UNIVERSIDADE FEDERAL DE MINAS GERAIS
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Stabilization and $\mathcal{L}_2$-gain performance techniques applied to LPV sampled-data systems with gain-scheduled control

Técnicas para estabilização e desempenho $\mathcal{L}_2$ aplicadas a sistemas LPV amostrados com escalonamento de ganho

Belo Horizonte
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Supervisor: Prof. Dr. Luciano Antonio Frezzatto Santos

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To my family and friends.

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Abstract

This work presents new sufficient conditions for stabilization and controller synthesis to sampled-data LPV systems. The adopted control strategy considers gain-scheduling state-feedback controllers subject to induced time-varying input delay. The stabilization and $\mathcal{L}_2$-gain performance are investigated in the sense of Lyapunov by three different approaches. After the proposition of a Lyapunov function and with the application of Wirtinger's inequality, the derived conditions are rewritten in terms of linear matrix inequalities. Aiming at the enlargement of the aperiodic sampling time and at the minimization of the $\mathcal{L}_2$-gain, the effectiveness of the developed methodologies is assessed through numerical simulations of LPV systems available in the control literature.

Keywords: LPV systems; Sampled-data control; Gain scheduling; $\mathcal{L}_2$-gain performance; Lyapunov theory.

Resumo

Este trabalho apresenta novas condições suficientes para a estabilização e síntese de controladores para sistemas LPV amostrados. A estratégia de controle adotada considera controladores por realimentação de estados com escalonamento de ganho, sujeitos a um atraso induzido de tempo variante. A estabilização e o desempenho $\mathcal{L}_2$ são investigados no sentido de Lyapunov, por três abordagens distintas. A partir da definição de uma funcão de Lyapunov e com o emprego da desigualdade de Wirtinger, as condições obtidas são reescritas em termos de desigualdades matriciais lineares. Visando à maximização do intervalo de amostragem aperiódico e à minimização do ganho $\mathcal{L}_2$, a eficácia das metodologias desenvolvidas é avaliada com simulações numéricas de sistemas LPV disponíveis na literatura.

Palavras-chaves: Sistemas LPV; Controle amostrado; Ganho escalonado; Ganho $\mathcal{L}_2$; Teoria de Lyapunov.

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List of Abbreviations and Acronyms

BMI Bilinear matrix inequality

LMI Linear matrix inequality

LPV Linear parameter-varying

LTI Linear time-invariant

MASP Maximum allowable sampling period

SDP Semidefinite programming

ZOH Zero-order holder

List of Symbols

$\sum$	Summation.
$\mathbb{R}^n$	Set of n-dimension real vectors.
$\mathbb{R}^{n\times m}$	Set of $n \times m$ real matrices.
$\mathbf{I}_n$	Identity matrix of size n.
$\mathbf{0}_{n\times m}$	Null matrix of dimension $n \times m$.
A^{-1}	Inverse matrix of A.
A^T	Transpose matrix of A.
A^H	Hermitian operator, defined as $A^H = A + A^T$.
$A \succ 0$ ($A \succeq 0$)	Positive-definite (semidefinite) matrix.
$A \prec 0$ ($A \preceq 0$)	Negative-definite (semidefinite) matrix.
$rank(A)$	Rank of matrix A.
$\in$	Belongs to.
$\|v\|$	Euclidean norm of vector v.
$\mathcal{L}_2$	Space of absolutely integrable functions.
$*$	Symmetric term inside matrices.
$diag(A, B)$	Block-diagonal matrix composed by the blocks A and B.
$\dot{x}(t)$	First time derivative of $x(t)$.
$\otimes$	Kronecker product operator.
$\iff$	Material biconditional (denoted as "iff").
$\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$	Real function $\mathbf{f}$ mapping n inputs to m outputs.

Contents

1 Introduction

Over the past years, the study of sampled-data systems has been motivated by the growing interest for, among others, networked control and embedded systems (Walsh; Ye, 2001; Hespanha *et al.*, 2007; You; Xie, 2013; Tognetti; Calliero, 2017). Sampled-data systems are abstractions of dynamical models, whose dynamics from the controller perspective are updated at sampling instants only. As reported in Goebel *et al.* (2009), in a typical control scenario, a continuous-time system is controlled with a sampled-data controller, which can be then implemented as a digital controller.

A key topic regarding the implementation of a digital controller is the appropriate determination of the sampling times. According to Longo *et al.* (2013), the sampling times are often periodic in the classical sampled-data control, meaning that the information exchange usually happens instantaneously and on predetermined time instants. On the contrary, in real control applications, due to the interconnection and to the spatial distribution of sensors and actuators, not only the sampling times cannot be equally spaced, but there also is an inherent time-varying delay in the information exchange.

Therefore, in the framework of sampled-data systems, a more reliable control strategy should admit aperiodic sampling intervals. Although larger sampling intervals are advantageous in order to save resources (processing time, sensors and data transfer, for instance) and to provide more economical solutions, the systems must remain stable in closed-loop (Hooshmandi *et al.*, 2018). As a result, determining an upper bound of the allowable sampling periods, for which the closed-loop system stability is still guaranteed, is of greatest importance.

In control applications, some dynamical systems can be represented with linear parameter-varying (LPV) models (Leith; Leithead, 2000; Rugh; Shamma, 2000; Lacerda *et al.*, 2011; Rotondo *et al.*, 2013; Rodrigues *et al.*, 2018). In LPV systems, the bounds of the scheduling parameters exist and are known, whereas the bounds of the time derivatives of the scheduling parameters can be unspecified. A purpose of representing systems with LPV models is the obtention of (polytopic) linear time-invariant (LTI)-like models. It then allows the application of powerful control design tools, usually dedicated to linear systems, to LPV systems (Rugh; Shamma, 2000).

As reported in Palmeira *et al.* (2018), the stability analysis, the controller synthesis and the filter project of closed-loop systems are then made simpler, provided that the arising problems can be addressed in the framework of linear-matrix inequality (LMI) conditions and of convex optimization. Several works have successfully applied LMI conditions to the design of controllers and filters for LPV systems, see, for instance, the works of Boyd *et al.* (1993), Gahinet and Apkarian (1994), Xie *et al.* (1996), de

Souza (2019), Kim *et al.* (2020), Bahmani and Rahmani (2020), Aguiar *et al.* (2020) and references therein.

A control strategy with increasing interest in the context of LPV systems is the gain-scheduled control (Blanchini; Miani, 2003; de Caigny *et al.*, 2010; Pandey; de Oliveira, 2019; Sadeghzadeh, 2019). Its applications are wide, covering from aerospace and automotive systems (Alcala *et al.*, 2018; Li *et al.*, 2019; Liu *et al.*, 2020) up to energy conversion system (Mani *et al.*, 2020) and to cloud computing systems (Saikrishna *et al.*, 2017). According to Palmeira *et al.* (2018), gain-scheduling controllers have been largely used in the control literature since they are more suitable to incorporate the variation of the scheduling parameters and provide more accurate control actions.

An outstanding feature of gain-scheduled control is the possibility of considering performance constraints, written as well in terms of LMI-based conditions (Rugh; Shamma, 2000). The motivation for imposing such constraints lies in guaranteeing a desired behavior to the closed-loop system with the controller to be designed. In consonance with Mohammadpour and Scherer (2012), Briat (2015), two commonly used performance criteria are the disturbance rejection (minimization of the $\mathcal{L}_2$-gain) and the minimization of the system energy (also known as $\mathcal{H}_2$ guaranteed cost).

In the literature, several sampled-data control strategies formulated for LPV systems can be found. The available approaches can be grouped, mainly, in three major areas: emulation, approximate discretization, and direct sampled-data. In the framework of emulation control, a continuous-time controller is first designed and then discretized for obtaining a sampled-data controller (Tóth *et al.*, 2010). The application of such method to the control of LPV systems suffers from two main drawbacks: the existence of the discretization error when discretizing the continuous-time controller, and the assumption that the scheduling parameters do not vary in the intersample. In the approximate discretization approach, the sampled-data controller to be synthesized requires the LPV model to be discretized (Lam; Zhou, 2008; de Caigny *et al.*, 2010). In this case, since the time dependence of the scheduling parameters is neglected during the sampling intervals, the approximate discretization is not usually suitable for the control of LPV systems. In the direct sampled-data approach, LPV systems are controlled with an induced time-varying input delay, which corresponds to the sampled states or outputs of the system (Ramezanifar *et al.*, 2012; Gomes da Silva Jr *et al.*, 2018; Hooshmandi *et al.*, 2018). This method can cope with LPV systems modeled as continuous-time systems, and it also does not require the designed controller to be discretized. Additionally, the scheduling parameters can vary with time, under the assumptions that the scheduling parameters are bounded and have limited variation rates and that such bounds are known.

1.1 Objectives

The main objective of this work is the proposition of new sufficient LMI conditions to stabilization and controller synthesis for sampled-data LPV systems, whose scheduling parameters are bounded and have known variation rate. With an input delay strategy, gain-scheduling state-feedback controllers are designed and obtained from a semidefinite programming (SDP) problem subject to LMI constraints. The attained conditions are derived after a Lyapunov function adapted from Hooshmandi *et al.* (2018). For comparison purposes, the developed conditions are applied also to benchmark systems, so that the proposed approaches can be validated.

The motivation behind this dissertation lies in investigating the role of the sampling time on the performance of closed-loop sampled-data systems. Differently from other similar works available in the control literature, the considered sampled-data systems are represented in terms of LPV models.

The contributions proposed in this dissertation regard the obtention of less conservative LMI conditions, with respect to the constraints imposed in similar works of the control literature. For example, Geromel and Souza (2015) employ a standard constant Lyapunov function to the synthesis of state-feedback control laws to sampled-data systems, which are recast as hybrid systems. Gomes da Silva Jr *et al.* (2018) implement gain-scheduled state-feedback controllers to sampled-data LPV systems, using a Lyapunov function with more terms than the one adopted in Geromel and Souza (2015). To attain the proposed conditions, Gomes da Silva Jr *et al.* (2018) apply Jensen's inequality to provide an upper-bound to an integral quadratic term arisen in the derivation process. Hooshmandi *et al.* (2018) extend the results obtained in Gomes da Silva Jr *et al.* (2018) as a more general Lyapunov function is used. Similarly to Gomes da Silva Jr *et al.* (2018), Jensen's inequality is also exploited by Hooshmandi *et al.* (2018) to introduce an upper-bound to an intermediate integral quadratic term.

The conditions attained in this dissertation are more relaxed due to the usage of Wirtinger's inequality (Seuret; Gouaisbaut, 2013) and due to the inclusion of a new term in the Lyapunov function candidate, which considers the integral of the internal states $x(t)$ of the system. Therefore, stabilizing control actions with larger sample periods and lower upper-bounds to the $\mathcal{L}_2$-gain performance are synthesized.

1.2 Text organization

The remaining of this dissertation is structured as follows:

- In Chapter 2, the mathematical background regarding gain-scheduled control synthesis applied to sampled-data LPV systems is presented. Concepts such as stability analysis in the sense of Lyapunov are introduced, and then extended for closed-loop sampled-data systems with guaranteed $\mathcal{L}_2$-gain cost.

- In Chapter 3, the design of gain-scheduling controllers for LPV systems in continuous time is addressed with an iterative approach, leading to the derivation of new sufficient LMI conditions.
- Chapter 4 extends the conditions proposed in Chapter 3 by incorporating slack variables and by avoiding the use of an iterative procedure. A possible extension of the attained conditions to quasi-LPV systems is also discussed.
- In Chapter 5, numerical examples are used to evaluate the effectiveness of the methodologies developed in Chapters 3 and 4, when compared to other methods from the literature.
- Chapter 6 concludes this dissertation and provides future research directions.

2 Mathematical background

This chapter presents theoretic aspects relevant for this dissertation. The stability concept in the sense of Lyapunov is introduced as a technique for assessing the asymptotic stability of dynamic continuous-time LPV systems. The adopted performance criterion for the controller synthesis is also established.

Mathematical results available in the scientific literature, e.g. Schur complement, Finsler's lemma and Wirtinger's inequality, are given as well for completeness. By means of the Lyapunov theory and with the aid of such mathematical tools, stabilizing conditions can be formulated as SDP problems, written in terms of LMIs.

2.1 Polytopic systems

In the framework of LPV models, a usual realization of the system dynamics is made in terms of polytopic systems. A polytopic model is a representation of a system with multiple operating conditions (Rugh; Shamma, 2000). It is assumed that the full range of system modes are available among a finite set of leading operation points, also called vertex systems (Rotondo *et al.*, 2013). In fact, a polytopic system is described by the convex combination of its vertices, under the assumption that the weighting factor η of each vertex is available in an N-dimensional space Λ_N. Multiple definitions can be found for Λ_N in the scientific literature (Boyd *et al.*, 1994; Cloosterman *et al.*, 2010; de Caigny *et al.*, 2011; Gomes da Silva Jr *et al.*, 2018), although the most common approach is to define Λ_N as the unit simplex described as

$$\Lambda_N = \left\{ \eta \in \mathbb{R}^N \mid \sum_{i=1}^{N} \eta_i = 1,\ \eta_i \geq 0,\ i = 1, \dots, N \right\}. \tag{2.1}$$

In the scientific literature, the polytopic description of systems is widely employed in several frameworks. As part of the Takagi-Sugeno fuzzy modeling, for instance, the system dynamics can be evaluated alongside the defined membership functions. On the other hand, in the LPV systems, the system behavior can be assessed with respect to the scheduling parameters. Both membership functions and scheduling parameters replace the role of the weighting factor η for polytopic system.

In terms of LPV systems, the weighting factor $\eta(t)$ is called *scheduling parameter*, depends on time and varies along the operation of the system. Thus, the matrices of any LPV system can be generically denoted as $X(\eta(t))$, with $\eta(t) \in \Lambda_N$.

There is a conceptual distinction between LPV systems and quasi-LPV systems. According to Rotondo *et al.* (2013), in the framework of LPV systems, the scheduling parameters rely only on exogenous signals, such as the time. It implies that the values of the scheduling parameters are known *a priori*, possibly alongside the values of the time derivative of $\eta(t)$. On the contrary, in quasi-LPV systems, the scheduling parameters can depend on both exogenous and endogenous signals. Endogenous signals are system variables, such as the internal states $x(t)$. Since the dynamics of system variables are captured only in real time, the values of scheduling parameters (or of their time derivative) is generally unknown to quasi-LPV systems.

A recurring representation of parameter-dependent matrices in the literature is the affine one. In this representation the matrices $X(\zeta(t))$ are such that

$$\mathbf{X}(\zeta(t)) = \mathbf{X}_0 + \textstyle\sum_{i=1}^{N} \zeta_i(t)\mathbf{X}_i, \tag{2.2}$$

with $\mathbf{X}_i$ constant matrices, for $i = 0, \dots, N$. Any polytopic matrix discussed in this dissertation is assumed to be affine with respect to the scheduling parameters. It is possible to recast affine matrices (2.2) as polytopic matrices, as discussed in the following example.

Example 2.1. *Let an LPV model be described by the differential equation*

$$\dot{x}(t) = \boldsymbol{A}(\zeta(t))x(t), \tag{2.3}$$

with $x(t) \in \mathbb{R}^{n_x}$ the state vector, and $\boldsymbol{A}(\zeta(t)) \in \mathbb{R}^{n_x \times n_x}$ the affine matrix

$$\boldsymbol{A}(\zeta(t)) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \zeta(t) \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \tag{2.4}$$

in which $\zeta(t) = sin(t)$, such that $-1 \leq \zeta(t) \leq 1$. It is clear that the system dynamics rely on the value of $\zeta(t)$. The vertices of the LPV system (2.3) are:

$$\begin{aligned} \zeta(t) = 1 \Rightarrow \boldsymbol{A}_1 &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \\ \zeta(t) = -1 \Rightarrow \boldsymbol{A}_2 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \end{aligned} \tag{2.5}$$

with $\boldsymbol{A}(\eta(t)) = \eta(t)\boldsymbol{A}_1 + (1 - \eta(t))\boldsymbol{A}_2$ and $\eta(t) = (1 + \zeta(t))/2$, for instance. The eigenvalues λ of the vertices of matrices (2.5) are $\lambda = 1$ (when $\zeta(t) = 1$) and $\lambda = -1$ (when $\zeta(t) = -1$). Therefore, according to the traditional control theory, two different system behaviors can be identified to the LPV system (2.3): as $\zeta(t) \to 1$, the LPV system behaves as an unstable system. On the other hand, as $\zeta(t) \to -1$, the LPV system operates in a stable regime (Khalil, 2002).

For values of $\zeta(t)$ such that $|\zeta(t)| \neq 1$, the LPV system (2.3) can be described by both unstable and stable modes. One behavior should overweight the other, depending on the value of $\zeta(t)$.

An alternative definition of Λ_N is employed in this dissertation to guarantee convexity of the parameters $\eta(t)$. Similarly to Hooshmandi *et al.* (2018), the adopted representation of Λ_N is described as

$$\Lambda_N = \Big\{ \eta(t) \in \mathbb{R}^N \mid \eta(t) \in \Delta_\eta,\ \dot{\eta}(t) \in \Delta_{\dot{\eta}} \Big\}, \tag{2.6}$$

being Δ_η and $\Delta_{\dot{\eta}}$ compact admissible sets of the parameter $\eta(t)$ and its derivative $\dot{\eta}(t)$, which are defined by

$$\begin{aligned} \Delta_\eta &= \Big\{ \eta(t) \in \mathbb{R}^N : \underline{\eta}_i \leq \eta_i(t) \leq \overline{\eta}_i,\ i = 1, \ldots, N \Big\}, \\ \Delta_{\dot{\eta}} &= \Big\{ \dot{\eta}(t) \in \mathbb{R}^N : |\dot{\eta}_i(t)| \leq v_i,\ i = 1, \ldots, N \Big\}, \end{aligned} \tag{2.7}$$

with $\underline{\eta}_i$ and $\overline{\eta}_i$, respectively, the lower and upper bounds of $\eta_i(t)$, and v_i the maximum absolute value for the variation rate of $\eta_i(t)$.

The sets Δ_η and $\Delta_{\dot{\eta}}$ are *a priori* known. In other words, the domain of discourse associated to the LPV model (2.3) should be given. Such domain of discourse can be thus understood as a convex polyhedron whose vertices are either the bounds of the parameters $\eta_i(t)$ or the bounds of the time derivatives $\dot{\eta}_i(t)$ (Rotondo *et al.*, 2013).

Recovering the conceptual difference between LPV and quasi-LPV systems, it is straightforward to check that the bounds $\underline{\eta}_i$, $\overline{\eta}_i$ and v_i are given when it comes to LPV models. However, these values might be undetermined for quasi-LPV systems. If that is the case, in order for the scheduling parameters to be described with the compact sets Δ_η and $\Delta_{\dot{\eta}}$, theoretic bounds are imposed to the scheduling parameters.

2.2 Stability analysis

Several concepts of stability are available in the scientific literature. Just to cite a few of them, one can consider the input-to-output stability (IOS), input-to-state stability (ISS), bounded-input bounded-output (BIBO) stability and stability in the sense of Lyapunov (Sontag, 1989; Sontag; Wang, 1999; Khalil, 2002). Some definitions of stability are applicable to LTI systems only, whereas others can be extended to nonlinear systems.

In this dissertation, the stability analysis is evaluated in the sense of Lyapunov due to its extensive usage in control theory (Lyapunov, 1992). A system is said to be asymptotically stable in the sense of Lyapunov if there is an energy-like positive Lyapunov function $W(x,t)$, such that its derivative with respect to time $\dot{W}(x,t) = \frac{dW(x,t)}{dt}$ is negative for all values of t. Let the dynamics of a system be described by the differential equation

$$\dot{x}(t) = \mathbf{f}(x(t), u(t), w(t)), \tag{2.8}$$

with $x(t) \in \mathbb{R}^{n_x}$ the state, $u(t) \in \mathbb{R}^{n_u}$ the control input, and $w(t) \in \mathbb{R}^{n_w}$ the disturbance vectors. $\mathbf{f} : \mathbb{R}^{n_x+n_u+n_w} \mapsto \mathbb{R}^{n_x}$ is a function which maps $x(t)$, $u(t)$, and $w(t)$ to $\dot{x}(t)$.

On the one hand, system (2.8) is asymptotically stable if, provided $w(t) = 0$,

$$\lim_{t\to\infty} x(t) = 0, \ \forall\, x(0) \in \mathbb{R}^{n_x}. \tag{2.9}$$

On the other hand, system (2.8) is asymptotically stable in the sense of Lyapunov if the sufficient conditions presented in Theorem 2.1 are satisfied.

Theorem 2.1. *(Kolmanovskii; Myshkis, 1999) Given class-$\mathcal{K}$ functions a_1, a_2 and b, if there exists a continuous and differentiable Lyapunov function $W(x,t)$ such that, for all $x(t) \neq 0$,*

1. $0 < a_1(\|x\|) \leq W(x,t) \leq a_2(\|x\|)$, (2.10)

2. $\dot{W}(x,t) \leq -b(\|x\|) < 0$, (2.11)

then the origin is an asymptotically stable equilibrium point of the system (2.8), in the sense of Lyapunov.

The proof of Theorem 2.1 is omitted and can be found in Kolmanovskii and Myshkis (1999). An implementation of Theorem 2.1 is given as follows: the LPV system (2.3) is asymptotically stable in the sense of Lyapunov if there exists a Lyapunov function $W(x) = x^T(t)Px(t)$ such that the following LMIs are satisfied:

1. $W(x) > 0 \Longleftrightarrow P = P^T \succ 0$,

2. $\dot{W}(x) < 0 \Longleftrightarrow \mathbf{A}^T(\eta(t))P + P\mathbf{A}(\eta(t)) \preceq -\epsilon\mathbf{I} \prec 0$,

with ϵ a given positive scalar. Candidate class-$\mathcal{K}$ functions a_1, a_2 and b ensuring asymptotic stability of (2.3) are written as

$$\begin{aligned} a_1(\|x\|) &= \mu_1 x^T(t)x(t), \\ a_2(\|x\|) &= \mu_2 x^T(t)x(t), \\ b(\|x\|) &= \epsilon x^T(t)x(t), \end{aligned} \tag{2.12}$$

with scalars $\mu_1 > 0$ and $\mu_2 > 0$ assumed to be the smallest and the largest eigenvalues of matrix P, respectively.

2.3 $\mathcal{L}_2$-gain

In addition to the criteria established in Theorem 2.1 for ensuring asymptotic stability to LPV systems, performance criteria can be also imposed. The performance criteria guarantee, for instance, a desired behavior to LPV systems with a controller to be designed.

Two commonly used performance criteria for closed-loop systems in control literature are the disturbance rejection (minimization of the $\mathcal{L}_2$-gain) and the minimization of the system energy (also known as $\mathcal{H}_2$ guaranteed cost) (Boyd *et al.*, 1994; Mohammadpour; Scherer, 2012; Briat, 2015). These performance criteria consider a cost function J to be minimized.

In this dissertation, the adopted performance criterion is the $\mathcal{L}_2$-gain, whose cost function J can be defined as

$$J = \lim_{t\to\infty} \int_0^t \Big(y^T(s)y(s) - \gamma^2 w^T(s)w(s)\Big)ds = \|y(t)\|^2 - \gamma^2\|w(t)\|^2, \tag{2.13}$$

with $y(t) \in \mathbb{R}^{n_y}$ the output, $w(t) \in \mathbb{R}^{n_w}$ the disturbance vectors, and γ the value for the $\mathcal{L}_2$-gain. Consider a class of LPV systems described by the following set of differential equations:

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}(\eta(t))x(t) + \mathbf{B}_1(\eta(t))w(t), \\ y(t) &= \mathbf{C}(\eta(t))x(t), \end{aligned} \tag{2.14}$$

with $x(t) \in \mathbb{R}^{n_x}$ the state vector, and $\mathbf{A}(\eta(t))$, $\mathbf{B}_1(\eta(t))$, $\mathbf{C}(\eta(t))$ polytopic matrices with compatible dimensions, as defined in (2.2).

In the framework of LPV systems (2.14), two different definitions for the $\mathcal{L}_2$-gain are available: the finite $\mathcal{L}_2$-gain and the induced $\mathcal{L}_2$-gain norm. These are presented in Definitions 2.1 and 2.2, respectively (Briat, 2015).

Definition 2.1. *LPV systems (2.14) are said to be stable with a finite $\mathcal{L}_2$-gain if, for $x(0) = 0$, there exists a constant $\gamma \geq 0$ such that*

$$\|y(t)\|^2 \leq \gamma^2\|w(t)\|^2, \tag{2.15}$$

with $w(t) \in \mathcal{L}_2$.

Definition 2.2. *For LPV systems (2.14), γ^* corresponds to the induced $\mathcal{L}_2$-gain norm and is given by*

$$\gamma^* = \sup_{w(t)\neq 0} \frac{\|y(t)\|}{\|w(t)\|}, \tag{2.16}$$

with $w(t) \in \mathcal{L}_2$.

The calculation of γ, which defines the finite $\mathcal{L}_2$-gain cost, is not easily achievable. As a result, the determination of γ is usually addressed in an alternative way: by means of the upper bound of the $\mathcal{L}_2$-gain cost. An upper bound of γ can be determined as part of SDP problems considering the analysis of a Lyapunov function, as shown in Theorem 2.2.

Theorem 2.2. *(Briat, 2015) If there exists a continuous and differentiable Lyapunov function $W(x,t)$ such that, for $x(0) = 0$ and for $w(t) \in \mathcal{L}_2$,*

1. $0 < a_1(\|x\|) \leq W(x,t) \leq a_2(\|x\|)$, $x(t) \neq 0$, (2.17)

2. $\dot{W}(x,t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) \leq -b(\|x\|) < 0,\ x(t) \neq 0,$ (2.18)

then the origin is, in the sense of Lyapunov, an asymptotically stable equilibrium point of the system (2.14), whose $\mathcal{L}_2$-gain is finite and bounded by γ.

Proof of Theorem 2.2. Condition (2.17) is satisfied by imposing the Lyapunov function to be positive definite. By integrating condition (2.18) from $t = 0$ to $t \to \infty$, one has that

$$-W(x,0) + \lim_{t\to\infty} W(x,t) + \|y(t)\|^2 - \gamma^2\|w(t)\|^2 \leq 0. \tag{2.19}$$

Provided that $x(0) = 0$ and that the LPV system (2.14) is asymptotically stable ($x(t) \to 0$ as $t \to \infty$), $W(x,0) = \lim_{t\to\infty} W(x,t) = 0$. Thus, (2.19) reduces to

$$\|y(t)\|^2 - \gamma^2\|w(t)\|^2 \leq 0 \Longleftrightarrow \|y(t)\|^2 \leq \gamma^2\|w(t)\|^2. \tag{2.20}$$

According to Definition 2.1, (2.20) guarantees that γ is an upper bound of the finite $\mathcal{L}_2$-gain. This concludes the proof. □

An application of Theorem 2.2 is given as follows: the LPV system (2.14) is asymptotically stable in the sense of Lyapunov with finite $\mathcal{L}_2$-gain bounded by γ if there exists a Lyapunov function $W(x) = x^T(t)Px(t)$ such that the following LMIs are satisfied:

1. $W(x) > 0 \Longleftrightarrow P = P^T \succ 0,$

2. $\dot{W}(x) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) < 0$

   $\Big(\dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)\Big) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) < 0$

   $x^T(t)\Big(\mathbf{A}^T(\eta(t))P + P\mathbf{A}(\eta(t)) + \mathbf{C}^T(\eta(t))\mathbf{C}(\eta(t))\Big)x(t) + x^T(t)\Big(P\mathbf{B}_1(\eta(t))\Big)w(t) + w^T(t)\Big(\mathbf{B}_1^T(\eta(t))P\Big)x(t) + w^T(t)(-\gamma^2\mathbf{I}_{n_w})w(t) < 0$

   $$\begin{bmatrix} \mathbf{A}^T(\eta(t))P + P\mathbf{A}(\eta(t)) + \mathbf{C}^T(\eta(t))\mathbf{C}(\eta(t)) & * \\ \mathbf{B}_1^T(\eta(t))P & -\gamma^2\mathbf{I}_{n_w} \end{bmatrix} \prec 0$$

   Taking the Schur complement (described in Section 2.4) of the term $\mathbf{C}^T(\eta(t))\mathbf{C}(\eta(t))$, one has that

   $$\begin{bmatrix} \mathbf{A}^T(\eta(t))P + P\mathbf{A}(\eta(t)) & * & * \\ \mathbf{B}_1^T(\eta(t))P & -\gamma^2\mathbf{I}_{n_w} & * \\ \mathbf{C}(\eta(t)) & \mathbf{0} & -\mathbf{I}_{n_y} \end{bmatrix} \prec 0.$$

Since the $\mathcal{L}_2$-gain performance criterion intends to minimize the finite $\mathcal{L}_2$-gain bounded by γ, the adopted objective function of the arising SDP problem is to minimize γ.

2.4 Schur complement

In the derivation process of conditions ensuring the feasibility of Theorem 2.1, obtaining nonlinear inequalities is common (Boyd *et al.*, 1994). At first, these terms cannot be considered in the framework of LMI conditions. However, when these terms are available in quadratic form, this problem can be solved by means of the Schur complement.

The Schur complement is a key tool in matrix analysis, whose main interest lies in its property of linearizing a product of variables, as shown in Lemma 2.1.

Lemma 2.1. *(Boyd* et al.*, 1994) If M is a symmetric matrix defined as*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ * & C \end{bmatrix}, \tag{2.21}$$

then

- $M \succ 0$ *if and only if* $A \succ 0$ *and* $M|A \succ 0$,
- $M \succ 0$ *if and only if* $C \succ 0$ *and* $M|C \succ 0$,
- $M \prec 0$ *if and only if* $A \prec 0$ *and* $M|A \prec 0$,
- $M \prec 0$ *if and only if* $C \prec 0$ *and* $M|C \prec 0$,

with $M|A = C - B^T A^{-1} B$ *and* $M|C = A - BC^{-1}B^T$.

Proof of Lemma 2.1. The symmetric matrix M can be represented in terms of the product of three matrices, as shown in

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} \mathbf{I} & A^{-1}B \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & M|A \end{bmatrix} \begin{bmatrix} \mathbf{I} & A^{-1}B \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \mathcal{T}^T \mathcal{M} \mathcal{T}, \tag{2.22}$$

with $\mathcal{M} = diag(A,\ M|A)$ and $\mathcal{T} = \begin{bmatrix} \mathbf{I} & A^{-1}B \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$. Notice that the equivalence expressed in (2.22) requires matrix A to be nonsingular.

Necessity: assuming positiveness of matrix M, the diagonal terms of M must be positive-definite matrices. Hence, there exists an inverse of A. Since $M \succ 0$ and $\mathcal{T}$ is of full rank, it is clear that $\mathcal{M} \succ 0$. Provided that $\mathcal{M}$ is a block-diagonal matrix, $\mathcal{M} \succ 0$ if and only if its diagonal blocks, A and $M|A$, are positive definite.

Sufficiency: assuming positiveness of the block-diagonal matrix $\mathcal{M}$, its diagonal blocks A and $M|A$ must be positive definite. It is straightforward to check that there exists A^{-1}. Being $\mathcal{M} \succ 0$ and having $\mathcal{T}$ full rank, then M is a positive-definite matrix.

The proof for any other assertion presented in Lemma 2.1 is omitted and follows a similar procedure as the one followed in this proof. This concludes the proof. □

2.5 Finsler's lemma

The Finsler' lemma is a well-known tool in the framework of SDP problems. As presented in Lemma 2.2, it allows interchanging the representation of LMI conditions, possibly with the introduction of new terms, called slack variables.

Lemma 2.2. *(Oliveira; Skelton, 2001) Let* $\omega \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{m\times n}$ *(rank*$(\mathcal{B}) < n$*), and* $\mathcal{B}^{\perp}$ *be defined such that* $\mathcal{B}\mathcal{B}^{\perp} = 0$. *Then, the following four statements are equivalent:*

1. $\omega^T \mathcal{Q}\omega < 0, \ \forall\ \omega \neq 0 : \mathcal{B}\omega = 0$
2. $\mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp} \prec 0$
3. $\exists \mu \in \mathbb{R} : \mathcal{Q} - \mu\mathcal{B}^T\mathcal{B} \prec 0$
4. $\exists \mathcal{X} \in \mathbb{R}^{n\times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T\mathcal{X}^T \prec 0$

By means of Lemma 2.2, more flexible stability conditions can be obtained due to the inclusion of slack variables.

2.6 Jensen's inequality

In the derivation process of sufficient LMIs ensuring stability in the sense of Lyapunov for LPV systems, obtaining integral inequalities is not unusual (Shao, 2009; Sun *et al.*, 2010; Hooshmandi *et al.*, 2018). Integral quadratic terms, such as

$$\ell(\dot{\omega}) = \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du, \tag{2.23}$$

where R is a symmetric positive-definite matrix held constant in the interval $[a, b]$ and $\omega(u)$ is a continuously differentiable function, are not suitable for deriving LMIs.

This problem can be overcome by means of the Jensen's inequality, which provides a lower bound of (2.23). The Jensen's inequality is presented in the next Lemma.

Lemma 2.3. *(Gu* et al.*, 2003). Given a constant symmetric positive-definite matrix* R*, the following inequality is verified for every function* $\omega(u)$ *continuously differentiable on the interval* $[a,b] \to \mathbb{R}^n$*:*

$$\ell(\dot{\omega}) = \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du \geq \frac{1}{b-a}\Omega_1^T R\Omega_1, \tag{2.24}$$

where $\Omega_1 = \int_a^b \dot{\omega}(u)du = \omega(b) - \omega(a)$.

2.7 Wirtinger's inequality

Although Jensen's inequality solves the problem related to integral inequalities, it also introduces an inherent conservativeness into the derived LMIs (Briat, 2011). Such restrictive conditions arise from the gap between these integral inequalities and the lower bound of (2.23).

Less conservative results, leading to more accurate lower bounds for (2.23), have been attained with the aid of Wirtinger's inequality (Seuret; Gouaisbaut, 2013) and of the auxiliary-function based integral inequality (Park *et al.*, 2015). All the mentioned inequalities are particular cases of Bessel-Legendre inequality (Seuret; Gouaisbaut, 2018), which is shown to introduce the closest lower bound of (2.23). However, as an alternative to Jensen's inequality, this dissertation considers the application of Wirtinger's inequality, as formulated in Lemma 2.4.

Lemma 2.4. *(Seuret; Gouaisbaut, 2013) Given a constant symmetric positive-definite matrix R, the following inequality is verified for every function $\omega(u)$ continuously differentiable in the interval $[a,b] \to \mathbb{R}^n$:*

$$\ell(\dot{\omega}) = \int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2, \qquad (2.25)$$

where $\Omega_1 = \omega(b) - \omega(a)$ and $\Omega_2 = \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(u)du$.

Comparing the right-hand side of (2.24) and of (2.25), it is straightforward to verify that the Jensen's inequality is a special case of the Wirtinger's inequality, when Ω_2 is assumed to be zero. Thanks to the additional term $\frac{3}{b-a}\Omega_2^T R \Omega_2$, the Wirtinger's inequality manages to provide a tighter lower bound to (2.23) with respect to the Jensen's inequality.

In similar works of the control literature, Jensen's inequality is used as a lower bound to (2.23) (Hooshmandi *et al.*, 2018; Gomes da Silva Jr *et al.*, 2018). In the upcoming chapters of this dissertation, the Wirtinger's inequality will be used instead when deriving sufficient LMI conditions ensuring (2.18). As a result, one of the attained contributions of this dissertation is the proposition of less conservative constraints to the optimization problem, if compared to the ones available in other works from the literature.

3 Iterative approach

This chapter introduces an iterative approach for the synthesis of sampled-data state-feedback controllers for LPV systems in continuous time. This problem is addressed by means of a Lyapunov function, after which sufficient conditions are derived in terms of LMIs, ensuring system stabilization and minimization of a performance criterion. Finally, the computational aspects regarding the implementation of the iterative approach are explored. The approach presented in this chapter is an extension of the work available in Hooshmandi *et al.* (2018). Since the iterative approach is compared with the slack approaches to be defined in Chapter 4, the numerical simulation results are postponed to Chapter 5.

3.1 Problem statement

Consider LPV systems described by the following set of differential equations:

$$\begin{aligned} \dot{x}(t) =& \mathbf{A}(\eta(t))x(t) + \mathbf{B}_1(\eta(t))w(t) + \mathbf{B}_2(\eta(t))u(t) \\ y(t) =& \mathbf{C}(\eta(t))x(t) + \mathbf{D}_1(\eta(t))w(t) + \mathbf{D}_2(\eta(t))u(t) \end{aligned} \tag{3.1}$$

in which $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$ denote the state and the control input vectors, with $n_u \leq n_x$. $w(t) \in \mathbb{R}^{n_w}$ represents the disturbance vector, assumed to be in $\mathcal{L}_2$. $y(t) \in \mathbb{R}^{n_y}$ is the output vector. $\mathbf{A}(\eta(t))$, $\mathbf{B}_1(\eta(t))$, $\mathbf{B}_2(\eta(t))$, $\mathbf{C}(\eta(t))$, $\mathbf{D}_1(\eta(t))$, and $\mathbf{D}_2(\eta(t))$, with compatible dimensions, are given matrices in the form of (2.2), which depend on the scheduling parameter vector $\eta(t) \in \mathbb{R}^N$.

The adopted LPV models consider both the bounds of the parameters $\eta_i(t)$ and the bounds of the time derivatives $\dot{\eta}_i(t)$ to be known. It is then straightforward to check that the domain of discourse associated to the scheduling parameters $\eta(t)$ fits for simplexes formulated in (2.6).

Having introduced the LPV system (3.1) to be controlled, it is important to examine the design details of the adopted control strategy. As discussed in Chapter 1, this dissertation considers the application of gain-scheduled sampled-data state-feedback control law in the form

$$u(t) = u(t_n) = K(\eta(t_n))x(t_n), \ t \in [t_n, \ t_{n+1}) \tag{3.2}$$

to stabilize system (3.1) in a closed-loop, meaning that the control input (3.2) is held constant between two successive sampling instants by means of a zero-order holder (ZOH). In this scenario, the control signal can be recast as an induced delayed signal, whose delay $\tau(t)$ is given by

$$\tau(t) = t - t_n \leq T_m, \ t \in [t_n, \ t_{n+1}), \tag{3.3}$$

in which T_m is the chosen maximum allowable sampling period (MASP). The induced delay $\tau(t)$ is the time elapsed since the last sampling instant t_n, and it cannot exceed T_m.

Supposing the structure of the gain-scheduled control law is given by (3.2), the closed-loop LPV dynamics (3.1) are then written as

$$\begin{aligned}\dot{x}(t) =& \mathbf{A}(\eta(t))x(t) + \mathbf{B}_1(\eta(t))w(t) + \mathbf{B}_2(\eta(t))K(\eta(t_n))x(t_n),\\ y(t) =& \mathbf{C}(\eta(t))x(t) + \mathbf{D}_1(\eta(t))w(t) + \mathbf{D}_2(\eta(t))K(\eta(t_n))x(t_n).\end{aligned} \tag{3.4}$$

It is remarkable that the system matrices and the gain matrix in (3.4) do not share a uniform representation. Such mismatch can be circumvented by adopting an expanded parameter vector $\rho(t) = \begin{bmatrix}\eta^T(t_n) & \delta^T(t)\end{bmatrix}^T$, where $\delta(t) = \eta(t) - \eta(t_n)$ is the uncertainty between the real continuous parameters $\eta(t)$ and the sampled parameters $\eta(t_n)$.

The expanded parameter vector $\rho(t)$ is defined in the space Θ, so that

$$\Theta = \left\{\rho \in \mathbb{R}^{2N} : \rho(t) \in \Delta_\rho,\ \dot{\rho}(t) \in \Delta_{\dot{\rho}}\right\}, \tag{3.5}$$

in which Δ_ρ and $\Delta_{\dot{\rho}}$ are compact sets as defined in (2.7). Taking into account the expanded parameter vector $\rho(t)$, the closed-loop dynamics (3.4) can be recast as follows:

$$\begin{aligned}\dot{x}(t) =& \mathbf{A}(\rho(t))x(t) + \mathbf{B}_1(\rho(t))w(t) + \mathbf{B}_2(\rho(t))K(\eta(t_n))x(t_n),\\ y(t) =& \mathbf{C}(\rho(t))x(t) + \mathbf{D}_1(\rho(t))w(t) + \mathbf{D}_2(\rho(t))K(\eta(t_n))x(t_n).\end{aligned} \tag{3.6}$$

In this dissertation, stabilization and controller synthesis for LPV models (3.6) are carried out within the framework of SDP problems. As discussed in Sections 2.2 and 2.3, the derived conditions ensuring stabilization and minimization of the $\mathcal{L}_2$-gain can be written in terms of LMIs.

Remark 3.1. *(Gomes da Silva Jr* et al.*, 2015) In order to cope with the representation of the scheduling parameter $\eta(t)$ with two components, $\eta(t_n)$ and $\delta(t)$, the compact sets Δ_ρ and $\Delta_{\dot{\rho}}$ are defined as follows:*

$$\begin{aligned}\Delta_\rho &= \left\{\rho(t) \in \mathbb{R}^{2N} : \underline{\eta}_i \leq \rho_i(t) \leq \overline{\eta}_i,\ |\rho_{i+N}(t)| \leq T_m v_i,\ i = 1, \ldots, N\right\},\\ \Delta_{\dot{\rho}} &= \left\{\dot{\rho}(t) \in \mathbb{R}^{2N} : \dot{\rho}_i(t) = 0,\ |\dot{\rho}_{i+N}(t)| \leq v_i,\ i = 1, \ldots, N\right\},\end{aligned} \tag{3.7}$$

where $\rho_i(t)$ and $\rho_{i+N}(t)$ stand, respectively, for $\eta_i(t_n)$ and $\delta_i(t)$, for $i = 1, \ldots, N$, and T_m is the chosen MASP. The determination of such bounds to $\eta_i(t_n)$ and $\delta_i(t)$ is borrowed from Gomes da Silva Jr et al. *(2015) and is presented below. For the sake of simplicity, this demonstration is done for a single scheduling parameter $\eta_i(t)$, over the interval $t \in [t_n,\ t_{n+1})$.*

The scheduling parameter $\eta_i(t)$ can be decomposed as follows:

$$\eta_i(t) = \eta_i(t_n) + \delta_i(t) \Rightarrow \dot{\eta}_i(t) = \dot{\delta}_i(t), \tag{3.8}$$

where $\eta_i(t_n)$ and $\delta_i(t)$ stand for the sampled and the strictly continuous components of the scheduling parameter $\eta_i(t)$, respectively.

By assumption, the bounds of $\eta_i(t_n)$ consist of the bounds imposed to $\eta_i(t)$, such that

$$\underline{\eta}_i \leq \eta_i(t_n) \leq \overline{\eta}_i. \tag{3.9}$$

From (3.8), it is straightforward to certify that $\dot{\eta}_i(t_n) = 0$ and that $|\dot{\eta}_i(t)| \equiv |\dot{\delta}_i(t)| \leq v_i$. Considering $|\dot{\eta}_i(t)| \leq v_i$ and using the mean-value theorem, one has that

$$|\eta_i(t) - \eta_i(t_n)| \leq v_i(t - t_n) \leq v_i T_m. \tag{3.10}$$

Since $\delta_i(t) = \eta_i(t) - \eta_i(t_n)$, (3.10) is equivalent to

$$|\delta_i(t)| \leq v_i T_m. \tag{3.11}$$

3.2 Definition of a Lyapunov function

In Sections 2.2 and 2.3, it is shown that both stabilization and controller synthesis consider the stability in the sense of Lyapunov.

The Lyapunov function $W(x,t)$ to be defined might consider not only parameter-dependent Lyapunov functions $P(\eta(t))$, but also a non-exhaustive number of additional terms (Kolmanovskii; Myshkis, 1999; Cloosterman *et al.*, 2010; Souza, 2013; Gomes da Silva Jr *et al.*, 2018).

As a result, the choice of an appropriate Lyapunov function $W(x,t)$, which fulfills the requirements presented in Theorem 2.1 (or in Theorem 2.2), is usually a challenging task. For the sake of simplicity, the Lyapunov function $W(x,t)$ adopted in this dissertation is adapted from the one proposed in Hooshmandi *et al.* (2018).

Consider the following time-dependent Lyapunov function:

$$W(x,t) = V(x) + V_0(x,t) = V(x) + \sum_{i=1}^{3} V_i(x,t), \tag{3.12}$$

in which[1]

$$V(x) = x^T(t)\overline{P}(\rho)x(t) \tag{3.13}$$

$$V_1(x,t) = (t_{n+1} - t)\int_{t_n}^{t} \begin{bmatrix} \dot{x}(q) \\ x(t_n) \end{bmatrix}^T \begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix} \begin{bmatrix} \dot{x}(q) \\ x(t_n) \end{bmatrix} dq \tag{3.14}$$

$$V_2(x,t) = (t_{n+1} - t)\begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix}^T \begin{bmatrix} \overline{X}_1(\rho) & \overline{X}_2^T(\rho) - \overline{X}_1(\rho) \\ \overline{X}_2(\rho) - \overline{X}_1(\rho) & \overline{X}_1(\rho) - \overline{X}_2^H(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix} \tag{3.15}$$

$$V_3(x,t) = (t_{n+1} - t)(t - t_n)\nu^T(t)\overline{F}(\eta(t_n))\nu(t) \tag{3.16}$$

[1] The symbol A^H is a short-hand form for $A + A^T$.

with $\nu(t) = \frac{1}{\tau(t)}\int_{t_n}^{t} x(q)dq$, and $\overline{P}(\rho)$, $\overline{F}(\eta(t_n))$, $\overline{E}_1(\eta(t_n))$, $\overline{E}_3(\eta(t_n))$, $\overline{X}_1(\rho) \in \mathbb{R}^{n_x \times n_x}$ symmetric matrices, and $\overline{E}_2(\eta(t_n))$, $\overline{X}_2(\rho) \in \mathbb{R}^{n_x \times n_x}$.

Depending on the approach to be developed for the stabilization and for the controller synthesis, different assumptions might be made for matrices $\overline{P}(\rho)$, $\overline{X}_1(\rho)$, and $\overline{X}_2(\rho)$. These premises are discussed in the next sections of this dissertation.

One of the main contributions from this dissertation is the inclusion of the term $V_3(x,t)$ in the Lyapunov function (3.12). The decision making behind such component consists of three goals:

1. As it will be further shown in Section 3.2.2, a subproduct of the derived LMI conditions ensuring (2.18) is a component related to the integral of the states. This component comes from the application of the Wirtinger's inequality, in which Ω_2 (see Section 2.4) presents some dependence on $\int_a^b w(u)du$. As a result, the new term $V_3(x,t)$ is introduced to deal with such term.

2. In order for the Lyapunov function $W(x,t)$ to be recast in the framework of a looped functional (see upcoming Remark 3.3), the proposed term $V_3(x,t)$ should respect the boundary condition $V_3(x,t_n) = V_3(x,t_{n+1}) = 0$.

3. Assuming that the Lyapunov function $W(x,t)$ contains a looped functional, the analysis of $\dot{W}(x,t)$ can be made simpler if it shows an affine dependence of $\dot{W}(x,t)$ with respect to the variable t. If that is the case, sufficient conditions ensuring (2.18) can be attained by evaluating $\dot{W}(x,t)$ at the extreme values of the variable $t \in [t_n,\ t_{n+1})$, i.e., at $t = t_n$ and at $t = t_{n+1}$.

Several alternatives to $V_3(x,t)$ were exploited in place of (3.16), as displayed below. All of them failed in the derivation process of LMI conditions ensuring negativeness of $\dot{W}(x,t)$, considering at least one of the three desired properties.

- $V_3(x,t) = \left(\int_{t_n}^{t} x^T(s)ds\right)\overline{F}(\eta(t_n))\left(\int_{t_n}^{t} x(s)ds\right)$ does not satisfy the boundary condition $V_3(x,t_n) = V_3(x,t_{n+1}) = 0$. There is no relationship between $V_3(x,t_n) \geq 0$ and $V_3(x,t_{n+1}) \geq 0$.

- Both $V_3(x,t) = \nu^T(t)\overline{F}(\eta(t_n))\nu(t)$ and $V_3(x,t) = (t-t_n)\nu^T(t)\overline{F}(\eta(t_n))\nu(t)$ do not satisfy the boundary condition $V_3(x,t_n) = V_3(x,t_{n+1}) = 0$, since $V_3(x,t_n) = 0$ and $V_3(x,t_{n+1}) \geq 0$.

- The time derivative of any functional $V_3(x,t) = (t_{n+1}-t)\nu^T(t)\overline{F}(\eta(t_n))\nu(t)$, $V_3(x,t) = (t_{n+1}-t)(t-t_n)\left(\int_{t_n}^{t} x^T(s)ds\right)\overline{F}(\eta(t_n))\left(\int_{t_n}^{t} x(s)ds\right)$, or $V_3(x,t) = (t_{n+1}-t)(t-t_n)\nu^T(t)\overline{F}(\rho)\nu(t)$ is not affine with respect to the variable t. Their derivatives should provide terms that are either rational or quadratic with respect to t.

Remark 3.2. *Notice that the choice of matrices* $\overline{F}(\eta(t_n))$*,* $\overline{E}_1(\eta(t_n))$*,* $\overline{E}_2(\eta(t_n))$*, and* $\overline{E}_3(\eta(t_n))$ *dependent only on the sampled-data scheduling parameter vector* $\eta(t_n)$ *is of greatest importance. Admitting* $\overline{E}_1(\eta(t_n))$*,* $\overline{E}_2(\eta(t_n))$*, and* $\overline{E}_3(\eta(t_n))$ *to depend on* $\eta(t_n)$ *implies that these matrices are constant in the interval* $t \in [t_n,\ t_{n+1})$*. It is a necessary requirement for applying the Wirtinger's inequality in (3.30). Besides, provided that* $\overline{F}(\eta(t_n))$ *is also constant in the interval* $t \in [t_n,\ t_{n+1})$*, one has* $\dot{\overline{F}}(\eta(t_n)) = 0$ *and, as a result, the time derivative of (3.16) is still affine with respect to* t*, as it will be further shown.*

Remark 3.3. *(Seuret; Gouaisbaut, 2013) The adopted Lyapunov function (3.12) is a looped functional, provided that* $V_1(x,t) = V_2(x,t) = V_3(x,t) = 0$ *for all* $t = t_n$ *and for all* $t = t_{n+1}$*. At any jump instant, it is then guaranteed that* $W(x,t) \equiv V(x)$*. The interest behind looped functionals lies in the fact that expansive jumps are allowed within a sample interval, as long as the storage function* $W(x,t)$ *accordingly accommodates the jumps and reaches* $V(x)$ *in the next sampling time* t_{n+1}*. If a monotonically decreasing* $V(x)$ *is considered, then system stability is ensured.*

By adopting the Lyapunov function (3.12), deriving conditions guaranteeing the feasibility of Theorems 2.1 and 2.2 is possible. Before moving on to one of the main results attained by this dissertation, two preliminary steps are of interest: the development of sufficient conditions ensuring both $W(x,t) > 0$ and $\dot{W}(x,t) < 0$. These steps are taken in the Subsections 3.2.1 and 3.2.2.

3.2.1 Positiveness of adopted Lyapunov function

As discussed in Khalil (2002), the direct method of Lyapunov stability theory usually requires the Lyapunov function (3.12) to be positive, apart from the equilibrium point $x(t) = 0$. Typically, this requirement is translated into the imposition of positiveness for all terms of the Lyapunov function $W(x,t)$, so that

$$V(x) > 0,\ V_1(x,t) > 0,\ V_2(x,t) > 0,\ V_3(x,t) > 0 \Rightarrow W(x,t) > 0. \tag{3.17}$$

Although the conditions presented in (3.17) ensure positiveness of the Lyapunov function (3.12), they also introduce an inherent conservativeness. If that approach is followed, every matrix available in the Lyapunov function (3.12) would have to be definite positive.

More relaxed conditions could be attained if one imposes positiveness to a sum of terms from the Lyapunov function (3.12), as performed in

$$V(x) + V_2(x,t) > 0,\ V_1(x,t) > 0,\ V_3(x,t) > 0 \Rightarrow W(x,t) > 0. \tag{3.18}$$

Although (3.17) and (3.18) introduce sufficient conditions ensuring positiveness of $W(x,t)$, it is important to reconsider that the adopted Lyapunov function (3.12) is a looped functional. According to Seuret (2012), the positiveness of the Lyapunov function (3.12) is made simpler, as it can be evaluated with the following Lemma.

Lemma 3.1. *(Seuret, 2012) Let a continuously differentiable Lyapunov function $V(x) > 0$ and class-$\mathcal{K}$ functions a_1, a_2 be such that*

$$a_1(\|x\|) \leq V(x) \leq a_2(\|x\|). \tag{3.19}$$

As a result, for $t_{n+1} - t_n \leq T_m$, with T_m a given positive scalar, the following two statements are equivalent:

1. *The increment of the Lyapunov function is strictly negative for all $t_{n+1} > t_n$, such that*

$$\Delta_n V(x) = V(x(t_{n+1})) - V(x(t_n)) < 0. \tag{3.20}$$

2. *There exists a continuous and differentiable looped functional $V_0(x,t)$, such that both conditions below apply:*

   - $V_0(x, t_n) = V_0(x, t_{n+1})$
   - $\dot{W}(x,t) = \frac{d}{dt}\Big(V(x) + V_0(x,t)\Big) < 0$

   If one of the statements above apply, then the solutions of the closed-loop system (3.6) are asymptotically stable.

The proof of Lemma 3.1 is omitted and can be found in Seuret (2012). In Subsection 3.2.2, the negativeness of the adopted Lyapunov function (3.12) will be imposed. Provided that this Lyapunov function is also a looped functional, the second statement of Lemma 3.1 applies. Therefore, by means of Lemma 3.1, the condition

$$V(x) > 0 \Longleftrightarrow \overline{P}(\rho) = \overline{P}^T(\rho) \succ 0 \Rightarrow W(x,t) > 0 \tag{3.21}$$

is a sufficient condition ensuring positiveness of the adopted Lyapunov function (3.12).

3.2.2 Negativeness of time derivative of adopted Lyapunov function

Considering also the generic representation of (3.12), the time derivative of the adopted Lyapunov function (3.12) can be written as

$$\dot{W}(x,t) = \dot{V}(x) + \sum_{i=1}^{3} \dot{V}_i(x,t) < 0, \tag{3.22}$$

with

$$\dot{V}(x) = x^T(t)\dot{\overline{P}}(\rho)x(t) + \Big(\dot{x}^T(t)\overline{P}(\rho)x(t)\Big)^H \tag{3.23}$$

$$\begin{aligned}\dot{V}_1(x,t) = &-\int_{t_n}^{t}\begin{bmatrix}\dot{x}(q)\\x(t_n)\end{bmatrix}^T\begin{bmatrix}\overline{E}_1(\eta(t_n)) & *\\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n))\end{bmatrix}\begin{bmatrix}\dot{x}(q)\\x(t_n)\end{bmatrix}dq\\ &+(t_{n+1}-t)\begin{bmatrix}\dot{x}(t)\\x(t_n)\end{bmatrix}^T\begin{bmatrix}\overline{E}_1(\eta(t_n)) & *\\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n))\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\x(t_n)\end{bmatrix}\end{aligned} \tag{3.24}$$

$$\begin{aligned}\dot{V}_2(x,t) = &-\begin{bmatrix}x(t)\\x(t_n)\end{bmatrix}^T\begin{bmatrix}\overline{X}_1(\rho) & \overline{X}_2^T(\rho)-\overline{X}_1(\rho)\\ \overline{X}_2(\rho)-\overline{X}_1(\rho) & \overline{X}_1(\rho)-\overline{X}_2^H(\rho)\end{bmatrix}\begin{bmatrix}x(t)\\x(t_n)\end{bmatrix}\\ &+(t_{n+1}-t)\left(\begin{bmatrix}\dot{x}(t)\\0\end{bmatrix}^T\begin{bmatrix}\overline{X}_1(\rho) & \overline{X}_2^T(\rho)-\overline{X}_1(\rho)\\ \overline{X}_2(\rho)-\overline{X}_1(\rho) & \overline{X}_1(\rho)-\overline{X}_2^H(\rho)\end{bmatrix}\begin{bmatrix}x(t)\\x(t_n)\end{bmatrix}\right)^H\\ &+(t_{n+1}-t)\begin{bmatrix}x(t)\\x(t_n)\end{bmatrix}^T\begin{bmatrix}\dot{\overline{X}}_1(\rho) & \dot{\overline{X}}_2^T(\rho)-\dot{\overline{X}}_1(\rho)\\ \dot{\overline{X}}_2(\rho)-\dot{\overline{X}}_1(\rho) & \dot{\overline{X}}_1(\rho)-\dot{\overline{X}}_2^H(\rho)\end{bmatrix}\begin{bmatrix}x(t)\\x(t_n)\end{bmatrix}\end{aligned} \tag{3.25}$$

$$\begin{aligned}\dot{V}_3(x,t) = &-(t-t_n)\nu^T(t)\overline{F}(\eta(t_n))\nu(t)+(t_{n+1}-t)\nu^T(t)\overline{F}(\eta(t_n))\nu(t)\\ &+(t_{n+1}-t)(t-t_n)\Big(\dot{\nu}^T(t)\overline{F}(\eta(t_n))\nu(t)\Big)^H\end{aligned} \tag{3.26}$$

Taking into account that

$$\begin{aligned}\dot{\nu}(t) &= -\frac{1}{\tau^2(t)}\int_{t_n}^{t}x(q)dq+\frac{1}{\tau(t)}\Big(x(t)-x(t_n)\Big)=\frac{1}{\tau(t)}\Big(x(t)-x(t_n)-\nu(t)\Big)\\ &=\frac{1}{\tau(t)}M_3\xi(t),\end{aligned} \tag{3.27}$$

with

$$M_3=\begin{bmatrix}\mathbf{I} & -\mathbf{I} & -\mathbf{I}\end{bmatrix},\ \xi(t)=\begin{bmatrix}x^T(t) & x^T(t_n) & \nu^T(t)\end{bmatrix}^T, \tag{3.28}$$

then (3.26) can be rewritten as

$$\begin{aligned}\dot{V}_3(x,t) = &-(t-t_n)\nu^T(t)\overline{F}(\eta(t_n))\nu(t)+(t_{n+1}-t)\nu^T(t)\overline{F}(\eta(t_n))\nu(t)\\ &+(t_{n+1}-t)\Big(\xi^T(t)M_3^T\overline{F}(\eta(t_n))\nu(t)\Big)^H.\end{aligned} \tag{3.29}$$

An upper bound for the integral term in (3.24) can be found by means of Lemma 2.4, where $R=\begin{bmatrix}\overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n))\\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n))\end{bmatrix}$ is held constant between two successive sampling instants and $\dot{\omega}(t)=\begin{bmatrix}\dot{x}^T(t) & x^T(t_n)\end{bmatrix}^T$:

$$-\int_{t_n}^{t}\begin{bmatrix}\dot{x}(q)\\x(t_n)\end{bmatrix}^T R\begin{bmatrix}\dot{x}(q)\\x(t_n)\end{bmatrix}dq\leq\xi^T(t)\left\{-\frac{1}{\tau(t)}(\Omega_1^*)^TR(\Omega_1^*)-\frac{3}{\tau(t)}(\Omega_2^*)^TR(\Omega_2^*)\right\}\xi(t) \tag{3.30}$$

in which $\Omega_1^*=M_1+\tau(t)M_t$ and $\Omega_2^*=M_2$, with

$$M_1=\begin{bmatrix}\mathbf{I} & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\ M_t=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix},\ M_2=\begin{bmatrix}\mathbf{I} & \mathbf{I} & -2\mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{3.31}$$

Matrices M_1, M_t, and M_2 are obtained as follows:

$$\begin{aligned}\Omega_1^*\xi(t) &= \omega(t) - \omega(t_n) = \begin{bmatrix} x(t) - x(t_n) \\ \tau(t)x(t_n) \end{bmatrix} - \begin{bmatrix} x(t_n) - x(t_n) \\ \tau(t_n)x(t_n) \end{bmatrix} \\ &= \begin{bmatrix} x(t) - x(t_n) \\ 0 \end{bmatrix} + \tau(t) \begin{bmatrix} 0 \\ x(t_n) \end{bmatrix} \\ &= \Big(M_1 + \tau(t)M_t\Big)\xi(t)\end{aligned} \tag{3.32}$$

$$\begin{aligned}\Omega_2^*\xi(t) &= \omega(t) + \omega(t_n) - \frac{2}{t-t_n}\int_{t_n}^{t} w(u)du \\ &= \begin{bmatrix} x(t) - x(t_n) \\ \tau(t)x(t_n) \end{bmatrix} + \begin{bmatrix} x(t_n) - x(t_n) \\ \tau(t_n)x(t_n) \end{bmatrix} - \frac{2}{t-t_n}\int_{t_n}^{t} \begin{bmatrix} x(u) - x(t_n) \\ \tau(u)x(t_n) \end{bmatrix} du \\ &= \begin{bmatrix} x(t) - x(t_n) \\ \tau(t)x(t_n) \end{bmatrix} - \frac{2}{t-t_n}\frac{t-t_n}{2}\begin{bmatrix} 2\nu(t) - 2x(t_n) \\ \tau(t)x(t_n) \end{bmatrix} \\ &= \begin{bmatrix} x(t) - x(t_n) \\ \tau(t)x(t_n) \end{bmatrix} - \begin{bmatrix} 2\nu(t) - 2x(t_n) \\ \tau(t)x(t_n) \end{bmatrix} = \begin{bmatrix} x(t) + x(t_n) - 2\nu(t) \\ 0 \end{bmatrix} \\ &= M_2\xi(t)\end{aligned} \tag{3.33}$$

An upper bound to the right-hand side of (3.30) can be obtained by using the quadratic relation

$$\Big(R\Omega_i^* + \tau(t)\overline{N}_i(\rho)\Big)^T R^{-1}\Big(R\Omega_i^* + \tau(t)\overline{N}_i(\rho)\Big) \succeq 0, \tag{3.34}$$

for $i = 1, 2$, and with

$$\overline{N}_i(\rho) = \begin{bmatrix} \overline{N}_{i1}(\rho) & \overline{N}_{i2}(\rho) & \overline{N}_{i3}(\rho) \\ \overline{N}_{i4}(\rho) & \overline{N}_{i5}(\rho) & \overline{N}_{i6}(\rho) \end{bmatrix},$$

an upper bound for (3.30) is given by

$$\begin{aligned}&-\int_{t_n}^{t} \begin{bmatrix} \dot{x}(q) \\ x(t_n) \end{bmatrix}^T R \begin{bmatrix} \dot{x}(q) \\ x(t_n) \end{bmatrix} dq \leq \\ &\quad \xi^T(t)\Bigg\{ -\frac{1}{\tau(t)}(\Omega_1^*)^T R(\Omega_1^*) - \frac{3}{\tau(t)}(\Omega_2^*)^T R(\Omega_2^*)\Bigg\}\xi(t) \leq \\ &\quad \xi^T(t)\Bigg\{ \Big[\overline{N}_1^T(\rho)(\Omega_1^*)\Big]^H + \tau(t)\overline{N}_1^T(\rho)R^{-1}\overline{N}_1(\rho) \\ &\qquad + 3\Big[\overline{N}_2^T(\rho)(\Omega_2^*)\Big]^H + 3\tau(t)\overline{N}_2^T(\rho)R^{-1}\overline{N}_2(\rho)\Bigg\}\xi(t)\end{aligned} \tag{3.35}$$

Replacing (3.29) and (3.35) in (3.22), the resulting terms can be then grouped based on their dependence on time:

$$\dot{W}(x,t) \leq \left\{ \pi_1 \right\} + (t_{n+1} - t)\left\{ \pi_2 \right\}$$
$$+ (t - t_n)\left\{ \pi_3 + \xi^T(t)\overline{N}_1^T(\rho)\begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}^{-1} \overline{N}_1\xi(t) + \right.$$
$$\left. 3\xi^T(t)\overline{N}_2^T(\rho)\begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}^{-1} \overline{N}_2\xi(t) \right\} < 0, \quad (3.36)$$

with

$$\pi_1 = x^T(t)\dot{\overline{P}}(\rho)x(t) + \left(\dot{x}^T(t)\overline{P}(\rho)x(t)\right)^H + \xi^T(t)\left[\overline{N}_1^T(\rho)M_1\right]^H\xi(t)$$
$$- \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix}^T \begin{bmatrix} \overline{X}_1(\rho) & \overline{X}_2^T(\rho) - \overline{X}_1(\rho) \\ \overline{X}_2(\rho) - \overline{X}_1(\rho) & \overline{X}_1(\rho) - \overline{X}_2^H(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix} + \xi^T(t)\left[3\overline{N}_2^T(\rho)M_2\right]^H\xi(t) \quad (3.37)$$

$$\pi_2 = \begin{bmatrix} \dot{x}(t) \\ x(t_n) \end{bmatrix}^T \begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t_n) \end{bmatrix} + \left(\xi^T(t)M_3^T\overline{F}(\eta(t_n))\nu(t)\right)^H$$
$$+ \left(\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}^T \begin{bmatrix} \overline{X}_1(\rho) & \overline{X}_2^T(\rho) - \overline{X}_1(\rho) \\ \overline{X}_2(\rho) - \overline{X}_1(\rho) & \overline{X}_1(\rho) - \overline{X}_2^H(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix} \right)^H \quad (3.38)$$
$$+ \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix}^T \begin{bmatrix} \dot{\overline{X}}_1(\rho) & \dot{\overline{X}}_2^T(\rho) - \dot{\overline{X}}_1(\rho) \\ \dot{\overline{X}}_2(\rho) - \dot{\overline{X}}_1(\rho) & \dot{\overline{X}}_1(\rho) - \dot{\overline{X}}_2^H(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix} + \nu^T(t)\overline{F}(\eta(t_n))\nu(t)$$

$$\pi_3 = \xi^T(t)\left[\overline{N}_1^T(\rho)M_t\right]^H\xi(t) - \nu^T(t)\overline{F}(\eta(t_n))\nu(t) \quad (3.39)$$

Remark 3.4. *The inequality (3.36) is not itself written in terms of LMI constraints. The derivation of LMI conditions is further explored in the upcoming sections of this chapter and also in Chapter 4.*

Remark 3.5. *The upper bound for (3.22) is affine with respect to t. Since the Lyapunov function (3.12) is a looped functional, the feasibility of condition (3.22) can be guaranteed by evaluating (3.36) at the extreme values of $t \in [t_n,\ t_{n+1})$ (Seuret; Gouaisbaut, 2013).*

3.3 Controller synthesis

The upcoming Theorem 3.1 provides sufficient conditions for designing sampled-data gain-scheduling state-feedback controllers with $\mathcal{L}_2$-gain guaranteed cost. It is important to emphasize that the provided conditions are actually in a bilinear matrix inequality

(BMI) structure. Therefore, in an attempt to circumvent the optimization problem described in terms of BMIs, a two-step iterative procedure is applied. That is why the first proposed stabilizing methodology is also known as *iterative approach*. The setup of such iterative approach for obtaining the desired controllers is discussed after the theorem development.

Furthermore, in this iterative approach, the gain-scheduling controllers are synthesized based on the Lyapunov matrix $\overline{P}(\rho)$, thus one must impose $\overline{P}(\rho) = \overline{P}(\eta(t_n))$. Additionally, for linearization purposes, $\overline{X}_2(\rho) \equiv 0$ and $\overline{X}_1(\rho) = \overline{X}_1(\eta(t_n))$ are adopted for the Lyapunov function (3.12).

Theorem 3.1. *Given scalars $T_m > 0$ and λ, if there exist symmetric positive-definite matrices $Q(\rho) = Q(\eta(t_n))$, $\Lambda_2(\eta(t_n)) \in \mathbb{R}^{n_x \times n_x}$, $\Gamma_1(\eta(t_n))$, $\Lambda_1(\eta(t_n)) \in \mathbb{R}^{2n_x \times 2n_x}$, matrices $Y(\eta(t_n)) \in \mathbb{R}^{n_u \times n_x}$, $\overline{N}_1(\rho)$, $\overline{N}_2(\rho) \in \mathbb{R}^{2n_x \times 3n_x + n_w}$, $L(\rho)$, $G(\rho)$, $\Upsilon(\rho) \in \mathbb{R}^{2n_x \times 2n_x}$, and a scalar $\gamma > 0$ such that*

$$\begin{bmatrix} \Pi_1 + T_m\Pi_2 & * & * \\ T_m L_5 & -T_m\Lambda_1(\eta(t_n)) & * \\ \Phi & \boldsymbol{0} & -\gamma\boldsymbol{I} \end{bmatrix} \prec 0 \tag{3.40}$$

$$\begin{bmatrix} \Pi_1 + T_m\Pi_3 & * & * & * \\ T_m\overline{N}_1(\rho) & -T_m\Gamma_1(\eta(t_n)) & * & * \\ 3T_m\overline{N}_2(\rho) & \boldsymbol{0} & -3T_m\Gamma_1(\eta(t_n)) & * \\ \Phi & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I} \end{bmatrix} \prec 0 \tag{3.41}$$

$$\begin{bmatrix} \Lambda_1(\eta(t_n)) + \Big(\Upsilon^T(\rho)\big(\overline{Q}(\rho) + L^T(\rho)\big)\Big)^H & * \\ -L^T(\rho) + G^T(\rho)\Upsilon(\rho) & \Gamma_1(\eta(t_n)) - G^H(\rho) \end{bmatrix} \prec 0 \tag{3.42}$$

for all $\rho \in \Theta$, in which[2]

$$\Pi_1 = \Pi_1^0 - \gamma L_8^T L_8,\ \Pi_1^0 = \Big(L_4^T L_1\Big)^H - \lambda L_2^T\overline{Q}(\rho)L_0L_2 + \Big(\overline{N}_1^T(\rho)M_1^* + 3\overline{N}_2^T(\rho)M_2^*\Big)^H$$

$$\Pi_2 = \lambda\Big(L_6^T L_0 L_2\Big)^H + \Big(L_7^T\Lambda_2(\eta(t_n))L_3\Big)^H + L_3^T\Lambda_2(\eta(t_n))L_3$$

$$\Pi_3 = \Big(\overline{N}_1^T(\rho)M_t^*\Big)^H - L_3^T\Lambda_2(\eta(t_n))L_3$$

$$M_1^* = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ M_2^* = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} & -2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ M_t^* = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ S_1 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

$$L_0 = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix},\ L_1 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ L_2 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ L_3 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

$$L_4 = \begin{bmatrix} \boldsymbol{A}(\rho)Q(\rho) & \boldsymbol{B}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} & \boldsymbol{B}_1(\rho) \end{bmatrix}$$

[2] The symbol $\otimes$ denotes the product of Kronecker.

$$L_5 = \begin{bmatrix} \boldsymbol{A}(\rho)Q(\rho) & \boldsymbol{B}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} & \boldsymbol{B}_1(\rho) \\ \boldsymbol{0} & Q(\rho) & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

$$L_6 = \begin{bmatrix} \boldsymbol{A}(\rho)Q(\rho) & \boldsymbol{B}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} & \boldsymbol{B}_1(\rho) \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \; L_7 = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

$$L_8 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \; \overline{Q}(\rho) = \boldsymbol{I}_2 \otimes Q(\rho), \; \Phi = \begin{bmatrix} \boldsymbol{C}(\rho)Q(\rho) & \boldsymbol{D}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} & \boldsymbol{D}_1(\rho) \end{bmatrix}$$

then, the origin is an asymptotically stable equilibrium point of system (3.6) with aperiodic samplings lower than T_m and with a gain-scheduled sampled-data state-feedback control law given by $K(\eta(t_n)) = Y(\eta(t_n))Q^{-1}(\rho)$. Furthermore, γ is an upper bound to the $\mathcal{L}_2$-gain of the closed-loop system.

Proof of Theorem 3.1. Consider the Lyapunov function (3.12), with $\overline{P}(\rho) = \overline{P}(\eta(t_n))$, $\overline{X}_1(\rho) = \overline{X}_1(\eta(t_n))$, and $\overline{X}_2(\rho) \equiv 0$. Therefore,

$$\overline{E}(\eta(t_n)) = \begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}, \; \overline{X}(\rho) = \begin{bmatrix} \overline{X}_1(\rho) & -\overline{X}_1(\rho) \\ -\overline{X}_1(\rho) & \overline{X}_1(\rho) \end{bmatrix}. \tag{3.43}$$

In order to simultaneously ensure closed-loop stability for systems (3.6) and guaranteed $\mathcal{L}_2$-gain cost, conditions presented in Theorem 2.2 must be met. As shown in Section 3.2.1, the positiveness of the adopted Lyapunov function (3.12) is guaranteed if and only if the LMI (3.21) holds.

By retaking both (3.36) and the closed-loop systems (3.6), condition (2.18) can be rewritten as

$$\begin{aligned} \overline{\xi}^T(t)\Bigg\{ & \left[\left(S_4^T\overline{P}(\rho)L_1\right)^H - L_2^T\overline{X}(\rho)L_2 + \frac{1}{\gamma}\phi^T\phi - \gamma L_8^T L_8 \right] + \left(N_1^T(\rho)\Omega_1^* + 3N_2^T(\rho)\Omega_2^*\right)^H \\ & + (t_{n+1} - t)\left[S_5^T\overline{E}(\eta(t_n))S_5 + \left(S_6^T\overline{X}(\rho)L_2\right)^H + \left(L_7^T\overline{F}(\eta(t_n))L_3\right)^H + L_3^T\overline{F}(\eta(t_n))L_3 \right] \\ & + (t - t_n)\left[N_1^T(\rho)\overline{E}^{-1}(\eta(t_n))N_1(\rho) + 3N_2^T(\rho)\overline{E}^{-1}(\eta(t_n))N_2(\rho) - L_3^T\overline{F}(\eta(t_n))L_3 \right] \Bigg\}\overline{\xi}(t) < 0, \end{aligned} \tag{3.44}$$

with $\overline{\xi}(t) = \begin{bmatrix} \xi^T(t) & w^T(t) \end{bmatrix}^T$ and

$$S_4 = \begin{bmatrix} \mathbf{A}(\rho) & \mathbf{B}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{B}_1(\rho) \end{bmatrix}$$

$$S_5 = \begin{bmatrix} \mathbf{A}(\rho) & \mathbf{B}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{B}_1(\rho) \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$S_6 = \begin{bmatrix} \mathbf{A}(\rho) & \mathbf{B}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{B}_1(\rho) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\phi = \begin{bmatrix} \mathbf{C}(\rho) & \mathbf{D}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{D}_1(\rho) \end{bmatrix}$$

Remark that (3.44) is a particular case of (3.36), when $\dot{\overline{P}}(\rho) = \dot{\overline{X}}_1(\rho) = 0$. By choosing

$$\begin{gathered}\overline{P}(\rho) = Q^{-1}(\rho),\ \overline{X}_1(\rho) = \lambda Q^{-1}(\rho),\\ \overline{E}(\eta(t_n)) = \Lambda_1^{-1}(\eta(t_n)),\ \overline{F}(\eta(t_n)) = Q^{-1}(\rho)\Lambda_2(\eta(t_n))Q^{-1}(\rho),\end{gathered} \tag{3.45}$$

applying the congruence transformation

$$\tilde{\Omega} = diag\Big(\mathbf{I}_3 \otimes Q(\rho),\ \mathbf{I}_{n_w}\Big)$$

to both sides of (3.44) yields

$$\begin{aligned}&\left[\Big(L_4^T L_1\Big)^H - \lambda L_2^T \overline{Q}(\rho) L_0 L_2 + \Big(\overline{N}_1^T(\rho) M_1^* + 3\overline{N}_2^T(\rho) M_2^*\Big)^H + \frac{1}{\gamma}\Phi^T\Phi - \gamma L_8^T L_8\right]\\ &+ (t_{n+1} - t)\left[L_5^T \Lambda_1^{-1}(\eta(t_n)) L_5 + \lambda\Big(L_6^T L_0 L_2\Big)^H + \Big(L_7^T \Lambda_2(\eta(t_n)) L_3\Big)^H + L_3^T \Lambda_2(\eta(t_n)) L_3\right]\\ &+ (t - t_n)\Bigg[\Big(\overline{N}_1^T(\rho) M_t^*\Big)^H - L_3^T \Lambda_2(\rho) L_3 + \overline{N}_1^T(\rho)\overline{Q}^{-1}(\rho)\Lambda_1(\eta(t_n))\overline{Q}^{-1}(\rho)\overline{N}_1(\rho)\\ &\qquad + 3\overline{N}_2^T(\rho)\overline{Q}^{-1}(\rho)\Lambda_1(\eta(t_n))\overline{Q}^{-1}(\rho)\overline{N}_2(\rho)\Bigg] \prec 0.\end{aligned} \tag{3.46}$$

In order to obtain (3.46), the equivalence

$$\Omega_i^* \tilde{\Omega} \equiv \overline{Q}(\rho)\Omega_i^* \tag{3.47}$$

is considered, for $i = 1, 2$, with $\Omega_1^* = M_1^* + \tau(t) M_t^*$ and $\Omega_2^* = M_2^*$. Notice that matrices M_1, M_2 and M_t, exploited in (3.32) and in (3.33), are sub-matrices of M_1^*, M_2^* and M_t^*. The * notation denotes that the expanded vector $\overline{\xi}(t)$ is used instead of $\xi(t)$, such that $\Omega_1^*\overline{\xi}(t) \equiv \Omega_1\xi(t)$ and $\Omega_2^*\overline{\xi}(t) \equiv \Omega_2\xi(t)$.

The equivalence (3.47) is only possible because matrices Ω_1^* and Ω_2^* are composed of constant sub-matrices independent with respect to the disturbances $w(t)$. Consequently, the changes of variables

$$\overline{N}_i(\rho) = \overline{Q}(\rho) N_i(\rho)\tilde{\Omega}, \tag{3.48}$$

for $i = 1, 2$, are adopted.

Condition (3.46) presents two strongly nonlinear terms: $\overline{Q}^{-1}(\rho)\Lambda_1(\eta(t_n))\overline{Q}^{-1}(\rho)$ and $\Lambda_1^{-1}(\eta(t_n))$. Provided that the two terms rely on the decision matrix $\Lambda_1(\eta(t_n))$, they cannot be both linearized at once. In order to overcome such limitation, Hooshmandi *et al.* (2018) discuss a two-step iterative procedure, whose computational aspects are available in Section 3.5. Within the proposed iterative procedure, the following upper-bound

$$\Gamma_1^{-1}(\eta(t_n)) \succeq \overline{Q}^{-1}(\rho)\Lambda_1(\eta(t_n))\overline{Q}^{-1}(\rho) \tag{3.49}$$

is adopted for the product of variables. Thus, the resulting terms in (3.46) can then be grouped based on their dependence on time:

$$\left[\Pi_1 + \frac{1}{\gamma}\Phi^T\Phi\right] + (t_{n+1} - t)\left[\Pi_2 + L_5^T\Lambda_1^{-1}(\eta(t_n))L_5\right] \\ + (t - t_n)\left[\Pi_3 + \overline{N}_1^T\Gamma_1^{-1}(\eta(t_n))\overline{N}_1 + 3\overline{N}_2^T\Gamma_1^{-1}(\eta(t_n))\overline{N}_2\right] \prec 0 \quad (3.50)$$

Since (3.50) is affine with respect to t and provided that (3.12) is a looped functional, it is sufficient to ensure that (3.50) holds for both $t = t_n$ and $t = t_{n+1}$. By evaluating (3.50) at the end points of the sampling interval and by applying Schur complements, conditions (3.40) and (3.41) are achieved.

Inequality (3.49) is still non convex, because of the product of multiple decision variables. Hooshmandi *et al.* (2018) introduces a linearization strategy for condition (3.49), as presented in the following lemma.

Lemma 3.2. *(Hooshmandi* et al.*, 2018) Let $\Gamma \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix, $\Lambda \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and Q, L, G be $\mathbb{R}^{n\times n}$ matrices. If there exists a matrix $\Upsilon \in \mathbb{R}^{n\times n}$, then the following two statements are equivalent:*

1. $\Lambda - Q^T\Gamma^{-1}Q \prec 0$ (3.51)
2. $\begin{bmatrix} \Lambda + \left(\Upsilon^T(Q + L^T)\right)^H & -L + \Upsilon^T G \\ * & \Gamma - G^H \end{bmatrix} \prec 0$ (3.52)

Furthermore, choosing $\Upsilon = -\Gamma^{-1}Q$, the two previous statements are identical.

The proof of Lemma 3.2 is given in Appendix A. By applying Lemma 3.2, the constraint (3.42) ensures that (3.49) holds for all $t \in [t_n,\ t_{n+1})$. It is assumed that matrix $\Upsilon(\rho)$ is known when computing the control gain $K(\eta(t_n))$.

The proposed changes of variables (3.45) must be accordingly reflected in the condition ensuring positiviness of the Lyapunov function (3.12). Applying the congruence transformation $Q(\rho)$ to both sides of (3.21), $Q(\rho) \succ 0$ is obtained, which is a sufficient condition to certify that $W(x,t) > 0$. This completes the proof. □

Remark 3.6. *As discussed in Section 2.1, it is assumed that any matrix presented in Theorem 3.1 (and all upcoming results) is polytopic with respect to the scheduling parameters $\rho(t)$ (or $\eta(t_n)$) and that the scheduling parameters are described by the simplex (3.5). These assumptions are valid to both system matrices and decision matrices.*

Remark 3.7. *In Theorem 3.1, the proposed LMI conditions are obtained as the sum of terms with affine dependence on the expanded vector $\rho(t)$, on the sampled-data vector $\eta(t_n)$, or on neither scheduling parameters. An homogenization procedure is performed on the entries of the proposed LMI conditions, such that all matrix polynomials have the* *same degree of dependence on the scheduling parameters. The homogenization procedure follows the homogeneous polynomially parameter-dependent solutions discussed in Oliveira and Peres (2007), which is implemented by the MATLAB package ROLMIP (Agulhari* et al.*, 2019). This package defines a finite set of LMI conditions to be tested, given polynomial structures to the involved problem matrices. Since any polytopic matrix is thus represented with a homogeneous polynomial function, it is guaranteed that the number of LMI conditions in Theorem 3.1 are finite.*

3.4 Stabilization

The stabilization of closed-loop LPV systems (3.6) is addressed in light of the discussion held in Section 2.2. In this section, LMI conditions ensuring asymptotic stability for (3.6) are derived, with gain-scheduling state-feedback controllers (3.2) to be synthesized.

Theorem 3.2. *Given scalars $T_m > 0$ and λ, if there exist symmetric positive-definite matrices $Q(\rho) = Q(\eta(t_n))$, $\Lambda_2(\eta(t_n)) \in \mathbb{R}^{n_x \times n_x}$, $\Gamma_1(\eta(t_n))$, $\Lambda_1(\eta(t_n)) \in \mathbb{R}^{2n_x \times 2n_x}$, and matrices $Y(\eta(t_n)) \in \mathbb{R}^{n_u \times n_x}$, $\overline{N}_1(\rho)$, $\overline{N}_2(\rho) \in \mathbb{R}^{2n_x \times 3n_x}$, $L(\rho)$, $G(\rho)$, $\Upsilon(\rho) \in \mathbb{R}^{2n_x \times 2n_x}$ satisfying the following conditions:*

$$\begin{bmatrix} \Pi_1^0 + T_m\Pi_2 & * \\ T_m L_5 & -T_m\Lambda_1(\eta(t_n)) \end{bmatrix} \prec 0 \tag{3.53}$$

$$\begin{bmatrix} \Pi_1^0 + T_m\Pi_3 & * & * \\ T_m\overline{N}_1(\rho) & -T_m\Gamma_1(\eta(t_n)) & * \\ 3T_m\overline{N}_2(\rho) & \boldsymbol{0} & -3T_m\Gamma_1(\eta(t_n)) \end{bmatrix} \prec 0 \tag{3.54}$$

$$\begin{bmatrix} \Lambda_1(\eta(t_n)) + \Big(\Upsilon^T(\rho)\big(\overline{Q}(\rho) + L^T(\rho)\big)\Big)^H & * \\ -L^T(\rho) + G^T(\rho)\Upsilon(\rho) & \Gamma_1(\eta(t_n)) - G^H(\rho) \end{bmatrix} \prec 0 \tag{3.55}$$

for all $\rho \in \Theta$, in which

$$\Pi_1^0 = \Big(L_4^T L_1\Big)^H - \lambda L_2^T\overline{Q}(\rho)L_0L_2 + \Big(\overline{N}_1^T(\rho)M_1^* + 3\overline{N}_2^T(\rho)M_2^*\Big)^H$$

$$\Pi_2 = \lambda\Big(L_6^T L_0 L_2\Big)^H + \big(L_7^T\Lambda_2(\eta(t_n))L_3\big)^H + L_3^T\Lambda_2(\eta(t_n))L_3$$

$$\Pi_3 = \Big(\overline{N}_1^T(\rho)M_t^*\Big)^H - L_3^T\Lambda_2(\eta(t_n))L_3$$

$$M_1^* = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ M_2^* = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} & -2\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ M_t^* = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix},\ S_1 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$$

$$L_0 = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix},\ L_1 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ L_2 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix},\ L_3 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

$$L_4 = \begin{bmatrix} \boldsymbol{A}(\rho)Q(\rho) & \boldsymbol{B}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} \end{bmatrix}, \ L_5 = \begin{bmatrix} \boldsymbol{A}(\rho)Q(\rho) & \boldsymbol{B}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} \\ \boldsymbol{0} & Q(\rho) & \boldsymbol{0} \end{bmatrix}$$

$$L_6 = \begin{bmatrix} \boldsymbol{A}(\rho)Q(\rho) & \boldsymbol{B}_2(\rho)Y(\eta(t_n)) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \ L_7 = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}$$

$$\overline{Q}(\rho) = \boldsymbol{I}_2 \otimes Q(\rho)$$

then, the origin is an asymptotically stable equilibrium point of system (3.6) with aperiodic samplings lower than T_m. The synthesized gain-scheduled state-feedback control gain can be recovered with $K(\eta(t_n)) = Y(\eta(t_n))Q^{-1}(\rho)$.

Proof of Theorem 3.2. Taking the Lyapunov function (3.12), the asymptotic stability in the sense of Lyapunov for the closed-loop LPV systems (3.6), with $w(t) = 0$, is ensured if conditions (2.10) and (2.11) are respected for all $t \in [t_n,\ t_{n+1})$.

The steps of the proof follow the exact same lines as the proof of Theorem 3.1. In fact, by removing the last rows and columns of conditions (3.40) and (3.41), one gets precisely (3.53) and (3.54), which are sufficient conditions ensuring the feasibility of

$$\begin{bmatrix} \Pi_1^0 \end{bmatrix} + (t_{n+1} - t)\begin{bmatrix} \Pi_2 + L_5^T \Lambda_1^{-1}(\eta(t_n))L_5 \end{bmatrix} \\ + (t - t_n)\begin{bmatrix} \Pi_3 + \overline{N}_1^T \Gamma_1^{-1}(\eta(t_n))\overline{N}_1 + 3\overline{N}_2^T \Gamma_1^{-1}(\eta(t_n))\overline{N}_2 \end{bmatrix} \prec 0, \quad (3.56)$$

with Π_1^0, Π_2, and Π_3 as defined in both Theorems 3.1 and 3.2. Any other inequality in Theorem 3.1 remains unchanged. □

3.5 Computational aspects

In this chapter, an iterative approach is introduced for the synthesis of gain-scheduled control laws for LPV systems (3.6). The attained results in Section 3.3 originally lead to conditions outside of the framework of LMIs. In fact, the obtained relations are BMI constraints, provided the product of multiple decision variables, as in (3.46).

Due to the use of the upper-bound (3.49), the posed BMI problem can be solved as part of an iterative procedure, whose initial feasible solution is $\Upsilon_0 = -\Gamma_1^{-1}\overline{Q}(\rho)$ (Hooshmandi *et al.*, 2018). The value of Γ_1 can be obtained from the inequality

$$\Lambda_1(\eta(t_n)) + \varepsilon^2\Gamma_1 - 2\varepsilon\overline{Q}(\rho) \prec 0, \quad (3.57)$$

with ε some given positive scalar and Γ_1 a parameter-independent matrix during initialization only. The constraint (3.57) can replace condition (3.42) in the first iteration of the iterative procedure, as shown in the proof below.

Proof. (Hooshmandi *et al.*, 2018) Since by assumption Γ_1 is a positive-definite matrix, the following quadratic relation is true:

$$\left(\varepsilon\mathbf{I} - \Gamma_1^{-1}\overline{Q}(\rho)\right)^T \Gamma_1 \left(\varepsilon\mathbf{I} - \Gamma_1^{-1}\overline{Q}(\rho)\right) \succeq 0. \tag{3.58}$$

Expanding the relation (3.58) and adding $\Lambda_1(\eta(t_n))$ to both sides of this condition, one has that

$$\Lambda_1(\eta(t_n)) + \varepsilon^2\Gamma_1 - 2\varepsilon\overline{Q}(\rho) + \overline{Q}(\rho)\Gamma_1^{-1}\overline{Q}(\rho) \succeq \Lambda_1(\eta(t_n)) \tag{3.59}$$

$$\Lambda_1(\eta(t_n)) - \overline{Q}(\rho)\Gamma_1^{-1}\overline{Q}(\rho) \preceq \Lambda_1(\eta(t_n)) + \varepsilon^2\Gamma_1 - 2\varepsilon\overline{Q}(\rho) \tag{3.60}$$

Imposing the right-hand side of (3.60) to be negative definite, the condition (3.57) is clearly satisfied. Moreover, it is straightforward to verify that (3.57) is an upper bound to the relation (3.49). For initialization purposes, the LMI constraint (3.57) can therefore replace the condition (3.42). □

The numerical solution for the conditions derived in Theorem 3.1 is achieved by an iterative procedure, which is developed in Algorithm 3.1. With a few changes, Algorithm 3.1 can be also applied for computing the solution for the conditions presented in Theorem 3.2.

Algorithm 3.1 Iterative procedure for synthesizing gain-scheduled $\mathcal{L}_2$-gain sampled-data controllers.

Initialization:
1) Adopt a value for maximum allowable sampling period T_m.
2) Set $\varepsilon = 1$, $\lambda_0 = 1$, $\lambda = 1$, $\gamma_0 = 10$, $\epsilon = 0.01$, $k_{max} = 20$, and $k = 1$.
3) Given ε and λ, minimize γ under conditions (3.40), (3.41) and (3.57) for computing $Q(\rho)$ and Γ_1.
4) Set $\overline{Q}(\rho) = \mathbf{I}_2 \otimes Q(\rho)$, $\Upsilon_0 = -\Gamma_1^{-1}\overline{Q}(\rho)$, and $\gamma_1 = \gamma$.
Iterative procedure:
While $k < k_{max}$ or $|\gamma_k - \gamma_{k-1}| > \epsilon$ **do**
5) Given $\Upsilon_{k-1}(\rho)$ and λ_{k-1}, minimize γ under conditions (3.40)–(3.42) for determining $Q(\rho)$, $\Lambda_1(\eta(t_n))$, $\Lambda_2(\eta(t_n))$, $\Gamma_1(\eta(t_n))$, $G(\rho)$, $L(\rho)$, $Y(\eta(t_n))$, $\overline{N}_1(\rho)$, and $\overline{N}_2(\rho)$.
6) Set $Q_{k-1}(\rho) = Q(\rho)$, $\Lambda_{1,k-1}(\eta(t_n)) = \Lambda_1(\eta(t_n))$, $\Lambda_{2,k-1}(\eta(t_n)) = \Lambda_2(\eta(t_n))$, $\Gamma_{1,k-1}(\eta(t_n)) = \Gamma_1(\eta(t_n))$, $G_{k-1}(\rho) = G(\rho)$, $L_{k-1}(\rho) = L(\rho)$, $Y_{k-1}(\eta(t_n)) = Y(\eta(t_n))$, $\overline{N}_{1,k-1}(\rho) = \overline{N}_1(\rho)$, and $\overline{N}_{2,k-1}(\rho) = \overline{N}_2(\rho)$.
7) Given $Q_{k-1}(\rho)$, $\Lambda_{1,k-1}(\eta(t_n))$, $\Lambda_{2,k-1}(\eta(t_n))$, $\Gamma_{1,k-1}(\eta(t_n))$, $G_{k-1}(\rho)$, $L_{k-1}(\rho)$, $Y_{k-1}(\eta(t_n))$, $\overline{N}_{1,k-1}(\rho)$, and $\overline{N}_{2,k-1}(\rho)$, minimize γ under conditions (3.40)–(3.42) to obtain $\Upsilon(\rho)$ and λ.
8) Set $\Upsilon_k(\rho) = \Upsilon(\rho)$, $\gamma_k = \gamma$, and $\lambda_k = \lambda$.
9) Set $k = k + 1$.
End

4 Non-iterative approaches

This chapter presents two slack variable-based procedures for the design of sampled-data state-feedback controllers for LPV systems in continuous time. The same problem stated and Lyapunov function defined in Chapter 3 are considered, and it is shown how the attained results can be then extended to quasi-LPV systems. A few computational aspects ensuring the implementation of the slack approaches are presented. The motivation behind slack techniques is the deliberate attempt to develop less conservative conditions for stabilization and for minimization of the $\mathcal{L}_2$-gain cost criterion, by introducing extra degrees of freedom to the LMI conditions and by decoupling the synthesis of the control gain from the Lyapunov matrix $\overline{P}(\rho)$. The derivation of more relaxed conditions exploits Finsler's lemma, discussed in Section 2.5. Another substantial benefit of the proposed slack approaches is the possibility of synthesizing control laws in the form (3.2) without using an iterative procedure. The numerical results for the proposed approaches are presented in Chapter 5.

4.1 Full approach

In this section and in the subsequent Section 4.2, Finsler's lemma is used to introduce extra variables to the problem of designing sampled-data gain-scheduled state-feedback controllers for systems (3.6). Using such variables enables the decoupling of the Lyapunov matrix $\overline{P}(\rho)$ from the controller gain (the synthesis is performed in terms of the slack variables), which allows the adoption of a more general structure to $\overline{P}(\rho)$. Similar relaxed structures can be adopted to matrices $\overline{X}_1(\rho)$ and $\overline{X}_2(\rho)$ in (3.12).

The key point for deriving the proposed conditions is determining an upper-bound to the term

$$-\frac{1}{\tau(t)}\xi^T(t)\left\{(\Omega_1^*)^T R(\Omega_1^*) + 3(\Omega_2^*)^T R(\Omega_2^*)\right\}\xi(t), \tag{4.1}$$

which arises from $\dot{V}_1(x,t)$, as given in (3.30). In this section, the upper bound obtained through relation (3.34) is employed and, in Section 4.2, an alternative upper bound for (4.1) is proposed.

4.1.1 Controller synthesis

In Theorem 4.1, a design approach for sampled-data gain-scheduled state-feedback control laws (3.2) for LPV systems (3.6) with $\mathcal{L}_2$-gain performance is proposed.

Theorem 4.1. *The origin is an asymptotically stable equilibrium point of system (3.6) with aperiodic samplings lower than T_m if, given a scalar $T_m > 0$ and a set of real scalars α_1, α_2, β_1, β_2, $\delta_1 > 0$, δ_2, ζ_1, ζ_2, $\mu_1 > \frac{1}{2}$, $\mu_2 > \frac{1}{2}$, ϕ_2, θ_2, λ_2, $\omega_2 > 0$, there exist symmetric positive-definite matrices $P(\rho), E_1(\eta(t_n)), E_3(\eta(t_n)), F(\eta(t_n)) \in \mathbb{R}^{n_x \times n_x}$, a symmetric matrix $X_1(\rho) \in \mathbb{R}^{n_x \times n_x}$, matrices $G(\eta(t_n)), E_2(\eta(t_n)), X_2(\rho) \in \mathbb{R}^{n_x \times n_x}$, $Z(\eta(t_n)) \in \mathbb{R}^{n_u \times n_x}$, $N_{ij}(\rho) \in \mathbb{R}^{n_x \times n_x}$, for $i = 1, 2,\ j = 1, \ldots, 6$, minimizing γ subject to the following LMIs:*

$$\begin{bmatrix} \Psi_{11} & * & * & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * & * & * \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} & * & * \\ \Psi_{51} & \Psi_{52} & \Psi_{53} & \Psi_{54} & \Psi_{55} & * \\ \Psi_{61} & \Psi_{62} & \Psi_{63} & \Psi_{64} & \Psi_{65} & \Psi_{66} \end{bmatrix} \prec 0 \tag{4.2}$$

$$\begin{bmatrix} \Upsilon_{11} & * & * & * & * & * & * & * & * & * \\ \Upsilon_{21} & \Upsilon_{22} & * & * & * & * & * & * & * & * \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} & * & * & * & * & * & * & * \\ \Upsilon_{41} & \Upsilon_{42} & \Upsilon_{43} & \Upsilon_{44} & * & * & * & * & * & * \\ \Upsilon_{51} & \Upsilon_{52} & \Upsilon_{53} & \Upsilon_{54} & \Upsilon_{55} & * & * & * & * & * \\ \Upsilon_{61} & \Upsilon_{62} & \Upsilon_{63} & \Upsilon_{64} & \Upsilon_{65} & \Upsilon_{66} & * & * & * & * \\ \Upsilon_{71} & \Upsilon_{72} & \Upsilon_{73} & \Upsilon_{74} & \Upsilon_{75} & \Upsilon_{76} & \Upsilon_{77} & * & * & * \\ \Upsilon_{81} & \Upsilon_{82} & \Upsilon_{83} & \Upsilon_{84} & \Upsilon_{85} & \Upsilon_{86} & \Upsilon_{78} & \Upsilon_{88} & * & * \\ \Upsilon_{91} & \Upsilon_{92} & \Upsilon_{93} & \Upsilon_{94} & \Upsilon_{95} & \Upsilon_{96} & \Upsilon_{97} & \Upsilon_{98} & \Upsilon_{99} & * \\ \Upsilon_{10'1} & \Upsilon_{10'2} & \Upsilon_{10'3} & \Upsilon_{10'4} & \Upsilon_{10'5} & \Upsilon_{10'6} & \Upsilon_{10'7} & \Upsilon_{10'8} & \Upsilon_{10'9} & \Upsilon_{10'10} \end{bmatrix} \prec 0 \tag{4.3}$$

for all $\rho \in \Theta$, in which

$$\Psi_{11} = \dot{P}(\rho) - X_1(\rho) + T_m \dot{X}_1(\rho) + \Big(N_{11}(\rho) + 3N_{21}(\rho) + \alpha_1 \boldsymbol{A}(\rho) G(\eta(t_n))\Big)^H$$

$$\Psi_{21} = -\Big(X_2(\rho) - X_1(\rho)\Big) + T_m\Big(\dot{X}_2(\rho) - \dot{X}_1(\rho)\Big) - N_{11}(\rho) + 3N_{21}(\rho) + N_{12}^T(\rho) + 3N_{22}^T(\rho) + \zeta_1 \boldsymbol{A}(\rho) G(\eta(t_n)) + \alpha_1\Big(\boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^T$$

$$\Psi_{22} = -\Big(X_1(\rho) - X_2^H(\rho)\Big) + T_m\Big(\dot{X}_1(\rho) - \dot{X}_2^H(\rho)\Big) + T_m E_3(\eta(t_n)) + \Big(-N_{12}(\rho) + 3N_{22}(\rho) + \zeta_1 \boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^H$$

$$\Psi_{31} = T_m F(\eta(t_n)) - 6N_{21}(\rho) + N_{13}^T(\rho) + 3N_{23}^T(\rho) + \beta_1 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Psi_{32} = -T_m F(\eta(t_n)) - 6N_{22}(\rho) - N_{13}^T(\rho) + 3N_{23}^T(\rho) + \beta_1 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Psi_{33} = -T_m F(\eta(t_n)) - 6N_{23}^H(\rho)$$

$$\Psi_{41} = P(\rho) + T_m X_1(\rho) - \alpha_1 G^T(\eta(t_n)) + \delta_1 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Psi_{42} = T_m E_2^T(\eta(t_n)) + T_m\Big(X_2^T(\rho) - X_1(\rho)\Big) - \zeta_1 G^T(\eta(t_n)) + \delta_1 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Psi_{43} = -\beta_1 G^T(\eta(t_n))$$

$$\Psi_{44} = T_m E_1(\eta(t_n)) - \delta_1 G^H(\eta(t_n))$$

$$\Psi_{51} = \mu_1 \boldsymbol{C}(\rho) G(\eta(t_n)),\ \Psi_{52} = \mu_1 \boldsymbol{D}_2(\rho) Z(\eta(t_n)),\ \Psi_{53} = \Psi_{54} = \boldsymbol{0}_{n_y \times n_x}$$

$$\Psi_{55} = (1 - 2\mu_1)\boldsymbol{I}_{n_y}$$

$$\Psi_{61} = \alpha_1 \boldsymbol{B}_1^T(\rho),\ \Psi_{62} = \zeta_1 \boldsymbol{B}_1^T(\rho),\ \Psi_{63} = \beta_1 \boldsymbol{B}_1^T(\rho)$$

$$\Psi_{64} = \delta_1 \boldsymbol{B}_1^T(\rho),\ \Psi_{65} = \mu_1 \boldsymbol{D}_1^T(\rho),\ \Psi_{66} = -\gamma^2 \boldsymbol{I}_{n_w}$$

$$\Upsilon_{11} = \dot{P}(\rho) - X_1(\rho) + \Big(N_{11}(\rho) + 3N_{21}(\rho) + \alpha_2 \boldsymbol{A}(\rho) G(\eta(t_n))\Big)^H$$

$$\Upsilon_{21} = -\Big(X_2(\rho) - X_1(\rho)\Big) - N_{11}(\rho) + 3N_{21}(\rho) + N_{12}^T(\rho) + 3N_{22}^T(\rho) + T_m N_{14}(\rho)$$
$$+\zeta_2 \boldsymbol{A}(\rho) G(\eta(t_n)) + \alpha_2\Big(\boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^T$$

$$\Upsilon_{22} = -\Big(X_1(\rho) - X_2^H(\rho)\Big) + T_m N_{15}^H(\rho) + \Big(-N_{12}(\rho) + 3N_{22}(\rho) + \zeta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^H$$

$$\Upsilon_{31} = -6N_{21}(\rho) + N_{13}^T(\rho) + 3N_{23}^T(\rho) + \beta_2 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Upsilon_{32} = -6N_{22}(\rho) - N_{13}^T(\rho) + 3N_{23}^T(\rho) + T_m N_{16}^T(\rho) + \beta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Upsilon_{33} = -T_m F(\eta(t_n)) - 6N_{23}^H(\rho)$$

$$\Upsilon_{41} = T_m N_{11}(\rho) + \delta_2 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Upsilon_{42} = T_m N_{12}(\rho) + \delta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Upsilon_{43} = T_m N_{13}(\rho),\ \Upsilon_{44} = -T_m E_1(\eta(t_n))$$

$$\Upsilon_{51} = T_m N_{14}(\rho) + \phi_2 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Upsilon_{52} = T_m N_{15}(\rho) + \phi_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Upsilon_{53} = T_m N_{16}(\rho),\ \Upsilon_{54} = -T_m E_2(\eta(t_n)),\ \Upsilon_{55} = -T_m E_3(\eta(t_n))$$

$$\Upsilon_{61} = 3T_m N_{21}(\rho) + \theta_2 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Upsilon_{62} = 3T_m N_{22}(\rho) + \theta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Upsilon_{63} = 3T_m N_{23}(\rho),\ \Upsilon_{64} = \Upsilon_{65} = \boldsymbol{0}_{n_x \times n_x},\ \Upsilon_{66} = -3T_m E_1(\eta(t_n))$$

$$\Upsilon_{71} = 3T_m N_{24}(\rho) + \lambda_2 \boldsymbol{A}(\rho) G(\eta(t_n))$$

$$\Upsilon_{72} = 3T_m N_{25}(\rho) + \lambda_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))$$

$$\Upsilon_{73} = 3T_m N_{26}(\rho),\ \Upsilon_{74} = \Upsilon_{75} = \boldsymbol{0}_{n_x \times n_x},\ \Upsilon_{76} = -3T_m E_2(\eta(t_n)),\ \Upsilon_{77} = -3T_m E_3(\eta(t_n))$$

$$\Upsilon_{81} = P(\rho) + \omega_2 \boldsymbol{A}(\rho) G(\eta(t_n)) - \alpha_2 G^T(\eta(t_n))$$

$$\Upsilon_{82} = \omega_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n)) - \zeta_2 G^T(\eta(t_n))$$

$$\Upsilon_{83} = -\beta_2 G^T(\eta(t_n)),\ \Upsilon_{84} = -\delta_2 G^T(\eta(t_n)),\ \Upsilon_{85} = -\phi_2 G^T(\eta(t_n))$$

$$\Upsilon_{86} = -\theta_2 G^T(\eta(t_n)),\ \Upsilon_{87} = -\lambda_2 G^T(\eta(t_n)),\ \Upsilon_{88} = -\omega_2 G^H(\eta(t_n))$$

$$\Upsilon_{91} = \mu_2 \boldsymbol{C}(\rho) G(\eta(t_n)),\ \Upsilon_{92} = \mu_2 \boldsymbol{D}_2(\rho) Z(\eta(t_n))$$

$$\Upsilon_{93} = \Upsilon_{94} = \Upsilon_{95} = \Upsilon_{96} = \Upsilon_{97} = \Upsilon_{98} = \boldsymbol{0}_{n_y \times n_x},\ \Upsilon_{99} = (1 - 2\mu_2)\boldsymbol{I}_{n_y}$$

$$\Upsilon_{10'1} = \alpha_2 \boldsymbol{B}_1^T(\rho),\ \Upsilon_{10'2} = \zeta_2 \boldsymbol{B}_1^T(\rho),\ \Upsilon_{10'3} = \beta_2 \boldsymbol{B}_1^T(\rho),\ \Upsilon_{10'4} = \delta_2 \boldsymbol{B}_1^T(\rho)$$
$$\Upsilon_{10'5} = \phi_2 \boldsymbol{B}_1^T(\rho),\ \Upsilon_{10'6} = \theta_2 \boldsymbol{B}_1^T(\rho),\ \Upsilon_{10'7} = \lambda_2 \boldsymbol{B}_1^T(\rho),\ \Upsilon_{10'8} = \omega_2 \boldsymbol{B}_1^T(\rho)$$
$$\Upsilon_{10'9} = \mu_2 \boldsymbol{D}_1^T(\rho),\ \Upsilon_{10'10} = -\gamma^2 \boldsymbol{I}_{n_w}$$

with a gain-scheduled sampled-data control gain given by $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$*, and* γ *as an upper bound to the* $\mathcal{L}_2$*-gain of the closed-loop system.*

Proof of Theorem 4.1. Consider the time-dependent Lyapunov function (3.12). As shown in Section 3.2.1, the positiveness of the adopted Lyapunov function (3.12) is guaranteed if the condition (3.21) holds.

By retaking (3.36), condition (2.18) can be rewritten as

$$\dot{W}(x,t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) \leq \left\{ \pi_1 + y^T(t)y(t) - \gamma^2 w^T(t)w(t) \right\}$$
$$+ (t_{n+1} - t)\left\{ \pi_2 \right\} + (t - t_n)\left\{ \pi_3 + \xi^T(t)\overline{N}_1^T(\rho) \begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}^{-1} \overline{N}_1 \xi(t) + \right.$$
$$\left. 3\xi^T(t)\overline{N}_2^T(\rho) \begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}^{-1} \overline{N}_2 \xi(t) \right\} < 0, \quad (4.4)$$

where π_1, π_2, and π_3 are defined, respectively, in (3.37), (3.38), and (3.39).

Provided that (4.4) is affine with respect to t, adopting $t = t_n$, with $t_{n+1} - t_n \leq T_m$, the feasibility is guaranteed if

$$\left\{ \pi_1 + y^T(t)y(t) - \gamma^2 w^T(t)w(t) \right\} + T_m \left\{ \pi_2 \right\} < 0 \quad (4.5)$$

is satisfied.

Defining

$$\mathcal{Q}_1 = \begin{bmatrix} Q_{11}^1 & * & * & * & * & * \\ Q_{21}^1 & Q_{22}^1 & * & * & * & * \\ Q_{31}^1 & Q_{32}^1 & Q_{33}^1 & * & * & * \\ Q_{41}^1 & Q_{42}^1 & \mathbf{0} & Q_{44}^1 & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_y} & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{I}_{n_w} \end{bmatrix},\ \overline{\xi}_1(t) = \begin{bmatrix} x(t) \\ x(t_n) \\ \nu(t) \\ \dot{x}(t) \\ y(t) \\ w(t) \end{bmatrix}, \quad (4.6)$$

in which

$$
\begin{aligned}
Q_{11}^1 &= \dot{\overline{P}}(\rho) - \overline{X}_1(\rho) + T_m\dot{\overline{X}}_1(\rho) + \left(\overline{N}_{11}(\rho) + 3\overline{N}_{21}(\rho)\right)^H \\
Q_{21}^1 &= -\left(\overline{X}_2(\rho) - \overline{X}_1(\rho)\right) + T_m\left(\dot{\overline{X}}_2(\rho) - \dot{\overline{X}}_1(\rho)\right) - \overline{N}_{11}(\rho) + 3\overline{N}_{21}(\rho) + \overline{N}_{12}^T(\rho) + 3\overline{N}_{22}^T(\rho) \\
Q_{22}^1 &= -\left(\overline{X}_1(\rho) - \overline{X}_2^H(\rho)\right) + T_m\left(\dot{\overline{X}}_1(\rho) - \dot{\overline{X}}_2^H(\rho)\right) + T_m\overline{E}_3(\eta(t_n)) + \left(3\overline{N}_{22}(\rho) - \overline{N}_{12}(\rho)\right)^H \\
Q_{31}^1 &= T_m\overline{F}(\eta(t_n)) - 6\overline{N}_{21}(\rho) + \overline{N}_{13}^T(\rho) + 3\overline{N}_{23}^T(\rho) \\
Q_{32}^1 &= -\,T_m\overline{F}(\eta(t_n)) - 6\overline{N}_{22}(\rho) - \overline{N}_{13}^T(\rho) + 3\overline{N}_{23}^T(\rho) \\
Q_{33}^1 &= -\,T_m\overline{F}(\eta(t_n)) - 6\overline{N}_{23}^H(\rho) \\
Q_{41}^1 &= \overline{P}(\rho) + T_m\overline{X}_1(\rho) \\
Q_{42}^1 &= T_m\overline{E}_2^T(\eta(t_n)) + T_m\left(\overline{X}_2^T(\rho) - \overline{X}_1(\rho)\right) \\
Q_{44}^1 &= T_m\overline{E}_1(\eta(t_n))
\end{aligned}
$$

one can notice that

$$
(4.5) \iff \overline{\xi}_1^T(t)\mathcal{Q}_1\overline{\xi}_1(t) < 0 \;\forall\; \overline{\xi}_1(t) \neq 0 : \mathcal{B}_1\overline{\xi}_1(t) = 0, \tag{4.7}
$$

with

$$
\mathcal{B}_1 = \begin{bmatrix} \mathbf{A}(\rho) & \mathbf{B}_2(\rho)K(\eta(t_n)) & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{B}_1(\rho) \\ \mathbf{C}(\rho) & \mathbf{D}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{D}_1(\rho) \end{bmatrix}. \tag{4.8}
$$

By recovering the implications (i) and (iv) of Lemma 2.2, (4.7) implies that there exists a matrix

$$
\mathcal{X}_1 = \begin{bmatrix} G_1^1(\rho) & J_1^1(\rho) \\ G_2^1(\rho) & J_2^1(\rho) \\ G_3^1(\rho) & J_3^1(\rho) \\ G_4^1(\rho) & J_4^1(\rho) \\ G_5^1(\rho) & J_5^1(\rho) \\ G_6^1(\rho) & J_6^1(\rho) \end{bmatrix} \tag{4.9}
$$

such that

$$
\mathcal{Q}_1 + \mathcal{X}_1\mathcal{B}_1 + \mathcal{B}_1^T\mathcal{X}_1^T \prec 0. \tag{4.10}
$$

On the other hand, adopting $t = t_{n+1}$ and since $t_{n+1} - t_n \leq T_m$, if the condition

$$
\begin{aligned}
&\left\{\pi_1 + y^T(t)y(t) - \gamma^2 w^T(t)w(t)\right\} + T_m\left\{\pi_3 + \xi^T(t)\overline{N}_1^T(\rho)\begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}^{-1}\overline{N}_1\xi(t)\right. \\
&\qquad \left. + 3\xi^T(t)\overline{N}_2^T(\rho)\begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}^{-1}\overline{N}_2\xi(t)\right\} < 0
\end{aligned} \tag{4.11}
$$

holds, then the feasibility of (4.4) is ensured.

Defining

$$\mathcal{Q}_2 = \begin{bmatrix} Q_{11}^2 & * & * & * & * & * & * & * & * & * \\ Q_{21}^2 & Q_{22}^2 & * & * & * & * & * & * & * & * \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^1 & * & * & * & * & * & * & * \\ Q_{41}^2 & Q_{42}^2 & Q_{43}^2 & Q_{44}^1 & * & * & * & * & * & * \\ Q_{51}^2 & Q_{52}^2 & Q_{53}^2 & Q_{54}^2 & Q_{55}^2 & * & * & * & * & * \\ Q_{61}^2 & Q_{62}^2 & Q_{63}^2 & \mathbf{0} & \mathbf{0} & Q_{66}^2 & * & * & * & * \\ Q_{71}^2 & Q_{72}^2 & Q_{73}^2 & \mathbf{0} & \mathbf{0} & Q_{76}^2 & Q_{77}^2 & * & * & * \\ Q_{81}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_y} & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2\mathbf{I}_{n_w} \end{bmatrix}, \tag{4.12}$$

in which

$$\begin{aligned}
Q_{11}^2 &= \dot{\overline{P}}(\rho) - \overline{X}_1(\rho) + \Big(\overline{N}_{11}(\rho) + 3\overline{N}_{21}(\rho)\Big)^H \\
Q_{21}^2 &= -\Big(\overline{X}_2(\rho) - \overline{X}_1(\rho)\Big) + T_m\overline{N}_{14}(\rho) - \overline{N}_{11}(\rho) + 3\overline{N}_{21}(\rho) + \overline{N}_{12}^T(\rho) + 3\overline{N}_{22}^T(\rho) \\
Q_{22}^2 &= -\Big(\overline{X}_1(\rho) - \overline{X}_2^H(\rho)\Big) + T_m\overline{N}_{15}^H(\rho) + \Big(-\overline{N}_{12}(\rho) + 3\overline{N}_{22}(\rho)\Big)^H \\
Q_{31}^2 &= -6\overline{N}_{21}(\rho) + \overline{N}_{13}^T(\rho) + 3\overline{N}_{23}^T(\rho) \\
Q_{32}^2 &= T_m\overline{N}_{16}^T(\rho) - 6\overline{N}_{22}(\rho) - \overline{N}_{13}^T(\rho) + 3\overline{N}_{23}^T(\rho) \\
Q_{33}^2 &= -T_m\overline{F}(\eta(t_n)) - 6\overline{N}_{23}^H(\rho) \\
Q_{41}^2 &= T_m\overline{N}_{11}(\rho),\ Q_{42}^2 = T_m\overline{N}_{12}(\rho),\ Q_{43}^2 = T_m\overline{N}_{13}(\rho) \\
Q_{44}^2 &= -T_m\overline{E}_1(\eta(t_n)) \\
Q_{51}^2 &= T_m\overline{N}_{14}(\rho),\ Q_{52}^2 = T_m\overline{N}_{15}(\rho),\ Q_{53}^2 = T_m\overline{N}_{16}(\rho) \\
Q_{54}^2 &= -T_m\overline{E}_2(\eta(t_n)) \\
Q_{55}^2 &= -T_m\overline{E}_3(\eta(t_n)) \\
Q_{61}^2 &= 3T_m\overline{N}_{21}(\rho),\ Q_{62}^2 = 3T_m\overline{N}_{22}(\rho),\ Q_{63}^2 = 3T_m\overline{N}_{23}(\rho) \\
Q_{66}^2 &= -3T_m\overline{E}_1(\eta(t_n)) \\
Q_{71}^2 &= 3T_m\overline{N}_{24}(\rho),\ Q_{72}^2 = 3T_m\overline{N}_{25}(\rho),\ Q_{73}^2 = 3T_m\overline{N}_{26}(\rho) \\
Q_{76}^2 &= -3T_m\overline{E}_2(\eta(t_n)) \\
Q_{77}^2 &= -3T_m\overline{E}_3(\eta(t_n)) \\
Q_{81}^2 &= \overline{P}(\rho)
\end{aligned}$$

and

$$\mathcal{B}_2^{\perp} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{A}(\rho) & \mathbf{B}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1(\rho) \\ \mathbf{C}(\rho) & \mathbf{D}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_1(\rho) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{4.13}$$

the following relation holds, by applying Schur complements to (4.11):

$$(4.11) \iff \mathcal{B}_2^{\perp T} \mathcal{Q}_2 \mathcal{B}_2^{\perp} \prec 0. \tag{4.14}$$

Carrying the equivalence of (ii) and (iv) of Lemma 2.2, (4.14) suggests the existence of a matrix

$$\mathcal{X}_2 = \begin{bmatrix} G_1^2(\rho) & J_1^2(\rho) \\ G_2^2(\rho) & J_2^2(\rho) \\ G_3^2(\rho) & J_3^2(\rho) \\ G_4^2(\rho) & J_4^2(\rho) \\ G_5^2(\rho) & J_5^2(\rho) \\ G_6^2(\rho) & J_6^2(\rho) \\ G_7^2(\rho) & J_7^2(\rho) \\ G_8^2(\rho) & J_8^2(\rho) \\ G_9^2(\rho) & J_9^2(\rho) \\ G_{10}^2(\rho) & J_{10}^2(\rho) \end{bmatrix} \tag{4.15}$$

such that

$$\mathcal{Q}_2 + \mathcal{X}_2 \mathcal{B}_2 + \mathcal{B}_2^T \mathcal{X}_2^T \prec 0, \tag{4.16}$$

with

$$\mathcal{B}_2 = \begin{bmatrix} \mathbf{A}(\rho) & \mathbf{B}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{B}_1(\rho) \\ \mathbf{C}(\rho) & \mathbf{D}_2(\rho)K(\eta(t_n)) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{D}_1(\rho) \end{bmatrix}. \tag{4.17}$$

The feasibility of both conditions (4.10) and (4.16) implies that condition (2.18) holds for all $t \in [t_n,\ t_{n+1})$. Nonetheless, one can verify that the expansion of (4.10) and (4.16) yields the product of multiple decision variables. The resulting problem is therefore nonconvex.

In order to achieve the linearization of the product of variables in (4.10) and (4.16), the first step consists of applying congruence transformations $\tilde{\Omega}_1^T$ and $\tilde{\Omega}_1$ to the left and to the right of (4.10), and of pre- and post-multiplying (4.16) with $\tilde{\Omega}_2^T$

and $\tilde{\Omega}_2$, respectively, in which

$$\begin{aligned}\tilde{\Omega}_1 &= diag\Big(\mathbf{I}_4 \otimes G(\eta(t_n)),\ \mathbf{I}_{n_y},\ \mathbf{I}_{n_w}\Big),\\ \tilde{\Omega}_2 &= diag\Big(\mathbf{I}_8 \otimes G(\eta(t_n)),\ \mathbf{I}_{n_y},\ \mathbf{I}_{n_w}\Big).\end{aligned} \tag{4.18}$$

The sole application of the congruence transformations (4.18) is not enough for linearizing relations (4.10) and (4.16), since the products of decision variables remain. However, by properly choosing the slack variables in (4.9) and in (4.15), and by considering some changes of variables, the linearization process is achieved. Therefore, the second step is choosing

$$\begin{gathered}G_1^i(\rho) = \alpha_i G^{-T}(\eta(t_n)),\ G_2^i(\rho) = \zeta_i G^{-T}(\eta(t_n)),\ G_3^i(\rho) = \beta_i G^{-T}(\eta(t_n)),\\ G_4^i(\rho) = \delta_i G^{-T}(\eta(t_n)),\ G_5^2(\rho) = \phi_2 G^{-T}(\eta(t_n)),\\ G_6^2(\rho) = \theta_2 G^{-T}(\eta(t_n)),\ G_7^2(\rho) = \lambda_2 G^{-T}(\eta(t_n)),\ G_8^2(\rho) = \omega_2 G^{-T}(\eta(t_n)),\\ J_5^1(\rho) = \mu_1 \mathbf{I}_{n_y},\ J_9^2(\rho) = \mu_2 \mathbf{I}_{n_y},\end{gathered} \tag{4.19}$$

for $i = 1, 2$. For linearization purposes, any other block variable in (4.9) and in (4.15) is chosen to be a null matrix. Furthermore, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are chosen to simultaneously depend on $G(\eta(t_n))$ since both matrices multiply to the right the controller gain matrix, $K(\eta(t_n))$.

As a third step, adopting the following changes of variables

$$\begin{gathered}P(\rho) = G^T(\eta(t_n))\overline{P}(\rho)G(\eta(t_n)),\ E_1(\eta(t_n)) = G^T(\eta(t_n))\overline{E}_1(\eta(t_n))G(\eta(t_n)),\\ E_2(\eta(t_n)) = G^T(\eta(t_n))\overline{E}_2(\eta(t_n))G(\eta(t_n)),\ E_3(\eta(t_n)) = G^T(\eta(t_n))\overline{E}_3(\eta(t_n))G(\eta(t_n)),\\ X_1(\rho) = G^T(\eta(t_n))\overline{X}_1(\rho)G(\eta(t_n)),\ X_2(\rho) = G^T(\eta(t_n))\overline{X}_2(\rho)G(\eta(t_n)),\\ F(\eta(t_n)) = G^T(\eta(t_n))\overline{F}(\eta(t_n))G(\eta(t_n)),\ N_{ij}(\rho) = G^T(\eta(t_n))\overline{N}_{ij}(\rho)G(\eta(t_n)),\\ Z(\eta(t_n)) = K(\eta(t_n))G(\eta(t_n)),\end{gathered} \tag{4.20}$$

for $i = 1, 2$, $j = 1, \ldots, 6$, one can verify that the resulting relations are linear and that

$$\begin{aligned}\tilde{\Omega}_1^T(4.10)\tilde{\Omega}_1 &\iff (4.2),\\ \tilde{\Omega}_2^T(4.16)\tilde{\Omega}_2 &\iff (4.3).\end{aligned} \tag{4.21}$$

Thus, (4.2) and (4.3) provide sufficient conditions for satisfying (2.18), for all $t \in [t_n,\ t_{n+1})$, with a control gain $K(\eta(t_n))$ given by

$$K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n)). \tag{4.22}$$

Considering the changes of variables given in (4.20), the condition (3.21), which ensures (2.17), must be accordingly adapted. Applying the congruence transformation matrix

$$\tilde{\Omega}_3 = G(\eta(t_n)), \tag{4.23}$$

in a similar way to what is performed in (4.21), one can show that

$$\tilde{\Omega}_3^T(3.21)\tilde{\Omega}_3 \iff P(\rho) \succ 0. \tag{4.24}$$

Condition (4.24) ensures that (2.17) is satisfied for all $t \in [t_n,\ t_{n+1})$, which completes the proof. □

4.1.2 Stabilization

In the absence of disturbances $w(t)$, the problem of stabilization of closed-loop LPV systems (3.6) is addressed similarly to what is performed in Theorem 3.2. In this section, LMI conditions ensuring asymptotic stability for (3.6) are derived with the aid of Finsler's lemma, as presented in the following theorem.

Theorem 4.2. *The origin is an asymptotically stable equilibrium point of system (3.6) with aperiodic samplings lower than T_m if, given a scalar $T_m > 0$ and a set of real scalars α_1, α_2, β_1, β_2, $\delta_1 > 0$, δ_2, ζ_1, ζ_2, ϕ_2, θ_2, λ_2, $\omega_2 > 0$, there exist symmetric positive-definite matrices $P(\rho), E_1(\eta(t_n)), E_3(\eta(t_n)),\ F(\eta(t_n)) \in \mathbb{R}^{n_x \times n_x}$, a symmetric matrix $X_1(\rho) \in \mathbb{R}^{n_x \times n_x}$, matrices $G(\eta(t_n)), E_2(\eta(t_n)),\ X_2(\rho) \in \mathbb{R}^{n_x \times n_x}$, $Z(\eta(t_n)) \in \mathbb{R}^{n_u \times n_x}$, $N_{ij}(\rho) \in \mathbb{R}^{n_x \times n_x}$, for $i = 1, 2,\ j = 1, \ldots, 6$, satisfying the following LMIs:*

$$\begin{bmatrix} \Psi_{11} & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{bmatrix} \prec 0 \tag{4.25}$$

$$\begin{bmatrix} \Upsilon_{11} & * & * & * & * & * & * & * \\ \Upsilon_{21} & \Upsilon_{22} & * & * & * & * & * & * \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} & * & * & * & * & * \\ \Upsilon_{41} & \Upsilon_{42} & \Upsilon_{43} & \Upsilon_{44} & * & * & * & * \\ \Upsilon_{51} & \Upsilon_{52} & \Upsilon_{53} & \Upsilon_{54} & \Upsilon_{55} & * & * & * \\ \Upsilon_{61} & \Upsilon_{62} & \Upsilon_{63} & \Upsilon_{64} & \Upsilon_{65} & \Upsilon_{66} & * & * \\ \Upsilon_{71} & \Upsilon_{72} & \Upsilon_{73} & \Upsilon_{74} & \Upsilon_{75} & \Upsilon_{76} & \Upsilon_{77} & * \\ \Upsilon_{81} & \Upsilon_{82} & \Upsilon_{83} & \Upsilon_{84} & \Upsilon_{85} & \Upsilon_{86} & \Upsilon_{78} & \Upsilon_{88} \end{bmatrix} \prec 0 \tag{4.26}$$

for all $\rho \in \Theta$, in which the matrix blocks in (4.25) and in (4.26) are identical to the matrix blocks defined in (4.2) and in (4.3), with a gain-scheduling sampled-data state-feedback controller given by $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$.

The proof of Theorem 4.2 follows the same lines as Theorem 4.1 and is omitted.

4.2 Simplified approach

As mentioned in Section 4.1, a different upper bound to (3.30) can be proposed with the introduction of less matrices. One advantage of such procedure is reducing the number of scalar variables on the proposed design conditions, when compared to Theorem 4.1. In fact, the denomination *simplified* method is given due to the reduced number of decision matrices, with respect to the number of decision matrices available in the full approach. Such reduction on the number of scalar variables implies that less exhaustive scalar searches have to be carried out when control gains are being synthesized.

4.2.1 Controller synthesis

In the following theorem, a simplified approach to the one presented in Theorem 4.1 is proposed.

Theorem 4.3. *The origin is an asymptotically stable equilibrium point of system (3.6) with aperiodic samplings lower than* T_m *if, given a scalar* $T_m > 0$ *and a set of real scalars* α_1, α_2, β_1, β_2, $\delta_1 > 0$, $\delta_2 > 0$, ζ_1, ζ_2, $\mu_1 > \frac{1}{2}$, $\mu_2 > \frac{1}{2}$, *there exist symmetric positive-definite matrices* $P(\rho), E_1(\eta(t_n)), E_3(\eta(t_n))$, $F(\eta(t_n)) \in \mathbb{R}^{n_x \times n_x}$, *a symmetric matrix* $X_1(\rho) \in \mathbb{R}^{n_x \times n_x}$, *matrices* $G(\eta(t_n)), E_2(\eta(t_n))$, $X_2(\rho) \in \mathbb{R}^{n_x \times n_x}$, $Z(\eta(t_n)) \in \mathbb{R}^{n_u \times n_x}$, *minimizing* γ *subject to the following LMIs:*

$$\begin{bmatrix} A_{11} & * & * & * & * & * \\ A_{21} & A_{22} & * & * & * & * \\ A_{31} & A_{32} & A_{33} & * & * & * \\ A_{41} & A_{42} & A_{43} & A_{44} & * & * \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & * \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \prec 0 \tag{4.27}$$

$$\begin{bmatrix} B_{11} & * & * & * & * & * \\ B_{21} & B_{22} & * & * & * & * \\ B_{31} & B_{32} & B_{33} & * & * & * \\ B_{41} & B_{42} & B_{43} & B_{44} & * & * \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & * \\ B_{61} & B_{62} & B_{63} & B_{64} & B_{65} & B_{66} \end{bmatrix} \prec 0 \tag{4.28}$$

for all $\rho \in \Theta$, *in which*

$$\begin{aligned}
A_{11} =& \dot{P}(\rho) - X_1(\rho) + T_m \dot{X}_1(\rho) - \frac{4}{T_m} E_1(\eta(t_n)) + \alpha_1 \Big(\boldsymbol{A}(\rho) G(\eta(t_n))\Big)^H \\
A_{21} =& -\Big(X_2(\rho) - X_1(\rho)\Big) + T_m \Big(\dot{X}_2(\rho) - \dot{X}_1(\rho)\Big) - \frac{2}{T_m} E_1(\eta(t_n)) - E_2(\eta(t_n)) \\
& + \zeta_1 \boldsymbol{A}(\rho) G(\eta(t_n)) + \alpha_1 \Big(\boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^T \\
A_{22} =& -\Big(X_1(\rho) - X_2^H(\rho)\Big) + T_m \Big(\dot{X}_1(\rho) - \dot{X}_2^H(\rho)\Big) + T_m E_3(\eta(t_n)) - \frac{4}{T_m} E_1(\eta(t_n)) \\
& + E_2^H(\eta(t_n)) + \zeta_1 \Big(\boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^H \\
A_{31} =& T_m F(\eta(t_n)) + \frac{6}{T_m} E_1(\eta(t_n)) + \beta_1 \boldsymbol{A}(\rho) G(\eta(t_n)) \\
A_{32} =& -T_m F(\eta(t_n)) + \frac{6}{T_m} E_1(\eta(t_n)) + \beta_1 \boldsymbol{B}_2(\rho) Z(\eta(t_n)) \\
A_{33} =& -T_m F(\eta(t_n)) - \frac{12}{T_m} E_1(\eta(t_n)) \\
A_{41} =& P(\rho) + T_m X_1(\rho) - \alpha_1 G^T(\eta(t_n)) + \delta_1 \boldsymbol{A}(\rho) G(\eta(t_n))
\end{aligned}$$

$$
\begin{aligned}
A_{42} =& T_m E_2^T(\eta(t_n)) + T_m\Big(X_2^T(\rho) - X_1(\rho)\Big) - \zeta_1 G^T(\eta(t_n)) + \delta_1 \boldsymbol{B}_2(\rho) Z(\eta(t_n))\\
A_{43} =& -\beta_1 G^T(\eta(t_n)),\ A_{44} = \ T_m E_1(\eta(t_n)) - \delta_1 G^H(\eta(t_n))\\
A_{51} =& \mu_1 \boldsymbol{C}(\rho) G(\eta(t_n)),\ A_{52} = \mu_1 \boldsymbol{D}_2(\rho) Z(\eta(t_n)),\ A_{53} = A_{54} = \boldsymbol{0}_{n_y \times n_x}\\
A_{55} =& (1-2\mu_1)\boldsymbol{I}_{n_y}\\
A_{61} =& \alpha_1 \boldsymbol{B}_1^T(\rho),\ A_{62} = \boldsymbol{B}_1^T(\rho),\ A_{63} = \beta_1 \boldsymbol{B}_1^T(\rho),\ A_{64} = \delta_1 \boldsymbol{B}_1^T(\rho)\\
A_{65} =& \mu_1 \boldsymbol{D}_1^T(\rho),\ A_{66} = -\gamma^2 \boldsymbol{I}_{n_w}
\end{aligned}
$$

$$
\begin{aligned}
B_{11} =& \dot{P}(\rho) - X_1(\rho) - \frac{4}{T_m} E_1(\eta(t_n)) + \alpha_2\Big(\boldsymbol{A}(\rho) G(\eta(t_n))\Big)^H\\
B_{21} =& -\Big(X_2(\rho) - X_1(\rho)\Big) - \frac{2}{T_m} E_1(\eta(t_n)) - E_2(\eta(t_n)) + \zeta_2 \boldsymbol{A}(\rho) G(\eta(t_n))\\
&+\alpha_2\Big(\boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^T\\
B_{22} =& -\Big(X_1(\rho) - X_2^H(\rho)\Big) - \frac{4}{T_m} E_1(\eta(t_n)) + E_2^H(\eta(t_n)) - T_m E_3(\eta(t_n))\\
&+\zeta_2\Big(\boldsymbol{B}_2(\rho) Z(\eta(t_n))\Big)^H\\
B_{31} =& \frac{6}{T_m} E_1(\eta(t_n)) + \beta_2 \boldsymbol{A}(\rho) G(\eta(t_n))\\
B_{32} =& \frac{6}{T_m} E_1(\eta(t_n)) + \beta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n))\\
B_{33} =& -T_m F(\eta(t_n)) - \frac{12}{T_m} E_1(\eta(t_n))\\
B_{41} =& P(\rho) + \delta_2 \boldsymbol{A}(\rho) G(\eta(t_n)) - \alpha_2 G^T(\eta(t_n))\\
B_{42} =& \delta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n)) - G^T(\eta(t_n)),\ B_{43} = \ -\beta_2 G^T(\eta(t_n)),\ B_{44} = -\delta_2 G^H(\eta(t_n))\\
B_{51} =& \mu_2 \boldsymbol{C}(\rho) G(\eta(t_n)),\ B_{52} = \mu_2 \boldsymbol{D}_2(\rho) Z(\eta(t_n)),\ B_{53} = B_{54} = \boldsymbol{0}_{n_y \times n_x}\\
B_{55} =& (1-2\mu_2)\boldsymbol{I}_{n_y}\\
B_{61} =& \alpha_2 \boldsymbol{B}_1^T(\rho),\ B_{62} = \boldsymbol{B}_1^T(\rho),\ B_{63} = \beta_2 \boldsymbol{B}_1^T(\rho)\\
B_{64} =& \delta_2 \boldsymbol{B}_1^T(\rho),\ B_{65} = \mu_2 \boldsymbol{D}_1^T(\rho),\ B_{66} = -\gamma^2 \boldsymbol{I}_{n_w}
\end{aligned}
$$

with a gain-scheduled sampled-data control gain given by $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$*, and with* γ *as an upper bound to the* $\boldsymbol{\mathcal{L}}_2$*-gain of the closed-loop system.*

Proof of Theorem 4.3. Retake the time-dependent Lyapunov function (3.12). The positiveness of the adopted Lyapunov function (3.12) is guaranteed if the condition (3.21) holds. This is the same condition used in the full approach before adopting any change of variables, available in Theorem 4.1.

An alternative upper bound for (4.1) can be computed as follows. Firstly, (4.1) is expanded, taking into account matrices Ω_1^* and Ω_2^*, which are given in (3.31), and $R = \begin{bmatrix} \overline{E}_1(\eta(t_n)) & \overline{E}_2^T(\eta(t_n)) \\ \overline{E}_2(\eta(t_n)) & \overline{E}_3(\eta(t_n)) \end{bmatrix}$:

$$
\begin{aligned}
&-\frac{1}{\tau(t)}\xi^T(t)\Big\{(\Omega_1^*)^T R(\Omega_1^*) + 3(\Omega_2^*)^T R(\Omega_2^*)\Big\}\xi(t) = \\
&-\frac{1}{\tau(t)}\xi^T(t)\begin{bmatrix} 4\overline{E}_1(\eta(t_n)) & 2\overline{E}_1(\eta(t_n)) + \tau(t)\overline{E}_2^T(\eta(t_n)) & -6\overline{E}_1(\eta(t_n)) \\ * & 4\overline{E}_1(\eta(t_n)) - \tau(t)\overline{E}_2^H(\eta(t_n)) + \tau^2(t)\overline{E}_3(\eta(t_n)) & -6\overline{E}_1(\eta(t_n)) \\ * & * & 12\overline{E}_1(\eta(t_n)) \end{bmatrix}\xi(t) = \\
&-\xi^T(t)\begin{bmatrix} \frac{4}{\tau(t)}\overline{E}_1(\eta(t_n)) & \frac{2}{\tau(t)}\overline{E}_1(\eta(t_n)) + \overline{E}_2^T(\eta(t_n)) & -\frac{6}{\tau(t)}\overline{E}_1(\eta(t_n)) \\ * & \frac{4}{\tau(t)}\overline{E}_1(\eta(t_n)) - \overline{E}_2^H(\eta(t_n)) + \tau(t)\overline{E}_3(\eta(t_n)) & -\frac{6}{\tau(t)}\overline{E}_1(\eta(t_n)) \\ * & * & \frac{12}{\tau(t)}\overline{E}_1(\eta(t_n)) \end{bmatrix}\xi(t)
\end{aligned}
\tag{4.29}
$$

Secondly, using the relation $\tau(t) = t - t_n \leq T_m$ on the terms of (4.29) which are function of $\frac{1}{\tau(t)}$, (4.29) can be rewritten as

$$
-\frac{1}{\tau(t)}\xi^T(t)\Big\{(\Omega_1^*)^T R(\Omega_1^*) + 3(\Omega_2^*)^T R(\Omega_2^*)\Big\}\xi(t) \leq \xi^T(t)(M_1^N + \tau(t)M_t^N)\xi(t), \tag{4.30}
$$

with

$$
\begin{aligned}
M_1^N &= \begin{bmatrix} -\frac{4}{T_m}\overline{E}_1(\eta(t_n)) & -\frac{2}{T_m}\overline{E}_1(\eta(t_n)) - \overline{E}_2^T(\eta(t_n)) & \frac{6}{T_m}\overline{E}_1(\eta(t_n)) \\ * & -\frac{4}{T_m}\overline{E}_1(\eta(t_n)) + \overline{E}_2^H(\eta(t_n)) & \frac{6}{T_m}\overline{E}_1(\eta(t_n)) \\ * & * & -\frac{12}{T_m}\overline{E}_1(\eta(t_n)) \end{bmatrix} \\
M_t^N &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -\overline{E}_3(\eta(t_n)) & \mathbf{0} \\ * & * & \mathbf{0} \end{bmatrix}
\end{aligned}
\tag{4.31}
$$

Therefore, a possible upper bound for (3.30) can be given by

$$
-\int_{t_n}^{t}\begin{bmatrix}\dot{x}(q) \\ x(t_n)\end{bmatrix}^T R \begin{bmatrix}\dot{x}(q) \\ x(t_n)\end{bmatrix} dq \leq \xi^T(t)(M_1^N + \tau(t)M_t^N)\xi(t) \tag{4.32}
$$

Replacing (3.29), (4.32), and (3.22) in (2.18), the resulting terms can be then grouped based on their dependence on time:

$$
\begin{aligned}
\dot{W}(x,t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) \leq &\Big\{\pi_1^S + y^T(t)y(t) - \gamma^2 w^T(t)w(t)\Big\} \\
&+ (t_{n+1} - t)\Big\{\pi_2^S\Big\} + (t - t_n)\Big\{\pi_3^S\Big\} < 0,
\end{aligned}
\tag{4.33}
$$

with

$$
\begin{aligned}
\pi_1^S &= x^T(t)\dot{\overline{P}}(\rho)x(t) + \big(\dot{x}^T(t)\overline{P}(\rho)x(t)\big)^H \\
&\quad - \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix}^T \begin{bmatrix} \overline{X}_1(\rho) & \overline{X}_2^T(\rho) - \overline{X}_1(\rho) \\ \overline{X}_2(\rho) - \overline{X}_1(\rho) & \overline{X}_1(\rho) - \overline{X}_2^H(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_n) \end{bmatrix} + \xi^T(t)M_1^N\xi(t), \\
\pi_2^S &= \pi_2, \\
\pi_3^S &= -\nu^T(t)\overline{F}(\eta(t_n))\nu(t) + \xi^T(t)M_t^N\xi(t),
\end{aligned}
$$

and π_2 as defined in (3.38). As previously performed, (4.33) is affine with respect to t, hence it is sufficient to ensure that (2.18) holds for both $t = t_n$ and $t = t_{n+1}$. Adopting $t = t_n$ and since $t_{n+1} - t_n \leq T_m$, the feasibility of (4.33) is guaranteed if

$$\left\{\pi_1^S + y^T(t)y(t) - \gamma^2 w^T(t)w(t)\right\} + T_m\left\{\pi_2^S\right\} < 0 \tag{4.34}$$

is satisfied.

Defining

$$\mathcal{Q}_1^S = \begin{bmatrix} Q_{11}^1 & * & * & * & * & * \\ Q_{21}^1 & Q_{22}^1 & * & * & * & * \\ Q_{31}^1 & Q_{32}^1 & Q_{33}^1 & * & * & * \\ Q_{41}^1 & Q_{42}^1 & \mathbf{0} & Q_{44}^1 & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_y} & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2\mathbf{I}_{n_w} \end{bmatrix}, \tag{4.35}$$

in which

$$\begin{aligned}
Q_{11}^1 &= \dot{\overline{P}}(\rho) - \overline{X}_1(\rho) + T_m\dot{\overline{X}}_1(\rho) - \frac{4}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{21}^1 &= -\left(\overline{X}_2(\rho) - \overline{X}_1(\rho)\right) + T_m\left(\dot{\overline{X}}_2(\rho) - \dot{\overline{X}}_1(\rho)\right) - \frac{2}{T_m}\overline{E}_1(\eta(t_n)) - \overline{E}_2(\eta(t_n)) \\
Q_{22}^1 &= -\left(\overline{X}_1(\rho) - \overline{X}_2^H(\rho)\right) + T_m\left(\dot{\overline{X}}_1(\rho) - \dot{\overline{X}}_2^H(\rho)\right) + T_m\overline{E}_3(\eta(t_n)) - \frac{4}{T_m}\overline{E}_1(\eta(t_n)) + \overline{E}_2^H(\eta(t_n)) \\
Q_{31}^1 &= T_m\overline{F}(\eta(t_n)) + \frac{6}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{32}^1 &= -T_m\overline{F}(\eta(t_n)) + \frac{6}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{33}^1 &= -T_m\overline{F}(\eta(t_n)) - \frac{12}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{41}^1 &= \overline{P}(\rho) + T_m\overline{X}_1(\rho) \\
Q_{42}^1 &= T_m\overline{E}_2^T(\eta(t_n)) + T_m\left(\overline{X}_2^T(\rho) - \overline{X}_1(\rho)\right) \\
Q_{44}^1 &= T_m\overline{E}_1(\eta(t_n))
\end{aligned}$$

one can notice that

$$(4.34) \iff \overline{\xi}_1^T(t)\mathcal{Q}_1^S\overline{\xi}_1(t) < 0 \;\forall\; \overline{\xi}_1(t) \neq 0 : \mathcal{B}_1\overline{\xi}_1(t) = 0, \tag{4.36}$$

with $\mathcal{B}_1$ and $\overline{\xi}_1(t)$ as defined in (4.8) and (4.6), respectively.

By recovering the implications (i) and (iv) of Lemma 2.2, (4.36) implies that there exists a matrix

$$\mathcal{X}_1 = \begin{bmatrix} G_1^1(\rho) & J_1^1(\rho) \\ G_2^1(\rho) & J_2^1(\rho) \\ G_3^1(\rho) & J_3^1(\rho) \\ G_4^1(\rho) & J_4^1(\rho) \\ G_5^1(\rho) & J_5^1(\rho) \\ G_6^1(\rho) & J_6^1(\rho) \end{bmatrix} \tag{4.37}$$

such that

$$\mathcal{Q}_1^S + \mathcal{X}_1\mathcal{B}_1 + \mathcal{B}_1^T\mathcal{X}_1^T \prec 0. \tag{4.38}$$

Adopting $t = t_{n+1}$ and since $t_{n+1} - t_n \leq T_m$, if

$$\left\{\pi_1^S + y^T(t)y(t) - \gamma^2 w^T(t)w(t)\right\} + T_m\left\{\pi_3^S\right\} < 0 \tag{4.39}$$

holds, then the feasibility of (3.36) is ensured.

Defining

$$\mathcal{Q}_2^S = \begin{bmatrix} Q_{11}^2 & * & * & * & * & * \\ Q_{21}^2 & Q_{22}^2 & * & * & * & * \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^1 & * & * & * \\ Q_{41}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_y} & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2\mathbf{I}_{n_w} \end{bmatrix}, \tag{4.40}$$

in which

$$\begin{aligned}
Q_{11}^2 &= \dot{\overline{P}}(\rho) - \overline{X}_1(\rho) - \frac{4}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{21}^2 &= -\left(\overline{X}_2(\rho) - \overline{X}_1(\rho)\right) - \frac{2}{T_m}\overline{E}_1(\eta(t_n)) - \overline{E}_2(\eta(t_n)) \\
Q_{22}^2 &= -\left(\overline{X}_1(\rho) - \overline{X}_2^H(\rho)\right) - \frac{4}{T_m}\overline{E}_1(\eta(t_n)) + \overline{E}_2^H(\eta(t_n)) - T_m\overline{E}_3(\eta(t_n)) \\
Q_{31}^2 &= Q_{32}^2 = \frac{6}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{33}^2 &= -T_m\overline{F}(\eta(t_n)) - \frac{12}{T_m}\overline{E}_1(\eta(t_n)) \\
Q_{41}^2 &= \overline{P}(\rho)
\end{aligned}$$

one can notice that

$$(4.39) \iff \overline{\xi}_1^T(t)\mathcal{Q}_2^S\overline{\xi}_1(t) < 0 \; \forall \; \overline{\xi}_1(t) \neq 0 : \mathcal{B}_2\overline{\xi}_1(t) = 0, \tag{4.41}$$

with $\mathcal{B}_2 = \mathcal{B}_1$. Carrying the equivalence of (i) and (iv) of Lemma 2.2, (4.41) suggests the existence of a matrix

$$\mathcal{X}_2 = \begin{bmatrix} G_1^2(\rho) & J_1^2(\rho) \\ G_2^2(\rho) & J_2^2(\rho) \\ G_3^2(\rho) & J_3^2(\rho) \\ G_4^2(\rho) & J_4^2(\rho) \\ G_5^2(\rho) & J_5^2(\rho) \\ G_6^2(\rho) & J_6^2(\rho) \end{bmatrix} \tag{4.42}$$

such that

$$\mathcal{Q}_2^S + \mathcal{X}_2\mathcal{B}_2 + \mathcal{B}_2^T\mathcal{X}_2^T \prec 0. \tag{4.43}$$

The feasibility of both conditions (4.38) and (4.43) implies that condition (2.18) holds for all $t \in [t_n,\ t_{n+1})$. Nonetheless, the expansion of (4.38) and (4.43) yields the product of multiple decision variables. The resulting problem is therefore nonconvex. A similar linearization procedure to the one presented in the proof of Theorem 4.1 is adopted.

The linearization of the product of variables in (4.38) and (4.43) is a three-step process. The first step consists of applying congruence transformations $\tilde{\Omega}_1^T$ and $\tilde{\Omega}_1$ to the left and to the right of (4.38), and of pre- and post-multiplying (4.43) with $\tilde{\Omega}_2^T$ and $\tilde{\Omega}_2$, respectively, in which

$$\tilde{\Omega}_1 = \tilde{\Omega}_2 = diag(\mathbf{I}_4 \otimes G(\eta(t_n)),\ \mathbf{I}_{n_y},\ \mathbf{I}_{n_w}). \tag{4.44}$$

The second step is to choose

$$\begin{gathered} G_1^i(\rho) = \alpha_i G^{-T}(\eta(t_n)),\ G_2^i(\rho) = \zeta_i G^{-T}(\eta(t_n)), \\ G_3^i(\rho) = \beta_i G^{-T}(\eta(t_n)),\ G_4^i(\rho) = \delta_i G^{-T}(\eta(t_n)), \\ J_5^i(\rho) = \mu_i \mathbf{I}_{n_y}, \end{gathered} \tag{4.45}$$

for $i = 1, 2$. For linearization purposes, $\tilde{\Omega}_1$ is chosen to be equal to $\tilde{\Omega}_2$, and any other block variable in (4.37) and in (4.42) is chosen to be a null matrix. As a third step, adopting the following changes of variables

$$\begin{gathered} P(\rho) = G^T(\eta(t_n))\overline{P}(\rho)G(\eta(t_n)),\ E_1(\eta(t_n)) = G^T(\eta(t_n))\overline{E}_1(\eta(t_n))G(\eta(t_n)), \\ E_2(\eta(t_n)) = G^T(\eta(t_n))\overline{E}_2(\eta(t_n))G(\eta(t_n)),\ E_3(\eta(t_n)) = G^T(\eta(t_n))\overline{E}_3(\eta(t_n))G(\eta(t_n)), \\ X_1(\rho) = G^T(\eta(t_n))\overline{X}_1(\rho)G(\eta(t_n)),\ X_2(\rho) = G^T(\eta(t_n))\overline{X}_2(\rho)G(\eta(t_n)), \\ F(\eta(t_n)) = G^T(\eta(t_n))\overline{F}(\eta(t_n))G(\eta(t_n)), \\ Z(\eta(t_n)) = K(\eta(t_n))G(\eta(t_n)), \end{gathered} \tag{4.46}$$

one can verify that the resulting relations are linear and that

$$\begin{gathered} \tilde{\Omega}_1^T(4.38)\tilde{\Omega}_1 \iff (4.27), \\ \tilde{\Omega}_2^T(4.43)\tilde{\Omega}_2 \iff (4.28). \end{gathered} \tag{4.47}$$

Thus, (4.27) and (4.28) provide sufficient conditions for satisfying (2.18), for all $t \in [t_n,\ t_{n+1})$, with a control gain $K(\eta(t_n))$ as in (4.22).

With the changes of variables performed in (4.46), the condition (3.21), which ensures (2.17), must be accordingly adapted. The application of the congruence transformation matrix

$$\tilde{\Omega}_3 = G(\eta(t_n)), \tag{4.48}$$

in an equivalent way to what is made in (4.47), yields

$$\tilde{\Omega}_3^T(3.21)\tilde{\Omega}_3 \iff P(\rho) \succ 0. \tag{4.49}$$

Condition (4.49) ensures that (2.17) is satisfied for all $t \in [t_n,\ t_{n+1})$, which completes the proof. □

4.2.2 Stabilization

Similar to the stabilization approach proposed in Theorem 4.2, in this section stabilization conditions based on the procedure reported in Theorem 4.3 are developed.

Theorem 4.4. *Given a scalar $T_m > 0$ and a set of real scalars α_1, α_2, β_1, β_2, $\delta_1 > 0$, $\delta_2 > 0$, ζ_1, ζ_2, if there exist symmetric positive-definite matrices $P(\rho), E_1(\eta(t_n)), E_3(\eta(t_n))$, $F(\eta(t_n)) \in \mathbb{R}^{n_x \times n_x}$, a symmetric matrix $X_1(\rho) \in \mathbb{R}^{n_x \times n_x}$, matrices $G(\eta(t_n)), E_2(\eta(t_n))$, $X_2(\rho) \in \mathbb{R}^{n_x \times n_x}$, $Z(\eta(t_n)) \in \mathbb{R}^{n_u \times n_x}$, satisfying the following LMIs:*

$$\begin{bmatrix} A_{11} & * & * & * \\ A_{21} & A_{22} & * & * \\ A_{31} & A_{32} & A_{33} & * \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \prec 0 \tag{4.50}$$

$$\begin{bmatrix} B_{11} & * & * & * \\ B_{21} & B_{22} & * & * \\ B_{31} & B_{32} & B_{33} & * \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} \prec 0 \tag{4.51}$$

for all $\rho \in \Theta$, in which

$$\begin{aligned}
A_{11} =& \dot{P}(\rho) - X_1(\rho) + T_m\dot{X}_1(\rho) - \frac{4}{T_m}E_1(\eta(t_n)) + \alpha_1\Big(\boldsymbol{A}(\rho)G(\eta(t_n))\Big)^H \\
A_{21} =& -\Big(X_2(\rho) - X_1(\rho)\Big) + T_m\Big(\dot{X}_2(\rho) - \dot{X}_1(\rho)\Big) - \frac{2}{T_m}E_1(\eta(t_n)) - E_2(\eta(t_n)) \\
&+\zeta_1\boldsymbol{A}(\rho)G(\eta(t_n)) + \alpha_1\Big(\boldsymbol{B}_2(\rho)Z(\eta(t_n))\Big)^T \\
A_{22} =& -\Big(X_1(\rho) - X_2^H(\rho)\Big) + T_m\Big(\dot{X}_1(\rho) - \dot{X}_2^H(\rho)\Big) + T_mE_3(\eta(t_n)) - \frac{4}{T_m}E_1(\eta(t_n)) \\
&+E_2^H(\eta(t_n)) + \zeta_1\Big(\boldsymbol{B}_2(\rho)Z(\eta(t_n))\Big)^H \\
A_{31} =& T_mF(\eta(t_n)) + \frac{6}{T_m}E_1(\eta(t_n)) + \beta_1\boldsymbol{A}(\rho)G(\eta(t_n)) \\
A_{32} =& -T_mF(\eta(t_n)) + \frac{6}{T_m}E_1(\eta(t_n)) + \beta_1\boldsymbol{B}_2(\rho)Z(\eta(t_n)) \\
A_{33} =& -T_mF(\eta(t_n)) - \frac{12}{T_m}E_1(\eta(t_n)) \\
A_{41} =& P(\rho) + T_mX_1(\rho) - \alpha_1G^T(\eta(t_n)) + \delta_1\boldsymbol{A}(\rho)G(\eta(t_n)) \\
A_{42} =& T_mE_2^T(\eta(t_n)) + T_m\Big(X_2^T(\rho) - X_1(\rho)\Big) - \zeta_1G^T(\eta(t_n)) + \delta_1\boldsymbol{B}_2(\rho)Z(\eta(t_n)) \\
A_{43} =& -\beta_1G^T(\eta(t_n)),\ A_{44} = \ T_mE_1(\eta(t_n)) - \delta_1G^H(\eta(t_n))
\end{aligned}$$

$$\begin{aligned}
B_{11} =& \dot{P}(\rho) - X_1(\rho) - \frac{4}{T_m}E_1(\eta(t_n)) + \alpha_2\Big(\boldsymbol{A}(\rho)G(\eta(t_n))\Big)^H \\
B_{21} =& -\Big(X_2(\rho) - X_1(\rho)\Big) - \frac{2}{T_m}E_1(\eta(t_n)) - E_2(\eta(t_n)) + \zeta_2\boldsymbol{A}(\rho)G(\eta(t_n)) \\
&+\alpha_2\Big(\boldsymbol{B}_2(\rho)Z(\eta(t_n))\Big)^T \\
B_{22} =& -\Big(X_1(\rho) - X_2^H(\rho)\Big) - \frac{4}{T_m}E_1(\eta(t_n)) + E_2^H(\eta(t_n)) - T_mE_3(\eta(t_n)) \\
&+\zeta_2\Big(\boldsymbol{B}_2(\rho)Z(\eta(t_n))\Big)^H\Big)^H
\end{aligned}$$

$$
\begin{aligned}
B_{31} =& \frac{6}{T_m} E_1(\eta(t_n)) + \beta_2 \boldsymbol{A}(\rho) G(\eta(t_n)) \\
B_{32} =& \frac{6}{T_m} E_1(\eta(t_n)) + \beta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n)) \\
B_{33} =& - T_m F(\eta(t_n)) - \frac{12}{T_m} E_1(\eta(t_n)) \\
B_{41} =& P(\rho) + \delta_2 \boldsymbol{A}(\rho) G(\eta(t_n)) - \alpha_2 G^T(\eta(t_n)) \\
B_{42} =& \delta_2 \boldsymbol{B}_2(\rho) Z(\eta(t_n)) - G^T(\eta(t_n)) \\
B_{43} =& - \beta_2 G^T(\eta(t_n)), \ B_{44} = -\delta_2 G^H(\eta(t_n))
\end{aligned}
$$

then the origin is an asymptotically stable equilibrium point of system (3.6) with aperiodic samplings lower than T_m, provided a gain-scheduling sampled-data controller given by $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$.

The proof of Theorem 4.4 comes straight from the LMI conditions derived in Theorem 4.3. If the two last rows and columns from (4.27) and (4.28) are removed, the $\mathcal{L}_2$-gain cost is disregarded and conditions (4.50) and (4.51) arise.

4.3 Extension to quasi-LPV systems

During the development of stabilizing LMI conditions for LPV systems (3.6), terms with dependence on $\dot{\rho}(t)$ are derived (see (3.23) and (3.25)). If the decision matrices $\overline{P}(\rho)$, $\overline{X}_1(\rho)$ or $\overline{X}_2(\rho)$ are functions only of the sampled-data component $\eta(t_n)$ of the expanded parameter vector $\rho(t)$, the conditions of Theorems 4.1–4.4 can then be applied for synthesizing gain-scheduling state-feedback controllers for quasi-LPV systems as well.

On the contrary, if matrices $\overline{P}(\rho)$, $\overline{X}_1(\rho)$ or $\overline{X}_2(\rho)$ depend also on the continuous-time component $\delta(t)$ of $\rho(t)$, additional restrictions should be imposed for quasi-LPV systems (3.6). As discussed in the work of Palmeira *et al.* (2021), the purposes of these additional conditions are to ensure that the system trajectories $x(t)$ remain inside a specified domain $\mathcal{D} \subseteq \mathbb{R}^{n_x}$, for given initial conditions $x_0 = x(0)$ also inside the domain, and to ensure that the bounds of the time-derivative of the scheduling parameters $\eta(t)$, which become dependent of endogenous signals (system's states, for instance), are respected. By enclosing the variation of the states $x(t)$, the time-derivative of the scheduling parameters $\eta(t)$ can be bounded.

In light of the input-to-state and input-to-output stability theories (Khalil, 2002), the theoretic bounds of $\dot{\eta}(t)$ can be satisfied if some conditions are met, namely:

- System trajectories $x(t)$ remain inside a domain $\mathcal{D}$, for any given initial conditions $x_0 \in \mathcal{D}$,

- The domain $\mathcal{D}$ can be estimated by means of a domain of attraction $\Omega_c\Big(W(x,t),c\Big) = \{x(t) \in \mathcal{D} \mid W(x,t) < c\}$, inside of which all the trajectories initiating at any $x_0 \in \mathcal{D}$ approach the origin as $t \to \infty$ (El Ghaoui; Scorletti, 1996).

- The intrinsic relation

$$\dot{\eta}_i(t) = \frac{d}{dt}\eta_i(t) = \frac{\partial \eta_i(t)}{\partial x(t)}\frac{d}{dt}x(t) = \frac{\partial \eta_i(t)}{\partial x(t)}\dot{x}(t), \tag{4.52}$$

  for $i = 1, \ldots, N$, is considered to bind the scheduling parameters $\eta(t)$ to the system dynamics $\dot{x}(t)$.

- As discussed by Palmeira *et al.* (2021), the Lyapunov function $W(x,t)$ (see (3.12)) should be such that $V_0(x,t)$ is positive definite in the intersample $t \in (t_n,\ t_{n+1})$.

Extending the conditions proposed in Theorems 4.1–4.4 is beyond the scope of this dissertation. Nonetheless, the discussion above points to suggestions of future works to be developed in the area.

4.4 Computational aspects

Two slack variable-based approaches are developed, in Sections 4.1 and 4.2, for the design of gain-scheduling controllers for (quasi-)LPV systems (3.6). The proposed theorems are composed of LMI-based conditions, under the assumption that some scalar parameters are given.

The choice of proper scalars is a demanding assignment, provided the high number of slack variables suggested in Theorems 4.1–4.4. In an attempt to reduce the search space for the scalars, the first choice for such parameters is made in agreement with the values displayed in Table 1. Unless otherwise stated, the values given in Table 1 for the slack variables are also considered in numerical simulations.

Table 1 – First-choice values for the scalar parameters in Theorems 4.1–4.4.

Method	Scalar parameters
Theorem 4.1	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \delta_2 = \phi_2 = \theta_2 = \lambda_2 = 0$ $\delta_1 = \zeta_1 = \zeta_2 = \mu_1 = \mu_2 = \omega_2 = 1$
Theorem 4.2	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \delta_2 = \phi_2 = \theta_2 = \lambda_2 = 0$ $\delta_1 = \zeta_1 = \zeta_2 = \omega_2 = 1$
Theorem 4.3	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ $\delta_1 = \delta_2 = \zeta_1 = \zeta_2 = \mu_1 = \mu_2 = 1$
Theorem 4.4	$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ $\delta_1 = \delta_2 = \zeta_1 = \zeta_2 = 1$

5 Numerical simulations

In order to demonstrate the effectiveness of the gain-scheduled control strategies developed in this dissertation, some numerical examples, borrowed from the recent literature, are presented.

The numerical solution of the proposed optimization problems, addressed in terms of LMIs, considers the usage of SDP. The computational packages SeDuMi (Sturm, 1999) and Mosek (MOSEK ApS, 2015), for solving convex optimization problems, and YALMIP (Lofberg, 2004) and ROLMIP (Agulhari *et al.*, 2019), for describing the LMI conditions, were exploited. All programming was performed in Matlab.

The calculation of matrices $\dot{P}(\rho)$, $\dot{X}_1(\rho)$, and $\dot{X}_2(\rho)$ was required for the LMIs derived in Theorems 4.1–4.4. The ROLMIP package is used for computing the referred matrices, by providing the variation rate bounds of the expanded parameter vector $\rho(t)$.

It is assumed that the states $x(t)$ are either measured or estimated at least at each sampling instant t_n. Due to the availability of $x(t_n)$, the sampled-data scheduling parameters $\eta(t_n)$ can be computed, allowing the determination of the control law (3.2).

Furthermore, the next sampling instants t_{n+1} are arbitrarily chosen from the range $t_n < t_{n+1} < t_n + T_m$, in which T_m is the maximum allowable sampling period.

For comparison purposes, other techniques from the literature were also implemented under the same framework. As discussed in Appendix B, these techniques can be recast in the structure of the iterative approach (see Chapter 3) or of the full approach (see Chapter 4). Hence, the approaches proposed in this dissertation can be no more conservative than the ones available in the literature.

5.1 Example 1 – LPV system

Consider the LPV system proposed in Gomes da Silva Jr *et al.* (2018), which is composed of

$$\begin{aligned} \mathbf{A}(\eta) &= \begin{bmatrix} 0 & 1 \\ 0.1 & 0.4+0.6\eta(t) \end{bmatrix}, \ \mathbf{B}_1(\eta) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \ \mathbf{B}_2(\eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{C}(\eta) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{D}_1(\eta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{D}_2(\eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{5.1}$$

The scheduling parameter $\eta(t)$ is such that

$$\eta(t) = \sin(\varrho t), \ |\eta(t)| \leq 1, \ |\dot{\eta}(t)| \leq \varrho, \tag{5.2}$$

with ϱ a given bound for the variation rate of $\eta(t)$.

Gain-scheduling sampled-data controllers for the LPV system (5.1) are synthesized, adopting the procedures discussed in Sections 3.5 and 4.4.

In order to assess the key role of the bound ϱ for the variation rate of $\eta(t)$, the stabilization issue of the LPV system (5.1) is addressed first. As summarized in Table 2, the stabilizing LMI conditions developed in this dissertation achieved larger maximum bounds on the aperiodic sampling periods of system (5.1), if compared to the methodology devised in Hooshmandi *et al.* (2018). With reference to Gomes da Silva Jr *et al.* (2018)[1], the simplified approach was not able to provide larger admissible MASPs T_m. Among the proposed stabilizing approaches, the largest maximum admissible sampling period T_m was attained with the full approach, whereas the smallest MASP was obtained from the application of the simplified approach. Moreover, it is shown that the larger the bound ϱ, the smaller the maximum aperiodic sampling time T_m.

Table 2 also presents an analysis of algorithmic complexity for the computational problems formulated in Theorems 3.2, 4.2, and 4.4. The proposed analysis considers the number of LMI lines to be solved as part of an optimization problem, alongside the number of involved decision variables. For the sake of comparison, the analysis of complexity is extended to the stabilization problems described by Hooshmandi *et al.* (2018) and adapted from Gomes da Silva Jr *et al.* (2018). Due to the addition of a new term in the Lyapunov function (3.12) and to the application of Wirtinger's inequality in (3.24), the complexity of the proposed conditions is higher than the complexity of the compared methods.

Table 2 – Maximum bound on the aperiodic sampling period of system (5.1) for different variation rates $|\dot{\eta}(t)|$, number of LMI lines (NoL), and number of decision variables (NoV).

Bounds on $\|\dot{\eta}(t)\|$	0.2	0.6	1	NoL	NoV
Hooshmandi *et al.* (2018)[Th. 4.2]	0.806	0.795	0.781	172–176	242–65
Gomes da Silva Jr *et al.* (2018)	1.362	1.173	0.851	344	96
Theorem 3.2	1.675	1.213	0.908	196–216	376–65
Theorem 4.2	1.775	1.600	1.200	584	270
Theorem 4.4	0.900	0.852	0.790	392	72

The effectiveness of the $\mathcal{L}_2$-gain controller synthesis techniques implemented in this dissertation is evaluated with respect to two scenarios. In the first scenario, an upper bound to the $\mathcal{L}_2$-gain cost is given, on the purpose of enlarging the maximum allowable sampling periods T_m. Secondly, provided a maximum allowable sampling period T_m, the interest lies in obtaining improved (smaller) upper bounds to the $\mathcal{L}_2$-gain. The design of sampled-data gain-scheduling control laws by means of the solution of optimization problems, discussed in Sections 3.5 and 4.4, can be exploited in both scenarios.

[1] The LMI conditions reported in Gomes da Silva Jr *et al.* (2018) were adapted to cope with the problem of stabilization.

First, let the $\mathcal{L}_2$-gain be bounded by $\gamma = 15$. Since (5.1) is an LPV system, the LMI conditions derived in Theorems 3.1, 4.1 and 4.3 apply without loss of generality to the synthesis of gain-scheduled controllers, and can be compared with other results available in the control literature. The attained MASPs are compiled in Table 3. The proposed methodologies, apart from Theorem 4.3, obtained larger maximum aperiodic sampling periods for system (5.1), if compared to the framework implemented in Gomes da Silva Jr *et al.* (2018). Since the proposed conditions in Theorems 3.1 and 4.1 contain the conditions reported in Gomes da Silva Jr *et al.* (2018), they cannot be more conservative and, therefore, one should expect such outcome. Furthermore, the largest MASP was attained with the application of the non-iterative full approach, presented in Theorem 4.1.

Table 3 – Maximum aperiodic sampling periods for system (5.1) with $\gamma = 15$ and $|\dot{\eta}(t)| \leq 0.2$.

Gomes da Silva Jr *et al.* (2018)[Th. 1]	1.349
Theorem 3.1	1.659
Theorem 4.1	1.772
Theorem 4.3	0.873

On the other hand, given a maximum allowable sampling period $T_m = 1.772$ s, and scalar parameters $\alpha_1 = 0.3$, $\alpha_2 = 0$, $\beta_1 = 200$, $\delta_1 = 600$, $\delta_2 = 1.3$, $\omega_2 = 0.8$, the upper bound for the $\mathcal{L}_2$-gain of system (5.1) is computed as $\gamma = 1.6816$ in the framework of Theorem 4.1, and the designed control gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, where

$$\begin{aligned} Z(\eta(t_n)) &= -\begin{bmatrix} 35.3522 & 30.6238 \end{bmatrix} + \begin{bmatrix} -13.3177 & 0.4113 \end{bmatrix} \eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix} 190.6147 & -13.7213 \\ 10.5027 & 38.4071 \end{bmatrix} + \begin{bmatrix} 26.8895 & 1.0793 \\ 5.6136 & -7.9877 \end{bmatrix} \eta(t_n) \end{aligned} \tag{5.3}$$

The LPV system (5.1) is simulated with the control law (5.3), given a zero initial condition, a disturbance input $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 1.772$ s. Figures 1 and 2 outline, respectively, the outputs $y(t)$ response, and the control signal $u(t)$. The sampling periods are shown in Figure 3, where each stem indicates when the sampling occurred and its amplitude represents the time elapsed from the last sampling. The figures portray the stability of system (5.1) in closed-loop with the control law (5.3), even in the presence of external disturbances $w(t)$. The induced $\mathcal{L}_2$-gain norm for the closed-loop system is $\gamma^\star = 0.4362$, which is below the reported upper bound $\gamma = 1.6816$.

In the second scenario, the maximum allowable sampling period is chosen as $T_m = 1.349$ s. Table 4 summarizes the results attained from the optimization problems subject to LMI constraints derived in Theorems 3.1, 4.1 and 4.3. Improved upper bounds for the $\mathcal{L}_2$-gain of system (5.1) were obtained with the full and the iterative approaches. Notice that the controllers synthesized with the simplified approach did not manage to

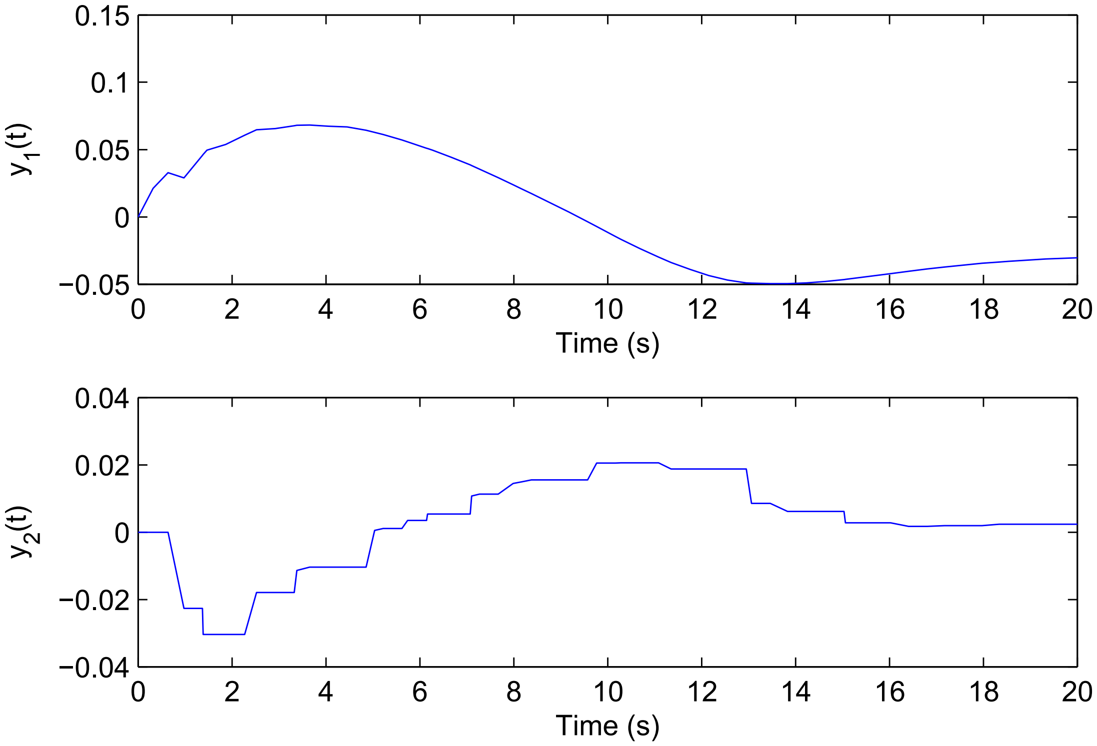

Figure 1 – Outputs response for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 1.772$ s for system (5.1).

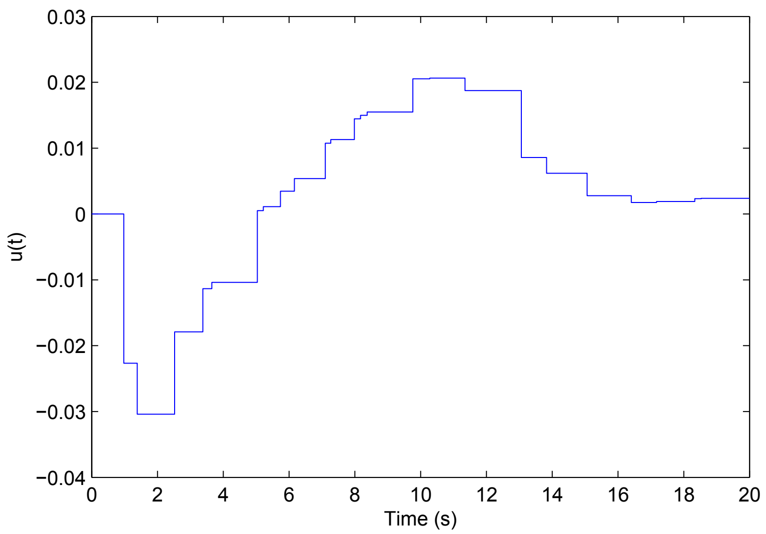

Figure 2 – Control signal for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 1.772$ s for system (5.1).

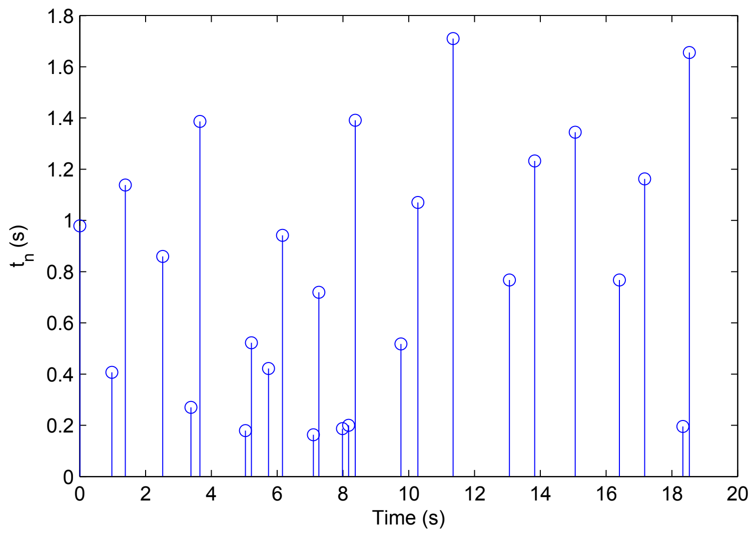

Figure 3 – Aperiodic sampling time with MASP $T_m = 1.772$ s for system (5.1).

stabilize the closed-loop system with $T_m = 1.349$ s. According to Table 3, the MASP ensuring closed-loop stability with the simplified approach is $T_m = 0.873$ s.

Table 4 – Upper bounds for the $\mathcal{L}_2$-gain of system (5.1) for $T_m = 1.349$ s and $|\dot{\eta}(t)| \leq 0.2$.

Gomes da Silva Jr *et al.* (2018)[Th. 1]	6.3007
Theorem 3.1	1.2000
Theorem 4.1	0.4660
Theorem 4.3	Infeasible

In order to compare the performance of the proposed approaches with other works from the literature, a time-domain simulation of the closed-loop system (5.1) is performed, with an initial condition $x(0) = 0$, a disturbance input $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 1.349$ s. The following control laws are applied: for Gomes da Silva Jr *et al.* (2018)[Th. 1], with $\epsilon = 0.3$, the control law $K(\eta(t_n))$ is

$$K(\eta(t_n)) = -\begin{bmatrix} 0.1508 & 0.7422 \end{bmatrix} + \begin{bmatrix} 0.0033 & -0.5522 \end{bmatrix} \eta(t_n). \tag{5.4}$$

In the framework of Theorem 3.1, the designed control gain is $K(\eta(t_n)) = Y(\eta(t_n))Q^{-1}(\eta(t_n))$, with

$$\begin{aligned} Y(\eta(t_n)) &= -\begin{bmatrix} 0.4468 & 0.1418 \end{bmatrix} + \begin{bmatrix} 0.1566 & -0.0598 \end{bmatrix} \eta(t_n) \\ Q(\eta(t_n)) &= \begin{bmatrix} 3.0658 & -0.2907 \\ -0.2907 & 0.2329 \end{bmatrix} + \begin{bmatrix} -0.2768 & 0.0369 \\ 0.0369 & -0.0228 \end{bmatrix} \eta(t_n), \end{aligned} \tag{5.5}$$

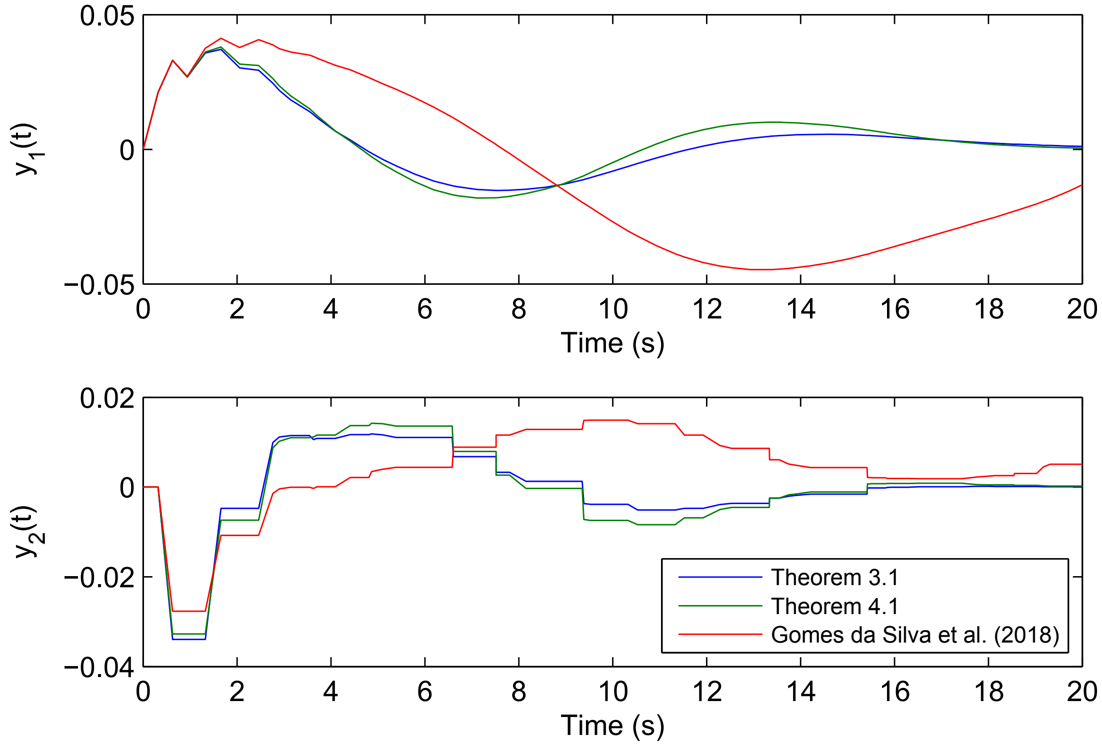

Figure 4 – Comparison of the outputs response for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 1.349$ s for system (5.1).

whereas, for Theorem 4.1, given $\alpha_1 = \alpha_2 = 0.5$ and $\delta_1 = 2.5$, the designed control gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, in which

$$\begin{aligned} Z(\eta(t_n)) &= -\begin{bmatrix} 0.5970 & 0.4254 \end{bmatrix} - \begin{bmatrix} 0.3043 & 0.0228 \end{bmatrix}\eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix} 1.7912 & -0.2028 \\ 0.2285 & 0.5549 \end{bmatrix} + \begin{bmatrix} 0.1427 & -0.0122 \\ 0.1145 & -0.1437 \end{bmatrix}\eta(t_n) \end{aligned} \tag{5.6}$$

Figures 4 and 5 show a comparison of the outputs $y(t)$ response and of the control signal $u(t)$, respectively. The aperiodic sampling instants are depicted in Figure 6. The figures illustrate the closed-loop stability of system (5.1), given gain-scheduling controllers synthesized with the approaches proposed in this dissertation, despite the existing disturbance signals. The induced $\mathcal{L}_2$-gain norms for the closed-loop system are $\gamma^* = 0.1432$ (Theorem 3.1), $\gamma^* = 0.1567$ (Theorem 4.1), and $\gamma^* = 0.3378$ (Gomes da Silva Jr *et al.*, 2018), which are below the respective upper bounds reported in Table 4.

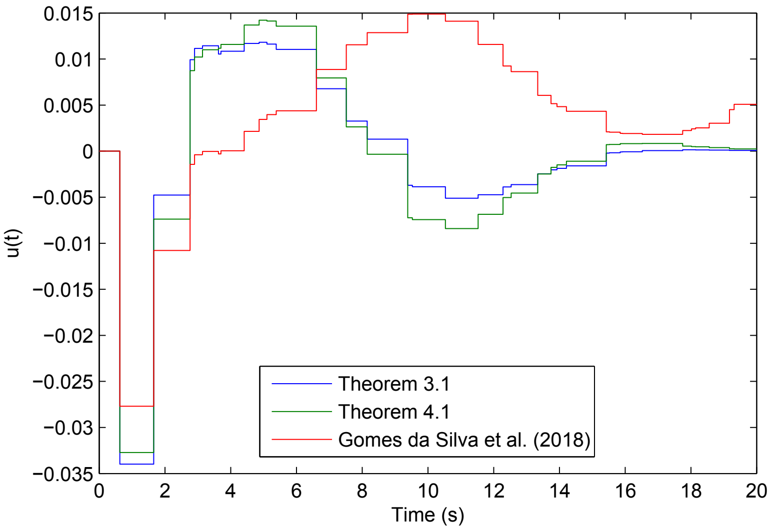

Figure 5 – Comparison of the control signal for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 1.349$ s for system (5.1).

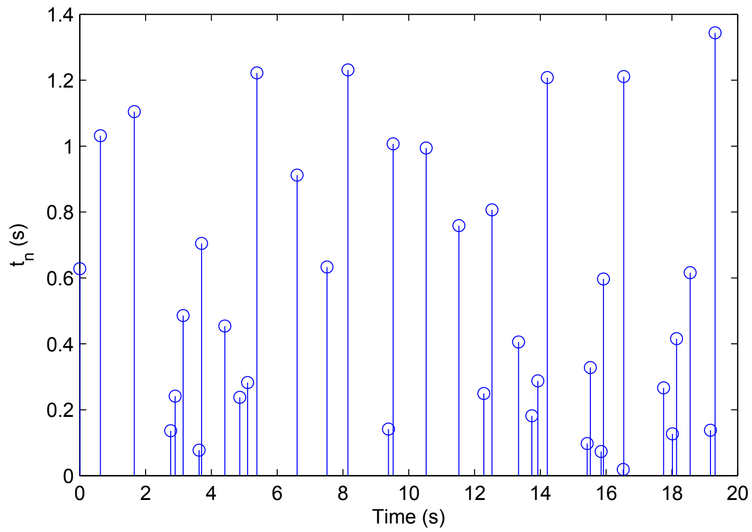

Figure 6 – Aperiodic sampling time with MASP $T_m = 1.349$ s for system (5.1).

5.2 Example 2 – Inverted pendulum on a cart

As discussed in Section 4.3, the conditions proposed in Theorems 4.1–4.4 can be applied to design gain-scheduled controllers for quasi-LPV systems if matrices $\overline{P}(\rho)$, $\overline{X}_1(\rho)$, and $\overline{X}_2(\rho)$ are made independent of the continuous-time parameter $\delta(t)$ in the parameter vector $\rho(t)$. In this regard, this example and the following one explore how the proposed design conditions can be employed in the framework of quasi-LPV systems.

Consider the following quasi-LPV model of an inverted pendulum on a cart, borrowed from Hooshmandi *et al.* (2018):

$$\begin{aligned}\mathbf{A}(\eta) &= \begin{bmatrix} 0 & 1 \\ 12.63 - 4.66\eta(t) & 0 \end{bmatrix},\ \mathbf{B}_1(\eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{B}_2(\eta) = \begin{bmatrix} 0 \\ -0.077 - 0.098\eta(t) \end{bmatrix} \\ \mathbf{C}(\eta) &= \begin{bmatrix} 1 & 0 \end{bmatrix},\ \mathbf{D}_1(\eta) = 0,\ \mathbf{D}_2(\eta) = 0.006 + 0.002\eta(t)\end{aligned} \tag{5.7}$$

where $x_1(t) \in [-\pi/3,\ \pi/3]$ is the angle of the pendulum with respect to the vertical axis, $x_2(t)$ is the angular velocity of the pendulum, and

$$\eta(t) = \left(1 - \frac{1}{1 + \exp(-7[x_1(t) - \pi/4])}\right) \times \left(\frac{1}{1 + \exp(-7[x_1(t) + \pi/4])}\right),\ 0 \leq \eta(t) \leq 1. \tag{5.8}$$

The theoretic bounds of $\dot{\eta}(t)$ are such that $|\dot{\eta}(t)| \leq \varrho$, with ϱ a given scalar.

Gain-scheduled control laws are designed for system (5.7) with the application of the $\mathcal{L}_2$-gain control theorems formulated in this dissertation. The achieved controllers performances are contrasted with the ones obtained from the framework developed in Hooshmandi *et al.* (2018).

Tables 5 and 6 present the computed upper bounds to the $\mathcal{L}_2$-gain of closed-loop system (5.7) for different maximum allowable sampling periods T_m and for different theoretic bounds ϱ for $|\dot{\eta}(t)|$, respectively. Decision matrices of the optimization problems to be solved are assumed to depend only on the sampled-data component $\eta(t_n)$ of the parameter vector $\rho(t)$.

For a given MASP T_m, the attained results show improved performance for any of the proposed approaches, if compared to the methodology implemented in Hooshmandi *et al.* (2018). Among the control strategies developed in this dissertation, the smaller upper bounds to the $\mathcal{L}_2$-gain were obtained with the full approach. Furthermore, the full approach was also able to enlarge the maximum allowable sampling period to values up to $T_m = 0.45$ s, which is three times the MASP obtained in Hooshmandi *et al.* (2018). Notice that the reported upper bounds to the $\mathcal{L}_2$-gain remained steady as the theoretic bounds of $|\dot{\eta}(t)|$ increased. In this scenario, provided that the considered decision matrices do not depend on the continuous-time component of $\eta(t)$, no major variation to the upper bounds of the $\mathcal{L}_2$-gain was expected.

The inverted pendulum system (5.7) is simulated, provided zero initial conditions, a disturbance input $w(t) = e^{-t}\sin(2\pi t)$, $T_m = 0.15$ s and $|\dot{\eta}(t)| \leq 0.1$. The following

Table 5 – Upper bounds for the $\mathcal{L}_2$-gain of system (5.7) for several MASPs T_m and $|\dot{\eta}(t)| \leq 0.1$. Decision matrices of Lyapunov function (3.12) depend only on $\eta(t_n)$.

T_m (s)	Hooshmandi *et al.* (2018)[Th. 4.2]	Th. 3.1	Th. 4.1	Th. 4.3
0.01	0.159	0.052	0.049	0.108
0.05	0.161	0.072	0.083	0.112
0.10	0.166	0.098	0.097	0.121
0.15	0.433	0.108	0.106	0.195
0.20	Infeasible	0.585	0.123	0.382
0.25	Infeasible	Infeasible	0.143	1.102
0.30	Infeasible	Infeasible	0.227	Infeasible
0.35	Infeasible	Infeasible	0.313	Infeasible
0.40	Infeasible	Infeasible	0.373	Infeasible
0.45	Infeasible	Infeasible	0.760	Infeasible

Table 6 – Upper bounds for the $\mathcal{L}_2$-gain of system (5.7) for different variation rates $|\dot{\eta}(t)|$ and $T_m = 0.10$ s. Decision matrices of Lyapunov function (3.12) depend only on $\eta(t_n)$.

Bounds on $\lvert\dot{\eta}(t)\rvert$	0.3	0.5	0.8
Hooshmandi *et al.* (2018)[Th. 4.2]	0.171	0.177	0.193
Theorem 3.1	0.100	0.101	0.104
Theorem 4.1	0.098	0.100	0.103
Theorem 4.3	0.124	0.127	0.132

control laws were designed: for Hooshmandi *et al.* (2018)[Th. 4.2], the control law $K(\eta(t_n))$ is

$$K(\eta(t_n)) = \begin{bmatrix} 480.16 & 120.88 \end{bmatrix} - \begin{bmatrix} 260.27 & 70.27 \end{bmatrix} \eta(t_n). \tag{5.9}$$

In the framework of Theorem 3.1, the obtained control gain is $K(\eta(t_n)) = Y(\eta(t_n))Q^{-1}(\eta(t_n))$, with

$$\begin{aligned} Y(\eta(t_n)) &= \begin{bmatrix} 4.9383 & 563.1864 \end{bmatrix} - \begin{bmatrix} 2.8813 & 129.2475 \end{bmatrix} \eta(t_n) \\ Q(\eta(t_n)) &= \begin{bmatrix} 0.3921 & -1.6866 \\ -1.6866 & 13.9880 \end{bmatrix} + \begin{bmatrix} 0.3501 & -0.8526 \\ -0.8526 & 7.7447 \end{bmatrix} \eta(t_n), \end{aligned} \tag{5.10}$$

whereas, for Theorem 4.1, given $\alpha_1 = 6.9$, $\alpha_2 = 5.5$, $\delta_1 = 0.54$, and $\delta_2 = 0.5$, the designed control gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, in which

$$\begin{aligned} Z(\eta(t_n)) &= \begin{bmatrix} -8.1340 & 5.2187 \end{bmatrix} + \begin{bmatrix} 0.7161 & -2.4828 \end{bmatrix} \eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix} 0.0764 & -0.1377 \\ -0.1389 & 2.6067 \end{bmatrix} + \begin{bmatrix} 0.0049 & -0.0199 \\ -0.0599 & 0.6082 \end{bmatrix} \eta(t_n). \end{aligned} \tag{5.11}$$

As for Theorem 4.3, given $\alpha_1 = \alpha_2 = -0.5$, $\delta_1 = \delta_2 = 0.2625$, the synthesized

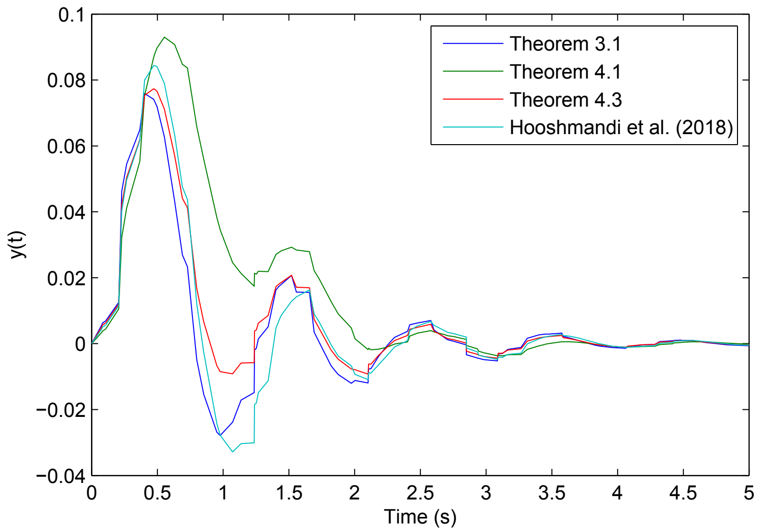

Figure 7 – Comparison of the outputs response for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.15$ s for system (5.7).

control gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, with

$$\begin{aligned} Z(\eta(t_n)) &= \begin{bmatrix} 29.9469 & -32.1000 \end{bmatrix} + \begin{bmatrix} -17.5699 & 52.5012 \end{bmatrix} \eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix} 0.2737 & -0.9873 \\ -0.6560 & 3.2504 \end{bmatrix} + \begin{bmatrix} 0.0719 & -0.0797 \\ 0.0694 & 0.4713 \end{bmatrix} \eta(t_n). \end{aligned} \tag{5.12}$$

Figures 7 and 8 depict the output $y(t)$ response and the control signal $u(t)$, respectively. The aperiodic sampling instants are presented in Figure 9. It is shown that the closed-loop system (5.7) is stable with the synthesized gain-scheduling controllers, even with the presence of a perturbation signal $w(t)$. Moreover, the induced $\mathcal{L}_2$-gain norms for the closed-loop system are $\gamma^* = 0.0948$ (Theorem 3.1), $\gamma^* = 0.1402$ (Theorem 4.1), $\gamma^* = 0.1002$ (Theorem 4.3), and $\gamma^* = 0.1103$ (Hooshmandi *et al.*, 2018), which are below the respective upper bounds reported in Table 5.

5.3 Example 3 – Chaotic Lorenz attractor

The following system describes a quasi-LPV model of the Lorenz system with an input term:

$$\begin{aligned} \dot{x}_1(t) &= -ax_1(t) + ax_2(t) + u(t) \\ \dot{x}_2(t) &= cx_1(t) - x_2(t) - \eta(t)dx_3(t) + w(t) \\ \dot{x}_3(t) &= \eta(t)dx_2(t) - bx_3(t) \\ y_1(t) &= x_1(t),\ \ y_2(t) = x_2(t),\ \ y_3(t) = x_3(t), \end{aligned} \tag{5.13}$$

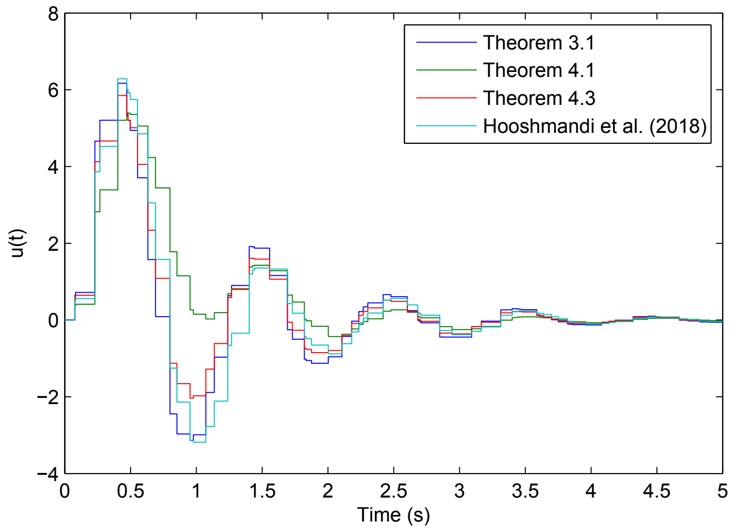

Figure 8 – Comparison of the control signal for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.15$ s for system (5.7).

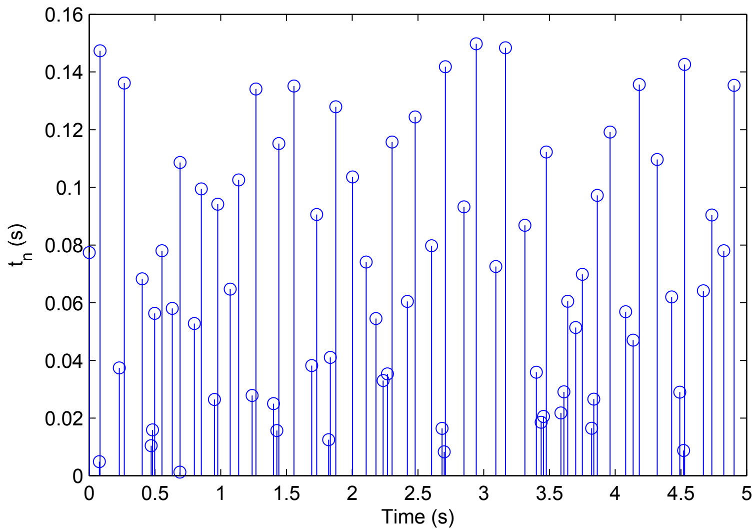

Figure 9 – Aperiodic sampling time with MASP $T_m = 0.15$ s for system (5.7).

where

$$\eta(t) = \frac{x_1(t)}{d},\ \eta(t) \in [-1,\ 1] \tag{5.14}$$

is the scheduling parameter, with $|x_1(t)| \leq d$. The time derivative of the scheduling parameter $\eta(t)$ is theoretically bounded by $|\dot{\eta}(t)| \leq e$, with e a given positive value.

With the purpose of assessing the $\mathcal{L}_2$-gain cost performance for system (5.13), output $y(t)$ and disturbance $w(t)$ vectors are added to the Lorenz equations originally presented in Wu *et al.* (2014). Adopting $x(t) = y(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$, the quasi-LPV system corresponding to the Lorenz equations (5.13) can be described by

$$\begin{aligned} \dot{x}(t) &= \big(\mathbf{A}_0 + \mathbf{A}_1\eta(t)\big)x(t) + \mathbf{B}_1 w(t) + \mathbf{B}_2 u(t) \\ y(t) &= \mathbf{C}x(t), \end{aligned} \tag{5.15}$$

with

$$\mathbf{A}_0 = \begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix},\ \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -d \\ 0 & d & 0 \end{bmatrix},\ \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{C} = \mathbf{I}_3 \tag{5.16}$$

The values chosen for the scalar parameters in the chaotic Lorenz system (5.13) are $a = 10$, $b = \frac{8}{3}$, and $c = 28$ (Wu *et al.*, 2014). It is assumed that the state $x_1(t)$ varies in the range $|x_1(t)| \leq d = 25$. The theoretic upper bound of $|\dot{\eta}(t)|$ is arbitrarily set as $e = 1.2$.

Following the procedures discussed in Sections 3.5 and 4.4, gain-scheduling sampled-data controllers for the Lorenz attractor (5.13) are designed. The obtained results are compared with the methodology developed in Gomes da Silva Jr *et al.* (2018), with respect to the same scenarios considered in Section 5.1.

In the first scenario, for a fixed upper bound to the $\mathcal{L}_2$-gain set to $\gamma = 15$, the maximum allowable sampling period T_m is estimated. The approach of Gomes da Silva Jr *et al.* (2018) can be directly compared with the $\mathcal{L}_2$-gain control theorems presented in this dissertation, if the decision matrices of the optimization problem are assumed to depend only on the sampled-data component $\eta(t_n)$ of the parameter vector $\rho(t)$. The attained MASPs are presented in Table 7. Similarly to the results presented in Section 5.1, the full and the iterative approaches led into MASPs of system (5.13) larger than the one obtained with the framework implemented in Gomes da Silva Jr *et al.* (2018).

Table 7 – Maximum aperiodic sampling periods of system (5.13) for $\gamma = 15$. Decision matrices of Lyapunov function (3.12) depend only on $\eta(t_n)$.

Gomes da Silva Jr *et al.* (2018)[Th. 1]	0.042
Theorem 3.1	0.051
Theorem 4.1	0.070
Theorem 4.3	0.039

Considering the maximum allowable sampling period $T_m = 0.070$ s, and given scalar parameters $\alpha_1 = \alpha_2 = 5$, $\beta_1 = 160.5$, $\delta_1 = 2.5$, the upper bound for the $\mathcal{L}_2$-gain of system (5.13) is computed as $\gamma = 0.7593$, and the synthesized control gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, where

$$\begin{aligned} Z(\eta(t_n)) &= -\begin{bmatrix} 3.1120 & 12.4397 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 6.3001 \end{bmatrix} \eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix} 0.5645 & -0.1847 & 0 \\ -0.4849 & 1.5882 & 0 \\ 0 & 0 & 1.1951 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0.5765 \\ 0 & 0 & -0.0766 \\ -0.0073 & 0.0829 & 0 \end{bmatrix} \eta(t_n) \end{aligned} \tag{5.17}$$

The Lorenz attractor system (5.15) is simulated with the control gain (5.17), provided zero initial conditions, a disturbance input $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.070$ s.

Figures 10, 11 and 12 illustrate the outputs $y(t)$ response, the control signal $u(t)$ and the aperiodic sampling instants, respectively. The figures certify that the closed-loop system remains stable, despite the effect of a disturbance signal. Notice that the considered MASP $T_m = 0.070$ s is 116% larger than the MASP obtained in Wu *et al.* (2014), computed as $T_m = 0.0347$ s. Moreover, it is worth stating the induced $\mathcal{L}_2$-gain norm for the closed-loop system is $\gamma^* = 0.4938$, which is below the attained upper bound of $\gamma = 0.7593$.

In the second scenario, given a maximum allowable sampling period $T_m = 0.0347$ s, the upper bounds to the $\mathcal{L}_2$-gain are assessed. Table 8 presents the obtained results when $\eta(t_n)$-dependent decision matrices are considered in the optimization problem. With the application of the approaches developed in this dissertation, improved $\mathcal{L}_2$-gain performance was obtained, regarding other works available in the control literature.

Table 8 – Upper bounds for the $\mathcal{L}_2$-gain of system (5.13) for $T_m = 0.0347$ s. Decision matrices of Lyapunov function (3.12) depend only on $\eta(t_n)$.

Gomes da Silva Jr *et al.* (2018)[Th. 1]	0.344
Theorem 3.1	0.111
Theorem 4.1	0.096
Theorem 4.3	0.198

Choosing an initial condition $x(0) = 0$, a disturbance input $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.0347$ s, the quasi-LPV representation (5.15) of the Lorenz attractor is simulated with the control law obtained from the second scenario, considering decision matrices relying on the sampled-data parameter $\eta(t_n)$. The following control laws were synthesized: for Gomes da Silva Jr *et al.* (2018)[Th. 1], given $\epsilon = 39.4$, the control gain

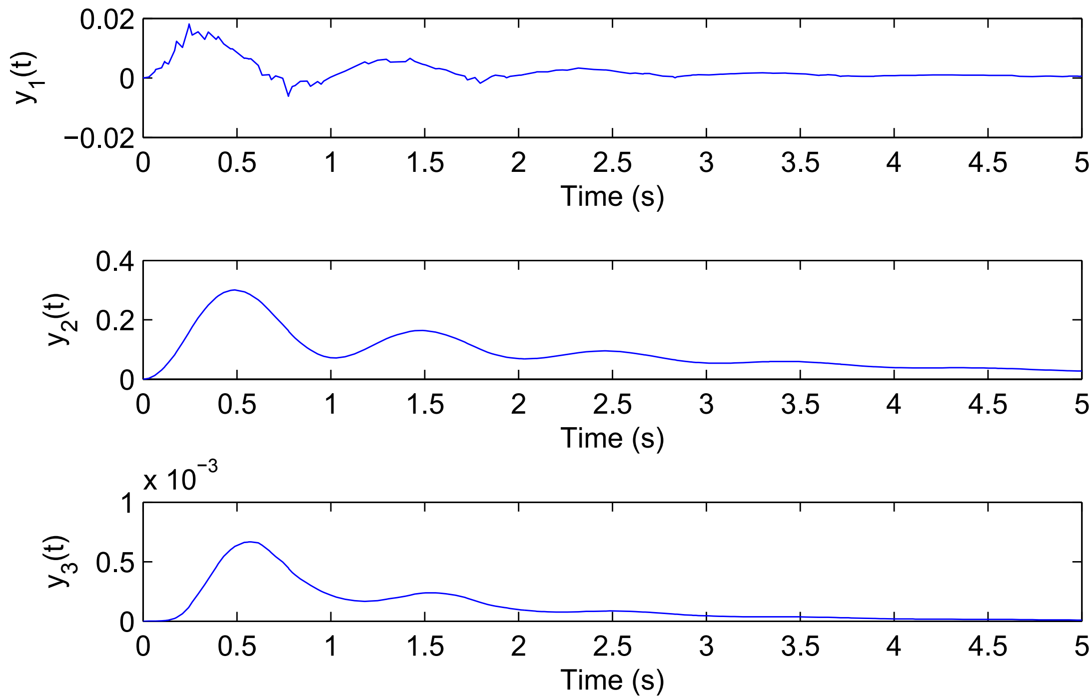

Figure 10 – Outputs response for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.070$ s for system (5.13).

is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, with

$$Z(\eta(t_n)) = -\begin{bmatrix}14.7734 & 3.6928 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 13.8718\end{bmatrix}\eta(t_n)$$
$$G(\eta(t_n)) = \begin{bmatrix}0.8310 & -0.6374 & 0\\ -0.3988 & 1.0633 & 0\\ 0 & 0 & 1.1406\end{bmatrix} + \begin{bmatrix}0 & 0 & 0.1164\\ 0 & 0 & -0.0299\\ -0.0081 & 0.0102 & 0\end{bmatrix}\eta(t_n). \tag{5.18}$$

In the framework of Theorem 3.1, the designed control law is $K(\eta(t_n)) = Y(\eta(t_n))Q^{-1}(\eta(t_n))$, in which

$$Y(\eta(t_n)) = -\begin{bmatrix}14.4457 & 18.1911 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 20.8076\end{bmatrix}\eta(t_n)$$
$$Q(\eta(t_n)) = \begin{bmatrix}0.8446 & -0.5240 & 0\\ -0.5240 & 1.3438 & 0\\ 0 & 0 & 1.2072\end{bmatrix} + \begin{bmatrix}0 & 0 & 0.1672\\ 0 & 0 & -0.0811\\ 0.1672 & -0.0811 & 0\end{bmatrix}\eta(t_n). \tag{5.19}$$

For Theorem 4.1, given $\alpha_1 = \alpha_2 = 0.3$, $\beta_1 = 300$, $\delta_1 = 2.5$, the obtained control

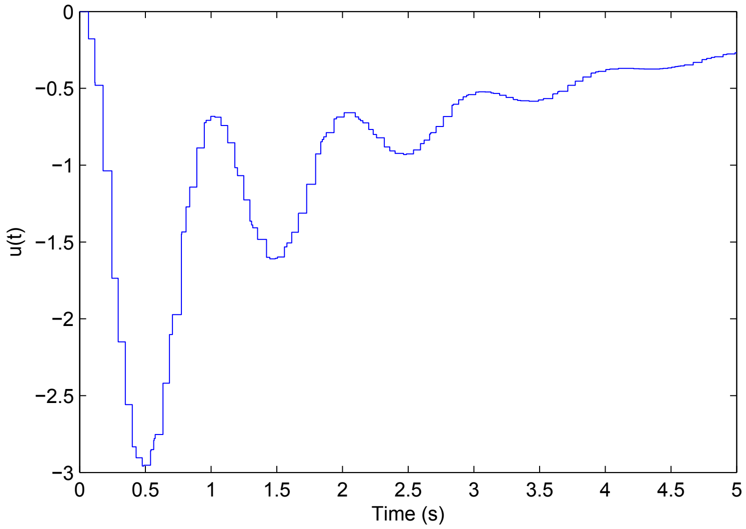

Figure 11 – Control signal for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.070$ s for system (5.13).

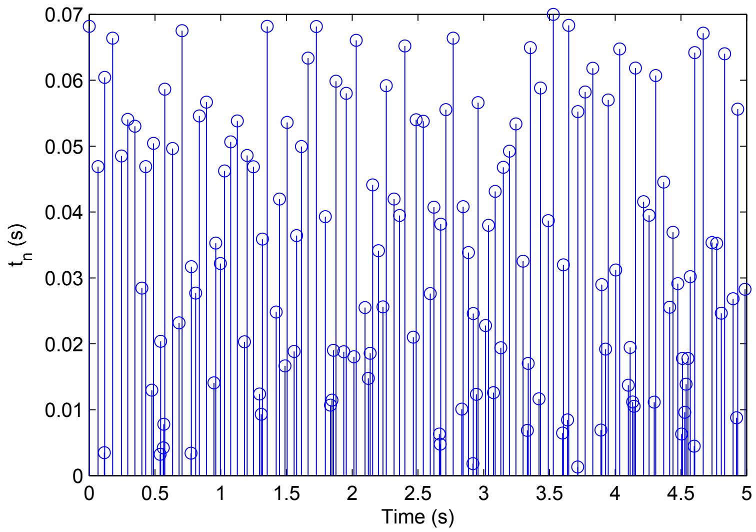

Figure 12 – Aperiodic sampling time with MASP $T_m = 0.070$ s for system (5.13).

gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, with

$$\begin{aligned} Z(\eta(t_n)) &= -\begin{bmatrix}10.4937 & 23.0132 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & -4.3544\end{bmatrix}\eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix}0.8580 & -0.3810 & 0 \\ -0.4329 & 1.3940 & 0 \\ 0 & 0 & 1.0087\end{bmatrix} + \begin{bmatrix}0 & 0 & 0.1887 \\ 0 & 0 & 0.0459 \\ -0.0144 & 0.0430 & 0\end{bmatrix}\eta(t_n). \end{aligned} \tag{5.20}$$

As for Theorem 4.3, given $\alpha_1 = \alpha_2 = 25$, $\beta_1 = 120.5$, and $\delta_1 = 4.5$, the designed control gain is $K(\eta(t_n)) = Z(\eta(t_n))G^{-1}(\eta(t_n))$, with

$$\begin{aligned} Z(\eta(t_n)) &= -\begin{bmatrix}27.2052 & 60.6153 & -0.0515\end{bmatrix} + \begin{bmatrix}0.0344 & -0.0459 & 37.0535\end{bmatrix}\eta(t_n) \\ G(\eta(t_n)) &= \begin{bmatrix}4.2609 & -2.5420 & -0.0018 \\ -6.0265 & 10.8976 & 0.0018 \\ -0.0009 & 0.0006 & 7.0869\end{bmatrix} + \begin{bmatrix}0 & -0.0001 & 3.4729 \\ 0 & 0.0011 & -2.1871 \\ -0.0387 & 0.0837 & -0.0005\end{bmatrix}\eta(t_n). \end{aligned} \tag{5.21}$$

Figures 13 and 14 depict the outputs $y(t)$ response and the control signal $u(t)$, respectively. The aperiodic sampling instants are outlined in Figure 15. The figures demonstrate that the sampled-data controllers designed with the proposed approaches led the closed-loop system (5.13) to stability even in the presence of external disturbances. The induced $\mathcal{L}_2$-gain norms for the closed-loop system are $\gamma^* = 0.087$ (Theorem 3.1), $\gamma^* = 0.089$ (Theorem 4.1), $\gamma^* = 0.193$ (Theorem 4.3), and $\gamma^* = 0.112$ (Gomes da Silva Jr *et al.*, 2018), which are below the respective upper bounds reported in Table 8.

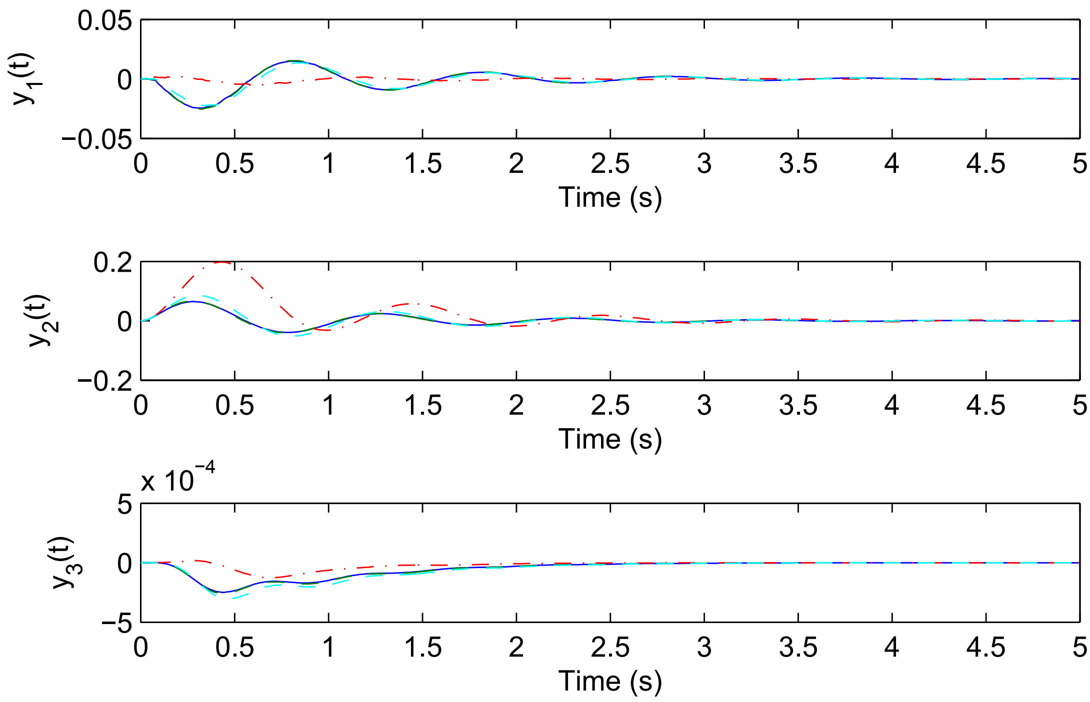

Figure 13 – Comparison of the outputs response for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.0347$ s for system (5.13): Theorem 3.1 in blue solid line, Theorem 4.1 in green dashed line, Theorem 4.3 in red dot-dashed line, and Gomes da Silva Jr *et al.* (2018)[Th. 1] in cyan dashed line.

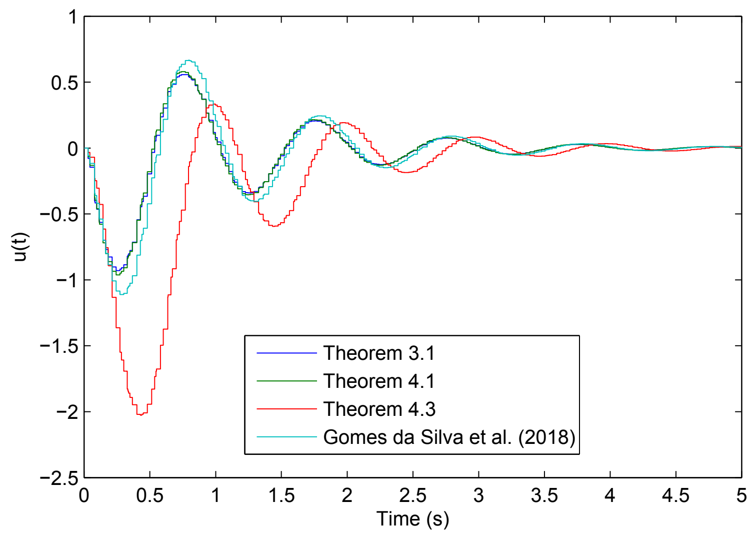

Figure 14 – Comparison of the control signal for $x(0) = 0$, $w(t) = e^{-t}\sin(2\pi t)$ and $T_m = 0.0347$ s for system (5.13).

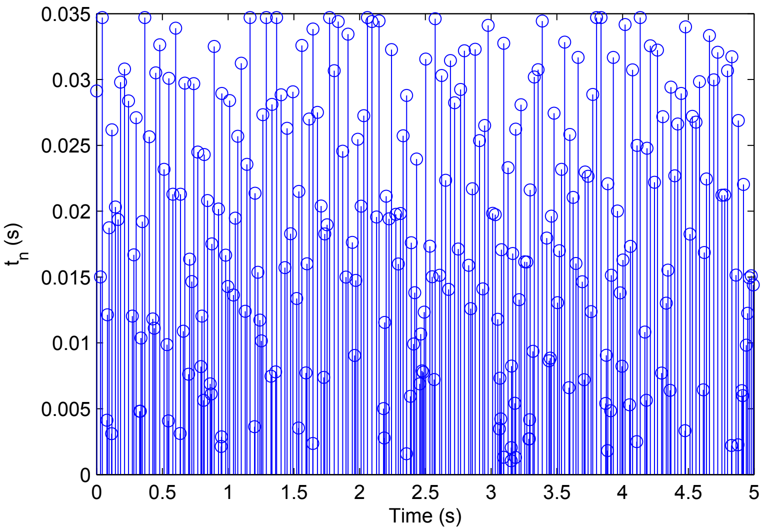

Figure 15 – Aperiodic sampling time with MASP $T_m = 0.0347$ s for system (5.13).

6 Conclusion

In this dissertation, sufficient LMI conditions were proposed to stabilization and $\mathcal{L}_2$-gain performance of sampled-data nonlinear systems, represented in terms of LPV systems. The provided conditions guarantee asymptotic stability and $\mathcal{L}_2$-gain cost in closed-loop systems, with gain-scheduled state-feedback controllers to be synthesized.

The process of obtaining LMI constraints consisted in applying the Lyapunov theory after a suitable Lyapunov function is defined. As a first contribution of this dissertation, the use of Wirtinger's inequality and the inclusion of an additional term to the Lyapunov function aided in reducing the conservativeness of the attained LMI conditions. On the purpose of enlarging the aperiodic sampling time T_m and of minimizing the $\mathcal{L}_2$-gain performance in closed-loop, three different approaches were presented and validated through numerical examples.

The iterative method stood for a stabilizing technique based on a two-step procedure, which originally imposed BMI conditions. Despite the need for following an iterative and time-consuming procedure, the implementation of this approach provided larger MASPs T_m and improved the $\mathcal{L}_2$-gain performance, if compared to some results existing in the literature.

In attempt to circumvent iterative procedures, the full approach was developed. It relies on slack variables introduced with the usage of Finsler's lemma. The establishment of the full approach is the second contribution made in this dissertation. Among the methodologies presented in this text, the full method led to the obtention of the largest aperiodic sampling times T_m and smallest $\mathcal{L}_2$-gain costs in all examples. A drawback of this method is the proper choice of several scalar parameters, which must be given prior to the execution of the optimization problem.

Moreover, another non-iterative approach was presented: the simplified one. It also employs Finsler's lemma, bringing slack variables to the optimization problem created. Not only setting the scalar variables is required, but also the simplified approach did not reveal itself suitable for stabilizing sampled-data nonlinear systems. The achieved results were less satisfactory than the ones obtained with the iterative approach.

As a final contribution of this dissertation, it is shown that the iterative and full approaches provided less conservative results with respect to similar works available in the control literature (for instance, Gomes da Silva Jr *et al.* (2018) and Hooshmandi *et al.* (2018)). It was possible because the Lyapunov function considered in this dissertation contains the Lyapunov function adopted therein, and because more powerful tools were exploited, such as Wirtinger's inequality. Additionally, it was shown that both methodologies, (Gomes da Silva Jr *et al.*, 2018) and (Hooshmandi *et al.*, 2018), are particular cases of the proposed approaches.

6.1 Future works

The contributions made in this dissertation to the stabilization and $\mathcal{L}_2$-gain performance of sampled-data nonlinear systems can be improved in at least six directions. As outlined in Chapter 1, there has been a growing interest in networked control. On the one hand, the approaches developed in this work are suitable for sampled-data nonlinear systems, and thus the proposed methodologies can be applied to real-world networked systems. On the other hand, a scheduling policy to access the network has yet to be formulated, in an approach known as co-design.

In Chapter 5, it is assumed that the next sampling periods t_{n+1} are randomly chosen from the range $t_n < t_{n+1} < t_n + T_m$. Other techniques could be employed as well for determining the next sampling periods t_{n+1}. For example, Wang and Lemmon (2010) consider an event-triggering data transmission strategy in which states updates are broadcast only when needed. The determination of when such updates should happen requires the definition of an appropriate criterion.

The numerical simulations in Chapter 5 showed the potential of the full approach for stabilizing LPV systems in closed-loop. However, the application of this method requires several scalar parameters to be given. Since a proper choice of these parameters is a demanding task, a couple of structural simplifications (such as imposing $\alpha_1 \equiv \alpha_2$) and the development of computational tools, able to narrow the search of these scalars, can be considered.

As discussed in Chapter 4, another possible research direction is extending the proposed design conditions to cope with quasi-LPV systems, without imposing conservative choices for the decision matrices $\overline{P}(\rho)$, $\overline{X}_1(\rho)$ or $\overline{X}_2(\rho)$.

In the framework of time-delay systems, the dynamics of sampled-data nonlinear systems should be changed to cope with the inherent dependence on delayed signals. For instance, a time-delayed LPV model (3.1) can be written as

$$\begin{aligned} \dot{x}(t) =& \mathbf{A}(\eta(t))x(t) + \mathbf{A}_S(\eta(t))x(t_n) + \mathbf{B}_1(\eta(t))w(t) + \mathbf{B}_2(\eta(t))u(t) \\ y(t) =& \mathbf{C}(\eta(t))x(t) + \mathbf{D}_1(\eta(t))w(t) + \mathbf{D}_2(\eta(t))u(t) \end{aligned} \tag{6.1}$$

where $\mathbf{A}_S(\eta(t))$, with compatible dimensions, is a component binding the dynamics of $x(t)$ to a delayed version of $x(t)$. Even though the inclusion of $\mathbf{A}_S(\eta(t))$ in the derived LMIs is straightforward, determining $\mathbf{A}_S(\eta(t))$ is not trivial when sampled-data nonlinear systems are recast as LPV models with the sector nonlinearity method.

Furthermore, as proposed in Seuret and Gouaisbaut (2013), more relaxed conditions can be obtained if null terms are included in the LMI derivation process.

6.2 Related publication

The publication related to the contributions of this dissertation is listed below:

- Oliveira, V.; Frezzatto, L. Gain-scheduled control of nonlinear sampled-data systems: A Wirtinger-based approach. *Proceedings of the XXIII Brazilian Conference on Automation*, v. 2, n. 1, November 2020.

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Appendix

APPENDIX A – Proof of Lemma 3.2

Necessity: applying a Schur complement (see Section 2.4) to (3.51) provides

$$\Xi = \begin{bmatrix} \Lambda & Q^T \\ Q & \Gamma \end{bmatrix}. \tag{A.1}$$

Pre- and post-multiplying (A.1) respectively with Ω^T and Ω, where $\Omega = \begin{bmatrix} \mathbf{I}_n & \Upsilon^T \end{bmatrix}^T$, one has that

$$\Omega^T \Xi \Omega = \Lambda + \Big(Q^T \Upsilon\Big)^H + \Upsilon^T \Gamma \Upsilon \tag{A.2}$$

$$\Omega^T \Xi \Omega = \Big(\Lambda - Q^T \Gamma^{-1} Q\Big) + \Big(Q + \Gamma \Upsilon\Big)^T \Gamma^{-1} \Big(Q + \Gamma \Upsilon\Big) \tag{A.3}$$

From (3.51), it is assumed that $\Lambda - Q^T \Gamma^{-1} Q \prec 0$. The negativeness of (A.3) is ensured if, for instance, Υ is chosen as $\Upsilon = -\Gamma^{-1} Q$. In other words, it means that there exists Υ such that

$$\Omega^T \Xi \Omega \prec 0. \tag{A.4}$$

If the null terms

$$\Upsilon^T L^T - \Upsilon^T L^T + L\Upsilon - L\Upsilon + \Upsilon^T G \Upsilon - \Upsilon^T G \Upsilon + \Upsilon^T G \Upsilon - \Upsilon^T G \Upsilon = 0. \tag{A.5}$$

are added to (A.4), the obtained terms can be grouped as follows:

$$\mathbf{I}_n \Big(\Lambda + \Upsilon^T Q + Q^T \Upsilon + \Upsilon^T L^T + L\Upsilon\Big) \mathbf{I}_n + \mathbf{I}_n \Big(-L + \Upsilon^T G\Big) \Upsilon \\ + \Upsilon^T \Big(-L^T + G^T \Upsilon\Big) \mathbf{I}_n + \Upsilon^T \Big(\Gamma - G - G^T\Big) \Upsilon \prec 0 \tag{A.6}$$

$$\mathbf{I}_n \bigg(\Lambda + \Big(\Upsilon^T (Q + L^T)\Big)^H\bigg) \mathbf{I}_n + \mathbf{I}_n \Big(-L + \Upsilon^T G\Big) \Upsilon \\ + \Upsilon^T \Big(-L^T + G^T \Upsilon\Big) \mathbf{I}_n + \Upsilon^T \Big(\Gamma - G^H\Big) \Upsilon \prec 0 \tag{A.7}$$

$$\Omega^T \begin{bmatrix} \Lambda + \Big(\Upsilon^T (Q + L^T)\Big)^H & -L + \Upsilon^T G \\ * & \Gamma - G^H \end{bmatrix} \Omega \prec 0 \tag{A.8}$$

From (A.8), it is possible to verify that (3.51) implies (3.52).

Sufficiency: applying the transformation matrices $\Omega^T = \begin{bmatrix} \mathbf{I}_n & \Upsilon^T \end{bmatrix}$ and Ω respectively to the left and to the right of (3.52) and rearranging the obtained terms, one gets

$$\Big(\Lambda - Q^T \Gamma^{-1} Q\Big) + \Big(Q + \Gamma \Upsilon\Big)^T \Gamma^{-1} \Big(Q + \Gamma \Upsilon\Big) \prec 0 \tag{A.9}$$

$$\left(\Lambda - Q^T\Gamma^{-1}Q\right) \prec -\left(Q + \Gamma\Upsilon\right)^T\Gamma^{-1}\left(Q + \Gamma\Upsilon\right) \tag{A.10}$$

Provided that Γ is a positive-definite matrix, the left-hand side of (A.10) is negative definite for all Υ. Notice that, if $\Upsilon = -\Gamma^{-1}Q$, the relation $\Lambda - Q^T\Gamma^{-1}Q \prec 0$ remains true and the equivalency between (A.4) and (A.10) becomes clear. This concludes the proof.

APPENDIX B – Recovery of theoretical results from control literature with the iterative and full approaches

B.1 Theorem 3.1 contains Hooshmandi *et al.* (2018)[Th. 4.2]

Since Theorem 3.1 extends the main results attained in Hooshmandi *et al.* (2018), recovering the LMI conditions presented in Hooshmandi *et al.* (2018)[Th. 4.2] requires the execution of a few steps:

- Disposal of the term $V_3(x,t)$ from the definition of the Lyapunov function (3.12). Hence, the obtained Lyapunov function is not a function of $\nu(t)$.
- Application of Jensen's inequality (see Section 2.6) instead of Wirtinger's inequality (see Section 2.7) in the LMI derivation procedure.
- Removal of any row and column corresponding to $\nu(t)$ or containing $\overline{N}_2(\rho)$ from the derived LMIs. Notice that the terms associated with $\overline{N}_2(\rho)$ are subproducts of the application of Wirtinger's inequality.

B.2 Theorem 4.1 contains Hooshmandi *et al.* (2018)[Th. 4.2]

Theorem 4.1 is a non-iterative adaptation of Theorem 3.1. In order for the LMI constraints proposed in Hooshmandi *et al.* (2018)[Th. 4.2] to be retrieved having Theorem 4.1 as a starting point, the same steps presented in Section B.1 apply. However, since Theorem 4.1 is itself a non-iterative approach, other changes must be introduced:

- The decision variables from the Lyapunov function (3.12) are defined such that

$$\dot{\overline{P}}(\rho) = 0,\ \overline{X}_1(\rho) = \lambda \overline{P}(\rho),\ \dot{\overline{X}}_1(\rho) = 0,\ \overline{X}_2(\rho) = 0,\ \dot{\overline{X}}_2(\rho) = 0, \tag{B.1}$$

  where λ is a scalar to be determined as part of the iterative procedure discussed in Hooshmandi *et al.* (2018).

- The scalar parameters in (4.19) are chosen to be null, apart from

$$\alpha_1 = \alpha_2 = \mu_1 = \mu_2 = 1. \tag{B.2}$$

- The rows and columns corresponding to $\dot{x}(t)$ are eliminated.

- By performing the relaxation (3.49), the derived conditions are actually BMIs. As a result, the iterative procedure discussed in Section 3.5 must be adopted.

B.3 Theorem 4.1 contains Gomes da Silva Jr *et al.* (2018)[Th. 1]

In order to retrieve the sufficient LMI conditions presented in Gomes da Silva Jr *et al.* (2018)[Th. 1], the following choices must take place:

- The component $V_3(x,t)$ should be dropped from the definition of the considered Lyapunov function in this dissertation. Thus, the obtained Lyapunov function does not depend on $\nu(t)$.

- The decision variables from the Lyapunov function (3.12) are set as

$$\begin{gathered}\overline{X}_1(\rho) = F(\rho),\ \dot{\overline{X}}_1(\rho) = \dot{F}(\rho),\ \overline{X}_2(\rho) = G(\rho),\ \dot{\overline{X}}_2(\rho) = \dot{G}(\rho),\\ \overline{E}_1(\rho) = R(\rho),\ \overline{E}_2(\rho) = 0,\ \overline{E}_3(\rho) = X(\rho),\ \overline{F}(\rho) = 0.\end{gathered} \tag{B.3}$$

- The decision variables introduced in Theorem 4.1 are chosen as

$$\overline{N}_1(\rho) = Q(\rho),\ \overline{N}_2(\rho) = 0,\ G(\rho) = Y(\rho). \tag{B.4}$$

- Any scalar parameters in (4.19) are set to zero, except

$$\alpha_1 = \alpha_2 = \epsilon,\ \delta_1 = \omega_2 = \mu_1 = \mu_2 = 1, \tag{B.5}$$

  where ϵ is a given positive scalar, as discussed in Gomes da Silva Jr *et al.* (2018).

- The rows and columns corresponding to $\nu(t)$ or containing $\overline{N}_2(\rho)$ are removed from the LMI conditions of Theorem 4.1.