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Baum-Bott indices for curves of singularities

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Abstract. Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n$ by curves such that the components of its singular locus are curves C_i and points p_j. We compute the Baum-Bott indices $BB_\varphi(\mathcal{F}, C_i)$ in terms of the main invariants of $\mathcal{F}$ and C_i. We also determine the sum of the $BB_\varphi(\mathcal{F}, p_i)$ in terms of the same invariants. When φ corresponds to the determinant, the latter result generalizes, from special to all holomorphic foliations, a formula for the number of isolated singularities of $\mathcal{F}$, counted with multiplicities.

Keywords: holomorphic foliations, non-isolated singularities, Milnor's number.

Mathematical subject classification: Primary: 32S65, 58K45.

1 Introduction

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n$ and Σ its singular scheme. We call $\mu(\mathcal{F}, p)$ and $BB_\varphi(\mathcal{F}, p)$ the Milnor number and the Baum-Bott indices of $\mathcal{F}$ at a closed point p of $\mathbb{P}^n$, respectively. In the case of germs of vector fields v on $(\mathbb{C}^n, 0)$ with isolated singularity $0 \in \mathbb{C}^n$, there is a result due to M. Soares (cf. [11]) that gives an upper bound for the Poincaré-Hopf index $PH(v, 0)$. E. Esteves and I. Vainsencher (cf. [5]) give another proof of M. Soares' bounds, using the intersection theory. F. Bracci and T. Suwa (cf. [2]) proved that Baum-Bott indices vary continuously under smooth deformations of holomorphic foliations. Moreover, they reported in the case where the singular set contains non-isolated singularities and, in some cases, some formulas are available but, in general, explicit computation of the residues is rather difficult. More precisely, let v be a vector field defined on a complex manifold M and S, be a compact connected component of $\mathrm{Sing}(v)$. Take a cellular tube $\mathcal{T}$ around S (cf. [12]) such that $\mathrm{Sing}(v) \cap \overline{\mathcal{T}} = S$. Thus, if v is everywhere transverse to $\partial\mathcal{T}$ then $PH(v, S) = \chi(S)$, the Euler-Poincaré characteristic of S.

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We will consider the case where $\Sigma := \mathrm{Sing}(\mathcal{F})$ contains smooth curves or singular complete intersection curves and some isolated closed points. Let C be one of these curves. The aim of this paper is to introduce an effective method to compute the equivalence for $BB_\varphi(\mathcal{F}, C)$, under smooth deformations. These numbers $BB_\varphi(\mathcal{F}, C)$ must be understood in the following way: let $\mathcal{F}_t$ be a generic deformation of $\mathcal{F}$ such that $\mathcal{F}_0 = \mathcal{F}$, $\deg(\mathcal{F}) = \deg(\mathcal{F}_t)$ and $\mathrm{Sing}\big(\mathcal{F}_t\big) = \{p_1^t, \dots, p_{s_t}^t\}$ for $0 < |t| < \epsilon$ where p_i^t are closed points of M. Thus,

$$BB_\varphi(\mathcal{F}, C) = \lim_{t\to 0} \sum_{\lim p_i^t \in C} BB_\varphi(\mathcal{F}_t, p_i^t). \tag{1}$$

As a direct consequence, we obtain the explicit formula and the lower bound for the Minor number $\mu(\mathcal{F}, C)$. Furthermore, we determine the number of closed points of Σ, counted with multiplicities. We would like to emphasize that the method can be directly generalized to varieties of any dimension contained in Σ. The method consists of constructing a suitable deformation which allows us to calculate explicitly these indices. In Section 4, more precisely Lemma 4.2, all the procedures will be treated more details.

Let Y be a smooth projective scheme Y and X a projective subscheme of Y. Let us consider the blowup morphism of Y along X, denoted by $\pi_X : \widetilde{Y} \to Y$ with exceptional divisor $E_X := \pi^{-1}(X)$. We will denote by $\ell := \mathrm{mult}_{E_X}(\pi_X^*\mathcal{F})$ the *order of annulment* of the pull-back foliation $\pi_X^*\mathcal{F}$ at E_X. The foliation $\mathcal{F}$ will be called *special* along X if the induced foliation $\widetilde{\mathcal{F}}$, defined in $\widetilde{Y}$, obtained via $\pi_X^*\mathcal{F}$, has E_X as an invariant set, and $\mathrm{Sing}(\widetilde{\mathcal{F}})$ meets E_X at isolated singularities at most. In [4], the authors determine the number of isolated singularities of $\mathcal{F}$, counted with multiplicities, when $\mathcal{F}$ is special along each curve contained in Σ. In this paper, we just relaxed this hypothesis basically allowing singular curves as non-dicritical component of Σ. Now, we prove the following result:

Theorem 1.1. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, of degree k, such that its singular locus Σ is the disjoint union of irreducible curves $C_1, \dots, C_r$ and closed points $p_1, \dots, p_s$. Assume each C_i is either smooth, or a singular scheme theoretic complete intersection; and also a non-dicritical component of Σ. Then*

(i) $\sum_{i=0}^{s} \mu(\mathcal{F}, p_i) = \sum_{j=0}^{n} k^j + \sum_{j=1}^{r} \nu(\mathcal{F}, C_j) - \sum_{j=1}^{r} \mathcal{A}_j$ *where*

$$\nu(\mathcal{F}, C_j) := (\ell_j + 1)^{n-2}$$
$$\times\Big[-\chi(C_j)(\ell_j^2 + \ell_j + 1) + (n+1)d_j\ell_j^2 - (k-1)d_j(n\ell_j + 1)\Big],$$

with $d_j = \deg(C_j), \chi(C_j) = 2-2g_j+\sum_{i=1}^{l_j}(b_{i,j}-1)$, g_j *is the arithmetic genus of* C_j, $b_{i,j}$ *the number of branches of* $q_{i,j} \in C_j$, $q_{i,j}$ *are the singular points of* C_j, $\ell_j = \mathrm{mult}_{E_{C_j}}(\pi^*_{C_j}\mathcal{F})$ *and* $\mathcal{A}_j$ *is the number of embedded closed points of* C_j, *counted with multiplicities.*

(ii) $\mu(\mathcal{F}, C_j) = -\nu(\mathcal{F}, C_j) + \mathcal{A}_j \geq -\nu(\mathcal{F}, C_j)$.

The statement (ii) of the Theorem 1.1 gives a lower bound for $\mu(\mathcal{F}, C_j)$ which is always attained if the foliation is especial along C_j or C_j has no embedded closed points. Moreover, the number $-\nu(\mathcal{F}, C_j)$, when $\ell_j = 0$, coincides with the number of isolated singularities, counted with multiplicities, of any holomorphic foliation by curves in $\mathbb{P}^n$ of degree k having C_j as invariant set.

2 Preliminaries

2.1 Multiplicities

We now describe the multiplicity of $\mathcal{F}$ at an irreducible curve $C \subset \Sigma$. By a holomorphic change of coordinates, C can be locally given as $z_1 = \ldots = z_{n-1} = 0$. In this neighborhood, $\mathcal{F}$ is described by the following vector field,

$$\mathcal{D}_{\mathcal{F}} = f_1(z)\,\frac{\partial}{\partial z_1} + \ldots + f_n(z)\,\frac{\partial}{\partial z_n}. \tag{2}$$

Therefore, one may write the local sections in (2) as

$$f_i(z) = \sum_{j=1}^{n-1} z_j\, f_{i,j}(z). \tag{3}$$

If all $f_{i,j}$ also vanish on the $z_n - axis$, for $j = 1, \ldots, n-1$, we can apply (3) again to all of them. Thus, the function f_i can be rewritten as

$$f_i(z) = \sum_{j\leq k} z_j z_k\, f_{i,j,k}(z).$$

We will repeat this process, until we find some function $f_{i,a}(z)$, with $a := (a_1, \ldots, a_{n-1})$, which does not vanish on the $z_n - axis$, the function f_i will be of the form

$$f_i(z) = \sum_{|a|=m_i} z_1^{a_1} \cdots z_{n-1}^{a_{n-1}}\, f_{i,a}(z) \tag{4}$$

where $|a| := a_1 + \ldots + a_{n-1}$. We will denote by $m_i = \text{mult}_C(f_i)$. As done in [4], we define the *multiplicity* of $\mathcal{F}$ at C as

$$m_C(\mathcal{F}) = min\{m_1, \ldots, m_n\}. \tag{5}$$

In order to simplify our analysis, we need the following lemma.

Lemma 2.1. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation in $\mathbb{P}^n$ with $C \subset \text{Sing}(\mathcal{F})$ where C is a regular curve. Then, for each point $p \in C$ exists a neighborhood U of p and a holomorphic coordinates $w \in \mathbb{C}^n$ such that $p(0) = 0 \in \mathbb{C}^n$ and $U \cap C = (w_1 = \ldots = w_{n-1} = 0)$ and $\mathcal{F}$ is defined in U by the following vector field*

$$\mathcal{D}_{\mathcal{F}} = P_1(w)\frac{\partial}{\partial w_1} + \ldots + P_n(w)\frac{\partial}{\partial w_n}$$

where $P_i(w)$ given as in (4) with

(i) $\text{mult}_C(P_i) = min\{m_1, \ldots, m_{n-1}\}$ *for* $1 \leq i \leq n-1$,

(ii) $\text{mult}_C(P_n) = m_C(\mathcal{F})$.

Proof. It is enough to consider $\mathcal{F}$ is described by the vector field as in (2). Let $A = (a_{ij}) \in GL(n, \mathbb{C})$ be a matrix such that $a_{in} = 0$ for $1 \leq i \leq n-1$. Consequently $B = A^{-1} = (b_{ij})$ has the same property, i.e., $b_{in} = 0$ for $1 \leq i \leq n-1$. Thus, the linear transformation $z = Aw$ preserves the n-th coordinate axis. Then,

$$\dot{w}_i = \sum_{j=1}^{n-1} b_{ij}\dot{z}_j = \sum_{j=1}^{n-1} b_{ij} f_j \circ Aw = P_i(w)$$

for $1 \leq i \leq n-1$ and

$$\dot{w}_n = \sum_{j=1}^{n} b_{nj}\dot{z}_j = \sum_{j=1}^{n} b_{ij} f_j \circ Aw = P_n(w).$$

In this way, adjusting some coefficients a_{ij}, if necessary, the foliation $\mathcal{F}$ is described by the following vector field

$$\mathcal{D}_{\mathcal{F}} = P_1(w)\frac{\partial}{\partial w_1} + \ldots + P_n(w)\frac{\partial}{\partial w_n} \tag{6}$$

with each P_i in the conditions of the Lemma. □

By now, we start by recalling some elementary fact about blowing-up's. See [8] for more details. Let $z = (z_i) \in \mathbb{C}^n$ be affine local coordinates as given in the Lemma 2.1. For each $j = 1, \ldots, n-1$ let $U_j = (z_j \neq 0)$ and $\widetilde{U}_j = \pi_C^{-1}(U_j)$. In $\widetilde{U}_j$ we introduce coordinates $u \in \mathbb{C}^n$ and $\pi := \pi_C$ has the following expressions

$$\sigma(u) = \pi|_{\widetilde{U}_j} = z = (z_1, z_2, \ldots, z_{n-1}, z_n)$$

where $z_i = u_i$ for $i = j$ or $i = n$, and $z_i = u_j u_i$ for $i = 1, \ldots, j-1, \hat{j}, j+1, \ldots, n-1$.

Now, we will describe briefly the behavior of $\mathcal{F}$ under blowup along this curve and hence we obtain the equations of the induced foliation $\widetilde{\mathcal{F}}$. For more details, see [4]. We will assume $\mathcal{F}$ is defined by the following vector field $\sum_{i=1}^n P_i(z)\frac{\partial}{\partial z_i}$ where each P_i is given as in (4) with $\mathrm{mult}_C(P_i) = m_1$, for $1 \leq i \leq n-1$ and $\mathrm{mult}_C(P_n) = m_n$ with $m_n \leq m_1$.

In the open set $\widetilde{U}_1$, we have that

$$\begin{aligned}\dot{u}_1 &= \sum_{|a|=m_1} u_1^{a_1}(u_1u_2)^{a_2}\cdots(u_1u_{n-1})^{a_{n-1}} P_{1,a}(\sigma(u))\\ &= u_1^m \sum_{|a|=m_i} u_2^{a_2}\cdots u_{n-1}^{a_{n-1}} P_{i,a}(\sigma(u)).\end{aligned}$$

But, $P_{1,a}(\sigma(u)) = P_{1,a}(0, \ldots, 0, u_n) + u_1\widetilde{P}_{i,a}(u) = p_{i,a}(u_n) + u_1\widetilde{P}_{i,a}(u)$ and hence

$$\dot{u}_1 = u_1^m \left(\sum_{|a|=m_i} u_2^{a_2}\cdots u_{n-1}^{a_{n-1}} p_{1,a}(u_n) + u_1\widetilde{P}_1(u)\right) \tag{7}$$

for some functions $\widetilde{P}_1(u)$. In this same way, we have that

$$\dot{u}_n = u_1^{m_n} \left(\sum_{|a|=m_n} u_2^{a_2}\cdots u_{n-1}^{a_{n-1}} p_{n,a}(u_n) + u_1\widetilde{P}_n(u)\right). \tag{8}$$

For $2 \leq i \leq n-1$, since $z_i = u_1u_i$, we have that $\dot{z}_i = \dot{u}_1u_i + u_1\dot{u}_i$ and thus

$$\dot{u}_i = u_1^{m-1}\big(g_i(u) - u_1 g_1(u) + u_1\widetilde{P}_i(u)\big) \tag{9}$$

where $g_i(u) := \sum_{|a|=m_i} u_2^{a_2}\cdots u_{n-1}^{a_{n-1}} p_{i,a}(u_n)$ and for some functions $\widetilde{P}_i(u)$.

With the equations (7), (8) and (9) we obtain the vector field which described in $\widetilde{U}_1$ the foliation $\pi^*\mathcal{F}$,

$$\begin{aligned}\mathcal{D}_{\pi^*\mathcal{F}} &= u_1^{m_1}\Big(g_1(u) + u_1\widetilde{P}_1(u)\Big)\frac{\partial}{\partial u_1}\\ &+ \sum_{i=2}^{n-1} u_1^{m_1-1}\Big(h_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i}\\ &+ u_1^{m_n}\Big(g_n(u) + u_1\widetilde{P}_n(u)\Big)\frac{\partial}{\partial u_n}\end{aligned} \tag{10}$$

where $h_i(u) := g_i(u) - u_i g_1(u)$ for $2 \leq i \leq n-1$.

According to the possible values of m_1 and m_n, we can divide the equation (10) by an adequate power of u_1, and thus, we obtain the vector field which defines the foliation $\widetilde{\mathcal{F}}$. As in the case of isolated singularity, the curve C is called nondicritical if the exceptional divisor E is invariant by $\widetilde{\mathcal{F}}$; otherwise, C is called dicritical. The nondicritical case can be divided into three different situations:

(i) $m_n + 1 = m_1$ with $h_{i_0} \not\equiv 0$ for some i_0.

Dividing (10) by $u_1^{m_n}$ we get the vector field defining $\widetilde{\mathcal{F}}$ which is

$$\begin{aligned}\mathcal{D}_{\widetilde{\mathcal{F}}} &= u_1\Big(g_1(u) + u_1\widetilde{P}_1(u)\Big)\frac{\partial}{\partial u_1}\\ &+ \sum_{i=2}^{n-1} \Big(h_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i}\\ &+ \Big(g_n(u) + u_1\widetilde{P}_n(u)\Big)\frac{\partial}{\partial u_n}.\end{aligned} \tag{11}$$

If $h_i \equiv 0$, for some i, then the leaves of $\widetilde{\mathcal{F}}$, when restricted to E, are contained in the hyperplane $u_i = k_i$, where k_i is a constant.

(ii) $m_n + 1 < m_1$

Now, dividing again (10) by $u_1^{m_n}$ we get

$$\begin{aligned}\mathcal{D}_{\widetilde{\mathcal{F}}} &= u_1^{m_1-m_n}\Big(g_1(u) + u_1\widetilde{P}_1(u)\Big)\frac{\partial}{\partial u_1}\\ &+ \sum_{i=2}^{n-1} u_1^{m_1-m_n-1}\Big(h_i(u) + u_1\widetilde{P}_i(u)\Big)\frac{\partial}{\partial u_i}\\ &+ \Big(g_n(u) + u_1\widetilde{f}_n(u)\Big)\frac{\partial}{\partial u_n}.\end{aligned} \tag{12}$$

In this situation, the leaves of of $\widetilde{\mathcal{F}}$, when restricted to E, are contained in the hyperplane given by $u_i = k_i$ for $i = 1, \ldots, n-1$, where each k_i is also a constant.

(iii) $m_n = m_1$ and $h_{i_0} \not\equiv 0$ for some i_0.

Dividing (10) by $u_1^{m_1-1}$ we get

$$\begin{aligned} \mathcal{D}_{\widetilde{\mathcal{F}}} &= u_1\left(g_1(u) + u_1\widetilde{P}_1(u)\right)\frac{\partial}{\partial u_1} \\ &\quad + \sum_{i=2}^{n-1}\left(h_i(u) + u_1\widetilde{P}_i(u)\right)\frac{\partial}{\partial u_i} \\ &\quad + u_1\left(g_n(u) + u_1\widetilde{f}_n(u)\right)\frac{\partial}{\partial u_n}. \end{aligned} \tag{13}$$

Now, the leaves of of $\widetilde{\mathcal{F}}$, when restricted to E, are contained in the hyperplane $u_i = k_i$, if $h_i \equiv 0$ and $u_n = k_n$. Furthermore, if the situations (ii) or (iii) occurs, the new non-isolated singularities will appear in the singular locus $\mathrm{Sing}(\widetilde{\mathcal{F}})$.

On the other hand, the dicritical case corresponds to a single situation described as:

(i) $m_1 = m_n$ and $h_i \equiv 0$ for all $2 \leq i \leq n-1$.

Dividing (10) by $u_1^{m_n}$ we get

$$\begin{aligned} \mathcal{D}_{\widetilde{\mathcal{F}}} &= \left(g_1(u) + u_1\widetilde{P}_1(u)\right)\frac{\partial}{\partial u_1} + \sum_{i=2}^{n-1}\widetilde{P}_i(u)\frac{\partial}{\partial u_i} \\ &\quad + \left(g_n(u) + u_1\widetilde{P}_n(u)\right)\frac{\partial}{\partial u_n}. \end{aligned} \tag{14}$$

In this time, the exceptional divisor E is not an invariant set of $\widetilde{\mathcal{F}}$. But, the foliation $\widetilde{\mathcal{F}}$ is transverse to E except at the hypersurface locally given by $g_1(u) = 0$.

Keeping this notation, we have the following Lemma:

Lemma 2.2. *The following hold:*

(i) $m_C(\mathcal{F}) = \min\{m_1, m_n\}$;

(ii) *if ℓ is the integer such that*

$$\mathcal{L}_{\widetilde{\mathcal{F}}} \cong \pi^*\mathcal{L}_{\mathcal{F}} \otimes \mathcal{O}_{\widetilde{Y}}(\ell E)$$

where $\mathcal{L}_{\widetilde{\mathcal{F}}}$ and $\mathcal{L}_{\mathcal{F}}$ are the tangent bundle of $\widetilde{\mathcal{F}}$ and $\mathcal{F}$, respectively, then

$$\ell = \begin{cases} \min\{m_1 - 1, m_n\} & \textit{if } C \textit{ is nondicritical} \\ m_1 = m_n & \textit{if } C \textit{ is dicritical.} \end{cases}$$

The number $\ell = \operatorname{mult}_E(\pi^*\mathcal{F})$ will be called by the order of annulment of $\pi^*\mathcal{F}$ at E.

(iii) *If $\mathcal{F}$ is special along C then $m_n + 1 = m_1$, $\ell \geq 1$ and $h_i \not\equiv 0$ for all $2 \leq i \leq n-1$.*

2.2 Chern classes

Let $\pi : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$, $n \geq 3$, be the blowup of $\mathbb{P}^n$ along a regular curve C, with exceptional divisor E. Set $\mathcal{N} := \mathcal{N}_{C/\mathbb{P}^n}$ and $\rho := \pi|_E$. Since $E \cong \mathbb{P}(\mathcal{N})$, recall that $A(E)$ is generated as an $A(C)$-algebra by the Chern class

$$\zeta := c_1(\mathcal{O}_{\mathcal{N}}(-1))$$

with the single relation

$$\begin{aligned} \zeta^{n-1} - \rho^* c_1(\mathcal{N})\zeta^{n-2} + \ldots + (-1)^{n-1}\rho^* c_{n-2}(\mathcal{N})\zeta \\ + (-1)^{n-1}\rho^* c_{n-1}(\mathcal{N}) = 0. \end{aligned} \tag{15}$$

The normal bundle $\mathcal{N}_{E/\widetilde{\mathbb{P}}^n}$ agrees with the tautological bundle $\mathcal{O}_{\mathcal{N}}(-1)$, and hence

$$\zeta = c_1(\mathcal{N}_{E/\widetilde{\mathbb{P}}^n}). \tag{16}$$

If $\iota : E \hookrightarrow \widetilde{\mathbb{P}}^n$ is the inclusion map, we also get

$$\iota_*(\zeta^i) = (-1)^i E^{i+1}. \tag{17}$$

Given that

$$\int_E \rho^* c_i(\mathcal{N})\zeta^{n-i-1} = (-1)^{n-i-1} \int_C c_i(\mathcal{N}) = 0$$

for $i \geq 2$, we have

$$\begin{aligned}\int_E \zeta^{n-1} &= \int_E \rho^* c_1(\mathcal{N})\zeta^{n-2} = (-1)^n \int_C c_1(\mathcal{N}) \\ &= (-1)^n \int_C c_1(\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}_C) - c_1(C) \\ &= (-1)^n \Big[(n+1)d - 2 + 2g \Big]\end{aligned} \tag{18}$$

where g is the genus and d is the degree of C_{red}.

From Porteous Theorem (cf. [9], [6] or [4]), we have

$$c_1(\widetilde{\mathbb{P}^n}) - \pi^* c_1(\mathbb{P}^n) = -(n-2)E \tag{19}$$

and for $j \geq 2$,

$$\begin{aligned}c_j(\widetilde{\mathbb{P}^n}) &= \pi^* c_j(\mathbb{P}^n) + \sum_{m=0}^{j-1} (-1)^{j-1-m} \Gamma_{1,j} \cdot \rho^* c_m(\mathcal{N}) E^{j-m} \\ &\quad + \sum_{m=0}^{j-2} (-1)^{j-2-m} \Gamma_{2,j} \cdot \rho^* c_m(\mathcal{N}) \rho^* c_1(C) E^{j-1-m}\end{aligned} \tag{20}$$

where $\Gamma_{i,j} = \left(\binom{n-1-m}{j-i-m} - \binom{n-1-m}{j-i+1+m}\right)$, $c_j(\widetilde{\mathbb{P}^n})$ and $c_j(\mathbb{P}^n)$ are the Chern Classes of tangent bundles of $\widetilde{\mathbb{P}}^n$ and $\mathbb{P}^n$, respectively. Here we are assuming that $\binom{p}{q} = 0$ if $p < q$ or $q < 0$.

3 Special foliations along regular curves

In this section, we will consider a one-dimensional holomorphic foliation $\mathcal{F}$ defined in $\mathbb{P}^n$ such that its singular locus is the disjoint union of the irreducible smooth curve C and some closed points p_i. Moreover, $\mathcal{F}$ will be special along this curve. Thus,

$$\Sigma = C \cup \{p_1, \ldots, p_s\}.$$

Our goal is to compute the numbers $\sum_{j=1}^{s} BB_\varphi(\mathcal{F}, p_j)$ as well $BB_\varphi(\mathcal{F}, C)$. For an isolated singularity p the Baum-Bott residue can be defined in terms of the Grothendieck residue as follows. Let $I = (i_1, \ldots, i_n)$ with $i_j \geq 0$ for $j = 1, \ldots, n$. Let $|I| = i_1 + 2i_2 + \ldots + ni_n$ be the height of I. For a $k \times k$ matrix M and $r = 1, \ldots, k$ let $c_r(M)$ be the r-th symmetric function of the eigenvalues of M, i.e., $c_1(M) = \text{trace}(M), \ldots, c_k(M) = \det(M)$. For a multi-index I set $\varphi(M) := (c_1)^{i_1}(c_2)^{i_2} \cdots (c_n)^{i_n}$. Let $z \in \mathbb{C}^n$ be a system of local coordinates on $\mathbb{P}^n$ defined on an open set U_p such that $p \in U_p$ and $p = 0 \in \mathbb{C}^n$. Since p is an isolated singularity, then for any $z \in U_p \setminus \{p\}$, there exists (at least one) i such that $f_i(z) \neq 0$, given in (2). Thus, let

$$J(f)(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{1 \leq i, j \leq n}$$

be the jacobian matrix of $f(z) = (f_1(z), \ldots, f_n(z))$.

Then, the Baum-Bott indices for an isolated singularity p is defined in terms of the Grothendieck residue as

$$BB_\varphi(\mathcal{F}, p) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_R \frac{\varphi(J(f)(z))}{f_1(z)\cdots f_n(z)} dz_1 \wedge \ldots \wedge z_n$$

where $R = \{z \in U_p : |f_i(z)| = \epsilon, i = 1, \ldots, n\}$, $|I| = i_1 + 2i_2 + \ldots + ni_n = n$ for some small $\epsilon > 0$ with the orientation $d(\arg f_1) \wedge \ldots \wedge d(\arg f_n) > 0$.

In order to calculate these indices, we make the blowup $\pi : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$ centered at C, and thus, we obtain an induced foliation $\widetilde{\mathcal{F}}$ on $\widetilde{\mathbb{P}}^n$ which has only isolated singularities as well the exceptional divisor E as an invariant set. In this way, we can assume that $\mathrm{Sing}(\widetilde{\mathcal{F}}) = \{\widetilde{q}_1, \ldots, \widetilde{q}_l\}$. We begin the proof of the Theorem 1.1 with the following proposition:

Proposition 3.1. *Let $\mathcal{G}$ be a holomorphic foliation of degree k defined in $\mathbb{P}^n$ with singular locus* $\mathrm{Sing}(\mathcal{G}) = \{q_1, \ldots, q_r\}$, *$q_i$ are isolated closed points. Then*

$$\sum_{j=1}^{r} BB_\varphi(\mathcal{G}, q_j) = \gamma_\varphi$$

where $\gamma_\varphi = \prod_{j=1}^{n} \gamma_j^{i_j}$ with $\gamma_j = \sum_{i=0}^{j} \binom{n+1}{j-i}(k-1)^i$.

Proof. Let us consider the virtual normal bundle $\nu_{\mathcal{G}} = T\mathbb{P}^n - \mathcal{L}_{\mathcal{G}}$ where $\mathcal{L}_{\mathcal{G}}$ is the tangent bundle of $\mathcal{G}$. By [1], we have that

$$\sum_{j=1}^{r} BB_\varphi(\mathcal{G}, q_j) = \int_{\mathbb{P}^n} c_1^{i_1}(\nu_{\mathcal{G}}) \cdot c_2^{i_2}(\nu_{\mathcal{G}}) \cdots c_n^{i_n}(\nu_{\mathcal{G}})$$

where $c_i(\nu_{\mathcal{G}}) = \sum_{j=0}^{i} c_{i-j}(T\mathbb{P}^n) \cdot c_1^j(\mathcal{L}_{\mathcal{G}}^*) = \gamma_i h^i$, h is the hyperplane class of $\mathbb{P}^n$. Therefore,

$$\sum_{j=1}^{r} BB_\varphi(\mathcal{G}, q_j) = \int_{\mathbb{P}^n} \gamma_1^{i_1} \cdots \gamma_n^{i_n} h^{|I|} = \gamma_\varphi$$

because $|I| = i_1 + 2i_2 + \ldots + ni_n = n$. □

From Lemma 2.2, we calculate the Chern class of the invertible sheaf $\mathcal{L}_{\widetilde{\mathcal{F}}}$, the tangent bundle of the foliation $\widetilde{\mathcal{F}}$. Given that

$$\mathcal{L}_{\widetilde{\mathcal{F}}} \cong \pi^*\mathcal{L}_{\mathcal{F}} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^n}(\ell E)$$

where $\ell = m_C(\mathcal{F})$, we have

$$c_1(\mathcal{L}_{\widetilde{\mathcal{F}}}) = \pi^* c_1(\mathcal{L}_{\mathcal{F}}) + \ell E. \tag{21}$$

We need the following result in order to obtain the proof of Theorem 1.1:

Theorem 3.2. *Let $\mathcal{F}$ has degree k, and multiplicity ℓ at C; let C has genus g and degree d; and let* $\mathrm{Sing}(\widetilde{\mathcal{F}}|_E) = \{\widetilde{q}_1, \ldots, \widetilde{q}_l\}$. *Then*

$$\begin{aligned}\sum_{i=1}^{l} \mu(\widetilde{\mathcal{F}}|_E, \widetilde{q}_i) \;&=\; ((2-2g)\Big[1 + (\ell+1) + (\ell+1)^2 + \ldots + (\ell+1)^{n-3}\Big] \\ &\quad + (\ell+1)^{n-2}\Big[(2-2g)(\ell+1) - (n+1)d\ell + (k-1)d(n-1)\Big].\end{aligned}$$

Proof. See [4]. □

Theorem 3.3. *Let $\mathcal{F}$ has degree k, and multiplicity ℓ at C; let C has genus g and degree d; and let* $\mathrm{Sing}(\widetilde{\mathcal{F}}) = \{\widetilde{p}_1, \ldots, \widetilde{p}_r\}$. *Then*

$$\sum_{i=1}^{r} BB_\varphi(\widetilde{\mathcal{F}}, \widetilde{p}_i) = \gamma_\varphi + \eta_\varphi(\mathcal{F}, C) \tag{22}$$

where

$$\eta_\varphi(\mathcal{F}, C) := (-1)^{\sum i_j}\alpha_\varphi\left[(n+1)d - \chi(C) + \frac{i_1}{\alpha_1(\ell)}\gamma_1 d - \sum_{j=2}^{n}\frac{i_j}{\alpha_j(\ell)}\Omega_j\right]$$

with

$$\Omega_j := \big[\alpha_j'(\ell)(k-1)d + \beta_{1,j}(\ell)\big((n+1)d - \chi(C)\big) + \beta_{0,j}(\ell)\chi(C) - \ell^{j-1}(n+1)d\big],$$

$\alpha_j(\ell) = \sum_{i=0}^{j}\left[\binom{n-1}{i-1} - \binom{n-1}{i}\right]\ell^{j-i}$, $\alpha_j'(\ell) = \sum_{i=0}^{j}\binom{j-i}{1}\left[\binom{n-1}{i-1} - \binom{n-1}{i}\right]\ell^{j-i-1}$, $\alpha_\varphi = \prod_{j=1}^{n}\alpha_j^{i_j}(\ell)$ *and* $\beta_{a,j}(\ell) := \sum_{i=2}^{j}\left[\binom{n-1-a}{i-2} - \binom{n-1-a}{i-1}\right]\ell^{j-1}$.

Proof. Let us consider the virtual normal bundle $\nu_{\widetilde{\mathcal{F}}} = T\widetilde{\mathbb{P}}^n - \mathcal{L}_{\widetilde{\mathcal{F}}}$. From [1], we have that

$$\sum_{i=1}^{r} BB_{\varphi}(\widetilde{\mathcal{F}}, \widetilde{p}_i) = \int_{\widetilde{\mathbb{P}}^n} c_1^{i_1}(\nu_{\widetilde{\mathcal{F}}}) \cdots c_n^{i_n}(\nu_{\widetilde{\mathcal{F}}}) \tag{23}$$

where $c_j(\nu_{\widetilde{\mathcal{F}}}) = \sum_{i=0}^{j} c_i(T\widetilde{\mathbb{P}}^n) \cdot c_1^{j-i}(\mathcal{L}^*_{\widetilde{\mathcal{F}}})$. From (21), we obtain that

$$\begin{aligned} c_1^{j-i}(\mathcal{L}^*_{\widetilde{\mathcal{F}}}) &= \sum_{m=0}^{j-i} \binom{j-i}{m} \pi^* c_1^m(\mathcal{L}^*_{\mathcal{F}})(-\ell)^{j-i-m} \cdot E^{j-i-m} \\ &= (-\ell E)^{j-i} + (j-i)\pi^* c_1(\mathcal{L}^*_{\mathcal{F}})(-\ell E)^{j-i-1} + \ldots + \pi^* c_1^{j-i}(\mathcal{L}^*_{\mathcal{F}}). \end{aligned}$$

The expression for $c_j(T\widetilde{\mathbb{P}}^n)$ is obtained from (20). Given that C is an one-dimensional variety, it follows that

$$\int_{\widetilde{\mathbb{P}}^n} c_{\beta_1}(T\widetilde{\mathbb{P}}^n) \cdot \pi^* c_{\beta_2}(\mathcal{N}) \cdot \pi^* c_1^{\beta_3}(C) \cdot \pi^* c_1^{\beta_4}(\mathcal{L}^*_{\mathcal{F}}) \cdot E^{n-\beta_1-\beta_2-\beta_3-\beta_4} = 0 \tag{24}$$

if

$$2 \leq \beta_1 + \beta_2 + \beta_3 + \beta_4 \leq n-1.$$

Therefore, for $i \geq 2$, we only need to focus on these terms,

$$\begin{aligned} c_i(T\widetilde{\mathbb{P}}^n) \cdot c_1^{j-i}(\mathcal{L}^*_{\widetilde{\mathcal{F}}}) = {} & (-1)^{j-1}\left[\binom{n-1}{i-1} - \binom{n-1}{i}\right](\ell)^{j-i} \cdot E^j \\ & + (-1)^j \binom{j-i}{1}\left[\binom{n-1}{i-1} - \binom{n-1}{i}\right]\ell^{j-i-1}\pi^* c_1(\mathcal{L}^*_{\mathcal{F}}) \cdot E^{j-1} \\ & + (-1)^j \left[\binom{n-2}{i-2} - \binom{n-2}{i-1}\right]\ell^{j-i}\pi^* c_1(\mathcal{N}) \cdot E^{j-1} \\ & + (-1)^j \left[\binom{n-1}{i-2} - \binom{n-1}{i-1}\right]\ell^{j-i}\pi^* c_1(C) \cdot E^{j-1} \\ & + \pi^* c_i(T\mathbb{P}^n)\pi^* c_1^{j-i}(\mathcal{L}^*_{\mathcal{F}}) + \ldots \end{aligned}$$

In the case where $i = 1$, we can concentrate in the following terms

$$\begin{aligned} c_1(T\widetilde{\mathbb{P}}^n) \cdot c_1^{j-1}(\mathcal{L}^*_{\widetilde{\mathcal{F}}}) = {} & \pi^* c_1(T\mathbb{P}^n) \cdot \pi^* c_1^{j-1}(\mathcal{L}^*_{\mathcal{F}}) + (-\ell)^{j-1}\pi^* c_1(T\mathbb{P}^n) \cdot E^{j-1} \\ & - (n-2)(j-1)(-\ell)^{j-2}\pi^* c_1(T^*_{\mathcal{F}}) \cdot E^{j-1} \\ & - (n-2)(-\ell)^{j-1} E^j + \ldots \end{aligned}$$

Summing up the cases $i = 0$, $i = 1$ and $i \geq 2$, we obtain that

$$\begin{aligned} c_j(\nu_{\widetilde{\mathcal{F}}}) \;=\; & \sum_{i=0}^{j} \pi^* c_i(T\mathbb{P}^n)\cdot \pi^* c_1^{j-i}(\mathcal{L}^*_{\mathcal{F}}) \\ & + (-1)^{j-1}\sum_{i=0}^{j}\left[\binom{n-1}{i-1}-\binom{n-1}{i}\right](\ell)^{j-i}E^j \\ & + (-1)^{j}\sum_{i=0}^{j-1}\binom{j-i}{1}\left[\binom{n-1}{i-1}-\binom{n-1}{i}\right](\ell)^{j-i-1}\pi^* c_1(\mathcal{L}^*_{\mathcal{F}})E^{j-1} \\ & (-1)^{j}\sum_{i=2}^{j}\left[\binom{n-2}{i-2}-\binom{n-2}{i-1}\right](\ell)^{j-i}\rho^* c_1(\mathcal{N})E^{j-1} \\ & (-1)^{j}\sum_{i=2}^{j}\left[\binom{n-1}{i-2}-\binom{n-1}{i-1}\right](\ell)^{j-i}\rho^* c_1(C)E^{j-1} \\ & (-1)^{j-1}(\ell)^{j-1}\pi^* c_1(T\mathbb{P}^n)\cdot E^{j-1}+\ldots \end{aligned} \tag{25}$$

for $j \geq 2$. Note that $\sum_{i=0}^{j}\pi^* c_i(T\mathbb{P}^n)\cdot\pi^* c_1^{j-i}(\mathcal{L}^*_{\mathcal{F}}) = \gamma_j\pi^* h^j$. For $j = 1$, we have that

$$c_1(\nu_{\widetilde{\mathcal{F}}}) = \pi^* c_1(T\mathbb{P}^n) + \pi^* c_1(T^*_{\mathcal{F}}) - (n-2+\ell)E = \gamma_1\pi^* h + \alpha_1(0)E. \tag{26}$$

In order to simplify our notation, we will rewrite (25) in the following way

$$c_j(\nu_{\widetilde{\mathcal{F}}}) = \gamma_j\pi^* h^j + (-1)^{j-1}\alpha_j(\ell)E^j + (-1)^j\delta_j E^{j-1} + \ldots$$

where

$$\delta_j := \left(\alpha'_j(\ell)\pi^* c_1(\mathcal{L}^*_{\mathcal{F}}) + \beta_{1,j}(\ell)\rho^* c_1(\mathcal{N}) + \beta_{0,j}(\ell)\rho^* c_1(TC) - \ell^{j-1}\pi^* c_1(T\mathbb{P}^n)\right).$$

Now, to apply (23) we must calculate $c_j^{i_j}(\nu_{\widetilde{\mathcal{F}}})$. And for this, we get the binomial theorem. Always keeping in our mind (24), the non-vanishing terms are to concern with the following terms

$$\begin{aligned} c_j^{i_j}(\nu_{\widetilde{\mathcal{F}}}) \;&=\; \gamma_j^{i_j}\pi^* h^{ji_j} + (-1)^{(j-1)i_j}\alpha_j^{i_j}(\ell)E^{ji_j} \\ &\quad + (-1)^{(j-1)(i_j-1)+j}(i_j)\alpha_j^{i_j-1}(\ell)E^{j(i_j-1)}\delta_j\cdot E^{j-1}+\ldots \\ &=\; \gamma_j^{i_j}\pi^* h^{ji_j} + (-1)^{(j-1)i_j}\alpha_j^{i_j}(\ell)E^{ji_j} \\ &\quad + (-1)^{(j-1)i_j+1}(i_j)\alpha_j^{i_j-1}(\ell)\delta_j E^{ji_j-1}+\ldots \end{aligned}$$

Finally, for $j = 1$, we have that

$$\begin{aligned} c_1^{i_1}(\nu_{\widetilde{\mathcal{F}}}) &= [\gamma_1\pi^*h + \alpha_1(\ell)E]^{i_1} \\ &= \gamma_1^{i_1}\pi^*h^{i_1} + \ldots + i_1\gamma_1\pi^*h[\alpha_1(\ell)E]^{i_1-1} + [\alpha_1(\ell)E]^{i_1}. \end{aligned}$$

Thus,

$$\begin{aligned} \prod_{j=1}^{n} c_j^{i_j}(\nu_{\widetilde{\mathcal{F}}}) &= \prod_{j=1}^n \gamma_j^{i_j}\pi^*h^n + (-1)^{n-\sum i_j}\prod_{j=1}^n \alpha_j^{i_j}\cdot E^n \\ &\quad - (-1)^{n-\sum i_j}\sum_{j=2}^{n} i_j\alpha_j^{i_j-1}\delta_j\left[\alpha_1^{i_1}\ldots\widehat{\alpha_j^{i_j}}\ldots\alpha_n^{i_n}\right]E^{n-1} \\ &\quad + (-1)^{n-\sum i_j} i_1\gamma_1\alpha_1^{i_1-1}\pi^*h\left[\alpha_2^{i_2}\ldots\alpha_n^{i_n}\right]E^{n-1} + \ldots \end{aligned}$$

where $\alpha_j := \alpha_j(\ell)$. It follows that

$$\begin{aligned} \prod_{j=1}^{n} c_j^{i_j}(\nu_{\widetilde{\mathcal{F}}}) = \gamma_\varphi\pi^*h^n + (-1)^{n-\sum i_j}\alpha_\varphi E^n - (-1)^{n-\sum i_j}\alpha_\varphi\sum_{j=2}^{n}\frac{i_j}{\alpha_j}\delta_j E^{n-1} \\ + (-1)^{n-\sum i_j}\alpha_\varphi\frac{i_1}{\alpha_1}\gamma_1\pi^*hE^{n-1} + \ldots \end{aligned}$$

with $\alpha_\varphi := \alpha_1^{i_1}\cdots\alpha_n^{i_n}$. Given that $\int_{\widetilde{\mathbb{P}}^n} E^n = \int_E \zeta^{n-1}$, from (18), we get

$$\begin{aligned} \int_{\widetilde{\mathbb{P}}^n}\delta_j E^{n-1} &= \int_E \delta_j E^{n-2} = (-1)^n\int_C \pi_*\delta_j \\ &= (-1)^n\big[\alpha_j'(\ell)(k-1)d + \beta_{1,j}(\ell)((n+1)d - \chi(C)) \\ &\quad + \beta_{0,j}(\ell)\chi(C) - \ell^{j-1}(n+1)d\big]. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\widetilde{\mathbb{P}}^n}\prod_{j=1}^{n} c_j^{i_j}(\nu_{\widetilde{\mathcal{F}}}) &= \gamma_\varphi + (-1)^{\sum i_j}\alpha_\varphi\Bigg\{(n+1)d - \chi(C) + \frac{i_1}{\alpha_1}\gamma_1 d \\ &\quad - \sum_{j=2}^{n}\frac{i_j}{\alpha_j}\Big[\alpha_j'(\ell)(k-1)d + \beta_{1,j}(\ell)\big((n+1)d - \chi(C)\big) \\ &\quad + \beta_{0,j}(\ell)\chi(C) - \ell^{j-1}(n+1)d\Big]\Bigg\}. \end{aligned}$$

Then, we conclude that

$$\sum_{j=1}^{r} BB_\varphi(\widetilde{\mathcal{F}}, \widetilde{p}_j) = \gamma_\varphi + \eta_\varphi(\mathcal{F}, C). \tag{27}$$

□

Example 3.4. *For sake of example, we expand the formula (27) for three cases of our interest, that are, $\varphi = c_1^n$, $\varphi = c_1 c_{n-1}$ and $\varphi = c_n$. Then, in the first case, we have $i_1 = n$ therefore,*

$$\begin{aligned}\sum_{j=1}^{r} BB_{c_1^n}(\widetilde{\mathcal{F}}, \widetilde{p_j}) &= \gamma_1^n + (-1)^n \alpha_1^n(\ell)[(n+1)d - \chi(C)] + (-1)^n n d \gamma_1 \alpha_1^{n-1}(\ell) \\ &= \gamma_1^n + [(n+1)d - \chi(C)](n+\ell-2)^n - n d \gamma_1 (n+\ell-2)^{n-1}.\end{aligned}$$

In the second case, we have that $i_1 = i_{n-1} = 1$ and $i_j = 0$ for others cases. Thus,

$$\begin{aligned}\sum_{j=1}^{r} BB_{c_1 c_{n-1}}(\widetilde{\mathcal{F}}, \widetilde{p}_j) &= \gamma_1 \gamma_{n-1} + \alpha_1(\ell)\alpha_{n-1}(\ell)\big[(n+1)d - \chi(C)\big] \\ &\quad - \alpha_1(\ell)\Big[\alpha'_{n-1}(\ell)(k-1)d + \beta_{1,n-1}(\ell)\big((n+1)d - \chi(C)\big) \\ &\quad + \beta_{0,n-1}(\ell)\chi(C) - \ell^{n-2}(n+1)d\Big] + \gamma_1 \alpha_{n-1}(\ell) d.\end{aligned}$$

But, $\alpha_{n-1}(\ell) = \sum_{j=0}^{n-2}(1+\ell)^j - (1+\ell)^{n-1}$, $\beta_{1,n-1}(\ell) = \sum_{j=0}^{n-3}(1+\ell)^j - (1+\ell)^{n-2} + \ell^{n-2}$ and $\beta_{0,n-1}(\ell) = \sum_{j=0}^{n-3}(n-3-j)(1+\ell)^j - (1+\ell)^{n-2} + \ell^{n-2}$ which result that

$$\begin{aligned}\sum_{j=1}^{r} BB_{\varphi}(\widetilde{\mathcal{F}}, \widetilde{p}_j) &= (n+k)\Bigg[\sum_{i=0}^{n-1}(n-i)k^i + d\Bigg(\sum_{j=0}^{n-2}(1+\ell)^j - (1+\ell)^{n-1}\Bigg)\Bigg] \\ &\quad - (n+\ell-2)\Bigg\{\big[(1+\ell)^{n-2}(1-\ell) - \ell^{n-2}\big]\big[(n+1)d - \chi(C)\big] \\ &\quad - (k-1)d\Bigg(\sum_{j=0}^{n-3}(j+1)(1+\ell)^j - (n-1)(1+\ell)^{n-2}\Bigg) \\ &\quad - \chi(C)\Bigg(\sum_{j=0}^{n-3}(n-3-j)(1+\ell)^i - (1+\ell)^{n-2} + \ell^{n-2}\Bigg) \\ &\quad + \ell^{n-2}(n+1)d\Bigg\}.\end{aligned}$$

for $\varphi = c_1 c_{n-1}$.

In the last case, $\varphi = c_n$*, that is,* $i_n = 1$*, therefore*

$$\sum_{j=1}^{r} \mu(\widetilde{\mathcal{F}}, \widetilde{p}_j) = \gamma_n - \alpha_n(\ell)[(n+1)d - \chi(C)] + \alpha_n'(\ell)(k-1)d$$
$$+ \beta_{1,n}(\ell)\big[(n+1)d - \chi(C)\big] + \beta_{0,n}(\ell)\chi(C) - \ell^{n-1}(n+1)d.$$

But, $\gamma_n = \sum_{j=0}^{n} k^j$, $\alpha_n(\ell) = (1-\ell)(1+\ell)^{n-1}$, $\beta_{1,n}(\ell) = (1-\ell)(1+\ell)^{n-2}+\ell^{n-1}$, $\beta_{0,n}(\ell) = \sum_{j=0}^{n-2}(1+\ell)^j - (1+\ell)^{n-1} + \ell^{n-1}$. *It follows that*

$$\sum_{i=1}^{r} \mu(\widetilde{\mathcal{F}}, \widetilde{p}_i) = \sum_{j=0}^{n} k^j + (2-2g)\Big(1 + (\ell+1) + \ldots + (\ell+1)^{n-3}\Big) +$$
$$+ (\ell+1)^{n-2}\Big((n+1)d(\ell^2-\ell) - (2-2g)\ell^2 - (k-1)d(n\ell - n + 2)\Big).$$

Therefore, this result agrees with the result of [4] when $\varphi = c_n$.

Example 3.5. *Let* $\mathcal{F}$ *be a one-dimensional holomorphic foliation of degree* $k \geq 2$ *on* $\mathbb{P}^3$*, defined on the affine open set* $U_3 = \{[\xi_i] \in \mathbb{P}^3 \mid \xi_3 \neq 0\}$ *by the vector field*

$$\mathcal{D}_{\mathcal{F}} = \Big(\sum_{j=0}^{k} a_{1,j} z_1^{k-j} z_2^j\Big)\frac{\partial}{\partial z_1} + \Big(\sum_{j=0}^{k} a_{2,j} z_1^{k-j} z_2^j\Big)\frac{\partial}{\partial z_2} + \Big(\sum_{j=0}^{k-1} z_1^{k-j-1} z_2^j R_j(z)\Big)\frac{\partial}{\partial z_3}$$

where $z_i = \xi_{i-1}/\xi_0$ *and* $R_j(z) = r_{0,j} + \sum_{i=1}^{3} r_{i,j} \cdot z_i$ *are affine linear functions for* $0 \leq j \leq k-1$.

Let C *be the curve defined by* $\xi_0 = \xi_1 = 0$ *which is the curve of singularities of* $\mathcal{F}$ *and we blowup* $\mathbb{P}^3$ *along it. We will assume that the polynomial* $\varphi_i(\zeta) = \sum_{j=0}^{k} a_{i,j}\zeta^j$ *do not have any common root for* $i = 1, 2$*. This hypothesis is fundamental in order to* $\mathcal{F}$ *be special along* C*. Thus,* $\ell = k-1$, $\chi(C) = 2$ *and* $d = \deg(C) = 1$*. Therefore, in the open* $\widetilde{U}_1$*, with coordinates* $u \in \mathbb{C}^3$ *such that*

$$\sigma(u) = (u_1, u_1u_2, u_3) = (z_1, z_2, z_3) \in \mathbb{C}^3 \tag{28}$$

the induced foliation $\widetilde{\mathcal{F}}$ *is described by the following vector field*

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = u_1\varphi_1(u_2)\frac{\partial}{\partial u_1} + \big[\varphi_2(u_2) - u_2\varphi_1(u_2)\big]\frac{\partial}{\partial u_2}$$
$$+ \big[r_0(u_2) + u_3 r_3(u_2) + \widetilde{R}(u)\big]\frac{\partial}{\partial u_3}$$

where $r_i(u_2) = \sum_{j=0}^{k-1} r_{i,j} u_2^j$ *and some function* $\widetilde{R}$ *which is null on the exceptional divisor. It is not hard to see that, in the affine open set* U_3*, the foliation* $\widetilde{\mathcal{F}}$*, when restricted to the exceptional divisor* E*, which is given by* $u_1 = 0$*, has* $k+1$ *isolated singularities, namely,* $\widetilde{p}_{1,j} = \{[0 : 0 : \beta_j : 1] \times [\zeta_j : 1]\}$ *where* ζ_j *is a root of* $\psi(t) := \varphi_2(t) - t\varphi_1(t)$ *and* $\beta_j = -r_0(\zeta_j)/r_3(\zeta_j)$ *for* $j = 1, \ldots, k+1$*. In these points, the eigenvalues of the linear part of* $\mathcal{D}_{\widetilde{\mathcal{F}}}$ *are* $\lambda_{1,j}^1 = \varphi_1(\zeta_j)$*,* $\lambda_{1,j}^2 = \psi'(\zeta_j)$ *and* $\lambda_{1,j}^3 = r_3(\zeta_j)$*.*

After the change of coordinates, for the affine chart $\xi_2 \neq 0$*, with coordinates* $x_1 = z_1/z_3$*,* $x_2 = z_2/z_3$ *and* $x_3 = 1/z_3$*,* $\mathcal{F}$ *is described by the vector field*

$$\mathcal{D}_{\mathcal{F}} = \sum_{i=1}^{2} \left(\sum_{j=0}^{k} a_{1,j} x_1^{k-j} x_2^j \right) \frac{\partial}{\partial z_i} - \sum_{i=1}^{3} x_i g(x) \frac{\partial}{\partial z_i}$$

where $g(x) = \sum_{j=0}^{k-1} x_1^{k-j-1} x_2^j [r_{0,j} x_3 + r_{1,j} x_1 + r_{2,j} x_2 + r_{3,j}]$*.*

Again, with the same relations (28), i.e., $\sigma(u) = x \in \mathbb{C}^3$*, we get the expression for* $\mathcal{D}_{\widetilde{\mathcal{F}}}$*,*

$$\mathcal{D}_{\widetilde{\mathcal{F}}} = u_1[\varphi_1(u_2) - \delta(u)] \frac{\partial}{\partial u_1} + \psi(u_2) \frac{\partial}{\partial u_2} - u_3 \delta(u) \frac{\partial}{\partial u_3}$$

where $\delta(u) = u_3 r_0(u_2) + r_1(u_2) u_1 + u_1 u_2 r_2(u_2) + r_3(u_2)$*.*

Therefore, $\widetilde{\mathcal{F}}$ *has additional* $2k+2$ *isolated singularities. Namely,* $\widetilde{p}_{2,j} = \{[0 : 0 : 1 : 0] \times [\zeta_j : 1]$ *and* $\widetilde{p}_{3,j} = [v_j : \zeta_j v_j : 1 : 0]$ *where* $v_j = \frac{a_1(\zeta_j) - r_3(\zeta_j)}{r_1(\zeta_j) + \zeta_j r_2(\zeta_j)}$ *for* $j = 1, \ldots, k+1$*. Then,* $\widetilde{\mathcal{F}}$ *has* $3(k+1)$ *isolated singularities in* $\widetilde{\mathbb{P}}^n$*, counted with multiplicities.*

The eigenvalues of the linear part of $\mathcal{D}_{\widetilde{\mathcal{F}}}$ *at* $\widetilde{p}_{2,j}$ *are* $\lambda_{2,j}^1 = \varphi_1(\zeta_j) - r_3(\zeta_j)$*,* $\lambda_{2,j}^2 = \psi'(\zeta_j)$ *and* $\lambda_{2,j}^3 = -r_3(\zeta_j)$ *while the eigenvalues at* $\widetilde{p}_{3,j}$ *are* $\lambda_{3,j}^1 = -(\varphi_1(\zeta_j) - r_3(\zeta_j))$*,* $\lambda_{3,j}^2 = \psi'(\zeta_j)$ *and* $\lambda_{3,j}^3 = -\varphi_1(\zeta_j)$*.*

Then, given that

$$\sum_{i=1}^{3} BB_{c_1^3}(\widetilde{\mathcal{F}}, \widetilde{p}_{i,j}) = 27, \sum_{i=1}^{3} BB_{c_1 c_2}(\widetilde{\mathcal{F}}, \widetilde{p}_{i,j}) = 12, \forall j,$$

we obtain

$$\sum_{i=1}^{3} \sum_{j=1}^{k+1} BB_{c_1^3}(\widetilde{\mathcal{F}}, \widetilde{p}_{i,j}) = 27(k+1), \sum_{i=1}^{3} \sum_{j=1}^{k+1} BB_{c_1 c_2}(\widetilde{\mathcal{F}}, \widetilde{p}_{i,j}) = 12(k+1)$$

and

$$\sum_{i=1}^{3}\sum_{j=1}^{k+1} BB_{c_n}(\widetilde{\mathcal{F}}, \widetilde{p}_{i,j}) = \sum_{i=1}^{3}\sum_{j=1}^{k+1} \mu(\widetilde{\mathcal{F}}, \widetilde{p}_{i,j}) = 3k+3.$$

These results agree with the number obtained by Theorem 3.3, as shown in the last example, taking $\ell = k-1$, $g = 0$ and $d = 1$.

Corollary 3.6. *Let $\mathcal{F}$ has degree k, and multiplicity ℓ at C; let C has genus g and degree d. Then*

(i) $\sum_{i=1}^{s} BB_\varphi(\mathcal{F}, p_i) = \gamma_\varphi + \eta_\varphi(\mathcal{F}, C) - \sum_{\widetilde{p}_j\in E} BB_\varphi(\widetilde{\mathcal{F}}, \widetilde{p}_j)$.
In particular, $\sum_{i=1}^{s} \mu(\mathcal{F}, p_i) = \sum_{j=0}^{n} k^j + \nu(\mathcal{F}, C)$;

(ii) $BB_\varphi(\mathcal{F}, C) = -\eta_\varphi(\mathcal{F}, C) + \sum_{\widetilde{p}_j\in E} BB_\varphi(\widetilde{\mathcal{F}}, \widetilde{p}_j)$.
In particular, $\mu(\mathcal{F}, C) = -\nu(\mathcal{F}, C)$.

Proof. Just note that

$$\sum_{i=1}^{s} BB_\varphi(\mathcal{F}, p_i) = \sum_{i=1}^{r} BB_\varphi(\widetilde{\mathcal{F}}, \widetilde{p}_i) - \sum_{\widetilde{p}_j\in E} BB_\varphi(\widetilde{\mathcal{F}}, \widetilde{p}_j)$$

because π is an isomorphism away for C, $\sum_{i=1}^{s} BB_\varphi(\mathcal{F}, p_i) + BB_\varphi(\mathcal{F}, C) = \gamma_\varphi$ and recall Theorems 3.2 and 3.3 and Example 3.4. □

4 A method to compute Baum-Bott indices

In this section, we will introduce a method to calculate the Baum-Bott indices. This procedure is quite effective when the singular locus of the foliation has more than one curve. As consequence, we obtain the proof of the Theorem 1.1. In order to do it, we need the following result:

Proposition 4.1. *Let $\mathcal{F}$ be a holomorphic foliation defined in $\mathbb{P}^n$ with $n \geq 2$. Assume that exists a regular curve C invariant by $\mathcal{F}$ such that* $\mathrm{Sing}(\mathcal{F}) \cap C = \{q_1, \ldots, q_l\}$. *Then,*

$$\sum_{j=1}^{l} \mu(\mathcal{F}, q_j) = (k-1)d + 2 - 2g = -\nu(\mathcal{F}, C)|_{\ell=0}$$

where $\deg(C) = d$ *and g is the genus of C.*

Proof. Let us consider the normal virtual bundle $\nu_{\mathcal{F}} = TC - \mathcal{L}_{\mathcal{F}}$. Then,

$$\sum_{j=1}^{l} \mu(\mathcal{F}, q_j) = \int_C c_1(\nu_{\mathcal{F}})$$

where $c_1(\nu_{\mathcal{F}}) = c_1(TC) + c_1(\mathcal{L}_{\mathcal{F}}^*)$. Therefore, follows the result. □

Now, if the singular locus Σ contains r disjoint curves C_i then we can admit, reordering if necessary, that $\ell_i := \text{mult}_{E_{C_i}}(\pi_{C_i}^* \mathcal{F}) \geq 1$ for $i \leq a \leq r$ and $\ell_i = 0$ for $a < i \leq r$.

Lemma 4.2. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, of degree k, such that its singular locus is the disjoint union of irreducible curves $C_1, \ldots, C_r$ and closed points $p_1, \ldots, p_s$. Assume each C_i is either smooth, or a singular scheme theoretic complete intersection; and also a non-dicritical component of* $\text{Sing}(\mathcal{F})$ *for* $1 \leq i \leq r$.

Then there exists a one-parameter family of holomorphic foliations by curves denoted by $\mathcal{F}_t$ defined in $\mathbb{P}^n$ with $t \in D(0, \epsilon)$, for some $\epsilon > 0$ sufficiently small such that

(i) $\mathcal{F}_0 = \mathcal{F}$ *and* $\deg(\mathcal{F}_t) = \deg(\mathcal{F}), \forall t \in D(0, \epsilon)$;

(ii) $\text{Sing}(\mathcal{F}_t) = \cup_{i=1}^{a} C_i^t \cup \{q_1^t, \ldots, q_{s_t}^t\}$, *where C_i^t is a regular irreducible projective curve such that* $\deg(C_i) = \deg(C_i^t)$ *and q_j^t are closed points for $t \neq 0$;*

(iii) *$\mathcal{F}_t$ is special along C_i^t with* $\text{mult}_{E_{C_i^t}}(\pi_{C_i^t}^* \mathcal{F}_t) = \text{mult}_{E_{C_i}}(\pi_{C_i}^* \mathcal{F})$, *for $i \leq a \leq r$;*

(iv) *C_i^t is an invariant set of $\mathcal{F}_t$ for $i > a$, that is, if* $\ell_i = \text{mult}_{E_{C_i}}(\pi_{C_i}^* \mathcal{F}) = 0$;

(v) *Let $Y_0 = \mathbb{P}^n$ and $\{\pi_i^t\}_{i=1}^{a}$ be the sequence of blowups $\pi_i^t : Y_i \to Y_{i-1}$ centered at C_i^t and $E_i^t = (\pi_i^t)^{-1}(C_i^t)$ be the exceptional divisor of each blowup. Then*

$$\sum_{j=i}^{s} BB_{\varphi}(\mathcal{F}, p_j) = \gamma_{\varphi} + \sum_{j=1}^{a} \eta_{\varphi}(\mathcal{F}, C_j) - \lim_{t \to 0} \sum_{i=1}^{a} \sum_{\lim \widetilde{q}_j^t \in E_i^t} BB_{\varphi}(\widetilde{\mathcal{F}}_t, \widetilde{q}_j^t)$$
$$- \lim_{t \to 0} \sum_{i=a+1}^{r} \sum_{\lim q_j^t \in C_i^t} BB_{\varphi}(\mathcal{F}_t, q_j^t)$$

where $\widetilde{\mathcal{F}}_t$ is the foliation induced by $\mathcal{F}_t$ via π_a^t, the singular locus of $\widetilde{\mathcal{F}}_t$ is $\mathrm{Sing}(\widetilde{\mathcal{F}}_t) = \{\widetilde{q}_j^t,\ j = 1, \ldots, \widetilde{s}_t\}$, *$\widetilde{q}_j^t$ are closed points. In particular,*

$$\sum_{i=0}^{s} \mu(\mathcal{F}, p_i) = \sum_{j=0}^{n} k^j + \sum_{j=1}^{r} \nu(\mathcal{F}, C_j) - \mathcal{A}_i;$$

where $\mathcal{A}_i$ is the number of embedded closed points of C_i, counted with multiplicities.

(vi) *If $\ell_j \geq 1$ then*

$$BB_\varphi(\mathcal{F}, C_i) = -\eta_\varphi(\mathcal{F}, C_i) + \lim_{t\to 0}\Bigg[\sum_{\lim \widetilde{q}_j^t \in E_i^t} BB_\varphi(\widetilde{\mathcal{F}}_t, \widetilde{q}_j^t)\Bigg].$$

If $\ell_j = 0$ then

$$BB_\varphi(\mathcal{F}, C_i) = \lim_{t\to 0}\Bigg[\sum_{\lim \widetilde{q}_j^t \in C_i^t} BB_\varphi(\mathcal{F}_t, q_j^t)\Bigg].$$

Anyway,

$$\mu(\mathcal{F}, C_i) = -\nu(\mathcal{F}, C_i) + \mathcal{A}_i, \forall i = 1, \ldots, r.$$

Proof. As before, we will begin our analysis with the case where the singular locus of $\mathcal{F}$ contains only one curve denoted by C. In an affine standard chart of $\mathbb{P}^n$, C is given as the zeros of the polynomials $f_1, \ldots, f_{n-1}$. Let $p \in C$ be a regular point of C and $z \in \mathbb{C}^n$ be the holomorphic coordinate such that $z(p) = 0$. There exist a neighborhood U of $0 \in \mathbb{C}^n$ such that $C \cap U$ is a regular curve. Let us consider the vectorial polynomial function

$$G(z) = \big(f_1(z), \ldots, f_{n-1}(z)\big).$$

Without loss of generality, we can admit $\mathrm{Det}(M) \neq 0$ where M is the jacobian matrix of G relatives to variables $z_1, \ldots, z_{n-1}$ in the neighborhood U_1 of $0 \in \mathbb{C}^{n-1}$. Shrinking U, if necessary, we can admit the holomorphic function $F(z) = (G(z), z_n)$ is a local biholomorphism in $U \subset \mathbb{C}^n$ onto an open set $V \subset \mathbb{C}^n$ such that the image of $U \cap C$ by F is the w_n-axis restricted to V. In this way, we may fix coordinates $w = F(z)$. Thus we describe the push-forward $F_*\mathcal{F}$ in V, as in (4), by the vector field

$$\mathcal{D}_{F_*\mathcal{F}} = P_1(w)\frac{\partial}{\partial w_1} + \cdots + P_n(w)\frac{\partial}{\partial w_n} \tag{29}$$

where

$$P_i(w) = \sum_{|a|=m_i} w_1^{a_1} \cdots w_{n-1}^{a_{n-1}} P_{i,a}(w) \tag{30}$$

where $a = (a_1, \ldots, a_{n-1})$, with at least one $P_{i,a}$ not vanishing in the w_n-axis, i.e, $m_i = \mathrm{mult}_C(P_i)$. The foliation $\mathcal{F}$ is given by the polynomial vector field

$$\mathcal{D}_{\mathcal{F}} = Q_1(z)\frac{\partial}{\partial z_1} + \ldots + Q_n(z)\frac{\partial}{\partial w_n}$$

where $Q(z) = (Q_1(z), \ldots, Q_n(z))$ are obtained by the system $P \circ F(z) = DF(z) \cdot Q(z)$, with $P = (P_1, \ldots, P_n)$ and $DF(z)$ is the jacobian matrix of F.

Solving this system by the Cramer's rule, we obtain that

$$Q_i(z) = \frac{\mathrm{Det}(A_i)}{\mathrm{Det}(M)}$$

where one gets A_i replacing the i-th column of DF by the vector column $P \circ F(z)$. In particular,

$$Q_n(z) = \frac{P_n \circ F(z) \cdot \mathrm{Det}(M)}{\mathrm{Det}(M)} = P_n \circ F(z).$$

Therefore, after the normalizing by the factor $\mathrm{Det}(M)$, one may describe $\mathcal{F}$ in U by the following vector field

$$\mathcal{D}_{\mathcal{F}} = \sum_{i=1}^{n-1} \mathrm{Det}(A_i(z))\frac{\partial}{\partial z_i} + P_n \circ F(z) \cdot \mathrm{Det}(M)\frac{\partial}{\partial z_n}. \tag{31}$$

As all the components of $\mathcal{D}_{\mathcal{F}}$ are polynomials, using Hartogs' Extension Theorem, we can consider $\mathcal{D}_{\mathcal{F}}$ defined in $\mathbb{C}^n$.

Let C_t be the projective closure of the common zeros locus of the polynomials $f_i + th_i$ with $\deg(h_i) \leq \deg(f_i)$ for $1 \leq i \leq n-1$, which assures that $\deg(C_t) = \deg(C)$. So we may adjust the g_i and find $\epsilon > 0$ sufficiently small such that $C_t \cap \mathbb{C}^n$ is regular for $0 < |t| < \epsilon$. See [4] for more details. And we may shrink ϵ even more in order to have $\chi(C_t) = \chi(C) + \sum_{i=1}^{l}(b_i - 1)$(cf. [10]). As was done for a regular point of C, let $w = F_t(z) = (G_t(z), z_n)$ be a local biholomorphism where $G_t(z) = G(z) + t(h_1(z), \ldots, h_{n-1}(z))$. Thus, the image of $C_t \cap \mathbb{C}^n$ by F_t is the $w_n - axis$. Let $\mathcal{G}_t$ be a holomorphic foliation by curves described by the vector field

$$\mathcal{D}_{\mathcal{G}_t} = \sum_{i=1}^{n-1} \mathrm{Det}(A_i^t(z))\frac{\partial}{\partial z_i} + P_n \circ F_t(z) \cdot \mathrm{Det}(M_t)\frac{\partial}{\partial z_n}. \tag{32}$$

where (32) is obtained from (31), replacing f_i by $f_i + th_i$. By construction, $\mathcal{F} = \mathcal{G}_0$, $C_t \subset \mathrm{Sing}(\mathcal{G}_t)$ and $\deg\big(\mathcal{F}\big) = \deg\big(\mathcal{G}_t\big)$ for all t.

Let $\mathcal{F}_t$ be the one-parameter family of holomorphic foliation defined locally via its push-forward $(F_t)_*(\mathcal{F}_t)$ as follows

$$\mathcal{D}_{(F_t)_*(\mathcal{F}_t)} = \mathcal{D}_{(F_t)_*\mathcal{G}_t} + t\sum_{i=1}^{n} g_i(w)\frac{\partial}{\partial w_i} \tag{33}$$

where $g_i(w) = \sum_{|a|=q_i} w_1^{a_1}\cdots w_{n-1}^{a_{n-1}} g_{i,a}(w)$, with $g_{i,a}$ are constants for $1 \le i \le n-1$, $g_{n,a}(w) = r_{0,a} + \sum_{i=1}^{n} r_{i,a}w_i$ are affine linear functions and $q_i = \mathrm{mult}_C(g_i)$ such that

$$q_1 = q_2 = \ldots = q_{n-1} = q_n + 1 = \ell + 1. \tag{34}$$

The behavior of C_t in relation to $\mathcal{F}_t$ varies to according to $\ell = 0$ or not. To be more precise, if $\ell \ge 1$ then, by (33) and (34), the curve C_t is contained in the singular locus of $\mathcal{F}_t$ since $\mathcal{D}_{(F_t)_*(\mathcal{F}_t)}$ is identically zero on the w_n-axis. But, if $\ell = 0$ then C_t is an invariant set by $\mathcal{F}_t$ because $\mathcal{D}_{(F_t)_*(\mathcal{F}_t)}$, when restricted to the w_n-axis, has the following form $t(r_{0,0} + r_{n,0}w_n)\frac{\partial}{\partial w_n}$ since $q_n = 0$.

Firstly, we will assume $\ell \ge 1$. Adjusting the functions g_i and shrinking ϵ, if necessary, we can admit that $\mathcal{F}_t$ is special along C_t and $\mathrm{Sing}\big(\mathcal{F}_t\big) = C_t \cup \{q_1^t, \ldots, q_{s_t}^t\}$ where each q_i^t is a closed point. By construction, $\deg(\mathcal{F}) = \deg(\mathcal{F}_t)$ and $\mathrm{mult}_C(\pi^*\mathcal{F}) = \mathrm{mult}_{C_t}(\pi_t^*(\mathcal{F}_t))$ for all t such that $0 < |t| < \epsilon$ where π_t is the blowup of $\mathbb{P}^n$ centered at C_t. Moreover, the functions g_i and h_i are chosen in order to $\deg(\mathcal{F}) = \deg(\mathcal{F}_t)$. Thus, $\mathcal{F}_t$ satisfy the items (i), (ii) and (iii) of the Lemma.

If $\mathcal{F}$ is special along C, then

$$\sum_{i=1}^{s} \mu(\mathcal{F}, p_i) = \sum_{i=1}^{s_t} \mu(\mathcal{F}_t, q_i^t)$$

for t sufficiently small, that is, C has no embedded points (cf. [4], Lemma 4.1).

Consequently, by [2], we have that

$$\sum_{i=1}^{s} BB_\varphi(\mathcal{F}, p_i) = \lim_{t\to 0} \sum_{\lim q_i^t \notin E_t} BB_\varphi(\mathcal{F}_t, q_i^t).$$

Or equivalently, by Theorem 3.3,

$$\sum_{i=1}^{s} BB_\varphi(\mathcal{F}, p_i) = \gamma_\varphi + \eta_\varphi(\mathcal{F}, C) - \lim_{t\to 0} \sum_{\lim q_i^t \in E_t} BB_\varphi(\widetilde{\mathcal{F}}_t, \widetilde{q}_i^{\,t}).$$

But,

$$\lim_{t\to 0}\sum_{\lim q_i^t\in E_t} BB_\varphi(\widetilde{\mathcal{F}}_t,\widetilde{q_i}^t)=\lim_{t\to 0}\sum_{q_i^t\in E_t} BB_\varphi(\widetilde{\mathcal{F}}_t,\widetilde{q_i}^t)+\lim_{t\to 0}\sum_{q_i^t\in A_t} BB_\varphi(\widetilde{\mathcal{F}}_t,\widetilde{q_i}^t)$$

where $A_t=\{q_i^t\notin E_t|\lim_{t\to 0}q_i^t\in E_t\}$. In particular, for $\varphi=c_n$, by Theorems 3.2, 3.3 and Corolary 3.6, we have

$$\sum_{i=1}^{s}\mu(\mathcal{F},p_i)=\sum_{i=1}^{s}BB_{c_n}(\mathcal{F},p_i)=\sum_{i=0}^{n}k^i+\nu(\mathcal{F},C)-\mathcal{A}$$

where $\mathcal{A}$ is the number of embedded closed points of C, counted with multiplicity. Therefore, $\mathcal{F}_t$ also satisfy the item (v) of the Lemma when $\ell\geq 1$. Now, in order to verify the item (vi) is enough to observe that $\sum BB_\varphi(\mathcal{F},p_i)+\sum BB_\varphi(\mathcal{F},C)=\gamma_\varphi$ (cf. [3], pg. 103).

Now, let us consider the case which $\ell=0$. From (33), we have

$$\mathcal{D}_{(F_t)_*(\mathcal{F}_t)}=\mathcal{D}_{(F_t)_*\mathcal{G}_t}+t\sum_{i=1}^{n}g_i(w)\frac{\partial}{\partial w_i}$$

where $g_i(w)=\sum_{j=1}^{n-1}g_{i,j}w_j$ for $1\leq i\leq n-1$ and $g_n(w)=r_{n,0}+\sum_{j=1}^{n}r_{n,j}w_j$. Thus, the w_n-axis is an invariant set by $(F_t)_*(\mathcal{F}_t)$ which implies $\mathcal{F}_t$ has C_t as an invariant set. Again, adjusting the functions g_i and shrinking ϵ , if necessary, we can admit that $\mathrm{Sing}(\mathcal{F}_t)=\{q_i^t,\ldots,q_{s_t}^t\}$, that is, the singular set of $\mathcal{F}_t$ consist exclusively by closed points. Therefore, $\mathcal{F}_t$ satisfy the items (i), (ii) and (iv) of the Lemma. By [2], we have that

$$BB_\varphi(\mathcal{F},C)=\lim_{t\to 0}\sum_{\lim q_i^t\in C}BB_\varphi(\mathcal{F}_t,q_i^t).$$

Thus, by Proposition 4.1, there exits $-\nu(\mathcal{F}_t,C_t)|_{\ell=0}=\chi(C_t)+(k-1)d=\chi(C)-\sum_{i=1}^{l}(b_i-1)+(k-1)d=-\nu(\mathcal{F},C)|_{\ell=0}$ isolated singularities of $\mathcal{F}_t$ in C_t. Therefore,

$$BB_{c_n}(\mathcal{F},C)=\mu(\mathcal{F},C)=-\nu(\mathcal{F},C)+\mathcal{A}.$$

where $\mathcal{A}$ is as before. As consequence,

$$\sum_{i=1}^{s}BB_\varphi(\mathcal{F},p_i)=\gamma_\varphi-\lim_{t\to 0}\sum_{\lim q_i^t\in C_t}BB_\varphi(\mathcal{F}_t,q_i^t).$$

In particular,

$$\sum_{i=1}^{s}\mu(\mathcal{F},p_i)=\sum_{i=1}^{s}BB_{c_n}(\mathcal{F},p_i)=\sum_{i=0}^{n}k^i+\nu(\mathcal{F},C)-\mathcal{A}.$$

Thus, $\mathcal{F}_t$ satisfy the items (v) and (vi) of the Lemma, in the case which $\ell=0$. Now, we will suppose that exist others curves in $\mathrm{Sing}(\mathcal{F})$, namely, $C_1,\ldots,C_r$. Let $F_t(z)=F_1^t\circ\ldots F_r^t(z)$ be the composition of biholomorphism F_i^t such that the image of C_i^t by $F_1^t\circ\ldots F_i^t(z)$ is just the w_n^i-axis in some coordinate system $w^i\in\mathbb{C}^n$. Thus, we apply the same steps of the case where $\mathrm{Sing}(\mathcal{F})$ has a unique curve. Then, with the sequence of blowups $\{\pi_i^t\}$, we just apply successively the proof of Theorem 3.3 noticing that $E_i^t\cdot E_j^t=0$ if $i\neq j$ because the curves C_i and C_j are disjoint. □

Example 4.3. *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation in $\mathbb{P}^3$, of degree k, defined in the open affine set $U_3:=\{[\xi]\in\mathbb{P}^3|\xi_3\neq 0\}$ by the vector field:*

$$\mathcal{D}_{\mathcal{F}}=\sum_{i=1}^{3}\sum_{j=0}^{k}a_{i,j}x_1^{k-j}x_2^{j}\frac{\partial}{\partial x_i}$$

where $x_i=\xi_{i-1}/\xi_3$, for $i=1,2,3$. Given that $\mathrm{cod}_{\mathbb{C}}(\mathrm{Sing}\mathcal{F})\geq 2$, we can admit that the three polynomials $\varphi_i(\varsigma)=\sum_{j=0}^{k}a_{i,j}\varsigma^j$ do not have any roots in common. Eventually, φ_1 and φ_2 may have some root in common. The curve $C\subset\mathbb{P}^3$ defined as $\xi_0=\xi_1=0$ is contained in the singular locus $\mathrm{Sing}(\mathcal{F})$. Thus, we have that $\ell=\mathrm{mult}_C(\pi_C^*\mathcal{F})=k-1\geq 0$, $\chi(C)=2$ and $d=\deg(C)=1$.

We will begin our analysis with the case which $\ell=0$. By the Lemma 4.2, let $\mathcal{G}_t$ be a family of foliation given in U_3 as follows

$$\mathcal{D}_{\mathcal{G}_t}=\mathcal{D}_{\mathcal{F}}+t(d_0+d_1x_3)\frac{\partial}{\partial x_3}$$

for some non-null constants d_0 and d_1. Thus, for $t\neq 0$, we have $\mathrm{Sing}(\mathcal{G}_t)=\{q_i^t,1\leq i\leq 4\}$ where $q_1^t=[0:0:(-d_0):d_1]$, $q_2^t=[0:0:1:0]$, $q_3^t=[u_1:\varsigma_1u_1:1:0]$ and $q_4^t=[u_2:\varsigma_2u_2:1:0]$ where ς_i are the root of the quadratic equation $\varphi_2(\varsigma)-\varsigma\varphi_1(\varsigma)=0$ and $u_i=(\varphi_1(\varsigma_i)-td_1)/\varphi_3(\varsigma_i)$ for $i=1,2$. Then, $\varphi_1(\varsigma_i)=0$ if only if $\lim_{t\to 0}u_i(t)=0$. Therefore, C will have a one embedded closed point if φ_2 and φ_1 have a root in common. Consequently, C has at most one point in this situation.

If $\varphi_1(\varsigma)$ and $\varphi_2(\varsigma)$ have distinct roots which is equivalent to

$$[a_{11}a_{22}-a_{12}a_{21}]\neq 0\ \ \text{then}\ \ BB_{c_3}(\mathcal{F},C)=-\nu(\mathcal{F},C)=2.$$

Furthermore, the eigenvalues $\{\lambda_i\}$ *of* $\mathcal{G}_t$ *at* q_1^t *are* $\lambda_1 = \frac{a_{11}+a_{22}+\sqrt{\Delta}}{2}$, $\lambda_2 = \frac{a_{11}+a_{22}-\sqrt{\Delta}}{2}$ *and* $\lambda_3 = td_1$ *with* $\Delta = (a_{11}-a_{22})^2 + 4a_{12}a_{21}$ *while the eigenvalues* $\{\mu_i\}$ *at* q_2^t *are* $\mu_1 = \lambda_1 - td_1$, $\mu_2 = \lambda_2 - td_1$ *and* $\mu_3 = -\lambda_3 = -td_1$. *Consequently,*

$$\begin{aligned} BB_{c_1^3}(\mathcal{F}, C) &= \lim_{t\to 0}\left[BB_{c_1^3}(\mathcal{G}_t, q_1^t) + BB_{c_1^3}(\mathcal{G}_t, q_2^t)\right] \\ &= -\frac{(\lambda_1+\lambda_2)^4}{\lambda_1^2\lambda_2^2} + 12\frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2}, \end{aligned}$$

$$\begin{aligned} BB_{c_1c_2}(\mathcal{F}, C) &= \lim_{t\to 0}\left[BB_{c_1c_2}(\mathcal{G}_t, q_1^t) + BB_{c_1c_2}(\mathcal{G}_t, q_2^t)\right] \\ &= 8 + 2\frac{\lambda_1}{\lambda_2} + 2\frac{\lambda_2}{\lambda_1}. \end{aligned}$$

In [2], Suwa and Bracci considered the foliation $\mathcal{F}_1$, *described in* U_3 *as follows*

$$\mathcal{D}_{\mathcal{F}_1} = x_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3}.$$

As given in Lemma 4.2, let $\mathcal{H}_t$ *be family of holomorphic foliation described as follows*

$$\mathcal{D}_{\mathcal{H}_t} = \mathcal{D}_{\mathcal{F}_1} + t\left(ax_2\frac{\partial}{\partial x_1} + bx_2\frac{\partial}{\partial x_2} + (d_0 + d_1x_3)\frac{\partial}{\partial x_3}\right)$$

with non-null constants d_0, d_1, a *and* b *such that* $a \neq b$. *The singularities of* $\mathcal{H}_t$ *are* $p_1^t = q_1^t$, $p_2^t = q_2^t$, $p_3^t = [u_1 : \varsigma_1 u_1 : 1 : 0]$ *and* $p_4^t = [u_2 : \varsigma_2 u_2 : 1 : 0]$ *where* $u_j = (1 + at\varsigma_j - td_1)/\varsigma_j$, ς_j *are the root of the quadratic equation* $ta\varsigma^2 + (1-bt)\varsigma - 1 = 0$ *for* $j = 1, 2$. *Thus,*

$$u_1 = \frac{at(1+tb-\sqrt{\Delta_1}) - 2ad_1t^2}{-(1-tb)-\sqrt{\Delta_1}} \quad \text{and} \quad u_2 = \frac{at(1+tb+\sqrt{\Delta_1}) - 2ad_1t^2}{-(1-tb)+\sqrt{\Delta_1}}$$

with $\Delta_1 = (1-tb)^2 + 4at$. *But, in this time,* $\lim_{t\to 0} p_3^t = p_2^t \in C$, *that is,* C *has one embedded closed point. Consequently,* $\mu(\mathcal{F}, C) = -\nu(\mathcal{F}, C) + 1 = 3$ *and more generally,*

$$BB_\varphi(\mathcal{F}_1, C) = \lim_{t\to 0}\sum_{j=1}^{3} BB_\varphi(\mathcal{H}_t, p_j^t) = \gamma_\varphi - \lim_{t\to 0} BB_\varphi(\mathcal{H}_t, p_4^t).$$

As $\lim_{t\to 0} p_4^t = [1:1:1:0]$, *we obtain*

$$\lim_{t\to 0} BB_{c_1^3}(\mathcal{H}_t, p_4^t) = 27; \lim_{t\to 0} BB_{c_1c_2}(\mathcal{H}_t, p_4^t) = 9.$$

Therefore, $BB_{c_1^3}(\mathcal{F}_1, C) = \gamma_{c_1^3} - 27 = 37$ *and* $BB_{c_1c_2}(\mathcal{F}_1, C) = \gamma_{c_1c_2} - 9 = 15$.

Now, we will consider the case which $\ell \geq 1$. *As before, let* $\mathcal{F}_t$ *be the family of holomorphic foliation given also in* U_3 *by the vector field*

$$\mathcal{D}_{\mathcal{F}_t} = \mathcal{D}_{\mathcal{F}} + t\sum_{i=1}^{3} h_j(x)\frac{\partial}{\partial x_i}$$

where

$$h_i(x) = \sum_{j=0}^{k} h_{i,j} x_1^{k-j} x_2^j, \ \ for \ \ i = 1, 2$$

and

$$h_3(x) = \sum_{j=0}^{k-1} x_1^{k-j-1} x_2^j (r_{0,j} + r_{1,j} x_3).$$

The polynomials h_1 *and* h_2 *are chosen so that the singular locus of* $\mathcal{F}_t$, *when restricted to* U_3, *consists only of the* x_3*-axis.*

After the change of coordinates, in the affine chart $U_2 = \{[\xi] \in \mathbb{P}^3 |, \xi_2 \neq 0\}$, *with coordinates* $u_1 = x_1/x_3$, $u_2 = x_2/x_3$ *and* $u_3 = 1/x_3$, *we have that*

$$\mathcal{D}_{\mathcal{F}_t} = \sum_{i=1}^{2}\sum_{j=0}^{k}[a_{i,j} + t h_{i,j}] u_1^{k-j} u_2^j \frac{\partial}{\partial u_i} - \sum_{i=1}^{3} u_i g(u)\frac{\partial}{\partial u_i}$$

where $g(u) = \sum_{j=0}^{k} a_{3,j} u_1^{k-j} u_2^j + t\sum_{j=0}^{k-1} u_1^{k-j-1} u_2^j (r_{0,j} u_3 + r_{1,j})$.

In the open set $\widetilde{U}_1$ *with coordinates* $\varsigma \in \mathbb{C}^3$ *such that* $\sigma(\varsigma) = (\varsigma_1, \varsigma_1\varsigma_2, \varsigma_3) = (u_1, u_2, u_3)$, *we get*

$$\begin{aligned}\mathcal{D}_{\widetilde{\mathcal{F}}_t} &= \varsigma_1\Big[\varphi_1(\varsigma_2) + t h_1(\varsigma_2) - \psi(\varsigma)\Big]\frac{\partial}{\partial \varsigma_1} \\ &\quad + \Big[(\varphi_2(\varsigma_2) - \varsigma_2\varphi_1(\varsigma_2)) + t(h_2(\varsigma_2) - \varsigma_2 h_1(\varsigma_2))\Big]\frac{\partial}{\partial \varsigma_2} - \varsigma_3\psi(\varsigma)\frac{\partial}{\partial \varsigma_3}\end{aligned}$$

where $h_i(\varsigma_2) = \sum_{j=0}^{k} h_{i,j}\varsigma_2^j$ *and* $\psi(\varsigma) = \varsigma_1 a_3(\varsigma_2) + t\sum_{j=0}^{k-1}(r_{0,j}\varsigma_3 + r_{1,j})\varsigma_2^j$.

The singular locus of $\mathcal{F}_t$, *for* $t \neq 0$, *consists of the closed points* $p_i^t = [u_i^t : \zeta_i^t u_i^t : 1 : 0]$ *where*

$$u_i^t = \frac{\varphi_1(\zeta_i^t) + th_1(\zeta_i^t) - t\sum_{j=0}^{k-1} r_{1,j}(\zeta_i^t)^j}{\varphi_3(\zeta_i^t)}$$

which ζ_i^t *are roots of* $(\varphi_2(\zeta) - \zeta\varphi_1(\zeta)) + t(h_2(\zeta) - \zeta h_1(\zeta)) = 0$ *for* $j = 1, \ldots, k+1$. *Therefore,* $\lim_{t\to 0} u_i^t = \varphi_1(\zeta_i)/\varphi_3(\zeta_i)$, *with* $\zeta_i = \lim_{t\to 0}\zeta_i^t$. *Consequently,* $p_i = \lim_{t\to 0} p_i^t$ *is an embedded closed point of* C *if only if* $\varphi_1(\zeta_i) = 0$, *i.e., if* ζ_i *is a root of* $\varphi_1(\zeta)$ *and* $\varphi_2(\zeta)$ *simultaneously. It follows that* C *has at most* k *embedded closed points. Furthermore, given that* $-\nu(\mathcal{F}, C) = k^3 + k^2$ *we obtain that*

$$k^3 + k^2 \leq \mu(\mathcal{F}, C) \leq k^3 + k^2 + k$$

or equivalently $\mathrm{Sing}(\mathcal{F})$ *has at least one isolated closed point for all* $k \geq 1$.

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