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ANALYSIS AND DESIGN OF DISTRIBUTED PROTOCOLS FOR MULTI-AGENT SYSTEMS SUBJECT TO INPUT SATURATION AND TIME-VARYING DELAYS

Thales Costa Silva

Belo Horizonte, Brazil
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A thesis submitted to the Graduate Program in Electrical Engineering of the Federal University of Minas Gerais, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering.

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"Analysis And Design Of Distributed Protocols For Multi-agent Systems Subject To Input Saturation And Time-varying Delays"

Thales Costa Silva

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"*A hero is no braver than an ordinary man,*
but he is brave five minutes longer."

Ralph Waldo Emerson

To my love, Carolina,
and my family.

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Abstract

This work deals with the problem of coordination of multi-agent systems, with interconnected systems subject to time-varying delays and input saturation. The proposed approach consists in deriving sufficient conditions for the asymptotic convergence of the network, based on the Lyapunov-Krasovskii framework. Two tasks are tackled in this manuscript: the consensus and the formation-containment. The consensus is investigated in the discrete- and continuous-time domains, considering directed networks composed of agents modeled as linear systems of arbitrary order, subject to input saturation and time-varying delays. The main results are sufficient conditions for consensus analysis and the design of feedback gains for distributed consensus protocols for multi-agent systems. Besides, in this context, it is also proposed a strategy to compute a region from which the consensus is always guaranteed. On the formation-containment problem, the node dynamics of the network are extended to distinct systems modeled by the Euler-Lagrange equation, also subject to bounded inputs and time-varying delays. The main results on the formation-containment are the design of new control algorithms and sufficient conditions to ensure the convergence of the network. The control algorithms are designed as distributed dynamic controllers, in such a way that the number of neighbors of each agent is decoupled from the bound of the control inputs. Finally, the utilization of the proposed methods is illustrated through numerical examples, and it is shown that they can improve related recent works from the literature that deal with similar issues.

Keywords: Sistemas multi-agentes. Lyapunov-Krasovskii. Saturação. Atrasos. Sistemas Euler-Lagrangea. Consensus.

Resumo

Este trabalho aborda o problema de coordenação em sistemas multiagentes, com agentes sujeitos a atrasos e a saturação nos atuadores. A abordagem proposta consiste em obter condições suficientes para a convergência assintótica dos sistemas interconectados, por meio da teoria de Lyapunov-Krasovskii. Neste manuscrito, são abordadas duas tarefas diferentes, o consenso e o controle de formação-contenção. O consenso é estudado levando em consideração redes direcionadas compostas por agentes descritos por modelos lineares de ordem arbitrária, sujeitos a saturação na entrada e a atrasos variantes no tempo. Neste contexto, o estudo é realizado em tempo discreto e em tempo contínuo. Os resultados obtidos para o problema de consenso são: condições suficientes que garantem a convergência; e o projeto dos ganhos para protocolos distribuídos para sistemas multiagentes. Além disso, neste contexto também é proposto uma estratégia para calcular a região a partir da qual o consenso é sempre garantido. No estudo do problema de formação-contenção, a dinâmica dos nós da rede é estendida para sistemas possivelmente diferentes, modelados pela equação de Euler-Lagrange, igualmente sujeitos a restrições nos atuadores e a atrasos variantes no tempo. Os principais resultados alcançados neste contexto são o um novo algoritmo para o projeto de controle e condições suficientes que garantem a convergência dos sistemas interconectados. Os controladores projetados são dinâmicos e distribuídos, além de possibilitar o desacoplamento dos vizinhos de cada sistema do limite dos atuadores. O uso dos métodos propostos é ilustrado por meio de exemplos numéricos, e a comparação com resultados recentes encontrados na literatura sugere que a abordagem proposta pode levar a resultados menos conservadores.

Palavras-chave: Multi-agent systems. Lyapunov-Krasovskii. Saturation. Delays. Euler-Lagrange systems. Consensus.

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$\mathbb{R}$	-	set of real numbers
$\mathbb{N}$	-	set of positive integers
$\lVert\cdot\rVert$	-	Euclidean norm
$\lVert x_t\rVert_C$	-	Induced norm $\sup_{\sigma\in[t-\bar{\tau},t]}\lVert x(\sigma)\rVert$
$*$	-	symmetric entry of a symmetric matrix
M', M^{-1}	-	transpose and inverse of matrix M
$M_{(r)}$	-	rth element (row) of the vector (matrix) M
$M'_{(r)}$	-	transpose of the rth row of the matrix M
$M > 0$ ($M \geq 0$)	-	matrix M is positive definite (semi-definite)
$M < 0$ ($M \leq 0$)	-	matrix M is negative definite (semi-definite)
$\mathrm{diag}(A, B)$	-	block diagonal matrix with matrices A and B in main diagonal
$(\boldsymbol{a}\vert\boldsymbol{b})$	-	row-wise concatenation $\begin{bmatrix}\boldsymbol{a}' & \boldsymbol{b}'\end{bmatrix}'$ of vectors $\boldsymbol{a} \in \mathbb{R}^m$ and $\boldsymbol{b} \in \mathbb{R}^n$
$A \otimes B$	-	Kronecker product between matrices A and B
$\mathrm{Co}\{\cdot\}$	-	convex hull
$\lambda_{\max}(M)$	-	biggest eigenvalue a the matrix M
$\lambda_{\min}(M)$	-	smallest eigenvalue a the matrix M
I_m	-	identity matrix of dimension m
$\mathbf{1}_n$	-	vector of ones in $\mathbb{R}^n$
$\mathbf{0}_n$	-	vector of zeros in $\mathbb{R}^n$
0	-	matrix of zeros with appropriate dimensions
$\mathcal{C}_\tau^n$	-	set of continuous or discrete functions on the interval $[-\tau, 0]$, in $\mathbb{R}^n$
$\mathcal{G}$	-	graph
$\mathcal{V}$	-	vertex set of a graph $\mathcal{G}$
$\mathcal{E}$	-	edges set of a graph $\mathcal{G}$
$\mathcal{A}$	-	adjacency matrix associated with a graph $\mathcal{G}$
$\mathcal{D}$	-	degree matrix associated with a graph $\mathcal{G}$
L	-	Laplacian Matrix associated with a graph $\mathcal{G}$
τ_i(t)	-	ith agent's input time-varying delay
τ	-	constant input delay
μ_m	-	bound for time-varying delay
$\varphi_i(t_0)$	-	initial condition for ith agent
$\delta[\boldsymbol{z}(t)]$	-	time-derivative $\delta[\boldsymbol{z}(t)] = \dot{\boldsymbol{z}}(t)$ for $t \in \mathbb{R}$, or time-shift $\delta[\boldsymbol{z}(t)] = \dot{\boldsymbol{z}}(t+1)$ for $t \in \mathbb{N}$
$\mathcal{S}$	-	domain of consensus
$\Theta(x)$	-	dead-zone function $\Theta(x) = x - \mathrm{sat}(x)$
$\mathbb{I}_i$	-	vector of dimension $\mathbb{R}^n$ with 1 on ith position and 0 elsewhere

Contents

Chapter 1

Introduction

The wide range of applications for networked systems that work cooperatively have made the research of these arrangements remarkably attractive. This type of system is desirable since it provides features like robustness to individual failure, network scalability, and the possibility to attain tasks otherwise very hard for single systems (for example, carry a heavy and sparse load). That is to say, there are situations where it is simpler to develop solutions for a group of systems than for a single one.

These networked systems are often called multi-agent systems, where each agent is an individual entity able to make decisions and interact with the environment. Agents in such networks are required to operate harmoniously with each other in order to achieve a global level objective, usually having limited access to information and using simple rules. It is possible to divide the control approach for multi-agent systems into two large groups, according to the kind of information that each agent has access to and the group communication. Namely, there are *centralized* and *decentralized* methods. In the former, the actions of each agent are planned by a central controller which has access to information on the whole group. Whereas in the latter, the agents share information only with a subset of the group, called neighbors, and use local information to plan their actions.

Although the centralized approach may be applicable to ensure the accomplishment of various tasks, it is not scalable in general because the complexity of these systems grows fast with the number of agents. On the other hand, decentralized methods impose that each agent has rules that use only local information, in this way, it may be possible to add agents without a great increase in the complexity of the algorithm to be performed by the networked system.

In the field of networked systems, some of the most important problems that have been investigated are: the so-called consensus problem (Saber and Murray, 2003), in which the agents must reach an agreement on pre-specified variables; the containment problem (Ji et al., 2008), in which the agents are required to move into a referred convex hull; the distributed parameter estimation (Gubner, 1993); and the problem of formation (Lewis and Tan, 1997), where the main objective is that each agent keeps a desired relative position, distance, or orientation in relation to neighbor agents. Within these topics, the consensus problem is probably the most investigated one, partly due to the applications that can be improved from the development of this topic (see the studies by Qin et al. (2017) and Dorri, Kanhere, and Jurdak (2018), and references therein for recent advances).

From the topics listed above, a very general and challenging problem in multi-agent systems is the formation-containment. The main purpose of this problem is to stabilize multiple leaders in a formation and move a group of followers towards the convex hull constituted by the leaders. An

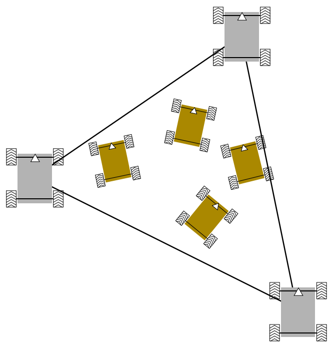


Figure 1.1: Formation-containment representation.

example of this desired objective is illustrated in Figure 1.1, where the vehicles in gray and brown represent leaders and follower agents, respectively. As a matter of fact, the problems of leaderless and the leader following consensus can be seen as particular cases of the formation-containment problem (Dong et al., 2015). In fact, by selecting a single leader the formation-containment problem becomes the consensus in which the followers have to synchronize their states with the states of the leader, and by considering some leaders with no followers the problem can be reduced to the leaderless consensus problem. The formation-containment not only extends other problems in multi-agent systems, but it also has many applications, for example, in the maneuver of a team of vehicles it might be required that only a portion of the group be aware of the environment and target position.

In the context of multi-agent systems, there is a growing body of literature investigating issues such as consensus (Savino, Souza, and Pimenta, 2018; Savino et al., 2020; Alves Neto, Mozelli, and Macharet, 2020), tracking in multi-robots settings (Pacheco, Pimenta, and Raffo, 2020; Yao, Ding, and Ge, 2018; Chu et al., 2018), teleoperation (Souza et al., 2008; Lima et al., 2019; Yuan, Wang, and Guo, 2019; Kebria et al., 2020), line formation (or convoy) (Neto, Mozelli, and Souza, 2019; Borkar, Borkar, and Sinha, 2019; Souza et al., 2020), segregation in robotic swarms (Ferreira Filho and Pimenta, 2020; Inácio, Macharet, and Chaimowicz, 2019; Santos, Pimenta, and Chaimowicz, 2014), input saturation (Jesus et al., 2014; Deng and Yang, 2017; Li and Lin, 2017), and challenges related to implementation like event-triggered (Dimarogonas, Frazzoli, and Johansson, 2012; Cui et al., 2018; Wang et al., 2017), intermittent communication (Li and Liu, 2018a; Savino et al., 2016; Yang, Wang, and Ni, 2013), nonlinear systems (Araújo, Torres, and Palhares, 2019; Gu et al., 2019; Li and Liu, 2018b), signed networks (Altafini, 2013; Valcher and Misra, 2014) and heterogeneous systems (Li and Liu, 2018a; Zheng and Wang, 2015; Wen et al., 2016). In this large field, few relevant works can be found where input saturation and time-delay are addressed together, although this is not a rare situation in physical networked systems. In fact, on the field of interconnected systems there are issues that arise from both, the interconnection and from each system itself. In the following the main difficulties that emerge on the study of multi-agent systems are summarized.

1.1 Seminal Works

The cooperative control of multi-agent systems has its roots in statistical analysis (Winkler, 1968; Degroot, 1974) and computer science (Tsitsiklis, Bertsekas, and Athans, 1986). The authors in Winkler (1968), Degroot (1974), and Tsitsiklis, Bertsekas, and Athans (1986) started to use ideas and terms that later composed the lexicon of multi-agent systems that lead the research line on consensus, which is closely related to formation control, flocking, and other tasks on multi-agent systems. Winkler (1968) suggested multiple methods to formally combine several probability distributions to obtain a single distribution to be used as a basis for decision-making, a problem which he referred to as "consensus problem". Degroot (1974) proposed a technique for a group of individuals to reach an agreement on an estimation of an unknown quantity given that each individual had its probability distribution of this quantity. While Tsitsiklis, Bertsekas, and Athans (1986) proposed a model for asynchronous distributed computation and showed that distributed versions of a large class of gradient-like algorithms preserve convergence properties of their centralized counterparts.

Another field that preceded and influenced the research area of cooperative control in multi-agent systems was the computer graphics modeling of flocks of birds and herds of land animals. Inspired by the motion of groups of individuals in nature, the objective was to simulate the emergent behavior that occurs in large groups of animals, without using a central controller. One example is the alignment and movement at the same speed that these groups tend to attain (Okubo, 1986). A remarkable characteristic is that in these models each individual is designed to react to its neighborhood with simple rules. Reynolds (1987) proposed an algorithm to simulate this behavior in a three-dimensional space and established his results using simulations, without convergence proofs. Then, Vicsek et al. (1995) proposed a simpler model for particles moving on the plane, a particular case of Reynolds' results. In their paper Vicsek et al. (1995), provide a variety of simulations to show that the agents eventually move in the same direction in absence of centralized control and with the varying neighborhood. The theoretical explanation of the results of Vicsek et al. (1995) was given later by Jadbabaie, Lin, and Morse (2003).

More specifically in the context of control of multi-agent systems, the studies started to arise more frequently later in the 90's with the problem of formation control, inheriting the theoretical development of statistics and graphical modeling (Sugihara and Suzuki, 1996; Suzuki and Yamashita, 1999; Singh et al., 2000; Liu, Passino, and Polycarpou, 2001; Mesbahi and Hadaegh, 2001). At this time the main objective was to establish protocols to make a group of robots cooperatively perform a given task. Particularly, Suzuki and Yamashita (1999) studied algorithms that distribute groups of robots to form geometrical patterns with rigorous proofs of convergence. Singh et al. (2000) suggested an adaptive control system for multiple unmanned aerial vehicles asymptotically track the desired shape. While Liu, Passino, and Polycarpou (2001) proposed distributed models of interaction for members of groups of systems required to behave as a swarm. The authors in Liu, Passino, and Polycarpou (2001) established simple rules in such a way the swarm convergence is guaranteed whenever the systems follow these rules. By this time an interesting interplay between Lyapunov theory and algebraic graph theory was shown in the works by Mesbahi and Hadaegh (2001) and Fax and Murray (2002). After that, the study of distributed networked systems using this framework became standard.

1.2 State of the Art

In the past twenty years, considerable research has been made on multi-agent systems (Suzuki and Yamashita, 1999; Singh et al., 2000; Mesbahi and Hadaegh, 2001; Fax and Murray, 2002; Olfati-Saber and Murray, 2004), mostly motivated by the potential capacity to solve several problems more efficiently if compared to single controlled systems. However, to fully enjoy these benefits some drawbacks intrinsically associated with networked systems have to be solved. The next sections present various research issues in multi-agent systems and some recent approaches to handle these problems.

1.2.1 Network Issues

Connectivity maintenance

The connectivity of the network topology is critical in guaranteeing the desired behavior of the agents and, more often than not, the connectivity is assumed *a priori*. However, it was shown in Ji and Egerstedt (2007) that in scenarios where the topology is state-dependent, for example, a network in which agents can only exchange information with other agents within some range, under the traditional consensus algorithm the connectivity could be lost even if the network topology is initially connected. Yang, Shi, and Constantinescu (2019) proposed a control strategy to preserve the connectivity of networked Euler-Lagrange systems subject to time-delays. Their methodology is formulated as a strict minimization of an energy-like function of the neighbors' states of each agent. In the work of Su and Lin (2019) the problem of consensus with a virtual leader and agents with double-integrator dynamics, previously studied (Olfati-Saber, 2006), is revisited and extended, and the proposed control strategy can maintain initially existent connections and new ones created during the network evolution, hence the requirement of initially connected topology is relaxed.

Quantization

In practical systems, the finite sampling capacity implies that no perfect state information can be processed. Without exact neighbors information, the multi-agent system might not attain convergence, and depending on the information precision the convergence might be attainable only to a certain range, which can prevent some tasks to be performed with a desirable performance, for example, the load balancing (Li et al., 2014). To deal with that sort of problem Li, Ho, and Li (2018) investigated an adaptive control strategy on the consensus problem, considering logarithmic quantizer measurements. The distributed adaptive controller is designed to handle quantized information among agents, which are modeled as structured uncertainties.

Distance dependent disturbance

An interesting observation made by Cucker and Smale (2007) is that in some networks the *influence* between two agents is related to the distance between them. Specifically, to address this aspect, Cucker and Smale (2007) modeled the multi-agent system in such a way the closer two agents are, the bigger the influence among them. After that, the extended idea of modeling non-ideal measurements associated with the distance among agents came along, which also extended the information modeling with additive noise and reliability to a more appropriate representation of multi-agent systems (Ren, Beard, and Kingston, 2005; Huang et al., 2010). One recent strategy to deal with this particularity is proposed in Mo, Guo, and Yu (2019), where a mean-square H_∞ formation problem, also subject to external disturbances, is investigated.

Packet loss

Due to non-ideal environment and communication channels, networked systems are almost always subject to data packet dropout. On the literature there are three main strategies used to deal with the analysis of stability under this problem, namely: i) the formulation as input delay (Souza et al., 2012; Souza et al., 2016; Zhao et al., 2017) in such a way the delayed information encompasses the time between two valid data packets; ii) the use of hybrid systems to handle deterministic packet losses (Chen and Zheng, 2012; Sui et al., 2018), such that each agent is composed of an uncontrolled (when loss happens) and a controlled system (when there is no packet loss); and iii) by representing the input random packet losses as an input stochastic sampled-data that fulfill a Bernoulli distribution (Zhang et al., 2017; Sui et al., 2018).

Time-delay

Time-delay is an important feature to take into account on system stabilization. It is almost inevitable and its presence can lead to undesirable performance and even to instability. Its presence is critical in multi-agent systems, as first shown by Olfati-Saber and Murray (2004), it can prevent the network to reach convergence even for agents with simple dynamics. Olfati-Saber and Murray have started the analysis of consensus in multi-agent systems with constant uniform time-delays and then their research was expanded for different types of delays, for example, considering multiple time-varying delays (Sun, Wang, and Xie, 2008) and nonuniform delays, in other words delays that are distinct over the network (Bliman and Ferrari-Trecate, 2008). Nevertheless, it is important to notice that time-delays not necessarily lead to degradation in performance, as it was shown in Michael et al. (2014) and Edwards et al. (2020), it can be used to induce behaviors on flocking or even to *improve* the convergence of networked systems.

Recent strategies to handle delays are often based on the Lyapunov-Krasovskii framework together with the derivation of LMI conditions to verify the network convergence. Some examples are the works in Savino et al. (2016), Zhou et al. (2018), and Fattahi and Afshar (2019).

1.2.2 Node Issues

Energy constraint

In some applications, each agent of the network may have a limited source of energy, which makes its autonomy limited. One promising strategy to design efficient networks that have been gaining popularity is event-triggered control and measurement. Roughly speaking, the main objective is to alleviate the assumption of continuous and periodic controller updates. That is, efficiently update the controllers' states at specific events (Nowzari, Garcia, and Cortés, 2019). There are several studies on event-based control for multi-agent systems, see for example the works by Wang, Yu, and Sun (2018), Nowzari, Garcia, and Cortés (2019), and Yi et al. (2019) and references therein for recent approaches.

Input saturation

Input saturation is a bound imposed by physical limitations in the actuators of the agents, and as such, it is naturally part of almost every real-world system. This constraint causes a nonlinear behavior in the closed-loop system that can lead to performance degradation, the occurrence of limit cycles, and instability, which may prevent the multi-agent system to attain the desirable goal. Two main strategies are generally used to deal with saturated control systems (Tarbouriech and Turner, 2009): the first one is to design the controller without considering the limits of the actuators and then appropriately tune it and/or introduce modifications to minimize saturation effects (Oliveira et al., 2013). In this approach, the system behaves as a linear one whenever the limits of the actuators are not reached and the modifications, if any, become active only in the presence of saturation. The second strategy is to take into account the limits of actuators *a priori* while trying to ensure performance requirements. This approach usually allows a clear trade-off between performance and the region of operation (when global stabilization is not guaranteed) (Paim et al., 2002; Castelan et al., 2006). It is worth mentioning that within this strategy there are promising approaches to deal with saturation, for example, Zhang, Wang, and Li (2013) presents a distributed model predictive control to deal with actuator saturation and derive linear matrix inequalities conditions to solve it. Cheng et al. (2015) also suggest model predictive control protocols for multi-agent systems with double-integrators dynamics. Yi et al. (2019) have studied the problem from the perspective of event-triggered control on single integrators dynamics, and in Sakthivel et al. (2018) conditions for the analysis of a stochastic fault-tolerant control subject to input saturation is proposed as LMIs, to study the leader-following consensus.

Model uncertainty

Uncertainties and unmodeled dynamics are typical sources of undesired behavior and even instability in the control of dynamic systems. Naturally, not all control strategies are robust to these characteristics, and there are some controllers more sensitive than others. On networked systems, these uncertainties have a critical impact, since a "bad" behaving node would propagate its dynamics through the network, and could disrupt the accomplishment of the group task. It is noteworthy that most works on multi-agent systems are usually concerned with agents with precisely known dynamics. However, there is a research line in which the primary interest is to derive robust control strategies for multi-agent systems, see for instance the works by Li, Soh, and Xie (2018) and by Khalili et al. (2018) and references therein.

Nonlinear dynamics

Most real-world systems would be better modeled using nonlinear differential equations. However, more often than not, linear control techniques are applied to stabilize nonlinear systems, even though linear representations are generally valid only in the vicinity of the linearization point (Slotine and Li, 1991). Following this tendency, the research on multi-agent systems started considering agents with integrators and double-integrators dynamics, then it was expanded to finite-order linear systems, and only a small portion of the field was involved with nonlinear dynamics. Only over the past ten years, the results have been largely extended to nonlinear systems (Chung and Slotine, 2007; Ren, 2009; Nuño et al., 2011; Nuño et al., 2012). Some recent approaches to handle nonlinear nodes on networked systems are proposed by Zhou et al. (2018), Gong and Lan (2018), and Chen et al. (2019b).

This research fits into the context of multi-agent systems with time-varying delays and input saturation. The main purpose is to extend the research area by deriving analysis conditions and designing appropriate controllers such that the networked system reaches convergence. The problem is tackled considering linear and nonlinear node dynamics, and input and communication delays. The contributions in each context are summarized at the end of this chapter and specifically discussed in Chapters 2 and 3, where they are also contrasted with similar studies found in the literature.

One should note that this section is limited to the presentation of problems associated with multi-agent systems, and has not addressed the challenge of different tasks, nor specific control strategies. The chosen topics are by no means complete–their main purpose is to provide a *feeling* of the fundamental problems of the field and to indicate some recent approaches to address them.

1.3 Contributions Overview

This work investigates a scenario that has started to be studied only recently in the field of multi-agent systems, nevertheless it is an important one as it considers scenarios closer to real-world by considering models for the agents with saturating inputs and time-varying delays. New conditions to analyze the convergence and design ot feedback controllers for the agents of the network are proposed.

In this research, two different tasks of cooperative control are studied, the consensus and formation-containment, each one with distinct node dynamics. The consensus problem is addressed in the continuous- and discrete-time domains considering saturated linear agents. The consensus is translated into a stability analysis problem, with saturation represented as a dead-zone function. In this context, input time-delays are considered nonuniform over the network and possibly non-differentiable within a closed set (Savino, Souza, and Pimenta, 2014). On the formation-containment problem, the node dynamics are considered nonlinear and modeled as Euler-Lagrange systems subject to input saturation. In this context, the time-delays that affect the network are considered in the communication channels and are also nonuniform, meaning that they can vary independently over the network.

The main improvements to the field that this work provides come in twofold: in the context of consensus with linear agents i) firstly, it is proposed a method to study the consensus in terms of the stability of linear multi-agent systems subject to input saturation and input time-varying delays, based on the transformation carried by Sun and Wang (2009). Then, it is proposed systematic procedures to study the stability and to synthesize the gains of controllers for the agents in continuous- and discrete-time multi-agent systems. In both scenarios, the methods can maximize the size of the region in which the consensus is always reachable; and ii) regarding the formation-containment with Euler-Lagrange systems, it is proposed new distributed dynamic controllers that take into account the communication delays and provide a bounded input for each agent. The control strategy is elaborated such that it is applicable on networks composed of heterogeneous systems with the same number of degrees of freedom. Then, it is proposed an LMI test that provides the convergence analysis by considering some controller gains and upper-bounds on the delays.

The remaining of this text is divided in the following way:

- **Chapter 2:** In this chapter the proposed methods for the consensus analysis and gains design on saturated linear systems are presented, as well as mathematical proofs of these results and numerical examples.

- **Chapter 3:** In this chapter, it is presented the design of distributed dynamic controllers, along with sufficient conditions for the networked Euler-Lagrange systems subject to input saturation and communication delays attain the formation-containment.

- **Chapter 4:** Finally, final considerations about the results are made, and a perspective of future developments is shown.

Chapter 2

Multi-Agent Systems with General Linear Dynamics

This chapter investigates the consensus problem subject to saturating inputs and time-delays. The problem is examined in the continuous- and discrete-time domains. In the continuous-time, the question is tackled employing the Lyapunov-Krasovskii methodology, the approach presented in this text is an extension of the results presented in the Master Thesis (Silva, 2019). In the discrete-time domain, two different approaches are given, one based on the Lyapunov-Krasovskii framework and another developed by writing the multi-agent delayed system as an equivalent switching system. The primary points presented in this chapter are i) the use of sectorial non-linearity to represent the saturation, ii) the transformation of the consensus problem into a stability problem on disagreement variables, and finally, iv) the formulation of stability and gain design conditions.

An important highlight on the notation used through this chapter is that because it will be presented two approaches, considering continuous- and discrete-time multi-agent systems, and part of the theory is shared among the two, a portion of the notation will also be given jointly. More specifically, the definitions and manipulations presented up to Section 2.1.6 are introduced together. Particularly, until Section 2.1.6 the independent variable t might represent the continuous- ($t \in \mathbb{R}$) or the discrete-time ($t \in \mathbb{N}$) variable. For this reason, I will use the operator

$$\delta[\boldsymbol{x}(t)]$$

to refer to the time-derivative in the continuous-time domain, i.e., $\delta[\boldsymbol{x}(t)] = \dot{\boldsymbol{x}}(t)$ with $t \in \mathbb{R}$, while in discrete-time domain to refer to the time-shift, i.e., $\delta[\boldsymbol{x}(t)] = \boldsymbol{x}(t+1)$ with $t \in \mathbb{N}$. In the sections in which this notation might insert some ambiguity, I will indicate specifically which context is being considered.

2.1 Introduction and Preliminaries

It is noteworthy that the first works on multi-agent systems were mainly concerned with agents modeled by linear models (Suzuki and Yamashita, 1999; Saber and Murray, 2003; Olfati-Saber and Murray, 2004; Ren, Beard, and Kingston, 2005; Cucker and Smale, 2007). Despite the fact that this approach can be used to model a rather small set of real world-systems, interesting problems can be solved from this perspective (see for example the results in Cheng et al. (2015), Chen et al. (2017a), Li, Soh, and Xie (2018), and Yi et al. (2019)), and as it gives flexibility and eases the system analysis, it remains a valuable tool to access the stability and stabilization of real-world systems. With this in mind, this chapter presents new contributions to address the consensus of linear systems subject to input saturation and time delays.

Consensus is one of the fundamental problems in multi-agent systems, as pointed out by Ren and Beard (2008). That is because consensus algorithms have applications in several problems of multi-agent systems, for example, the problem of rendezvous (Cortes, Martinez, and Bullo, 2006; Dong and Huang, 2018), formation control (Porfiri, Roberson, and Stilwell, 2007; Ren, Beard, and Atkins, 2007), and sensor networks (Spanos and Murray, 2005; Mohammadi and Asif, 2015). Consensus means to reach an agreement on a joint variable value, where agents share information and update its variable according to some rule until the network converges. Formally, the definition of consensus can be stated as:

Definition 2.1. *A multi-agent system with n agents and state variables $\boldsymbol{x}_i(t) \in \mathbb{R}^m$, where $i \in \{1, \ldots, n\}$ is the index of the agents, asymptotically reaches consensus on these variables if, for all $i \neq j$,*

$$\lim_{t\to\infty} (\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) = 0. \tag{2.1}$$

A consensus protocol is the interaction rule among agents that aims to drive the system toward consensus. The classical protocol discussed in this text was introduced by Saber and Murray (2003), and for the ith agent it has the following form:

$$\boldsymbol{u}_i(t) = -\sum_{j=1}^{n} a_{ij} K\Big(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\Big), \tag{2.2}$$

where $\boldsymbol{u}_i(t) \in \mathbb{R}^p$ is the input signal and $K \in \mathbb{R}^{p\times m}$ is a constant gain matrix. This is a simple distributed protocol that uses only local information with a linear relationship between the states of agents. The constant a_{ij} is the (i, j) entry of the adjacency matrix (formally presented below in Section 2.1.4), and in the consensus protocol (2.2) it selects only the information provided by the agents which the ith agent interacts with. Note that this constant is equal to zero for non-neighbor agents.

This chapter is concerned with the problem of consensus subject to constrained control signals according to the following assumption:

Assumption 2.1. *All the control inputs of agents, $\boldsymbol{u}_i(t)$, are constrained by symmetric limits $-u_{\max,i(k)} \leq u_{i(k)}(t) \leq u_{\max,i(k)}$, for all $k \in \{1, \ldots, p\}$, in which $u_{\max,i(k)} \in \mathbb{R}$ is a scalar limit of the kth input in the ith agent, and the subscript in parenthesis selects the kth element of the vector.*

Without loss of generality, the analysis is carried by assuming that $u_{\max,i(k)} = u_{\max}$, for all i, k. This is done to ease notation, however, one can observe that the proposed conditions are checked in each of the np inputs in the network. Thus, to recover the general instance the upper-bound for each input can be set individually.

Assumption 2.1 portrays a practical constraint present in most real systems which can lead to performance degradation or even to instability. It has been studied in different scenarios (see the work of Bernstein and Michel (1995) for a chronological review on saturation until 1995, and the book of Tarbouriech et al. (2011) on the topic of stability and stabilization of linear systems with input saturation). A consequence of the nonlinear behavior that Assumption 2.1 represents is that it might be impossible to steer the agents from some initial condition to the origin, when using static state-feedback control without additional assumptions on the dynamics of the agents, e.g., open-loop stability, as it is shown in Schmitendorf and Barmish (1980). Hence, with stabilizing feedback and agents potentially open-loop unstable, it is necessary to compute a region of the state space where for any initial condition exists bounded controls that stabilize the agents.

Associated with the saturation phenomenon, the problem of input time-delay is also considered. It is assumed that the delay occurs locally on each agent i. This is stated in the form of the following assumption:

Assumption 2.2. *The consensus protocol has access to all neighbors states instantaneously, but the controller action is delayed.*

This problem has been studied from different points of view to find maximum delays that allow consensus achievement, for example, the works by Olfati-Saber and Murray (2004), Savino et al. (2013), Wen et al. (2016), You et al. (2016), and Zhao et al. (2017). In the consensus protocol (2.2) the input delay can be represented with the following adaptation:

$$\boldsymbol{u}_i(t) = -\sum_{j=1}^{n} a_{ij} K\Big(\boldsymbol{x}_i(t-\tau_i(t)) - \boldsymbol{x}_j(t-\tau_i(t))\Big), \tag{2.3}$$

where the function $\tau_i(t)$ maps the time-varying delay of the ith agent. In light of this, we define a region for time-delay systems from which the consensus can be attained.

Definition 2.2. *The domain of consensus is a region $\mathcal{S} \subset \mathcal{C}^{nm}_{t_0,\bar{\tau}}$ in which for any $\varphi(t_0) \in \mathcal{S}$, the multi-agent system achieves consensus. The domain of consensus can be written in the following way,*

$$\mathcal{S} = \Big\{\varphi(t_0) \in \mathcal{C}^{nm}_{t_0,\bar{\tau}} : \lim_{t\to\infty} \boldsymbol{x}_i(t) = \boldsymbol{x}_j(t),\ \forall i,j \in \mathcal{V},\ i \neq j\Big\}, \tag{2.4}$$

where $\bar{\tau}$ denotes the maximum delay upper-bound, $\mathcal{C}^{nm}_{t_0,\bar{\tau}}$ represents the set of states of order nm on the interval $[t_0 - \bar{\tau}, t_0]$, and $\varphi(t_0) \in \mathcal{C}^{nm}_{t_0,\bar{\tau}}$ is a continuous- (discrete-) time signal of state variables of the agents on $[t_0 - \bar{\tau}, t_0]$.

In general, the exact computation of the region of convergence is a challenging task, even for delay-free single agent systems (Tarbouriech et al., 2011). Typically, this problem is tackled by estimating a subset of $\mathcal{S}$ less conservative as possible.

As shown in the seminal studies in Olfati-Saber and Murray (2004) and Saber and Murray (2003), depending on the time-delay some trajectories might converge or diverge to consensus. In this work, the objective is to investigate the scenario in which both, saturation and time-delays, might be present. To the best of the author's knowledge, this type of restriction have started to be studied just recently with works in continuous-time domain in You et al. (2016), Yanumula, Kar, and Majhi (2017) and in Ding, Zheng, and Guo (2018), and there is only one work addressing saturation and time-delays simultaneously on discrete-time multi-agent systems by Zhang, Saberi, and Stoorvogel (2020). Nevertheless, the approach presented here extend those results in significant aspects. Namely, the work in You et al. (2016)

i) deals only with the problem of leader-following consensus, while our approach can be used on problems with and without leader;

ii) even though they said that the approach is appropriated for global consensus, no guarantees is given in order to ensure that only systems with this characteristic will be analyzed. Thus, there might be cases where their conditions are satisfied and global consensus cannot be reached (this is illustrated in Example 2.1);

iii) considers undirected switching communication topologies, while the network communication topology in this work is directed, but fixed;

iv) the proposed conditions are matrix inequalities but are not linear, which makes it difficult to find optminal solutions.

The work in Ding, Zheng, and Guo (2018)

i) also is concerned only with the problem of leader-following consensus, while we deal with consensus with and without leader;

ii) formulates the problem of consensus analysis with bilinear matrix inequalities. Although this formulation might have some advantages, it is a nonconvex problem and in general there is no unified method to solve it;

iii) considers *buffers* over the network used to synchronize the input delays, in such a way that the control action is updated only after *all* agents receive a confirmation signal. In other words, their approach assumes that the ith agent receives the confirmation signal from all the other agents, leading to a centralized type of communication. The approach presented here is more general, considering decentralized, distinct time-delays.

In the discrete-time domain, Zhang, Saberi, and Stoorvogel (2020)

i) investigate the convergence of agents subject to input saturation and unknown nonuniform constant delays, here we consider time-varying delays;

ii) assume agents having open-loop stable dynamics. This facilitates the study since it is not required to compute regions of guaranteed convergence. In the approach proposed here no such assumption is made;

iii) derive an upper bound for the delay tolerance and, therefore, ensure the stability for a set of networks with different constant delays. Here, similarly, it is considered nonuniform time-delays based on upper-bounds of their values.

2.1.1 Review of Lyapunov-Krasovskii Methodology

A fundamental assumption of systems modeled by ordinary differential equations (difference equations in the discrete-time domain), such as

$$\delta[x(t)] = f(t, x(t)),$$

is that the future state variables of the system are completely determined by the current state values. That is, the values of the state $x(t)$ can be established once the initial condition $x(0)$ is known. However, in the class of delayed systems the future evolution of state variables not only depends on the current value $x(t)$, but also on historical values $x_t = x(t+\eta)$, with $\eta \in [-\tau, 0]$. The Lyapunov-Krasovskii framework is a generalization of the Lyapunov second method to attest the stability of time-delay systems, and its main theorem can be expressed as follows:

Lemma 2.1 (Lyapunov-Krasovskii (Gu, Kharitonov, and Chen (2003)))**.** *Suppose $f : \mathbb{R} \times \mathcal{C}^m_{-\bar{\tau}} \to \mathbb{R}^m$ and that u, v, and $w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, and $u(0) = v(0) = 0$. The time-delay system $\delta[x(t)] = f(t, x_t)$ has a uniformly stable equilibrium point if there exists a continuous functional $V(x_t) : \mathbb{R} \times \mathcal{C}^m_{-\bar{\tau}} \to \mathbb{R}_+$ which is positive definite*

$$u(\|x_t\|) \leq V(x_t) \leq v(\|x_t\|_C),$$

and its derivative (forward difference for discrete-time systems) along the solution of the system is such that

$$\dot{V}(x_t) \leq -w(\|x_t\|) \quad \Big(\text{or } \Delta V(x_t) \leq -w(\|x_t\|) \text{ in the discrete-time domain}\Big),$$

where x_t represents the system states $x(\sigma)$ with $\sigma \in [t - \bar{\tau}, t]$, $\|x_t\|_C = \sup_{\sigma \in [t-\bar{\tau}, t]} \|x(\sigma)\|$, and the constant $\bar{\tau}$ represents the maximum value of the delay.

As in the second method of Lyapunov, the main difficulty with the application of the above theorem is the choice of an appropriate functional $V(x_t)$. Moreover, even for single linear agent systems subject to time-varying delay, without saturation, it is still challenging to put together an exact Lyapunov-Krasovskii functional that gives necessary and sufficient stability conditions. Hence, several strategies have been proposed in the literature to cope with the conservatism of the approach. Usually, the technique is to add information about the delay. See for example the works in Wang et al. (2018) and in Fridman (2014) and references therein for some standard strategies.

2.1.2 Dynamic Network

Consider a multi-agent system consisting of n agents subject to saturation and input time-delay, in which the open-loop dynamics of the ith agent is given by

$$\delta[\boldsymbol{x}_i(t)] = A\boldsymbol{x}_i(t) + B\,\mathrm{sat}(\boldsymbol{u}_i(t)), \tag{2.5}$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^m$ is the agent state variable, $\boldsymbol{u}_i(t) \in \mathbb{R}^p$ the input control subject to delay, and A and B are real constant known matrices of appropriate dimensions. Initial conditions for the ith agent are given by

$$\varphi_i(t_0) \in \mathcal{C}^m_{\max\{\tau_i(t)\}},$$

in which $\mathcal{C}^m_{\max\{\tau_i(t)\}}$ represents the set of states in the continuous-time domain of order m on the interval $[t_0 - \max\{\tau_i(t)\}, t_0]$. The time-varying delay that affects the ith agent's input is implicit in (2.5) (with input given as (2.3)) and is given by $\tau_i(t)$, as in Savino et al. (2016), and it is modeled by

$$\tau_i(t) = \tau + \mu_i(t), \ \forall i \in \{1, \dots, n\},$$

where τ is a constant value and $\mu_i(t)$ a possibly non-differentiable time-varying perturbation that satisfies $|\mu_i(t)| \leq \mu_m \leq \tau$ with μ_m known, hence the delay belongs to an interval given by $\tau_i(t) \in [\tau - \mu_m, \tau + \mu_m]$.

The function sat$(\cdot)$ maps the saturation as:

$$\begin{aligned} &\mathrm{sat}(\boldsymbol{u}_i) = [\mathrm{sat}(u_{i(1)}) \cdots \mathrm{sat}(u_{i(p)})]', \ \text{with} \\ &\mathrm{sat}(u_{i(r)}) = \mathrm{sign}(u_{i(r)}) \min(u_{\max}, |u_{i(r)}|), \end{aligned} \tag{2.6}$$

for all $r \in \{1, \dots, p\}$, and the scalar $u_{\max}$ represents the limit of the actuators. The subscript in parenthesis $M_{(r)}$ denotes the rth element or the rth row, if M is a vector or a matrix, respectively, and $M'_{(r)}$ represents the transpose of its rth row. Without loss of generality, it is assumed that all inputs are independent and have the same maximum value. Observe that due to the saturation it is impossible to steer the agents' states from some initial conditions $\varphi_i(t_0)$ to the origin with static state-feedback control, if the open-loop system in (2.5) is unstable, i.e., having some eigenvalues of A with i) positive real part in the continuous-time domain or ii) outside the unit circle in the discrete-time domain (Tingshu Hu, 2001; Schmitendorf and Barmish, 1980).

2.1.3 Problem Statement

In the present chapter we address the following problems:
Problem 1. Given the state-feedback matrix K in (2.3), determine whether the multi-agent system with dynamics given by (2.5) reaches consensus, with agents subject to input time-delay and input saturation. In addition, establish a region from where the agents always attain the consensus.
Problem 2. Design the stabilizing state-feedback matrix K in (2.3) for the agents subject to input time-delays and input saturation of the multi-agent system with dynamics given by (2.5) to guarantee consensus, along with an estimate of the domain of consensus.

2.1.4 Graph Theory

In this investigation, the interaction between agents is represented using graphs. This abstraction allows to represent the communication complexity as essentially combinatorial: it identifies which pairs of agents have interactions and to what degree. To illustrate this idea consider the group of five range sensors in Figure 2.1, where each dot on the left side represents a sensor and the gray area represents its range. The communication topology is abstracted on the right side as a graph, where the nodes represent the sensors and an edge exists only if a sensor can perceive a neighbor sensor, the arrows represent the direction of the information flow. The algebraic

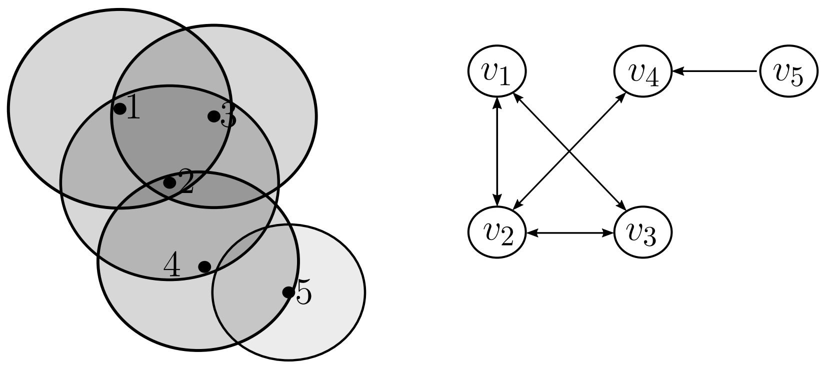


FIGURE 2.1: Graph representation of agents interaction.

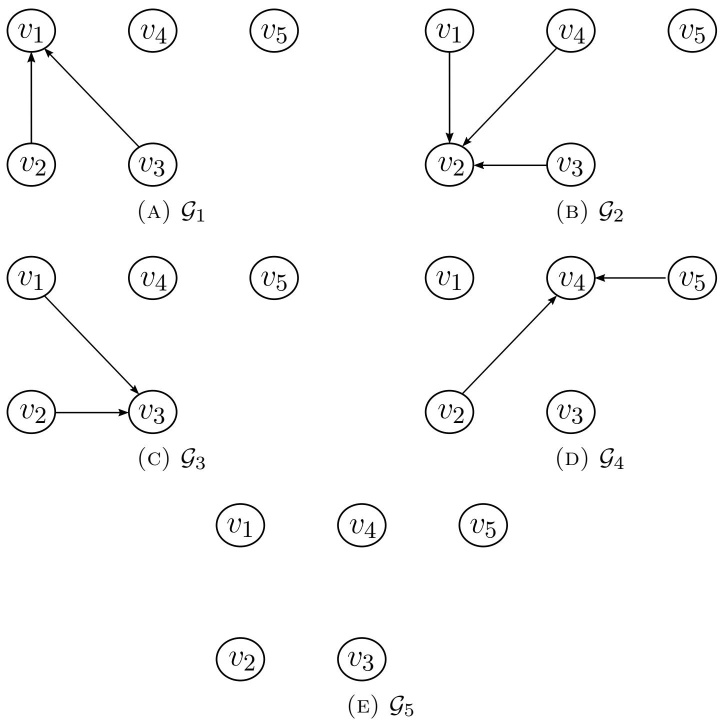
theory of graphs to model interactions between systems has been used by several authors in the context of networked systems. Mesbahi and Hadaegh (2001) were one of the first authors to use it in the multi-agent systems context and then this representation became popular, most references in this text represent the multi-agent system communication in this way.

A *finite, directed, simple graph* is built upon a finite set called *vertex set* denoted by $\mathcal{V} = \{v_1, \ldots, v_n\}$ (Mesbahi and Egerstedt, 2010). To formally define a graph $\mathcal{G}$ we use a particular subset of the Cartesian product $\mathcal{V} \times \mathcal{V}$, named *edges* of $\mathcal{G}$, and assign the symbol $\mathcal{E}$ to it. Each element of the edge set, e_{ij}, represents a directed information flow in the multi-agent system. Thus, set $\mathcal{E}$ is built by choosing ordered pairs that represent two agents that interact with each other. In each element $e_{ij} = (v_i, v_j)$, the first argument v_i is the parent node (the arrow's tail) and the second argument v_j is the child node (the arrow's head), formally we have: $\mathcal{E} = \{(v_i, v_j) : v_j \text{ receives information from } v_i\}$. The neighborhood of the vertex v_i is the set of all vertices that v_i receives information from, specifically, $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Additionally, an adjacent node, or neighbor of v_i, is a node v_j that belongs to the neighborhood $\mathcal{N}_i$. The adjacency matrix $\mathcal{A}$ is defined by assigning values to its elements according to the edge set:

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \text{ or } \nexists\, e_{ji}, \\ 1, & \text{if } e_{ji} \in \mathcal{E}. \end{cases}$$

Then, the idea of moving from node to node through the edges comes naturally using the notion of adjacency. Thereby, a *path* related to the graph $\mathcal{G}$ is given by a sequence of nodes $\{v_1, \ldots, v_n\}$ such that for all $k \in \{1, \ldots, n-1\}$, the nodes v_k and v_{k+1} are adjacent and v_{k+1} is a child node. Finally, the degree matrix $\mathcal{D}$ associated with the graph $\mathcal{G}$ is a diagonal matrix with elements given by $d_{ii} = \sum_{j=1}^{n} a_{ij}$, and the Laplacian matrix associated with the graph is defined as $L = \mathcal{D} - \mathcal{A}$.

The idea of sub-graph of a graph $\mathcal{G}$ is used during the text to facilitate notation. In general, a sub-graph of a graph $\mathcal{G}$ is a graph constituted of a subset of nodes and edges of $\mathcal{G}$. In this manuscript, the interest relies on sub-graphs composed of a node v_i along with all the other nodes and the so-called incoming edges of v_i. A sub-graph is denoted by the same symbol of the primary graph with a subscript, for example, $\mathcal{G}_i$ is the sub-graph of the ith node associated with the graph $\mathcal{G}$. Thus, the sub-graph of the ith node is a graph $\mathcal{G}_i = (\mathcal{V}, \mathcal{E}_i)$, where $\mathcal{E}_i = \{(v_i, v_j) \in \mathcal{E} : \forall v_j \in \mathcal{V}\}$. To illustrate this idea, the graph formed in Figure 2.1 is shown in Figure 2.2 with its sub-graphs.



(A) $\mathcal{G}_1$ (B) $\mathcal{G}_2$ (C) $\mathcal{G}_3$ (D) $\mathcal{G}_4$ (E) $\mathcal{G}_5$

FIGURE 2.2: Sub-graphs formed by the nodes of Figure 2.1.

To ensure the convergence the following assumption is necessary:

Assumption 2.3. *The graph $\mathcal{G}$ associated with the communication topology of the network have a directed spanning tree.*

A directed tree is a directed graph where all nodes but one, which is called root and has no parent node, has exactly one parent, and there is a path from the root to any other node. A spanning tree of a directed graph is a directed tree composed of the graph edges that connect all the nodes of the graph.

2.1.5 Tree-Type Transformation

The tree-type transformation, proposed in Sun and Wang (2009), is used to translate the consensus problem into a stability problem. Under Assumption 2.3, defining the disagreement of state variables as follows

$$\boldsymbol{z}_i(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_{i+1}(t),$$

for $i = 1, \ldots, n-1$. The disagreements of states for all $i \in \{1, \ldots, n-1\}$ are written in one equation by using the transformation,

$$\boldsymbol{z}(t) = (U \otimes I_m)\boldsymbol{x}(t), \tag{2.7}$$

where $\boldsymbol{z}(t) = [\boldsymbol{z}_1(t)' \ldots \boldsymbol{z}_{n-1}(t)']'$, $\boldsymbol{x}(t) = [\boldsymbol{x}_1(t)' \ldots \boldsymbol{x}_n(t)']'$, the symbol $\otimes$ represents the Kronecker product between two matrices, and I_m is an identity matrix of order m and the matrix U is given by

$$U = \begin{bmatrix} \mathbf{1}_{n-1} & -I_{n-1} \end{bmatrix}.$$

The transformation from the disagreement back to original states is given by

$$\boldsymbol{x}(t) = \mathbf{1}_n \otimes \boldsymbol{x}_1(t) + (W \otimes I_m)\boldsymbol{z}(t), \tag{2.8}$$

where the matrix W is defined as,

$$W = \begin{bmatrix} \mathbf{0}'_{n-1} \\ -I_{n-1} \end{bmatrix},$$

with $\mathbf{1}_{n-1}$ and $\mathbf{0}_{n-1}$ being column vectors of size $n-1$, in which all elements are ones and zeros, respectively. Hence, the Definition 2.1 of the consensus in the network can be formulated according to the following lemma:

Lemma 2.2. *(Sun and Wang, 2009). A multi-agent system with n agents and state variables $\boldsymbol{x}_i(t) \in \mathbb{R}^m$, where $i \in \{1, \ldots, n\}$ is the index of the agents, asymptotically reaches consensus if, and only if*

$$\lim_{t \to \infty} \boldsymbol{z}(t) = 0. \tag{2.9}$$

Therefore, the stability of a system in the new coordinates $\boldsymbol{z}(t)$ is investigated to access the convergence of the network into consensus.

2.1.6 Consensus as a Stability Problem

In this section, the consensus is rewritten as a stability problem using the tree-type transformation. Note that the multi-agent system (2.5) can be represented compactly by using the Kronecker product, as

$$\delta[\boldsymbol{x}(t)] = (I_n \otimes A)\boldsymbol{x}(t) + (I_n \otimes B)\,\mathrm{sat}(\boldsymbol{u}(t)), \tag{2.10}$$

where,

$$\begin{aligned}\boldsymbol{u}(t) &= \begin{bmatrix}\boldsymbol{u}_1(t)' & \cdots & \boldsymbol{u}_n(t)'\end{bmatrix}', \text{ and} \\ \boldsymbol{x}(t) &= \begin{bmatrix}\boldsymbol{x}_1(t)' & \cdots & \boldsymbol{x}_n(t)'\end{bmatrix}'.\end{aligned} \tag{2.11}$$

Defining the vector $\mathbb{I}_i \in \mathbb{R}^n$ with 1 in its ith position and 0 elsewhere, for example for $i = 1$, $\mathbb{I}_1 = [1 \quad 0 \quad \cdots \quad 0]'$, makes it possible to rewrite (2.11) as,

$$\boldsymbol{u}(t) = \sum_{i=1}^{n} \mathbb{I}_i \otimes \boldsymbol{u}_i(t). \tag{2.12}$$

Additionally, the following equalities hold:

$$\sum_{j=1}^{n} a_{ij}\boldsymbol{x}_j(t - \tau_i(t)) = (\mathbb{I}_i'\mathcal{A}_i \otimes I_m)\boldsymbol{x}(t - \tau_i(t)), \tag{2.13}$$

$$\sum_{j=1}^{n} a_{ij}\boldsymbol{x}_i(t - \tau_i(t)) = (\mathbb{I}_i'\mathcal{D}_i \otimes I_m)\boldsymbol{x}(t - \tau_i(t)), \tag{2.14}$$

where the subscript on the adjacency matrix $\mathcal{A}_i$ and on the degree matrix $\mathcal{D}_i$ refers to the sub-graph related to ith agent, as introduced in Section 2.1.4. Applying the consensus protocol

$$\boldsymbol{u}_i(t) = -\sum_{j=1}^{n} a_{ij}K\Big(\boldsymbol{x}_i(t - \tau_i(t)) - \boldsymbol{x}_j(t - \tau_i(t))\Big),$$

in (2.12), then using (2.13) and (2.14) and recalling that $L_i = \mathcal{D}_i - \mathcal{A}_i$, we get

$$\begin{aligned}\boldsymbol{u}(t) =& -\sum_{i=1}^{n} \mathbb{I}_i \otimes \sum_{j=1}^{n} a_{ij}K(\boldsymbol{x}_i(t - \tau_i(t)) - \boldsymbol{x}_j(t - \tau_i(t))) \\ =& -\sum_{i=1}^{n} (\mathbb{I}_i \otimes K)(\mathbb{I}_i'L_i \otimes I_m)\boldsymbol{x}(t - \tau_i(t)).\end{aligned} \tag{2.15}$$

By using the mixed-product property for the Kronecker product, (2.15) becomes,

$$\begin{aligned}\boldsymbol{u}(t) =& -\sum_{i=1}^{n} (\mathbb{I}_i\mathbb{I}_i'L_i \otimes K)\boldsymbol{x}(t - \tau_i(t)) \\ =& -\sum_{i=1}^{n} (L_i \otimes K)\boldsymbol{x}(t - \tau_i(t))\end{aligned}$$

Finally, the closed-loop multi-agent system (2.10) can be written in a stacked form as,

$$\delta[\boldsymbol{x}(t)] =(I_n \otimes A)\boldsymbol{x}(t) - (I_n \otimes B)\,\mathrm{sat}\Big(\sum_{i=1}^{n} (L_i \otimes K)\boldsymbol{x}(t - \tau_i(t))\Big). \tag{2.16}$$

Finally, the operator δ is applied on equation (2.7) and afterwards, the dynamics (2.16) is substituted in $\delta[\boldsymbol{x}(t)]$, this gives us

$$\delta[\boldsymbol{z}(t)] = (U\otimes I_m)(I_n\otimes A)\boldsymbol{x}(t) - (U\otimes I_m)(I_n\otimes B)\,\mathrm{sat}\left(\sum_{i=1}^{n}(L_i\otimes K)\boldsymbol{x}(t-\tau_i(t))\right). \quad (2.17)$$

Replacing (2.8) in (2.17),

$$\begin{aligned}\delta[\boldsymbol{z}(t)] =&(U\otimes I_m)(I_n\otimes A)\Big((\mathbf{1}_n\otimes x_1(t)) + (W\otimes I_m)\boldsymbol{z}(t)\Big)\\ &-(U\otimes I_m)(I_n\otimes B)\,\mathrm{sat}\left(\sum_{i=1}^{n}(L_i\otimes K)\Big((\mathbf{1}_n\otimes x_1(t-\tau_i(t))) + (W\otimes I_m)\boldsymbol{z}(t-\tau_i(t))\Big)\right),\end{aligned}$$

then, through Kronecker properties,

$$\begin{aligned}\delta[\boldsymbol{z}(t)] =&(U\mathbf{1}_n)\otimes(Ax_1(t)) + (UW\otimes A)\boldsymbol{z}(t)\\ &-(U\otimes B)\,\mathrm{sat}\left(\sum_{i=1}^{n}(L_i\mathbf{1}_n)\otimes(Kx_1(t-\tau_i(t))) + (L_iW\otimes K)\boldsymbol{z}(t-\tau_i(t))\right),\end{aligned}$$

noticing that $U\mathbf{1}_n = 0$, $L_i\mathbf{1}_n = 0$, and $UW = I_{n-1}$, the following disagreement system is obtained

$$\delta[\boldsymbol{z}(t)] = (I_{n-1}\otimes A)\boldsymbol{z}(t) - (U\otimes B)\,\mathrm{sat}\left(\sum_{i=1}^{n}(L_iW\otimes K)\boldsymbol{z}(t-\tau_i(t))\right). \quad (2.18)$$

To handle the saturation, we study the stability analysis and the stabilization of (2.18) as a Lur'e problem satisfying a sector condition (Khalil, 2002). Following Tarbouriech et al. (2011), let us consider the dead-zone function

$$\Theta(\boldsymbol{g}(t)) = \boldsymbol{g}(t) - \mathrm{sat}(\boldsymbol{g}(t)), \quad (2.19)$$

with

$$\boldsymbol{g}(t) = \sum_{i=1}^{n}(L_iW\otimes K)\boldsymbol{z}(t-\tau_i(t)),$$

then summing and subtracting $(U\otimes B)\boldsymbol{g}(t)$ from the right-hand side of (2.18), it is possible to represent the closed-loop multi-agent system as

$$\begin{aligned}\delta[\boldsymbol{z}(t)] =& (I_{n-1}\otimes A)\boldsymbol{z}(t) - \sum_{i=1}^{n}(U\otimes B)(L_iW\otimes K)\boldsymbol{z}(t-\tau_i(t))\\ &+(U\otimes B)\Theta\left(\sum_{i=1}^{n}(L_iW\otimes K)\boldsymbol{z}(t-\tau_i(t))\right). \quad (2.20)\end{aligned}$$

Thus, as presented in Lemma 2.2, the stability of (2.20) implies that the multi-agent system reaches consensus. Moreover, because of the dead-zone function $\Theta(\boldsymbol{g}(t))$, it is possible to define a polyhedral set and a sector condition to assist the the stability analysis and the stabilization of the multi-agent system, specifically:

Lemma 2.3 (Generalized Sector Condition (Tarbouriech and Silva Jr., 2005))**.** *For a given auxiliary signal $\boldsymbol{\omega}(t)$ and saturation limits $u_{\max}$, define the set*

$$\mathbb{S}(\boldsymbol{\omega}(t), u_{\max}) = \{\boldsymbol{u}(t) \in \mathbb{R}^{np} : |(\boldsymbol{u}(t) - \boldsymbol{\omega}(t))_{(\ell)}| \leq u_{\max},\ \textit{for}\ \ell = 1, ..., np\}.$$

If $\boldsymbol{u}(t)$ and $\boldsymbol{\omega}(t)$ belongs to $\mathbb{S}(\boldsymbol{\omega}(t), u_{\max})$, then $\Theta\Big(\boldsymbol{u}(t)\Big)' T\Big[\Theta\Big(\boldsymbol{u}(t)\Big) + \boldsymbol{\omega}(t)\Big] \leq 0$ is satisfied for any positive diagonal matrix $T \in \mathbb{R}^{np \times np}$, with $\Theta(\boldsymbol{u}(t))$ given in (2.19).

2.2 Consensus of Continuous-Time Systems

This section addresses the problems of consensus analysis and design of the gains with the consensus protocol (2.3). The main results presented in this section are extensions of the work in the Master Thesis in Silva (2019), some important distinctions are; i) the saturation here is modeled as a dead-zone function, which allows to tackle it as a Lur'e problem satisfying a sector condition. Because of that, the stability and design conditions are given directly as LMIs, while in Silva (2019) they were bilinear matrix inequalities and needed to be solved as iterative linear matrix inequalities; ii) the Lyapunov-Krasovskii functional candidate is built taking into account more information about delayed state variables of each agent, which possibly diminish the conservativeness of the approach, and iii) it is proposed a methodology to characterize the region of convergence by considering current and delayed states to find a set of initial conditions as large as possible.

An appealing aspect of (2.20) is that it can be analyzed using robust control methodologies. Particularly, in this section we address the stability analysis and stabilization of the continuous-time version of (2.20) by employing the following Lyapunov-Krasovskii functional candidate inspired by the studies in Souza (2013), Savino, Souza, and Pimenta (2014), Savino et al. (2016), and Silva (2019):

$$\begin{aligned} V(\varphi(t)) =& \chi(t)'\bar{P}\chi(t) + \int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+s}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{S}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds d\theta + \int_{-\mu_m}^{\mu_m}\int_{t+s-\tau}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{Z}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds \\ &+ \int_{-\tau}^{0}\int_{t+s}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{R}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds + (\tau+\mu_m)\sum_{i=1}^{n}\int_{-\tau-\mu_m}^{0}\int_{t+s}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{M}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds, \end{aligned} \tag{2.21}$$

with $\chi(t)' = \Big[\boldsymbol{z}(t)' \quad \int_{t-\tau}^{t} \boldsymbol{z}(\epsilon)'d\epsilon\Big]$,

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ * & \bar{P}_3 \end{bmatrix},$$

$\varphi(t) \in \mathcal{C}^{nm}_{t,\tau+\mu_m}$, and recalling that in continuous-time $\delta[x(t)] = \dot{x}(t)$. Moreover, the main differences between the functional candidate (2.21) and the ones in Savino et al. (2016), Savino, Souza, and Pimenta (2014), and Silva (2019) are the inclusion of the n integrals of derivative of states over $[-\tau-\mu_m, 0]$ and the exclusion of a single integral term over $[-\tau, 0]$. The motivation behind these changes is to include delay information for each agent and dissociate delayed from current states. These modifications allowed the use of the Generalized Sector Condition (Lemma 2.3), the characterization of decoupled regions for delayed and current states, as well as less conservatism in the investigated examples. Besides, if (2.21) is a Lyapunov-Krasovskii functional, then we can calculate an estimate of $\mathcal{S}$ as the associated level set as:

$$\mathcal{L}_V = \{\varphi(t_0) \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : V(\varphi_0) \leq 1\}, \tag{2.22}$$

such that $\mathcal{L}_V \subset \mathcal{S}$.

2.2.1 Stability Analysis

The following theorem presents sufficient conditions to certify consensus in the multi-agent system (2.10).

Theorem 2.1. *Let the feedback gain matrix K and the scalars τ, μ_m, and ϱ be given. The multi-agent system (2.5) with consensus protocol (2.3) subject to saturating inputs and time-varying delays $\tau_i(t) \in [\tau - \mu_m, \tau + \mu_m]$ with $\tau > 0$, $0 \leq \mu_m \leq \tau$, reaches consensus if there exist symmetric positive definite matrices P_1, P_3, R, S, M, and $Z \in \mathbb{R}^{m(n-1)\times m(n-1)}$, matrices $P_2 \in \mathbb{R}^{m(n-1)\times m(n-1)}$, $G \in \mathbb{R}^{np\times nm(n-1)}$, a diagonal positive definite matrix $H \in \mathbb{R}^{np\times np}$, and a non-singular matrix $F \in \mathbb{R}^{m(n-1)\times m(n-1)}$, such that the following inequalities are satisfied:*

$$\begin{bmatrix} \Phi & \mu_m\Gamma \\ * & -\mu_m Z \end{bmatrix} < 0, \tag{2.23}$$

$$\begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0, \tag{2.24}$$

and

$$\begin{bmatrix} \hat{M} & -\hat{M} & 0 \\ * & \hat{M} & \left(\bar{L}(I_n \otimes F') - G\right)'_{(r)} \\ * & * & u^2_{\max} \end{bmatrix} \geq 0, \quad \forall r \in \{1, \ldots, np\}, \tag{2.25}$$

where

$$\Phi = \Phi_P + \Phi_Z + \Phi_R + \Phi_S + \Phi_{\text{sat}} + \Phi_M + \Phi_F, \tag{2.26}$$

$$\Gamma = \begin{bmatrix} \bar{B}F' \\ \varrho\bar{B}F' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Phi_Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 2\mu_m Z & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Phi_P = \begin{bmatrix} P_2+P_2' & P_1 & -P_2 & 0 & P_3 & 0 \\ * & 0 & 0 & 0 & P_2 & 0 \\ * & * & 0 & 0 & -P_3 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}, \Phi_R = \begin{bmatrix} -\frac{4}{\tau}R & 0 & -\frac{2}{\tau}R & 0 & \frac{6}{\tau^2}R & 0 \\ * & \tau R & 0 & 0 & 0 & 0 \\ * & * & -\frac{4}{\tau}R & 0 & \frac{6}{\tau^2}R & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & \frac{-12}{\tau^3}R & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Phi_S = \begin{bmatrix} -2S & 0 & 0 & 0 & \frac{2}{\tau}S & 0 \\ * & \frac{\tau^2}{2}S & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & -\frac{2}{\tau^2}S & 0 \\ * & * & * & * & * & 0 \end{bmatrix}, \Phi_{\text{sat}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & -G' \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & -2H \end{bmatrix},$$

$$\Phi_M = \begin{bmatrix} -nM & 0 & 0 & \mathcal{I}\hat{M} & 0 & 0 \\ * & n(\tau+\mu_m)^2 M & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -\hat{M} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Phi_F = \begin{bmatrix} \bar{A}F' + F\bar{A}' & \varrho F\bar{A}' - F' & \bar{B}F' & 0 & 0 & (U\otimes B)H \\ * & -\varrho(F+F') & \varrho\bar{B}F' & 0 & 0 & \varrho(U\otimes B)H \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

and

$$\begin{aligned} \bar{A} &= I_{n-1} \otimes A, \\ \bar{B} &= -ULW \otimes BK, \\ \bar{L} &= \begin{bmatrix} L_1 W \otimes K & L_2 W \otimes K & \cdots & L_n W \otimes K \end{bmatrix}, \\ \mathcal{I} &= \mathbf{1}_n' \otimes I_{m(n-1)} \in \mathbb{R}^{m(n-1)\times nm(n-1)}, \\ \hat{M} &= I_n \otimes M \in \mathbb{R}^{nm(n-1)\times nm(n-1)}. \end{aligned}$$

Furthermore, a region from which the consensus is always attainable is given by $\mathcal{L}_V = \{\varphi(t_0) \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : V(\varphi_0) \leq 1\}$.

Proof. In the following, it is demonstrated that if the conditions in the theorem hold the origin of the system in (2.20) is an asymptotically stable equilibrium point, which implies, from Lemma 2.2, that the multi-agent system (2.10) reaches consensus.

In this regard, it is used the Lyapunov-Krasovskii functional candidate proposed in (2.21), with $\bar{P} = F^{-1}P(F^{-1})'$, $\bar{R} = F^{-1}R(F^{-1})'$, $\bar{S} = F^{-1}S(F^{-1})'$, $\bar{M} = F^{-1}M(F^{-1})'$, and $\bar{Z} =$

$F^{-1}Z(F^{-1})'$. Therefore, by requiring $P > 0$, $R > 0$, $S > 0$, $M > 0$ and $Z > 0$, the positivity of the Lyapunov-Krasovskii functional candidate is guaranteed. Moreover, the functions $u(\cdot)$ and $v(\cdot)$ in Lemma 2.1 can be built as $u(\|\boldsymbol{z}(t)\|) = \lambda_{\min}(P_1)\|\boldsymbol{z}(t)\|^2$ and $v(\|\varphi(t)\|) = \rho \sup_{\eta\in[t-\tau-\mu_m,t]}\|\dot{\boldsymbol{z}}(\eta)\|^2 + \lambda_{\max}(P)\sup_{\eta\in[t-\tau,t]}\|[\boldsymbol{z}(t)' \quad \tau\boldsymbol{z}(\eta)']\|^2$, with

$$\rho = \frac{\tau^2}{2}\lambda_{\max}(R) + 2\mu_m\tau\lambda_{\max}(Z) + \frac{\tau^2}{6}\lambda_{\max}(S) + \frac{n(\tau+\mu_m)^2}{2}\lambda_{\max}(M).$$

Next we show that the negativity of the functional derivative is guaranteed if the conditions in the theorem hold.

Using the Newton-Leibniz formula and (2.20), we have the following null term

$$\begin{aligned}
0 &= 2\Lambda(t)\left[\dot{\boldsymbol{z}}(t) - \bar{A}\boldsymbol{z}(t) - \sum_{i=1}^{n}\bar{B}_i\boldsymbol{z}(t-\tau_i(t)) - (U\otimes B)\Theta(\boldsymbol{g}(t))\right] \\
&= 2\Lambda(t)\left[\dot{\boldsymbol{z}}(t) - \bar{A}\boldsymbol{z}(t) - \sum_{i=1}^{n}\bar{B}_i\boldsymbol{z}(t-\tau) + \sum_{i=1}^{n}\bar{B}_i\int_{-\tau_i(t)}^{-\tau}\dot{\boldsymbol{z}}(t+\epsilon)d\epsilon - (U\otimes B)\Theta(\boldsymbol{g}(t))\right] \\
&= 2\Lambda(t)\left[\dot{\boldsymbol{z}}(t) - \bar{A}\boldsymbol{z}(t) - \bar{B}\boldsymbol{z}(t-\tau) - (U\otimes B)\Theta(\boldsymbol{g}(t))\right] + v(t), \qquad (2.27)
\end{aligned}$$

where,

$$\Lambda(t) = -\boldsymbol{z}(t)'F^{-1} - \varrho\dot{\boldsymbol{z}}(t)'F^{-1}, \qquad (2.28)$$

$$\bar{A} = I_{n-1}\otimes A, \qquad (2.29)$$

$$\bar{B}_i = -(UL_iW)\otimes BK, \qquad (2.30)$$

$$\bar{B} = -(ULW)\otimes BK, \qquad (2.31)$$

$$v(t) = \sum_{i=1}^{n}\int_{-\tau_i(t)}^{-\tau} 2\Lambda(t)\bar{B}_i\dot{\boldsymbol{z}}(t+\epsilon)d\epsilon, \qquad (2.32)$$

and ϱ is a free constant for adjustment. Moreover, assuming $Z > 0$ we have that[1]

$$\begin{aligned}
v(t) \leq & \sum_{i=1}^{n}\int_{-\tau_i(t)}^{-\tau}\Big(\Lambda(t)\bar{B}_i\Big)n\bar{Z}^{-1}\Big(\Lambda(t)\bar{B}_i\Big)'d\epsilon + \sum_{i=1}^{n}\int_{-\tau_i(t)}^{-\tau}\dot{\boldsymbol{z}}(t+\epsilon)'\frac{\bar{Z}}{n}\dot{\boldsymbol{z}}(t+\epsilon)d\epsilon \\
\leq & \sum_{i=1}^{n}\mu_m\Big(\Lambda(t)\bar{B}_i\Big)n\bar{Z}^{-1}\Big(\Lambda(t)\bar{B}_i\Big)' + \int_{t-\tau-\mu_m}^{t-\tau+\mu_m}\dot{\boldsymbol{z}}(\epsilon)'\bar{Z}\dot{\boldsymbol{z}}(\epsilon)d\epsilon. \qquad (2.33)
\end{aligned}$$

Therefore, defining $\hat{F} = F^{-1}$ to alleviate notation, then combining (2.27) and (2.33) it yields

$$\begin{aligned}
0 \leq & -2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\dot{\boldsymbol{z}}(t) + 2\boldsymbol{z}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) - 2\boldsymbol{z}(t)'\hat{F}\dot{\boldsymbol{z}}(t) + 2\boldsymbol{z}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau) \\
& + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t)) + 2\boldsymbol{z}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t)) \\
& + \sum_{i=1}^{n}\mu_m\Big(\Lambda(t)\bar{B}_i\Big)nZ^{-1}\Big(\Lambda(t)\bar{B}_i\Big)' + \int_{t-\tau-\mu_m}^{t-\tau+\mu_m}\dot{\boldsymbol{z}}(\epsilon)'Z\dot{\boldsymbol{z}}(\epsilon)d\epsilon. \qquad (2.34)
\end{aligned}$$

[1] We used the inequality $2\boldsymbol{a}'\boldsymbol{b} \leq \boldsymbol{a}'X\boldsymbol{a} + \boldsymbol{b}'X^{-1}\boldsymbol{b}$ where $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^{m(n-1)}$ and $X > 0 \in \mathbb{R}^{m(n-1)\times m(n-1)}$.

The functional candidate (2.21) time-derivative is given by

$$\begin{aligned}\dot{V}(\varphi(t)) =&\chi(t)'\bar{P}\dot{\chi}(t) + \dot{\chi}(t)'\bar{P}\chi(t) + \tau\dot{\boldsymbol{z}}(t)'\bar{R}\dot{\boldsymbol{z}}(t) - \int_{t-\tau}^{t}\dot{\boldsymbol{z}}(\epsilon)'\bar{R}\dot{\boldsymbol{z}}(\epsilon)d\epsilon + \frac{\tau^2}{2}\dot{\boldsymbol{z}}(t)'\bar{S}\dot{\boldsymbol{z}}(t)\\ &- \int_{-\tau}^{0}\int_{t+s}^{t}\dot{\boldsymbol{z}}(\epsilon)'\bar{S}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds + 2\mu_m\dot{\boldsymbol{z}}(t)'\bar{Z}\dot{\boldsymbol{z}}(t) - \int_{t-\tau-\mu_m}^{t-\tau+\mu_m}\dot{\boldsymbol{z}}(\epsilon)'\bar{Z}\dot{\boldsymbol{z}}(\epsilon)d\epsilon\\ &+ n(\tau+\mu_m)^2\dot{\boldsymbol{z}}(t)'\bar{M}\dot{\boldsymbol{z}}(t) - (\tau+\mu_m)\sum_{i=1}^{n}\int_{t-\tau-\mu_m}^{t}\dot{\boldsymbol{z}}(\epsilon)'\bar{M}\dot{\boldsymbol{z}}(\epsilon)d\epsilon,\end{aligned}$$

then, adding in it with (2.34) we have

$$\begin{aligned}\dot{V}(\varphi(t)) \leq&\chi(t)'\bar{P}\dot{\chi}(t) + \dot{\chi}(t)'\bar{P}\chi(t) + \tau\dot{\boldsymbol{z}}(t)'\bar{R}\dot{\boldsymbol{z}}(t) - \int_{t-\tau}^{t}\dot{\boldsymbol{z}}(\epsilon)'\bar{R}\dot{\boldsymbol{z}}(\epsilon)d\epsilon + \frac{\tau^2}{2}\dot{\boldsymbol{z}}(t)'\bar{S}\dot{\boldsymbol{z}}(t)\\ &- \int_{-\tau}^{0}\int_{t+s}^{t}\dot{\boldsymbol{z}}(\epsilon)'\bar{S}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds + 2\mu_m\dot{\boldsymbol{z}}(t)'\bar{Z}\dot{\boldsymbol{z}}(t) + n(\tau+\mu_m)^2\dot{\boldsymbol{z}}(t)'\bar{M}\dot{\boldsymbol{z}}(t)\\ &- (\tau+\mu_m)\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\dot{\boldsymbol{z}}(\epsilon)'\bar{M}\dot{\boldsymbol{z}}(\epsilon)d\epsilon - 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\dot{\boldsymbol{z}}(t) + 2\boldsymbol{z}(t)'\hat{F}\bar{A}\boldsymbol{z}(t)\\ &+ 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) - 2\boldsymbol{z}(t)'\hat{F}\dot{\boldsymbol{z}}(t) + 2\boldsymbol{z}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau)\\ &+ 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t)) + 2\boldsymbol{z}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t))\\ &+ \sum_{i=1}^{n}\mu_m\Big(\Lambda(t)\bar{B}_i\Big)n\bar{Z}^{-1}\Big(\Lambda(t)\bar{B}_i\Big)'.\end{aligned}$$

Finally, applying Lemma A.1 on the integral with $\bar{S}$, Lemma A.2 on the integral with term $\bar{R}$, and Lemma A.3 on each integral term with $\bar{M}$, assuming that $\boldsymbol{g}(t) \in \mathbb{S}(\boldsymbol{\omega}(t), u_{\max})$ and applying the inequality $-2\Theta(\boldsymbol{g}(t))'T[\Theta(\boldsymbol{g}(t)) + \boldsymbol{\omega}(t)] \geq 0$ from Lemma 2.3, with $\boldsymbol{\omega}(t) = \bar{G}\hat{\boldsymbol{z}}(t-\boldsymbol{\tau}(t))'$ in which $\bar{G} \in \mathbb{R}^{np\times m(n-1)}$ is a matrix variable and

$$\hat{\boldsymbol{z}}(t-\boldsymbol{\tau}(t))' = \Big[\boldsymbol{z}(t-\tau_1(t))' \quad \boldsymbol{z}(t-\tau_2(t))' \quad \cdots \quad \boldsymbol{z}(t-\tau_n(t))'\Big],$$

yields:

$$\begin{aligned}\dot{V}(\varphi(t)) \leq&\chi(t)'\bar{P}\dot{\chi}(t) + \dot{\chi}(t)'\bar{P}\chi(t) + \tau\dot{\boldsymbol{z}}(t)'\bar{R}\dot{\boldsymbol{z}}(t) - \frac{1}{\tau}\int_{t-\tau}^{t}\dot{\boldsymbol{z}}(\epsilon)'d\epsilon\bar{R}\int_{t-\tau}^{t}\dot{\boldsymbol{z}}(\epsilon)d\epsilon - \frac{3}{\tau}\Omega'\bar{R}\Omega\\ &+ \frac{\tau^2}{2}\dot{\boldsymbol{z}}(t)'\bar{S}\dot{\boldsymbol{z}}(t) + 2\mu_m\dot{\boldsymbol{z}}(t)'\bar{Z}\dot{\boldsymbol{z}}(t) - \frac{2}{\tau^2}\int_{-\tau}^{0}\int_{t+s}^{t}\dot{\boldsymbol{z}}(\epsilon)'d\epsilon ds\bar{S}\int_{-\tau}^{0}\int_{t+s}^{t}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds\\ &+ n(\tau+\mu_m)^2\dot{\boldsymbol{z}}(t)'\bar{M}\dot{\boldsymbol{z}}(t) - \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\dot{\boldsymbol{z}}(\epsilon)'d\epsilon\bar{M}\int_{t-\tau_i(t)}^{t}\dot{\boldsymbol{z}}(\epsilon)d\epsilon - 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\dot{\boldsymbol{z}}(t)\\ &+ 2\boldsymbol{z}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) - 2\boldsymbol{z}(t)'\hat{F}\dot{\boldsymbol{z}}(t) + 2\boldsymbol{z}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau)\\ &+ 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t)) + 2\boldsymbol{z}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t))\\ &- 2\Theta\Big(\boldsymbol{g}(t)\Big)'T\Big[\Theta\Big(\boldsymbol{g}(t)\Big) + \boldsymbol{\omega}(t)\Big] + \sum_{i=1}^{n}\mu_m\Big(\Lambda(t)\bar{B}_i\Big)n\bar{Z}^{-1}\Big(\Lambda(t)\bar{B}_i\Big)'\end{aligned}$$

$$
\begin{aligned}
&\leq \chi(t)'\bar{P}\dot{\chi}(t) + \dot{\chi}(t)'\bar{P}\chi(t) + \tau\dot{\boldsymbol{z}}(t)'\bar{R}\dot{\boldsymbol{z}}(t) - \frac{1}{\tau}\Big(\boldsymbol{z}(t) - \boldsymbol{z}(t-\tau)\Big)'\bar{R}\Big(\boldsymbol{z}(t) - \boldsymbol{z}(t-\tau)\Big) \\
&\quad - \frac{3}{\tau}\Omega'\bar{R}\Omega + \frac{\tau^2}{2}\dot{\boldsymbol{z}}(t)'\bar{S}\dot{\boldsymbol{z}}(t) - 2\boldsymbol{z}(t)'\bar{S}\boldsymbol{z}(t) + \frac{4}{\tau}\boldsymbol{z}(t)'\bar{S}\int_{t-\tau}^{t}\boldsymbol{z}(\epsilon)d\epsilon \\
&\quad - \frac{2}{\tau^2}\int_{t-\tau}^{t}\boldsymbol{z}(\epsilon)'d\epsilon\bar{S}\int_{t-\tau}^{t}\boldsymbol{z}(\epsilon)d\epsilon + n(\tau+\mu_m)^2\dot{\boldsymbol{z}}(t)'\bar{M}\dot{\boldsymbol{z}}(t) \\
&\quad - \sum_{i=1}^{n}\Big(\boldsymbol{z}(t) - \boldsymbol{z}(t-\tau_i(t))\Big)'\bar{M}\Big(\boldsymbol{z}(t) - \boldsymbol{z}(t-\tau_i(t))\Big) + 2\mu_m\dot{\boldsymbol{z}}(t)'\bar{Z}\dot{\boldsymbol{z}}(t) - 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\dot{\boldsymbol{z}}(t) \\
&\quad + 2\boldsymbol{z}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{A}\boldsymbol{z}(t) - 2\boldsymbol{z}(t)'\hat{F}\dot{\boldsymbol{z}}(t) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t)) \\
&\quad + 2\boldsymbol{z}(t)'\hat{F}(U\otimes B)\Theta(\boldsymbol{g}(t)) + 2\boldsymbol{z}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau) + 2\varrho\dot{\boldsymbol{z}}(t)'\hat{F}\bar{B}\boldsymbol{z}(t-\tau) \\
&\quad - 2\Theta\Big(\boldsymbol{g}(t)\Big)'T\Big[\Theta\Big(\boldsymbol{g}(t)\Big) + \boldsymbol{\omega}(t)\Big] + \sum_{i=1}^{n}\mu_m\Big(\Lambda(t)\bar{B}_i\Big)n\bar{Z}^{-1}\Big(\Lambda(t)\bar{B}_i\Big)' \\
&\leq \Upsilon(t)'\hat{\Phi}\Upsilon(t) + \sum_{i=1}^{n}\mu_m\Upsilon(t)'\Gamma_i n\bar{Z}^{-1}\Gamma_i'\Upsilon(t) \\
&= \Upsilon(t)'\left(\sum_{i=1}^{n}\left(\frac{1}{n}\hat{\Phi} + \mu_m\Gamma_i n\bar{Z}^{-1}\Gamma_i'\right)\right)\Upsilon(t),
\end{aligned} \tag{2.35}
$$

with

$$
\begin{aligned}
\Upsilon(t)' &= \Big[\boldsymbol{z}(t)' \quad \dot{\boldsymbol{z}}(t)' \quad \boldsymbol{z}(t-\tau)' \quad \hat{\boldsymbol{z}}(t-\boldsymbol{\tau}(t))' \quad \textstyle\int_{t-\tau}^{t}\boldsymbol{z}(\epsilon)'d\epsilon \quad \Theta(\boldsymbol{g}(t))'\Big], \\
\hat{\boldsymbol{z}}(t-\boldsymbol{\tau}(t))' &= \Big[\boldsymbol{z}(t-\tau_1(t))' \quad \boldsymbol{z}(t-\tau_2(t))' \quad \cdots \quad \boldsymbol{z}(t-\tau_n(t))'\Big] \\
\Gamma_i' &= \Big[(\hat{F}\bar{B}_i)' \quad \varrho(\hat{F}\bar{B}_i)' \quad 0 \quad 0 \quad 0 \quad 0\Big],
\end{aligned}
$$

$\hat{\Phi}$ defined with similar structure as (2.26), however with matrices $\bar{P}$, $\bar{S}$, $\bar{Z}$, $\bar{R}$, $\bar{M}$ and

$$
\hat{\Phi}_{sat} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & -\bar{G}'T' \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & -2T \end{bmatrix},
$$

instead of P, S, Z, R, M, and Φ_{sat}.

Finally, $\dot{V}(\varphi(t)) \leq -w(\|\boldsymbol{z}(t)\|) = -\alpha\|\boldsymbol{z}(t)\|^2 < 0$ for any $\Upsilon(t) \neq 0$, for some $\alpha > 0$ sufficiently small, is achieved whenever the term inside parenthesis in equation (2.35) is negative definite, which from Schur complement is equivalent to

$$
\sum_{i=1}^{n}\begin{bmatrix} \frac{1}{n}\hat{\Phi} & \mu_m\Gamma_i \\ * & \frac{-\mu_m}{n}Z \end{bmatrix} = \begin{bmatrix} \hat{\Phi} & \mu_m\Gamma \\ * & -\mu_m\bar{Z} \end{bmatrix} < 0. \tag{2.36}
$$

The inequality (2.36) becomes identical to condition (2.23) after pre- and post-multiplying it with

$$
\mathrm{diag}(F, F, \cdots, F, T^{-1}, F)
$$

and its transpose respectively, and making the change of variables $H = T^{-1}$, $G = \bar{G}F'$.

Therefore, if conditions (2.23) and (2.24) of Theorem 2.1, and $\boldsymbol{g}(t) \in \mathbb{S}(\boldsymbol{\omega}(t), u_{\max})$ hold we have $V(\varphi(t)) > 0$ and $\dot{V}(\varphi(t)) < 0$, which implies that the multi-agent system reaches consensus asymptotically.

Henceforth, we show that if inequality (2.25) is satisfied then $\boldsymbol{g}(t) \in \mathbb{S}(\boldsymbol{\omega}(t), u_{\max})$, and from the generalized sector condition in Lemma 2.3, $-2\Theta(\boldsymbol{g}(t))'T[\Theta(\boldsymbol{g}(t)) + \boldsymbol{\omega}(t)] \geq 0$ holds. In addition, it is demonstrated that all trajectories $\varphi(t)$ starting from $\mathcal{L}_V$ remains within $\mathcal{L}_V$ and go to the origin.

The inequality (2.25) implies that whenever the trajectories of the multi-agent system $\varphi(t)$ belong to $\mathcal{L}_V$, the inputs belong to the generalised sector condition $\mathbb{S}(\boldsymbol{\omega}(t), u_{\max})$, in which $\mathcal{L}_V$ represents a level set of the Lyapunov-Krasovskii functional. That is

$$\left|\Big((\bar{L} - \bar{G})\hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))\Big)_{(r)}\right| \leq u_{\max}, \ \forall \varphi(t) \in \mathcal{L}_V,$$

for $r = 1, \ldots, np$, where in Lemma 2.3, $\boldsymbol{u}(t) = \bar{L}\hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))$ and $\boldsymbol{\omega}(t) = \bar{G}\hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))$. The previous inequality holds whenever

$$\begin{aligned}\Big((\bar{L} - \bar{G})\hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))\Big)'_{(r)} u_{\max}^{-2} \Big((\bar{L} - \bar{G})\hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))\Big)_{(r)} \\ \leq (\hat{\boldsymbol{z}}(t) - \hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t)))'(I_n \otimes \bar{M})(\hat{\boldsymbol{z}}(t) - \hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))) \leq 1\end{aligned}$$

which, by Schur complement and pre- and post-multiplying the result by $\mathrm{diag}((I_n \otimes F), 1)$ and its transpose, respectively, becomes the inequality (2.25). Furthermore, we have for all trajectories originating from $\varphi_0 \in \mathcal{L}_V$,

$$(\hat{\boldsymbol{z}}(t) - \hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t)))'(I_n \otimes \bar{M})(\hat{\boldsymbol{z}}(t) - \hat{\boldsymbol{z}}(t - \boldsymbol{\tau}(t))) \leq V(\varphi_0) \leq 1,$$

where the left-hand side inequality is obtained by applying Lemma A.1 in the integral terms with $\bar{M}$ of the Lyapunov-Krasovskii functional candidate and noticing that $\int_{t-\tau_i(t)}^{t} \dot{\boldsymbol{z}}(\epsilon)d(\epsilon) = \boldsymbol{z}(t) - \boldsymbol{z}(t - \tau_i(t))$. Hence, with condition (2.25), we ensure that $\boldsymbol{g}(t)$ belongs to $\mathbb{S}(\boldsymbol{\omega}(t), u_{\max})$, consequently the condition in Lemma 2.3 is satisfied. Therefore, because $\mathcal{L}_V$ is a level set of $V(\varphi(t))$ and the inequalities $V(\varphi(t)) > 0$ and $\dot{V}(\varphi(t)) < 0$ hold, the set $\mathcal{L}_V$ is positively invariant, which concludes the proof. □

Theorem 2.1 presents sufficient conditions to guarantee consensus of multi-agent system with agents described by (2.5), particularly it characterizes the multi-agent system stability described as equation (2.20), which results in consensus. Additionally it gives a procedure to compute a region from which the consensus is always attainable.

It is important to notice that part of the development of the demonstration of Theorem 2.1 is based on (Savino, Souza, and Pimenta, 2014; Savino et al., 2016) and (Souza, 2013). In particular, the Lyapunov-Krasovskii functional candidate, the null-term to take into account the dynamics of the system, and the bounds of time-delays are similar to the ones in those studies.

Remark 2.1. The adjustment parameter ϱ is an attempt to add a degree of freedom to the conditions, which is added by means of the structure in $\Lambda(t)$ in (2.28). However, no systematic procedure to tune it was developed. In all examples its value is explicitly given, and, in general, they were found by searching the best result in a given bounded region.

2.2.2 Stabilization

In the next theorem, we present conditions to design stabilizing state feedback matrices for the agents in the continuous-time domain, along with a region of ensured convergence. The results are extensions of the previous one based on the Lyapunov-Krasovskii methodology and are derived by selecting a particular structure for a matrix variable.

Theorem 2.2. *Let the scalars τ, μ_m, and ϱ be given. The multi-agent system (2.5) with consensus protocol (2.3) subject to saturating inputs and time-varying delays $\tau_i(t) \in [\tau - \mu_m, \tau + \mu_m]$ with $\tau > 0$, $0 \leq \mu_m \leq \tau$, reaches consensus with state feedback matrix $K{=}\bar{K}(F_m^{-1})'$, if there exist symmetric positive definite matrices P_1, P_3, R, S, M, and $Z \in \mathbb{R}^{m(n-1)\times m(n-1)}$, matrices $P_2 \in \mathbb{R}^{m(n-1)\times m(n-1)}$, $G \in \mathbb{R}^{np\times nm(n-1)}$, $\bar{K} \in \mathbb{R}^{p\times m}$, a diagonal positive definite matrix $H \in \mathbb{R}^{np\times np}$, and a non-singular matrix $\bar{F} = I_{n-1} \otimes F_m \in \mathbb{R}^{m(n-1)\times m(n-1)}$ with $F_m \in \mathbb{R}^{m\times m}$, such that the following conditions are satisfied:*

$$\begin{bmatrix} \Phi & \mu_m\Gamma \\ * & -\mu_m Z \end{bmatrix} < 0, \tag{2.37}$$

$$\begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0, \tag{2.38}$$

and

$$\begin{bmatrix} \hat{M} & -\hat{M} & 0 \\ * & \hat{M} & (\hat{L}-G)'_{(r)} \\ * & * & u_{\max}^2 \end{bmatrix} \geq 0, \quad \forall r \in \{1,\ldots,np\}, \tag{2.39}$$

where

$$\Phi = \Phi_P + \Phi_Z + \Phi_R + \Phi_S + \Phi_{\text{sat}} + \Phi_M + \Phi_{\bar{F}}, \tag{2.40}$$

$$\Gamma = \begin{bmatrix} \hat{B} \\ \varrho\hat{B} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Phi_Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 2\mu_m Z & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Phi_P = \begin{bmatrix} P_2 + P_2' & P_1 & -P_2 & 0 & P_3 & 0 \\ * & 0 & 0 & 0 & P_2 & 0 \\ * & * & 0 & 0 & -P_3 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}, \Phi_R = \begin{bmatrix} -\frac{4}{\tau}R & 0 & -\frac{2}{\tau}R & 0 & \frac{6}{\tau^2}R & 0 \\ * & \tau R & 0 & 0 & 0 & 0 \\ * & * & -\frac{4}{\tau}R & 0 & \frac{6}{\tau^2}R & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & \frac{-12}{\tau^3}R & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Phi_S = \begin{bmatrix} -2S & 0 & 0 & 0 & \frac{2}{\tau}S & 0 \\ * & \frac{\tau^2}{2}S & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & -\frac{2}{\tau^2}S & 0 \\ * & * & * & * & * & 0 \end{bmatrix}, \Phi_{\text{sat}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & -G' \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & -2H \end{bmatrix},$$

$$\Phi_M = \begin{bmatrix} -nM & 0 & 0 & \mathcal{I}\hat{M} & 0 & 0 \\ * & n(\tau+\mu_m)^2 M & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -\hat{M} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Phi_{\bar{F}} = \begin{bmatrix} \bar{A}\bar{F}' + \bar{F}\bar{A}' & \varrho\bar{F}\bar{A}' - \bar{F}' & \hat{B} & 0 & 0 & (U\otimes B)H \\ * & -\varrho(\bar{F}+\bar{F}') & \varrho\hat{B} & 0 & 0 & \varrho(U\otimes B)H \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

and

$$\begin{aligned} \bar{A} =& I_{n-1}\otimes A, \\ \hat{B} =& -ULW\otimes B\bar{K}, \\ \hat{L} =& \Big[L_1W\otimes\bar{K} \quad L_2W\otimes\bar{K} \quad \cdots \quad L_nW\otimes\bar{K}\Big], \\ \mathcal{I} =& \mathbf{1}_n'\otimes I_{m(n-1)} \in \mathbb{R}^{m(n-1)\times nm(n-1)}, \\ \hat{M}=& I_n\otimes M \in \mathbb{R}^{nm(n-1)\times nm(n-1)}. \end{aligned}$$

Furthermore, a region from which the consensus is always attainable is given by $\mathcal{L}_V = \{\varphi(t_0) \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : V(\varphi_0) \le 1\}$.

Proof. The proof follows directly from the results of Theorem 2.1 as an extension of those conditions. Replacing F by the structured variable $\bar{F} = I_{n-1}\otimes F_m$, $F_m \in \mathbb{R}^{m\times m}$ allows us to benefit from products $\bar{L}(I_n\otimes\bar{F}')$ and $\bar{B}\bar{F}'$. Specifically, those products appear in Γ on inequality (2.23) and in Φ_F on equation (2.26), which is replaced by $\Phi_{\bar{F}}$ in (2.40). Observe that, because of the chosen structure, the products result in:

$$\begin{aligned} \bar{B}\bar{F}' &= (ULW\otimes BK)(I_{n-1}\otimes F_m) = ULW\otimes BKF_m', \\ \bar{L}(I_n\otimes\bar{F}') &= \Big[L_1W\otimes KF_m' \quad L_2W\otimes KF_m' \quad \cdots \quad L_nW\otimes KF_m'\Big]. \end{aligned}$$

Finally, substituting the product KF_m', by the variable $\bar{K}$ yields to conditions (2.37) and (2.39), and the state feedback for each agent can be retrieved from $K = \bar{K}(F_m^{-1})'$, which concludes the proof. □

Theorems 2.1 and 2.2 provide sufficient conditions to guarantee the convergence in the form of feasibility problems, consequently, the region provided by those conditions might not be the largest one. In this regard, we propose next a procedure to assist the maximization of such regions.

2.2.3 Characterization of the Set of Initial Conditions

Because the Lyapunov-Krasovskii candidate in (2.21) depends on the derivative of current and delayed states, and these values might not affect the safe initialization of the network, it is presented here a characterization of the region of guaranteed convergence for the multi-agent system. The motivation for distinguishing initial regions is to jointly compute distinct regions for the current states, delayed states, and derivative of state variables. Specifically, the estimate of the set of admissible initial conditions is such that:

$$\varphi(t_0) = \left\{ \varphi \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : \varphi \in \mathcal{Y}, \dot{\varphi} \in \mathcal{B}(r_1) \right\},$$

where $\mathcal{Y}$ is an ellipsoidal set and $\mathcal{B}(r_1)$ is a ball that contains current and delayed first derivative of the state variables, respectively. These sets, which allow to characterize the region of convergence, are defined in the next proposition.

Proposition 2.1. *If (2.21) is a Lyapunov-Krasovskii functional of the closed-loop multi-agent system (2.20), then initial states φ_0 verifying*

$$\varphi(t_0) = \left\{ \varphi \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : \varphi \in \mathcal{Y}, \dot{\varphi} \in \mathcal{B}(r_1) \right\}, \tag{2.41}$$

with

$$\mathcal{Y} = \left\{ \varphi \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : \chi' P \chi \leq 1 - \gamma \right\} \tag{2.42}$$

$$\mathcal{B}(r_1) = \left\{ \dot{\varphi} \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : \sup_{\eta \in [-\tau-\mu_m]} \|\dot{\boldsymbol{z}}(\eta)\| \leq r_1 \right\} \tag{2.43}$$

where

$$\gamma = \rho \sup_{\eta \in [-\tau-\mu_m]} \|\dot{\boldsymbol{z}}(\eta)\|^2,$$

$$\rho = \frac{\tau^2}{2}\lambda_{\max}(\bar{R}) + 2\mu_m \tau \lambda_{\max}(\bar{Z}) + \frac{\tau^2}{6}\lambda_{\max}(\bar{S}) + \frac{n(\tau+\mu_m)^2}{2}\lambda_{\max}(\bar{M}) \tag{2.44}$$

with

$$\sup_{\eta \in [-\tau-\mu_m]} \|\dot{\boldsymbol{z}}(\eta)\| \leq r_1, \ \textit{and } 0 \leq \rho r_1^2 < 1, \tag{2.45}$$

do not leave the estimate of the domain of consensus $\mathcal{L}_V$ and converge asymptotically.

Proof. The demonstration is based on the ideas proposed in Lemma 3 in Castro et al. (2020). Provided that (2.21) is a Lyapunov-Krasovskii functional, we can define a level set given by

$$\mathcal{L}_V = \{\varphi \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : \chi' P \chi \leq 1 - \gamma\},$$

by using the terms on $\bar{S}$, $\bar{Z}$, $\bar{R}$, and $\bar{M}$ of (2.21) we get

$$\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+s}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{S}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds d\theta + \int_{-\mu_m}^{\mu_m}\int_{t+s-\tau}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{Z}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds$$
$$+ \int_{-\tau}^{0}\int_{t+s}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{R}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds + \sum_{i=1}^{n}\int_{-\tau-\mu_m}^{0}\int_{t+s}^{t} \dot{\boldsymbol{z}}(\epsilon)'\bar{M}\dot{\boldsymbol{z}}(\epsilon)d\epsilon ds \leq \rho \sup_{\eta\in[-\tau-\mu_m]} \|\dot{\boldsymbol{z}}(\eta)\|^2,$$

with ρ given in (2.44). To ensure that $0 \leq \gamma < 1$ the inequalities (2.45) are required to be satisfied. Since (2.21) is a Lyapunov-Krasovskii functional, with the conditions in this proposition $\mathcal{L}_V$ is a level set of the functional, satisfying $\mathcal{L}_V = \{\varphi(t_0) \in \mathcal{C}^{nm}_{t_0,\tau+\mu_m} : V(\varphi(t_0)) \leq 1\}$, thus all trajectories starting from $\mathcal{L}_V$ remains in $\mathcal{L}_V$ and attain consensus asymptotically. □

The characterization in Proposition 2.1 gives flexibility when estimating the region of convergence when compared, for example, with the method proposed in Silva (2019). Thus, it is possible to explicitly choose which of the sets $\mathcal{Y}$ and $\mathcal{B}(r_1)$ are more significant, allowing to possibly compute less conservative estimates for the required region. For example, it is possible to cast a region in which all current and delayed variables states are inside a ball, as well as the initial time-derivative of the states. It is also possible to ignore their time-derivatives and choose the best solution for current and delayed state variables. Moreover, the structure of $\chi' P \chi$ can be explored to compute a region for current states only, by letting delayed states equals zero the ellipsoid $\mathcal{Y}$ is reduced to $\mathcal{Y} = \{\varphi : \boldsymbol{z}(t)' P_1 \boldsymbol{z}(t) \leq 1\}$.

Maximization of the allowed initial conditions

With the conditions proposed in Theorems 2.1 and 2.2, we are interested in finding a set of initial conditions as large as possible included in the domain of consensus. To this end, inspired by the development in Castro et al. (2020), we propose a maximization of an ellipsoid within the region $\mathcal{L}_V$, which can be obtained by solving the following optimization problem:

$$\begin{aligned} &\min \operatorname{trace}\left(\sum_{j=1}^{5}\alpha_j W_j\right) \\ &\text{s.t.} \begin{cases} a) & \text{LMIs (2.23), (2.24) and (2.25) from Theorem 2.1,} \\ & \text{or (2.37), (2.38) and (2.39) from Theorem 2.2.} \\ & \\ b) & \begin{bmatrix} W_j & I \\ I & Y + Y' - X_j \end{bmatrix} \geq 0, \end{cases} \end{aligned} \tag{2.46}$$

where α_j are scalar weights, W_j are variable matrices with appropriated dimensions, $Y = F$ for Theorem 2.1 or $Y = \bar{F}$ for Theorem 2.2, and X_j corresponds to the jth matrix in the set $\{P, R, S, M, Z\}$.

Remark 2.2. In order to handle the nonlinear matrix inequalities on the estimation for the domain of consensus, the inequality $FX_j^{-1}F' \geq F + F' - X_j$ was used multiple times. That was done because, from the Lyapunov-Krasovskii candidate (2.21), it was chosen $\bar{X}_j = (F^{-1})' X_j F^{-1}$. Although this can insert some conservatism, it leads to a linear optimization problem for synthesizing the gains of the agents and determine a region from which the consensus is always attainable.

2.2.4 Numerical Examples

This section shows examples to illustrate the use of Theorems 2.1 and 2.2. The methodology is compared with the works in You et al. (2016), Ding, Zheng, and Guo (2018), and Silva (2019), in which similar conditions are assumed on the consensus problem. In addition, it is discussed the numerical complexity of the approach.

Example 2.1. *In this example, the analysis conditions of You et al. (2016) and Silva (2019) are contrasted with the result proposed here, because of this, a similar structure in the example is examined. Consider a multi-agent system composed of five agents with directed communication topology, as shown in Figure 2.3, with Laplacian matrix given by*

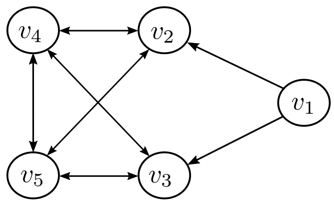


FIGURE 2.3: Example 2.1–Graph representation of agents interaction.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

This is a leader-following problem–all followers have to synchronize their states with the states of the leader (v_1). It is assumed that the followers are modeled by,

$$\begin{bmatrix} \dot{x}_i^1(t) \\ \dot{x}_i^2(t) \\ \dot{x}_i^3(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \\ x_i^3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \operatorname{sat}\Big(\boldsymbol{u}_i(t)\Big),$$

numbered $i = 2, \ldots, 5$, where $x_i^l(t)$ denote the lth element of $\boldsymbol{x}_i(t)$, and the model of the leader is given by,

$$\begin{bmatrix} \dot{x}_1^1(t) \\ \dot{x}_1^2(t) \\ \dot{x}_1^3(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} x_1^1(t) \\ x_1^2(t) \\ x_1^3(t) \end{bmatrix}.$$

The matrix gain K in the consensus protocol (2.2), is

$$K = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.7 \end{bmatrix},$$

and the limit of actuators $u_{max} = 2.5$.

In the studies in You et al. (2016) and Silva (2019), it is assumed synchronous time-varying delays $\tau_i(t)$, satisfying $0 \leq \tau_i(t) \leq \tau$ with $\dot{\tau}_i(t) \leq \mu_m < 1$, where τ and μ_m are positive constants. Table 2.1 shows the maximum allowed constant part for the delays representation (τ_i) obtained by Theorem 4.1 in Silva (2019) with $\varrho = 0.4$, Theorem 1 in You et al. (2016), and Theorem 2.1 with $\varrho = 0.7$, considering different values for the bound of delays time-varying part.

TABLE 2.1: Example 2.1–Comparison of maximum input delay τ_{max} for various time-varying parts μ_m.

	$\mu_m = 0.01$	$\mu_m = 0.05$	$\mu_m = 0.10$
You et al. (2016)	0.35	0.34	0.34
Theorem 4.1 in Silva (2019)	0.43	0.34	0.23
Theorem 2.1	0.45	0.35	0.24

It is shown that as the time-varying part of the delay (μ_m) increases the approach of You et al. (2016) guarantees biggest values for the constant part (τ) in comparison to the values obtained by Silva (2019) and Theorem 2.1, while the approach proposed here appears to have slightly less conservatism as the one in Silva (2019). However, note that the conditions in Silva (2019) and in Theorem 2.1 guarantee consensus for non-uniform and non-differentiable delays within the set $[\tau - \mu_m, \tau + \mu_m]$, while You et al. (2016) consider synchronous and differentiable delays. The conditions presented here are more general and might be applied to a larger number of systems. Besides, unlike You et al. (2016), the approaches in Silva (2019) and in Theorem 2.1 give a region from which the consensus is guaranteed. Furthermore, the conditions in You et al. (2016) are nonlinear, which in general are hard to solve.

Example 2.2. *In this example, the numerical complexity of the proposed conditions is investigated. The computational algorithms to solve LMIs have polynomial time in the worst-case scenario with the number of decision variables[2] (Boyd et al., 1994). Some methods have obtained a computational complexity of $O(NL)$ (Potra and Wright, 2000; Monteiro and Zhang, 1998), while some older ones $O(N^{1.5}L)$ (Monteiro and Zhang, 1998; Boyd et al., 1994), where N is the number of decision variables and L the number of constraints. Theorems 2.1 and 2.2 have symmetric and full matrices variables, in symmetric matrix variables the number of variables is given by $N = k(k+1)/2$, in which k is the number of lines of the matrix, and in full*

[2]In general, a solution of a semidefinite programming problem cannot be obtained in a finite number of interactions. Hence, an interior-point method for a semidefinite programming is said to be "polynomial" with complexity $O(n^{\omega}L)$, if there is a constant ω such that the distance to the optimum is reduced by a factor of $2^{-O(L)}$ in at most $O(n^{\omega}L)$ interactions, where L is the size of the problem data. Sometimes, in the context of Control Theory, L is accounted as the number of Lyapunov equations with same variables (i.e., number of linear constraints), e.g., $A_i'P + PA_i < 0$, $i = 1, \ldots, L$, where A_i are given matrices and P is a symmetric positive definite matrix variable (Boyd et al., 1994; Potra and Wright, 2000).

TABLE 2.2: Example 2.2–Contrast of the number of variables in the problem of Example 2.1.

	Number of Variables
You et al. (2016)	13
Theorem 4.1 in Silva (2019)	759
Theorem 2.1	1056

matrices they are given by $N = k^2$. Hence, we have the following equations for the number of variables in each theorem:

$$N_{\textit{Theorem 2.1}} = 5m^2(n-1)^2 + 3m(n-1) + \frac{1}{2}((np)^2 + np) + npm(n-1), \quad (2.47)$$

$$N_{\textit{Theorem 2.2}} = 4m^2(n-1)^2 + 3m(n-1) + \frac{1}{2}m(m+1) + \frac{1}{2}((np)^2 + np) + npm(n-1). \quad (2.48)$$

Therefore, the number of variables of the problems grows quadratically with the number of agents in networks with agents with constant dimensions. Hence, if we increase the number of vertices n in our multi-agent system by a factor of 10 *the running time can be expected to increase at least by a hundred if a method with $O(NL)$ is considered. Thus, our approach might be restricted to networks with a relatively small number of agents (when compared to studies on swarms of robots in which the interest hinges in networks with hundreds or even thousand vertices). Nevertheless, some authors consider algorithms with complexity equal or lower than $O(N^3)$ to be scalable in general, see for example Newman (2010). In Table 2.2 it is shown the number of variables of the method proposed here, in You et al. (2016), and in Silva (2019) for the multi-agent system of Example 2.1. It is worth to also mention that in practice, to simply attest the running time based on the complexity of the approach may not give an accurate view. For instance, it is necessary that the entire problem fits in the fast access computer memory. If, for any reason, the computer starts to store the problem variable in a slow access memory (e.g., a hard disk), the performance will be substantially reduced.*

Example 2.3. *In this example the gain design conditions are contrasted with the methods proposed by Ding, Zheng, and Guo (2018) and by Silva (2019). Because the study in Ding, Zheng, and Guo (2018) is concerned only with the leader-following problem, this task will be considered. Consider a multi-agent system consisting of four agents with directed communication topology, represented in Figure 2.4, with Laplacian matrix given by*

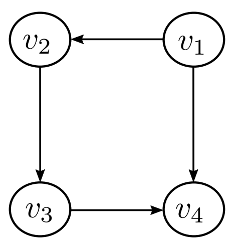


FIGURE 2.4: Example 2.3–Communication topology of multi-agent system.

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$

All agents are subject to input saturation and time-varying delays, the saturation limit is u_{max}=10 *and the ith agent dynamics is*

$$\begin{bmatrix} \dot{x}_i^1(t) \\ \dot{x}_i^2(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 1 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} x_i^1(t) \\ x_i^2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{sat}\Big(\boldsymbol{u}_i(t)\Big),$$

note that the open-loop dynamics is unstable.

The optimization problem (2.46), with $\varrho = 5.7$ *and* $\alpha_j = 0$ *for* $j = 2, \ldots, 5$*, Theorem* 4.1 *in Silva (2019), and Theorem* 2 *in Ding, Zheng, and Guo (2018) were used to compute the guaranteed domain of consensus for increasing values of the upper bound of the delay interval. Table*

TABLE 2.3: Example 2.3– Radius of the ball inscribed in the domain of consensus for different delay intervals.

$[\tau - \mu_m, \tau + \mu_m]$	$[0.02, 0.1]$	$[0.02, 0.2]$	$[0.02, 0.3]$	$[0.02, 0.4]$
Theorem 2 (Ding, Zheng, and Guo, 2018)	9.20	9.13	5.24	–
Theorem 4.1 (Silva, 2019)	17.80	17.23	16.62	16.00
Theorem 2.2	60.14	35.41	20.31	17.95

2.3 shows that Theorem 2.2 and Theorem 4.1 *in Silva (2019) guarantee consensus for intervals of delay for which the method of Ding, Zheng, and Guo (2018) was not able to ensure stability. In addition, Theorem 2.2 achieves a larger domain of consensus than both other approaches for all analyzed sets. Besides, the methods proposed here and in Silva (2019) are appropriated for consensus analysis with or without a leader and non-uniform delays, while Ding, Zheng, and Guo (2018) considered only the problem of leader-following with uniform delays.*

Another interesting highlight is that the studies made by Silva (2019) and Ding, Zheng, and Guo (2018) compute the region of guaranteed convergence provided by the biggest ball inscribed into the intersection of all the invariant sets calculated directly by the Lyapunov-Krasovskii functional. As shown in Proposition 2.1, this might take into account regions that might not be of interest for starting the systems. Also, to search for ellipsoidal sets seems to be much less conservative, as shown in Figure 2.5A where the ball computed with the approach in Silva (2019) and an ellipsoidal set calculated using the optimization problem (2.46) are contrasted, with delays $\tau_i(t) \in [0.02, 0.2]$.

To illustrate the evolution of the network, Figure 2.5B shows the trajectories of the multi-agent system with initial conditions belonging to the computed region of convergence, using the consensus protocol matrix gain $K = \begin{bmatrix} 0.0 & 0.258 \end{bmatrix}$, *designed with optimization problem (2.46), with* $\varrho = 5.7$ *and* $\alpha_j = 0$ *for* $j = 2, \ldots, 5$.

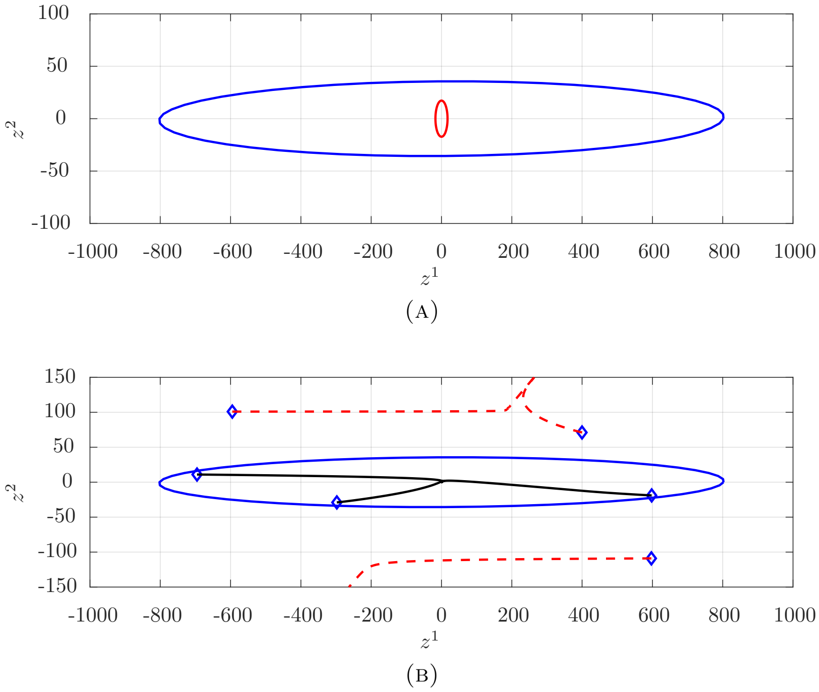


FIGURE 2.5: Example 2.3– The domain of consensus for $\tau_i(t) \in [0.02, 0.2]$: (A) calculated using Theorem 4.1 (Silva, 2019) (in red), and using Theorem 2.2 (in blue); (B) relative trajectories of the agents for starting outside (in red) and inside (in black) of the domain of consensus calculated with Theorem 2.2.

2.3 Consensus of Discrete-Time Systems: Lyapunov-Krasovskii Approach

This and the next sections investigate the consensus problem subject to time-delays and saturating inputs in the discrete-time domain. Two approaches are provided to tackle this problem: In the first approach, presented in this section, the Lyapunov-Krasovskii framework is applied in the same spirit as in the continuous-time domain proposed in Section 2.2. The second one, presented in Section 2.4, is based on the switching scheme: by converting the delayed multi-agent system into an augmented system with the delay modeled as a switching function. The main results are presented in form of theorems, providing sufficient conditions for the network asymptotic convergence and design gains that drive the network towards consensus. As in Section 2.2, there are also proposed optimization procedures in the form of LMIs to estimate the domain of consensus as large as possible.

Let us first present the result based on the Lyapunov-Krasovskii framework. The next Lemma, borrowed from Castro et al. (2020), is instrumental in the formulation of this result. It provides an inequality that can be used to insert information about delayed states of each agent in the Lyapunov-Krasovskii analysis, in such a way that the new delayed terms are negative.

Lemma 2.4. *(Castro et al., 2020). For any constant matrix $M = M' > 0$ and integer $c \in [\underline{\tau}, \bar{\tau}]$, with $\bar{\tau} \geq \underline{\tau} \geq 0$, the inequality*

$$(\bar{\tau} - \underline{\tau}) \sum_{j=-\bar{\tau}}^{-\underline{\tau}-1} \boldsymbol{y}_j' M \boldsymbol{y}_j \geq \Omega_c' M \Omega_c$$

holds, where $\boldsymbol{y}_j = \boldsymbol{x}_j - \boldsymbol{x}_{j-1}$ for all $j \in [-\bar{\tau}+1, 0]$, and $\Omega_c = \boldsymbol{x}_{-\underline{\tau}} - 2\boldsymbol{x}_{-c} + \boldsymbol{x}_{-\bar{\tau}}$ for all $c \in [\underline{\tau}, \bar{\tau}]$.

In this section, we address the stability analysis and stabilization of the discrete-time version of (2.20) by employing the following Lyapunov-Krasovskii functional candidate:

$$V(\varphi(k)) = \sum_{v=1}^{4} V_v(\varphi(k)) \tag{2.49}$$

with

$$\begin{aligned}
V_1(\varphi(k)) &= \boldsymbol{z}_k' \bar{P} \boldsymbol{z}_k, \\
V_2(\varphi(k)) &= \sum_{j=k-\bar{\tau}}^{k-1} \boldsymbol{z}_k' \bar{Q} \boldsymbol{z}_k, \\
V_3(\varphi(k)) &= \sum_{\ell=1-\bar{\tau}}^{0} \sum_{j=k+\ell}^{k} \boldsymbol{y}_j' \bar{Z}_1 \boldsymbol{y}_j, \\
V_4(\varphi(k)) &= \bar{\tau} \sum_{i=1}^{n} \sum_{\ell=1-\bar{\tau}}^{0} \sum_{j=k+\ell}^{k} \boldsymbol{y}_j' \bar{Z}_2 \boldsymbol{y}_j,
\end{aligned}$$

and $\varphi(k) \in \mathcal{C}_{k,\bar{\tau}}^{nm}$. The term $V_4(\varphi(k))$ is added to take into account information of delayed states of each agent employing Lemma 2.4. This accomplishes two things, it allows to model the saturation as a Lur'e problem and it helps to decrease the conservatism of the estimate of the region of convergence. Besides, if (2.49) is a Lyapunov-Krasovskii functional of the multi-agent system (2.20), then an estimate of $\mathcal{S}$ can be computed as the level set

$$\mathcal{L}_V = \{\varphi(t_0) \in \mathcal{C}_{k_0,\bar{\tau}}^{nm} : V(\varphi(t_0)) \leq 1\}, \tag{2.50}$$

such that $\mathcal{L}_V \subset \mathcal{S}$.

2.3.1 Stability Analysis

The following theorem presents sufficient conditions to certify consensus of the multi-agent system (2.10) on the discrete-time domain.

Theorem 2.3. *Let the feedback gain matrix K and the scalars $\bar{\tau}$, ϱ, and $u_{\max}$ be given. The multi-agent system (2.5) with consensus protocol (2.3) subject to saturating inputs and time-varying delays reaches consensus if there exist nonsingular matrix $S \in \mathbb{R}^{m(n-1)\times m(n-1)}$, a matrix $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{R}^{np\times(2n+1)m(n-1)}$, symmetric positive definite matrices P, Q, Z_1, and $Z_2 \in \mathbb{R}^{m(n-1)\times m(n-1)}$, and a diagonal positive definite matrix $Y \in \mathbb{R}^{np\times np}$, such that the* *following inequalities are satisfied:*

$$\begin{bmatrix} \Pi & \bar{\tau}\Gamma \\ * & -\bar{\tau}Z_1 \end{bmatrix} < 0, \tag{2.51}$$

and

$$\begin{bmatrix} \begin{bmatrix} \hat{Z}_2 & -\hat{Z}_2 & 0 \\ * & \hat{Z}_2 & 0 \\ * & * & Z_1 \end{bmatrix} & \left(\begin{bmatrix} 0 & \bar{L}(I_n \otimes S') & 0 \end{bmatrix} - G\right)'_{(r)} \\ * & u_{\max}^2 \end{bmatrix} \geq 0, \tag{2.52}$$

for all $r \in \{1, \ldots, np\}$, *where*

$$\Pi = \Pi_{PQ} + \Pi_Q + \Pi_{Z_1} + \Pi_{Z_2} + \Pi_S + \Pi_{\text{sat}}, \tag{2.53}$$

$$\Gamma = \begin{bmatrix} \varrho\bar{B}S' \\ \bar{B}S' \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \Pi_{Z_1} = \begin{bmatrix} \bar{\tau}Z_1 & -\bar{\tau}Z_1 & 0 & 0 & 0 \\ * & \bar{\tau}Z_1 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Pi_{Z_2} = \begin{bmatrix} n\bar{\tau}^2 Z_2 & -n\bar{\tau}^2 Z_2 & 0 & 0 & 0 \\ * & n\bar{\tau}^2 Z_2 - nZ_2 & 2\mathcal{I}\hat{Z}_2 & -nZ_2 & 0 \\ * & * & -4\hat{Z}_2 & 2\hat{Z}_2\mathcal{I}' & 0 \\ * & * & * & -nZ_2 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Pi_{PQ} = \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ * & Q-P & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & 0 \end{bmatrix}, \Pi_{\text{sat}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & -G_1' \\ * & * & 0 & 0 & -G_2' \\ * & * & * & 0 & -G_3' \\ * & * & * & * & -2Y \end{bmatrix},$$

$$\Pi_S = \begin{bmatrix} -\varrho(S+S') & -S' + \varrho(\bar{A}-\bar{B})S' & 0 & 0 & \varrho(U \otimes B)S' \\ * & (\bar{A}-\bar{B})S' + S(\bar{A}-\bar{B})' & 0 & 0 & (U \otimes B)S' \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} \tag{2.54}$$

and,

$$\begin{aligned} \bar{A} &= I_{n-1} \otimes A, \\ \bar{B} &= ULW \otimes BK, \\ \bar{L} &= \begin{bmatrix} L_1 W \otimes K & L_2 W \otimes K & \cdots & L_n W \otimes K \end{bmatrix}, \\ \mathcal{I} &= \mathbf{1}_n' \otimes I_{m(n-1)} \in \mathbb{R}^{m(n-1) \times nm(n-1)}, \\ \hat{Z}_2 &= I_n \otimes Z_2 \in \mathbb{R}^{nm(n-1) \times nm(n-1)}. \end{aligned}$$

Furthermore, a region from which the consensus is always attainable is given by $\mathcal{L}_V$ given in (2.50).

Proof. The demonstration follows some of the ideas of the proof of Theorem 2.1. Consider the Lyapunov-Krasovskii functional candidate in (2.49) with $P = F^{-1}\bar{P}(F^{-1})'$, $Q = F^{-1}\bar{Q}(F^{-1})'$, $Z_1 = F^{-1}\bar{Z}_1(F^{-1})'$, and $Z_2 = F^{-1}\bar{Z}_2(F^{-1})'$. Therefore, with $P > 0$, $Q > 0$, $Z_1 > 0$, and $Z_2 > 0$, the positivity of the Lyapunov-Krasovskii functional above is guaranteed with functions $u(\|\boldsymbol{z}(k)\|) = \lambda_{\min}(P)\|\boldsymbol{z}(k)\|^2$ and

$$v(\|\varphi(k)\|) = \lambda_{\max}(P)\|\boldsymbol{z}(k)\|^2 + \bar{\rho}_1 \sup_{\eta\in[k-\bar{\tau},k-1]} \|\boldsymbol{z}_\eta\|^2 + \bar{\rho}_2 \sup_{\eta\in[k+1-\bar{\tau},k]} \|\boldsymbol{y}_\eta\|^2$$

in Lemma 2.1, with

$$\begin{aligned}
\bar{\rho}_1 &= \bar{\tau}\lambda_{\max}(Q),\\
\bar{\rho}_2 &= 0.5\bar{\tau}(\bar{\tau}+1)(\lambda_{\max}(Z_1) + n\bar{\tau}\lambda_{\max}(Z_2)).
\end{aligned}$$

Next we show that the negativity of the functional forward difference is guaranteed if the conditions in the theorem hold. Using the discrete counterpart of the Newton-Leibniz formula and (2.20), we have the following null-term

$$\begin{aligned}
0 &= 2\Lambda(k)\left[\boldsymbol{z}(k+1) - \bar{A}\boldsymbol{z}(k) + \sum_{i=1}^{n} \bar{B}_i \boldsymbol{z}(k-\tau_i(k)) - (U\otimes B)\Theta(\boldsymbol{g}(k))\right]\\
&= 2\Lambda(k)\left[\boldsymbol{z}(k+1) - \bar{A}\boldsymbol{z}(k) + \sum_{i=1}^{n} \bar{B}_i \boldsymbol{z}(k) - (U\otimes B)\Theta(\boldsymbol{g}(k)) - \sum_{i=1}^{n}\bar{B}_i \sum_{\ell=k-\tau_i(k)+1}^{k} \boldsymbol{y}_\ell\right]\\
&= 2\Lambda(k)\Big[\boldsymbol{z}(k+1) - \bar{A}\boldsymbol{z}(k) + \bar{B}\boldsymbol{z}(k) - (U\otimes B)\Theta(\boldsymbol{g}(k))\Big] + v(k),
\end{aligned} \tag{2.55}$$

where, $\boldsymbol{y}_\ell = \boldsymbol{z}_\ell - \boldsymbol{z}_{\ell-1}$,

$$\Lambda(k) = -\boldsymbol{z}(k)'F - \boldsymbol{z}(k+1)'\varrho F, \tag{2.56}$$

$$\bar{B}_i = (UL_iW)\otimes BK, \tag{2.57}$$

$$\bar{B} = (ULW)\otimes BK, \tag{2.58}$$

$$v(k) = -\sum_{i=1}^{n}\sum_{\ell=k-\tau_i(k)+1}^{k} 2\Lambda(k)\bar{B}_i\boldsymbol{y}_\ell. \tag{2.59}$$

Moreover, assuming $\bar{Z}_1 > 0$ we have that[3]

$$\begin{aligned}
v(k) \leq& \sum_{i=1}^{n}\sum_{\ell=k-\tau_i(k)+1}^{k} \Big(\Lambda(k)\bar{B}_i\Big) n\bar{Z}_1^{-1}\Big(\Lambda(k)\bar{B}_i\Big)' + \sum_{i=1}^{n}\sum_{\ell=k-\tau_i(k)+1}^{k} \boldsymbol{y}_\ell' \frac{\bar{Z}_1}{n}\boldsymbol{y}_\ell\\
\leq& \sum_{i=1}^{n} \bar{\tau}\Big(\Lambda(k)\bar{B}_i\Big) n\bar{Z}_1^{-1}\Big(\Lambda(k)\bar{B}_i\Big)' + \sum_{\ell=k+1-\bar{\tau}}^{k} \boldsymbol{y}_\ell'\bar{Z}_1\boldsymbol{y}_\ell.
\end{aligned} \tag{2.60}$$

[3]We used the inequality $-2\boldsymbol{a}'\boldsymbol{b} \leq \boldsymbol{a}'X\boldsymbol{a} + \boldsymbol{b}'X^{-1}\boldsymbol{b}$ where $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^{m(n-1)}$ and $X > 0 \in \mathbb{R}^{m(n-1)\times m(n-1)}$.

Combining (2.55) and (2.60) it yields

$$
\begin{aligned}
0 \leq & -2\boldsymbol{z}(k+1)'\varrho F\boldsymbol{z}(k+1) + 2\boldsymbol{z}(k)'F\bar{A}\boldsymbol{z}(k) + 2\boldsymbol{z}(k+1)'\varrho F\bar{A}\boldsymbol{z}(k) - 2\boldsymbol{z}(k)'F\boldsymbol{z}(k+1) \\
& - 2\boldsymbol{z}(k)'F\bar{B}\boldsymbol{z}(k) - 2\boldsymbol{z}(k+1)'\varrho F\bar{B}\boldsymbol{z}(k) + 2\boldsymbol{z}(k+1)'\varrho F(U\otimes B)\Theta(\boldsymbol{g}(k)) \\
& + 2\boldsymbol{z}(k)'F(U\otimes B)\Theta(\boldsymbol{g}(k)) + \sum_{i=1}^{n}\bar{\tau}\Big(\Lambda(k)\bar{B}_i\Big)n\bar{Z}^{-1}\Big(\Lambda(k)\bar{B}_i\Big)' + \sum_{\ell=k+1-\bar{\tau}}^{k}\boldsymbol{y}_\ell'\bar{Z}_1\boldsymbol{y}_\ell. \qquad (2.61)
\end{aligned}
$$

To calculate the functional candidate (2.49) forward difference consider,

$$
\Delta V_1(\boldsymbol{z}(k)) = \boldsymbol{z}(k+1)'\bar{P}\boldsymbol{z}(k+1) - \boldsymbol{z}'\bar{P}\boldsymbol{z}, \qquad (2.62)
$$

$$
\begin{aligned}
\Delta V_2(\boldsymbol{z}(k)) &= \sum_{j=k+1-\bar{\tau}}^{k}\boldsymbol{z}_j'\bar{Q}\boldsymbol{z}_j - \sum_{j=k-\bar{\tau}}^{k-1}\boldsymbol{z}_j'\bar{Q}\boldsymbol{z}_j \\
&= \boldsymbol{z}(k)'\bar{Q}\boldsymbol{z}(k) - \boldsymbol{z}(k-\bar{\tau})'\bar{Q}\boldsymbol{z}(k-\bar{\tau}), \qquad (2.63)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_3(\boldsymbol{z}(k)) &= \sum_{\ell=1-\bar{\tau}}^{0}\left[\sum_{j=k+1+\ell}^{k+1}\boldsymbol{y}_j'\bar{Z}_1\boldsymbol{y}_j - \sum_{j=k+\ell}^{k}\boldsymbol{y}_j'\bar{Z}_1\boldsymbol{y}_j\right] \\
&= \sum_{\ell=1-\bar{\tau}}^{0}\Bigg[\boldsymbol{y}(k+1)'\bar{Z}_1\boldsymbol{y}(k+1) - \boldsymbol{y}(k+\ell)'\bar{Z}_1\boldsymbol{y}(k+\ell) \\
&\quad + \sum_{j=k+1+\ell}^{k}\boldsymbol{y}_j'\bar{Z}_1\boldsymbol{y}_j - \sum_{j=k+\ell+1}^{k}\boldsymbol{y}_j'\bar{Z}_1\boldsymbol{y}_j\Bigg] \\
&= \bar{\tau}\boldsymbol{y}(k+1)'\bar{Z}_1\boldsymbol{y}(k+1) - \sum_{\ell=k+1-\bar{\tau}}^{k}\boldsymbol{y}_\ell'\bar{Z}_1\boldsymbol{y}_\ell, \qquad (2.64)
\end{aligned}
$$

and lastly

$$
\begin{aligned}
\Delta V_4(\boldsymbol{z}(k)) &= \bar{\tau}\sum_{i=1}^{n}\sum_{\ell=1-\bar{\tau}}^{0}\left[\sum_{j=k+1+\ell}^{k+1}\boldsymbol{y}_j'\bar{Z}_2\boldsymbol{y}_j - \sum_{j=k+\ell}^{k}\boldsymbol{y}_j'\bar{Z}_2\boldsymbol{y}_j\right] \\
&= \bar{\tau}^2 n\boldsymbol{y}(k+1)'\bar{Z}_2\boldsymbol{y}(k+1) - \bar{\tau}\sum_{i=1}^{n}\sum_{\ell=k+1-\bar{\tau}}^{k}\boldsymbol{y}_\ell'\bar{Z}_2\boldsymbol{y}_\ell. \qquad (2.65)
\end{aligned}
$$

Then, by applying Lemma 2.4 in the ith term of the sum in (2.65), with $c = \tau_i(k)$, we get

$$
-\bar{\tau}\sum_{\ell=k+1-\bar{\tau}}^{k}\boldsymbol{y}_\ell'\bar{Z}_2\boldsymbol{y}_\ell \leq -(\boldsymbol{z}(k) - 2\boldsymbol{z}(k-\tau_i(k)) + \boldsymbol{z}(k-\bar{\tau}))'\bar{Z}_2(\boldsymbol{z}(k) - 2\boldsymbol{z}(k-\tau_i(k)) + \boldsymbol{z}(k-\bar{\tau})). \qquad (2.66)
$$

Finally, as in the demonstration of Theorem 2.1, we first assume that $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$ and show afterward that condition (2.52) ensures this inclusion. Additionally, adding the result of Lemma 2.3, the null term (2.34), with equations (2.62), (2.63), (2.64), and (2.66) the

Lyapunov-Krasovskii candidate forward difference becomes

$$\begin{aligned}\Delta V(\boldsymbol{z}(k)) \leq & \Upsilon(k)'\hat{\Pi}\Upsilon(k) + \sum_{i=1}^{n} \bar{\tau}\Upsilon(k)'\Gamma_i n\bar{Z}_1^{-1}\Gamma_i'\Upsilon(k) \\ = & \Upsilon(k)'\left(\sum_{i=1}^{n}\left(\frac{1}{n}\hat{\Pi} + \bar{\tau}\Gamma_i n\bar{Z}_1^{-1}\Gamma_i'\right)\right)\Upsilon(k),\end{aligned} \tag{2.67}$$

with

$$\Upsilon(k)' = \left[\boldsymbol{z}(k+1)' \quad \boldsymbol{z}(k)' \quad \hat{\boldsymbol{z}}(k-\boldsymbol{\tau}(k))' \quad \boldsymbol{z}(k-\bar{\tau})' \quad \Theta'\right],$$

$$\hat{\boldsymbol{z}}(k-\boldsymbol{\tau}(k))' = \left[\boldsymbol{z}(k-\tau_1(k))' \quad \boldsymbol{z}(k-\tau_2(k))' \cdots \boldsymbol{z}(k-\tau_n(k))'\right],$$

$$\Gamma_i = \begin{bmatrix} \varrho F\bar{B}_i \\ F\bar{B}_i \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and $\hat{\Pi}$ defined as in (2.26), however with the matrices $\bar{P}$, $\bar{Q}$, $\bar{Z}_1$, $\bar{Z}_2$, and

$$\hat{\Pi}_{\text{sat}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & -\bar{G}_1'T' \\ * & * & 0 & 0 & -\bar{G}_2'T' \\ * & * & * & 0 & -\bar{G}_3'T' \\ * & * & * & * & -2T \end{bmatrix},$$

instead of P, Q, Z_1, Z_2, and Π_{sat}, respectively.

Finally, $\Delta V(\varphi(k)) \leq -w(\|\boldsymbol{z}(k)\|) = -\alpha\|\boldsymbol{z}(k)\|^2 < 0$ for any $\Upsilon(k) \neq 0$, and for some sufficiently small $\alpha < 0$, is achieved whenever the parenthesis term in equation (2.67) is negative definite, which from Schur complement is equivalent to

$$\sum_{i=1}^{n}\begin{bmatrix} \frac{1}{n}\hat{\Pi} & \bar{\tau}\Gamma_i \\ * & \frac{-\bar{\tau}}{n}\bar{Z}_1 \end{bmatrix} = \begin{bmatrix} \hat{\Pi} & \bar{\tau}\Gamma \\ * & -\bar{\tau}\bar{Z}_1 \end{bmatrix} < 0. \tag{2.68}$$

The inequality (2.68) becomes identical to condition (2.51) after pre- and post-multiplying with

$$\text{diag}(F^{-1}, F^{-1}, F^{-1}, \cdots, F^{-1}, T^{-1}, F^{-1})$$

and its transpose respectively, and making the change of variables $S = F^{-1}$, $Y = T^{-1}$, $G_1 = \bar{G}_1(F^{-1})'$, $G_2 = \bar{G}_2(F^{-1})'$, $G_3 = \bar{G}_3(F^{-1})$.

Therefore, if condition (2.51) of Theorem 2.3 and $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$ hold we have $V(\varphi(k)) > 0$ and $\Delta V(\varphi(k)) < 0$, which ensures that the multi-agent system reaches consensus asymptotically.

Next, it is shown that if inequality (2.52) is satisfied then $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$, and from the generalized sector condition in Lemma 2.3, $-2\Theta\big(\boldsymbol{u}(k)\big)'T\big[\Theta\big(\boldsymbol{u}(k)) + \boldsymbol{\omega}(k)\big] \geq 0$ holds. In addition, all trajectories of $\varphi(k)$ starting from $\mathcal{L}_V$ remains within $\mathcal{L}_V$.

Inequality (2.52) implies that whenever the trajectories of the multi-agent system $\varphi_0(k)$ belong to $\mathcal{L}_V$, the inputs of the agents will belong to the set $\mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$, in which $\mathcal{L}_V$ represents a level set of the Lyapunov-Krasovskii functional. That is

$$\left| \left(\left\{ \begin{bmatrix} 0 & \bar{L} & 0 \end{bmatrix} - G \right\} \chi \right)_{(r)} \right| \leq u_{\max}, \ \forall \boldsymbol{z}(k) \in \mathcal{L}_V,$$

for $r = 1, \ldots, np$, where $\boldsymbol{u}(k) = \bar{L}\hat{\boldsymbol{z}}(k - \boldsymbol{\tau}(k))$, $\boldsymbol{\omega}(k) = G\chi$, with

$$\chi = \begin{bmatrix} \hat{\boldsymbol{z}}(k)' & \hat{\boldsymbol{z}}(k - \boldsymbol{\tau}(k))' & \boldsymbol{y}(k+1-\bar{\tau})' \end{bmatrix}'.$$

The previous inequality holds whenever

$$0 \leq \left(\left\{ \begin{bmatrix} 0 & \bar{L} & 0 \end{bmatrix} - G \right\} \chi \right)_{(r)}' u_{\max}^{-2} \left(\left\{ \begin{bmatrix} 0 & \bar{L} & 0 \end{bmatrix} - G \right\} \chi \right)_{(r)} \leq \chi' \begin{bmatrix} \hat{\bar{Z}}_2 & -\hat{\bar{Z}}_2 & 0 \\ * & \hat{\bar{Z}}_2 & 0 \\ * & * & \bar{Z}_1 \end{bmatrix} \chi$$

which, by Schur complement and pre- and post-multiplying the result by $\operatorname{diag}\left(F^{-1}, (I_n \otimes F^{-1}), F^{-1}, 1\right)$ and its transpose, respectively, becomes similar to inequality (2.52). Furthermore, we have that all trajectories $\varphi_0(k)$ originating from $\varphi_0 \in \mathcal{L}_V$ verifies:

$$\begin{aligned} \chi' \begin{bmatrix} \hat{\bar{Z}}_2 & -\hat{\bar{Z}}_2 & 0 \\ * & \hat{\bar{Z}}_2 & 0 \\ * & * & \bar{Z}_1 \end{bmatrix} \chi =& (\hat{\boldsymbol{z}}(k) - \hat{\boldsymbol{z}}(k - \boldsymbol{\tau}(k)))' \hat{\bar{Z}}_2 (\hat{\boldsymbol{z}}(k) - \hat{\boldsymbol{z}}(k - \boldsymbol{\tau}(k))) \\ &+ \boldsymbol{y}(k+1-\bar{\tau})' \bar{Z}_1 \boldsymbol{y}(k+1-\bar{\tau}) \\ \leq& V(\varphi_0) \leq 1. \end{aligned}$$

Hence, with condition (2.52), we ensure that $\boldsymbol{u}(k)$ belongs to $\mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$, consequently the condition of Lemma 2.3 is satisfied. Because $\mathcal{L}_V$ is a level set of V and the inequalities $V(\boldsymbol{z}(k)) > 0$ and $\Delta V(\boldsymbol{z}(k)) < 0$ hold, the multi-agent systems converges asymptotically.

□

2.3.2 Stabilization

In the next theorem, we present conditions to design stabilizing state feedback matrices for the agents along with a region of ensured convergence. The results in the next theorem are extensions of the previous one.

Theorem 2.4. *Let the scalars $\bar{\tau}$, ϱ and $u_{\max}$ be given. The multi-agent system (2.5) with consensus protocol (2.3) subject to input saturation and time-varying delays reaches consensus with state feedback matrix $K = \bar{K}(F_m^{-1})'$, if there exist symmetric positive definite matrices P, Q, Z_1, and $Z_2 \in \mathbb{R}^{m(n-1)\times m(n-1)}$, matrices $\bar{F} = I_{n-1} \otimes F_m \in \mathbb{R}^{m(n-1)\times m(n-1)}$ with $F_m \in \mathbb{R}^{m\times m}$, $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{R}^{np\times(2n+1)m(n-1)}$, $\bar{K} \in \mathbb{R}^{p\times m}$, a diagonal positive definite matrix*

$H \in \mathbb{R}^{np\times np}$, *such that the following inequalities are satisfied:*

$$\begin{bmatrix} \bar{\Pi} & \bar{\tau}\bar{\Gamma} \\ * & -\bar{\tau}Z_1 \end{bmatrix} < 0, \tag{2.69}$$

and

$$\begin{bmatrix} \begin{bmatrix} \hat{Z}_2 & -\hat{Z}_2 & 0 \\ * & \hat{Z}_2 & 0 \\ * & * & Z_1 \end{bmatrix} & \left(\begin{bmatrix} 0 & \hat{L} & 0 \end{bmatrix} - G\right)'_{(r)} \\ * & u_{\max}^2 \end{bmatrix} \geq 0,\ \forall r \in \{1,\dots,np\}, \tag{2.70}$$

where

$$\bar{\Pi} = \Pi_P + \Pi_Q + \Pi_{Z_1} + \Pi_{Z_2} + \Pi_{\bar{F}} + \Pi_{\text{sat}}, \tag{2.71}$$

$$\bar{\Gamma} = \begin{bmatrix} \varrho\bar{B} \\ \bar{B} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \Pi_{Z_1} = \begin{bmatrix} \bar{\tau}Z_1 & -\bar{\tau}Z_1 & 0 & 0 & 0 \\ * & \bar{\tau}Z_1 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Pi_{Z_2} = \begin{bmatrix} n\bar{\tau}^2Z_2 & -n\bar{\tau}^2Z_2 & 0 & 0 & 0 \\ * & n\bar{\tau}^2Z_2 - nZ_2 & 2\mathcal{I}\hat{Z}_2 & -nZ_2 & 0 \\ * & * & -4\hat{Z}_2 & 2\hat{Z}_2\mathcal{I}' & 0 \\ * & * & * & -nZ_2 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Pi_{PQ} = \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ * & Q-P & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & 0 \end{bmatrix}, \Pi_{\text{sat}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & -G_1 \\ * & * & 0 & 0 & -G_2 \\ * & * & * & 0 & -G_3 \\ * & * & * & * & -2H \end{bmatrix},$$

$$\Pi_{\bar{F}} = \begin{bmatrix} -\varrho(\bar{F}+\bar{F}') & -\bar{F}' + \varrho(\bar{A}\bar{F}' - \bar{B}) & 0 & 0 & \varrho(U\otimes B)\bar{F}' \\ * & \bar{A}\bar{F}' - \bar{B} + \bar{F}\bar{A}' - \bar{B}' & 0 & 0 & (U\otimes B)\bar{F}' \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} \tag{2.72}$$

and,

$$\begin{aligned}
\bar{A} =& I_{n-1} \otimes A, \\
\bar{B} =& ULW \otimes B\bar{K}, \\
\hat{L} =& \begin{bmatrix} L_1W\otimes\bar{K} & L_2W\otimes\bar{K} & \cdots & L_nW\otimes\bar{K} \end{bmatrix}, \\
\mathcal{I} =& \mathbf{1}_n' \otimes I_{m(n-1)} \in \mathbb{R}^{m(n-1)\times nm(n-1)}, \\
\hat{Z}_2 =& I_n \otimes Z_2 \in \mathbb{R}^{nm(n-1)\times nm(n-1)}.
\end{aligned}$$

Furthermore, a region from which the consensus is always attainable is given by $\mathcal{L}_V$ given in (2.50).

Proof. In the same spirit of Theorem 2.2, the demonstration of Theorem 2.4 follows from inequalities (2.51) and (2.52). Replacing S by the structured variable $\bar{F} = I_{n-1} \otimes F_m$, $F_m \in \mathbb{R}^{m\times m}$ allows us to benefit from products $\bar{L}(I_n \otimes \bar{F}')$ and $\bar{B}\bar{F}'$. Observe that, because of the chosen structure, the products result respectively in:

$$\bar{B}\bar{F}' = (ULW \otimes BK)(I_{n-1} \otimes F_m) = ULW \otimes BKF_m',$$
$$\bar{L}(I_n \otimes \bar{F}') = \begin{bmatrix} L_1W \otimes \bar{K}F_m' & L_2W \otimes \bar{K}F_m' \cdots L_nW \otimes \bar{K}F_m' \end{bmatrix}.$$

Finally, replacing the product KF_m', by the variable $\bar{K}$ yields to conditions (2.69) and (2.70), and the state feedback for each agent can be retrieved from $K = \bar{K}(F^{-1})'$, which concludes the proof. □

As the results in continuous-time, Theorems 2.3 and 2.4 provide sufficient conditions to guarantee the convergence in the form of feasibility problems, hence, the region provided might not be the largest one. Following the same strategy as in Section 2.2.3, we propose next a procedure to assist in the maximization of such regions and to represent them more precisely.

2.3.3 Characterization of the Set of Initial Conditions

Inspired by the approaches in Castro et al. (2020), we present next a characterization of the region of guaranteed convergence for the multi-agent system. This procedure permits to jointly compute distinct regions for the current states, the couplings between current and previous states, and delayed states. Specifically, the estimate of set of admissible initial conditions is such that:

$$\varphi(k_0) = \{\varphi \in \mathcal{C}_{\bar{\tau}}^{nm} : \boldsymbol{z}_0 \in \mathcal{C}, \boldsymbol{y}_0 \in \mathcal{Y}, \bar{\varphi}_0 \in \mathcal{B}(r_1, r_2)\}, \tag{2.73}$$

where $\mathcal{C}$ and $\mathcal{Y}$ are ellipsoidal sets confining allowed values of $\boldsymbol{z}(k_0)$ and $\boldsymbol{y}(k_0)$, and the set $\mathcal{B}(r_1, r_2)$ contains the delayed states $\bar{\varphi}_0 \in \mathcal{C}_{\bar{\tau},1}^{nm}$, which $\mathcal{C}_{\bar{\tau},1}^{nm}$ representes sequences of states on the interval $[k_0 - \bar{\tau}, k_0 - 1]$. The sets $\mathcal{C}$, $\mathcal{Y}$, and $\mathcal{B}(r_1, r_2)$, which characterize the set of admissible initial conditions, are defined in the next proposition considering the multi-agent system (2.5).

Proposition 2.2. *If the functional (2.49) is a Lyapunov-Krasovskii functional of the multi-agent system (2.20), then initial sequences φ_0 verifying (2.73) with*

$$\begin{aligned} \mathcal{C} =& \{\boldsymbol{z}_0 \in \mathbb{R}^{m(n-1)\times m(n-1)} : V_1(\boldsymbol{z}_0) \leq 1 - \gamma\},\\ \mathcal{Y} =& \{\boldsymbol{y}_0 \in \mathbb{R}^{m(n-1)\times m(n-1)} : \boldsymbol{y}_0' J \boldsymbol{y}_0\},\\ \mathcal{B}(r_1, r_2) =& \{\bar{\varphi}_0 \in \mathcal{C}_{\bar{\tau},1}^{nm} : \|\bar{\varphi}_0\| \leq r_1, \|\Delta\bar{\varphi}_0\| \leq r_2\},\\ \gamma =& \rho_1\|\bar{\varphi}_0\|^2 + \rho_2\|\Delta\bar{\varphi}_0\|^2 + \boldsymbol{y}_0' J \boldsymbol{y}_0, \end{aligned}$$

where $\Delta\bar{\varphi}_0 = \{\boldsymbol{z}(k_0 + i) - \boldsymbol{z}(k_0 + i - 1) : i = -\bar{\tau}, \ldots, -1\}$ is the sequence variation and $\|\bar{\varphi}_0\| = \max_{i\in[-\tau,-1]}\|\boldsymbol{z}(k_0 + i) - \boldsymbol{z}(k_0 + i - 1)\|$, and r_1 and r_2 are such that $0 \leq \rho_1 r_1^2 + \rho_2 r_2^2 + \boldsymbol{y}_0' J \boldsymbol{y}_0 <$

1, *with*

$$J = \bar{\tau} Z_1 + n\bar{\tau}^2 Z_2, \tag{2.74}$$

$$\rho_1 = \bar{\tau}\lambda_{\max}(Q), \tag{2.75}$$

$$\rho_2 = 0.5(\bar{\tau}-1)\Big(\bar{\tau}\lambda_{\max}(Z_1) + n\bar{\tau}^2\lambda_{\max}(Z_2)\Big), \tag{2.76}$$

are such that the trajectories of the multi-agent system do not leave the region of attraction and converges asymptotically.

Proof. In the same lines of demonstration of Proposition 2.1, where the sets of interest are defined by employing an upper-bound of the Lyapunov-Krasovskii functional, we can define the following upper-bound

$$V(\varphi(k_0)) = \sum_{v=1}^{4} V_v(\varphi(k_0)) \leq V_1(\boldsymbol{z}(0)) + \rho_1\|\bar{\varphi}_0\|^2 + \rho_2\|\Delta\bar{\varphi}_0\|^2 + \boldsymbol{y}_0' J \boldsymbol{y}_0 < 1,$$

on the Lyapunov-Krasovskii functional give in (2.21). In which J, ρ_1, and ρ_2 are given in (2.74), (2.75), and (2.76). To guarantee $0 \leq \gamma < 1$ the inequality $0 \leq \rho_1 r_1^2 + \rho_2 r_2^2 + \boldsymbol{y}_0' J \boldsymbol{y}_0 < 1$ is required to be satisfied. □

As pointed out in the works by Castro et al. (2020), Silva et al. (2020), and Silva et al. (2018b), the main advantage of employing characterizations such as the one in Proposition 2.2 is the flexibility it gives when estimating specific regions for states in different time-instants, allowing to possibly compute less conservative sets. For example, it is possible to cast the region of convergence inside a ball, to compute the largest possible set for transition states $\boldsymbol{y}_0$, or even to calculate a region of convergence considering only current states, with all previous states being zero.

An important observation is that unlike the result presented in Castro et al. (2020), Proposition 2.2 is based on a distinct augmented Lyapunov-Krasovskii function which takes into account the difference between states of agents in multi-agent systems interconnected through a directed graph. Hence, the characterization we propose not only is incommensurable with the one by Castro et al. (2020) but it is applied to a different problem.

Maximization of the allowed initial conditions

In possession of the conditions presented in Theorems 2.3 and 2.4, we consider the maximization of an ellipsoid within the region $\mathcal{L}_V$ to find a set of initial conditions as large as possible, which can be obtained by solving the following optimization problem based on Castro et al. (2020).

$$\min \operatorname{trace}\left(\sum_{j=1}^{4}\alpha_j H_j\right)$$
$$\text{s.t.}\begin{cases} a) & \text{LMIs (2.51) and (2.52) from Theorem 2.3,} \\ & \text{or (2.69) and (2.70) from Theorem 2.4.} \\ \\ b) & \begin{bmatrix} H_j & I \\ I & Z+Z'-X_j \end{bmatrix} \geq 0, \end{cases} \tag{2.77}$$

where α_j are scalar weights, $H_j \in \mathbb{R}^{m(n-1)\times m(n-1)}$ are variable matrices, $Z = S$ for Theorem 2.3 or $Z = \bar{F}$ for Theorem 2.4, and X_j corresponds to the jth matrix in the set $\{P, Q, Z_1, Z_2\}$.

2.4 Consensus of Discrete-Time Systems: Augmented System Approach

This section explores the consensus problem subject to time delays and saturating inputs in the discrete-time domain by converting the delayed multi-agent system in (2.20) into an augmented switching system with the delay modeled as a switching function. As in Sections 2.2 and 2.3, it is also proposed an optimization procedure in the form of LMIs to compute estimates of the domain of consensus as large as possible.

The procedure presented is based on the framework proposed in Silva et al. (2018a) to take into account possible bounded time delay variations. This method allows us to introduce more information about the delay, and as such, it might reduce the computational burden by reducing the number of variables of the problem. To further reduce the computational burden, in this section we also assume homogeneous varying delays in the network (i.e., $\tau_i(k) = \tau(k)$ for all $i \in \mathcal{V}$).

A bound on the time-delay variation is such that,

$$|\tau(k+1) - \tau(k)| \leq \Delta\tau_{\max} \leq \bar{\tau},$$

where $\Delta\tau_{\max}$ represents the biggest transition between two successive instants, this allows us to define the set of all possible values of the next delay as

$$\mathcal{C}(\tau) = \{\tau^+ : \tau^+ \in \{\max(0, \tau - \Delta\tau_{\max}), \max(0, \tau - \Delta\tau_{\max}) + 1, ..., \min(\bar{\tau}, \tau + \Delta\tau_{\max})\}\},$$

where τ^+ is a shorthand for $\tau(k+1)$.

Inspired by the studies in Hetel, Daafouz, and Iung (2008), Silva et al. (2018a), and in Souza et al. (2019) for single agent systems, the delayed multi-agent system is represented by an augmented switching system with the delay acting as the switching function. In our context, the discrete-time version of the multi-agent system (2.20) is represented by

$$\bar{z}(k+1) = \mathbb{A}\bar{z}(k) + \mathbb{B}\Theta\Big(\Gamma\bar{z}(k)\Big), \tag{2.78}$$

where $\bar{\boldsymbol{z}}(k) = \begin{bmatrix}\boldsymbol{z}(k)' & \boldsymbol{z}(k-1)' & \cdots & \boldsymbol{z}(k-\bar{\tau})'\end{bmatrix}'$ is the augmented vector taking into account delayed states, $\mathbb{B} = \begin{bmatrix}(U \otimes B)' & 0'\end{bmatrix}'$ and the matrices $\mathbb{A}$ and Γ switches according to the current delay as follow,

$$\mathbb{A} = \left[\begin{array}{cccc|c} (I_{n-1}\otimes A) - (U \otimes B)\Gamma_{0,\tau_k} & -(U \otimes B)\Gamma_{1,\tau_k} & \cdots & -(U \otimes B)\Gamma_{\bar{\tau}-1,\tau_k} & -(U \otimes B)\Gamma_{\bar{\tau},\tau_k} \\ \hline \multicolumn{4}{c|}{I_{\bar{\tau}m(n-1)}} & 0 \end{array}\right]$$
$$\Gamma = \begin{bmatrix}\Gamma_{0,\tau_k} & \Gamma_{1,\tau_k} & \cdots & \Gamma_{\bar{\tau},\tau_k}\end{bmatrix},$$

where

$$\Gamma_{j,\tau_k} = \begin{cases} (LW \otimes K), & \text{if } j = \tau_k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.79}$$

Observe that the system (2.78) is equivalent to (2.20), with augmented states $\boldsymbol{z}(k-1)$, $\boldsymbol{z}(k-2)$, ..., $\boldsymbol{z}(k-\bar{\tau})$, and the delay as the switching function. Moreover, to write (2.79) we take advantage of homogeneous delays, i.e., in this case it is possible to carry out the sum $\sum_{i=1}^{n}(L_i W \otimes K)\boldsymbol{z}(k-\tau(k)) = (LW \otimes K)\boldsymbol{z}(k-\tau(k))$.

2.4.1 Stability Analysis

The following theorem presents sufficient conditions to verify the consensus of the discrete-time multi-agent system (2.5) represented as the augmented system (2.78).

Theorem 2.5. *Let the feedback matrix $K \in \mathbb{R}^{p\times m}$ and scalars $u_{\max}$ and $\bar{\tau}$ be given. The multi-agent system (2.5) with consensus protocol (2.3) subject to input saturation and time-varying delay reaches consensus if there exist a diagonal positive definite matrix $S \in \mathbb{R}^{np\times np}$, symmetric positive definite matrices $R_\tau \in \mathbb{R}^{m(n-1)(\bar{\tau}+1)\times m(n-1)(\bar{\tau}+1)}$, matrices $Z \in \mathbb{R}^{np\times m(n-1)(\bar{\tau}+1)}$, and $Q_\tau \in \mathbb{R}^{m(n-1)(\bar{\tau}+1)\times m(n-1)(\bar{\tau}+1)}$, $\tau = 0, 1, \ldots, \bar{\tau}$, such that the following conditions are satisfied:*

$$\begin{bmatrix} -R_{\tau^+} & \mathbb{A}_\tau Q_\tau & \mathbb{B}S \\ * & R_\tau - Q_\tau - Q_\tau' & -Z' \\ * & * & -2S \end{bmatrix} < 0, \tag{2.80}$$

and

$$\begin{bmatrix} Q_\tau + Q_\tau' - R_\tau & Q_\tau \Gamma_{(\ell)}' - Z_{(\ell)}' \\ * & u_{\max}^2 \end{bmatrix} \geq 0, \tag{2.81}$$

for $\tau = 0, \ldots, \bar{\tau}$; $\tau^+ \in \mathcal{C}(\tau)$; $\ell = 1, \ldots, np$, where

$$\mathbb{A}_\tau = \left[\begin{array}{cccc|c} (I_{n-1}\otimes A) - (U \otimes B)\Gamma_{0,\tau_k} & -(U \otimes B)\Gamma_{1,\tau_k} & \cdots & -(U \otimes B)\Gamma_{\bar{\tau}-1,\tau_k} & -(U \otimes B)\Gamma_{\bar{\tau},\tau_k} \\ \hline \multicolumn{4}{c|}{I_{\bar{\tau}m(n-1)}} & 0 \end{array}\right],$$
$$\mathbb{B} = \begin{bmatrix}(U \otimes B)' & 0'\end{bmatrix}',$$

and for $k = 0, \dots, \bar{\tau}$, *if* $k = \tau$, *then* $\Gamma_{k,\tau} = (LW \otimes K)$. *Otherwise we have* $\Gamma_{k,\tau} = 0$. *Furthermore, a region from which the consensus is always attainable is given by the level set* $\mathcal{L}_V = \{\bar{\boldsymbol{z}}(k) : \bar{\boldsymbol{z}}(k)' P(\tau(k)) \bar{\boldsymbol{z}}(k) \leq 1\}$.

Proof. The proof follows the same lines of the proof of Theorem 1 (Hetel, Daafouz, and Iung, 2008) and Theorem 6 in Silva et al. (2018a) for single agent systems. In the following we show that if the conditions in the theorem hold, then the system in (2.20) is asymptotically stable which implies, from Definition 2.2, that the multi-agent system (2.5) reaches consensus. To this end we choose the following augmented Lyapunov functional candidate:

$$V(\bar{\boldsymbol{z}}(k)) = \bar{\boldsymbol{z}}(k)' P(\tau_k) \bar{\boldsymbol{z}}(k).$$

Therefore, with $P(\tau(k)) > 0$, and functions in Lemma 2.1 $u(\|\bar{\boldsymbol{z}}(k)\|) = \lambda_{\min}(P(\tau_k))\|\bar{\boldsymbol{z}}(k)\|^2$ and $v(\|\bar{\boldsymbol{z}}(k)\|) = \lambda_{\max}(P(\tau_k))\|\bar{\boldsymbol{z}}(k)\|^2$, the positivity of the Lyapunov functional candidate above is guaranteed. Next we show that the negativity of $\Delta V(\bar{\boldsymbol{z}}(k))$ along the trajectories of (2.78) is guaranteed if the conditions in the theorem hold. Indeed, we have that

$$\begin{aligned}\Delta V(\bar{\boldsymbol{z}}(k)) =& \bar{\boldsymbol{z}}(k+1)' P(\tau(k+1)) \bar{\boldsymbol{z}}(k+1) - \bar{\boldsymbol{z}}(k)' P(\tau(k)) \bar{\boldsymbol{z}}(k) \\ =& \Big[\mathbb{A}\bar{\boldsymbol{z}}(k) + \mathbb{B}\Theta\Big(\Gamma\bar{\boldsymbol{z}}(k)\Big)\Big]' P(\tau(k+1)) \Big[\mathbb{A}\bar{\boldsymbol{z}}(k) + \mathbb{B}\Theta\Big(\Gamma\bar{\boldsymbol{z}}(k)\Big)\Big] - \bar{\boldsymbol{z}}(k)' P(\tau(k)) \bar{\boldsymbol{z}}(k).\end{aligned}$$

Assuming that $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$ and applying the generalized sector condition in Lemma 2.3, with $\boldsymbol{\omega}(k) = G\bar{\boldsymbol{z}}(k)$ yields

$$\begin{aligned}\Delta V(\bar{\boldsymbol{z}}(k)) \leq& \Big[\mathbb{A}\bar{\boldsymbol{z}}(k) + \mathbb{B}\Theta\Big(\Gamma\bar{\boldsymbol{z}}(k)\Big)\Big]' P(\tau(k+1)) \Big[\mathbb{A}\bar{\boldsymbol{z}}(k) + \mathbb{B}\Theta\Big(\Gamma\bar{\boldsymbol{z}}(k)\Big)\Big] \\ &- \bar{\boldsymbol{z}}(k)' P(\tau(k)) \bar{\boldsymbol{z}}(k) - 2\Theta(\Gamma\bar{\boldsymbol{z}}(k))' T \Big(\Theta(\Gamma\bar{\boldsymbol{z}}(k) + G\bar{\boldsymbol{z}}(k)\Big) \\ \leq& \xi_k' \Pi \xi_k,\end{aligned} \tag{2.82}$$

with $\xi_k' = \Big[\bar{\boldsymbol{z}}(k)' \quad \Theta\Big(\Gamma\bar{\boldsymbol{z}}(k)\Big)'\Big]$ and

$$\Pi = \begin{bmatrix} \mathbb{A}' P(\tau(k+1)) \mathbb{A} - P(\tau(k)) & \mathbb{A}' P(\tau(k+1)) \mathbb{B} - G'T' \\ * & -2T + \mathbb{B}' P(\tau(k+1)) \mathbb{B} \end{bmatrix}.$$

By applying Schur's complement, left- and right-multiplying the resulting matrix by $\mathrm{diag}(I, Q_\tau, T^{-1})$, the negativity of (2.82) is guaranteed by requiring

$$\begin{bmatrix} -P_{\tau^+}^{-1} & \mathbb{A}_\tau Q_\tau & \mathbb{B}T^{-1} \\ * & -Q_\tau P_\tau Q_\tau & -Q_\tau G' \\ * & * & -2T^{-1} \end{bmatrix} < 0.$$

Finally, by applying the upper bound $-Q_\tau P_\tau Q_\tau \leq P_\tau^{-1} - Q_\tau' - Q_\tau$, and the change of variables $R_\tau = P_\tau^{-1}$, $S = T^{-1}$, and $Z_\tau' = Q_\tau G'$, one obtains the inequality (2.80).

Therefore, if $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$ and the condition (2.80) of Theorem 2.5 hold, we have $0 < u(\|\bar{\boldsymbol{z}}(k)\|) \leq V(\bar{\boldsymbol{z}}(k)) \leq v(\|\bar{\boldsymbol{z}}(k)\|)$ and $\Delta V(\bar{\boldsymbol{z}}(k)) \leq -w(\|\bar{\boldsymbol{z}}(k)\|) = -\alpha\|\bar{\boldsymbol{z}}(k)\|^2 < 0$ for some sufficient small α, which implies that the multi-agent system reaches consensus asymptotically.

Henceforth, we show that if inequality (2.81) is satisfied then $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$, and the inequality in Lemma 2.3 is always satisfied. The inequality (2.81) implies that whenever the trajectories of the multi-agent system belong to $\mathcal{L}_V$, $\boldsymbol{u}(k) \in \mathbb{S}(\boldsymbol{\omega}(k), u_{\max})$ is satisfied, where $\mathcal{L}_V$ represents a level set of the Lyapunov function given by

$$\mathcal{L}_V = \{\bar{\boldsymbol{z}}(k) : \bar{\boldsymbol{z}}(k)' P(\tau(k)) \bar{\boldsymbol{z}}(k) \leq 1\}.$$

This is equivalent to

$$|(\Gamma_{(\ell)} - G_{(\ell)}) \bar{\boldsymbol{z}}(k)| \leq u_{\max}, \ \forall \bar{\boldsymbol{z}}(k) \in \mathcal{L}_V,$$

for $\ell = 1, \ldots, np$. The previous inequality holds whenever

$$\Big(\bar{\boldsymbol{z}}(k)'(\Gamma'_{(\ell)} - G'_{(\ell)})\Big)\Big((\Gamma_{(\ell)} - G_{(\ell)})\bar{\boldsymbol{z}}(k)\Big) \leq u^2_{\max} \bar{\boldsymbol{z}}(k)' P(\tau(k)) \bar{\boldsymbol{z}}(k),$$
$$\bar{\boldsymbol{z}}(k)'\Big(P(\tau(k)) - \Big(\Gamma'_{(\ell)} - G'_{(\ell)}\Big) u^{-2}_{\max} \Big(\Gamma_{(\ell)} - G_{(\ell)}\Big)\Big)\bar{\boldsymbol{z}}(k) \geq 0,$$

which, by Schur's complement, left- and right-multiplying the resulting matrix by $\operatorname{diag}(Q_\tau, 1)$, and applying the inequality $Q_\tau P_\tau Q_\tau \geq -P_\tau^{-1} + Q' + Q$, have the positivity guaranteed by

$$\begin{bmatrix} -P_\tau^{-1} + Q'_\tau + Q_\tau & Q_\tau \Gamma'_{(\ell)} - Q_\tau G'_{(\ell)} \\ * & u^2_{\max} \end{bmatrix} \geq 0, \ \text{ for } \ell = 1, \ldots, np; \ \tau = 0, \ldots, \bar{\tau}. \tag{2.83}$$

Finally, the inequality (2.83) becames equal to (2.81) with the change of variables $R_\tau = P_\tau^{-1}$, and $Z'_\tau = Q_\tau G'$. Hence, if (2.81) is satisfied the condition of Lemma 2.3 is also satisfied, and whenever (2.80) is jointly satisfied, $\mathcal{L}_V$ is a level set of the Lyapunov functional and we have $V(\bar{\boldsymbol{z}}(k)) > 0$ and $\Delta V(\bar{\boldsymbol{z}}(k)) < 0$. Thus all the trajectories of $\bar{\boldsymbol{z}}(k)$ starting from $\mathcal{L}_V$ remains within $\mathcal{L}_V$ and go to origin. □

2.4.2 Stabilization

In the following theorem, we extend the previous result and present new conditions to synthesize stabilizing feedback controls for the agents. The asymptotic convergence and gain design conditions are given separately because a particular structure is chosen in the matrices variables of the latter. This choice would introduce some conservatism in the stability conditions if the same structure of the design conditions were considered. In fact, as it is shown in the numerical examples, this particular choice makes the design conditions using the augmented approach *more* conservative than the conditions derived through the Lyapunov-Krasovskii approach, even though in the former no sum inequalities is used. We provide design conditions based on Theorem 2.5 as follows.

Theorem 2.6. *Let the scalars $\bar{\tau}$ and $u_{\max}$ be given. The multi-agent system (2.5) with consensus protocol (2.3) subject to input saturation and time-varying delay reaches consensus with state feedback matrix $K = \bar{K} Q_m^{-1}$, if there exist a diagonal positive definite matrix $S \in \mathbb{R}^{np \times np}$, symmetric positive definite matrices $R_\tau \in \mathbb{R}^{m(n-1)(\bar{\tau}+1) \times m(n-1)(\bar{\tau}+1)}$, for $\tau = 0, 1, \ldots, \bar{\tau}$, and* *matrices* $\bar{Q} = I_{(\bar{\tau}+1)(n-1)} \otimes Q_m \in \mathbb{R}^{m(n-1)(\bar{\tau}+1)\times m(n-1)(\bar{\tau}+1)}$ *with* $Q_m \in \mathbb{R}^{m\times m}$*,* $\bar{K} \in \mathbb{R}^{p\times m}$*, and* $Z \in \mathbb{R}^{np\times m(n-1)(\bar{\tau}+1)}$*, such that the following conditions are satisfied:*

$$\begin{bmatrix} -R_{\tau^+} & \hat{\mathbb{A}}\bar{Q} - \mathbb{B}\hat{\Gamma}_\tau & \mathbb{B}S \\ * & R_\tau - \bar{Q} - \bar{Q}' & -Z' \\ * & * & -2S \end{bmatrix} < 0, \tag{2.84}$$

and

$$\begin{bmatrix} \bar{Q} + \bar{Q}' - R_\tau & \hat{\Gamma}'_{(\ell)} - Z'_{(\ell)} \\ * & u^2_{\max} \end{bmatrix} \geq 0, \tag{2.85}$$

for $\tau = 0, \ldots, \bar{\tau}$*;* $\tau^+ \in \mathcal{C}(\tau)$*;* $\ell = 1, \ldots, np$*, where*

$$\hat{\mathbb{A}} = \left[\begin{array}{ccc|c} (I_{n-1}\otimes A) & 0 & \cdots & 0 \\ \hline & I_{\bar{\tau}m(n-1)} & \cdots & 0 \end{array}\right],$$
$$\mathbb{B} = \begin{bmatrix} (U \otimes B)' & 0' \end{bmatrix}',$$
$$\hat{\Gamma}_\tau = \begin{bmatrix} \hat{\Gamma}_{0,\tau} & \hat{\Gamma}_{1,\tau} & \cdots & \hat{\Gamma}_{\bar{\tau},\tau} \end{bmatrix},$$

and for $k = 0, \ldots, \bar{\tau}$*, if* $k = \tau$*, then* $\hat{\Gamma}_{k,\tau} = (LW \otimes \bar{K})$*. Otherwise we have* $\hat{\Gamma}_{k,\tau} = 0$*. Furthermore, a region from which the consensus is always attainable is given by the level set* $\mathcal{L}_V = \{\bar{z}(k) : \bar{z}(k)'P(\tau(k))\bar{z}(k) \leq 1\}$*.*

Proof. The conditions (2.84) and (2.85) of Theorem 2.6 are derived directly from the inequalities (2.80) and (2.81), by replacing the variables Q_τ, for all $\tau \in \{1, \ldots, \bar{\tau}\}$, with the structured variable $\bar{Q} = I_{(\bar{\tau}+1)(n-1)} \otimes Q_m$, with $Q_m > 0 \in \mathbb{R}^{m\times m}$. The product $\Gamma\bar{Q}$ in (2.84) and in (2.85) leads to

$$\Gamma\bar{Q} = \Big[\Gamma_0\Big(I_{n-1} \otimes Q_m\Big) \quad \Gamma_1\Big(I_{n-1} \otimes Q_m\Big) \quad \cdots \quad \Gamma_{\bar{\tau}}\Big(I_{n-1} \otimes Q_m\Big)\Big],$$

where the jth element is given by,

$$\begin{aligned} \Gamma_j\Big(I_{n-1} \otimes Q_m\Big) &= -\Big(LW \otimes K\Big)\Big(I_{n-1} \otimes Q_m\Big) \\ &= -LWI_{n-1} \otimes KQ_m. \end{aligned}$$

Defining $\bar{K} = KQ_m$ and $\hat{\Gamma}_{k,\tau} = LW \otimes \bar{K}$, the state feedback matrices for the agents can be computed by $K = \bar{K}Q_m^{-1}$, which completes the demonstration. □

Theorems 2.5 and 2.6 present sufficient conditions to guarantee consensus in the multi-agent systems with agents described by (2.78), and additionally, they give a procedure to compute a region from which the consensus is always attainable. We propose next a simple procedure to maximize the domain of consensus along with the given conditions.

Maximization of the allowed initial conditions

With the conditions presented in Theorems 2.5 and 2.6, as in the previous approaches, we consider the maximization of an ellipsoid included in the level set $\mathcal{L}_V$, which is obtained by solving the following optimization problem.

$$\begin{aligned} &\min \operatorname{trace}(H) \\ &\text{s.t.} \begin{cases} a) & \text{LMIs (2.80) and (2.81) from Theorem 2.5,} \\ & \text{or (2.84) and (2.85) from Theorem 2.6.} \\ b) & \begin{bmatrix} H & I \\ I & R_\tau \end{bmatrix} \geq 0, \text{ for } \tau = 0, ..., \bar{\tau}, \end{cases} \end{aligned} \tag{2.86}$$

where $H \in \mathbb{R}^{m(n-1)(\bar{\tau}+1)\times m(n-1)(\bar{\tau}+1)}$.

An interesting aspect of this method is that the computation of the region included in the domain of consensus readily gives an estimate for current and delayed states. Hence, with the region in the augmented space $\bar{z}(k)$ it is possible to define the essential states for a given problem, and specific characterization of separate regions is not necessary.

2.4.3 Numerical Examples

In this section, we contrast the conservatism associated with the time-delay between our results and recent ones found in the literature. We also present two simulated experiments illustrating the findings in Sections 2.3 and 2.4. In the first example, we address a multi-agent system with scalar agents to facilitate the comprehension of the computed region of attraction in the method offered in Section 2.4. In the second simulation, the focus is to exemplify the conservatism of the region of convergence using the method proposed in Section 2.3.

Example 2.4. *Consider a multi-agent system in the discrete-time domain with two agents that communicate with each other, and have dynamics given by (2.5), with input bound $u_{\max} = 0.5$ and*

$$A = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & \frac{\sqrt{3}}{2} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.87}$$

Firstly, to compare the proposed conditions with the result in Zhang, Saberi, and Stoorvogel (2020) we consider the design approaches, Theorems 2.4 and 2.6. The solutions are presented in Table 2.4, in which Theorem 2.4 was considered with $\varrho = 5$.

Method	$\bar{\tau}_{\max}$
Theorem 1 in Zhang, Saberi, and Stoorvogel (2020)	4
Theorem 2.6	8
Theorem 2.4	10

TABLE 2.4: Example 2.4–Comparison of maximum delay for design conditions of the multi-agent system given in (2.87).

One can see that Theorems 2.6 and 2.4 achieve less conservative results in this example. However, contrary to what was expected, Theorem 2.6 showed to be more conservative than Theorem 2.4. The tests suggest that this might be due to the structure variable necessary to derive the synthesis conditions, the specific structures have shown to be more taxing in Theorem 2.6 than in 2.4. As evidence, Table 2.5 presents solutions for the stability conditions, which employ more variables. In both, Theorems 2.3 and 2.5, it was considered $K = \begin{bmatrix} 0.5 & 0.02 \end{bmatrix}$.

Method	$\bar{\tau}_{\max}$
Theorem 2.3	15
Theorem 2.5	18

TABLE 2.5: Example 2.4–Comparison of maximum delay for synthesis conditions of the multi-agent system given in (2.87).

As shown, the approach used in Theorem 2.5 indeed leads to less conservatism. However, this advantage might be lost in the extension to design conditions.

Example 2.5. *Consider a multi-agent system composed of four agents with directed communication topology illustrated in Figure 2.6, which has the following Laplacian matrix:*

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

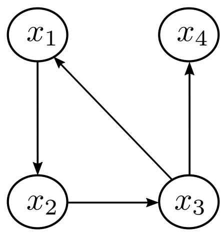


FIGURE 2.6: Example 2.5–Communication topology of multi-agent system.

In this scenario, it is considered the following dynamics for the agents:

$$x_i(k+1) = 1.2x_i(k) + 0.5\,\mathrm{sat}(u(k)),$$

for $i = 1, \ldots, 4$, *subjected to a maximum input delay of* $\bar{\tau} = 1$, *with biggest allowed transition* $\Delta\tau_{\max} = 1$, *and input saturation* $u_{\max} = 1$.

Designing the controller gains by solving the optimization problem (2.86) gives the gain $K = 0.476$. *Note that in this case, the domain of consensus has order six. To facilitate the understanding of such a region it is performed the projection into the planes* $z_i(k) \times z_i(k-1)$, $i = 1, \ldots, 3$, *and an ellipsoid is inflated in the intersection of these projections (shown in blue line in Figure 2.7). Because the proposed approach readily takes into account the region of convergence for current and delayed states, the order of the domain of consensus increase with* *the upper bound of delay. For instance, if in this scenario the agents were subject to* $\bar{\tau} = 3$*, the domain of consensus would have order* 12 *and it might be harder to visualize such a region.*

The ellipsoidal region for the domain of consensus is presented in Figure 2.7. To illustrate the evolution of the network and demonstrate the low conservatism of our approach, Figure 2.7 shows trajectories of the multi-agent system with initial conditions inside (in black line) and outside (in red line) of the computed domain of consensus. Note that the trajectories starting outside the domain of consensus do not achieve consensus.

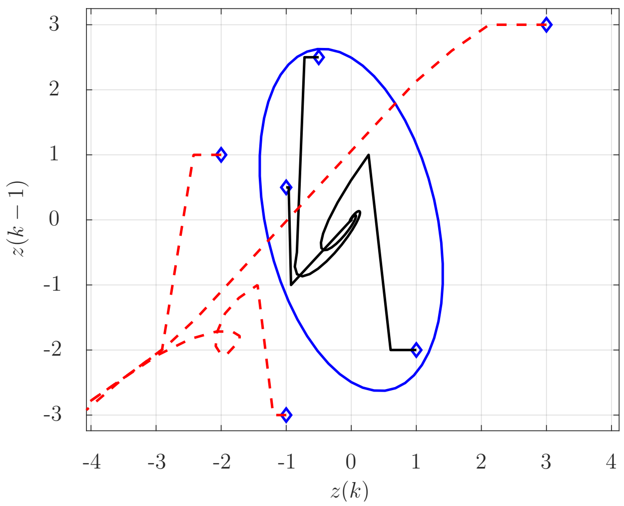


FIGURE 2.7: Example 2.5–The domain of consensus calculated using Theorem 2.6 (in blue), and relative trajectories of the agents starting outside (in red) and inside (in black) of the calculated domain of consensus.

Example 2.6. *In this example, consider that the matrices of the agents in (2.5) are given by*

$$A = \begin{bmatrix} 1.01 & 0.2 \\ 0.0 & -0.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \tag{2.88}$$

with input saturation $u_{\max} = 0.5$*, and that the multi-agent system is composed of ten agents, with communication topology represented by the following Laplacian matrix:*

$$L = \begin{bmatrix} 3 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 6 & -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 6 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 6 & -1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 & 5 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & 5 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 7 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 \end{bmatrix}.$$

For this network, we exemplify the characterization of the region of guaranteed convergence by using the synthesis conditions of Theorem 2.4, with parameter $\varrho = 5$*, through the optimization* *problem given in Section 2.2. We design three regions: (1) the biggest set for transition states* $\boldsymbol{y}_0$*; (2) the region in which* $\mathcal{Y}$ *and* $\mathcal{B}(r_1, r_2)$ *define a ball with radius* $r = r_1 = r_2 = \|\boldsymbol{y}_0\|$*; and (3) the biggest set for current states* $\boldsymbol{z}_0$*.*

To define a region for (1), we set $r_1 = r_2 = 0$*, and* $\alpha_2 = 0$*, which gives us* $\|\boldsymbol{y}_0\| < 1/\sqrt{(\bar{\tau}\lambda_{\max}(Z_1) + n\bar{\tau}^2\lambda_{\max}(Z_2))}$*. The regions* $\mathcal{C}_{(1)}$ *and* $\mathcal{Y}_{(1)}$ *in this scenario are represented in Figure 2.8 with solid blue line. For the region of convergence (2), we obtain the condition* $r < 1/\sqrt{(\rho_1 + \rho_2 + \rho_3)}$*, where* $\rho_3 = (\bar{\tau}\lambda_{\max}(Z_1) + n\bar{\tau}^2\lambda_{\max}(Z_2))$*. Computing the sets with* $\alpha_j = 1$*, for* $j = 1, \ldots, 4$*, give us the regions* $\mathcal{C}_{(2)}, \mathcal{Y}_{(2)}$*, and* $\mathcal{B}_{(2)}$*, represented in Figure 2.8 in dashed red line. Finally, to compute the region (3) we set* $\alpha_j = 0$*, for* $j = 2, \ldots, 4$*,* $\alpha_1 = 1$*, and* $r_1 = r_2 = 0$*, which give us the region* $\mathcal{C}_{(3)}$*, depicted in Figure 2.8 in solid black line.*

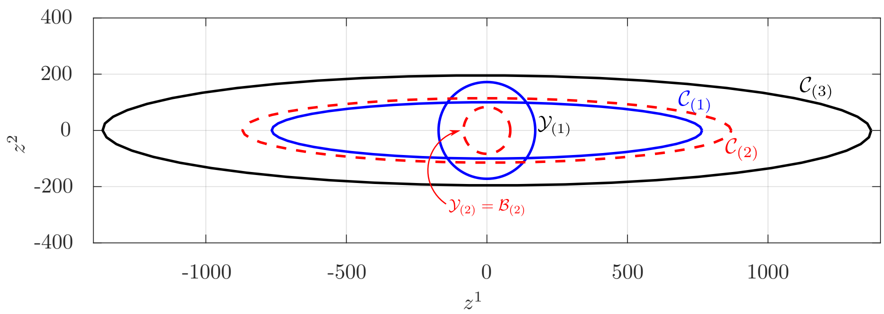


FIGURE 2.8: Example 2.6–Geometrical characterization of the domain of consensus computed using Lemma 2.2. The sets $\mathcal{C}_{(1)}$ and $\mathcal{Y}_{(1)}$ in solid blue line, $\mathcal{C}_{(2)}$, $\mathcal{Y}_{(2)}$, and $\mathcal{B}_{(2)}$ in dashed red line, and $\mathcal{C}_{(3)}$ in solid black line.

It is possible to see through the three different representations of the region of convergence that there is a trade-off between the sets associated with each initializing state. Consequently, this representation gives flexibility in structuring a specific set of initial conditions for a given problem, which might lead to less conservative estimate.

Example 2.7. *A numerical complexity analysis of the two proposed approaches of discrete-time systems is provided. The theoretical advantage of the augmented approach is that, because an equivalent form of general quadratic Lyapunov-Krasovskii functional is used, and no conservative inequalities were employed to derive linear conditions, e.g., Jensen's inequality (Kim, 2016), those conditions should lead to nonconservative conditions. In fact, for single-agent systems with time varying delays, the switching augmented system approach is equivalent to employ the most general form of a Lyapunov-Krasovskii functional candidate that involves sums of quadratic terms (Hetel, Daafouz, and Iung, 2008). The drawback of the method is that the dimension of the problem increases with the delay upper-bound. Thus, the Lyapunov-Krasovskii methodology offered in Theorems 2.3 and 2.4, although possibly more conservative, might suit better multi-agent systems. To better understand this growth, it is analyzed the number of variables of the LMI problem, as in Example 2.2, and the two methods in the discrete-time domain are contrasted.*

The number of variables in each theorem is given by:

$$
\begin{aligned}
N_{Theorem\ 2.3} =& 3m^2(n-1)^2 + 2m(n-1) + np(n+2)m(n-1) + np, \\
N_{Theorem\ 2.4} =& 2m^2(n-1)^2 + 2m(n-1) + m^2 + 2m(m+1) + np(n+2)m(n-1) \\
& + np + pm, \\
N_{Theorem\ 2.5} =& (1+\bar{\tau})^3 m^2(n-1)^2 + (1+\bar{\tau})^2 m(n-1) + (1+\bar{\tau})npm(n-1) + np, \\
N_{Theorem\ 2.6} =& \frac{1}{2}\Big((1+\bar{\tau})^3 m^2(n-1)^2 + (1+\bar{\tau})^2 m(n-1)\Big) + m^2 \\
& + mp + (1+\bar{\tau})npm(n-1) + np.
\end{aligned}
$$

As it can be seen, the number of variables grows roughly the same way in both methods regarding the number of agents in the network and their dimension. However, the conditions in Theorems 2.5 and 2.6 grow cubically as the upper-bound delay increases. This can be highly restrictive depending on the network to be studied and expected delays. Besides, it should be highlighted again that, because the structure on the variable matrices, Theorem 2.6 shows no improvement regarding the conservatism reduction. To give a glimpse of the effect caused by dependence on the time-delay, Figure 2.9 depicts the number of variables in different scenarios.

2.5 Chapter Conclusions

In this chapter, it was proposed new conditions to analyze and design consensus protocols for multi-agent systems under communication topology described by directed graphs, and inputs subject to saturation and time-delays. The obtained results were investigated in the continuous- and discrete-time domains. The analysis and design of the distributed protocols were formulated in such a way that the region in which the consensus is ensured is maximized.

In all presented approaches, the saturation was modeled with the use of a dead-zone function. With this, it was possible to employ a sector condition that permits the inclusion of more variables in the problem, which led to less conservative estimates of the region of convergence when compared with other saturation representations, for example, the polytopic model in Silva (2019). In addition, the dead-zone function eases the manipulation of the equations of the system because the sector feature of the non-linearity is explored, allowing the use of tools developed for the analysis of uncertain linear systems.

In the continuous-time domain, the time-varying delays were assumed to be specified by intervals and it was not assumed the existence of the delays derivative. This reduces the conservatism of the conditions, in the sense that it allows the methodology to be applicable to scenarios where the variation of the delays is not known and might even be random. Another key feature is that the used representation for delays fits well in the context of multi-agent systems, once it allows representing arbitrary delays for each agent within the same set and, consequently, to represent the whole networked system in a compact form.

In the discrete-time domain, two approaches were given. As a first result, it was proposed a technique based on the Lyapunov-Krasovskii framework for discrete-time multi-agent systems. With this method, we were able to derive conditions in which the computational burden does

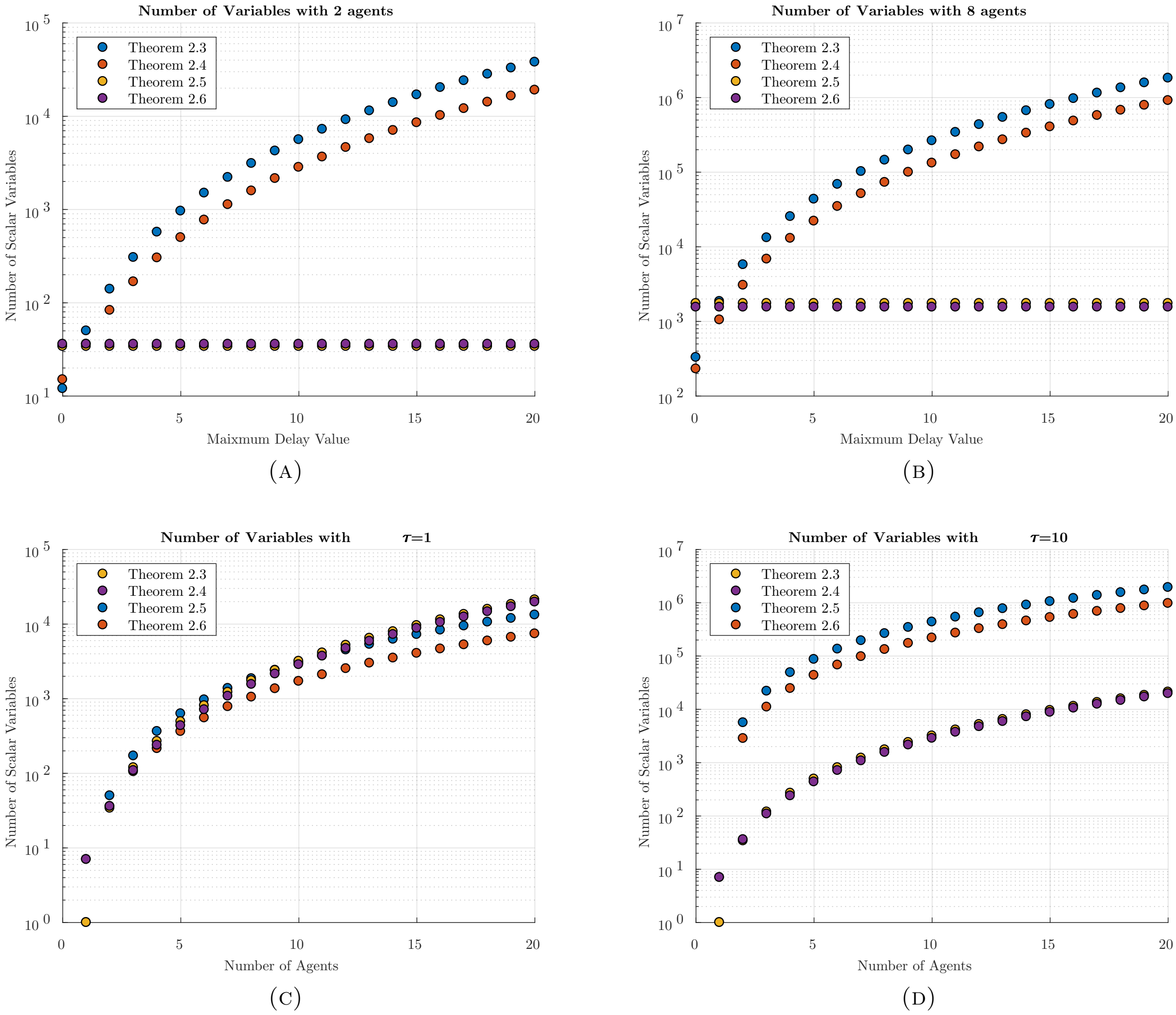


FIGURE 2.9: Example 2.7– Growth of the number of variables in Theorems 2.3, 2.4, 2.5, and 2.6 for different values of delays (Figures 2.9A and 2.9A) and agents in the network (Figures 2.9C and 2.9D), with second-order agents and single input ($m = 2$, and $p = 1$).

not increase with delays, while still managed to obtain less conservative conditions than previous studies from the literature for the numerical problems considered.

With the goal of reducing the conservatism of the first approach, we rewrote the problem as the study of convergence of an augmented switching system subject to input saturation. This transformation allowed the use a Lyapunov functional that depends on the modes of the switched system, which is equivalent to use a general delay-dependent Lyapunov-Krasovskii functional (Hetel, Daafouz, and Iung, 2008). As pointed by Hetel, Daafouz, and Iung (2008), this functional might represent the most general form that can be obtained using a sum of quadratic terms. The main drawback of the proposed approach is that the computational complexity of the conditions might increase rapidly as the delay upper bound increases, despite this value be considered known a priori. This is a trade-off between conservatism and computational burden. An interesting finding is that despite the trade-off between the two methods, the conservatism of the synthesis conditions of the augmented system approach seems to be higher than the conditions from the Lyapunov-Krasovskii approach.

In the development process of the stability conditions some possibly conservative approaches were made–and, naturally, they propagated to the synthesis conditions. These decisions may have led to conservative results, in the sense that there may exist conditions that ensure the convergence with larger region for consensus with tighter input saturation, and with larger time-delays. This is the price to be paid for the use of LMIs, if in one hand they concede the possibility to formulate sufficient conditions for convergence that can be easily solved using efficient well known methods, in another hand, more often than not, it is necessary to insert some conservatism on the conditions.

Finally, the numerical examples showed that our approach is less conservative and/or more general than recent similar approaches found in the literature for the considered examples. More specifically, our results can deal with the problem of consensus with and without leader, with directed communication topology, non-differentiable input time-delays, and provide a region in which the consensus is guaranteed.

Chapter 3

Formation-Containment of Euler-Lagrange Systems

This chapter addresses the formation-containment on interconnected systems modeled by using the Euler-Lagrange formalism, which are subject to input saturation and time-varying delays in the communication channels. The main results are the design of distributed control algorithms and sufficient conditions to ensure the multi-agent system convergence. The design and convergence analysis of leaders and followers are addressed separately. This is possible because the problem can be seen as a composition of two other sub-problems. First, the design of distributed protocols for leaders is presented and, in the sequence, the design for followers. Lastly, a theorem addressing the complete formation-containment problem is proposed. Finally, the use of the approach is illustrated using numerical examples in different scenarios.

3.1 Introduction and Preliminaries

The main objective in this chapter on the cooperative control of nonlinear systems subject to communication delays and input saturation is to generalize the approach that has usually been made in the literature, by providing a framework to handle both problems simultaneously. Given that delays and input saturation are part of the nature of networked systems, we aim to derive convergence guarantees by taking into account these characteristics.

In this chapter, an approach for the problem of formation-containment of networked Euler-Lagrange systems is proposed, which has started to be studied recently in the literature. For example, in Xu et al. (2020a) the time-varying formation is studied using finite-time estimators for velocity and states of leaders, and neural networks are used to handle uncertainties. In Xu et al. (2020b) the transient and steady-state performance of the formation-containment convergence is investigated along with unknown control directions. In the same lines, Hua et al. (2021) propose the fixed-time convergence by employing slide-mode strategy, which is also able to deal with parametric uncertainties and input disturbances. Zhou and Chen (2020) proposed an approach based on adaptive obervers to solve the formation-containment, but modeling leaders using linear dynamics and only followers using the Euler-Lagrange formalism. In addition, considering similar conditions as the ones in this chapter can be found in the work of Yang, Shi, and Constantinescu (2019) and Hernández-Guzmán and Orrante-Sakanassi (2019) on the *consensus* problem. In Yang, Shi, and Constantinescu (2019) it is studied the connectivity-preserving problem for consensus, and their approach have significant distinctions from ours. Particularly, they are mainly concerned with the connectivity-preserving consensus, which is an interesting problem *per se* and present some challenges associated with the implementation on real-world systems. For this reason, for example, they assume that each agent of the network is initially at rest. Our proposed framework is appropriated for a more general problem than the consensus and, since we are not concerned with the problem of connectivity-preserving, no assumptions related with the behavior of the agents are necessary. Hernández-Guzmán and Orrante-Sakanassi (2019) are also concerned with the consensus problem, they proposed an integration action in the controllers of Euler-Lagrange systems considering communication delays, but input saturation was not analyzed. Moreover, the approach used by Yang, Shi, and Constantinescu (2019) and Hernández-Guzmán and Orrante-Sakanassi (2019) to deal with the delays among agents corresponds to the one proposed by Nuño and collaborators on previous years (Nuño et al., 2011; Nuño et al., 2012; Nuño, Sarras, and Basanez, 2013; Nuño and Ortega, 2018).

Here, the communication delays are handled in a different way leading to an LMI test that relates the delays upper bounds with some parameters of the controllers. As a result, it is shown via numerical examples that the proposed conditions are less conservative in ensuring convergence on multi-agent systems subject to communication delays. More precisely, the main contributions in this chapter can be highlighted as:

i) The dynamics of all agents in the multi-agent system are nonlinear and modeled by Euler-Lagrange equations which, as it is well known, can be used to represent numerous mechanical and electrical systems. Most of the literature on formation-containment control is focused on linear dynamics, with some recent exceptions, (Mei, Ren, and Ma, 2012; Chen et al., 2017b; Li et al., 2018; Chen et al., 2019a; Chen et al., 2019b).

ii) The actuators limits are considered along with the nonlinear dynamics. This consideration is pertinent because it eases the implementation of the control strategy on real systems. Moreover, in addition to the control performance degradation caused by saturation, there is also the risk of thermal and mechanical failure when the actuators constraints are neglected. This premise was previously studied in a related context by Li et al. (2018) and Ren (2009), but with some important distinctions from our approach, namely: i) the formation-containment problem is more general than the consensus problem, ii) we propose a strategy to deal with input saturation and communication delays simultaneously, iii) different from Li et al. (2018) our dynamic controller does not rely on the inertia and Coriolis matrices and, iv) in contrast with both approaches, the control strategy proposed here does not depend on relative velocity measurements.

iii) The delay in the communication among agents is explicitly considered. Communication is a necessary condition for distributed control, and delays in the communication of networked systems are almost an inherent feature that might prevent the system from achieving the desired formation, thus it is desirable to consider its effects on the analysis. It is worth mentioning that the majority of works that approach this problem consider the self and neighbors states subject to the *same* delay, but it is easy to note that this consideration does not represent a realistic case. In fact, the results of Nuño and collaborators (Nuño et al., 2011; Nuño et al., 2012; Nuño, Sarras, and Basanez, 2013; Nuño and Ortega, 2018) seem to contribute with the least conservative conditions in communication delays in networked Euler-Lagrange systems. On the numerical examples we show that, even though we also consider the input saturation, the proposed conditions are less conservative than the ones in their approach.

iv) The proposed distributed control strategy does not depend on the knowledge of the agent's inertia matrix or the Coriolis forces, and it does not rely on the neighbors velocity information, which reduces the communication burden on the network.

3.1.1 Nonlinear Dynamics

The majority of real-world systems are intrinsically nonlinear, and the linearization often does not provide a desirable representation when the systems do not operate in the vicinity of the linearization point (Slotine and Li, 1991). Even though in the previous section we provided a rigorous definition of the region in which the convergence is ensured, we had assumed that the agents' models are well represented by the linear dynamics. We acknowledge that this might be a strong assumption, and in order to draw a methodology closer to realistic systems, it is now assumed nonlinear dynamics for the networked nodes.

From an engineering perspective it is of great interest to model mechanical and electrical systems, and if we look from the cooperative control point of view the set of systems of interest is even more specific–it essentially encompasses the control and coordination of groups of robots, unnamed aerial vehicles, and sensors. We, thereby, draw upon the large body of interest of the field, by addressing nonlinear Euler-Lagrange systems with input saturation, represented by

$$M_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_i + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_i + \boldsymbol{g}_i(\boldsymbol{q}_i) = \text{sat}(\boldsymbol{u}_i), \quad \forall i \in \{1, \dots, M\}, \tag{3.1}$$

where the vectors $\boldsymbol{q}_i \in \mathbb{R}^m$ are the generalized coordinates, the inertia matrix $M_i(\boldsymbol{q}_i) \in \mathbb{R}^{m\times m}$ is symmetric and positive-definite, the Coriolis and centripetal effects are represented with the matrix $C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i) \in \mathbb{R}^{m\times m}$, and the vector $\boldsymbol{g}_i(\boldsymbol{q}_i) \in \mathbb{R}^m$ maps the gravitational effects. The input signal produced by the actuators are represented by $\boldsymbol{u}_i \in \mathbb{R}^m$ and the element-wise function $\text{sat}(\cdot)$ maps the bound on the input, that is $||\text{sat}(\boldsymbol{u}_i)||_\infty \leq u_{\max\, i}$ where the scalar $u_{\max\, i}$ represents the actuator limit and the saturation function is defined as

$$\begin{aligned} &\text{sat}\,(\boldsymbol{u}_i) = [\text{sat}(u_{i(1)}) \dots \text{sat}(u_{i(m)})]', \text{ with} \\ &\text{sat}\,(u_{i(r)}) = \text{sign}(u_{i(r)}) \min(u_{\max\, i}, |u_{i(r)}|), \quad \forall r \in \{1, \dots, m\}, \end{aligned} \tag{3.2}$$

in which the subscript inside the parenthesis represents the rth element of the vector $\boldsymbol{u}_i$. Here, for simplicity, we slightly abuse notation using the same notation for vector saturation function and scalar function. The agents dynamics are assumed to be possible distinct and to have the following properties of Euler-Lagrange systems (Spong, Hutchinson, and Vidyasagar, 2005):

Property 1. The matrix $\dot{M}_i(\boldsymbol{q}_i) - 2C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)$ is skew-symmetric, that is

$$\boldsymbol{z}'\Big(\dot{M}_i(\boldsymbol{q}_i) - 2C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\Big)\boldsymbol{z} = 0, \text{ for any vector } \boldsymbol{z} \in \mathbb{R}^m.$$

Property 2. There are positive constants a_i, b_i such that

$$0 < I_m a_i \leq M_i(\boldsymbol{q}_i) \leq I_m b_i,\ \forall \boldsymbol{q}_i \in \mathbb{R}^m.$$

Property 3. There exist positive constants c_i, such that

$$||C_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{z}|| \leq c_i ||\boldsymbol{z}|| ||\boldsymbol{y}||,\ \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^m.$$

In addition to considering nonlinear dynamics for the nodes, we take a step further in relation to the consensus problem and approach a more general problem. We address the problem of formation-containment (Mei, Ren, and Ma, 2012; Chen et al., 2019b), which is a composition of the formation problem, where a group of agents are required to maintain a given formation, and the containment problem, where the purpose is to guarantee that the agents move into a given region. To achieve the so-called formation-containment, the agents of the network are split into two groups, one is required to achieve a relative formation and the other is required to move into the region formed by the first one. With this strategy, it is possible to generalize some other approaches of cooperative control, in the sense that some problems can be solved as a particular case of the formation-containment. To make this more clear, consider for example the case where all agents are required to perform a formation described by a single point and no agent is required to move "into" this region. In this scenario, the behavior of the multi-agent system is analogous to the leaderless consensus. Now consider the case in which the group that is required to achieve a relative formation is reduced to a single agent, that is just one leader, and there are agents in the group of followers. The relative formation would be "completed" from the beginning (as it is made by only one leader), and, as the followers are required to move into the given formation, they will synchronize with the leader. This gives an identical behavior as the leader following problem.

Henceforth, the group of agents required to achieve the relative formation will be called *leaders*, and will be indexed in the set $\mathcal{V}_L = \{1, \ldots, N\}$, and the agents required to move into this formation will be called *followers*, indexed in $\mathcal{V}_F = \{N+1, \ldots, M\}$. Because the overall network behavior can be seen as a composition of two other behaviors, even though they happen simultaneously, we will address each one separately. The formal definition of the leaders convergence is given as,

Definition 3.1. *The leaders formation with a desired constant configuration is achieved if the leaders states converge asymptotically to a constant relative configuration, that is,*

$$\lim_{t\to\infty} ||\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t) - \boldsymbol{\delta}_{ij}|| = 0, \tag{3.3}$$

$$\lim_{t\to\infty} ||\dot{\boldsymbol{q}}_i(t)|| = 0, \quad \forall i, j \in \mathcal{V}_L, \tag{3.4}$$

in which the variable $\boldsymbol{q}_i(t) \in \mathbb{R}^m$ represents the generalized coordinates of the ith Euler-Lagrange system, the constant vector $\boldsymbol{\delta}_{ij} \in \mathbb{R}^m$ denotes the desired relative configuration on the generalized coordinates, and $\mathcal{V}_L$ represents the leaders set.

It is assumed that the formation-containment is achieved if the followers states converge to the convex hull spanned by the leaders states. Specifically, a convex hull is defined as,

Definition 3.2. *The convex hull of a finite set of points $X = \{x_1, \ldots, x_n\}$ is denoted by,*

$$\mathrm{Co}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in X, \quad \forall \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1 \right\},$$

where $\alpha_i \in \mathbb{R}$.

With this characterization, the formal definition of the followers convergence is stated as,

Definition 3.3. *The followers containment into the leaders formation is achieved if the followers states converge asymptotically into the convex hull spanned by the leaders, that is, there exist constant values α_{ij} such that*

$$\lim_{t\to\infty} \left\| \boldsymbol{q}_i(t) - \sum_{j=1}^{N} \alpha_{ij} \boldsymbol{q}_j(t) \right\| = 0, \quad \alpha_{ij} \geq 0, \quad \sum_{j=1}^{N} \alpha_{ij} = 1 \quad \forall i \in \mathcal{V}_F, \quad \forall j \in \mathcal{V}_L, \tag{3.5}$$

in which the variable $\boldsymbol{q}_i(t) \in \mathbb{R}^m$ represents the generalized coordinates of the ith Euler-Lagrange system, $\mathcal{V}_F$ represents the followers set and $\mathcal{V}_L$ represents the leaders set.

Therefore, the problem of formation-containment is said to be solved when the convergence of leaders and followers occur simultaneously, that is,

Definition 3.4. *The distributed saturated input control,* $\mathrm{sat}\,(\boldsymbol{u}_i)$ *for all $i \in \{1, \cdots, M\}$, is said to solve the formation-containment problem of the networked Euler-Lagrange system (3.1), if all leaders converge asymptotically to the desired formation and, simultaneously, all followers move into the convex hull spanned by the configuration of leaders, that is, the limits (3.3), (3.4), and (3.5) hold simultaneously.*

Next, it is shown explicitly how the interactions among leaders and followers is assumed to happen.

3.1.2 Communication Topology on the Formation-Containment

The network communication topology on the formation-containment problem is represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. According to the sets of leaders and followers, the adjacency matrix of the multi-agent system can be partitioned into the block lower triangular form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_L & 0 \\ \mathcal{A}_{LF} & \mathcal{A}_F \end{bmatrix},$$

in which $\mathcal{A}_L \in \mathbb{R}^{N\times N}$ denotes the communication topology among leaders, $\mathcal{A}_F \in \mathbb{R}^{M-N\times M-N}$ denotes the communication topology among followers, and $\mathcal{A}_{LF} \in \mathbb{R}^{M-N\times N}$ denotes the topology of information flow from leaders to followers. Recall, from the definition of the degree matrix in Section 2.1.4 that the elements of the degree matrix are given by $d_{ii} = \sum_{j=1}^{n} a_{ij}$. Hence, as a result of the structure of $\mathcal{A}$ above, the information flow from leaders to followers is reflected into the degree matrix of the followers, and consequently, into the Laplacian matrix of the followers L_F. Moreover, it is assumed that the Laplacian matrix associated to the leaders communication topology L_L and to the followers L_F, are both symmetric. This means, accordingly to the definition of the Laplacian matrix on Section 2.1.4, that the communication topology among agents of the same group (leaders or followers) is undirected. Besides, because leaders do not receive any information from followers and only some followers receive information from leaders, the matrix L_{LF} is non-symmetric.

To ensure the convergence the following assumptions are necessary:

Assumption 3.1. *The graphs $\mathcal{G}$ and the subgraph $\mathcal{G}_L$ associated with the communication topology of the network and of the leaders, respectively, have a directed spanning tree.*

Assumption 3.2. *There exists a path from at least one leader to each follower.*

The following lemma is used to derive the main results on the containment of followers:

Lemma 3.1 (Mei, Ren, and Ma (2012))**.** *Under the Assumptions 3.1 and 3.2, the matrix L_F^{-1} is positive definite and, additionally, each entry of the matrix $-L_F^{-1}L_{LF}$ is non-negative and each row of $-L_F^{-1}L_{LF}$ has sum equal to one.*

In fact, Assumption 3.2 is a consequence of Assumption 3.1–if Assumption 3.2 is not met, then $\mathcal{G}$ will have no spanning tree. Also, the result in Lemma 3.1 follows from the fact that a spanning tree in $\mathcal{G}$ is equivalent to the Laplacian matrix L having a single zero eigenvalue and all other eigenvalues with positive real part (Ren and Beard, 2008). Because the communication among leaders happens only among them and $\mathcal{G}_L$ is required to have a spanning tree, the single zero eigenvalue is related to L_L.

3.1.3 Communication Channels Subject to Delay

On the study of nonlinear cooperative control, one of the problems that we are concerned with is how a delayed information exchanged by the nodes affects the network convergence, different from the problem of input delay which was studied in Chapter 2. The objective is to represent time-delays in the network itself. More specifically, the time spent between sending and receiving information might prevent a multi-agent system to behave in a desired way. We aim to be able to assert for which delays values the convergence is guaranteed. When analysing this aspect we assume that each agent access its own states instantaneously, without input delay. Thus, the study is carried under the following assumption:

Assumption 3.3. *Each agent has access to its neighbors states subject to communication delay, but it access its own states instantaneously. Moreover, the time-delay derivative $\dot{\tau}_{ji}(t)$ is bounded for all $i, j \in \mathcal{V}_L \cup \mathcal{V}_F$, that is, there exists some unknown positive scalar $\hat{\tau}$ such that $|\dot{\tau}_{ji}(t)| \leq \hat{\tau}$.*

3.1.4 Saturation on the Formation-Containment

When considering saturation limits on Euler-Lagrange systems it is imperative to assume a compatibility between the saturation limits, $u_{\max\, i}$, and the steady-state generalized forces. This assumption has been made in works on similar scenarios (Morabito, Teel, and Zaccarian, 2004; Hashemzadeh, Hassanzadeh, and Tavakoli, 2013), and is formalized on the following assumption:

Assumption 3.4. *Given the gravitational vector $\boldsymbol{g}(\boldsymbol{q}_i)$ of the Euler-Lagrange agents (3.1) and the saturation limits $u_{max\,i}$, for $i = 1, \ldots, M$, the following inequality always holds:*

$$\sup_{\boldsymbol{q}_i \in \mathbb{R}^m} \|\boldsymbol{g}(\boldsymbol{q}_i)\|_\infty < u_{max\,i}, \quad \forall i \in \{1, \ldots, M\}.$$

Note that Assumption 3.4 is necessary for the systems to be stabilized under saturated inputs in some configurations. To make this clear, note from equation (3.1) that for $\ddot{\boldsymbol{q}}_i = \dot{\boldsymbol{q}}_i = 0$ the input is required to fully compensate the gravitational effects. Hence, Assumption 3.4 requires that the saturation limit is large enough so the input can stabilize the agents at any configuration.

3.2 Leaders Distributed Formation Control

The role of leaders on the formation control is, in a precise description, to establish the region in which the followers are required to stay. This kind of task can be employed, for example, when some systems have more sensing capabilities and could assist a subset of systems, the followers, to move in the environment. In this work, this region is characterized by the convex hull defined by the leaders' configuration, and the relative configuration among leaders can be adjusted by changing a control parameter.

In order to the leaders modeled by the Euler-Lagrange equation subject to input saturation,

$$M_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_i + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_i + \boldsymbol{g}_i(\boldsymbol{q}_i) = \mathrm{sat}(\boldsymbol{u}_i),$$

to attain the desired constant formation in a distributed fashion, inspired by Li et al. (2018) we propose the following distributed dynamic controller, constituted by the dynamic part

$$D_i\ddot{\boldsymbol{\eta}}_i(t) = k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=1}^{N} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t - \tau_{ji}(t)) - \boldsymbol{\delta}_{ij}\Big) - k_{3i}\dot{\boldsymbol{\eta}}_i(t), \quad \forall i \in \mathcal{V}_L, \tag{3.6}$$

and algebraic portion, that provides the ith leader input as

$$\boldsymbol{u}_i = -k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i(t)) - k_{4i}\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i(t)) + \boldsymbol{g}_i(\boldsymbol{q}_i), \quad \forall i \in \mathcal{V}_L, \tag{3.7}$$

where $\mathcal{V}_L = \{1, \ldots, N\}$ is the index set of leaders, $\tilde{\boldsymbol{q}}_i = \boldsymbol{q}_i(t) - \boldsymbol{\eta}_i(t)$ is the coupling between the ith agent and its dynamic controller, the desired relative configuration is represented by the constant $\boldsymbol{\delta}_{ij} \in \mathbb{R}^m$, the state of the ith dynamic controller is represented by $\boldsymbol{\eta}_i(t) \in \mathbb{R}^m$, the matrix $D_i \in \mathbb{R}^{m\times m}$ is constant and positive definite, $k_{1i}, k_{2i}, k_{3i}, k_{4i}, \beta_{1i}$, and β_{2i} are positive constants. To account for the input saturation the control is designed based on a continuous function $\sigma(\cdot) : \mathbb{R} \to (-1, 1)$ which is applied element-wise and satisfies the following properties:

- $\lim_{x\to-\infty} \sigma(x) = -1$;
- $\lim_{x\to\infty} \sigma(x) = 1$;
- $\sigma(0) = 0$;
- $\sigma(\cdot)$ is a strictly increasing function,

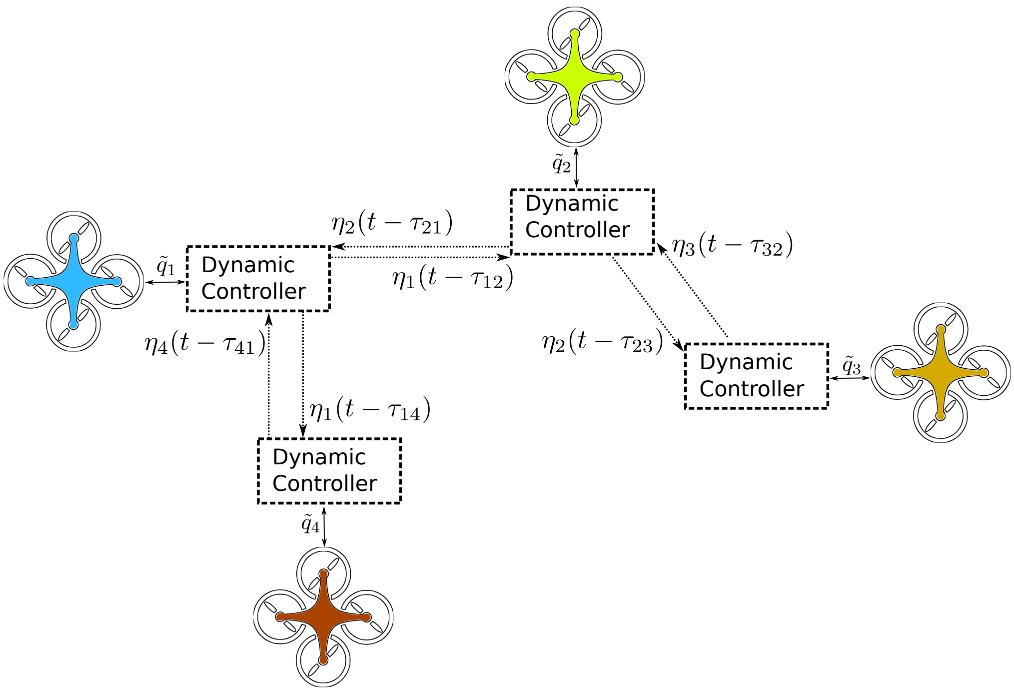


FIGURE 3.1: Representation of the proposed dynamic controller in (3.6) as proxies for communication among agents.

- its time-derivative with respect to its argument is bounded.

An example of such a function is the hyperbolic tangent. Note that the input (3.7) is bounded by $k_{1i} + k_{4i} + ||\boldsymbol{g}(\boldsymbol{q}_i)||_\infty$, and the gains can be adjusted in order to fulfill the actuators limit, i.e., satisfying the saturation constraint. Moreover, the input (3.7) is chosen in such a way to synchronize the agent and its controller, while adding a damping on the agent's dynamics.

With this approach, the networked system is coupled by the dynamic controllers of each agent, such that the controllers of neighbors exchange information with each other. The dynamic controller can be seen as a proxy for the interaction among agents, Figure 3.1 illustrates a scheme with four agents, in which the dynamic controllers in the boxes are given in (3.6), and the time dependency of delays $\tau_{ji}(t)$ is not shown. Thus, the communication delay of the network is represented in (3.6) by the delayed neighbor state of the controller $\boldsymbol{\eta}_j(t-\tau_{ji}(t))$, where the variable $\tau_{ji}(t)$ represents a time-varying delay which is assumed to be bounded by a known positive constant $|\tau_{ji}(t)| \leq \bar{\tau}_j$, and have the first derivative $\dot{\tau}_{ji}$ bounded by an unkown value.

Remark 3.1. Note that the local dynamic controller (3.6) can be seen as a second-order linear system with an aggregated input $\boldsymbol{u}_{ci}(t)$, as

$$D_i \ddot{\boldsymbol{\eta}}_i(t) + k_{3i} \dot{\boldsymbol{\eta}}_i(t) = \boldsymbol{u}_{ci}(t). \tag{3.8}$$

It is a well known result from the literature that the above system is input to state stable (Khalil, 2002) for appropriate choices of D_i and k_{3i}. That is, the system has a globally asymptotically stable equilibrium point in absence of external input, $\boldsymbol{u}_{ci}(t) = 0$, and for bounded inputs the states of the system remain bounded.

In the next lemma, it is proposed a sufficient condition for the leaders to reach the desired static formation, with undirect communication topology between leaders.

Lemma 3.2. *The N leaders modeled by Euler-Lagrange equation (3.1), with dynamic controller (3.6), bounded input (3.7), and time-varying communication delays with bounded first derivative, reach asymptotically the static relative configuration,*

$$\lim_{t\to\infty} ||\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t) - \boldsymbol{\delta}_{ij}|| = 0, \lim_{t\to\infty} ||\dot{\boldsymbol{q}}_i(t)|| = 0,$$

for all $i, j \in \mathcal{V}_L$, if for given positive constants $\bar{\tau}_i$, k_{2i}, and k_{3i}, there exist positive definite matrices $P_i \in \mathbb{R}^{m\times m}$ for $i = 1, \cdots, N$, such that the following linear matrix inequality holds,

$$\begin{bmatrix} \Xi_1 & \Xi_2 \\ * & \Xi_3 \end{bmatrix} < 0, \tag{3.9}$$

with

$$\Xi_1 = \text{diag}\left\{-2k_{31}I + l_{11}\bar{\tau}_1 P_1, \quad -2k_{32}I + l_{22}\bar{\tau}_2 P_2, \quad \cdots, \quad -2k_{3N}I + l_{NN}\bar{\tau}_N P_N\right\}, \tag{3.10}$$

$$\Xi_2 = \text{diag}\left\{\Xi_{21}, \quad \Xi_{22}, \quad \cdots, \quad \Xi_{2N}\right\}, \tag{3.11}$$

$$\Xi_{2k} = \underbrace{\begin{bmatrix} -k_{2k}I & -k_{2k}I & \cdots & -k_{2k}I \end{bmatrix}}_{l_{kk}\ times},$$

$$\Xi_3 = \text{diag}\left\{\Xi_{31}, \quad \Xi_{32}, \quad \cdots, \quad \Xi_{3N}\right\}, \tag{3.12}$$

the terms Ξ_{3k} are arranged considering the network topology, according to the following algorithm,

---

Structure of Ξ_{3k}

$\Xi_{3k} \leftarrow []$
  for $j = 1, \ldots, N$:
    if $a_{kj} \neq 0$:
      $\Xi_{3k} \leftarrow \text{diag}\left(\Xi_{3k}, \left[-P_j/\bar{\tau}_j\right]\right)$
end

The term $\bar{\tau}_i$ denotes the upper bound on the time-varying delays associated with the ith agent, $[]$ is an empty matrix (i.e., the only linear map is $[] : \{0\} \to \{0\}$), and l_{ii} is the ith diagonal element of the degree matrix.

Proof. The proof consists of two main parts, firstly it is shown conditions that guarantee bounds on the states of the dynamic controllers and Euler-Lagrange systems. Secondly, it is shown that if these conditions are satisfied, then the states of each dynamic controller asymptotically synchronizes over the network and the states of each system also synchronize with its associated dynamic controller, leading to the desired convergence on the target formation.

Consider the following Lyapunov-Krasovskii functional candidate,

$$V_L = V_1 + V_2 + V_3,$$

with

$$\begin{aligned}
V_1 =& \sum_{i=1}^{N} \Big\{ \dot{\boldsymbol{\eta}}_i' D_i \dot{\boldsymbol{\eta}}_i + \dot{\boldsymbol{q}}_i' M_i(q_i) \dot{\boldsymbol{q}}_i \Big\}, \\
V_2 =& \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} k_{2i} (\boldsymbol{\eta}_i - \boldsymbol{\eta}_j - \boldsymbol{\delta}_{ij})' (\boldsymbol{\eta}_i - \boldsymbol{\eta}_j - \boldsymbol{\delta}_{ij}) + 2 \sum_{i=1}^{N} \int_0^{\tilde{\boldsymbol{q}}_i} k_{1i} \sigma(\beta_{1i} \varepsilon) d\varepsilon, \\
V_3 =& \sum_{i=1}^{N} \left\{ l_{ii} \int_{-\bar{\tau}_i}^{0} \int_{t+\theta}^{t} \dot{\boldsymbol{\eta}}_i(\varepsilon)' P_i \dot{\boldsymbol{\eta}}_i(\varepsilon) d\varepsilon d\theta \right\}.
\end{aligned}$$

The positivity of the Lyapunov-Krasovskii functional is immediate. Next we derive the conditions for the negativity of the functional derivative. The time-derivative of V_1 along the trajectory of the system (3.1) and its controller (3.6) is given by,

$$\begin{aligned}
\dot{V}_1 =& \sum_{i=1}^{N} \left\{ 2\dot{\boldsymbol{q}}_i' M_i(\boldsymbol{q}_i) \ddot{\boldsymbol{q}}_i + \dot{\boldsymbol{q}}_i' \dot{M}_i(\boldsymbol{q}_i) \dot{\boldsymbol{q}}_i + 2\dot{\boldsymbol{\eta}}_i' D_i \ddot{\boldsymbol{\eta}}_i \right\} \\
=& 2 \sum_{i=1}^{N} \dot{\boldsymbol{q}}_i' M_i(\boldsymbol{q}_i) \left\{ M_i(\boldsymbol{q}_i)^{-1} \Big(- C(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i) \dot{\boldsymbol{q}}_i - k_{1i} \sigma(\beta_{1i} \tilde{\boldsymbol{q}}_i) - k_{4i} \sigma(\beta_{2i} \dot{\boldsymbol{q}}_i) \Big) \right\} \\
&+ 2 \sum_{i=1}^{N} \dot{\boldsymbol{\eta}}_i' D_i \left\{ D_i^{-1} \Big(k_{1i} \sigma(\beta_{1i} \tilde{\boldsymbol{q}}_i) - k_{2i} \sum_{j=1}^{N} a_{ij} \Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t - \tau_{ji}(t)) - \boldsymbol{\delta}_{ij} \Big) \right. \\
&\left. - k_{3i} \dot{\boldsymbol{\eta}}_i \Big) \right\} + \sum_{i=1}^{N} \dot{\boldsymbol{q}}_i' \dot{M}_i(\boldsymbol{q}_i) \dot{\boldsymbol{q}}_i \\
=& 2 \sum_{i=1}^{N} \Big\{ - k_{1i} \dot{\boldsymbol{q}}_i' \sigma(\beta_{1i} \tilde{\boldsymbol{q}}_i) - k_{4i} \dot{\boldsymbol{q}}_i' \sigma(\beta_{2i} \dot{\boldsymbol{q}}_i) + k_{1i} \dot{\boldsymbol{\eta}}_i' \sigma(\beta_{1i} \tilde{\boldsymbol{q}}_i) - k_{3i} \dot{\boldsymbol{\eta}}_i' \dot{\boldsymbol{\eta}}_i \\
&- k_{2i} \sum_{j=1}^{N} a_{ij} \dot{\boldsymbol{\eta}}_i' \Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t - \tau_{ji}(t)) - \boldsymbol{\delta}_{ij} \Big) \Big\} \\
=& 2 \sum_{i=1}^{N} \Big\{ - k_{1i} \dot{\tilde{\boldsymbol{q}}}_i' \sigma(\beta_{1i} \tilde{\boldsymbol{q}}_i) - k_{4i} \dot{\boldsymbol{q}}_i' \sigma(\beta_{2i} \dot{\boldsymbol{q}}_i) - k_{3i} \dot{\boldsymbol{\eta}}_i' \dot{\boldsymbol{\eta}}_i \\
&- k_{2i} \sum_{j=1}^{N} a_{ij} \dot{\boldsymbol{\eta}}_i' \Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t) - \boldsymbol{\delta}_{ij} \Big) - k_{2i} \sum_{j=1}^{N} a_{ij} \dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon \Big) \Big\},
\end{aligned} \tag{3.13}$$

where we have used the Property 1 in Section 3.1.1 (i.e., the matrix $\dot{M}_i(\boldsymbol{q}_i) - 2C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)$ is skew-symmetric), the fact that the inertia matrices are positive-definite, and the fact that the delayed term can be written as,

$$\boldsymbol{\eta}_j(t-\tau_{ji}(t)) = \boldsymbol{\eta}_j(t) - \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon. \tag{3.14}$$

Taking the time-derivative of V_2 gives us,

$$\dot{V}_2 = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}k_{2i}(\dot{\boldsymbol{\eta}}_i - \dot{\boldsymbol{\eta}}_j)'(\boldsymbol{\eta}_i - \boldsymbol{\eta}_j - \boldsymbol{\delta}_{ij}) + 2\sum_{i=1}^{N} k_{1i}\dot{\tilde{\boldsymbol{q}}}_i'\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i).$$

Recall that the communication among leaders is considered undirected, which imply symmetry on the block Laplacian matrix L_L. Hence, if there exists $\dot{\boldsymbol{\eta}}_i'\boldsymbol{\eta}_j$ with $a_{ij} \neq 0$, then there also exists $\dot{\boldsymbol{\eta}}_j'\boldsymbol{\eta}_i$ with $a_{ij} \neq 0$. Thus, we can write

$$\dot{V}_2 = 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}k_{2i}\dot{\boldsymbol{\eta}}_i'(\boldsymbol{\eta}_i - \boldsymbol{\eta}_j - \boldsymbol{\delta}_{ij}) + 2\sum_{i=1}^{N} k_{1i}\dot{\tilde{\boldsymbol{q}}}_i'\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i). \tag{3.15}$$

The derivative of V_3 is given by,

$$\begin{aligned}\dot{V}_3 =& \sum_{i=1}^{N}\left\{ l_{ii}\bar{\tau}_i\dot{\boldsymbol{\eta}}_i'P_i\dot{\boldsymbol{\eta}}_i - l_{ii}\int_{t-\bar{\tau}_i}^{t}\dot{\boldsymbol{\eta}}_i(\varepsilon)'P_i\dot{\boldsymbol{\eta}}_i(\varepsilon)d\varepsilon \right\}\\ \leq& \sum_{i=1}^{N}\left\{ l_{ii}\bar{\tau}_i\dot{\boldsymbol{\eta}}_i'P_i\dot{\boldsymbol{\eta}}_i - \sum_{j=1}^{N} a_{ij}\int_{t-\tau_{ij}(t)}^{t}\dot{\boldsymbol{\eta}}_i'(\varepsilon)P_i\dot{\boldsymbol{\eta}}_i(\varepsilon)d\varepsilon \right\},\end{aligned}$$

and using the symmetry of undirected graphs and applying Lemma A.3 on the integral term yields

$$\dot{V}_3 \leq \sum_{i=1}^{N}\left\{ l_{ii}\bar{\tau}_i\dot{\boldsymbol{\eta}}_i'P_i\dot{\boldsymbol{\eta}}_i - \sum_{j=1}^{N} a_{ij}\frac{1}{\bar{\tau}_j}\int_{t-\tau_{ji}(t)}^{t}\dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon P_j\int_{t-\tau_{ji}(t)}^{t}\dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \right\}. \tag{3.16}$$

Thus, the time-derivative of the Lyapunov-Krasovskii candidate is given by,

$$\begin{aligned}\dot{V}_L \leq& -2\sum_{i=1}^{N} k_{4i}\dot{\boldsymbol{q}}_i'\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) + \sum_{i=1}^{N}\Bigg\{ -2k_{3i}\dot{\boldsymbol{\eta}}_i'\dot{\boldsymbol{\eta}}_i + l_{ii}\bar{\tau}_i\dot{\boldsymbol{\eta}}_i'P_i\dot{\boldsymbol{\eta}}_i\\ &-\sum_{j=1}^{N} a_{ij}\frac{1}{\bar{\tau}_j}\int_{t-\tau_{ji}(t)}^{t}\dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon P_j\int_{t-\tau_{ji}(t)}^{t}\dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon - 2k_{2i}\sum_{j=1}^{N} a_{ij}\dot{\boldsymbol{\eta}}_i'\int_{t-\tau_{ji}(t)}^{t}\dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \Bigg\}\\ =& -2\sum_{i=1}^{N} k_{4i}\dot{\boldsymbol{q}}_i'\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) + \Upsilon(t)'\Xi\Upsilon(t),\end{aligned}$$

where the vector $\Upsilon(t)$ is arranged according the following algorithm:

Algorithm 1 Structure of $\Upsilon(t)$ (3.17)

$\Upsilon(t) \leftarrow \left[\dot{\boldsymbol{\eta}}_1' \cdots \dot{\boldsymbol{\eta}}_N'\right]'$
  for $i = 1, \ldots, N$:
    for $j = 1, \ldots, N$:
      if $a_{ij} \neq 0$:
        $\Upsilon(t) \leftarrow \left(\Upsilon(t) | \left[\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon\right]'\right)$
end

In which the operation $(\boldsymbol{a}|\boldsymbol{b})$, with vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, denotes the row-wise concatenation $\left[\boldsymbol{a}' \quad \boldsymbol{b}'\right]' \in \mathbb{R}^{2n}$.

Hence,

$$\Upsilon(t)' = \left[\dot{\boldsymbol{\eta}}_1' \cdots \dot{\boldsymbol{\eta}}_N' \quad \int_{t-\tau_{21}(t)}^{t} \dot{\boldsymbol{\eta}}_2(\varepsilon)' d\varepsilon \cdots \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' d\varepsilon \cdots \int_{t-\tau_{(N-1)N}(t)}^{t} \dot{\boldsymbol{\eta}}_{(N-1)}(\varepsilon)' d\varepsilon\right],$$

where the terms on $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' d\varepsilon$ are defined only if $(v_i, v_j) \in \mathcal{E}$, and

$$\Xi = \begin{bmatrix} \Xi_1 & \Xi_2 \\ * & \Xi_3 \end{bmatrix},$$

where Ξ_1, Ξ_2, and Ξ_3 are defined in (3.10), (3.11), and (3.12).

Therefore, if the conditions in our statement hold we have $V_L > 0$ and $\dot{V}_L < 0$ for non-zero variables $\dot{\boldsymbol{\eta}}_i$, $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon$, and $\dot{\boldsymbol{q}}_i$. However, the function $\dot{V}_L$ might be zero for non-zero variables $\boldsymbol{\eta}_i$ and $\tilde{\boldsymbol{q}}_i$, in which V_L is defined, thus the Lyapunov's direct method is applied to attest asymptotic convergence. Nevertheless, it follows that V_L is bounded and consequently $\dot{\boldsymbol{q}}_i$, $\dot{\boldsymbol{\eta}}_i$, $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_i(\varepsilon) d\varepsilon$ are also bounded. According to leaders and controllers dynamics, (3.1) and (3.6), Properties 2, 3, and Assumption 3.4, we have that $\ddot{\boldsymbol{q}}_i$ and $\ddot{\boldsymbol{\eta}}_i$ are also bounded, with the assumption of boundedness of $\dot{\tau}_{ji}$, we get that $\ddot{V}_L$ is bounded. Thus, $\dot{V}_L$ is uniformly continuous and, invoking Barbalat's lemma, we conclude that $\lim_{t\to\infty} \dot{V}_L = 0$, $\lim_{t\to\infty} \dot{\boldsymbol{\eta}}_i(t) = 0$, $\lim_{t\to\infty} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_i(\varepsilon) d\varepsilon = 0$, and $\lim_{t\to\infty} \dot{\boldsymbol{q}}_i(t) = 0$.

Henceforth, we want to show the synchronization of the controllers over the network and the convergence of the leaders to constant relative configurations. To this end we show that: i) the leaders acceleration $\ddot{\boldsymbol{q}}_i$ goes to zero, which implies that $\lim_{t\to\infty} \tilde{\boldsymbol{q}}_i = 0$, and ii) the controller acceleration $\ddot{\boldsymbol{\eta}}_i$ also goes to zero. Therefore, from (3.6) we have $\lim_{t\to\infty} \Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t - \tau_{ji}(t)) - \boldsymbol{\delta}_{ij}\Big) = 0$.

In order to show that the acceleration $\ddot{\boldsymbol{q}}_i$ asymptotically converges to zero, note that its time-derivative is given by

$$\dddot{\boldsymbol{q}}_i = -M_i^{-1}(\boldsymbol{q}_i)\Big[\dot{C}_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_i + C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\ddot{\boldsymbol{q}}_i - \dot{\boldsymbol{u}}_i\Big] - \dot{M}_i^{-1}(\boldsymbol{q}_i)\Big[C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\ddot{\boldsymbol{q}}_i - \boldsymbol{u}_i\Big],$$

and recalling Properties 1-3, the boundedness of $\dot{\boldsymbol{q}}_i$ and $\ddot{\boldsymbol{q}}_i$, the previous equation suggests that $\dddot{\boldsymbol{q}}_i$ is bounded for bounded $\dot{M}_i^{-1}$ and $\dot{C}_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_i$, which is a realistic assumption for Euler-Lagrange systems with prismatic and revolute joints (Ghorbel, Srinivasan, and Spong, 1998;

Mulero-Martínez, 2007; Nuño, Basañez, and Ortega, 2009). Hence, the Barbalat's lemma is used once again to show that that $\lim_{t\to\infty} \ddot{\boldsymbol{q}}_i(t) = 0$, and finally, from the leaders dynamics we conclude that $\lim_{t\to\infty} \tilde{\boldsymbol{q}}_i(t) = 0$.

We have that $\lim_{t\to\infty} \tilde{\boldsymbol{q}}_i(t) = 0$ and $\lim_{t\to\infty} \dot{\boldsymbol{\eta}}_i(t) = 0$. Therefore, with similar arguments used to show the convergence of $\ddot{\boldsymbol{q}}_i(t)$, we also get that $\lim_{t\to\infty} \ddot{\boldsymbol{\eta}}_i(t) = 0$. Thus, from the controller (3.6), we obtain that $\lim_{t\to\infty} \left(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t)) - \boldsymbol{\delta}_{ij}\right) = 0$. Hence,

$$\lim_{t\to\infty} ||\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t) - \boldsymbol{\delta}_{ij}|| = 0,$$

which completes the proof. □

The previous result presents an LMI test for the convergence of multiple Euler-Lagrange systems into a constant relative configuration. The proposed condition is designed taking into account bounded inputs and communication delays on the multi-agent system and it establishes a relationship between some controller parameters and the upper bounds of the communication delays. Specifically, from the controller parameters $k_{1i}, k_{2i}, k_{3i}, k_{4i}, \beta_{1i}, \beta_{2i}$, only k_{2i} and k_{3i} have to be adjusted in such a way that the conditions in Lemma 3.2 are satisfied and they are not directly associated with the input bounds. That is to say, the remainder parameters have no influence on the given convergence conditions as long as they are positive and k_{1i} and k_{4i} are chosen such that the actuators limits are satisfied. Furthermore, the proposed LMI condition is independent of the model of each agent in the sense that it does not depend on the inertia nor Coriolis matrices of the agents, it depends only on the number of degrees of freedom of each system, which can be used for analysis of heterogeneous multi-agent systems.

3.2.1 Numerical Examples

This section shows an example to illustrate the use of Lemma 3.2, which is a particular case of the problem under analysis. The motivation is to compare the proposed result with the specific research line that study this case, particularly the result is compared with a partial result in Nuño and Ortega (2018).

Example 3.1. *For comparison of the conditions on the communication delay we consider the problem studied by Nuño and Ortega (2018), which is a particular case of the results presented here, namely, the leaderless consensus without input-saturation, that is, the set of followers is empty and the leaders must converge to the same point. We consider a team of four manipulators with two degrees of freedom, in which the inertia and Coriolis matrices are given by*

$$M_i(\boldsymbol{q}_i) = \begin{bmatrix} h_{1i} + 2h_{2i}c_{2i} & h_{3i} + h_{2i}c_{2i} \\ h_{3i} + h_{2i}c_{2i} & h_{3i} \end{bmatrix}, \quad C_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i) = h_{2i} \begin{bmatrix} -s_{2i}\dot{q}_{2i} & -s_{2i}(\dot{q}_{1i} + \dot{q}_{2i}) \\ s_{2i}\dot{q}_{1i} & 0 \end{bmatrix},$$

and the gravity vector,

$$\boldsymbol{g}(\boldsymbol{q}_i) = \begin{bmatrix} \frac{g}{l_{2i}} h_{3i}c_{12i} + \frac{g}{l_{1i}}(h_{1i} - h_{3i})c_{1i} \\ \frac{g}{l_{2i}} h_{3i}c_{12i} \end{bmatrix},$$

where $h_{1i} = l_{2i}^2 m_{2i} + l_{1i}^2(m_{1i} + m_{2i})$, $h_{2i} = l_{1i}l_{2i}m_{2i}$, *and* $h_{3i} = l_{2i}^2 m_{2i}$. *We used the short notation,* c_{2i}, s_{2i}, *and* c_{12i}, *for* $\cos(q_{2i})$, $\sin(q_{2i})$, *and* $\cos(q_{1i} + q_{2i})$, *respectively. The joint position of the kth link of the manipulator i is represented by* q_{ki}. *The variables* l_{ki}, *and* m_{ki} *are the respective length and mass of each link. The communication topology of the network is represented on Figure 3.2, which results in the following Laplacian matrix,*

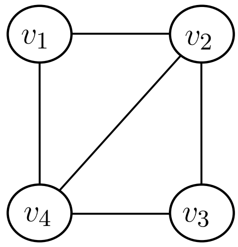


FIGURE 3.2: Example 3.1–Graph representation of the agents interaction.

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

We compute the maximum upper bound for delays with parameters $k_{2i} = 5$, $k_{3i} = 12$ *and for* $k_{2i} = 1.5$, $k_{3i} = 6.5$ *for all agents. Note from the conditions in Lemma 3.2 that only elements of the Laplacian matrix, the gains* k_{2i}, k_{3i}, *and the bounds of delays are necessary to attest the convergence. No physical information related to the dynamics of the agents is needed, besides the fact that they are Euler-Lagrange systems and their degrees of freedom. Thus, using the conditions in Lemma 3.2 and the Proposition 1 of Nuño and Ortega (2018) we find the values shown on Table 3.1.*

TABLE 3.1: Example 3.1–Maximum bounds for the communication delays.

(k_{2i}, k_{3i})	$(5, 12)$	$(1.5, 6.5)$
(Nuño and Ortega, 2018)		
$\bar{\tau}_1$	1.00	1.71
$\bar{\tau}_2$	0.59	0.90
$\bar{\tau}_3$	1.00	1.71
$\bar{\tau}_4$	0.59	0.90
Lemma 3.2		
$\bar{\tau}_1$	1.19	2.16
$\bar{\tau}_2$	0.79	1.44
$\bar{\tau}_3$	1.19	2.16
$\bar{\tau}_4$	0.79	1.44

It is shown that Lemma 3.2 guarantees convergence for biggest delays bounds in the communication. Moreover, the conditions proposed here are appropriated for multi-agent systems with input-saturation and can be used in formation-containment problems. In this sense, the results presented here are more general and might be applied to a larger class of systems.

3.3 The Followers Distributed Containment Control

In this section it is proposed a dynamic controller that ensures the convergence of all followers into the convex hull spanned by the leaders.

The distributed dynamic controller for the followers is designed as,

$$\begin{aligned} D_i\ddot{\boldsymbol{\eta}}_i &= k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=1}^{M} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) - k_{3i}\dot{\boldsymbol{\eta}}_i, \quad \forall i \in \mathcal{V}_F, & (3.18)\\ \boldsymbol{u}_i &= -k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{4i}\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) + \boldsymbol{g}_i(\boldsymbol{q}_i), \quad \forall i \in \mathcal{V}_F, & (3.19)\end{aligned}$$

where $\mathcal{V}_F = \{N+1,\ldots,M\}$ is the index set of followers, $\tilde{\boldsymbol{q}}_i = \boldsymbol{q}_i(t) - \boldsymbol{\eta}_i(t)$ for all $i \in \mathcal{V}_F$. Note that the main difference between the dynamic controller for leaders and followers is the presence of the relative desired configuration $\boldsymbol{\delta}_{ij}$ and that some leaders belong to the neighborhood of followers, which is represented in (3.18) by the sum over the leaders and followers set, $\mathcal{V}_L$ and $\mathcal{V}_F$. It is important to stress that this feature leads to a directed communication topology, since it is assumed that no leader receives information from followers. On the next lemma we show that the chosen input for the controllers is bounded and, therefore, under Remark 3.1 (input-state stability) the states of the dynamic controller are bounded.

Lemma 3.3. *The states of distributed dynamic controller (3.8) with input*

$$\boldsymbol{u}_{ci}(t) = k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=1}^{M} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big), \quad \forall i \in \mathcal{V}_F, \qquad (3.20)$$

are bounded, and in addition $\lim_{t\to\infty}\dot{\boldsymbol{\eta}}_i = 0$ *and* $\lim_{t\to\infty}\int_{t-\tau_{ji}(t)}^{t}\dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon = 0$ *for all* $i,j \in \mathcal{V}_F$, *if there are matrices* $P_i \in \mathbb{R}^{m\times m}$ *and constants* $\bar{\tau}_i$, k_{2i}, *and* k_{3i} *for* $i \in \{N+1,\cdots,M\}$, *such that the following linear matrix inequality holds,*

$$\begin{bmatrix}\hat{\Xi}_1 & \hat{\Xi}_2\\ * & \hat{\Xi}_3\end{bmatrix} < 0,$$

with

$$\begin{aligned}\hat{\Xi}_1 = \quad & \operatorname{diag}\Big\{ -2k_{3N+1}I + l^F_{N+1,N+1}\bar{\tau}_{N+1}P_{N+1}, \quad -2k_{3N+2}I + l^F_{N+2,N+2}\bar{\tau}_{N+2}P_{N+2}, \quad \cdots\\ & \cdots, -2k_{3M}I + l^F_{M,M}\bar{\tau}_M P_M\Big\},\end{aligned} \qquad (3.21)$$

$$\hat{\Xi}_2 = \operatorname{diag}\Big\{\hat{\Xi}_{2N+1}, \quad \hat{\Xi}_{2N+2}, \quad \cdots, \quad \hat{\Xi}_{2M}\Big\}, \qquad (3.22)$$

$$\hat{\Xi}_{2k} = \underbrace{\Big[-k_{2k}I \quad -k_{2k}I \quad \cdots \quad -k_{2k}I\Big]}_{l^F_{kk}\ \textit{times}},$$

$$\hat{\Xi}_3 = \operatorname{diag}\Big\{\hat{\Xi}_{3(N+1)}, \quad \hat{\Xi}_{3(N+2)}, \quad \cdots, \quad \hat{\Xi}_{3M}\Big\}, \qquad (3.23)$$

the terms $\hat{\Xi}_{3k}$ *are arranged considering the network topology, according the following algorithm,*

Structure of $\hat{\Xi}_{3k}$

$\hat{\Xi}_{3k} \leftarrow []$

  for $j = N+1, \ldots, M$:

    if $a_{kj} \neq 0$:

      $\hat{\Xi}_{3k} \leftarrow \operatorname{diag}\left(\hat{\Xi}_{3k}, \left[-P_j/\bar{\tau}_j\right]\right)$

end

The variable $\bar{\tau}_i$ *denotes the upper bound on the time-varying delays associated with the ith agent,* $[]$ *is an empty matrix (i.e., the only linear map is* $[] : \{0\} \to \{0\}$*), and* $l_{ii}^F = \sum_{j=N+1}^{M} a_{ij}$ *is the number of neighbors of the ith follower that are also followers.*

Proof. Let the input of the dynamic controller (3.8) be given by (3.20), setting apart the sum over the followers and leaders the input can be written as,

$$
\begin{aligned}
\boldsymbol{u}_{ci}(t) =& k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=N+1}^{M} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) \\
& - k_{2i}\sum_{j=1}^{N} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) \\
=& k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=N+1}^{M} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) - k_{2i}\sum_{j=1}^{N} a_{ij}\boldsymbol{\eta}_i(t) \\
& + k_{2i}\sum_{j=1}^{N} a_{ij}\boldsymbol{\eta}_j(t-\tau_{ji}(t)).
\end{aligned}
$$

We have that $k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_1)$ is bounded and from Lemma 3.2 we get that $\boldsymbol{\eta}_j$, for all $j \in \mathcal{V}_L$, is also bounded. Thus, given that the dynamic controller is input to state stable (Remark 3.1) and the sum of bounded functions is bounded, we now show that the controller's states remain bounded for

$$
\bar{\boldsymbol{u}}_{ci}(t) = -k_{2i}\sum_{j=N+1}^{M} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) - k_{2i}\sum_{j=1}^{N} a_{ij}\boldsymbol{\eta}_i(t). \tag{3.24}
$$

Consider the following Lyapunov functional candidate,

$$
W = W_1 + W_2,
$$

with

$$W_1 = \sum_{i=N+1}^{M} \left\{ \dot{\boldsymbol{\eta}}_i' D_i \dot{\boldsymbol{\eta}}_i + k_{2i} \sum_{j=1}^{N} a_{ij} \boldsymbol{\eta}_i' \boldsymbol{\eta}_i + \frac{k_{2i}}{2} \sum_{j=N+1}^{M} a_{ij} (\boldsymbol{\eta}_i - \boldsymbol{\eta}_j)' (\boldsymbol{\eta}_i - \boldsymbol{\eta}_j) \right\},$$

$$W_2 = \sum_{i=N+1}^{M} \sum_{j=N+1}^{M} a_{ij} \left\{ \int_{-\bar{\tau}_i}^{0} \int_{t+\theta}^{t} \dot{\boldsymbol{\eta}}_i(\varepsilon)' P_i \dot{\boldsymbol{\eta}}_i(\varepsilon) d\varepsilon d\theta \right\}.$$

The time-derivative of W_1 along the trajectories of the controller (3.8) with input (3.24) is given by

$$\begin{aligned} \dot{W}_1 = \sum_{i=N+1}^{M} \Bigg\{ & 2\dot{\boldsymbol{\eta}}_i' \Bigg(- k_{2i} \sum_{j=N+1}^{M} a_{ij} \Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t - \tau_{ji}(t)) \Big) - k_{2i} \sum_{j=1}^{N} a_{ij} \boldsymbol{\eta}_i(t) - k_{3i} \dot{\boldsymbol{\eta}}_i(t) \Bigg) \\ & + 2k_{2i} \sum_{j=1}^{N} a_{ij} \dot{\boldsymbol{\eta}}_i' \boldsymbol{\eta}_i + 2k_{2i} \sum_{j=N+1}^{M} a_{ij} \dot{\boldsymbol{\eta}}_i' (\boldsymbol{\eta}_i - \boldsymbol{\eta}_j) \Bigg\}, \end{aligned}$$

rewriting the delayed term as in (3.14), and using the symmetry of the Laplacian matrix provided by the undirect communication topology among agents of the same group yields,

$$\dot{W}_1 = \sum_{i=N+1}^{M} \left\{ - 2k_{2i} \dot{\boldsymbol{\eta}}_i' \sum_{j=N+1}^{M} a_{ij} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon - 2\dot{\boldsymbol{\eta}}_i' k_{3i} \dot{\boldsymbol{\eta}}_i(t) \right\}.$$

Taking the time-derivative of W_2 and using the same arguments used on the development of equation (3.16),

$$\dot{W}_2 \leq \sum_{i=N+1}^{M} \sum_{j=N+1}^{M} a_{ij} \left\{ \bar{\tau}_i \dot{\boldsymbol{\eta}}_i{}' P_i \dot{\boldsymbol{\eta}}_i - \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' P_j \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon \right\},$$

then applying Lemma A.3 give us,

$$\dot{W}_2 \leq \sum_{i=N+1}^{M} \sum_{j=N+1}^{M} a_{ij} \left\{ \bar{\tau}_i \dot{\boldsymbol{\eta}}_i{}' P_i \dot{\boldsymbol{\eta}}_i - \frac{1}{\bar{\tau}_j} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' d\varepsilon P_j \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon \right\}.$$

Finally, adding $\dot{W}_1$ and $\dot{W}_2$,

$$\begin{aligned} \dot{W} \leq \sum_{i=N+1}^{M} \Bigg\{ & - 2\dot{\boldsymbol{\eta}}_i' k_{3i} \dot{\boldsymbol{\eta}}_i(t) + l_{ii}^F \bar{\tau}_i \dot{\boldsymbol{\eta}}_i{}' P_i \dot{\boldsymbol{\eta}}_i - 2k_{2i} \dot{\boldsymbol{\eta}}_i' \sum_{j=N+1}^{M} a_{ij} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon \\ & - \sum_{j=N+1}^{M} a_{ij} \frac{1}{\bar{\tau}_j} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' d\varepsilon P_j \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon \Bigg\} \\ = & \hat{\Upsilon}(t)' \hat{\Xi} \hat{\Upsilon}(t), \end{aligned}$$

where the vector $\hat{\Upsilon}(t)$ is built following the Algorithm 3.17, for $i = N+1, \ldots, M$ and $j = N+1, \ldots, M$. Hence,

$$\hat{\Upsilon}(t)' = \Bigg[\dot{\boldsymbol{\eta}}'_{N+1} \cdots \dot{\boldsymbol{\eta}}'_{M} \int_{t-\tau_{(N+2)(N+1)}(t)}^{t} \dot{\boldsymbol{\eta}}_{N+2}(\varepsilon)'d\varepsilon \cdots \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_{j}(\varepsilon)'d\varepsilon \cdots \\ \cdots \int_{t-\tau_{(M-1)M}(t)}^{t} \dot{\boldsymbol{\eta}}_{(M-1)}(\varepsilon)'d\varepsilon\Bigg],$$

where the terms on $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon$ are defined only if $(v_i, v_j) \in \mathcal{E}$, and

$$\hat{\Xi} = \begin{bmatrix} \hat{\Xi}_1 & \hat{\Xi}_2 \\ * & \hat{\Xi}_3 \end{bmatrix},$$

where $\hat{\Xi}_1$, $\hat{\Xi}_2$, and $\hat{\Xi}_3$ are defined in (3.21), (3.22), and (3.23). Therefore, with appropriated choices of k_{2i}, k_{3i}, and D_i, the proposed condition is satisfied and the time derivative of W is negative, and consequently W is bounded and, from its definition, we have that $\boldsymbol{\eta}_i$, $\dot{\boldsymbol{\eta}}_i$, and $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon$ are bounded for all $i, j \in \mathcal{V}_F$. In addition, note that the time-derivative of $\dot{W}$ also depend on $\ddot{\boldsymbol{\eta}}_i$ and $\dot{\tau}_i$, thus from the controller dynamics (3.18) and the assumption of bounded $\dot{\tau}_i$ we have that $\dot{W}$ is uniformly continuous. Finally, by invoking Barbalat's lemma we conclude that $\lim_{t\to\infty} \dot{W} = 0$, and as a result $\lim_{t\to\infty} \dot{\boldsymbol{\eta}}_i = 0$, and $\lim_{t\to\infty} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon = 0$ for all $i, j \in \mathcal{V}_F$. □

The following lemma presents sufficient conditions that ensures the convergence of the followers into the convex hull formed by the leaders.

Lemma 3.4. *Each of the $M-N$ followers modeled by the Euler-Lagrange equation (3.1), with dynamic controller (3.18), bounded input (3.19), and time-varying communication delays with bounded first derivative, converges to the convex hull spanned by the leaders asymptotically, that is*

$$\lim_{t\to\infty} \left\|\boldsymbol{q}_i(t) - \sum_{j=1}^{N} \alpha_{ij}\boldsymbol{q}_j(t)\right\| = 0, \quad \alpha_{ij} \geq 0, \quad \sum_{j=1}^{N} \alpha_{ij} = 1, \quad \forall i \in \mathcal{V}_F, \quad \forall j \in \mathcal{V}_L,$$

if for given constants $\bar{\tau}_i$, k_{2i}, and k_{3i}, there exist positive definite matrices $P_i \in \mathbb{R}^{m\times m}$ for $i = N+1, \cdots, M$, such that the following linear matrix inequality holds,

$$\begin{bmatrix} \bar{\Xi}_1 & \bar{\Xi}_2 \\ * & \bar{\Xi}_3 \end{bmatrix} < 0, \tag{3.25}$$

with

$$\bar{\Xi}_1 = \operatorname{diag}\Big\{ -2k_{3N+1}I + l^F_{N+1,N+1}\bar{\tau}_{N+1}P_{N+1}, \quad -2k_{3N+2}I + l^F_{N+2,N+2}\bar{\tau}_{N+2}P_{N+2}, \quad \cdots \\ \cdots - 2k_{3M}I + l^F_{MM}\bar{\tau}_M P_M\Big\}, \tag{3.26}$$

$$\bar{\Xi}_2 = \operatorname{diag}\left\{\bar{\Xi}_{2N+1}, \quad \bar{\Xi}_{2N+2}, \quad \cdots, \quad \bar{\Xi}_{2M}\right\}, \tag{3.27}$$

$$\bar{\Xi}_{2k} = \underbrace{\begin{bmatrix}-k_{2k}I & -k_{2k}I & \cdots & -k_{2k}I\end{bmatrix}}_{l^F_{kk}\ times},$$

$$\bar{\Xi}_3 = \operatorname{diag}\left\{\bar{\Xi}_{3(N+1)}, \quad \bar{\Xi}_{3(N+2)}, \quad \cdots, \quad \bar{\Xi}_{3M}\right\}, \tag{3.28}$$

the terms $\bar{\Xi}_{3k}$ are arranged considering the network topology, according the following algorithm,

---

Structure of $\bar{\Xi}_{3k}$

$\bar{\Xi}_{3k} \leftarrow []$

for $j = N+1, \ldots, M:$

  if $a_{kj} \neq 0:$

    $\bar{\Xi}_{3k} \leftarrow \operatorname{diag}\left(\bar{\Xi}_{3k}, \left[-P_j/\bar{\tau}_j\right]\right)$

end

The term $\bar{\tau}_i$ denotes the upper bound on the time-varying delays associated with the ith agent, $[]$ is an empty matrix (i.e., the only linear map is $[] : \{0\} \to \{0\}$), $l^F_{ii} = \sum_{N+1}^{M} a_{ij}$ is the number of neighbors of the ith follower that are also followers, and k_{2i}, k_{3i} are positive constants.

Proof. The proof follows a similar procedure as the proof of Lemma 3.2. Hence, for the sake of brevity some steps are not explicitly shown.

Consider the following Lyapunov-Krasovskii functional candidate,

$$V_F = V_4 + V_5 + V_6,$$

with

$$\begin{aligned}
V_4 =& \sum_{i=N+1}^{M} \left\{\dot{\boldsymbol{\eta}}_i' D_i \dot{\boldsymbol{\eta}}_i + \dot{\boldsymbol{q}}_i' M_i(\boldsymbol{q}_i)\dot{\boldsymbol{q}}_i\right\}, \\
V_5 =& \frac{1}{2} \sum_{i=N+1}^{M} \sum_{j=N+1}^{M} a_{ij} k_{2i} (\boldsymbol{\eta}_i - \boldsymbol{\eta}_j)'(\boldsymbol{\eta}_i - \boldsymbol{\eta}_j) + 2 \sum_{i=N+1}^{M} \int_0^{\tilde{\boldsymbol{q}}_i} k_{1i}\sigma(\beta_{1i}\varepsilon)' d\varepsilon, \\
V_6 =& \sum_{i=N+1}^{M} \sum_{j=N+1}^{M} a_{ij} \left\{ \int_{-\bar{\tau}_j}^{0} \int_{t+\theta}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' P_j \dot{\boldsymbol{\eta}}_j(\varepsilon) d\varepsilon d\theta \right\}.
\end{aligned}$$

Following the same ideas in the proof of Lemma 3.2 to obtain (3.13), we have the time-derivative of V_4 given by,

$$\begin{aligned}\dot{V}_4 =& \sum_{i=N+1}^{M} \Big\{ 2\dot{\boldsymbol{q}}_i' M_i(\boldsymbol{q}_i)\ddot{\boldsymbol{q}}_i + \dot{\boldsymbol{q}}_i'\dot{M}_i(\boldsymbol{q}_i)\dot{\boldsymbol{q}}_i + 2\dot{\boldsymbol{\eta}}_i' D_i \ddot{\boldsymbol{\eta}}_i \Big\} \\ =&2\sum_{i=N+1}^{M} \Bigg\{ - k_{1i}\dot{\tilde{\boldsymbol{q}}}_i'\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{4i}\dot{\boldsymbol{q}}_i'\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) - k_{3i}\dot{\boldsymbol{\eta}}_i'\dot{\boldsymbol{\eta}}_i \\ &-k_{2i}\sum_{j=1}^{M} a_{ij}\dot{\boldsymbol{\eta}}_i'\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t)\Big) - k_{2i}\sum_{j=1}^{M} a_{ij}\dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \Bigg\},\end{aligned}$$

where we have used the Property 1 and written the delayed term as (3.14). It is possible to isolate the communication from leaders and write,

$$\begin{aligned}\dot{V}_4 =&2\sum_{i=N+1}^{M} \Bigg\{ - k_{1i}\dot{\tilde{\boldsymbol{q}}}_i'\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{4i}\dot{\boldsymbol{q}}_i'\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) - k_{3i}\dot{\boldsymbol{\eta}}_i'\dot{\boldsymbol{\eta}}_i \\ &-k_{2i}\sum_{j=1}^{N} a_{ij}\dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon - k_{2i}\sum_{j=N+1}^{M} a_{ij}\dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \\ &-\sum_{j=1}^{N} a_{ij}k_{2i}\dot{\boldsymbol{\eta}}_i'\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t)\Big) - \sum_{j=N+1}^{M} a_{ij}k_{2i}\dot{\boldsymbol{\eta}}_i'\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t)\Big) \Bigg\}.\end{aligned}$$

Through the same reasoning used to obtain (3.15), the time-derivative of V_5 is given by,

$$\dot{V}_5 = \sum_{i=N+1}^{M} \Bigg\{ \sum_{j=N+1}^{M} a_{ij} 2k_{2i}\dot{\boldsymbol{\eta}}_i'(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t)) + 2k_{1i}\dot{\tilde{\boldsymbol{q}}}_i'\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) \Bigg\},$$

and similar to the manipulation in (3.16), the derivative of V_6 is given by applying Lemma A.3 and using the symmetry of the communication topology among followers,

$$\dot{V}_6 \leq \sum_{i=N+1}^{M}\sum_{j=N+1}^{M} a_{ij}\Bigg\{ \bar{\tau}_i\dot{\boldsymbol{\eta}}_i' P_i\dot{\boldsymbol{\eta}}_i - \frac{1}{\bar{\tau}_j}\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' d\varepsilon P_j \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \Bigg\}.$$

Therefore, the time-derivative of the Lyapunov-Krasovskii candidate is as the follow,

$$\begin{aligned}\dot{V}_F \leq& \sum_{i=N+1}^{M} \Bigg\{ - 2k_{4i}\dot{\boldsymbol{q}}_i'\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) - 2\sum_{j=1}^{N} a_{ij}\Bigg(k_{2i}\dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon + k_{2i}\dot{\boldsymbol{\eta}}_i'\big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t)\big)\Bigg)\Bigg\} \\ &+ \sum_{i=N+1}^{M} \Bigg\{ - 2k_{3i}\dot{\boldsymbol{\eta}}_i'\dot{\boldsymbol{\eta}}_i + l_{ii}^F\bar{\tau}_i\dot{\boldsymbol{\eta}}_i' P_i\dot{\boldsymbol{\eta}}_i - \sum_{j=N+1}^{M} a_{ij}\Bigg(2k_{2i}\dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \\ &+\frac{1}{\bar{\tau}_j}\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)' d\varepsilon P_j \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \Bigg)\Bigg\} \\ \leq& \sum_{i=N+1}^{M} \Bigg\{ - 2k_{4i}\dot{\boldsymbol{q}}_i'\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) - 2\sum_{j=1}^{N} a_{ij}\Bigg(k_{2i}\dot{\boldsymbol{\eta}}_i' \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon + k_{2i}\dot{\boldsymbol{\eta}}_i'\big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t)\big)\Bigg)\Bigg\} \\ &+\bar{\Upsilon}(t)'\bar{\Xi}\bar{\Upsilon}(t),\end{aligned}$$

where the vector $\hat{\Upsilon}(t)$ is built following the Algorithm 3.17, for $i = N+1, \ldots, M$ and $j = N+1, \ldots, M$. Hence,

$$\bar{\Upsilon}(t)' = \Big[\dot{\boldsymbol{\eta}}'_{N+1} \cdots \dot{\boldsymbol{\eta}}'_{M} \int_{t-\tau_{(N+2)(N+1)}(t)}^{t} \dot{\boldsymbol{\eta}}'_{N+2}(\varepsilon)d\varepsilon \cdots \\ \cdots \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon \cdots \int_{t-\tau_{(M-1)M}(t)}^{t} \dot{\boldsymbol{\eta}}_{M-1}(\varepsilon)'d\varepsilon\Big],$$

where the terms on $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon$ are defined only if $(v_i, v_j) \in \mathcal{E}$, and

$$\bar{\Xi} = \begin{bmatrix} \bar{\Xi}_1 & \bar{\Xi}_2 \\ * & \bar{\Xi}_3 \end{bmatrix},$$

where $\bar{\Xi}_1$, $\bar{\Xi}_2$, and $\bar{\Xi}_3$ are defined in equations (3.26), (3.27), and (3.28).

Therefore, if the conditions of Lemma 3.4 hold we have[1]

$$\begin{aligned} \dot{V}_F \leq & -2\sum_{i=N+1}^{M}\sum_{j=1}^{N} a_{ij}\left(k_{2i}\dot{\boldsymbol{\eta}}'_i \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon + k_{2i}\dot{\boldsymbol{\eta}}'_i(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t))\right) \\ \leq & \sum_{i=N+1}^{M}\sum_{j=1}^{N} a_{ij}k_{2i}\Bigg(2\dot{\boldsymbol{\eta}}'_i\dot{\boldsymbol{\eta}}_i + \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \\ & +(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t))'(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t))\Bigg), \end{aligned}$$

and integrating both sides yields

$$\begin{aligned} V_F(t) - V_F(0) \leq \sum_{i=N+1}^{M}\sum_{j=1}^{N} a_{ij}k_{2i}\int_0^t \Bigg(& 2\dot{\boldsymbol{\eta}}_i(\theta)'\dot{\boldsymbol{\eta}}_i(\theta) + \int_{\theta-\tau_{ji}(\theta)}^{\theta} \dot{\boldsymbol{\eta}}_j(\varepsilon)'d\varepsilon \int_{\theta-\tau_{ji}(\theta)}^{\theta} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon \\ & +\Big(\boldsymbol{\eta}_i(\theta) - \boldsymbol{\eta}_j(\theta)\Big)'\Big(\boldsymbol{\eta}_i(\theta) - \boldsymbol{\eta}_j(\theta)\Big)\Bigg)d\theta. \end{aligned}$$

From Lemma 3.2 we have that, $\boldsymbol{\eta}_j$, and $\int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon$ are bounded for all $j \in \mathcal{V}_L$, and from Lemma 3.3 (noticing that the conditions of the Lemma 3.4 encompass those from Lemma 3.3,) we have that $\boldsymbol{\eta}_i$, $\dot{\boldsymbol{\eta}}_i$, and $\int_{t-\tau_{ij}(t)}^{t} \dot{\boldsymbol{\eta}}_i(\varepsilon)d\varepsilon$ are bounded for all $i \in \mathcal{V}_F$. Thus, $V_F(t)$ is bounded and from its definition we obtain that $\dot{\boldsymbol{q}}_i$ and $\tilde{\boldsymbol{q}}_i$ are also bounded for all $i \in \mathcal{V}_F$. In order to show the boundedness of $\boldsymbol{q}_i$ note that

$$||\tilde{\boldsymbol{q}}_i(t)|| = ||\boldsymbol{q}_i(t) - \boldsymbol{\eta}_i(t)|| \leq C,$$

for some constant $C \in \mathbb{R}$. As $\boldsymbol{\eta}_i(t)$ is bounded, the above inequality implies that $\boldsymbol{q}_i(t)$ is also bounded for all $i \in \mathcal{V}_F$.

Therefore, from the followers and their controller dynamics (3.1) and (3.18), Properties 2 and 3, and Assumptions 3.4 and 3.3, we get that $\ddot{\boldsymbol{q}}_i$ and $\ddot{\boldsymbol{\eta}}_i$ are also bounded. Thus, we have that $\dot{\boldsymbol{q}}_i$ and $\dot{\boldsymbol{\eta}}_i$ are uniformly continuous for all followers. Then, by invoking Barbalat's lemma

[1] We used the inequality $-2\boldsymbol{a}'\boldsymbol{b} \leq \boldsymbol{a}'X\boldsymbol{a} + \boldsymbol{b}'X^{-1}\boldsymbol{b}$ where $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $X > 0 \in \mathbb{R}^{m\times m}$.

and the result in Lemma 3.3, we conclude that $\lim_{t\to\infty} \dot{\boldsymbol{\eta}}_i(t) = 0$, $\lim_{t\to\infty} \int_{t-\tau_{ji}(t)}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon = 0$, and $\lim_{t\to\infty} \dot{\boldsymbol{q}}_i(t) = 0$ for all $i, j \in \mathcal{V}_F$.

Through the same reasoning of the proof of Lemma 3.2 we have that $\lim_{t\to\infty} \ddot{\boldsymbol{\eta}}_i(t) = 0$, $\lim_{t\to\infty} \ddot{\boldsymbol{q}}_i(t) = 0$, $\lim_{t\to\infty} \tilde{\boldsymbol{q}}_i(t) = 0$. Lastly, considering that $\lim_{t\to\infty} \int_{t-\tau_{ij}(t)}^{t} \dot{\boldsymbol{\eta}}_i(\varepsilon)d\varepsilon = 0$, the terms that take into account the interaction among agents can be rewritten in the following way,

$$\begin{aligned}\sum_{i=N+1}^{M}\sum_{j=1}^{M} a_{ij}k_{2i}\Big(\boldsymbol{\eta}_i(t)-\boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) &= \sum_{i=N+1}^{M}\sum_{j=1}^{M} a_{ij}k_{2i}\left(\boldsymbol{\eta}_i(t)-\boldsymbol{\eta}_j(t) + \int_{t-\tau_{ji}}^{t} \dot{\boldsymbol{\eta}}_j(\varepsilon)d\varepsilon\right)\\ &= \sum_{i=N+1}^{M}\sum_{j=1}^{M} a_{ij}k_{2i}\Big(\boldsymbol{\eta}_i(t)-\boldsymbol{\eta}_j(t)\Big)\\ &= \Big((K_2L_F)\otimes I_m\Big)\boldsymbol{\eta}_F + \Big((K_2L_{LF})\otimes I_m\Big)\boldsymbol{\eta}_L,\end{aligned}$$

with $\boldsymbol{\eta}_F = \begin{bmatrix}\boldsymbol{\eta}'_{N+1} & \cdots & \boldsymbol{\eta}'_M\end{bmatrix}'$, $\boldsymbol{\eta}_L = \begin{bmatrix}\boldsymbol{\eta}'_1 & \cdots & \boldsymbol{\eta}'_N\end{bmatrix}'$ and $K_2 = \mathrm{diag}\{k_{2N+1}, \cdots, k_{2M}\}$. Note from Assumption 3.2 that L_F is positive definite, then we can pre-multiply the previous equation to obtain

$$\Big((K_2L_F)\otimes I_m\Big)^{-1}\Big(((K_2L_F)\otimes I_m)\boldsymbol{\eta}_F + ((K_2L_{LF})\otimes I_m)\boldsymbol{\eta}_L\Big) = \boldsymbol{\eta}_F + \Big((L_F^{-1}L_{LF})\otimes I_m\Big)\boldsymbol{\eta}_L, \quad (3.29)$$

which can be seen as an error between the followers' dynamics and the linear combination of the leaders' dynamics. Finally, from the controller dynamics (3.18), $\lim_{t\to\infty} \tilde{\boldsymbol{q}}_i(t) = 0$, $\lim_{t\to\infty} \ddot{\boldsymbol{\eta}}_i(t) = 0$, and $\lim_{t\to\infty} \dot{\boldsymbol{\eta}}_i(t) = 0$, we have that

$$\lim_{t\to\infty}\Big[\boldsymbol{\eta}_F + \Big((L_F^{-1}L_{LF})\otimes I_m\Big)\boldsymbol{\eta}_L\Big] = 0.$$

Recall from Lemma 3.1 that each entry of the matrix $-L_F^{-1}L_{LF}$ is non-negative and each of its rows has sum equal to one, thus each row of (3.29) is analogous to the argument of the limit (3.5) that is,

$$\lim_{t\to\infty}\left\|\boldsymbol{q}_i(t) - \sum_{j=1}^{N}\alpha_{ij}\boldsymbol{q}_j(t)\right\| = 0, \quad \alpha_{ij} \geq 0, \quad \sum_{j=1}^{N}\alpha_{ij} = 1, \quad \forall i \in \mathcal{V}_F, \quad \forall j \in \mathcal{V}_L,$$

which completes the proof. □

The LMI condition proposed in Lemma 3.4 provides a sufficient test for the convergence analysis of agents modeled as Euler-Lagrange systems into a convex hull spanned by a set of Euler-Lagrange systems. Similarly to the Lemma 3.2, this condition is also designed taking into account bounded inputs and communication delays. It also establishes a relationship between some controller parameters and the upper bound of the communication delays, leaving some other parameters free for adjustment. Namely, only the controller parameters k_{2i} and k_{3i} have to be adjusted in such a way that the condition in Lemma 3.4 is satisfied. Furthermore, as pointed out in Section 3.2, the independence of the model makes the proposed condition appropriated for the convergence analysis of heterogeneous multi-agent systems.

3.4 Formation-Containment Control

In Sections 3.2 and 3.3 there were established sufficient conditions for stabilization of leaders into a desired formation and the convergence of followers into the convex hull spanned by the leaders, both modeled by Euler-Lagrange equations subject to input saturation and with communication delays. With these results we now enunciate the main result of this chapter, stated in the next theorem.

Theorem 3.1. *Consider a multi-agent system composed of N leaders and $M-N$ followers, subject to time-varying delays on the communication among agents $\tau_{ij}(t)$, with $\dot{\tau}_{ij}(t)$ bounded for all $i,j \in \mathcal{V}_L \cup \mathcal{V}_F$, in which all agents are modeled by the Euler-Lagrange equation (3.1) subject to input-saturation. The distributed dynamic controller*

$$
\begin{aligned}
D_i\ddot{\boldsymbol{\eta}}_i(t) &= k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=1}^{N} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t)) - \boldsymbol{\delta}_{ij}\Big) - k_{3i}\dot{\boldsymbol{\eta}}_i(t), \quad \forall i \in \mathcal{V}_L,\\
\boldsymbol{u}_i &= -k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i(t)) - k_{4i}\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i(t)) + \boldsymbol{g}_i(\boldsymbol{q}_i), \quad \forall i \in \mathcal{V}_L,
\end{aligned}
$$

for leaders, and the distributed dynamic controller for followers,

$$
\begin{aligned}
D_i\ddot{\boldsymbol{\eta}}_i &= k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{2i}\sum_{j=1}^{M} a_{ij}\Big(\boldsymbol{\eta}_i(t) - \boldsymbol{\eta}_j(t-\tau_{ji}(t))\Big) - k_{3i}\dot{\boldsymbol{\eta}}_i, \quad \forall i \in \mathcal{V}_F,\\
\boldsymbol{u}_i &= -k_{1i}\sigma(\beta_{1i}\tilde{\boldsymbol{q}}_i) - k_{4i}\sigma(\beta_{2i}\dot{\boldsymbol{q}}_i) + \boldsymbol{g}_i(\boldsymbol{q}_i), \quad \forall i \in \mathcal{V}_F,
\end{aligned}
$$

lead the network asymptotically to the formation-containment with maximum inputs given by $k_{1i} + k_{4i} + ||\boldsymbol{g}(\boldsymbol{q}_i)||_\infty$ for all $i \in \mathcal{V}_L \cup \mathcal{V}_F$, if for given constants $\bar{\tau}_i$, k_{2i}, and k_{3i}, there exist positive definite matrices $P_i \in \mathbb{R}^{m\times m}$ for $i = 1, \cdots, M$, such that the following linear matrix inequality holds,

$$
\begin{bmatrix} \Phi_1 & \Phi_2 \\ * & \Phi_3 \end{bmatrix} < 0,
$$

with

$$
\Phi_1 = \operatorname{diag}\Big\{-2k_{31}I + l_{11}\bar{\tau}_1 P_1, \quad -2k_{32}I + l_{22}\bar{\tau}_2 P_2, \quad \cdots, \quad -2k_{3M}I + l_{MM}\bar{\tau}_M P_M\Big\},
$$

$$
\Phi_2 = \operatorname{diag}\Big\{\Phi_{21}, \quad \Phi_{22}, \quad \cdots, \quad \Phi_{2M}\Big\},
$$

$$
\Phi_{2k} = \underbrace{\begin{bmatrix} -k_{2k}I & -k_{2k}I & \cdots & -k_{2k}I \end{bmatrix}}_{l_{kk}\ \textit{times}},
$$

$$
\Phi_3 = \operatorname{diag}\Big\{\Phi_{31}, \quad \Phi_{32}, \quad \cdots, \quad \Phi_{3M}\Big\},
$$

the terms Φ_{3k} are arranged considering the network topology, according the following algorithm, *Structure of* Φ_{3k}

$\Phi_{3k} \leftarrow []$
  for $j = 1, \ldots, M$:
    if $a_{kj} \neq 0$:
      $\Phi_{3k} \leftarrow \text{diag}\left(\Phi_{3k}, \left[-P_j/\bar{\tau}_j\right]\right)$
end

The variable $\bar{\tau}_i$ *denotes the upper bound on the time-varying delays associated with the ith agent,* $\tilde{\boldsymbol{q}}_i = \boldsymbol{q}_i - \boldsymbol{\eta}_i$, a_{ij} *is the* (i, j) *entry of the adjacency matrix,* $\sigma(\cdot) : \mathbb{R} \to (-1, 1)$ *is a strictly increasing continuous function,* $[]$ *is an empty matrix (i.e., the only linear map is* $[] : \{0\} \to \{0\}$*),* $l_{ii} = \sum_{j=1}^{N} a_{ij}$ *for* $i \in \mathcal{V}_L$*, and* $l_{ii} = \sum_{j=N+1}^{M} a_{ij}$ *for* $i \in \mathcal{V}_F$*, and* k_{1i}, k_{2i}, k_{3i}, k_{4i}, β_{1i}*, and* β_{2i} *are positive constants.*

Proof. The result of the theorem follows directly from the stabilization of leaders into a desired constant formation, demonstrated on Lemma 3.2, and the convergence of followers into the convex hull spanned by the leaders, established on Lemma 3.4. The conditions of the theorem follows from the composition of the conditions in Lemmas 3.2 and 3.4. Finally, the bound of the input signal is ensured by the boundedness of the gravitational vector $\|\boldsymbol{g}(\boldsymbol{q})\|_{\infty}$ and the inequality $\|\sigma(\cdot)\|_{\infty} \leq 1$. □

The above theorem presents conditions to achieve the formation-containment of multiple Euler-Lagrange systems in a distributed fashion by combining the behavior of leaders and followers subject to input saturation and communication delays under the proposed control strategy. It is noteworthy that in the convergence proof of Lemma 3.4 it is shown that the states of followers and their dynamic controllers are bounded during the process of convergence of leaders, which ensures that the followers cannot go to infinity in finite time. Moreover, because the theorem is a composition of Lemmas 3.2 and 3.4, the observations made previously are still valid.

3.4.1 Numerical Examples

In this section it is shown examples to illustrate the use of the proposed method in the formation-containment in multi-agent systems with non-identical agents. One of the examples provided is used in others works on the formation-containment problem, but because the premises and conditions are rather unlike to the proposed ones, they are not directly contrasted.

Example 3.2. *In order to illustrate the effectiveness of the proposed strategy on the formation-containment problem we consider the same example studied by Chen et al. (2019b),Chen et al. (2016) and Li et al. (2018). It is considered a group of ten satellites with six leaders,* $\mathcal{V}_L = \{1, \cdots, 6\}$*, and four followers* $\mathcal{V}_F = \{7, \cdots, 10\}$*. The dynamics of each system is described* *as,*

$$\ddot{x}_i - 2\omega_0\dot{y}_i - \omega_0^2 x_i - \dot{\omega}_0 y_i + \frac{\mu_e(R_0 + x_i)}{R_i^3} - \frac{\mu_e}{R_0^2} = \frac{u_{oix}}{m_i},$$
$$\ddot{y}_i + 2\omega_0\dot{x}_i - \omega_0^2 y_i + \dot{\omega}_0 x_i + \frac{\mu_e y_i}{R_i^3} = \frac{u_{oiy}}{m_i},$$
$$\ddot{z}_i + \frac{\mu_e z_i}{R_i^3} = \frac{u_{oiz}}{m_i},$$

in which x_i, y_i, and z_i are the coordinates of the center of mass on the global frame, R_0 is the orbit radius, m_i is the mass of the satellite, μ_e is the gravitation constant, $\omega_0 = \sqrt{\mu_e/R_0^3}$ is the orbit angular velocity, R_i is the distance between the satellite and the geocenter, and the control input is given by the variables u_{oix}, u_{oiy}, and u_{oiz}. Note that this model is equivalent to the Euler-Lagrange equation, with $\boldsymbol{q}_i = [x_i, y_i, z_i]'$, $\boldsymbol{u}_i = [u_{oix}, u_{oiy}, u_{oiz}]'$,

$$C_i = 2m_i \begin{bmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \boldsymbol{g}_i = m_i \begin{bmatrix} -\omega_0 x_i - \dot{\omega}_0 y_i + \mu_e(R_0 + x_i)/R_i^3 - \mu_e/R_0^2 \\ -\omega_0 y_i + \mu_e y_i/R_i^3 + \dot{\omega}_0 x_i \\ \mu_e z_i/R_i^3 \end{bmatrix}.$$

In this numerical example, the networked system follows the same initial orbit elements given in Chen et al. (2019b). We also consider distinct mass for the systems, for the systems $i \in \{1, 2, 3, 7, 8, 9\}$, we consider $m_i = 35kg$ and for $i \in \{4, 5, 6, 10\}$, $m_i = 40kg$ and assume that all input limits are given by $u_{max} = 10N$. The communication topology is represented in Figure 3.3, and it gives us the following Laplacian matrix,

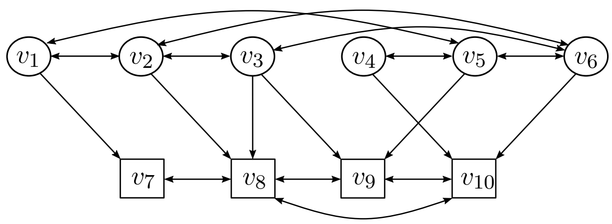


FIGURE 3.3: Example 3.2–Graph representation of the agents interaction.

$$L = \begin{bmatrix} L_L & 0 \\ L_{LF} & L_F \end{bmatrix},$$

where,

$$L_{LF} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad L_F = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & 3 \end{bmatrix},$$

$$L_L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}.$$

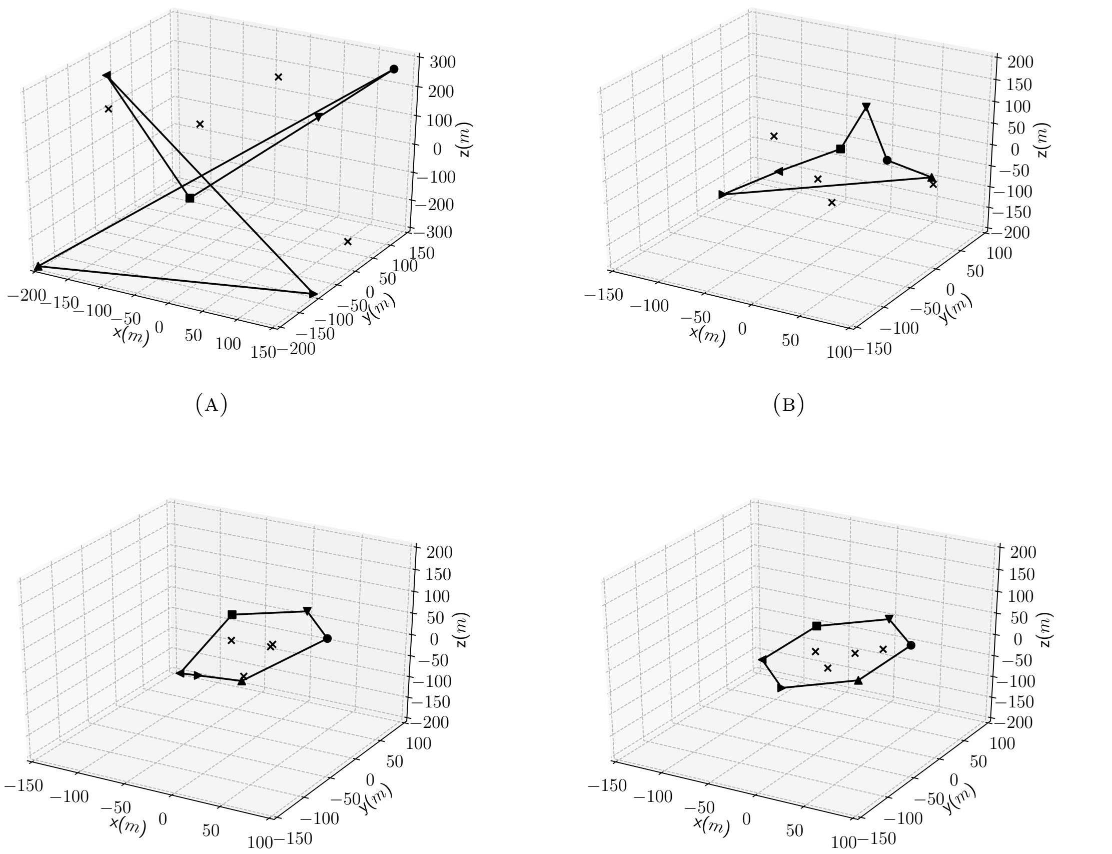
The desired relative configuration for the leaders is given by,

$$\begin{aligned} \boldsymbol{\delta}_{15} = &\ [50, 150, 0]', & \boldsymbol{\delta}_{21} =&[-50, 50, 0]', & \boldsymbol{\delta}_{23} =&[-50, 150, 0]', \\ \boldsymbol{\delta}_{36} =&[100, -100, 0]', & \boldsymbol{\delta}_{54} =&[50, -50, 0]', & \boldsymbol{\delta}_{56} =&[50, -150, 0]', \end{aligned}$$

and the initial conditions are given by,

$$\begin{aligned} \boldsymbol{q}_1(0) &= [150, 100, 300]', & \boldsymbol{q}_2(0) &= [150, -100, 300]', \\ \boldsymbol{q}_3(0) &= [-200, -200, -300]', & \boldsymbol{q}_4(0) &= [-150, -100, 300]', \\ \boldsymbol{q}_5(0) &= [150, -100, -300]', & \boldsymbol{q}_6(0) &= [-150, 100, -300]', \\ \boldsymbol{q}_7(0) &= [-200, -20, 100]', & \boldsymbol{q}_8(0) &= [-15, 100, 200]', \\ \boldsymbol{q}_9(0) &= [150, -10, -200]', & \boldsymbol{q}_{10}(0) &= [15, -100, 200]', \\ \dot{\boldsymbol{q}}_i(0) &= [0, 0, 0]', & \forall i &= 1, \ldots, 10. \end{aligned}$$

We use the Lemmas 3.2 and 3.4 with gains $k_{2i} = 3$ *and* $k_{3i} = 5$*, for all* $i \in \mathcal{V}_L \cup \mathcal{V}_F$ *to obtain the maximum delay with guaranteed convergence. We obtain* $\bar{\tau}_8 = 0.83$*,* $\bar{\tau}_i = 0.64$ *for* $i \in \{9, 10\}$*,* $\bar{\tau}_i = 0.58$ *for* $i \in \{2, 5, 6\}$*,* $\bar{\tau}_i = 0.51$ *for* $i \in \{1, 3, 7\}$*,* $\bar{\tau}_4 = 0.50$*. The time-delays was set to* $\tau_i(t) = \bar{\tau}_i(1 + \sin(t))/2$*, and the remain parameters used in the simulation was* $(D_i, k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i}) = (\mathrm{diag}(50, 50, 50), 8.0, 2.0, 1.0, 20.0)$ *with function* $\sigma(\cdot) = \tanh(\cdot)$ *for all* $i \in \mathcal{V}_L \cup \mathcal{V}_F$*. Figure 3.4 shows the trajectories of the network, the agents positions are represented by symbols, where* $\times$ *represents the followers and the other symbols represent the leaders positions, and the final desired convex hull spanned by the leaders is highlighted in black solid line (a video of the simulation can be seen at* `https://youtu.be/lJvKzvM_5SE`*). Figure 3.5 shows the control inputs of the systems and Figure 3.6 depicts the revolute of the joint coordinates.*

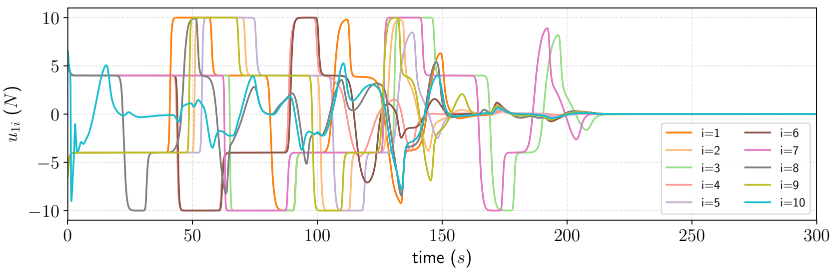


(A) (B)

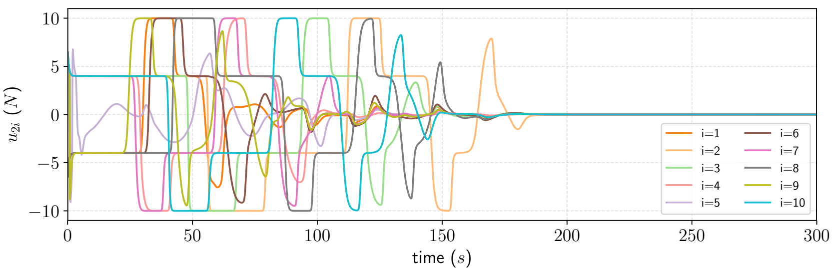
(C) (D)

FIGURE 3.4: Example 3.2–Snapshots of the networked system trajectories: (A) at $t = 0s$; (B) at $t = 75s$; (C) at $t = 150s$; and (D) at $t = 300s$. The symbol $\times$ represents the followers, while the others symbols represent the leaders positions and the black line links the leaders that shape the final convex hull.

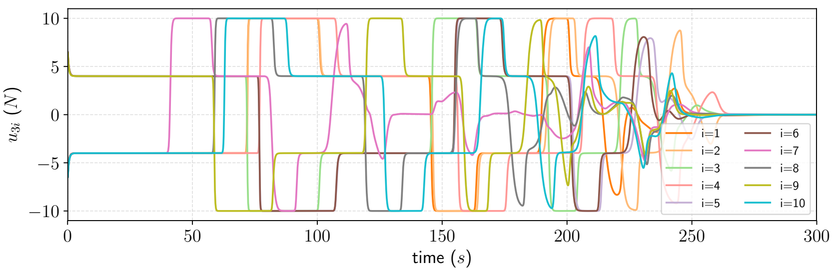
According to Figure 3.4, we can see that the multi-agent system subject to input saturation and communication delays achieves the formation-containment. That is, the leaders move to the desired relative positions and the followers move into the convex hull spanned by the leaders. It can be viewed in Figure 3.5 that the agents control inputs do not exceed $10N$*, which is the input bound. This illustrate the effectiveness of the proposed approach.*



(A)

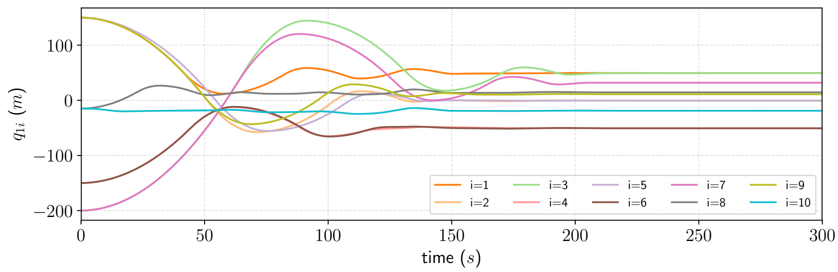


(B)

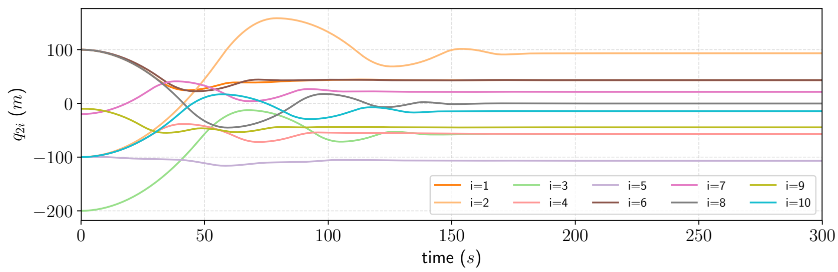


(C)

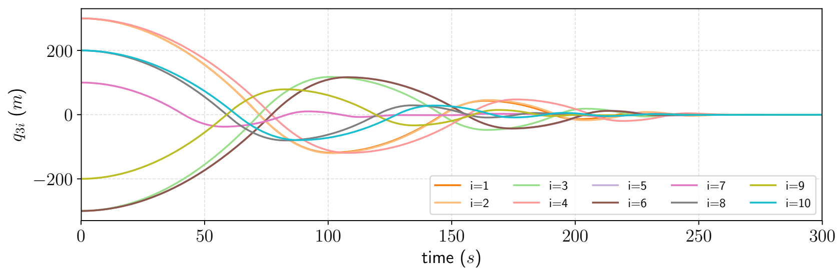
FIGURE 3.5: Example 3.2–Input signal of each agents' generalized coordinate.



(A)



(B)



(C)

FIGURE 3.6: Example 3.2–Position of each coordinate of the agents.

Example 3.3. *The purpose of this example is to discuss the tuning of the control parameters. One should note that the proposed conditions certify the asymptotic convergence of the network into the formation-containment. However, because no performance is required to be met, the tuning of some parameters remains open.*

The parameters to be tuned in the proposed strategy are $k_{1i}, k_{2i}, k_{3i}, k_{4i}, \beta_{1i}$*, and* β_{2i}*. While the gains* k_{2i} *and* k_{3i} *have fixed values for a given problem, since the conditions are needed to be satisfied according to the relation between their values and the time delays, the sum of* k_{1i} *and* k_{4i} *is required to be set according to the input saturation. Hence, there is a degree of freedom in the selection of* k_{1i} *and* k_{4i}*. Observe that the gains* k_{1i} *and* k_{4i} *adjust the signal associated to the states of the agents and their dynamic controllers* $\tilde{\mathbf{q}}_i$*, and the local damping in the agent dynamics. The gains* β_{1i} *and* β_{2i} *appear as arguments of the bounded function* $\sigma(\cdot)$*, along with the signals* $\tilde{\mathbf{q}}_i$ *and* $\dot{\mathbf{q}}_i$*, respectively. Naturally, the adjustment of their values affects directly the time the controllers spend saturated. Since the signal* $\tilde{\mathbf{q}}_i$ *is related with the synchronization of the states of the agents with their controllers, and* $\dot{\mathbf{q}}_i$ *is associated with a local damping, the* *closed-loop systems are rather sensitive to the selection of high values for* β_{1i}*, and* β_{2i}*, as it is exemplified next.*

Firstly, consider a multi-agent system with 20 *agents, in which* 10 *are leaders and the other* 10 *are followers interconnected according to the Laplacian matrix in (3.30), and they are required to converge to attain the consensus, i.e.,* $\boldsymbol{\delta}_{ij} = 0$ *for all* i, j*. The dynamics of each agent is given by two-link manipulators as presented in Example 3.1 with actuators limit of* $u_{\max} = 2$*, with physical parameters given in Table 3.2. Theorem 3.1 was used with* $k_{2i} = 5$ *and* $k_{3i} = 12$ *for all* $i \in \mathcal{V}_L \cup \mathcal{V}_F$*. The minimum upper-bound for the communication delays computed was* $\bar{\tau} = 0.19$*, and this value was used fixed in all communication links.*

$$L = \begin{bmatrix} L_L & 0 \\ L_{LF} & L_F \end{bmatrix}, \tag{3.30}$$

where,

$$L_{LF} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \end{bmatrix},$$

$$L_L = \begin{bmatrix} 5 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 6 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 5 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 6 & -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & 8 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & 7 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 4 \end{bmatrix},$$

$$L_F = \begin{bmatrix} 10 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 11 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 8 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 11 & -1 & -1 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & 10 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 9 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 12 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & -1 & -1 & 12 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 10 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 12 \end{bmatrix}.$$

TABLE 3.2: Example 3.3–Physical parameters of the agents. m_k and l_k, for $k \in \{1, 2\}$, are the mass and the length of each link.

	$i \in \{1, 2, 3, 7, 8, 9, 16, 17, 18, 19\}$	$i \in \{4, 5, 6, 10, 11, 12, 13, 14, 15, 20\}$
(m_1, m_2)	$(2, 0.32)$	$(3.95, 0.62)$
(l_1, l_2)	$(0.38, 0.38)$	$(0.38, 0.38)$

This setting is studied with several values for the parameters of the controllers, which are presented in Table 3.3 and displayed in Figure 3.7, in which the simulations were performed with $D_i = 50$ *and the function* $\sigma(\cdot) = \tanh(\cdot)$ *for all agents.*

TABLE 3.3: Example 3.3–Distinct parameters satisfying conditions in Theorem 3.1 with $k_{2i} = 5$ and $k_{3i} = 12$.

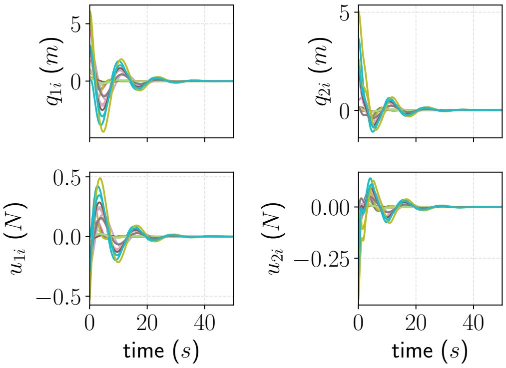
	Values	Figure
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(1.0, 1.0, 1.0, 1.0)$	Figure 3.7A
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(1.7, 0.3, 0.1, 0.1)$	Figure 3.7B
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(0.3, 1.7, 0.1, 0.1)$	Figure 3.7C
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(1.0, 1.0, 30.0, 0.1)$	Figure 3.7D
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(1.0, 1.0, 0.1, 30)$	Figure 3.7E
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(1.7, 0.3, 0.1, 30)$	Figure 3.7F

In a second setting, consider the same configuration given in Example 3.2, with parameters of the controllers represented in Table 3.4. The simulations are displayed in Figures 3.8 and 3.9, all of them were done considering $\bar{\tau} = 0.5$ *in every communication link and again with* $D_i = 50$ *and the function* $\sigma(\cdot) = \tanh(\cdot)$ *for all agents.*

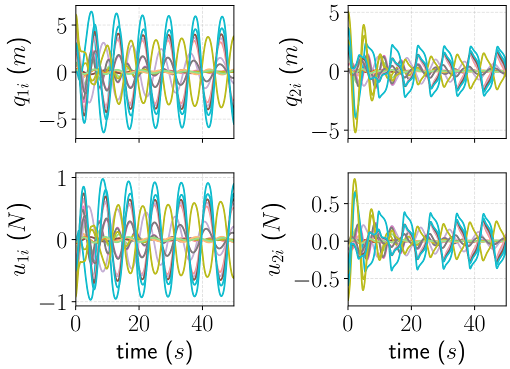
TABLE 3.4: Example 3.3–Distinct parameters satisfying conditions in Theorem 3.1 with $k_{2i} = 3$ and $k_{3i} = 5$.

	Values	Figure
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(5.0, 5.0, 1.0, 1.0)$	Figure 3.8A
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(8.0, 2.0, 0.1, 0.1)$	Figure 3.8B
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(2.0, 8.0, 0.1, 0.1)$	Figure 3.8C
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(5.0, 5.0, 20.0, 0.1)$	Figure 3.8D
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(5.0, 5.0, 0.1, 20.0)$	Figure 3.9A
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$	$(8.0, 2.0, 0.1, 20.0)$	Figure 3.9B

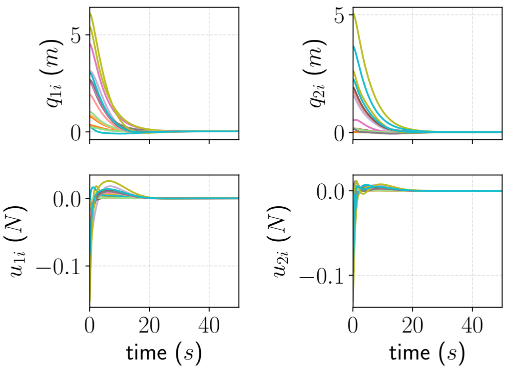
The simulations corroborate with the asymptotic convergence of the network once the conditions of the proposed approach are satisfied. However, the rate of convergence varies as a function of the control parameters. In both examples, the natural choice of 50% *of the input limit for each gain,* k_{1i}*, and* k_{4i}*, together with* $\beta_{1i} = \beta_{2i} = 1$ *leads to a reasonable rate of convergence (depict in Figures 3.7A, and 3.8A), when compared with other more polarized choices, even though this might not conduct to the fastest stabilization. From the simulations, it is also possible to realize that the gains* β_{1i}*,* β_{2i} *are highly associated with the actuator activity (Figures 3.7D, 3.7E, 3.7F, 3.8D, 3.9A, and 3.9B), and that for fast stabilization it seems to be best to select a high value for* β_{2i} *(Figures 3.7D, and 3.8D).*



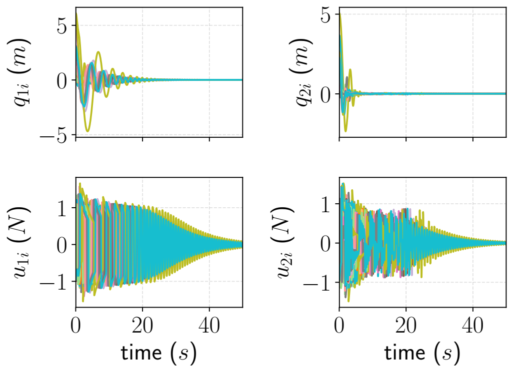
(A)
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(1.0, 1.0, 1.0, 1.0).



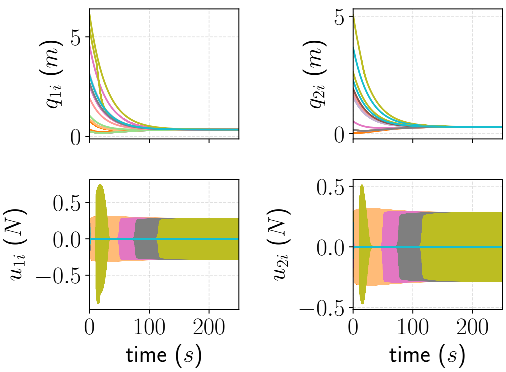
(B)
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(1.7, 0.3, 0.1, 0.1).



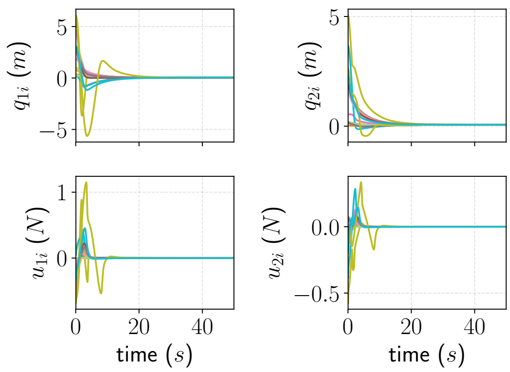
(C)
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(0.3, 1.7, 0.1, 0.1).



(D)
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(1.0, 1.0, 30.0, 0.1).

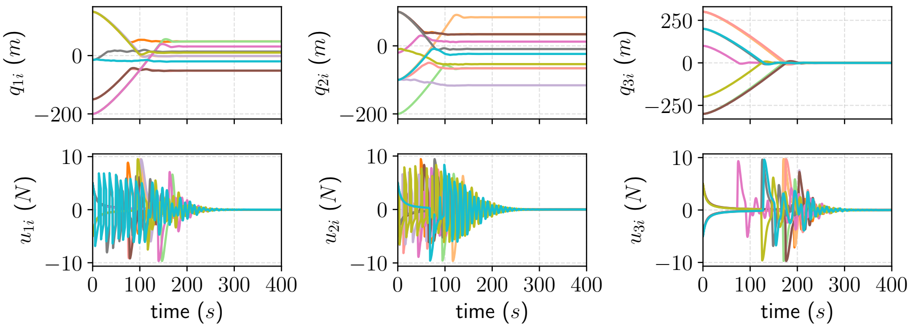


(E)
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(1.0, 1.0, 0.1, 30).

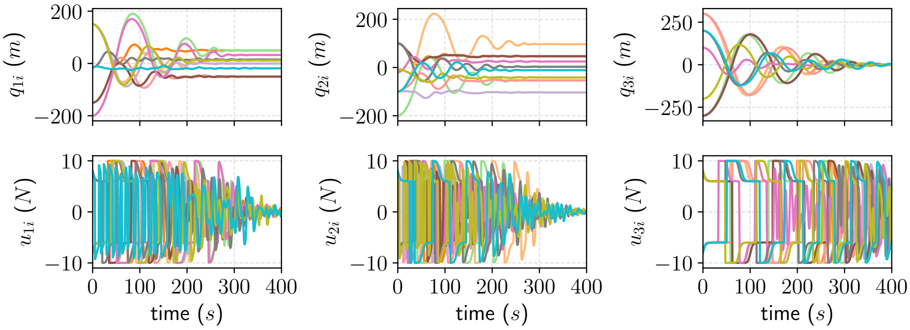


(F)
$(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(1.7, 0.3, 0.1, 30).

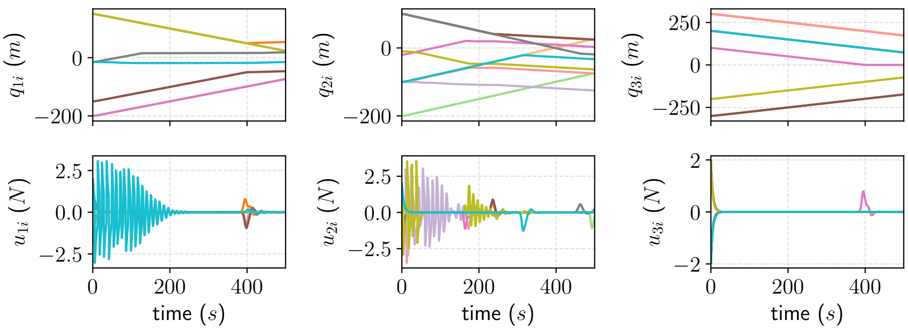
FIGURE 3.7: Example 3.3–Several simulations with distinct control parameters in the proposed approach in a network with 20 agents.



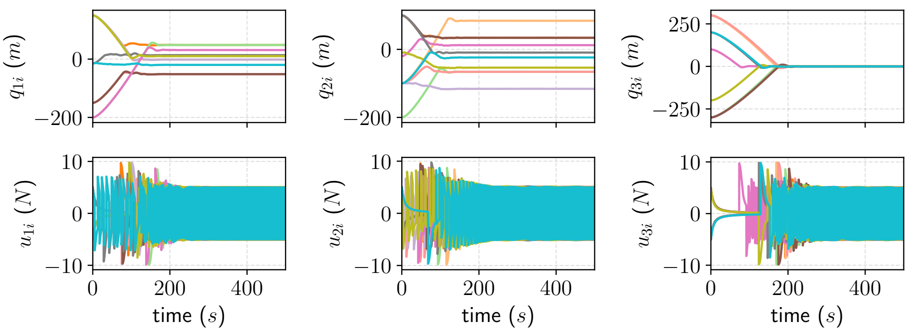
(A) $(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=$(5.0, 5.0, 1.0, 1.0)$.



(B) $(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=$(8.0, 2.0, 0.1, 0.1)$.

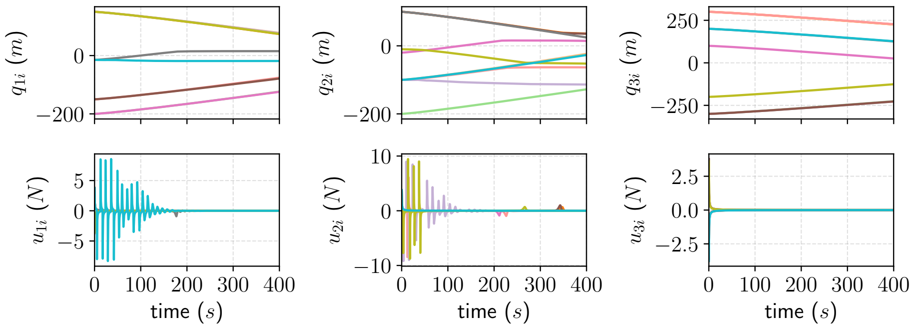


(C) $(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=$(2.0, 8.0, 0.1, 0.1)$.

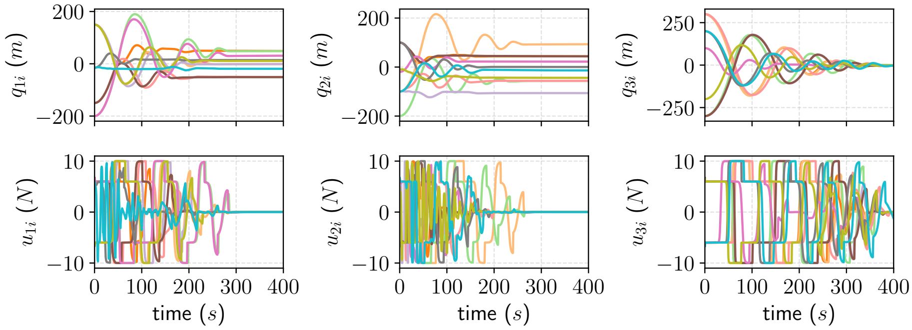


(D) $(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=$(5.0, 5.0, 20.0, 0.1)$.

FIGURE 3.8: Example 3.3–Several simulations with distinct control parameters in the proposed approach in the network of Example 3.2.



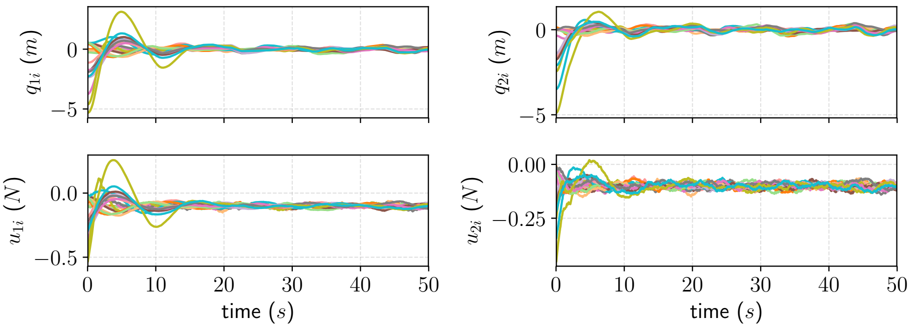
(A) $(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(5.0, 5.0, 0.1, 20.0).



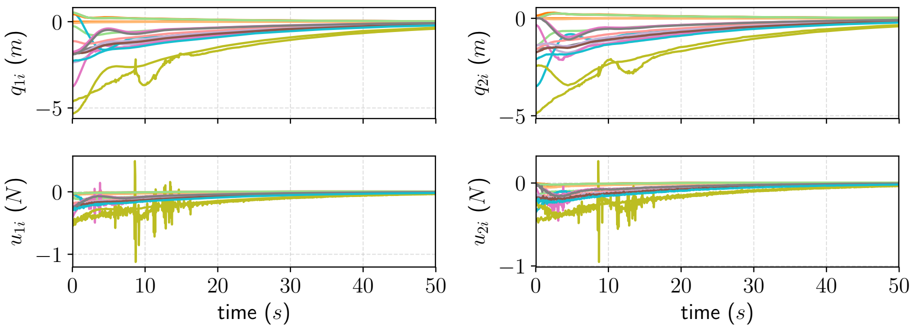
(B) $(k_{1i}, k_{4i}, \beta_{1i}, \beta_{2i})$=(8.0, 2.0, 0.1, 20.0).

FIGURE 3.9: Example 3.3–Several simulations with distinct control parameters in the proposed approach in the network of Example 3.2.

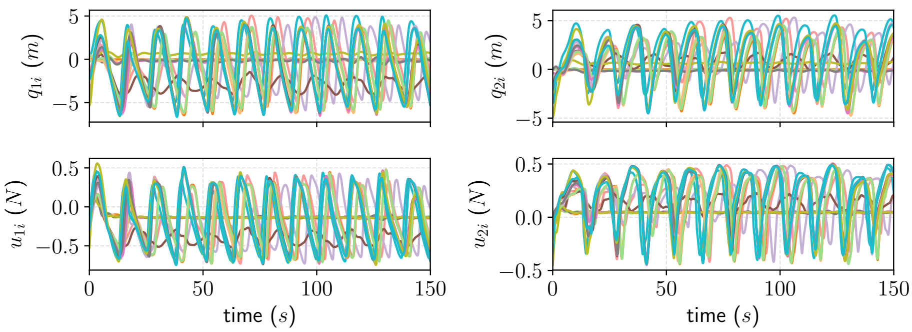
Example 3.4. *This example provides some simulations to illustrate the robustness of the control strategy against uncertainties. Note that no theoretical robustness analysis was made, this example aims to set an expectation of the network behavior under certain types of uncertainties. To examine the robustness, three types of additive uncertainties are considered, namely, i) locally in the actuator of the agents, ii) in the inertia and Coriolis matrices, and iii) in the communication channels, e.g., together with the information from neighbors. To study those configurations, the communication topology represented by the Laplacian matrix in (3.30) is considered, with nodes as two-link manipulators dynamics as the ones given in Example 3.3, with parameters given in Table 3.2 and gains of the controllers* $(k_{1i}, k_{2i}, k_{3i}, k_{4i}, \beta_{1i}, \beta_{2i}) = (1.0, 5.0, 12.0, 1.0, 0.1, 0.1)$*. In all three scenarios, the noise is considered to be a random value within a set* $[-\varepsilon, \varepsilon]$*, in which* $\varepsilon \in \mathbb{R}^2$ *is a constant bound for the disturbance. The three scenarios are depicted in Figure 3.10.*



(A) Scenario i): input uncertainties with $\varepsilon = 0.2$.



(B) Scenario ii) additive uncertainties in the systems matrices with $\varepsilon = 1.0$.



(C) Scenario iii) additive uncertainties in the communication channels with $\varepsilon = 2.0$.

FIGURE 3.10: Example 3.4–Several simulations with uncertainties in (A) the input, (B) the systems matrices, and in (C) the communication channels considering the network of two-link manipulators in Example 3.3.

From these simulations, we can see that the network attains the desired formation in scenario i) and ii), and in iii) the network keeps oscilating. Interestingly, the multi-agent system worked better with matrices uncertainties than the other two types considered. A likely explanation is that the control strategy does not need information about those matrices, thus an unknown behavior in their dynamics would not disturb the stabilization too much. Even though, it is considered that the strategy exactly compensates the gravitational forces. From the simulations, the strategy seems to be more sensitive to uncertainties in the actuators, even though it handled well values *of ε with 10% of the actuators limits. Contrary to what Figure 3.10c might suggest, the multi-agent system attained convergence to a neighborhood of the desired formation for values of the elements of the vector ε less than 1.5 (as it is shown in Figure 3.11), the purpose of showing the oscillation is to display an interesting behavior. The other two scenarios have not shown this type of trajectory.*

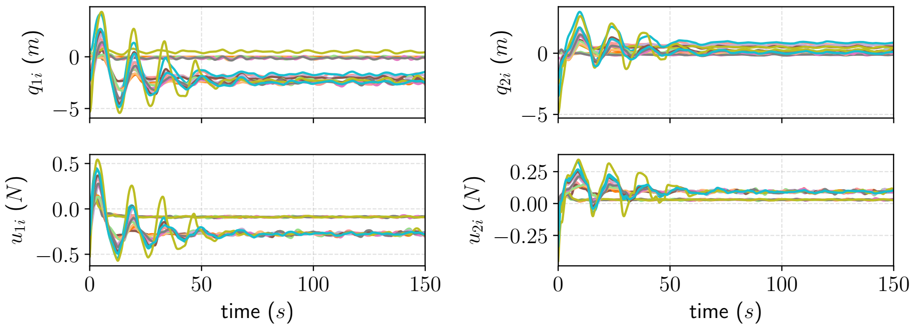


FIGURE 3.11: Example 3.4–Simulations with uncertainties in the communication channels considering the network of two-link manipulators in Example 3.3 with $\varepsilon = 1.5$.

3.5 Chapter Conclusions

In this chapter we investigated the formation-containment problem of heterogeneous Euler-Lagrange systems with directed communication topology (from leaders to followers). The communication channels of the network are subject to nonuniform time-varying delays, and the agents are subject to input saturation. With the use of a dynamic controller, which does not rely on relative velocity measurements neither on the inertia and Coriolis matrices, conditions for the convergence analysis are derived. The conditions are appropriated for agents with distinct dynamics, with the same number of degrees of freedom. Despite the fact the distributed controllers do not depend on relative measurements of velocity, they do rely on active communication among agents, since they depend on the information exchange of controllers states, which can not be directly measured by neighbors.

The communication time-varying delays were assumed to have an upper bound associated with each agent of the network. Moreover, it was not assumed the knowledge of the delays derivative, only its existence, a desirable feature since it might be hard to evaluate the variation of multiple delays on multi-agent systems. The input saturation was dealt with by using a strictly increasing function $\sigma(\cdot) : \mathbb{R} \to (-1, 1)$ in such a way the input upper and lower bounds can be adjusted by choosing appropriated parameters on the controller. It is noteworthy that the parameters associated with the controller bounds are not necessary in the proposed conditions for the convergence analysis, which gives flexibility on the control tuning.

Finally, it was shown in the numerical examples that the proposed strategy outperforms recent works on guaranteeing the consensus with time-delays, even though the presented strategy solves a more general problem than the classical consensus and simultaneously considers input saturation and communication delays.

Chapter 4

Final Discussion

This work has presented new sufficient conditions for the convergence analysis and design of distributed protocols for multi-agent systems with linear and nonlinear agents. Two different problems were addressed, the consensus and the formation-containment, both subject to time-delay and saturating inputs. The investigation was carried mainly utilizing the Lyapunov-Krasovskii and Barbalat's frameworks, deriving less conservative LMIs conditions than those from the literature that approach similar problems.

The contributions to the consensus problem were introduced in Chapter 2. In this context, it was considered linear saturated agents with input time-delay. Input saturation was handled as a Lur'e problem, by considering a dead-zone function along with a sector condition. This representation allowed the use of tools developed for the analysis of linear systems. Firstly, it was demonstrated a technique to translate the consensus into a stability problem of saturated linear systems subject to time-delay. The appealing aspect of this transformation is that it allows the consensus to be studied by using different methods. With the equivalent system, it was derived results in the study of consensus in the continuous- and discrete-time domains.

In the continuous-time domain, the time-delay was such that its values belong to an interval with a possible non-zero lower-bound. Moreover, it was considered non-differentiable time-varying delays, a desirable feature since it is hard to measure the variation of delays and it might even be random. This feature also fits well in the context of multi-agent systems, because it allows representing arbitrary delays for each agent within the same set and, consequently, to represent the whole networked system in a compact form. This is evident in the upper-bound choice of inequality (2.33). Numerical experiments have shown that our method is less conservative and provides better estimates for the region of guaranteed convergence, but at the expense of an increase in the number of decision variables.

In the discrete-time domain, the study was carried employing two different approaches, by rewriting the multi-agent system as an augmented switching system and by using the Lyapunov-Krasovskii approach. As discussed in Chapter 2, the motivation to model the delayed network as an augmented switching system comes from an attempt to provide nonconservative conditions, as for single-agent systems without saturation this method is equivalent to using a general, delay dependent, Lyapunov-Krasovskii functional candidate. The numerical tests corroborate with the expectation in the stability study, and suggest a decrease in the conservatism when compared with the Lyapunov-Krasovskii approach. However, the procedure to obtain synthesis conditions introduced too much conservatism, to the point that the conditions derived using the Lyapunov-Krasovskii methodology became less conservative. Another drawback of the augmented approach is that the computational complexity grows with the delay upper-bound. To try to cope with this problem, it was considered homogeneous delays in the network and possible bounds on the delay variation. Nevertheless, even with those considerations, the computational complexity may be an obstacle in using this method in high-dimensional systems subject to large values of delays.

In all approaches, in the continuous- and discrete-time domain, the consensus analysis, and gain design were developed in such a way the convergence is locally ensured, and the search for the largest estimate of the domain of consensus is posed as a convex optimization problem. The numerical examples suggest that the offered approaches outperform those few works that have studied similar premises in the considered examples.

In Chapter 3 the approach taken in Chapter 2 for continuous-time multi-agent systems was expanded in two main ways: the general problem of formation-containment was studied and the agents were modeled by nonlinear Euler-Lagrange equations. Moreover, in this context, the delays are considered on the communication channels of the network, instead of input delays, which together with the input saturation it might represent more realistic scenarios. The communication delays are handled in a similar fashion as the input delays on the consensus problem. To handle the input limits the control signals were bounded using a continuous strictly increasing function such that $\sigma(\cdot) : \mathbb{R} \to (-1, 1)$. Thereby, the input limits could be adjusted with appropriate parameters. The main improvements offered in Chapter 3 in relation to similar works from the literature can be highlighted as: the possibility to handle input saturation and communication time-varying delays simultaneously, the use of dynamic controllers which does not rely on relative velocity measurements, nor on the system matrices, and a procedure to perform offline convergence analysis that does not depend on the parameters of the models of the agents . In addition to that, the proposed strategy surpasses recent works on guaranteeing the consensus with time-delays, even though a more general problem is being considered.

It should be stressed that in order to derive LMIs conditions, some possible conservative inequalities were used during the development of the results in this text, but despite this trade-off, it was shown that the proposed methods outperform recent works found in the literature in the considered examples. This suggests that there is potential room for improvements in the sense of diminishing the conservatism of the proposed conditions.

4.1 Future Research

Possible lines to further investigate the problem of time-delay and saturation in multi-agent systems are:

- To design an anti-windup structure to mitigate the performance degradation due to saturation (Tarbouriech et al., 2011). The objective of this strategy is to avoid the growth of the integral action of the controller by compensating saturated input signals. This structure could enlarge the linear region of operation of the agents. Thus, it would guarantee performance requirements for larger sets and improve the overall behavior of the multi-agent system.

- To guarantee the operation of the linear multi-agent system inside a safety region by employing Control Barrier Functions (Ames et al., 2019). This approach could lead to less conservatism in the estimate of the domain of consensus, since the required constraints alleviate, for example, the necessity of a positive definite function. An interesting aspect of this strategy is that the safety region can be cast into an invariant set without rendering its sublevel sets invariant, which could be less restrictive than the proposed methodology.

- To formulate the formation-containment problem for varying formations. This formulation will allow to address formations on multi-agent systems that are able to track a trajectory, maintaining the followers states within a region established by a set of leaders. Similar problems were recently studied considering networked Euler-Lagrange systems (Ge et al., 2016), but employing computed-torque control strategy, which makes the systems dynamics similar to double-integrators. Hence, the strategy is highly dependent on the knowledge of system parameters. Santiaguillo-Salinas and Aranda-Bricaire (2017) studied the time-varying formation with networked differential-drive mobile robots. Deng et al. (2020) have also tackled a related problem, taking into consideration general linear agents and using event-triggered control strategy to guarantee the tracking. The main idea is to try to extend the results presented in Chapter 3, from static formations to time-varying ones.

- To combine two stabilizing controllers to mitigate the performance degradation due to saturation and impose a local convergence criterion. The idea is to address the global asymptotic convergence while satisfactorily meeting local performance requirements. The approach could initially be similar to the anti-windup on linear systems (Oliveira et al., 2013; Tarbouriech et al., 2011), where the main idea is to mitigate the saturation effects. At first, the method could be to combine a local controller based on the computed-torque (Feedback linearization) with an exponential rate convergence, and the controller proposed in Chapter 3, which does not depend on the inertia nor Coriolis matrices and has global convergence guarantees.

- To consider uncertain agent dynamics subject to external disturbances. Consequently, investigate more realistic scenarios on multi-agent systems. This might enlarge and facilitates the real-world applications. The main motivation for this approach is to develop completely model-independent controllers, which could be achieved with the controller proposed on Chapter 3 if the gravitational compensation was considered a bounded uncertainty. There are some promising alternatives to be tried: the first one to be considered is to combine the integral action proposed by Hernández-Guzmán and Orrante-Sakanassi (2019) along with the control strategy proposed in Chapter 3. Another interesting approach to perform the robustification is to use sliding mode control, which has low sensitivity to plant parameter variations and disturbances (Chen et al., 2019a; Karimi, 2012).

4.2 List of Publications

The publications related to the contributions of this thesis are listed below:

T. C. Silva et al. (2021). "Regional consensus in discrete-time multi-agent systems subject to time-varying delays and saturating actuators". In: *Submited to International Journal of Control*

T. C. Silva, F. O. Souza, and L. C. A. Pimenta (2021). "Consensus in multi-agent systems subject to input saturation and time-varying delays". In: *International Journal of Systems Science* 52.7, pp. 1479–1498. DOI: 10.1080/00207721.2020.1860267

T. C. Silva, F. O. Souza, and L. C. A. Pimenta (2020). "Distributed formation-containment control with Euler-Lagrange systems subject to input saturation and communication delays". In: *International Journal of Robust and Nonlinear Control* 30.7, pp. 2999–3022. ISSN: 1049-8923. DOI: 10.1002/rnc.4919

Appendix A

Mathematical Tools

In this section some results are presented on the stability analysis and on linearization of the stability conditions. The following results are used to derive the main conclusions drawn in this manuscript.

A.1 Auxiliary Lemmas

In many problems, the relation between variables may appear non-linearly throughout the development of the stability conditions. Fortunately, many of them can be linearized using algebraic techniques. Here, we highlight some important techniques used in this manuscript:

Lemma A.1. *(Sun, Liu, and Chen, 2009). For any constant matrix $M = M^T > 0$ and scalars $t > t - \tau \geq 0$ such that the following integrations are well defined, then*

$$\int_{-\tau}^{0}\int_{t+s}^{t} z^T(\epsilon)Mz(\epsilon)d\epsilon ds \geq \frac{2}{\tau^2}\int_{-\tau}^{0}\int_{t+s}^{t} z^T(\epsilon)d\epsilon ds M\int_{-\tau}^{0}\int_{t+s}^{t} z(\epsilon)d\epsilon ds.$$

Lemma A.2. *(Seuret and Gouaisbaut, 2013). For any constant matrix $M = M^T > 0$ and scalars $t > t - \tau \geq 0$ such that the following integrations are well defined, then*

$$\int_{t-\tau}^{t} \dot{z}^T(\epsilon)M\dot{z}(\epsilon)d\epsilon \geq \frac{1}{\tau}\int_{t-\tau}^{t} \dot{z}^T(\epsilon)d\epsilon M\int_{t-\tau}^{t} \dot{z}(\epsilon)d\epsilon + \frac{3}{\tau}\Omega^T M\Omega,$$

with

$$\Omega = z(t-\tau) + z(t) - \frac{2}{\tau}\int_{t-\tau}^{t} z(\epsilon)d\epsilon.$$

Lemma A.3. *(Gu, Kharitonov, and Chen, 2003). For any constant matrix $M = M^T > 0$ and scalars $t > t - \bar{\tau} \geq 0$ such that the following integrations are well defined, then*

$$\int_{t-\bar{\tau}}^{t} \dot{z}^T(\epsilon)M\dot{z}(\epsilon)d\epsilon \geq \frac{1}{\bar{\tau}}\int_{t-\bar{\tau}}^{t} \dot{z}^T(\epsilon)d\epsilon M\int_{t-\bar{\tau}}^{t} \dot{z}(\epsilon)d\epsilon.$$

Lemma A.4 (Schur Complement (Gu, Kharitonov, and Chen, 2003))**.** *For any constant symmetric matrix*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix},$$

the following statements are equivalent:

1) $S < 0$,

2) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,

3) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

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