# PÓLYA URN MODELS WITH REINFORCEMENT FUNCTIONS AND TIME DEPENDENT FITNESS

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Tese apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais como requisito parcial para a obtenção do grau de Doutor em Matemática.

Orientador: Remy de Paiva Sanchis

Belo Horizonte

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Aos oito dias do mês de setembro de 2020, às 13h00, em reunião pública virtual na Plataforma Google Meet pelo link meet.google.com/igb-xfph-ogo (conforme mensagem eletrônica da Pró-Reitoria de Pós-Graduação de 26/03/2020, com orientações para a atividade de defesa de tese durante a vigência da Portaria nº 1819), reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Cristiano Santos Benjamin**, intitulada: "*Pólya Urn Models With Reinforcement Functions And Time Dependent Fitness"*, requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Rémy de Paiva Sanchis, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente, sem a presença do aluno, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 08 de setembro de 2020.

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*““Knowing” can be useful, but learning not to know creates a powerful openness that is inconceivable until it is experienced.”*

(Peter Ralston in The Book of Not Knowing: Exploring the True Nature of Self, Mind, and Consciousness)

# Resumo

Vantagem acumulativa é um fenômeno observado em vários sistemas onde há competição por recursos. Por exemplo, duas empresas podem competir por clientes. Quanto mais clientes uma empresa possui, mais popular ela será, e quanto mais popular ela for, mais clientes ela atrairá. Este fenômeno, a capacidade que recursos acumulados tem para promover a acumulação de mais recursos, aparece na literatura sobre vários nomes, tais como vantagem acumulativa [Price, 1976], fixação preferencial [Barabasi and Albert, 1999], "O rico fica mais rico" [DiPrete and Eirich, 2008], processos com feedback [Drinea et al., 2002, Oliveira, 2009], entre outros.

O modelo mais antigo com este fenômeno é o processo de urna de Pólya, o qual foi introduzido por Eggenberger e Pólya [Eggenberger and Pólya, 1923] como um modelo para doenças contagiosas. Depois disso, o modelo foi largamente estudado e aplicado [Mahmoud, 2008, Pemantle, 2007].

No modelo de urna de Pólya, nós temos uma urna com bolas coloridas. Então sorteamos uma bola da urna e a colocamos de volta junto com outra bola da mesma cor. Note que se tivermos $x$ bolas de uma certa cor, aquela cor será sorteada com probabilidade proporcional a $f(x) = x$, onde $f$ é chamada de função de reforço. Em [Khanin and Khanin, 2001] os autores introduzem não linearidade para o processo, fazendo $f(x) = x^\alpha$ com $\alpha > 0$. Depois, em [Oliveira, 2008], Oliveira generaliza o modelo para qualquer função positiva $f$.

Além da vantagem cumulativa, uma característica observada e reconhecida em competições é o fitness, que se refere a habilidade intrínsica que um agente possui para acumular recursos e que não depende da quantidade de recursos já acumulados [Borgs et al., 2007, Dereich and Ortgiese, 2014].

Processos de urna de Pólya não linear com fitness foi estudado em [Jiang et al., 2016]. No modelo estudado pelos autores, temos uma urna com duas cores, 1 e 2. Se a urna contém $x$ bolas da cor $i$, então sorteamos uma bola daquela cor com probabilidade proporcional a $f_i(x) = r_i x^\alpha$, onde $r_1, r_2, \alpha > 0$. Note que, ao contrário dos casos estudados em [Khanin and Khanin, 2001, Oliveira, 2008], os autores

em [Jiang et al., 2016] permitem cada cor ter sua própria função de reforço $f_1, f_2$. Mas esta não foi a primeira vez que um modelo com funções de reforço diferentes para cada agente foi considerada (veja [Collevecchio et al., 2013]). Se $r_2 > r_1$ então as bolas da cor 2 tem uma vantagem (o maior fitness) que não depende da quantidade de bolas já presentes na urna. Trazendo de volta o exemplo das duas empresas competindo por clientes, podemos pensar no fitness sendo a qualidade do serviço oferecido pelas duas empresas. Os autores em [Jiang et al., 2016] provaram que se $0 < \alpha < 1$ então o agente com maior fitness será o vencedor da competição com probabilidade 1, isto é, existe um tempo aleatório $s$ finito quase certamente tal que, a partir desse tempo, o agente com maior fitness sempre terá a maior fração de recursos. Dizemos que o ganhador alcançou liderança final.

No modelo descrito acima, o agente que inicia com o menor fitness não tem nenhuma chance de ganhar a competição. Mas no mundo real, o agente pode mudar ao passar do tempo. Ele pode melhorar suas habilidades, aumentando o fitness. Neste trabalho mostramos que, mesmo iniciando a competição com um fitness menor, se o agente aumenta o seu fitness suficientemente rápido (mesmo permanecendo menor que o fitness do outro agente) então a probabilidade de não perder se torna positiva.

Os modelos "Bolas nas caixas" são modelos de urnas onde temos duas ou mais urnas, o qual a partir daqui chamaremos de caixas, e cada caixa contém bolas. Aqui a cor não vai ter nenhum papel. Colocamos bolas nas caixas seguindo alguma regra probabilística. A quantidade de bolas em cada caixa no modelo "bolas nas caixas" se associa com a quantidade de bolas de cada cor no modelo urna de Pólya. Ao longo deste trabalho, iremos usar a terminologia do modelo "bolas nas caixas" em vez do tradicional modelo urna de Pólya.

O processo que vamos estudar inicia com $X_1(0) = a_1$ e $X_2(0) = a_2$ bolas nas caixas 1 e 2 respectivamente, onde $a_1$ e $a_2$ são inteiros positivos. A probabilidade de adicionarmos uma bola na caixa $s$ no tempo $t+1$ é proporcional a $f_s(X_s(t), t)$, onde $f_s$ é uma função de reforço temporal e $X_s(t)$ é a quantidade de bolas na caixa $s$ no tempo $t$. No caso onde as funções de reforço não dependem do tempo, as chamamos de funções de reforço atemporal.

Como nós vimos, em [Jiang et al., 2016] eles estudaram o processo com as funções de reforço atemporal

$$f_1(i,t) = r_1 i^{\alpha}, \qquad f_2(i,t) = r_2 i^{\alpha}.$$

Neste trabalho, estamos particularmente interessados nas funções de reforço temporal

$$f_1(i,t) = r_1(t) i^{\alpha}, \qquad f_2(i,t) = r_2(t) i^{\alpha}$$

onde $0 < r_1(t) \leq r_2(t)$ para todo $t \geq 0$. Note que podemos definir $r(t) = r_2(t)/r_1(t)$ e podemos redefinir $f_1, f_2$, para ter um processo com a mesma distribuição do processo acima, como

$$f_1(i,t) = i^\alpha, \qquad f_2(i,t) = r(t)i^\alpha.$$

De fato, dado que no tempo $t$ a caixa 1 tem $x$ bolas e a caixa 2 tem $y$ bolas, a probabilidade de adicionar uma bola na caixa 1 no tempo $t+1$ será

$$\frac{r_1(t)x^\alpha}{r_1(t)x^\alpha + r_2(t)y^\alpha} = \frac{r_1(t)/r_1(t)x^\alpha}{r_1(t)/r_1(t)x^\alpha + r_2(t)/r_1(t)y^\alpha} = \frac{x^\alpha}{x^\alpha + r(t)y^\alpha},$$

e o mesmo se aplica para a probabilidade de adicionar uma bola na caixa 2.

Se $r(t) = 1$ para todo $t \geq 0$, então estamos no caso estudado em [Khanin and Khanin, 2001] e há uma probabilidade positiva da caixa 1 não perder. Mas se $r(t) = r > 1$ para todo $t \geq 0$, então estamos no caso estudado em[Jiang et al., 2016], e a probabilidade da caixa 1 não perder se torna zero. Estamos interessados no caso onde $r(t) > 1$ mas $r(t) \to 1$. No exemplo das duas empresas competindo por clientes, a empresa 1 inicia com o menor fitness, mas com o passar do tempo, ela melhora seu fitness se aproximando cada vez mais do fitness da empresa 2. Mostramos que, dependendo do quão rápido $r(t)$ converge para 1, a probabilidade da empresa 1 não perder pode ser positiva ou zero.

Um caso particular e interessante das funções de reforço temporal $f_1(i,t) = i^\alpha$ e $f_2(i,t) = r(t)i^\alpha$ é o caso onde $r(t) = (1+(t+1)^{-\beta})$ com $\beta > 0$. Para qualquer $0 < \alpha < 1$ fixo, temos uma transição de fase em $\beta$, dependendo de $\alpha$. Para $0 < \alpha < 1/2$ temos que se $\beta < 1/2$ então a caixa 2 alcança liderança final com probabilidade 1 e para $\beta \geq 1/2$ há infinitas mudanças de liderança (há uma sequência de tempos tal que a caixa 1 terá mais bolas que a caixa 2 e há uma outra sequência de tempos tal que a caixa 2 terá mais bolas que a caixa 1) com probabilidade 1. Para $\alpha = 1/2$ temos que se $\beta \leq 1/2$ então a caixa 2 alcança liderança final com probabilidade 1 e para $\beta > 1/2$ há infinitas mudanças de liderança com probabilidade 1. Finalmente, para $1/2 < \alpha < 1$ temos que para $\beta \leq 1 - \alpha$ a probabilidade que a caixa 1 alcança liderança final é 0 e para $\beta > 1 - \alpha$ esta probabilidade é positiva. Este é o nosso principal resultado e é provado no Teorema 2.12. Veja Figure 1.1 para um resumo dos resultados enunciados aqui.

Uma das ferramentas principais utilizadas no estudo deste modelo é o *exponential embedding* [Khanin and Khanin, 2001, Jiang et al., 2016, Oliveira, 2008, Davis, 1990], mas a adição do tempo nas funções de reforço faz com que não seja possível utilizar esta ferramenta. Então, a fim de obter os resultados para as funções de reforço temporais, precisamos primeiro generalizar alguns resultados encontrados em [Khanin and Khanin, 2001, Jiang et al., 2016] para um conjunto maior de funções de reforço atemporais. Então, através de um acoplamento, obtemos resultados para as funções de reforço temporais.

A generalização das funções de reforço atemporais estudadas em [Jiang et al., 2016] para um conjunto maior de pares $(f_1, f_2)$, embora seja um passo intermediário para provar o nosso resultado principal, é interessante por si só. Devido à técnica de *exponential embedding*, somos capazes de provar resultados bem precisos. Por exemplo, um caso interessante neste conjunto maior de funções de reforço atemporais é $f_1(i) = i^\alpha, f_2(i) = r_i i^\alpha$ com $r_i \searrow 1$ quando $i \to +\infty$. Em particular, podemos tomar $r_i = (1 + i^{-\beta})$ com $\beta > 0$ (este é o modelo utilizado no acoplamento para provar nosso resultado principal). O parâmetro $\beta$ nos diz o quão rápido $r_i$ converge para 1. Tomarmos um $\beta$ menor significa mais vantagem para a caixa 2. Para $\alpha < 1/2$ e $\beta = 1/2$, a vantagem da caixa 2 não é o suficiente e a caixa 1 ultrapassa a caixa 2 infinitas vezes. Mas, o que acontece se diminuirmos a velocidade da convergência, multiplicando uma função $h(i)$ (tal que $h(i) \to +\infty$ mas $h(i) = o(i^\delta)$ para todo $\delta > 0$) a $i^{-1/2}$? Digo, o que acontece se definirmos $r_i = 1 + h(i)i^{-1/2}$? Neste caso, $r_i$ irá para 1 um pouco mais lento mas ainda assim, mais rápido do que $1 + i^{-\beta}$ para qualquer $\beta < 1/2$. Definindo

$$h(i) = C\sqrt{(1-2\alpha)\log\log\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)},$$

para $C > 1$, é suficiente para fazer a caixa 2 alcançar liderença final com probabilidade 1, mas se tomarmos $0 < C < 1$, então, haverá infinitas trocas de liderança. Este é o resultado do corolário 2.4. A nossa técnica não permite chegar a uma conclusão para $C = 1$.

Fazemos o mesmo tipo de pergunta em outros pontos $(\alpha, \beta)$ onde temos uma transição de fase. Veja os Corolários 2.6, 2.10 e 2.11.

Quando $f_1(i) = i^\alpha$ e $f_2(i) = r_i i^\alpha$ com $0 < \alpha \leq 1/2$ e $r_i \searrow r \geq 1$, somos capazes de encontrar o comportamento assintótico de $X_1(t)/X_2(t)$ (veja o Teorema 2.13). Se $r_i = 1 + i^{-\beta}$ embora $X_1(t)/X_2(t)$ convirja para 1 q.c. quando $t \to +\infty$, temos que $(X_2 - X_1)(t) \to +\infty$. O Teorema 2.14 nos diz exatamente o quão rápido isto acontece.

**Palavras-chave:** Bolas em caixas, Modelos de urna de Pólya, ligação preferencial, fitness.

# Abstract

In this work we study a Pólya urn model with temporal reinforcement functions, i.e., the probability of adding a ball of color $s$ at time $t+1$ will be proportional to a function of the amount of balls of that color at time $t$ and the time $t$ itself. Specifically, the probability will be proportional to $f_s(x,t)$ where $x$ is the amount of balls of that color at time $t$ and $f_s$ is a positive function associated with the color $s$.

We are particularly interested in the Pólya urn model with two colors, 1 and 2, and the temporal reinforcement functions $f_1(x,t) = x^\alpha, f_2(x,t) = (1+(t+1)^{-\beta})x^\alpha$ where $0 < \alpha < 1$ and $\beta > 0$. We find three regimes depending on $(\alpha, \beta)$. If $0 < \alpha < 1/2$ and $0 < \beta < 1/2$ then the color 2 wins the competition with probability 1, i.e., from a random time on, we always have more balls of color 2 than of color 1. If $0 < \alpha < 1/2$ and $\beta \geq 1/2$ then there will be endless leadership changes. For $\alpha = 1/2$ we have a similar result but the regime is split between $\beta \leq 1/2$ and $\beta > 1/2$. For $1/2 < \alpha < 1$ we have a phase transition on $\beta = 1 - \alpha$. If $\beta \leq 1 - \alpha$ then the color 1 will lose with probability 1, but if $\beta > 1 - \alpha$ then there is a positive probability of bin 1 win the competition. To deal with this problem, because of dependence on time, we were not able to use the exponential embedding, which is a classic tool analysing such models. To prove the results we needed first to generalize some results present in the literature and then, to define a coupling with our model.

**Keywords:** Balls into Bins, Pólya urn models, preferrential attachment, fitness.

# Contents

# Chapter 1

# Introduction

Cumulative advantage is a phenomenon observed in several systems where there is competition for resources. For instance, two companies can compete for customers. The more customers a company has, the more popular it will be, and the more popular the company is, the more customers it will atract. This phenomenon, the capacity that accumulated resources have to promote accumulation of more resources, appears in the literature under various names such as cumulative advantage [Price, 1976], preferential attachment [Barabasi and Albert, 1999], "the rich get richer" [DiPrete and Eirich, 2008], processes with feedback [Drinea et al., 2002, Oliveira, 2009], among others.

The oldest model with this phenomenon is the Pólya Urn process, which was introduced by Eggenberger and Pólya [Eggenberger and Pólya, 1923] as a model for contagious diseases. Later the model was widely studied and applied [Mahmoud, 2008, Pemantle, 2007].

In the Pólya urn model, we have an urn with colored balls. Then we draw a ball at random from the urn and put it back together with another ball of the same color. Note that if we have $x$ balls of some color, that color is drawn with probability proportional to $f(x) = x$, where $f$ is called a reinforcement function. In [Khanin and Khanin, 2001] the authors introduced nonlinearity to the model, making $f(x) = x^\alpha$ with $\alpha > 0$. Later, in [Oliveira, 2008], Oliveira generalizes the model for any positive function $f$.

Beyond cumulative advantage, an observed and recognized characteristics in competitions is fitness, which refers to the inherent ability of an agent to accumulate resources that does not depend on the amount of resources already accumulated [Borgs et al., 2007, Dereich and Ortgiese, 2014].

Non-linear Pólya urn processes with fitness were analyzed in [Jiang et al., 2016]. In the model studied by the authors, we have an urn with two colors, 1 and 2. If the urn contains $x$ balls of color $i$, then we draw a ball of that color with probability proportional to $f_i(x) = r_i x^\alpha$, where $r_1, r_2, \alpha > 0$. Note that unlike the cases studied in [Khanin and Khanin, 2001, Oliveira, 2008], the authors in [Jiang et al., 2016] allowed each color to have its own reinforcement functions $f_1, f_2$, but it was not the first time a model with different reinforcement functions was considered (see [Collevecchio et al., 2013]). If $r_2 > r_1$ then the balls of color 2 have one advantage (the greater fitness) that does not depend on the amount of balls of color 2 already present in the urn. Bringing back the example of the two companies competing for customers, we can think the fitness being the quality of service offered by the company. The authors in [Jiang et al., 2016] proved that if $0 < \alpha < 1$ then the agent with greater fitness will win the competition with probability 1, i.e., there exists a finite random time $s$ such that, from that time, the fittest agent will always have the greater fraction of resources. We say that the winner reaches eventual leadership.

The model described above says that the agent starting with smaller fitness has no chance to win the competition. However, in reality, the agent can change over time. It can improve its abilities, increasing its fitness. In this work we show that even starting the competition with smaller fitness, if the agent increases its fitness fast enough (even remaining smaller), the probability of not losing becomes positive.

"Balls into bins" models are urn models where we have two or more urns, which from here on, we are going to call bins, and each bin contains balls. Here the colors play no role. We put balls into the bins following some probability rule.

The amount of balls in each bin in the "balls into bins" model associates with the amount of balls of each color in the Polya urn model. Thoughout this work, we are going to use the terminology of "ball into bins" model instead the traditional Pólya urn model.

The proccess we are going to study will start with $X_1(0) = a_1$ and $X_2(0) = a_2$ balls in the bins 1 and 2 respectively, where $a_1$ and $a_2$ are positive integers. The probability of adding a ball in bin $s$ at time $t+1$ is proportional to $f_s(X_s(t), t)$, where $f_s$ is a temporal reinforcement function and $X_s(t)$ is the amount of balls in bin $s$ at time $t$. In the case where the reinforcement functions do not depend on time, we call them atemporal reinforcement functions.

As we saw, in [Jiang et al., 2016], they studied the process with atemporal reinforcement functions

$$f_1(i,t) = r_1 i^\alpha, \qquad f_2(i,t) = r_2 i^\alpha.$$

In this work, we are particularly interested in the temporal reinforcement functions

$$f_1(i,t) = r_1(t) i^\alpha, \qquad f_2(i,t) = r_2(t) i^\alpha$$

where $0 < r_1(t) \leq r_2(t)$ for all $t \geq 0$. Note we can define $r(t) = r_2(t)/r_1(t)$ and we can redefine $f_1, f_2$, to have a process with same distribution, as

$$f_1(i,t) = i^\alpha, \qquad f_2(i,t) = r(t)i^\alpha.$$

Indeed, given that at time $t$ bin 1 has $x$ balls and bin 2 has $y$ balls, the probability of adding a ball in bin 1 at time $t+1$ will be

$$\frac{r_1(t)x^\alpha}{r_1(t)x^\alpha + r_2(t)y^\alpha} = \frac{r_1(t)/r_1(t)x^\alpha}{r_1(t)/r_1(t)x^\alpha + r_2(t)/r_1(t)y^\alpha} = \frac{x^\alpha}{x^\alpha + r(t)y^\alpha},$$

and the same applies for the probability of adding a ball to bin 2.

If $r(t) = 1$ for all $t \geq 0$, then we are in the case studied in [Khanin and Khanin, 2001] and there is a positive probability of bin 1 not losing. But if $r(t) = r > 1$ for all $t \geq 0$, then we are in the case studied in [Jiang et al., 2016], and the probability of bin 1 not losing becomes zero. We are interested in the case where $r(t) > 1$ but $r(t) \to 1$. In the example of two companies competing for customers, the company 1 start with smaller fitness but along the time it improves its fitness getting closer and closer from the fitness of company 2. We show that, depending on how fast $r(t)$ converges to 1, the probability of company 1 not losing can be positive or zero.

This is not the first time the Pólya urn model with the rule changing over time is studied. In [Athreya, 1969, Pemantle, 1990, Sidorova, 2018b] the authors studied the linear Pólya urn model where the amount of ball added to the urn at each drawn changes over time. In [Sidorova, 2018a] Sidorova studied the same model for the reinforcement function $f(x) = x^\alpha$ with $\alpha > 1$. But in those models, the reinforcement function does not change over time. To the best of our knowledge, this is the first time the model is studied with a temporal reinforcement function.

A particular and interesting case of temporal reinforcement functions $f_1(i,t) = i^\alpha$ and $f_2(i,t) = r(t)i^\alpha$ is the case where $r(t) = (1 + (t+1)^{-\beta})$ with $\beta > 0$. For any fixed $0 < \alpha \leq 1$ we have a phase transition on $\beta$, depending on $\alpha$. For $0 < \alpha < 1/2$ we have that if $\beta < 1/2$ then bin 2 reaches eventual leadership with probability 1 and for $\beta \geq 1/2$ there are endless leadership changes (there is a sequence of time such that bin 1 will have more balls then bin 2 and there is another sequence of time such that bin 2 will have more balls then bin 1) with probability 1. For $\alpha = 1/2$ we have that if $\beta \leq 1/2$ then bin 2 reaches eventual leadership with probability 1 and for $\beta > 1/2$ there are endless leadership changes with probability 1. Finally, for $1/2 < \alpha \leq 1$ we have that for $\beta \leq 1 - \alpha$ the probability that bin 1 reaches eventual leadership is 0 and for $\beta > 1 - \alpha$ that probability is positive. This is our main result and is proved in

Theorem 2.12. See Figure 1.1 for an overview of the results stated here.

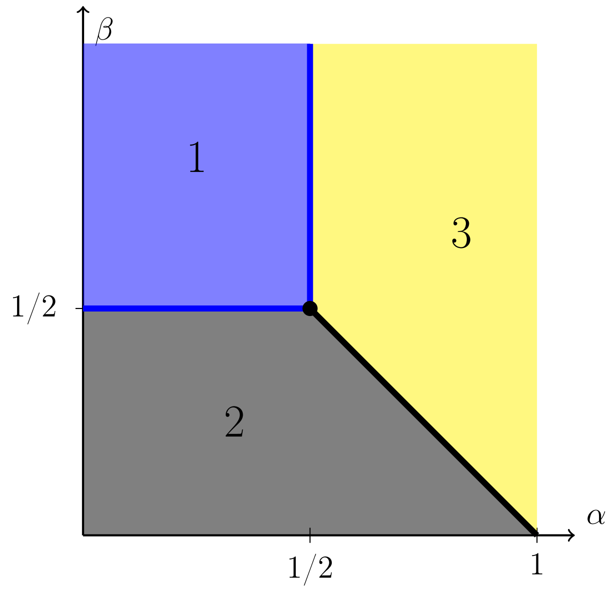


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**Figure 1.1.** The three phases for $\alpha > 0$ and $\beta > 0$. **1)** Endless leadership changes. **2)** Bin 2 reaches eventual leadership with probability 1. **3)** Bin 1 reaches eventual leadership with positive probability.

One of the main tools used to study this model is the exponential embedding [Khanin and Khanin, 2001, Jiang et al., 2016, Oliveira, 2008, Davis, 1990], but the addition of time in the reinforcement functions makes this tool useless. So, in order to get results for the temporal reinforcement functions, we need first generalize some results found in [Khanin and Khanin, 2001, Jiang et al., 2016] to a wider class of atemporal reinforcement functions. Then, through a coupling, we obtain results for the temporal reinforcement functions.

The generalization of the atemporal reinforcement functions studied in [Jiang et al., 2016] to a larger class of pairs $(f_1, f_2)$, although it is an intermediate step to prove our main results, is interesting for itself. Due to the exponential embedding, we were able to prove very accurate results. For example, an interesting case for that wider class of atemporal reinforcement functions is $f_1(i) = i^\alpha, f_2(i) = r_i i^\alpha$ with $r_i \searrow 1$ as $i \to +\infty$. In particular, we can take $r_i = (1 + i^{-\beta})$ with $\beta > 0$ (this is the model used in the coupling to prove our main results). The parameter $\beta$ tells us how fast $r_i$ converges to 1. Smaller $\beta$ means more advantage for bin 2. For $\alpha < 1/2$ and $\beta = 1/2$, the advantage for bin 2 is not enough and the bin 1 overtakes bin 2 infinity often. But, what happens if we slow down the speed of convergence multiplying $i^{-1/2}$ by a function $h(i)$ (such that $h(i) \to +\infty$ and $h(i) = o(i^\delta)$ for all $\delta > 0$)? More specifically, what happens if we define $r_i = 1 + h(i)i^{-1/2}$? In this case, $r_i$ goes to 1 a bit slower but yet faster than $1 + i^{-\beta}$ for any $\beta < 1/2$. Defining

$$h(i) = C\sqrt{(1-2\alpha)\log\log\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)},$$

for $C > 1$, is enough to make bin 2 reaches eventual leadership with probability 1, but if we take $0 < C < 1$, then, there are endless leadership changes. That is what Corollary 2.4 deals with. What happens if $C = 1$ is still an open problem.

We also ask the same type of question about other points $(\alpha, \beta)$ where we have a phase transition. See Corollaries 2.6, 2.10 and 2.11.

When $f_1(i) = i^\alpha$ and $f_2(i) = r_i i^\alpha$ with $0 < \alpha \leq 1/2$ and $r_i \searrow r \geq 1$, we are able to find the assymptotic behaviour of $X_1(t)/X_2(t)$ (see Theorem 2.13). If $r_i = 1 + i^{-\beta}$ although $X_1(t)/X_2(t)$ converges to 1 a.s. as $t \to +\infty$, we have $(X_2 - X_1)(t) \to +\infty$. Theorem 2.14 gives us the assymptotic behaviour of the difference.

In the next chapter, we give the precise statements of our results and in Chapter 3 we present the tools needed for the proofs. Finally, in Chapter 4 we give all the proofs.

# Chapter 2

# The model and results

In this chapter we are going to define the model formally and to present our results. Our process will be $(X_1(t, f_1, f_2, a_1, a_2), X_2(t, f_1, f_2, a_1, a_2))_{t\geq 0}$, where $f_1, f_2, a_1, a_2$ are given. Throughout the text, we will omit the arguments $f_1, f_2, a_1, a_2$ when they are implicit from the context.

Let $a_1$ and $a_2$ be positive integers and $f_1, f_2 : \mathbb{N} \times \mathbb{N} \to (0, \infty)$. Let $X_1(0) = a_1$ and $X_2(0) = a_2$. The model is defined recursively by the following probabilities

$$\mathbb{P}[X_1(t+1) = x+1, X_2(t+1) = y | X_1(t) = x, X_2(t) = y] = \frac{f_1(x,t)}{f_1(x,t) + f_2(y,t)}, \quad (2.1)$$

$$\mathbb{P}[X_1(t+1) = x, X_2(t+1) = y+1 | X_1(t) = x, X_2(t) = y] = \frac{f_2(y,t)}{f_1(x,t) + f_2(y,t)}. \quad (2.2)$$

In most cases in this dissertation, the reinforcement functions do not depend on time, i.e., they are atemporal reinforcement functions. And when this is the case, we omit their second argument. For instance, the following definition is for such kind of reinforcement functions.

**Definition 2.1.** *Let $f_1, f_2 : \mathbb{N} \to (0, \infty)$ be such that $f_1(i) \leq f_2(i)$ for all $i \in \mathbb{N}$. Define*

$$s_n^2 := \sum_{i=1}^{n} \left( \frac{1}{f_1(i)^2} + \frac{1}{f_2(i)^2} \right) \quad \textit{and} \quad t_n^2 := 2 \log_2 s_n^2,$$

*where $\log_2$ is given by*

$$\log_2 x := \begin{cases} 1 & \textit{if } x \geq 0, \log x \leq e \\ \log\log x & \textit{if } \log x > e. \end{cases} \quad (2.3)$$

*We say that* $(f_1, f_2)$ *is GLIL (Good for Law of Iterated Logarithm) if*

1. $\sum_{n=1}^{+\infty} (s_n t_n f_1(n))^{-3} < +\infty,$
2. $\lim_{n\to+\infty} s_n = +\infty$ *and*
3. $\limsup_{n\to+\infty} \frac{s_{n+1}}{s_n} < +\infty$

*Also, we say that* $(f_1, f_2)$ *is GKTS (Good for Kolmogorov Two Series theorem) if*

i. $\sum_{i=1}^{+\infty} \frac{1}{f_2(i)} = +\infty$ *and*

ii. $\lim_{n\to+\infty} s_n < +\infty.$

In the case where $f_1(i) = f_2(i) = i^\alpha$, if $0 < \alpha \leq 1/2$ then the pair $(f_1, f_2)$ is GLIL. If $1/2 < \alpha \leq 1$ then $(f_1, f_2)$ is $GKTS$. Theorem 2.2 deal with the set of GLIL pairs and tell us sufficient conditions for endless leadership changes to happen with probability 1 and sufficient conditions for bin 2 to reach eventual leadership with probability 1.

**Theorem 2.2.** *Let* $(f_1, f_2)$ *be GLIL (see Definition 2.1). Define*

$$a_n = \sum_{i=1}^{n} \left( \frac{1}{f_1(i)} - \frac{1}{f_2(i)} \right) \tag{2.4}$$

*and*

$$b_n = s_n t_n \tag{2.5}$$

*where* $s_n$ *and* $t_n$ *are given by*

$$s_n^2 := \sum_{i=1}^{n} \left( \frac{1}{f_1(i)^2} + \frac{1}{f_2(i)^2} \right) \quad \textit{and} \quad t_n^2 := 2 \log_2 s_n^2,$$

*and* $\log_2$ *is defined in* (2.3). *Also, let* $I = \liminf_{n\to+\infty} a_n/b_n$ *and* $S = \limsup_{n\to+\infty} a_n/b_n$*. Then*

1. *If* $1 < I \leq S$ *then* $X_2(t) - X_1(t) \to +\infty$ *a.s. and therefore bin 2 reaches eventual leadership with probability 1.*
2. *If* $I \leq S < 1$ *then* $\liminf_{t\to+\infty} X_2(t) - X_1(t) = -\infty$ *and* $\limsup_{t\to+\infty} X_2(t) - X_1(t) = +\infty$ *a.s. and this implies that there are endless leadership changes with probability 1.*

The theorem above does not consider all cases of GLIL functions. There is the case where $I \leq 1 \leq S$, and what happens on these cases is still an open problem. Now, an important corollary of the above theorem.

**Corollary 2.3.** *Let* $0 < \alpha \leq 1/2$ *and define* $f_1(i) = i^\alpha$, $f_2(i) = r_i i^\alpha$, *where* $r_i \searrow r \geq 1$. *Then* $(f_1, f_2)$ *is GLIL. Let* $r_i = 1 + ci^{-\beta}$ *with* $c, \beta > 0$. *If* $\beta < 1/2$ *or* $\beta = \alpha = 1/2$ *then* $I = S = +\infty$ *and if* $\beta > 1/2$ *or* $\alpha < 1/2 = \beta$ *then* $I = S = 0$.

Analyzing $I$ and $S$ in the above corollary when $\alpha < 1/2$, if $\beta < 1/2$ we have $I = S = +\infty$ and if $\beta \geq 1/2$ then $I = S = 0$. When we decrease the value of $\beta$, we are decreasing the speed of the convergence $r_i \searrow 1$, hence increasing the advantage of bin 2. Note that decreasing $\beta$ from $1/2$ to $1/2 - \delta$ is the same as multiply $i^{-1/2}$ by $h(i) = i^\delta$. But, it does not matter how small $\delta > 0$ is, we always change the regime. So, what about taking $h(i)$ such that $h(i) \to +\infty$ and $h(i) = o(i^\delta)$ for all $\delta > 0$?

**Corollary 2.4.** *Let* $0 < \alpha < 1/2$ *and define* $f_1(i) = i^\alpha$, $f_2(i) = r_i i^\alpha$ *where* $r_i = 1 + i^{-1/2}h(i)$ *and*

$$h(i) = C\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}. \tag{2.6}$$

*If* $C > 1$ *then bin 2 reaches eventual leadership with probability 1. If* $0 < C < 1$ *then there are endless leadership changes with probability 1.*

**Corollary 2.5.** *Let* $0 < \alpha < 1/2$, $r_i \searrow 1$ *a sequence such that*

$$\liminf_{i\to+\infty} \frac{r_i - 1}{i^{-1/2}\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}} = C_1$$

$$\limsup_{i\to+\infty} \frac{r_i - 1}{i^{-1/2}\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}} = C_2$$

*and define* $f_1(i) = i^\alpha$ *and* $f_2(i) = r_i i^\alpha$. *If* $C_1 > 1$ *bin 2 reaches eventually leadership with probability 1 and if* $C_2 < 1$ *there are endless leadership changes with probability 1.*

In the phase diagram of $\alpha$ and $\beta$ (see Figure 1.1), the most interesting point is $\alpha = \beta = 1/2$. It is in the boundary of all three regimes and we know it belongs to the regime where the probability of bin 2 reaches eventual leadership is 1. What happens if we change the speed of convergence of the fitness multiplying it by a function $h(i)$ or change how fast the reinforcement functions goes to infinity multiplying $\sqrt{i}$ by $g(i)$?

Let $f_1(i) = \sqrt{i}$ and $f_2(i) = (1 + h(i)i^{-1/2})\sqrt{i}$. Let's consider the role of $h(i)$. As we said before, it is responsible for increasing or decreasing the speed at which $r_i$

approaches 1. If it decreases the speed, bin 2 will have more advatage, so, since without $h(i)$ the probability of bin 2 reaches eventual leadership is already 1, then, deacresing the speed, the probability of bin 2 reaches eventual leadership remains 1. Now, if $h(i)$ increases the speed, it makes bin 2 to have less advantage. The following corollary tells us how fast $h(i)$ should go to zero in order to change the regime.

**Corollary 2.6.** *Let $f_1(i) = \sqrt{i}, f_2(i) = (1 + h(i)i^{-1/2})\sqrt{i}$, where*

$$h(i) = C\sqrt{\frac{\log_2 2\log i}{\log i}}.$$

*If $C > 1$ then bin 2 reaches eventual leadership with probability 1 and if $C < 1$ there are endless leadership changes with probability 1.*

The next result deal with the set of GKTS pairs, and tell us necessary and sufficient condition for bin 1 reaches eventual leadership with positive probability.

**Theorem 2.7.** *Let $(f_1, f_2)$ be GKTS (see Definition 2.1), $A_1$ be the event $\{$bin 1 reaches eventual leadership$\}$ and $A_2$ be the event $\{$bin 2 reaches eventual leadership$\}$. Then $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$. Also $\mathbb{P}(A_1) > 0$ if and only if*

$$\sum_{i=1}^{+\infty}\left(\frac{1}{f_1(i)} - \frac{1}{f_2(i)}\right) < +\infty.$$

**Corollary 2.8.** *Let $1/2 < \alpha \leq 1$, $f_1(i) = i^\alpha$ and $f_2(i) = (1 + ci^{-\beta})i^\alpha$ where $c, \beta > 0$. Let $A_1$ and $A_2$ as in Theorem 2.7. Then, $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$. Also $\mathbb{P}(A_1) > 0$ if and only if $\beta > 1 - \alpha$. In particular, when $\beta = 1 - \alpha$ the probability of bin 1 reaching eventual leadership is 0. But if we define $f_2(i) = (1 + h(i)i^{\alpha-1})i^\alpha$ where $h(i)$ is such that*

$$\sum_{i=1}^{+\infty}\frac{h(i)}{i} < +\infty,$$

*then the probability of bin 1 reaching eventual leadership becomes positive.*

For example, if we take $h(i) = (\log i)^{-2}$, then the probability of bin 1 reaching eventual leadership is positive. Now we make a remark about the continuity of $\mathbb{P}$(bin 1 reaches eventual leadership) when $\beta$ is equal to $1 - \alpha$. We prove it in Chapter 4 using Chebyshev's inequality.

**Remark 2.9.** *Let $1/2 < \alpha \leq 1$, $f_1(i) = i^\alpha$ and $f_2(i) = (1 + i^{-\beta})i^\alpha$. Let $\mathbb{P}_\beta$ be the probability of the process $(X_1(t), X_2(t))$ with $\beta > 0$. Let $A = \{$bin 1 reaches eventual* *leadership*}. *Then*

$$\lim_{\beta \to 1-\alpha} \mathbb{P}_\beta(A) = 0.$$

When $f_1(i) = i^\alpha$ and $f_2(i) = (1+i^{-\beta})i^\alpha$ with $\beta > 1/2$, Corollary 2.3 tells us that if $\alpha \leq 1/2$ then there are endless leadership changes with probability 1, but Corollary 2.8 tells us that for $\alpha > 1/2$ the probability of the occurence of endless leadership changes becomes 0. The role of $\alpha$ here is the speed at which $f_1$ and $f_2$ goes to infinity. Changing the power from 1/2 to an $\alpha$ greater and 1/2 is the same as mutiply $f_1(i) = \sqrt{i}$ and $f_2(i) = (1 + i^{-\beta})\sqrt{i}$ by $g(i) = i^{\alpha-1/2}$, and that is enough to change the probability of happens endless leadership changes from 1 to 0. But what about takeing $g(i)$ such that $g(i)$ goes to infinity slower than any positive power of $i$? For example, $g(i) = \log i$? The answer is in Corollary 2.10 bellow.

**Corollary 2.10.** *Let $f_1(i) = g(i)\sqrt{i}$ and $f_2(i) = (1 + i^{-\beta})g(i)\sqrt{i}$ with $\beta > 1/2$. Then the probability of endless leadership changes is 1 if*

$$\sum_{i=1}^{+\infty} \frac{1}{g(i)^2 i} = +\infty$$

*and 0 otherwise.*

For instance, if we take $g(i) = \sqrt{\log i}$ then the probability of endless leadership changes will be 1, but taking $g(i) = \log i$, this probability will be 0.

Returning to the case where $\alpha = \beta = 1/2$, let $f_1(i) = g(i)i^{1/2}$ and $f_2(i) = (1 + i^{-1/2})g(i)i^{1/2}$. With $\beta = 1/2$, we can change regime both decreasing or increasing the speed as $f_1$ and $f_2$ goes to infinity. For example, taking $g(i) = i^{-\delta}$ or $g(i) = i^\delta$ with any $\delta > 0$. But let's see what happens with some choices of $g(i)$ with $g(i) = o(i^\delta)$ for any $\delta > 0$ and $g(i)$ with $i^{-\delta} = o(g(i))$ for any $\delta > 0$.

In the next corollary, we will see that even $g(i) = \exp(-\log^\delta i)$ with $0 < \delta < 1$ is not enough to change from "bin 2 reaches eventual leadership with probability 1" regime to "endless leadership changes" regime. On the other hand, it is enough to take $g(i) = \log^\delta i$ with $\delta > 1$ to change from "bin 2 reaches eventual leadership with probability 1" regime to "bin 1 reaches eventual leadership with positive probability" regime.

**Corollary 2.11.** *Let $f_1(i) = g(i)i^{1/2}$ and $f_2(i) = (1 + i^{-1/2})g(i)i^{1/2}$. If $g(i) = \exp(-\log^\delta i)$ with $0 < \delta < 1$ then bin 2 reaches eventual leadership with probability 1. If $g(i) = \log^\delta i$ with $\delta > 1$ then bin 1 reaches eventual leadership with positive probability.*

Now, we state our main result about the temporal reinforcement functions.

**Theorem 2.12.** *Let* $f_1(i,t) = i^\alpha$ *and* $f_2(i,t) = (1+(t+1)^{-\beta})i^\alpha$ *for* $0 < \alpha < 1$ *and* $\beta > 0$*. Then we have the following regimes:*

1. *if* $0 < \alpha < 1/2$ *and* $\beta < 1/2$*, or* $\alpha = 1/2$ *and* $\beta \leq 1/2$*, or* $1/2 < \alpha < 1$ *and* $\beta \leq 1-\alpha$ *then bin 2 reaches eventual leadership with probability 1.*

2. *if* $0 < \alpha < 1/2$ *and* $\beta \geq 1/2$*, or* $\alpha = 1/2$ *and* $\beta > 1/2$ *then there are endless leadership changes with probability 1.*

3. *If* $1/2 < \alpha < 1$ *and* $\beta > 1-\alpha$ *then bin 1 reaches eventual leadership with positive probability.*

Returning to the atemporal reinforcement functions, the next two theorems are results about the model with $f_1(i) = i^\alpha$ and $f_2(i) = r_i i^\alpha$ with $0 < \alpha \leq 1/2$ and $r_i \searrow r \geq 1$. In [Khanin and Khanin, 2001] they show that for $r_i \equiv 1$, $X_1(t)$ and $X_2(t)$ have the same order, i.e $\frac{X_1(t)}{X_2(t)} \to 1$. We are interested in what happens when $r_i \searrow r \geq 1$.

**Theorem 2.13.** *Let* $f_1(i) = i^\alpha$ *and* $f_2(i) = r_i i^\alpha$ *for* $0 < \alpha \leq 1/2$ *and* $r_i \searrow r \geq 1$*. Then* $\frac{X_1(t)}{X_2(t)} \to r^{-\frac{1}{1-\alpha}}$ *a.s. as* $t \to +\infty$*.*

There is a special case where we can compute the asymptotic behaviour of $(X_1 - X_2)(t)$ as $t \to +\infty$.

**Theorem 2.14.** *Let* $f_1(i) = i^\alpha$ *and* $f_2(i) = (1+i^{-\beta})i^\alpha$ *for* $0 < \alpha \leq 1/2$ *and* $0 < \beta < 1/2$*. Then*

$$(X_2 - X_1)(t) \sim \frac{t^{1-\beta}}{2^{1-\beta}(1-\alpha-\beta)}. \tag{2.7}$$

# Chapter 3

# Preliminares

Here, we present some definitions, techniques and basic results that will be useful in the proofs of our theorems.

## 3.1 Asymptotics

- Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences of real numbers. As $n \to +\infty$, we say:

$$\begin{aligned}
a_n &= O(b_n), \quad \text{if } \limsup_{n\to+\infty} |a_n/b_n| < +\infty, \\
a_n &= o(b_n), \quad \text{if } \lim_{n\to+\infty} a_n/b_n = 0, \\
a_n &= \Omega(b_n), \quad \text{if } b_n = O(a_n), \\
a_n &= \Theta(b_n), \quad \text{if } a_n = O(b_n) \text{ and } b_n = O(a_n).
\end{aligned}$$

- The expression "$a_n \sim b_n$ as $n \to +\infty$" means $\lim_{n\to+\infty} a_n/b_n = 1$ and "$a_n \asymp b_n$" means $a_n = \Theta(b_n)$.

- Whenever $\{X_n\}_{n\geq 1}$ is a sequence of random variables defined in the same probability space, we also use the asymptotics notation stated above. Sometimes there is a little confusion in these cases, because when we deal with random variables, we have more than one type of convergence (see [Janson, 2011] for a detailed account of the asymptotic notation in probability).

  In this work we always use the almost surely convergence. For instance, whenever we write $X_n = O(a_n)$, it should be understood that there exists an event $A$ of probability 1 such that for each $\omega \in A$ we have $X_n(\omega) = O(a_n)$.

## 3.2 Classical results

In this section we present some classical results that will be very useful in our proofs. First we state Kolmogorov three and two series theorems, whose proofs can be found in [Durrett, 1996].

**Theorem 3.1** (Kolmogorov three series). *Let $\xi_1, \xi_2, ...$ be independent. Let $A > 0$ and $Y_i = \xi_i \mathbb{1}_{\{\{|\xi_i| \leq A\}\}}$. In order that $\sum_{n=1}^{+\infty} \xi_n$ converges a.s., it is necessary and sufficient that*

*i)* $\sum_{n=1}^{+\infty} P(|\xi_n| > A) < +\infty$,

*ii)* $\sum_{n=1}^{+\infty} \mathbb{E}Y_n$ *converges, and*

*iii)* $\sum_{n=1}^{+\infty} Var\ (Y_n) < +\infty$.

**Theorem 3.2** (Kolmogorov two series). *Let $\xi_1, \xi_2, ...$ be independent random variables with expected values $\mathbb{E}\xi_n = \mu_n$ and variances $Var\ \xi_n = \sigma_n^2$, such that $\sum_{n=1}^{+\infty} \mu_n$ and $\sum_{n=1}^{+\infty} \sigma_n^2$ converges. Then $\sum_{n=1}^{+\infty} \xi_n$ converges almost surely.*

If $X$ is a random variable, we write $X \overset{d}{=} \exp(\lambda)$ to say that $X$ is exponentially distributed with rate $\lambda$, meaning that

$$\mathbb{P}(X > t) = e^{-\lambda t} \qquad (t \geq 0).$$

This distribution has a density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

It follows that $\mathbb{E}X = \lambda^{-1}$ and Var $X = \lambda^{-2}$. Now we give three results about exponential distribution.

**Proposition 3.3.** *The exponential distribution has memory loss, that is, if $\eta \overset{d}{=} exp(\lambda)$ and $t, s \geq 0$, then*

$$P(\eta > s + t | \eta > s) = P(\eta > t).$$

*Proof.* It follows easily by simple computations

$$\begin{aligned} P(\eta > s + t | \eta > s) &= \frac{P(\eta > s + t \cap \eta > s)}{P(\eta > s)} \\ &= \frac{P(\eta > s + t)}{P(\eta > s)} \end{aligned}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$
$$= e^{-\lambda t} = P(\eta > t).$$

□

**Proposition 3.4.** *If $U$ and $V$ are two independent exponential random variables with parameters $u$ and $v$ respectively, then*

$$P(U < V) = \frac{u}{u+v}.$$

*Proof.* Let $f$ and $g$ be the density functions of $U$ and $V$. Since $U$ and $V$ are independent, the density function $h$ of the random vector $(U, V)$ is given by $h(x, y) = f(x)g(y)$. Hence

$$\begin{aligned} P(U < V) &= \int_0^{+\infty} \int_x^{+\infty} ue^{uvx} ve^{-vy} dy dx \\ &= \int_0^{+\infty} ue^{-ux} e^{-vx} dx \\ &= \int_0^{+\infty} ue^{-(u+v)x} dx \\ &= \left[ -\frac{u}{u+v} e^{-(u+v)x} \right]_0^{+\infty} \\ &= \frac{u}{u+v}. \end{aligned}$$

□

**Proposition 3.5.** *Let $(\lambda_i)_{i\geq 1}$ be a sequence of positive real numbers and let $(\eta_i)_{i\geq 1}$ be a sequence of independent random variables, where for each $i$, $\eta_i \stackrel{d}{=} exp(\lambda_i)$. Then*

*i) If $\sum_{i=1}^{+\infty} \lambda_i^{-1} < +\infty$, then $\sum_{i=1}^{+\infty} \eta_i < +\infty$ a.s..*

*ii) If $\sum_{i=1}^{+\infty} \lambda_i^{-1} = +\infty$, then $\sum_{i=1}^{+\infty} \eta_i = +\infty$ a.s..*

*Proof. i)* The first part follows easily from Theorem 3.2 by noting that $\mathbb{E}\eta_i = \lambda_i^{-1}$, $\text{Var}\,\eta_i = \lambda_i^{-2}$ and that $\sum_{i=1}^{+\infty} \lambda_i^{-1} < +\infty$ implies $\sum_{i=1}^{+\infty} \lambda_i^{-2} < +\infty$.

To prove *ii)*, we break in two cases. The first one when $\lambda_i \not\to +\infty$ and the second one when $\lambda_i \to +\infty$. In the first case, there exists $M > 0$ and a subsequence $\lambda_{i_j}$ such that $\lambda_{i_j} < M$ for all $j$. Hence

$$\sum_{i=1}^{+\infty} P(\eta_i > 1) = \sum_{i=1}^{+\infty} e^{-\lambda_i} \geq \sum_{j=1}^{+\infty} e^{-\lambda_{i_j}} > \sum_{j=1}^{+\infty} e^{-M} = +\infty,$$

and the divergence we want follows from Theorem 3.1.

For the second case, define $Y_i = \eta_i \mathbb{1}_{\{|\eta_i|<1\}}$. Let $N \in \mathbb{N}$ such that if $i \geq N$ then $\lambda_i^{-1} < 1/2$. So, if $i \geq N$

$$\begin{aligned} EY_i &= \frac{1}{\lambda_i} - e^{-\lambda_i}\left(\frac{1}{\lambda_i}+1\right) \\ (\text{using } e^x > 2x) &\geq \frac{1}{\lambda_i} - \frac{1}{2\lambda_i}\left(\frac{1}{\lambda_i}+1\right) \\ &= \frac{1}{2\lambda_i}\left(1-\frac{1}{\lambda_i}\right) \\ (\text{using } \lambda_i^{-1} < 1/2) &> \frac{1}{4\lambda_i}. \end{aligned}$$

Then

$$\sum_{i=1}^{+\infty} \mathbb{E}Y_i \geq \sum_{i=N}^{+\infty} \mathbb{E}Y_i > \frac{1}{4}\sum_{i=N}^{+\infty} \lambda_i^{-1} = +\infty$$

by the hypothesis from *ii)*. So, also by Theorem 3.1, $\sum_{i=1}^{+\infty} \eta_i = +\infty$ almost surely. $\square$

## 3.3 Exponential embedding

The first technique we present is known as exponential embedding (or Rubin representation) and it was first introduced by Davis [Davis, 1990]. The exponential embedding is a countinuous-time process $N(t)$ with state space $(\mathbb{N} \cup \{+\infty\})^2$ and initial state $(a_1, a_2)$. To define the process let $\{\eta_i^{(s)}\}_{i\geq 1, s=1,2}$ be independent random variables such that

$$\eta_i^{(s)} \stackrel{d}{=} \exp(f_s(i)). \tag{3.1}$$

For $s \in \{1, 2\}$, define

$$N_s(t) = \sup\left\{ n \in \mathbb{N} : \sum_{j=a_s}^{n-1} \eta_j^{(s)} \leq t \right\}$$

and let $N(t) = (N_1(t), N_2(t))$ for $t \geq 0$. Consider the set

$$\mathcal{T} = \bigcup_{s=1}^{2} \bigcup_{n=a_s}^{\infty} \left\{ \sum_{j=a_s}^{n-1} \eta_j^{(s)} \right\}$$

of arrival times and let $\{0 = T_0, T_1, T_2, ...\}$ be the increasing ordering of $\mathcal{T}$ (up to the first accumulation point, whenever it exists). The following theorem relates the process

$N(T_n)$ with the original process $(X_1(n), X_2(n))$.

**Theorem 3.6** (Rubin). *The proccess* $(N_1(T_n), N_2(T_n))_{n\in\mathbb{N}}$ *has the same distribution as* $(X_1(n, f_1, f_2, a_1, a_2), X_2(n, f_1, f_2, a_1, a_2))_{n\in\mathbb{N}}$.

*Proof.* The proof of this theorem can be found in [Oliveira, 2008] but we reproduce it here. First note that $T_1 = \min\{\eta_{a_1}^{(1)}, \eta_{a_2}^{(2)}\}$. If $T_1 = \eta_{a_1}^{(1)}$ then the first ball goes to bin 1, otherwise the first ball goes to bin 2. So

$$\mathbb{P}(N_1(T_1) = a_1 + 1, N_2(T_1) = a_2) = \mathbb{P}(\eta_{a_1}^{(1)} < \eta_{a_2}^{(2)}) = \frac{f_1(a_1)}{f_1(a_1) + f_2(a_2)} \tag{3.2}$$

where the last equality comes from Proposition 3.4. And for the same reason

$$\mathbb{P}(N_1(T_1) = a_1, N_2(T_1) = a_2 + 1) = \frac{f_2(a_2)}{f_1(a_1) + f_2(a_2)}.$$

More generally, let $t \in \mathbb{R}^+$ and $b_s \geq a_s$ for each $s \in \{1, 2\}$. Conditioning on

$$\forall s \in \{1, 2\} \qquad \sum_{j=a_s}^{b_s - 1} \eta_j^{(s)} \leq t < \sum_{j=a_s}^{b_s} \eta_j^{(s)},$$

from the memory loss of exponentials (Proposition 3.3), one can deduce that the first arrival after time $t$ at a given bin $s$ will happen at a $\exp(f(b_s))$-distributed time, independently for different bins. This almost takes us back to the situation of (3.2), with $b_s$ replacing $a_s$, and we can similarly deduce that bin $s$ gets the next ball with the desired probability,

$$\frac{f_s(b_s)}{f_1(b_1) + f_2(b_2)}.$$

□

Although the exponential embedding looks more artificial and less transparent than the original process, it is much easier to analyse it mathematically. This simplicity is due to the independence of random variables $\{\eta_j^{(s)}\}$.

## 3.4 The Law of Iterated Logarithm

Another important tool that we will use is a version of the Law of Iterated Logarithm, due to Wittmann [Wittmann, 1985]. This theorem is only useful to the GLIL reinforcement functions, because we need $\sum_{i\in\mathbb{N}} f(i)^{-2} = +\infty$ to use it, as we will see in Lemma 3.8.

**Theorem 3.7** (Law of Iterated Logarithm)**.** *Let $\{\xi_i\}_{i\geq 1}$ be a sequence of independent random variables with*

$$E(\xi_i) = 0, \qquad Var\ (\xi_i) < +\infty \quad (i \geq 1).$$

*We denote for any $n \geq 1$*

$$S_n := \sum_{i=1}^{n} \xi_i, \qquad s_n^2 := Var\ (S_n), \qquad t_n^2 := 2\log_2 s_n^2,$$

*where $\log_2$ was defined in (2.3). If*

*(i)* $\sum_{n=1}^{+\infty} (s_n t_n)^{-p} \mathbb{E}(|\xi_n|^p) < +\infty$ *for some* $2 < p \leq 3$,

*(ii)* $\lim_{n\to+\infty} s_n = +\infty$ *and*

*(iii)* $\limsup_{n\to+\infty} \frac{s_{n+1}}{s_n} < +\infty$,

*then*

$$\limsup_{n\to+\infty} \frac{S_n}{s_n t_n} = 1 \ \ a.s.$$

*and*

$$\liminf_{n\to+\infty} \frac{S_n}{s_n t_n} = -1 \ \ a.s.$$

We use the above theorem to prove some essential lemmas that we state here and prove at Section 4.5. The first lemma will be used in the proof of Theorem 2.2.

**Lemma 3.8.** *Assume $(f_1, f_2)$ is GLIL, let $\eta_i^{(s)}$ be as in (3.1), and let*

$$s_n^2 := \sum_{i=1}^{n} \left( \frac{1}{f_1(i)^2} + \frac{1}{f_2(i)^2} \right) \quad \text{and} \quad t_n^2 := 2\log_2 s_n^2.$$

*Also, define $\xi_i := \eta_i^{(1)} - \eta_i^{(2)}$ and*

$$S_n := \sum_{i=1}^{n} \xi_i. \tag{3.3}$$

*Then*

$$\limsup_{n\to+\infty} \frac{S_n - \mathbb{E}S_n}{s_n t_n} = 1 \ \ a.s.$$

*and*

$$\liminf_{n\to+\infty} \frac{S_n - \mathbb{E}S_n}{s_n t_n} = -1 \ \ a.s.$$

We now present a lemma required in the proof of Theorem 2.13.

**Lemma 3.9.** *Let* $f_1(i) = i^\alpha$, $f_2(i) = r_i i^\alpha$ *where* $r_i \searrow r \geq 1$. *Let* $s \in \{1, 2\}$ *and*

$$S_{s,n} = \sum_{i=1}^{n} \eta_i^{(s)}, \quad s_{s,n}^2 := Var\ (S_{s,n}), \quad t_{s,n}^2 := 2\log_2 s_{s,n}^2. \tag{3.4}$$

*Then*

$$\limsup \frac{S_{s,n} - \mathbb{E}S_{s,n}}{s_{s,n} t_{s,n}} = 1$$

*and*

$$\liminf \frac{S_{s,n} - \mathbb{E}S_{s,n}}{s_{s,n} t_{s,n}} = -1.$$

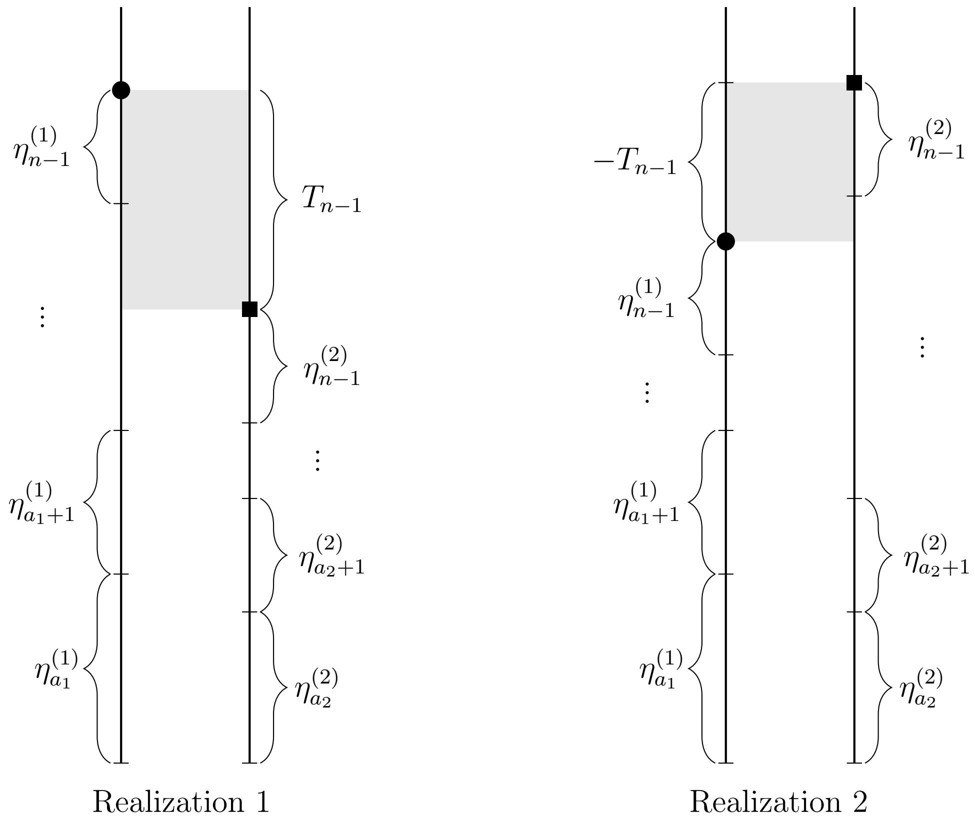


**Figure 3.1.** Two realization of the Rubin process. In the circle mark ● bin 1 get the $n$-th ball. In the square mark ■ bin 2 get the $n$-ball. In realization 1 $T_{n-1} > 0$, wich means bin 2 get the $n$-th ball before bin 1 get the $n$-th ball. In realization 2 $T_{n-1} < 0$, wich means bin 1 get the $n$-th ball before bin 2 get the $n$-th ball.

From now on, we are going to use $T_n$ and $Y$ to represent

$$T_n = \sum_{i=X_1(0)}^{n} \eta_i^{(1)} - \sum_{i=X_2(0)}^{n} \eta_i^{(2)} \tag{3.5}$$

and

$$Y = T_n - \sum_{i=1}^{n} \eta_i^{(1)} - \eta_i^{(2)}. \tag{3.6}$$

We can write $Y$ as

$$Y := \sum_{i=X_1(0)}^{X_2(0)-1} \eta_i^{(1)} + \sum_{i=X_2(0)}^{X_1(0)-1} -\eta_i^{(2)} - \sum_{i=1}^{\max\{X_1(0),X_2(0)\}-1} \eta_i^{(1)} - \eta_i^{(2)}$$

to see it is a finite sum of random variables. Also $T_n$ can be written as

$$T_n := Y + \sum_{i=1}^{n} \eta_i^{(1)} - \eta_i^{(2)}.$$

See Figure 3.1 to see the meaning of $T_n$.

# Chapter 4

# Proofs

Now, it's time to prove our results. To improve the presentation, we state again each result before proving it. In the first section we prove the results about the leadership change for atemporal reinforcement functions, which are Theorems 2.2 and 2.7 and their corollaries. In Section 4.2 we prove our main result, first proving Lemmas 4.1 and 4.2 and then proving Theorem 2.12. After that, in Section 4.3 we prove Theorem 2.13, that deals with the order of $X_i(t)$ for a particular case of atemporal reinforcement functions. In Section 4.4 we prove Theorem 2.14, which is about the order of the difference $(X_2 - X_1)(t)$. And finally, in Section 4.5, we prove the Lemmas 3.8 and 3.9, which are used in some proofs throughout this chapter.

## 4.1 The leadership change

**Theorem 2.2.** *Let $(f_1, f_2)$ be GLIL (see Definition 2.1). Define*

$$a_n = \sum_{i=1}^{n} \left( \frac{1}{f_1(i)} - \frac{1}{f_2(i)} \right) \tag{2.4}$$

*and*

$$b_n = s_n t_n \tag{2.5}$$

*where $s_n$ and $t_n$ are given by*

$$s_n^2 := \sum_{i=1}^{n} \left( \frac{1}{f_1(i)^2} + \frac{1}{f_2(i)^2} \right) \quad \text{and} \quad t_n^2 := 2\log_2 s_n^2,$$

*and $\log_2$ is defined in* (2.3). *Also, let $I = \liminf_{n\to+\infty} a_n/b_n$ and $S = \limsup_{n\to+\infty} a_n/b_n$. Then*

1. *If* $1 < I \le S$ *then* $X_2(t) - X_1(t) \to +\infty$ *a.s. and therefore bin 2 reaches eventual leadership with probability 1.*

2. *If* $I \le S < 1$ *then* $\liminf_{t\to+\infty} X_2(t) - X_1(t) = -\infty$ *and* $\limsup_{t\to+\infty} X_2(t) - X_1(t) = +\infty$ *a.s. and this implies that there are endless leadership changes with probability 1.*

*Proof.* First, define

$$w_n^{(\epsilon)\pm} = a_n - (1 \pm \epsilon) b_n.$$

Note that $1 < I$ if and only if there exists $\epsilon > 0$ such that for large enough $n$ we have

$$\begin{aligned} &(1 + 2\epsilon) < \frac{a_n}{b_n} \\ \Longrightarrow\ &\epsilon b_n < a_n - (1+\epsilon)\, b_n = w_n^{(\epsilon)+} \end{aligned}$$

and since $b_n \to +\infty$ as $n \to +\infty$, it follows that $1 < I$ implies $w_n^{(\epsilon)+} \to +\infty$ for some $\epsilon > 0$. A similar argument allows us to conclude that $S < 1$ implies $w_n^{(\epsilon)-} \to -\infty$ as $n \to +\infty$ for some $0 < \epsilon < 1$. So, for the first part of the theorem, we will use in the proof that $w_n^{(\epsilon)+} \to +\infty$ for some $\epsilon > 0$ and for the second part of the theorem we will use $w_n^{(\epsilon)-} \to -\infty$ for some $0 < \epsilon < 1$.

Let $T_n$ as in (3.5) and note that $T_{n-1} > 0$ means that when the second bin got the $n$-th ball, the first bin had less than $n$ balls (see Figure 3.1). On the other side, $T_{n-1} < 0$ means that when the first bin got the $n$-th ball, the second bin had less than $n$ balls. We will show that when $w_n^{(\epsilon)+} \to +\infty$ for some $\epsilon > 0$ then $T_n \to +\infty$ a.s. and if $w_n^{(\epsilon)-} \to -\infty$ for some $0 < \epsilon < 1$, then $\liminf_{n\to+\infty} T_n = -\infty$ and $\limsup_{n\to+\infty} T_n = +\infty$ a.s..

Let $S_n$, $s_n$ and $t_n$ as in Lemma 3.8. It follows from that lemma

$$\limsup_{n\to+\infty} \frac{S_n - \mathbb{E}S_n}{s_n t_n} = 1 \tag{4.1}$$

and

$$\liminf_{n\to+\infty} \frac{S_n - \mathbb{E}S_n}{s_n t_n} = -1 \tag{4.2}$$

almost surely.

For the first part of the theorem, let $\epsilon > 0$ such that $\lim_{n\to+\infty} w_n^{(\epsilon)+} = +\infty$. It follows from (4.2) that there exists a random variable $n_0 \in \mathbb{N}$ such that with probability 1, if $n > n_0$ then

$$-(1+\epsilon) s_n t_n + \mathbb{E}S_n \le S_n.$$

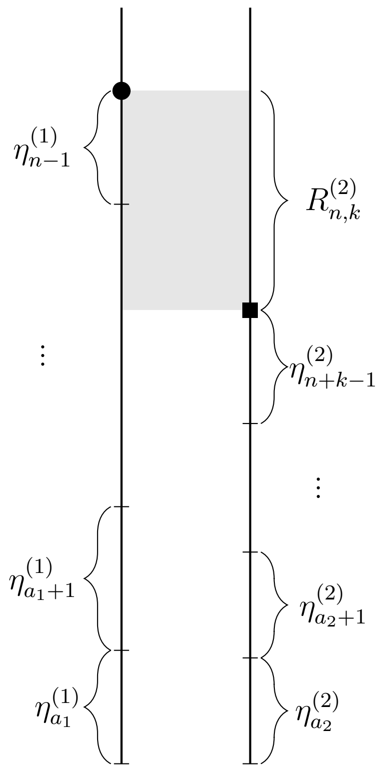


**Figure 4.1.** The circle mark ● is where bin 1 get the $n$-th ball. The square mark ■ is where bin 2 get the $(n+k)$-th ball.

But the left side of inequality above is exactly $w_n^{(\epsilon)+}$, and since $\lim_{n\to+\infty} w_n^{(\epsilon)+} = +\infty$, it follows that $S_n \to +\infty$ a.s.. Since $T_n = S_n + Y$ (see (3.6)) and $Y$ is finite almost surely, it follows that $T_n \to +\infty$ a.s..

Observe that $T_n \to +\infty$ implies not only that $X_2(t) \geq X_1(t)$ for large enough $t$, but also that $X_2(t) - X_1(t) \to +\infty$. Indeed, let $\{\eta_i^{(s)}\}_{i\geq 1, s=1,2}$ as in (3.1) and, given $k \in \mathbb{N}$, define $B_{n,k}^{(s)} = \sum_{j=n+1}^{n+k} \eta_j^{(s)}$ and $R_{n,k}^{(2)} = T_{n-1} - B_{n-1,k}^{(2)}$ (see Figure 4.1). Note that $R_{n,k}^{(2)} > 0$ implies that the second bin got the $(n+k)$-th ball before the first bin has gotten the $n$-th ball. If $R_{n,k}^{(2)} \to +\infty$ then for large enough $t$ we will have $X_2(t) - X_1(t) \geq k$ and since $k$ is arbitrary, it follows that $X_2(t) - X_1(t) \to +\infty$. But indeed $R_{n,k}^{(2)} \to +\infty$ because $\lim_{n\to+\infty} T_n = +\infty$ and $B_{n,k}^{(s)} \to 0$ (since it is a finite sum of random variables that converge to zero almost surely).

For the second part of the theorem, let $0 < \epsilon < 1$ such that $\lim_{n\to+\infty} w_n^{(\epsilon)-} = -\infty$. It follows from (4.2) that there exists a random sequence $n_1, n_2, ...$ such that almost surely $n_k \to +\infty$ as $k \to +\infty$ and

$$S_{n_k} \leq -(1-\epsilon) s_{n_k} t_{n_k} + \mathbb{E} S_{n_k} \quad \text{(for all } k \in \mathbb{N}\text{)} \tag{4.3}$$

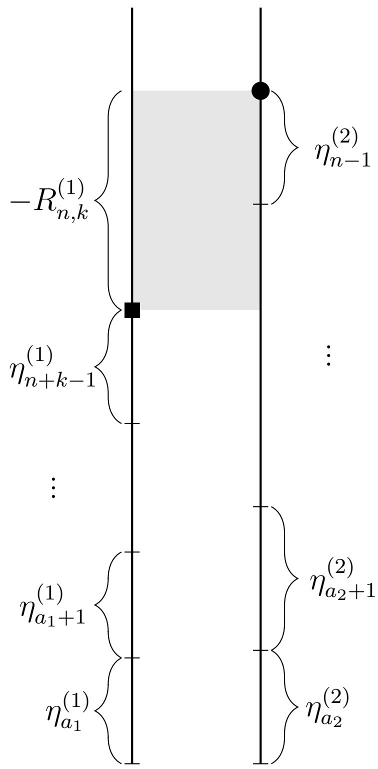


**Figure 4.2.** The square mark ■ is where bin 1 get the $(n+k)$-th ball. The circle mark ● is where bin 2 get the $n$-th ball.

and it follows from (4.1) that there exists another random sequence $m_1, m_2, \dots$ such that almost surely $m_k \to +\infty$ as $k \to +\infty$ and

$$S_{m_k} \geq (1-\epsilon) s_{m_k} t_{m_k} + \mathbb{E} S_{m_k} \quad \text{(for all } k \in \mathbb{N}). \tag{4.4}$$

The right hand side of (4.3) is exactly $w^{(\epsilon)-}_{n_k}$, and since $w^{(\epsilon)-}_{n_k} \to -\infty$, it follows that $\liminf_{n\to+\infty} S_n = -\infty$ a.s.. The right hand side of (4.4) clearly goes to infinity, since $s_n \to +\infty$ (see Item 2 of Definition 2.1) and $\mathbb{E}S_n \geq 0$ for all $n \geq 1$. This implies that $\limsup_{n\to+\infty} S_n = +\infty$. The same is true for $T_n$ since $T_n = S_n + Y$. Defining $R^{(1)}_{n,k} = T_{n-1} + B^{(1)}_{n-1,k}$ it follows easily that $\liminf_{n\to+\infty} R^{(1)}_{n,k} = -\infty$ and $\limsup_{n\to+\infty} R^{(2)}_{n,k} = +\infty$ a.s. (same argument used to prove that $R^{(2)}_{n,k} \to +\infty$ a.s.), where $B^{(1)}_{n-1,k}$ and $R^{(2)}_{n,k}$ were defined in the first part of the proof. Hence, for any $k \in \mathbb{N}$, with probability one, there exists infinity many times $t$ such that $X_1(t) - X_2(t) \geq k$ and infinity many times $t$ such that $X_2(t) - X_1(t) \geq k$. Since $k$ is arbitrary, we have that almost surely $\liminf_{t\to+\infty} X_2(t) - X_1(t) = -\infty$ and $\limsup_{t\to+\infty} X_2(t) - X_1(t) = +\infty$. And that implies endless leadership changes. $\square$

**Corollary 2.3.** *Let* $0 < \alpha \leq 1/2$ *and define* $f_1(i) = i^\alpha$, $f_2(i) = r_i i^\alpha$, *where* $r_i \searrow r \geq 1$.

*Then* $(f_1, f_2)$ *is GLIL. Let* $r_i = 1 + ci^{-\beta}$ *with* $c, \beta > 0$*. If* $\beta < 1/2$ *or* $\beta = \alpha = 1/2$ *then* $I = S = +\infty$ *and if* $\beta > 1/2$ *or* $\alpha < 1/2 = \beta$ *then* $I = S = 0$*.*

*Proof.* Given $0 < \alpha \leq 1/2$, $f_1(i) = i^\alpha$ and $f_2(i) = r_i i^\alpha$ where $r_i \searrow r \geq 1$, we need to check if $(f_1, f_2)$ is GLIL. First note that $f_1(i) \leq f_2(i)$ for all $i \in \mathbb{N}$, since $r_i \geq 1$. Now we note that

$$s_n^2 = \sum_{i=1}^{n} \frac{1 + r_i^2}{r_i^2 i^{2\alpha}} \sim \begin{cases} \frac{2n^{1-2\alpha}}{1-2\alpha} \text{ if } \alpha < 1/2 \\ 2 \log n \text{ if } \alpha = 1/2. \end{cases}$$

and for both $\alpha < 1/2$ and $\alpha = 1/2$ it follows that $\lim_{n \to +\infty} s_n = +\infty$ and $\lim_{n \to +\infty} \frac{s_{n+1}}{s_n} = 1 < +\infty$. For the condition (1) of the Definition 2.1

$$(s_n t_n f_1(n))^{-3} = \frac{1}{s_n^3 \sqrt{(2 \log_2 s_n^2)^3}} \frac{1}{n^{3\alpha}}.$$

Now, we separate in two cases. If $\alpha = 1/2$ then

$$\frac{1}{s_n^3 \sqrt{(2 \log_2 s_n^2)^3}} \frac{1}{n^{3\alpha}} \asymp \frac{1}{(\log n)^3 \sqrt{(2 \log_2 s_n^2)^3}} \frac{1}{n^{3/2}},$$

which is summable. Now, if $\alpha < 1/2$ then

$$\begin{aligned} \frac{1}{s_n^3 \sqrt{(2 \log_2 s_n^2)^3}} \frac{1}{n^{3\alpha}} &\asymp \frac{1}{n^{3/2-3\alpha} \sqrt{(2 \log_2 s_n^2)^3}} \frac{1}{n^{3\alpha}} \\ &= \frac{1}{\sqrt{(2 \log_2 s_n^2)^3}} \frac{1}{n^{3/2}}, \end{aligned}$$

which is also summable. Hence the pair $(f_1, f_2)$ is GLIL. Now, if $r_i = 1 + ci^{-\beta}$ for $\beta > 0$ then

$$\begin{aligned} a_n &= \sum_{i=1}^{n} \frac{1}{(1 + ci^{-\beta}) i^{\alpha+\beta}} \\ &\sim \begin{cases} \frac{n^{1-\alpha-\beta}}{1-\alpha-\beta} & \text{if } 1 - \alpha - \beta > 0 \\ \log n & \text{if } 1 - \alpha - \beta = 0 \\ \text{constant} & \text{if } 1 - \alpha - \beta < 0 \end{cases} \end{aligned}$$

and

$$b_n = \sqrt{\sum_{i=1}^{n} \frac{1 + r_i^2}{r_i^2 i^{2\alpha}}} \sqrt{2 \log_2 s_n^2}$$

$$\sim \begin{cases} \frac{2n^{1/2-\alpha}}{\sqrt{1-2\alpha}}\sqrt{\log_2 \frac{2n^{1-2\alpha}}{1-2\alpha}} & \text{if } \alpha < 1/2 \\ 2\sqrt{\log(n)\log_2 2\log n} & \text{if } \alpha = 1/2. \end{cases}$$

If ($\alpha < 1/2$ and $\beta \geq 1/2$) or ($\alpha = 1/2$ and ($\beta > 1/2$) then $I = S = 0$, otherwise, $I = S = +\infty$. □

**Corollary 2.4.** *Let $0 < \alpha < 1/2$ and define $f_1(i) = i^\alpha$, $f_2(i) = r_i i^\alpha$ where $r_i = 1 + i^{-1/2}h(i)$ and*

$$h(i) = C\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}. \tag{2.6}$$

*If $C > 1$ then bin 2 reaches eventual leadership with probability 1. If $0 < C < 1$ then there are endless leadership changes with probability 1.*

*Proof.* As in the proof of Corollary 2.3, we have

$$b_n \sim \frac{2n^{1/2-\alpha}}{\sqrt{1-2\alpha}}\sqrt{\log_2\left(\frac{2n^{1-2\alpha}}{1-2\alpha}\right)}.$$

If $x = C\sqrt{1-2\alpha}, y = 2/(1-2\alpha)$ and $z = 1-2\alpha$, then

$$\begin{aligned}
a_n &= \sum_{i=1}^n \frac{1}{i^\alpha} - \frac{1}{(1+i^{-1/2}x\sqrt{\log_2 yi^z})i^\alpha} \\
&= \sum_{i=1}^n \frac{i^{-1/2}x\sqrt{\log_2 yi^z}}{(1+i^{-1/2}x\sqrt{\log_2 yi^z})i^\alpha} \\
&= \sum_{i=1}^n \frac{x\sqrt{\log_2 yi^z}}{(1+i^{-1/2}x\sqrt{\log_2 yi^z})i^{\alpha+1/2}} \\
&\sim x\sum_{i=1}^n \frac{\sqrt{\log_2 yi^z}}{i^{\alpha+1/2}} \\
&\sim x\int_k^n \frac{\sqrt{\log_2 yi^z}}{i^{\alpha+1/2}}di \\
&= x\left[\frac{\sqrt{\log_2 yi^z}i^{1/2-\alpha}}{1/2-\alpha}\right]_k^n - \int_k^n \frac{di}{(1-2\alpha)y^2 i^{1/2+\alpha}\log(yi^z)\sqrt{\log_2 yi^z}} \\
&= x\left[\frac{\sqrt{\log_2 yi^z}i^{1/2-\alpha}}{1/2-\alpha}\right]_k^n - O\left(\int_k^n \frac{di}{i^{1/2+\alpha}}\right) \\
&= x\left[\frac{\sqrt{\log_2 yi^z}i^{1/2-\alpha}}{1/2-\alpha}\right]_k^n - O\left(\left[\frac{i^{1/2-\alpha}}{1/2-\alpha}\right]_k^n\right)
\end{aligned}$$

$$\sim x\frac{2n^{1/2-\alpha}\sqrt{\log_2 yn^z}}{1-2\alpha}$$
$$= C\frac{2n^{1/2-\alpha}}{\sqrt{1-2\alpha}}\sqrt{\log_2\left(\frac{2n^{1-2\alpha}}{1-2\alpha}\right)}.$$

Hence,

$$I = S = \lim_{n\to+\infty}\frac{a_n}{b_n} = C$$

and the result follows from Theorem 2.2. □

**Corollary 2.5.** *Let $0<\alpha<1/2$, $r_i \searrow 1$ a sequence such that*

$$\liminf_{i\to+\infty}\frac{r_i-1}{i^{-1/2}\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}} = C_1$$

$$\limsup_{i\to+\infty}\frac{r_i-1}{i^{-1/2}\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}} = C_2$$

*and define $f_1(i)=i^\alpha$ and $f_2(i)=r_i i^\alpha$. If $C_1>1$ bin 2 reaches eventually leadership with probability 1 and if $C_2<1$ there are endless leadership changes with probability 1.*

*Proof.* We prove the result when $C_1>1$. For $C_2<1$ the proof is almost the same. The hypothesis implies that if $C=(C_1+1)/2$ then there exists $i_0\in\mathbb{N}$ such that for $i\ge i_0$ we have

$$r_i > 1 + Ci^{-1/2}\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)} =: r_i'.$$

First note that for both $f_1(i)=i^\alpha$, $f_2(i)=r_i i^\alpha$ and $f_1(i)=i^\alpha$, $f_2(i)=r_i' i^\alpha$ we have

$$b_n \sim \frac{2n^{1/2-\alpha}}{\sqrt{1-2\alpha}}\sqrt{\log_2\left(\frac{2n^{1-2\alpha}}{1-2\alpha}\right)}.$$

Also, using $a_n$ for $f_1(i)=i^\alpha$, $f_2(i)=r_i i^\alpha$ and $a_n'$ for $f_1(i)=i^\alpha$, $f_2(i)=r_i' i^\alpha$ we note

$$a_n' = \sum_{i=1}^{n}\frac{1}{i^\alpha} - \frac{1}{\left(1+Ci^{-1/2}\sqrt{(1-2\alpha)\log_2\left(\frac{2i^{1-2\alpha}}{1-2\alpha}\right)}\right)i^\alpha}$$
$$< \tilde{C} + \sum_{i=1}^{n}\frac{1}{i^\alpha} - \frac{1}{r_i i^\alpha} = \tilde{C} + a_n$$

where $\tilde{C}$ is a positive constant. Then

$$\liminf_{n\to+\infty} \frac{a_n}{b_n} > \liminf_{n\to+\infty} \frac{a'_n - \tilde{C}}{b_n} = C > 1,$$

where the equality above comes from Corollary 2.4 proof. Now, it follows from Theorem 2.2 the desired result. □

**Corollary 2.6.** *Let $f_1(i) = \sqrt{i}$, $f_2(i) = (1 + h(i)i^{-1/2})\sqrt{i}$, where*

$$h(i) = C\sqrt{\frac{\log_2 2\log i}{\log i}}.$$

*If $C > 1$ then bin 2 reaches eventual leadership with probability 1 and if $C < 1$ there are endless leadership changes with probability 1.*

*Proof.* As we saw in the proof of Corollary 2.3, in this case

$$b_n \sim 2\sqrt{\log(n)\log_2 2\log n}.$$

Defining

$$g(x) = 2\sqrt{\log(x)\log_2 2\log x},$$

we have

$$\begin{aligned} g'(x) &= \frac{1}{\sqrt{\log(x)\log_2 2\log x}}\left(\frac{1}{x}\log_2 2\log x + \log x\frac{1}{\log(2\log x)\log(x)x}\right) \\ &= \frac{1}{x}\frac{1}{\sqrt{\log(x)\log_2 2\log x}}\left(\log_2 2\log x + \frac{1}{\log(2\log x)}\right). \end{aligned}$$

Now, note that

$$\begin{aligned} a_n &= \sum_{i=1}^{n} \frac{1}{\sqrt{i}} - \frac{1}{(1+h(i)i^{-1/2})\sqrt{i}} \\ &= \sum_{i=1}^{n} \frac{h(i)}{i} \\ &\sim C\int_k^n \frac{1}{i}\sqrt{\frac{\log_2 2\log i}{\log i}}di \\ &= C\int_k^n \frac{1}{i}\frac{\log_2 2\log i}{\sqrt{\log(i)\log_2 2\log i}}di \\ &\sim C\int_k^n g'(i)di \end{aligned}$$

$$\sim Cg(n) \sim Cb_n.$$

Hence, it follows that

$$I = S = \lim \frac{a_n}{b_n} = C$$

and the result follows from Theorem 2.2. □

**Theorem 2.7.** *Let $(f_1, f_2)$ be GKTS (see Definition 2.1), $A_1$ be the event {bin 1 reaches eventual leadership} and $A_2$ be the event {bin 2 reaches eventual leadership}. Then $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$. Also $\mathbb{P}(A_1) > 0$ if and only if*

$$\sum_{i=1}^{+\infty} \left( \frac{1}{f_1(i)} - \frac{1}{f_2(i)} \right) < +\infty.$$

*Proof.* Let $\eta_i^{(s)}$ be as in (3.1), define $\xi_i := \eta_i^{(1)} - \eta_i^{(2)}$ and let

$$S_n := \sum_{i=1}^{n} \xi_i. \tag{4.5}$$

Note that

$$\mathbb{E}S_n = \sum_{i=1}^{n} \left( \frac{1}{f_1(i)} - \frac{1}{f_2(i)} \right) \quad \text{and} \quad (\text{Var } S_n)^2 = s_n^2 = \sum_{i=1}^{n} \left( \frac{1}{f_1(i)^2} + \frac{1}{f_2(i)^2} \right).$$

Since $(f_1, f_2)$ is GKTS,

$$\lim_{n\to+\infty} \text{Var } S_n = \lim_{n\to+\infty} s_n < +\infty.$$

It follows from Kolmogorov's two-series theorem that $S_n - \mathbb{E}S_n$ converges almost surely as $n \to +\infty$. Let $S = \lim_{n\to+\infty} S_n - \mathbb{E}S_n$.

If $\mathbb{E}S_n \to +\infty$ then $S_n \to +\infty$, since $S$ is finite with probability 1. Hence, $T_n = S_n + Y \to +\infty$. As we conclude in the proof of Theorem 2.2, $T_n \to +\infty$ implies that $X_2(t) - X_1(t) \to +\infty$ as $t \to +\infty$. Since this happens with probability 1, it follows that $\mathbb{P}(A_2) = 1$ and $\mathbb{P}(A_1) = 0$.

Now, if $\lim_{n\to+\infty} \mathbb{E}S_n < +\infty$, then $\lim_{n\to+\infty} S_n = S + \lim_{n\to+\infty} \mathbb{E}S_n$, which is a finite random variable. Hence, $T = \lim_{n\to+\infty} T_n$ is a finite random variable with probability 1, since $T_n = S_n + Y$. The same argument we used to prove that $T_n \to +\infty$ implies $(X_2 - X_1)(t) \to +\infty$ works for proving that $T > 0$ implies $(X_2 - X_1)(t) \to +\infty$ and $T < 0$ implies $(X_1 - X_2)(t) \to +\infty$.

Note that if $\xi_1$ and $\xi_2$ are two independent random variable and for all $a < b$ we have $\mathbb{P}(\xi_1 \in (a,b)) > 0$, then for all $a < b$ we have $\mathbb{P}(\xi_1 + \xi_2 \in (a,b)) > 0$. Also, if $\mathbb{P}(\xi_1 = a) = 0$ for all $a \in \mathbb{R}$ then $\mathbb{P}(\xi_1 + \xi_2 = a) = 0$ for all $a \in \mathbb{R}$.

Let $n > \max\{a_1, a_2\}$. Since $T = (T - (\eta_n^{(1)} - \eta_n^{(2)})) + (\eta_n^{(1)} - \eta_n^{(2)})$, $(T - (\eta_n^{(1)} - \eta_n^{(2)}))$ and $(\eta_n^{(1)} - \eta_n^{(2)})$ are independent and $(\eta_n^{(1)} - \eta_n^{(2)})$ is such that $\mathbb{P}(\eta_n^{(1)} - \eta_n^{(2)} \in (a,b)) > 0$ for all $a < b$ and $\mathbb{P}(\eta_n^{(1)} - \eta_n^{(2)} = a) > 0$ for all $a \in \mathbb{R}$, it follows that $\mathbb{P}(T < 0) > 0$ and $\mathbb{P}(T = 0) = 0$. Hence, $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$ and $\mathbb{P}(A_1) > 0$. □

**Corollary 2.8.** *Let $1/2 < \alpha \leq 1$, $f_1(i) = i^\alpha$ and $f_2(i) = (1 + ci^{-\beta})i^\alpha$ where $c, \beta > 0$. Let $A_1$ and $A_2$ as in Theorem 2.7. Then, $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$. Also $\mathbb{P}(A_1) > 0$ if and only if $\beta > 1 - \alpha$. In particular, when $\beta = 1 - \alpha$ the probability of bin 1 reaching eventual leadership is 0. But if we define $f_2(i) = (1 + h(i)i^{\alpha-1})i^\alpha$ where $h(i)$ is such that*

$$\sum_{i=1}^{+\infty} \frac{h(i)}{i} < +\infty,$$

*then the probability of bin 1 reaching eventual leadership becomes positive.*

*Proof.* First note that $(f_1, f_2)$ is GKTS. Indeed we have

$$\sum_{i=1}^{+\infty} \frac{1}{f_2(i)} = \sum_{i=1}^{+\infty} \frac{1}{(1 + ci^{-\beta})i^\alpha} > \sum_{i=1}^{+\infty} \frac{1}{(1 + c)i^\alpha} = +\infty$$

and

$$(\lim_{n \to +\infty} s_n)^2 = \sum_{i=1}^{+\infty} \frac{1 + (1 + ci^{-\beta})^2}{(1 + ci^{-\beta})^2 i^{2\alpha}} < \sum_{i=1}^{+\infty} \frac{2}{i^{2\alpha}} < +\infty$$

since $2\alpha > 1$. Now observe that

$$\sum_{i=1}^{+\infty} \frac{1}{(1 + c)i^{\alpha+\beta}} < \sum_{i=1}^{+\infty} \left( \frac{1}{f_1(i)} - \frac{1}{f_2(i)} \right) = \sum_{i=1}^{+\infty} \frac{1}{(1 + ci^{-\beta})i^{\alpha+\beta}} < \sum_{i=1}^{+\infty} \frac{1}{i^{\alpha+\beta}}$$

and the first and last series are convergent if and only if $\beta > 1 - \alpha$. The result follows now from Theorem 2.7. □

**Remark 2.9.** *Let $1/2 < \alpha \leq 1$, $f_1(i) = i^\alpha$ and $f_2(i) = (1 + i^{-\beta})i^\alpha$. Let $\mathbb{P}_\beta$ be the probability of the process $(X_1(t), X_2(t))$ with $\beta > 0$. Let $A = \{$bin 1 reaches eventual leadership$\}$. Then*

$$\lim_{\beta \to 1-\alpha} \mathbb{P}_\beta(A) = 0.$$

*Proof.* When $\beta \to (1 - \alpha)^-$, the result is trivial, because $\beta \leq 1 - \alpha$ implies $\mathbb{P}_\beta(A) = 0$. Now, if $\beta > 1 - \alpha$, we saw in the proof of Theorem 2.7 that $T_n \to T$ where $T$ is a finite random variable. Also we noted that $A = \{T < 0\}$. Then

$$\begin{aligned}\mathbb{P}_\beta(A) &= \mathbb{P}_\beta(T < 0)\\ &= \mathbb{P}_\beta(T - \mathbb{E}T < -\mathbb{E}T).\end{aligned}$$

Note that, making $a = \max\{a_1, a_2\}$,

$$\begin{aligned}\mathbb{E}T = \sum_{i=a}^{+\infty} \frac{1}{(1+i^{-\beta})i^{\alpha+\beta}} + \sum_{i=a_1}^{a-1} \frac{1}{(1+i^{-\beta})i^{\alpha}} - \sum_{i=a_2}^{a-1} \frac{1}{i^\alpha}\\ \to +\infty \text{ as } \beta \to (1-\alpha)^+.\end{aligned}$$

Hence, we can assume $\mathbb{E}T > 0$. Retuning to $\mathbb{P}_\beta(A)$, we have

$$\begin{aligned}\mathbb{P}_\beta(A) &= \mathbb{P}_\beta(T < 0)\\ &= \mathbb{P}_\beta(T - \mathbb{E}T < -\mathbb{E}T)\\ &\le \mathbb{P}_\beta(|T - \mathbb{E}T| > \mathbb{E}T)\\ &\le \frac{\text{Var } T}{(\mathbb{E}T)^2}.\end{aligned}$$

We already saw that $\mathbb{E}T \to +\infty$ as $\beta \to (1-\alpha)^+$ and if Var $T$ is bounded as $\beta \to (1-\alpha)^+$ then

$$\frac{\text{Var } T}{(\mathbb{E}T)^2} \to 0 \text{ as } \beta \to (1-\alpha)^+.$$

Indeed

$$\begin{aligned}\text{Var } T &= \sum_{i=a_1}^{+\infty} \frac{1}{(1+i^{-\beta})^2 i^{2\alpha}} + \sum_{i=a_2}^{+\infty} \frac{1}{i^{2\alpha}}\\ &\to \sum_{i=a_1}^{+\infty} \frac{1}{(1+i^{1-\alpha})^2 i^{2\alpha}} + \sum_{i=a_2}^{+\infty} \frac{1}{i^{2\alpha}} \text{ as } \beta \to (1-\alpha)^+\\ &< +\infty.\end{aligned}$$

□

**Corollary 2.10.** *Let $f_1(i) = g(i)\sqrt{i}$ and $f_2(i) = (1+i^{-\beta})g(i)\sqrt{i}$ with $\beta > 1/2$. Then the probability of endless leadership changes is 1 if*

$$\sum_{i=1}^{+\infty} \frac{1}{g(i)^2 i} = +\infty$$

*and 0 otherwise.*

*Proof.* If we name $r_i = 1 + i^{-\beta}$ then

$$s_n^2 = \sum_{i=1}^{n} \frac{1 + r_i^2}{r_i^2 h(i)^2 i},$$

and $\lim_{n\to+\infty} s_n < +\infty$ if and only if

$$\sum_{i=1}^{+\infty} \frac{1}{h(i)^2 i} < +\infty.$$

When the series is divergent, $(f_1, f_2)$ is GLIL and we can use Theorem 2.2. In this case we have

$$a_n = \sum_{i=1}^{n} \frac{1}{r_i h(i) i^{1/2+\beta}}$$

and

$$b_n = \sqrt{\sum_{i=1}^{n} \left(\frac{1 + r_i^2}{r_i^2 h(i)^2 i}\right) 2 \log_2 \sum_{i=1}^{n} \frac{1 + r_i^2}{r_i^2 h(i)^2 i}}.$$

Since $\beta > 1/2$ we have $a_n$ converges and $b_n \to +\infty$. Then

$$I = S = \lim_{n\to+\infty} \frac{a_n}{b_n} = 0.$$

Theorem 2.2 give us that the probability of happens endless leadership changes is 1.

Now, when

$$\sum_{i=1}^{+\infty} \frac{1}{h(i)^2 i} < +\infty,$$

$(f_1, f_2)$ is GKTS and

$$\sum_{i=1}^{+\infty} \frac{1}{f_1(i)} - \frac{1}{f_2(i)} = \sum_{i=1}^{+\infty} \frac{1}{r_i h(i) i^{1/2+\beta}} < +\infty.$$

It allows us to use Theorem 2.7 to conclude not only the probability of happens endless leadership changes is 0 but also that the probability of bin 1 reaches eventual leadership is positive, which was the expected behavior. □

**Corollary 2.11.** *Let* $f_1(i) = g(i) i^{1/2}$ *and* $f_2(i) = (1 + i^{-1/2}) g(i) i^{1/2}$. *If* $g(i) = \exp(-\log^{\delta} i)$ *with* $0 < \delta < 1$ *then bin 2 reaches eventual leadership with probability*

*1. If $g(i) = \log^\delta i$ with $\delta > 1$ then bin 1 reaches eventual leadership with positive probability.*

*Proof.* Let $0 < \delta < 1$, $g(i) = \exp(-\log^\delta i)$, $r_i = 1 + i^{-1/2}$, $f_1(i) = g(i)i^{1/2}$ and $f_2(i) = r_i g(i) i^{1/2}$. One can prove that $(f_1, f_2)$ is GLIL. Let's compute $\lim a_n/b_n$ in order to use Theorem 2.2.

$$\begin{aligned} a_n &= \sum_{i=1}^n \frac{1}{r_i g(i) i} \\ &\sim \sum_{i=1}^n \frac{1}{r_i g(i) i} \\ &\sim \sum_{i=1}^n \frac{\exp(\log^\delta i)}{i} \\ &\sim \int_k^n \frac{\exp(\log^\delta x)}{x} dx \\ &\sim \frac{1}{\delta} \exp(\log^\delta n) \log^{1-\delta} n. \end{aligned}$$

Also

$$\begin{aligned} b_n &= \sqrt{\sum_{i=1}^n \frac{1+r_i^2}{r_i^2 g(i)^2 i}} \sqrt{2\log_2\left(\sum_{i=1}^n \frac{1+r_i^2}{r_i^2 g(i)^2 i}\right)} \\ &\sim 2\sqrt{\sum_{i=1}^n \frac{\exp(2\log^\delta i)}{i}} \sqrt{\log_2\left(\sum_{i=1}^n \frac{\exp(2\log^\delta i)}{i}\right)}. \end{aligned}$$

But, note that

$$\begin{aligned} \sum_{i=1}^n \frac{\exp(2\log^\delta i)}{i} &\sim \int_k^n \frac{\exp(2\log^\delta x)}{x} dx \\ &\sim \frac{1}{2\delta} \exp(2\log^\delta x) \log^{1-\delta} x. \end{aligned}$$

So

$$\begin{aligned} b_n &\sim 2\sqrt{\sum_{i=1}^n \frac{\exp(2\log^\delta i)}{i}} \sqrt{\log_2\left(\sum_{i=1}^n \frac{\exp(2\log^\delta i)}{i}\right)} \\ &\sim 2\sqrt{\frac{1}{2\delta} \exp(2\log^\delta n) \log^{1-\delta} n} \sqrt{\log_2\left(\frac{1}{2\delta} \exp(2\log^\delta n) \log^{1-\delta} n\right)} \end{aligned}$$

$$\sim \exp(\log^\delta n)\sqrt{\frac{2}{\delta}}\sqrt{\log^{1-\delta} n}\sqrt{\log\log^\delta n}.$$

Hence

$$\frac{a_n}{b_n} \sim \frac{\frac{1}{\delta}\exp(\log^\delta n)\log^{1-\delta} n}{\exp(\log^\delta n)\sqrt{\frac{2}{\delta}}\sqrt{\log^{1-\delta} n}\sqrt{\log\log^\delta n}}$$
$$= \frac{1}{\sqrt{2\delta}}\frac{\sqrt{\log^{1-\delta} n}}{\sqrt{\log\log^\delta n}} \to +\infty \text{ as } n \to +\infty.$$

By Theorem 2.2, it follows that bin 2 reaches eventual leadership with probability 1.

Now, let $\delta > 1$, $g(i) = \log^\delta i$, $r_i = 1 + i^{-1/2}$, $f_1(i) = g(i)i^{1/2}$ and $f_2(i) = r_i g(i) i^{1/2}$. In this case, $(f_1, f_2)$ is GKTS. In order to use Theorem 2.7, let's compute $a_n$.

$$a_n = \sum_{i=1}^{n} \frac{1}{r_i g(i) i}$$
$$\leq \sum_{i=1}^{n} \frac{1}{\log^\delta i}.$$

Since $\delta > 1$, it follows that $\lim_{n\to+\infty} a_n < +\infty$. So, by Theorem 2.7, bin 1 reaches eventual leaderhsip with positive probability. □

## 4.2 Main result

Now, we are going to prove our main result about temporal reinforcement functions. First we need two lemmas.

**Lemma 4.1.** *Let $(f_1, f_2)$ and $(\tilde{f}_1, \tilde{f}_2)$ be two pairs of reinforcement functions and $(X_1, X_2)$ and $(\tilde{X}_1, \tilde{X}_2)$ be the processes associated with the pairs $(f_1, f_2)$ and $(\tilde{f}_1, \tilde{f}_2)$ respectivelly, with initial conditions $(a_1, a_2)$. For each $t \geq 0$, let $B_t \subset \mathbb{N} \times \mathbb{N}$. If*

$$\frac{f_2(y,t)}{f_1(x,t) + f_2(y,t)} \leq \frac{\tilde{f}_2(y,t)}{\tilde{f}_1(x,t) + \tilde{f}_2(y,t)} \tag{4.6}$$

*for all $t \geq 0$ and $(x, y) \in B_t$, then there exists a coupling between the processes such that, in this coupling, on the event $\cap_{t\geq 0}\{(X_1(t), X_2(t)) \in B_t\}$, $X_2(t) \leq \tilde{X}_2(t)$ for all $t \geq 0$.*

*Proof.* First, we are going to build the coupling. For that, let $\{U_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{N}}$ be independent random variables uniformly distributed on $[0, 1]$. The process can be seen as a

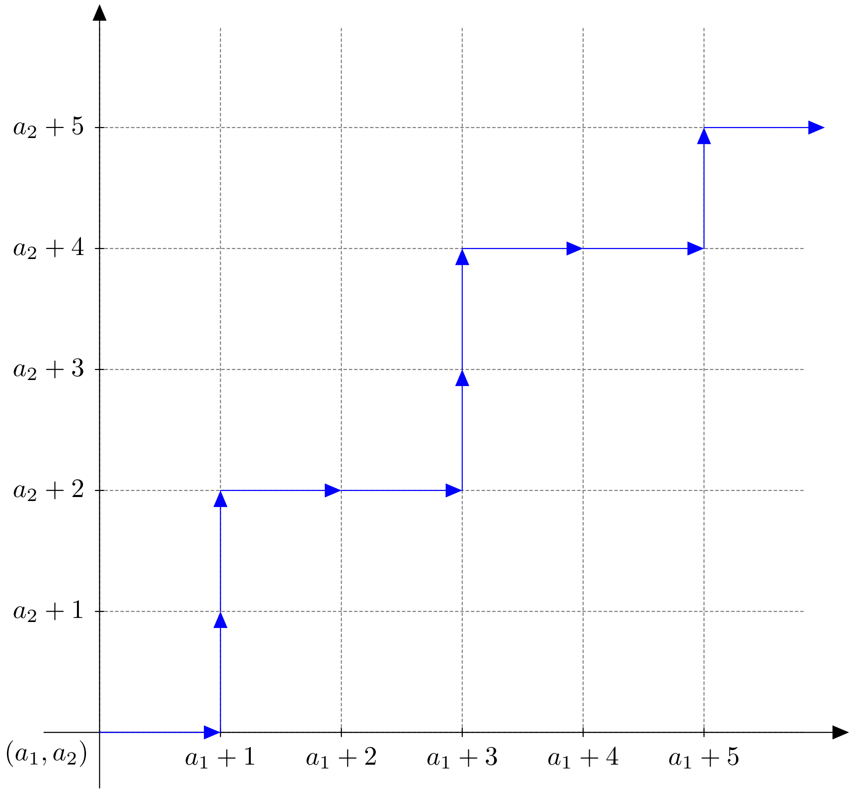


**Figure 4.3.** Realization of $\{U_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{N}}$. The arrows are based on $\tilde{f}_1(i,t) = \tilde{f}_2(i,t) = i^\alpha$ and the rule described in (4.7).

random walk on $\mathbb{Z}^2$ (see Figure 4.3), where we start on $(a_1, a_2)$ and we move either up or to the right accordingly to $\{U_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{N}}$. Assuming $X_1(t) = x$ and $X_2(t) = y$, if

$$U_{x,y} > \frac{f_2(y,t)}{f_1(x,t) + f_2(y,t)} \tag{4.7}$$

then $X_1(t+1) = x+1$ and $X_2(t+1) = y$, otherwise $X_2(t+1) = y+1$ and $X_1(t+1) = x$. We do the same with $(\tilde{X}_1, \tilde{X}_2)$. Now, we assume we are in the event $\cap_{t\geq 0}\{(X_1(t), X_2(t)) \in B_t\}$ and we use induction to prove that $X_2(t) \leq \tilde{X}_2(t)$ for all $t \geq 0$. For $t = 0$, it is obvious, since $X_2(0) = \tilde{X}_2(0) = a_2$. Let $t \geq 0$ and assume $X_2(t) \leq \tilde{X}_2(t)$. If $X_2(t) < \tilde{X}_2(t)$ then it follows that $X_2(t+1) \leq \tilde{X}_2(t+1)$ because only 1 ball is added each time. Now, if $X_2(t) = \tilde{X}_2(t)$, note that $X_1(t) = \tilde{X}_1(t)$ since $X_1(t) + X_2(t) = \tilde{X}_1(t) + \tilde{X}_2(t)$ (the intitial condition is the same for both processes). Let $x \coloneqq X_1(t)$ and $y \coloneqq X_2(t)$. Since we are in the event $\{(X_1(t), X_2(t)) \in B_t\}$,

$$\frac{f_2(y,t)}{f_1(x,t) + f_2(y,t)} \leq \frac{\tilde{f}_2(y,t)}{\tilde{f}_1(x,t) + \tilde{f}_2(y,t)}.$$

If

$$U_{x,y} \leq \frac{f_2(y,t)}{f_1(x,t) + f_2(x,t)}$$

then $X_2(t+1) = \tilde{X}_2(t+1) = y+1$. If

$$\frac{f_2(y,t)}{f_1(x,t) + f_2(x,t)} < U_{x,y} \leq \frac{\tilde{f}_2(y,t)}{\tilde{f}_1(x,t) + \tilde{f}_2(x,t)}$$

then $X_2(t+1) = y$ and $\tilde{X}_2(t+1) = y+1$. Finally, if

$$\frac{\tilde{f}_2(y,t)}{\tilde{f}_1(x,t) + \tilde{f}_2(x,t)} < U_{x,y}$$

then $X_2(t+1) = \tilde{X}_2(t+1) = y$. In all three cases, we have $X_2(t+1) \leq \tilde{X}_2(t+1)$. We can see the coupling in Figure 4.4.

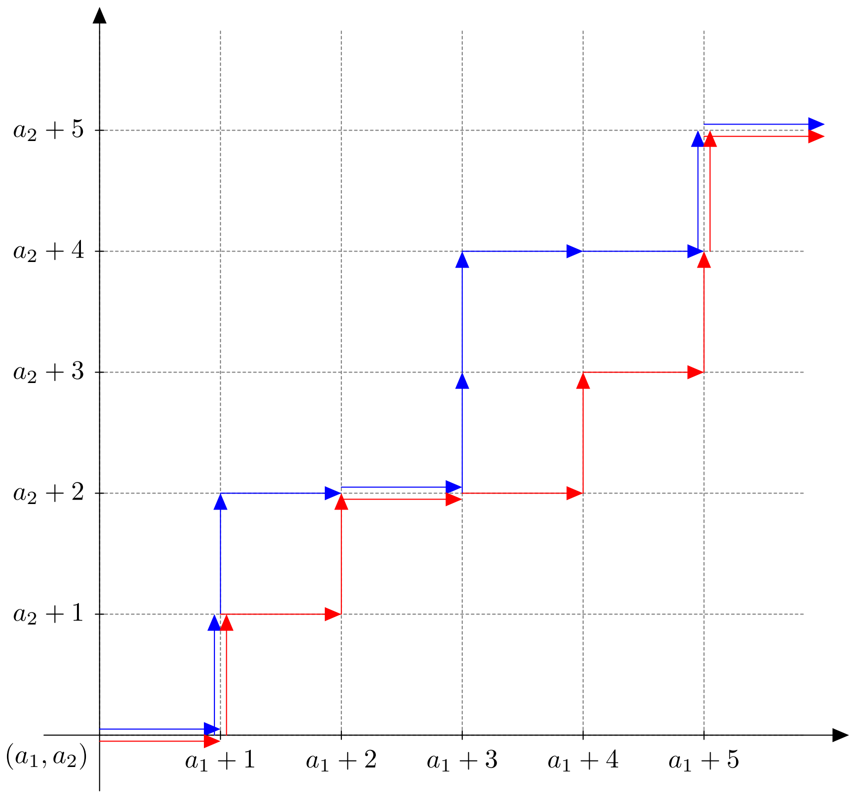


**Figure 4.4.** Realization of $\{U_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{N}}$. The blue arrows are based on $(\tilde{f}_1, \tilde{f}_2)$ and the red arrows are based on $(f_1, f_2)$.

□

**Lemma 4.2.** *Let* $(r(t))_{t\geq 0}$ *be a sequence such that* $r(t) \geq 1$ *for all* $t \geq 0$*, let* $f_1(i,t) = i^\alpha$*,* $f_2(i,t) = r(t)i^\alpha$ *where* $0 < \alpha < 1$ *and let* $a_1, a_2$ *be positive integers. Then, there exists a finite random variable* $s$ *such that, almost surely, for all* $t \geq s$*,* $X_2(t, f_1, f_2, a_1, a_2) \geq t/3$*.*

*Proof.* This follows from a coupling with the model $\tilde{f}_1(i,t) = \tilde{f}_2(i,t) = i^\alpha$. Observe that

$$\frac{\tilde{f}_2(y,t)}{\tilde{f}_1(x,t) + \tilde{f}_2(y,t)} = \frac{y^\alpha}{x^\alpha + y^\alpha} \leq \frac{r(t)y^\alpha}{x^\alpha + r(t)y^\alpha} = \frac{f_2(y,t)}{f_1(x,t) + f_2(y,t)} \tag{4.8}$$

for any $x, y \geq 1$, $t \geq 0$. Then, it follows from Lemma 4.1 that $\tilde{X}_2(t) \leq X_2(t)$ for all $t \geq 0$.

Since $\tilde{X}_2(t) \sim t/2$ (see [Khanin and Khanin, 2001], Proposition 3), it follows that there exists a finite random time $s$ such that

$$X_2(t) \geq t/3 \qquad \text{for all } t \geq s. \tag{4.9}$$

□

**Theorem 2.12.** *Let* $f_1(i,t) = i^\alpha$ *and* $f_2(i,t) = (1 + (t+1)^{-\beta})i^\alpha$ *for* $0 < \alpha < 1$ *and* $\beta > 0$*. Then we have the following regimes:*

1. *if* $0 < \alpha < 1/2$ *and* $\beta < 1/2$*, or* $\alpha = 1/2$ *and* $\beta \leq 1/2$*, or* $1/2 < \alpha < 1$ *and* $\beta \leq 1 - \alpha$ *then bin 2 reaches eventual leadership with probability 1.*
2. *if* $0 < \alpha < 1/2$ *and* $\beta \geq 1/2$*, or* $\alpha = 1/2$ *and* $\beta > 1/2$ *then there are endless leadership changes with probability 1.*
3. *If* $1/2 < \alpha < 1$ *and* $\beta > 1 - \alpha$ *then bin 1 reaches eventual leadership with positive probability.*

*Proof.* 1. To prove the first part of the theorem, let $A_s$ be the event $\{X_2(t) \geq t/3;\ \forall\ t \geq s\}$. Since $(1 + (t+1)^{-\beta}) \geq 1$ for all $t \geq 0$, it follows from Lemma 4.2 that

$$\mathbb{P}\left(\bigcup_{s\geq 1} A_s\right) = 1.$$

So, we only need to prove the result for the events $\{A_s\}_{s\geq 1}$.

Let $\alpha, \beta$ be such that $0 < \alpha < 1/2$ and $\beta < 1/2$ or $\alpha = 1/2$ and $\beta \leq 1/2$ or $1/2 < \alpha < 1$ and $\beta < 1 - \alpha$ and let $f_1(i,t) = i^\alpha$ and $f_2(i,t) = (1 + (t+1)^{-\beta})i^\alpha$. Also, let $s \geq 1$ and define $w = \max\{6, s+1\}$, $\tilde{f}_1(i,t) = i^\alpha$ and $\tilde{f}_2(i,t) = (1 + (wi)^{-\beta})i^\alpha$.

If $t \le s$ then $wy \ge w \ge s+1 \ge t+1$ for all $y \ge 1$. If $t > s$ and $y \ge t/3$,

$$wy \ge w\frac{t}{3} \ge 2t \ge t+1.$$

In both cases $wy \ge t+1$. This implies

$$\begin{aligned} \frac{f_2(y,t)}{f_1(x,t)+f_2(y,t)} &= \frac{(1+(t+1)^{-\beta})y^\alpha}{x^\alpha+(1+(t+1)^{-\beta})y^\alpha} \\ &\ge \frac{(1+(wy)^{-\beta})y^\alpha}{x^\alpha+(1+(wy)^{-\beta})y^\alpha} = \frac{\tilde{f}_2(y,t)}{\tilde{f}_1(x,t)+\tilde{f}_2(y,t)}. \end{aligned}$$

For all $(x,y) \in B_t$ where $B_t \coloneqq \mathbb{N}\times\mathbb{N}$ if $t \le s$ and $B_t \coloneqq \{(x,y) \in \mathbb{N}\times\mathbb{N} : y \ge t/3\}$ for $t > s$. So, it follows from Lemma 4.1 that $X_2(t) \ge \tilde{X}_2(t)$ for all $t \ge 0$ in the event $\cap_{t\ge 0}\{(X_1(t), X_2(t)) \in B_t\}$, but note that $A_s = \cap_{t\ge 0}\{(X_1(t), X_2(t)) \in B_t\}$. By Corollaries 2.3 and 2.8 $\tilde{X}_2(t) - \tilde{X}_1(t) \to +\infty$. Since $X_1(t) \le \tilde{X}_1(t)$ and $X_2(t) \ge \tilde{X}_2(t)$ it follows that $X_2(t) - X_1(t) \to +\infty$ almost surely. It means bin 2 reaches eventual leadership with probability 1.

2. For the second part of the theorem, we define a coupling with two models, $\tilde{f}_1(i,t) = \tilde{f}_2(i,t) = i^\alpha$ and $\hat{f}_1(i,t) = i^\alpha$, $\hat{f}_2(i,t) = (1+(ci)^{-\beta})i^\alpha$, where $c = 1/a_2$. Let $(X_1, X_2)$ be the process associated with $(f_1, f_2)$, $(\tilde{X}_1, \tilde{X}_2)$ the process associated with $(\tilde{f}_1, \tilde{f}_2)$ and $(\hat{X}_1, \hat{X}_2)$ the process associated with $(\hat{f}_1, \hat{f}_2)$. We prove, using Lemma 4.1, that (in a coupling) $\tilde{X}_2(t) \le X_2(t) \le \hat{X}_2(t)$ for all $t \ge 0$. We know from [Khanin and Khanin, 2001] that there are endless leadership changes for $(\tilde{X}_1, \tilde{X}_2)$ and we know from Corollary 2.3 that also we have endless leadership changes for $(\hat{X}_1, \hat{X}_2)$.

   Let $(t_k)_{k\ge 1}$ be the sequence of times such that $\tilde{X}_2(t_k) \ge \tilde{X}_1(t_k)$. Since $X_1(t) + X_2(t) = \tilde{X}_1(t) + \tilde{X}_2(t)$ for all $t \ge 0$, we have $X_2(t_k) \ge \tilde{X}_2(t_k) \ge \tilde{X}_2(t_k) \ge X_1(t_k)$. Also, let $(t_n)_{n\ge 1}$ be the sequence of times such that $\hat{X}_2(t_n) \le \hat{X}_1(t_n)$. Then $X_2(t_n) \le \hat{X}_2(t_n) \le \hat{X}_1(t_n) \le X_1(t_n)$, because we also have $X_1(t) + X_2(t) = \hat{X}_1(t) + \hat{X}_2(t)$ for all $t \ge 0$. Hence, we have endless leadership changes for $(X_1, X_2)$ too.

   To prove that $\tilde{X}_2(t) \le X_2(t) \le \hat{X}_2(t)$, note that

$$\frac{y^\alpha}{x^\alpha+y^\alpha} \le \frac{(1+(t+1)^{-\beta})y^\alpha}{x^\alpha+(1+(t+1)^{-\beta})y^\alpha} \le \frac{(1+(cy)^{-\beta})y^\alpha}{x^\alpha+(1+(cy)^{-\beta})y^\alpha}$$

   for all $t \ge 0$ and $x, y \ge 1$. The remaining follows from Lemma 4.1.

3. For the third and last part of the theorem, we do a coupling with the model $\hat{f}_1(i,t) = i^\alpha$, $\hat{f}_2(i,t) = (1 + (ci)^{-\beta})i^\alpha$ with $c = 1/a_2$. As we saw in the second part of the proof, if $(X_1, X_2)$ is the process associated with $(f_1, f_2)$ and $(\hat{X}_1, \hat{X}_2)$ is the process associated with $(\hat{f}_1, \hat{f}_2)$, then $X_2(t) \le \hat{X}_2(t)$ for all $t \ge 0$ (through coupling). Hence $X_1(t) \ge \hat{X}_1(t)$ for all $t \ge 0$. Since, in the process $(\hat{X}_1, \hat{X}_2)$, there is a positive probability of bin 1 reaches eventual leadership, we have in that event, for large enough $t$, $X_1(t) \ge \hat{X}_1(t) \ge \hat{X}_2(t) \ge X_2(t)$, then the event "bin 1 reaches eventual leadership in the process $(X_1, X_2)$" contains the event "bin 1 reaches eventual leadership in the process $(\hat{X}_1, \hat{X}_2)$". Hence, its probability is also positive.

□

## 4.3 The order of $X_i(t)$

**Theorem 2.13.** *Let $f_1(i) = i^\alpha$ and $f_2(i) = r_i i^\alpha$ for $0 < \alpha \le 1/2$ and $r_i \searrow r \ge 1$. Then $\frac{X_1(t)}{X_2(t)} \to r^{-\frac{1}{1-\alpha}}$ a.s. as $t \to +\infty$.*

*Proof.* Fix $\epsilon > 0$ and $0 < \delta < r^{-\frac{1}{1-\alpha}}$. Let $S_{1,n}$, $s_{1,n}$ and $t_{1,n}$ as in Lemma 3.9. By Lemma 3.9, for large enough $n$

$$S_{1,n\delta} \le (1+\epsilon)s_{1,n\delta}t_{1,n\delta} + \mathbb{E}S_{1,n\delta}$$

and

$$S_{1,n} \ge -(1+\epsilon)s_{1,n}t_{1,n} + \mathbb{E}S_{1,n}.$$

Thus

$$\begin{aligned} S_{1,n} - S_{1,n\delta} &\ge \sum_{i=n\delta}^{n} \frac{1}{i^\alpha} - (1+\epsilon)(s_{1,n}t_{1,n} + s_{1,n\delta}t_{1,n\delta}) \\ &\sim \frac{1-\delta^{1-\alpha}}{1-\alpha} n^{1-\alpha}. \end{aligned} \tag{4.10}$$

Let $S_n$, $s_n$ and $t_n$ as in Lemma 3.8. By Lemma 3.8 for large enough $n$

$$S_n \le (1+\epsilon)s_n t_n + \mathbb{E}S_n.$$

Since $Y = T_n - S_n$ then for large enough $n$

$$T_n = S_n + Y \le (1+\epsilon)s_n t_n + \mathbb{E}S_n + Y.$$

Note that $\delta < r^{-\frac{1}{1-\alpha}}$ implies $\frac{r-1}{r} < 1 - \delta^{1-\alpha}$. So, let $n_0$ be such that

$$C_{n_0} := \sup_{i > n_0} \frac{r_i - 1}{r_i} < 1 - \delta^{1-\alpha}.$$

Thus

$$\begin{aligned} T_n &\leq \mathbb{E}S_n + (1+\epsilon)s_n t_n + Y \\ &= \sum_{i=1}^{n} \frac{r_i - 1}{r_i i^\alpha} + (1+\epsilon)s_n t_n + Y \\ &= \sum_{i=1}^{n_0} \frac{r_i - 1}{r_i i^\alpha} + \sum_{i=n_0+1}^{n} \frac{r_i - 1}{r_i i^\alpha} + (1+\epsilon)s_n t_n + Y \qquad (4.11) \\ &\leq \sum_{i=1}^{n_0} \frac{r_i - 1}{r_i i^\alpha} + C_{n_0} \sum_{i=n_0+1}^{n} \frac{1}{i^\alpha} + (1+\epsilon)s_n t_n + Y \\ &\begin{cases} \sim C_{n_0} \frac{n^{1-\alpha}}{1-\alpha} & \text{if } C_{n_0} > 0 \\ = o\left(\frac{n^{1-\alpha}}{1-\alpha}\right) & \text{if } C_{n_0} = 0 \end{cases} \end{aligned}$$

Note that since $C_{n_0} < 1 - \delta^{1-\alpha}$, it follows from (4.10) and (4.11) that for large enough $n$

$$T_n < S_{1,n} - S_{1,n\delta}.$$

This means that for large enough $n$, when the bin 2 get the $(n+1)$-th ball, the bin 1 has at least $n\delta + 1$ balls (see Figure 4.5) and then, for large enough $t$

$$\frac{X_1(t)}{X_2(t)} \geq \frac{X_2(t)\delta}{X_2(t)} = \delta.$$

Since $\delta < r^{-\frac{1}{1-\alpha}}$ is arbritary, we have

$$\liminf_{t \to +\infty} \frac{X_1(t)}{X_2(t)} \geq r^{-\frac{1}{1-\alpha}}. \qquad (4.12)$$

To prove that $\limsup_{t\to+\infty} \frac{X_1(t)}{X_2(t)} \leq r^{-\frac{1}{1-\alpha}}$, we need to split in two cases. First assume $r > 1$ and let $r^{-\frac{1}{1-\alpha}} < \delta < 1$ and $\lambda = \delta^{-1}$. Note that $\lambda > 1$. Let $S_{2,n}$, $s_{2,n}$ and $t_{2,n}$ as in Lemma 3.9. We can conclude that $S_{2,n}$ obey the Law of Iterated Logarithm (Lemma 3.9) and then, for large enough $n$

$$S_{2,n\lambda} \leq \mathbb{E}S_{2,n\lambda} + (1+\epsilon)s_{2,n\lambda}t_{2,n\lambda}$$

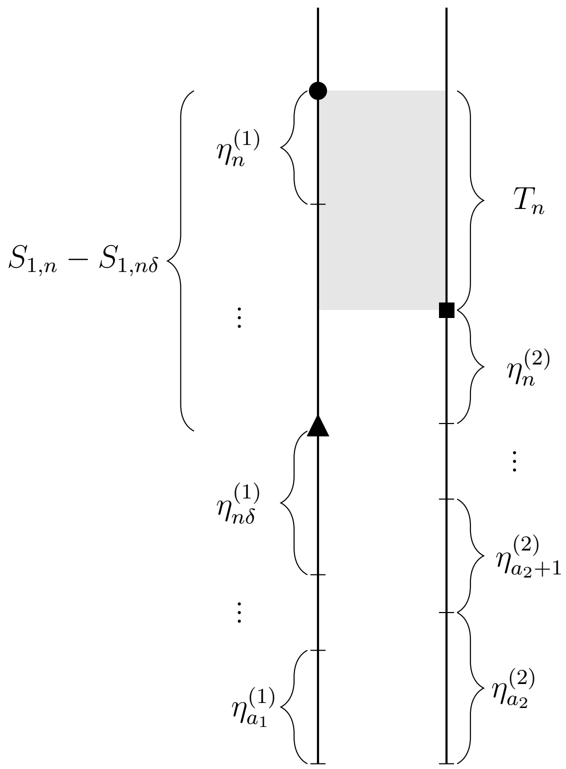


**Figure 4.5.** The triangle mark ▲ is where bin 1 get the $(n\delta+1)$-th ball. The circle mark ● is where bin 1 get the $(n+1)$-th ball. The square mark ■ is where bin 2 get the $(n+1)$-th ball.

and

$$S_{2,n} \geq \mathbb{E}S_{2,n} - (1+\epsilon)s_{2,n\lambda}t_{2,n\lambda}.$$

Then

$$\begin{aligned} S_{2,n\lambda} - S_{2,n} &\leq \sum_{i=n}^{n\lambda} \frac{1}{r_i i^\alpha} + (1+\epsilon)(s_{2,n}t_{2,n} + s_{2,n\lambda}t_{2,n\lambda}) \\ &\leq \frac{1}{r}\sum_{i=n}^{n\lambda} \frac{1}{i^\alpha} + (1+\epsilon)(s_{2,n}t_{2,n} + s_{2,n\lambda}t_{2,n\lambda}) \qquad (4.13) \\ &\sim \frac{\lambda^{1-\alpha}-1}{r}\frac{n^{1-\alpha}}{1-\alpha}. \end{aligned}$$

Also, for large enough $n$

$$\begin{aligned} T_n &\geq \mathbb{E}S_n - (1+\epsilon)s_n t_n + Y \\ &= \sum_{i=1}^{n} \frac{r_i - 1}{r_i i^\alpha} - (1+\epsilon)s_n t_n + Y \end{aligned}$$

$$\geq \frac{r-1}{r}\sum_{i=1}^{n}\frac{1}{i^{\alpha}} - (1+\epsilon)s_n t_n + Y \tag{4.14}$$
$$\sim \frac{r-1}{r}\frac{n^{1-\alpha}}{1-\alpha}.$$

Since $\delta > r^{-\frac{1}{1-\alpha}}$ implies $\lambda^{1-\alpha} < r$, it follows from (4.13) and (4.14) that for large enough $n$

$$S_{2,n\lambda} - S_{2,n} < T_n$$

and this implies that for large enough $n$, when the bin 1 get the $n$-th ball, the bin 2 has at least $n\lambda$ balls and then, for large enough $t$,

$$\frac{X_1(t)}{X_2(t)} \leq \frac{X_1(t)}{\lambda X_1(t)} = \lambda^{-1} = \delta.$$

And since $r^{-\frac{1}{1-\alpha}} < \delta < 1$ is arbritrary, we have

$$\limsup_{t\to+\infty}\frac{X_1(t)}{X_2(t)} \leq r^{-\frac{1}{1-\alpha}}. \tag{4.15}$$

Combining (4.12) with (4.15) we conclude

$$\lim_{t\to+\infty}\frac{X_1(t)}{X_2(t)} = r^{-\frac{1}{1-\alpha}}.$$

Now, assume $r = 1$. Let $r_0 = \sup_{i\geq 1} r_i$, $\delta < 1$ and $\epsilon > 0$, then, for large enough $n$,

$$S_{2,n\delta} \leq \mathbb{E}S_{2,n\delta} + (1+\epsilon)s_{2,n\delta}t_{2,n\delta}$$

and

$$S_{2,n} \geq \mathbb{E}S_{2,n} - (1+\epsilon)s_{2,n}t_{2,n}.$$

So,

$$S_{2,n} - S_{2,n\delta} \geq \sum_{i=n\delta}^{n}\frac{1}{r_i i^{\alpha}} - (1+\epsilon)(s_{2,n}t_{2,n} + s_{2,n\delta}t_{2,n\delta})$$
$$\geq \frac{1}{r_0}\sum_{i=n\delta}^{n}\frac{1}{i^{\alpha}} - (1+\epsilon)(s_{2,n}t_{2,n} + s_{2,n\delta}t_{2,n\delta}) \tag{4.16}$$
$$\sim \frac{1-\delta^{1-\alpha}}{r_0}\frac{n^{1-\alpha}}{1-\alpha}.$$

Also, for large enough $n$ we have

$$\begin{aligned} -T_n &\le -\mathbb{E}S_n + (1+\epsilon)s_n t_n - Y \\ &\le (1+\epsilon)s_n t_n - Y. \end{aligned} \tag{4.17}$$

Since $(1+\epsilon)s_n t_n - Y = o(n^{1-\alpha})$, follows from (4.16) and (4.17) that for large enough $n$ we have

$$-T_n < S_{2,n} - S_{2,n\delta}$$

and this implies that for large enough $n$, when the bin 1 get the $n$-th ball, the bin 2 has at least $n\delta$ balls and then, for large enough $t$

$$\frac{X_1(t)}{X_2(t)} \le \frac{X_1(t)}{X_1(t)\delta} = \delta^{-1}.$$

And since $\delta < 1$ is arbritrary, we have

$$\limsup_{t\to+\infty} \frac{X_1(t)}{X_2(t)} \le 1. \tag{4.18}$$

Combining (4.12) (with $r = 1$) with (4.18) we conclude

$$\lim_{t\to+\infty} \frac{X_1(t)}{X_2(t)} = 1.$$

□

## 4.4 The order of $(X_2 - X_1)(t)$

**Theorem 2.14.** *Let $f_1(i) = i^\alpha$ and $f_2(i) = (1 + i^{-\beta})i^\alpha$ for $0 < \alpha \le 1/2$ and $0 < \beta < 1/2$. Then*

$$(X_2 - X_1)(t) \sim \frac{t^{1-\beta}}{2^{1-\beta}(1-\alpha-\beta)}. \tag{2.7}$$

*Proof.* Let $T_n$ as in (3.5). Using Lemma 3.8, we can see that

$$T_n = \frac{n^{1-\alpha-\beta}}{1-\alpha-\beta} + o\left(n^{1/2-\alpha+\epsilon}\right),$$

for any $\epsilon > 0$. Also, defining $S_{1,n}$ and $S_{2,n}$ as in Lemma 3.9, using that lemma we conclude

$$S_{1,n} = \frac{n^{1-\alpha}}{1-\alpha} + o\left(n^{1/2-\alpha+\epsilon}\right)$$

and
$$S_{2,n} = \frac{n^{1-\alpha}}{1-\alpha} + o\left(n^{1/2-\alpha+\epsilon}\right).$$
for any $\epsilon > 0$. Note that $T_n = S_{1,n} - S_{2,n} + Y$, where $Y$ was defined at (3.6). Since $T_n \to +\infty$ as $n \to +\infty$, we have that $S_{1,n}$ will be greater than $S_{2,n}$ for large $n$. Let $j_n = \max\{i \in \mathbb{N} : S_{2,n+i} < S_{1,n}\}$ . Clearly

$$|S_{1,n} - S_{2,n+j_n}| \leq |S_{2,n+j_n+1} - S_{2,n+j_n}| = \eta^{(2)}_{n+j_n+1} \to 0$$

as $n \to +\infty$, so

$$\begin{aligned} S_{2,n+j_n} - S_{2,n} &= S_{1,n} + (S_{2,n+j_n} - S_{1,n}) - S_{2,n} \\ &= S_{1,n} + o(1) - S_{2,n} = \frac{n^{1-\alpha-\beta}}{1-\alpha-\beta} + o\left(n^{1/2-\alpha+\epsilon}\right). \end{aligned}$$

Also
$$S_{2,n+j_n} - S_{2,n} = \frac{(n+j_n)^{1-\alpha}}{1-\alpha} - \frac{n^{1-\alpha}}{1-\alpha} + o\left(n^{1/2-\alpha+\epsilon}\right).$$
Observe that $j_n$ is the amount of balls bin 2 has more than bin 1 when bin 1 get the $n$-th ball, that is, $j_n \sim (X_2 - X_1)(2n + j_n)$. We can find the asymptotic behaviour of $j_n$ solving the equation

$$\frac{(n+j_n)^{1-\alpha} - n^{1-\alpha}}{1-\alpha} = \frac{n^{1-\alpha-\beta}}{1-\alpha-\beta} + o\left(n^{1/2-\alpha+\epsilon}\right).$$

Note that

$$\begin{aligned} \frac{(n+j_n)^{1-\alpha} - n^{1-\alpha}}{1-\alpha} &= \frac{1}{1-\alpha} n^{1-\alpha}\left(\left(1+\frac{j_n}{n}\right)^{1-\alpha} - 1\right) \\ &= \frac{1}{1-\alpha} n^{1-\alpha}(1-\alpha)\frac{j_n}{n}(1+o(1)) \\ &= \frac{j_n}{n^\alpha}(1+o(1)) \\ &= \frac{n^{1-\alpha-\beta}}{1-\alpha-\beta} + o\left(n^{1/2-\alpha+\epsilon}\right). \end{aligned}$$

Taking $\epsilon < 1/2 - \beta$ we conclude

$$j_n \sim \frac{n^{1-\beta}}{1-\alpha-\beta}.$$

That implies

$$(X_2 - X_1)(2n + j_n) \sim \frac{n^{1-\beta}}{1 - \alpha - \beta}.$$

Finally, for $n \in \mathbb{N}$, let $t \in \mathbb{N}$ such that

$$2n + j_n \leq t \leq 2(n+1) + j_{n+1}. \tag{4.19}$$

Note that $n \leq X_1(t) \leq n + 1$ and $n + j_n \leq X_2(t) \leq n + 1 + j_{n+1}$. Then

$$j_n - 1 \leq (X_2 - X_1)(t) \leq 1 + j_{n+1}.$$

Dividing the inequality above by $a_t := t^{1-\beta}/(1 - \alpha - \beta)$

$$\frac{j_n - 1}{a_n}\frac{a_n}{a_t} \leq \frac{(X_2 - X_1)(t)}{a_t} \leq \frac{1 + j_{n+1}}{a_n}\frac{a_n}{a_t}. \tag{4.20}$$

Also, raising the terms of inequalities (4.19) to $1 - \beta$ and dividing by $n^{1-\beta}$ we get

$$\frac{(2n + j_n)^{1-\beta}}{n^{1-\beta}} \leq \frac{t^{1-\beta}}{n^{1-\beta}} \leq \frac{(2(n+1) + j_{n+1})^{1-\beta}}{n^{1-\beta}}. \tag{4.21}$$

Making $n \to +\infty$ in inequalities (4.21) we conclude $a_t/a_n = t^{1-\beta}/n^{1-\beta} \to 2^{1-\beta}$. This together inequalities (4.20) implies

$$\frac{(X_2 - X_1)(t)}{t^{1-\beta}} \to \frac{1}{2^{1-\beta}(1 - \alpha - \beta)}$$

as $t \to +\infty$. □

## 4.5 The proof of lemmas 3.8 and 3.9

Finally, in this section we prove the two essential lemmas that come from the Law of Iterated Logarithm. The first one was used in the proof of Theorem 2.2.

**Lemma 3.8.** *Assume* $(f_1, f_2)$ *is GLIL, let* $\eta_i^{(s)}$ *be as in* (3.1), *and let*

$$s_n^2 := \sum_{i=1}^{n} \left( \frac{1}{f_1(i)^2} + \frac{1}{f_2(i)^2} \right) \quad \textit{and} \quad t_n^2 := 2 \log_2 s_n^2.$$

*Also, define* $\xi_i := \eta_i^{(1)} - \eta_i^{(2)}$ *and*

$$S_n := \sum_{i=1}^{n} \xi_i. \tag{3.3}$$

*Then*

$$\limsup_{n\to+\infty} \frac{S_n - \mathbb{E}S_n}{s_n t_n} = 1 \ a.s.$$

*and*

$$\liminf_{n\to+\infty} \frac{S_n - \mathbb{E}S_n}{s_n t_n} = -1 \ a.s.$$

*Proof.* We only need to check if $(Z_i)_{i\geq 1}$ given by $Z_i = X_i - \mathbb{E}X_i$ satisfies the three conditions of Theorem 3.7. The conditions (ii) and (iii) follow from Items 2 and 3 of Definition 2.1 (GLIL definition).

For (i) we observe that using Minkowski inequality

$$\begin{aligned}
\mathbb{E}(|Z_n|^3)^{\frac{1}{3}} &= \left\| \eta_n^{(1)} - \eta_n^{(2)} - \frac{1}{f_1(n)} + \frac{1}{f_2(n)} \right\|_3 \\
&\leq \|\eta_n^{(1)}\|_3 + \|\eta_n^{(2)}\|_3 + \frac{1}{f_1(n)} + \frac{1}{f_2(n)} \\
&= \frac{\sqrt[3]{3!}}{f_1(n)} + \frac{\sqrt[3]{3!}}{f_2(n)} + \frac{1}{f_1(n)} + \frac{1}{f_2(n)} \\
&\leq \frac{2(\sqrt[3]{3!}+1)}{f_1(n)}.
\end{aligned}$$

Hence

$$(s_n t_n)^{-3}\mathbb{E}(|Z_n|^3) \leq 8(\sqrt[3]{3!}+1)^3 (s_n t_n f_1(n))^{-3}.$$

Since the right side of the above inequality is sumable (by Item 1 of Definition 2.1), it follows that (i) holds. So, by Theorem 3.7 we conclude the desired result. □

The second one was used in the proof of Theorem 2.7.

**Lemma 3.9.** *Let* $f_1(i) = i^\alpha$, $f_2(i) = r_i i^\alpha$ *where* $r_i \searrow r \geq 1$. *Let* $s \in \{1, 2\}$ *and*

$$S_{s,n} = \sum_{i=1}^{n} \eta_i^{(s)}, \quad s_{s,n}^2 := Var\ (S_{s,n}), \quad t_{s,n}^2 := 2\log_2 s_{s,n}^2. \tag{3.4}$$

*Then*

$$\limsup \frac{S_{s,n} - \mathbb{E}S_{s,n}}{s_{s,n} t_{s,n}} = 1$$

*and*

$$\liminf \frac{S_{s,n} - \mathbb{E}S_{s,n}}{s_{s,n}t_{s,n}} = -1.$$

*Proof.* We will prove for $s = 1$. For $s = 2$ the proof is similar. We will use the Theorem 3.7 on $\xi_i = \eta_i^{(1)} - \frac{1}{i^\alpha}$. Conditions $(ii)$ and $(iii)$ are easily verifiable. For condition (i), observe that

$$\begin{aligned}\mathbb{E}\left|\eta_i^{(1)} - \frac{1}{i^\alpha}\right|^3 &= i^\alpha \int_0^{+\infty} \left|x - \frac{1}{i^\alpha}\right|^3 e^{-i^\alpha x} dx \\ &= \frac{2(6-e)}{e i^{3\alpha}}.\end{aligned}$$

Then if $\alpha < 1/2$

$$\begin{aligned}(s_{1,n}t_{1,n})^{-3}\mathbb{E}(|\xi_n|^3) &\asymp \frac{1}{n^{3/2-3\alpha}t_{1,n}^3}\frac{1}{n^{3\alpha}} \\ &= \frac{1}{t_{1,n}^3}\frac{1}{n^{3/2}},\end{aligned}$$

which is sumable. If $\alpha = 1/2$ then

$$\begin{aligned}(s_{1,n}t_{1,n})^{-3}\mathbb{E}(|S_{2,n}|^3) &\asymp \frac{1}{t_{1,n}^3 \log n}\frac{1}{n^{3\alpha}} \\ &= \frac{1}{t_{1,n}^3 \log n}\frac{1}{n^{3/2}},\end{aligned}$$

which is also sumable. Then, by Theorem 3.7, we have de desired result. □

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