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Vector fields on $\mathfrak{gl}_{m|n}(\mathbb{C})$-flag supermanifolds [☆]

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ABSTRACT

The main result of this paper is the computation of the Lie superalgebras of holomorphic vector fields on complex flag supermanifolds, introduced by Yu.I. Manin. We prove that with several exceptions any holomorphic vector field is fundamental with respect to the natural action of the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$.

1. Introduction

It is a classical result that all holomorphic vector fields on a flag manifold in $\mathbb{C}^n$ are fundamental for the natural action of the general linear Lie group $\mathrm{GL}_n(\mathbb{C})$. More precisely the Lie algebra of holomorphic vector fields on a flag manifold is isomorphic to $\mathfrak{pgl}_n(\mathbb{C})$. A similar statement holds with some exceptions for flag manifolds which are isotropic with respect to a non-degenerate symmetric or skew-symmetric bilinear form in $\mathbb{C}^n$. These results were obtained by A.L. Onishchik in 1959, see [1] for details.

In [8] Yu.I. Manin constructed four series of complex compact homogeneous supermanifolds corresponding to four series of classical linear Lie superalgebras: $\mathfrak{gl}_{m|n}(\mathbb{C})$,

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$\mathfrak{osp}_{m|n}(\mathbb{C})$, $\pi\mathfrak{sp}_n(\mathbb{C})$ and $\mathfrak{q}_n(\mathbb{C})$, see [6] for precise definitions. The present paper is devoted to the calculation of the Lie superalgebras of holomorphic vector fields on complex flag supermanifolds corresponding to the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. It turns out that under some restrictions on the flag type all global holomorphic vector fields are fundamental with respect to the natural action of the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. In case of super-Grassmannians the similar result was obtained in [9].

In the present paper we study flag supermanifold $\mathbf{F}^{m|n}_{k|l}$ of type $k|l$ in the vector superspace $\mathbb{C}^{m|n}$. Here we put $k = (k_1, \dots, k_r)$ and $l = (l_1, \dots, l_r)$ such that

$$\begin{gathered} 0 \leq k_r \leq \dots \leq k_1 \leq m, \quad 0 \leq l_r \dots \leq l_1 \leq n \quad \text{and} \\ 0 < k_r + l_r < \dots < k_1 + l_1 < m + n. \end{gathered} \tag{1}$$

The number r is called the *length* of $\mathbf{F}^{m|n}_{k|l}$. The idea of the proof is to use results of [9] and the following fact. For $r > 1$ the supermanifold $\mathbf{F}^{m|n}_{k|l}$ is the total space of a holomorphic superbundle with base space isomorphic to the super-Grassmannian $\mathbf{F}^{m|n}_{k_1|l_1}$ and the fiber isomorphic to a flag supermanifold of length $r-1$. The projection of this superbundle is equivariant with respect to the natural actions of the Lie supergroup $\mathrm{GL}_{m|n}(\mathbb{C})$ on the total space and base space of $\mathbf{F}^{m|n}_{k|l}$.

Let $p : \mathcal{M} \to \mathcal{B}$ be a morphism of supermanifolds. A vector field v defined on $\mathcal{M}$ is said to be *projectable* with respect to p if there is a vector field v_1 on $\mathcal{B}$ such that

$$p^*(v_1(f)) = v(p^*(f))$$

for any $f \in \mathcal{O}_{\mathcal{B}}$. A vector field v on $\mathcal{M}$ is called *vertical* if it is projected to 0. If p is a projection of a superbundle, then every projectable vector field v is projected to a unique vector field v_1. In [2] the following statement was proven. If $p : \mathcal{M} \to \mathcal{B}$ is the projection of a superbundle with fiber $\mathcal{S}$ with $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$, that is, any global holomorphic function on $\mathcal{S}$ is constant, then every vector field on $\mathcal{M}$ is projectable with respect to p. Denote by $\mathfrak{v}(\mathcal{M})$ the Lie superalgebra of holomorphic vector fields on $\mathcal{M}$. If $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$, we have a map

$$\mathcal{P} : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\mathcal{B}).$$

This map is a Lie superalgebra homomorphism, and its kernel $\operatorname{Ker}\mathcal{P}$ is the set of all vertical vector fields.

Consider the superbundle $\mathbf{F}^{m|n}_{k|l}$. The space of global holomorphic functions $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0)$ was computed in [14]. It was shown that $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$ under some restrictions on the flag type $k|l$. Therefore, in general all holomorphic vector fields on $\mathcal{M}$ are projectable to the super-Grassmannian $\mathcal{B} = \mathbf{F}^{m|n}_{k_1|l_1}$ and we have the following homomorphism of Lie superalgebras

$$\mathcal{P} : \mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \to \mathfrak{v}(\mathbf{F}^{m|n}_{k_1|l_1}).$$

From the equivariance of p with respect to the actions of $\mathrm{GL}_{m|n}(\mathbb{C})$ it follows that the natural Lie algebra homomorphisms

$$\mu : \mathfrak{gl}_{m|n}(\mathbb{C}) \to \mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \quad \text{and} \quad \mu_{\mathcal{B}} : \mathfrak{gl}_{m|n}(\mathbb{C}) \to \mathfrak{v}(\mathbf{F}^{m|n}_{k_1|l_1})$$

satisfy the relation $\mu_{\mathcal{B}} = \mathcal{P} \circ \mu$. Assuming that the homomorphism $\mu_{\mathcal{B}}$ is surjective, in other words assuming that

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k_1|l_1}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}),$$

we see that $\mathcal{P}$ is also surjective. The main goal of this paper is to prove that $\mathcal{P}$ is injective. Then $\mathcal{P}$ is invertible and we have

$$\mu = \mathcal{P}^{-1} \circ \mu_{\mathcal{B}}.$$

Therefore,

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

The main result of this paper was announced in [15] in case $0 < k_r < \ldots < k_1 < m$ and $0 < l_r < \ldots < l_1 < n$ and the idea of the proof was sketched in [12] also in this case. Here we give the proof in general case. Our main result is the following.

Theorem. *Assume that $r > 1$ and that we have the following restrictions on the flag type:*

$$\begin{aligned} &(k_i, l_i) \neq (k_{i-1}, 0),\ (0, l_{i-1}),\ \ i \geq 2; \\ &(k_{i-1}, k_i | l_{i-1}, l_i) \neq (1, 0 | l_{i-1}, l_{i-1} - 1),\ (1, 1 | l_{i-1}, 1),\ \ i \geq 1; \\ &(k_{i-1}, k_i | l_{i-1}, l_i) \neq (k_{i-1}, k_{i-1} - 1 | 1, 0),\ (k_{i-1}, 1 | 1, 1),\ \ i \geq 1; \\ &k|l \neq (0, \ldots, 0 | n, l_2, \ldots, l_r),\ \ k|l \neq (m, k_2, \ldots, k_r | 0, \ldots, 0). \end{aligned}$$

Then

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

If $k|l = (0, \ldots, 0 | n, l_2, \ldots, l_r)$ or $k|l = (m, k_2, \ldots, k_r | 0, \ldots, 0)$, then

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq W_{mn} \Subset (\textstyle\bigwedge(\xi_1, \ldots, \xi_{mn}) \otimes \mathfrak{pgl}_n(\mathbb{C})),$$

where $W_{mn} = \operatorname{Der} \bigwedge(\xi_1, \ldots, \xi_{mn})$.

2. Preliminaries

2.1. Flag supermanifolds

A detailed introduction to the theory of supermanifolds can be found in [3,5,7,8]. Throughout we will restrict our attention to the complex-analytic version of the theory. Recall that a *complex-analytic superdomain of dimension* $s|t$ is a $\mathbb{Z}_2$-graded locally ringed space of the form

$$\mathcal{U} = (\mathcal{U}_0, \mathcal{F}_{\mathcal{U}_0} \otimes_{\mathbb{C}} \bigwedge(t)),$$

where $\mathcal{F}_{\mathcal{U}_0}$ is the sheaf of holomorphic functions on an open set $\mathcal{U}_0 \subset \mathbb{C}^s$ and $\bigwedge(t)$ is the Grassmann algebra with t generators. A *complex-analytic supermanifold* of dimension $s|t$ is a $\mathbb{Z}_2$-graded locally ringed space that is locally isomorphic to a complex-analytic superdomain of dimension $s|t$. We will denote a supermanifold by $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$, where $\mathcal{M}_0$ is the underlying complex-analytic manifold and $\mathcal{O}_{\mathcal{M}}$ is the structure sheaf.

Let us give an explicit description of a flag supermanifold in terms of charts and local coordinates (see also [8,14,13]). Let us take two non-negative integers $m, n \in \mathbb{Z}$ and two sets of non-negative integers

$$k = (k_1, \dots, k_r) \quad \text{and} \quad l = (l_1, \dots, l_r)$$

such that (1) holds. The underlying space of the supermanifold $\mathbf{F}^{m|n}_{k|l}$ is the product $\mathbf{F}^m_k \times \mathbf{F}^n_l$ of two flag manifolds of types $k = (k_1, \dots, k_r)$ and $l = (l_1, \dots, l_r)$ in the vector spaces $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Let us fix two subsets

$$I_{s\bar{0}} \subset \{1, \dots, k_{s-1}\} \quad \text{and} \quad I_{s\bar{1}} \subset \{1, \dots, l_{s-1}\},$$

where $k_0 := m$ and $l_0 := n$, such that $|I_{s\bar{0}}| = k_s$, and $|I_{s\bar{1}}| = l_s$, for any $s = 1, \dots, r$. We put $I_s := (I_{s\bar{0}}, I_{s\bar{1}})$ and $I := (I_1, \dots, I_r)$. We assign the following $(k_{s-1} + l_{s-1}) \times (k_s + l_s)$-matrix

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ \mathrm{H}_s & Y_s \end{pmatrix}, \quad s = 1, \dots, r, \tag{2}$$

to any I_s. We assume that the matrices $X_s = (x^s_{ij})$ and $Y_s = (y^s_{ij})$ from (2) are of size $(k_{s-1} \times k_s)$ and $(l_{s-1} \times l_s)$, respectively. We also assume that Z_{I_s} contains the identity submatrix $E_{k_s + l_s}$ of size $(k_s + l_s) \times (k_s + l_s)$ in the lines with numbers $i \in I_{s\bar{0}}$ and $k_{s-1} + i$, $i \in I_{s\bar{1}}$. For example in case

$$I_{s\bar{0}} = \{k_{s-1} - k_s + 1, \dots, k_{s-1}\}, \quad I_{s\bar{1}} = \{l_{s-1} - l_s + 1, \dots, l_{s-1}\}$$

the matrix Z_{I_s} has the following form:

$$Z_{I_1} = \begin{pmatrix} X_s & \Xi_s \\ E_{k_s} & 0 \\ \mathrm{H}_s & Y_s \\ 0 & E_{l_s} \end{pmatrix}.$$

Here E_q is the identity matrix of size $q \times q$. For simplicity of notation we use here the same letters X_s, Y_s, Ξ_s and H_s as in (2).

Further we consider elements from $X_s = (x^s_{ij})$ and $Y_s = (y^s_{ij})$, which are not contained in the identity matrix, as the coordinate functions on $\mathbb{C}^{(k_{s-1}-k_s)\times k_s|0}$ and $\mathbb{C}^{(l_{s-1}-l_s)\times l_s|0}$, respectively. These coordinate functions are assumed to be even. The elements of the matrices $\Xi_s = (\xi^s_{ij})$ and $\mathrm{H}_s = (\eta^s_{ij})$, which are not contained in the identity matrix, are assumed to be the coordinate functions on $\mathbb{C}^{0|(k_{s-1}-k_s)\times l_s}$ and $\mathbb{C}^{0|(l_{s-1}-l_s)\times k_s}$, respectively. They assumed to be odd. Summing up, the matrices (2) determine the superdomain $\mathcal{U}_I$ with even coordinates x^s_{ij} and y^s_{ij}, and odd coordinates ξ^s_{ij} and η^s_{ij}.

Let us define the transition functions between two superdomains $\mathcal{U}_I$ and $\mathcal{U}_J$ that correspond to $I = (I_s)$ and $J = (J_s)$, respectively, by the following formulas:

$$Z_{J_1} = Z_{I_1} C^{-1}_{I_1 J_1}, \quad Z_{J_s} = C_{I_{s-1} J_{s-1}} Z_{I_s} C^{-1}_{I_s J_s}, \quad s \geq 2. \tag{3}$$

Here $C_{I_s J_s}$ is an invertible submatrix in Z_{I_s} that consists of the lines with numbers $i \in J_{s\bar{0}}$ and $k_{s-1} + i$, where $i \in J_{s\bar{1}}$. In other words, we choose the matrix $C_{I_s J_s}$ in such a way that Z_{J_s} contains the identity submatrix $E_{k_s + l_s}$ in lines with numbers $i \in J_{s\bar{0}}$ and $k_{s-1} + i$, where $i \in J_{s\bar{1}}$. These charts and transition functions define a supermanifold that we denote by $\mathbf{F}^{m|n}_{k|l}$. This supermanifold we will call the *flag supermanifold* of type $k|l$. In case $r = 1$ this supermanifold is called the *super-Grassmannian* and is denoted by $\mathbf{Gr}_{m|n,k|l}$.

Further, the underlying space $U_{I_{\bar{0}}} \times V_{I_{\bar{1}}}$ of the superdomain $\mathcal{U}_I$ is a chart on $\mathbf{F}^m_k \times \mathbf{F}^n_l$. Indeed, we can take the non-trivial elements (i.e., those are not contained in the identity submatrix) from X_s and Y_s as standard local coordinates in $\mathbf{F}^m_k$ and $\mathbf{F}^n_l$, respectively. So we have the following atlas on $\mathbf{F}^m_k \times \mathbf{F}^n_l$:

$$\{U_I = U_{I_{\bar{0}}} \times U_{I_{\bar{1}}}\}$$

with charts parametrized by $I = (I_s)$. The reduction of the transition functions (3) leads to the usual transition function for $\mathbf{F}^m_k \times \mathbf{F}^n_l$. Therefore the underlying space of $\mathbf{F}^{m|n}_{k|l}$ is $\mathbf{F}^m_k \times \mathbf{F}^n_l$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ be a complex-analytic supermanifold. Denote by $\mathcal{T} = \mathcal{D}er\,(\mathcal{O}_{\mathcal{M}})$ the sheaf of vector fields on $\mathcal{M}$. It is a sheaf of Lie superalgebras with respect to the following multiplication

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX.$$

The global sections of $\mathcal{T}$ are called *holomorphic vector fields* on $\mathcal{M}$. They form a complex Lie superalgebra that we denote by $\mathfrak{v}(\mathcal{M})$. This Lie superalgebra is finite dimensional in case when $\mathcal{M}_0$ is compact. Let us sketch a proof of this statement. The sheaf $\mathcal{O}_{\mathcal{M}}$ possesses the following filtration

$$\mathcal{O}_{\mathcal{M}} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \cdots ,$$

where $\mathcal{J}$ is the subsheaf of ideal in $\mathcal{O}_{\mathcal{M}}$ generated by odd elements. This filtration induces the following filtration in $\mathcal{T}$:

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \mathcal{T}_{(1)} \supset \cdots ,$$

where

$$\mathcal{T}_{(p)} := \{v \in \mathcal{T} \mid v(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{J}^p,\ \ v(\mathcal{J}) \subset \mathcal{J}^{p+1}\}.$$

Consider the following exact sequence.

$$0 \to \mathcal{T}_{(p+1)} \longrightarrow \mathcal{T}_{(p)} \longrightarrow \mathcal{T}_{(p)}/\mathcal{T}_{(p+1)} \to 0.$$

Clearly the sheaf $\mathcal{T}_{(p)}/\mathcal{T}_{(p+1)}$ is a locally free sheaf of $\mathcal{O}_{\mathcal{M}}/\mathcal{J}$-modules on the usual complex manifold $\mathcal{M}_0$. Now our statement follows from two facts. First of all $\mathcal{T}_{(p)} = \{0\}$ for enough large p. Secondly, it is well-known that the sheaf cohomology of any coherent sheaf (and in particular of a locally free sheaf) on a compact complex manifold is finite dimensional.

The goal of this paper is to compute the Lie superalgebra $\mathfrak{v}(\mathcal{M})$, when $\mathcal{M}$ is a flag supermanifold of type $k|l$ in $\mathbb{C}^{m|n}$. As usual we denote by $\mathfrak{gl}_{m|n}(\mathbb{C})$ the general Lie superalgebra of the superspace $\mathbb{C}^{m|n}$. It consists of the following matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{where} \quad A \in \mathfrak{gl}_m(\mathbb{C}) \quad \text{and} \quad B \in \mathfrak{gl}_n(\mathbb{C}).$$

Denote by $\mathrm{GL}_{m|n}(\mathbb{C})$ the Lie supergroup of the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. (See [16] for more information about Lie supergroups.) In [8] an action of $\mathrm{GL}_{m|n}(\mathbb{C})$ on the supermanifold $\mathbf{F}^{m|n}_{k|l}$ was defined. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

be a coordinate matrix of $\mathrm{GL}_{m|n}(\mathbb{C})$. This meant that we consider elements of the matrices L_{11} and L_{22} as even coordinates, and elements of L_{12} and L_{22} as odd coordinates of a certain superdomain. In addition we assume that $\det L_{11} \neq 0$ and $\det L_{22} \neq 0$. The action of $\mathrm{GL}_{m|n}(\mathbb{C})$ on $\mathbf{F}^{m|n}_{k|l}$ in coordinates is given by the following formulas:

$$\begin{aligned} &(L,(Z_{I_1},\dots,Z_{I_r}))\longmapsto(\tilde Z_{J_1},\dots,\tilde Z_{J_r}),\quad \text{where}\\ &\tilde Z_{J_1}=LZ_{I_1}C_1^{-1},\quad \tilde Z_{J_s}=C_{s-1}Z_{I_s}C_s^{-1}. \end{aligned}\tag{4}$$

Here C_1 is an invertible submatrix in LZ_{I_1} that consists of the lines with numbers $i\in J_{1\bar 0}$ and $m+i$, where $i\in J_{1\bar 1}$, and C_s, where $s\geq 2$, is an invertible submatrix in $C_{s-1}Z_{I_s}$ that consists of the lines with numbers $i\in J_{s\bar 0}$ and $k_{s-1}+i$, where $i\in J_{\bar 1 s}$. This Lie supergroup action induces a Lie superalgebra homomorphism

$$\mu:\mathfrak{gl}_{m|n}(\mathbb{C})\to\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}).$$

In case $r=1$ in [9, Lemma 1] it was proven that $\operatorname{Ker}\mu=\langle E_{m+n}\rangle$, where E_{m+n} is the identity matrix of size $m+n$. In general case $r>1$ we also have $\operatorname{Ker}\mu=\langle E_{m+n}\rangle$ and the proof is similar to [9, Lemma 1]. We see that μ induces an injective homomorphism of Lie superalgebras

$$\bar\mu:\mathfrak{gl}_{m|n}(\mathbb{C})/\langle E_{m+n}\rangle\to\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}).$$

We will show that with some exceptions this homomorphism is an isomorphism.

2.2. Superbundles and projectable vector fields

Recall that a *morphism of complex-analytic* supermanifolds $\mathcal{M}$ to $\mathcal{N}$ is a pair $f=(f_0,f^*)$, where $f_0:\mathcal{M}_0\to\mathcal{N}_0$ is a holomorphic map and $f^*:\mathcal{O}_\mathcal{N}\to(f_0)_*(\mathcal{O}_\mathcal{M})$ is a homomorphism of sheaves of superalgebras.

Definition 1. A *superbundle* is a set $(\mathcal{M},\mathcal{B},p,\mathcal{S})$, where $\mathcal{S}$ is the fiber, $\mathcal{B}$ is the base space, $\mathcal{M}$ is the total space and $p=(p_0,p^*):\mathcal{M}\to\mathcal{B}$ is the projection, such that there exists an open covering $\{U_i\}$ on $\mathcal{B}_0$, isomorphisms $\psi_i:(p_0^{-1}(U_i),\mathcal{O}_\mathcal{M})\to(U_i,\mathcal{O}_\mathcal{B})\times\mathcal{S}$ and the following diagram is commutative:

$$\begin{tikzcd}
(p_0^{-1}(U_i),\mathcal{O}_\mathcal{M}) \arrow[r, "\psi_i"'] \arrow[d, "p"'] & (U_i,\mathcal{O}_\mathcal{B})\times\mathcal{S} \arrow[d, "pr"] \\
(U_i,\mathcal{O}_\mathcal{B}) \arrow[r, "id"'] & (U_i,\mathcal{O}_\mathcal{B})
\end{tikzcd}$$

where pr is the natural projection.

Usually we will denote a superbundle $(\mathcal{M},\mathcal{B},p,\mathcal{S})$ just by $\mathcal{M}$.

Remark. From the form of transition functions (3) it follows that for $r>1$ the supermanifold $\mathbf{F}^{m|n}_{k|l}$ is a superbundle with base $\mathbf{Gr}_{m|n,k_1|l_1}$ and fiber $\mathbf{F}^{k_1|l_1}_{k'|l'}$, where $k'=(k_2,\dots,k_r)$

and $l' = (l_2, \dots, l_r)$. In local coordinates introduced above the projection p is given by

$$(Z_1, Z_2, \dots Z_n) \longmapsto (Z_1).$$

Moreover, formulas (4) tell us that the projection p is equivariant with respect to the action of the supergroup $\mathrm{GL}_{m|n}(\mathbb{C})$ on $\mathbf{F}^{m|n}_{k|l}$ and $\mathbf{Gr}_{m|n,k_1|l_1}$.

Let $p = (p_0, p^*) : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds.

Definition 2. A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called *projectable* with respect to p, if there exists a vector field $v_1 \in \mathfrak{v}(\mathcal{N})$ such that

$$p^*(v_1(f)) = v(p^*(f)) \quad \text{for all} \quad f \in \mathcal{O}_{\mathcal{N}}.$$

In this case we say that v *is projected to* v_1.

Projectable vector fields form a super Lie subalgebra $\overline{\mathfrak{v}}(\mathcal{M})$ in $\mathfrak{v}(\mathcal{M})$. In case if p is a projection of a superbundle, the homomorphism $p^* : \mathcal{O}_{\mathcal{N}} \to p_*(\mathcal{O}_{\mathcal{M}})$ is injective. Hence, any projectable vector field v is projected into unique vector field $v_1 = \mathcal{P}(v)$ and we have the following map

$$\mathcal{P} : \overline{\mathfrak{v}}(\mathcal{M}) \to \mathfrak{v}(\mathcal{N}), \quad v \mapsto v_1.$$

It is a homomorphism of Lie superalgebras. A vector field $v \in \mathfrak{v}(\mathcal{M})$ is called *vertical*, if $\mathcal{P}(v) = 0$. Vertical vector fields form an ideal $\operatorname{Ker} \mathcal{P}$ in $\overline{\mathfrak{v}}(\mathcal{M})$.

We will need the following proposition proved in [2].

Proposition 1. *Let $p : \mathcal{M} \to \mathcal{B}$ be the projection of a superbundle with fiber $\mathcal{S}$. Assume that $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$, i.e. any global holomorphic function is constant. Then any holomorphic vector field from $\mathfrak{v}(\mathcal{M})$ is projectable with respect to p and we have a homomorphism of Lie superalgebras $\nu : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\mathcal{B})$.*

Let $p : \mathcal{M} \to \mathcal{B}$ be a superbundle with fiber $\mathcal{S}$. We define the sheaf $\mathcal{W}$ on $\mathcal{B}_0$ in the following way. We assign to any open set $U \subset \mathcal{B}_0$ the set of all vertical vector fields on the supermanifold $(p_0^{-1}(U), \mathcal{O}_{\mathcal{M}})$. In [12] the following proposition was proven.

Proposition 2. *Assume that $\mathcal{S}_0$ is compact. Then $\mathcal{W}$ is a locally free sheaf of $\mathcal{O}_{\mathcal{B}}$-modules and* $\dim \mathcal{W} = \dim \mathfrak{v}(\mathcal{S})$.

Clearly, the Lie algebra $\mathcal{W}(\mathcal{B}_0)$ coincides with the ideal of all vertical vector fields in $\mathfrak{v}(\mathcal{M})$. We describe the corresponding to $\mathcal{W}$ graded sheaf. Consider the following filtration in $\mathcal{O}_{\mathcal{B}}$

$$\mathcal{O}_{\mathcal{B}} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \dots$$

where $\mathcal{J}$ is the sheaf of ideals in $\mathcal{O}_{\mathcal{B}}$ generated by odd elements. We have the corresponding graded sheaf of superalgebras

$$\tilde{\mathcal{O}}_{\mathcal{B}} = \bigoplus_{p\geq 0}(\tilde{\mathcal{O}}_{\mathcal{B}})_p, \quad \text{where} \quad (\tilde{\mathcal{O}}_{\mathcal{B}})_p = \mathcal{J}^p/\mathcal{J}^{p+1}.$$

Putting $\mathcal{W}_{(p)} = \mathcal{J}^p\mathcal{W}$ we get the following filtration in $\mathcal{W}$:

$$\mathcal{W} = \mathcal{W}_{(0)} \supset \mathcal{W}_{(1)} \supset \dots . \tag{5}$$

We define the $\mathbb{Z}$-graded sheaf of $\mathcal{F}_{\mathcal{B}_0}$-modules by

$$\tilde{\mathcal{W}} = \bigoplus_{p\geq 0}\tilde{\mathcal{W}}_p, \quad \text{where} \quad \tilde{\mathcal{W}}_p = \mathcal{W}_{(p)}/\mathcal{W}_{(p+1)}. \tag{6}$$

Here $\mathcal{F}_{\mathcal{B}_0}$ is the structure sheaf of the underlying space $\mathcal{B}_0$. The $\mathbb{Z}_2$-grading in $\mathcal{W}_{(p)}$ induces the $\mathbb{Z}_2$-grading in $\tilde{\mathcal{W}}_p$. Using Proposition 2 we get the following result.

Proposition 3. *Assume that $\mathcal{S}_0$ is compact. Then $\tilde{\mathcal{W}}_0$ is a locally free sheaf of $\mathcal{F}_{\mathcal{B}_0}$-modules. Any fiber of the corresponding vector bundle is isomorphic to $\mathfrak{v}(\mathcal{S})$.*

2.3. The Borel–Weil–Bott Theorem

To calculate the Lie superalgebra of vector fields we will use the Borel–Weil–Bott Theorem, see for example [1] for details. This theorem permits to compute cohomology with values in a holomorphic homogeneous bundle over a flag manifold. For completeness we formulate it here, adapting to our notations and agreements.

Let $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ be the underlying space of $\mathrm{GL}_{m|n}(\mathbb{C})$, P be a parabolic subgroup in G and R be the reductive part of P. Assume that $\mathbf{E}_\varphi \to G/P$ is the homogeneous vector bundle corresponding to a representation φ of P in $E = (\mathbf{E}_\varphi)_P$. Denote by $\mathcal{E}_\varphi$ the sheaf of holomorphic section of this vector bundle. In the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})_{\bar{0}} \simeq \mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C})$ we fix the Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, where

$$\mathfrak{t}_1 = \{\mathrm{diag}(\mu_1, \dots, \mu_m)\} \quad \text{and} \quad \mathfrak{t}_2 = \{\mathrm{diag}(\lambda_1, \dots, \lambda_n)\},$$

the following system of positive roots:

$$\Delta^+ = \Delta_1^+ \cup \Delta_2^+,$$

where

$$\Delta_1^+ = \{\mu_i - \mu_j,\ i < j\} \quad \text{and} \quad \Delta_2^+ = \{\lambda_p - \lambda_q,\ p < q\},$$

and the following system of simple roots $\Phi = \Phi_1 \cup \Phi_2$, where

$$\Phi_1 = \{\alpha_1, \dots, \alpha_{n-1}\},\ \ \alpha_i = \mu_i - \mu_{i+1}, \quad \Phi_2 = \{\beta_1, \dots, \beta_{n-1}\},\ \ \beta_p = \lambda_p - \lambda_{p+1}.$$

Denote by $\mathfrak{t}^*(\mathbb{R})$ a real subspace in $\mathfrak{t}^*$ spanned by μ_j and λ_i. Consider the scalar product $(\ ,\ )$ in $\mathfrak{t}^*(\mathbb{R})$ such that the vectors μ_j, λ_i form an orthonormal basis. An element $\gamma \in \mathfrak{t}^*(\mathbb{R})$ is called *dominant* if $(\gamma, \alpha) \geq 0$ for all $\alpha \in \Delta^+$.

Theorem 1 *(Borel–Weil–Bott). Assume that the representation $\varphi : P \to \mathrm{GL}(E)$ is completely reducible and $\lambda_1, \dots, \lambda_s$ are highest weights of $\varphi|R$. Then the G-module $H^0(G/P, \mathcal{E}_\varphi)$ is isomorphic to the sum of irreducible G-modules with highest weights $\lambda_{i_1}, \dots, \lambda_{i_t}$, where λ_{i_a} are dominant highest weights.*

2.4. Holomorphic functions on flag supermanifolds

Holomorphic functions on homogeneous supermanifolds and in particular on flag supermanifolds were studied in [14]. It is well-known that any holomorphic function on a connected compact complex manifold is constant. This statement is false for a supermanifold with a connected compact underlying space. However in case of flag supermanifolds the following theorem holds true:

Theorem 2. *(See [14].) Consider the flag supermanifold $\mathcal{M} = \mathbf{F}^{m|n}_{k|l}$. Assume that*

$$\begin{aligned} (k|l) &\neq (m, \dots, m, k_{s+2}, \dots, k_r | l_1, \dots, l_s, 0, \dots, 0), \\ (k|l) &\neq (k_1, \dots, k_s, 0, \dots, 0 | n, \dots, n, l_{s+2}, \dots, l_r), \end{aligned} \tag{7}$$

for any $s \geq 0$. Then $\mathcal{O}_\mathcal{M}(\mathcal{M}_0) = \mathbb{C}$. In other words under conditions (7) any holomorphic function on $\mathbf{F}^{m|n}_{k|l}$ is constant.

Otherwise

$$\mathbf{F}^{m|n}_{k|l} \simeq (\mathrm{pt}, \textstyle\bigwedge(mn)) \times (\mathbf{F}^m_k \times \mathbf{F}^n_l)$$

and $\mathcal{O}_\mathcal{M}(\mathcal{M}_0) = \bigwedge(mn)$, where $\bigwedge(mn)$ is the Grassmann algebra with mn generators.

3. Vector fields on flag supermanifolds

3.1. Vector fields on super-Grassmannians

Recall that a supermanifold $\mathcal{M}$ is called *homogeneous* if it possesses a transitive action of a certain Lie supergroup. An action of a Lie supergroup on a supermanifold $\mathcal{M}$ is called *transitive* if the underlying action of the Lie group is transitive on the underlying space $\mathcal{M}_0$ of $\mathcal{M}$ and the corresponding action of the Lie superalgebra is transitive, see [14] for details. From previous sections it can be deduced that $\mathbf{F}^{m|n}_{k|l}$ is a $\mathrm{GL}_{m|n}(\mathbb{C})$-homogeneous superspace. Indeed, the action of $\mathrm{GL}_{m|n}(\mathbb{C})$ on $\mathbf{F}^{m|n}_{k|l}$ is given by formulas (4). Clearly the underlying action is transitive: it is just the standard action of $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $\mathbf{F}^m_k \times \mathbf{F}^n_l$. The corresponding action of the Lie superalgebra is also transitive. This can be explicitly commuted in coordinates.

The action (4) induces the Lie algebra homomorphism

$$\mu : \mathfrak{gl}_{m|n}(\mathbb{C}) \to \mathfrak{v}(\mathbf{Gr}_{m|n,k|l}).$$

The kernel of this homomorphism is equal to $\langle E_{m+n}\rangle$, [9, Lemma 1]. Further we will use the following notation:

$$\mathfrak{pgl}_{m|n}(\mathbb{C}) := \mathfrak{gl}_{m|n}(\mathbb{C})/\langle E_{m+n}\rangle.$$

The Lie superalgebra of holomorphic vector fields on super-Grassmannian $\mathbf{Gr}_{m|n,k|l}$ was computed in [4,9–11].

Theorem 3. *The homomorphism* $\mu : \mathfrak{gl}_{m|n}(\mathbb{C}) \to \mathfrak{v}(\boldsymbol{Gr}_{m|n,k|l})$ *is almost always surjective and*

$$\mathfrak{v}(\boldsymbol{Gr}_{m|n,k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

The exceptional cases are the following.

1.1 *For the super-Grassmannian* $\mathbf{Gr}_{2|2,1|1}$ *we have*

$$\mathfrak{v}(\mathbf{Gr}_{2|2,1|1}) \simeq \mathfrak{psl}_{2|2}(\mathbb{C}) \ni \mathfrak{sl}_2(\mathbb{C}),$$

where $\mathfrak{psl}_{2|2}(\mathbb{C}) = \mathfrak{sl}_{2|2}(\mathbb{C})/< E_4 >$.

1.2 *For* $\mathbf{Gr}_{1|n,0|n-1} \simeq \mathbf{Gr}_{n|1,n-1|0} \simeq \mathbf{Gr}_{n|1,1|1} \simeq \mathbf{Gr}_{1|n,1|1}$, $n > 2$, *we have*

$$\mathfrak{v}(\mathbf{Gr}_{1|n,0|n-1}) \simeq W_n = \mathrm{Der} \bigwedge(\zeta_1, \dots, \zeta_n).$$

1.3 *In the degenerate case* $\mathbf{Gr}_{m|n,0|n} \simeq \mathbf{Gr}_{m|n,m|0}$ *we have*

$$\mathfrak{v}(\mathbf{Gr}_{m|n,0|n}) \simeq W_{mn} = \mathrm{Der} \bigwedge(\zeta_1, \dots, \zeta_{mn}).$$

1.4 *For* $\mathbf{Gr}_{2|2,0|1} \simeq \mathbf{Gr}_{2|2,1|0} \simeq \mathbf{Gr}_{2|2,1|2} \simeq \mathbf{Gr}_{2|2,2|1}$ *we have*

$$\mathfrak{v}(\mathbf{Gr}_{2|2,0|1}) \simeq \tilde{\mathbf{H}}_4 \ni \langle z\rangle,$$

where $\mathrm{ad}\, z$ *acts on the Lie superalgebra of Cartan type* $\tilde{\mathbf{H}}_4$ *as the grading operator.*

In case

$$0 < k < m \quad \text{and} \quad 0 < l < n, \quad (m|n, k|l) \neq (2|2, 1|1),$$

the Lie superalgebra of vector fields was computed in [9]. Results 1.1 and 1.2 of Theorem 3 were obtained in [4] (see also [9] for an explicit description of the Lie superalgebra) and [10,11], respectively. Result 1.3 of Theorem 3 is obvious. Result 1.4 of Theorem 3 follows from arguments in [11], Proof of Theorem 2.6. Note that in the statement of Theorem 2.6 in [11] and also in [9, Theorem 7] the Lie superalgebra of vector fields in case 1.4 was pointed incorrectly.

We will need an explicit description of the Lie superalgebra of holomorphic vector fields on $\mathbf{Gr}_{2|2,1|1}$, case 1.1 of Theorem 3, in the following local chart

$$\begin{pmatrix} x & \xi \\ 1 & 0 \\ \eta & y \\ 0 & 1 \end{pmatrix}.$$

The image of $\mathfrak{gl}_{2|2}(\mathbb{C})_{\bar{0}}$ with respect to the homomorphism μ from Theorem 3 is given by:

$$\begin{aligned}
&\mu(E_{11}) = x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}, \quad \mu(E_{12}) = \frac{\partial}{\partial x}, \quad \mu(E_{22}) = -x\frac{\partial}{\partial x} - \eta\frac{\partial}{\partial \eta}, \\
&\mu(E_{21}) = -x^2\frac{\partial}{\partial x} - x\eta\frac{\partial}{\partial \eta} - x\xi\frac{\partial}{\partial \xi} + \xi\eta\frac{\partial}{\partial y}, \quad \mu(E_{34}) = \frac{\partial}{\partial y}, \\
&\mu(E_{43}) = -y^2\frac{\partial}{\partial y} - y\xi\frac{\partial}{\partial \xi} - y\eta\frac{\partial}{\partial \eta} - \xi\eta\frac{\partial}{\partial x}, \quad \mu(E_{33}) = y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \eta}, \\
&\mu(E_{44}) = -y\frac{\partial}{\partial y} - \xi\frac{\partial}{\partial \xi}.
\end{aligned} \tag{8}$$

The image of $\mathfrak{gl}_{2|2}(\mathbb{C})_{\bar{1}}$ with respect to the homomorphism μ from Theorem 3 is given by:

$$\begin{aligned}
&\mu(E_{14}) = \frac{\partial}{\partial \xi}, \quad \mu(E_{32}) = \frac{\partial}{\partial \eta}, \quad \mu(E_{13}) = \eta\frac{\partial}{\partial x} + y\frac{\partial}{\partial \xi}, \\
&\mu(E_{31}) = \xi\frac{\partial}{\partial y} + x\frac{\partial}{\partial \eta}, \quad \mu(E_{23}) = -x\eta\frac{\partial}{\partial x} - xy\frac{\partial}{\partial \xi} + y\eta\frac{\partial}{\partial y}, \\
&\mu(E_{41}) = -y\xi\frac{\partial}{\partial y} - xy\frac{\partial}{\partial \eta} + x\xi\frac{\partial}{\partial x}, \quad \mu(E_{24}) = -x\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial y}, \\
&\mu(E_{42}) = -y\frac{\partial}{\partial \eta} + \xi\frac{\partial}{\partial x}.
\end{aligned} \tag{9}$$

Additional holomorphic on $\mathbf{Gr}_{2|2,1|1}$ vector fields are

$$\eta\frac{\partial}{\partial\xi},\quad \xi\frac{\partial}{\partial\eta}. \tag{10}$$

A direct computation shows that

$$\mathfrak{v}(\mathbf{Gr}_{2|2,1|1})\simeq \mathfrak{pgl}_{2|2}(\mathbb{C})_{\bar 0}\oplus \mathfrak{pgl}_{2|2}(\mathbb{C})_{\bar 1}\oplus\langle \eta\frac{\partial}{\partial\xi},\ \xi\frac{\partial}{\partial\eta}\rangle \tag{11}$$

as $\mathfrak{gl}_{2|2}(\mathbb{C})_{\bar 0}$-modules.

Let us give an explicit description of the Lie superalgebra of holomorphic vector fields on $\mathbf{Gr}_{2|2,1|2}$, case 1.4 of Theorem 3 in the following local chart

$$\begin{pmatrix} x & \xi_1 & \xi_2\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{12}$$

The definition of the Lie superalgebra $\tilde{\mathbf{H}}_4$ can be found in [6]. For completeness we remind it here. We have $\tilde{\mathbf{H}}_4\subset \operatorname{Der}\bigwedge(\theta_1,\dots,\theta_4)$ and $\tilde{\mathbf{H}}_4$ consists of all elements in the form:

$$D_f=\sum_{i=1}^{4}\frac{\partial f}{\partial\theta_i}\frac{\partial}{\partial\theta_i},\quad f\in\bigwedge(\theta_1,\dots,\theta_4),\quad f(0)=0.$$

The Lie superalgebra $\tilde{\mathbf{H}}_4$ is $\mathbb{Z}$-graded and in chosen chart the image of an injective homomorphism $\tilde{\mathbf{H}}_4\to\mathfrak{v}(\mathbf{Gr}_{2|2,1|2})$ is given by the following vector fields:

$$\begin{aligned}
(\tilde{\mathbf{H}}_4)_{-1}&=\Big\langle \frac{\partial}{\partial\xi_1},\quad \frac{\partial}{\partial\xi_2},\quad x\frac{\partial}{\partial\xi_1},\quad x\frac{\partial}{\partial\xi_2}\Big\rangle;\\
(\tilde{\mathbf{H}}_4)_{0}&=\Big\langle \frac{\partial}{\partial x},\quad x\frac{\partial}{\partial x}+\xi_1\frac{\partial}{\partial\xi_1},\quad x\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial\xi_2},\quad \xi_1\frac{\partial}{\partial\xi_2},\\
&\qquad\quad \xi_2\frac{\partial}{\partial\xi_1},\quad x\xi_1\frac{\partial}{\partial\xi_1}+x\xi_2\frac{\partial}{\partial\xi_2}+x^2\frac{\partial}{\partial x}\Big\rangle;\\
(\tilde{\mathbf{H}}_4)_{1}&=\Big\langle \xi_1\frac{\partial}{\partial x},\quad \xi_2\frac{\partial}{\partial x},\quad x\xi_1\frac{\partial}{\partial x}+\xi_1\xi_2\frac{\partial}{\partial\xi_2},\quad x\xi_2\frac{\partial}{\partial x}-\xi_1\xi_2\frac{\partial}{\partial\xi_1}\Big\rangle;\\
(\tilde{\mathbf{H}}_4)_{2}&=\Big\langle \theta=\xi_1\xi_2\frac{\partial}{\partial x}\Big\rangle;
\end{aligned} \tag{13}$$

The $\mathbb{Z}$-graded operator mentioned in Theorem 3 is given by:

$$z=\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}.$$

We will call the super-Grassmannians from 1.1–1.4 of Theorem 3 *exceptional*. Note that the super-Grassmannian $\mathbf{Gr}_{0|n,0|l} \simeq \mathbf{Gr}_{n|0,l|0}$ is just usual Grassmannians isomorphic to $\mathbf{Gr}_{n,l}$. It is well-known that

$$\mathfrak{v}(\mathbf{Gr}_{n,l}) \simeq \mathfrak{pgl}_n(\mathbf{C}),$$

see [1] for details.

3.2. Vector fields on flag supermanifolds. Main case

Assume that $r > 1$. From now on we use the following notations:

$$\mathcal{M} = \mathbf{F}^{m|n}_{k|l}, \quad \mathcal{B} = \mathbf{Gr}_{m|n,k_1|l_1} \quad \text{and} \quad \mathcal{S} = \mathbf{F}^{k_1|l_1}_{k'|l'},$$

where $k' = (k_2, \dots, k_r)$ and $l' = (l_2, \dots, l_r)$. If $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$, then by Proposition 1 the projection of the superbundle $\mathcal{M} \to \mathcal{B}$ determines the homomorphism of Lie superalgebras

$$\mathcal{P} : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\mathcal{B}).$$

This projection is $\mathrm{GL}_{m|n}(\mathbb{C})$-equivariant. Hence for the natural Lie superalgebra homomorphisms $\mu : \mathfrak{gl}_{m|n}(\mathbb{C}) \to \mathfrak{v}(\mathcal{M})$ and $\mu_{\mathcal{B}} : \mathfrak{gl}_{m|n}(\mathbb{C}) \to \mathfrak{v}(\mathcal{B})$ we have

$$\mu_{\mathcal{B}} = \mathcal{P} \circ \mu.$$

By Theorem 3, the homomorphisms $\mu_{\mathcal{B}}$ and hence the homomorphism $\mathcal{P}$ is almost always surjective. We will prove that $\mathcal{P}$ is injective. Hence,

$$\mu = \mathcal{P}^{-1} \circ \mu_{\mathcal{B}} \tag{14}$$

is surjective and

$$\mathfrak{v}(\mathcal{M}) \simeq \mathfrak{gl}_{m|n}(\mathbb{C}) / \langle E_{m+n} \rangle.$$

In previous section we constructed a locally free sheaf $\tilde{\mathcal{W}}$ on $\mathcal{B}_0$. The sheaf $\mathcal{W}$ possesses the natural action of the Lie group $G = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, because G is the underlying space of $\mathrm{GL}_{m|n}(\mathbb{C})$. This action preserves the filtration (5) and induces the action in the sheaf $\tilde{\mathcal{W}}$. Hence the vector bundle $\mathbf{W}_0 \to \mathcal{B}_0 = G/H$, where H is the stationary subgroup of a point $o \in \mathcal{B}_0$, corresponding to the locally free sheaf $\tilde{\mathcal{W}}_0$ is homogeneous. Our goal now is to give a description of the homogeneous bundle $\mathbf{W}_0$. Such bundles are determined by the representation of H in the fiber $(\mathbf{W}_0)_o$, see [1] for details.

Consider the local chart on the super-Grassmannian $\mathcal{B}$ corresponding to

$$I_{1\bar{0}} = \{m - k_1 + 1, \dots, m\} \quad \text{and} \quad I_{1\bar{1}} = \{n - l_1 + 1, \dots, n\}. \tag{15}$$

The coordinate matrix Z_{I_1} in this case has the following form

$$Z_{I_1} = \begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ \mathrm{H}_1 & Y_1 \\ 0 & E_{l_1} \end{pmatrix}. \tag{16}$$

Denote by o the point in $\mathcal{B}_0$ defined by the following equations:

$$X_1 = Y_1 = \Xi_1 = \mathrm{H}_1 = 0.$$

Then $\mathcal{B}_0$ is naturally isomorphic to G/H, where H is the stabilizer of o. An easy computation shows that H contains all matrices in the following form:

$$\begin{pmatrix} A_1 & 0 & 0 & 0 \\ C_1 & B_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & C_2 & B_2 \end{pmatrix}, \tag{17}$$

where

$$A_1 \in \mathrm{GL}_{m-k_1}(\mathbb{C}),\ A_2 \in \mathrm{GL}_{n-l_1}(\mathbb{C}),\ B_1 \in \mathrm{GL}_{k_1}(\mathbb{C}) \text{ and } B_2 \in \mathrm{GL}_{l_1}(\mathbb{C}).$$

The reductive part R of H is given by the following equations

$$C_i = 0, \quad i = 1, 2.$$

Let us compute the representation ψ of H in the fiber $(\mathbf{W}_0)_o$ of $\mathbf{W}_0$ over the point o. We identify $(\mathbf{W}_0)_o$ with the Lie superalgebra of holomorphic vector fields $\mathfrak{v}(\mathcal{S})$, see Proposition 3. Let us choose an atlas on $\mathcal{M}$ defined by $I_1 = (I_{1\bar{0}}, I_{s\bar{1}})$, see (15), and by certain I_s, $s = 2, \dots, r$. In notations (16) and (17) the group H acts in the chart on the super-Grassmannian $\mathcal{B}$ defined by Z_{I_1} in the following way:

$$\begin{pmatrix} A_1 & 0 & 0 & 0 \\ C_1 & B_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & C_2 & B_2 \end{pmatrix} \begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ \mathrm{H}_1 & Y_1 \\ 0 & E_{l_1} \end{pmatrix} = \begin{pmatrix} A_1 X_1 & A_1 \Xi_1 \\ C_1 X_1 + B_1 & C_1 \Xi_1 \\ A_2 \mathrm{H}_1 & A_2 Y_1 \\ C_2 \mathrm{H}_1 & C_2 Y_1 + B_2 \end{pmatrix}.$$

(Note that a chart on $\mathcal{B}$ is defined only by Z_{I_1}, and a chart on the whole flag supermanifold $\mathcal{M}$ is defined by Z_{I_s}, where $s = 1, \dots, r$.) Further, for Z_{I_2} we have

$$\begin{aligned}&\begin{pmatrix} C_1X_1+B_1 & C_1\Xi_1\\ C_2\,\mathrm{H}_1 & C_2Y_1+B_2\end{pmatrix}\begin{pmatrix} X_2 & \Xi_2\\ \mathrm{H}_2 & Y_2\end{pmatrix}=\\ &=\begin{pmatrix}(C_1X_1+B_1)X_2+C_1\Xi_1\,\mathrm{H}_2 & (C_1X_1+B_1)\Xi_2+C_1\Xi_1Y_2\\ C_2\,\mathrm{H}_1\,X_2+(C_2Y_1+B_2)\,\mathrm{H}_2 & C_2\,\mathrm{H}_1\,\Xi_2+(C_2Y_1+B_2)Y_2\end{pmatrix}.\end{aligned} \tag{18}$$

Note that the local coordinates of Z_{I_s}, $s \geq 2$, can be interpreted as local coordinates on the fiber $\mathcal{S}$ of the superbundle $\mathcal{M}$. To obtain the action of H in the fiber $(\mathbf{W}_0)_o$ in these coordinates we put

$$X_1 = Y_1 = 0 \quad \text{and} \quad \Xi_1 = \mathrm{H}_1 = 0$$

in (18) and modify Z_{I_s}, $s \geq 3$, accordingly. We see that the nilradical of H and the subgroup $\mathrm{GL}_{m-k_1}(\mathbb{C}) \times \mathrm{GL}_{n-l_1}(\mathbb{C})$ in R act trivially on $\mathcal{S}$ and that the subgroup $\mathrm{GL}_{k_1}(\mathbb{C}) \times \mathrm{GL}_{l_1}(\mathbb{C}) \subset R$ acts in the natural way. In other words the action of H in $\mathcal{S}$ over o is given by the following formulas:

$$\begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}\begin{pmatrix} X_2 & \Xi_2\\ \mathrm{H}_2 & Y_2\end{pmatrix}=\begin{pmatrix} B_1X_2 & B_1\Xi_2\\ B_2\,\mathrm{H}_2 & B_2Y_2\end{pmatrix}. \tag{19}$$

This means that H acts as the underlying space of the Lie supergroup $\mathrm{GL}_{k_1|l_1}(\mathbb{C})$ on the flag supermanifold $\mathcal{S}$, see (4). Furthermore assume that

$$\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{gl}_{k_1|l_1}(\mathbb{C})/\langle E_{k_1+l_1}\rangle = \left\{\begin{pmatrix} Z_1 & T_1\\ T_2 & Z_2\end{pmatrix} + < E_{k_1+l_1} >\right\},$$

where $Z_1 \in \mathfrak{gl}_{k_1}(\mathbb{C})$ and $Z_2 \in \mathfrak{gl}_{l_1}(\mathbb{C})$. Then the induced action of the Lie group $\mathrm{GL}_{k_1}(\mathbb{C}) \times \mathrm{GL}_{l_1}(\mathbb{C})$ on $(\mathbf{W}_0)_o = \mathfrak{v}(\mathcal{S})$ coincides with the adjoint action of the underlying Lie group of the Lie supergroup $\mathrm{GL}_{k_1|l_1}(\mathbb{C})$. More precisely, we have

$$\begin{aligned}&\begin{pmatrix} B_1 & 0\\ 0 & B_2\end{pmatrix}\left(\begin{pmatrix} Z_1 & T_1\\ T_2 & Z_2\end{pmatrix} + < E_{k_1+l_1} >\right)\begin{pmatrix} B_1^{-1} & 0\\ 0 & B_2^{-1}\end{pmatrix}=\\ &\begin{pmatrix} B_1Z_1B_1^{-1} & B_1T_1B_2^{-1}\\ B_2T_2B_1^{-1} & B_2Z_2B_2^{-1}\end{pmatrix} + < E_{k_1+l_1} >,\end{aligned} \tag{20}$$

where $B_1 \in \mathrm{GL}_{k_1}(\mathbb{C})$ and $B_2 \in \mathrm{GL}_{l_1}(\mathbb{C})$.

Denote by Ad_{k_1} and Ad_{l_1} the adjoint representations of $\mathrm{GL}_{k_1}(\mathbb{C})$ and $\mathrm{GL}_{l_1}(\mathbb{C})$ on $\mathfrak{sl}_{k_1}(\mathbb{C})$ and $\mathfrak{sl}_{l_1}(\mathbb{C})$, respectively, and by ρ_{k_1} and ρ_{l_1} the standard representations of $\mathrm{GL}_{k_1}(\mathbb{C})$ and $\mathrm{GL}_{l_1}(\mathbb{C})$ in $\mathbb{C}^{k_1}$ and $\mathbb{C}^{l_1}$, respectively. We denote by 1 the one dimensional trivial representation of $\mathrm{GL}_{k_1}(\mathbb{C}) \times \mathrm{GL}_{l_1}(\mathbb{C})$. The following lemma follows from (20).

Lemma 1. *The representation ψ of H in the fiber $(\mathbf{W}_0)_o = \mathfrak{v}(\mathcal{S})$ is completely reducible. The nilradical of H acts trivially in $(\mathbf{W}_0)_o$. If $\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{gl}_{k_1|l_1}(\mathbb{C})/\langle E_{k_1+l_1}\rangle$, then*

$$\psi|R = \begin{cases} \mathrm{Ad}_{k_1} + \mathrm{Ad}_{l_1} + \rho_{k_1} \otimes \rho^*_{l_1} + \rho_{l_1} \otimes \rho^*_{k_1} + 1 \ \textit{for}\ k_1, l_1 > 0, \\ \mathrm{Ad}_{k_1} \ \ \textit{for}\ k_1 > 0,\ l_1 = 0, \\ \mathrm{Ad}_{l_1} \ \ \textit{for}\ k_1 = 0,\ l_1 > 0. \end{cases} \tag{21}$$

Further we will use the chart on $\mathbf{F}^{m|n}_{k|l}$ defined by $I_s = I_{s\bar{0}} \cup I_{s\bar{1}}$, where $I_{1\bar{i}}$ is as above, and

$$I_{s\bar{0}} = \{k_{s-1} - k_s + 1, \dots, k_{s-1}\}, \quad I_{s\bar{1}} = \{l_{s-1} - l_s + 1, \dots, l_{s-1}\}$$

for $s \geq 2$. The coordinate matrices of this chart have the following form

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ E_{k_s} & 0 \\ \mathrm{H}_s & Y_s \\ 0 & E_{l_s} \end{pmatrix}, \quad s = 1, \dots k,$$

where again the local coordinates are

$$X_s = (x^s_{ij}),\ Y_s = (y^s_{ij}),\ \Xi_s = (\xi^s_{ij}) \text{ and } \mathrm{H}_s = (\eta^s_{ij}).$$

We denote this chart by $\mathcal{U}$ and the corresponding chart on $\mathcal{B}$ by $\mathcal{U}_{\mathcal{B}}$. In other words, $\mathcal{U}_{\mathcal{B}}$ is given by the coordinate matrix (16).

Lemma 2. *The vector fields* $\frac{\partial}{\partial \xi^1_{ij}}$ *and* $\frac{\partial}{\partial \eta^1_{ij}}$ *are fundamental. That is, they are induced by the natural action of* $\mathrm{GL}_{m|n}(\mathbb{C})$ *on* $\mathcal{M}$.

Proof. Let us prove this statement for example for the vector field $\frac{\partial}{\partial \xi^1_{11}}$. This vector field corresponds to the one-parameter subgroup $\exp(\tau E_{1,a})$, where $a = m + n - l_1 + 1$ and τ is an odd parameter. Indeed, the action of this subgroup is given by

$$\begin{pmatrix} X_1 & \Xi_1 \\ E_{k_1} & 0 \\ \mathrm{H}_1 & Y_1 \\ 0 & E_{l_1} \end{pmatrix} \mapsto \begin{pmatrix} X_1 & \tilde{\Xi}_1 \\ E_{k_1} & 0 \\ \mathrm{H}_1 & Y_1 \\ 0 & E_{l_1} \end{pmatrix} \quad \text{and} \quad Z_{I_s} \mapsto Z_{I_s},\ s \geq 2,$$

where

$$\tilde{\Xi}_1 = \begin{pmatrix} \tau + \xi^1_{11} & \dots & \xi^1_{1l_1} \\ \vdots & \ddots & \vdots \\ \xi^1_{m-k_1,1} & \dots & \xi^1_{m-k_1,l_1} \end{pmatrix}. \qquad \square$$

Let us choose a basis v_i, where $i = 1, \dots \dim(\mathfrak{v}(\mathcal{S}))$, in $\mathfrak{v}(\mathcal{S})$. Any holomorphic vertical vector field on $\mathcal{M}$ can be written uniquely in the form

$$w = \sum_q f_q v_q, \tag{22}$$

where f_q are holomorphic functions on $\mathcal{U}$ depending only on coordinates from Z_{I_1}. We will need the following two lemmas:

Lemma 3. *If* $\operatorname{Ker}\mathcal{P} \neq \{0\}$, *then* $\dim \mathcal{W}_{(0)}(\mathcal{B}_0) > \dim \mathcal{W}_{(1)}(\mathcal{B}_0)$.

Note that since $\mathcal{B}_0$ is compact, $\dim \mathcal{W}_{(i)}(\mathcal{B}_0) < \infty$ for all i.

Proof. By definition we have the inclusion of sheaves $\mathcal{W}_{(1)} \hookrightarrow \mathcal{W}_{(0)}$ and hence we have the inclusion of the vector spaces of global sections

$$\mathcal{W}_{(1)}(\mathcal{B}_0) \hookrightarrow \mathcal{W}_{(0)}(\mathcal{B}_0).$$

Therefore we need to show that there exists a vector field $v \in \mathcal{W}_{(0)}(\mathcal{B}_0)$ such that $v \notin \mathcal{W}_{(1)}(\mathcal{B}_0)$. Consider a vector field $w \in \mathcal{W}_{(1)}(\mathcal{B}_0)$ written in the form (22). Assume that there is a function f_q that depends for example on odd coordinate ξ^1_{ij}. Then $w = \xi^1_{ij} w' + w''$, where w' and w'' are local vertical vector fields and their coefficients (22) do not depend on ξ^1_{ij}, and $w' \neq 0$. Using Lemma 2 and the fact that $\operatorname{Ker}\mathcal{P}$ is an ideal in $\mathfrak{v}(\mathcal{M})$, we see that

$$w' = [\frac{\partial}{\partial \xi^1_{ij}}, w] \in \operatorname{Ker}\mathcal{P}.$$

In particular, w' is a global vertical vector field. In this way we can exclude all odd coordinates ξ^1_{ij} and η^1_{ij}. Therefore there exists a vector field v from $\operatorname{Ker}\mathcal{P}$ such that $v \in \mathcal{W}_{(0)}(\mathcal{B}_0)$ but $v \notin \mathcal{W}_{(1)}(\mathcal{B}_0)$. □

Lemma 4. *Assume that* $\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{gl}_{k_1|l_1}(\mathbb{C})/\langle E_{k_1+l_1}\rangle$. *Then we have*

$$\widetilde{\mathcal{W}_0}(\mathcal{B}_0) \simeq \begin{cases} \mathbb{C}, & 0 < k_1 < m,\ 0 < l_1 < n; \\ \mathfrak{r}_1 \oplus \mathfrak{r}_2 \oplus \mathbb{C}, & 1 < k_1 = m,\ 0 < l_1 < n; \\ \mathfrak{r}_3 \oplus \mathfrak{r}_4 \oplus \mathbb{C}, & 0 < k_1 < m,\ 1 < l_1 = n; \\ \mathfrak{r}_2 \oplus \mathbb{C}, & 1 = k_1 = m,\ 0 < l_1 < n; \\ \mathfrak{r}_3 \oplus \mathbb{C}, & 0 < k_1 < m,\ 1 = l_1 = n; \\ \{0\}, & 0 < k_1 < m,\ 0 = l_1 \leq n,\ \textit{or} \\ & 0 = k_1 \leq m,\ 0 < l_1 < n,\ \textit{or} \\ & 0 = k_1 < m,\ 1 = l_1 \leq n,\ \textit{or} \\ & 1 = k_1 \leq m,\ 0 = l_1 < n; \\ \mathfrak{r}_1, & 1 < k_1 = m,\ 0 = l_1 < n; \\ \mathfrak{r}_4, & 0 = k_1 < m,\ 1 < l_1 = n, \end{cases} \tag{23}$$

where $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}_3$, $\mathfrak{r}_4$ *are irreducible* $\mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$*-modules with the highest weights* $\mu_1 - \mu_m$, $\mu_1 - \lambda_n$, $\lambda_1 - \mu_m$ *and* $\lambda_1 - \lambda_n$ *respectively. The trivial* 1*-dimensional module* $\mathbb{C}$ *corresponds to the highest weight* 0*.*

Proof. We compute the vector space of global sections of $\mathbf{W}_0$ using the Borel–Weil–Bott Theorem 1. The representation ψ of H in $(\mathbf{W}_0)_o$ is described in Lemma 1. From (21) it follows that the highest weights of ψ have the form:

- $\mu_{m-k_1+1} - \mu_m$, $\mu_{m-k_1+1} - \lambda_n$, $\lambda_{n-l_1+1} - \mu_m$, $\lambda_{n-l_1+1} - \lambda_n$, 0 for $k_1, l_1 > 1$;
- $\mu_m - \lambda_n$, $\lambda_{n-l_1+1} - \mu_m$, $\lambda_{n-l_1+1} - \lambda_n$, 0 for $k_1 = 1$, $l_1 > 1$;
- $\mu_{m-k_1+1} - \mu_m$, $\mu_{m-k_1+1} - \lambda_n$, $\lambda_n - \mu_m$, 0 for $k_1 > 1$, $l_1 = 1$;
- $\mu_m - \lambda_n$, $\lambda_n - \mu_m$, 0 for $k_1 = 1$, $l_1 = 1$;
- $\mu_{m-k_1+1} - \mu_m$ for $k_1 > 1$, $l_1 = 0$;
- $\lambda_{n-l_1+1} - \lambda_n$ for $k_1 = 0$, $l_1 > 1$.

(Note that for $k_1 = 1$, $l_1 = 0$ and $k_1 = 0$, $l_1 = 1$ the representation space of ψ is trivial.) Therefore the dominant highest weights of ψ have the following form:

- 0, if $0 < k_1 < m$ and $0 < l_1 < n$;
- 0, $\mu_1 - \mu_m$, $\mu_1 - \lambda_n$, if $1 < k_1 = m$, $0 < l_1 < n$;
- 0, $\mu_1 - \lambda_n$, if $1 = k_1 = m$, $0 < l_1 < n$;
- 0, $\lambda_1 - \lambda_n$, $\lambda_1 - \mu_m$, if $0 < k_1 < m$, $1 < l_1 = n$;
- 0, $\lambda_1 - \mu_m$, if $0 < k_1 < m$, $1 = l_1 = n$;
- $\mu_1 - \mu_m$, if $1 < k_1 = m$, $0 = l_1 < n$;
- $\lambda_1 - \lambda_n$, if $0 = k_1 < m$, $1 < l_1 = n$.

We have no dominant weights in the following cases:

- $0 < k_1 < m$, $0 = l_1 \le n$;
- $0 = k_1 \le m$, $0 < l_1 < n$;
- $0 = k_1 < m$, $1 = l_1 \le n$;
- $1 = k_1 \le m$, $0 = l_1 < n$.

By Borel–Weil–Bott Theorem we get the result. □

We are ready to prove the following theorem.

Theorem 4. *Assume that* $r > 1$*. If*

$$\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}, \ \mathfrak{v}(\mathcal{S}) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}), \ (k_1, l_1) \neq (m, 0) \text{ and } (k_1, l_1) \neq (0, n),$$

then $\operatorname{Ker} \mathcal{P} = \{0\}$.

Proof. Consider the super-stabilizer $\mathcal{H} \subset \mathrm{GL}_{m|n}(\mathbb{C})$ of o. It contains all super-matrices of the following form:

$$\begin{pmatrix} A_1 & 0 & * & 0 \\ C_1 & B_1 & * & D_1 \\ * & 0 & A_2 & 0 \\ * & D_2 & C_2 & B_2 \end{pmatrix}, \tag{24}$$

where the size of all matrices is as in formula (17). Consider also the following Lie subsupergroup $\mathcal{L}$ in $\mathcal{H}$:

$$\begin{pmatrix} B_1 & D_1 \\ D_2 & B_2 \end{pmatrix}.$$

Clearly, $\mathcal{L} \simeq \mathrm{GL}_{k_1|l_1}(\mathbb{C})$. Repeating computations (18) for the super-matrix (24), we see that $\mathcal{L}$ acts on $\mathcal{S}$ in the natural way, see (4), and the $\mathfrak{l}$-module $(\mathbf{W}_0)_o \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$ is isomorphic to the adjoint $\mathfrak{l}$-module. Here $\mathfrak{l} \simeq \mathfrak{gl}_{k_1|l_1}(\mathbb{C})$ is the Lie superalgebra of $\mathcal{L}$.

Let $\pi : \mathcal{W} \to \widetilde{\mathcal{W}_0} = \mathcal{W}/\mathcal{W}_{(1)}$ be the natural map and $\pi_o : \mathcal{W} \to (\mathbf{W}_0)_o$ be the composition of π and of the evaluation map at the point o. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{W}(\mathcal{B}_0) \arrow[r, "{[X,\,\cdot\,]}"] \arrow[d, "\pi_o"'] & \mathcal{W}(\mathcal{B}_0) \arrow[d, "\pi_o"'] \\ (\mathbf{W}_0)_o \arrow[r, "{[X,\,\cdot\,]}"] & (\mathbf{W}_0)_o \end{tikzcd},$$

where $X \in \mathfrak{l}$. (Note that the vector space $\mathcal{W}(\mathcal{B}_0)$ is an ideal in $\mathfrak{v}(\mathcal{M})$ and in particular it is invariant with respect to the action of $\mathcal{L}$.) Denote by V the image $\pi_o(\mathcal{W}(\mathcal{B}_0))$. From the commutativity of this diagram it follows that

$$V \subset (\mathbf{W}_0)_o \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$$

is invariant with respect to the adjoint representation of $\mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$. Therefore, V is an ideal in $\mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$.

Let us describe ideals of the Lie superalgebra $\mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$, where $(k_1, l_1) \neq (1, 1)$, see [6] for details. (The Lie superalgebra $\mathfrak{pgl}_{1|1}(\mathbb{C})$ is nilpotent. We do not consider this case here because $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) \neq \mathbb{C}$ for $\mathcal{S} = \mathbf{F}^{1|1}_{k'|l'}$.) This Lie superalgebra contains two trivial ideals $I = \{0\}$, $\mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$ and it has one proper ideal

$$\mathfrak{psl}_{k_1|k_1}(\mathbb{C}) = \mathfrak{sl}_{k_1|k_1}(\mathbb{C})/\langle E_{2k_1}\rangle$$

for $k_1 = l_1$.

Clearly, we have $V \subset \mathrm{Im}(\gamma)$, where $\gamma : \widetilde{\mathcal{W}}_0(\mathcal{B}_0) \to (\mathbf{W}_0)_o$ is the evaluation map. By Lemma 4, we see that $\mathrm{Im}(\gamma)$ never coincides with $\mathfrak{pgl}_{k_1|l_1}(\mathbb{C})$ or $\mathfrak{psl}_{k_1|k_1}(\mathbb{C})$. Hence, $V = \{0\}$. In other words, all sections of $\pi(\mathcal{W}(\mathcal{B}_0))$ are equal to 0 at the point o. Since $\mathbf{W}_0$ is a homogeneous bundle, we get that $\pi(\mathcal{W}(\mathcal{B}_0))$ are equal to 0 at any point. Therefore, $\pi(\mathcal{W}(\mathcal{B}_0)) = \{0\}$ and

$$\mathcal{W}(\mathcal{B}_0)_{(0)} \simeq \mathcal{W}(\mathcal{B}_0)_{(1)}.$$

From Lemma 3 it follows that $\operatorname{Ker}\mathcal{P} = \{0\}$. □

Using Theorem 4 and formula (14), we get the following statement:

Theorem 5. *Assume that $r > 1$. If*

$$\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C},\ \mathfrak{v}(\mathbf{F}^{m|n}_{k_1|l_1}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}) \ \text{ and } \ \mathfrak{v}(\mathbf{F}^{k_1|l_1}_{k'|l'}) \simeq \mathfrak{pgl}_{k_1|l_1}(\mathbb{C}),$$

then

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

3.3. Vector fields on flag supermanifolds, some exceptional cases

3.3.1. The base $\mathcal{B}$ is an exceptional super-Grassmannian

Assume that $r > 1$, $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$ and $\mathcal{B} = \mathbf{F}^{m|n}_{k_1|l_1}$ is one of the following super-Grassmannians:

a) $\mathbf{F}^{m|n}_{k_1|l_1} = \mathbf{F}^{m|n}_{0|n}$ or $\mathbf{F}^{m|n}_{m|0}$, case 1.3 of Theorem 3.

b) $\mathbf{F}^{m|n}_{k_1|l_1} = \mathbf{F}^{2|2}_{1|2}$ or $\mathbf{F}^{2|2}_{2|1}$, case 1.4 of Theorem 3. (We do not consider super-Grassmannians $\mathbf{F}^{2|2}_{1|0}$ and $\mathbf{F}^{2|2}_{0|1}$ here, because in these cases $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) \neq \mathbb{C}$.)

c) $\mathbf{F}^{m|n}_{k_1|l_1} = \mathbf{F}^{1|n}_{0|n-1}$, where $n > 2$, or $\mathbf{F}^{m|1}_{m-1|0}$, where $m > 2$, case 1.2 of Theorem 3. This case we will consider in a separate paper.

d) $\mathbf{F}^{m|n}_{k_1|l_1} = \mathbf{F}^{2|2}_{1|1}$, case 1.1 of Theorem 3. In this case $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) \neq \mathbb{C}$. We do not consider this case here.

Case a. Without loss of generality we may consider only the case $\mathbf{F}^{m|n}_{k_1|l_1} = \mathbf{F}^{m|n}_{0|n}$. In this case the base space $\mathbf{F}^{m|n}_{0|n}$ is a superpoint, i.e. it is a superdomain with the underlying space $\{\mathtt{pt}\}$, one point, and with mn odd coordinates. Since $\mathbf{F}^{m|n}_{k|l}$ is a superbundle with the base space isomorphic to a superpoint, we have

$$\mathbf{F}^{m|n}_{k|l} = \mathbf{F}^{m|n}_{0|n} \times \mathbf{F}^{0|n}_{k'|l'}, \ \text{ where } k' = (0, \dots, 0) \ \text{ and } \ l' = (l_2, \dots, l_r).$$

Our goal now is to prove the following theorem.

Theorem 6. *Assume that $r > 1$ and $(k_1, l_1) = (m, 0)$ or $(k_1, l_1) = (0, n)$. Then*

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) = W_{mn} \inplus \Big(\bigwedge(mn) \otimes \mathfrak{pgl}_n(\mathbb{C}) \Big),$$

where $W_{mn} = \mathrm{Der}(\bigwedge(mn))$.

Proof. The result follows from the following facts:

$$\mathbf{F}^{m|n}_{k|l} = \mathbf{F}^{m|n}_{0|n} \times \mathbf{F}^{0|n}_{k'|l'}, \quad \mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}, \quad \mathcal{O}_{\mathcal{B}}(\mathcal{B}_0) = \bigwedge(mn),$$
$$\mathfrak{v}(\mathbf{F}^{m|n}_{0|n}) \simeq W_{mn}, \quad \mathfrak{v}(\mathbf{F}^{0|n}_{0|l'}) \simeq \mathfrak{pgl}_n(\mathbb{C}).$$

In more details, since $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$, we have a Lie superalgebra homomorphism

$$\mathcal{P} : \mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \to \mathfrak{v}(\mathbf{F}^{m|n}_{0|n}) \simeq W_{mn}.$$

Since the bundle projection $\mathbf{F}^{m|n}_{k|l} \to \mathbf{F}^{m|n}_{0|n}$ is just the projection to the first factor

$$\mathbf{F}^{m|n}_{k|l} = \mathbf{F}^{m|n}_{0|n} \times \mathbf{F}^{0|n}_{k'|l'} \to \mathbf{F}^{m|n}_{0|n},$$

all vector fields on $\mathbf{F}^{m|n}_{0|n}$ can be lifted to $\mathbf{F}^{m|n}_{k|l}$. The kernel of $\mathcal{P}$ is isomorphic to $\bigwedge(mn) \otimes \mathfrak{pgl}_n(\mathbb{C})$. The proof is complete. □

Case b. Assume that $r = 2$. Without loss of generality we may consider only the case $\mathbf{F}^{m|n}_{k_1|l_1} = \mathbf{F}^{2|2}_{1|2}$. Under restriction $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$ the fiber $\mathcal{S}$ can be one of the following super-Grassmannians:

$$\mathcal{S} = \mathbf{F}^{1|2}_{1|1} \text{ or } \mathbf{F}^{1|2}_{0|1}.$$

We have seen that $\mathfrak{v}(\mathbf{F}^{2|2}_{1|2}) \simeq \tilde{\mathbf{H}}_4 \niplus \langle z \rangle$, see (13), Theorem 3. A standard computation shows that the image of $\mathfrak{gl}_{2|2}(\mathbb{C})$ in $\mathfrak{v}(\mathbf{F}^{2|2}_{1|2})$ is

$$(\tilde{\mathbf{H}}_4)_{-1} \oplus (\tilde{\mathbf{H}}_4)_0 \oplus (\tilde{\mathbf{H}}_4)_1 \oplus \langle z \rangle \simeq \mathfrak{pgl}_{2|2}(\mathbb{C}).$$

Therefore,

$$\mathfrak{v}(\mathbf{F}^{2|2}_{1|2}) \simeq \mathfrak{pgl}_{2|2}(\mathbb{C}) \oplus \langle \theta \rangle, \tag{25}$$

as vector superspaces. (See (13) for the definition of θ.) By Theorem 2 we have $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$. Hence by Proposition 1 we have a homomorphism of Lie superalgebras

$$\mathcal{P} : \mathfrak{v}(\mathbf{F}^{2|2}_{k|l}) \to \mathfrak{v}(\mathbf{F}^{2|2}_{1|2}).$$

By Theorem 3 we see that $\mathfrak{v}(\mathcal{S}) \simeq \mathfrak{pgl}_{1|2}(\mathbb{C})$. Therefore by Theorem 4 the homomorphism $\mathcal{P}$ is injective. The vector fields from $\mathfrak{pgl}_{2|2}(\mathbb{C})$ are fundamental with respect to the action of the Lie superalgebra $\mathfrak{gl}_{2|2}(\mathbb{C})$. Hence they can be lifted to the flag supermanifold $\mathbf{F}^{2|2}_{k|l}$. Therefore we need to find $\mathcal{P}^{-1}(\theta)$. We will show that $\theta \notin \operatorname{Im}(\mathcal{P})$, i.e. θ cannot be lifted to $\mathbf{F}^{2|2}_{k|l}$.

Theorem 7. *We have*

$$\mathfrak{v}(\mathbf{F}^{2|2}_{(1,1)|(2,1)}) \simeq \mathfrak{pgl}_{2|2}(\mathbb{C}) \text{ and } \mathfrak{v}(\mathbf{F}^{2|2}_{(1,0)|(2,1)}) \simeq \mathfrak{pgl}_{2|2}(\mathbb{C}).$$

Proof. Consider the following chart on $\mathbf{F}^{2|2}_{(1,1)|(2,1)}$:

$$Z_{I_1} = \begin{pmatrix} x & \xi_1 & \xi_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z_{I_2} = \begin{pmatrix} 1 & 0 \\ \eta & y \\ 0 & 1 \end{pmatrix} \tag{26}$$

Assume that $w := \mathcal{P}^{-1}(\theta)$ is well-defined. Since all vector fields on $\mathbf{F}^{2|2}_{(1,1)|(2,1)}$ are projectable, in coordinates (26) w is equal to $\theta + v$, where $v = f\frac{\partial}{\partial y} + g\frac{\partial}{\partial \eta}$ is a vertical vector field and f, g are holomorphic functions in coordinates (26). Let us find f and g. We need the following fundamental vector fields on $\mathbf{F}^{2|2}_{(1,1)|(2,1)}$ written in coordinates (26):

$$\begin{gathered} E_{13} \longmapsto \frac{\partial}{\partial \xi_1}, \quad E_{14} \longmapsto \frac{\partial}{\partial \xi_2}, \quad E_{42} \longmapsto \xi_2\frac{\partial}{\partial x} + y\frac{\partial}{\partial \eta}, \\ E_{32} \longmapsto \xi_1\frac{\partial}{\partial x} - \frac{\partial}{\partial \eta}, \quad E_{34} \longmapsto -\xi_1\frac{\partial}{\partial \xi_2} - \frac{\partial}{\partial y}. \end{gathered} \tag{27}$$

Here we denote by E_{ij} the elementary matrix from $\mathfrak{gl}_{2|2}(\mathbb{C})$.

Since $\operatorname{Ker}\mathcal{P} = \{0\}$, using (27), we get

$$\begin{aligned} [\frac{\partial}{\partial \xi_1}, w] &= \xi_2\frac{\partial}{\partial x} + \frac{\partial f}{\partial \xi_1}\frac{\partial}{\partial y} + \frac{\partial g}{\partial \xi_1}\frac{\partial}{\partial \eta} = \xi_2\frac{\partial}{\partial x} + y\frac{\partial}{\partial \eta}; \\ [\frac{\partial}{\partial \xi_2}, w] &= -\xi_1\frac{\partial}{\partial x} + \frac{\partial f}{\partial \xi_2}\frac{\partial}{\partial y} + \frac{\partial g}{\partial \xi_2}\frac{\partial}{\partial \eta} = -\xi_1\frac{\partial}{\partial x} + \frac{\partial}{\partial \eta}. \end{aligned}$$

Hence,

$$\frac{\partial f}{\partial \xi_1} = 0, \quad \frac{\partial g}{\partial \xi_1} = y, \quad \frac{\partial f}{\partial \xi_2} = 0, \quad \frac{\partial g}{\partial \xi_2} = 1.$$

Furthermore,

$$[\xi_1\frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial y}, w] = \xi_1\frac{\partial}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial}{\partial \eta} = 0.$$

Hence, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial g}{\partial y} = -\xi_1$. Now we see that

$$\frac{\partial^2 g}{\partial \xi_1 \partial y} = -1, \quad \frac{\partial^2 g}{\partial y \partial \xi_1} = 1.$$

This is a contradiction. Therefore,

$$\mathcal{P}^{-1}(z) = \emptyset \text{ and } \mathfrak{v}(\mathbf{F}^{2|2}_{(1,1)|(2,1)}) \simeq \mathfrak{pgl}_{2|2}(\mathbb{C}).$$

The proof in the case $\mathbf{F}^{2|2}_{(1,0)|(2,1)}$ is similar. □

3.3.2. The fiber $\mathcal{S}$ is an exceptional super-Grassmannian

Assume that $r = 2$, $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) = \mathbb{C}$ and $\mathcal{S} = \mathbf{F}^{k_1|l_1}_{k_2|l_2}$ is one of the following super-Grassmannians:

a) $\mathcal{S} = \mathbf{F}^{2|2}_{1|1}$, case 1.1 of Theorem 3;
b) $\mathcal{S} = \mathbf{F}^{2|2}_{0|1}$, $\mathbf{F}^{2|2}_{1|0}$, $\mathbf{F}^{2|2}_{1|2}$ or $\mathbf{F}^{2|2}_{2|1}$, case 1.4 of Theorem 3;
c) $\mathcal{S} = \mathbf{F}^{1|l_1}_{0|l_1-1}$, $\mathbf{F}^{k_1|1}_{k_1-1|0}$, $\mathbf{F}^{1|l_1}_{1|1}$ or $\mathbf{F}^{k_1|1}_{1|1}$, where $n > 2$, case 1.2 of Theorem 3.
d) $\mathcal{S} = \mathbf{F}^{k_1|l_1}_{0|l_1}$ or $\mathbf{F}^{k_1|l_1}_{k_1|0}$, case 1.3 of Theorem 3. In both cases $\mathcal{O}_{\mathcal{S}}(\mathcal{S}_0) \neq \mathbb{C}$. We do not consider this case here.

Our goal now is to prove the following theorem.

Theorem 8. *Assume that $r = 2$ and the fiber $\mathcal{S}$ of the superbundle $\mathbf{F}^{m|n}_{k|l}$ is a super-Grassmannian of type* **a** *or* **b***. Then we have*

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

First of all let us compute the representation ψ of the stabilizer H in these cases. Formula (19) tells us that the action of H in $\mathcal{S}$ coincides with the restriction of this action on $\mathrm{GL}_{2|2}(\mathbb{C})_{\bar{0}}$. We need the following lemma:

Lemma 5. *The representation ψ of H in the fiber $(\mathbf{W}_0)_o$ is completely reducible and its highest weights are:*

1. $\mu_{m-1} - \mu_m$, $\lambda_{n-1} - \lambda_n$, $\mu_{m-1} - \lambda_n$, $\lambda_{n-1} - \mu_m$, 0, $\mu_{m-1} + \mu_m - \lambda_{n-1} - \lambda_n$, $\lambda_{n-1} + \lambda_n - \mu_{m-1} - \mu_m$, *in case* **a**.
2. $\mu_{m-1} - \mu_m$, $\lambda_{n-1} - \lambda_n$, $\mu_{m-1} - \lambda_n$, $\lambda_{n-1} - \mu_m$, 0, $\mu_{m-1} + \mu_m - \lambda_{n-1} - \lambda_n$, *in case* **b**, *super-Grassmannians* $\mathbf{F}^{2|2}_{0|1}$ *and* $\mathbf{F}^{2|2}_{1|2}$.
3. $\mu_{m-1} - \mu_m$, $\lambda_{n-1} - \lambda_n$, $\mu_{m-1} - \lambda_n$, $\lambda_{n-1} - \mu_m$, 0, $-\mu_{m-1} - \mu_m + \lambda_{n-1} + \lambda_n$, *in case* **b**, *super-Grassmannians* $\mathbf{F}^{2|2}_{1|0}$ *and* $\mathbf{F}^{2|2}_{2|1}$.

Proof. As in Section 3.2, we see that the nilradical of H and the subgroup $\mathrm{GL}_{m-2}(\mathbb{C})\times \mathrm{GL}_{n-2}(\mathbb{C})$ in H act trivially on $\mathcal{S}$. The subgroup $\mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})$ acts in the natural way. Consider Decomposition (11). We computed already highest weights of $\mathfrak{gl}_{2|2}(\mathbb{C})_{\bar 0}$-module $\mathfrak{pgl}_{2|2}(\mathbb{C})$. They are

$$\mu_{m-1}-\mu_m,\ \lambda_{n-1}-\lambda_n,\ \mu_{m-1}-\lambda_n,\ \lambda_{n-1}-\mu_m,\ 0. \tag{28}$$

Using the explicit description of $\mathfrak{v}(\mathbf{F}^{2|2}_{1|1})$ given by (8), (9) and (10), we get:

$$[\mu_{m-1}\mu(E_{11})+\mu_m\mu(E_{22})+\lambda_{n-1}\mu(E_{33})+\lambda_n\mu(E_{44}),\xi\frac{\partial}{\partial\eta}] = (\mu_{m-1}+\mu_m-\lambda_{n-1}-\lambda_n)\xi\frac{\partial}{\partial\eta};$$

$$[\mu_{m-1}\mu(E_{11})+\mu_m\mu(E_{22})+\lambda_{n-1}\mu(E_{33})+\lambda_n\mu(E_{44}),\eta\frac{\partial}{\partial\xi}] = (-\mu_{m-1}-\mu_m+\lambda_{n-1}+\lambda_n)\eta\frac{\partial}{\partial\xi}.$$

Here E_{ii}, where $i=1\dots4$, are elementary matrices from $\mathfrak{gl}_{2|2}(\mathbb{C})_{\bar 0}$. The result follows.

Let us prove the second statement. Consider $\mathbf{F}^{2|2}_{1|2}$ and decomposition (25) of $\mathfrak{v}(\mathbf{F}^{2|2}_{1|2})$. We see easily that the vector subspaces $\langle\theta\rangle$ and $\mathfrak{pgl}_{2|2}(\mathbb{C})$ are invariant with respect to the action of the Lie algebra $\mathfrak{pgl}_{2|2}(\mathbb{C})_{\bar 0}$. Again the vector space $\mathfrak{pgl}_{2|2}(\mathbb{C})$ was decomposed into a sum of irreducible representations, see (20). The highest weights of $\psi|\mathfrak{pgl}_{2|2}(\mathbb{C})$ are given by (28). Let us compute the highest weight of $\langle\theta\rangle$. The image of the Cartan subalgebra

$$\mathrm{diag}(\mu_{m-1},\mu_m)\times\mathrm{diag}(\lambda_{n-1},\lambda_n)$$

with respect to the homomorphism $\mu:\mathfrak{gl}_{2|2}(\mathbb{C})_{\bar 0}\longrightarrow\mathfrak{v}(\mathbf{F}^{2|2}_{1|2})$ in chart (12) is given by

$$\mu(E_{11})=x\frac{\partial}{\partial x}+\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2},\ \ \mu(E_{22})=-x\frac{\partial}{\partial x},$$
$$\mu(E_{33})=-\xi_1\frac{\partial}{\partial\xi_1},\ \ \mu(E_{44})=-\xi_2\frac{\partial}{\partial\xi_2}.$$

We have

$$[\mu_{m-1}\mu(E_{11})+\mu_m\mu(E_{22})+\lambda_{n-1}\mu(E_{33})+\lambda_n\mu(E_{44}),\theta] = (\mu_{m-1}+\mu_m-\lambda_{n-1}-\lambda_n)\theta.$$

The result follows.

Computations in the cases $\mathbf{F}^{2|2}_{2|1}$, $\mathbf{F}^{2|2}_{0|1}$ and $\mathbf{F}^{2|2}_{1|0}$ are similar. □

Proof of Theorem 8. First of all let us compute the vector space of global sections of the vector bundle $\mathbf{W}_0$ using Theorem 1. The dominant highest weights of the representation ψ are in case **a**:

1. 0 if $m > 2$ and $n > 2$;
2. 0, $\mu_1 - \mu_2$, $\mu_1 - \lambda_n$, $\mu_1 + \mu_2 - \lambda_{n-1} - \lambda_n$ for $m = 2$ and $n > 2$;
3. 0, $\lambda_1 - \lambda_2$, $\lambda_1 - \mu_m$, $\lambda_1 + \lambda_2 - \mu_{m-1} - \mu_m$ for $m > 2$ and $n = 2$.

In case **b** for $\mathcal{O}_{\mathcal{S}} \simeq \mathbf{F}^{2|2}_{1|2}$ or $\mathbf{F}^{2|2}_{0|1}$ the dominant highest weights of ψ are:

1. 0 for $m > 2$, $n > 2$;
2. 0, $\mu_1 - \mu_2$, $\mu_1 - \lambda_n$, $\mu_1 + \mu_2 - \lambda_{n-1} - \lambda_n$ for $m = 2$, $n > 2$;
3. 0, $\lambda_1 - \lambda_2$, $\lambda_1 - \mu_m$, for $m > 2$, $n = 2$.

In case **b** for $\mathcal{O}_{\mathcal{S}} \simeq \mathbf{F}^{2|2}_{2|1}$ or $\mathbf{F}^{2|2}_{1|0}$ the dominant highest weights of ψ are:

1. 0 for $m > 2$, $n > 2$;
2. 0, $\mu_1 - \mu_2$, $\mu_1 - \lambda_n$ for $m = 2$, $n > 2$;
3. 0, $\lambda_1 - \lambda_2$, $\lambda_1 - \mu_m$, $-\mu_{m-1} - \mu_m + \lambda_1 + \lambda_2$ for $m > 2$, $n = 2$.

We restrict all weights on the Cartan subalgebra of $\mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C}) \subset \mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$. By Theorem 1, in case **a** we have:

$$\widetilde{\mathcal{W}_0}(\mathcal{B}_0) = \begin{cases} \mathbb{C}, & m > 2,\ n > 2; \\ \mathbb{C} \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_3, & m = 2,\ n > 2; \\ \mathbb{C} \oplus \mathfrak{r}_4 \oplus \mathfrak{r}_5 \oplus \mathfrak{r}_6, & m > 2,\ n = 2. \end{cases}$$

Without loss of generality we consider only the case **b**, $\mathcal{O}_{\mathcal{S}} \simeq \mathbf{F}^{2|2}_{1|2}$ or $\mathbf{F}^{2|2}_{0|1}$. We have

$$\widetilde{\mathcal{W}_0}(\mathcal{B}_0) = \begin{cases} \mathbb{C}, & m > 2,\ n > 2; \\ \mathbb{C} \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_3, & m = 2,\ n > 2; \\ \mathbb{C} \oplus \mathfrak{r}_4 \oplus \mathfrak{r}_5, & m > 2,\ n = 2. \end{cases}$$

Here $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}_3$, $\mathfrak{r}_4$, $\mathfrak{r}_5$, $\mathfrak{r}_6$ are irreducible $\mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$-modules with highest weights $\mu_1 - \mu_2$, $\mu_1 - \lambda_n$, $\mu_1 + \mu_2 - \lambda_{n-1} - \lambda_n$, $\lambda_1 - \lambda_2$, $\lambda_1 - \mu_m$ and $\lambda_1 + \lambda_2 - \mu_{m-1} - \mu_m$, respectively, and $\mathbb{C}$ is the irreducible $\mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$-module with weight 0.

We use notations of Theorem 4. We have seen that V is invariant with respect to the action of Lie superalgebra $\mathfrak{pgl}_{2|2}(\mathbb{C})$. Consider the case **a**. In case $\widetilde{\mathcal{W}_0}(\mathcal{B}_0) = \mathbb{C}$, we have $V = \mathbb{C}$ or $\{0\}$. Since $\mathfrak{pgl}_{2|2}(\mathbb{C})$ does not have any 1-dimensional ideals, the trivial module $\mathbb{C}$ is not $\mathfrak{pgl}_{2|2}(\mathbb{C})$-invariant. Hence, $V = \{0\}$. Consider the case $\widetilde{\mathcal{W}_0}(\mathcal{B}_0) \simeq \mathbb{C} \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_3$. As in the proof of Theorem 4, we see that any combination of H-modules $\gamma(\mathbb{C})$, $\gamma(\mathfrak{r}_1)$,

$\gamma(\mathfrak{r}_2)$ and $\gamma(\mathfrak{r}_3)$ is not invariant with respect to $\mathfrak{pgl}_{2|2}(\mathbb{C})$, see explicit description (8), (9) and (10). Hence again $V = \{0\}$. We finish the proof similarly to Theorem 4.

Other cases are similar. □

3.4. Main result

We put $k_0 = m$, $l_0 = n$.

Theorem 9. *Assume that $r > 1$ and that we have the following restrictions on the flag type:*

$$\begin{aligned} &(k_i, l_i) \neq (k_{i-1}, 0),\ (0, l_{i-1}),\ i \geq 2;\\ &(k_{i-1}, k_i | l_{i-1}, l_i) \neq (1, 0 | l_{i-1}, l_{i-1} - 1),\ (1, 1 | l_{i-1}, 1),\ i \geq 1;\\ &(k_{i-1}, k_i | l_{i-1}, l_i) \neq (k_{i-1}, k_{i-1} - 1 | 1, 0),\ (k_{i-1}, 1 | 1, 1),\ i \geq 1;\\ &k|l \neq (0, \ldots, 0 | n, l_2, \ldots, l_r),\ k|l \neq (m, k_2, \ldots, k_r | 0, \ldots, 0). \end{aligned}$$

Then

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq \mathfrak{pgl}_{m|n}(\mathbb{C}).$$

If $k|l = (0, \ldots, 0 | n, l_2, \ldots, l_r)$ or $k|l = (m, k_2, \ldots, k_r | 0, \ldots, 0)$, then

$$\mathfrak{v}(\mathbf{F}^{m|n}_{k|l}) \simeq W_{mn} \Subset (\bigwedge(\xi_1, \ldots, \xi_{mn}) \otimes \mathfrak{pgl}_n(\mathbb{C})),$$

where $W_{mn} = \operatorname{Der} \bigwedge(\xi_1, \ldots, \xi_{mn})$.

Note that the flag supermanifolds $\mathbf{F}^{m|n}_{k|l}$ and $\mathbf{F}^{n|m}_{l|k}$ are isomorphic.

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