

Thus

$$\langle S_N(X), Y\rangle = \langle \mathrm{I\!I}(X, Y), N\rangle = \langle \overline{\nabla}_X Y, N\rangle = \langle -\overline{\nabla}_X N, Y\rangle,$$

because $\langle \mathrm{I\!I}(X, Y), N\rangle = \langle \overline{\nabla}_X Y - \nabla_X Y, N\rangle = \langle \overline{\nabla}_X Y - (\nabla_X Y)^T, N\rangle = \langle \overline{\nabla}_X Y, N\rangle$.

Now, knowing that $nH = \mathrm{tr}(S_N)$ and that each entry in the $S_N$ matrix is given by

$$\langle S_N(e_i), e_j\rangle = \langle \mathrm{I\!I}(e_i, e_j), N\rangle = \langle \overline{\nabla}_{e_i} e_j, N\rangle,$$

we can write $nH$ as follows:

$$nH = \sum_{i=1}^{n} \langle S_N(e_i), e_i\rangle = \sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} e_i, N\rangle.$$

**Definition 4.24.** *Let $M$ be a hypersurface of $\mathbb{R}^{n+1}$ and $p \in M$, we say that $p$ is an umbilic point of $S$, if in $p$, the principal curvatures coincide.*

**Definition 4.25.** *Let $M$ be a hypersurface of $\mathbb{R}^{n+1}$, we say that $M$ is an umbilic hypersurface, if all point $p \in M$ is umbilic.*

**Theorem 4.26.** *Let $x : M \to \mathbb{R}^{n+1}$ be an umbilic isometric immersion of a connected Riemannian manifold $M^n$ in $\mathbb{R}^{n+1}$. Then, $x(M)$ is an open subset of an affine hyperplane or sphere. In case the hypersurface is compact, the hypersurface is the sphere.*

*Proof.* See [12], exercise 6.c, page 183. □

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