Geom Dedicata (2016) 183:25–31
DOI 10.1007/s10711-016-0141-9

ORIGINAL PAPER

Rank two nilpotent co-Higgs sheaves on complex surfaces

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Received: 25 April 2015 / Accepted: 8 January 2016 / Published online: 19 January 2016

Abstract Let $(\mathcal{E}, \phi)$ be a rank two co-Higgs vector bundles on a Kähler compact surface X with $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$ nilpotent. If $(\mathcal{E}, \phi)$ is semi-stable, then one of the following holds up to finite étale cover: (1) X is uniruled. (2) X is a torus and $(\mathcal{E}, \phi)$ is strictly semi-stable. (3) X is a properly elliptic surface and $(\mathcal{E}, \phi)$ is strictly semi-stable.

Keywords co-Higgs bundles · Stable vector bundles · Holomorphic foliations

Mathematics Subject Classification (2010) Primary 14D20 · 53D18 · 14D06

1 Introduction

A generalised complex structure on a real manifold X of dimension $2n$, as defined by Hitchin [9], is a rank-$2n$ isotropic subbundle $E^{0,1} \subset (T_X \oplus T_X^*)^{\mathbb{C}}$ such that

1. $E^{0,1} \oplus \overline{E^{0,1}} = (T_X \oplus T_X^*)^{\mathbb{C}}$
2. $C^{\infty}(E^{0,1})$ is closed under the Courant bracket.

On a manifold with a generalized complex structure M. Gualtieri in [6] defined the notion of a generalized holomorphic bundle. More precisely, a generalized holomorphic bundle on a generalized complex manifold, is a vector bundle $\mathcal{E}$ with a differential operator $\overline{D}$: $C^{\infty}(\mathcal{E}) \longrightarrow C^{\infty}(\mathcal{E} \otimes E^{0,1})$ such that for all smooth function f and all section $s \in C^{\infty}(\mathcal{E})$ the following holds

Dedicated to Jose Seade, for his 60th birthday.

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1. $\overline{D}(fs) = \overline{\partial}(fs) + f\overline{D}(s)$
2. $\overline{D}^2 = 0$.

In the case of an ordinary complex structure and $\overline{D} = \overline{\partial} + \phi$, for operators

$$\overline{\partial} : C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E} \otimes \overline{T_X^*})$$

and

$$\phi : C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E} \otimes T_X),$$

the vanishing $\overline{D}^2 = 0$ means that $\overline{\partial}^2 = 0$, $\overline{\partial}\phi = 0$ and $\phi \wedge \phi = 0$. By a classical result of Malgrange the condition $\overline{\partial}^2 = 0$ implies that $\mathcal{E}$ is a holomorphic vector bundle. On the other hand, $\overline{\partial}\phi = 0$ implies that ϕ is a holomorphic global section

$$\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$$

which satisfies an integrability condition $\phi \wedge \phi = 0$. A co-Higgs sheaf on a complex manifold X is a sheaf $\mathcal{E}$ together with a section $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$(called a *Higgs fields*) for which $\phi \wedge \phi = 0$. General properties of co-Higgs bundles were studied in [8,15]. There is a motivation in physics for studying co-Higgs bundles, see [7,10] and [19].

There are no stable co-Higgs bundles with nonzero Higgs field on curves C of genus $g > 1$. (When $g = 1$, a co-Higgs bundle is the same thing as a Higgs bundle in the usual sense.) In fact, contracting with a holomorphic differential gives a non-trivial endomorphism of $\mathcal{E}$ commuting with ϕ which is impossible in the stable case [8,16]. S. Rayan showed in [14] the non-existence of stable co-Higgs bundles with non trivial Higgs field on K3 and general-type surfaces. In this note we prove the following result.

Theorem 1.1 *Let $(\mathcal{E}, \phi)$ be a rank two co-Higgs vector bundles on a Kähler compact surface X with $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$ nilpotent. If $(\mathcal{E}, \phi)$ is semi-stable, then one of the following holds up to finite étale cover:*

1. *X is uniruled.*
2. *X is a torus and $(\mathcal{E}, \phi)$ is strictly semi-stable.*
3. *X is a properly elliptic surface and $(\mathcal{E}, \phi)$ is strictly semi-stable.*

It follows direct of proof of Theorem 1.1 that the we can consider a more general classes of singular projective surfaces.

Theorem 1.2 *Let $(\mathcal{E}, \phi)$ be a rank two co-Higgs torsion-free sheaf on a normal projective surface X with $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$ nilpotent. If $(\mathcal{E}, \phi)$ is stable, then X is uniruled.*

Finally, in this work we consider a relation between co-Higgs bundles and Poisson geometry on $\mathbb{P}^1$-bundles. In [18] Polishchuk associated to each rank-2 co-Higgs bundle $(\mathcal{E}, \phi)$ a Poisson structure on its projectivized bundle $\mathbb{P}(\mathcal{E})$. This relation was explained by Rayan in [15] as follows:

let $Y := \mathbb{P}(\mathcal{E})$ and consider the natural projection $\pi : Y \to X$. The exact sequence

$$0 \longrightarrow T_{X|Y} \longrightarrow T_Y \longrightarrow \pi^* T_X \longrightarrow 0$$

implies that $T_{X|Y} \otimes \pi^* T_X \subset \bigwedge^2 T_Y$. Since $T_{X|Y} = \mathrm{Aut}(\mathbb{P}(\mathcal{E})) = \mathrm{Aut}(\mathcal{E})/\mathbb{C}^*$ we get that

$$\pi_*(T_{X|Y} \otimes \pi^* T_X) = \pi_* T_{X|Y} \otimes T_X = End_0(\mathcal{E}) \otimes T_X,$$

where $End_0(\mathcal{E})$ denotes the trace-free endomorphisms of $\mathcal{E}$. Therefore, we can associate a trace-free co-Higgs fields $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$ a bi-vector $\pi^*\phi \in H^0(X, T_{X|Y} \otimes \pi^*T_X) \subset H^0(X, \bigwedge^2 T_Y)$ on $\mathbb{P}(\mathcal{E})$. The co-Higgs condition $\phi \wedge \phi = 0$ implies that bi-vector $\pi^*\phi$ is integrable, see the introduction of [15]. The codimension one foliation on $\mathbb{P}(\mathcal{E})$ is the called *foliation by symplectic leaves* induced by Poisson struture.

We get an interisting consequence of the proof Theorem 1.2.

Corollary 1.1 *If $(\mathcal{E}, \phi)$ is locally free, stable and nilpotent , then the closure of the all leaves of the foliation by symplectic leaves on $\mathbb{P}(\mathcal{E})$ are rational surfaces.*

2 Semi-stable co-Higgs sheaves

Definition 2.1 A co-Higgs sheaf on a complex manifold X is a sheaf $\mathcal{E}$ together with a section $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$(called a Higgs fields) for which $\phi \wedge \phi = 0$.

Denote by $End_0(\mathcal{E}) := ker(tr : End(\mathcal{E}) \longrightarrow \mathcal{O}_X)$ the trace-free part of the endomorphism bundle of $\mathcal{E}$. Since

$$End(\mathcal{E}) = End_0(\mathcal{E}) \oplus \mathcal{O}_X$$

we have that $End(\mathcal{E}) \otimes T_X = (End_0(\mathcal{E}) \otimes T_X) \oplus T_X$. Thus, the Higgs field $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$ can be decomposed as $\phi = \phi_1 + \phi_2$, where ϕ_1 is the trace-free part and ϕ_2 is a global vector field on X. In particular, if the surface X has no global holomorphic vector fields, then every Higgs field is trace-free.

Definition 2.2 Let (X, ω) be a polarized Kähler compact manifold. We say that $(\mathcal{E}, \phi)$ is semi-stable if

$$\frac{c_1(\mathcal{F}) \cdot [\omega]}{rank(\mathcal{F})} \leq \frac{c_1(\mathcal{E}) \cdot [\omega]}{rank(\mathcal{E})}$$

for all coherent subsheaves $0 \neq \mathcal{F} \subsetneq \mathcal{E}$ satisfying $\Phi(\mathcal{F}) \subseteq \mathcal{F} \otimes T_X$, and stable if the inequality is strict for all such $\mathcal{F}$. We say that $(\mathcal{E}, \phi)$ is strictly semi-stable if $(\mathcal{E}, \phi)$ is semi-stable but non-stable.

3 Holomorphic foliations

Definition 3.1 Let X be a connected complex manifold. A one-dimensional holomorphic foliation is given by the following data:

1. an open covering $\mathcal{U} = \{U_\alpha\}$ of X;
2. for each U_α an holomorphic vector field ζ_α ;
3. for every non-empty intersection, $U_\alpha \cap U_\beta \neq \emptyset$, a holomorphic function

$$f_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta);$$

such that $\zeta_\alpha = f_{\alpha\beta}\zeta_\beta$ in $U_\alpha \cap U_\beta$ and $f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$.

We denote by $K_\mathcal{F}$ the line bundle defined by the cocycle $\{f_{\alpha\beta}\} \in \mathrm{H}^1(X, \mathcal{O}^*)$. Thus, a one-dimensional holomorphic foliation $\mathcal{F}$ on X induces a global holomorphic section $\zeta_\mathcal{F} \in \mathrm{H}^0(X, T_X \otimes K_\mathcal{F})$. The line bundle $K_\mathcal{F}$ is called the *canonical bundle* of $\mathcal{F}$. Two sections $\zeta_{\mathcal{F}}$ and $\eta_{\mathcal{F}}$ of $\mathrm{H}^0(X, T_X \otimes K_{\mathcal{F}})$ are equivalent, if there exists a never vanishing holomorphic function $\varphi \in \mathrm{H}^0(X, \mathcal{O}^*)$, such that $\zeta_{\mathcal{F}} = \varphi \cdot \eta_{\mathcal{F}}$. It is clear that $\zeta_{\mathcal{F}}$ and $\eta_{\mathcal{F}}$ define the same foliation. Thus, a holomorphic foliation $\mathcal{F}$ on X is an equivalence of sections of $H^0(X, T_X \otimes K_{\mathcal{F}})$.

4 Examples

4.1 Canonical example of split co-Higgs bundles

Here we will give an example which naturally generalizes the canonical example given by Rayan on [15, Chapter 6]. Let (X, ω) be a polarized Kähler compact manifold. Suppose that there exists a global section $\zeta \in H^0(X, Hom(N, TX \otimes L)) \simeq H^0(X, TX \otimes L \otimes N^*)$. Now, consider the vector bundle

$$\mathcal{E} = L \oplus N.$$

Define the following co-Higgs fields

$$\phi : L \oplus N \longrightarrow (T_X \otimes L) \oplus (T_X \otimes N) \in H^0(X, End(\mathcal{E}) \otimes T_X)$$

by $\phi(s, t) = (\zeta(t), 0)$. Since $\phi \circ \phi = 0 \in H^0(End(\mathcal{E}) \otimes T_X \otimes T_X)$ we get that $\phi \wedge \phi = 0$. Moreover, observe that the kernel of ϕ is the ϕ-invariant line bundle L. On the other hand, the line bundle L is destabilising only when

$$[2c_1(L) - c_1(\mathcal{E})] \cdot [\omega] = [c_1(L) - c_1(N)] \cdot [\omega] > 0.$$

That is, if

$$[c_1(L)] \cdot [\omega] > [c_1(N)] \cdot [\omega].$$

4.2 Co-Higgs bundles on ruled surfaces

Let C be a curve of genus $g > 1$. Now, consider the ruled surface $X := \mathbb{P}(K_C \oplus \mathcal{O}_C)$ and

$$\pi : \mathbb{P}(K_C \oplus \mathcal{O}_C) \longrightarrow C$$

the natural projection. Consider a Poisson structure on X given by a bivector $\sigma \in H^0(X, \wedge^2 T_X)$. Let $(\mathcal{E}, \phi)$ be a nilpotent Higgs bundle on C. S. Rayan showed in [15] that $(\pi^*\mathcal{E}, \sigma(\pi^*\phi))$ is a stable co-Higgs bundle on X.

4.3 Co-Higgs orbibundles on weighted projective spaces

Let w_0, w_1, w_2 be positive integers, set $|w| := w_0 + w_1 + w_2$. Assume that w_0, w_1, w_2 are relatively prime. Define an action of $\mathbb{C}^*$ in $\mathbb{C}^3 \setminus \{0\}$ by

$$\begin{aligned} \mathbb{C}^* \times (\mathbb{C}^3 \setminus \{0\}) &\longrightarrow (\mathbb{C}^3 \setminus \{0\}) \\ \lambda.(z_0, z_1, z_2) &\longmapsto (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2) \end{aligned} \tag{4.1}$$

and consider the weighted projective plane

$$\mathbb{P}(w_0, w_1, w_2) := (\mathbb{C}^3 \setminus \{0\}) / \sim$$

induced by the action above. We will denote this space by $\mathbb{P}(\omega)$. On $\mathbb{P}(\omega)$ we have an Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\omega)} \xrightarrow{\varsigma} \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}(\omega)}(\omega_i) \longrightarrow T\mathbb{P}(\omega) \longrightarrow 0,$$

where $\mathcal{O}_{\mathbb{P}(\omega)}$ is the trivial line orbibundle and $T\mathbb{P}(\omega) = \mathrm{Hom}(\Omega^1_{\mathbb{P}(\omega)}, \mathcal{O}_{\mathbb{P}(\omega)})$ is the tangent orbibundle of $\mathbb{P}(\omega)$. The map ς is given explicitly by $\varsigma(1) = (\omega_0 z_0, \omega_1 z_1, \omega_2 z_2)$. Now, let $(\mathcal{E}, \phi)$ be a co-Higgs orbibundle on $\mathbb{P}(\omega)$. Tensoring the Euler sequence by $End(\mathcal{E})$, we obtain

$$0 \longrightarrow End(\mathcal{E}) \longrightarrow \bigoplus_{i=0}^{2} End(\mathcal{E})(\omega_i) \longrightarrow End(\mathcal{E}) \otimes T\mathbb{P}(\omega) \longrightarrow 0.$$

Thus, the co-Higgs fields ϕ can be represented, in homogeneous coordinates, by

$$\phi = \phi_0 \otimes \frac{\partial}{\partial z_0} + \phi_1 \otimes \frac{\partial}{\partial z_1} + \phi_2 \otimes \frac{\partial}{\partial z_2},$$

where $\phi_i \in H^0(\mathbb{P}(\omega), End(\mathcal{E})(w_i))$, for all $i = 0, 1, 2$, and $\phi + \theta \otimes R_\omega$ define the same co-Higgs field as ϕ, where R_ω is the adapted radial vector field

$$R_\omega = \omega_0 z_0 \frac{\partial}{\partial z_0} + \omega_1 z_1 \frac{\partial}{\partial z_1} + +\omega_2 z_2 \frac{\partial}{\partial z_2},$$

with θ a endomorphism of $\mathcal{E}$. Suppose that

$$\mathcal{E} = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$$

and that there exists a stable ϕ for $\mathcal{E}$ such that $m_1 \geq m_2$. Then

$$|m_1 - m_2| \leq \max_{0 \leq i \neq j \leq 2} \{\omega_i + \omega_j\}.$$

In fact, this is a consequence of Bott's Formulae for weighted projective spaces. It follows from (see [5]) that

$$\mathrm{H}^0(\mathbb{P}(\omega), T\mathbb{P}(\omega) \otimes \mathcal{O}_\omega(k)) \simeq \mathrm{H}^0(\mathbb{P}(\omega), \Omega^1_{\mathbb{P}(\omega)}\left(\sum_{i=0}^{2} \omega_i + k\right)\right) \neq \emptyset$$

if and only if $k > -\max_{0 \leq i \neq j \leq 2} \{\omega_i + \omega_j\}$. This generalize the example given by S. Rayan in [14].

4.4 Co-Higgs bundles on two dimensional complex tori

Let X be a two dimensional complex torus and a co-Higgs bundle $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$. Then ϕ is equivalente to a pair of commutative endomorphism of $\mathcal{E}$. In fact, since the tangent bundle T_X is holomorphically trivial, we can take a trivialization by choosing two linearly independent global vector fields $v_1, v_2 \in H^0(X, T_X)$. Then, we can write

$$\phi = \phi_1 \otimes v_1 + \phi_2 \otimes v_2.$$

The condition $\phi \wedge \phi = 0$ implies that

$$\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1.$$

We have a canonical nilpotent co-Higgs bundle $(\mathcal{E}, \phi)$, where $\mathcal{E} = T_X = \mathcal{O} \oplus \mathcal{O}$ and

$$\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix},$$

where v is a global vector field on X.

5 Proof of Theorem

By using that the Higgs field $\phi \in H^0(X, End(\mathcal{E}) \otimes T_X)$ is nilpotent we have that $Ker(\phi) =: L$ is a well defined line bundle on X. Thus , we have a exact sequence

$$0 \to L \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes N \longrightarrow 0,$$

where the nilpotent Higgs field ϕ factors as

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E} \otimes T_X \\ \downarrow & & \uparrow \\ \mathcal{I}_Z \otimes N & \longrightarrow & L \otimes T_X. \end{array} \tag{5.1}$$

The morphism $\mathcal{I}_Z \otimes N \to L \otimes T_X$ induces a holomorphic foliation on X which induces a global section $\zeta_\phi \in H^0(X, T_X \otimes L \otimes N^*)$. Since $\det(\mathcal{E}) = L \otimes N$ we conclude that $L \otimes N^* = L^2 \otimes \det(\mathcal{E}^*)$. Then

$$\zeta_\phi \in H^0(X, T_X \otimes L^2 \otimes \det(\mathcal{E}^*)).$$

Let $K := L^2 \otimes \det(\mathcal{E}^*)$ the canonical bundle of the foliation $\mathcal{F}$ associated to the co-Higgs fields ϕ. If $\mathcal{E}$ is semi-stable then

$$[c_1(K)] \cdot [\omega] = [2c_1(L) - c_1(\mathcal{E})] \cdot [\omega] \leq 0$$

for some Kähler class ω. If $K \cdot [\omega] < 0$, it follows from [11] that K is not pseudo-efective [4]. It follows from Brunella's Theorem [1] that X is uniruled.

Now, suppose X is a normal projective surface and that $K \cdot H = 0$, for some H ample By Hodge index theorem we have that $K^2 \cdot H^2 \leq (K \cdot H)^2 = 0$, then $K^2 \leq 0$. Suppose that $K^2 < 0$. We have that $D = H + \epsilon K$ is a $\mathbb{Q}$-divisor ample for $0 < \epsilon << 1$, see [12, proposition 1.3.6]. Thus, we have that

$$K \cdot D = K \cdot H + \epsilon^2 K^2 = \epsilon^2 K^2 < 0.$$

By Bogomolov-McQuillan-Miyaoka's theorem [3] we conclude that X is uniruled. If $K^2 = 0$, then K is numerically trivial. This fact is well known, but for convenience of the reader we give a proof. Suppose that there exists $C \subset X$ such that $K \cdot C > 0$. Now, Consider the divisor $B = (H^2)C - (H \cdot C)H$. Then $B \cdot H = 0$ and $K \cdot B = (H^2)K \cdot C < 0$. Define $F = mK + B$ for $0 < m << 1$. Therefore $F \cdot H = 0$ and $F^2 > 0$. This is a contradiction by the Hodge index Theorem. In this case $\mathcal{E}$ is strictly semi-stable.

Now, we apply the classification, up to finite étale cover, of holomorphic foliations on projective surfaces with canonical bundle numerically trivial [2,13,17]. Therefore, up to finite étale cover, either:

(i) X is uniruled;
(ii) X is a torus;

(iii) $k(X) = 1$ and $X = B \times C$ with $g(B) \geq 2$, C is elliptic. That is, X is a sesquielliptic surface.

If X is Kähler and non-algebraic it follows from [2] that, up to finite étale cover, either X has a unique elliptic fibration or X is a torus.

6 Proof of Corollary 1.1

Since $(\mathcal{E}, \phi)$ is nilpotent and stable the co-Higgs fields induces a foliation $\mathcal{F}$ by rational curves on X. Now, consider the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$. Then the foliation by symplectic leaves $\mathcal{G}$ on $\mathbb{P}(\mathcal{E})$ is the pull-back of $\mathcal{F}$ by π. In particular, a closure of the a leaf of the foliation by symplectic leaves $\mathcal{G}$ is of type $\pi^{-1}(f(\mathbb{P}^1))$, where $f : \mathbb{P}^1 \to X$ is the uniformization of a rational leaf of $\mathcal{F}$. Clearly $\pi^{-1}(f(\mathbb{P}^1))$ is a rational surface.

Acknowledgments We are grateful to Arturo Fernandez-Perez, Renato Martins and Marcos Jardim for pointing out corrections. We are grateful to Henrique Bursztyn for interesting conversations about Poisson Geometry.

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