

Solving an Abstract Nonlinear Eigenvalue Problem by the Inverse Iteration Method

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Abstract Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces over \mathbb{R} , with X uniformly convex and compactly embedded into Y . The inverse iteration method is applied to solve the abstract eigenvalue problem $A(w) = \lambda \|w\|_Y^{p-q} B(w)$, where the maps $A : X \rightarrow X^*$ and $B : Y \rightarrow Y^*$ are homogeneous of degrees $p - 1$ and $q - 1$, respectively.

Keywords Eigenvalue problems · Inverse iteration · Quasilinear elliptic equations

Mathematics Subject Classification 47J05 · 47J25 · 35P30

1 Introduction

Many eigenvalue-type problems involving quasilinear elliptic equations are formulated as a functional equation of the form

$$A(w) = \lambda \|w\|_Y^{p-q} B(w), \quad (1)$$

where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces over \mathbb{R} , X compactly embedded into Y , and the maps $A : X \rightarrow X^*$ and $B : Y \rightarrow Y^*$ are homogeneous of degrees $p - 1 > 0$ and $q - 1 > 0$, respectively, that is:

$$(A1) \quad A(tw) = |t|^{p-2} t A(w) \text{ for all } t \in \mathbb{R};$$

$$(B1) \quad B(tw) = |t|^{q-2} t B(w) \text{ for all } t \in \mathbb{R}.$$

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We say that the pair $(\lambda, w) \in \mathbb{R} \times X \setminus \{0\}$ solves (1) if, and only if,

$$\langle A(w), v \rangle = \lambda \|w\|_Y^{p-q} \langle B(w), v \rangle, \quad \forall v \in X, \tag{2}$$

where we are using the notation $\langle f, v \rangle \stackrel{\text{def}}{=} f(v)$.

In this paper we apply the inverse iteration method to solve the abstract equation (1) by assuming the following additional hypotheses on the maps A and B :

- (A2) $\langle A(u), v \rangle \leq \|u\|_X^{p-1} \|v\|_X$ for all $u, v \in X$, with the equality occurring if, and only if, either $u = 0$ or $v = 0$ or $u = tv$, for some $t > 0$;
- (B2) $\langle B(u), v \rangle \leq \|u\|_Y^{q-1} \|v\|_Y$ for all $u, v \in Y$, with the equality occurring whenever $u = tv$, for some $t \geq 0$;
- (AB) for each $w \in Y \setminus \{0\}$ given, there exists at least one $u \in X \setminus \{0\}$ such that

$$\langle A(u), v \rangle = \langle B(w), v \rangle, \quad \forall v \in X.$$

We observe from (A1) and (B1) that (1) is homogeneous, that is: if (λ, w) solves (1) the same holds true for (λ, tw) , for all $t \neq 0$. Motivated by this intrinsic property of eigenvalue problems, we say that λ is an *eigenvalue* of (1) and that w is an *eigenvector* of (1) corresponding to λ or, for shortness, we simply say that (λ, w) is an *eigenpair* of (1).

Hypotheses (A2) and (B2) imply, respectively, that

$$\langle A(w), w \rangle = \|w\|_X^p, \quad \forall w \in X \tag{3}$$

and

$$\langle B(w), w \rangle = \|w\|_Y^q, \quad \forall w \in Y. \tag{4}$$

Thus, by choosing $v = w$ in (2), we see that

$$\lambda = \frac{\|w\|_X^p}{\|w\|_Y^q},$$

which shows that the eigenvalues of (1) are nonnegative. Actually, they are bounded from below by

$$\mu := \inf \{ \|w\|_X^p : w \in X \cap \mathbb{S}_Y \},$$

where $\mathbb{S}_Y := \{w \in Y : \|w\|_Y = 1\}$ is the unit sphere in Y .

We note that the compactness of the embedding $X \hookrightarrow Y$ (which we are assuming in this paper) implies that μ is positive and reached in \mathbb{S}_Y . Moreover, assuming in addition the conditions (A1), (A2), (B1), (B2) and (AB) we will show (see Proposition 2) that μ is an eigenvalue and that its corresponding eigenvectors are precisely the scalar multiple of those vectors in \mathbb{S}_Y at which μ is reached. Because of this, we refer to μ as the *first eigenvalue* of (1) and any of its corresponding eigenvectors as a *first eigenvector*.

As we will see, hypothesis (AB) allows us to construct, for each $w_0 \in \mathbb{S}_Y$, an inverse iteration sequence $\{w_0, w_1, w_2, \dots\} \subset X \cap \mathbb{S}_Y$ satisfying

$$\langle A(w_{n+1}), v \rangle = \lambda_n \langle B(w_n), v \rangle, \quad \forall v \in X$$

where $\lambda_n \geq \mu$.

Our main result in this paper is stated as follows.

Theorem 1 *Assume that X is uniformly convex and compactly embedded into Y , and that the maps $A : X \rightarrow X^*$ and $B : Y \rightarrow Y^*$ are continuous and satisfy the hypotheses (A1), (A2), (B1), (B2) and (AB). The sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\|w_{n+1}\|_X^p\}_{n \in \mathbb{N}}$ are nonincreasing and converge to the same limit λ , which is bounded from below by μ . Moreover, λ is an eigenvalue of (1) and there exists a subsequence $\{n_j\}_{j \in \mathbb{N}}$ such that both $\{w_{n_j}\}_{j \in \mathbb{N}}$ and $\{w_{n_j+1}\}_{j \in \mathbb{N}}$ converge in X to the same vector $w \in X \cap \mathbb{S}_Y$, which is an eigenvector corresponding to λ .*

The proof of this result is presented in Sect. 2 by combining two lemmas. In Lemma 4 we obtain, from the hypotheses (A2) and (B2), the monotonicity of the sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\|w_{n+1}\|_X^p\}_{n \in \mathbb{N}}$ as well as their convergence to λ .

In Lemma 5 we use the uniform convexity of X and the compactness of the embedding $X \hookrightarrow Y$ to guarantee the existence of a subsequence $\{w_{n_j}\}_{j \in \mathbb{N}}$ converging in X to a function $w \in X \cap \mathbb{S}_Y$. A delicate issue in the conclusion of the lemma is to show that the subsequence $\{w_{n_j+1}\}_{j \in \mathbb{N}}$ also converges to w in order to pass to the limit, as $j \rightarrow \infty$, in

$$\langle A(w_{n_j+1}), v \rangle = \lambda_{n_j} \langle B(w_{n_j}), v \rangle, \quad \forall v \in X.$$

For this we use the hypothesis (A2), which plays the same structural role that the Hölder's inequality plays in the quasilinear elliptic problems (we recall that the equality in the Hölder's inequality implies that the functions involved, raised to conjugate exponents, are proportional).

We conclude Sect. 2 by remarking that when λ is simple, meaning that its corresponding eigenvectors are scalar multiple of each other, then w and $-w$ are the only cluster points of the sequence $\{w_n\}_{n \in \mathbb{N}}$. Thus, in some concrete situations a suitable choice of $w_0 \in \mathbb{S}_Y$ guarantees that the vector w is the only cluster point of $\{w_n\}_{n \in \mathbb{N}}$ when λ is known to be simple. In such a situation one has $w_n \rightarrow w$.

It is important to highlight that the factor $\|w\|_Y^{p-q}$ in (1) makes it possible to formulate this problem as an eigenvalue problem by balancing two (possibly different) degrees of homogeneity. Our motivation for adopting this strategy came from Franzina and Lamberti (2010). In that paper, Franzina and Lamberti considered the following Dirichlet problem in an arbitrary domain Ω of \mathbb{R}^N with finite measure:

$$\begin{cases} -\Delta_p u = \lambda \|u\|_q^{p-q} |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5)$$

where $\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator,

$$1 \leq q < p^* := \begin{cases} \frac{Np}{N-p} & \text{if } 1 < p < N \\ \infty & \text{if } p \geq N, \end{cases}$$

and $\|\cdot\|_r$ denotes the norm of $L^r(\Omega)$ for $1 \leq r \leq \infty$ (we use this notation from now on). By introducing the factor $\|u\|_q^{p-q}$ at the right-hand side of (5), instead of simply incorporating it to the parameter λ , they were able to make (5) homogeneous, which is a feature of eigenvalue problems.

The Dirichlet problem (5) is a prototype for (1). Indeed, the concept of (weak) solution for (5) takes the form of (2) with

$$\langle A(u), v \rangle := \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \quad \text{and} \quad \langle B(u), v \rangle := \int_{\Omega} |u|^{q-2} u v \, dx. \quad (6)$$

In this setting, Y is the Lebesgue space $L^q(\Omega)$ endowed with the standard norm

$$\|u\|_q := \left(\int_{\Omega} |u|^q \, dx \right)^{\frac{1}{q}}$$

and X is the Sobolev space

$$W_0^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega)^N \quad \text{and} \quad u = 0 \text{ on } \partial\Omega \right\},$$

endowed with the norm

$$\|u\|_{W_0^{1,p}} := \|\nabla u\|_p = \left(\int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

which makes $W_0^{1,p}(\Omega)$ a uniformly convex Banach space.

The hypotheses (A1) and (B1) can be easily checked for the maps A and B defined in (6), whereas (A2) and (B2) are deduced from Hölder’s inequality. The compactness of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for $1 \leq q < p^*$, is a well-known fact as well as the continuity of the functions $A : W_0^{1,p}(\Omega) \rightarrow W_0^{-1,p'}(\Omega)$ and $B : L^q(\Omega) \rightarrow L^{q'}(\Omega)$. (It is usual to denote the dual space of $W_0^{1,p}(\Omega)$ by $W_0^{-1,p'}(\Omega)$, where $r' := \frac{r}{r-1}$ is the Hölder conjugate of $r > 1$, i. e. $\frac{1}{r} + \frac{1}{r'} = 1$.)

Property (AB) also holds true since A is surjective and $B(w) \in W_0^{-1,p'}(\Omega)$ for all $w \in L^q(\Omega)$. We refer the reader to Dinca et al. (2001), Lê (2006) where all the properties are proved.

The following facts regarding the eigenvalue problem (5) are well-known (see García 1987; Franzina and Lamberti 2010; Idogawa and Otani 1995; Ôtani 1988): there exists a sequence of eigenvalues tending to ∞ ; every eigenfunction belongs to $L^\infty(\Omega)$ and the first eigenfunctions do not change sign in Ω . (When dealing with

spaces of functions, the nomenclature “eigenfunction” seems to be more appropriate than “eigenvector”.)

The particular case $q = p$,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

has been extensively studied over the last three decades. Its first eigenvalue

$$\lambda_p := \min \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_p = 1 \right\} \tag{8}$$

is isolated and simple. Moreover, the first eigenfunctions are the only eigenfunctions that do not change sign in Ω . These and other properties of (7) can be verified in Anane (1987), Lindqvist (2008), Ôtani and Teshima (1988) and references therein.

When $p \neq 2$ the eigenvalue problem (7) is very difficult to be solved analytically and even numerically, since it loses the linear character of $p = 2$ and acquires the singular or degenerate term $|\nabla u|^{p-2}$. We remark that analytical expressions for the first eigenvalue are not known in general, not even for simple domains, such as squares, balls or triangles.

In Biezuner et al. (2009) the inverse iteration method was introduced to solve (7) in the particular domain: the unit ball $B_1 := \{x \in \mathbb{R}^N : |x| = 1\}$. Starting with $u_0 \equiv 1$ and exploring the radial structure of the Dirichlet problem at each iteration step, the authors proved that

$$\lim_{n \rightarrow \infty} \left(\frac{\|u_n\|_\infty}{\|u_{n+1}\|_\infty} \right)^{p-1} = \lambda_p \text{ and } \lim_{n \rightarrow \infty} \frac{u_n}{\|u_n\|_\infty} = u_p \text{ in } C^1(\overline{B_1}),$$

where $\|\cdot\|_\infty$ denotes the sup norm and u_p denotes the positive first eigenfunction such that $\|u_p\|_\infty = 1$.

They also conjectured that

$$\lim_{n \rightarrow \infty} \left(\frac{\|u_n\|_p}{\|u_{n+1}\|_p} \right)^{p-1} = \lambda_p \tag{9}$$

for a general bounded domain and presented some numerical experiments for the unit square as motivation to their conjecture.

The approach used in Biezuner et al. (2009), based on radial symmetry, was adapted in Ercole et al. (2015) to obtain the pair (λ_p, u_p) for a radially symmetric annulus.

Recently, in Hynd and Lindgren (2016), the authors considered, for a general bounded domain Ω , the sequence of iterates

$$\psi_n := (\lambda_p)^{\frac{n}{p-1}} u_n$$

where $u_0 \in L^p(\Omega)$ is given and $-\Delta_p u_{n+1} = |u_n|^{p-2} u_n$. By making use of the minimizing property (8) of λ_p , they proved the convergence, in $W_0^{1,p}(\Omega)$, of the

sequence $(\psi_n)_{n \in \mathbb{N}}$ to a function ψ . Then, under the assumption $\psi \not\equiv 0$, they concluded that ψ is a first eigenfunction and proved the conjecture (9) posed in Biezuner et al. (2009). They also showed that $\psi \not\equiv 0$ if either $u_0 \geq ke_p$ for some positive constant k or $u_0 \geq 0$, $u_0 \not\equiv 0$ and Ω is sufficiently smooth. (Here e_p denotes the positive eigenfunction such that $\|e_p\|_p = 1$.) It is simple to check that $u_0 \equiv 1$ leads to $\psi \not\equiv 0$.

We emphasize that the minimizing property (8) of the first eigenvalue λ_p plays a decisive role in the approach of Hynd and Lindgren (2016) and makes it applicable only to this eigenvalue.

The literature on the eigenvalue problem (5) in the case $q \neq p$ (which is shorter than in the case $q = p$) shows that there are some differences between the cases $1 \leq q < p$ and $p < q < p^*$ with respect to the properties of the first eigenvalue

$$\lambda_q := \min \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_q = 1 \right\} \tag{10}$$

(see García 1987; Ercole 2014; Franzina and Lamberti 2010; Kawohl 2000; Nazarov 2000).

In the cases $1 \leq q < p$ and $q = p$ some properties of the first eigenvalue problem are shared. For example, the first eigenvalue is simple and the first eigenfunctions are the only that do not change sign in Ω . Because of these properties, we can guarantee that our method is successful when it is used for the purpose of achieving a first eigenpair. In fact, if $1 \leq q \leq p$ and $u_0 \in L^q(\Omega) \setminus \{0\}$ is nonnegative, then $\lambda_n \rightarrow \lambda_q$ and $w_n \rightarrow e_q$ where e_q is the positive L^q -normalized eigenfunction (it is not necessary to pass to a subsequence).

When $p < q < p^*$ and Ω is a general bounded domain the simplicity of λ_q is not guaranteed nor the exclusivity of the first eigenfunctions with respect to have a definite sign. Thus, in this situation, we cannot guarantee that the eigenvalue λ , obtained when u_0 is nonnegative, coincides with λ_q . By the way, we think that our method could be used to investigate, at least numerically, the existence of positive eigenfunctions associated with $\lambda > \lambda_q$ for some domains.

Our first motivation, inspired by the papers (Biezuner et al. 2009; Hynd and Lindgren 2016), was to apply the inverse iteration method to (5). However, we realized that the arguments we had developed to deal with this problem depend only on the properties of the functions A and B defined in (6) combined with compactness. Thus, we arrived at the abstract eigenvalue problem (1) under the hypotheses (A1), (A2), (B1), (B2) and (AB).

We would like to emphasize that our abstract approach covers a large range of eigenvalue problems involving partial differential equations of quasilinear elliptic type and serves as a theoretical basis for a numerical treatment of them. For the sake of completeness, we present in Sect. 3 two more examples of such problems: the Dirichlet eigenproblem for the s -fractional p -Laplacian and a Steklov-type eigenvalue problem for the p -Laplacian involving a homogenous term of degree $q - 1$ on the boundary.

Our results in this paper complement those of Hynd and Lindgren (2017). In the first part of that paper the authors extend their own results presented in Hynd and Lindgren (2016) to an abstract setting, aiming to approximate the least Rayleigh quotient $\Phi(u) / \|u\|_Y^p$ where, according to our notation, $\Phi : X \rightarrow [0, \infty]$ is a functional

satisfying certain properties (among them, strict convexity and positive homogeneity of degree $p > 1$) and $X := \{u \in Y : \Phi(u) < \infty\}$. The authors reduce the problem of minimizing the Rayleigh quotient above to an equivalent subdifferential equation involving the subdifferentials of both functionals Φ and $\frac{1}{p} \|\cdot\|_Y^p$. Then, they apply an inverse iteration scheme to solve the subdifferential equation. Our approach, however, embraces eigenvalue problems that are not necessarily linked to least Rayleigh quotients. Moreover, it guarantees that the inverse iteration sequence always produces an eigenvalue.

2 The Results of Convergence

In this section we assume that X is uniformly convex, compactly embedded into Y and that $A : X \rightarrow X^*$ and $B : Y \rightarrow Y^*$ are continuous maps satisfying the hypotheses (A1), (A2), (B1), (B2) and (AB), stated in the Introduction.

We recall that

$$\mu := \inf \{ \|w\|_X^p : w \in X \cap \mathbb{S}_Y \}, \quad (11)$$

where $\mathbb{S}_Y := \{w \in Y : \|w\|_Y = 1\}$.

Proposition 2 *Let $\{w_n\}_{n \in \mathbb{N}} \subset X \cap \mathbb{S}_Y$ be a minimizing sequence of (11), that is: $\|w_n\|_Y = 1$ and $\|w_n\|_X \rightarrow \mu$. There exist a subsequence $\{w_{n_j}\}_{j \in \mathbb{N}}$ converging weakly in X to a vector $w \in X \cap \mathbb{S}_Y$ which reaches μ (i.e. $\|w\|_Y = 1$ and $\|w\|_X = \mu$). Moreover, μ is an eigenvalue of (1) and its corresponding eigenvectors are precisely the scalar multiple of those vectors where μ is reached.*

Proof Since $\{w_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X , there exist a subsequence $\{w_{n_j}\}_{j \in \mathbb{N}}$ and a vector $w \in X$ such that $w_{n_j} \rightharpoonup w$ in X (weak convergence) and $w_{n_j} \rightarrow w$ in Y . Here we have used that X is reflexive and compactly embedded into Y . The convergence $w_{n_j} \rightarrow w$ in Y implies that $\|w\|_Y = \lim_{j \rightarrow \infty} \|w_{n_j}\|_Y = 1$, whereas the weak convergence $w_{n_j} \rightharpoonup w$ in X yields

$$\|w\|_X \leq \lim_{j \rightarrow \infty} \|w_{n_j}\|_X = \mu^{1/p}.$$

Thus, since $\mu \leq \|w\|_X^p$ we conclude that $\mu = \|w\|_X^p$.

Now, let us prove that the pair (μ, w) solves (1). According to (AB) there exists $u \in X \setminus \{0\}$ such that

$$\langle A(u), v \rangle = \langle B(w), v \rangle, \quad \forall v \in X.$$

In view of (A1) we can rewrite this equation as

$$\langle A(\tilde{w}), v \rangle = \gamma \langle B(w), v \rangle, \quad \forall v \in X \quad (12)$$

where $\gamma := \|u\|_Y^{1-p}$ and $\tilde{w} := \|u\|_Y^{-1} u$ (so that $\tilde{w} \in X \cap \mathbb{S}_Y$). Taking $v = w$ in (12) we obtain, from (4) and (A2)

$$\gamma = \gamma \|w\|_Y^q = \gamma \langle B(w), w \rangle = \langle A(\tilde{w}), w \rangle \leq \|\tilde{w}\|_X^{p-1} \|w\|_X = \|\tilde{w}\|_X^{p-1} \mu^{1/p} \tag{13}$$

and taking $v = \tilde{w}$ in (12) we obtain, from (3) and (B2)

$$\|\tilde{w}\|_X^p = \langle A(\tilde{w}), \tilde{w} \rangle = \gamma \langle B(w), \tilde{w} \rangle \leq \gamma \|w\|_Y^{q-1} \|\tilde{w}\|_Y = \gamma.$$

It follows that

$$\mu \leq \|\tilde{w}\|_X^p \leq \gamma \leq \|\tilde{w}\|_X^{p-1} \mu^{1/p},$$

where the first inequality comes from the definition of μ . A simple analysis of these latter inequalities shows that all of them are, in fact, equalities. Thus,

$$\mu = \|\tilde{w}\|_X^p \quad \text{and} \quad \gamma = \mu.$$

Hence, (13) implies that

$$\langle A(\tilde{w}), w \rangle = \|\tilde{w}\|_X^{p-1} \|w\|_X$$

and then (A2) leads us to conclude that $\tilde{w} = w$ (note that $\|\tilde{w}\|_X = \|w\|_X = \mu$). Thus, (12) yields

$$\langle A(w), v \rangle = \mu \langle B(w), v \rangle, \quad \forall v \in X,$$

showing that (μ, w) is an eigenpair. Repeating the same arguments we can see that any other vector at which μ is reached is also an eigenvector corresponding to μ . In order to complete this proof we observe that if $u \in X$ is an eigenvector corresponding to μ then $u = tw$ for some $w \in X \cap \mathbb{S}_Y$ such that $\|w\|_X^p = \mu$. Indeed, we can pick $t = \|u\|_Y$ and $w = \|u\|_Y^{-1} u \in X \cap \mathbb{S}_Y$, since

$$\|u\|_X^p = \langle A(u), u \rangle = \mu \|u\|_Y^{p-q} \langle B(u), u \rangle = \mu \|u\|_Y^p$$

implies that $\|w\|_X^p = \mu$. □

Remark 3 The previous proof does not require of X to be uniformly convex. In fact, reflexivity is enough. However, when X is uniformly convex the minimizing subsequence $\{w_{n_j}\}_{j \in \mathbb{N}}$ converges strongly to w , since $\|w\|_X = \lim_{j \rightarrow \infty} \|w_{n_j}\|_X$.

Now, let us fix an arbitrary vector $w_0 \in \mathbb{S}_Y$. Thanks to property (AB), there exists $u_1 \in X \setminus \{0\}$ such that

$$\langle A(u_1), v \rangle = \langle B(w_0), v \rangle, \quad \forall v \in X.$$

Hence, by multiplying this equation by $\|u_1\|_Y^{1-p}$ and setting

$$w_1 := \|u_1\|_Y^{-1} u_1 \quad \text{and} \quad \lambda_1 := (\|u_1\|_Y)^{1-p}$$

we obtain

$$\langle A(w_1), v \rangle = \lambda_1 \langle B(w_0), v \rangle, \quad \forall v \in X.$$

Repeating inductively the above argument we construct the iteration sequence $\{w_n\}_{n \in \mathbb{N}} \subset X \cap \mathbb{S}_Y$ satisfying

$$\langle A(w_{n+1}), v \rangle = \lambda_n \langle B(w_n), v \rangle, \quad \forall v \in X, \tag{14}$$

where $\lambda_n = \|u_{n+1}\|_Y^{1-p}$.

We observe that

$$\lambda_n \geq \mu, \quad \forall n \in \mathbb{N}. \tag{15}$$

Indeed, since both w_n and w_{n+1} belong to \mathbb{S}_Y , by taking $v = w_{n+1}$ in (14) and using the definition of μ we find

$$\begin{aligned} \mu &\leq \|w_{n+1}\|_X^p = \langle A(w_{n+1}), w_{n+1} \rangle \\ &= \lambda_n \langle B(w_n), w_{n+1} \rangle \leq \lambda_n \|w_n\|_Y^{q-1} \|w_{n+1}\|_Y = \lambda_n. \end{aligned} \tag{16}$$

Lemma 4 *The sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\|w_{n+1}\|_X^p\}_{n \in \mathbb{N}}$ are nonincreasing and converge to the same limit λ . Moreover,*

$$\lambda \geq \mu. \tag{17}$$

Proof We can see from (16) that

$$\|w_{n+1}\|_X^p \leq \lambda_n \quad \forall n \in \mathbb{N}.$$

Taking $v = w_n$ in (14) we have

$$\lambda_n = \lambda_n \|w_n\|_Y^q = \lambda_n \langle B(w_n), w_n \rangle = \langle A(w_{n+1}), w_n \rangle \leq \|w_{n+1}\|_X^{p-1} \|w_n\|_X.$$

Hence,

$$\|w_{n+1}\|_X^p \leq \lambda_n \leq \|w_{n+1}\|_X^{p-1} \|w_n\|_X \leq (\lambda_n)^{\frac{p-1}{p}} (\lambda_{n-1})^{\frac{1}{p}}, \tag{18}$$

from which we obtain

$$\|w_{n+1}\|_X \leq \|w_n\|_X \quad \text{and} \quad \lambda_n \leq \lambda_{n-1}.$$

Since the numerical sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\|w_{n+1}\|_X^p\}_{n \in \mathbb{N}}$ are also bounded from below they are convergent. Thus, by making $n \rightarrow \infty$ in (18) we can see that both converge to the same limit, which we denote by λ . The inequality (17) follows directly from (15). □

Lemma 5 *There exist a subsequence $\{n_j\}_{j \in \mathbb{N}}$ and a vector $w \in X \cap \mathbb{S}_Y$ such that $w_{n_j} \rightarrow w$ in X . Moreover, (λ, w) is an eigenpair of (1) and $w_{n_{j+1}} \rightarrow w$ in X .*

Proof Since $\|w_n\|_X^p \rightarrow \lambda$, the compactness of the immersion $X \hookrightarrow Y$ guarantees the existence of a subsequence $\{w_{n_j}\}$ and an element $w \in X$ such that

$$w_{n_j} \rightharpoonup w \text{ (weakly) in } X, \quad w_{n_j} \rightarrow w \text{ (strongly) in } Y$$

and

$$\|w\|_X^p \leq \lim_{j \rightarrow \infty} \|w_{n_j}\|_X^p = \lambda. \tag{19}$$

We also have

$$\lambda_{n_j} \langle B(w_{n_j}), w \rangle = \langle Aw_{n_{j+1}}, w \rangle \leq \|w_{n_{j+1}}\|_X^{p-1} \|w\|_X \leq (\lambda_{n_j})^{\frac{p-1}{p}} \|w\|_X,$$

so that

$$\langle B(w_{n_j}), w \rangle \leq (\lambda_{n_j})^{-\frac{1}{p}} \|w\|_X \leq \lambda^{-\frac{1}{p}} \|w\|_X$$

The strong convergence $w_{n_j} \rightarrow w$ in Y implies that $w \in \mathbb{S}_Y$ and then the continuity of B yields

$$1 = \|w\|_Y^q = \langle B(w), w \rangle = \lim_{j \rightarrow \infty} \langle B(w_{n_j}), w \rangle \leq \lambda^{-\frac{1}{p}} \|w\|_X,$$

so that

$$\lambda \leq \|w\|_X^p.$$

This inequality, in view of (19), implies that $\lim_{j \rightarrow \infty} \|w_{n_j}\|_X^p = \|w\|_X^p = \lambda$. Hence, the uniform convexity of X allows us to conclude that $w_{n_j} \rightarrow w$ in X . Applying the same arguments to the sequence $\{w_{n_{j+1}}\}_{j \in \mathbb{N}}$, we can assume that there exist a subsequence $\{w_{n_{j_k}+1}\}_{k \in \mathbb{N}}$ and a point $\tilde{w} \in X \cap \mathbb{S}_Y$ such that

$$\|\tilde{w}\|_X^p = \lambda \quad \text{and} \quad w_{n_{j_k}+1} \rightarrow \tilde{w} \text{ in } X.$$

Since A and B are continuous, we can pass to the limit in

$$\langle A(w_{n_{j_k}+1}), v \rangle = \lambda_{n_{j_k}} \langle B(w_{n_{j_k}}), v \rangle, \quad v \in X,$$

in order to obtain

$$\langle A(\tilde{w}), v \rangle = \lambda \langle B(w), v \rangle.$$

This yields

$$\lambda = \lambda \|w\|_Y^q = \lambda \langle B(w), w \rangle = \langle A(\tilde{w}), w \rangle \leq \|\tilde{w}\|_X^{p-1} \|w\|_X = \lambda^{\frac{p-1}{p}} \lambda^{\frac{1}{p}} = \lambda,$$

showing thus that

$$\langle A(\tilde{w}), w \rangle = \|\tilde{w}\|_X^{p-1} \|w\|_X.$$

Since $\|\tilde{w}\|_X = \|w\|_X (= \lambda)$, our hypothesis (A2) implies that $\tilde{w} = w$, so that

$$\langle A(w), v \rangle = \lambda \langle B(w), v \rangle, \quad \forall v \in X.$$

This shows both that λ is an eigenvalue and that w is a corresponding eigenvector.

Note that our arguments show that w is the only cluster point of the subsequence $\{w_{n_{j+1}}\}_{j \in \mathbb{N}}$. This fact implies that $\{w_{n_{j+1}}\}$ also converges to w as claimed in the statement of the lemma. Actually, for any $m \in \mathbb{N}$ the shifted subsequence $\{w_{n_{j+m}}\}_{j \in \mathbb{N}}$ converges to w . □

Proof of Theorem 1 It follows from Lemmas 4 and 5. □

Remark 6 When we know in advance that λ is simple, in the sense that its corresponding eigenvectors are scalar multiple of each other, we have that w and $-w$ are the only cluster points of the sequence $\{w_n\}_{n \in \mathbb{N}}$.

Regarding the eigenvalue problem (5), when $q \in [1, p]$ and $w_0 \in L^q(\Omega) \setminus \{0\}$ is nonnegative, one has

$$w_n := \frac{u_n}{\|u_n\|_q} \rightarrow e_q \text{ in } W_0^{1,p}(\Omega) \text{ and } \lambda_n := \left(\frac{\|u_n\|_Y}{\|u_{n+1}\|_Y} \right)^{p-1} \rightarrow \lambda_q,$$

where e_q denotes the positive first eigenfunction such that $\|e_q\|_q = 1$ and λ_q is the first eigenvalue for (5), defined by (10). Indeed, as mentioned in the Introduction, when $q \in [1, p]$ the eigenvalue λ_q is simple and its eigenfunctions are the only that do not change sign in Ω . Hence, since $w_0 \geq 0$ a simple comparison principle guarantees that $w_n \geq 0$ for all $n \in \mathbb{N}$, the same occurring with the limit function w given by Theorem 1. Since w is a nonnegative eigenfunction corresponding to the eigenvalue λ , it must be strictly positive in Ω , according to the strong maximum principle (see Vázquez 1984). This fact implies that $\lambda = \lambda_q$ and then Remark 6 guarantees that $w_n \rightarrow e_q$ in $W_0^{1,p}(\Omega)$.

3 Two Concrete Examples

In this section we present two concrete examples of eigenvalue-type problems for which the results in the previous section apply. In both, Ω denotes a smooth bounded

domain of \mathbb{R}^N , $N \geq 2$. We anticipate that when $1 \leq q \leq p$ in both examples the first eigenvalue is simple and its eigenfunctions are the only that do not change sign. Thus, in this situation, it follows from the Strong Maximum Principle that the choice of an initial function w_0 nonnegative forces $\{\lambda_n\}$ to converge to the first eigenvalue and $\{w_n\}$ to converge to the only positive and normalized first eigenfunction.

3.1 Dirichlet Eigenproblem for the s -Fractional p -Laplacian

The results of Sect. 2 can be applied to the following fractional version of (5)

$$\begin{cases} (-\Delta_p)^s u = \lambda \|u\|_q^{p-q} |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

where $0 < s < 1 < p$, $\|\cdot\|_q$ denotes the standard norm of $L^q(\Omega)$,

$$1 \leq q < p_s^* := \begin{cases} \frac{Np}{N-ps} & \text{if } sp < N \\ \infty & \text{if } sp \geq N \end{cases}$$

and

$$(-\Delta_p)^s u := 2 \lim_{\epsilon \rightarrow 0^+} \int_{|x| \geq \epsilon} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy,$$

is the s -fractional p -Laplacian.

The usual space to deal with this problem is the fractional Sobolev space $W_0^{s,p}(\Omega)$ defined as the closure of $C_c(\Omega)$ with respect to the Gagliardo seminorm $[\cdot]_{s,p}$ in \mathbb{R}^N , whose expression, at a measurable function u of \mathbb{R}^N , is

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Thanks the fractional Poincaré inequality (see Brasco et al. 2014, Lemma 2.4) the Gagliardo seminorm $[\cdot]_{s,p}$ is really a norm in $W_0^{s,p}(\Omega)$.

It is well-known that $W_0^{s,p}(\Omega)$, endowed with the norm $[\cdot]_{s,p}$, is a Banach space uniformly convex. Moreover, $W_0^{s,p}(\Omega)$ is compactly embedded into $L^r(\Omega)$, for all $1 \leq r < p_s^*$.

The weak formulation of (20) is $A(u) = \lambda \|u\|_q^{p-q} B(u)$ where $A : W_0^{s,p}(\Omega) \rightarrow W_0^{-s,p'}(\Omega)$ is defined as

$$\langle A(u), v \rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+ps}} dx dy, \tag{21}$$

and $B : L^q(\Omega) \rightarrow L^{q'}(\Omega)$ is the map

$$\langle B(u), v \rangle = \int_{\Omega} |u|^{q-2} uv dx. \tag{22}$$

Therefore, by considering these maps, (20) takes the form (2) with $X = W_0^{s,p}(\Omega)$ and $Y = L^q(\Omega)$. It can be shown that the functions (21) and (22) satisfy the hypotheses (A1), (A2), (B1), (B2) and (AB). The proof of these claims as well as all of that made in this section on the fractional Sobolev space $W_0^{s,p}(\Omega)$ and the operator $(-\Delta_p)^s$ can be found in the papers (Brasco et al. 2014; Brasco and Franzina 2014; Nezza et al. 2012; Iannizzotto et al. 2016; Lindgren and Lindqvist 2014).

3.2 Steklov Eigenproblem for the p -Laplacian

Let us consider the following Steklov-type eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|_q^{p-q} |u|^{q-2} u & \text{on } \partial\Omega, \end{cases} \tag{23}$$

where $\frac{\partial}{\partial \nu}$ denotes the outer unit normal derivative along $\partial\Omega$ and

$$|u|_q := \left(\int_{\partial\Omega} |u|^q ds \right)^{\frac{1}{q}}$$

denotes the standard norm of the Banach space $L^q(\partial\Omega)$.

The appropriate space of solutions for (23) is the uniformly convex Sobolev space

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega)^N \right\}$$

endowed with the norm

$$\|u\|_{W^{1,p}} := \left(\|\nabla u\|_p^p + \|u\|_p^p \right)^{\frac{1}{p}}.$$

The embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is known as the *boundary trace operator* and associates $u \in W^{1,p}(\Omega)$ with its trace $u|_{\partial\Omega} \in L^q(\partial\Omega)$, which we denote here by u itself. This operator is compact if

$$1 \leq q < p_* := \begin{cases} \frac{p(N-1)}{N-p} & \text{if } p < N \\ \infty & \text{if } p \geq N \end{cases}$$

and just continuous when $q = p_*$ (see Bonder et al. 2003; Rossi 2005).

We say that $u \in W^{1,p}(\Omega)$ is a weak solution of (23), for some $\lambda \in \mathbb{R}$, if and only if,

$$\int_{\Omega} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla v + |u|^{p-2} uv \right) dx = \lambda |u|_q^{p-q} \int_{\partial\Omega} |u|^{q-2} uv ds, \quad \forall v \in W^{1,p}(\Omega).$$

Thus, (23) takes the form (2) with $X := (W^{1,p}(\Omega), \|u\|_{W_0^{1,p}})$, $Y := (L^q(\partial\Omega), |u|_q)$ and the maps $A : X \rightarrow X^*$ and $B : Y \rightarrow Y^*$ defined by

$$\langle A(u), v \rangle := \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_{\Omega} |u|^{p-2} uv dx, \quad \forall v \in W^{1,p}(\Omega)$$

and

$$\langle B(u), v \rangle := \int_{\partial\Omega} |u|^{q-2} uv ds, \quad \forall v \in L^q(\partial\Omega).$$

It is straightforward to check (see L e 2006) that A and B are continuous and satisfy the hypotheses (A1), (A2), (B1), (B2) and (AB).

For more details on the eigenvalue problem (23) we refer the reader to Bonder and Rossi (2001) (see also Auchmuty 2004 where properties and applications regarding the case $p = q = 2$ are provided).

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