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Pablo Almeida Gomes

Topics in Interacting Particle Systems and Anisotropic Percolation Models

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Orientador: Rémy de Paiva Sanchis.

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ATA DA CENTÉSIMA QUADRAGÉSIMA SEXTA DEFESA DE TESE DO ALUNO PABLO ALMEIDA GOMES, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 20 DE MAIO DE 2020.

Aos vinte dias do mês de maio de 2020, às 13h30 na reunião pública virtual meet.google.com/gss-hnye-zjq, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Pablo Almeida Gomes**, intitulada: "*Topics in Interacting Particle Systems and Anisotropic Percolation Models"*, requisito final para obtenção do Grau de doutor em Matemática. O código da reunião foi divulgado antecipadamente para a comunidade. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Rémy de Paiva Sanchis, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente em https://meet.google.com/tsg-bnyg-tmd, sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 20 de maio de 2020.

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Topics in Interacting Particle Systems and Anisotropic Percolation Models

PABLO ALMEIDA GOMES

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this thesis is dedicated to my wife

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Nós somos do Clube Atlético Mineiro
Jogamos com muita raça e amor
Vibramos com alegria nas vitórias
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Vencer, vencer, vencer
Este é o nosso ideal
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Nós somos Campeões dos Campeões
Somos o orgulho do esporte nacional

Lutar, Lutar, Lutar
Com toda nossa raça pra vencer
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Resumo

Nesta tese, apresentamos resultados em processos estocásticos, mais precisamente, em sistemas de partículas interagentes e em modelos de percolação anisotrópica.

O primeiro tópico de nossa análise é o processo de contato sob renovações, uma recente generalização do clássico processo de contato; analisamos o caso em que a distribuição dos intervalos, entre as renovações, tem cauda pesada, onde o decaimento é polinomial com expoente entre 0 e 1. Mostramos que neste caso, um fenômeno incomum é observado, há possibilidade de sobrevivência mesmo em grafos finitos: para cada expoente, há transição de fase de acordo com o tamanho do grafo, isto é, temos extinção quase certa se a quantidade de indivíduos é menor que o tamanho crítico e temos possibilidade de sobrevivência caso contrário. E, além disso, exibimos cotas inferior e superior bastante satisfatórias para o tamanho crítico.

O segundo tópico consiste em um sistema unidimensional e infinito de partículas, onde há uma partícula carregada que está sob a ação de uma força constante, que lhe provoca movimento e consequentemente interações com as demais partículas presentes no sistema. Como resultado, teoremas centrais do limite são estabelecidos, para a posição e velocidade da partícula carregada.

Por fim, analisamos o diagrama de fase do modelo de percolação anisotrópica de elos de Bernoulli independentes na rede hipercúbica d-dimensional, onde a probabilidade de um elo estar aberto, varia de acordo com sua direção. Dois resultados são obtidos: primeiro, estabelecemos que, na rede orientada, de certa forma, o diagrama de fase é similar ao do modelo na árvore d-regular; e segundo, estabelecemos que, se d é maior que 10, é válida uma conjectura envolvendo o expoente crítico de transição dimensional, que fora proposta por físicos.

Palavras-chave: Processo de contato. Percolação anisotrópica. Sistema de partículas. Transição de fase.

Abstract

This thesis consists of a presentation of four works, independent among themselves, developed during my doctoral period. The first two are about interacting particle systems, while the others two are about anisotropic percolation models.

The first one is a joint work with L.R. Fontes and R. Sanchis concerning the renewal contact process. We study an infection propagation on a finite population. We consider a finite graph where an individual is attached to each vertex. The population starts with a single infected individual and the infection propagates through neighbors according to marks of independent Poisson processes with rate $\lambda > 0$, each one associated to an edge. Given $\alpha \in (0, 1)$, for each individual, recovery occurs according to marks of a renewal process with heavy-tailed α-stable law, associated to its vertex. All processes are assumed to be independent. We explicitly give $V_{-}(\alpha)$ and $V_{+}(\alpha)$ such that $V_{+}(\alpha) - V_{-}(\alpha) < 1$, and for every $\lambda > 0$, almost surely, the infection is extinct if the total population size is less than $V_{-}(\alpha)$ and has positive probability to survive, if it is bigger than $V_{+}(\alpha)$.

In the second work, also in collaboration with L.R. Fontes and R. Sanchis, we establish central limit theorems for the position and velocity of the charged particle in a half-line mechanical particle system. A constant force F acts solely on the charged particle starting at origin, while all the other particles are force neutral, initially static and their interparticle distances are given by the family of i.i.d. positive random variables $\{\xi_i\}_{i\in\mathbb{N}}$. Let $\{\eta_i\}_{i\in\mathbb{N}}$ be a family of i.i.d. Bernoulli random variables with parameter p, the r.v. η_i determines the i-th neutral particle states, that can be sticky or elastic. Collisions between the charged particle and a sticky one are totally inelastic and the sticky particle mass is incorporated by the charged one, while collisions between an elastic particle and the charged one, as the name suggest, are perfectly elastic. We assume that neutral particles do not interact among themselves.

The third is a joint work with A. Pereira and R. Sanchis, we study anisotropic oriented percolation on $\mathbb{Z}^d$, $d \geq 4$. Independently of all others, an edge parallel to e_i is open with probability p_i, $i = 1, \ldots, d$. We show that if $p_1 + \cdots + p_d$ is strictly greater than one and each p_i is not too large, then percolation occurs.

The fourth and last, is a joint work with R. Sanchis and R.W.C. Silva. We consider independent anisotropic bond percolation on $\mathbb{Z}^d \times \mathbb{Z}^s$ where edges parallel to $\mathbb{Z}^d$ are open with probability $p < p_c(\mathbb{Z}^d)$ and edges parallel to $\mathbb{Z}^s$ are open with probability q, independently of all others. We prove that percolation occurs for $q \geq 8d^2(p_c(\mathbb{Z}^d) - p)$. This fact implies that the so-called *dimensional crossover* critical exponent, if it exists, is greater than 1. In particular, using known results, we provide a proof that, for $d \geq 11$, the crossover critical exponent exists and equals 1.

Keywords: Contact process. Anisotropic percolation. Particle systems. Phase transition.

Contents

Resumo Estendido

Nesta tese, apresento resultados obtidos junto a colaboradores, em pesquisas desenvolvidas durante meu doutorado. As pesquisas não abordam um tema principal em comum, consistem no estudo de tópicos em processos estocásticos, buscando solucionar questionamentos que surgiram, tanto em seminários quanto em conversas com colaboradoes, e que de alguma forma despertaram interesse ou se mostraram viáveis. Estes resultados foram divulgados em quatro artigos submetidos para publicação, cada capítulo desta tese contém um desses trabalhos, com modificações mínimas.

Os modelos estudados nos primeiros trabalhos tratam de sistemas de partículas interagentes, de natureza distintas. Já os dois últimos trabalhos são mais relacionados e estudam modelos de percolação anisotrópica. Abaixo é exibida uma breve introdução sobre os modelos envolvidos, além de uma descrição dos resultados.

Modelos de sistemas de partículas interagentes têm sido de grande interesse entre matemáticos nas últimas décadas. Para uma visão geral sobre o tema, veja os clássicos [Lig85] e [Dur95]. Em particular, o processo de contato introduzido por Harris em [Har74], é um destes modelos e vem sendo intensamente estudado. Se trata de um modelo de propagação de uma infecção, onde cada indivíduo é visto como um vértice de um dado grafo, que pode estar, em cada instante, infectado ou curado. Um indivíduo infectado pode propagar a infecção para um vizinho a uma taxa λ e pode se tornar curado com taxa 1. O processo de contato pode também ser interpretado como a evolução de uma população ao longo do tempo, onde um vértice pode estar "ocupado" (em correspondência com "infectado") ou "vazio" (em correspondência com "curado").

De maneira mais precisa, cada elo do grafo dado possui uma linha temporal, com início no instante 0 e que possui marcas de acordo com um processo de Poisson independente, com taxa λ. Nos instantes correspondentes a estas marcas, um indivíduo infectado, seja ele um dos vértices do respectivo elo, propaga a infecção para o indivíduo associado ao outro vértice do elo e que se encontra curado, o qual se torna infectado instantaneamente; caso ambos os indivíduos estejam curados ou infectados neste instante, o estado de ambos se mantém. Há também um mecanismo de cura similar: cada vértice possui uma linha temporal com marcas segundo um processo de Poisson independente e com taxa 1: nos instantes respectivos a tais marcas, o indivíduo associado a este vértice se torna curado caso esteja infectado ou, caso contrário, mantém o seu estado de curado.

Este modele exibe uma transição de fase de acordo com o parâmetro λ, precisamente: existe um ponto crítico $\lambda_c > 0$, tal que, se $\lambda > \lambda_c$, a infecção tem probabilidade positiva de sobreviver por todo o tempo e se $\lambda < \lambda_c$ a infecção é extinta quase certamente. Em [FMMV19] e [FMV20], os autores propõem uma variação deste modelo, conhecida na literatura por processo de contato sob renovações. Nesta variante, cada um dos processos de Poisson com taxa 1 que determinam os tempos de cura é substituído por um processo de renovação não estacionário, cujos tempos entre as renovações têm distribuição comum com cauda pesada, cujo decaimento é do tipo polinomial com índice α. Note que neste caso, o processo deixa de ser Markoviano. Quando $\alpha > 1$, como demonstrado em [FMV20], sob certa condição extra de monotonicidade, o modelo exibe a transição de fase usual para o parâmetro de infecção λ, isto é, a infecção é extinta quase certamente para valores suficientemente pequenos de λ. Entretanto, como exibido em [FMMV19], quando $\alpha < 1$, e sob algumas condições razoavelmente leves de regularidade, o modelo não exibe esta transição de fase, a saber, é estabelecido que, em qualquer grafo infinito e conexo, a infecção sobrevive com probabilidade estritamente positiva, qualquer que seja λ. Ter ponto crítico igual a 0 não é habitual em variantes do processo de contato, porém este fenômeno é também exibido em [Dur10] e [CD09], por exemplo.

Em [FMMV19], os autores notam um efeito de tunelamento: a infecção permanece em um indivíduo por um longo período de tempo, de acordo com um intervalo sem marca de cura, até que propague para um outro vizinho ainda não analisado, o qual possui um intervalo ainda maior, sem marcas de cura, e assim por diante. A existência desses intervalos tunelados, sugere que não há a necessidade de infinitos indivíduos para que a infecção sobreviva. Isto nos levou a estudar o Processo de Contato sob Renovações em grafos finitos, e a buscarmos uma nova transição de fase a respeito deste modelo, precisamente sobre a quantidade de vértices do grafo. O Capítulo 1 exibe o trabalho [FGS19] desenvolvido nesta pesquisa.

Como resultado, temos que fixado $0 < \alpha < 1$, damos explicitamente valores $V_-(\alpha)$ e $V_+(\alpha)$ tais que, para qualquer grafo conexo finito com número de vértices menor que $V_-(\alpha)$, a infecção é extinta quase certamente, independente da taxa λ e caso tenha mais vértices que $V_+(\alpha)$, a infecção tem probabilidade positiva de sobreviver, qualquer que seja λ. É ainda interessante mencionar que $V_+(\alpha) - V_-(\alpha) < 1$ e que esta diferença tende a 0, à medida que α tende a 1. Neste trabalho, uma regularidade maior na calda da distribuição dos tempos entre as renovações dos processos de cura é assumida como hipótese, precisamente é da forma $\mu(t, \infty) = L(t)t^{-\alpha}$, onde $L(\cdot)$ é uma função de variação lenta. Com essa regularidade temos mais facilidade de lidar com ocorrências de marcas de cura, em todos os indivíduos, proximamente concentradas, isto inviabiliza a sobrevivência da infecção e nos fornece $V_{-}(\alpha)$. Entretanto, é esperado que uma regularidade mais fraca seja suficiente para a obtenção da cota $V_{+}(\alpha)$.

À primeira vista pode ser surpreendente o fato do comportamento do modelo não depender da estrutura do grafo, mas note que, como estamos lidando com uma distribuição com cauda pesada na determinação dos tempos entre as marcas de cura, é esperado a ocorrência de tempos cada vez maiores. A ocorrência de tais grandes intervalos de tempo determina a possibilidade de que todos os indivíduos sejam infectados simultaneamente, desta forma, a estrutura do grafo passa a não desempenhar papel importante. E como veremos, tal ocorrência só depende da quantidade de indivíduos.

No Capítulo 2 estudamos um sistema de partículas interagentes de natureza distinta ao processo de contato, se trata de um sistema infinito, cujas partículas se movimentam de acordo com as leis da mecânica clássica (Newtoniana). Temos uma quantidade enumerável de partículas que estão inicialmente dispostas de maneira estática sobre o semi-eixo real não-negativo $[0, \infty)$, sobre o qual poderão vir a se mover ao decorrer do tempo. Sobre o sistema, atua uma força constante F. Um partícula carregada, sujeita à ação desta força, se encontra inicialmente disposta sobre a origem. Todas as demais partículas são neutras e portanto esta força não atua sobre elas. A posição inicial destas partículas neutras são definidas de acordo com a família i.i.d. de variáveis aleatórias estritamente positivas $\{\xi_i\}_{i\in\mathbb{N}}$ com média μ, precisamente, ξ_i denota a distância inicial entre a $i-$ésima partícula neutra e sua antecessora. Cada uma das partículas neutras, de maneira independente das demais, tem probabilidade $0 < p \leq 1$ de ser declarada "grudenta", neste caso, colisões com a partícula carregada são perfeitamente inelásticas e sua massa é então incorporada à partícula carregada, de acordo com as leis da mecânica Newtoniana. Com probabilidade $1 - p$, é declarada elástica e portanto tem interações elasticamente perfeitas com a partícula carregada durante a evolução do processo. É ainda assumido que partículas neutras não interagem entre si.

Este modelo foi introduzido em [FNV00], onde os autores estudam o comportamento da partícula carregada ao longo do tempo. As interações com as partículas neutras gera uma rede de forças opostas ao movimento provocado pela força F, esta situação, como mostrada pelos autores, tende ao equilíbrio, o que determina uma lei dos grande números para a velocidade instantânea da partícula carregada:

quase certamente converge para $\sqrt{F\mu/(2-p)}$.

Com respeito à literatura relacionada, modelos unidimensionais com agregação de massa e força gravitacional em sistemas finitos de partículas, foram estudados em [BMPZ98], [MP94] e [MP95]. Além disto, propriedades ergódicas de sistemas unidimensionais e semi-infinitos de natureza similar, porém com apenas interações elásticas ($p = 0$), foram estudadas anteriormente em [BMPZ98], [BPSS85], [PSS85], [PSV99] e [STV01], onde o comportamento limite da partícula carregada é determinado pela relação entre a pressão das partículas neutras e a força atuante no sistema ou pela posição inicial das partículas.

Nas observações finais do trabalho [FNV00], os autores sugerem que uma interessante continuidade deste trabalho seria obter um teorema central limite para a velocidade instantânea da partícula carregada. Exploramos esta sugestão e estabelecemos no trabalho [FGS20], teoremas centrais limites com flutuações gaussianas, para a posição e para a velocidade da partícula carregada.

Como mencionado anteriormente, os dois últimos trabalhos abrangem modelos de percolação anisotrópica. Percolação é um modelo simples, fácil de ser definido, porém, apresenta uma grande variedade de fenômenos interessantes, além de levantar questões extremamente duras de serem respondidas e com alto grau complexidade matemática. É ainda muito comum encontramos soluções elegantes e engenhosas através do uso de matemática elementar, o que faz deste modelo, ao meu ver, muito agradável de se trabalhar e apreciar.

O estudo matemático do modelo de percolação, foi introduzido por Broadbent e Hammersley [BH57], com o objetivo de modelar o fluxo de um fluido em um meio poroso cujos canais apresentam bloqueios aleatórios que impedem a passagem. Dado um grafo, esta modelagem, falando de maneira simplificada, busca estudar a estrutura de componentes conexas de subgrafos gerados de maneira aleatória. Comumente, este dado grafo é do tipo grade, podendo ser orientado ou não, e o subgrafo é obtido por seleção de elos ou vértices (os quais nomeamos abertos) de maneira independente. Nesta tese, os estudos presentes nos Capítulos 3 e 4, consideram modelos de percolação sobre elos independentes.

Algumas das principais questões da teoria de percolação recaem sobre a existência de um "aglomerado infinito", isto é, sobre a existência de uma componente conexa infinita no subgrafo resultante e também sobre a existência e determinação de expoentes críticos. Sugerimos ao leitor as obras clássicas [Gri99] e [BR06], as quais apresentam e abordam as principais questões e resultados da área.

Quando o grafo em questão tem como conjunto de vértices o conjunto $\mathbb{Z}^d$ e, como conjunto de elos (orientados ou não), pares entre primeiros vizinhos, o denotamos com certo abuso de notação, simplesmente por $\mathbb{Z}^d$ e até mesmo o mencionamos por rede $\mathbb{Z}^d$. Considerando $\mathbb{Z}^d$, se cada elo tem a mesma probabilidade p de estar aberto, o modelo exibe um ponto crítico $p_c(\mathbb{Z}^d)$: quase certamente, há um único aglomerado infinito se $p > p_c$ e, quase certamente, não existe nenhum aglomerado infinito quando $p < p_c$. Quando o grafo é a árvore $(d+1)$-regular, o modelo se trata de um processo de ramificação, o ponto crítico é facilmente calculado e é $1/d$. Um comportamento comum em percolação sobre grafos do tipo grade é que, à medida em que a dimensão cresce, de certa forma tendem a ter o comportamento de campo-médio, isto é, o ponto crítico é assintótico a $1/d$ em casos orientados e $1/2d$ nos casos não orientados. Este fato é constatado para a rede $\mathbb{Z}^d$, por Cox e Durrett [CD83] no caso orientado e de maneira independente nos trabalhos de Kesten [Kes90] e Gordon [Gor91], no caso sem orientação.

Uma grande quantidade de variações do modelo de percolação foram propostas ao longo do tempo, a maioria delas oriundas de análises de fenômenos físicos. Um dos principais ingredientes que acompanham estes fenômenos é a anisotropia do sistema. Em modelos de percolação anisotrópica, os elos são agrupados em classes e elos em classes distintas têm probabilidades distintas de estarem abertos. Um dos primeiros resultados em percolação anisotrópica aparece em 1964 no trabalho de Essam e Sykes [ES64]. Kesten em [Kes82], fornece o diagrama de fase completo para o modelo de percolação anisotrópica não orientada na rede quadrada ($\mathbb{Z}^2$), onde elos verticais estão abertos de maneira independente com probabilidade p, enquanto que horizontais com probabilidade q. Precisamente, há aglomerado infinito se $p + q > 1$ e não há se $p + q \leq 1$. Para $d \geq 3$, pouco é conhecido a respeito de diagramas de fase em modelos com anisotropia.

No Capítulo 3, buscamos constatar este efeito de campo-médio acima mencionado, em modelos com anisotropia. Note que, quando consideramos percolação independente sobre a árvore $(d+1)$-regular, onde os elos têm probabilidades $p_1, \ldots, p_d$ de estarem abertos (segundo uma ordenação dos elos pré-fixada), não há aglomerado infinito se $p_1 + \cdots + p_d < 1$ e há se $p_1 + \cdots + p_d > 1$. Em [GPS19] consideramos o modelo de percolação orientada em $\mathbb{Z}^d$, $d \geq 4$, onde cada elo com direção e_i, é aberto com probabilidade p_i, independente dos demais, para $i = 1, \ldots, d$. Obtemos como resultado que, para todo $d \geq 4$, se $p_1 + \cdots + p_d \geq 1 + \epsilon$ e cada p_i não é maior que $C\epsilon$, então quase certamente existe um aglomerado infinito. É interessante ressaltar que essa constante pode ser tomada próxima de 1 a medida que consideramos apenas dimensões suficientemente grandes ao invés de $d \geq 4$. Na demonstração deste resultado, transformamos nosso problema em uma análise de interseção de passeios aleatórios e esta análise é feita com métodos elementares em combinatória.

Uma condição que limita cada p_i se faz necessária, mas não sabemos o quão boa é a condição na ordem de ϵ, de fato, afirmamos a necessidade olhando para o caso isotrópico: como veremos, a cota inferior para $p_c(\mathbb{Z}^d)$ demonstrada em [CD83] nos diz que a ordem dessa condição é de pelo menos $\sqrt{\epsilon}$.

Outro tópico de estudo envolvendo anisotropia aparece na literatura da Física como *dimensional crossover*, o qual traduzimos por transição dimensional. Sobre a rede $\mathbb{Z}^{d+s} = \mathbb{Z}^d \times \mathbb{Z}^s$ com elos não orientados, consideramos percolação com a seguinte anisotropia: elos paralelos a $\mathbb{Z}^d$ são declarados abertos com probabilidade $p < p_c(\mathbb{Z}^d)$, enquanto elos paralelos a $\mathbb{Z}^s$ estão abertos com probabilidade q, de maneira independente dos demais. Este tópico de transição dimensional visa estudar efeitos provocados no sistema, quando p se aproxima de $p_c(\mathbb{Z}^d)$ e q se aproxima de 0.

Uma das questões acerca do tema, recai sobre a existência e cálculo de um expoente crítico ψ, conhecido na literatura como expoente crítico de transição dimensional. O expoente crítico ψ para percolação de elos em $\mathbb{Z}^{d+s}$ foi introduzido em [RS79], ele pode ser descrito da seguinte maneira: para cada parâmetro $p \leq p_c(\mathbb{Z}^d)$, seja $q_c(p)$ o ponto crítico tal que, quase certamente, não existe aglomerado infinito para valores de q menores que q_c e existe para valores maiores. É esperado que exista $\psi(d) > 0$, independente de s, tal que, $q_c(p)$ tenda a zero, à medida que $p \uparrow p_c$, assintoticamente como $|p_c - p|^{\psi}$.

Como mencionado anteriormente, o estudo de expoentes críticos é um dos tópicos de grande interesse da teoria de percolação, veja Capítulo 9 em [Gri99] para mais detalhes. Um dos principais expoentes em questão, é o expoente γ, o qual descreveremos brevemente abaixo. Considerando o modelo de percolação isotrópica com parâmetro $p < p_c$ sobre a rede $\mathbb{Z}^d$, o valor esperado para o tamanho da componente conexa aberta contendo a origem é denotada por $\chi(p)$. Acredita-se que existe $\gamma(d) > 0$, tal que $\chi(p)$ seja assintótico a $|p_c - p|^{-\gamma}$, quando $p \uparrow p_c$.

Conjectura-se que ambos expoentes ψ e γ existam e sejam iguais. Alguns artigos na literatura da Física, buscaram investigar esta conjectura através de simulações, como por exemplo [GCGR81, RS79, RC80], entretanto algumas dessas simulações se mostraram divergentes entre si. De maneira matematicamente rigorosa, o trabalho [SS17, SSc] estabelece que esta conjectura é verdadeira para $d = 1$ e também conclui que, caso ambos existam, então $\psi \leq \gamma$. Com a finalidade de dar continuidade a este trabalho, em [GSS19] mostramos que a conjectura é verdadeira para $d \geq 11$.

Na demonstração, utilizamos o resultado de que, para $d \geq 11$, o expoente crítico γ existe e é igual a 1, como pode ser visto em [FH17]. Utilizando a técnica de acoplamento dinâmico, obtemos uma cota inferior para $q_c(p)$, a partir da qual, juntamente com resultados acima mencionados, concluímos que $\psi = \gamma = 1$.

Segue abaixo uma listagem dos trabalhos científicos que foram produzidos durante o doutorado, em ordem cronológica.

Artigos aceitos para publicação

- **Contact Process under heavy-tailed renewals on finite graphs** (2019). *ArXiv:1907.00290*. Em colaboração com os professores Luiz Renato Fontes e Rémy Sanchis. A ser publicado em *Bernoulli*.
- **A note on the dimensional crossover critical exponent** (2019). *ArXiv:1912.08709*. Em colaboração com os professores Rémy Sanchis e Roger William Silva. A ser publicado em *Letters in Mathematical Physics*.
- **Anisotropic oriented percolation in high dimensions** (2019). *ArXiv: 1911.03775*. Em colaboração com os professores Alan Pereira e Rémy Sanchis. A ser publicado em: *Latin American Journal of Probability and Mathematical Statistics (ALEA)*.
- **Central limit theorems for a driven particle in a random medium with mass aggregation** (2020). *ArXiv: 2001.08719*. Em colaboração com os professores Luiz Renato Fontes e Rémy Sanchis. A ser publicado no volume especial, em homenagem ao professor Vladas Sidoravicius, da série *Progress in Probability*.

Chapter 1

Contact Process under Heavy-tailed Renewals on Finite Graphs

The contact process introduced by Harris in [Har74] is a model for the spread of an infection, where individuals, that can be infected or healthy, are identified with the vertices of a given graph. In this model every infected individual can propagate the infection to some neighbor at rate $\lambda > 0$ and it becomes healthy at rate 1. In each edge of the graph we place an independent Poisson point process with rate λ, the marks of such process determine instants when, if we have an infected and a healthy vertex in the corresponding edge, the infection propagates from the infected to the healthy vertex, at this instant both two vertices are declared infected. Also, to each vertex, we associate a Poisson point process with rate 1, this points determine instants when, if infected, the vertex change its status to healthy.

This model exhibits a phase transition according to parameter λ, namely: there exists a critical value $\lambda_c > 0$ such that, if λ is larger than λ_c the infection has positive probability to survive, and is almost surely extinct if $\lambda < \lambda_c$.

In [FMMV19] and [FMV20], the authors propose a variation of this model, namely the renewal contact process, where the Poisson point processes of rate 1 of the cure mechanism are replaced by renewal processes, with common heavy-tailed distribution μ of waiting times. As result, in [FMMV19], the authors conclude that, on any infinite connected graph, if $\mu(t, \infty) < t^{\alpha}$, for some $\alpha < 1$, under certain fairly mild regularity conditions, then $\lambda_c = 0$, that is, we have survival of the infection with positive probability for any $\lambda > 0$.

Since, under this circumstances, the renewal contact process does not exhibit phase transition on the infection parameter λ, we investigate the existence of a new kind of phase transition. We study the renewal contact process on finite connected graphs. Let $\alpha \in (0,1)$ and assume that the common distribution for the waiting times is of the form $\mu(t,\infty) = L(\cdot)t^{-\alpha}$, where $L(\cdot)$ is a slowly varying function. As result, we explicitly give $V_-(\alpha)$ such that, the process almost surely dies out for every λ if the graph size is less than $V_-(\alpha)$ and we also give $V_+(\alpha)$, such that, if the graph size is large than it, we have positive probability to survive for any λ. This bounds are quite sharp, in fact, $V_+(\alpha) - V_-(\alpha) < 1$, for every $\alpha \in (0,1)$.

Therefore, we conclude the existence of a finite critical value $k_c(\alpha) \in \mathbb{N}$; The infection survive with positive probability on any connected graph with size greater than or equal to k_c, for any λ. And we also have that, when $[V_-, V_+] \cap \mathbb{N} = \emptyset$, $k_c = \lceil V_+ \rceil$, otherwise we have that k_c can be either $\lfloor V_+ \rfloor$ or $\lfloor V_+ \rfloor + 1$.

1.1 The Model and Result

We consider versions of a renewal process $R = \{S_n = T_1 + \ldots + T_n;\ n \in \mathbb{N}\}$, with *waiting times* $\{T_i\}_{i\in\mathbb{N}}$ given as usual by i.i.d. non-negative random variables. Let U denote the associated *renewal measure*, given, we recall, by $U(B) = \sum_{n\geq 1} P(S_n \in B)$ for every Borel set $B \in \mathscr{B}(\mathbb{R})$, we will use the shorthand $U(t)$ for $U([0,t])$. For $t > 0$, let $N(t) = \sup\{n \in \mathbb{N}; S_n \leq t\}$ denote the number of renewals of R up to time t. We also consider the *current time* and *excess time* at t of R, given respectively by

$$C(t) = t - S_{N(t)} \text{ and } E(t) = S_{N(t)+1} - t.$$

In this chapter, we will take the common probability distribution μ of the waiting times in the basin of attraction of an α-stable law, that is,

$$\mu(t,\infty) = L(t)t^{-\alpha},\ t > 0, \tag{1.1}$$

where $L(\cdot)$ is a *slowly varying function*, and α is a parameter, in principle, in $(0,1]$, which in our context will be called *cure index*.

Let us recall two known results concerning renewal processes with such distribution, to be used below.

Theorem 1.1 (Theorem 1 in [Eri70]). *Let μ be as above with $1/2 < \alpha \leq 1$ and non-arithmetic. Then, for every $h > 0$, as $t \longrightarrow \infty$,*

$$U(t+h) - U(t) \sim \frac{C_\alpha h}{m(t)}, \tag{1.2}$$

where $C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}$ and $m(t) = \int_0^t \mu(x,\infty)dx$.

Remark 1.1. *When the probability distribution μ is arithmetic, then (1.2) holds provided h is the arithmetic span of μ. See Theorem 1.1 of [GL62].*

Remark 1.2. *The Relation (1.2) holds for $0 < \alpha \leq 1/2$ for both the arithmetic and the non-arithmetic cases (again, in the former case, provided h is the arithmetic span of μ) under extra conditions on μ. See Theorem 1.4 of [CD19].*

The second result is contained in the celebrated Dynkin-Lamperti Theorem, for which we refer again to [Eri70] (paragraph right below (9.1), Section 9), or to [Fel71], Chapter XIV.3.

Theorem 1.2. *Let μ be as above with $0 < \alpha < 1$. Then*

$$\lim_{t\to\infty} P\left(\frac{E(t)}{t} \leq x\right) = \int_0^x \frac{C'_\alpha}{y^\alpha (y+1)} dy, \ \forall x > 0,$$

where $C'_\alpha = [\Gamma(\alpha)\Gamma(1-\alpha)]^{-1}$.

We now define the Renewal Contact Process (RCP), denoted by $(\zeta_t)_{t\geq 0}$. Given a connected graph $G = (V, E)$, a cure index α as above, and an infection rate $\lambda > 0$, we construct the RCP on G graphically, *à la* Harris, as follows:

Let $\{T_n^x\}_{x\in V, n\in\mathbb{N}}$ be i.i.d. random variables with distribution μ as in (1.1), and let $\{X_n^e\}_{e\in E, n\in\mathbb{N}}$ be i.i.d. random variables with rate λ exponential distribution, independently of $\{T_n^x\}_{x\in V, n\in\mathbb{N}}$. It will be convenient later on to construct the underlying probability space $(\Omega, \mathcal{F}, P)$ as the product space $(\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$, with $\{T_n^x\}_{x\in V, n\in\mathbb{N}} \in (\Omega_1, \mathcal{F}_1, P_1)$ and $\{X_n^e\}_{e\in E, n\in\mathbb{N}} \in (\Omega_2, \mathcal{F}_2, P_2)$.

For $x \in V$, let R_x denote the renewal process with marks given by $\{S_n^x = T_1^x + \cdots + T_n^x;\ n \in \mathbb{N}\}$. In the rest of this chapter, R denotes any renewal process with the same distribution as R_x. Furthermore, for $e \in E$, R_e denotes the rate λ Poisson process given by $\{S_n^e = X_1^e + \cdots + X_n^e;\ n \in \mathbb{N}\}$. We will refer to R_x and R_e as the cure and infection processes and their arrivals as *cure marks* and *infection arrows* respectively. Throughout the text $E_x(\cdot)$, $C_x(\cdot)$, $E_e(\cdot)$ and $C_e(\cdot)$, denotes the *excess time* and *current time* of the process R_x, $x \in V$, and R_e, $e \in E$, respectively.

Given these processes, the RCP is constructed according to the usual recipe: if $s < t$ and $x, y \in V$, a path from (x, s) to (y, t) is a càdlàg function on $[s, t]$ for which there exist times $t_0 = s < t_1 < \cdots < t_n = t$ and $x_0 = x, x_1, \ldots, x_{n-1} = y$ in V such that it assumes x_i in $[t_i, t_{i+1})$, and

- $\langle x_i, x_{i+1}\rangle \in E,\ i = 0, \ldots, n-2;$
- $E_{\langle x_i, x_{i+1}\rangle}(t_i) = t_{i+1} - t_i,\ i = 0, \ldots, n-2;$

- $E_{x_i}(t_i) > t_{i+1} - t_i, \ i = 0, \dots, n-1.$

Informally, the first point ensures that the infection is transmitted only between neighboring vertices, the second that there is an infection arrow at every t_i and the third that there is no cure mark till the next transmission along the path. Figure 1.1 depicts a realization of the model.

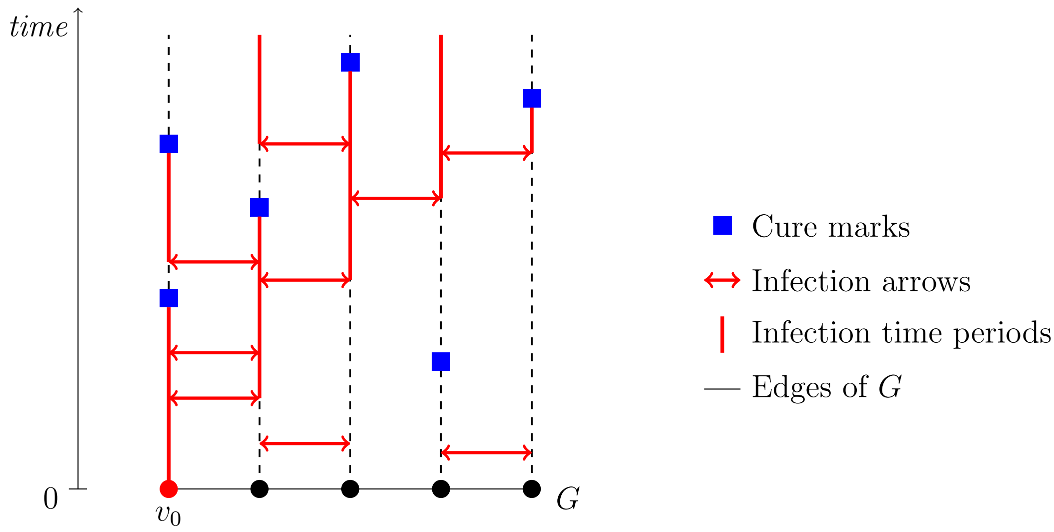

Figure 1.1: An illustration of the contact process

We define now, for each $t \geq 0$, the function of the state of the individuals, $\zeta_t : V \longrightarrow \{0, 1\}$. The model starts with a single infected individual, i.e., for some $v_0 \in V$, $\zeta_0(x) = 1$ iff $x = v_0$, and for $t > 0$, $\zeta_t(x) = 1$ iff there exists a path from $(v_0, 0)$ to (x, t). We say that the individual $x \in V$ is infected at time t, if $\zeta_t(x) = 1$, and healthy otherwise. We will abuse notation and denote the set of infected individuals at time t by $\zeta_t = \{x \in V; \ \zeta_t(x) = 1\}$.

The main result of this chapter is the following theorem.

Theorem 1.3. *Given $1/2 < \alpha < 1$, and any finite connected graph $G = (V, E)$, the RCP $(\zeta_t)_{t \geq 0}$ on G with cure index α is such that*

1. $P(\zeta_t \neq \emptyset, \forall t > 0) = 0,$ *if* $|V| < 2 + \frac{2\alpha - 1}{(1-\alpha)(2-\alpha)},$ $\forall \lambda > 0;$
2. $P(\zeta_t \neq \emptyset, \forall t > 0) > 0,$ *if* $|V| > \frac{1}{1-\alpha},$ $\forall \lambda > 0.$

Remark 1.3. *The two statements of Theorem 1.3 also hold when $0 < \alpha \leq 1/2$, except that in this case the first one is trivial, amounting to the claim that if V consists of a single point, then the process dies out for any λ (but, of course, there is no infection transmission in this case); and the second statement requires extra* *conditions on μ in order to allow us the use of the Strong Renewal Theorem, which is a key ingredient of our approach. See Remark 1.2 above. The second statement reads in this case: we have survival for $0 < \alpha < 1/2$ if $|V| \geq 2$, and for $\alpha = 1/2$ if $|V| \geq 3$.*

Remark 1.4.

1. *Note that the bounds in our theorem are quite sharp; indeed, writing $V_+(\alpha) = 1/(1-\alpha)$ and $V_-(\alpha) = 2+(2\alpha-1)/[(1-\alpha)(2-\alpha)]$, we have that $V_+ - V_- < 1$. Thus, if $[V_-, V_+] \cap \mathbb{N} = \emptyset$, then the model is well understood for every possible graph size $|V|$; otherwise, there is a single indeterminate case.*

2. *In terms of the notation introduced in the description of the chapter and in the above item, the bounds in Theorem 1.3 can be written as $V_-(\alpha) \leq k_c(\alpha)$ and $k_c(\alpha) \in \{\lfloor V_+(\alpha) \rfloor, \lfloor V_+(\alpha) \rfloor + 1\}$.*

1.2 Extinction

In this section we prove the first item of the Theorem 1.3. The idea consists in creating a sequence of disjoint random time intervals which, for the infection to survive, would be required to contain at least one mark of any of the cure processes. We then resort to a domination argument to show that we may find a subsequence of those intervals with bounded lengths, and the result readily follows from that.

Given $G = (V, E)$ and T as in Theorem 1.3, we start defining time intervals $(S_n, S_{n+1}]$. For this, recalling that $v_0 \in V$ is the single one initially infected individual, for each individual $x \in V$, let

$$X_{1,x} = \begin{cases} T_1^{v_0}, & \text{if} \quad x = v_0, \\ 0, & \text{if} \quad x \neq v_0. \end{cases}$$

Set $S_1 = X_1 = \max\{X_{1,x} \ ; \ x \in V\}$. Again, for each individual $x \in V$, let $W_{1,x} = X_1 - X_{1,x}$. And define, $x_1 = \arg\max\{X_{1,x} \ ; \ x \in V\}$.

Let us fix $t^* > 0$ as in Proposition 1.2 below. For a given $n \in \mathbb{N}$, we assume defined $X_{m,x}$, $W_{m,x}$, X_m, S_m, x_m, $m = 1, \ldots n$, $x \in V$, and set

$$X_{n+1,x} = \begin{cases} 0, & \text{if} \quad x = x_n, \\ E_x(S_n), & \text{if} \quad x \neq x_n \quad \text{and} \quad W_{n,x} \geq t^*, \\ E_x(S_n + t^*) + t^*, & \text{if} \quad x \neq x_n \quad \text{and} \quad W_{n,x} < t^*. \end{cases}$$

Analogously, we define $X_{n+1} = \max\{X_{n+1,x} \ ; \ x \in V\}$, $S_{n+1} = S_n + X_{n+1}$, for each individual $x \in V$, $W_{n+1,x} = X_{n+1} - X_{n+1,x}$, and also set $x_{n+1} =$

$\arg\max\{X_{n+1,x}\ ;\ x \in V\}$. Figure 1.2 below illustrates some of this random variables.

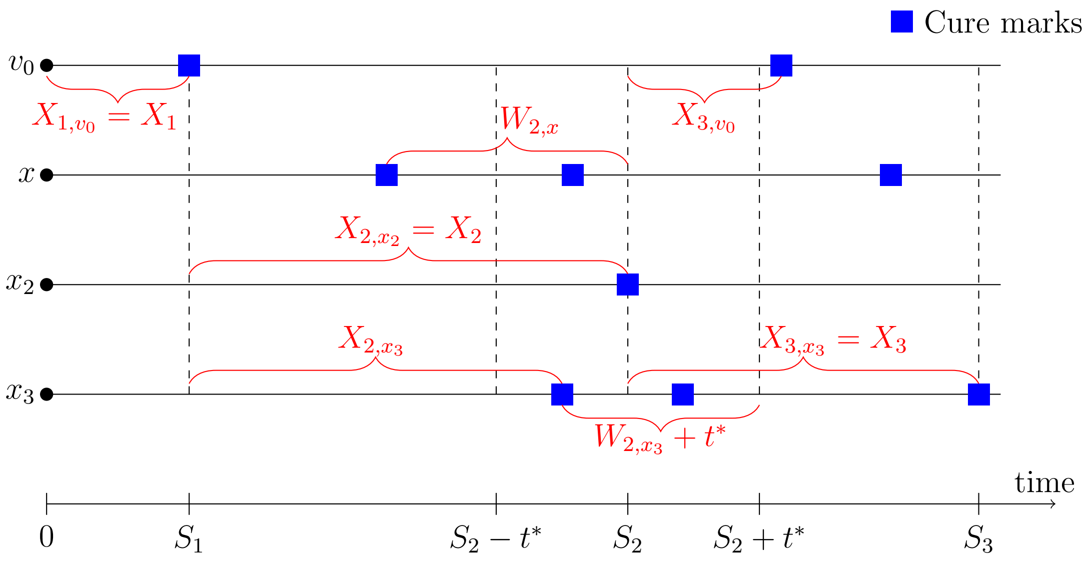

Figure 1.2: An illustration of the random variables $S_n, X_n, X_{n,x}$ and $W_{n,x}$.

The conditional distribution of $X_{n+1,x}$ on the past is given by

$$X_{n+1,x} \mid X_{m,y},\ 1 \le m \le n,\ y \in V$$
$$\sim \begin{cases} 0, & \text{if} \quad x = x_n, \\ E(W_{n,x}), & \text{if} \quad x \neq x_n \quad \text{and} \quad W_{n,x} \ge t^*, \\ E(W_{n,x} + t^*) + t^*, & \text{if} \quad x \neq x_n \quad \text{and} \quad W_{n,x} < t^*, \end{cases} \tag{1.3}$$

where, we recall, $E(\cdot)$ denotes the *excess time* of a renewal process R.

Observe that, by definition, in each interval $[S_n, S_{n+1}]$, every $x \in V$ has a cure mark, and thus, a necessary condition for the infection to survive is that in each one of these time intervals, there is at least one mark of some infection process R_e, $e \in E$. It readily follows that

$$P\left(\zeta_t \neq \emptyset, \forall t > 0 \;\middle|\; \left\{\lim_{n\to\infty} X_n = \infty\right\}^c\right) = 0. \tag{1.4}$$

We will show below that $P(\lim_{n\to\infty} X_n = \infty) = 0$ by resorting to a domination argument.

1.2.1 Domination

We will control the behavior of the random variables $(X_n)_{n\in\mathbb{N}}$ through Theorem 1.2 and two technical propositions, as follows.

Proposition 1.1. *Given $0 < \eta < 1$, there exists $t_\eta > 0$ such that*

$$P\left(\frac{E(t)}{t} > e^n\right) < \left(\frac{1+\eta}{e^\alpha}\right)^n, \ \forall n \in \mathbb{N}, \ \forall t > t_\eta.$$

Proof. We claim that there exists $t_\eta > 0$ such that

$$(1-\eta)^n < L(2^n t)/L(t) < (1+\eta)^n, \ \forall n \in \mathbb{N}, \ \forall t > t_\eta.$$

Indeed, since L is slowly-varying, we have that $\lim_{t\to\infty} L(2t)/L(t) = 1$; thus, there exists t_η where the claim is true for $n = 1$ and $t > t_\eta$. Let $s = 2^n t$, and write

$$\frac{L(2^{n+1}t)}{L(t)} = \frac{L(2^{n+1}t)}{L(2^n t)}\frac{L(2^n t)}{L(t)} = \frac{L(2s)}{L(s)}\frac{L(2^n t)}{L(t)}.$$

Since $s > t > t_\eta$, and supposing the claim is true for $t > t_\eta$ and a given $n \in \mathbb{N}$, then we have that the same is true for $n + 1$, and the claim follows by induction.

Fixing $t > 0$, and conditioning on the variable $C(t) = t - S_{N(t)}$, whose distribution function we denote by F_t, we have that

$$\begin{aligned}
P(E(t) > 2^n t) &= \int_0^t P(E(t) > 2^n t \mid C(t) = s)\, dF_t(s) \\
&= \int_0^t P(T > 2^n t + s \mid T > s)\, dF_t(s) \\
&= \int_0^t \frac{P(T > 2^n t + s)}{P(T > s)} dF_t(s) \\
&\le \int_0^t \frac{P(T > 2^n t)}{P(T > t)} dF_t(s) \\
&= \frac{L(2^n t)}{(2^n t)^\alpha} \div \frac{L(t)}{t^\alpha} \int_0^t dF_t(s) \\
&= \frac{L(2^n t)}{L(t)} \frac{1}{2^{\alpha n}}.
\end{aligned}$$

Hence, with the same t_η, the result follows directly from the claim above. ☐

Recalling the constant C'_α in the Theorem 1.2, let $M = |V| - 1$ and consider $Y_1, \ldots, Y_M$, independent random variables with common density

$$f(y) = \begin{cases} 0, & \text{if} \quad y \le 0, \\ \frac{C'_\alpha}{y^\alpha(1+y)}, & \text{if} \quad y > 0. \end{cases} \tag{1.5}$$

And let Y be the random variable

$$Y \equiv \max\{Y_i \ ; \ i = 1, \dots, M\}. \tag{1.6}$$

For technical reasons, that will be clear in the sequence, we claim that $\mathbb{E}[\log(Y)] < 0$. This is the content of the following lemma.

Lemma 1.1. *Let $1/2 < \alpha < 1$ and let Y be defined as in (1.6), if $M \in \mathbb{N}$ is such that*

$$M < 1 + \frac{2\alpha - 1}{(1-\alpha)(2-\alpha)},$$

then $\mathbb{E}[\log(Y)] < 0$.

Proof. Given $x > 0$, we have $P(Y \le x) = P(Y_1 \le x)^M$. Taking derivatives with respect to x, we get that Y has density $M{C'}_\alpha^{M-1} g(x)^{M-1} f(x)$, where $f(x)$ is given by (1.5) and

$$g(x) = \begin{cases} 0, & \text{if} \quad x \le 0, \\ \int_0^x \frac{1}{t^\alpha(1+t)} dt, & \text{if} \quad x > 0. \end{cases}$$

Hence, we have that

$$\mathbb{E}[\log(Y)] = M{C'}_\alpha^M \int_0^\infty \log(x) \frac{g(x)^{M-1}}{x^\alpha(x+1)} dx.$$

Making the change of variables $u = 1/x$, we get

$$\int_0^1 \log(x) \frac{g(x)^{M-1}}{x^\alpha(x+1)} dx = -\int_1^\infty \log(x) \frac{g(1/x)^{M-1}}{x^{1-\alpha}(x+1)} dx.$$

It readily follows that

$$\mathbb{E}[\log(Y)] = M{C'}_\alpha^M \int_1^\infty \frac{\log(x)}{x+1} \left[\frac{g(x)^{M-1}}{x^\alpha} - \frac{g(1/x)^{M-1}}{x^{1-\alpha}} \right] dx.$$

It is sufficient to show that the term in brackets is negative whenever $x > 1$. Note that if $M = 1$, this is equivalent to the obvious inequality $x^\alpha > x^{1-\alpha}$, $\forall x > 1$; otherwise, is equivalent to

$$\frac{g(x)}{g(1/x)} < x^{\frac{2\alpha-1}{M-1}}, \ \forall x > 1.$$

For simplicity, let $\beta = 1 - \alpha$. It follows from the hypothesis that, for $x > 1$,

$$x^{\frac{2\alpha-1}{M-1}} \ge x^{(1-\alpha)(2-\alpha)} = x^{\beta(\beta+1)}.$$

We define the auxiliary function $G : \mathbb{R} \to \mathbb{R}$, given by $G(x) = g(x) - x^{\beta(\beta+1)} g(1/x)$. Thus, we have that $G(1) = 0$ and its derivative is

$$G'(x) = \frac{1 + x^{\alpha+\beta^2} - \beta(\beta+1)g(1/x)x^{\beta^2}(1+x)}{x^{\alpha}(1+x)}. \tag{1.7}$$

Observe that, since for $0 < t \leq 1$ we have $1/(1+t) > 1 - t$, then, for all $x \geq 1$,

$$\begin{aligned} g(1/x) = \int_0^{1/x} \frac{1}{t^{\alpha}(1+t)} dt \quad &> \quad \frac{1}{\beta x^{\beta}} - \frac{1}{(\beta+1)x^{\beta+1}} \\ &= \quad \frac{x(\beta+1) - \beta}{x^{\beta+1}\beta(\beta+1)}. \end{aligned}$$

Hence $\beta(\beta+1)g(1/x) > [x(\beta+1) - \beta]/x^{\beta+1}$. To conclude that $G'(x) < 0$ for all $x > 1$, as stated in (1.7), it is enough to show that

$$\frac{x(\beta+1) - \beta}{x^{\beta+1}} > \frac{1 + x^{\alpha+\beta^2}}{x^{\beta^2}(1+x)}, \quad \forall x > 1.$$

Or equivalently,

$$\begin{aligned} & x^{\beta^2}(1+x)x(\beta+1) - x^{\beta^2}(1+x)\beta - x^{\beta+1}(1 + x^{\alpha+\beta^2}) \\ &\quad = x^{\beta^2}[(1+\beta)(x^2+x) - \beta(1+x) - (x^2 + x^{1+\beta-\beta^2})] \\ &\quad = x^{\beta^2}[x^2 + x + \beta(1+x)(x-1) - (x^2 + x^{1+\alpha\beta})] \\ &\quad = x^{\beta^2}[x + \beta(x+1)(x-1) - x^{1+\alpha\beta}] > 0, \ \forall x > 1, \end{aligned}$$

Since the last inequality is true, we have $G'(x) < 0$ for all $x > 1$, and since $G(1) = 0$, we conclude that $G(x) < 0$, for all $x > 1$. The proof is finished. $\square$

Note that, $\mathbb{E}[Y^t] < \infty \Leftrightarrow \mathbb{E}[Y_1^t] < \infty \Leftrightarrow t \in (-(1-\alpha), \alpha)$. Let $\Phi : (-(1-\alpha), \alpha) \to \mathbb{R}$ be defined by

$$\Phi(t) = \mathbb{E}[e^{t \log(Y)}] = \mathbb{E}[Y^t].$$

We observe that Φ is differentiable at 0, with $\Phi'(0) = \mathbb{E}[\log(Y)] < 0$ and $\Phi(0) = 1$. Hence, there exists $0 < \theta < \alpha$, with $\Phi(\theta) < 1$, that is $\mathbb{E}[Y^{\theta}] < 1$.

Let $N \in \mathbb{N}$ be such that $\log_2(N) \in \mathbb{N}$, and consider $a_j = j/N$, if $j = 0, \ldots, N^2$, and $a_j = N2^{(j-N^2)}$, if $j > N^2$. For each $j \in \mathbb{N}$, let $I_j = (a_{j-1}, a_j]$, and consider the following truncation of Y:

$$\overline{Y}_N = \sum_{j=1}^{N^2} a_j \mathbb{1}_{\{Y \in I_j\}}.$$

For given μ such that $\mathbb{E}[Y^\theta] < \mu < 1$, it follows by dominated convergence that, for N sufficiently large,

$$\mathbb{E}[(\overline{Y}_N)^\theta] < \mu < 1. \tag{1.8}$$

Let $a = (1+\eta)/2^\alpha$. From now on, we fix $0 < \eta < 1$ so that $a2^\theta < 1$. Given $\rho > 0$, for each $j \in \mathbb{N}$, we define

$$p_{N,\rho,j} = \begin{cases} P(Y \in I_j) + \rho, & \text{if} \quad j \leq N^2, \\ M a^{\log_2(N)+j-N^2-2}, & \text{if} \quad j > N^2. \end{cases} \tag{1.9}$$

We also define $C_{N,\rho} = \sum_{j\geq 1} p_{N,\rho,j}$. Recalling that (1.8) holds for N sufficiently large, and since $a2^\theta < 1$, then N and ρ can be chosen in such way that the following inequality is true

$$\begin{aligned} &\frac{1}{C_{N,\rho}} \left[\sum_{j=1}^{N^2} \left[a_j^\theta \left(P(Y \in I_j) + \rho \right) \right] + \sum_{j>N^2} a_j^\theta M a^{\log_2(N)+j-N^2-2} \right] \\ = \; &\frac{1}{C_{N,\rho}} \left[\mathbb{E}\left[(\overline{Y}_N)^\theta \right] + \rho \sum_{j=1}^{N^2} a_j^\theta + M \sum_{n \geq \log_2(N)-1} 2^{\theta(n+2)} a^n \right] \\ \leq \; &\frac{\mu}{C_{N,\rho}}. \end{aligned} \tag{1.10}$$

In the following, N and ρ are fixed and satisfy Inequality (1.10). In this case, we denote $C_{N,\rho}$ simply by C.

We define an auxiliary probability space, $([0,\infty), \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \sigma(I_j \; ; \; j \in \mathbb{N})$ and for each $j \in \mathbb{N}$, $\mathbb{P}(I_j) = p_j$, where

$$p_j = \frac{p_{N,\rho,j}}{C}. \tag{1.11}$$

Let $\tilde{Y} : [0,\infty) \longrightarrow (0,\infty)$, be a random variable in this space given by

$$\tilde{Y} = \sum_{j\geq 1} a_j \mathbb{1}_{I_j}. \tag{1.12}$$

It follows directly from (1.9), (1.10) and (1.11), that $\tilde{Y}$ satisfies $\mathbb{E}[\tilde{Y}^\theta] < \mu/C$.

We now apply Theorem 1.2 and Proposition 1.1 to establish our second technical proposition.

Proposition 1.2. *There exist $t^* > 0$ and $1/2 > \delta > 0$ such that, for each $t_1, \ldots, t_M > t^*$, whenever $V_1, \ldots, V_M$ are independent random variables with marginal distributions such that for $i = 1, \ldots, M$*

$$\textit{either} \quad V_i \sim \frac{E(t_i)}{t_i} \quad \textit{or} \quad V_i \sim \frac{E(t_i)}{t_i} + \delta,$$

and $V \equiv \max\{V_i \; ; \; i = 1, \ldots, M\}$, then $P(V \in I_j) < Cp_j$, $\forall j \in \mathbb{N}$.

Proof. Let $\{R_i\}_{1\le i\le M}$ be independent renewal processes with the same distribution as R and let $\{E_i(\cdot)\}_{1\le i\le M}$ denote their respective *excess time*.

If $1 \le j \le N^2$, then we use Theorem 1.2 to obtain

$$\lim_{t_1,\dots t_M\to\infty} P\left(\max_{1\le i\le M}\frac{E_i(t_i)}{t_i}\in I_j\right) = P(Y\in I_j) < P(Y\in I_j)+\rho = Cp_j.$$

Using the continuity of the limiting distribution of $E(t)/t$ as $t\to\infty$, it follows that, for $t_1,\dots,t_M$ large enough and δ small enough, $P(V\in I_j) < Cp_j$, $\forall j\le N^2$.

Recalling that $a_j = 2^{\log_2(N)+j-N^2}$ for all $j\ge N^2$ and that $a=(1+\eta)/2^\alpha$, it follows from Proposition 1.1 that if $t>t_\eta$, then for all $j>N^2$ we have that

$$P\left(\frac{E(t)}{t} > a_{j-2}\right) < a^{\log_2(N)+j-N^2-2}.$$

Observe that for all $j>N^2$ and δ small enough, we have for all possible cases of the marginal distributions of V_i, $i=1,\dots,M$ that

$$\begin{aligned} P(V\in I_j) &\le P(V_i > a_{j-1} \text{ for some } i=1,\dots,M) \\ &\le P\left(\frac{E_i(t_i)}{t_i} > a_{j-2} \text{ for some } i=1,\dots,M\right) \\ &\le Ma^{\log_2(N)+j-N^2-2} = Cp_j. \end{aligned}$$

□

In the next proposition, we finally obtain the above mentioned domination. For this, let $\{\tilde Y_m\}_{m\in\mathbb{N}}$ be i.i.d. random variables with the same distribution as $\tilde Y$ in (1.12).

Proposition 1.3. *Let $\tilde t = t^*/\delta$. Then, for every $n_0, m\in\mathbb{N}$,*

$$P\left(X_{n_0} > \tilde t,\ X_{n_0+1}\ge X_{n_0},\ \dots,\ X_{n_0+m}\ge X_{n_0}\right) \le C^m \mathbb{P}\left(\prod_{l=1}^m \tilde Y_l \ge 1\right).$$

Proof. For each $n\in\mathbb{N}$ and $x\in V$, we define

$$Z_{n+1,x} = \begin{cases} \dfrac{X_{n+1,x}}{W_{n,x}}, & \text{if} \quad W_{n,x}\ge t^*, \\ \dfrac{X_{n+1,x}-t^*}{W_{n,x}+t^*}+\dfrac{t^*}{\tilde t}, & \text{if} \quad W_{n,x} < t^*. \end{cases} \tag{1.13}$$

We set $Z_{n+1} = \max\{Z_{n+1,x}\ ;\ x \in V\}$. Since $\delta < 1/2$ — see Proposition 1.2 —, we have $2t^* < \tilde{t}$. Notice also that for each $x \in V$, we have $W_{n,x} \leq X_n$. Therefore, if $X_n > \tilde{t}$ then

$$Z_{n+1,x} = \begin{cases} \dfrac{X_{n+1,x}}{W_{n,x}} \geq \dfrac{X_{n+1,x}}{X_n}, & \text{if} \quad W_{n,x} \geq t^*, \\ \dfrac{X_{n+1,x} - t^*}{W_{n,x} + t^*} + \dfrac{t^*}{\tilde{t}} \geq \dfrac{X_{n+1,x} - t^*}{\tilde{t}} + \dfrac{t^*}{\tilde{t}} \geq \dfrac{X_{n+1,x}}{X_n}, & \text{if} \quad W_{n,x} < t^*. \end{cases}$$

Therefore, $Z_{n+1} \geq X_{n+1}/X_n$ whenever $X_n > \tilde{t}$. From whence we get that

$$\begin{aligned} &P\left(X_{n_0} > \tilde{t},\ X_{n_0+1} \geq X_{n_0},\ \ldots,\ X_{n_0+m} \geq X_{n_0}\right) \\ &\qquad = P\left(X_{n_0} > \tilde{t},\ \frac{X_{n_0+1}}{X_{n_0}} \geq 1,\ \ldots,\ \prod_{l=1}^{m} \frac{X_{n_0+l}}{X_{n_0+l-1}} \geq 1\right) \\ &\qquad\qquad \leq P\left(X_{n_0} > \tilde{t},\ Z_{n_0+1} \geq 1,\ \ldots,\ \prod_{l=1}^{m} Z_{n_0+l} \geq 1\right). \end{aligned} \tag{1.14}$$

Consider now the set $\Lambda = \left\{\gamma := (j_1, \ldots, j_m)\ ;\ \prod_{i=1}^{l} a_{j_i} \geq 1,\ \forall 1 \leq l \leq m\right\}$. We have that the last expression in (1.14) satisfies

$$\begin{aligned} &P\left(X_{n_0} > \tilde{t},\ Z_{n_0+1} \geq 1,\ \ldots,\ \prod_{l=1}^{m} Z_{n_0+m} \geq 1\right) \\ &\qquad \leq \sum_{\gamma \in \Lambda} P\left(X_{n_0} > \tilde{t},\ Z_{n_0+1} \in I_{j_1},\ \ldots,\ Z_{n_0+m} \in I_{j_m}\right) \\ &\qquad\qquad = \sum_{\gamma \in \Lambda} \left[\prod_{l=1}^{m} P\left(Z_{n_0+l} \in I_{j_l} \mid A_l\right)\right], \end{aligned} \tag{1.15}$$

where $A_l = \{X_{n_0} > \tilde{t},\ Z_{n_0+1} \in I_{j_1},\ \ldots,\ Z_{n_0+l-1} \in I_{j_{l-1}}\}$.

To simplify the notation, we define the random vector $\xi : \Omega \to \mathbb{R}^{(n_0+l-1)|V|}$, denoted by, $\xi = (\xi_{k,x})_{\{1 \leq k \leq n_0+l-1,\ x \in V\}}$, where $\xi_{k,x}(\omega) = X_{k,x}(\omega)$. Notice that A_l is measurable in the σ-algebra generated by ξ. Let ψ denote the deterministic function associating ξ to $(X_{n_0},\ Z_{n_0+1},\ \ldots,\ Z_{n_0+l-1})$, and make $\mathcal{R}_l = (\tilde{t}, \infty) \times I_{j_1} \times \ldots \times I_{j_{l-1}}$. Let $\tilde{F}$ denote the distribution function of ξ. Thus,

$$\begin{aligned} P\left(\{Z_{n_0+l} \in I_{j_l}\} \cap A_l\right) &= \int_{\psi^{-1}(\mathcal{R}_l)} P\left(Z_{n_0+l} \in I_{j_l} \mid \xi = y\right) d\tilde{F}(y) \\ &= \int_{\psi^{-1}(\mathcal{R}_l)} P\left(\max_{x \in V} Z_{n_0+l,x} \in I_{j_l} \,\middle|\, \xi = y\right) d\tilde{F}(y) \end{aligned} \tag{1.16}$$

Using the Markov Property described in (1.3) and the definition of Z_{n+1} given in (1.13), we obtain

$$P\left(\max_{x\in V} Z_{n_0+l,x} \in I_{j_l} \,\middle|\, \xi = y\right) = P\Big(\max\left\{V_x(y)\ ;\ x\in V,\ x\neq x_{n_0+l-1}\right\}\in I_{j_l}\Big), \tag{1.17}$$

where $V_x(y)$ are random variables with distribution

$$V_x(y) \stackrel{D}{=} \begin{cases} \dfrac{E\big(W_{n_0+l-1,x}(y)\big)}{W_{n_0+l-1,x}(y)}, & \text{if} \quad W_{n_0+l-1,x}(y)\geq t^*, \\ \dfrac{E\big(W_{n_0+l-1,x}(y)+t^*\big)}{W_{n_0+l-1,x}(y)+t^*}+\dfrac{t^*}{\tilde{t}}, & \text{if} \quad W_{n_0+l-1,x}(y)< t^*. \end{cases}$$

where $E(t)$ denotes the excess time of renewal process with distribution μ. Recalling that $\delta = t^*/\tilde{t}$, note that the variables $V_x(y)$, $x\in V\setminus\{x_{n_0+l-1}\}$, satisfy the conditions of Proposition 1.2 with $M=|V|-1$. Hence, for all $y\in\psi^{-1}(\mathcal{R}_l)$, we have

$$P\Big(\max\left\{V_x(y)\ ;\ x\in V,\ x\neq x_{n_0+l-1}\right\}\in I_{j_l}\Big) < Cp_{j_l}.$$

Replacing this in (1.16) and (1.17), we get

$$P\left(\{Z_{n_0+l}\in I_{j_l}\}\cap A_l\right) \leq \int_{\psi^{-1}(\mathcal{R}_l)} Cp_{j_l}\, d\tilde{F}(y) = Cp_{j_l} P(\xi\in\psi^{-1}(\mathcal{R}_l)) = Cp_{j_l}P(A_l).$$

Thus, (1.14) and (1.15) yield

$$\begin{aligned} P\left(X_{n_0}>\tilde{t},\ X_{n_0+1}\geq X_{n_0},\ \dots,\ X_{n_0+m}\geq X_{n_0}\right) & \\ \leq \sum_{\gamma\in\Lambda}\left[\prod_{l=1}^{m} P\left(Z_{n_0+l}\in I_{j_l}\mid A_l\right)\right] & \\ \leq \sum_{\gamma\in\Lambda}\left(\prod_{l=1}^{m} Cp_{j_l}\right) = C^m\sum_{\gamma\in\Lambda}\left(\prod_{l=1}^{m} p_{j_l}\right). & \end{aligned} \tag{1.18}$$

Recalling the definition of $\tilde{Y}$ in (1.12), since $\{\tilde{Y}_i\}_{i\in\mathbb{N}}$ are i.i.d. with same distribution as $\tilde{Y}$, we have that

$$\sum_{\gamma\in\Lambda}\left(\prod_{l=1}^{m} p_{j_l}\right) = \sum_{\gamma\in\Lambda}\mathbb{P}\left(\tilde{Y}_1 = a_{j_1},\dots,\tilde{Y}_m = a_{j_m}\right) \leq \mathbb{P}\left(\prod_{l=1}^{m}\tilde{Y}_l\geq 1\right), \tag{1.19}$$

and (1.18) and (1.19) yield the proof. □

1.2.2 Proof of Item (1) of Theorem 1.3

Since $\mathbb{E}[\tilde{Y}^\theta] \leq \mu/C$ — see paragraph of (1.12) — , we have

$$\mathbb{P}\left(\prod_{l=1}^{m} \tilde{Y}_l \geq 1\right) = \mathbb{P}\left(\prod_{l=1}^{m} \tilde{Y}_l^\theta \geq 1\right) \leq \mathbb{E}\left[\tilde{Y}^\theta\right]^m \leq \left(\frac{\mu}{C}\right)^m.$$

Therefore, recalling that $\mu < 1$, it follows from the Proposition 1.3 that

$$\begin{aligned} P\left(X_{n_0} > \tilde{t},\ X_{n_0+l} \geq X_{n_0},\ \forall l \in \mathbb{N}\right) \\ = \lim_{m\to\infty} P\left(X_{n_0} > \tilde{t},\ X_{n_0+1} \geq X_{n_0},\ \ldots,\ X_{n_0+m} \geq X_{n_0}\right) \\ \leq \lim_{m\to\infty} \mu^m = 0. \end{aligned}$$

Hence,

$$P\left(\lim_{n\to\infty} X_n = \infty\right) \leq P\left(\bigcup_{n_0 \geq 1} \left\{X_{n_0} > \tilde{t},\ X_{n_0+l} \geq X_{n_0},\ \forall l \in \mathbb{N}\right\}\right) = 0.$$

It follows, as noted above — see paragraph of (1.4) —, that $P(\zeta_t \neq \emptyset, \forall t > 0) = 0$ for every $\lambda > 0$.

1.3 Survival

In this section we prove the second item of the Theorem 1.3. The idea of the proof is to show that there exists a sequence of polynomially increasing time intervals, such that, with positive probability the following events take place: in each such interval, there exists an individual free of cure marks; each interval intersects the next, and in this intersection there exists a sub-polynomially sized interval where all individuals get infected. So if there exists a single infected individual at the beginning of the sequence, and the above events occur, then the infection survives forever.

Given the graph $G = (V, E)$ and a fixed α, let T denote a random variable with distribution μ. Once we have fixed the infection rate $\lambda > 0$, we start by choosing two constants as functions of λ and G that will be used in this section. Since $|V| > 1/(1-\alpha)$ we can choose $\epsilon > 0$ in such way that $\beta := |V|(1-\alpha-3\epsilon) > 1$ and since the graph $G = (V, E)$ is connected, there exists a *spanning cycle* $\tau = (e_1, e_2, \cdots, e_l)$, whose size l, by Euler's Theorem, is such that $l \leq 2|V|$ and then choose $\gamma > \max\{1, l/\lambda\}$. From now on, ϵ and γ are fixed.

With the objective to estimate the probability of existence of intervals without marks of the renewal process R, we derive the following corollary of Theorem 1.1.

Proposition 1.4. *There exists* $\hat{t}_1 > 0$ *such that for all* $t > \hat{t}_1$

$$P(E(t) \leq 1) \leq \frac{1}{t^{1-\alpha-\epsilon}}.$$

Proof. First consider μ non-arithmetic. Note that, given $t > 0$, $U(t+1) - U(t) = \sum_{n\geq 1} P(S_n \in (t, t+1])$, and let $M_t = |\{n \geq 1 \ ; \ S_n \in (t, t+1]\}|$ be the number of renewal marks of R in the interval $(t, t+1]$; then we have $U(t+1) - U(t) = \mathbb{E}(M_t)$. So, $P(E(t) \leq 1) = P(M_t \geq 1) \leq \mathbb{E}(M_t) = U(t+1) - U(t)$. Since $L(\cdot)$ is slowly varying, we find t_0 such that $L(t) \geq t^{-\epsilon/2}$ for all $t > t_0$. Thus, making $h = 1$ in Theorem 1.1, we get that

$$U(t+1) - U(t) \sim \frac{C_\alpha}{\int_0^t L(x) x^{-\alpha} dx} \leq \frac{C_\alpha}{\int_0^{t_0} L(x) x^{-\alpha} dx + \int_{t_0}^t 1/x^{\alpha + \frac{\epsilon}{2}} dx},$$

and thus may conclude that the left hand side is bounded above by $1/t^{1-\alpha-\epsilon}$ for all t sufficiently large.

If μ is arithmetic, we can assume, without loss of generality that its arithmetic span is equal to 1; in this case, Theorem 1.1 of [GL62] implies that, as t goes to infinity, $P(E(t) \leq 1)$ is also bounded above by $1/t^{1-\alpha-\epsilon}$, and the result follows. □

Noticing that if $E(t) \in (s, s+1]$, then necessarily $E(t+s) \leq 1$, we have the following corollary to the above proposition.

Corollary 1.1. *For all* $m \in \mathbb{N}$ *and for all* $t > \hat{t}_1$*, we have* $P(E(t) \leq m) \leq m/t^{1-\alpha-\epsilon}$.

Proof. It is enough to observe that

$$\begin{aligned} P(E(t) \leq m) &= \sum_{i=0}^{m-1} P(i < E(t) \leq i+1) \\ &\leq \sum_{i=0}^{m-1} P(E(t+i) \leq 1) \\ &\leq \sum_{i=0}^{m-1} \frac{1}{(t+i)^{1-\alpha-\epsilon}} \leq \frac{m}{t^{1-\alpha-\epsilon}}. \end{aligned}$$

□

We will use Corollary 1.1 to show that, with high probability, certain intervals with polynomially growing sizes are free of cure marks. For each $n \in \mathbb{N}$, let

$$b_n = \gamma \log(n) \quad \text{and} \quad c_n = \lceil b_n^{|V|(\alpha+\epsilon)+1} \rceil.$$

It follows that there exists n_0, such that $c_n b_n < n^\epsilon/2$, $\forall n \geq n_0$. Then, for each $n \geq n_0$, we define

$$t_n = \hat{t}_1 + \sum_{j=n_0}^{n} [j^\epsilon - c_j b_j].$$

It follows that $t_n \geq \sum_{j=n_0}^{n} j^\epsilon/2$, hence, for all n large enough we have $t_n > n$.

Consider now the event

$$A_n = \{\exists x \in V;\ E_x(t_n) > (n+1)^\epsilon\}. \tag{1.20}$$

Note that the interval $(t_n, t_n + (n+1)^\epsilon)$ intersects the interval $(t_{n+1}, t_{n+1} + (n+2)^\epsilon)$ and the length of this intersection is $c_{n+1}b_{n+1}$. In the proof, the intervals $(t_n, t_n + (n+1)^\epsilon)$ will be the intervals in which there is at least one vertex without cure mark, and in one of the c_{n+1} intervals of length b_{n+1} in the intersection all individuals will get infected.

The next proposition gives a lower bound for the probability of occurrence of the event A_n.

Proposition 1.5. *There exists $n_1 \in \mathbb{N}$, such that, for $n > n_1$, we have $P(A_n^c) \leq 1/n^\beta$, where $\beta = |V|(1 - \alpha - 3\epsilon) > 1$.*

Proof. Let us take n large enough so that $t_n > \max\{n, \hat{t}_1\}$. Then we can apply Corollary 1.1 to get

$$\begin{aligned} P(A_n^c) = P(E_x(t_n) \leq (n+1)^\epsilon, \forall x \in V) &\leq \left(\frac{\lceil (n+1)^\epsilon \rceil}{t_n^{1-\alpha-\epsilon}}\right)^{|V|} \\ &\leq \left(\frac{n^{2\epsilon}}{n^{(1-\alpha-\epsilon)}}\right)^{|V|} \\ &= \frac{1}{n^{|V|(1-\alpha-3\epsilon)}} \\ &= \frac{1}{n^\beta}. \end{aligned}$$

□

The next step is to show that, with high probability, at least one of the following c_n intervals with size b_n, is free of all cure marks R_x, $x \in V$. We begin with the following lemma:

Lemma 1.2. *There exists $\hat{t}_2 > 0$, such that, if $t > \hat{t}_2$, then, for all $s > 0$, we have $P(T > s + t | T > s) \geq 1/t^{\alpha+\epsilon}$.*

Proof. We start with the case $s \leq t$, where there exists t^* such that

$$P(T > t + s | T > s) \geq P(T > t + s) \geq P(T > 2t) = \frac{L(2t)}{(2t)^\alpha} \geq \frac{1}{t^{\alpha+\epsilon}},$$

for all $t > t^*$. For the other case, namely $s > t$, we have that

$$P(T > t + s | T > s) = \frac{P(T > t + s)}{P(T > s)} \geq \frac{P(T > 2s)}{P(T > s)} = \frac{L(2s)}{L(s)} \left(\frac{1}{2}\right)^\alpha.$$

Since $L(\cdot)$ is slowly varying, $L(2s)/L(s) \to 1$ as $s \to \infty$. It follows that there exists s^* such that, if $s > t > s^*$, then

$$P(T > t + s | T > s) \geq \frac{L(2s)}{L(s)} \left(\frac{1}{2}\right)^\alpha \geq \frac{1}{(s^*)^{\alpha+\epsilon}} > \frac{1}{t^{\alpha+\epsilon}}.$$

To conclude the proof, take $\hat{t}_2 = \max\{t^*, s^*\}$. □

Let $t_0 > 0$ be fixed, and consider the sub-σ algebra $\mathcal{F}_{t_0}$ of the underlying σ algebra of the model consisting of renewal events taking place up to time t_0. We have the following lemma.

Lemma 1.3. *Given $t_0 > 0$, then, for all $t > \hat{t}_2$ and all $x \in V$, almost surely*

$$P\left(E_x(t_0) > t \;\middle|\; \mathcal{F}_{t_0}\right) \geq \frac{1}{t^{\alpha+\epsilon}}.$$

Proof. Almost surely

$$\begin{aligned} P\left(E_x(t_0) > t \;\middle|\; \mathcal{F}_{t_0}\right)(\omega) &= P\left(T > t + \left(t_0 - S^x_{N(t_0)}(\omega)\right) \;\middle|\; T > t_0 - S^x_{N(t_0)}(\omega)\right) \\ &\geq \frac{1}{t^{\alpha+\epsilon}}, \end{aligned}$$

where we used Lemma 1.2 in the last passage. □

For $n > n_0$, we define $B_n = \{\exists j \in [0, c_n) \cap \mathbb{Z};\ E_x(t_n + jb_n) > b_n, \forall x \in V\}$. Observe that, on the occurrence of B_n it is assured that at least one of the c_n intervals of size b_n has no cure marks. Using the lemma above, we get an upper bound for the probability of B_n.

Proposition 1.6. *Let $n_2 = \inf\{n > n_0;\ b_n > \hat{t}_2\}$. If $n > n_2$, then $P(B_n^c) \leq 1/n^\gamma$.*

Proof. Let $C_{n,j} = \{\exists x \in V;\ E_x(t_n + jb_n) \le b_n\}$. Then we have

$$\begin{aligned} P(B_n^c) &= P(C_{n,j} \text{ occurs } \forall j = 0, \ldots, c_n - 1) \\ &= \prod_{j=0}^{c_n-1} P\left(C_{n,j} \,\middle|\, \bigcap_{i=0}^{j-1} C_{n,i}\right) \\ &= \prod_{j=0}^{c_n-1} \left[1 - P\left(E_x(t_n + jb_n) > b_n, \forall x \in V \,\middle|\, \bigcap_{i=0}^{j-1} C_{n,i}\right)\right]. \end{aligned}$$

Since the events $C_{n,i}$, where $0 \le i < j$, occur before $t_n + jb_n$, using the Lemma 1.3, we have $P(E(t_n + jb_n) > b_n | \cap_{i=0}^{j-1} C_{n,i}) \ge 1/b_n^{\alpha+\epsilon}$. Hence,

$$\begin{aligned} P(B_n^c) &= \prod_{j=0}^{c_n-1} \left[1 - P\left(E(t_n + jb_n) > b_n \,\middle|\, \bigcap_{i=0}^{j-1} C_{n,i}\right)^{|V|}\right] \\ &\le \prod_{j=0}^{c_n-1} \left(1 - \frac{1}{b_n^{|V|(\alpha+\epsilon)}}\right) \\ &= \left(1 - \frac{1}{b_n^{|V|(\alpha+\epsilon)}}\right)^{c_n} \le e^{-c_n/b_n^{|V|(\alpha+\epsilon)}} \le e^{-b_n} = \frac{1}{n^\gamma}. \end{aligned}$$

□

Recall the definition of a spanning cycle $\tau = (e_1, \ldots, e_l)$. Given $t > 0$, an *infection stairway* at time t is a sequence of random variables defined as:

$$Y_i^t = \begin{cases} t, & \text{if } i = 0 \\ Y_{i-1}^t + E_{e_i}(Y_{i-1}^t), & \text{if } 1 \le i \le l. \end{cases}$$

Observe that, since E_{e_i} corresponds to infection arrows along edge e_i and since for every pair $(x, y) \in V^2$, τ has a sub-path starting at x and ending at y, if at time t there is at least one infected individual in V, then, whenever $E_x(t) > Y_l^t - t$ for all $x \in V$, we will have that all individuals are infected at time Y_l^t.

We use the memoryless property of the exponential distribution to show that, with positive probability, in each one of the c_n intervals, we have the occurrence of an infection stairway.

Proposition 1.7. *Given $m > n_2 \in \mathbb{N}$, the event*

$$C_m := \bigcap_{n \ge m} \bigcap_{j=0}^{c_n-1} \left\{Y_l^{t_n+jb_n} - (t_n + jb_n) \le b_n\right\} \tag{1.21}$$

is such that $P(C_m) > 0$.

Proof. Observe that, for all t, the random variables $Y_i^t - Y_{i-1}^t$, $i = 1, 2, \ldots, l$, are i.i.d. exponentially distributed with rate λ. Hence,

$$P\left(Y_l^t - t \le b_n\right) = P\left(\sum_{i=1}^{l}(Y_i^t - Y_{i-1}^t) \le b_n\right) \ \ge \ P\left(\max_{1\le i\le l}(Y_i^t - Y_{i-1}^t) \le \frac{b_n}{l}\right)$$
$$= \left(1 - e^{\frac{-\lambda b_n}{l}}\right)^l.$$

It readily follows that

$$P\left(\bigcap_{n\ge m}\bigcap_{j=0}^{c_n-1}\left\{Y_l^{t_n+jb_n} - (t_n + jb_n) \le b_n\right\}\right) \ge \prod_{n\ge m}\left(1 - e^{\frac{-\lambda b_n}{l}}\right)^{lc_n},$$

and since $b_n = \gamma\log(n)$, taking logarithms, we obtain that, for some constant $c > 0$,

$$\log\left(\prod_{n\ge m}\left(1 - e^{\frac{-\lambda b_n}{l}}\right)^{lc_n}\right) \ > \ -cl\sum_{n\ge m} c_n e^{\frac{-\lambda b_n}{l}}$$
$$= \ -cl\sum_{n\ge m} c_n n^{-\frac{\gamma\lambda}{l}}.$$

Finally, since γ was chosen in such way that $\gamma\lambda > l$, and $c_n = \lceil (b_n)^{|V|(\alpha+\epsilon)+1}\rceil$, the latter sum in convergent and thus, the product above is positive. □

1.3.1 Proof of Item (2) of Theorem 1.3

Using the propositions above we can conclude the proof of Theorem 1.3 (2). Let us start with some definitions. For each $t > 0$, we say that a configuration $\omega \in \Omega$ is $t-$ bad, if there exist $s \ge t$ and $\{n_x \in \mathbb{N}, x \in V\}$, such that $S_{n_x}^x = s$, for all $x \in V$. This means that there is an instant in $[t, \infty)$ when every cure process R_x simultaneously has an arrival. We say that ω is bad if it is $0-$ bad, and is good otherwise.

Let $n_3 = n_1 \vee n_2$, where n_1 and n_2 are given in Proposition 1.5 and Proposition 1.6, respectively. Recall the event A_n defined in (1.20); given $m > n_3 \in \mathbb{N}$, we define $\tilde{A}_m = \cap_{n\ge m}A_n$; in words, $\tilde{A}_m$ is the event that there is at least one vertex without a cure mark in each of the intervals $(t_n; t_n + (n+1)^\epsilon)$, $n \ge m$. As $\cup_{n\ge m}(t_n, t_n+(n+1)^\epsilon) = (t_m; \infty)$, we have that $\{\omega \text{ is t-bad } \forall t > 0\}\cap\tilde{A}_m = \emptyset$. Now, since $\epsilon > 0$ was chosen in such way that $\beta = |V|(1-\alpha-3\epsilon) > 1$, it follows from Proposition 1.5 and the union bound that $P(\tilde{A}_m^c) \to 0$ as m goes to infinity. Let T be a random time such that $S_{n_x}^x = T$ for all x. Since $\{\omega \text{ is t-bad}, \forall t > 0\}\cap\tilde{A}_m = \emptyset$, if we suppose that $P(\omega \text{ is bad}) = 1$, then by the strong Markov property applied when the RCP hits T, we have that $P(\omega \text{ is t-bad}, \forall t > 0) = 1$, which in turn implies that $P(\tilde{A}_m) = 0$, in contradiction with what we just argued. Thus, we have that $P(\omega \text{ is good}) = p > 0$.

Recalling the event $C_{n,j}$ defined in the proof of Proposition 1.6, we have that $P(C_{n,0})$ goes to 0 as n goes to infinity. Now, we define $\tilde{B}_m = C^c_{m,0} \cap (\cap_{n>m} B_n)$; in words, $\tilde{B}_m$ is the event where there are no cure marks in $(t_m, t_m + b_m]$ for any $x \in V$, and also for each $n \geq m+1$ we may find $j \in [0, c_n) \cap \mathbb{Z}$ such that there are no cure marks in $(t_n + jb_n, t_m + (j+1)b_m]$ for any $x \in V$. Remembering that $\gamma > 1$, applying again the union bound and Propositions 1.5 and 1.6, we obtain

$$\begin{aligned} 1 - P(\{\omega \text{ is good}\} \cap \tilde{A}_m \cap \tilde{B}_m) \leq \\ P(\{\omega \text{ is bad}\}) + \sum_{n \geq m} P(A^c_n) + P(C_{m,0}) + \sum_{n>m} P(B^c_n) \leq \\ (1-p) + \sum_{n \geq m} \frac{1}{n^\beta} + P(C_{m,0}) + \sum_{n>m} \frac{1}{n^\gamma} < 1, \end{aligned} \tag{1.22}$$

for m large. We fix now $m > n_3 \in \mathbb{N}$ satisfying (1.22).

If at time t_m there exists an infected individual, and the events $\tilde{A}_m$, $\tilde{B}_m$ and C_m occur simultaneously, then the infection survives forever (C_m was defined in (1.21)). That is,

$$\{\zeta_t \neq \emptyset, \forall t > 0\} \supset \{\zeta_{t_m} \neq \emptyset\} \cap \tilde{A}_m \cap \tilde{B}_m \cap C_m.$$

Since the event $\{\omega \text{ is good}\} \cap \tilde{A}_m \cap \tilde{B}_m$ depends solely on the cure process, it may be written as $\Lambda \times \Omega_2$, where $\Lambda \in \mathcal{F}_1$ (recall the product construction of our underlying probability space, described below the statement of Theorem 1.2). Thus, from (1.22),

$$P(\{\omega \text{ is good}\} \cap \tilde{A}_m \cap \tilde{B}_m) = P_1(\Lambda) > 0.$$

Given the independence between the cure and infection processes, it is enough to argue that in the event $\{\omega \text{ is good}\}$, we have that $\zeta_s \neq \emptyset$ with positive probability for any s; but this follows from the fact that for every fixed s there is a strictly positive probability for the occurrence of an infection stairway up to time s and no cure mark for any of the vertices of V during that time.

Wrapping up, we may write

$$\begin{aligned} P(\{\zeta_t \neq \emptyset, \forall t > 0\}) \geq P(\{\zeta_{t_m} \neq \emptyset\} \cap \tilde{A}_m \cap \tilde{B}_m \cap C_m \cap \{\omega \text{ is good}\}) = \\ P(\{\zeta_{t_m} \neq \emptyset\} \cap \tilde{A}_m \cap \tilde{B}_m \cap \{\omega \text{ is good}\}) P(C_m) = \\ \int_\Lambda P_2(\zeta_{t_m}(\omega_1, \cdot) \neq \emptyset) dP_1(\omega_1)\, P(C_m) > 0. \end{aligned}$$

We note that by the lack of memory of the exponential distribution and independence of the cure and infection processes, we have that C_m is independent from ζ_{t_m}, $\tilde{A}_m$, $\tilde{B}_m$ and $\{\omega \text{ is good}\}$.

1.4 Discussion

As a continuation of this work, some interesting questions can be further analysed.

Noticing that $V_+(\alpha) = 1/(1-\alpha)$ goes to ∞ as α goes to 1, we can investigate the existence of a sequence $\alpha_n \uparrow 1$ such that, the renewal contact process on the graph $\mathbb{Z}$, where the distribution between the renewals on the vertices $\{-n, n\}$ have cure index α_n, almost surely is extinct for every λ. We can also investigate the existence of $\alpha_n \uparrow 1$, such that, the renewal contact process on $\mathbb{Z}$ as described above, does not survive on any finite subgraph, but can survive on the entire $\mathbb{Z}$.

We also draw the reader's attention to the fact that, on [FMMV19], the regularity conditions assumed on the distribution is weaker than the one at this chapter. We believe that, due to the similar nature of the proofs, to obtain Item 2 on Theorem 1.3, our regularity can be relaxed as in [FMMV19]. However, to prove the first item, we strongly use our regularity assumptions, and it appears harder to be relaxed, since we need a good control on the occurrence of close cure marks in each individual.

Chapter 2

Central Limit Theorems for a Driven Particle in a Random Medium with Mass Aggregation

In this chapter, we study an infinite mechanical particle system with mass aggregation, according to the usual Newtonian mechanics laws. In this model the tracer particle, initially at origin, is subject to a positive constant force F, and interacts with the field-neutral random media made of initially standing particles of two possible types. Each neutral particle, with probability $0 < p \leq 1$ and independently of all other particles, is declared to be perfectly inelastic; after the first interaction with the tracer particle it is "incorporated" into the tracer particle according to the usual Newtonian mechanics laws. With probability $1 - p$ each particle is declared to be perfectly elastic, it interacts elastically with the tracer particle during the evolution. Neutral particles do not interact among themselves.

This model was introduced in [FNV00] where the authors study the long-time behavior of the tracer particle. The interaction with neutral particles creates a net force opposite to the direction of the flow which therefore competes with the external force F, this situation, as stated by the authors, tends to an equilibrium, which provides a law of large numbers for its instantaneous velocity.

We establish central limit theorems for the position and velocity of the tracer particle. Our approach is similar to that of [FNV00], namely, we first prove CLT's for the corresponding objects of a modified process, where there are no recollisions. The results for the original process are established by showing that the differences between the actual and modified quantities are negligible in the relevant scales.

2.1 The Model and Results

We consider a system of infinitely many point like particles in the non-negative real semi-axis $[0,\infty)$. At time 0 the system is static, every particle has velocity 0. There is a distinguished particle of mass 2 initially at the origin; we will call it the *tracer particle (t.p.)* (referred to before as the charged particle). The remaining particles (referred to before as neutral particles) have mass 1.[1] Let $\{\xi_i\}_{i\in\mathbb{N}}$ denote a family of i.i.d. positive random variables, with an absolutely continuous distribution, and finite mean $\mathbb{E}\xi_1 = \mu < \infty$, representing the initial interparticle distances. In this way, $S_i = \xi_1 + \cdots + \xi_i$ denotes the position of the i-th particle initially in front of the t.p. at time 0. Moreover, given a parameter $p \in (0,1]$, and a family $\{\eta_i\}_{i\in\mathbb{N}}$ of i.i.d. Bernoulli random variables with success probability p, we say that the i-th particle is *sticky* if $\eta_i = 1$ and is *elastic* if $\eta_i = 0$. We assume $\{\xi_i\}_{i\in\mathbb{N}}$ and $\{\eta_i\}_{i\in\mathbb{N}}$ to be independent of one another.

A constant positive force F is turned on at time 0, and kept on. It acts solely on the tracer particle, producing in it an accelerated motion to the right. Collisions will thus take place in the system; we assume they occur only when involving the t.p., and suppose that all other particles do not interact among themselves. If at an instant $t > 0$, the t.p. collides with a sticky particle, then this is a perfectly inelastic collision, meaning that, upon collision, momentum is conserved and the energy of the two particle system is minimum, which in turn means that the t.p. incorporates the sticky particle, along with its mass, and the new velocity of the t.p. becomes (immediately after time t)

$$V(t^+) = \frac{M_t}{M_t+1}V_t, \tag{2.1}$$

where V_t and M_t are respectively the velocity and mass of t.p. at time t. However, if the t.p. collides with an elastic particle which is moving at velocity v at the time of the collision, say t, then we have a perfectly elastic collision, where energy and momentum are preserved, and in this case, immediately after time t, the t.p. and the elastic particle velocities become, respectively,

$$\begin{aligned} V(t^+) &= \frac{M_t-1}{M_t+1}V_t + \frac{2}{M_t+1}v \quad \text{and} \\ v' &= \frac{2M_t}{M_t+1}V_t - \frac{M_t-1}{M_t+1}v, \end{aligned} \tag{2.2}$$

where V_t and M_t are as above.

[1] The distinction of the initial mass of the t.p. with respect to the other particles, absent in [FNV00], is for convenience only; any positive initial mass for the t.p. would not change our results, but values 1 or below would require unimportant complications in our arguments.

For $t \geq 0$, let V_t and Q_t denote the velocity and position of the t.p. at time t, respectively. As argued in [FNV00], the stochastic process $(V_t, Q_t)_{t\geq 0}$ is well defined — see the discussion at the end of Section 2 of [FNV00]; in particular there a.s. are no multiple collisions or infinitely many recollisions in finite time intervals —, and is determined by $\{\xi_i, \eta_i \; ; \; i \in \mathbb{N}\}$. Therefore we consider the product sample space $\Omega = \{(0,\infty) \times \{0,1\}\}^{\mathbb{N}}$, and the usual product Borel σ-algebra, and the product probability measure $\mathbb{P} := \prod_{i\geq 1}[\mathbb{P}_{\xi_i} \otimes \mathbb{P}_{\eta_i}]$, where for $i \geq 1$, $\mathbb{P}_{\xi_i}$ and $\mathbb{P}_{\eta_i}$ denote the probability measures of ξ_i and η_i. We will make repeatedly make use of the notation

$$\bar{\xi}_i = \xi_i - \mu, \; \bar{\eta}_i = \eta_i - p.$$

From [FNV00], we know that $\mathbb{P}$-almost surely, the velocity of the t.p. converges to a(n explicit) limit. More precisely, we have the following result.

Theorem 2.1. *The stochastic process $(V_t, Q_t)_{t\geq 0}$ is such that*

$$\lim_{t\to\infty} V_t = \sqrt{\frac{F\mu}{2-p}} \quad \mathbb{P} - a.s.$$

From now on we denote the limit velocity $\sqrt{F\mu/(2-p)}$ by V_L. The purpose of this chapter is to show that the velocity V_t and position Q_t of the tracer particle satisfy central limit theorems. Our main results are as follows (where "$\Longrightarrow$" denotes convergence in distribution).

Theorem 2.2. *Let* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then, as* $t \to \infty$,

$$\frac{Q_t - tV_L}{\sqrt{t}} \Longrightarrow \mathcal{N}(0, \sigma_q^2),$$

where $\sigma_q > 0$.

Theorem 2.3. *Let* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then, as* $t \to \infty$,

$$\sqrt{t}(V_t - V_L) \Longrightarrow \mathcal{N}(0, \sigma_v^2),$$

where $\sigma_v > 0$.

2.2 Central Limit Theorems in a Modified Process

As mentioned in the Introduction, we first prove central limit theorem analogues of Theorems 2.2 and 2.3 for a modified process in which, when an elastic particle collides with the t.p., the elastic particle is annihilated and disappears from the system, and the velocity of the t.p. changes according to the formula (2.2), while collisions between the t.p. and sticky particles remain as in the original model. We denote the modified stochastic process by $(\bar{V}(t), \bar{Q}(t))_{t\geq 0}$, where $\bar{V}(t)$ and $\bar{Q}(t)$ are respectively the velocity and position of the t.p. in the modified system at time t.

In the modified model, for $i \geq 1$, the t.p. collides with the i-th particle only in the initial position of the latter particle, given by S_i; let us denote the instant when that collision occurs by $\bar{t}_i$, i.e., $\bar{Q}(\bar{t}_i) = S_i$. In this way, we can compute the $i-$th collision incoming and outgoing velocities $\bar{V}(\bar{t}_i)$ and $\bar{V}(\bar{t}_i^+)$, respectively, as follows. First note that, according the formulas (2.1) and (2.2), we have the following relations

$$\begin{aligned}
(a) \quad \bar{V}^2(\bar{t}_i\ ) &= \bar{V}^2(\bar{t}_{i-1}^+) + \frac{2F\xi_i}{M(\bar{t}_i)}; \\
(b) \quad \bar{V}^2(\bar{t}_i^+) &= \bar{V}^2(\bar{t}_i)\left[\frac{M(\bar{t}_i) + (\eta_i - 1)}{M(\bar{t}_i) + 1}\right]^2,
\end{aligned}$$

where $M(\bar{t}_i) = 2 + \sum_{l=1}^{i-1} \eta_l$.

Iterating this relations, we get for $i = 1, 2, \ldots$, that

$$\bar{V}^2(\bar{t}_i^+) = \sum_{j=1}^{i} \left[\frac{2F\xi_j}{M(\bar{t}_j)} \prod_{k=j}^{i} \left(\frac{M(\bar{t}_k) + (\eta_k - 1)}{M(\bar{t}_k) + 1}\right)^2\right]. \tag{2.3}$$

In [1], it is proved that, almost surely,

$$\lim_{t\to\infty} \bar{V}(t) = V_L.$$

Let us at this point set some notation. Given two random sequences $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$, we write $X_n = O(Y_n)$ if there almost surely exists $C > 0$, which may be a (proper) random variable, but does not depend on n, such that $|X_n| \leq CY_n$ for every $n \in \mathbb{N}$. And we say $X_n = o(Y_n)$ if X_n/Y_n almost surely converges to 0 as $n \to \infty$. For simplicity, along the rest of the chapter we denote $M(\bar{t}_i)$ by M_i. Notice that $M_1 = 2$ and $M_i = 2 + \sum_{k=1}^{i-1} \eta_k$, $i \geq 2$.

To obtain the central limit theorems for the modified process, we start with an estimate for the random term

$$X_{i,j} := \frac{1}{M_j} \prod_{k=j}^{i} \left(\frac{M_k + (\eta_k - 1)}{M_k + 1}\right)^2, \quad 1 \leq j \leq i \text{ and } i \in \mathbb{N}. \tag{2.4}$$

Given $\varepsilon > 0$, for each $m \in \mathbb{N}$ we define the event

$$A_{m,\varepsilon} = \left\{ X_{i,j} \in \left((1-\varepsilon)\frac{j^{\zeta-1}}{pi^{\zeta}}, (1+\varepsilon)\frac{j^{\zeta-1}}{pi^{\zeta}} \right), \quad \forall\, m \le j \le i \right\}, \tag{2.5}$$

where $\zeta := 2(2-p)/p$.

Lemma 2.1. *Let $X_{i,j}$ be as in (2.4), and $A_{m,\varepsilon}$ as in (2.5), where $\varepsilon > 0$ is otherwise arbitrary. Then we have that*

$$\lim_{m\to\infty} \mathbb{P}\left(A_{m,\varepsilon}\right) = 1.$$

Proof. We first Taylor-expand the logarithm to write

$$\prod_{k=j}^{i} \left(\frac{M_k + (\eta_k - 1)}{M_k + 1} \right)^2 = \exp\left\{ 2\sum_{k=j}^{i} \log\left(1 - \frac{2-\eta_k}{M_k+1}\right) \right\} =$$
$$\exp\left\{ -2\sum_{k=j}^{i} \left[\frac{2-p}{M_k+1} - \frac{\bar{\eta}_k}{M_k+1} \right] + O\left(\sum_{k=j}^{i} \left(\frac{2-\eta_k}{M_k+1} \right)^2 \right) \right\}. \tag{2.6}$$

Given $\delta > 0$, $m \in \mathbb{N}$, let $B_m^\delta = \{M_j \in ((1-\delta)pj, (1+\delta)pj)\,,\ \forall j \ge m\}$. It follows from the Law of Large Numbers that $P(B_m^\delta) \to 1$ a.s. as $m \to \infty$. In B_m^δ, we have

$$\sum_{k=1}^{\infty} \left(\frac{2-\eta_k}{M_k+1} \right)^2 \le \sum_{k=1}^{m-1} \left(\frac{2-\eta_k}{M_k+1} \right)^2 + \frac{4}{p^2(1-\delta)^2} \sum_{k=m}^{\infty} \frac{1}{k^2} < \infty. \tag{2.7}$$

Note also that

$$\sum_{k=j}^{i} \frac{1}{M_k+1} = \sum_{k=j}^{i} \left(\frac{1}{M_k+1} - \frac{1}{pk} \right) + \frac{1}{p}\left[\sum_{k=j}^{i} \frac{1}{k} - \int_j^i \frac{1}{x}dx \right] + \frac{1}{p}\int_j^i \frac{1}{x}dx. \tag{2.8}$$

Clearly the second term at the right-hand side of (2.8) goes to 0 as j and i goes to infinity. Let now $C_m = \left\{ |M_j + 1 - jp| \le j^{2/3},\ \forall j \ge m \right\}$. It follows from Law of the Iterated Logarithm that $\lim_{m\to\infty} \mathbb{P}(C_m) = 1$. In $B_m^\delta \cap C_m$ we have

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \left(\frac{1}{M_k+1} - \frac{1}{pk} \right) \right| &\le \left| \sum_{k=1}^{m-1} \left(\frac{1}{M_k+1} - \frac{1}{pk} \right) \right| + \frac{1}{p(1-\delta)} \sum_{k=m}^{\infty} \frac{|M_k+1-kp|}{k^2} \\ &\le \left| \sum_{k=1}^{m-1} \left(\frac{1}{M_k+1} - \frac{1}{pk} \right) \right| + \sum_{k=m}^{\infty} \frac{1}{k^{4/3}} < \infty. \end{aligned} \tag{2.9}$$

We also write

$$\sum_{k=1}^{\infty} \frac{\bar{\eta}_k}{M_k+1} = \sum_{k=1}^{\infty} \left[\bar{\eta}_k \left(\frac{1}{M_k+1} - \frac{1}{pk} \right) \right] + \sum_{k=1}^{\infty} \frac{\bar{\eta}_k}{pk}. \tag{2.10}$$

We may apply Kolmogorov's Two-series Theorem to obtain that $\sum_{k=1}^{\infty} \bar{\eta}_k/k$ converges a.s., and proceeding as in the estimation leading to (2.9), we may conclude that the first term in the right-hand side of (2.10) is also convergent in the event $B_m^\delta \cap C_m$.

To conclude, due to (2.6), (2.7), (2.8), (2.9) and (2.10), taking $\delta > 0$ sufficient small and m sufficient large, we have that, in the event $B_m^\delta \cap C_m$,

$$\prod_{k=j}^{i} \left(\frac{M_k + (\eta_k - 1)}{M_k+1} \right)^2 \in (1 \pm \varepsilon) \exp\left\{ -\zeta \int_j^i \frac{1}{x} dx \right\}. \tag{2.11}$$

Recalling now the definition of $X_{i,j}$ and $A_{m,\varepsilon}$ in (2.4) and (2.5), respectively, we have that (2.11) implies that $B_m^\delta \cap C_m \subset A_{m,\varepsilon}$, and the result follows. □

We now turn our attention to $S_n - \bar{t}_n V_L$, for which we will prove a central limit theorem, as a step to establish Theorem 2.2, as follows.

Proposition 2.1. *Let* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then, as* $n \to \infty$,

$$\frac{S_n - \bar{t}_n V_L}{\sqrt{n}} \Longrightarrow \mathcal{N}(0, \hat{\sigma}_q^2), \tag{2.12}$$

where $\hat{\sigma}_q > 0$.

The proof of this result consists of a number of steps which take most of this section.

From elementary physics relations, the time taken for the t.p. to go from S_{i-1} to S_i is given by

$$\bar{t}_i - \bar{t}_{i-1} = \frac{\bar{V}(\bar{t}_i) - \bar{V}(\bar{t}_{i-1}^+)}{F/M_i} = \frac{2\xi_i \left(\bar{V}(\bar{t}_i) - \bar{V}(\bar{t}_{i-1}^+)\right)}{2\xi_i F/M_i} = \frac{2\xi_i}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)}.$$

Thus, we may write

$$\begin{aligned} S_n - \bar{t}_n V_L &= \sum_{i=1}^{n} \left[\xi_i \left(1 - \frac{2V_L}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} \right) \right] \\ &= \sum_{i=1}^{n} \left[\xi_i \left(\frac{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+) - 2V_L}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} \right) \right] \\ &= \sum_{i=1}^{n} \left[\frac{2\xi_i \left(\bar{V}(\bar{t}_{i-1}^+) - V_L\right)}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} \right] + \sum_{i=1}^{n} \left[\xi_i \left(\frac{\bar{V}(\bar{t}_i) - \bar{V}(\bar{t}_{i-1}^+)}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} \right) \right] \end{aligned} \tag{2.13}$$

Note that

$$\frac{\bar{V}(\bar{t}_i) - \bar{V}(\bar{t}_{i-1}^+)}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} = \frac{2F\xi_i}{M_i \left(\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)\right)^2}. \tag{2.14}$$

Since $\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)$ converges to the constant $2V_L$, the Law of Large Numbers and (2.14) imply that

$$\sum_{i=1}^{n} \left[\xi_i \left(\frac{\bar{V}(\bar{t}_i) - \bar{V}(\bar{t}_{i-1}^+)}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} \right) \right] = O\left(\sum_{i=1}^{n} \frac{\xi_i^2}{i} \right). \tag{2.15}$$

Let $\tilde{S}_0 = 0$ and $\tilde{S}_k = \sum_{i=1}^{k} \xi_i^2$, $k \geq 1$. Assuming $\mathbb{E}\xi_1^2 < \infty$, we have that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\xi_i^2}{i} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\tilde{S}_i - \tilde{S}_{i-1}}{i} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{\tilde{S}_i}{i(i+1)} + \frac{\tilde{S}_n}{n^{3/2}} = o(1). \tag{2.16}$$

Noticing that $\bar{V}(\bar{t}_i) = \bar{V}(\bar{t}_{i-1}^+) + 2F\xi_i / \left[M_i \left(\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+) \right) \right]$, we find that

$$\frac{\bar{V}(\bar{t}_{i-1}^+) - V_L}{\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i)} = \frac{\bar{V}(\bar{t}_{i-1}^+) - V_L}{2V_L} + (V_L - \bar{V}(\bar{t}_{i-1}^+)) \left[\frac{1}{2V_L} - \frac{1}{\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i)} \right],$$

and the last parcel of the above sum is equal to

$$\begin{aligned}
&= (V_L - \bar{V}(\bar{t}_{i-1}^+)) \left[\frac{\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i) - 2V_L}{2V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))} \right] \\
&= (V_L - \bar{V}(\bar{t}_{i-1}^+)) \left[\frac{2\bar{V}(\bar{t}_{i-1}^+) - 2V_L}{2V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))} + \frac{\bar{V}(\bar{t}_i) - \bar{V}(\bar{t}_{i-1}^+)}{2V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))} \right] \\
&= -\frac{(\bar{V}(\bar{t}_{i-1}^+) - V_L)^2}{V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))} + \frac{(V_L - \bar{V}(\bar{t}_{i-1}^+))(\bar{V}(\bar{t}_i)^2 - \bar{V}(\bar{t}_{i-1}^+)^2)}{2V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))^2} \\
&= -\frac{(\bar{V}(\bar{t}_{i-1}^+) - V_L)^2}{V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))} + \frac{2F\xi_i}{M_i} \cdot \frac{V_L - \bar{V}(\bar{t}_{i-1}^+)}{2V_L(\bar{V}(\bar{t}_{i-1}^+) + \bar{V}(\bar{t}_i))^2}.
\end{aligned}$$

In particular, since $\bar{V}(\bar{t}_{i-1}^+)$ and $\bar{V}(\bar{t}_i)$ goes to V_L as i goes to infinity,

$$\begin{aligned}
&\sum_{i=1}^{n} \left[\frac{2\xi_i \left(\bar{V}(\bar{t}_{i-1}^+) - V_L \right)}{\bar{V}(\bar{t}_i) + \bar{V}(\bar{t}_{i-1}^+)} \right] = \\
&\qquad \sum_{i=1}^{n} \left[\frac{\xi_i \left(\bar{V}(\bar{t}_{i-1}^+) - V_L \right)}{V_L} \right] + O\left(\sum_{i=1}^{n} \left[\xi_i \left(\bar{V}(\bar{t}_{i-1}^+) - V_L \right)^2 \right] \right) + O\left(\sum_{i=1}^{n} \frac{\xi_i^2}{i} \right).
\end{aligned}$$

Proceeding in an analogous way, we obtain that

$$\sum_{i=1}^{n}\left[\frac{\xi_i\left(\bar V(\bar t_{i-1}^+)-V_L\right)}{V_L}\right]=$$
$$\sum_{i=1}^{n}\left[\frac{\xi_i\left(\bar V(\bar t_{i-1}^+)^2-V_L^2\right)}{2V_L^2}\right]+O\left(\sum_{i=1}^{n}\left[\xi_i\left(\bar V(\bar t_{i-1}^+)-V_L\right)^2\right]\right). \tag{2.17}$$

To simplify notation, for each $i\in\mathbb{N}$, we henceforth denote $\bar V(\bar t_i^+)$ simply by $\bar V_i$. The following lemma will be useful now; we postpone its proof till the end of this section.

Lemma 2.2. *Let* $\mathrm{Var}(\xi_1)=\sigma^2<\infty$ *and let* $\epsilon>0$. *The velocities* $\{\bar V_i\}_{i\in\mathbb{N}}$ *are such that* $\bar V_i-V_L=o(1/i^{1/2-\epsilon})$. *In particular,*

$$\frac{1}{\sqrt n}\sum_{i=1}^{n}\left[\xi_i\left(\bar V_{i-1}-V_L\right)^2\right]=o(1).$$

By (2.13) to (2.17) and Lemma 2.2, in order to establish Proposition 2.1 it is enough to show that as $n\to\infty$

$$\frac{1}{\sqrt n}\sum_{i=1}^{n}\xi_i(\bar V_{i-1}^2-V_L^2)\Longrightarrow\mathcal{N}(0,\tilde\sigma_q^2), \tag{2.18}$$

for some $\tilde\sigma_q>0$; we then of course have $\hat\sigma_q=\tilde\sigma_q/(2V_L^2)$. For that, the strategy we will follow is to expand the expression on the left of (2.18) into several terms, one of which depends only on the interparticle distances $\{\xi_i\}_{i\in\mathbb{N}}$, another one depending only on the stickiness indicator random variables $\{\eta_k\}_{k\in\mathbb{N}}$; for each of those terms we can apply Lindeberg-Feller's Central Limit Theorem; upon showing that the remaining terms are negligible, the result follows.

Recalling that $\zeta=2(2-p)/p$, (2.3) and (2.4), we start with

$$\frac{1}{\sqrt n}\sum_{i=1}^{n}\left[\xi_{i+1}(\bar V_i^2-V_L^2)\right]=\frac{2F}{\sqrt n}\sum_{i=1}^{n}\left[\xi_{i+1}\sum_{j=1}^{i}\left(\xi_jX_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right]+$$
$$\frac{2F\mu}{p\sqrt n}\sum_{i=1}^{n}\left[\xi_{i+1}\left(\frac1i\sum_{j=1}^{i}\left(\frac ji\right)^{\zeta-1}-\int_0^1x^{\zeta-1}dx\right)\right]. \tag{2.19}$$

The term on the left of expression within parentheses in the second term on the right hand side of (2.19) is a Riemann sum for the term to its right; we conclude that the full expression within parenthesis on the right hand side of (2.19) is an

$O(1/i)$, and we may thus conclude that the second term on the right-hand side of (2.19) is an $o(1)$, and proceed by dropping that term and focusing on the first term, which we write as follows.

$$\frac{2F}{\sqrt{n}}\sum_{i=1}^{n}\left[\xi_{i+1}\sum_{j=1}^{i}\left(\xi_j X_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right]=$$
$$\frac{2F}{\sqrt{n}}\sum_{i=1}^{n}\left[\bar{\xi}_{i+1}\sum_{j=1}^{i}\left(\xi_j X_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right]+\frac{2F\mu}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\xi_j X_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)$$
$$:= V_n + W_n. \quad (2.20)$$

Now writing

$$\sum_{j=1}^{i}\left(\xi_j X_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)=\sum_{j=1}^{i}\bar{\xi}_j X_{i,j}+\mu\sum_{j=1}^{i}\left(X_{i,j}-\frac{j^{\zeta-1}}{pi^{\zeta}}\right),$$

V_n given in (2.20) becomes

$$V_n=\frac{2F}{\sqrt{n}}\sum_{i=1}^{n}\left[\bar{\xi}_{i+1}\sum_{j=1}^{i}\left(\xi_j X_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right]=$$
$$\frac{2F}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\bar{\xi}_{i+1}\bar{\xi}_j X_{i,j}+\frac{2F\mu}{\sqrt{n}}\sum_{i=1}^{n}\left[\bar{\xi}_{i+1}\sum_{j=1}^{i}\left(X_{i,j}-\frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right]$$
$$=: V_{1,n}+V_{2,n}. \quad (2.21)$$

We will show in Lemmas 2.6 and 2.7 below that $V_{1,n}$ and $V_{2,n}$ are negligible.

Analogously, W_n given in (2.20) becomes

$$W_n=\frac{2F\mu}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\xi_j X_{i,j}-\mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right)=$$
$$\frac{2F\mu}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\bar{\xi}_j X_{i,j}+\frac{2F\mu^2}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(X_{i,j}-\frac{j^{\zeta-1}}{pi^{\zeta}}\right)$$
$$=: W_{1,n}+W_{2,n}, \quad (2.22)$$

and $W_{1,n}$ is further broken down into

$$\begin{aligned} W_{1,n} &= \frac{2F\mu}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\bar{\xi}_j X_{i,j} \\ &= \frac{2F\mu}{p\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j^{\zeta-1}}{i^{\zeta}}\bar{\xi}_j+\frac{2F\mu}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\bar{\xi}_j\left(X_{i,j}-\frac{j^{\zeta-1}}{pi^{\zeta}}\right) \\ &=: W_{3,n}+W_{4,n}. \end{aligned} \quad (2.23)$$

One may readily verify the conditions of Lindeberg-Feller's CLT to obtain

Lemma 2.3. *Let* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *For* $1 \le j \le n$, *set* $a_{j,n} = j^{\zeta-1} \sum_{i=j}^{n} \frac{1}{i^{\zeta}}$. *Then, as* $n \to \infty$,

$$W_{3,n} = \frac{2F\mu}{p\sqrt{n}} \sum_{j=1}^{n} a_{j,n} \bar{\xi}_j \Longrightarrow \mathcal{N}(0, \sigma_w^2),$$

where $\sigma_w = \frac{2F\mu}{p\sqrt{\zeta}}\sigma$.

In Lemma 2.8 below we show that $W_{4,n}$ is negligible.

Let us now focus on $W_{2,n}$. To alleviate notation, for each $1 \le j \le i$, set

$$Y_{i,j} = \log \left[\prod_{k=j}^{i} \left(\frac{M_k + (\eta_k - 1)}{M_k + 1} \right)^2 \right] = 2 \sum_{k=j}^{i} \log \left(1 - \frac{2 - \eta_k}{M_k + 1} \right), \tag{2.24}$$

thus $X_{i,j} = e^{Y_{i,j}}/M_j$, and therefore,

$$W_{2,n} = \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left(X_{i,j} - \frac{j^{\zeta-1}}{p i^{\zeta}} \right) =$$

$$\frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\left(\frac{1}{M_j} - \frac{1}{pj} \right) \frac{j^{\zeta}}{i^{\zeta}} \right] + \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\left(\frac{1}{M_j} - \frac{1}{pj} \right) \left(e^{Y_{i,j}} - \frac{j^{\zeta}}{i^{\zeta}} \right) \right] +$$

$$\frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{1}{pj} \left(e^{Y_{i,j}} - \frac{j^{\zeta}}{i^{\zeta}} \right) \right] =: Z_{1,n} + Z_{2,n} + Z_{3,n}. \tag{2.25}$$

Lemma 2.4. $Z_{2,n}$, *as defined in* (2.25), *is an* $o(1)$.

Proof. Note that, as defined in (2.24) and (2.25),

$$\begin{aligned} |Z_{2,n}| &= \left| \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\left(\frac{1}{M_j} - \frac{1}{pj} \right) \left(e^{Y_{i,j}} - \frac{j^{\zeta}}{i^{\zeta}} \right) \right] \right| \\ &= \left| \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta}}{i^{\zeta}} \left(\frac{1}{M_j} - \frac{1}{pj} \right) \left(\exp\left\{ Y_{i,j} + \zeta \int_j^i \frac{1}{x} dx \right\} - 1 \right) \right] \right| \\ &\le \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta}}{i^{\zeta}} \left| \frac{1}{M_j} - \frac{1}{pj} \right| \left| Y_{i,j} + \zeta \int_j^i \frac{1}{x} dx \right| \right]. \end{aligned} \tag{2.26}$$

For each $i \ge j \ge 1$, we define

$$R_{i,j} = Y_{i,j} + \zeta \int_j^i x^{-1} dx. \tag{2.27}$$

It follows from (2.26) that

$$|Z_{2,n}| = O\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\left[\frac{j^\zeta}{i^\zeta}\left|\frac{1}{M_j}-\frac{1}{pj}\right||R_{i,j}|\right]\right). \tag{2.28}$$

As we see in (2.6) and (2.24), $R_{i,j}$ can be written as

$$R_{i,j} = \zeta\int_j^i x^{-1}dx - 2\sum_{k=j}^{i}\frac{2-p}{M_k+1} + 2\sum_{k=j}^{i}\frac{\bar{\eta}_k}{M_k+1} + O\left(\sum_{k=j}^{i}\left(\frac{2-\eta_k}{M_k+1}\right)^2\right) =$$
$$\zeta\left[\int_j^i \frac{1}{x}dx - \sum_{k=j}^{i}\frac{1}{k}\right] + \sum_{k=j}^{i}\left(\frac{\zeta}{k}-\frac{2(2-p)}{M_k+1}\right) + 2\sum_{k=j}^{i}\left[\frac{\bar{\eta}_k}{M_k+1}-\frac{\bar{\eta}_k}{p(k-1)+3}\right] +$$
$$2\sum_{k=j}^{i}\frac{\bar{\eta}_k}{p(k-1)+3} + O\left(\sum_{k=j}^{i}\left(\frac{2-\eta_k}{M_k+1}\right)^2\right) := R^{(1)}_{i,j}+\cdots+R^{(5)}_{i,j}. \tag{2.29}$$

One readily checks by elementary deterministic estimation that for all $i \geq j \geq 1$, $|R^{(1)}_{i,j}|$ can be bounded above by $1/j$.

Let now $0 < \delta < 1/4$ be fixed. The Law of Large Numbers and the Law of the Iterated Logarithm, there a.s. exists $j_0 \in \mathbb{N}$ such that $|R^{(2)}_{i,j}|$, $|R^{(3)}_{i,j}|$ and $|R^{(5)}_{i,j}|$ are bounded above by $1/j^{1/2-\delta}$, for every $i \geq j \geq j_0$.

To study $|R^{(4)}_{i,j}|$, we apply Hoeffding's Inequality to obtain, for every $i \geq j \geq 1$,

$$\mathbb{P}\left(\left|\sum_{k=j}^{i}\frac{\bar{\eta}_k}{p(k-1)+3}\right| \geq \frac{1}{j^{1/2-\delta}}\right) \leq \exp\left\{-2\Big/\left(j^{1-2\delta}\sum_{k=j}^{i}\frac{1}{(p(k-1)+3)^2}\right)\right\}. \tag{2.30}$$

We next apply a variation of Lévy's Maximal Inequality, namely Proposition 1.1.2 in [DG99], combined with (2.30), to get that

$$\mathbb{P}\left(\max_{i\geq j}\left|\sum_{k=j}^{i}\frac{\bar{\eta}_k}{p(k-1)+3}\right| \geq \frac{3}{j^{1/2-\delta}}\right) =$$
$$\lim_{n\to\infty}\mathbb{P}\left(\max_{j\leq i\leq n}\left|\sum_{k=j}^{i}\frac{\bar{\eta}_k}{p(k-1)+3}\right| \geq \frac{3}{j^{1/2-\delta}}\right) \leq$$
$$3\lim_{n\to\infty}\max_{j\leq i\leq n}\mathbb{P}\left(\left|\sum_{k=j}^{i}\frac{\bar{\eta}_k}{p(k-1)+3}\right| \geq \frac{1}{j^{1/2-\delta}}\right) \leq$$
$$3\exp\left\{-2\Big/\left(j^{1-2\delta}\sum_{k=j}^{\infty}\frac{1}{(p(k-1)+3)^2}\right)\right\}. \tag{2.31}$$

Since the latter term is summable, we conclude that almost surely exists $j_0 \in \mathbb{N}$ such that $|R^{(4)}_{i,j}| \leq 3/j^{1/2-\delta}$, for every $i \geq j \geq j_0$. Collecting all the bounds, we find that a.s.

$$|R_{i,j}| \leq |R^{(1)}_{i,j}| + \cdots + |R^{(5)}_{i,j}| < 3/j^{1/2-\delta} \tag{2.32}$$

for every $i \geq j$ sufficiently large. Recalling that $M_j = 2 + \sum_{l=1}^{j-1} \eta_l$, we have, as consequence of the Law of the Iterated Logarithm and the Law of Large Numbers, that $|1/M_j - 1/(pj)| = o(1/j^{3/2-\delta})$, and the result follows from (2.28). □

It follows from (2.31) that $R_{i,j}$ is uniformly bounded in i, j by a proper random variable. We may thus write

$$\begin{aligned} Z_{3,n} &= \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{1}{pj} \left(e^{Y_{i,j}} - \frac{j^\zeta}{i^\zeta} \right) \right] = \frac{2F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \left(e^{R_{i,j}} - 1 \right) \right] \\ &= \frac{2F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{j^{\zeta-1}}{i^\zeta} R_{i,j} + O\left(\frac{2F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{j^{\zeta-1}}{i^\zeta} R_{i,j}^2 \right) =: Z'_{3,n} + \tilde{Z}_{3,n}. \end{aligned} \tag{2.33}$$

Since, almost surely, for every $i \geq j$ sufficiently large, we have the bound $|R^{(1)}_{i,j}| + |R^{(5)}_{i,j}| \leq 1/j^{2/3}$, it follows that

$$\frac{2F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \left(R^{(1)}_{i,j} + R^{(5)}_{i,j} \right) \right] = o(1).$$

Considering only the term $R^{(2)}_{i,j}$ of $R_{i,j}$ in (2.29), its contribution to $Z'_{3,n}$ in (2.33) is

$$\begin{aligned} &2(2-p)\frac{2F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \sum_{k=j}^{i} \left(\frac{1}{pk} - \frac{1}{M_k+1} \right) \right] = \\ &\qquad \zeta \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{i} \left[\left(\frac{1}{pk} - \frac{1}{M_k+1} \right) \sum_{j=1}^{k} \frac{j^{\zeta-1}}{i^\zeta} \right] = \\ &\qquad \frac{2F\mu^2}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{i} \left[\left(\frac{1}{pk} - \frac{1}{M_k} \right) \frac{k^\zeta}{i^\zeta} \right] + o(1) = -Z_{1,n} + o(1), \end{aligned} \tag{2.34}$$

where $Z_{1,n}$ is defined in (2.25). We may remark at this point that combining (2.34) and (2.25) drops $Z_{1,n}$ out of the overall computation.

Let us now estimate the contribution of $R_{i,j}^{(4)}$ to $Z'_{3,n}$ in (2.33), recalling that $M_k = 2 + \sum_{l=1}^{k-1} \eta_l$ and setting $\bar{M}_k = -\sum_{l=1}^{k} \bar{\eta}_l$:

$$\frac{4F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \sum_{k=j}^{i} \left(\frac{\bar{\eta}_k}{M_k+1} - \frac{\bar{\eta}_k}{p(k-1)+3} \right) \right] =$$

$$\frac{4F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \sum_{k=j}^{i} \frac{\bar{\eta}_k \bar{M}_{k-1}}{(M_k+1)(p(k-1)+3)} \right] =$$

$$\frac{4F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \sum_{k=j}^{i} \frac{\bar{\eta}_k \bar{M}_{k-1}}{(p(k-1)+3)^2} \right] +$$

$$\frac{4F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{j^{\zeta-1}}{i^\zeta} \sum_{k=j}^{i} \left(\frac{\bar{\eta}_k \bar{M}_{k-1}}{p(k-1)+3} \left(\frac{1}{M_k+1} - \frac{1}{p(k-1)+3} \right) \right) \right]$$

$$=: Z_{5,n} + Z_{6,n}. \tag{2.35}$$

Let us fix $0 < \alpha < 1/2$; the Law of the Iterated Logarithm and the Law of Large Numbers give us that

$$\left| \frac{\bar{\eta}_k \bar{M}_{k-1}}{p(k-1)+3} \left(\frac{1}{M_k+1} - \frac{1}{p(k-1)+3} \right) \right| = o\left(\frac{1}{k^{2-\alpha}} \right).$$

Since $0 < \alpha < 1/2$, it follows that $Z_{6,n} = o(1)$.

We will study the asymptotic behavior of $Z_{5,n}$ in Lemma 2.9.

We now estimate the contribution of $R_{i,j}^{(3)}$ to $Z'_{3,n}$ in (2.33):

$$\frac{4F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left[\frac{j^{\zeta-1}}{i^\zeta} \sum_{k=j}^{i-1} \frac{\bar{\eta}_k}{k} \right] = \frac{4F\mu^2}{p\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} \left[\frac{\bar{\eta}_k}{k} \sum_{j=1}^{k} \frac{j^{\zeta-1}}{i^\zeta} \right]$$

$$= \frac{4F\mu^2}{\zeta p\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} \frac{k^{\zeta-1}}{i^\zeta} \bar{\eta}_k + o(1) =: Z_{4,n} + o(1). \tag{2.36}$$

By a routine verification of the conditions of the Lindeberg-Feller CLT we get the following result.

Lemma 2.5. *As $n \to \infty$*

$$Z_{4,n} = \frac{4F\mu^2}{\zeta p\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} \frac{k^{\zeta-1}}{i^\zeta} \bar{\eta}_k \Longrightarrow \mathcal{N}(0, \sigma_z^2),$$

where $\sigma_z = 4F\mu^2 \sqrt{\frac{1-p}{p\zeta^3}}$.

Let us now estimate $\tilde{Z}_{3,n}$ in (2.33). From (2.32) it readily follows that

$$\frac{2F\mu^2}{p\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j^{\zeta-1}}{i^{\zeta}}R_{i,j}^2 = o(1),$$

and thus $\tilde{Z}_{3,n} = o(1)$.

So far we have argued that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\xi_{i+1}(\bar{V}_i^2 - V_L^2)\right] &= \big(W_{3,n} + Z_{4,n}\big) + \big(V_{1,n} + V_{2,n} + W_{4,n} + Z_{5,n}\big) + o(1) \\ &=: G_n + H_n + o(1), \end{aligned} \tag{2.37}$$

where $W_{3,n}, W_{4,n}, Z_{4,n}, V_{1,n}, V_{2,n}$ and $Z_{5,n}$ are defined, respectively, in (2.23), (2.36), (2.21) and (2.35). By the independence of $W_{3,n}$ and $Z_{4,n}$, we have by Lemmas 2.3 and 2.5 that $G_n \Longrightarrow \mathcal{N}(0, \tilde{\sigma}_q^2)$, where $\tilde{\sigma}_q^2 = \sigma_w^2 + \sigma_z^2$. To establish (2.18), it is then enough to show that $H_n = o(1)$, which we do in the following lemmas, one for each of the constituents of H_n.

Lemma 2.6. *Assume* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then* $V_{1,n} = o(1)$.

Proof. First fix $\delta > 0$. Given $\varepsilon > 0$, Lemma 2.1 states that exists $m \in \mathbb{N}$ such that $\mathbb{P}(A_{m,\varepsilon}^c) < \varepsilon/2$. Recall the definition of $X_{i,j}$ in (2.4), and that $\{X_{i,j},\ i \geq j \geq 1\}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are independent.

$$\mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\bar{\xi}_{i+1}\bar{\xi}_j X_{i,j}\right| > \delta\right) \leq \mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{i}\bar{\xi}_{i+1}\bar{\xi}_j X_{i,j}1_{A_{m,\varepsilon}}\right| > \delta\right) + \frac{\varepsilon}{2} \tag{2.38}$$

It follows from definition of $A_{m,\varepsilon}$ in (2.5) that $X_{i,j}1_{A_{m,\varepsilon}} \leq (1+\varepsilon)[j^{\zeta-1}/(pi^{\zeta})]$ for all $i \geq j \geq m$. Using this and by Markov's Inequality, we get that the first term on the right of (2.38) is bounded above by

$$\frac{1}{\delta^2 n}\sum_{i=1}^{n}\sum_{j=1}^{i}\mathbb{E}(\bar{\xi}_{i+1})^2\mathbb{E}(\bar{\xi}_j)^2\,\mathbb{E}(X_{i,j}^2 1_{A_{m,\varepsilon}}) = \frac{(1+\varepsilon)^2\sigma^4}{p^2\delta^2 n}\sum_{i=1}^{n}\sum_{j=m}^{i}\frac{j^{2\zeta-2}}{i^{2\zeta}} + o(1) = o(1).$$

Since $\delta > 0$ and $\varepsilon > 0$ are arbitrary, the combination of this inequality and (2.38) yields the result. □

Lemma 2.7. *Assume* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then* $V_{2,n} = o(1)$.

Proof. Arguing similarly as in the proof of Lemma 2.6, given $\delta > 0$ and $\varepsilon > 0$, we have that m large enough

$$\mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\bar{\xi}_{i+1}\sum_{j=1}^{i}\left(X_{i,j}-\frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right]\right|>\delta\right)\leq \\ \mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\bar{\xi}_{i+1}\sum_{j=1}^{i}\left(X_{i,j}-\frac{j^{\zeta-1}}{pi^{\zeta}}\right)1_{A_{m,\varepsilon}}\right]\right|>\delta\right)+\frac{\varepsilon}{2} \quad (2.39)$$

and since $|X_{i,j}(\omega) - j^{\zeta-1}/(pi^{\zeta})|1_{A_{m,\varepsilon}} \leq (\varepsilon j^{\zeta-1})/(pi^{\zeta})$ for all $i \geq j \geq m$, we get that the first term on the right of (2.39) is bounded above by

$$\frac{1}{\delta^2 n}\sum_{i=1}^{n}\left[\mathbb{E}(\bar{\xi}_{i+1})^2\,\mathbb{E}\left(\sum_{j=1}^{i}\left(X_{i,j}(\omega)-\frac{j^{\zeta-1}}{pi^{\zeta}}\right)1_{A_{m,\varepsilon}}\right)^2\right]\leq \\ \frac{\sigma^2}{\delta^2 n}\sum_{i=1}^{n}\mathbb{E}\left[\sum_{j=1}^{i}\left|X_{i,j}(\omega)-\frac{j^{\zeta-1}}{pi^{\zeta}}\right|1_{A_{m,\varepsilon}}\right]^2\leq\varepsilon^2\frac{\sigma^2}{\delta^2 n}\sum_{i=1}^{n}\left(\sum_{j=m}^{i}\frac{j^{\zeta-1}}{i^{\zeta}}\right)^2+o(1)\leq\frac{\varepsilon}{2},$$

as soon as n is large enough, and the result follows upon substitution in (2.39), since δ and ε are arbitrary. □

Lemma 2.8. *Assume* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then* $W_{4,n} = o(1)$.

Proof. Similar to the proof of Lemma 2.7. □

Lemma 2.9. *Assume* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then* $Z_{5,n} = o(1)$.

Proof. Changing the order of summation, we find that $Z_{5,n}$ is bounded by constant times

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} L_{k,n}\bar{\eta}_k \bar{M}_{k-1},$$

where $L_{k,n} = \frac{1}{k^2}\left(\sum_{j=1}^{k} j^{\zeta-1}\right)\left(\sum_{i=k}^{n}\frac{1}{i^{\zeta}}\right)$, which is bounded above by constant times $\frac{1}{k}$ uniformly in j and n. Now by Markov:

$$\mathbb{P}\left(|Z_{5,n}|\geq\delta\right)\leq\frac{\text{const}}{\delta^2 n}\sum_{k=1}^{n}\frac{1}{k^2}\mathbb{E}(\bar{M}_{k-1}^2)\leq\frac{\text{const}}{\delta^2}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}=o(1), \quad (2.40)$$

and we are done. □

We still owe a proof for Lemma 2.2.

Proof of Lemma 2.2. Since $\bar{V}_i^2 - V_L^2 = (\bar{V}_i - V_L)(\bar{V}_i + V_L)$ and almost surely $\bar{V}_i$ converges to V_L, to prove the first claim is enough to show that $(\bar{V}_i^2 - V_L^2) = o(1/i^{1/2-\epsilon})$. We write

$$\begin{aligned} \bar{V}_i^2 - V_L^2 &= 2F\sum_{j=1}^{i}[\xi_j X_{i,j}] - \frac{2F\mu}{p}\int_0^1 x^{\zeta-1}dx. \\ &= 2F\sum_{j=1}^{i}\left[\xi_j X_{i,j} - \mu\frac{j^{\zeta-1}}{pi^{\zeta}}\right] + \frac{2F\mu}{p}\left[\frac{1}{i}\sum_{j=1}^{i}\left(\frac{j}{i}\right)^{\zeta-1} - \int_0^1 x^{\zeta-1}dx\right]. \end{aligned}$$

The second term on the right-hand side of this equation is an $O(1/i)$. We break down the first term as follows

$$2F\sum_{j=1}^{i}\bar{\xi}_j\frac{j^{\zeta-1}}{pi^{\zeta}} + 2F\sum_{j=1}^{i}\bar{\xi}_j\left(X_{i,j} - \frac{j^{\zeta-1}}{pi^{\zeta}}\right) + 2F\mu\sum_{j=1}^{i}\left(X_{i,j} - \frac{j^{\zeta}}{pi^{\zeta}}\right). \tag{2.41}$$

Setting $\bar{S}_0 = 0$ and $\bar{S}_k := \sum_{l=1}^{k}\bar{\xi}_k$, $k \in \mathbb{N}$, we write the first term on the right of (2.41) as

$$\sum_{j=1}^{i}\left[(\bar{S}_j - \bar{S}_{j-1})\frac{j^{\zeta-1}}{pi^{\zeta}}\right] = \sum_{j=1}^{i-1}\left[\bar{S}_j\left(\frac{j^{\zeta-1}}{pi^{\zeta}} - \frac{(j+1)^{\zeta-1}}{pi^{\zeta}}\right)\right] + \frac{\bar{S}_i}{pi} = o(1/i^{1/2-\epsilon}),$$

where the last equality follows by the Law of the Iterated Logarithm.

Analogously, we write the second term on the right of (2.41) as

$$\sum_{j=1}^{i-1}[\bar{S}_j(X_{i,j} - X_{i,j+1})] + \sum_{j=1}^{i-1}\left[\bar{S}_j\left(\frac{(j+1)^{\zeta-1}}{pi^{\zeta}} - \frac{j^{\zeta-1}}{pi^{\zeta}}\right)\right] + \bar{S}_i\left(X_{i,i} - \frac{1}{ip}\right). \tag{2.42}$$

Recalling (2.4), one readily checks that that $|X_{i,j} - X_{i,j+1}| = O(|X_{i,j+1}|/(M_j+1))$. Given $\varepsilon > 0$, by Lemma 2.1 we a.s. find an $m \in \mathbb{N}$ such that $|X_{i,j+1}| \le (1+c)(j+1)^{\zeta-1}/(pi^{\zeta})$ for every $i \ge j \ge m$. Therefore, again by the Law of Large Numbers and the Law of the Iterated Logarithm, the three terms on (2.42) are $o(1/i^{1/2-\epsilon})$.

To deal with the third and last term on the right of (2.41), we may proceed similarly as in the analysis of $W_{2,n}$ above — recall (2.22), (2.25), (2.27) and (2.33). We write

$$\begin{aligned} \sum_{j=1}^{i}\left[X_{i,j} - \frac{j^{\zeta}}{pi^{\zeta}}\right] &= \sum_{j=1}^{i}\left[\frac{j^{\zeta}}{i^{\zeta}}\left(\frac{1}{M_j} - \frac{1}{pj}\right)\right] \\ &+ \sum_{j=1}^{i}\left[\frac{j^{\zeta}}{i^{\zeta}}\left(\frac{1}{M_j} - \frac{1}{pj}\right)(R_{i,j} + O(R_{i,j}^2))\right] + \sum_{j=1}^{i}\left[\frac{j^{\zeta-1}}{pi^{\zeta}}(R_{i,j} + O(R_{i,j}^2))\right]. \end{aligned} \tag{2.43}$$

In the proof of Lemma 2.4, we have shown that almost surely, for $i \geq j$ sufficiently large, $|R_{i,j}| \leq 1/j^{1/2-\epsilon}$, and we also argued that $(1/M_j - 1/(pj)) = o(1/j^{3/2-\epsilon})$. Using this estimates, we readly get that each of the terms on the right hand side of (2.43) is an $o(1/i^{1/2-\epsilon})$, for $0 < \epsilon < 1/4$, and thus, so is the left hand side of (2.43), and we are done with the first claim of the lemma.

To argue the last claim of the lemma, note that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\xi_i \left(\bar{V}_{i-1} - V_L \right)^2 \right] = o\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\xi_i}{i^{1-2\epsilon}} \right)$$
$$= o\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\mu}{i^{1-2\epsilon}} \right) + o\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\bar{\xi}_i}{i^{1-2\epsilon}} \right) = o(1/n^{1/2-2\epsilon}),$$

where the last equality holds by the hypothesis that ξ_1 has finite second moment and the Two Series Theorem, and we are done. □

Proceeding analogously as in the proof of Proposition 2.1, similarly breaking down the relevant quantities, we may also obtain a central limit theorem for the velocity of the t.p. on the modified process (at collision times), namely

Proposition 2.2. *Let* $\mathrm{Var}(\xi_1) = \sigma^2 < \infty$. *Then, as* $n \to \infty$,

$$\sqrt{n} \left(\bar{V}_n - V_L \right) \Longrightarrow \mathcal{N}(0, \hat{\sigma}_v^2),$$

where $\hat{\sigma}_v > 0$.

2.3 Central Limit Theorem for the Original Process

In this section, we prove our main results.

2.3.1 Proof of Theorem 2.2

For each, $i \in \mathbb{N}$, let t_i be the instant when the t.p. collides for the first time with the initial i−th particle in the line; more precisely, t_i is such that $Q(t_i) = S_i$. It is enough to show a CLT along (t_i), and for that it suffices to establish a version of Proposition 2.1 with barred quantities replaced by respective unbarred quantities, which ammounts to replacing $\bar{t}_n$ by t_n in (2.12), namely showing that $(S_n - t_n V_L)/\sqrt{n} \Longrightarrow \mathcal{N}(0, \hat{\sigma}_q^2)$. Theorem 2.2 readily follows with $\sigma_q^2 = \frac{V_L}{\mu} \hat{\sigma}_q^2$.

We use Proposition 2.1 and a comparison between $\bar{t}_i$ and t_i to conclude our proof. Due to Proposition 2.1, it is enough to argue that

$$\frac{t_n - \bar{t}_n}{\sqrt{n}} = o(1). \tag{2.44}$$

Let $s_1, s_2, \ldots$ be the instants when the t.p. recollides with a moving elastic particle, whose velocities will be, respectively, denoted by $v_1, v_2, \ldots$. As follows from the remarks in the Introduction on the fact that the dynamics is a.s. well defined — see paragraph right below (2.2) — these sequences are well defined, and $s_1, s_2, \ldots$ has no limit points. We also recall that, for each $l \in \mathbb{N}$, $V(s_l)$ and $V(s_l^+)$ denote the velocities of the t.p. immediately before and at the l-th recollision, respectively.

For each $j \in \mathbb{N}$ we define

$$\Delta(j) := \sum_{s_l \in [t_{j-1}, t_j]} \left[V^2(s_l) - V^2(s_l^+) \right] \quad \text{and} \quad \delta(j) := \sum_{s_l \in [t_{j-1}, t_j]} \left[V(s_l) - v_l \right]. \tag{2.45}$$

As follows from what has been pointed out in the above paragraph, these sums are a.s. well defined and consist of finitely many terms.

Let $v : [0, \infty) \longrightarrow \mathbb{R}$ denote the function that associates the position x to the velocity of the t.p. at x, that is, $v(x) = V(Q^{-1}(x))$. We analogously define $\bar{v} : [0, \infty) \longrightarrow \mathbb{R}$ for the modified process. We have that

$$t_n = \int_0^{S_n} \frac{1}{v(x)}\, dx \quad \text{and} \quad \bar{t}_n = \int_0^{S_n} \frac{1}{\bar{v}(x)}\, dx.$$

In this way, (2.44) becomes

$$\int_0^{S_n} \left(\frac{1}{v(x)} - \frac{1}{\bar{v}(x)} \right) dx = o(n^{1/2}),$$

and due to convergence of $v(x)$ and $\bar{v}(x)$, it is enough to argue that

$$\int_0^{S_n} \left(\bar{v}^2(x) - v^2(x) \right) dx = o(n^{1/2}).$$

Torricelli's equation, (2.1), (2.2) and (2.45), give us that, for each $i \in \mathbb{N}$, at position $x \in [S_{i-1}, S_i)$,

$$\bar{v}^2(x) - v^2(x) = \sum_{j=1}^{i-1} \left[\Delta(j) \prod_{k=j}^{i-1} \left(\frac{M_k + (\eta_k - 1)}{M_k + 1} \right)^2 \right] + \sum_{s_l \in [S_{i-1}, x)} \left(V^2(s_l) - V^2(s_l^+) \right). \tag{2.46}$$

Therefore, we have the following upper bound

$$\int_0^{S_n} \left(\bar{v}^2(x) - v^2(x)\right) dx \leq \sum_{i=1}^{n} \left[\xi_i \sum_{j=1}^{i-1} \left(\Delta(j) \prod_{k=j}^{i-1} \left(\frac{M_k + (\eta_k - 1)}{M_k + 1} \right)^2 \right) \right] + \sum_{i=1}^{n} \xi_i \Delta(i).$$

Turning back to (2.2), we have that,

$$V(s_j) - V(s_j^+) = V(s_j) - \left(\frac{M(s_j) - 1}{M(s_j) + 1} V(s_j) + \frac{2}{M(s_j) + 1} v_j \right) = \frac{2(V(s_j) - v_j)}{M(s_j) + 1}.$$

And therefore, again by the fact that $V(\cdot)$ is convergent, recalling (2.45), we have that

$$\Delta(j) = O\left(\frac{\delta(j)}{M_j + 1} \right);$$

moreover, recalling (2.4), we have that

$$\sum_{i=1}^{n} \left[\xi_{i+1} \sum_{j=1}^{i} \left(\Delta(j) \prod_{k=j}^{i} \left(\frac{M_k + (\eta_k - 1)}{M_k + 1} \right)^2 \right) \right] + \sum_{i=1}^{n} \xi_{i+1} \Delta(i+1) =$$
$$O\left(\sum_{i=1}^{n} \left[\xi_{i+1} \sum_{j=1}^{i} \delta(j) X_{i,j} \right] + \sum_{i=1}^{n} \frac{\xi_{i+1} \delta(i+1)}{i+1} \right). \quad (2.47)$$

By Lemma 2.1,

$$\sum_{i=1}^{n} \left[\xi_{i+1} \sum_{j=1}^{i} \delta(j) X_{i,j} \right] = \sum_{j=1}^{n} \left[\delta(j) \sum_{i=j}^{n} \xi_{i+1} X_{i,j} \right] = O\left(\sum_{j=1}^{n} \left[\delta(j) j^{\zeta - 1} \sum_{i=j}^{n} \frac{\xi_{i+1}}{i^{\zeta}} \right] \right).$$

Since $\mathbb{E}\xi^2 < \infty$, Borel-Cantelli lemma readily implies that for every $\epsilon > 0$, $\mathbb{P}(\xi_{n+1} > \epsilon\sqrt{n} \text{ i.o.}) = 0$. Thus,

$$\sum_{j=1}^{n} \left[\delta(j) j^{\zeta - 1} \sum_{i=j}^{n} \frac{\xi_{i+1}}{i^{\zeta}} \right] = O\left(\sum_{j=1}^{n} \left[\delta(j) j^{\zeta - 1} \sum_{i=j}^{n} \frac{\epsilon\sqrt{i}}{i^{\zeta}} \right] \right) =$$
$$\epsilon\sqrt{n} O\left(\sum_{j=1}^{n} \left[\delta(j) j^{\zeta - 1} \sum_{i=j}^{n} \frac{1}{i^{\zeta}} \right] \right) = \epsilon\sqrt{n} O\left(\sum_{j=1}^{n} \delta(j) \right),$$

and also

$$\sum_{i=1}^{n} \frac{\xi_{i+1} \delta(i+1)}{i+1} = O\left(\sum_{i=1}^{n} \delta(i+1) \right).$$

By Lemma 2.10, we are done, since $\epsilon > 0$ is arbitrary.

Lemma 2.10. *Let $\delta(j)$ as defined in (2.45). Almost surely,*

$$\sum_{j=1}^{\infty} \delta(j) < \infty. \tag{2.48}$$

Proof. This result is already contained more or less explicitly in [FNV00], in the argument to prove Theorem 2.1 — see discussion on page 803 of [FNV00]. For completeness and simplicity, circularity notwithstanding, we present an argument relying on Theorem 2.1 directly.

There a.s. exists a time T_0 such that there are no recollisions with standing particles met by the t.p. after T_0. This is because at large times, the velocity of the t.p. is close enough to V_L and its mass close enough to infinity, so that new collisions with standing elastic particles will give them velocity roughly $2V_L$, and thus they will be thence unreachable by the t.p. This means that we have only finitely many particles that recollide with the t.p.

We may also conclude by an elementary reasoning using Theorem 2.1 that if a particle collides infinitely often with the t.p., then its velocity may never exceed V_L. Let $u_1, u_2, \ldots$ denote the recollision times with such a particle, and $v(u_1), v(u_2), \ldots$, its velocity at such times, respectively. As we can deduce from (2.2), $v(u_{i+1}) > V(u_i)$; thus,

$$\sum_{i=1}^{\infty} [V(u_i) - v(u_i)] < \sum_{i=1}^{\infty} [v(u_{i+1}) - v(u_i)] \leq V_L,$$

and (2.48) follows. □

2.3.2 Proof of Theorem 2.3

By Proposition 2.2, and the convergences of both V_n and $\bar{V}_n$, and after similar considerations as at the beginning of Subsection 2.3.1, we find that it is enough to prove that

$$\sqrt{n}\left(\bar{V}_n^2 - V_n^2\right) = o(1) \tag{2.49}$$

(so that in the end we get that Theorem 2.3 holds with $\sigma_v^2 = \frac{\mu}{V_L}\hat{\sigma}_v^2$).

Recalling (2.46), we have that

$$\bar{V}_n^2 - V_n^2 = \bar{v}^2(S_n) - v^2(S_n) = \sum_{j=1}^{n-1}\left[\Delta(j)\prod_{k=j}^{i-1}\left(\frac{M_k + (\eta_k - 1)}{M_k + 1}\right)^2\right] + \Delta(n).$$

Proceeding similarly as in the proof of Theorem 2.2, we find that

$$\sqrt{n}\left(\bar{V}_n^2 - V_n^2\right) = O\left(\sqrt{n}\sum_{j=1}^{n}\left[\delta(j)\frac{j^{\zeta-1}}{n^{\zeta}}\right] + \frac{\delta(n+1)}{\sqrt{n}}\right).$$

By Lemma 2.10, given $\epsilon > 0$, there almost surely exists $j_0 \in \mathbb{N}$ such that $\sum_{j \geq j_0} \delta(j) \leq \epsilon/2$. Thus,

$$\sqrt{n} \sum_{j=1}^{n} \left[\delta(j) \frac{j^{\zeta-1}}{n^{\zeta}} \right] \leq \frac{1}{n^{\zeta-1/2}} \sum_{j=1}^{j_0} \delta(j) j^{\zeta-1} + \sum_{j > j_0} \delta(j) \leq \epsilon,$$

for n sufficiently large. Lemma 2.10 implies that $\delta(n) = o(1)$. Since $\epsilon > 0$ is arbitrary, (2.49) follows.

2.4 Discussion

A fact that, if known, would simplify comparisons between the auxiliary and original process, is the occurrence of only a finite number of recollisions between the tracer particle and each elastic particle. Note that in [FNV00], the authors observe that, the number of elastic particles for which this does not happen, is finite — see also the proof of Lemma 2.10. It would be very interesting to obtain a proof for this fact.

When we replace the common finite mean distribution of the interparticle distances by an one having infinity mean, it is expected that tracer particle velocity converges to infinity. This is not difficulty verify on the auxiliary process, but it seems hard to compare both process, since we will have infinitely many recollisions with each elastic particle.

Chapter 3

Anisotropic Oriented Percolation

A common feature of percolation models defined on lattices is that they exhibit, in some sense, a mean-field behavior as the dimension of the lattice grows. In the particular case of oriented percolation, the seminal paper by Cox and Durrett [CD83] shows that the asymptotic behavior of the critical point is $1/d$. A little earlier, Holley and Liggett [HL81] proved the occurrence of an analogous behavior for the high dimensional contact process critical rate and, recently, a similar result was proved for the contact process with random rates on a high dimensional percolation open cluster, see [Xue16]. For the non-oriented case, Kesten [Kes90] and Gordon [Gor91] independently showed that the critical point is asymptotically $1/2d$ and, in the last three decades, a rather complete mean-field picture of high dimensional non-oriented percolation has emerged; see [HH17] and references therein.

Several variants of percolation models have been proposed over time, many of which arose from the analysis of physical phenomena, and in this case, an ingredient that accompanies various phenomena is anisotropy of the system. On anisotropic percolation, edges on distinct classes have different probability to be open. One of the first results on anisotropic percolation appears in 1964 with the Essam and Sykes work [ES64]. Kesten in [Kes82] gives the complete phase diagram of non-oriented anisotropic percolation on the square lattice, where vertical edges are open with probability p while horizontal edges are open with probability q. For $d \geq 3$, little is understood about the phase diagram of anisotropic percolation. The results of [CD83, Kes90, Gor91] are statements about one point in the parameter domain, the isotropic point, whereas in [CLS14] the authors show that the critical surface is everywhere continuous for a particular setup in the non-oriented case.

At this chapter, we investigate the phase diagram of oriented anisotropic percolation on $\mathbb{Z}^d$, $d \geq 4$, where edges oriented along the $i-th$ direction are open independently of all others with probability p_i. We analyze the interior of the domain in a region containing the isotropic point, showing that the critical surface behaves nicely around this point and that the anisotropy introduced in the system does not create any unexpected behavior. More precisely, we show that if $p_1 + \cdots + p_d$ is strictly greater than one and each p_i is not too large, then an infinity open cluster occurs.

3.1 The Model and Result

We consider the anisotropic oriented edge percolation model in $\mathbb{Z}^d$. Let $\{e_1, \ldots, e_d\}$ be the set of canonical unit vectors of $\mathbb{Z}^d$. Given $0 < p_1, \ldots, p_d < 1$, we declare each edge $\langle x, x + e_i \rangle$ to be open independently of each other with probability p_i for $i = 1, \ldots, d$. We denote the corresponding probability measure simply by $\mathbb{P}$.

An *oriented path* of length n starting at the origin in $\mathbb{Z}^d$ is a path $(x_0, x_1, \ldots, x_n)$ such that $x_0 = 0$ and $x_i - x_{i-1} \in \{e_1, \ldots, e_d\}$ for $i = 1, \ldots, n$. Let C_0 be the open cluster of the origin, that is, the set of vertices $x \in \mathbb{Z}^d$ such that there is an open path from 0 to x; we let $|C_0|$ denote the size of C_0.

We are now ready to state our main theorem.

Theorem 3.1. *Let $\varepsilon > 0$ be of the form $\varepsilon = 10/n$, for some $n \in \mathbb{N}$. Let $d \geq 4$, and let $p_1, \ldots, p_d$ be non-negative numbers such that*

1. $p_1 + \cdots + p_d \geq 1 + \varepsilon$,
2. $\max_{1 \leq i \leq d} \left\{ \dfrac{p_i}{p_1 + \cdots + p_d} \right\} < \dfrac{\varepsilon}{10}$,

then

$$\mathbb{P}\big\{|C_0| = \infty\big\} > 0.$$

Remark: Note that if $p_1 + \cdots + p_d < 1$ then a straightforward branching process argument shows that $\mathbb{P}\big\{|C_0| = \infty\big\} = 0$. Note also that, although the result is non-asymptotic, it only makes sense for d of order $1/\varepsilon$. We will see in the course of the proof of Theorem 3.1 that the constant $1/10$ in Condition 2 could be made as close to 1 as we wish, as long as the dimension is taken big enough. We mention that, for the isotropic case, Theorem 3.1 gives the bound $p_c(\mathbb{Z}^d) \leq 1/d + 10/d^2$.

The strategy of the proof is similar to the one in Cox-Durret [CD83]: we build a martingale and prove the convergence to a positive limit showing that it has bounded second moment. The main difficulty here is to estimate the second moments of the martingale. We do this by converting the martingale problem into a random walk problem and comparing the asymmetric case with the symmetric one.

3.1.1 Proof of Main Result

In this section we prove the main result modulo a lemma which is stated in the course of the proof.

Proof of Theorem 3.1. We want to prove that the open cluster of the origin is infinite, which happens if and only if there exists an infinite open path starting at the origin. Thus our problem can be naturally converted into a counting of the number of open paths of length n starting on 0, as long as we have a good control of the second moment of this random variable.

For each $n \in \mathbb{N}$ let V_n be the set of the vertices in the n-th level , i.e.,

$$V_n = \{x \in \mathbb{Z}^d : x_1 + \cdots + x_d = n,\ x_i \geq 0\}, \tag{3.1}$$

and $\mathcal{C}_n$ to be the set of all possible oriented paths from the origin to V_n, i.e.,

$$\mathcal{C}_n = \{\gamma = (0 = v_0, v_1, v_2, \ldots, v_n) : v_j \in V_j \text{ and } v_j - v_{j-1} \in V_1,\ j = 1, \ldots, n\}. \tag{3.2}$$

We define X_n to be the random variable which counts the open paths from the origin up to level n, i.e.,

$$X_n := \sum_{\gamma \in \mathcal{C}_n} 1_{\{\gamma \text{ is open}\}}, \tag{3.3}$$

and write $\mu := \mathbb{E}[X_1] = p_1 + \cdots + p_d$. A simple calculation shows that $\mathbb{E}[X_n] = \mu^n$. Now, define

$$W_n := \frac{X_n}{\mu^n}. \tag{3.4}$$

We observe that $\{W_n\}_{n \in \mathbb{N}}$ is a positive martingale. For this, consider $x \in V_n$ and let $\mathcal{C}_x = \{\gamma \in \mathcal{C}_n : f(\gamma) = x\}$, where $f(\gamma)$ denotes the final vertex of γ. Define also the random variables

$$Y_x = \sum_{i=1}^{d} 1_{\{\langle x, x+e_i \rangle \text{ is open}\}} \text{ and } N_x = \sum_{\gamma \in \mathcal{C}_x} 1_{\{\gamma \text{ is open}\}},$$

where Y_x counts the number of oriented open edges leaving x and N_x counts the number of oriented open paths from 0 to x, respectively. Observe that

$$X_n = \sum_{x \in V_n} N_x \text{ and } X_{n+1} = \sum_{x \in V_n} N_x \cdot Y_x. \tag{3.5}$$

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$; observe that Y_x is independent of $\mathcal{F}_n$ for each $x \in V_n$ and that N_x is $\mathcal{F}_n$-measurable. We also have $\mathbb{E}[Y_x] = \mu$ for all x, so

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \sum_{x \in V_n} \mathbb{E}[N_x \cdot Y_x|\mathcal{F}_n] = \mu X_n.$$

Thus $\mathbb{E}[W_{n+1}|\mathcal{F}_n] = W_n$, as wanted.

Since W_n is a positive martingale, it converges to a non-negative random variable W. Assume for the moment the following lemma.

Lemma 3.1. *Let $\{W_n\}_n$ be as defined in (3.4). Then, under the conditions of Theorem 3.1 we have*

$$\sup_n \mathbb{E}[W_n^2] < \infty.$$

From this lemma it follows that W_n converges to W in L_1 and since $\mathbb{E}(W_n) = 1$ we have $\mathbb{P}(W > 0) > 0$. Noticing that $X_n > 0$ for all n in the event $\{W > 0\}$, the theorem follows. □

The proof of Lemma 3.1 in the anisotropic case requires more than a direct adaptation of Cox-Durret results. The next sections are dedicated to this work.

The structure of the remainder of the chapter is the following: in Section 3.2 we convert the martingale problem of Lemma 3.1 into a random walk problem, in Section 3.3 we compare asymmetric random walks with the symmetric case to finish the proof of Lemma 3.1 and in Section 3.4 we make some final remarks.

3.2 Preliminary Lemmas

In this section we state and prove four lemmas with the goal of converting the martingale problem of Lemma 3.1 into a random walk problem.

3.2.1 Open Paths Equivalencies

In the first lemma we give a criterion for bounding $\sup_m \mathbb{E}[W_m^2]$. To do that, we introduce the following notation. Given a path $\gamma = (0, v_1, \dots, v_n) \in \mathcal{C}_n$, we define, for each $i \in \{1, \dots, n\}$,

$$i(\gamma) := v_i \quad \text{and} \quad f(\gamma) := v_n. \tag{3.6}$$

Lemma 3.2. *Let*

$$a_n := \frac{1}{\mu^{2n}} \sum_{(\gamma_1,\gamma_2) \in \mathcal{C}_n^2, f(\gamma_1)=f(\gamma_2)} \mathbb{P}(\gamma_1, \gamma_2 \ \ \textit{open}). \tag{3.7}$$

Then

$$\sup_m \mathbb{E}[W_m^2] < \infty \quad \textit{iff} \quad \sum_{n=1}^{\infty} a_n < \infty.$$

Proof. By (3.5) we have

$$\mathbb{E}[X_{n+1}^2|\mathcal{F}_n] = \mathbb{E}\left[\sum_{(x,y)\in V_n^2} N_x Y_x N_y Y_y \middle| \mathcal{F}_n\right] = \mu^2 X_n^2 + (\mathrm{Var} Y_0) \sum_{x\in V_n} N_x^2.$$

Therefore

$$\mathbb{E}[W_{n+1}^2] = \frac{\mu^2 \mathbb{E}[X_n^2]}{\mu^{2n+2}} + \frac{(\mathrm{Var} Y_0)}{\mu^2} \frac{\mathbb{E}\left[\sum_{x\in V_n} N_x^2\right]}{\mu^{2n}}.$$

Using the definition of N_x we can see that $\mathbb{E}[N_x^2] = \sum_{(\gamma_1,\gamma_2)\in\mathcal{C}_x^2} \mathbb{P}(\gamma_1, \gamma_2 \ \text{ open})$ so

$$\begin{aligned} \frac{\mathbb{E}\left[\sum_{x\in V_n} N_x^2\right]}{\mu^{2n}} &= \frac{\sum_{x\in V_n} \sum_{(\gamma_1,\gamma_2)\in\mathcal{C}_x^2} \mathbb{P}(\gamma_1, \gamma_2 \ \text{ open})}{\mu^{2n}} \\ &= \frac{1}{\mu^{2n}} \sum_{(\gamma_1,\gamma_2)\in\mathcal{C}_n^2, f(\gamma_1)=f(\gamma_2)} \mathbb{P}(\gamma_1, \gamma_2 \ \text{ open}), \end{aligned}$$

which is exactly the definition of a_n. Hence

$$\mathbb{E}[W_{n+1}^2] = \mathbb{E}[W_n^2] + \frac{(\mathrm{Var} Y_0)}{\mu^2} a_n.$$

Iterating the recursion above, we have

$$\mathbb{E}[W_{n+1}^2] = \mathbb{E}[W_1^2] + \frac{(\mathrm{Var} Y_0)}{\mu^2} \sum_{j=1}^{n} a_j,$$

and the result follows. □

Given $m \in \mathbb{N}$, let

$$A_m := \{(\gamma_1, \gamma_2) \in \mathcal{C}_m^2 : i(\gamma_1) \neq i(\gamma_2) \quad \text{for all } i \neq m \quad \text{and} \quad f(\gamma_1) = f(\gamma_2)\} \tag{3.8}$$

be the set of the pair of paths of size m which meet on the their final vertices and define

$$b_m := \sum_{(\gamma_1,\gamma_2)\in A_m} \frac{\mathbb{P}(\gamma_1, \gamma_2 \ \text{ open})}{\mu^{2m}}. \tag{3.9}$$

Lemma 3.3. *Let a_n as defined in (3.7) and b_m as defined in (3.9). Then*

$$\sum_{n=1}^{\infty} a_n = \sum_{j=1}^{\infty} \left(\sum_{m=1}^{\infty} b_m \right)^j . \tag{3.10}$$

Proof. Recall that

$$a_n := \frac{1}{\mu^{2n}} \sum_{(\gamma_1,\gamma_2)\in\mathcal{C}_n^2, f(\gamma_1)=f(\gamma_2)} \mathbb{P}(\gamma_1, \gamma_2 \text{ open}). \tag{3.11}$$

We will show that the set $\{(\gamma_1,\gamma_2) \in \mathcal{C}_n^2 : f(\gamma_1) = f(\gamma_2)\}$ can be partitioned by the number of vertices in the intersection of γ_1 and γ_2. Let

$$I(\gamma_1,\gamma_2) = \{i : i(\gamma_1) = i(\gamma_2)\}$$

and writing $M = m_1 + \cdots + m_j$,

$$C(m_1,\ldots,m_j) := \big\{(\gamma_1,\gamma_2) \in \mathcal{C}_M^2 : I(\gamma_1,\gamma_2) = \{m_1, m_1+m_2, \ldots, m_1+\cdots+m_j\}\big\},$$

we have

$$\{(\gamma_1,\gamma_2) \in \mathcal{C}_M^2 : f(\gamma_1) = f(\gamma_2)\} = \bigsqcup_{j=1}^{n} \bigsqcup_{\substack{(m_1,\ldots,m_j)\in\mathbb{N}^j \\ m_1+\cdots+m_j=M}} C(m_1,\ldots,m_j). \tag{3.12}$$

Given two paths $\gamma_1 = (0, v_1, \ldots, v_n)$ and $\gamma_2 = (0, w_1, \ldots, w_m)$ we define the *concatenation* of γ_1 and γ_2 by $\gamma_1 \circ \gamma_2 := (0, v_1, \ldots, v_n, v_n + w_1, \ldots, v_n + w_m)$. Observe that, given a sequence of positive integers $(m_1,\ldots,m_j) \in \mathbb{N}^j$, and recalling (3.8), we have

$$C(m_1,\ldots,m_j) = \{(\gamma_1,\gamma_2) \in \mathcal{C}_M^2 : \gamma_1 = \gamma_{1,1} \circ \cdots \circ \gamma_{1,j}, \gamma_2 = \gamma_{2,1} \circ \cdots \circ \gamma_{2,j},$$
$$(\gamma_{1,k}, \gamma_{2,k}) \in A_{m_k}, \forall k = 1,\ldots,j\}.$$

Using (3.12) we can rewrite (3.11) as

$$\sum_{n=1}^{\infty} a_n = \sum_{j=1}^{\infty} \sum_{(m_1,\ldots,m_j)\in\mathbb{N}^j} \sum_{(\gamma_1,\gamma_2)\in C(m_1,\ldots,m_j)} \frac{\mathbb{P}(\gamma_1,\gamma_2 \text{ open})}{\mu^{2M}}. \tag{3.13}$$

From the definitions of $C(m_1,\dots,m_j)$ and b_m mentioned earlier, it follows that

$$\begin{aligned}\sum_{(\gamma_1,\gamma_2)\in C(m_1,\dots,m_j)} \frac{\mathbb{P}(\gamma_1,\gamma_2 \text{ open})}{\mu^{2M}} &= \sum_{\substack{(\gamma_{1,1}\circ\cdots\circ\gamma_{1,j},\gamma_{2,1}\circ\cdots\circ\gamma_{2,j})\\ (\gamma_{1,k},\gamma_{2,k})\in A_{m_k},k=1,\dots,j}} \prod_{k=1}^{j} \frac{\mathbb{P}(\gamma_{1,k},\gamma_{2,k} \text{ open})}{\mu^{2m_k}} \\ &= \prod_{k=1}^{j}\left[\sum_{(\gamma_1,\gamma_2)\in A_{m_k}} \frac{\mathbb{P}(\gamma_1,\gamma_2 \text{ open})}{\mu^{2m_k}}\right] \\ &= \prod_{k=1}^{j} b_{m_k}.\end{aligned}$$

Substituting the expression above in (3.13) we obtain

$$\sum_{n=1}^{\infty} a_n = \sum_{j=1}^{\infty} \sum_{(m_1,\cdots,m_j)\in\mathbb{N}^j} \left[\prod_{k=1}^{j} b_{m_k}\right] = \sum_{j=1}^{\infty}\left(\sum_{m=1}^{\infty} b_m\right)^j,$$

and the result follows. □

3.2.2 Converting the Problem to a Random Walk

Using the lemmas in the previous section, we have, so far,

$$\sup_m \mathbb{E}[W_m^2] < \infty \ \text{ iff } \ \sum_{m=1}^{\infty} b_m < 1;$$

we will now use random walks to compute b_n. In the remainder of the text, we will consider several independent random walks and we use $\mathbb{Q}$ to denote the probability on a space where they all live in harmony.

We say that $q = (q_1,\dots,q_d)$ is a (d-dimensional) *positive vector* if $0 \le q_1,\dots,q_d$ and $q_1+\cdots+q_d > 0$, and we say that q is a (d-dimensional) *probability vector* if it is a positive vector with $q_1+\cdots+q_d = 1$. Given a positive vector q we say that $\{S_n\}_n$ is the oriented random walk *associated to* q if

$$S_n = \xi_1 + \cdots + \xi_n, \tag{3.14}$$

where $\xi_1,\dots,\xi_n$ are i.i.d. random variables with

$$\mathbb{Q}(\xi_1 = e_i) = \frac{q_i}{q_1+\cdots+q_d}, \quad \text{for all } i = 1,\dots,d.$$

Given two independent random walks associated to q, $\{S_n^1\}_n$ and $\{S_n^2\}_n$ we define

$$\tau = \tau(\{S_n^1\}_n,\{S_n^2\}_n) := \inf\{n \ge 1 : S_n^1 = S_n^2\}. \tag{3.15}$$

Lemma 3.4. *Let b_m be as defined in (3.9), A_m be as in (3.8) and $\{S_n^1\}_n$, $\{S_n^2\}_n$ be two independent random walks associated to $p = (p_1, \dots, p_d)$. Then*

$$\sum_{m=1}^{\infty} b_m < 1 \quad \textit{iff} \quad \sum_{m=2}^{\infty} \mathbb{Q}(\tau = m) < 1 - \frac{1}{\mu}.$$

Proof. Observe that

$$b_1 = \sum_{(\gamma_1,\gamma_2)\in A_1} \frac{\mathbb{P}(\gamma_1, \gamma_2 \ \text{ open})}{\mu^2} = \sum_{i=1}^{d} \frac{\mathbb{P}(\langle 0, e_i\rangle, \langle 0, e_i\rangle \ \text{ open})}{\mu^2} = \sum_{i=1}^{d} \frac{p_i}{\mu^2} = \frac{1}{\mu}.$$

Also, for $m \geq 2$, we have

$$\begin{aligned} b_m &= \sum_{(\gamma_1,\gamma_2)\in A_m} \frac{\mathbb{P}(\gamma_1, \gamma_2 \ \text{ open})}{\mu^{2m}} \\ &= \sum_{(\gamma_1,\gamma_2)\in A_m} \mathbb{Q}\big((0, S_1^1, \cdots, S_m^1) = \gamma_1\big) \cdot \mathbb{Q}\big((0, S_1^2, \cdots, S_m^2) = \gamma_2\big) \\ &= \mathbb{Q}(\tau = m), \end{aligned}$$

thus

$$\sum_{m=1}^{\infty} b_m = \frac{1}{\mu} + \sum_{m=2}^{\infty} \mathbb{Q}(\tau = m),$$

and the result follows. □

Lemma 3.5. *Let $\{S_n^1\}_n$ and $\{S_n^2\}_n$ be two independent random walks associated to a positive vector q and let τ be as defined in (3.15). Then*

$$\sum_{m=1}^{\infty} \mathbb{Q}(\tau = m) \leq 1 - \frac{1}{\mu} \quad \textit{iff} \quad \sum_{k=1}^{\infty} \mathbb{Q}(S_k^1 = S_k^2) \leq \mu - 1.$$

Proof. Observe that

$$\mathbb{Q}(S_m^1 = S_m^2) = \sum_{n\leq m} \mathbb{Q}(\tau = n)\mathbb{Q}(S_{m-n}^1 = S_{m-n}^2), \tag{3.16}$$

so

$$\begin{aligned} \sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2) &= \sum_{m=1}^{\infty} \sum_{n\leq m} \mathbb{Q}(\tau = n)\mathbb{Q}(S_{m-n}^1 = S_{m-n}^2) \\ &= \sum_{n=1}^{\infty} \left[\mathbb{Q}(\tau = n) \sum_{m\geq 0} \mathbb{Q}(S_m^1 = S_m^2) \right] \\ &= \left(\sum_{n=1}^{\infty} \mathbb{Q}(\tau = n) \right) \left(1 + \sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2) \right), \end{aligned}$$

and therefore

$$\sum_{n=1}^{\infty} \mathbb{Q}(\tau = n) = \frac{\sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2)}{1 + \sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2)} = 1 - \frac{1}{1 + \sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2)}. \quad (3.17)$$

Finally, we have

$$1 - \frac{1}{1 + \sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2)} \leq 1 - \frac{1}{\mu} \quad \text{iff} \quad \sum_{m=1}^{\infty} \mathbb{Q}(S_m^1 = S_m^2) \leq \mu - 1,$$

and this finishes the proof. □

3.3 Analysis of the Random Walks

In this section, we will estimate the maximal probability of two i.i.d random walks meeting in a fixed time as a function of their parameters.

Given a probability vector $q = (q_1, \dots, q_d)$, let $\{S_n^1\}_n$ and $\{S_n^2\}$ be two independent random walks associated to q and define

$$\lambda(q) := \sum_{n=1}^{\infty} \mathbb{Q}(S_n^1 = S_n^2). \quad (3.18)$$

Combining the results of the previous sections, and observing that $\mathbb{Q}(\tau = 1) > 0$, we have

$$\text{If} \quad \lambda(q) \leq \mu - 1, \quad \text{then} \quad \sup_n \mathbb{E}[W_n^2] < \infty. \quad (3.19)$$

In this section, we investigate the behavior of $\lambda(q)$.

Theorem 3.2. *Let $d \geq 4$ and $4 \leq m \leq d$ be integers. Consider the d-dimensional probability vector $q^* = q_d^*(m) = (1/m, 1/m, \dots, 1/m, 0, \dots, 0)$, and let $\lambda(q)$ be as in (3.18). Then*

(a) $\lambda(q^*) \leq \dfrac{10}{m}$

(b) for all d-dimensional probability vectors q such that $\max_i q_i \leq \dfrac{1}{m}$ *we have*

$$\lambda(q) \leq \lambda(q^*).$$

We will prove Theorem 3.2 in the next sections, but before that we use it to prove Lemma 3.1.

Proof of Lemma 3.1. Let $q_i = \frac{p_i}{\sum p_i}$. Under the hypothesis of the Theorem 3.1, we have $q_i \le \varepsilon/10$ for $1 \le i \le d$; taking $m = 10/\varepsilon = n$ we obtain $q_i \le 1/m$. And taking $q = (q_1, \dots, q_d)$, it follows from Theorem 3.2 that $\lambda(q) \le 10/m = \varepsilon$. Finally, by (3.19) we have $\sup_n \mathbb{E}[W_n^2] < \infty$ and the lemma follows. □

3.3.1 Bounding the Probability of Meeting

In this subsection we prove Item (a) in Theorem 3.2. Observe that for all $d \ge m$ we have

$$\lambda(\underbrace{1/m, 1/m, \dots, 1/m}_{m}, \underbrace{0, \dots, 0}_{d-m}) = \lambda(\underbrace{1/m, 1/m, \dots, 1/m}_{m}).$$

For $m \le 3$, we have that $\lambda(q^*) = \infty$, so we let $m \ge 4$. Then

$$\begin{aligned} \lambda(1/m, 1/m, \dots, 1/m) &= \sum_{n=1}^{\infty} \sum_{\substack{(l_1,\dots,l_m)\in\mathbb{N}^m \\ l_1+\dots+l_m=n}} \binom{n}{l_1, \dots, l_m}^2 \frac{1}{m^{2n}} \\ &\le \sum_{n=1}^{\infty} \left[\max_{\substack{(l_1,\dots,l_m)\in\mathbb{N}^m \\ l_1+\dots+l_m=n}} \left\{ \binom{n}{l_1, \dots, l_m} \right\} \frac{1}{m^n} \right]. \end{aligned}$$

We will split the first sum in two parts, the first for $n \le m$ and the second for $n > m$, and bound each one separately.

Observe that for $n = 1, \dots, m$ the maximum inside the brackets is bounded by $n!$, so the sum for $n \le m$ is bounded by

$$\sum_{n=1}^{m} \frac{n!}{m^n} \le \frac{1}{m} + \frac{2}{m^2} + \sum_{n=3}^{m} \frac{3!}{m^3} < \frac{1}{m} + \frac{8}{m^2}. \tag{3.20}$$

Now, for each $j \ge 1$ and $0 \le \ell \le m-1$, we use Stirling's bounds,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}},$$

to obtain, for $n = jm + \ell$,

$$
\begin{aligned}
\max_{\substack{(l_1,\dots,l_m)\in\mathbb{N}^n\\ l_1+\dots+l_m=n}}\left\{\binom{n}{l_1,\dots,l_m}\right\} &= \frac{(jm+\ell)!}{[(j+1)!]^\ell (j!)^{m-\ell}}\\
&\le \frac{\sqrt{2\pi(jm+\ell)}\left(\frac{jm+\ell}{e}\right)^{jm+\ell} e^{\frac{1}{12(jm+\ell)}}}{\left[\sqrt{2\pi(j+1)}\left(\frac{j+1}{e}\right)^{j+1}\right]^\ell \times \left[\sqrt{2\pi j}\left(\frac{j}{e}\right)^j\right]^{m-\ell}}\\
&\le \frac{e^{\frac{1}{12n}}\sqrt{m}\cdot m^{jm+\ell}}{\left[\sqrt{2\pi j}\right]^{m-1}} \frac{\left(j+\frac{\ell}{m}\right)^{jm+\ell}}{(j+1)^{j\ell+\ell}\times j^{jm-j\ell}}.
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
\frac{\left(j+\frac{\ell}{m}\right)^{jm+\ell}}{(j+1)^{j\ell+\ell}\times j^{jm-j\ell}} &= \left(1-\frac{m-\ell}{m(j+1)}\right)^{(j+1)\ell}\left(1+\frac{\ell}{mj}\right)^{j(m-\ell)}\\
&\le \exp\left(-\frac{(m-\ell)}{m}\ell\right)\times \exp\left(\frac{\ell}{m}(m-\ell)\right)\\
&= 1,
\end{aligned}
$$

thus

$$
\max_{\substack{(l_1,\dots,l_m)\in\mathbb{N}^n\\ l_1+\dots+l_m=n}}\left\{\binom{n}{l_1,\dots,l_m}\right\} \le \frac{e^{\frac{1}{12m}}\cdot\sqrt{m}\cdot m^{mj+\ell}}{(\sqrt{2\pi})^{m-1}(\sqrt{j})^{m-1}}. \tag{3.21}
$$

Using the bound above we obtain

$$
\begin{aligned}
\lambda(q^*) &\le \frac{1}{m}+\frac{8}{m^2}+\frac{e^{\frac{1}{12m}}\cdot m\cdot\sqrt{m}}{(\sqrt{2\pi})^{m-1}}\sum_{j=1}^{\infty}\frac{1}{j^{\frac{m-1}{2}}}\\
&\le \frac{1}{m}+\frac{8}{m^2}+\frac{e^{\frac{1}{12m}}\cdot m\cdot\sqrt{m}}{(\sqrt{2\pi})^{m-1}}\left(1+\frac{2}{m-3}\right).
\end{aligned}
$$

For $m \ge 4$ we then have

$$
\frac{8}{m^2}\le\frac{2}{m} \quad\text{and}\quad \frac{e^{\frac{1}{12m}}\cdot m\cdot\sqrt{m}}{(\sqrt{2\pi})^{m-1}}\left(1+\frac{2}{m-3}\right)\le\frac{7}{m}, \tag{3.22}
$$

and the result follows.

3.3.2 Projecting the Random Walks

We now want to understand the behavior of a random walk in $\mathbb{Z}^d$. To do that we will split the random walk in $\mathbb{Z}^d$ in two "normalized" projections in $\mathbb{Z}^2$ and $\mathbb{Z}^{d-2}$ and consider the behavior of the two parts to determine the behavior of the main random walk.

Given two independent oriented d-dimensional random walks $\{S_n^1(q)\}_n$, $\{S_n^2(q)\}_n$ associated with the probability vector $q = (q_1, \ldots, q_d)$, for $i = 1, 2$, let $\{R_n^i(q)\}_n$ be two independent bi-dimensional oriented random walks associated with (q_1, q_2) and let $\{U_n^i(q)\}_n$ be two independent $(d-2)$-dimensional oriented random walks associated with $(q_3, \ldots, q_d)$. Writing, for $i = 1, 2$, $S_n^i(q) = (S_n^i(q)_1, \ldots, S_n^i(q)_d)$, we define two complementary bi-dimensional oriented new random walks, $\{\tilde{S}_n^1(q)\}_n$ and $\{\tilde{S}_n^2(q)\}_n$ coupled with $\{S_n^1(q)\}_n$ and $\{S_n^2(q)\}_n$ respectively, where

$$\tilde{S}_n^i(q) = \left(S_n^i(q)_1 + S_n^i(q)_2, S_n^i(q)_3 + \cdots + S_n^i(q)_d\right).$$

Clearly $\{\tilde{S}_n^1(q)\}_n$ and $\{\tilde{S}_n^2(q)\}_n$ are independents and have the same distribution of a random walk associated with the probability vector $\tilde{q} := (q_1+q_2, q_3+\cdots+q_d)$. We will omit the dependency on q until the proof of Theorem 3.2.

One can think of the those newly defined random walks defined above as psudo-projections of the original random walks and the next lemma will express the meeting probability of the first in term of the latter.

Lemma 3.6. *Let $\{S_n^1\}_n$ and $\{S_n^2\}_n$ be two independent random walks associated with the probability vector $q = (q_1, \ldots, q_d)$. Then*

$$\mathbb{Q}(S_n^1 = S_n^2) = \sum_{\substack{(j,k)\in\mathbb{N}^2 \\ j+k=n}} \mathbb{Q}(\tilde{S}_n^1 = \tilde{S}_n^2 = (j,k))\mathbb{Q}(R_j^1 = R_j^2)\mathbb{Q}(U_k^1 = U_k^2). \tag{3.23}$$

Proof. In fact

$$\mathbb{Q}(S_n^1 = S_n^2) = \sum_{\substack{(j,k)\in\mathbb{N}^2 \\ j+k=n}} \mathbb{Q}(S_n^1 = S_n^2, \tilde{S}_n^1 = \tilde{S}_n^2 = (j,k)). \tag{3.24}$$

Now, observe that fixed $(j,k) \in \mathbb{N}^2$ such that $j+k=n$, we have

$$\mathbb{Q}(S_n^1 = S_n^2, \tilde{S}_n^1 = \tilde{S}_n^2 = (j,k)) = \left[\binom{j+k}{j}^2 (q_1+q_2)^{2j} \cdot (q_3+\cdots+q_d)^{2k}\right] \cdot \left[\sum_{l=0}^{j} \binom{j}{l}^2 \left(\frac{q_1}{q_1+q_2}\right)^{2l} \left(\frac{q_2}{q_1+q_2}\right)^{2(j-l)}\right] \cdot \left[\sum_{l_3+\cdots+l_d=k} \binom{k}{l_3,\ldots,l_d}^2 \left(\frac{q_3}{q_3+\cdots+q_d}\right)^{2l_3} \cdots \left(\frac{q_d}{q_3+\cdots+q_d}\right)^{2l_d}\right] = \mathbb{Q}(\tilde{S}_n^1 = \tilde{S}_n^2 = (j,k)) \cdot \mathbb{Q}(R_j^1 = R_j^2) \cdot \mathbb{Q}(U_k^1 = U_k^2).$$

□

Next, we state and prove an elementary lemma which will be useful to bound the second term in (3.23).

Lemma 3.7. *For each $x \in [0,1]$ let $\{Z_n(x)\}_n$ be a random walk over the set of the integers*

$$Z_n(x) = \zeta_1(x) + \cdots + \zeta_n(x),$$

where $\{\zeta_i\}_{i\in\mathbb{N}}$ are i.i.d. random variables with

$$\mathbb{Q}(\zeta_i(x) = 0) = x$$

and

$$\mathbb{Q}(\zeta_i(x) = -1) = \mathbb{Q}(\zeta_i(x) = 1) = \frac{1-x}{2}.$$

Then, for each $n \in \mathbb{N}$ fixed, the function $F_n : [1/2, 1] \to [0,1]$ given by

$$F_n(x) = \mathbb{Q}(Z_n(x) = 0). \tag{3.25}$$

is increasing .

Proof. We want to prove that for $1/2 \le x \le y \le 1$ we have $F_n(y) - F_n(x) \ge 0$. To do that we will write F as a sum and analyze the terms of the sum separately. Define

$$Y_n(x) := \#\{i \in [n] : \zeta_i(x) = 0\}, \tag{3.26}$$

so we can write

$$F_n(x) = \sum_{j=0}^{n} \mathbb{Q}(Z_n(x) = 0 | Y_n(x) = j)\mathbb{Q}(Y_n(x) = j).$$

Let us now analyze each part of the sum. We first observe that

$$a_{n-j} := \mathbb{Q}(Z_n(x) = 0 | Y_n(x) = j) = \begin{cases} 0 & \text{if } n-j \equiv 1 \mod 2 \\ \binom{n-j}{(n-j)/2} \times \frac{1}{2^{n-j}} & \text{if } n-j \equiv 0 \mod 2 \end{cases}.$$

Observe also that a_{n-j} is well defined because it does not depend on n or j, but only on $n-j$. Tt is also easy to see that for any $0 \leq k \leq n/2$ we have $a_{2k} > a_{2k+2}$ and thus $a_{2k} \geq a_{2\ell}$ for all $0 \leq k \leq \ell \leq n/2$.

Now, let us analyze the behavior of the function g_j given by

$$g_j(x) := \mathbb{Q}(Y_n(x) = j) = \binom{n}{j} x^j (1-x)^{n-j}. \tag{3.27}$$

Taking the derivative, we have

$$g_j'(x) = \binom{n}{j} x^j (1-x)^{n-j} \left(\frac{j}{x} - \frac{n-j}{1-x} \right),$$

so that, for $|y-x|$ sufficiently small and $y > x > 1/2$, we have

$$g_j(y) - g_j(x) < 0, \text{ if } j \leq nx, \tag{3.28}$$

$$g_j(y) - g_j(x) > 0, \text{ if } j > nx. \tag{3.29}$$

Let $N = \{0 \leq j \leq n : j \equiv n \mod 2\}$; then

$$F_n(y) - F_n(x) = \sum_{j \in N} [g_j(y) - g_j(x)] a_{n-j},$$

and using the fact that $\{a_{2\ell}\}_{0 \leq \ell \leq n/2}$ is decreasing, and (3.28) and (3.29) we have

$$\begin{aligned} F_n(y) - F_n(x) &= \sum_{j \leq nx, j \in N} [g_j(y) - g_j(x)] a_{n-j} + \sum_{j > nx, j \in N} [g_j(y) - g_j(x)] a_{n-j} \\ &\geq \left[\sum_{j \leq nx, j \in N} [g_j(y) - g_j(x)] + \sum_{j > nx, j \in N} [g_j(y) - g_j(x)] \right] a_{n - \lfloor nx \rfloor} \\ &\geq \left[\sum_{j \in N} g_j(y) - \sum_{j \in N} g_j(x) \right] a_{n - \lfloor nx \rfloor}. \end{aligned}$$

Now note that $\sum_{j \in N} g_j(y)$ can be obtained as the sum of the expansion of two binomials

$$\sum_{j \in N} g_j(y) = \frac{1}{2}[(y - (1-y))^n + (y + (1-y))^n] = \frac{1 + (2y-1)^n}{2}$$

and hence $\sum_{j \in N} g_j(x)$ is an increasing function in $[1/2, 1]$. It follows that $F_n(y) - F_n(x) \geq 0$. □

Proof of Theorem 3.2. We want to show that, the value of λ is maximal when the positive entries of the vector are packed in m coordinates. Let

$$\mathcal{A} := \left\{ q = (q_1, \dots, q_d) : 0 \leq q_i \leq \frac{1}{m},\ \forall i = 1, \dots, d \text{ and } \sum q_i = 1 \right\},$$

and given $q \in \mathcal{A}$ let

$$B(q) := \# \left\{ i \in [d] : q_i \notin \{0, 1/m\} \right\}.$$

With that in mind we will define a packing algorithm $A : \mathcal{A} \to \mathcal{A}$ which increases the value of λ in each step.

Let $q \in \mathcal{A}$. If $B(q) = 0$, then define $A(q) = q^*$. If $B(q) > 0$, then necessarily $B(q) \geq 2$, and suppose without loss of generality that neither q_1 nor q_2 belongs to $\{0, 1/m\}$, so we define $A(q) = (q_1', \dots, q_d')$ where $q_i' = q_i$ for all $i \geq 3$, and

$$\begin{aligned} &\text{for } q_1 + q_2 \leq 1/m, \text{ we let } q_1' = q_1 + q_2 \text{ and } q_2' = 0, \\ &\text{for } q_1 + q_2 > 1/m, \text{ we let } q_1' = q_1 + q_2 - 1/m \text{ and } q_2' = 1/m, . \end{aligned}$$

We claim that the this algorithm has the following properties

1. if $B(q) = 0$ then $B(A(q)) = 0$;
2. if $B(q) > 0$ then $B(A(q)) < B(q)$;
3. $\lambda(A(q)) \geq \lambda(q)$.

Properties 1 and 2 follow from the definition and we proceed with the proof that Property 3 holds.

We now compare the probabilities of the random walks associated with q and $A(q)$. By Lemma 3.6 we have

$$\mathbb{Q}(S_n^1(q) = S_n^2(q)) = \sum_{\substack{(j,k) \in \mathbb{N}^2 \\ j+k=n}} \mathbb{Q}(\tilde{S}_n^1(q) = \tilde{S}_n^2(q) = (j,k)) \mathbb{Q}(R_j^1(q) = R_j^2(q)) \mathbb{Q}(U_k^1(q) = U_k^2(q)).$$

Observe that $\mathbb{Q}(U_k^1(q) = U_k^2(q))$ depends only on the last $d-2$ coordinates of q, so that

$$\mathbb{Q}\Big(U_k^1(q) = U_k^2(q)\Big) = \mathbb{Q}\Big(U_k^1(A(q)) = U_k^2(A(q))\Big).$$

Analogously, $\mathbb{Q}(\tilde{S}_n^1(q) = \tilde{S}_n^2(q) = (j,k))$ depends only on $q_1 + q_2$ and since this sum is invariant by A we have

$$\mathbb{Q}\Big(\tilde{S}_n^1(q) = \tilde{S}_n^2(q) = (j,k)\Big) = \mathbb{Q}\Big(\tilde{S}_n^1(A(q)) = \tilde{S}_n^2(A(q)) = (j,k)\Big).$$

We will now prove that

$$\mathbb{Q}\Big(R_j^1(q) = R_j^2(q)\Big) \leq \mathbb{Q}\Big(R_j^1(A(q)) = R_j^2(A(q))\Big), \tag{3.30}$$

and it will follow that

$$\lambda(q) = \sum_{n=1}^{\infty} \mathbb{Q}\Big(S_n^1(q) = S_n^2(q)\Big) \leq \sum_{n=1}^{\infty} \mathbb{Q}\Big(S_n^1(A(q)) = S_n^2(A(q))\Big) = \lambda(A(q)).$$

To prove (3.30) we observe that for each $q \in \mathcal{A}$, $R_n^1(q) - R_n^2(q)$ has the distribution of a lazy random walk $\{Z_n(x)\}_n$ as defined in Lemma 3.7 with $x = \frac{q_1^2+q_2^2}{(q_1+q_2)^2} \geq 1/2$. Analogously $R_n^1(A(q)) - R_n^2(A(q))$ has the same distribution as $\{Z_n(x')\}_n$, with $x' = \frac{(q_1')^2+(q_2')^2}{(q_1'+q_2')^2} \geq x$. Now, (3.30) follows from Lemma 3.7.

Finally, we observe that for all $q \in \mathcal{A}$ we have $A^d(q) = q^*$, where A^d is the d-th iterated of A. Hence, by Property 3, we get $\lambda(q^*) = \lambda(A^d(q)) \geq \lambda(q)$ and this finishes the proof.

□

3.4 Discussion

We do not know whether Condition 2 of Theorem 3.1, the upper bound on the probabilities p_i, is only a technical limitation or whether a large discrepancy on the anisotropy prevents the system to have a mean-field behavior even in arbitrary large dimensions. In any case, the isotropic case shows that some bound on the probabilities p_i must be required as we explain now.

One of the results in [CD83], states that the isotropic critical point satisfies $p_c(d) \geq 1/d + 1/(2d^3) + o(1/d^3)$. Let now each $p_i = 1/d_0 + 1/(3d_0^3)$, and take d_0 so that $p_i < p_c(d_0)$. In this case $\varepsilon = 1/(3d_0^2)$ and $p_i = \sqrt{3\varepsilon}(1+\varepsilon)$.

A natural related question is whether anisotropic non-oriented percolation has the same limiting critical surface behavior, i.e., under which conditions can we guarantee that critical surfaces stay close to the isotropic critical points in high dimensions.

Chapter 4

On the Dimensional Crossover Critical Exponent

Another subject on the study of the anisotropy arises in the physics literature as the dimensional crossover problem. The term crossover relates to the study of percolative systems on $(d + s)$-dimensional lattices, where the d-dimensional parameter p is close to $p_c(d)$ from below and the s-dimensional parameter q is small. Similar anisotropic ferromagnetic models have also been considered in the mathematical physics literature, see the works [FMMPV14], [FMMPV15] and [MPS02].

On this subject, the so-called dimensional crossover critical exponent ψ, for bond percolation on $\mathbb{Z}^{d+s}$ is introduced in [RS79], and it is expected to coincide with another critical exponent γ. They are briefly described as follow. Consider anisotropic bond percolation on $\mathbb{Z}^d \times \mathbb{Z}^s$ where edges parallel to $\mathbb{Z}^d$ are open with probability $p < p_c(\mathbb{Z}^d)$ and edges parallel to $\mathbb{Z}^s$ are open with probability q, independently of all others. For each parameter $p < p_c(\mathbb{Z}^d)$, let $q_c(p)$ be the critical point such that percolation occurs for values of q above $q_c(p)$ but does not occurs for any value below. Independent of the dimension s, it is believed that $q_c(p)$ goes to 0 as $p \uparrow p_c$ in the manner $|p - p_c|^{\psi}$. For isotropic percolation with parameter $p < p_c$ on $\mathbb{Z}^d$, the mean size of the open cluster containing the origin $\chi(p)$, is believed to diverges as $|p - p_c|^{-\gamma}$ as $p \uparrow p_c$.

In the work [SS17, SSc], the authors proven that, if γ and ψ exists, then $\psi \leq \gamma$. In this chapter, we give the upper bound $q_c(p) \leq 8d^2(p_c(\mathbb{Z}^d) - p)$. In particular, combining this to known results concerning the critical exponent γ (see [FH17] for details), we conclude that for $d \geq 11$, ψ exists and is equal to γ, as expected.

4.1 The Model and Results

We consider non-oriented anisotropic bond percolation on the graph $(\mathbb{Z}^{d+s}, E(\mathbb{Z}^{d+s}))$, where $E(\mathbb{Z}^{d+s})$ is the set of edges between nearest neighbors of $\mathbb{Z}^{d+s}$. We simplify notation and denote this graph by $\mathbb{Z}^{d+s} = \mathbb{Z}^d \times \mathbb{Z}^s$. An edge of $\mathbb{Z}^{d+s}$ is called a $\mathbb{Z}^d$-edge (respectively a $\mathbb{Z}^s$-edge) if it joins two vertices which differ only in their $\mathbb{Z}^d$ (respectively $\mathbb{Z}^s$) component. Probability is introduced as follows: given two parameters $p, q \in [0,1]$, we declare each $\mathbb{Z}^d$-edge open with probability p and each $\mathbb{Z}^s$-edge open with probability q, independently of all others. This model is described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{p,q})$ where $\Omega = \{0,1\}^{E(\mathbb{Z}^{d+s})}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder sets in Ω and $\mathbb{P}_{p,q} = \prod_{e \in E} \mu(e)$, where $\mu(e)$ is Bernoulli measure with parameter p or q according to e been a $\mathbb{Z}^d$- edge or a $\mathbb{Z}^s$-edge respectively.

Given two vertices $u, v \in \mathbb{Z}^{d+s}$, we say that u and v are connected in the configuration ω if there exists an open path in $\mathbb{Z}^{d+s}$ starting in u and ending in v. The event where v and u are connected is denoted by $\{\omega \in \Omega : v \leftrightarrow u \text{ in } \omega\}$ and we write $C(\omega) = \{u \in \mathbb{Z}^{d+s} : u \leftrightarrow 0 \text{ in } \omega\}$ for the open cluster containing the origin. We denote by $\theta(p,q) = \mathbb{P}_{p,q}(\omega \in \Omega : |C(\omega)| = \infty)$ the main macroscopic function in percolation theory and denote the mean size of the open cluster by $\chi(p,q) = \mathbb{E}_{p,q}(|C(\omega))|)$. Whenever necessary we shall write $\chi_p(d)$ and $p_c(d)$ for the expected cluster size and critical threshold on $\mathbb{Z}^d$ with a single parameter $p \in (0,1)$, respectively.

It is easy to see, by a standard coupling argument, that $\theta(p,q)$ is a monotone non-decreasing function of the parameters p and q. This enable us to define the function $q_c : [0,1] \to [0,1]$, where

$$q_c(p) = \sup\{q : \theta(p,q) = 0\}. \tag{4.1}$$

The function $q_c(p)$ is continuous and strictly decreasing (see [CLS14]) and we are interested in understanding its behavior as $p \uparrow p_c(d)$.

A major problem in percolation theory is the existence and determination of critical exponents. For instance, quantities such as $\chi_p(d)$ are believed to diverge as $p \uparrow p_c(d)$ in the manner of a power law in $|p - p_c(d)|$, whose exponent is called a critical exponent (see Chapter 9 in [Gri99] for details). More precisely, it is believed that there exists a $\gamma = \gamma(d) > 0$ such that

$$\chi_p(d) \approx |p - p_c(d)|^{-\gamma},$$

when $p \uparrow p_c(d)$. Here the relation $a(p) \approx b(p)$ means $l$og equivalence, i.e., $\frac{\log a(p)}{\log b(p)} \to 1$ when $p \uparrow p_c(d)$.

In [LS72] the authors introduce another critical exponent, the so-called dimensional crossover critical exponent for the Ising Model, which is related to the function in Equation (4.1). The same exponent is introduced in [RS79] for bond percolation and it is expected that:

Conjecture 4.1. *There exists a critical exponent* $\psi = \psi(d) > 0$*, depending only on* d*, such that*

$$q_c(p) \approx |p - p_c(d)|^{\psi}.$$

Moreover, if $\gamma(d)$ *exists, then* $\psi(d) = \gamma(d)$*.*

We highlight a few papers that investigate this matter. In [GCGR81] the authors examine bond percolation on $\mathbb{Z}^3 = \mathbb{Z}^2 \times \mathbb{Z}$. Here $\mathbb{Z}^2$-edges are open with probability p and $\mathbb{Z}$-edges are open with probability $q = Rp$, where R is the anisotropy parameter. By means of a simulation, the authors estimate $\psi(2)$ by 2.3 ± 0.1, which is compatible with the critical exponent $\gamma(2)$, which is expected to be $\frac{43}{18}$ (see [Sta81] for example). In [RS79] the authors study a percolation process on $\mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{Z}$ where $\mathbb{Z}^{d-1}$-edges are open with probability p and $\mathbb{Z}$-edges are open with probability $q = Rp$. Simulated data then indicate that in the limit $1/R \to 0$, the crossover exponent ψ is equal to 1 for all d. In the opposite limit $R \to 0$, their analysis suggests that $\psi(d-1) \neq \gamma(d-1)$. This result was later contradicted by Redner and Coniglio [RC80], where the authors argue the opposite relation, that is, $\psi(d-1) = \gamma(d-1)$.

In [SS17, SSc] the authors proved that, if $\gamma(d)$ and $\psi(d)$ exist, then $\psi(d) \leq \gamma(d)$. In this chapter we are concerned with the reversed inequality. The following theorem gives an upper bound for the critical curve $q_c(p)$ when p is sufficiently close to $p_c(d)$, providing a partial answer in that direction.

Theorem 4.1. *Consider an anisotropic bond percolation process on* $\mathbb{Z}^d \times \mathbb{Z}^s$*,* $d, s \geq 1$*, with parameters* (p, q) *and* $p_c(d) - p > 0$*, sufficiently small. If the pair* (p, q) *satisfies*

$$q \geq 8d^2(p_c(d) - p),$$

then there is a.s. an infinite open cluster in $\mathbb{Z}^{d+s}$*.*

Theorem 4.1 gives an upper bound for $q_c(p)$, *i.e.*, $q_c(p) \leq 8d^2(p_c(d) - p)$. This in turn gives

$$\frac{\log(q_c(p))}{\log(p_c(d) - p)} \geq \frac{\log(8d^2)}{\log(p_c(d) - p)} + 1.$$

Taking limits when $p \uparrow p_c(\mathbb{Z}^d)$ we obtain

$$\liminf_{p \uparrow p_c(d)} \frac{\log(q_c(p))}{\log |p - p_c(d)|} \geq 1. \tag{4.2}$$

In [SS17, SSc] it is shown that

$$\limsup_{p \uparrow p_c(d)} \frac{\log(q_c(p))}{\log |p - p_c(d)|} \leq \gamma(d),$$

whenever $\gamma(d)$ exists. The results in [FH17] imply that, for the nearest-neighbor percolation model in dimensions $d \geq 11$, the exponent $\gamma(d)$ exists and is equal to 1. This immediately implies that $\psi(d)$ exists and is equal to $\gamma(d)$ for all $d \geq 11$. We have just proved the following result.

Theorem 4.2. *Consider an anisotropic bond percolation process on $\mathbb{Z}^d \times \mathbb{Z}^s$, $d \geq 11, s \geq 1$, with parameters (p, q). Then the critical exponent $\psi(d)$ exists and is equal to $\gamma(d) = 1$. Hence Conjecture 4.1 is true in case $d \geq 11$.*

4.2 Proof of Theorem 1

It is sufficient to prove the theorem for the case $s = 1$. Let U be the set of unit vectors in $\mathbb{Z}^d$, so that $|U| = 2d$. We will denote the vertices of $\mathbb{Z}^{d+1}$ by $(\mathbf{u}, t)$ where $\mathbf{u}$ and t are the $\mathbb{Z}^d$ and $\mathbb{Z}$ component respectively. We also introduce the notation $[\mathbf{e}, t] \in E(\mathbb{Z}^{d+1})$ for the edge $\langle(\mathbf{u}_1, t), (\mathbf{u}_2, t)\rangle$ whenever $\mathbf{e} = \langle \mathbf{u}_1, \mathbf{u}_2\rangle \in E(\mathbb{Z}^d)$.

Consider now the multigraph obtained from the vertices of $\mathbb{Z}^{d+1} = \mathbb{Z}^d \times \mathbb{Z}$ where every $\mathbb{Z}$-edge is replaced by $2d$ other edges indexed by U. We denote this graph by $\mathbb{Z}_U^{d+1}$ and introduce percolation on it as follows: as before, every $\mathbb{Z}^d$-edge is open with probability p and every $\mathbb{Z}$-edge is open with probability $\bar{q}$ independently of all others, with $\bar{q}$ satisfying

$$(1 - q) = (1 - \bar{q})^{2d}. \tag{4.3}$$

It is clear that with these parameters, the distribution of the cluster of the origin in $\mathbb{Z}^{d+1}$ with law $\mathbb{P}_{p,q}$ is the same as that in $\mathbb{Z}_U^{d+1}$ with law $\mathbb{P}_{p,\bar{q}}$.

The proof consists of a dynamical coupling between two percolation processes. That is, for every configuration $\omega \in \{0,1\}^{E(\mathbb{Z}_U^{d+1})}$ with law $\mathbb{P}_{p,\bar{q}}$, we shall obtain a configuration $\omega' \in \{0,1\}^{E(\mathbb{Z}^d)}$ with law $\mathbb{P}_r$ on $\mathbb{Z}^d$ where $r = p + \bar{q}p(1-p)$. To accomplish this we construct a sequence $\{\eta(\mathbf{e})\}_{\mathbf{e}\in\mathbf{E}(\mathbb{Z}^\mathbf{d})}$ of independent 0-1 valued random variables with parameter r, a sequence $E_i = (A_i, B_i)$ of ordered pairs of subsets of $E(\mathbb{Z}^d)$, a sequence S_i of subsets of $\mathbb{Z}^{d+1}$ and a sequence S_i^π of subsets of $\mathbb{Z}^d$, with $i \in \mathbb{N}$. We proceed as follows.

Given $(\mathbf{u}, t) \in \mathbb{Z}_U^{d+1}$, we use the notation $\langle(\mathbf{u}, t), (\mathbf{u}, t+1)\rangle_v$ for the $\mathbb{Z}$-edge between $(\mathbf{u}, t)$ and $(\mathbf{u}, t+1)$ indexed by $v \in U$. We say there is a *v-hook* at vertex $(\mathbf{u}, t) \in \mathbb{Z}^{d+1}$ if the edges $\langle(\mathbf{u}, t), (\mathbf{u}, t+1)\rangle_v$ and $\langle(\mathbf{u}, t+1), (\mathbf{u}+v, t+1)\rangle$ are open. We consider an arbitrary, but fixed, ordering of $E(\mathbb{Z}^d)$ and let $(\mathbf{0}, 0)$ be the origin of $\mathbb{Z}^{d+1}$. We set $E_0 = (\emptyset, \emptyset)$, $S_0 = \{(\mathbf{0}, 0)\}$ and $S_0^\pi = \{\mathbf{0}\}$. Let $f_1 = \langle \mathbf{0}, \mathbf{0} + v\rangle$, $v \in U$, be the first $\mathbb{Z}^d$-edge, in the fixed ordering, incident to S_0^π. We set $\eta(f_1) = 1$ if exactly one of the following two conditions hold:

(a) $[f_1, 0]$ is open;

(b) $[f_1, 0]$ is closed and there is a *v-hook* at vertex $(\mathbf{0}, 0)$.

We set

$$E_1 = \begin{cases} (f_1, \emptyset) & \text{if} \quad \eta(f_1) = 1, \\ (\emptyset, f_1) & \text{if} \quad \eta(f_1) = 0, \end{cases}$$

If $\eta(f_1) = 1$, we set

$$S_1 = \begin{cases} S_0 \cup \{(v, 0)\} & \text{if condition (a) holds,} \\ S_0 \cup \{(v, 1)\} & \text{if condition (b) holds,} \end{cases}$$

and

$$S_1^\pi = S_0^\pi \cup \{v\}.$$

Whenever $\eta(f_1) = 0$, we set $S_1 = S_0$ and $S_1^\pi = S_0^\pi$.

Suppose the sequences $\{E_i\}, \{S_i\}$ and $\{S_i^\pi\}$ are defined up to the index $i = n-1$. We then define E_n, S_n and S_n^π as follows. At first, let $\pi : \mathbb{Z}^d \times \mathbb{Z} \to \mathbb{Z}$ be the projection of $\mathbb{Z}^d \times \mathbb{Z}$ onto $\mathbb{Z}$, that is, $\pi((\mathbf{u}, t)) = t$, and consider the bijection between S_{n-1} and S_{n-1}^π given by

$$\theta \colon S_{n-1} \longrightarrow S_{n-1}^\pi,$$
$$(\mathbf{u}, t) \longmapsto \mathbf{u}.$$

Let f_n be the earliest $\mathbb{Z}^d$-edge in the fixed ordering with the property that $f_n \cap S_{n-1}^\pi \neq \emptyset$, $f_n \cap (S_{n-1}^\pi)^c \neq \emptyset$ and $f_n \notin A_{n-1} \cup B_{n-1}$. Assume, with no loss of generality, that $f_n = \langle \mathbf{u}_{n-1}, \mathbf{u}_n \rangle$, where $\mathbf{u}_n = \mathbf{u}_{n-1} + v$ for some $v \in U$, with $\mathbf{u}_{n-1} \in S_{n-1}^\pi$ and $\mathbf{u}_{n-1} + v \in (S_{n-1}^\pi)^c$.

We set $\eta(f_n) = 1$ if exactly one of the following two conditions hold:

(a) $[f_n, \pi(\theta^{-1}(\mathbf{u}_{n-1}))]$ is open,

(b) $[f_n, \pi(\theta^{-1}(\mathbf{u}_{n-1}))]$ is closed and there is a *v-hook* at vertex $(\mathbf{u}_{n-1}, \pi(\theta^{-1}(\mathbf{u}_{n-1})))$.

We then set

$$E_n = \begin{cases} (A_{n-1} \cup f_n, B_{n-1}) & \text{if} \quad \eta(f_n) = 1, \\ (A_{n-1}, B_{n-1} \cup f_n) & \text{if} \quad \eta(f_n) = 0, \end{cases}$$

If $\eta(f_n) = 1$, we set

$$S_n = \begin{cases} S_{n-1} \cup \{(\mathbf{u}_n, \pi(\theta^{-1}(\mathbf{u}_{n-1})))\} & \text{if condition (a) holds,} \\ S_{n-1} \cup \{(\mathbf{u}_n, \pi(\theta^{-1}(\mathbf{u}_{n-1})) + 1)\} & \text{if condition (b) holds,} \end{cases}$$

and

$$S_n^\pi = S_{n-1}^\pi \cup \{\mathbf{u}_n\}.$$

In case $\eta(f_n) = 0$, we set $S_n = S_{n-1}$ and $S_n^\pi = S_{n-1}^\pi$.

Now that the dynamical coupling is well defined, we make some observations. By construction, there is a bijection between the sets A_n and $S_n \backslash \{(\mathbf{0}, 0)\}$ and the sets A_n, B_n and S_n are non-decreasing. So we can define $A_\infty := \lim A_n$, $B_\infty := \lim B_n$ and $S_\infty := \lim S_n$. We also observe that all edges in A_∞ form a connected set containing the origin of $\mathbb{Z}^d$ and also that S_∞ is a subset of the cluster of the origin of $\mathbb{Z}^{d+1}$ in the process with law $\mathbb{P}_{p,q}$.

To complete the description of the law on $\{0,1\}^{E(\mathbb{Z}^d)}$, we dispose of a collection of i.i.d. Bernoulli random variables $\{\eta(e)\}$, for all $e \in E(\mathbb{Z}^d) \backslash (A_\infty \cup B_\infty)$, with parameter $r = p + \bar{q}p(1-p)$, independent from all other random variables used previously.

Now, since the random variables $\{\eta(e)\}_{e \in E(\mathbb{Z}^d)}$ are independent, the probability measure generated by them is exactly the same as that of an independent bond percolation process on $\mathbb{Z}^d$ with parameter $r = p + \bar{q}p(1-p)$. This means that $P_{p,q}(|A_\infty| = \infty) = \theta(r)$.

Taking $q \geq 8d^2(p_c(d) - p)$, observing that $1 - p > 1/2$ and taking $(p_c(d) - p)$ sufficiently small, say $p \in (\frac{1}{2d}, p_c(d))$, we estimate

$$r = p + \bar{q}p(1-p) = p + \left[1 - (1-q)^{1/2d}\right] p(1-p) > p + \frac{q}{2d}\frac{1}{4d} \geq p_c(d).$$

To conclude, we observe that, for these values of q, we have

$$\theta(p,q) \geq \mathbb{P}_{p,q}(|S_\infty| = \infty) = P_{p,q}(|A_\infty| = \infty) = \theta(r) > 0.$$

4.3 Discussion

Under the hypothesis that $\psi(d)$ exists, the expression in (4.2) already shows that $\psi(d) \geq 1$. This bound should saturate when the dimension is above the so-called critical dimension for percolation ($d_c = 6$), but is not expected to be sharp for $2 \leq d \leq 5$.

An interesting feature of this dynamical coupling is that, with a minor modification, the same result holds for the bilayered graph $\mathbb{Z}^d \times \{0,1\}$ rather than the full graph $\mathbb{Z}^{d+1}$. One wonders if the bound obtained here is sharp for those bilayered graphs, even in low dimensions.

We also observe that the bound obtained in Theorem 1 is not directly related to $\chi_p(d)$ and we expect that the following result should be true, with a direct proof.

Conjecture 4.2. *Consider an anisotropic bond percolation process on $\mathbb{Z}^d \times \mathbb{Z}^s$, $d, s \geq 1$, with parameters (p, q) and $p_c(d) - p > 0$, sufficiently small. There exists a constant β such that if the pair (p, q) satisfies*

$$q > \frac{\beta}{\chi_p(d)},$$

then there is a.s. an infinite open cluster in $\mathbb{Z}^{d+s}$.

In mean-field, that is, when a d-regular tree is considered instead of $\mathbb{Z}^d$, is not difficult to prove that the conjecture above holds. To see this, let $\bar{\chi}_p(d) = \mathbb{E}\,|\{x \in C(\omega)\backslash\{0\}\ ;\ deg(x) = 1\}|$, i.e., the expected number of leaves of the open cluster containing the origin. Clearly $\bar{\chi}_p(d) = (1-p)^d \chi_p(d)$, and therefore, for $\beta = 1/(1-1/d)^d$, by a simple branching process comparison argument, we have that, if $q > \beta/\chi_p(d)$, then$q > 1/\bar{\chi}_p(d)$ and thus, almost sure there is an infinite open cluster.

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